Quadratic coefficients of Goulden–Rattan character polynomials

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Abstract. Goulden–Rattan polynomials give the exact value of the subdominant part of the normalized characters of the symmetric groups in terms of certain quantities (C_i) which describe the macroscopic shape of the Young diagram. The Goulden–Rattan positivity conjecture states that the coefficients of these polynomials are positive rational numbers with small denominators. We prove a special case of this conjecture for the coefficient of the quadratic term C_2^2 by applying certain bijections involving maps (i.e., graphs drawn on surfaces).

Keywords: characters of the symmetric groups, free cumulants, Kerov polynomials, Goulden–Rattan polynomials, maps

1 Introduction

1.1 Normalized characters

Characters are a basic tool of representation theory. After normalization, they are also useful in asymptotic problems.

If $k \leq n$ are natural numbers, then any permutation $\pi \in S_k$ can also be treated as an element of the larger symmetric group S_n by adding $n - k$ additional fixpoints. For any permutation $\pi \in S_k$ and any irreducible representation ρ^λ of the symmetric group S_n

which corresponds to the Young diagram λ, we define *the normalized character*

$$\Sigma_\pi(\lambda) = \begin{cases} n(n-1)\cdots(n-k+1)\frac{\operatorname{Tr}\rho^\lambda(\pi)}{\text{dimension of }\rho^\lambda} & \text{for } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Of particular interest are the character values on the cycles, therefore we will use the shorthand notation

$$\Sigma_k(\lambda) = \Sigma_{(1,2,\ldots,k)}(\lambda).$$

1.2   Free cumulants

Free cumulants are an important tool of free probability theory [VDN92] and random matrix theory [Voi91]. In the context of the representation theory of the symmetric groups they can be defined as follows, see [Bia03]. For a Young diagram λ we define its free cumulants $R_2(\lambda), R_3(\lambda), \ldots$ as

$$R_k(\lambda) = \lim_{s \to \infty} \frac{1}{s^k}\Sigma_{k-1}(s\lambda),$$

where the diagram $s\lambda$ is created from the diagram λ by dividing each box of λ into an $s \times s$ square.

The free cumulants are very helpful for studying asymptotic behaviour of the characters on a cycle of length k when the size of the Young diagram tends to infinity [Bia98].

1.3   Kerov character polynomials

Kerov [Ker00] formulated the following result: for each permutation π and any Young diagram λ, the normalized character $\Sigma_\pi(\lambda)$ is equal to the value of some polynomial $K_\pi(R_2(\lambda), R_3(\lambda), \ldots)$ (now called *the Kerov character polynomial*) with integer coefficients. The first published proof of this fact was provided by Biane [Bia03]. The Kerov character polynomial is *universal* because it does not depend on the choice of λ. We are interested in the values of the characters on cycles, therefore for $\pi = (1, 2, \ldots, k)$ we use the simplified notation

$$\Sigma_k = K_k(R_2, R_3, \ldots) \tag{1.1}$$

for such Kerov polynomials. The first few Kerov polynomials K_k are as follows:

$$K_1 = R_2,$$
$$K_2 = R_3,$$
$$K_3 = R_4 + R_2,$$
$$K_4 = R_5 + 5R_3,$$
$$K_5 = R_6 + 15R_4 + 5R_2^2 + 8R_2,$$
$$K_6 = R_7 + 35R_5 + 35R_3R_2 + 84R_3,$$
$$K_7 = R_8 + 180R_2 + 224R_2^2 + 14R_2^3 + 56R_3^2 + 469R_4 + 84R_2R_4 + 70R_6.$$

Kerov conjectured that the coefficients of the polynomial K_k are *non-negative* integers. Goulden and Rattan [GR07] found an explicit formula for the coefficients of the Kerov polynomial K_k; unfortunately, their formula was complicated and did not give any combinatorial interpretation to the coefficients. Later, Féray proved positivity [Fér09] and together with Dołęga and Śniady found a combinatorial interpretation of the coefficients [DFŚ10]. In this paper, we will use the combinatorial interpretation given by them in the special case of linear and square coefficients.

1.4 Goulden–Rattan conjecture

Goulden and Rattan [GR07] introduced a family of functions $C_2, C_3, \ldots$ on the set of Young diagrams given by $C_0 = 1$, $C_1 = 0$ and

$$C_k^\lambda = \frac{24}{k(k+1)(k+2)} \lim_{s \to \infty} \frac{1}{s^k} \left(\Sigma_{k+1}(s\lambda) - R_{k+2}(s\lambda) \right)$$

for $k \geq 2$.

