Hydrodynamic limit in the insertion tableau in the Robinson-Schensted-Knuth correspondence

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1 Introduction

tu bedzie wstep i pewnie odwołania do [RS16][RS15].

2 The main result

The main result of the article is the proof that the random variable describing the coordinates of the box with the fixed number w converges in probability to the function G(A)=[tu bedzie definicja].

Let $w \in [0,1]$ be a fixed number. Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of i.i.d. $U(0,1)$ random variable. For every $n \in \mathbb{N}$ let function $\mathrm{Pos}_n : \{n+1, n+2, \ldots\} \to \mathbb{N}^2$ describe coordinates of box with number w in the insertion tableau:

$$Pos_n(j) = \mathrm{box}_w(P(X_1, \ldots, X_n, w, X_{n+1}, \ldots, X_j))$$

for $j \in \{n+1, n+2, \ldots\}$. We consider the natural number $m \in \mathbb{N}$, which is approximately A times greater than n, which means $\lim_{n\to\infty} \frac{m}{n} = A \in (1, \infty)$. For example, let m be equal to $m = \lfloor An \rfloor$.

Theorem 1. *The random variable* $\mathrm{Pos}_n(m)$*, after scaling by* $\sqrt{w}\sqrt{n}$ *converges in probability to the limit* $G(A)$*. Namely*

$$\forall_{\epsilon>0} \lim_{n\to\infty} \mathbb{P}(\sup_{A\geq 1} ||G(A) - \frac{Pos_n([An])}{\sqrt{w}\sqrt{n}}|| > \epsilon) = 0$$

Proof. We apply the RSK algorithm to a random sequence of real numbers containing the number w and think about the coordinates of the box with the number w in the insertion tableau. Numbers greater than w do not affect the position of the box with the number w. So it is enough to consider only the subsequence containing numbers no greater than w.

The probability that the same number will occur twice in the sequence $(w, X_1, X_2, \ldots)$ is equal to 0. Without losing generality, we assume that all numbers $w, X_1, X_2, \ldots$ are different. Let the sequence $\{X'_j\}_{j=1}^{\infty}$ be the subsequence of the sequence $\{X_j\}_{j=1}^{\infty}$ containing all elements of the sequence $\{X_j\}_{j=1}^{\infty}$, which are less than w. The sequence $\{X'_j\}_{j=1}^{\infty}$ is a sequence of i.i.d. $U(0,w)$ random variables. Let n' and m' denote the numbers of element of sequences $\{X_j\}_{j=1}^{n}$, $\{X_j\}_{j=1}^{m}$ smaller than w. Then there is an equlity:

$$\begin{aligned}&\mathrm{box}_w(P(X_1, \ldots, X_n, w, X_{n+1}, \ldots, X_m)) \\ &\quad = \mathrm{box}_w(P(X_1, \ldots, X'_{n'}, w, X'_{n'+1}, \ldots, X'_{m'})).\end{aligned}$$

Namely

$$\begin{cases} n' = \#\{X_j | X_j < w, j \leq n\}, \\ m' = \#\{X_j | X_j < w, j \leq m\} = n' + \#\{X_j | X_j < w, n < j \leq m\}, \end{cases}$$

$$\begin{cases} n' = \#\{X_j | X_j < w, j \leq n\} \sim B(n, w), \\ m' - n' = \#\{X_j | X_j < w, n < j \leq m\} \sim B(m-n, w). \end{cases}$$

Moreover, the random variables n' and $m' - n'$ are independent, because the random variables $X_1, X_2, \ldots$ are independent.
From the Khinchin-Kolmogorov-Etemadi Strong Law of Large, we known that numbers $\frac{n'}{n}$ and $\frac{m'-n'}{m-n}$ converges almost surely to w. With probability 1 exists the limits:

$$\begin{cases} \lim_{n\to\infty} \frac{n'}{n} = w, \\ \lim_{n\to\infty} \frac{m'-n'}{m-n} = w. \end{cases}$$

