TTK4130 Formula Sheet

Part I
Modeling

1 Some useful models

1.1 Mass spring damper

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{1}$$
$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0 \tag{2}$$

Note: A driving force $F(t)$ could be included on the right hand side.

1.2 Capacitor and inductor equations

$$i_C(t) = C\frac{dv_C}{dt}(t) \tag{3}$$
$$v_L(t) = L\frac{di_L}{dt}(t) \tag{4}$$

1.3 Pendulum equation

$$\ddot{\theta} + \frac{g}{l}\sin(\theta) = 0 \tag{5}$$

For $\theta \ll 1$ we get the approximation

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \tag{6}$$

with period

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}} \tag{7}$$

2 Passivity

A system consisting of a parallel or feedback interconnection of passive subsystems, is itself passive.

Definition: If the following inequality is satisfied for all u and $T \geq 0$, then the system is passive.

$$\int_0^T y(t)u(t)dt \geq -E_0 \tag{8}$$

Note that if the roles of u and y are reversed, i.e. y is taken to be the input and u the output, then the inequality still holds.

Interpretation of this definition based on energy conservation: The product uy has dimension power, thus we can think about the integral as the energy supplied by u or, equivalently, the energy absorbed by the system.

1. If $\int_0^T y(t)u(t)dt \geq 0$, energy is only absorbed. This inequality will hold for a passive memoryless system (e.g. a circuit with only a resistor).
2. If $\int_0^T y(t)u(t)dt \geq -E_0$, the system can supply a limited amount of energy to the outside, due to initial conditions of energy storage elements, such as capacitors and inductors.
3. If $\int_0^T y(t)u(t)dt \to -\infty$, the system is active.

A system is passive iff its transfer function is positive real.

Definition of a positive real transfer function:

1. All poles of $H(s)$ have real parts less than or equal to zero.
2. $\operatorname{Re} H(j\omega) \geq 0 \ \forall \ \omega$ that are not poles of $H(s)$.
3. If $j\omega_0$ is a pole, it is simple and $\operatorname{Res}_{s=j\omega_0} H(s) = \lim_{s \to j\omega_0}(s - j\omega_0)H(s)$ is real and positive. If $H(s)$ has a pole at infinity, it is simple, and $R_\infty = \lim_{\omega \to \infty} \frac{H(j\omega)}{j\omega}$ exists, and is real and positive.

There is also the *storage function* approach for determining passivity. Consider a state space model $\dot{x} = f(x, u)$, $y = h(x)$. If there exists a storage function $V(x) \geq 0$ and dissipation function $g(x) \geq 0$, such that

$$\dot{V} = u^\top y - g(x), \tag{9}$$

then the system is passive.

Part II
Motors and actuators

Part III
Dynamics

3 Rigid body kinematics

3.1 Rotation matrices

Let $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ and $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ be the orthogonal bases of two coordinate frames. Then the coordinate transformation from frame b to frame a is given by

$$\mathbf{R}_b^a = \begin{pmatrix} \mathbf{b}_1^a & \mathbf{b}_2^a & \mathbf{b}_3^a \end{pmatrix} \tag{10}$$

That is,

$$\mathbf{v}^a = \mathbf{R}_b^a \mathbf{v}^b \tag{11}$$

The $\mathbf{R}_b^a$ matrix can also be thought of simply as a rotation of a vector. For a second, forget that the b frame exists, and think only about the a frame. A simple example is

$$\mathbf{b}_1^a = \mathbf{R}_b^a \mathbf{a}_1^a \tag{12}$$

which shows that the first basis vector of the a frame is rotated to the first basis vector of the b frame, when everything is referred to frame a. In this sense the matrix represents a rotation from a to b.

3.2 Homogeneous transformation matrices

$$T_b^a = \begin{pmatrix} R_b^a & r_{ab}^a \\ \mathbf{0}^\top & 1 \end{pmatrix} \tag{13}$$

where r_{ab}^a is the origin of frame b in a coordinates.

3.3 Differentiation of vectors and matrices

$$\frac{{}^a d}{dt}\vec{u} = \frac{{}^b d}{dt}\vec{u} + \vec{\omega}_{ab} \times \vec{u}, \tag{14}$$

Skew-symmetric form of coordinate vector

$$\mathbf{u}^\times := \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \tag{15}$$

Skew-symmetric form of angular velocity vector

$$(\boldsymbol{\omega}_{ab}^a)^\times = \dot{\mathbf{R}}_b^a (\mathbf{R}_b^a)^\top \tag{16}$$

Coordinate transformation

$$(\boldsymbol{\omega}_{ab}^a)^\times = \mathbf{R}_b^a (\boldsymbol{\omega}_{ab}^b)^\times \mathbf{R}_a^b, \quad \mathbf{R}_a^b = (\mathbf{R}_b^a)^{-1} = (\mathbf{R}_b^a)^\top \tag{17}$$

3.4 Kinematic differential equations

For Euler angles, when the middle rotation is $\frac{\pi}{2}$ radians, the E matrix in the kinematic differential equation is singular. This is because this rotation moves the third axis of rotation to the first axis of rotation, such that we lose a degree of freedom.

