

# TTK4130 Formula Sheet

## Part I

## Modeling

### 1 Some useful models

#### 1.1 Mass spring damper

$$m\ddot{x} + c\dot{x} + kx = 0 \quad (1)$$

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2x = 0 \quad (2)$$

Note: A driving force  $F(t)$  could be included on the right hand side.

#### 1.2 Capacitor and inductor equations

$$i_C(t) = C \frac{dv_C}{dt}(t) \quad (3)$$

$$v_L(t) = L \frac{di_L}{dt}(t) \quad (4)$$

#### 1.3 Pendulum equation

$$\ddot{\theta} + \frac{g}{l} \sin(\theta) = 0 \quad (5)$$

For  $\theta \ll 1$  we get the approximation

$$\ddot{\theta} + \frac{g}{l} \theta = 0 \quad (6)$$

with period

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{l}{g}} \quad (7)$$

## 2 Passivity

A system consisting of a parallel or feedback interconnection of passive subsystems, is itself passive.

**Definition:** If the following inequality is satisfied for all  $u$  and  $T \geq 0$ , then the system is passive.

$$\int_0^T y(t)u(t)dt \geq -E_0 \quad (8)$$

Note that if the roles of  $u$  and  $y$  are reversed, i.e.  $y$  is taken to be the input and  $u$  the output, then the inequality still holds.

Interpretation of this definition based on energy conservation: The product  $uy$  has dimension power, thus we can think about the integral as the energy supplied by  $u$  or, equivalently, the energy absorbed by the system.

1. If  $\int_0^T y(t)u(t)dt \geq 0$ , energy is only absorbed. This inequality will hold for a passive memoryless system (e.g. a circuit with only a resistor).
2. If  $\int_0^T y(t)u(t)dt \geq -E_0$ , the system can supply a limited amount of energy to the outside, due to initial conditions of energy storage elements, such as capacitors and inductors.
3. If  $\int_0^T y(t)u(t)dt \rightarrow -\infty$ , the system is active.

A system is passive iff its transfer function is positive real.

**Definition of a positive real transfer function:**

1. All poles of  $H(s)$  have real parts less than or equal to zero.
2.  $\text{Re } H(j\omega) \geq 0 \quad \forall \omega$  that are not poles of  $H(s)$ .
3. If  $j\omega_0$  is a pole, it is simple and  $\text{Res}_{s=j\omega_0} H(s) = \lim_{s \rightarrow j\omega_0} (s - j\omega_0)H(s)$  is real and positive. If  $H(s)$  has a pole at infinity, it is simple, and  $R_\infty = \lim_{\omega \rightarrow \infty} \frac{H(j\omega)}{j\omega}$  exists, and is real and positive.

There is also the *storage function* approach for determining passivity. Consider a state space model  $\dot{x} = f(x, u)$ ,  $y = h(x)$ . If there exists a storage function  $V(x) \geq 0$  and dissipation function  $g(x) \geq 0$ , such that

$$\dot{V} = u^\top y - g(x), \quad (9)$$

then the system is passive.

## Part II

# Motors and actuators

3 Electrical motors

4 Hydraulic motors

5 Friction

## Part III

# Electrical and mechanical

## 6 Capacitor

$$i(t) = C \frac{dv_C(t)}{dt} \quad (10)$$

## 7 Inductor

$$v(t) = L \frac{di_L(t)}{dt} \quad (11)$$

## 8 Rotating object

$$\omega = \frac{v}{r} \quad (12)$$

where  $r$  is the distance from axis,  $v$  is the tangential speed and  $\omega$  is the angular frequency of rotation.

## Part IV

# Dynamics

## 9 Rigid body kinematics

### 9.1 Rotation matrices

Let  $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$  and  $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$  be the orthogonal bases of two coordinate frames. Then the coordinate transformation from frame  $b$  to frame  $a$  is given by

$$\mathbf{R}_b^a = (\mathbf{b}_1^a \quad \mathbf{b}_2^a \quad \mathbf{b}_3^a) \quad (13)$$

That is,

$$\mathbf{v}^a = \mathbf{R}_b^a \mathbf{v}^b \quad (14)$$

The  $\mathbf{R}_b^a$  matrix can also be thought of simply as a rotation of a vector. For a second, forget that the  $b$  frame exists, and think only about the  $a$  frame. A simple example is

$$\mathbf{b}_1^a = \mathbf{R}_b^a \mathbf{a}_1^a \quad (15)$$

which shows that the first basis vector of the  $a$  frame is rotated to the first basis vector of the  $b$  frame, when everything is referred to frame  $a$ . In this sense the matrix represents a rotation from  $a$  to  $b$ .

