Quadratic coefficients of Goulden–Rattan character polynomials

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Abstract. Goulden–Rattan polynomials give the exact value of the subdominant part of the normalized characters of the symmetric groups in terms of certain quantities (C_i). The Goulden–Rattan positivity conjecture states that the coefficients of these polynomials are positive rational numbers with small denominators. We prove a special case of this conjecture for the coefficient of the quadratic term C_2^2 by applying certain bijections involving maps (i.e. graphs drawn on surfaces).

Keywords: characters of the symmetric groups, free cumulants, Kerov polynomials, Goulden–Rattan polynomials, maps

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1 Introduction

1.1 Normalized characters

Characters are a basic tool of representation theory. After normalization, they are also useful in asymptotic problems.

If $k \leq n$ are natural numbers, then any permutation $\pi \in S_k$ can also be treated as an element of the larger symmetric group S_n by adding $n-k$ additional fixpoints. For any permutation $\pi \in S_k$ and any irreducible representation ρ^λ of the symmetric group S_n which corresponds to the Young diagram λ, we define *the normalized character*

$$\Sigma_\pi(\lambda) = \begin{cases} n(n-1)\cdots(n-k+1)\frac{\operatorname{Tr}\rho^\lambda(\pi)}{\text{dimension of } \rho^\lambda} & \text{for } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Of particular interest are the character values on the cycles, therefore we will use the shorthand notation

$$\Sigma_k(\lambda) = \Sigma_{(1,2,\dots,k)}(\lambda).$$

1.2 Free cumulants

Free cumulants are an important tool of free probability theory [VDN92] and random matrix theory [Voi91]. In the context of the representation theory of the symmetric groups they can be defined as follows, see [Bia03]. For a Young diagram λ we define its free cumulants $R_2(\lambda), R_3(\lambda), \dots$ as

$$R_k(\lambda) = \lim_{s\to\infty} \frac{1}{s^k}\Sigma_{k-1}(s\lambda),$$

where the diagram $s\lambda$ is created from the diagram λ by dividing each box of λ into an $s \times s$ square.

The free cumulants are very helpful for studying asymptotic behaviour of the characters on a cycle of length k when the size of the Young diagram tends to infinity [Bia98].

1.3 Kerov character polynomials

Kerov formulated the following result: for each permutation π and any Young diagram λ, the normalized character $\Sigma_\pi(\lambda)$ is equal to the value of some polynomial

$K_\pi(R_2(\lambda), R_3(\lambda), \dots)$ (now called *the Kerov character polynomial*) with integer coefficients. The first published proof of this fact was provided by Biane [Bia03]. The Kerov character polynomial is *universal* because it does not depend on the choice of λ. We are interested in the values of the characters on cycles, therefore for $\pi = (1, 2, \dots, k)$ we use the simplified notation

$$\Sigma_k = K_k(R_2, R_3, \dots) \tag{1.1}$$

for such Kerov polynomials. The first few Kerov polynomials K_k are as follows:

$$\begin{aligned}
K_1 &= R_2, \\
K_2 &= R_3, \\
K_3 &= R_4 + R_2, \\
K_4 &= R_5 + 5R_3, \\
K_5 &= R_6 + 15R_4 + 5R_2^2 + 8R_2, \\
K_6 &= R_7 + 35R_5 + 35R_3R_2 + 84R_3, \\
K_7 &= R_8 + 180R_2 + 224R_2^2 + 14R_2^3 + 56R_3^2 + 469R_4 + 84R_2R_4 + 70R_6.
\end{aligned}$$

Kerov conjectured that the coefficients of the polynomial K_k are non-negative integers. Goulden and Rattan [GR07] found an explicit formula for the coefficients of the Kerov polynomial K_k; unfortunately, their formula was complicated and did not give any combinatorial interpretation to the coefficients. Later, Féray proved positivity [F0́9] and together with Dołęga and Śniady found a combinatorial interpretation of the coefficients [DFS10]. In this paper, we will use the combinatorial interpretation given by them in the special case for linear and square coefficients.

1.4 Goulden–Rattan conjecture

Goulden and Rattan [GR07] introduced a family of functions $C_2, C_3, \dots$ on the set of Young diagrams given by $C_0 = 1$, $C_1 = 0$ and

$$C_k^\lambda = \frac{24}{k(k+1)(k+2)} \lim_{s \to \infty} \frac{1}{s^k} \left(\Sigma_{k+1}^{s\lambda} - \Sigma_{k+2}^{s\lambda} \right)$$

for $k \geq 2$.

Śniady [Ś06] proved the explicit form of C_k (conjectured by Biane [Bia03]) as a polynomial in the free cumulants $R_2, R_3, \dots$ given by

$$C_k = \sum_{\substack{j_2, j_3, \dots \geq 0 \\ 2j_2 + 3j_3 + \cdots = k}} (j_2 + j_3 + \cdots)! \prod_{i \geq 2} \frac{\big((i-1)R_i\big)^{j_i}}{j_i!} \tag{1.2}$$

for $k \geq 2$. The aforementioned formula of Goulden and Rattan for the Kerov polynomials was naturally expressed in terms of these quantities $C_2, C_3, \dots$ [GR07]. More specifically, they constructed an explicit polynomial L_k with rational coefficients such that

$$K_k - R_{k+1} = L_k(C_2, C_3, \dots). \tag{1.3}$$

These polynomials are called *the Goulden–Rattan polynomials*. They formulated the following conjecture:

Goulden–Rattan conjecture. *The coefficients of the Goulden–Rattan polynomials are non-negative numbers with small denominators.*

