Semantics and Types - Assignment 5

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Task 5.1

Lemma 5.3 *If $[] \vdash t : \tau$ then either $t = c$ for some canonical form c, or $t \to t'$ for some term t'.*

$$\text{Case } \mathcal{T} = \textsc{T-Vnt}'\frac{\overset{\mathcal{T}_0}{[] \vdash t_0 : \tau_0}}{[] \vdash \langle l = t_0 \rangle : \langle l : \tau_0 \rangle}$$

As in the case for select, we consider only the $\textsc{T-Vnt}'$ rule. By IH on $\mathcal{T}_0$, the term t is already canonical or can take a step, which is exactly what we needed to show.

$$\text{Case } \mathcal{T} = \textsc{T-Case}\frac{\overset{\mathcal{T}_0}{[] \vdash t_0 : \langle (l_i : \tau_i)^{i \in 1 \dots n} \rangle} \quad \overset{\mathcal{T}_1}{([x_i \mapsto \tau_i] \vdash t_i : \tau)^{i \in 1 \dots n}}}{[] \vdash \textbf{case } t_0 \textbf{ of } (\langle l_i = x_i \rangle \Rightarrow t_i)^{i \in 1 \dots n} : \tau}$$

By IH on $\mathcal{T}_0$ either $t_0 \to t_0'$ and $t = \left(\textbf{case } t_0 \textbf{ of } (\langle l_i = x_i \rangle \Rightarrow t_i)^{i \in 1 \dots n} \to \textbf{case } t_0' \textbf{ of } (\langle l_i = x_i \rangle \Rightarrow t_i)^{i \in 1 \dots n}\right)$ by $\textsc{S-Case1}$ or t_0 is on canonical form. In the latter case, by Lemma 5.2(d), we must have $t_0 = \langle l = c_0 \rangle$, with $l = l_k$ for some k, and thus $\textbf{case } t_0 \textbf{ of } (\langle l_i = x_i \rangle \Rightarrow t_i)^{i \in 1 \dots n}$ reduces to $t_k[c_0/x_k]$ by $\textsc{T-Case}$.

Lemma 5.5 *If $[] \vdash t : \tau$ and $t \to t'$, then $[] \vdash t' : \tau$.*

$$\text{Case } \mathcal{S} = \textsc{S-Sel}\frac{}{\{(l_i = c_i)^{i \in 1 \dots n}\}.l_k \to c_k}$$

where $k \in 1 \dots n$. Since the typing derivation cannot end in $\textsc{T-Sub}$, it must end with a use of $\textsc{T-Sel}$.

$$\mathcal{T} = \textsc{T-Sel}\frac{\overset{\mathcal{T}_0}{[] \vdash \{(l_i = c_i)^{i \in 1 \dots n}\} : \{(l_i : \tau_i)^{i \in 1 \dots n}\}}}{[] \vdash \{(l_i = c_i)^{i \in 1 \dots n}\}.l_k : \tau_k}$$

where $k \in 1 \dots n$. Using Lemma 5.2(c) on $\mathcal{T}_0$ (where m is taken as n, and $c = (\{(l_i = c_i)^{i \in 1 \dots n}\}.l_k)$, we get that there exists an $i' \in 1 \dots n$ such that $l_i' = l_k$ and that $[] \vdash c_i : \tau_i$ (by some $\mathcal{T}_{00}$). But since labels in record types are required to be distinct, we must have that $i' = k$ and thus we also have that $[] \vdash c_i : \tau_k$ and using the Substitution Lemma on $\mathcal{T}_k$ and $\mathcal{T}_{00}$ we get $[] \vdash c_k : \tau_k$.

Task 5.2

We consider the system with just $\textsc{T-Var}$, $\textsc{T-Lam}$, $\textsc{T-App}$, $\textsc{T-Sub}$, $\textsc{ST-Refl}$, $\textsc{ST-Trans}$ and $\textsc{ST-Fun}$. We show, that the following type is admissible:

$$\textsc{T-Eta} : \frac{\Gamma \vdash \lambda x.(tx) : \tau}{\Gamma \vdash t : \tau} \; (x \notin FV(t))$$

We have that if $\Gamma[x \mapsto \tau'] \vdash t : \tau$ and $x \notin FV(t)$, then also $\Gamma \vdash t : \tau$. We start by showing the following Lemma.