Śniady [Śni06] proved the explicit form of C_k (conjectured by Biane [Bia03]) as a polynomial in the free cumulants $R_2, R_3, \ldots$ given by

$$C_k = \sum_{\substack{j_2, j_3, \ldots \geq 0 \\ 2j_2 + 3j_3 + \cdots = k}} (j_2 + j_3 + \cdots)! \prod_{i \geq 2} \frac{\left((i-1)R_i \right)^{j_i}}{j_i!} \tag{1.2}$$

for $k \geq 2$. The aforementioned formula of Goulden and Rattan for the Kerov polynomials was naturally expressed in terms of these quantities $C_2, C_3, \ldots$ [GR07]. More specifically,

they constructed an explicit polynomial L_k with rational coefficients such that

$$K_k - R_{k+1} = L_k(C_2, C_3, \ldots). \tag{1.3}$$

These polynomials are called *the Goulden–Rattan polynomials*. They formulated the following conjecture:

Goulden–Rattan conjecture. *The coefficients of the Goulden–Rattan polynomials are non-negative numbers with small denominators.*

The first few Goulden–Rattan polynomials are as follows [GR07]:

$$K_1 - R_2 = 0$$
$$K_2 - R_3 = 0$$
$$K_3 - R_4 = C_2,$$
$$K_4 - R_5 = \frac{5}{2}C_3,$$
$$K_5 - R_6 = 5C_4 + 8C_2,$$
$$K_6 - R_7 = \frac{35}{4}C_5 + 42C_3,$$
$$K_7 - R_8 = 14C_6 + \frac{469}{3}C_4 + \frac{203}{3}C_2^2 + 180C_2.$$

Linear coefficients of the Goulden–Rattan polynomials are non-negative, because they are equal to certain scaled coefficients of the Kerov polynomial:

$$[C_j]L_k = \frac{1}{j-1}[R_j]K_k.$$

In this paper we will prove that the coefficient of C_2^2 is non-negative. We hope that edge sliding will also be useful tool in proving non-negativity of the square coefficients $[C_iC_j]L_k$. The next step towards the proof of Goulden–Rattan conjecture would be to understand the cubic coefficients $[C_iC_jC_u]L_k$; we hope that our methods will still be applicable there, nevertheless, there seem to be some difficulties related to the inclusion-exclusion principle.

1.5   Graphs on surfaces, maps and expanders

We will consider graphs drawn on an oriented surface. Each face of such a graph has some number of edges ordered cyclically by going along the boundary of the face and

touching it with the right hand. We will call it *clockwise boundary direction*. If we use the left hand and visit the edges in the opposite order, we will call it counterclockwise boundary direction.

By *a map* we mean a bipartite graph drawn without intersections on an oriented and connected surface with minimal genus. The maps which we consider have a fixed choice of colouring the vertices, i.e., each vertex is coloured black or white, with the edges connecting the vertices of the opposite colours. An example of a map is shown in Figure 1.
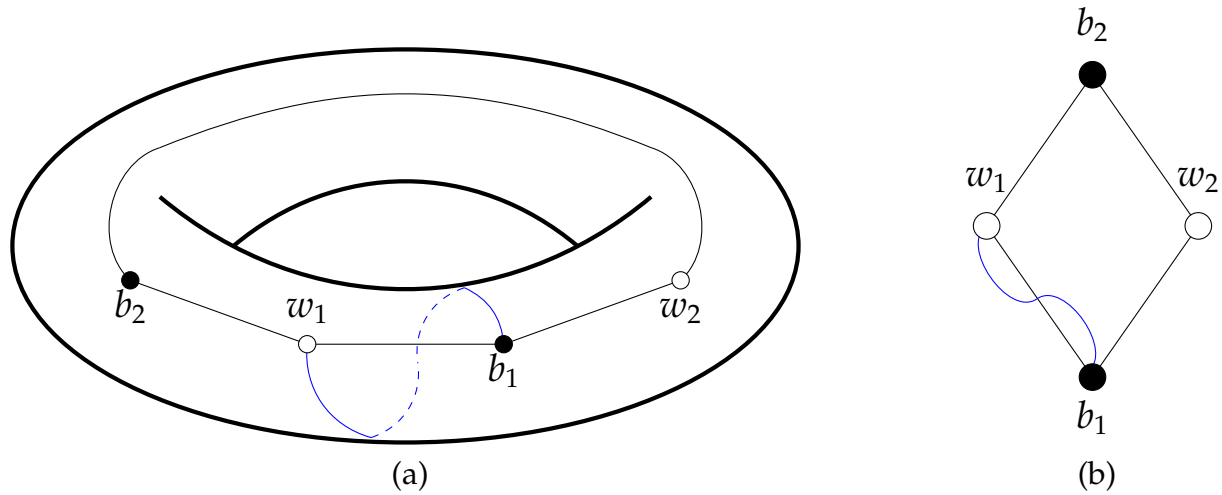

