Séminaire Lotharingien de Combinatoire XX (2020)
Article #YY, 17 pp.

Proceedings of the 32^{nd} *Conference on Formal Power Series and Algebraic Combinatorics (Ramat Gan)*

Dynamic limit of Robinson–Schensted–Knuth algorithm

Mikołaj Marciniak*[1]

[1] *Interdisciplinary PhD School "Academia Copernicana", Faculty of Mathematics and Computer Science, Nicolaus Copernicus University in Toruń, ul. Chopina 12/18, 87-100 Toruń, Poland*

Abstract. We investigate the evolution in time of the position of a fixed number in the insertion tableau when the Robinson–Schensted–Knuth algorithm is applied to a sequence of random numbers. When the length of the sequence tends to infinity, a typical trajectory after scaling converges in probability to some deterministic curve.

Keywords: RSK algorithm, bumping route, random Young tableaux, limit shape

1 Introduction

A full version of this paper will be published elsewhere.

1.1 Notations

A *partition* of a natural number n is a break up of n into a sum $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ are positive integer numbers. The vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is usually used to denote a partition. Let $\lambda \vdash n$ denote that λ is a partition of a number n. A *Young diagram* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a finite collection of boxes arranged in left-justified rows with the row length λ_j of the j-th row. Thus the Young diagram λ is a graphical interpretation of the partition λ. A *Young tableau* is a Young diagram filled with numbers. If the entries strictly decrease along each column from top to bottom and weakly increase along each row from left to right, a tableau is called *semistandard*. A *standard Young tableau* is a semistandard Young tableau with n boxes which contains all numbers $1, 2, \ldots, n$. Figure 1 shows examples of a Young diagram and of a standard Young tableau.

The *Robinson–Schensted–Knuth algorithm RSK* is a bijective algorithm which takes a finite sequence of numbers as the input and returns a pair of Young tableaux (P, Q) with the same shape $\lambda \vdash n$. The semistandard tableau P is called an *insertion tableau*,

*mikolaj@mat.umk.pl. Mikołaj Marciniak was partially supported by Narodowe Centrum Nauki, grant number 2017/26/A/ST1/00189.

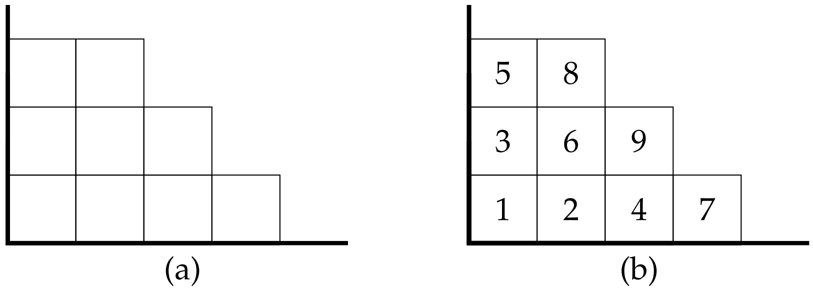


Figure 1: (a) The Young diagram of shape $(4,3,2) \vdash 9$ and (b) a standard Young tableau of shape $(4,3,2) \vdash 9$.

and the standard tableau Q is called a *recording tableau*. In particular, the RSK algorithm assigns to any permutation σ a pair of standard Young tableaux (P, Q). A detailed description of the RSK algorithm can be found in [Rom15, Chapter 1.6].

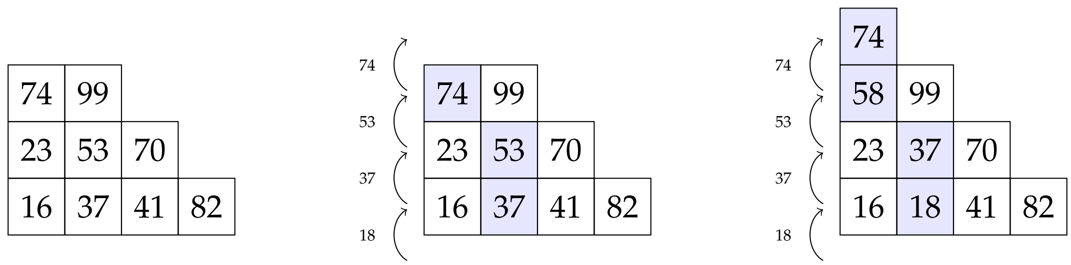
The RSK algorithm is based on applying the *insertion step* to successive numbers from a given finite sequence $\{X_j\}_{j=1}^{n}$. The insertion step takes as input the previously obtained tableau $P(X_1, X_2, \dots, X_{j-1})$ and the next number X_j from the sequence. It produces as the output a new tableau $P(X_1, X_2, \dots, X_j)$ with shape "increased" by one box; this tableau is obtained in the following way, see Figure 2. The RSK-insertion step starts in the first row with the number $x := X_j$. The insertion step consists of inserting the number x into the leftmost box in this row containing a number y greater than x. Move to the next row with the number $x := y$ and repeat the action. At some row we are forced to insert the number at the end of the row, which will end the insertion step. The collection of rearranged boxes is called the *bumping route*.

We can say that the boxes with numbers are moved along the bumping route during an RSK insertion step. In this article we will investigate the position of the box with a selected number and how its position is changing over time.

1.2 Motivations

The RSK algorithm is an important tool in algebraic combinatorics, especially in the context of Littlewood–Richardson coefficients and plactic monoid [FH91].

For many years mathematicians have been studying the asymptotic behavior of the insertion tableau when we apply the Robinson–Schensted–Knuth algorithm to a random input. In the following paragraphs we will see several examples of such considerations. The RSK algorithm applied to the sequence of independent and identically distributed random variables with the uniform distribution on the interval $(0, 1)$ gener-



(a) (b) (c)

Figure 2: (a) The original tableau P. (b) The highligted boxes form the bumping route which corresponds to an insertion of the number 18. The numbers next to the arrows describe the rearrenged boxes with the numbers.(c) The output of RSK insertion step.

ates the *Plancherel measure* on Young diagrams [Rom15]. The Plancherel measure is an important element of the representation theory because it describes how the left regular representation decomposes into irreducible componennts [FH91, Chapter 3.3].

