Semantics and Types - Assignment 1

Mikkel Willén
bmq419

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1 Task 1

We extend the IMP language with

$$\text{c} ::= \ \ldots \mid \textbf{repeat } c_0 \textbf{ until } b$$

The big-step rules for defining the formal semantics of **repeat**-loops are:

$$\textsc{EC-RepeatT}\frac{\langle c_0, \sigma\rangle \downarrow \sigma' \quad \langle b, \sigma'\rangle \downarrow \textbf{true}}{\langle \textbf{repeat } c_0 \textbf{ until } b, \sigma\rangle \downarrow \sigma'}$$

$$\textsc{EC-RepeatF}\frac{\langle c_0, \sigma\rangle \downarrow \sigma'' \quad \langle b, \sigma''\rangle \downarrow \textbf{false} \quad \langle \textbf{repeat } c_0 \textbf{ until } b, \sigma''\rangle \downarrow \sigma'}{\langle \textbf{repeat } c_0 \textbf{ until } b, \sigma\rangle \downarrow \sigma'}$$

We now prove the semantic equivalence:

$$\underbrace{\textbf{repeat } c_0 \textbf{ until } b}_{c} \sim \underbrace{c_0;\ \textbf{if } b \textbf{ then skip else repeat } c_0 \textbf{ until } b}_{c'}$$

We assume that we have a derivation $\mathcal{E}$ of $\langle c, \sigma\rangle \downarrow \sigma'$ and we must construct an $\mathcal{E}'$ of $\langle c', \sigma\rangle \downarrow \sigma'$. Looking at the rules we just made, we see that $\mathcal{E}$ must have one of the two following shapes:

$$\text{Case } \mathcal{E} = \textsc{EC-RepeatT} : \frac{\overset{\mathcal{E}_0}{\langle c_0, \sigma\rangle \downarrow \sigma'} \quad \overset{\mathcal{E}_1}{\langle b, \sigma'\rangle \downarrow \textbf{true}}}{\langle \textbf{repeat } c_0 \textbf{ until } b, \sigma\rangle \downarrow \sigma'}$$

Here we can build $\mathcal{E}'$ as follows:

$$\mathcal{E}' = \textsc{EC-Seq}\frac{\overset{\mathcal{E}_0}{\langle c_0, \sigma\rangle \downarrow \sigma'} \quad \textsc{EC-IfT}\dfrac{\overset{\mathcal{E}_1}{\langle b, \sigma'\rangle \downarrow \textbf{true}} \quad \textsc{EC-Skip}\dfrac{}{\langle \textbf{skip}, \sigma\rangle \downarrow \sigma}}{\langle \textbf{if } b \textbf{ then skip else } c, \sigma'\rangle \downarrow \sigma'}}{\langle c_0;\ \textbf{if } b \textbf{ then skip else } c, \sigma\rangle \downarrow \sigma'}$$

$$\text{Case } \mathcal{E} = \textsc{EC-RepeatF}\frac{\overset{\mathcal{E}_0}{\langle c_0, \sigma\rangle \downarrow \sigma''} \quad \overset{\mathcal{E}_1}{\langle b, \sigma''\rangle \downarrow \textbf{false}} \quad \overset{\mathcal{E}_2}{\langle c, \sigma''\rangle \downarrow \sigma'}}{\langle c, \sigma\rangle \downarrow \sigma'}$$

Here we can build $\mathcal{E}'$ as follows:

$$\mathcal{E}' = \textsc{EC-Seq}\frac{\overset{\mathcal{E}_0}{\langle c_0, \sigma\rangle \downarrow \sigma''} \quad \textsc{EC-IfF}\dfrac{\overset{\mathcal{E}_1}{\langle b, \sigma''\rangle \downarrow \textbf{false}} \quad \overset{\mathcal{E}_2}{\langle c, \sigma''\rangle \downarrow \sigma'}}{\langle \textbf{if } b \textbf{ then skip else } c, \sigma''\rangle \downarrow \sigma'}}{\langle c_0;\ \textbf{if } b \textbf{ then skip else } c, \sigma\rangle \downarrow \sigma'}$$

