Hydrodynamic limit shape in the Young tableau

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1 Introduction

tu bedzie streszczenie i pewnie odwołania do [RS16][RS15].

2 Hydrodynamic limit shape

Theorem 1. *Let $w \in [0,1]$ be a fixed number. Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of i.i.d. $U(0,1)$. Let $A \in (1,\infty)$, $n \in \mathbb{N}$ and let function $\mathrm{Pos}_n : \{n+1, n+2, \ldots\} \to \mathbb{N}^2$ describe position of box with number z:*

$$Pos_n(m) = \mathrm{box}_w(P(X_1, \ldots, X_n, w, X_{n+1}, \ldots, X_j))$$

for $j \in \{n+1, n+2, \ldots\}$. We consider the natural number $m \in \mathbb{N}$, which is approximately A times greater than n. For example, let m be equal to $m = \lfloor An \rfloor$. Then the $\mathrm{Pos}_n(m)$ function, after scaling by $\sqrt{w}\sqrt{n}$ goes in probability to a certain limit shape. Namely

$$\exists_{G:[1,\infty)\to\mathbb{R}_+^2} \forall_{\epsilon>0} \lim_{n\to\infty} \mathbb{P}(\sup_{1\leq A} ||G(A) - \frac{Pos_n([An])}{\sqrt{w}\sqrt{n}}|| > \epsilon) = 0$$

Proof. The probability that the same number will occur twice in the sequence $(w, X_1, X_2, \ldots)$ is equal to 0. Without losing generality, we assume that all numbers $w, X_1, X_2, \ldots$ are different. Let the sequence $\{X'_j\}_{j=1}^{\infty}$ be a subsequence of the sequence $\{X_j\}_{j=1}^{\infty}$ containing all elements of the sequence $\{X_j\}_{j=1}^{\infty}$, which are less than w. The sequence $\{X'_j\}_{j=1}^{\infty}$ is a sequence of i.i.d $U(0,w)$. Let n' and m' denote the numbers of element of sequences $\{X_j\}_{j=1}^n$, $\{X_j\}_{j=1}^m$ smaller than w. Namely

$$\begin{cases} n' = \#\{X_j | X_j < w, j \leq n\}, \\ m' = \#\{X_j | X_j < w, j \leq m\} = n' + \#\{X_j | X_j < w, n < j \leq m\}, \end{cases}$$

$$\begin{cases} n' = \#\{X_j | X_j < w, j \leq n\} \sim B(n,w), \\ m' - n' = \#\{X_j | X_j < w, n < j \leq m\} \sim B(m-n,w). \end{cases}$$

Moreover, the random variables n' and $m'-n'$ are independent, because the random variables $X_1, X_2, \ldots$ are independent.
From the Khinchin-Kolmogorov-Etemadi Strong Law of Large, we known that numbers $\frac{n'}{n}$ and $\frac{m'-n'}{m-n}$ converges almost surely to w. With probability 1 exists the limits:

$$\begin{cases} \lim_{n\to\infty} \frac{n'}{n} = w, \\ \lim_{n\to\infty} \frac{m'-n'}{m-n} = w. \end{cases}$$

Therefore, with probability 1 also exist the limit:

$$\begin{aligned} \lim_{n \to\infty} \frac{m'}{n'} &= \lim_{n\to\infty} 1 + \frac{m'-n'}{n'} \\ &= 1 + \lim_{n\to\infty} \frac{m'-n'}{m-n} \frac{1}{\frac{n'}{n}} \frac{m-n}{n} \\ &= 1 + \lim_{n\to\infty} \frac{m'-n'}{m-n} \frac{1}{\frac{n'}{n}} \left(\frac{\lfloor An \rfloor}{n} - 1\right) \\ &= 1 + \frac{w}{w}(A-1) \\ &= A. \end{aligned}$$

We have:

$$\begin{aligned} &\mathrm{box}_w(P(X_1, \ldots, X_n, z, X_{n+1}, \ldots, X_m)) \\ &\quad = \mathrm{box}_w(P(X_1, \ldots, X'_{n'}, z, X'_{n'+1}, \ldots, X'_{m'})), \end{aligned}$$

where the sequence $\{X'_j\}_{j=1}^{\infty}$ is a sequence of i.i.d. $U(0,z)$.

We define the function $z : \{1, 2, \ldots m'\} \cup \{m', n' + \frac{1}{2}\} \to [0,1]$, that assigns to the number t, the $t-th$ largest number among $X'_1, X'_2, \ldots, X'_{m'}$.

$$z(j) = \begin{cases} \min(\{X'_1, X'_2, \ldots X'_{m'}\} \setminus \{z(1), z(2), \ldots z(j-1)\}) & j \in \{1, 2, \ldots, m'\} \\ w & j = m' + 1 \\ \frac{z(n') + z(n'+1)}{2} & j = n' + \frac{1}{2} \end{cases}$$

The sequence $\{z(j)\}_{j=1}^{m'}$ contains all elements of the sequence $\{X'_j\}_{j=1}^{m'}$ in the ascending order. The sequence $\{X'_j\}_{j=1}^{m'}$ is a random permutation Π with uniform distribution of the sequence $\{z(j)\}_{j=1}^{m'}$. The sequence $\{z(j)\}_{j=1}^{m'}$ will be called a random increasing sequence with a uniform distribution on the interval $[0, 1]$. Let $\Pi = (\Pi_1, \Pi_2, \dots, \Pi_{m'})$. Then:

$$\{X'_j\}_{j=1}^{m'} = \{z(\Pi_j)\}_{j=1}^{m'} = z \circ \Pi,$$

where Π is the function that acts separately on every element of the sequence $\{z(j)\}_{j=1}^{m'}$.

