Good Afternoon.

Advanced algorithms and data structures

Lecture 5: Hashing

Jacob Holm (jaho@di.ku.dk)

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Today's Lecture

Hashing
- Hashing fundamentals
- Application: Unordered sets/Hashing with chaining
- Application: Signatures
- Practical hash functions
- Application: Coordinated sampling

Preliminaries

Notation:

For $n \in \mathbb{N}$:

$$\begin{aligned}[n] &= \{0, \ldots, n-1\} \\ [n]_+ &= \{1, \ldots, n-1\}\end{aligned}$$

Iverson bracket:

$$[\text{condition}] = \begin{cases} 1 & \text{if condition is true} \\ 0 & \text{if condition is false} \end{cases}$$

For a random variable X:

$$\begin{aligned} \mu_X &= \mathbb{E}[X] && \text{(expectation)} \\ \mathrm{Var}[X] &= \mathbb{E}[(X-\mu_X)^2] && \text{(variance)} \\ \sigma_X &= \sqrt{\mathrm{Var}[X]} && \text{(std. deviation)} \end{aligned}$$

Inequalities:

Expectation of indicator variable X:

$$\mathbb{E}[X] = \Pr[X = 1]$$

Linearity of expectation:

$$\mathbb{E}\Big[\sum_i X_i\Big] = \sum_i \mathbb{E}[X_i]$$

Sum of pairwise indep. variances:

$$\mathrm{Var}\Big[\sum_i X_i\Big] = \sum_i \mathrm{Var}[X_i]$$

Union bound:

$$\Pr[A \cup B] \le \Pr[A] + \Pr[B]$$

Markovs Inequality: For $X \geq 0, t > 0$

$$\Pr[X \ge t] \le \frac{\mathbb{E}[X]}{t} = \frac{\mu_X}{t}$$

Chebyshevs Inequality: For $t > 0$

$$\Pr\Big[|X - \mu_X| \ge t\sigma_X\Big] \le \frac{1}{t^2}$$

AADS Lecture 5 (Hashing), Part 1

Hashing fundamentals

Hash function

Given a (typically large) universe U of keys, and a positive integer m.

Definition

A *random hash function* $h : U \to [m]$ is a randomly chosen function from some family of functions mapping U to $[m]$. Equivalently, it is a function h such that for each $x \in U$, $h(x) \in [m]$ is a random variable.

Cryptographic "hash functions" such as MD5, SHA-1, and SHA-256 are not *random* hash functions, and do not have most of the properties we want here. Do not confuse them!

Hash function

When discussing random hash functions, we usually care about

1. Space (*seed size*) needed to represent h.
2. Time needed to calculate $h(x)$ given $x \in U$.
3. Properties of the random variable.

Hash function types

Definition

A hash function $h : U \to [m]$ is *truly random* if the variables $h(x)$ for $x \in U$ are independent and uniform in $[m]$.

Impractical, why? Space! Require $|U| \log_2 m$ bits to represent.

Definition

A random hash function $h : U \to [m]$ is *universal* if, for all $x \neq y \in U$: $\Pr_h[h(x) = h(y)] \leq \frac{1}{m}$.

Definition

A random hash function $h : U \to [m]$ is *strongly universal* (a.k.a. 2-independent) if,

- Each key is hashed uniformly into $[m]$.
  (i.e. $\forall x \in U, q \in [m] : \Pr_h[h(x) = q] = \frac{1}{m}$)
- Any two distinct keys hash independently.

Or equivalently, if for all $x \neq y \in U$, and $q, r \in [m]$:
$\Pr_h[h(x) = q \wedge h(y) = r] = \frac{1}{m^2}$.

There are $m^{|U|}$ possible functions from U to $[m]$, so it takes at least $\log_2(m^{|U|}) = |U| \log_2 m$ bits to store which one we picked.

We use $\Pr_h[\cdots]$ or $\Pr_h[\cdots]$ instead of just $\Pr[\cdots]$ to make it clear that the probability is based on the random choice of h, and *not* on e.g. x, y being chosen at random.

For many purposes c-approximately universal hash functions for some small constant c are enough. We will see examples of such functions a little later today.

Hash function types

Definition

A hash function $h : U \to [m]$ is *truly random* if the variables $h(x)$ for $x \in U$ are independent and uniform in $[m]$.

Impractical, why? Space! Require $|U| \log_2 m$ bits to represent.