Lemma S. *If $\mathcal{T}$ is a derivation of $\Gamma \vdash t : \tau$, then for some τ', there exists a derivation $\mathcal{T}'$ of $\Gamma \vdash t : \tau'$, where $\mathcal{T}'$ does not end in a use of $\textsc{T-Sub}$, and a derivation $\mathcal{ST}$ of $\tau' \leq \tau$.*

We do this by induction on derivations:

$$\text{Case } \mathcal{T} = \textsc{T-Var}\frac{}{\Gamma \vdash x : \tau} \; (\Gamma(x) = \tau)$$

Since the typing derivation $\mathcal{T}'$ cannot end in T-Sub, must must end with T-Var.

$$\mathcal{T}' = \text{T-Var}\frac{}{\Gamma \vdash x : \tau'} \; (\Gamma(x) = \tau')$$

We have that x maps to some τ_0 in Γ, and the lookup of this in the same Γ must have the same type, so we have that $\tau' = \tau$ which is true by ST-Refl.

$$\text{Case } \mathcal{T} = \text{T-Lam}\frac{\overset{\mathcal{T}_0}{\Gamma[x \mapsto \tau_1] \vdash t_0 : \tau_2}}{\Gamma \vdash \lambda x.t_0 : \tau_1 \to \tau_2}$$

Since the typing derivation $\mathcal{T}'$ cannot end in T-Sub, must must end with T-Lam.

$$\mathcal{T}' = \text{T-Lam}\frac{\overset{\mathcal{T}_0'}{\Gamma[x \mapsto \tau_1] \vdash t_0 : \tau_2'}}{\Gamma \vdash \lambda x.t_0 : \tau_1 \to \tau_2'}$$

We have that $\tau = \tau_1 \to \tau_2$ and $\tau' = \tau_1 \to \tau_2'$. By IH on $\mathcal{T}_0$ with $\mathcal{T}_0'$ we have that $\tau_2' \leq \tau_2$ and then by ST-Fun with ST-Refl on t_1 we have that $\tau_1 \to \tau_2' \leq \tau_1 \to \tau_2$.

$$\text{Case } \mathcal{T} = \text{T-App}\frac{\overset{\mathcal{T}_0}{\Gamma \vdash t_1 : \tau_1 \to \tau_2} \quad \overset{\mathcal{T}_1}{\Gamma \vdash t_2 : \tau_1}}{\Gamma \vdash t_1 t_2 : \tau_2}$$

Since the typing derivation $\mathcal{T}'$ cannot end in T-Sub, must must end with T-App.

$$\mathcal{T}' = \text{T-App}\frac{\overset{\mathcal{T}_0'}{\Gamma \vdash t_1 : \tau_1 \to \tau_2'} \quad \overset{\mathcal{T}_1'}{\Gamma \vdash t_2 : \tau_1}}{\Gamma \vdash t_1 t_2 : \tau_2'}$$

We have that $\tau = \tau_2$ and $\tau' = \tau_2'$. By IH on $\mathcal{T}_0$ with $\mathcal{T}_0'$ we have that $\tau_1 \to \tau_2 \leq \tau_1 \to \tau_2'$ and then by ST-Fun backwards, we have that $\tau_2' \leq \tau_2$.

$$\text{Case } \mathcal{T} = \text{T-Sub}\frac{\overset{\mathcal{T}'}{\Gamma \vdash t : \tau'} \quad \overset{\mathcal{ST}}{\tau' \leq \tau}}{\Gamma \vdash t : \tau}$$

The derivation is trivially true.

Now we have

$$\text{T-Eta}\frac{\text{T-Lam}\frac{\text{T-App}\frac{\overset{\mathcal{T}_0}{\Gamma[x \mapsto \tau_1] \vdash t : \tau_1 \to \tau_2} \quad \Gamma[x \mapsto \tau_1] \vdash t_2 : \tau_1}{\Gamma[x \mapsto \tau_1] \vdash tx : \tau_2}}{\Gamma \vdash \lambda x.(tx) : \tau}}{\Gamma \vdash t : \tau} \; (x \notin FV(t))$$

then by (*strengthening*) on $\mathcal{T}_0$ (where $\tau' = \tau_1$, $\tau = \tau_1 \to \tau_2$ and $x \notin FV(t)$) we have that T-Eta holds.

