Linear Programming

Pawel Winter

Diet Problem (after Chvatal)

- Every day Polly needs:
  - 2000 kcal,
  - 55g protein,
  - 800mg calcium.
- She will get other stuff (e.g., iron and vitamins) by taking pills. Not that this could not be included in the model – she just wants to keep it simple.
- She wants a diet that will meet the requirements while being neither expensive nor boring.

Value and Price per Serving

Food	Energy (kcal)	Protein (g)	Calcium (mg)	Price per serving
Oatmeal	110	4	2	3
Chicken	205	32	12	24
Eggs	160	13	54	13
Whole milk	160	8	285	9
Cherry pie	420	4	22	20
Pork with beans	260	14	80	19

10 portions of pork with beans would cover her needs!
And would cost only 190. But ...

Limits to What Polly Can Stomach

- Oatmeal: at most 4 servings a day.
- Chicken: at most 3 servings a day.
- Eggs: at most 2 servings a day.
- Milk: at most 8 servings a day.
- Cherry pie: at most 2 servings a day.
- Pork with beans: at most 2 servings a day.

8 servings of milk and 2 servings of cherry pie would meet her needs. Boring but she could stomach it. Especially since it would cost 112. Can she find a less expensive diet?

Variables

- X_1 : number of oatmeal servings.
- X_2 *:* number of chicken servings.
- X_3 *:* number of eggs servings.
- X_4 *:* number of milk servings.
- X_5 *:* number of cherry pie servings.
- X_6 *:* number of pork and pie servings.

Linear Constraints

$$
\begin{array}{ccccccccccccccc}
x_1 & & & & & & & & & & & & \le & 4 \\
& & x_2 & & & & & & & & & & \le & 3 \\
& & & & x_3 & & & & & & & & \le & 2 \\
& & & & & & x_4 & & & & & & \le & 8 \\
& & & & & & & & x_5 & & & & \le & 2 \\
& & & & & & & & & & x_6 & & \le & 2 \\
110x_1 & + & 205x_2 & + & 160x_3 & + & 160x_4 & + & 420x_5 & + & 260x_6 & & \ge & 2000 \\
4x_1 & + & 32x_2 & + & 13x_3 & + & 8x_4 & + & 4x_5 & + & 14x_6 & & \ge & 55 \\
2x_1 & + & 12x_2 & + & 54x_3 & + & 285x_4 & + & 22x_5 & + & 80x_6 & & \ge & 800 \\
x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & & \ge & 0
\end{array}
$$

Linear Programming Problem

$$
\begin{array}{lcccccccccccccc}
min & 3x_1 & + & 24x_2 & + & 13x_3 & + & 9x_4 & + & 20x_5 & + & 19x_6 & & \\
s.t. & x_1 & & & & & & & & & & & \leq & 4 \\
 & & & x_2 & & & & & & & & & \leq & 3 \\
 & & & & & x_3 & & & & & & & \leq & 2 \\
 & & & & & & & x_4 & & & & & \leq & 8 \\
 & & & & & & & & & x_5 & & & \leq & 2 \\
 & & & & & & & & & & & x_6 & \leq & 2 \\
 & 110x_1 & + & 205x_2 & + & 160x_3 & + & 160x_4 & + & 420x_5 & + & 260x_6 & \geq & 2000 \\
 & 4x_1 & + & 32x_2 & + & 13x_3 & + & 8x_4 & + & 4x_5 & + & 14x_6 & \geq & 55 \\
 & 2x_1 & + & 12x_2 & + & 54x_3 & + & 285x_4 & + & 22x_5 & + & 80x_6 & \geq & 800 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0
\end{array}
$$

http://www-neos.mcs.anl.gov/CaseStudies/dietpy/WebForms/

Linear Objective Function

$$\begin{array}{ll} \textit{min} & 3\mathrm{x}_1+24\mathrm{x}_2+13\mathrm{x}_3+9\mathrm{x}_4+20\mathrm{x}_5+19\mathrm{x}_6 \\ \textit{s.t.} & \text{linear constraints} \end{array}$$

- The value of the **objective function** for a particular set of values for x_1, x_2, x_3, x_4, x_5, x_6 is called its **objective value**.
- If a particular set of values for x_1, x_2, x_3, x_4, x_5, x_6, satisfies all constraints, it is said to be a **feasible solution**.
- The set of all feasible solutions is called the **feasible region**. It can be shown to be convex.
- A feasible solution that has the minimum (or maximum) objective value is called an **optimal solution**.

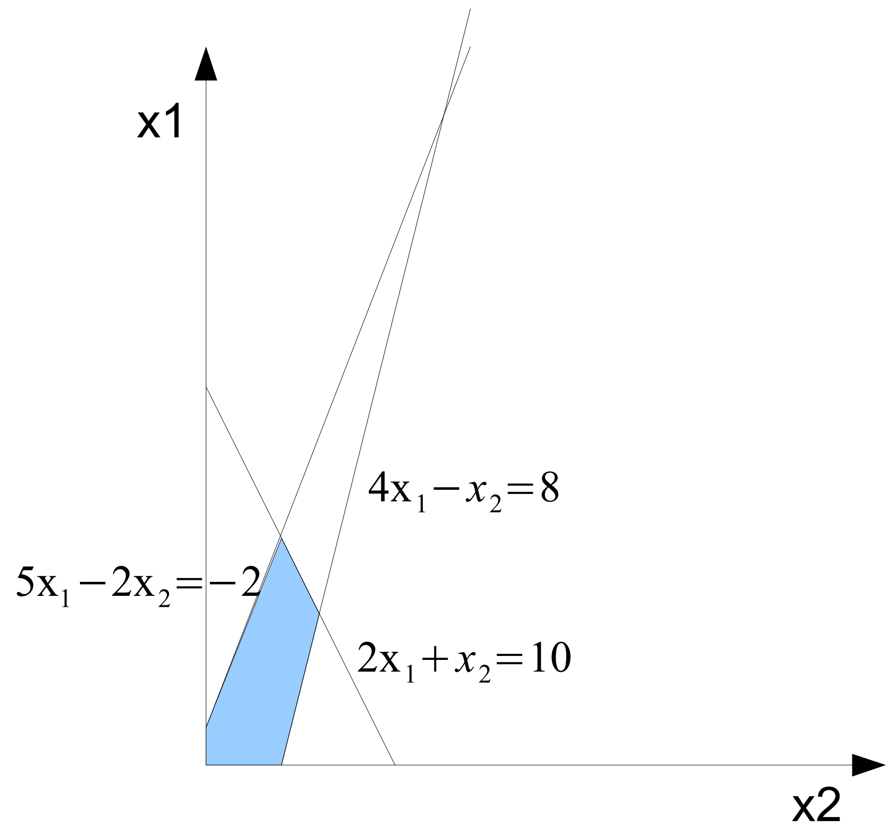
General LP Problem

$$\begin{aligned} &min \quad \sum_{j=1}^{n} c_j x_j \\ &s.t. \quad m \text{ linear constraints} \end{aligned}$$

- Minimization or maximization of a **linear** objective function with *n* real-valued variables.
- An optimal solution must satisfy *m* **linear** constraints (inequalities or equalities).
- Strict inequalities are not allowed.
- ”programming” in ”linear programming” does not refer to any code. It was chosen before computer programming was born.