### 9.2 Homogeneous transformation matrices

$$T_b^a = \begin{pmatrix} R_b^a & r_{ab}^a \\ \mathbf{0}^\top & 1 \end{pmatrix} \quad (16)$$

where  $r_{ab}^a$  is the origin of frame  $b$  in  $a$  coordinates.

### 9.3 Differentiation of vectors and matrices

$$\frac{{}^a d}{dt} \vec{u} = \frac{{}^b d}{dt} \vec{u} + \vec{\omega}_{ab} \times \vec{u}, \quad (17)$$

Skew-symmetric form of coordinate vector

$$\mathbf{u}^\times := \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \quad (18)$$

Skew-symmetric form of angular velocity vector

$$(\boldsymbol{\omega}_{ab}^a)^\times = \dot{\mathbf{R}}_b^a (\mathbf{R}_b^a)^\top \quad (19)$$

Coordinate transformation

$$(\boldsymbol{\omega}_{ab}^a)^\times = \mathbf{R}_b^a (\boldsymbol{\omega}_{ab}^b)^\times \mathbf{R}_a^b, \quad \mathbf{R}_a^b = (\mathbf{R}_b^a)^{-1} = (\mathbf{R}_b^a)^\top \quad (20)$$

## 9.4 Kinematic differential equations

For Euler angles, when the middle rotation is  $\frac{\pi}{2}$  radians, the  $E$  matrix in the kinematic differential equation is singular. This is because this rotation moves the third axis of rotation to the first axis of rotation, such that we lose a degree of freedom.

## 9.5 Coordinate systems

Cylindrical coordinates  $(r, \theta, z)$ :

$$x = r \cos(\theta) \quad (21)$$

$$y = r \sin(\theta) \quad (22)$$

$$z = z \quad (23)$$

$$r = \sqrt{x^2 + y^2} \quad (24)$$

$$\theta = \begin{cases} 0 & \text{if } x = 0 \text{ and } y = 0 \\ \arctan(\frac{y}{x}) & \text{if } x > 0 \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \end{cases} \quad (25)$$

$$dV = dx dy dz = r dr d\theta dz \quad (26)$$

Spherical coordinates  $(r, \varphi, \theta)$ ,  $\varphi$  angle from  $z$ -axis:

$$x = r \sin(\varphi) \cos(\theta) \quad (27)$$

$$y = r \sin(\varphi) \sin(\theta) \quad (28)$$

$$z = r \cos(\varphi) \quad (29)$$

$$r = \sqrt{x^2 + y^2 + z^2} \quad (30)$$

$$\theta \text{ defined as above.} \quad (31)$$

$$dV = dx dy dz = r^2 \sin(\varphi) dr d\varphi d\theta \quad (32)$$

Note that  $r \geq 0$ ,  $0 \leq \theta \leq 2\pi$  and  $0 \leq \varphi \leq \pi$  (typical definition). The definition of  $\theta$  above has the range  $(-\pi, \pi]$ , to obtain only positive results  $2\pi$  can be added to negative values.

Add position, velocity and acceleration in Cartesian, cylindrical and spherical coordinates. Or at least the general formulas.

## 9.6 The center of mass

The mass of a rigid body  $b$  is

$$m = \int_b dm = \int_b \rho(x, y, z) dV \quad (33)$$

The center of mass  $\vec{r}_c$  is defined as

$$\vec{r}_c = \frac{1}{m} \int_b \vec{r}_p dm \quad (34)$$

where  $\vec{r}_p$  is the position of a mass element  $dm$  that is fixed in frame  $b$ . The  $x$ -coordinate of the center of mass is given by

$$x_c = \frac{1}{m} \iiint_b x_p \rho(x, y, z) dV \quad (35)$$

The definitions for  $y$  and  $z$  are exactly the same. Typically  $(x_p, y_p, z_p) = (x, y, z)$ .

## 9.7 Other useful formulas

Relation between linear and angular velocity

$$v = \omega r \quad (36)$$

# 10 Newton-Euler equations of motion

## 10.1 Kinetic energy

$$\mathcal{T} = \frac{1}{2} m (\mathbf{v}_c^b)^\top \mathbf{v}_c^b + \frac{1}{2} (\boldsymbol{\omega}_{ib}^b)^\top \mathbf{M}_{b/c}^b \boldsymbol{\omega}_{ib}^b \quad (37)$$

The subscript  $c$  denotes the center of mass and the superscript  $b$  denotes a coordinate vector/matrix in frame  $b$ .  $\boldsymbol{\omega}_{ib}^b$  is the angular velocity of frame  $b$  relative to frame  $i$ .  $\mathbf{M}_{b/c}^b$  is the inertia matrix of  $b$  about  $c$ , i.e. the inertia matrix of the rigid body about the center of mass.