Task 3

a)

We write out the typing rules for **binary-sum** type.

$$\text{T-BSum}: \frac{\Gamma \vdash t_0 : \tau_1 + \tau_2 \quad \Gamma \vdash t_1 : \tau_1 \quad \Gamma \vdash t_2 : \tau_2}{\Gamma \vdash \textbf{case } t_0 \textbf{ of } \textbf{inl}(x_1) \Rightarrow t_1, \; \textbf{inr}(x_2) \Rightarrow t_2 : \tau_1 + \tau_2}$$

$$\text{T-InL}: \frac{\Gamma \vdash t_0 : \tau_1 + \tau_2}{\Gamma \vdash \textbf{inl}(t_0) : \tau_1} \qquad \text{T-InR}: \frac{\Gamma \vdash t_0 : \tau_1 + \tau_2}{\Gamma \vdash \textbf{inr}(t_0) : \tau_2}$$

b)

$$\text{CT-BSum}: \dfrac{\hat\Gamma \vdash^{i} t_0 : \hat\tau_0 \,|^{i''}\, C_0 \quad \hat\Gamma \vdash^{i''} t_1 : \hat\tau_1 \,|^{i'''}\, C_1 \quad \hat\Gamma \vdash^{i'''} t_2 : \hat\tau_2 \,|^{i'}\, C_2}{\hat\Gamma \vdash^{i} \textbf{case}\ t_0\ \textbf{of}\ \textbf{inl}(x_1) \Rightarrow t_1,\ \textbf{inr}(x_2) \Rightarrow t_2 : \hat\tau_1 + \hat\tau_2 \,|^{i'}\, C_0, C_1, C_2, \hat\tau_0 \doteq \hat\tau_1 + \hat\tau_2}$$

$$\text{CT-InL}: \dfrac{\hat\Gamma \vdash^{i} t_0 : \hat\tau_0 \,|^{i'}\, C_0}{\hat\Gamma \vdash^{i} \textbf{inl}(t_0) : \boxed{i'} \,|^{i'+2}\, C_0, \hat\tau_0 \doteq \boxed{i'} + \boxed{i'+1}}$$

$$\text{CT-InR}: \dfrac{\hat\Gamma \vdash^{i} t_0 : \hat\tau_0 \,|^{i'}\, C_0}{\hat\Gamma \vdash^{i} \textbf{inr}(t_0) : \boxed{i'+1} \,|^{i'+2}\, C_0, \hat\tau_0 \doteq \boxed{i'} + \boxed{i'+1}}$$

c)

$$\text{CT-Lam}\,\dfrac{\text{CT-BSum}\,\dfrac{\text{CT-Var}\,\dfrac{}{[x \mapsto \boxed{0}] \vdash^{1} x : \boxed{0} \,|^{1''}} \qquad \text{CT-InL}_x \qquad \text{CT-InL}_z}{[x \mapsto \boxed{0}] \vdash^{0+1} \textbf{case}\ x\ \textbf{of}\ \textbf{inl}(y) \Rightarrow x, \textbf{inr}(z) \Rightarrow \textbf{inl}(z) : \hat\tau_0 \,|^{0'}\, C_0}}{[] \vdash^{0} \lambda x.\textbf{case}\ x\ \textbf{of}\ \textbf{inl}(y) \Rightarrow x, \textbf{inr}(z) \Rightarrow \textbf{inl}(z) : \boxed{0} \to \hat\tau_0 \,|^{0'}\, C_0}$$

$$\text{CT-InL}_x\,\dfrac{\text{CT-Var}\,\dfrac{}{[x \mapsto \boxed{0}] \vdash^{1} x : \boxed{1'} \,|^{1'}}}{[x \mapsto \boxed{0}] \vdash^{1} \textbf{inl}(y) : \boxed{1'} \,|^{1'+2}\, \hat\tau_0 \doteq \boxed{1'} + \boxed{1'+1}}$$

$$\text{CT-InL}_z\,\dfrac{\text{CT-Var}\,\dfrac{}{[x \mapsto \boxed{0}] \vdash^{2} z : \boxed{2'} \,|^{2'}}}{[x \mapsto \boxed{0}] \vdash^{2} \textbf{inl}(z) : \boxed{2'} \,|^{2'+2}\, \hat\tau_0 \doteq \boxed{1'} + \boxed{1'+1}}$$

We find that the expression has type $\boxed{0} \to \boxed{1} + \boxed{2}$ and we have that $\boxed{0} \doteq \boxed{1}$ and $\boxed{2} \doteq \boxed{2}$.

d)

$\boxed{2} \doteq \boxed{2},\ \boxed{0} \doteq \boxed{1},\ \boxed{0} \to \boxed{1} + \boxed{2}$
$\rightsquigarrow$ (by CS-Triv)
$\boxed{0} \doteq \boxed{1},\ \boxed{0} \to \boxed{1} + \boxed{2}$
$\rightsquigarrow$ (by CS-ElimL
$\boxed{0} \to \boxed{1} + \boxed{2}, \boxed{0} \doteq^{\checkmark} \boxed{1}$
$=$
$\boxed{1} \to \boxed{1} + \boxed{2}, \boxed{0} \doteq^{\checkmark} \boxed{1}$
and we are done.