We now assume that we have a derivation $\mathcal{E}'$ of $\langle c', \sigma\rangle \downarrow \sigma'$ and we must construct an $\mathcal{E}$ of $\langle c, \sigma\rangle \downarrow \sigma'$. Looking at the rules for **sequence**-commands, we see we only have one case:

$$\mathcal{E}' = \textsc{EC-Seq}\frac{\overset{\mathcal{E}'_0}{\langle c_0, \sigma\rangle \downarrow \sigma''} \quad \overset{\mathcal{E}'_1}{\langle \textbf{if } b \textbf{ then skip else } c, \sigma''\rangle \downarrow \sigma'}}{\langle c_0;\ \textbf{if } b \textbf{ then skip else } c, \sigma\rangle \downarrow \sigma'}$$

Then, since there are two execution rules for **if**-commands, $\mathcal{E}_1'$ must have one of the two following shapes:

$$\text{Case } \mathcal{E}_1' = \text{EC-IfT}\dfrac{\overset{\mathcal{E}_{1,0}'}{\langle b,\sigma''\rangle \downarrow \textbf{true}} \quad \text{EC-Skip}\dfrac{}{\langle \textbf{skip},\sigma''\rangle\downarrow\sigma'}}{\langle \textbf{if } b \textbf{ then skip else } c,\sigma''\rangle\downarrow\sigma'}$$

Since the execution rule for **skip** does not change σ, so we have that $\sigma'' = \sigma'$ and we can build $\mathcal{E}$ as follows:

$$\mathcal{E} = \text{EC-RepeatT}\dfrac{\overset{\mathcal{E}_0'}{\langle c_0,\sigma\rangle\downarrow\sigma'} \quad \overset{\mathcal{E}_1'}{\langle b,\sigma'\rangle\downarrow\textbf{true}}}{\langle \textbf{repeat } c_0 \textbf{ until } b,\sigma\rangle\downarrow\sigma'}$$

$$\text{Case } \mathcal{E}_1' = \text{EC-IfF}\dfrac{\overset{\mathcal{E}_{1,0}'}{\langle b,\sigma''\rangle \downarrow \textbf{false}} \quad \overset{\mathcal{E}_{1,1}'}{\langle c,\sigma''\rangle\downarrow\sigma'}}{\langle \textbf{if } b \textbf{ then skip else } c,\sigma''\rangle\downarrow\sigma'}$$

Here we can build $\mathcal{E}$ as follows:

$$\mathcal{E} = \text{EC-RepeatF}\dfrac{\overset{\mathcal{E}_0'}{\langle c_0,\sigma\rangle\downarrow\sigma''} \quad \overset{\mathcal{E}_{1,0}'}{\langle b,\sigma''\rangle\downarrow\textbf{false}} \quad \overset{\mathcal{E}_{1,1}'}{\langle c,\sigma''\rangle\downarrow\sigma'}}{\langle c,\sigma\rangle\downarrow\sigma'}$$

2 Task 2

We show the left-to-right direction of the following semantic equivalence:

$$\underbrace{\text{x } := \bar{0};\ \textbf{if } y = \bar{2} \textbf{ then } x := x + \bar{1} \textbf{ else skip}}_{c} \sim \underbrace{\textbf{if } \bar{2} = y \textbf{ then } x := \bar{1} \textbf{ else } x := \bar{0}}_{c'}$$

We assums that we have a derivation $\mathcal{E}$ of $\langle c,\sigma\rangle\downarrow\sigma'$ and we must construct an $\mathcal{E}'$ of $\langle c',\sigma\rangle\downarrow\sigma'$. Looking at the execution rule for the **sequence**-command we can see $\mathcal{E}$ only have one case:

$$\mathcal{E} = \text{EC-Seq}\dfrac{\text{EC-Assign}\dfrac{\text{EA-Num}\dfrac{\mathcal{E}_0}{\langle\bar{0},\sigma\rangle\downarrow 0}}{\langle x := \bar{0},\sigma\rangle\downarrow\sigma[x\mapsto 0]} \quad \overset{\mathcal{E}_1}{\langle \textbf{if } y = \bar{2} \textbf{ then } x := x + \bar{1} \textbf{ else skip}, \sigma[x\mapsto 0]\rangle\downarrow\sigma'}}{\langle x := \bar{0};\ \textbf{if } y = \bar{2} \textbf{ then } x := x+\bar{1} \textbf{ else skip }\rangle\downarrow\sigma'}$$