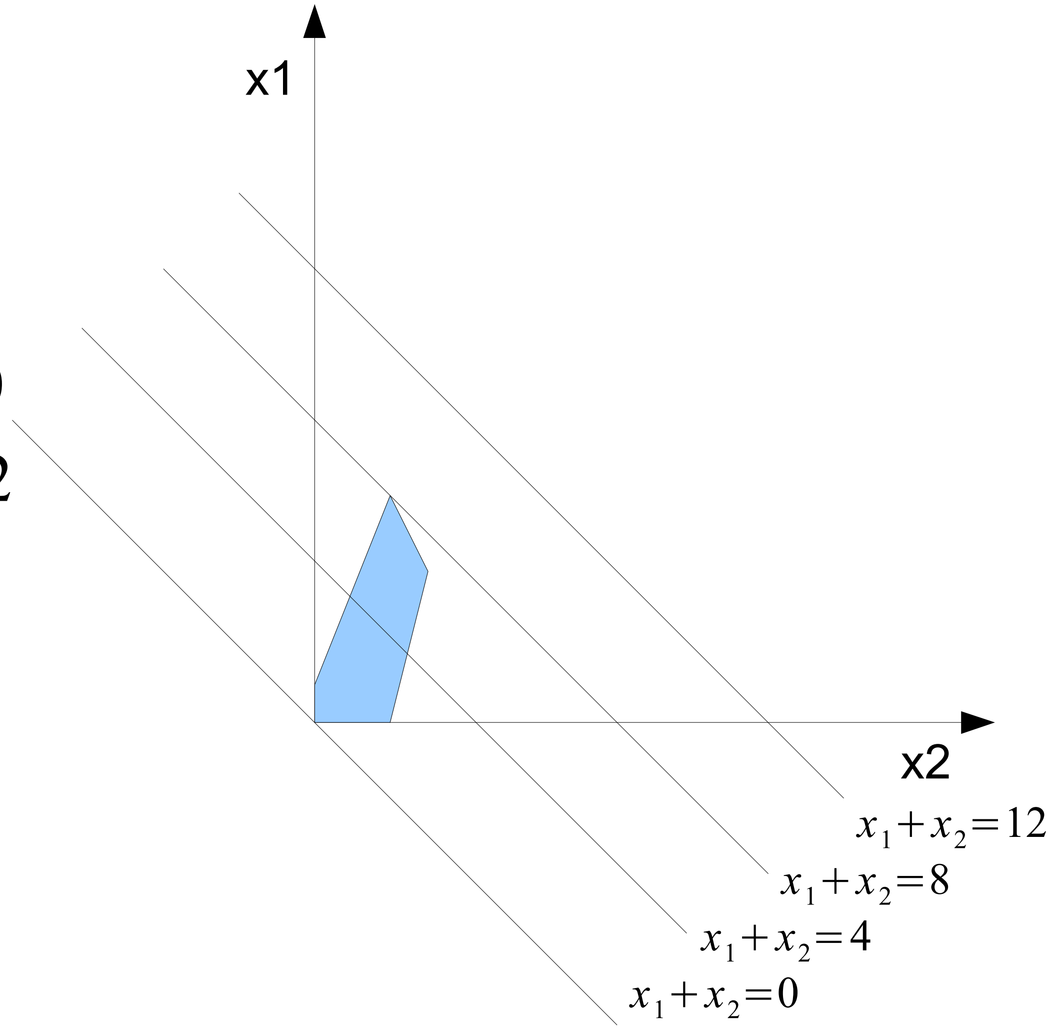
Geometric Interpretation

$$
\begin{array}{lccccc}
max & x_1 & + & x_2 & & \\
s.t. & 4x_1 & - & x_2 & \leq & 8 \\
 & 2x_1 & + & x_2 & \leq & 10 \\
 & 5x_1 & - & 2x_2 & \geq & -2 \\
 & x_1, & & x_2 & \geq & 0
\end{array}
$$



Geometric Interpretation

$$
\begin{array}{llllll}
max & x_1 & + & x_2 & & \\
s.t. & 4x_1 & - & x_2 & \leq & 8 \\
 & 2x_1 & + & x_2 & \leq & 10 \\
 & 5x_1 & - & 2x_2 & \geq & -2 \\
 & x_1, & & x_2 & \geq & 0
\end{array}
$$



Geometric Interpretation in R^d

- *d* variables.
- Each constraint defines a half-space in R^d. The set of feasible solutions is the intersection of these half-spaces, called **simplex**. It is convex. Can be unbounded or empty.
- The set of points in which the objective function has the same value *z* is a **hyperplane**.
- The value of the objective function increases or decreases as the hyperplane is translated.
- If the set of feasible solutions is bounded and not empty, then there is an optimal solution in an extreme vertex of the simplex.

General Idea Behind SIMPLEX Algorithm

- SIMPLEX starts with a feasible solution corresponding to some vertex of the simplex. Section 29-5 shows how to find such a vertex (or decide that the feasible region is empty).
- SIMPLEX keeps ”jumping” from a vertex of the simplex to a new vertex if it offers a feasible solution that is better (or at least not worse). We will indicate how SIMPLEX ”jumps”. We will mention how to avoid ”cycling” when SIMPLEX jumps through feasible solutions with the same objective value.
- When no more ”jumps” are possible, we will give the intuition that SIMPLEX is in an optimal vertex (or the LP is unbounded).
- We will show that the number of jumps is finite (at most equal to the number of simplex vertices).

History of LP

- L.V. Kantorovich pointed out in 1939 the importance of restricted classes of LP problems. Nobel Prize in 1975
- T.C. Koopmans realized in 1947 the importance of LP for the analysis of classical economic theories. Nobel Prize in 1975.
- G.B. Dantzig designed in 1947 the simplex method to solve LP for U.S. Air Force. Not a polynomial algorithm!
- Many applications followed over the years. Work of Kantorovich was rediscovered in 1950's.
- L.G. Khachian discovered first polynomial algorithm in 1979. Terribly slow. Not strongly polynomial.
- N. Karmarkar discovered second polynomial algorithm in 1984. Practical. Not strongly polynomial.

Applications of LP

- Scheduling problems: airline wishes to schedule its flight crews on all flights while using as few crew members as possible.
- Manufacturing problems: How much of each type of product should be produced subject to technical and financial constraints.
- Location problems: Locating drills to maximize the amount of oil that will be extracted under given budget constraints.
- Many network and graph problems can be formulated as LP; dimensionig telecommunication and distribution networks.
- Allocation of financial assets to maximize profit or minimize risk.
- Integer linear programming problems.

