

Limit shapes of position routes

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1 Introduction

Dan Romik and Piotr Sniady in [1] were considering limit shape of bumping routes obtained from insertion tableau, when applying an RSK insertion step with a fixed input $z \in [0, 1]$ to an existing insertion tableau P_n , where P_n is the result of n previous insertion steps applying to n i.i.d. $U(0, 1)$ random inputs X_1, \dots, X_n .

In [2] they were considering limit shape of jeu de taquin path obtained from insertion tableau made from n i.i.d. $U(0, 1)$ random inputs X_1, \dots, X_n prepending a fixed number $z \in [0, 1]$.

They were looking into in the first case to $P((X_1, \dots, X_n, z))$ and in the second case to $P((z, X_1, \dots, X_n))$. We can merge this cases and consider $P((X_1, \dots, X_n, z, X_{n+1}, \dots, X_m))$, where m is approximately A times greater than n for any $A \in (1, \infty]$.

After applying an RSK insertion steps consecutive for numbers X_1, \dots, X_n, z in insertion tableau will appear box with number z . During applying an RSK insertion steps consecutive for numbers X_{n+1}, \dots, X_m box with number z can be sliding by the bumping roots. We would like to prove, that exist macroscopic limit shape of path describing the position of box with number z .

2 Position path

Theorem 1. *Let's take a fixed number $z \in [0, 1]$. Let $\{X_j\}_{j=1}^\infty$ will be a sequence of i.i.d. $U(0, 1)$. Let $A \in (1, \infty)$, $n \in \mathbb{N}$ and let function $Pos_n : \{n+1, n+2, \dots\} \rightarrow \mathbb{N}^2$ describe position of box with number z :*

$$Pos_n(j) = (a_j^{(n)}, b_j^{(n)}) = box_z(P(X_1, \dots, X_n, z, X_{n+1}, \dots, X_j))$$

for $j \in \{n+1, n+2, \dots\}$. $\exists_{G:[1, \infty) \rightarrow \mathbb{R}_+^2} \forall_{\epsilon > 0}$

$$\lim_{n \rightarrow \infty} \mathbb{P}(\sup_{1 \leq A} \|G(A) - \frac{Pos_n([An])}{\sqrt{A}\sqrt{n}}\| > \epsilon) = 0$$

Proof. The probability that in our sequence there will be a repetition of numbers or numbers z is equal to 0. Without losing generality, we assume that all numbers z, X_1, X_2, \dots are different. Numbers greater than z do not affect to the position of the box with the number z , so we are only considering elements of the sequence $\{X_j\}_{j=1}^\infty$ of less than z . Let the sequence $\{X'_j\}_{j=1}^\infty$ be a subsequence of the sequence $\{X_j\}_{j=1}^\infty$ containing all elements of the sequence $\{X_j\}_{j=1}^\infty$, which are less than z . Let $m = [An]$. We have:

$$box_z(P(X_1, \dots, X_n, z, X_{n+1}, \dots, X_m)) = box_z(P(X_1, \dots, X'_{n'}, z, X'_{n'+1}, \dots, X'_{m'}))$$

where the sequence $\{X'_j\}_{j=1}^\infty$ is a sequence of i.i.d. $U(0, z)$ and

$$\begin{cases} \lim_{n \rightarrow \infty} \frac{m}{n} = \lim_{n \rightarrow \infty} \frac{[An]}{n} = A \\ n' = \#\{X_j | X_j < z, j \leq n\} = Binom(n, z) \\ m' = \#\{X_j | X_j < z, j \leq m\} = Binom(m, z) \end{cases}$$