The first few Goulden–Rattan polynomials are as follows [GR07]:

$$\begin{aligned}
K_1 - R_2 &= 0 \\
K_2 - R_3 &= 0 \\
K_3 - R_4 &= C_2, \\
K_4 - R_5 &= \frac{5}{2} C_3, \\
K_5 - R_6 &= 5C_4 + 8C_2, \\
K_6 - R_7 &= \frac{35}{4} C_5 + 42 C_3, \\
K_7 - R_8 &= 14 C_6 + \frac{469}{3} C_4 + \frac{203}{3} C_2^2 + 180 C_2.
\end{aligned}$$

Linear coefficients of the Goulden–Rattan polynomials are non-negative, because they are equal to certain scaled coefficients of the Kerov polynomial:

$$[C_j] L_k = \frac{1}{j-1} [R_j] K_k.$$

In this paper we will prove that the coefficient of C_2^2 is non-negative. Using the same methods we plan to prove in the future that any square coefficients $[C_i C_j] L_k$ is

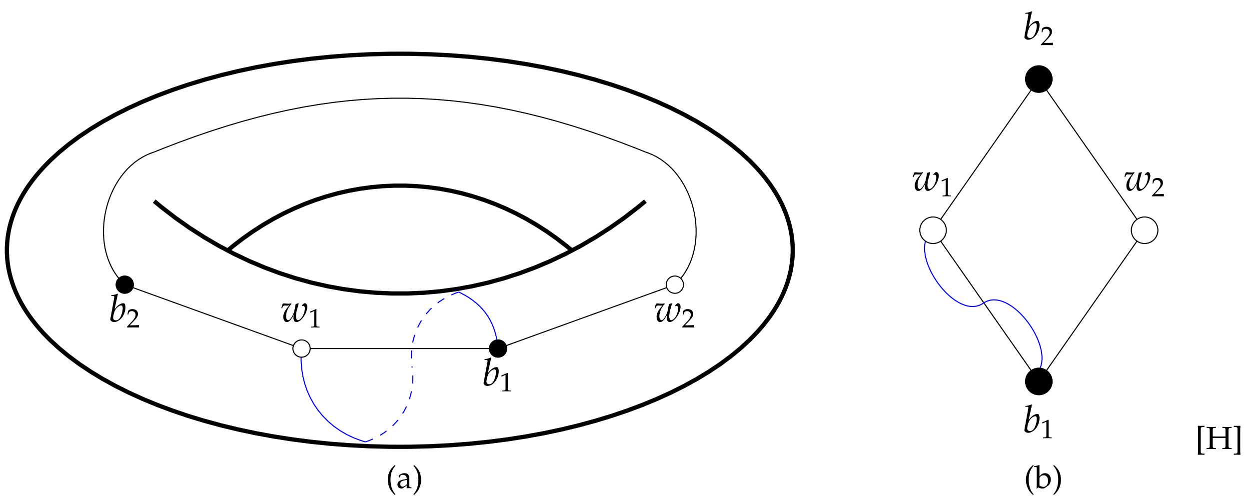


Figure 1: (a) An example of a map with 4 vertices and 5 edges drawn on a torus. (b) The same map drawn for simplicity on the plane.

non-negative. The next step towards the proof of Goulden–Rattan conjecture would be to understand the cubic coefficients $[C_iC_jC_u]L_k$; we hope that our methods will still be applicable there, nevertheless there seem to be some difficulties related to the inclusion-exclusion principle.

1.5 Graphs on surfaces, maps and expanders

We will consider graphs drawn on an oriented surface. Each face of such a graph has some number of edges ordered cyclically by going along the boundary of the face and touching it with the right hand. We will call it *clockwise direction*. If we use the left hand and visit the edges in the opposite order, we will call it counterclockwise direction.

By *a map* we mean a bipartite graph drawn without intersections on an oriented and connected surface with minimal genus. The maps which we consider have fixed a choice of colouring the vertices. , i.e., each vertex is coloured black or white, with the edges connecting the vertices of the opposite colours. An example of a map is shown in Figure 1.

An expander [Ś19, Appendix A.1] is a map with the following properties.

- It has a distinguished edge (known as the root) and one face.

- Each black vertex is assigned a natural number, known as a weight, such that each non-empty proper subset of the set of black vertices has more white neighbours than the sum of its weights.
- The sum of all weights is equal to the number of white vertices.

The map from Figure 1 is an expander if each black vertex has weight 1 (any choice of the root is valid).

Using the Euler characteristic we get

$$2 - 2g = \chi = V - k + 1 \tag{1.4}$$

where g denotes the genus of the surface, V denotes the number of vertices and k denotes the number of edges.

1.6 Combinatorial interpretation of the Kerov polynomial coefficients

The following two theorems Theorem 1 and Theorem 3 give a combinatorial interpretation to the linear and square coefficients of the Kerov character polynomials [DFS10, Theorem 1.2, Theorem 1.3]. The first is as follows.

Theorem 1. *For all integers $l \geq 2$ and $k \geq 1$ the coefficient $[R_l]K_k$ is equal to the number of pairs (σ_1, σ_2) of permutations $\sigma_1, \sigma_2 \in S(k)$ such that $\sigma_1 \circ \sigma_2 = (1, 2, \ldots, k)$ and such that σ_2 consists of one cycle and σ_1 consists of $l - 1$ cycles.*

The expanders are a graphical interpretation of these pairs of permutations. There is a natural bijection between such pairs of permutations (σ_1, σ_2) and the expanders with 1 face, 1 black vertex, $l - 1$ white vertices and k edges. Additionally, one edge is selected as the root and the unique black vertex has a weight $l - 1$. More precisely:

- The edges are numbered $1, 2, \ldots, k$. The edge with number 1 is selected as the root.
- *The clockwise angular order of the edges on a given vertex* (i.e. order of edges ending at this vertex around it) correspond to a cycle of a permutation depending on the colour of this vertex, i.e. σ_1 for white and σ_2 for black (in this case we have unique cycle of the permutation σ_2).
- The unique face corresponds to the unique cycle of the permutation $(1, 2, \ldots, k)$.