The Ulam–Hammersley problem appears in [BDJ99] and concerning the typical length of the longest increasing subsequence in a random permutation corresponds to the problem concerning the typical length of the first row in the Young tableau obtained by the RSK algorithm from the sequence of independent random variables $\{X_j\}_{j=1}^{n}$ with the uniform distribution $U(0,1)$ on the interval $(0,1)$.

Logan and Shepp [LS77] and Vershik and Kerov [KV86] described the limit shape of the insertion tableau $P(X_1, X_2, \dots, X_n)$ obtained when we apply the RSK algorithm to a random finite sequence.

Romik and Śniady [RŚ16] considered the limit shape of the bumping routes obtained from the insertion tableau $P(X_1, X_2, \dots, X_n, w)$, when applying an RSK insertion step with a fixed number w to an existing insertion tableau obtained from a random finite sequence. In [RŚ15] they considered also the limit shape of *jeu de taquin* obtained from the recording tableau $Q(w, X_1, X_2, \dots, X_n)$ made from a random finite sequence preceded by a fixed number w.

1.3 The main problem

This paper also concerns the asymptotic behavior of the insertion tableau when we apply the RSK algorithm to a random input. *What can we say about the evolution over time of the insertion tableau from the viewpoint of box dynamics, when we apply the RSK algorithm to a sequence of independent random variables with the uniform distribution $U(0,1)$? How do the boxes move in the insertion tableau? If we investigate the scaled position of a box with a fixed number, will we get a deterministic limit, when the number of boxes tends to infinity?*

More specifically, we consider the insertion tableau $P(X_1, X_2, \ldots, X_n, w, X_{n+1} \ldots, X_m)$ obtained by the RSK algorithm applied to a random finite sequence containing a fixed number w at some index. The box with this fixed number w is being bumped by the RSK insertion step along the bumping routes. We will describe the scaled limit position of the box with the number w depending on the ratio of the numbers m and n.

2 Tools

2.1 The function $G(x)$

We define the functions $F_{SC}, \Omega_\star, u, v, G$ by:

$$\begin{aligned}
F_{SC}(x) &= \frac{1}{2} + \frac{1}{\pi}\left(\frac{x\sqrt{4-x^2}}{4} + \sin^{-1}\left(\frac{x}{2}\right)\right) && (-2 \le x \le 2),\\
\Omega_\star(x) &= \frac{2}{\pi}\left(\sqrt{4-x^2} + x\sin^{-1}\left(\frac{x}{2}\right)\right) && (-2 \le x \le 2),\\
u(x) &= F_{SC}^{-1}(x),\\
v(x) &= \Omega_\star(u(x)),\\
G(x) &= \left(\frac{v(x)+u(x)}{2}, \frac{v(x)-u(x)}{2}\right).
\end{aligned}$$

Figure 3 shows the graph of the function $G(x)$.

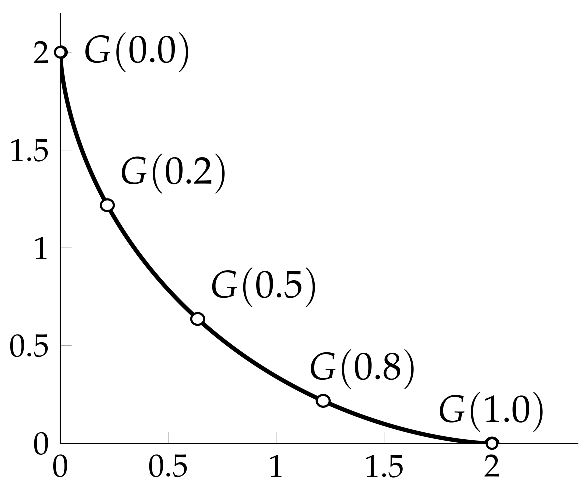


Figure 3: The graph of the function $G(x)$ with specified values for numbers $x = 0.0$, $x = 0.2$, $x = 0.5$, $x = 0.8$ and $x = 1.0$.

The function F_{SC} is the cumulative distribution function of the *semicircle distribution*. The function $\Omega_\star$ is the limit shape of the Young Tableau obtained from the *Plancherel growth process*. This curve is called the Logan–Sheep–Vershik–Kerov curve. The function $\big(u(x), v(x)\big)$ is a special parameterisation of the function $\Omega_\star(x)$ and describes the limit position of the new box in the RSK insertion step applied with the number x to the random Young Tableau. The function $G(x)\colon [0,1] \to [0,2]^2$ is the function $\big(u(x), v(x)\big)$ rotated by 45 degrees (see [RŚ15, Chapter 5.1]). Moreover the function $G(x)$ is continuous and increasing in the first coordinate and decreasing in the second coordinate.

2.2 The result of Romik and Śniady

In the proof of Theorem 2 we will need the following result of Romik and Śniady [RŚ15, Theorem 5.1].

Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of independent random variables with the uniform distribution $U(0,1)$. Let $\square_n(x) \in \mathbb{N}^2$ denote the position of the new box in the insertion tableau $P(X_1, \dots, X_n, x)$ when we apply the RSK insertion step for the number $x \in [0,1]$ to the previously obtained tableau $P(X_1, \dots, X_n)$.

Theorem 1. *For each x the position $\square_n(x)$, after scaling by $\sqrt{n}$, converges in probability to a specific point $G(x) \in [0,2]^2$, when n tends to infinity:*

$$\frac{\square_n(x)}{\sqrt{n}} \xrightarrow{p} G(x).$$

2.3 The partial order on the plane

We define *the partial order $\prec$ on the plane* as follows: $(x_1, y_1) \prec (x_2, y_2)$ if and only if $x_1 \leq x_2$ and $y_1 \geq y_2$.