3.5 Coordinate systems

Cylindrical coordinates (r, θ, z):

$$x = r\cos(\theta) \tag{18}$$
$$y = r\sin(\theta) \tag{19}$$
$$z = z \tag{20}$$
$$r = \sqrt{x^2 + y^2} \tag{21}$$
$$\theta = \begin{cases} 0 \text{ if } x = 0 \text{ and } y = 0 \\ \arctan(\frac{y}{x}) \text{ if } x > 0 \\ \arctan(\frac{y}{x}) + \pi \text{ if } x < 0 \text{ and } y \geq 0 \\ \arctan(\frac{y}{x}) - \pi \text{ if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} \text{ if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} \text{ if } x = 0 \text{ and } y < 0 \end{cases} \tag{22}$$
$$dV = dxdydz = rdrd\theta dz \tag{23}$$

Spherical coordinates (r, φ, θ), φ angle from z-axis:

$$x = r\sin(\varphi)\cos(\theta) \tag{24}$$
$$y = r\sin(\varphi)\sin(\theta) \tag{25}$$
$$z = r\cos(\varphi) \tag{26}$$
$$r = \sqrt{x^2 + y^2 + z^2} \tag{27}$$
$$\theta \text{ defined as above.} \tag{28}$$
$$dV = dxdydz = r^2\sin(\varphi)drd\varphi d\theta \tag{29}$$

Note that $r \geq 0$, $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi$ (typical definition). The definition of θ above has the range $(-\pi, \pi]$, to obtain only positive results 2π can be added to negative values.

Next I list the position, velocity and acceleration vectors for these coordinate systems. Cylindrical first:

$$\vec{r} = r\vec{e}_r + z\vec{e}_z \tag{30}$$
$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + \dot{z}\vec{e}_z \tag{31}$$
$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta + \ddot{z}\vec{e}_z \tag{32}$$

Spherical:

$$\vec{r} = r\vec{e}_r \tag{33}$$

$$\vec{v} = \dot{r}\vec{e}_r + r\,\dot{\theta}\vec{e}_\theta + r\,\dot{\varphi}\,\sin\theta\vec{e}_\varphi \tag{34}$$

$$\vec{a} = \left(a - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta\right)\vec{e}_r \tag{35}$$

$$+\left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta\right)\vec{e}_\theta \tag{36}$$

$$+\left(r\ddot{\varphi}\,\sin\theta + 2\dot{r}\,\dot{\varphi}\,\sin\theta + 2r\,\dot{\theta}\,\dot{\varphi}\,\cos\theta\right)\vec{e}_\varphi \tag{37}$$

3.6 The center of mass

The mass of a rigid body b is

$$m = \int_b dm = \int_b \rho(x,y,z)dV \tag{38}$$

The center of mass $\vec{r}_c$ is defined as

$$\vec{r}_c = \frac{1}{m}\int_b \vec{r}_p dm \tag{39}$$

where $\vec{r}_p$ is the position of a mass element dm that is fixed in frame b. The x–coordinate of the center of mass is given by

$$x_c = \frac{1}{m}\iiint\limits_b x_p\rho(x,y,z)dV \tag{40}$$

The definitions for y and z are exactly the same. Typically $(x_p, y_p, z_p) = (x, y, z)$.

3.7 Other useful formulas

Relation between linear and angular velocity

$$v = \omega r \tag{41}$$

4 Newton-Euler equations of motion

4.1 Kinetic energy

$$\mathcal{T} = \frac{1}{2}m(\mathbf{v}_c^b)^\top\mathbf{v}_c^b + \frac{1}{2}(\boldsymbol{\omega}_{ib}^b)^\top\mathbf{M}_{b/c}^b\boldsymbol{\omega}_{ib}^b \tag{42}$$

The subscript c denotes the center of mass and the superscript b denotes a coordinate vector/matrix in frame b. $\boldsymbol{\omega}_{ib}^b$ is the angular velocity of frame b relative to frame i. $\mathbf{M}_{b/c}^b$ is the inertia matrix of b about c, i.e. the inertia matrix of the rigid body about the center of mass.