Definition

A random hash function $h : U \to [m]$ is *c-approximately universal* if, for all $x \neq y \in U$: $\Pr_h[h(x) = h(y)] \leq \frac{c}{m}$.

Definition

A random hash function $h : U \to [m]$ is *c-approximately strongly universal* if,

- Each key is hashed c-approximately uniformly into $[m]$.
  (i.e. $\forall x \in U, q \in [m] : \Pr_h[h(x) = q] \leq \frac{c}{m}$)
- Any two distinct keys hash independently.

Implying that for all $x \neq y \in U$, and $q, r \in [m]$:
$\Pr_h[h(x) = q \wedge h(y) = r] \leq$? (See Assignment 3 exercise 3.2).

There are $m^{|U|}$ possible functions from U to $[m]$, so it takes at least $\log_2(m^{|U|}) = |U| \log_2 m$ bits to store which one we picked.

We use $\Pr_h[\cdots]$ or $\Pr\limits_h[\cdots]$ instead of just $\Pr[\cdots]$ to make it clear that the probability is based on the random choice of h, and *not* on e.g. x, y being chosen at random.

For many purposes c-approximately universal hash functions for some small constant c are enough. We will see examples of such functions a little later today.

AADS Lecture 5 (Hashing), Part 2

Application:
Unordered sets/Hashing with chaining

Unordered sets

Maintain a set S of at most n keys from some unordered universe U, under

$\textsc{insert}(x, S)$ Insert key x into S.

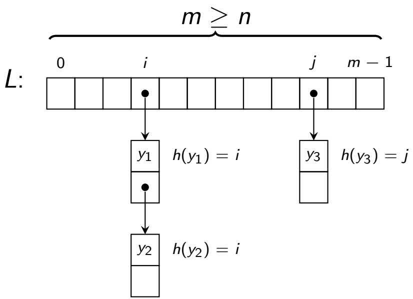
$\textsc{delete}(x, S)$ Delete key x from S.

$\textsc{member}(x, S)$ Return $x \in S$.

We could use some form of balanced tree to store S, but they usually take $\mathcal{O}(\log n)$ or $\mathcal{O}(\log \log U)$ time per operation, and we want each operation to run in expected constant time.

Hashing with chaining

Idea: Pick $m \geq n$ and a *universal* $h : U \to [m]$.
Store array L, where
$L[i] =$ linked list over $\{y \in S \mid h(y) = i\}$.



Then $x \in S \iff x \in L[h(x)]$.

Each operation takes $\mathcal{O}(|L[h(x)]| + 1)$ time.

Hashing with chaining

Theorem

For $x \notin S$, $\mathbb{E}_h\left[\left|L[h(x)]\right|\right] \leq 1$

Proof.

$$
\begin{aligned}
\mathbb{E}_h\left[\left|L[h(x)]\right|\right] &= \mathbb{E}_h\left[\left|\{y \in S \mid h(y) = h(x)\}\right|\right] \\
&= \mathbb{E}_h\left[\sum_{y \in S} [h(y) = h(x)]\right] \\
&= \sum_{y \in S} \mathbb{E}_h\left[[h(y) = h(x)]\right] \\
&= \sum_{y \in S} \Pr_h[h(y) = h(x)] \\
&\leq |S| \tfrac{1}{m} \leq \tfrac{n}{m} \leq 1
\end{aligned}
$$

$\square$

By definition of $L[i] := \{y \in S \mid h(y) = i\}$.
Here we use the *Iverson Bracket* notation

$$
[P] := \begin{cases} 1 & \text{if } P \text{ is true} \\ 0 & \text{if } P \text{ is false} \end{cases}
$$

This can often be used as a shorthand for an indicator variable.
In this case $[h(y) = h(x)]$ becomes an indicator variable for the event $h(y) = h(x)$.
Linearity of expectation.
Expectation of indicator variable
Since $x \notin S$ and $y \in S$, we have $x \neq y$.
Then by definition of a universal hash function $h : U \to [m]$, $\Pr_h[h(y) = h(x)] \leq \frac{1}{m}$.

AADS Lecture 5 (Hashing), Part 3

Application: Signatures

Application: Signatures

Problem: Assign a unique "signature" to each $x \in S \subseteq U$, $|S| = n$.

Solution: Use universal hash function $s : U \to [n^3]$.