LP in Standard Form

- Maximization of a linear function.
- *n* non-negative real-valued variables.
- *m* linear inequalities (”less than or equal to”).

$$
\begin{array}{lll}
max & \sum_{j=1}^{n} c_j x_j & \\
s.t. & \sum_{j=1}^{n} a_{ij} x_j \le b_i & for \quad i=1,2,\ldots,m \\
 & x_j \ge 0 & for \quad j=1,2,\ldots,n
\end{array}
$$

Converting LP into Standard Form

$$
\begin{array}{llllll}
\min & -2\mathrm{x}_1 & + & 3\mathrm{x}_2 & & \\
s.t. & x_1 & + & x_2 & = & 7 \\
 & x_1 & - & 2\mathrm{x}_2 & \leq & 4 \\
 & & & x_1 & \geq & 0
\end{array}
$$

- Minimization LP is converted to an equivalent maximization problem by negating the coefficients of the objective function.

$$
\begin{array}{llllll}
\max & 2\mathrm{x}_1 & - & 3\mathrm{x}_2 & & \\
s.t. & x_1 & + & x_2 & = & 7 \\
 & x_1 & - & 2\mathrm{x}_2 & \leq & 4 \\
 & & & x_1 & \geq & 0
\end{array}
$$

Converting LP into Standard Form

$$\begin{array}{llllll} max & 2x_1 & - & 3x_2 & & \\ s.t. & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0 \end{array}$$

- Every variable x_j without the non-negativity constraint is replaced by two non-negative variables x'_j and x''_j and each occurrence of x_j is replaced by $x'_j - x''_j$.

$$\begin{array}{llllllll} max & 2x_1 & - & 3x'_2 & + & 3x''_2 & & \\ s.t. & x_1 & + & x'_2 & - & x''_2 & = & 7 \\ & x_1 & - & 2x'_2 & + & 2x''_2 & \leq & 4 \\ & x_1, & & x'_2, & & x''_2 & \geq & 0 \end{array}$$

Converting LP into Standard Form

$$\begin{array}{llllllll} max & 2x_1 & - & 3x'_2 & + & 3x''_2 & & \\ s.t. & x_1 & + & x'_2 & - & x''_2 & = & 7 \\ & x_1 & - & 2x'_2 & + & 2x''_2 & \leq & 4 \\ & x_1, & & x'_2, & & x''_2 & \geq & 0 \end{array}$$

- Each equality constraint is replaced by a pair of ”opposite” inequality constraints.

$$\begin{array}{llllllll} max & 2x_1 & - & 3x'_2 & + & 3x''_2 & & \\ s.t. & x_1 & + & x'_2 & - & x''_2 & \leq & 7 \\ & x_1 & + & x'_2 & - & x''_2 & \geq & 7 \\ & x_1 & - & 2x'_2 & + & 2x''_2 & \leq & 4 \\ & x_1, & & x'_2, & & x''_2 & \geq & 0 \end{array}$$

Converting LP into Standard Form

$$
\begin{array}{llllllll}
max & 2x_1 & - & 3x'_2 & + & 3x''_2 & & \\
s.t. & x_1 & + & x'_2 & - & x''_2 & \leq & 7 \\
 & x_1 & + & x'_2 & - & x''_2 & \geq & 7 \\
 & x_1 & - & 2x'_2 & + & 2x''_2 & \leq & 4 \\
 & x_1, & & x'_2, & & x''_2 & \geq & 0
\end{array}
$$

- Inequalities are ”turned around” by multiplying both sides by -1.

$$
\begin{array}{llllllll}
max & 2x_1 & - & 3x'_2 & + & 3x''_2 & & \\
s.t. & x_1 & + & x'_2 & - & x''_2 & \leq & 7 \\
 & -x_1 & - & x'_2 & + & x''_2 & \leq & -7 \\
 & x_1 & - & 2x'_2 & + & 2x''_2 & \leq & 4 \\
 & x_1, & & x'_2, & & x''_2 & \geq & 0
\end{array}
$$

Converting LP into a Standard Form

$$
\begin{array}{llllllll}
max & 2x_1 & - & 3x'_2 & + & 3x''_2 & & \\
s.t. & x_1 & + & x'_2 & - & x''_2 & \leq & 7 \\
 & -x_1 & - & x'_2 & + & x''_2 & \leq & -7 \\
 & x_1 & - & 2x'_2 & + & 2x''_2 & \leq & 4 \\
 & x_1, & & x'_2, & & x''_2 & \geq & 0
\end{array}
$$

- Renaming the variables

$$
\begin{array}{llllllll}
max & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
s.t. & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
 & x_1, & & x_2, & & x_3 & \geq & 0
\end{array}
$$

LP in Standard Form

- *n* variables, *m* constraints

$$
\begin{array}{llcccl}
max & \sum_{j=1}^{n} c_j x_j & & & & \\
s.t. & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & for & i = 1, 2, \ldots, m \\
 & x_j & \geq & 0 & for & j = 1, 2, \ldots, n
\end{array}
$$

Slack Variables (Overskudsvariable)

- Consider one of the constraints, for example

$$2x_1 + 3x_2 + x_3 \leq 5$$

- For every feasible solution s_1, s_2, s_3 , the value of the left-hand side is at most the value of the right-hand side.
- Often there can be a **slack** between these two values.
- Denote the slack by x_4.
- By requiring that $x_4 \geq 0$, we can replace the inequality by the equality

$$2x_1 + 3x_2 + x_3 + x_4 = 5$$

Slack Variables

$$\begin{aligned}
max \quad & \sum_{j=1}^{n} c_j x_j \\
s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad && for \quad i = 1, 2, \ldots, m \\
& x_j \geq 0 && for \quad j = 1, 2, \ldots, n
\end{aligned}$$

$$\begin{aligned}
max \quad & \sum_{j=1}^{n} c_j x_j \\
s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i \quad && for \quad i = 1, 2, \ldots, m \\
& x_j \geq 0 && for \quad j = 1, 2, \ldots, n+m
\end{aligned}$$

Standard to Slack Form - Example

$$
\begin{array}{lcccccccc}
\max & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
s.t. & x_1 & + & x_2 & - & x_3 & \leq & 7 \\
 & -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & \leq & 4
\end{array}
$$

$$
\begin{array}{lcccccccccccc}
\max & 2x_1 & - & 3x_2 & + & 3x_3 & & & & & & & \\
s.t. & x_1 & + & x_2 & - & x_3 & + & x_4 & & & & = & 7 \\
 & -x_1 & - & x_2 & + & x_3 & & & + \; x_5 & & & = & -7 \\
 & x_1 & - & 2x_2 & + & 2x_3 & & & & + & x_6 & = & 4
\end{array}
$$

$$
\begin{array}{ccccccccc}
z & = & 0 & + & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3
\end{array}
$$

LP in Slack Form

$$
\begin{array}{lllll}
max & \sum_{j=1}^{n} c_j x_j & & & \\
s.t. & \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} & = & b_i & for \quad i=1,2,\dots,m \\
 & x_j & \geq & 0 & for \quad j=1,2,\dots,n+m
\end{array}
$$

$$
\begin{array}{lllll}
max & z & = & \sum_{j=1}^{n} c_j x_j & \\
s.t. & x_{n+i} & = & b_i - \sum_{j=1}^{n} a_{ij} x_j & for \quad i=1,2,\dots,m \\
 & x_j & \geq & 0 & for \quad j=1,2,\dots,n+m
\end{array}
$$

$$
\begin{array}{llll}
z & = & 0 + \sum_{j=1}^{n} c_j x_j & \\
x_{n+i} & = & b_i - \sum_{j=1}^{n} a_{ij} x_j & for \quad i=1,2,\dots,m
\end{array}
$$

Basic Solutions

- Any solution of LP in standard form yields a solution of LP in the corresponding slack form (with the same objective value) and vice versa.
- Setting right-hand side variables of the slack form to 0 yields a **basic solution.**
- Left-hand side variables are called **basic**. Right-hand side variables are called **nonbasic**.
- The basic variables are said to constitute a **basis.**
- Note that a basic solution does not need to be feasible.