4.2 Inertia matrix

$$\mathbf{M}^b_{b/c} = -\int_b (\mathbf{r}^b)^\times (\mathbf{r}^b)^\times dm = \int_b \left[(\mathbf{r}^b)^\top \mathbf{r}^b \mathbf{I} - \mathbf{r}^b (\mathbf{r}^b)^\top \right] dm \tag{43}$$

Fun facts: Swap the b superscripts with i on the right hand side to get $\mathbf{M}^i_{b/c}$. $\mathbf{M}^b_{b/c}$ is positive definite, since the kinetic energy $\mathcal{T} \geq 0$. Note that the integral above is a triple integral of a 3×3-matrix. About a specified axis the formula reduces to

$$I = \int_b (\mathbf{r}^b)^\top \mathbf{r}^b dm \tag{44}$$

4.3 Parallel axis theorem

The inertia matrix of b about a point o is given by

$$\mathbf{M}^b_{b/o} = \mathbf{M}^b_{b/c} - m(\mathbf{r}^b_g)^\times (\mathbf{r}^b_g)^\times = \mathbf{M}^b_{b/c} + m \left[(\mathbf{r}^b_g)^\top \mathbf{r}^b_g \mathbf{I} - \mathbf{r}^b_g (\mathbf{r}^b_g)^\top \right] \tag{45}$$

where $\mathbf{r}^b_g$ is the vector from the point o to the center of mass c. If o is the origin this corresponds to $\mathbf{r}^b_c$. In its simplest form with two parallel axes, the formula reduces to

$$I = I_c + md^2 \tag{46}$$

where I_c is the moment of inertia about the axis through the center of mass and d is the distance between the axes.

4.4 Other useful formulas

Relationships between torque, angular momentum and angular velocity

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{47}$$

$$\vec{\tau} = \frac{d\vec{L}}{dt} \tag{48}$$

$$\vec{\tau} = I\dot{\omega} \tag{49}$$

$$P = \tau\omega \tag{50}$$

5 Lagrangian dynamics

Lagrangian

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathcal{T}(\mathbf{q}, \dot{\mathbf{q}}, t) - \mathcal{U}(\mathbf{q}) \tag{51}$$

Lagrange's equation of motion

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i \tag{52}$$

Generalized force

$$Q_i = \sum_{k=1}^{N} \frac{\partial \vec{r}_k}{\partial q_i} \cdot \vec{F}_k \tag{53}$$

Part IV

Balance equations

6 Kinematics of flow

6.1 Material derivative

Let $\mathbf{x}$ be the position of some fluid particle, with velocity $\dot{\mathbf{x}} = \mathbf{v}$. Further, let φ be some quantity related to the particle, e.g. its temperature. The material derivative of φ is then defined as

$$\frac{D\varphi}{Dt} = \frac{\partial \varphi}{\partial t} + \mathbf{v}^\top \nabla \varphi \tag{54}$$

7 Mass, momentum and energy balances

7.1 Mass balance

Level of tank

$$\frac{d}{dt}(\rho A h) = w_1 - w_2 \tag{55}$$

$$\dot{h} = \frac{1}{\rho A}(w1 - w2) \tag{56}$$

Remember that the pressure at the bottom of a (open) tank is $\rho g h$.

7.2 Momentum balance

Bernoulli's equation for stationary frictionless flow along a streamline for an incompressible fluid

$$\frac{1}{2}(v_2^2 - v_1^2) + \frac{p_2 - p_1}{\rho} + (z_2 - z_1)g = 0 \tag{57}$$

See page 426-427 for relevant examples on momentum balances.

7.3 Energy balance

The total energy of a volume element dV is

$$\rho e dV, \tag{58}$$

where $e = u + \frac{1}{2}v^2 + \phi$, i.e. the sum of specific internal, kinetic and potential energy. Internal energy, enthalpy, heat capacities and temperature have the following relationships

$$h = c_p T \tag{59}$$

$$u = c_v T \tag{60}$$

Note that all quantities (except temperature) are specific (per unit mass) in these equations. The specific enthalpy is given by

$$h = u + \frac{p}{\rho}, \tag{61}$$

where u is the internal energy. See page 443-445 for relevant examples of energy balances.

Part V
Simulation

We are concerned with solving the IVP

$$\dot{y} = f(y, t), \quad y(t_0) = y_0 \tag{62}$$

The Jacobian of the system is defined as

$$J = \frac{\partial f}{\partial y}(y, t) \tag{63}$$

Note that the Jacobian is A for a linear, time-invariant system $\dot{x} = Ax + Bu$.