Since there is only one face, the root determines the numbering of all edges. We can reformulate Theorem 1.

Theorem 2. *For all integers $l \geq 2$ and $k \geq 1$ the coefficient $[R_l]K_k$ is equal to the number of expanders with k edges, $l-1$ white vertices and 1 black vertex with the weight $l-1$.*

Similarly we use the second theorem for square coefficients.

Theorem 3. *For all integers $l_1, l_2 \geq 2$ and $k \geq 1$ the coefficient $[R_{l_1} R_{l_2}]K_k$ is equal to the number of triples (σ_1, σ_2, q) with the following properties.*

- *$\sigma_1, \sigma_2 \in S_k$ and $\sigma_1 \circ \sigma_2 = (1, 2, \cdots, k)$.*
- *σ_1 consists of two cycles and σ_2 consists of $l_1 + l_2 - 2$ cycles.*
- *The function q associates the numbers l_1 and l_2 to the cycles of σ_2 such that for a cycle of σ_2 with number l exist at least l cycles of σ_1 which intersect nontrivially this cycle.*

Similarly, we can also reformulate Theorem 3.

Theorem 4. *For all integers $l_1, l_2 \geq 2$ and $k \geq 1$ the coefficient $[R_{l_1} R_{l_2}]K_k$ is equal to the number of expanders with k edges, $l_1 + l_2 - 2$ white vertices and 2 black vertices with weights $l_1 - 1, l_2 - 1$.*

1.7 Relationship between coefficients of Goulden–Rattan polynomials and coefficients of Kerov polynomials

The formula (1.2) allows us to express (C_i) in terms of free cumulants; we see that the coefficients of the terms $R_i R_j$ and R_{i+j} in the expressions $C_i C_j$ and C_{i+j} are given by

$$\begin{aligned} C_i C_j &= (i-1)(j-1) R_i R_j + 0 R_{i+j} + (\text{sum of other terms}), \\ C_{i+j} &= 2(i-1)(j-1) R_i R_j + (i+j-1) R_{i+j} + (\text{sum of other terms}), \qquad \text{for } i \neq j. \\ C_{2j} &= (j-1)^2 R_j^2 + (2j-1) R_{2j} + (\text{sum of other terms}) \end{aligned}$$

Moreover, any product $C_{i_1} C_{i_2} \cdots C_{i_t}$ of at least $t \geq 3$ factors does not contain any of the terms $C_i C_j$, C_{i+j} and C_{2j}. It follows that the square coefficients of the Goulden–Rattan polynomial are related to the coefficients of the Kerov polynomial via

$$\begin{aligned}\left.\frac{\partial^2 L_k}{\partial C_i \partial C_j}\right|_{0=C_1=C_2=\cdots} &= \frac{1}{(i-1)(j-1)} \left.\frac{\partial^2 K_k}{\partial R_i \partial R_j}\right|_{0=R_1=R_2=\cdots} - 2\left.\frac{\partial L_k}{\partial C_{i+j}}\right|_{0=C_1=C_2=\cdots} \\ &= \frac{1}{(i-1)(j-1)} \left.\frac{\partial^2 K_k}{\partial R_i \partial R_j}\right|_{0=R_1=R_2=\cdots} - \frac{2}{(i+j-1)} \left.\frac{\partial K_k}{\partial R_{i+j}}\right|_{0=R_1=R_2=\cdots};\end{aligned}$$

Whereas the quadratic coefficients are related via

$$\begin{aligned}\left.\frac{\partial^2 L_k}{\partial C_j^2}\right|_{0=C_1=C_2=\cdots} &= \frac{1}{(j-1)^2} \left.\frac{\partial^2 K_k}{\partial R_j^2}\right|_{0=R_1=R_2=\cdots} - \left.\frac{\partial L_k}{\partial C_{2j}}\right|_{0=C_1=C_2=\cdots} \\ &= \frac{1}{(j-1)^2} \left.\frac{\partial^2 K_k}{\partial^2 R_j}\right|_{0=R_1=R_2=\cdots} - \frac{1}{(2j-1)} \left.\frac{\partial K_k}{\partial R_{2j}}\right|_{0=R_1=R_2=\cdots};\end{aligned}$$

Thus, we obtain the explicit formula for the square coefficients of the Goulden–Rattan polynomial:

$$[C_j^2]L_k = \frac{1}{(j-1)^2}[R_j^2]K_k - \frac{1}{2j-1}[R_{2j}]K_k \tag{1.5}$$

and

$$[C_i C_j]L_k = \frac{1}{(i-1)(j-1)}[R_i R_j]K_k - \frac{2}{i+j-1}[R_{i+j}]K_k \qquad \text{for } i \neq j. \tag{1.6}$$

2 The main result

Let $Y_k(u)$ denote the set of expanders with k edges, $u-1$ white vertices and one black vertex. Let $X_k(i,j)$ denote the set of expanders with k edges, $i+j-2$ white vertices and two black vertices with weights $i-1$ and $j-1$. Using Theorem 2 and Theorem 4 we can also reformulate the Goulden–Rattan conjecture for the square coefficients in terms of expanders, as follows.

