Semantics and Types - Assignment 4

Mikkel Willén
bmq419

March 6, 2024

We extend the FUN syntax with the following term:

$$t ::= 5dots \mid \textbf{min}\ x \geq t_0.\ t_1$$

We define the big-step semantics as the following:

$$\text{E-MinT} : \frac{t_0 \downarrow \overline{n_0} \quad t_1[\overline{n_0}/x] \downarrow \textbf{true}}{\textbf{min}\ x \geq t_0.\ t_1 \downarrow \overline{n_0}}$$

$$\text{E-MinF} : \frac{t_0 \downarrow \overline{n_0} \quad t_1[\overline{n_0}/x] \downarrow \textbf{false} \quad \textbf{min}\ x \geq \overline{n_0 + 1}.\ t_1 \downarrow c}{\textbf{min}\ x \geq t_0.\ t_1 \downarrow c}$$

and the small-step semantics as:

$$\text{S-Min1} : \frac{t_0 \to t_0'}{\textbf{min}\ x \geq t_0.\ t_1 \to \textbf{min}\ x \geq t_0'.\ t_1}$$

$$\text{S-Min} : \frac{}{\textbf{min}\ x \geq \overline{n_0}.\ t_1 \to \textbf{if}\ t_1[\overline{n_0}/x]\ \textbf{then}\ \overline{n_0}\ \textbf{else}\ \textbf{min}\ x \geq \overline{n_0 + 1}.\ t_1}$$

and the typing rule as:

$$\text{T-Min} : \frac{\Gamma \vdash t_0 : \textbf{int} \quad \Gamma[x \mapsto \textbf{int}] \vdash t_1 : \textbf{bool}}{\Gamma \vdash \textbf{min}\ x \geq t_0.\ t_1 : \textbf{int}}$$

Task 4.1

Theorem 4.2 *If $t \downarrow c$, then $t \to^* c$.*

Proof. By induction on the big-step derivation.

$$\text{Case } \mathcal{E} = \text{E-MinT} \frac{\overset{\mathcal{E}_0}{t_0 \downarrow \overline{n_0}} \quad \overset{\mathcal{E}_1}{t_1[\overline{n_0}/x] \downarrow \textbf{true}}}{\textbf{min}\ x \geq t_0.\ t_1 \downarrow \overline{n_0}}$$

Here $t = (\textbf{min}\ x \geq t_0.\ t_1)$ and $c = \overline{n_0}$. By IH on $\mathcal{E}_0$ we get $\mathcal{SS}_0$ of $t_0 \to^* \overline{n_0}$, and we can then, by step-wise use of S-Min1, get

$$\textbf{min}\ x \geq t_0.\ t_1 \to^* \textbf{min}\ x \geq \overline{n_0}.\ t_1$$

Now, by step-wise use of S-Min we get

$$\textbf{min}\ x \geq \overline{n_0}.\ t_1 \to^* \textbf{if}\ t_1[\overline{n_0}/x]\ \textbf{then}\ \overline{n_0}\ \textbf{else}\ \textbf{min}\ x \geq \overline{n_0 + 1}.\ t_1$$

By IH on $\mathcal{E}_1$ we get $\mathcal{SS}_1$ of $t_1[\overline{n_0}/x] \to^* \textbf{true}$ and we can use S-IfT to get

$$\textbf{if true then}\ \overline{n_0}\ \textbf{else}\ \textbf{min}\ x \geq \overline{n_0 + 1}.\ t_1 \to^* \overline{n_0}$$

Concatenating these together with **Lemma 4.1** we get the desired

$$\textbf{min } x \geq t_0.\ t_1 \downarrow c \rightarrow^* \overline{n_0}$$

$$\text{Case } \mathcal{E} = \text{E-MinF}\,\dfrac{\overset{\mathcal{E}_0}{t_0 \downarrow \overline{n_0}} \quad \overset{\mathcal{E}_1}{t_1[\overline{n_0}/x] \downarrow \textbf{false}} \quad \overset{\mathcal{E}_2}{\textbf{min } x \geq \overline{n_0+1}.\ t_1 \downarrow c'}}{\textbf{min } x \geq t_0.\ t_1 \downarrow c'}$$

Here $t = (\textbf{min } x \geq t_0.\ t_1)$ and $c = c'$. By IH on $\mathcal{E}_0$ we get $\mathcal{SS}_0$ of $t_0 \rightarrow^* \overline{n_0}$, and we can then, by step-wise use of S-Min1, get

$$\textbf{min } x \geq t_0.\ t_1 \rightarrow^* \textbf{min } x \geq \overline{n_0}.\ t_1$$

