Limit shapes of position routes

Mikolaj Marciniak and Piotr Sniady

1 Introduction

Dan Romik and Piotr Sniady in [1] were considering limit shape of bumping routes obtained from insertion tableau, when applying an RSK insertion step with a fixed input $z \in [0,1]$ to an existing insertion tableau P_n, where P_n is the result of n previous insertion steps applying to n i.i.d. $U(0,1)$ random inputs $X_1, \dots, X_n$.

In [2] they were considering limit shape of jeu de taquin path obtained from insertion tableau made from n i.i.d. $U(0,1)$ random inputs $X_1, \dots, X_n$ prepending a fixed number $z \in [0,1]$.

They were looking into in the first case to $P((X_1, \dots, X_n, z))$ and in the second case to $P((z, X_1, \dots, X_n))$. We can merge this cases and consider $P((X_1, \dots, X_n, z, X_{n+1}, \dots, X_m))$, where m is approximately A times greater than n for any $A \in (1, \infty]$.

After applying an RSK insertion steps consecutive for numbers $X_1, \dots, X_n, z$ in insertion tableau will appear box with number z. During applying an RSK insertion steps consecutive for numbers $X_{n+1}, \dots, X_m$ box with number z can be sliding by the bumping roots. We would like to prove, that exist macroscopic limit shape of path describing the position of box with number z.

2 Position path

Theorem 1. *Let's take a fixed number* $z \in [0,1]$. *Let* $\{X_j\}_{j=1}^{\infty}$ *will be a sequence of i.i.d.* $U(0,1)$. *Let* $A \in (1,\infty)$, $n \in \mathbb{N}$ *and let function* $Pos_n : \{n+1, n+2, \ldots\} \to \mathbb{N}^2$ *describe position of box with number z:*

$$Pos(n) = (a_n, b_n) = box_z(P(X_1, \ldots, X_n, z, X_{n+1}, \ldots, X_m))$$

for $m = [An]$. $\exists_{G:[1,\infty)\to\mathbb{R}_+^2} \forall_{\epsilon>0}$

$$lim_{n\to\infty}\mathbb{P}(sup_{1\leq A}||G(A) - \frac{Pos(n)}{\sqrt{A}\sqrt{n}}||> \epsilon) = 0$$

Proof. The probability that in our sequence there will be a repetition of numbers or numbers z is equal to 0. Without losing generality, we assume that all numbers $z, X_1, X_2, \ldots$ are different. Numbers greater than z do not affect to the position of the box with the number z, so we are only considering elements of the sequence $\{X_j\}_{j=1}^{\infty}$ of less than z. Let the sequence $\{X_j'\}_{j=1}^{\infty}$ be a subsequence of the sequence $\{X_j\}_{j=1}^{\infty}$ containing all elements of the sequence $\{X_j\}_{j=1}^{\infty}$, which are less than z. We have:

$$box_z(P(X_1, \ldots, X_n, z, X_{n+1}, \ldots, X_m)) = box_z(P(X_1, \ldots, X_{n'}', z, X_{n'+1}', \ldots, X_{m'}'))$$

where the sequence $\{X_j'\}_{j=1}^{\infty}$ is a sequence of i.i.d. $U(0,z)$ and

$$\begin{cases} \lim_{n\to\infty} \frac{m}{n} = \lim_{n\to\infty} \frac{[An]}{n} = A \\ n' = \#\{X_j | X_j < z, j \leq n\} = Binom(n,z) \\ m' = \#\{X_j | X_j < z, j \leq m\} = Binom(m,z) \end{cases}$$

Moreover, similarly as in [] we consider the sequence $\{z(j)\}_{j=1}^{m'}$, which is a sequence containing all elements of the sequence $\{X_j'\}_{j=1}^{m'}$ in ascending order. The sequence $\{z(j)\}_{j=1}^{m'}$ is random increasing sequence with a uniform distribution $U(0,z)$ and random variables $X_1', \ldots, x_{m'}'$ have a uniform distribution, therefore the sequence $\{z(j)\}_{j=1}^{m'}$ is a random permutation Π of sequence $\{X_j'\}_{j=1}^{m'}$, where random variables Π is permutation of numbers $\{1, 2, \ldots, m'\}$ and also have a uniform distribution. For each $j \leq m'$, we have $X_j' = z(\Pi(j))$. For write legibility we put

$$\begin{cases} z(m'+1) = z \\ S_n' = z(n' + \frac{1}{2}) = \frac{z(n') + z(n'+1)}{2} \end{cases}$$

Let $\Pi_0 = (\Pi(1), \ldots, \Pi(n'), m'+1)$ and

$$\Pi^{-1}(j)\uparrow = \begin{cases} \Pi^{-1}(j) & j \le n' \\ \Pi^{-1}(j)+1 & n' < j \le m' \\ n'+1 & j = m'+1 \end{cases}$$