Conjecture 1. *Let $i \neq j$ be natural numbers. Then*

$$(2j-1)\ \|X_k(j,j)\| \geq (j-1)^2\ \|Y_k(2j)\|$$

and

$$(i+j-1)\ \|X_k(i,j)\| \geq 2(i-1)(j-1)\ \|Y_k(i+j)\|$$

for any natural number k.

These inequalities are equivalent to the positivity of the coefficients $[C_j^2]L_k$ and $[C_iC_j]L_k$ respectively. In this text we prove only the first inequality in the special case $j=2$. We hope to present a proof of Conjecture 1 in its general form in a future paper.

Using Equation (1.5) we can calculate several examples of the coefficient of C_2^2 of the Goulden–Rattan polynomials

$$\begin{aligned}
[C_2^2]L_4 &= 0-0=0,\\
[C_2^2]L_5 &= 5-\frac{1}{3}\cdot 15=0,\\
[C_2^2]L_6 &= 0-0=0,\\
[C_2^2]L_7 &= 224-\frac{1}{3}\cdot 469=\frac{203}{3},\\
[C_2^2]L_8 &= 0-0=0.
\end{aligned}$$

Note that if k is even then $[C_2^2]L_k=0$ because there does not exist an expander with 4 vertices and an even number of edges, since $2-2g=2j-k+1$ by Equation (1.4). Thus we can assume that the number of edges k is odd and $k\geq 5$.

Let

$$X_k = X_k(2,2), \tag{2.1}$$
$$Y_k = Y_k(4). \tag{2.2}$$

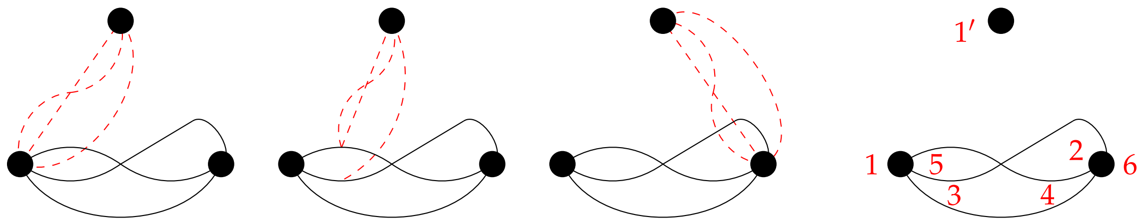
The set X_k consists of expanders with 2 black vertices and 2 white vertices such that each black vertex is connected with both white vertices; each black vertex necessarily has weight equal to 1. The set Y_k consists of expanders with one black vertex (which necessarily has the weight 3) connected with all 3 white vertices. From now on we will omit the weights of the black vertices.

The main goal of this paper is to prove the following:

Theorem 5. *The inquality*

$$3\|X_k\| \geq \|Y_k\|$$

is true for any natural number k.



(a) (b) (c) (d)

Figure 2: An example of edge sliding. (a) A graph with the special edges dashed and coloured red. (b) The graph during the edge sliding. Both ends of each special edge are assigned the clockwise direction. (c) The graph after the edge sliding. (d) The graph without the special edges with the numbered order of the corners.

3 Maps, Expanders, and edge sliding

In this section we provide some necessary background details for the proof of Theorem 5.

3.1 Edge sliding

We define *the edge sliding* on a graph as follows. We start from a graph G drawn on an oriented and connected surface with a selected set of *special edges*. Further, we assume that some ends (at the vertices) of special edges (not necessarily all) are assigned either the clockwise or counterclockwise direction.

By *the rest of the graph* $G^\star$ we will understand the graph G with the special edges removed. The graph $G^\star$ is drawn on the same surface, even if its genus is not minimal. A *corner* of $G^\star$ is an inner angle in a vertex between two neighboring edges of $G^\star$. All ends of the special edges are located inside certain corners of $G^\star$.

Let $\sigma_\star$ be the permutation on the set of the corners of $G^\star$ such that each cycle of $\sigma_\star$ corresponds to the corners which belong to some face of $G^\star$, arranged in the clockwise cyclic order (i.e., by going along the boundary of the face of $G^\star$, touching it with the right-hand side). For the example on Figure 2d we have $\sigma_\star = (1')(1,2,3,4,5,6)$.

We assume that each corner of the graph $G^\star$ contains the ends of the ends of special edges in the following counterclockwise angular order.

- First, some number of the ends of special edges with the clockwise direction.
- Second, some number of the ends of special edges with no assigned direction.
- Finally, some number of the ends of special edges with the counterclockwise direction.

Furthermore, we assume that there is no corner c of $G^\star$ such that the corner c contains the end of the special edge with the direction clockwise and the corner $\sigma_\star(c)$ contains the end of the special edge with the direction counterclockwise.

The output of the edge sliding is defined as the graph G in which the ends of the special edges in a corner c which have an assigned direction are slid to the next corner $\sigma_\star(c)$ if the direction is clockwise and to the previous corner $\sigma_\star^{-1}(c)$ if the direction is counterclockwise (see Figure 2 for an example). More precisely, after the edge sliding each corner c contains the ends of special edges in the following counterclockwise angular order.

- First, some number of the ends of special edges which previously appeared in the same order in the corner $\sigma_\star^{-1}(c)$ with the counterclockwise direction.
- Second, some number of the ends of special edges which previously appeared in the same order in the corner c without the direction.
- Finally, some number of the ends of special edges which previously appeared in the same order in the corner $\sigma_\star(c)$ with the clockwise direction.

