Semantics and Types - Assignment 2

Mikkel Willén
bmq419

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Task 2.2

Theorem 2.19 *If* $\langle c, \sigma\rangle \downarrow \sigma'$, *then* $\langle c, \sigma\rangle \to^* \langle \mathbf{skip}, \sigma'\rangle$.

$$\text{Case } \mathcal{E} = \cfrac{\overset{\mathcal{E}_0}{\langle c_0, \sigma\rangle \downarrow \sigma''} \quad \overset{\mathcal{E}_1}{\langle c_1, \sigma''\rangle \downarrow \sigma'}}{\langle c_0; c_1, \sigma\rangle \downarrow \sigma'}$$

By IH on $\mathcal{E}_0$ we get

$$\langle c_0, \sigma\rangle \to^* \langle \mathbf{skip}, \sigma''\rangle$$

Then by iterated SC-Seq1 we get

$$\langle c_0; c_1, \sigma''\rangle \to^* \langle c_1, \sigma''\rangle$$

which we recognize to be $\mathcal{E}_1$. By IH on $\mathcal{E}_1$ we get

$$\langle c_1, \sigma''\rangle \to^* \langle \mathbf{skip}, \sigma'\rangle$$

Concatenating these we get

$$\langle c_0; c_1, \sigma\rangle \to^* \langle \mathbf{skip}, \sigma'\rangle$$

Lemma 2.20 *Given a derivation* $\mathcal{S}$ *of* $\langle c, \sigma\rangle \to \langle c', \sigma'\rangle$, *and a derivation* $\mathcal{E}'$ *of* $\langle c', \sigma'\rangle \downarrow \sigma''$, *then there also exists a derivation* $\mathcal{E}$ *of* $\langle c, \sigma\rangle \downarrow \sigma''$.

$$\text{Case } \mathcal{S} = \cfrac{\overset{\mathcal{S}_0}{\langle c_0, \sigma\rangle \to \langle c_0', \sigma'\rangle}}{\langle c_0; c_1, \sigma\rangle \to \langle c_0'; c_1, \sigma'\rangle}$$

We have $c = (c_0; c_1)$ and $c' = (c_0'; c_1)$. Since there is only one big-step rule for **sequence** commands, $\mathcal{E}'$ must have the shape:

$$\mathcal{E}' = \cfrac{\overset{\mathcal{E}_0'}{\langle c_0', \sigma'\rangle \downarrow \sigma_2'} \quad \overset{\mathcal{E}_1'}{\langle c_1, \sigma_2'\rangle \downarrow \sigma_1'}}{\langle c_0'; c_1, \sigma'\rangle \downarrow \sigma_1'}$$

where $\sigma'' = \sigma_1'$. By IH on $\mathcal{S}_0$ and $\mathcal{E}_0'$ we get a big-step derivation of $\mathcal{E}_0$ of $\langle c_0, \sigma\rangle \downarrow \sigma_2'$ which we can use to construct $\mathcal{E}$ as follows

$$\mathcal{E} = \cfrac{\overset{\mathcal{E}_0}{\langle c_0, \sigma\rangle \downarrow \sigma_2'} \quad \overset{\mathcal{E}_1}{\langle c_1, \sigma_2'\rangle \downarrow \sigma''}}{\langle c_0; c_1, \sigma\rangle \downarrow \sigma''}$$

$$\text{Case } \mathcal{S} = \cfrac{}{\langle \mathbf{skip}; c_1, \sigma\rangle \to \langle c_1, \sigma\rangle}$$

Here $c = (\mathbf{skip}; c_1)$, $c' = c_1$ and $\sigma' = \sigma$. Since $\sigma' = \sigma$ we can use the supplied big-step derivation $\mathcal{E}'$ directly to construct $\mathcal{E}$ using EC-Skip

$$\mathcal{E} = \cfrac{\langle \mathbf{skip}, \sigma\rangle \downarrow \sigma \quad \langle c_1, \sigma\rangle \downarrow \sigma''}{\langle \mathbf{skip}; c_1, \sigma\rangle \downarrow \sigma''}$$

Task 2.3

$$\text{Case } \mathcal{S} = \dfrac{}{\langle \textbf{while } b \textbf{ do } c_0, \sigma\rangle \to \langle \textbf{if } b \textbf{ then } (c_0; \textbf{while } b \textbf{ do } c_0) \textbf{ else skip}, \sigma\rangle}$$