Now, since there are two execution rules for **if**-commands, $\mathcal{E}_1$ must have one of the two following shapes: (note $\sigma_i = \sigma[x\mapsto 0]$)

$$\text{Case } \mathcal{E}_1 = \text{EC-IfF}\dfrac{\text{EB-EqF}\dfrac{\text{EA-Loc}\dfrac{\mathcal{E}_{1,0}}{\langle y,\sigma_i\rangle\downarrow\sigma_i(y)} \quad \text{EA-Num}\dfrac{\mathcal{E}_{1,1}}{\langle\bar{2},\sigma_i\rangle\downarrow 2}}{\langle y = \bar{2},\sigma_i\rangle\downarrow\textbf{false}} \quad \text{EC-Skip}\dfrac{}{\langle\textbf{skip},\sigma_i\rangle\downarrow\sigma_i}}{\langle \textbf{if } y = \bar{2} \textbf{ then } x := x+\bar{1} \textbf{ else skip},\sigma_i\rangle\downarrow\sigma'}$$

We see that $\sigma' = \sigma_i$ and we can build $\mathcal{E}'$ as follows:

$$\mathcal{E}' = \text{EC-IfF}\dfrac{\text{EB-EqF}\dfrac{\text{EA-Num}\dfrac{\mathcal{E}_{1,1}}{\langle\bar{2},\sigma\rangle\downarrow 2} \quad \text{EA-Loc}\dfrac{\mathcal{E}_{1,0}}{\langle y,\sigma\rangle\downarrow\sigma(y)}}{\langle\bar{2} = y,\sigma\rangle\downarrow\textbf{false}} \quad \text{EC-Assign}\dfrac{\text{EA-Num}\dfrac{\mathcal{E}_0}{\langle\bar{0},\sigma\rangle\downarrow 0}}{\langle x := \bar{0},\sigma\rangle\downarrow\sigma[x\mapsto 0]}}{\langle\textbf{if } \bar{2} = y \textbf{ then } x := \bar{1} \textbf{ else } x := \bar{0},\sigma\rangle\downarrow\sigma[x\mapsto 0]}$$

$$\text{Case } \mathcal{E}_1 = \text{EC-IfT}\dfrac{\text{EB-EqT}\dfrac{\text{EA-Loc}\dfrac{\mathcal{E}_{1,0}}{\langle y,\sigma_i\rangle\downarrow\sigma_i(y)} \quad \text{EA-Num}\dfrac{\mathcal{E}_{1,1}}{\langle\bar{2},\sigma_i\rangle\downarrow 2}}{\langle y = \bar{2},\sigma_i\rangle\downarrow\textbf{true}} \quad \overset{\mathcal{E}_2}{\langle x+1,\sigma\rangle\downarrow\sigma[x\mapsto 1]}}{\langle \textbf{if } y = \bar{2} \textbf{ then } x := x+\bar{1} \textbf{ else skip},\sigma_i\rangle\downarrow\sigma'}$$

$$\mathcal{E}_2 = \text{EC-Assign}\,\dfrac{\text{EA-Plus}\,\dfrac{\text{EA-Loc}\,\dfrac{}{\langle x, \sigma_i \rangle \downarrow 0} \quad \text{EA-Num}\,\dfrac{}{\langle \bar{1}, \sigma_i \rangle \downarrow 1}}{\langle x := x + 1, \sigma \rangle \downarrow 1}}{\langle x + 1, \sigma \rangle \downarrow \sigma[x \mapsto 1]}$$

We see that $\sigma' = \sigma[x \mapsto 1]$ and we can build $\mathcal{E}'$ as follows:

$$\mathcal{E}' = \text{EC-IfT}\,\dfrac{\text{EB-EqT}\,\dfrac{\text{EA-Num}\,\dfrac{\mathcal{E}_{1,1}}{\langle \bar{2}, \sigma \rangle \downarrow 2} \quad \text{EA-Loc}\,\dfrac{\mathcal{E}_{1,0}}{\langle y, \sigma \rangle \downarrow \sigma(y)}}{\langle \bar{2} = y, \sigma \rangle \downarrow \mathbf{true}} \quad \text{EC-Assign}\,\dfrac{\text{EA-Num}\,\dfrac{}{\langle \bar{1}, \sigma \rangle \downarrow 1}}{\langle x := \bar{1}, \sigma \rangle \downarrow \sigma[x \mapsto 1]}}{\langle \mathbf{if}\ \bar{2} = y\ \mathbf{then}\ x := \bar{1}\ \mathbf{else}\ x := \bar{0}, \sigma \rangle \downarrow \sigma[x \mapsto 1]}$$