Figure 1: (a) An example of a map with 4 vertices and 5 edges drawn on a torus. (b) The same map drawn for simplicity on the plane.

An expander [Śni19, Appendix A.1] is a map with the following properties.

- It has a distinguished edge (known as the root) and one face.

- Each black vertex is assigned a natural number, known as a weight, such that each non-empty proper subset of the set of black vertices has more white neighbours than the sum of its weights.

- The sum of all weights is equal to the number of white vertices.

The map from Figure 1 is an expander if each black vertex has weight 1 (any choice of the root is valid).

Using the Euler characteristic we get

$$2 - 2g = \chi = V - k + 1 \tag{1.4}$$

where g denotes the genus of the surface, V denotes the number of vertices and k denotes the number of edges.

1.6   Combinatorial interpretation of the Kerov polynomial coefficients

The following two theorems Theorem 1 and Theorem 3 give a combinatorial interpretation to the linear and square coefficients of the Kerov character polynomials [DFŚ10, Theorem 1.2, Theorem 1.3]. The first is as follows.

Theorem 1. *For all integers $l \geq 2$ and $k \geq 1$ the coefficient $[R_l]K_k$ is equal to the number of pairs (σ_1, σ_2) of permutations $\sigma_1, \sigma_2 \in S(k)$ such that $\sigma_1 \circ \sigma_2 = (1, 2, \ldots, k)$ and such that σ_2 consists of one cycle and σ_1 consists of $l - 1$ cycles.*

The expanders are a graphical interpretation of these pairs of permutations. There is a natural bijection between such pairs of permutations (σ_1, σ_2) and the expanders with one face, one black vertex, $l - 1$ white vertices and k edges. Additionally, one edge is selected as the root and the unique black vertex has a weight $l - 1$. More precisely:

- The edges are numbered $1, 2, \ldots, k$. The edge with number 1 is selected as the root.

- *The counterclockwise angular cyclic order of the edges on a given vertex* (i.e., the order of edges ending at this vertex around it) corresponds to a cycle of a permutation depending on the colour of this vertex, i.e., σ_1 for white and σ_2 for black (in this case we have a unique cycle of the permutation σ_2).

- The unique face corresponds to the unique counterclockwise boundary cycle of the permutation $(1, 2, \ldots, k)$.

Since there is only one face, the root determines the numbering of all edges. We can reformulate Theorem 1 as follows. (See Figure 2a for an example.)

Theorem 2. *For all integers $l \geq 2$ and $k \geq 1$ the coefficient $[R_l]K_k$ is equal to the number of expanders with k edges, $l - 1$ white vertices and 1 black vertex with the weight $l - 1$.*
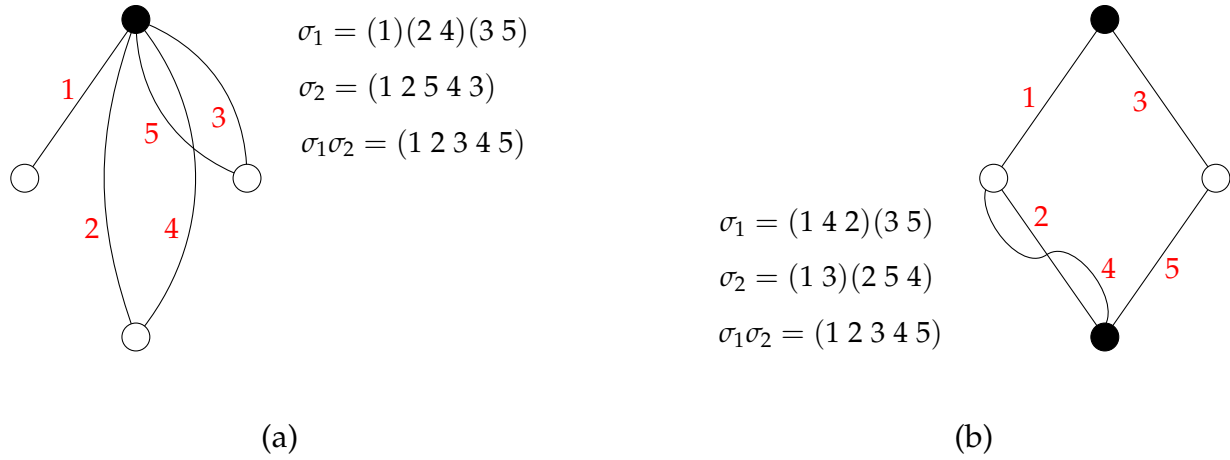
$\sigma_1 = (1)(2\,4)(3\,5)$