Moreover, the output of the edge sliding has opposite assignments of directions to both ends of each special edge.

The edge sliding is a bijection from the set of graphs drawn on an oriented and connected surface with a selected set of special edges which satisfy the edge sliding conditions to itself. Edge sliding is an invertible transformation, with the inverse also given by edge sliding.

In addition, it is easy to see that the edge sliding on a graph does not change the number of faces of this graph.

3.2 The set X_k of maps

We consider any map from the set X_k of maps. Any such map has one face and an odd number of edges $k \geq 5$. We denote the black vertices by b_1, b_2 and the white vertices by w_1, w_2. There is at least one edge between each pair of the vertices of different colours. Of course, $\deg(b_1) + \deg(b_2) = k$ is an odd number. Without loss of generality we may assume that $\deg(b_1) > 0$ is an odd number and $\deg(b_2) > 0$ is an even number. Let $k_1, k_2 > 0$ denote the numbers of edges which connect the vertex b_1 with the vertices w_1, w_2, respectively. As $\deg(b_1) = k_1 + k_2$ is an odd number, then without loss of generality we may assume that k_1 is even and k_2 is odd. For example, the unique (up to choice of the root) map from the set X_5 is shown in Figure 3a.

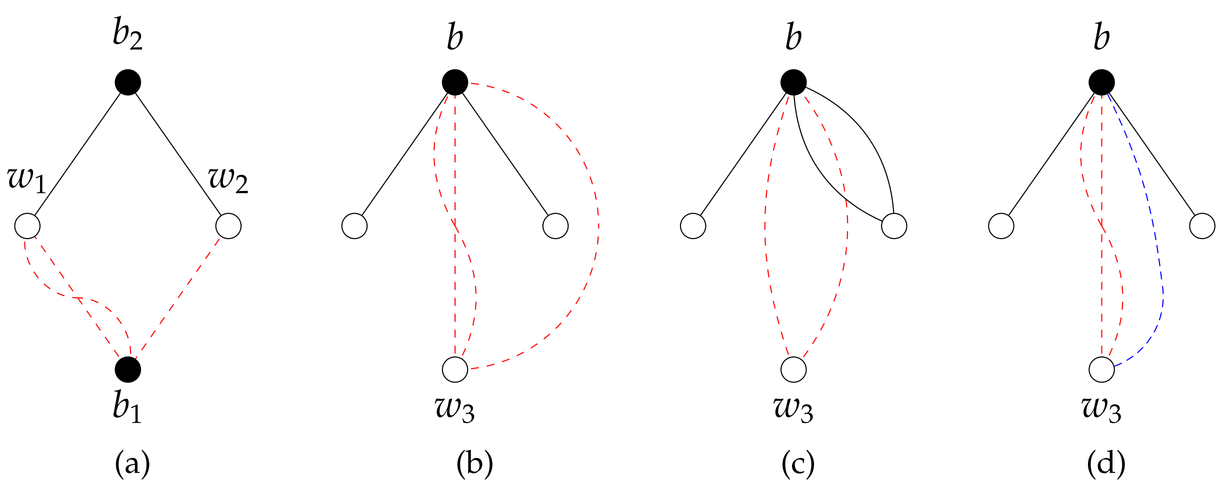


Figure 3: Examples of maps from the sets (a) X_5, (b) T_5^{odd}, (c) T_5^{even}, (d) T_5^{rest}. The root and the directions are not marked (the blue edge is assigned the opposite direction).

3.3 The set Y_k of maps

Let σ be the cycle that encodes the permutation of corners on the unique face of G (this is just $\sigma_\star$ for $G^\star$ with no special edges). We will say that *the vertex w_j is the descendant of the vertex w_i* (we denote it by $w_i \to w_j$) if using the clockwise order of the corners on the unique face of the map we can move (by walking along the edges and holding them with the right hand) in two steps from a certain corner c_i of the vertex w_i to a certain corner c_j of the vertex w_j, i.e., $\sigma^2(c_i) = c_j$.

We consider any map from the set Y_k. Any such map has one face and an odd number of edges $k \geq 5$. We denote the black vertex by b and the white vertices by w_1, w_2, w_3. We will write the set Y_k as a union of three sets which will be defined below.

Let $Y_k^{\text{odd}} \subseteq Y_k$ be the set of maps for which there exists an odd degree white vertex (let us say it is w_3) which has the other two white vertices as descendants , i.e., $w_3 \to w_1$ and $w_3 \to w_2$. Let T_k^{odd} be the set of all maps from the set Y_k^{odd} with a distinguished vertex w_3 with this property and a fixed set of edges of the vertex w_3 as the set of special edges, such that the edge ends at the vertex w_3 have no assigned direction, and the edge ends at the vertex b have assigned clockwise direction. The unique (up to choice of the root) map from the set T_5^{odd} is shown in Figure 3b. Clearly

$$|T_k^{\text{odd}}| \geq |Y_k^{\text{odd}}|. \tag{3.1}$$

Let $Y_k^{\text{even}} \subseteq Y_k$ be the set of maps such that there exists an even degree white vertex (let us say it is w_3) which has the other two white vertices as descendants , i.e., $w_3 \to w_1$ and $w_3 \to w_2$. Let T_k^{even} be the set of all the maps from the set Y_k^{even} with a distinguished vertex w_3 with this property and a fixed set of edges of the vertex w_3 as the set of special edges, such that the edge ends at the vertex w_3 have no assigned direction, and the edge ends at the vertex b have assigned clockwise direction. The unique (up to choice of the root) map from the set T_5^{even} is shown in Figure 3c. Clearly

$$|T_k^{\text{even}}| \geq |Y_k^{\text{even}}|. \tag{3.2}$$

Let $Y_k^{\text{rest}} \subseteq Y_k$ be the set of maps not included in the sets Y_k^{odd} and Y_k^{even}, i.e.