SIMPLEX - Example

- LP problem in standard form:

$$
\begin{array}{lccccccc}
max & 3x_1 & + & x_2 & + & 2x_3 & & \\
s.t. & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
 & 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
 & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}
$$

SIMPLEX – Example Continued

- LP in slack form:

$$
\begin{aligned}
z &= 0 + 3x_1 + x_2 + 2x_3 \\
x_4 &= 30 - x_1 - x_2 - 3x_3 \\
x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
x_6 &= 36 - 4x_1 - x_2 - 2x_3
\end{aligned}
$$

- Set all **nonbasic** variables (right-hand side) to 0.
- Compute values of **basic** variables: x_4=30, x_5=24, x_6=36.
- Compute the objective value z (= 0).
- This gives the feasible basic solution (0,0,0,30,24,36).
- It is feasible; not always the case – we were lucky.

SIMPLEX: 1. Pivoting

- Can x_1 be increased without violating feasibility?

$$
\begin{array}{ccccccccc}
z & = & 0 & + & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

- If x_1 is increased to 1, then x_4=29, x_5=22, x_6=32 while z=3. (1,0,0,29,22,32) is a feasible solution.
- If x_1 is increased to 2, then x_4=28, x_5=20, x_6=28 while z=6. (2,0,0,28,20,28) is a feasible solution.
- If x_1 is increased to 3, then x_4=27, x_5=18, x_6=24 while z=9. (3,0,0,27,18,24) is a feasible solution.

SIMPLEX: 1. Pivoting

- Can x_1 be increased without violating feasibility? By how much?

$$\begin{array}{ccccccccc} z & = & 0 & + & 3x_1 & + & x_2 & + & 2x_3 \\ x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\ x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\ x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 \end{array}$$

- If x_1 is increased beyond 30 then x_4 becomes negative.
- If x_1 is increased beyond 12 then x_5 becomes negative.
- If x_1 is increased beyond 9 then x_6 becomes negative.
- Constraint defining x_6 is **binding**.

SIMPLEX: 1. Pivoting

- So x_1 can be increased to 9 without losing feasibility. The feasible solution is (9,0,0,21,6,0) and z = 27.
- We will now rewrite the slack form to an equivalent slack form with x_1, x_4, x_5 as basic variables and with (9,0,0,21,6,0) being its feasible basic solution.
- This rewriting is called **pivoting.**
- Binding constraint defining x_6 is rewritten so that it has x_1 on its left-hand side.
- All occurences of x_1 in other constraints and in the objective function are replaced by the right-hand side of the binding constraint.

SIMPLEX: 1. Pivoting

$$
\begin{array}{ccccccccc}
z & = & 0 & + & 3\left(9-x_2/4-x_3/2-x_6/4\right) & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & \left(9-x_2/4-x_3/2-x_6/4\right) & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2\left(9-x_2/4-x_3/2-x_6/4\right) & - & 2x_2 & - & 5x_3 \\
x_1 & = & 9 & - & x_2/4 & - & x_3/2 & - & x_6/4
\end{array}
$$

$$
\begin{array}{ccccccccc}
z & = & 27 & + & x_2/4 & + & x_3/2 & - & 3x_6/4 \\
x_4 & = & 21 & - & 3x_2/4 & - & 5x_3/2 & + & x_6/4 \\
x_5 & = & 6 & - & 3x_2/2 & - & 4x_3 & + & x_6/2 \\
x_1 & = & 9 & - & x_2/4 & - & x_3/2 & - & x_6/4
\end{array}
$$

New basic variables: $x_1=9$, $x_4=21$, $x_5=6$

New objective value $z = 27$

New feasible basic solution: (9,0,0,21,6,0)

SIMPLEX: 2. Pivoting

- Can x_3 be increased without violating feasibility? By how much?

$$
\begin{array}{ccccccccc}
z & = & 27 & + & x_2/4 & + & x_3/2 & - & 3x_6/4 \\
x_4 & = & 21 & - & 3x_2/4 & - & 5x_3/2 & + & x_6/4 \\
x_5 & = & 6 & - & 3x_2/2 & - & 4x_3 & + & x_6/2 \\
x_1 & = & 9 & - & x_2/4 & - & x_3/2 & - & x_6/4
\end{array}
$$

- If x_3 is increased beyond 42/5 then $x_4 < 0$.
- If x_3 is increased beyond 3/2 then $x_5 < 0$.
- If x_3 is increased beyond 18 then $x_1 < 0$.
- Constraint defining x_5 is binding.

SIMPLEX: 2. Pivoting

$$
\begin{array}{ccccccccc}
z & = & 27 & + & x_2/4 & + & \frac{1}{2}(3/2-3\mathrm{x}_2/8+x_6/8-x_5/4) & - & 3\mathrm{x}_6/4 \\
x_4 & = & 21 & - & 3\mathrm{x}_2/4 & - & \frac{5}{2}(3/2-3\mathrm{x}_2/8+x_6/8-x_5/4) & + & x_6/4 \\
x_3 & = & 3/2 & - & 3\mathrm{x}_2/8 & - & x_5/4 & + & x_6/8 \\
x_1 & = & 9 & - & x_2/4 & - & \frac{1}{2}(3/2-3\mathrm{x}_2/8+x_6/8-x_5/4) & - & x_6/4
\end{array}
$$

$$
\begin{array}{ccccccccc}
z & = & 111/4 & + & x_2/16 & - & x_5/8 & - & 11\mathrm{x}_6/16 \\
x_4 & = & 69/4 & + & 3\mathrm{x}_2/16 & + & 5\mathrm{x}_5/8 & - & x_6/16 \\
x_3 & = & 3/2 & - & 3\mathrm{x}_2/8 & - & x_5/4 & + & x_6/8 \\
x_1 & = & 33/4 & - & x_2/16 & + & x_5/8 & - & 5\mathrm{x}_6/16
\end{array}
$$

New basic variables: $x_1=33/4$, $x_3=3/2$, $x_4=69/4$

New objective value z = 27.75

New feasible basic solution: (33/4, 0, 3/2, 69/4, 0, 0)

SIMPLEX: 3. Pivoting

- Can x_2 be increased without violating feasibility? By how much?

$$
\begin{array}{ccccccccc}
z & = & 111/4 & + & x_2/16 & - & x_5/8 & - & 11x_6/16 \\
x_4 & = & 69/4 & + & 3x_2/16 & + & 5x_5/8 & - & x_6/16 \\
x_3 & = & 3/2 & - & 3x_2/8 & - & x_5/4 & + & x_6/8 \\
x_1 & = & 33/4 & - & x_2/16 & + & x_5/8 & - & 5x_6/16
\end{array}
$$

- If x_2 is increased then x_4 also increases.
- If x_2 is increased beyond 4 then $x_3 < 0$.
- If x_2 is increased beyond 132 then $x_1 < 0$.
- Constraint defining x_3 is binding.