The function $G(x)$ is *increasing in the relation $\prec$*, i.e. if $x_1 \leq x_2$ then $G(x_1) \prec G(x_2)$. Likewise from a property of the RSK insertion step for each natural number $n \in \mathbb{N}$, the function $\square_n(x)$ is increasing in the relation $\prec$, i.e. if $x_1 \leq x_2$ then $\square_n(x_1) \prec \square_n(x_2)$.

2.4 The random increasing sequence

Let $w \in (0,1]$ be a fixed number and let $\{X'_j\}_{j=1}^{m'}$ be a finite sequence of independent random variables with the uniform distribution $U(0,w)$. We define the function z : $\{1, 2, \dots m'\} \to [0,w]$ that assigns to a number $t \in \{1, 2, \dots, m'\}$, the t-th smallest number among $X'_1, X'_2, \dots, X'_{m'}$. The sequence $\{z(j)\}_{j=1}^{m'}$ is called the order statistic of the sequece $\{X'_j\}_{j=1}^{m'}$.

The sequence $\{z(j)\}_{j=1}^{m'}$ contains all elements of the sequence $\{X'_j\}_{j=1}^{m'}$ in the ascending order. The sequence $\{z(j)\}_{j=1}^{m'}$ will be called a *random increasing sequence with the uniform distribution on the interval* $[0,w]$. In addition, the sequence $\{X'_j\}_{j=1}^{m'}$ is a some permutation Π of the sequence $\{z(j)\}_{j=1}^{m'}$. The permutation Π is the random permutation with the uniform distribution. Let $\Pi = (\Pi_1, \Pi_2, \ldots, \Pi_{m'})$. Then

$$\left\{X'_j\right\}_{j=1}^{m'} = \left\{z(\Pi_j)\right\}_{j=1}^{m'} = z \circ \Pi,$$

where z is the function that acts pointwise on every element of the permutation Π. We see that the finite sequence of independent random variables with the uniform distribution $U(0,w)$ have the same distribution as the finite random increasing sequence with the uniform distribution on the interval $[0,w]$ permuted by the random permutation with the uniform distribution.

3 The main result

Our main result is Theorem 2 describing the asymptotic behavior of the box with a fixed number. It states that when the number of boxes tends to infinity then the (scaled down) trajectory of the box converges in probability to the curve $H\colon [1,\infty) \to \mathbb{R}_+^2$ given by

$$H(A) := \sqrt{A}\, G\left(\frac{1}{A}\right).$$

The same curve H also happens to be the limit shape of the bumping routes [RŚ16] in the RSK algorithm. Figure 4 shows the graph of the curve H and the experimentally determined trajectory of the box with the number $w = 0.5$.

More specifically, let $w \in (0,1]$ be a fixed number. Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of independent random variables with the uniform distribution $U(0,1)$. For every $n \in \mathbb{N}$ we define the function $\mathrm{Pos}_n : \{n+1, n+2, \ldots\} \to \mathbb{N}^2$ by:

$$\mathrm{Pos}_n(j) = \mathrm{box}_w\Big(P\left(X_1, \ldots, X_n, w, X_{n+1}, \ldots, X_j\right)\Big)$$

for $j \in \{n+1, n+2, \ldots\}$, where for a tableau P we denote by $\mathrm{box}_w(P) \in \mathbb{N}^2$ the coordinates of the box with the number w.

Theorem 2. *For each number $A \in [1,\infty)$ the random variable $\mathrm{Pos}_n(\lfloor An \rfloor)$, after scaling by $\sqrt{wn}$, converges in probability to the limit $H(A)$, when n tends to infinity. In other words, for each $\epsilon > 0$:*

$$\lim_{n\to\infty} \mathbb{P}\left(\left\| \frac{\mathrm{Pos}_n(\lfloor An \rfloor)}{\sqrt{wn}} - H(A) \right\| > \epsilon \right) = 0.$$