$$Y_k^{\text{rest}} = Y_k \setminus (Y_k^{\text{odd}} \cup Y_k^{\text{even}}). \tag{3.3}$$

Consider some map $m \in Y_k^{\text{rest}}$. Obviously $w_1 \to w_2 \to w_3 \to w_1$ or the other way around. Without loss of generality we may assume that $w_1 \to w_2 \to w_3 \to w_1$ and as a consequence $w_1 \not\leftarrow w_2 \not\leftarrow w_3 \not\leftarrow w_1$.

Lemma. *The map m has a white vertex of odd degree, greater than* 1.

Proof. By contradiction, suppose this is not the case. The map m has at least one odd degree white vertex, because $\deg(w_1) + \deg(w_2) + \deg(w_3) = k$ is odd. Without loss of generality we may assume that $\deg(w_1)$ is odd. Since

$$\deg(w_1) + \deg(w_2) + \deg(w_3) = k > 3 = 1 + 1 + 1,$$

it follows that $\deg(w_1) = 1$ and $\deg(w_2), \deg(w_3)$ are even, because m does not have a white vertex with odd degree greater than 1. The vertex w_1 is a leaf and thus has a unique corner which we denote by c_1.

Naturally $\sigma^2(c_1)$ is a corner of the vertex w_2. Note that σ^2 is a permutation of the corners of the white vertices which has only one cycle. The corners of the white vertices can be labelled 1, 2, 3 according to the names of the vertices they are in. If a corner c has the label a, its descendant $\sigma^2(c)$ has either the label a or $1 + a \bmod 3$. There is only one corner which has the label 1, so the corner labels (arranged in the cyclic order according to the unique cycle of σ^2) are $(1, 2, \dots, 2, 3, \dots, 3)$. Since there exists only one corner c_2 of the white vertex w_2 such that $\sigma^2(c_2)$ is a corner of the vertex w_3, then the clockwise angular cyclic order of the edges around the black vertex b is as follows: one edge connected to the vertex w_1, a certain number of edges connected to the vertex w_2, a certain number of edges connected to the vertex w_3, as there exists a unique corner $c_0 = \sigma(c_2)$ of the vertex b such that $\sigma(c_0)$ is the corner of the vertex w_3 and $\sigma^{-1}(c_0)$ is the corner of the vertex w_2.

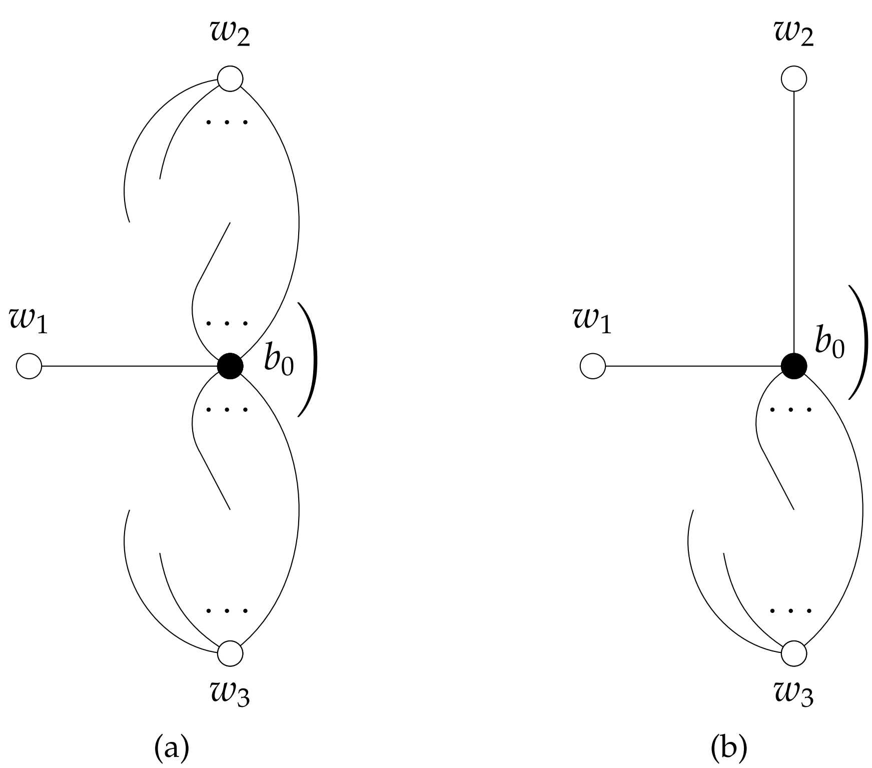


Figure 4: (a) The picture before replace (b) The picture after replace

It is easy to see that we can replace all edges of the white vertex w_2 by a single edge and get a map with 4 vertices, one face and an even number of edges. We get a contradiction, because such a map does not exist (see Equation (1.4)). Therefore, the map m has a white vertex with an odd degree greater than 1. □

We now fix as special the set of edges between the vertices b and w_3. Furthermore the ends of special edges in the vertex w_3 have no assigned direction. To all ends of special edges in the vertex b we assign the direction such that among them is an even number with clockwise direction and an odd number with counterclockwise direction. This is always possible, e.g. for a single edge with counterclockwise direction.