$\sigma_2 = (1\,2\,5\,4\,3)$

$\sigma_1\sigma_2 = (1\,2\,3\,4\,5)$

$\sigma_1 = (1\,4\,2)(3\,5)$

$\sigma_2 = (1\,3)(2\,5\,4)$

$\sigma_1\sigma_2 = (1\,2\,3\,4\,5)$

(a)   (b)

Figure 2: Examples of expander with 5 edges and their corresponding pair of permutations σ_1, σ_2 such that $\sigma_1\sigma_2 = (1\,2\,3\,4\,5)$. The root is assigned the number 1. (a) The expander with one black vertex and three white vertices. (b) The expander with two black vertices and two white vertices.

Similarly we use the second theorem [DFŚ10, Theorem 1.3] for square coefficients.

Theorem 3. *For all integers $l_1, l_2 \geq 2$ and $k \geq 1$ the coefficient $[R_{l_1}R_{l_2}]K_k$ is equal to the number of triples (σ_1, σ_2, q) with the following properties.*

- *The permutations $\sigma_1, \sigma_2 \in S_k$ fulfill the equality $\sigma_1 \circ \sigma_2 = (1, 2, \cdots, k)$.*

- *The permutation σ_1 consists of two cycles and the permutation σ_2 consists of $l_1 + l_2 - 2$ cycles.*

- *The function q associates the numbers l_1 and l_2 to the two cycles of σ_2. Furthermore, for each cycle c of σ_2 there exist at least $q(c)$ cycles of σ_1 which intersect nontrivially c.*

Analogously, we can also reformulate Theorem 3. (See Figure 2b for an example.)

Theorem 4. *For all integers $l_1, l_2 \geq 2$ and $k \geq 1$ the coefficient $[R_{l_1}R_{l_2}]K_k$ is equal to the number of expanders with k edges, $l_1 + l_2 - 2$ white vertices and 2 black vertices with weights $l_1 - 1, l_2 - 1$.*

1.7   Relationship between coefficients of Goulden–Rattan polynomials and coefficients of Kerov polynomials

The formula (1.2) allows us to express (C_i) in terms of free cumulants; we see that the coefficients of the terms $R_i R_j$, R_{i+j} and R_{2j} in the expressions $C_i C_j$, C_{i+j} and C_{2j} are given for $i \neq j$ by

$$
\begin{aligned}
C_i C_j &= (i-1)(j-1)R_i R_j + 0R_{i+j} + \text{(sum of other terms)}, \\
C_{i+j} &= 2(i-1)(j-1)R_i R_j + (i+j-1)R_{i+j} + \text{(sum of other terms)}, \\
C_{2j} &= (j-1)^2 R_j^2 + (2j-1)R_{2j} + \text{(sum of other terms)}.
\end{aligned}
$$

Moreover, any product $C_{i_1} C_{i_2} \cdots C_{i_t}$ of at least $t \geq 3$ factors does not contain any of the terms $C_i C_j$, C_{i+j} and C_{2j}. It follows that the square coefficients of the Goulden–Rattan polynomial are related to the coefficients of the Kerov polynomial via

$$
\begin{aligned}
\left. \frac{\partial^2 L_k}{\partial C_i \partial C_j} \right|_{0=C_1=C_2=\cdots} &= \frac{1}{(i-1)(j-1)} \left. \frac{\partial^2 K_k}{\partial R_i \partial R_j} \right|_{0=R_1=R_2=\cdots} - 2 \left. \frac{\partial L_k}{\partial C_{i+j}} \right|_{0=C_1=C_2=\cdots} \\
&= \frac{1}{(i-1)(j-1)} \left. \frac{\partial^2 K_k}{\partial R_i \partial R_j} \right|_{0=R_1=R_2=\cdots} - \frac{2}{(i+j-1)} \left. \frac{\partial K_k}{\partial R_{i+j}} \right|_{0=R_1=R_2=\cdots}
\end{aligned}
$$