8 Stability functions

The stability of a numerical method is ensured if $|R(h\lambda_i)| \leq 1$ for all eigenvalues λ_i.

8.1 ERK methods

$$R_E(h\lambda) = \det\left[I - h\lambda(A - \mathbf{1}b^\top)\right], \text{ where } \mathbf{1} = (1, ..., 1)^\top \tag{64}$$

Note that $R_E(h\lambda)$ will be a polynomial in $h\lambda$ of order less than or equal to σ (the number of stages).

8.2 IRK methods

$$R(h\lambda) = \left[1 + h\lambda b^\top (I - h\lambda A)^{-1}\mathbf{1}\right] \tag{65}$$

$$R(h\lambda) = \frac{\det\left[I - h\lambda(A - \mathbf{1}b^\top)\right]}{\det(I - h\lambda A)} \tag{66}$$

9 Stability of RK methods

9.1 Aliasing

The *Nyquist frequency* is half of the sampling rate

$$\omega_{\text{Nyquist}} = \frac{1}{2} \cdot \frac{2\pi}{h}, \text{ where } h \text{ is the step size.} \tag{67}$$

Two systems oscillating at a low frequency $\omega < \omega_{\text{Nyquist}}$ and a high frequency $\omega + 2k\frac{\pi}{h} > \omega_{\text{Nyquist}}$ (k integer) will intercept at all sampling points, and therefore a solver will not be able to distinguish them. More specifically, the solver will believe that the system with higher frequency is the system with lower frequency, when fitting the curve.

9.2 A- and L-stability

Definition: A method is A-stable if $|R(h\lambda)| \leq 1 \;\forall\; \operatorname{Re}\lambda \leq 0$.
This definitions means that an A-stable method is stable for all stable test systems $\dot{y} = \lambda y$. Note also that no ERK method can be A-stable, since $|R_E(h\lambda)| \to \infty$ as $|\lambda| \to \infty$.
Definition: A method is L-stable if it is A-stable and $|R(j\omega h)| \to 0$ when $\omega \to \infty \;\forall$ systems $\dot{y} = j\omega y$.
A-stable methods can suffer from aliasing for systems with fast dynamics (faster than Nyquist frequency), whereas an L-stable method will simply damp out these fast dynamics. This means that the L-stable method might give a better qualitative representation of what the actual solution looks like.

9.3 Stiffly accurate methods and algebraic stability

Definition: A method is stiffly accurate if

$$\det(A) \neq 0 \text{ and } b = A^\top [0, 0, ..., 1]^\top \tag{68}$$

Note: A-stable and stiffly accurate $\implies$ L-stable.
Definition: A method is algebraically stable if

$$M = \operatorname{diag}(b)A + (\operatorname{diag}(b)A)^\top + bb^\top \tag{69}$$

is positive semi-definite. Note: Algebraically stable $\implies$ A-stable.

10 DAEs

A fully implicit ODE, $F(\dot{x}, x, u) = 0$ is a DAE if $\frac{\partial F}{\partial \dot{x}}$ is rank deficient (note that the partial derivative is with respect to $\dot{x}$, not x).

Method for finding index (one way to go about it):

- Differentiate algebraic equation(s) $g(x, z)$ until you can solve for the algebraic variable(s).
- The DAE system is now index 1. If you differentiated p times in the previous step, the index is $p + 1$.

11 Advanced topics

11.1 Automatic adjustment of step size

The step size h can be selected so that the desired accuracy is obtained. Variable-step methods are useful for stiff systems (large spread in eigenvalues of Jacobian) and systems with strong nonlinearities (eigenvalues of Jacobian of linearization change a lot for each time step).

Idea: Estimate local error and adjust h such that the local error is less than the specified tolerance.

Implementation:

1. Compute the next iteration with two different methods: y_{n+1} with a method of order p and $\hat{y}_{n+1}$ with a method of order $\hat{p} = p + 1$.
2. The local exact solution is then

$$y_L(t_n; t_{n+1}) = y_{n+1} + e_{n+1} = \hat{y}_n + \hat{e}_{n+1} \tag{70}$$

   with $e_{n+1} = O(h^{p+1})$ and $\hat{e}_{n+1} = O(h^{p+2})$.
3. Since $\hat{e}_{n+1} \ll e_{n+1}$, we get the following

$$y_{n+1} - \hat{y}_n = e_{n+1} - \hat{e}_{n+1} \approx e_{n+1} \tag{71}$$

   h can then be chosen such that the local error e_{n+1} is as small as desired.