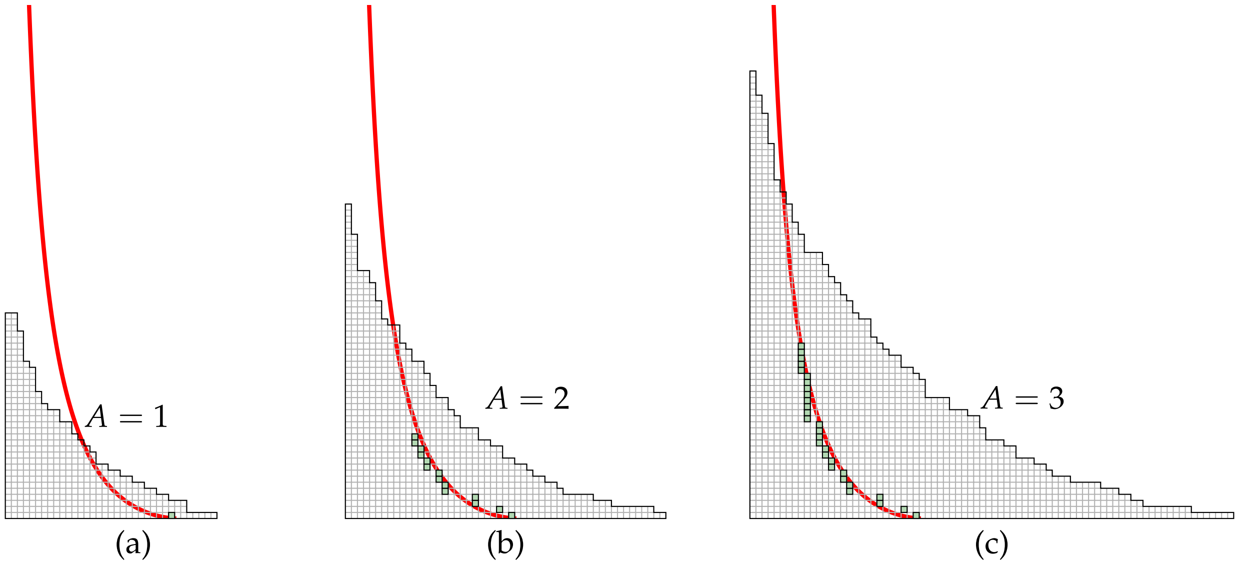


Figure 4: (a) The initial shape of the insertion tableau $P(X_1, \dots, X_n, w)$ immediately after the new box with the number w was added (the higlighted box in the bottom row) for $n = 400$ and $w = 0.5$. (b) The shape of the insertion tableau $P(X_1, \dots, X_n, w, X_{n+1}, \dots, X_{\lfloor An \rfloor})$ at the time parameter $A = 2$. The highlighted boxes indicate the trajectory of the box with the number w. The red smooth curve is the plot of H. (c) Analogous picture for $A = 3$.

Proof. We apply the RSK algorithm to a random sequence of real numbers containing the number w and investigate the position of the box with the number w in the insertion tableau. An insertion step applied to a number greater that w does not change the position of the number w in the tableau, so it is enough to consider only the subsequence containing numbers not greater than w.

Now we will use this observation in the proof. Let $m = \lfloor An \rfloor$. The probability that the same number occurs twice in the sequence $(w, X_1, X_2, \dots)$ is equal to 0, hence without losing generality we assume that the numbers $w, X_1, X_2, \dots$ are all different. Let $\{X'_j\}_{j=1}^{\infty}$ be the subsequence of the sequence $\{X_j\}_{j=1}^{\infty}$ containing all elements of the sequence $\{X_j\}_{j=1}^{\infty}$, which are less than w. The sequence $\{X'_j\}_{j=1}^{\infty}$ is a sequence of independent random variables with the uniform distribution $U(0, w)$.

Let $n' = n'(n)$ and $m' = m'(n)$ denote the number of elements, respectively, of the sequences $\{X_j\}_{j=1}^{n}$, $\{X_j\}_{j=1}^{m}$ which are smaller than w. Then there is an equality:

$$\begin{aligned} \mathrm{Pos}_n\left(\lfloor An \rfloor\right) &= \mathrm{box}_w\left(P\left(X_1, \dots, X_n, w, X_{n+1}, \dots, X_m\right)\right) \\ &= \mathrm{box}_w\left(P\left(X_1, \dots, X'_{n'}, w, X'_{n'+1}, \dots, X'_{m'}\right)\right). \end{aligned}$$

The random variable $n' = \sum_{j=1}^{n} [X_j < w]$ counts how many numbers from the sequence $\{X_j\}_{j=1}^{n}$ are less than w, so n' is a random variable with the binomial distribution with parameters n and w. We denote it $n' \sim B(n, w)$. Likewise the random variable $m' - n'$ counts how many numbers from the sequence $\{X_j\}_{j=n'+1}^{m'}$ are less than w, so $m' - n' \sim B(m - n, w)$. Moreover, the random variables n' and $m' - n'$ are independent, because the random variables $X_1, X_2, \ldots$ are independent.

From the Strong Law of Large Numbers [Dur19, Theorem 2.4.1] we know that if n tends to infinity, then the following limits exist almost surely:

$$\lim_{n\to\infty} \frac{n'}{n} = \mathbb{E}[X_j < w] = \mathbb{P}(X_j < w) = w,$$
$$\lim_{n\to\infty} \frac{m' - n'}{m - n} = \mathbb{E}[X_j < w] = \mathbb{P}(X_j < w) = w.$$

Therefore, also the following limits exist almost surely

$$\begin{aligned}
\lim_{n\to\infty} \frac{m'}{n'} &= \lim_{n\to\infty} 1 + \frac{m' - n'}{n'} \\
&= 1 + \lim_{n\to\infty} \frac{m' - n'}{m - n} \frac{1}{\frac{n'}{n}} \frac{m - n}{n} \\
&= 1 + \lim_{n\to\infty} \frac{m' - n'}{m - n} \frac{1}{\frac{n'}{n}} \left(\frac{m}{n} - 1 \right) \\
&= 1 + \frac{w}{w} (A - 1) \\
&= A
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} \frac{m'}{n} &= \lim_{n\to\infty} \frac{n'}{n} \frac{m'}{n'} \\
&= wA.
\end{aligned}$$

We define the function $z : \{1, 2, \ldots m'\} \cup \{m' + 1, n' + \frac{1}{2}\} \to [0, 1]$ that assigns to a number $t \in \{1, 2, \ldots, m'\}$, the t-th smallest number among $X'_1, X'_2, \ldots, X'_{m'}$ and additionally $z(m' + 1) = w$ and $z\left(n' + \frac{1}{2}\right) = \frac{z(n') + z(n'+1)}{2}$.

From a property of the random increasing sequence with the uniform distribution (Section 2.4) we have:

$$\left\{X'_j\right\}_{j=1}^{m'} = \left\{z(\Pi_j)\right\}_{j=1}^{m'} = z \circ \Pi,$$

where $\{z(j)\}_{j=1}^{m'}$ is the random increasing sequence and Π is a random permutation of range m' with the uniform distribution. The function z acts pointwise on every element of the permutation Π. Similarly, the function z acts on a Young tableau by acting on each box individually. Then

$$\begin{aligned}
\mathrm{Pos}_n(m) &= \mathrm{box}_w\left(P\left(X'_1,\dots,X'_{n'},w,X'_{n'+1},\dots,X'_{m'}\right)\right)\\
&= \mathrm{box}_w\left(P\Big(z(\Pi_1),\dots,z\left(\Pi_{n'}\right),w,z\left(\Pi_{n'+1}\right),\dots,z\left(\Pi_{m'}\right)\Big)\right)\\
&= \mathrm{box}_w\left(z\circ P\left(\Pi_1,\dots,\Pi_{n'},m'+1,\Pi_{n'+1},\dots,\Pi_{m'}\right)\right)\\
&= \mathrm{box}_{m'+1}\left(P\left(\Pi_1,\dots,\Pi_{n'},m'+1,\Pi_{n'+1},\dots,\Pi_{m'}\right)\right).
\end{aligned}$$