Now, by step-wise use of S-Min we get

$$\textbf{min } x \geq \overline{n_0}.\ t_1 \rightarrow^* \textbf{if } t_1[\overline{n_0}/x] \textbf{ then } \overline{n_0} \textbf{ else min } x \geq \overline{n_0+1}.\ t_1$$

By IH on $\mathcal{E}_1$ we get $\mathcal{SS}_1$ of $t_1[\overline{n_0}/x] \rightarrow^* \textbf{false}$ and we can use S-IfF to get

$$\textbf{if false then } \overline{n_0} \textbf{ else min } x \geq \overline{n_0+1}.\ t_1 \rightarrow^* \textbf{min } x \geq \overline{n_0+1}.\ t_1$$

Now by IH on $\mathcal{E}_2$ we get $\mathcal{SS}_2$ of $\textbf{min } x \geq \overline{n_0+1}.\ t_1 \rightarrow^* c'$. Concatenating these together with **Lemma 4.1** we get the desired

$$\textbf{min } x \geq t_0.\ t_1 \downarrow c \rightarrow^* c'$$

Lemma 4.4 *If $t \rightarrow t'$ and $t' \downarrow c$, then $t \downarrow c$.*

Proof. By induction on the first derivation.

$$\text{Case } \mathcal{S} = \text{S-Min1}\,\dfrac{\overset{\mathcal{S}_0}{t_0 \rightarrow t_0'}}{\textbf{min } x \geq t_0.\ t_1 \rightarrow \textbf{min } x \geq t_0'.\ t_1}$$

We have $t = (\textbf{min } x \geq t_0.\ t_1)$, which means we have two possibilities for $\mathcal{E}'$ depending on which of the two big-step rules are used:

$$\text{Subcase } \mathcal{E}' = \text{E-MinT}\,\dfrac{\overset{\mathcal{E}_0'}{t_0' \downarrow \overline{n_0}} \quad \overset{\mathcal{E}_1'}{t_1[\overline{n_0}/x] \downarrow \textbf{true}}}{\textbf{min } x \geq t_0'.\ t_1 \downarrow \overline{n_0}}$$

By IH on $\mathcal{S}_0$ with $\mathcal{E}_1$ we get get $\mathcal{E}_1$ of $t_1[\overline{n_0}/x] \downarrow \textbf{true}$ and we can construct $\mathcal{E}$ as

$$\mathcal{E} = \text{E-MinT}\,\dfrac{\overset{\mathcal{E}_0}{t_0' \downarrow \overline{n_0}} \quad \overset{\mathcal{E}_1}{t_1[\overline{n_0}/x] \downarrow \textbf{true}}}{\textbf{min } x \geq t_0'.\ t_1 \downarrow \overline{n_0}}$$

$$\text{Subcase } \mathcal{E}' = \text{E-MinF}\,\dfrac{\overset{\mathcal{E}_0'}{t_0' \downarrow \overline{n_0}} \quad \overset{\mathcal{E}_1'}{t_1[\overline{n_0}/x] \downarrow \textbf{false}} \quad \overset{\mathcal{E}_2'}{\textbf{min } x \geq \overline{n_0+1}.\ t_1 \downarrow c'}}{\textbf{min } x \geq t_0'.\ t_1 \downarrow c'}$$

By IH on $\mathcal{S}_0$ with $\mathcal{E}_1$ we get get $\mathcal{E}_1$ of $t_1[\overline{n_0}/x] \downarrow \textbf{false}$ and we can construct $\mathcal{E}$ as

$$\mathcal{E} = \text{E-MinF}\,\dfrac{\overset{\mathcal{E}_0}{t_0' \downarrow \overline{n_0}} \quad \overset{\mathcal{E}_1}{t_1[\overline{n_0}/x] \downarrow \textbf{false}} \quad \overset{\mathcal{E}_2}{\textbf{min } x \geq \overline{n_0+1}.\ t_1 \downarrow c'}}{\textbf{min } x \geq t_0'.\ t_1 \downarrow c'}$$

Task 4.2

Let the term $\texttt{MIN}(x,t)$ of type **int** $\to$ **int** be defined as:

$$\texttt{MIN}(x,t) \equiv \mathbf{rec}\ f.\ \lambda x.\ \underbrace{\mathbf{if}\ t\ \mathbf{then}\ x\ \mathbf{else}\ f(x+\overline{1})}_{t_0'}$$

where t is any boolean term potentially containing an integer variable x.