Similarly the function z act on Young tableau by acting on each box individually. Then

$$\begin{aligned}
&\mathrm{box}_w(P(X'_1, \dots, X'_{n'}, w, X'_{n'+1}, \dots, X'_{m'})) \\
&\quad = \mathrm{box}_w(P(z(\Pi_1), \dots, z(\Pi_{n'}), w, z(\Pi_{n'+1}), \dots, z(\Pi_{m'}))) \\
&\quad = \mathrm{box}_w(z \circ P(\Pi_1, \dots, \Pi_{n'}, m'+1, \Pi_{n'+1}, \dots, \Pi_{m'})) \\
&\quad = \mathrm{box}_{m'+1}(P(\Pi_1, \dots, \Pi_{n'}, m'+1, \Pi_{n'+1}, \dots, \Pi_{m'})).
\end{aligned}$$

We define the permutation $\Pi{\uparrow} = (\Pi_1, \dots, \Pi_{n'}, m'+1, \Pi_{n'+1}, \dots, \Pi_{m'})$ as a natural extension of the permutation Π. In additional we will use the fact that for any permutation $\Pi{\uparrow}$ the insertion tableau of $\Pi{\uparrow}$ is equal to the recording tableau of $\Pi{\uparrow}^{-1}$:

$$P(\Pi{\uparrow}) = Q(\Pi{\uparrow}^{-1}).$$

Therefore:

$$\begin{aligned}
&\mathrm{box}_{m'+1}(P(\Pi_1, \dots, \Pi_{n'}, m'+1, \Pi_{n'+1}, \dots, \Pi_{m'})) \\
&\quad = boxi_{m'+1}(P(\Pi{\uparrow})) \\
&\quad = \mathrm{box}_{m'+1}(Q(\Pi{\uparrow}^{-1})) \\
&\quad = \mathrm{box}_{m'+1}(Q(\Pi_1^{-1}{\uparrow}, \dots, \Pi_{m'}^{-1}{\uparrow}, n'+1)).
\end{aligned}$$

Now, using the function z, we will try to get the sequence of i.i.d. $U(0, 1)$

$$\begin{aligned}
&\mathrm{box}_{m'+1}(Q(\Pi_1^{-1}{\uparrow}, \dots, \Pi_{m'}^{-1}{\uparrow}, n'+1)) \\
&\quad = \mathrm{box}_{m'+1}(Q(\Pi_1^{-1}{\uparrow}, \dots, \Pi_{m'}^{-1}{\uparrow}, n'+\frac{1}{2})) \\
&\quad = \mathrm{box}_{m'+1}(Q(\Pi_1^{-1}, \dots, \Pi_{m'}^{-1}, n'+\frac{1}{2})) \\
&\quad = \mathrm{box}_{m'+1}(z \circ Q(\Pi_1^{-1}, \dots, \Pi_{m'}^{-1}, n'+\frac{1}{2})) \\
&\quad = \mathrm{box}_{m'+1}(Q(z(\Pi_1^{-1}), \dots, z(\Pi_{m'}^{-1}), z(n'+\frac{1}{2}))).
\end{aligned}$$

Π is the random permutation with uniform distribution, so Π^{-1} is also random permutation with uniform distribution, then the sequence $z \circ \Pi^{-1} = (z(\Pi_1^{-1}), \ldots, z(\Pi_{m'}^{-1}))$ is a sequence of i.i.d. $U(0, w)$. We define the random variable S_n and the sequence $\{Y_j\}_{j=1}^{m'}$.

$$\begin{cases} Y_j = \frac{z(\Pi_j^{-1})}{w} & for\ j \in 1, 2, \ldots, m' \\ S_n = \frac{z(n' + \frac{1}{2})}{w} \end{cases}$$

The sequence $\{Y_j\}_{j=1}^{m'}$ is a sequence of i.i.d $U(0, 1)$. In additional random variable S_n converges almost surely to $\frac{1}{Aw}$:

$$\begin{aligned} \lim_{n \to \infty} S_n &= \lim_{n \to \infty} \frac{z(n' + \frac{1}{2})}{w} \\ &= \lim_{n \to \infty} \frac{1}{w} \frac{z(n') + z(n' + 1)}{2} \\ &= \lim_{n \to \infty} \frac{z(n')}{w} + \lim_{n \to \infty} \frac{z(n' + 1) - z(n')}{2w} \\ &= \frac{1}{w} \lim_{n \to \infty} z(n') \\ &= \frac{1}{w} \lim_{n \to \infty} \frac{n'}{m'} \\ &= \frac{1}{Aw}. \end{aligned}$$

Therefore

$$\begin{aligned} &\mathrm{box}_{m'+1}(Q(z(\Pi_1^{-1}), \ldots, z(\Pi_{m'}^{-1}), z(n' + \frac{1}{2}))) \\ &= \mathrm{box}_{m'+1}(Q(\frac{z(\Pi_1^{-1})}{w}, \ldots, \frac{z(\Pi_{m'}^{-1})}{w}, \frac{z(n' + \frac{1}{2})}{w}) \\ &= \mathrm{box}_{m'+1}(Q(Y_1, \ldots, Y_{m'}, S_n)). \end{aligned}$$

□

References

[RS15] Dan Romik and Piotr 'Sniady. Jeu de taquin dynamics on infinite young tableaux and second class particles. 2015.

[RS16] Dan Romik and Piotr 'Sniady. Limit shapes of bumping routes in the robinson-schensted correspondence. *Random Struct. Algorithms*, 48:171–182, 2016.