Since $\hat{y}_{n+1}$ is computed with a higher-order method than y_{n+1}, it would make sense to use that for the next iteration instead, this is called local extrapolation. Whichever solution is chosen as $\hat{y}_{n+1}$ is called the *embedded solution*.

11.2 Event detection

Let the event be given by

$$g(y, t) = 0 \tag{72}$$

e.g. a bouncing ball hitting the floor (crossing the x-axis). By checking for sign changes in g for each iteration, the time $t_n + \alpha$ of the event can be found by solving

$$g[y_n(\alpha), t_n + \alpha h] = 0 \tag{73}$$

for $\alpha \in [0, 1]$, where $y_n(\alpha)$ is the *dense output* found with interpolation (see page 565).

11.3 Multistep methods

A one-step method only uses the previous value y_n to compute y_{n+1}. A multistep method, on the other hand, uses y_{n-1}, y_{n-2}, etc. as well. The scheme looks like this:

$$\begin{aligned} y_{n+1} = {} & \alpha_1 y_n + \alpha_2 y_{n-1} + \dots \\ & + h(\beta_0 f(y_{n+1}, t_{n+1}) + \beta_1 f(y_n, t_n) + \beta_2 f(y_{n-1}, t_{n-1}) + \dots) \end{aligned} \tag{74}$$

The parameters/weights are derived by curve fitting polynomials to the previous time steps. The known stability concepts from one-step methods apply to multistep methods as well.

Part VI
Modelica reference

Modelica is an object-oriented, equation-based modeling language. This section shows some example models. An important thing to note is that you need the same number of equations as the number of variables.

```
model Oscillator "Descriptive comment"
    constant Real m = 1; // This comment is ignored by the compiler.
    parameter Real c = 1, k = 1;
    Real x(start=1); // 'start' is a hint to the compiler.
    Real vx;
equation
    der(x) = vx;
    m*der(vx) + c*vx + k*x = 0;
    // Could also add algebraic equations to make it a DAE.
intial equation
    vx = 0; // This is an actual constraint, unlike 'start'.
end Oscillator;
```

Next up, we look at a model which inherits from `Oscillator`, by use of the keyword `extends`:

```
model DrivenOscillator
    extends Oscillator;
    Real F;
equation
    F = sin(time);
    m*der(vx) + c*vx + k*x = F;
end DrivenOscillator;
```

The third example deals with the use of connectors.

```
connector Pin "Electrical pin";
    Modelica.SIunits.Voltage v; // Potential/energy level
    flow Modelica.SIunits.Current i; // Flow
end Pin;

partial model TwoPin // partial because of 6 var. and only 5 eq.
    Pin p, n;
    Modelica.SIunits.Voltage v; // Voltage across component
    Modelica.SIunits.Current i; // Current through component
equation
    v = p.v - n.v;
    p.i + n.i = 0 // KCL
    i = p.i // Current into component
end TwoPin;

model Inductor
    extends TwoPin;
    parameter Modelica.SIunits.Inductance L;
equation
    v = L * der(i);
end Inductor;
```

Declaring your own type (based on `Real`) in Modelica is done in the following way:

```
type Voltage = Real(unit = "V", min = 0, max = 5.0);
```

The final example shows the use of the `replaceable` and `redeclare` keywords.

```
model System
    replaceable Sensor sensor;
end System;

model SystemVariation
    extends System(
        redeclare Cheapsensor sensor
    );
end SystemVariation;
```

Part VII

Tables

12 Inverse of 3x3 matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \frac{1}{\det} \begin{pmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{pmatrix} \tag{75}$$

13 Trig functions

$$\cot(\theta) = \frac{1}{\tan(\theta)} \tag{76}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} \tag{77}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} \tag{78}$$

$$\sin(\theta) = \frac{\text{motstående}}{\text{hypotenus}} \tag{79}$$

$$\cos(\theta) = \frac{\text{hosliggende}}{\text{hypotenus}} \tag{80}$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \tag{81}$$

14 Trig identities

$$\cos(\theta)\sin(\theta) = \frac{1}{2}\sin(2\theta) \tag{82}$$

$$\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta) \tag{83}$$

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \tag{84}$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \tag{85}$$

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \tag{86}$$

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin \tag{87}$$

$$\tag{88}$$

15 Geometric series

$$\sum_{k=0}^{n-1} ar^k = a\left(\frac{1 - r^n}{1 - r}\right) \tag{89}$$

16 Partial integration

$$\int u\,dv = uv - \int v\,du \tag{90}$$