We show by induction on derivations, that

$$\underbrace{\mathbf{min}\ x \geq t_0.\ t_1}_{t} \downarrow c \Rightarrow \underbrace{\texttt{MIN}(x,t_1)t_0}_{t'} \downarrow c$$

Looking at the execution rules for **min**-commands, we see that $\mathcal{E}$ must have one of the two following shapes

$$\text{Case } \mathcal{E} = \text{E-MinT}\frac{\overset{\mathcal{E}_0}{t_0' \downarrow \overline{n_0}} \quad \overset{\mathcal{E}_1}{t_1[\overline{n_0}/x] \downarrow \mathbf{true}}}{\mathbf{min}\ x \geq t_0'.\ t_1 \downarrow \overline{n_0}}$$

Since $\mathcal{E}_1$ evaluates to **true**, we can construct $\mathcal{E}'$ as follows:

$$\mathcal{E}' = \text{E-App}\frac{\text{E-Rec}\dfrac{\text{E-Lam}\dfrac{}{\lambda x.\ t_0' \downarrow \lambda x.\ t_0'}}{\mathbf{rec}\ f.\lambda x.\ t_0' \downarrow \lambda x.\ t_0'} \quad t_0 \downarrow \overline{n_0} \quad \text{E-IfT}\dfrac{t_1 \downarrow \mathbf{true} \quad \text{E-Num}\dfrac{}{\overline{n_0} \downarrow \overline{n_0}}}{\dfrac{\mathbf{if}\ t_1\ \mathbf{then}\ \overline{n_0}\ \mathbf{else}\ f(x+\overline{1}) \downarrow \overline{n_0}}{t_0'[\overline{n_0}/x] \downarrow \overline{n_0}}}}{\texttt{Min}(x,t_1)t_0 \downarrow \overline{n_0}}$$

$$\text{Case } \mathcal{E} = \text{E-MinF}\frac{\overset{\mathcal{E}_0}{t_0 \downarrow \overline{n_0}} \quad \overset{\mathcal{E}_1}{t_1[\overline{n_0}/x] \downarrow \mathbf{false}} \quad \overset{\mathcal{E}_2}{\mathbf{min}\ x \geq \overline{n_0+1}.\ t_1 \downarrow c'}}{\mathbf{min}\ x \geq t_0.\ t_1 \downarrow c'}$$

Since $\mathcal{E}_1$ evaluates to **false**, we can construct $\mathcal{E}'$ as follows:

$$\mathcal{E}' = \text{E-App}\frac{\text{E-Rec}\dfrac{\text{E-Lam}\dfrac{}{\lambda x.\ t_0' \downarrow \lambda x.\ t_0'}}{\mathbf{rec}\ f.\lambda x.\ t_0' \downarrow \lambda x.\ t_0'} \quad t_0 \downarrow \overline{n_0} \quad \text{E-IfF}\dfrac{t_1 \downarrow \mathbf{false} \quad \overset{\mathcal{E}_2'}{f(x+\overline{1}) \downarrow c''}}{\dfrac{\mathbf{if}\ t_1\ \mathbf{then}\ \overline{n_0}\ \mathbf{else}\ f(x+\overline{1}) \downarrow c''}{t_0'[\overline{n_0}/x] \downarrow c''}}}{\texttt{Min}(x,t_1)t_0 \downarrow c''}$$

We see $\mathcal{E}_2$ is on the form of t and $\mathcal{E}_2'$ is on the form of t' and by IH on $\mathcal{E}_2$ with $\mathcal{E}_2'$ we get $c'' = c'$.

Task 4.4

Lemma 4.8 *If* $[] \vdash t : \tau$*, then either* $t = c$ *for some canonical form* c*, or* $\exists\, t'$ *such that* $t \to t'$*.*

Proof. By induction on the typing derivation $\mathcal{T}$ of $[] \vdash t : \tau$:

$$\text{Case } \mathcal{T} = \text{T-Min}\frac{\overset{\mathcal{T}_0}{[] \vdash t_0 : \mathbf{int}} \quad \overset{\mathcal{T}_1}{[x \mapsto \mathbf{int}] \vdash t_1 : \mathbf{bool}}}{[] \vdash \mathbf{min}\ x \geq t_0.\ t_1 : \mathbf{int}}$$

Here $t = (\mathbf{min}\ x \geq t_0.\ t_1 : \mathbf{int})$. By IH on $\mathcal{T}_0$ we have that either $t_0 = c_0$ or $t_0 \to t_0'$. In the latter case, the entire **min** term can take a step with S-Min1. In the former case we must have $t_0 = c_0 = \overline{n_0}$, since no other canonical forms have type **int**, so $t = (\mathbf{min}\ x \geq \overline{n_0}.\ t_1)$, which steps to $t' = (\mathbf{if}\ t_1[\overline{n_0}/x]\ \mathbf{then}\ \overline{n_0}\ \mathbf{else}\ \mathbf{min}\ x \geq \overline{n_0+1}.\ t_1)$ by S-Min.