Then by a "union bound"

$$\begin{aligned}
\Pr_s[\exists \{x, y\} \subseteq S \mid s(x) = s(y)] &\le \sum_{\{x,y\} \subseteq S} \Pr_s[s(x) = s(y)] \\
&\le \frac{\binom{n}{2}}{n^3} \\
&< \frac{1}{2n}
\end{aligned}$$

Thus with "high probability" we have no collisions.

AADS Lecture 5 (Hashing), Part 4

Practical hash functions

Multiply-mod-prime

Let $U = [u]$ and pick prime $p \geq u$. For any $a, b \in [p]$, and $m < u$, let $h^m_{a,b} : U \to [m]$ be

$$h^m_{a,b}(x) = ((ax + b) \bmod p) \bmod m$$

Is this a random hash function? NO!

Multiply-mod-prime

Let $U = [u]$ and pick prime $p \geq u$. For any $a, b \in [p]$, and $m < u$, let $h^m_{a,b} : U \to [m]$ be

$$h^m_{a,b}(x) = ((ax + b) \bmod p) \bmod m$$

Choose $a, b \in [p]$ independently and uniformly at random, and let $h(x) := h^m_{a,b}(x)$.
Then $h : U \to [m]$ is a 2-approximately strongly universal hash function.

Multiply-shift

Let $U = [2^w]$ and $m = 2^\ell$. For any odd $a \in [2^w]$ define

$$h_a(x) := \left\lfloor \frac{(ax) \bmod 2^w}{2^{w-\ell}} \right\rfloor$$

Choose odd $a \in [2^w]$ uniformly at random, and let
$h(x) := h_a(x)$.
Then $h : U \to [m]$ is a 2-approximately universal hash function.

(Assignment 3 exercise 3.4 asks you to show that it is not c-approximately strongly universal for any constant c).

Multiply-shift, C

For $U = [2^{64}]$ the C code looks like this:

```
#include<stdint.h>
uint64_t hash(uint64_t x, uint64_t l, uint64_t a)
{
  return (a*x) >> (64-l);
}
```

Strong Multiply-shift

Let $U = [2^w]$ and $m = 2^\ell$, and pick $\bar{w} \geq w + \ell - 1$. For any pair $(a, b) \in [2^{\bar{w}}]^2$ define

$$h_{a,b}(x) := \left\lfloor \frac{(ax + b) \bmod 2^{\bar{w}}}{2^{\bar{w}-\ell}} \right\rfloor$$

Choose $a, b \in [2^{\bar{w}}]$ independently and uniformly at random, and let $h(x) := h_{a,b}(x)$.
Then $h : U \to [m]$ is a strongly universal hash function.

Strong Multiply-shift, C

For $\ell \leq w = 32$ and $\bar{w} = 64$ we have $U = [2^{32}]$ and the C code looks like this:

```
#include<stdint.h>
uint32_t hash(uint32_t x, uint32_t l,
              uint64_t a, uint64_t b)
{
  return (a*x+b) >> (64-l);
}
```

AADS Lecture 5 (Hashing), Part 5

Application: Coordinated sampling

Application: Coordinated sampling

Suppose we have a bunch of *agents* that each observe some set of events from some universe U. Let $A_i \subseteq U$ denote the set of events seen by agent i, and suppose $|A_i|$ is large so only a small sample $S_i \subseteq A_i$ is actually stored.

If each agent independently just samples a random subset of the seen events, there is very little chance that two agents that see an event make the same decision.
$\implies$ The samples are incomparable.

Coordinated sampling means that all agents that see an event make the same decision about whether to store it.
$\implies$ Samples can be combined, i.e.

- $S_i \cup S_j$ is a sample of $A_i \cup A_j$
- $S_i \cap S_j$ is a sample of $A_i \cap A_j$

Application: Coordinated sampling

Let $h : U \to [m]$ be a strongly universal hash function, and let $t \in \{0, \ldots, m\}$. Send h and t to all the agents.

Each agent samples $x \in U$ iff $h(x) < t$.

Thus if an agent sees the set $A \subseteq U$, the set $S_{h,t}(A) := \{x \in A \mid h(x) < t\}$ is sampled. Note that

- $S_{h,t}(A_i) \cup S_{h,t}(A_j) = S_{h,t}(A_i \cup A_j)$
- $S_{h,t}(A_i) \cap S_{h,t}(A_j) = S_{h,t}(A_i \cap A_j)$

Each $x \in A$ is sampled with probability $\Pr_h[h(x) < t] = \frac{t}{m}$.
Why? Strong universality $\implies$ $h(x)$ uniform in $[m]$

For any $A \subseteq U$, $\mathbb{E}_h[|S_{h,t}(A)|] = |A| \cdot \frac{t}{m}$.