Therefore, with probability 1 also exist the limit:

$$\begin{aligned} \lim_{n \to \infty} \frac{m'}{n'} &= \lim_{n\to\infty} 1 + \frac{m'-n'}{n'} \\ &= 1 + \lim_{n\to\infty} \frac{m'-n'}{m-n} \frac{1}{\frac{n'}{n}} \frac{m-n}{n} \\ &= 1 + \lim_{n\to\infty} \frac{m'-n'}{m-n} \frac{1}{\frac{n'}{n}} (\frac{\lfloor An \rfloor}{n} - 1) \\ &= 1 + \frac{w}{w}(A-1) \\ &= A. \end{aligned}$$

where the sequence $\{X'_j\}_{j=1}^{\infty}$ is a sequence of i.i.d. $U(0, z)$.

We define the function $z : \{1, 2, \ldots m'\} \cup \{m', n' + \frac{1}{2}\} \to [0, 1]$, that assigns to the number t, the $t - th$ largest number among $X'_1, X'_2, \ldots, X'_{m'}$.

$$z(j) = \begin{cases} \min(\{X'_1, X'_2, \ldots X'_{m'}\} \setminus \{z(1), z(2), \ldots z(j-1)\}) & j \in \{1, 2, \ldots, m'\} \\ w & j = m' + 1 \\ \frac{z(n') + z(n'+1)}{2} & j = n' + \frac{1}{2} \end{cases}$$

The sequence $\{z(j)\}_{j=1}^{m'}$ contains all elements of the sequence $\{X'_j\}_{j=1}^{m'}$ in the ascending order. The sequence $\{X'_j\}_{j=1}^{m'}$ is a random permutation Π with uniform distribution of the sequence $\{z(j)\}_{j=1}^{m'}$. The sequence $\{z(j)\}_{j=1}^{m'}$ will be called a random increasing sequence with a uniform distribution on the interval $[0, 1]$. Let $\Pi = (\Pi_1, \Pi_2, \ldots, \Pi_{m'})$. Then:

$$\{X'_j\}_{j=1}^{m'} = \{z(\Pi_j)\}_{j=1}^{m'} = z \circ \Pi,$$

where Π is the function that acts separately on every element of the sequence $\{z(j)\}_{j=1}^{m'}$.

Similarly the function z act on Young tableau by acting on each box individually. Then

$$\begin{aligned}
&\mathrm{box}_w(P(X'_1, \ldots, X'_{n'}, w, X'_{n'+1}, \ldots, X'_{m'})) \\
&\;= \mathrm{box}_w(P(z(\Pi_1), \ldots, z(\Pi_{n'}), w, z(\Pi_{n'+1}), \ldots, z(\Pi_{m'}))) \\
&\;= \mathrm{box}_w(z \circ P(\Pi_1, \ldots, \Pi_{n'}, m' + 1, \Pi_{n'+1}, \ldots, \Pi_{m'})) \\
&\;= \mathrm{box}_{m'+1}(P(\Pi_1, \ldots, \Pi_{n'}, m' + 1, \Pi_{n'+1}, \ldots, \Pi_{m'})).
\end{aligned}$$

We define the permutation $\Pi{\uparrow} = (\Pi_1, \ldots, \Pi_{n'}, m' + 1, \Pi_{n'+1}, \ldots, \Pi_{m'})$ as a natural extension of the permutation Π. In additional we will use the fact that for any permutation $\Pi{\uparrow}$ the insertion tableau of $\Pi{\uparrow}$ is equal to the recording tableau of $\Pi{\uparrow}^{-1}$:

$$P(\Pi{\uparrow}) = Q(\Pi{\uparrow}^{-1}).$$

Therefore:

$$\begin{aligned}
&\mathrm{box}_{m'+1}(P(\Pi_1, \ldots, \Pi_{n'}, m' + 1, \Pi_{n'+1}, \ldots, \Pi_{m'})) \\
&\;= boxi_{m'+1}(P(\Pi{\uparrow})) \\
&\;= \mathrm{box}_{m'+1}(Q(\Pi{\uparrow}^{-1})) \\
&\;= \mathrm{box}_{m'+1}(Q(\Pi_1^{-1}{\uparrow}, \ldots, \Pi_{m'}^{-1}{\uparrow}, n' + 1)).
\end{aligned}$$