We have $c = (\textbf{while } b \textbf{ do } c_0)$, $c' = (\textbf{if } b \textbf{ then } (c_0; \textbf{while } b \textbf{ do } c_0) \textbf{ else skip})$ and $\sigma' = \sigma$. By Theorem 2.2 we have that $c \sim c'$ and since there are two big-step rules for **conditionals**, $\mathcal{E}'$ can have the two following shapes:

$$\text{Subcase } \mathcal{E}' = \dfrac{\overset{\mathcal{E}'_0}{\langle b, \sigma\rangle \downarrow \textbf{true}} \quad \dfrac{\overset{\mathcal{E}'_1}{\langle c_0, \sigma\rangle \downarrow \sigma'} \quad \overset{\mathcal{E}'_2}{\langle \textbf{while } b \textbf{ do } c_0, \sigma'\rangle \downarrow \sigma''}}{\langle c_0; \textbf{while } b \textbf{ do } c_0, \sigma\rangle \downarrow \sigma''}}{\langle \textbf{if } b \textbf{ then } (c_0; \textbf{while } b \textbf{ do } c_0) \textbf{ else skip}, \sigma\rangle \downarrow \sigma''}$$

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}_0}{\langle b, \sigma\rangle \downarrow \textbf{true}} \quad \overset{\mathcal{E}_1}{\langle c_0, \sigma\rangle \downarrow \sigma'} \quad \overset{\mathcal{E}_2}{\langle \textbf{while } b \textbf{ do } c_0, \sigma'\rangle \downarrow \sigma''}}{\langle \textbf{while } b \textbf{ do } c_0, \sigma\rangle \downarrow \sigma''}$$

$$\text{Subcase } \mathcal{E}' = \dfrac{\overset{\mathcal{E}'_0}{\langle b, \sigma\rangle \downarrow \textbf{false}} \quad \dfrac{}{\langle \textbf{skip}, \sigma\rangle \downarrow \sigma''}}{\langle \textbf{if } b \textbf{ then } (c_0; \textbf{while } b \textbf{ do } c_0) \textbf{ else skip}, \sigma\rangle \downarrow \sigma''}$$

$$\mathcal{E} = \dfrac{\overset{\mathcal{E}_0}{\langle b, \sigma\rangle \downarrow \textbf{false}}}{\langle \textbf{while } b \textbf{ do } c_0, \sigma\rangle \downarrow \sigma''}$$

where $\sigma'' = \sigma$.

Task 2.4

(No rules for t: already fully executed)

$$\text{SB-EQ1}: \dfrac{\sigma \vdash a_0 \to a'_0}{\sigma \vdash a_0 = a_1 \to a'_0 = a_1} \qquad \text{SB-EQ2}: \dfrac{\sigma \vdash a_1 \to a'_1}{\sigma \vdash \overline{n_0} = a_1 \to \overline{n_0} = a'_1}$$

$$\text{SB-EQT}: \dfrac{}{\sigma \vdash \overline{n} = \overline{n} \to \textbf{true}} \qquad \text{SB-EQF}: \dfrac{}{\sigma \vdash \overline{n} = \overline{n} \to \textbf{false}}\ (n_0 \neq n_1)$$

$$\text{SB-LEQ1}: \dfrac{\sigma \vdash a_0 \to a'_0}{\sigma \vdash a_0 \leq a_1 \to a'_0 \leq a_1} \qquad \text{SB-LEQ2}: \dfrac{\sigma \vdash a_1 \to a'_1}{\sigma \vdash \overline{n_0} \leq a_1 \to \overline{n_0} \leq a'_1}$$

$$\text{SB-LEQT}: \dfrac{}{\sigma \vdash \overline{n_0} \leq \overline{n_1} \to \textbf{true}}\ (n_0 \leq n_1) \qquad \text{SB-LEQF}: \dfrac{}{\sigma \vdash \overline{n_0} \leq \overline{n_1} \to \textbf{false}}\ (n_0 > n_1)$$

$$\text{SB-NEG1}: \dfrac{\sigma \vdash b \to b'}{\sigma \vdash \neg b \to \neg b'}$$

$$\text{SB-NEGT}: \dfrac{}{\neg\textbf{true} \to \textbf{false}} \qquad \text{SB-NEGT}: \dfrac{}{\neg\textbf{false} \to \textbf{true}}$$

Task 2.5

Theorem 2.15 *If* $\langle b, \sigma\rangle \downarrow t$ *then* $\sigma \vdash b \to^* t$.

$$\text{Case } \mathcal{E} = \frac{\overset{\mathcal{E}_0}{\langle b_0, \sigma\rangle \downarrow \textbf{true}} \quad \overset{\mathcal{E}_1}{\langle b_1, \sigma\rangle \downarrow t}}{\langle b_0 \wedge b_1, \sigma\rangle \downarrow t}$$