## 10.2 Inertia matrix

$$\mathbf{M}_{b/c}^b = - \int_b (\mathbf{r}^b)^\times (\mathbf{r}^b)^\times dm = \int_b \left[ (\mathbf{r}^b)^\top \mathbf{r}^b \mathbf{I} - \mathbf{r}^b (\mathbf{r}^b)^\top \right] dm \quad (38)$$

Fun facts: Swap the  $b$  superscripts with  $i$  on the right hand side to get  $\mathbf{M}_{b/c}^i$ .  $\mathbf{M}_{b/c}^b$  is positive definite, since the kinetic energy  $\mathcal{T} \geq 0$ . Note that the integral above is a triple integral of a  $3 \times 3$ -matrix. About a specified axis the formula reduces to

$$I = \int_b (\mathbf{r}^b)^\top \mathbf{r}^b dm \quad (39)$$

### 10.3 Parallel axis theorem

The inertia matrix of  $b$  about a point  $o$  is given by

$$\mathbf{M}_{b/o}^b = \mathbf{M}_{b/c}^b - m(\mathbf{r}_g^b)^\times (\mathbf{r}_g^b)^\times = \mathbf{M}_{b/c}^b + m \left[ (\mathbf{r}_g^b)^\top \mathbf{r}_g^b \mathbf{I} - \mathbf{r}_g^b (\mathbf{r}_g^b)^\top \right] \quad (40)$$

where  $\mathbf{r}_g^b$  is the vector from the point  $o$  to the center of mass  $c$ . If  $o$  is the origin this corresponds to  $\mathbf{r}_c^b$ . In its simplest form with two parallel axes, the formula reduces to

$$I = I_c + md^2 \quad (41)$$

where  $I_c$  is the moment of inertia about the axis through the center of mass and  $d$  is the distance between the axes.

### 10.4 Other useful formulas

Relationships between torque, angular momentum and angular velocity

$$\vec{\tau} = \vec{r} \times \vec{F} \quad (42)$$

$$\vec{\tau} = \frac{d\vec{L}}{dt} \quad (43)$$

$$\vec{\tau} = I\dot{\omega} \quad (44)$$

$$P = \tau\omega \quad (45)$$

## 11 Lagrangian dynamics

Lagrangian

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathcal{T}(\mathbf{q}, \dot{\mathbf{q}}, t) - \mathcal{U}(\mathbf{q}) \quad (46)$$

Lagrange's equation of motion

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i \quad (47)$$

Generalized force

$$Q_i = \sum_{k=1}^N \frac{\partial \vec{r}_k}{\partial q_i} \cdot \vec{F}_k \quad (48)$$



## Part V

# Balance equations

## 12 Kinematics of flow

### 12.1 Material derivative

Let  $\mathbf{x}$  be the position of some fluid particle, with velocity  $\dot{\mathbf{x}} = \mathbf{v}$ . Further, let  $\varphi$  be some quantity related to the particle, e.g. its temperature. The material derivative of  $\varphi$  is then defined as

$$\frac{D\varphi}{Dt} = \frac{\partial\varphi}{\partial t} + \mathbf{v}^\top \nabla \varphi \quad (49)$$

## 13 Mass, momentum and energy balances

### 13.1 Mass balance

Level of tank

$$\frac{d}{dt}(\rho Ah) = w_1 - w_2 \quad (50)$$

$$\dot{h} = \frac{1}{\rho A}(w_1 - w_2) \quad (51)$$

### 13.2 Momentum balance

Bernoulli's equation for stationary frictionless flow along a streamline for an incompressible fluid

$$\frac{1}{2}(v_2^2 - v_1^2) + \frac{p_2 - p_1}{\rho} + (z_2 - z_1)g = 0 \quad (52)$$

### 13.3 Energy balance

Internal energy, enthalpy, heat capacities and temperature

$$h = c_p T \quad (53)$$

$$u = c_v T \quad (54)$$

Note that all quantities (except temperature) are specific (per unit mass) in these equations.

