Notes on Lemmas 26.1 and 26.4 in CLRS

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1 Introduction

These notes for the course Advanced Algorithms and Data Structures fix a bug in the proof of Lemma 26.1 (this bug is no longer present in the fifth printing of the third edition of the CLRS course book) and simplify the proof of Lemma 26.4 in CLRS.

2 Fixing the bug in the proof of Lemma 26.1

The proof of flow conservation for $f \uparrow f'$ in CLRS is incorrect (unless you have the fifth printing or later) since $(f \uparrow f')(u,v) = f(u,v) + f'(u,v) - f'(v,u)$ is only guaranteed to hold when $(u,v) \in E$. We can fix this bug as follows. Let $u \in V \setminus \{s,t\}$ be given. Then

$$
\begin{aligned}
&\sum_{(u,v)\in E} (f \uparrow f')(u,v) - \sum_{(v,u)\in E} (f \uparrow f')(v,u) \\
&= \sum_{(u,v)\in E} (f(u,v) + f'(u,v) - f'(v,u)) - \sum_{(v,u)\in E} (f(v,u) + f'(v,u) - f'(u,v)) \\
&= \sum_{(u,v)\in E} (f'(u,v) - f'(v,u)) + \sum_{(v,u)\in E} (f'(u,v) - f'(v,u)) \\
&= \sum_{(u,v)\in E_f} f'(u,v) - \sum_{(v,u)\in E_f} f'(v,u) \\
&= 0,
\end{aligned}
$$

where the second equality follows from flow conservation of f and from re-ordering terms, and the last equality follows from flow conservation of f'. Note that we are allowed to limit the sums in the first line to edges of E since by definition, $(f \uparrow f')(x,y) = 0$ when $(x,y) \notin E$.

3 A simplified proof of Lemma 26.4

We extend the CLRS definition of net flow from cuts to arbitrary pairs of subsets of V as follows. For any $A \subseteq V$ and $B \subseteq V$,

$$f(A,B) = \sum_{u\in A}\sum_{v\in B} f(u,v) - f(v,u).$$

Now we proceed with the proof of Lemma 26.4. The definition of net flow above immediately implies $f(S, S) = 0$. Hence

$$f(S,T) = f(S,S) + f(S,T) = f(S,V) = \sum_{u \in S} \sum_{v \in V} (f(u,v) - f(v,u)).$$

On the right-hand side, every $u \in S \setminus \{s\}$ contributes 0 to the sum due to flow conservation. Thus,

$$f(S,T) = \sum_{u \in \{s\}} \sum_{v \in V} (f(u,v) - f(v,u)) = \sum_{v \in V} (f(s,v) - f(v,s)) = |f|,$$

as desired.

If one is comfortable using the notation for net flow all the way through the proof, we can fit the proof into one line:

$$f(S,T) = f(S,S) + f(S,T) = f(S,V) = f(S \setminus \{s\}, V) + f(\{s\}, V) = f(\{s\}, V) = |f|.$$