We denote the inverse permutation to Π by $\Pi^{-1}=\left(\Pi_1^{-1},\dots,\Pi_{m'}^{-1}\right)$ and we define the permutation

$$\Pi{\uparrow}=\left(\Pi_1,\dots,\Pi_{n'},m'+1,\Pi_{n'+1},\dots,\Pi_{m'}\right)$$

as a natural extension of the permutation Π. Then $\Pi{\uparrow}^{-1}=\left(\Pi{\uparrow}_1^{-1},\dots,\Pi{\uparrow}_{m'+1}^{-1}\right)$ where

$$\Pi{\uparrow}_j^{-1}=\begin{cases}\Pi_j^{-1} & \text{if } j\le n,\\ \Pi_j^{-1}+1 & \text{if } n<j\le m',\\ n+1 & \text{if } j=m'+1.\end{cases}$$

In addition, we will use the fact [FH91] that for any permutation $\Pi{\uparrow}$ the insertion tableau of $\Pi{\uparrow}$ is equal to the recording tableau of the inverse permutation $\Pi{\uparrow}^{-1}$:

$$P\left(\Pi{\uparrow}\right)=Q\left(\Pi{\uparrow}^{-1}\right).$$

Therefore

$$\begin{aligned}
\mathrm{Pos}_n(m) &= \mathrm{box}_{m'+1}\left(P\left(\Pi_1,\dots,\Pi_{n'},m'+1,\Pi_{n'+1},\dots,\Pi_{m'}\right)\right)\\
&= \mathrm{box}_{m'+1}\left(Q\left(\Pi{\uparrow}^{-1}\right)\right)\\
&= \mathrm{box}_{m'+1}\left(Q\left(\Pi{\uparrow}_1^{-1},\dots,\Pi{\uparrow}_{m'}^{-1},n'+1\right)\right).
\end{aligned}$$

Now, using the function z, we will get the sequence of independent random variables with uniform distribution $U(0,1)$

$$\begin{aligned}
\mathrm{Pos}_n\,(m) &= \mathrm{box}_{m'+1}\left(Q\left(\Pi{\uparrow}_1^{-1},\dots,\Pi{\uparrow}_{m'}^{-1},n'+1\right)\right)\\
&= \mathrm{box}_{m'+1}\left(Q\left(\Pi{\uparrow}_1^{-1},\dots,\Pi{\uparrow}_{m'}^{-1},n'+\frac{1}{2}\right)\right)\\
&= \mathrm{box}_{m'+1}\left(Q\left(\Pi_1^{-1},\dots,\Pi_{m'}^{-1},n'+\frac{1}{2}\right)\right)\\
&= \mathrm{box}_{m'+1}\left(z\circ Q\left(\Pi_1^{-1},\dots,\Pi_{m'}^{-1},n'+\frac{1}{2}\right)\right)\\
&= \mathrm{box}_{m'+1}\left(Q\left(z\left(\Pi_1^{-1}\right),\dots,z\left(\Pi_{m'}^{-1}\right),z\left(n'+\frac{1}{2}\right)\right)\right).
\end{aligned}$$

The permutation Π is a random permutation with the uniform distribution, so Π^{-1} is also a random permutation with the uniform distribution. If we act with a random permutation on a random increasing sequence with the uniform distribution we will get a sequence of independent random variables with the uniform distribution, thus the sequence

$$z\circ\Pi^{-1} = \left(z\left(\Pi_1^{-1}\right),\dots,z\left(\Pi_{m'}^{-1}\right)\right)$$

is a sequence of independent random variables with the uniform distribution $U(0,w)$. We define the random variable T_n and the sequence $\{Y_j\}_{j=1}^{m'}$:

$$\begin{aligned}
Y_j &= \frac{z\left(\Pi_j^{-1}\right)}{w} \qquad \text{for } j\in 1,2,\dots,m',\\
T_n &= \frac{z\left(n'+\frac{1}{2}\right)}{w}.
\end{aligned}$$

The sequence $\{Y_j\}_{j=1}^{m'}$ is a sequence of independent random variables with the uniform distribution $U(0,1)$, and the random variable T_n converges almost surely to $\frac{1}{A}$:

$$\begin{aligned}
\lim_{n\to\infty} T_n &= \lim_{n\to\infty}\frac{z\left(n'+\frac{1}{2}\right)}{w}\\
&= \lim_{n\to\infty}\frac{1}{w}\frac{z\left(n'\right)+z\left(n'+1\right)}{2}\\
&= \lim_{n\to\infty}\frac{z\left(n'\right)}{w}+\lim_{n\to\infty}\frac{z\left(n'+1\right)-z\left(n'\right)}{2w}\\
&= \lim_{n\to\infty}\frac{n'}{m'}+0\\
&= \frac{1}{A}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathrm{Pos}_n(m) &= \mathrm{box}_{m'+1}\left(Q\left(z\left(\Pi_1^{-1}\right),\ldots,z\left(\Pi_{m'}^{-1}\right),z\left(n'+\frac{1}{2}\right)\right)\right)\\
&= \mathrm{box}_{m'+1}\left(Q\left(\frac{z\left(\Pi_1^{-1}\right)}{w},\ldots,\frac{z\left(\Pi_{m'}^{-1}\right)}{w},\frac{z\left(n'+\frac{1}{2}\right)}{w}\right)\right)\\
&= \mathrm{box}_{m'+1}\left(Q(Y_1,\ldots,Y_{m'},T_n)\right).\\
&= \square_{m'}(T_n),
\end{aligned}$$

where almost surely $T_n \to T = \frac{1}{A} \in (0,1)$, when n tends to infinity.