Now, using the function z, we will try to get the sequence of i.i.d. $U(0, 1)$

$$\begin{aligned}
&\mathrm{box}_{m'+1}(Q(\Pi_1^{-1}{\uparrow}, \ldots, \Pi_{m'}^{-1}{\uparrow}, n' + 1)) \\
&\;= \mathrm{box}_{m'+1}(Q(\Pi_1^{-1}{\uparrow}, \ldots, \Pi_{m'}^{-1}{\uparrow}, n' + \frac{1}{2})) \\
&\;= \mathrm{box}_{m'+1}(Q(\Pi_1^{-1}, \ldots, \Pi_{m'}^{-1}, n' + \frac{1}{2})) \\
&\;= \mathrm{box}_{m'+1}(z \circ Q(\Pi_1^{-1}, \ldots, \Pi_{m'}^{-1}, n' + \frac{1}{2})) \\
&\;= \mathrm{box}_{m'+1}(Q(z(\Pi_1^{-1}), \ldots, z(\Pi_{m'}^{-1}), z(n' + \frac{1}{2}))).
\end{aligned}$$

Π is the random permutation with uniform distribution, so Π^{-1} is also random permutation with uniform distribution, then the sequence $z \circ \Pi^{-1} = (z(\Pi_1^{-1}), \ldots, z(\Pi_{m'}^{-1}))$ is a sequence of i.i.d. $U(0, w)$. We define the random variable S_n and the sequence $\{Y_j\}_{j=1}^{m'}$.

$$\begin{cases} Y_j = \frac{z(\Pi_j^{-1})}{w} \quad for \ j \in 1, 2, \ldots, m' \\ S_n = \frac{z(n' + \frac{1}{2})}{w} \end{cases}$$

The sequence $\{Y_j\}_{j=1}^{m'}$ is a sequence of i.i.d $U(0, 1)$. In additional random variable S_n converges almost surely to $\frac{1}{Aw}$:

$$\begin{aligned}
\lim_{n \to \infty} S_n &= \lim_{n\to\infty} \frac{z(n' + \frac{1}{2})}{w} \\
&= \lim_{n\to\infty} \frac{1}{w} \frac{z(n') + z(n' + 1)}{2} \\
&= \lim_{n\to\infty} \frac{z(n')}{w} + \lim_{n\to\infty} \frac{z(n' + 1) - z(n')}{2w} \\
&= \frac{1}{w} \lim_{n\to\infty} z(n') \\
&= \frac{1}{w} \lim_{n\to\infty} \frac{n'}{m'} \\
&= \frac{1}{Aw}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathrm{box}_{m'+1}(Q(z(\Pi_1^{-1}), \ldots, z(\Pi_{m'}^{-1}), z(n' + \frac{1}{2}))) \\
&= \mathrm{box}_{m'+1}(Q(\frac{z(\Pi_1^{-1})}{w}, \ldots, \frac{z(\Pi_{m'}^{-1})}{w}, \frac{z(n' + \frac{1}{2})}{w}) \\
&= \mathrm{box}_{m'+1}(Q(Y_1, \ldots, Y_{m'}, S_n)).
\end{aligned}$$

□

References

[RS15] Dan Romik and Piotr 'Sniady. Jeu de taquin dynamics on infinite young tableaux and second class particles. 2015.

[RS16] Dan Romik and Piotr 'Sniady. Limit shapes of bumping routes in the robinson-schensted correspondence. *Random Struct. Algorithms*, 48:171–182, 2016.