## 14 Gas dynamics

## Part VI

# Simulation

We are concerned with solving the IVP

$$\dot{y} = f(y, t), \quad y(t_0) = y_0 \quad (55)$$

The Jacobian of the system is defined as

$$J = \frac{\partial f}{\partial y}(y, t) \quad (56)$$

Note that the Jacobian is  $A$  for a linear, time-invariant system  $\dot{x} = Ax + Bu$ .

## 15 Stability functions

The stability of a numerical method is ensured if  $|R(h\lambda_i)| \leq 1$  for all eigenvalues  $\lambda_i$ .

### 15.1 ERK methods

$$R_E(h\lambda) = \det \left[ I - h\lambda(A - \mathbf{1}b^\top) \right], \quad \text{where } \mathbf{1} = (1, \dots, 1)^\top \quad (57)$$

Note that  $R_E(h\lambda)$  will be a polynomial in  $h\lambda$  of order less than or equal to  $\sigma$  (the number of stages).

### 15.2 IRK methods

$$R(h\lambda) = \left[ 1 + h\lambda b^\top (I - h\lambda A)^{-1} \mathbf{1} \right] \quad (58)$$

$$R(h\lambda) = \frac{\det [I - h\lambda(A - \mathbf{1}b^\top)]}{\det(I - h\lambda A)} \quad (59)$$

## 16 Stability of RK methods

### 16.1 Aliasing

The *Nyquist frequency* is half of the sampling rate

$$\omega_{\text{Nyquist}} = \frac{1}{2} \cdot \frac{2\pi}{h}, \quad \text{where } h \text{ is the step size.} \quad (60)$$

Two systems oscillating at a low frequency  $\omega < \omega_{\text{Nyquist}}$  and a high frequency  $\omega + 2k\frac{\pi}{h} > \omega_{\text{Nyquist}}$  ( $k$  integer) will intercept at all sampling points, and therefore a solver will not be able to distinguish them. More specifically, the solver will believe that the system with higher frequency is the system with lower frequency, when fitting the curve.

## 16.2 A- and L-stability

**Definition:** A method is A-stable if  $|R(h\lambda)| \leq 1 \quad \forall \quad \text{Re } \lambda \leq 0$ .

This definition means that an A-stable method is stable for all stable test systems  $\dot{y} = \lambda y$ . Note also that no ERK method can be A-stable, since  $|R_E(h\lambda)| \rightarrow \infty$  as  $|\lambda| \rightarrow \infty$ .

**Definition:** A method is L-stable if it is A-stable and  $|R(j\omega h)| \rightarrow 0$  when  $\omega \rightarrow \infty \quad \forall$  systems  $\dot{y} = j\omega y$ .

A-stable methods can suffer from aliasing for systems with fast dynamics (faster than Nyquist frequency), whereas an L-stable method will simply damp out these fast dynamics. This means that the L-stable method might give a better qualitative representation of what the actual solution looks like.

## 16.3 Stiffly accurate methods and algebraic stability

**Definition:** A method is stiffly accurate if

$$\det(A) \neq 0 \text{ and } b = A^\top [0, 0, \dots, 1]^\top \quad (61)$$

Note: A-stable and stiffly accurate  $\implies$  L-stable.

**Definition:** A method is algebraically stable if

$$M = \text{diag}(b)A + (\text{diag}(b)A)^\top + bb^\top \quad (62)$$

is positive semi-definite. Note: Algebraically stable  $\implies$  A-stable.

## 17 DAEs

A fully implicit ODE,  $F(\dot{x}, x, u) = 0$  is a DAE if  $\frac{\partial F}{\partial \dot{x}}$  is rank deficient (note that the partial derivative is with respect to  $\dot{x}$ , not  $x$ ).

Method for finding index (one way to go about it):

- Differentiate algebraic equation(s)  $g(x, z)$  until you can solve for the algebraic variable(s).
- The DAE system is now index 1. If you differentiated  $p$  times in the previous step, the index is  $p + 1$ .

## 18 Advanced topics

### 18.1 Automatic adjustment of step size

The step size  $h$  can be selected so that the desired accuracy is obtained. Variable-step methods are useful for stiff systems (large spread in eigenvalues of Jacobian) and systems with strong nonlinearities (eigenvalues of Jacobian of linearization change a lot for each time step).

Idea: Estimate local error and adjust  $h$  such that the local error is less than the specified tolerance.