Let

$$G_n(x) := \frac{\square_{m'}(x)}{\sqrt{Anw}}.$$

The function $\square_{m'}$ is increasing in the order $\prec$ (Section 2.3). Hence G_n also is increasing in the order $\prec$, i.e. if $x_1 \le x_2$, then

$$G_n(x_1) \prec G_n(x_2). \tag{3.1}$$

From Theorem 1 for each $x \in (0,1)$ the random variable $G_n(x)$ converges in probability to the limit $G(x)$, when n tends to infinity. Therefore

$$G_n(x) = \sqrt{\frac{m'}{n}\frac{1}{Aw}}\,\frac{\square_{m'}(x)}{\sqrt{m'}} \xrightarrow{p} \sqrt{\frac{Aw}{Aw}}\,G(x) = G(x).$$

Using little-o notation this means that for each $x \in [0,1]$ and for each $\epsilon > 0$ occur:

$$\mathbb{P}\Big(\|G_n(x) - G(x)\| \le \epsilon\Big) = 1 - o(1). \tag{3.2}$$

If $\|G_n(x) - G(x)\| \le \epsilon$ then also

$$(-\epsilon,\epsilon) \prec G_n(x) - G(x) \prec (\epsilon,-\epsilon). \tag{3.3}$$

Let $\epsilon > 0$.
The function $G(x)$ is a continuous function in the point $T \in (0,1)$. This means that there exists $\delta > 0$ that if $|x - T| < 2\delta$ then $\|G(x) - G(T)\| < \epsilon$. Thus

$$(-\epsilon,\epsilon) \prec G(x) - G(T) \prec (\epsilon,-\epsilon). \tag{3.4}$$

In addition, almost surely $T_n \to T$. Then also $T_n \xrightarrow{p} T$. Using little-o notation we have:

$$\mathbb{P}(T-\delta < T_n < T+\delta) = 1 - o(1). \tag{3.5}$$

Furthermore if $T_n < T+\delta$ and $\|G_n(T+\delta) - G(T+\delta)\| \leq \epsilon$ then:

$$\begin{aligned}
G_n(T_n) - G(T) &\prec G_n(T+\delta) - G(T) && (3.1)\\
&= G_n(T+\delta) - G(T+\delta) + G(T+\delta) - G(T)\\
&\prec (\epsilon, -\epsilon) + (\epsilon, -\epsilon) && (3.3,3.4)\\
&= (2\epsilon, -2\epsilon).
\end{aligned}$$

Analogously if $T-\delta < T_n$ and $\|G_n(T-\delta) - G(T-\delta)\| \leq \epsilon$ then:

$$\begin{aligned}
G_n(T_n) - G(T) &\succ G_n(T-\delta) - G(T) && (3.1)\\
&= G_n(T-\delta) - G(T-\delta) + G(T-\delta) - G(T)\\
&\succ (-\epsilon, \epsilon) + (-\epsilon, \epsilon) && (3.3,3.4)\\
&= (-2\epsilon, 2\epsilon).
\end{aligned}$$

Thus if $\|G_n(T+\delta) - G(T+\delta)\| \leq \epsilon$ and $\|G_n(T-\delta) - G(T-\delta)\| \leq \epsilon$ and $|T_n - t| < \delta$ then:

$$\|G_n(T_n) - G(T)\| < \|(2\epsilon, -2\epsilon)\| < 4\epsilon.$$

Thereby:

$$\begin{aligned}
&\mathbb{P}\Big(\|G_n(T_n) - G(T)\| > 4\epsilon \Big) \leq\\
&\leq \mathbb{P}\Big(\|G_n(T+\delta) - G(T+\delta)\| > \epsilon \quad \text{or} \quad \|G_n(T-\delta) - G(T-\delta)\| > \epsilon \quad \text{or} \quad |T_n - T| \geq \delta \Big)\\
&\leq \mathbb{P}\Big(\|G_n(T+\delta) - G(T+\delta)\| > \epsilon \Big) + \mathbb{P}\Big(\|G_n(T-\delta) - G(T-\delta)\| > \epsilon \Big) + \mathbb{P}\Big(|T_n - T| \geq \delta \Big)\\
&= o(1). && (3.2,3.5)
\end{aligned}$$

Then for each $A \in [1, \infty)$ and each $\epsilon > 0$ we have:

$$\lim_{n\to\infty} \mathbb{P}\left(\|G_n(T_n) - G(T)\| > 4\frac{\epsilon}{4\sqrt{A}} \right) = 0,$$

$$\lim_{n\to\infty} \mathbb{P}\left(\left\| \frac{\mathrm{Pos}_n\left(\lfloor An \rfloor\right)}{\sqrt{wn}} - H\left(A\right) \right\| > \epsilon \right) = 0.$$

$\square$

4 The uniform convergence on the interval

Theorem 3. *For each number $A \in [1, R]$ the random variable $\mathrm{Pos}_n(\lfloor An \rfloor)$, after scaling by $\sqrt{wn}$, converges in probability to the limit $H(A)$, when n tends to infinity. Moreover, the convergence is uniform, i.e. for each $\epsilon > 0$*

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{A\in[1,R]} \left\| \frac{\mathrm{Pos}_n\left(\lfloor An \rfloor\right)}{\sqrt{wn}} - H\left(A\right) \right\| > \epsilon \right) = 0.$$