SIMPLEX: 3.Pivoting

$$\begin{array}{lllllllll}
z & = & 111/4 & + & \frac{1}{16}(4-8x_3/3-2x_5/3+x_6/3) & - & x_5/8 & - & 11x_6/16 \\
x_4 & = & 69/4 & + & \frac{1}{16}(4-8x_3/3-2x_5/3+x_6/3) & + & 5x_5/8 & - & x_6/16 \\
x_2 & = & 4 & - & 8x_3/3 & - & 2x_5/3 & + & x_6/3 \\
x_1 & = & 33/4 & - & \frac{1}{16}(4-8x_3/3-2x_5/3+x_6/3) & + & x_5/8 & - & 5x_6/16
\end{array}$$

$$\begin{array}{lllllllll}
z & = & 28 & - & x_3/6 & - & x_5/6 & - & 2x_6/3 \\
x_4 & = & 18 & - & x_3/2 & + & x_5/2 & + & 0x_6 \\
x_2 & = & 4 & - & 8x_3/3 & - & 2x_5/3 & + & x_6/3 \\
x_1 & = & 8 & + & x_3/6 & + & x_5/6 & - & x_6/3
\end{array}$$

New basic variables: $x_1=8$, $x_2=4$, $x_4=18$

New objective value z = 28

New feasible basic solution: (8, 4, 0, 18, 0, 0) is optimal

SIMPLEX

- SIMPLEX starts with a slack form corresponding to some feasible basic solution and iterates:
  - Select a nonbasic variable x_e with $c_e > 0$. If no such x_e exists, SIMPLEX terminates. We will show that the feasible basic solution is optimal.
  - Select a basic variable x_l whose constraint most severely limits the nonbasic variable x_e. Ties are broken arbitrarily. LP is unbounded if no such x_l exists.
  - Pivot.

SIMPLEX – Open Issues

- How to decide that LP is feasible?
- What to do if the initial basic solution is infeasible?
- How to select entering and leaving variables?
- How to decide that LP is unbounded?
- Does SIMPLEX terminate?
- Does it terminate with an optimal solution?

Termination

- SIMPLEX computes a feasible basic solution during each iteration.
- When does SIMPLEX terminate?
  - When all coefficients in the objective function are negative.
  - When it becomes obvious that LP is unbounded.

SIMPLEX: LP is Unbounded

- Can x_2 be increased without violating feasibility? By how much?

$$
\begin{array}{rcccccccc}
z & = & 111/4 & + & x_2/16 & - & x_5/8 & - & 11x_6/16 \\
x_4 & = & 69/4 & + & 3x_2/16 & + & 5x_5/8 & - & x_6/16 \\
x_3 & = & 3/2 & + & 3x_2/8 & - & x_5/4 & + & x_6/8 \\
x_1 & = & 33/4 & + & x_2/16 & + & x_5/8 & - & 5x_6/16
\end{array}
$$

- If x_2 is increased then x_4 also increases.
- If x_2 is increased then x_3 also increases.
- If x_2 is increased then x_1 also increases.
- No constraint is binding; LP is unbounded.

Termination

- The number of basic solutions is finite: Number of basic variables is *m.* They are selected from among *m*+*n* variables. This can be done in

$$\binom{m+n}{m}=\frac{(n+m)!}{n!m!}$$

  ways
- Each basic solution has exactly one objective value. If the objective value increases at each iteration, we will eventually end up with a solution where the coefficients of the objective function are all negative (or we will realize that the LP is unbounded).
- Is it possible that the objective value does not change? YES.
- Is it possible that the same basic solution appears twice? YES

Degeneracy

$$
\begin{aligned}
z &= 0 + x_1 + x_2 + x_3 \\
x_4 &= 8 - x_1 - x_2 \\
x_5 &= x_2 - x_3
\end{aligned}
$$

$$
\begin{aligned}
z &= 8 + x_3 - x_4 \\
x_1 &= 8 - x_2 - x_4 \\
x_5 &= x_2 - x_3
\end{aligned}
$$

$$
\begin{aligned}
z &= 8 + x_2 - x_4 - x_5 \\
x_1 &= 8 - x_2 - x_4 \\
x_3 &= x_2 - x_5
\end{aligned}
$$

Degeneracy

$$
\begin{aligned}
z &= 8 + x_2 - x_4 - x_5 \\
x_1 &= 8 - x_2 - x_4 \\
x_3 &= x_2 - x_5
\end{aligned}
$$

$$
\begin{aligned}
z &= 16 - x_1 - 2\mathrm{x}_4 - x_5 \\
x_2 &= 8 - x_1 - x_4 \\
x_3 &= 8 - x_1 - x_4 - x_5
\end{aligned}
$$

Cycling

- Is it possible to get the same basis (slack form) more than once? Than SIMPLEX has a problem.
- It is in fact possible. This happens even if we use specific rules for selecting entering and leaving variables at each iteration, such as
  - The entering variable will always be a nonbasic variable with the largest positive coefficient in the z-row.
  - If two or more basic variables compete for leaving the basis, then the candidate with the smallest subscript is made to leave.

Cycling - Example

$$
\begin{aligned}
z &= 0 + 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
x_5 &= 0 - 0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\
x_6 &= 0 - 0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\
x_7 &= 1 - x_1
\end{aligned}
$$

$$
\begin{aligned}
z &= 0 + 53x_2 + 41x_3 - 204x_4 - 20x_5 \\
x_1 &= 0 + 11x_2 + 5x_3 - 18x_4 - 2x_5 \\
x_6 &= 0 - 4x_2 - 2x_3 + 8x_4 + x_5 \\
x_7 &= 1 - 11x_2 - 5x_3 + 18x_4 + 2x_5
\end{aligned}
$$

$$
\begin{aligned}
z &= 0 + 14.5x_3 - 98x_4 - 6.75x_5 - 13.25x_6 \\
x_1 &= 0 - 0.5x_3 + 4x_4 + 0.75x_5 - 2.75x_6 \\
x_2 &= 0 - 0.5x_3 + 2x_4 + 0.25x_5 - 0.25x_6 \\
x_7 &= 1 + 0.5x_3 - 4x_4 - 0.75x_5 - 13.25x_6
\end{aligned}
$$

Cycling - Example

$$
\begin{aligned}
z &= 0 + 18x_4 + 15x_5 - 93x_6 - 29x_1 \\
x_2 &= 0 - 2x_4 - 0.5x_5 + 2.5x_6 + x_1 \\
x_3 &= 0 + 8x_4 + 1.5x_5 - 5.5x_6 - 2x_1 \\
x_7 &= 1 - x_1
\end{aligned}
$$

$$
\begin{aligned}
z &= 0 + 10.5x_5 - 70.5x_6 - 20x_1 - 9x_2 \\
x_3 &= 0 - 0.5x_5 + 4.5x_6 + 2x_1 - 4x_2 \\
x_4 &= 0 - 0.25x_5 + 1.25x_6 + 0.5x_1 - 0.5x_2 \\
x_7 &= 1 - x_1
\end{aligned}
$$