Thus we have an unbiased estimate $|A| \approx \frac{m}{t} \cdot |S_{h,t}(A)|$.

How good is this estimate, i.e. what can we say about the *relative error* $:= \left|\frac{\text{estimated value}}{\text{actual value}} - 1\right|$?

$$
\begin{aligned}
\mathop{\mathbb{E}}_h[|S_{h,t}(A)|] &= \mathop{\mathbb{E}}_h\left[\sum_{x \in A} [h(x) < t]\right] \\
&= \sum_{x \in A} \mathop{\mathbb{E}}_h[[h(x) < t]] \\
&= \sum_{x \in A} \Pr_h[h(x) < t] \\
&= \sum_{x \in A} \frac{t}{m} \\
&= |A| \cdot \frac{t}{m}
\end{aligned}
$$

Concentration bound

Lemma

Let $X = \sum_{a \in A} X_a$ *where the* X_a *are pairwise independent* 0–1 *variables.*
Let $\mu = \mathbb{E}[X]$*. Then* $\mathrm{Var}[X] \le \mu$*, and for any* $q > 0$*,*

$$\Pr[|X - \mu| \ge q\sqrt{\mu}] \le \frac{1}{q^2}$$

Proof (not curriculum).

For $a \in A$ let $p_a = \Pr[X_a = 1]$. Then $p_a = \mathbb{E}[X_a]$ and

$$\begin{aligned}
\mathrm{Var}[X_a] &= \mathbb{E}[(X_a - p_a)^2] = (1 - p_a)(0 - p_a)^2 + p_a(1 - p_a)^2 \\
&= (p_a^2 + p_a(1 - p_a))(1 - p_a) = p_a(1 - p_a) \le p_a \\
\mathrm{Var}[X] &= \mathrm{Var}\left[\sum_{a \in A} X_a\right] = \sum_{a \in A} \mathrm{Var}[X_a] \le \sum_{a \in A} p_a = \mu
\end{aligned}$$

Finally, since $\sigma_X = \sqrt{\mathrm{Var}[X]} \le \sqrt{\mu}$ we get:

$$\begin{aligned}
\Pr[|X - \mu| \ge q\sqrt{\mu}] &\le \Pr[|X - \mu| \ge q\sigma_X] \\
&\le \frac{1}{q^2} \qquad \text{(Chebyshev's ineq.)} \ \Box
\end{aligned}$$

Application: Coordinated sampling

Let's apply this lemma to the estimate $|A| \approx \frac{m}{t}|S_{h,t}(A)|$ from our coordinated sampling.

Let $X = |S_{h,t}(A)|$ and for $a \in A$ let $X_a = [h(a) < t]$. Then $X = \sum_{a\in A} X_a$ and for any $a, b \in A$, X_a and X_b are independent. Also, let $\mu = \mathbb{E}_h[X] = \frac{t}{m}|A|$.

Then for any $q > 0$,

$$\begin{aligned}
\Pr_h\left[\left|\frac{\frac{m}{t}|S_{h,t}(A)|}{|A|} - 1\right| \geq q \cdot \frac{1}{\sqrt{\frac{t}{m}|A|}}\right]
&= \Pr_h\left[\left||S_{h,t}(A)| - \tfrac{t}{m}|A|\right| \geq q \cdot \sqrt{\tfrac{t}{m}|A|}\right] \\
&= \Pr_h\left[|X - \mu| \geq q \cdot \sqrt{\mu}\right] \\
&\leq \frac{1}{q^2}
\end{aligned}$$

We needed strong universality in two places for this to work.

Where? *h* must be uniform to get unbiased estimate, and pairwise independent for the lemma.

Summary

Todays topic was hashing, and we have covered

- What is a random hash function, and what properties do we want.
- Two applications of universal hashing — unordered sets and signatures.
- Some concrete universal or strongly universal hash functions.
- An application of strongly universal hashing — coordinated sampling.
- Next time: NP-completeness
- Later: An ordered set data structure that is not comparison based, and an application of hash tables.