Lemma 4.11 *If* $t \to t'$ *(by* $\mathcal{S}$*) and* $[] \vdash t : \tau$ *(by* $\mathcal{T}$*), then also* $[] \vdash t' : \tau$ *(by some* $\mathcal{T}'$*).*

Proof. By induction on the derivation $\mathcal{S}$.

$$\text{Case } \mathcal{S} = \text{S-Min1}\frac{\overset{\mathcal{S}_0}{t_0 \to t_0'}}{\mathbf{min}\ x \geq t_0.\ t_1 \to \mathbf{min}\ x \geq t_0'.\ t_1}$$

Since $t = (\mathbf{min}\ x \geq t_0.\ t_1)$ the typing derivation $\mathcal{T}$ must be:

$$\mathcal{T} = \textsc{T-Min}\frac{\overset{\mathcal{T}_0}{[] \vdash t_0 : \mathbf{int}} \quad \overset{\mathcal{T}_1}{[x \mapsto \mathbf{int}] \vdash t_1 : \mathbf{bool}}}{[] \vdash \mathbf{min}\ x \geq t_0.\ t_1 : \mathbf{int}}$$

By IH on $\mathcal{S}_0$ with $\mathcal{T}_0$ we get a derivation $\mathcal{T}_0'$ of $[] \vdash t_0' : \mathbf{int}$, and we can construct the $\mathcal{T}'$ as:

$$\mathcal{T}' = \textsc{T-Min}\frac{\overset{\mathcal{T}_0'}{[] \vdash t_0' : \mathbf{int}} \quad \overset{\mathcal{T}_1}{[x \mapsto \mathbf{int}] \vdash t_1 : \mathbf{bool}}}{[] \vdash \mathbf{min}\ x \geq t_0'.\ t_1 : \mathbf{int}}$$

$$\text{Case } \mathcal{S} = \textsc{S-Min}\frac{}{\mathbf{min}\ x \geq \overline{n_0}.\ t_1 \to \mathbf{if}\ t_1[\overline{n_0}/x]\ \mathbf{then}\ \overline{n_0}\ \mathbf{else}\ \mathbf{min}\ x \geq \overline{n_0 + 1}.\ t_1}$$

Here $t = (\mathbf{min}\ x \geq \overline{n_0}.\ t_1)$ and $t' = (\mathbf{if}\ t_1[\overline{n_0}/x]\ \mathbf{then}\ \overline{n_0}\ \mathbf{else}\ \mathbf{min}\ x \geq \overline{n_0 + 1}.\ t_1)$, and the typing derivation for t must look like:

$$\mathcal{T} = \textsc{T-Min}\frac{\textsc{T-Num}\frac{}{[] \vdash \overline{n_0} : \mathbf{int}} \quad \overset{\mathcal{T}_1}{[x \mapsto \mathbf{int}] \vdash t_1 : \mathbf{bool}}}{[] \vdash \mathbf{min}\ x \geq \overline{n_0}.\ t_1 : \mathbf{int}}$$

We can construct $\mathcal{T}'$ as:

$$\mathcal{T}' = \textsc{T-If}\frac{\overset{\mathcal{T}_0'}{[x \mapsto \mathbf{int}] \vdash t_1 : \mathbf{bool}} \quad \textsc{T-Num}\frac{}{[] \vdash \overline{n_0} : \mathbf{int}} \quad \overset{\mathcal{T}_2'}{[x \mapsto \mathbf{int}] \vdash \mathbf{min}\ x \geq \overline{n_0 + 1} : \mathbf{int}}}{\mathbf{if}\ t_1[\overline{n_0}/x]\ \mathbf{then}\ \overline{n_0}\ \mathbf{else}\ \mathbf{min}\ x \geq \overline{n_0 + 1}.\ t_1 : \mathbf{int}}$$

since both cases of the **if** term ends in type **int**.