Let T_k^{rest} be the set of all the maps from the set Y_k^{rest} with a distinguished white vertex denoted by w_3 with a fixed choice of the set of special edges together with the directions of their ends satisfying the conditions just mentioned. The unique (up to choice of the root) example of the map from the set T_5^{rest} is shown in Figure 3d. Clearly

$$|T_k^{\text{rest}}| \geq |Y_k^{\text{rest}}|. \tag{3.4}$$

4 Proof of main result

4.1 Goal

In this section we will construct three bijections which shows the cardinality of the corresponding sets is equal. Using these equalities and the definitions of these sets we prove Theorem 5.

4.2 Three bijections

The first bijection. We start from a map $m \in X_k$. As the set of special edges we select all edges of the vertex b_1. Recall that we have assumed $\deg(b_1)$ is odd. The ends of the special edges at the vertex b_1 have no assigned direction and the other ends have assigned the direction counterclockwise. We apply the edge sliding on the map m. Then we change the colour of the black vertex b_1 to white and its name to w_3, and the name of the vertex b_2 to b. Of course, the degree of the vertex w_3 does not change and is odd. In addition, $w_3 \to w_1$ and $w_3 \to w_2$, because any map from the set X_k has at least one edge between each pair of the vertices of different colours. We obtain a map from the set T_k^{odd}. (At all times one of the edges is selected as the root.) Moreover, each map from the set T_k^{odd} can be produced. Such a transformation is a bijection between the set X_k and the set T_k^{odd}, since the edge sliding is reversible. Figure 5 shows an example of this bijection for $k = 5$. Thus

$$|X_k| = |T_k^{\mathrm{odd}}|. \tag{4.1}$$

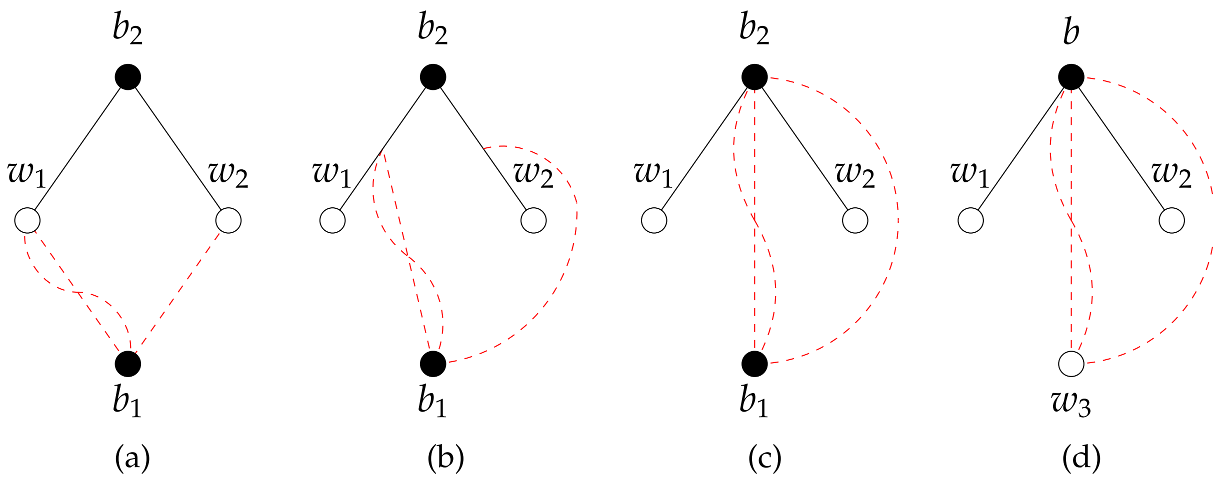


Figure 5: The example of the first bijection for the 5 edged map. (a) The map from the set X_5. (b) The map during the edge sliding. (c) The map after the edge sliding. (d) The map from the set Y_5^{odd}.

The second bijection. We start from a map $m \in X_k$. As the set of special edges we select all edges of the vertex b_2. Recall that we have assumed $\deg(b_2)$ is even. The ends of the special edges at the vertex b_2 have no assigned direction and the other ends have assigned the direction counterclockwise. We apply the edge sliding on the map m. Then we change the colour of the black vertex b_2 to white and its name to w_3, and the name of the vertex b_1 to b. Of course, the degree of the vertex w_3 does not change and is even. In addition, $w_3 \to w_1$ and $w_3 \to w_2$, because any map from the set X_k has at least one edge between each pair of the vertices of different colours. We obtain a map from the set T_k^{even}. (At all times one of the edges is selected as the root.) Moreover, each map from the set T_k^{even} can be produced. Such a transformation is a bijection between the set X_k

and the set T_k^{even}, since the edge sliding is reversible. Figure 6 shows an example of this bijection for $k = 5$. Thus