We will use the fact stated in [] that for any permutation Π_0 the insertion tableau of Π_0 is equal to the recording tableau of Π_0^{-1}:

$$P(\Pi_0) = Q(\Pi_0^{-1})$$

Therefore:

$$\begin{aligned}
Pos(n) &= box_z(P(X_1, \ldots, X_n, z, X_{n+1}, \ldots, X_m)) \\
&= box_z(P(X'_1, \ldots, X'_{n'}, z, X'_{n'+1}, \ldots, X'_{m'})) \\
&= box_z(P(z(\Pi(1)), \ldots, z(\Pi(n')), z, z(\Pi(n'+1)), \ldots, z(\Pi(m')))) \\
&= box_z(z \circ P(\Pi(1), \ldots, \Pi(n'), m'+1, \Pi(n'+1), \ldots, \Pi(m'))) \\
&= box_{m'+1}(P(\Pi(1), \ldots, \Pi(n'), m'+1, \Pi(n'+1), \ldots, \Pi(m'))) \\
&= box_{m'+1}(P(\Pi_0)) \\
&= box_{m'+1}(Q(\Pi_0^{-1})) \\
&= box_{m'+1}(Q(\Pi^{-1}(1)\uparrow, \ldots, \Pi^{-1}(m')\uparrow, n'+1)) \\
&= box_{m'+1}(Q(\Pi^{-1}(1)\uparrow, \ldots, \Pi^{-1}(m')\uparrow, n'+\frac{1}{2})) \\
&= box_{m'+1}(z \circ Q(\Pi^{-1}(1), \ldots, \Pi^{-1}(m'), n'+\frac{1}{2})) \\
&= box_{m'+1}(Q(Y'_1, \ldots, Y'_{m'}, S'_n)) \\
&= box_{m'+1}(Q(zY_1, \ldots, zY_{m'}, zS_n)) \\
&= box_{m'+1}(Q(Y_1, \ldots, Y_{m'}, S_n))
\end{aligned}$$

since the sequence $\{Y'_j\}_{j=1}^{m'}$ is the random increasing sequence with a uniform distribution $U(0,z)$ permutated by the random permutation with a uniform distribution. Thus $\{Y'_j\}_{j=1}^{m'}$ is a sequence of i.i.d $U(0,z)$. In addition, we define $Y_j = \frac{Y'_j}{z}$ for each j and $S_n = \frac{S'_n}{z}$. We receive that the sequence $\{Y_j\}_{j=1}^{m'}$ is a sequence of i.i.d. $U(0,1)$, $m' = Binom(m,z)$ and with probability 1 there exist the limit:

$$\begin{aligned}\lim_{n\to\infty} S_n &= \lim_{n\to\infty} \frac{S_n'}{z} \\ &= \lim_{n\to\infty} \frac{z}{z}\frac{n'}{m'} \\ &= \lim_{n\to\infty} \frac{\frac{Binom(n,z)}{n}}{\frac{Binom(m,z)}{m}}\frac{n}{m} \\ &= \frac{1}{A}\end{aligned}$$

$S_n \xrightarrow{p} S = \frac{1}{A}$. In accordance with the markings from [1] we have:

$$Pos_(n) = box_{m'+1}(Q(Y_1, \ldots, Y_{m'}, S_n)) = \quad \square_{m'}(S_n)\,,$$

where for each x: $\frac{\square_{m'}(x)}{\sqrt{m'}} \xrightarrow{p} W(x) = (\frac{v(x)+u(x)}{2}, \frac{v(x)-u(x)}{2})$ as $m' \to \infty$, since $W(x)$ is the Lipschitz function. Let $W_{m'}(x) = \frac{Pos(n)}{\sqrt{m'}}$. We have:

$$\begin{cases} S_n \xrightarrow{p} S \\ W_{m'}(x) \xrightarrow{p} W(x)\end{cases}$$

Therefore $\forall_{\epsilon>0}\exists_{n_0}\forall_{n>n_0}\forall_{S\in[0,1]}$

$$\begin{cases} P(S_n > S-\epsilon) = 1 \\ P(S_n < S+\epsilon) = 1 \\ P(W_{m'}(s-\epsilon) > W(S-\epsilon) - \epsilon(1,-1)) = 1 \\ P(W_{m'}(s+\epsilon) < W(S+\epsilon) + \epsilon(1,-1)) = 1\end{cases}$$

$P(W_{m'}(S_n) > W_{m'}(S-\epsilon)) > W(S-\epsilon) - \epsilon(1,-1) > W(S) - 2\epsilon(1,-1)) = 1$

$P(W_{m'}(S_n) < W_{m'}(S+\epsilon)) < W(S+\epsilon) + \epsilon(1,-1) < W(S) + 2\epsilon(1,-1)) = 1$

Then $P(W(S) - 2\epsilon(1,-1) < W_{m'}(S_n) < W(S) + 2\epsilon(1,-1)) = 1$

Thereby $W_{m'}(S_n) \xrightarrow{p} W(S)$ In addition:

$$\frac{Pos(n)}{\sqrt{A}\sqrt{n}} = \frac{\sqrt{m'}W_{m'}(S_n)}{\sqrt{A}\sqrt{n}} = \sqrt{\frac{[An]}{An}\frac{Binom([An],z)}{[An]}}W_{m'}(S_n) \xrightarrow{p} \sqrt{z}W(S)$$

Therefore there exist the fucntion $G(A) = \sqrt{z}W(\frac{1}{A})$ that

$lim_{n\to\infty}\mathbb{P}(sup_{1\leq A}||G(A) - \frac{Pos(n)}{\sqrt{A}\sqrt{n}}|| > \epsilon|(1,-1)| > \epsilon) = 0$ $\square$

References

[1] Dan Romik, Piotr Śniady *Limit shapes of bumping routes in the Robinson-Schensted correspondence.*

[2] Dan Romik, Piotr Śniady *Jeu de taquin dynamics on infinite Young tableaux and second class particles.* The Annals of Probability 2015, Vol. 43, No. 2: 719-724.