Cycling - Example

$$
\begin{aligned}
z &= 0 + 24x_6 + 22x_1 - 93x_2 - 21x_3 \\
x_4 &= 0 - x_6 - 0.5x_1 + 1.5x_2 + 0.5x_3 \\
x_5 &= 0 + 9x_6 + 4x_1 - 8x_2 - 2x_3 \\
x_7 &= 1 - x_1
\end{aligned}
$$

$$
\begin{aligned}
z &= 0 + 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
x_5 &= 0 - 0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\
x_6 &= 0 - 0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\
x_7 &= 1 - x_1
\end{aligned}
$$

Cycling

- **Claim**: If SIMPLEX fails to terminate then it cycles.
- **Proof:** Suppose that SIMPLEX does not cycle but it fails to terminate. So it must generate infinite number of different slack forms.
- However, the number of different bases is finite. If we can show that a slack form for a given basis is unique then we have a contradiction. Proof in CLRS.

Avoiding Cycling

- Perturb input slightly so that it is impossible to have two basic solutions with the same objective value. The perturbation must be such that basic variables in the optimal solutions of the original and perturbed problems are the same.
- Always choose the entering and leaving variables with the smallest indicies.

Small Indicies

$$
\begin{aligned}
z &= 0 + 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
x_5 &= 0 - 0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\
x_6 &= 0 - 0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\
x_7 &= 1 - x_1
\end{aligned}
$$

$$
\begin{aligned}
z &= 0 + 53x_2 + 41x_3 - 204x_4 - 20x_5 \\
x_1 &= 0 + 11x_2 + 5x_3 - 18x_4 - 2x_5 \\
x_6 &= 0 - 4x_2 - 2x_3 + 8x_4 + x_5 \\
x_7 &= 1 - 11x_2 - 5x_3 + 18x_4 + 2x_5
\end{aligned}
$$

$$
\begin{aligned}
z &= 0 + 14.5x_3 - 98x_4 - 6.75x_5 - 13.25x_6 \\
x_1 &= 0 - 0.5x_3 + 4x_4 + 0.75x_5 - 2.75x_6 \\
x_2 &= 0 - 0.5x_3 + 2x_4 + 0.25x_5 - 0.25x_6 \\
x_7 &= 1 + 0.5x_3 - 4x_4 - 0.75x_5 - 13.25x_6
\end{aligned}
$$

Small Indicies

$$
\begin{array}{lcccccccccc}
z & = & 0 & + & 18x_4 & + & 15x_5 & - & 93x_6 & - & 29x_1 \\
x_2 & = & 0 & - & 2x_4 & - & 0.5x_5 & + & 2.5x_6 & + & x_1 \\
x_3 & = & 0 & + & 8x_4 & + & 1.5x_5 & - & 5.5x_6 & - & 2x_1 \\
x_7 & = & 1 & & & & & & & - & x_1
\end{array}
$$

$$
\begin{array}{lcccccccccc}
z & = & 0 & + & 10.5x_5 & - & 70.5x_6 & - & 20x_1 & - & 9x_2 \\
x_3 & = & 0 & - & 0.5x_5 & + & 4.5x_6 & + & 2x_1 & - & 4x_2 \\
x_4 & = & 0 & - & 0.25x_5 & + & 1.25x_6 & + & 0.5x_1 & - & 0.5x_2 \\
x_7 & = & 1 & & & & & - & x_1 & &
\end{array}
$$

Small Indicies

$$
\begin{aligned}
z &= 0 + 24x_6 + 22x_1 - 93x_2 - 21x_3 \\
x_5 &= 0 + 9x_6 + 4x_1 - 8x_2 - 2x_3 \\
x_4 &= 0 - x_6 - 0.5x_1 + 1.5x_2 + 0.5x_3 \\
x_7 &= 1 - x_1
\end{aligned}
$$

$$
\begin{aligned}
z &= 0 - 20x_6 - 27x_2 + x_3 - 44x_4 \\
x_5 &= 0 + x_6 + 4x_2 + 2x_3 - 8x_4 \\
x_1 &= 0 - 2x_6 + 3x_2 + x_3 - 2x_4 \\
x_7 &= 1 + 2x_6 - 3x_2 - x_3 + 2x_4
\end{aligned}
$$

$$
\begin{aligned}
z &= 1 - 18x_6 - 30x_2 - 42x_4 - x_7 \\
x_5 &= 2 + 5x_6 - 2x_2 - 4x_4 - 2x_7 \\
x_1 &= 1 - x_7 \\
x_3 &= 1 + 2x_6 - 3x_2 + 2x_4 - x_7
\end{aligned}
$$

Infeasible First Basic Solution

$$
\begin{array}{llllll}
max & 2x_1 & - & x_2 & & \\
s.t. & 2x_1 & - & x_2 & \leq & 2 \\
 & x_1 & - & 5x_2 & \leq & -4
\end{array}
$$

$$
\begin{array}{lllllll}
z & = & 0 & + & 2x_1 & - & x_2 \\
x_3 & = & 2 & - & 2x_1 & + & x_2 \\
x_4 & = & -4 & - & x_1 & + & 5x_2
\end{array}
$$

- x_1 and x_2 is set to 0 in the first basic solution.
- This solution is infeasible since x_4 = -4.

Auxiliary LP

- We will define a related auxiliary LP.
- This auxiliary LP is always feasible and bounded.
- Optimal value of this auxiliary LP will indicate if the original LP is feasible.
- If original LP is feasible, then the slack form of the auxiliary LP will yield a feasible basic solution to the original LP (and the corresponding slack form).

Auxiliary Linear Program

- L: LP in standard form:

$$\begin{array}{lrcll} max & \sum_{j=1}^{n} c_j x_j & & & \\ s.t. & \sum_{j=1}^{n} a_{ij} x_j & \leq & b_i & for \quad i=1,2,\ldots,m \\ & x_j & \geq & 0 & for \quad j=1,2,\ldots,n \end{array}$$

- L_{aux}: Auxiliary LP:

$$\begin{array}{lrcll} max & -x_0 & & & \\ s.t. & \sum_{j=1}^{n} a_{ij} x_j - x_0 & \leq & b_i & for \quad i=1,2,\ldots,m \\ & x_j & \geq & 0 & for \quad j=0,1,2,\ldots,n \end{array}$$

- L_{aux} is bounded and feasible.

INITIALIZE_SIMPLEX

- Let l be the index of the minimum b_l (most negative).
- Special pivot: x_0 enters the basis, x_l leaves the basis.
- Keep pivoting in standard way until SIMPLEX terminates.
  - If the returned objective value is 0, then use the returned slack form (with x_0 removed) as a starting point for original LP problem.
  - If the returned objective value is negative, then the original LP problem is infeasible.