$$|X_k| = |T_k^{\text{even}}|. \tag{4.2}$$

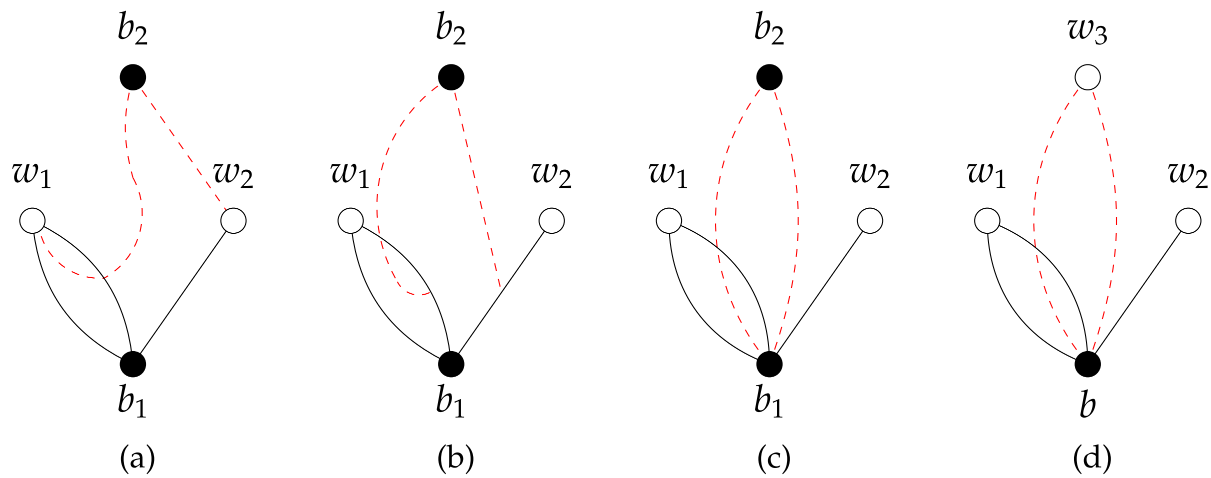


Figure 6: (a)—(d) The example of the second bijection for the 5 edged map.

The Third bijection. We start from a map $m \in X_k$. As the set of special edges we select all edges of the vertex b_1. Recall that we have assumed $\deg(b_1)$ is odd. The ends of the special edges at the vertex b_1 have no assigned direction, the ends at the vertex w_1 have assigned the direction counterclockwise and at the vertex w_2 have the direction clockwise. We apply the edge sliding on the map m. Then we change the colour of the black vertex b_1 to white and its name to w_3, and the name of the vertex b_2 to b. Of course, the degree of the vertex w_3 does not change and is odd. In addition, $w_3 \to w_1$ and $w_2 \to w_3$, because any map from the set X_k has at least one edge between each pair of the vertices of different colours. We do not necessarily obtain a map from the set T_k^{rest} (it may be that we obtain a map from set T_k^{odd}), but it can be seen that each map from the set T_k^{rest} can be produced. (At all times one of the edges is selected as the root.) Such a transformation is a bijection between the set X_k and some superset of the set T_k^{odd}, since the edge sliding is reversible. Figure 7 shows an example of this bijection for $k = 5$. Thus

$$|X_k| \geq |T_k^{\text{rest}}|. \tag{4.3}$$

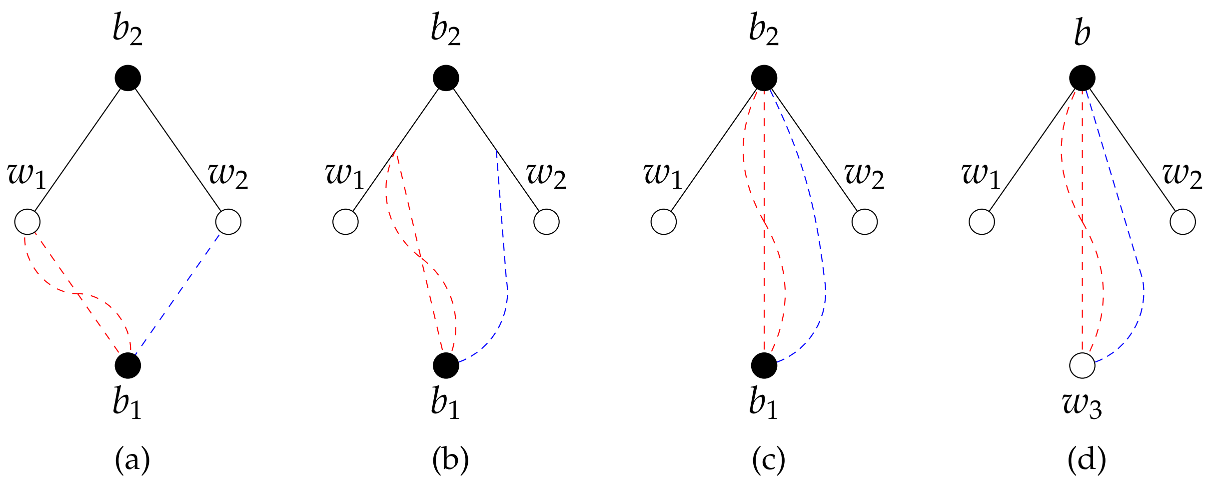


Figure 7: (a) - (d) The example of the third bijection for the 5 edged map.

4.3 The conclusion of the proof

We can now proceed to the proof of Theorem 5, we have:

$$\begin{aligned}
3[C_2^2]L_k &= 3[R_2^2]K_k - [R_4]K_k && \text{by (1.5)}\\
&= 3|X_k| - |Y_k| && \text{by (2.1), (2.2)}\\
&\geq |T_k^{\text{odd}}| + |T_k^{\text{even}}| + |T_k^{\text{rest}}| - |Y_k| && \text{by (4.1), (4.2), (4.3)}\\
&\geq |Y_k^{\text{odd}}| + |Y_k^{\text{even}}| + |Y_k^{\text{rest}}| - |Y_k| && \text{by (3.1), (3.2), (3.4)}\\
&= |Y_k^{\text{odd}} \cap Y_k^{\text{even}}| && \text{by (3.3)}\\
&\geq 0.
\end{aligned}$$

□

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