Implementation:

1. Compute the next iteration with two different methods:  $y_{n+1}$  with a method of order  $p$  and  $\hat{y}_{n+1}$  with a method of order  $\hat{p} = p + 1$ .
2. The local exact solution is then

$$y_L(t_n; t_{n+1}) = y_{n+1} + e_{n+1} = \hat{y}_n + \hat{e}_{n+1} \quad (63)$$

with  $e_{n+1} = O(h^{p+1})$  and  $\hat{e}_{n+1} = O(h^{p+2})$ .

3. Since  $\hat{e}_{n+1} \ll e_{n+1}$ , we get the following

$$y_{n+1} - \hat{y}_n = e_{n+1} - \hat{e}_{n+1} \approx e_{n+1} \quad (64)$$

$h$  can then be chosen such that the local error  $e_{n+1}$  is as small as desired.

Since  $\hat{y}_{n+1}$  is computed with a higher-order method than  $y_{n+1}$ , it would make sense to use that for the next iteration instead, this is called local extrapolation. Whichever solution is chosen as  $\hat{y}_{n+1}$  is called the *embedded solution*.

## 18.2 Event detection

Let the event be given by

$$g(y, t) = 0 \quad (65)$$

e.g. a bouncing ball hitting the floor (crossing the  $x$ -axis). By checking for sign changes in  $g$  for each iteration, the time  $t_n + \alpha$  of the event can be found by solving

$$g[y_n(\alpha), t_n + \alpha h] = 0 \quad (66)$$

for  $\alpha \in [0, 1]$ , where  $y_n(\alpha)$  is the *dense output* found with interpolation (see page 565).

## 18.3 Multistep methods

A one-step method only uses the previous value  $y_n$  to compute  $y_{n+1}$ . A multistep method, on the other hand, uses  $y_{n-1}$ ,  $y_{n-2}$ , etc. as well. The scheme looks like this:

$$y_{n+1} = \alpha_1 y_n + \alpha_2 y_{n-1} + \dots + h(\beta_0 f(y_{n+1}, t_{n+1}) + \beta_1 f(y_n, t_n) + \beta_2 f(y_{n-1}, t_{n-1}) + \dots) \quad (67)$$

The parameters/weights are derived by curve fitting polynomials to the previous time steps. The known stability concepts from one-step methods apply to multistep methods as well.

## Part VII

# Modelica reference

Potentially add some stuff from exercise 4, like user-defined types. Look at connectors.

Modelica is an object-oriented, equation-based modeling language. This section shows some example models. An important thing to note is that you need the same number of equations as the number of variables.

```
model Oscillator "Descriptive comment"
  constant Real m = 1; // This comment is ignored by the compiler.
  parameter Real c = 1, k = 1;
  Real x(start=1); // 'start' is a hint to the compiler.
  Real vx;
equation
  der(x) = vx;
  m*der(vx) + c*vx + k*x = 0;
  // Could also add algebraic equations to make it a DAE.
initial equation
  vx = 0; // This is an actual constraint, unlike 'start'.
end Oscillator;
```

Next up, we look at a model which inherits from `Oscillator`, by use of the keyword `extends`:

```
model DrivenOscillator
  extends Oscillator;
  Real F;
equation
  F = sin(time);
  m*der(vx) + c*vx + k*x = F;
end DrivenOscillator;
```

## Part VIII

## Tables

### 19 Inverse of 3x3 matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \frac{1}{\det} \begin{pmatrix} ei - fh & -(bi - ch) & bf - ce \\ -(di - fg) & ai - cg & -(af - cd) \\ dh - eg & -(ah - bg) & ae - bd \end{pmatrix} \quad (68)$$

### 20 Trig functions

$$\cot(\theta) = \frac{1}{\tan(\theta)} \quad (69)$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} \quad (70)$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} \quad (71)$$

### 21 Trig derivatives

$$\frac{d}{dt} \tan(x) = \sec^2(x) \quad (72)$$

$$\frac{d}{dt} \csc(x) = -\csc(x) \cot(x) \quad (73)$$

$$\frac{d}{dt} \sec(x) = \sec(x) \tan(x) \quad (74)$$

$$\frac{d}{dt} \cot(x) = -\csc^2(x) \quad (75)$$

### 22 Trig identities

$$\cos(\theta) \sin(\theta) = \frac{1}{2} \sin(2\theta) \quad (76)$$

$$\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta) \quad (77)$$

$$\sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta) \quad (78)$$

$$\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta) \quad (79)$$

$$\sin(\alpha - \beta) = \sin(\alpha) \cos(\beta) - \cos(\alpha) \sin(\beta) \quad (80)$$

$$\cos(\alpha - \beta) = \cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta) \quad (81)$$