Proof. From proof of Theorem 2 we know that $G_n(T_n)$ converges in probability to the limit $G(T)$, when n tends to infinity for $T = \frac{1}{A} \in [\frac{1}{R}, 1]$.
The function $G(x)$ is also increasing in the order $\prec$. In addition $G(x)$ is the continuous function on the closed interval $[\frac{1}{R}, 1]$. Then from the Heine-Cantor theorem the function $G(x)$ is uniformly continuous on the interval $[\frac{1}{R}, 1]$. This means that for each $\epsilon > 0$ there exists $\delta > 0$ that for each $x \in [\frac{1}{R}, 1]$ if $|x - T| < \delta$ then $\|G(x) - G(T)\| < \epsilon$. Thus

$$(-\epsilon, \epsilon) \prec G(x) - G(T) \prec (\epsilon, -\epsilon).$$

Let $\delta > 0$. We split the interval $[\frac{1}{R}, 1]$ into the p equal intervals of lengths $\frac{1-\frac{1}{R}}{p} < \delta$ for a some natural number p. Let $x_0, x_1, \ldots, x_n$ denote the ends of these intervals. Let $T_n \in [\frac{1}{R}, 1]$. We choose a number n_0 so that

$$T_n, T \in (x_k, x_{k+1}) \text{ for a some number } k \in \{0, 1, \ldots, p-1\}$$

and

$$||G_n(x_k) - G(x_k)|| < \epsilon \text{ for all } n > n_0 \text{ and all } k \in \{0, 1, \ldots, p\}.$$

Thereby

$$\begin{aligned} G_n(T_n) - G(T) &\prec G_n(x_{k+1}) - G(T) && (3.1) \\ &\prec (\epsilon, -\epsilon) + G(x_{k+1}) - G(T) \\ &\prec (\epsilon, -\epsilon) + (\epsilon, -\epsilon) && (3.1) \\ &= (2\epsilon, -2\epsilon). \end{aligned}$$

Analogously

$$\begin{aligned} G_n(T_n) - G(T) &\succ G_n(x_k) - G(T) && (3.1) \\ &\succ (-\epsilon, \epsilon) + G(x_k) - G(T) \\ &\succ (-\epsilon, \epsilon) + (-\epsilon, \epsilon) && (3.1) \\ &= (-2\epsilon, 2\epsilon). \end{aligned}$$

Thus

$$\|G_n(T_n) - G(T)\| < \|(2\epsilon, -2\epsilon)\| < 4\epsilon.$$

Obviously if $T_n = x_k$ for a some number $k \in 0, 1, \ldots, p$ then the inequality also is fulfil. Then for each $A \in [1, R]$ and each $\epsilon > 0$ we have:

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{T\in[\frac{1}{R},1]} \sqrt{A}\, \|G_n(T_n) - G(T)\| > 4\sqrt{A}\epsilon \right) = 0.$$

Using the inequality $A \le R$ we obtain

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{T\in[\frac{1}{R},1]} \sqrt{A}\, \|G_n(T_n) - G(T)\| > 4\sqrt{R}\epsilon \right) = 0,$$

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{A\in[1,R]} \left\| \frac{\mathrm{Pos}_n\left(\lfloor An \rfloor\right)}{\sqrt{wn}} - H\left(A\right) \right\| > 4\sqrt{R}\epsilon \right) = 0.$$

Thus for each $\epsilon > 0$ occur

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{A\in[1,R]} \left\| \frac{\mathrm{Pos}_n\left(\lfloor An \rfloor\right)}{\sqrt{wn}} - H\left(A\right) \right\| > \epsilon \right) = 0.$$

□

5 Asymptotic behavior of the function $H(A)$

Let $A = \frac{1}{x}$. In this chapter we will describe the asymptotic behavior of the function $H(\frac{1}{x})$ when x tends to zero. For this purpose, we will use the formulas given in Section 2.1.

$$H(1/x) = \sqrt{\frac{1}{x}} G(x) = \frac{1}{2\sqrt{x}} \Big(v(x) + u(x), v(x) - u(x) \Big).$$

We define the function $f(x) : [0, 2] \to [0, \sqrt{\frac{3\pi}{2}}]$ by

$$f(x) = \sqrt[3]{\frac{3\pi}{2} F_{sc}(x^2 - 2)}.$$

The function F_{sc} is inversible, so the function f also is inversible and the above formula can be transformed into a form

$$f^{-1}(y) = \sqrt{F_{sc}^{-1}\left(\frac{2}{3\pi} y^3\right) + 2}.$$

Therefore for $x \in [0,1]$ we have:

$$F_{sc}^{-1}(x) = \left(f^{-1}\left(\sqrt[3]{\frac{3\pi}{2} x} \right) \right)^2 - 2.$$

Next we determine the taylor series of the function $f(x)$.

$$f(x) = x - \frac{1}{40}x^3 - \frac{39}{22400}x^5 - O(x^7).$$

We use some formulas and determine the series of inverse function $f^{-1}(y)$.

$$f^{-1}(y) = y + \frac{1}{40}y^3 + \frac{81}{22400}y^5 + O(y^7).$$

Of course $O\big(f^{-1}(y)\big) = O(y)$. Then for $x \in [0,1]$ occurs

$$\begin{aligned} F_{sc}^{-1}(x) &= \left(\left(\sqrt[3]{\frac{3\pi}{2} x} \right) + \frac{1}{40} \left(\sqrt[3]{\frac{3\pi}{2} x} \right)^3 + \frac{81}{22400} \left(\sqrt[3]{\frac{3\pi}{2} x} \right)^5 + O\left(\sqrt[3]{\frac{3\pi}{2} x} \right)^7 \right)^2 - 2 \\ &= -2 + \left(\frac{3\pi}{2} x \right)^{\frac{2}{3}} + \frac{1}{20} \left(\frac{3\pi}{2} x \right)^{\frac{4}{3}} + \frac{11}{1400} \left(\frac{3\pi}{2} x \right)^{\frac{6}{3}} + O\left(x^{\frac{8}{3}} \right) \end{aligned}$$

Additionally we determine the taylor series of the function $\Omega_\star(t^2-2) + t^2 - 2$

$$\Omega_\star(t^2-2) + t^2 - 2 = \frac{4}{3\pi}t^3 + \frac{1}{30\pi}t^5 + \frac{3}{1120\pi}t^7 + O\left(t^9\right)$$

