Signal Output of a Linear Time-Variant System

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We define the output of a Linear Time-Variant (LT) system, illustrated in Fig 1, by the following integral equation

$$y(t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} h(\tau;t)x(t-\tau)\,d\tau. \tag{1}$$

Further, we define the Fourier Transform ($\mathcal{F}$) of a possibly multivariate funciton, with respect to one of its input variable, as

$$\mathcal{F}_{\tau\to f}\left\{\,h(\tau;t)\,\right\} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} h(\tau;t)e^{-j2\pi f\tau}\,d\tau = H(f;t), \tag{2}$$

and the Inverse Fourier Transform ($\mathcal{F}^{-1}$) of a possibly multivariate function, with respect to one of its input variables, as

$$\mathcal{F}^{-1}_{f\to\tau}\left\{\,H(f;t)\,\right\} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} H(f;t)e^{j2\pi f\tau}\,df = h(\tau;t). \tag{3}$$

With these three definitions, we are able to prove the following relationship between the Inverse Foruier Transform of the product $X(f)\cdot H(f;t)$, and the time-domain output from the LT system $y(t)$:

$$\begin{aligned}
\int_{-\infty}^{\infty} H(f;t)X(f)e^{j2\pi ft}\,df &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(w;t)e^{-j2\pi fw}\,dw\,X(f)e^{j2\pi ft}\,df = \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(w;t)e^{-j2\pi fw}X(f)e^{2\pi ft}\,dw\,df = \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(w;t)e^{-j2\pi fw}X(f)e^{2\pi ft}\,df\,dw = \\
&= \int_{-\infty}^{\infty} h(w;t)\int_{-\infty}^{\infty} X(f)e^{-j2\pi fw}e^{2\pi ft}\,df\,dw = \\
&= \int_{-\infty}^{\infty} h(w;t)\int_{-\infty}^{\infty} X(f)e^{j2\pi f(t-w)}\,df\,dw = \\
&= \int_{-\infty}^{\infty} h(w;t)x(t-w)\,dw \stackrel{\text{def}}{=} y(t).
\end{aligned} \tag{4}$$

We conclude the following important relation for LT systems

$$\underline{\underline{y(t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} h(\tau;t)x(t-\tau)\,d\tau = \int_{-\infty}^{\infty} H(f;t)X(f)e^{j2\pi ft}\,df.}} \tag{5}$$

$$x(t) \longrightarrow \boxed{h(\tau;t)} \longrightarrow y(t)$$

Figure 1: A general Linear and Time-Variant (LT) system.