By IH on $\mathcal{E}_0$ we get $\mathcal{SS}_0$ of $\sigma \vdash b_0 \to^* \textbf{true}$. Using SB-And1 we get

$$\sigma \vdash b_0 \wedge b_1 \to \textbf{true} \wedge b_1$$

which we can concatenate with SB-AndT to obtain $\sigma \vdash \textbf{true} \wedge b_1 \to^* b_1$. By IH on $\mathcal{E}_1$ we get $\sigma \vdash b_1 \to^* t$ to obtain

$$\langle b_0 \wedge b_1, \sigma\rangle \to^* t$$

$$\text{Case } \mathcal{E} = \frac{\overset{\mathcal{E}_0}{\langle b_0, \sigma\rangle \downarrow \textbf{false}}}{\langle b_0 \wedge b_1, \sigma\rangle \downarrow \textbf{false}}$$

By IH on $\mathcal{E}_0$ we get $\sigma \vdash b_0 \to^* \textbf{false}$. Using SB-And1 we get

$$\sigma \vdash b_0 \wedge b_1 \to^* \textbf{false} \wedge b_1$$

which we can concatenate with SB-AndF we get $\sigma \vdash \textbf{false} \wedge b_1 \to^* \textbf{false}$ and we have

$$\langle b_0 \wedge b_1, \sigma\rangle \to^* t$$

Lemma 2.16 *If* $\sigma \vdash b \to b'$ *and* $\langle b', \sigma\rangle \downarrow t$ *then* $\langle b, \sigma\rangle \downarrow t$.

$$\text{Case } \mathcal{S} = \text{SB-AND1}\frac{\overset{\mathcal{S}_0}{\sigma \vdash b_0 \to b_0'}}{\sigma \vdash b_0 \wedge b_1 \to b_0' \wedge b_1}$$

So $b = (b_0 \wedge b_1)$ and $b' = (b_0' \wedge b_1)$. Since there are two big-step rules for **conjunctions**, $\mathcal{E}'$ must have one of the two following shapes:

$$\text{Subcase } \mathcal{E}' = \text{EB-ANDT}\frac{\overset{\mathcal{E}_0'}{\langle b_0', \sigma\rangle \downarrow \textbf{true}} \quad \overset{\mathcal{E}_1'}{\langle b_1, \sigma\rangle \downarrow t}}{\langle b_0' \wedge b_1, \sigma\rangle \downarrow t}$$

By IH on $\mathcal{S}_0$ with $\mathcal{E}_0'$ we get a derivation $\mathcal{E}_0$ of $\langle b_0, \sigma\rangle \downarrow \textbf{true}$ and we can construct $\mathcal{E}$ as

$$\mathcal{E} = \text{EB-ANDT}\frac{\overset{\mathcal{E}_0}{\langle b_0, \sigma\rangle \downarrow \textbf{true}} \quad \overset{\mathcal{E}_1}{\langle b_1, \sigma\rangle \downarrow t}}{\langle b_0 \wedge b_1, \sigma\rangle \downarrow t}$$

$$\text{Subcase } \mathcal{E}' = \text{EB-ANDF}\frac{\overset{\mathcal{E}_0'}{\langle b_0', \sigma\rangle \downarrow \textbf{false}}}{\langle b_0' \wedge b_1, \sigma\rangle \downarrow \textbf{false}}$$

By IH on $\mathcal{S}_0$ with $\mathcal{E}_0'$ we get a derivation $\mathcal{E}_0$ of $\langle b_0, \sigma\rangle \downarrow \textbf{false}$ and we can construct $\mathcal{E}$ as

$$\mathcal{E} = \text{EB-ANDF}\frac{\overset{\mathcal{E}_0}{\langle b_0, \sigma\rangle \downarrow \textbf{false}}}{\langle b_0 \wedge b_1, \sigma\rangle \downarrow \textbf{false}}$$

$$\text{Case } \mathcal{S} = \text{SB-ANDT}\frac{}{\sigma \vdash \textbf{true} \wedge b_1 \to b_1}$$

So $b = (\mathbf{true} \wedge b_1)$ and $b' = b_1$. Since there is only one possible big-step rule for evaluation of b', namely EB-Cst, we must have

$$\mathcal{E}' = \text{EB-Cst}\frac{}{\langle b_1, \sigma \rangle \downarrow b_1}$$

and we can construct

$$\mathcal{E} = \text{EB-Cst}\frac{}{\langle b_1, \sigma \rangle \downarrow b_1}$$

$$\text{Case } \mathcal{S} = \text{SB-andF}\frac{}{\sigma \vdash \mathbf{false} \wedge b_1 \to \mathbf{false}}$$

So $b = (\mathbf{false} \wedge b_1)$ and $b' = \mathbf{false}$. Since there is only one possible big-step rule for evaluation of b', namely EB-Cst, we must have

$$\mathcal{E}' = \text{EB-Cst}\frac{}{\langle b_0, \sigma \rangle \downarrow \mathbf{false}}$$

and we can construct

$$\mathcal{E} = \text{EB-Cst}\frac{}{\langle b_0, \sigma \rangle \downarrow \mathbf{false}}$$