Moreover, similarly as in [] we consider the sequence $\{z(j)\}_{j=1}^{m'}$, which is a sequence containing all elements of the sequence $\{X'_j\}_{j=1}^{m'}$ in ascending order. The sequence $\{z(j)\}_{j=1}^{m'}$ is random increasing sequence with a uniform distribution $U(0, z)$ and random variables $X'_1, \dots, x'_{m'}$ have a uniform distribution, therefore the sequence $\{z(j)\}_{j=1}^{m'}$ is a random permutation Π of sequence $\{X'_j\}_{j=1}^{m'}$, where random variables Π is permutation of numbers $\{1, 2, \dots, m'\}$ and also have a uniform distribution. For each $j \leq m'$, we have $X'_j = z(\Pi(j))$. For write legibility we put

$$\begin{cases} z(m' + 1) = z \\ z(m' + \frac{1}{2}) = \frac{z(n') + z(n'+1)}{2} \\ zS_n = z(n' + \frac{1}{2}) = \frac{z(n') + z(n'+1)}{2} \end{cases}$$

Let $\Pi_0 = (\Pi(1), \dots, \Pi(n'), m' + 1)$ and

$$\Pi^{-1}(j) \uparrow = \begin{cases} \Pi^{-1}(j) & j \leq n' \\ \Pi^{-1}(j) + 1 & n' < j \leq m' \\ n' + 1 & j = m' + 1 \end{cases}$$

We will use the fact stated in \square that for any permutation Π_0 the insertion tableau of Π_0 is equal to the recording tableau of Π_0^{-1} :

$$P(\Pi_0) = Q(\Pi_0^{-1})$$

Therefore:

$$\begin{aligned} Pos_n(m) &= box_z(P(X_1, \dots, X_n, z, X_{n+1}, \dots, X_m)) \\ &= box_z(P(X'_1, \dots, X'_{n'}, z, X'_{n'+1}, \dots, X'_{m'})) \\ &= box_z(P(z(\Pi(1)), \dots, z(\Pi(n')), z, z(\Pi(n' + 1)), \dots, z(\Pi(m')))) \\ &= box_z(z \circ P(\Pi(1), \dots, \Pi(n'), m' + 1, \Pi(n' + 1), \dots, \Pi(m'))) \\ &= box_{m'+1}(P(\Pi(1), \dots, \Pi(n'), m' + 1, \Pi(n' + 1), \dots, \Pi(m'))) \\ &= box_{m'+1}(P(\Pi_0)) \\ &= box_{m'+1}(Q(\Pi_0^{-1})) \\ &= box_{m'+1}(Q(\Pi^{-1}(1) \uparrow, \dots, \Pi^{-1}(m') \uparrow, n' + 1)) \\ &= box_{m'+1}(Q(\Pi^{-1}(1) \uparrow, \dots, \Pi^{-1}(m') \uparrow, n' + \frac{1}{2})) \\ &= box(m' + 1)(z \circ Q(\Pi^{-1}(1) \uparrow, \dots, \Pi^{-1}(m') \uparrow, n' + \frac{1}{2})) \\ &= box_{m'+1}(Q(zY_1, \dots, zY_{m'}, zS_n)) \\ &= box_{m'+1}(Q(Y_1, \dots, Y_{m'}, S_n)) \end{aligned}$$

since the sequence $\{zY_j\}_{j=1}^{m'}$ is the random increasing sequence with a uniform distribution permuted by the random permutation with a uniform distribution. We receive that the sequence $\{Y_j\}_{j=1}^{m'}$ is a sequence of i.i.d. $U(0, 1)$, $m' = Binom([An], z)$ and with probability 1 we have:

$$\begin{aligned} \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \frac{n'}{m'} \\ &= \lim_{n \rightarrow \infty} \frac{\frac{Binom(n, z)}{n}}{\frac{Binom(m, z)}{m}} \frac{n}{m} \\ &= \frac{1}{A} \end{aligned}$$

\square

References

- [1] Dan Romik, Piotr Śniady *Limit shapes of bumping routes in the Robinson-Schensted correspondence.*
- [2] Dan Romik, Piotr Śniady *Jeu de taquin dynamics on infinite Young tableaux and second class particles.* The Annals of Probability 2015, Vol. 43, No. 2: 719-724.