Thereby

$$\begin{aligned}
&v(x)+u(x)\\
&=\Omega_\star\big(u(x)\big)+u(x)\\
&=\Omega_\star\big(F_{sc}^{-1}(x)\big)+F_{sc}^{-1}(x)\\
&=\Omega_\star\left(\left(f^{-1}\left(\sqrt[3]{\frac{3\pi}{2}x}\right)\right)^2-2\right)+\left(f^{-1}\left(\sqrt[3]{\frac{3\pi}{2}x}\right)\right)^2-2\\
&=\frac{4}{3\pi}\left(f^{-1}\left(\sqrt[3]{\frac{3\pi}{2}x}\right)\right)^3+\frac{1}{30\pi}\left(f^{-1}\left(\sqrt[3]{\frac{3\pi}{2}x}\right)\right)^5+\frac{3}{1120\pi}\left(f^{-1}\left(\sqrt[3]{\frac{3\pi}{2}x}\right)\right)^7+O\left(x^{\frac{9}{3}}\right)\\
&=O\left(x^{\frac{9}{3}}\right)+\frac{4}{3\pi}\left(\left(\frac{3\pi}{2}x\right)^{\frac{1}{3}}+\frac{1}{40}\left(\frac{3\pi}{2}x\right)^{\frac{3}{3}}+\frac{81}{22400}\left(\frac{3\pi}{2}x\right)^{\frac{5}{3}}+O\left(x^{\frac{7}{3}}\right)\right)^3+\ldots\\
&=O\left(x^{\frac{9}{3}}\right)+\frac{4}{3\pi}\left(\left(\frac{3\pi}{2}x\right)^{\frac{1}{3}}+\frac{1}{40}\left(\frac{3\pi}{2}x\right)^{\frac{3}{3}}+\frac{81}{22400}\left(\frac{3\pi}{2}x\right)^{\frac{5}{3}}\right)^3+\ldots\\
&=2x+\frac{1}{5}\left(\frac{3\pi}{2}\right)^{\frac{2}{3}}x^{\frac{5}{3}}+\frac{1}{28}\left(\frac{3\pi}{2}\right)^{\frac{4}{3}}x^{\frac{7}{3}}+O\left(x^3\right)
\end{aligned}$$

and

$$\begin{aligned}
&v(x)-u(x)\\
&=v(x)+u(x)-2u(x)\\
&=v(x)+u(x)-2F_{sc}^{-1}(x)\\
&=2x+\frac{1}{5}\left(\frac{3\pi}{2}\right)^{\frac{2}{3}}x^{\frac{5}{3}}+\frac{1}{28}\left(\frac{3\pi}{2}\right)^{\frac{4}{3}}x^{\frac{7}{3}}+4-2\left(\frac{3\pi}{2}x\right)^{\frac{2}{3}}-\frac{1}{10}\left(\frac{3\pi}{2}x\right)^{\frac{4}{3}}-\frac{11}{700}\left(\frac{3\pi}{2}x\right)^{\frac{6}{3}}+O\left(x^{\frac{8}{3}}\right)
\end{aligned}$$

Thus

$$\frac{v(x)+u(x)}{2\sqrt{x}}=\sqrt{x}+\frac{1}{10}\left(\frac{3\pi}{2}\right)^{\frac{2}{3}}x^{\frac{7}{6}}+\frac{1}{56}\left(\frac{3\pi}{2}\right)^{\frac{4}{3}}x^{\frac{11}{6}}+O\left(x^{\frac{15}{6}}\right)$$

and

$$\begin{aligned}
&\frac{v(x)-u(x)}{2\sqrt{x}}\\
&=\frac{2}{\sqrt{x}}-\left(\frac{3\pi}{2}\right)^{\frac{2}{3}}x^{\frac{1}{6}}+x^{\frac{3}{6}}-\left(\frac{3\pi}{2}\right)^{\frac{4}{3}}\frac{x^{\frac{5}{6}}}{20}+\left(\frac{3\pi}{2}\right)^{\frac{2}{3}}\frac{x^{\frac{7}{6}}}{10}-\left(\frac{3\pi}{2}\right)^{\frac{6}{3}}\frac{11x^{\frac{9}{6}}}{1400}+\left(\frac{3\pi}{2}\right)^{\frac{4}{3}}\frac{x^{\frac{11}{6}}}{56}+O\left(x^{\frac{13}{6}}\right)
\end{aligned}$$

References

[BDJ99] Jinho Baik, Percy Deift, and Kurt Johansson, *On the distribution of the length of the longest increasing subsequence of random permutations*, J. Amer. Math. Soc. **12** (1999), no. 4, 1119–1178. MR 1682248 (2000e:05006)

[Dur19] Rick Durrett, *Probability—theory and examples*, Cambridge Series in Statistical and Probabilistic Mathematics, vol. 49, Cambridge University Press, Cambridge, 2019, Fifth edition of [MR1068527]. MR 3930614

[FH91] William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics. MR 1153249

[KV86] Sergei V. Kerov and Anatol M. Vershik, *The characters of the infinite symmetric group and probability properties of the Robinson-Schensted-Knuth algorithm*, SIAM J. Algebraic Discrete Methods **7** (1986), no. 1, 116–124. MR 819713 (87e:22014)

[LS77] B. F. Logan and L. A. Shepp, *A variational problem for random Young tableaux*, Advances in Math. **26** (1977), no. 2, 206–222. MR 1417317 (98e:05108)

[Rom15] Dan Romik, *The surprising mathematics of longest increasing subsequences*, Institute of Mathematical Statistics Textbooks, vol. 4, Cambridge University Press, New York, 2015. MR 3468738

[RŚ15] Dan Romik and Piotr Śniady, *Jeu de taquin dynamics on infinite Young tableaux and second class particles*, Ann. Probab. **43** (2015), no. 2, 682–737. MR 3306003

[RŚ16] ______, *Limit shapes of bumping routes in the Robinson-Schensted correspondence*, Random Structures Algorithms **48** (2016), no. 1, 171–182. MR 3432576