INITIALIZE_SIMPLEX - Example

$$
\begin{array}{lclclclcl}
z & = & 0 & & & & & - & x_0 \\
x_3 & = & 2 & - & 2\mathrm{x}_1 & + & x_2 & + & x_0 \\
x_4 & = & -4 & - & x_1 & + & 5\mathrm{x}_2 & + & x_0
\end{array}
$$

$$
\begin{array}{lclclclcl}
z & = & & & & & & - & (x_4 + 4 + x_1 - 5\mathrm{x}_2) \\
x_3 & = & 2 & - & 2\mathrm{x}_1 & + & x_2 & + & (x_4 + 4 + x_1 - 5\mathrm{x}_2) \\
x_0 & = & 4 & + & x_1 & - & 5\mathrm{x}_2 & + & x_4
\end{array}
$$

$$
\begin{array}{lclclclcl}
z & = & -4 & - & x_1 & + & 5\mathrm{x}_2 & - & x_4 \\
x_3 & = & 6 & - & x_1 & - & 4\mathrm{x}_2 & + & x_4 \\
x_0 & = & 4 & + & x_1 & - & 5\mathrm{x}_2 & + & x_4
\end{array}
$$

feasible!!

INITIALIZE_SIMPLEX - Example

$$
\begin{array}{ccccccccc}
z & = & -4 & - & x_1 & + & 5\mathrm{x}_2 & - & x_4 \\
x_3 & = & 6 & - & x_1 & - & 4\mathrm{x}_2 & + & x_4 \\
x_0 & = & 4 & + & x_1 & - & 5\mathrm{x}_2 & + & x_4
\end{array}
$$

$$
\begin{array}{ccccccccc}
z & = & -4 & - & x_1 & + & 5\left(4/5+x_1/5+x_4/5-x_0/5\right) & - & x_4 \\
x_3 & = & 6 & - & x_1 & - & 4\left(4/5+x_1/5+x_4/5-x_0/5\right) & + & x_4 \\
x_2 & = & 4/5 & + & x_1/5 & - & x_0/5 & + & x_4/5
\end{array}
$$

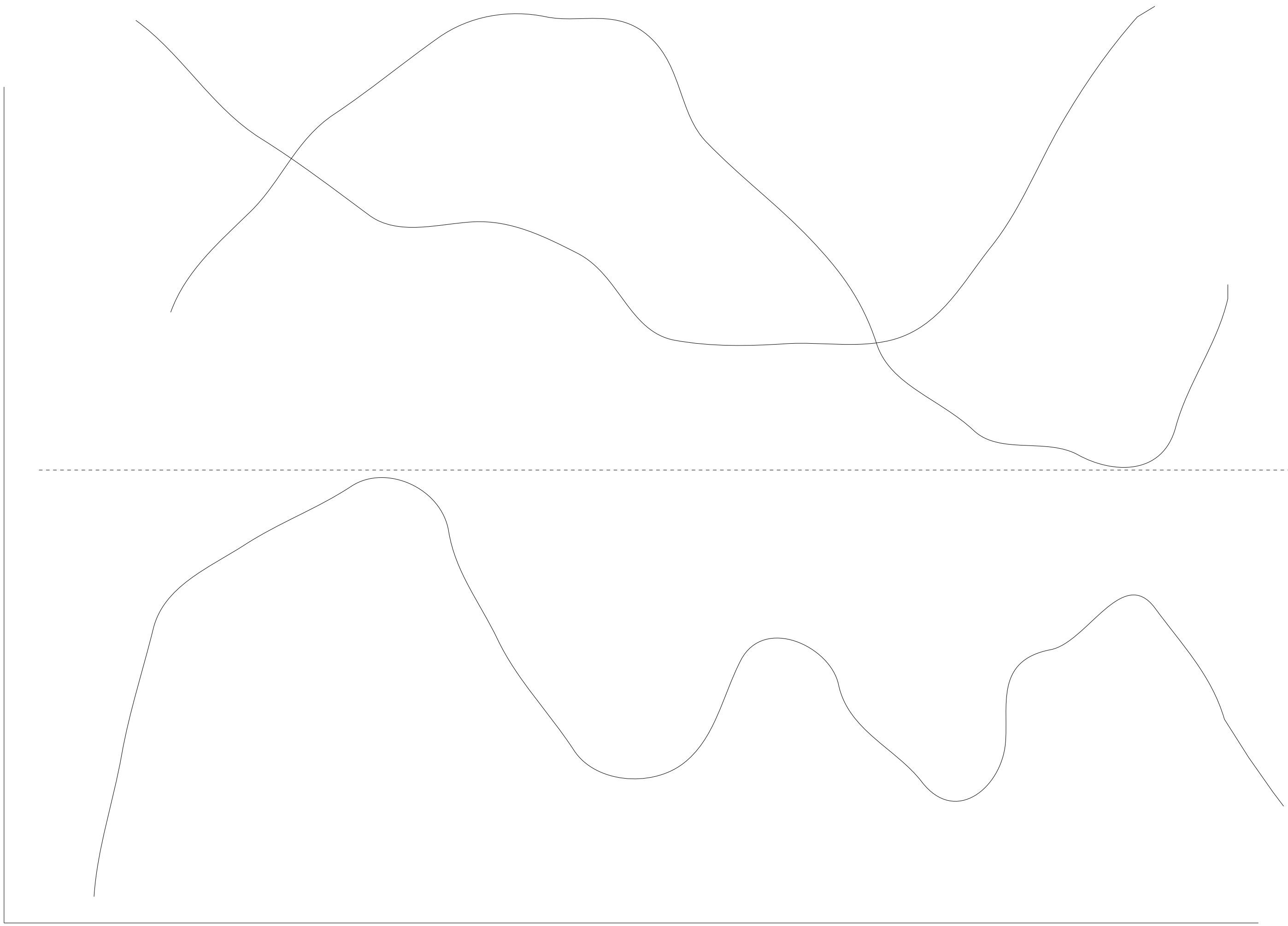
INITIALIZE_SIMPLEX - Example

$$\begin{array}{lllllllll}
z & = & 0 & & & - & x_0 & & \\
x_3 & = & 14/5 & - & 9x_1/5 & + & 4x_0/5 & + & x_4/5 \\
x_2 & = & 4/5 & + & x_1/5 & - & x_0/5 & + & x_4/5
\end{array}$$

$$\begin{array}{lllllllll}
z & = & 0 & + & 2x_1 & - & \left(4/5 + x_1/5 - x_0/5 + x_4/5\right) & & \\
x_3 & = & 14/5 & - & 9x_1/5 & + & 4x_0/5 & + & x_4/5 \\
x_2 & = & 4/5 & + & x_1/5 & - & x_0/5 & + & x_4/5
\end{array}$$

$$\begin{array}{lllllll}
z & = & -4/5 & + & 9x_1/4 & - & x_4/5 \\
x_3 & = & 14/5 & - & 9x_1/5 & + & x_4/5 \\
x_2 & = & 4/5 & + & x_1/5 & + & x_4/5
\end{array}$$

Duality



Upper Bounds on Maximization LP

$$
\begin{array}{lccccccccc}
max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
s.t. & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
 & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
 & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}
$$

- Multiply second constraint by 5/3:

$$\frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}$$

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq \frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}$$

Upper Bounds on Maximization LP

$$
\begin{array}{lcccccccccc}
\max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
s.t. & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
 & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
 & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}
$$

- Add the second constraint to the third constraint:

$$4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58$$

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq 4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58$$

Upper Bounds on Maximization LP

$$\begin{array}{lccccccccc} max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\ s.t. & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\ & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\ & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

- Construct a linear combination of the constraints using nonnegative multipliers y_1, y_2, and y_3:

$$y_1(x_1 - x2 - x_3 + 3x_4) + y_2(5x_1 + x_2 + 3x_3 + 8x_4) + y_3(-x_1 + 2x_2 + 3x_3 - 5x_4) \leq y_1 + 55y_2 + 3y_3$$

$$(y_1 + 5y_2 - y_3)x_1 + (-y_1 + y_2 + 2y_3)x_2 + (-y_1 + 3y_2 + 3y_3)x_3 + (3y_1 + 8y_2 - 5y_3)x_4 \leq y_1 + 55y_2 + 3y_3$$

- Left-hand side will be an upper bound for the LP if the coefficient at each x_j is at least as big as the corresponding coefficient in the objective function

$$y_1 + 5y_2 - y_3 \geq 4 \qquad -y_1 + y_2 + 2y_3 \geq 1 \qquad 3y_1 + 8y_2 - 5y_3 \geq 3 \qquad -y_1 + 3y_2 + 3y_3 \geq 5$$

- Any set of nonnegative multipliers y_i satisfying these inequalities also satisfy $4x_1 + x_2 + 5x_3 + 3x_4 \leq y_1 + 55y_2 + 3y_3$
- Good upper bound: minimize right-hand side s.t. constraints.

Good Upper Bound

$$
\begin{array}{lccccccc}
min & y_1 & + & 55y_2 & + & 3y_3 & & \\
s.t. & y_1 & + & 5y_2 & - & y_3 & \geq & 4 \\
 & -y_1 & + & y_2 & + & 2y_3 & \geq & 1 \\
 & -y_1 & + & 3y_2 & + & 3y_3 & \geq & 5 \\
 & 3y_1 & + & 8y_2 & - & 5y_3 & \geq & 3 \\
 & y_1, & & y_2, & & y_3 & \geq & 0
\end{array}
$$

LP in Standard Form and Its Dual

$$
\begin{array}{lcccccccc}
\max & 3x_1 & + & x_2 & + & 2x_3 & & \\
s.t. & x_1 & + & x_2 & + & 3x_3 & \leq & 30 \\
 & 2x_1 & + & 2x_2 & + & 5x_3 & \leq & 24 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 36 \\
 & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}
$$

$$
\begin{array}{lcccccccc}
\min & 30y_1 & + & 24y_2 & + & 36y_3 & & \\
s.t. & y_1 & + & 2y_2 & + & 4y_3 & \geq & 3 \\
 & y_1 & + & 2y_2 & + & y_3 & \geq & 1 \\
 & 3y_1 & + & 5y_2 & + & 2y_3 & \geq & 2 \\
 & y_1 & , & y_2 & , & y_3 & \geq & 0
\end{array}
$$

LP in Standard Form and Its Dual

$$
\begin{aligned}
max \quad & \sum_{j=1}^{n} c_j x_j \\
s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad && for \quad i = 1, 2, \dots, m \\
& x_j \geq 0 && for \quad j = 1, 2, \dots, n
\end{aligned}
$$

$$
\begin{aligned}
min \quad & \sum_{i=1}^{m} b_i y_i \\
s.t. \quad & \sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad && for \quad j = 1, 2, \dots, n \\
& y_i \geq 0 && for \quad i = 1, 2, \dots, m
\end{aligned}
$$

Weak Duality

- ***x****: feasible solution to the primal LP.
- ***y****: feasible solution to the dual LP.
- Claim $\sum_{j=1}^{n} c_j x_j^* \leq \sum_{i=1}^{m} b_i y_i^*$
- Proof:

$$
\begin{aligned}
\sum_{j=1}^{n} c_j x_j^* &\leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i^*\right) x_j^* && \text{from the dual} \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j^*\right) y_i^* \\
&\leq \sum_{i=1}^{m} b_i y_i^* && \text{from the primal}
\end{aligned}
$$

Importance of Weak Duality

- ***x****: feasible solution to the primal LP.
- ***y****: feasible solution to the dual LP.
- If

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

then ***x**** and ***y**** are optimal solutions to the primal and to the dual LPs, respectively.

Final Feasible Basic Solution and Corresponding Dual Solution

$$
\begin{array}{llcccccccc}
max & z & = & 28 & - & x_3/6 & - & x_5/6 & - & 2x_6/3 \\
s.t. & x_4 & = & 18 & - & x_3/2 & + & x_5/2 & + & 0x_6 \\
 & x_2 & = & 4 & - & 8x_3/3 & - & 2x_5/3 & + & x_6/3 \\
 & x_1 & = & 8 & + & x_3/6 & + & x_5/6 & - & x_6/3
\end{array}
$$

Basic variables: x_1=8, x_2=4, x_4=18

Objective value z = 28

$$
y_i = \begin{cases} -c'_{n+i} & \text{if}\,(n+i)\in N \\ 0 & \text{otherwise} \end{cases}
$$

y_1 = 0 (since x_4 is basic), y_2 = 1/6, y_3 = 2/3

$$
\begin{array}{lcccccccc}
min & 30y_1 & + & 24y_2 & + & 36y_3 & & \\
s.t. & y_1 & + & 2y_2 & + & 4y_3 & \geq & 3 \\
 & y_1 & + & 2y_2 & + & y_3 & \geq & 1 \\
 & 3y_1 & + & 5y_2 & + & 2y_3 & \geq & 2 \\
 & y_1 & , & y_2 & , & y_3 & \geq & 0
\end{array}
$$

Feasible Solution to the Dual

$$
\begin{array}{lccccccc}
min & 30y_1 & + & 24y_2 & + & 36y_3 & & \\
s.t. & y_1 & + & 2y_2 & + & 4y_3 & \geq & 3 \\
 & y_1 & + & 2y_2 & + & y_3 & \geq & 1 \\
 & 3y_1 & + & 5y_2 & + & 2y_3 & \geq & 2 \\
 & y_1 & , & y_2 & , & y_3 & \geq & 0
\end{array}
$$

$y_1 = 0$ (since x_4 is basic)
$y_2 = 1/6$
$y_3 = 2/3$

Objective value: 30 x 0 + 24 x 1/6 + 36 x 2/3 = 28

1 x 0 + 2 x 1/6 + 4 x 2/3 = 3
1 x 0 + 2 x 1/6 + 1 x 2/3 = 1
3 x 0 + 5 x 1/6 + 2 x 2/3 = 13/6

Practical Implications

- If $m >> n$ in the primal, then the number of constraint in the dual will be much smaller than in the primal.
- Number of pivots in SIMPLEX is usually less than $1.5m$ and only rarely is higher than $3m$.
- Number of pivots increases very slowly with n.
- Solving dual will in such cases be more efficient.