for $i \neq j$. Whereas the quadratic coefficients are related via

$$
\begin{aligned}
\left. \frac{\partial^2 L_k}{\partial C_j^2} \right|_{0=C_1=C_2=\cdots} &= \frac{1}{(j-1)^2} \left. \frac{\partial^2 K_k}{\partial R_j^2} \right|_{0=R_1=R_2=\cdots} - \left. \frac{\partial L_k}{\partial C_{2j}} \right|_{0=C_1=C_2=\cdots} \\
&= \frac{1}{(j-1)^2} \left. \frac{\partial^2 K_k}{\partial^2 R_j} \right|_{0=R_1=R_2=\cdots} - \frac{1}{(2j-1)} \left. \frac{\partial K_k}{\partial R_{2j}} \right|_{0=R_1=R_2=\cdots}
\end{aligned}
$$

for any natural number j. Thus, we obtain the explicit formula for the square coefficients of the Goulden–Rattan polynomial:

$$
[C_j^2]L_k = \frac{1}{(j-1)^2}[R_j^2]K_k - \frac{1}{2j-1}[R_{2j}]K_k \tag{1.5}
$$

and

$$
[C_i C_j]L_k = \frac{1}{(i-1)(j-1)}[R_i R_j]K_k - \frac{2}{i+j-1}[R_{i+j}]K_k \qquad \text{for } i \neq j. \tag{1.6}
$$

2 The main result

Let $Y_k(u)$ denote the set of expanders with k edges, $u - 1$ white vertices and one black vertex. Let $X_k(i, j)$ denote the set of expanders with k edges, $i + j - 2$ white vertices and two black vertices with weights $i - 1$ and $j - 1$. Using Theorem 2 and Theorem 4 we can also reformulate the Goulden–Rattan conjecture for the square coefficients in terms of expanders, as follows.

Conjecture 1. *Let $i \neq j$ be natural numbers. Then*

$$(2j - 1) \ \|X_k(j, j)\| \geq (j - 1)^2 \ \|Y_k(2j)\|$$

and

$$(i + j - 1) \ \|X_k(i, j)\| \geq 2(i - 1)(j - 1) \ \|Y_k(i + j)\|$$

for any natural number k.

These inequalities are equivalent to the positivity of the coefficients $[C_j^2]L_k$ and $[C_i C_j]L_k$ respectively. In this text we prove only the first inequality in the special case $j = 2$. We hope to present a proof of Conjecture 1 in its general form in a future paper.

Using Equation (1.5) we can calculate several examples of the coefficient of C_2^2 of the Goulden–Rattan polynomials

$$[C_2^2]L_4 = 0 - 0 = 0,$$
$$[C_2^2]L_5 = 5 - \frac{1}{3} \cdot 15 = 0,$$
$$[C_2^2]L_6 = 0 - 0 = 0,$$
$$[C_2^2]L_7 = 224 - \frac{1}{3} \cdot 469 = \frac{203}{3},$$
$$[C_2^2]L_8 = 0 - 0 = 0.$$

Note that if k is even then $[C_2^2]L_k = 0$ because there does not exist an expander with 4 vertices and an even number of edges, since $2 - 2g = 2j - k + 1$ by Equation (1.4). Additionally, also $[C_2^2]L_1 = 0$ and $[C_2^2]L_3 = 0$. Thus we can assume that the number of edges k is odd and $k \geq 5$.

Let

$$X_k = X_k(2,2), \tag{2.1}$$

$$Y_k = Y_k(4). \tag{2.2}$$

The set X_k consists of expanders with 2 black vertices and 2 white vertices such that each black vertex is connected with both white vertices; each black vertex necessarily has weight equal to 1. The set Y_k consists of expanders with one black vertex (which necessarily has the weight 3) connected with all 3 white vertices. From now on we will omit the weights of the black vertices.

The main goal of this paper is to prove the following:

Theorem 5. *The inquality*

$$3\|X_k\| \geq \|Y_k\|$$

is true for any natural number k.

3 Maps, expanders, and edge sliding

In this section we provide some necessary background details for the proof of Theorem 5.

3.1 Edge sliding

We define *the edge sliding* on a graph as follows. We start from a graph G drawn on an oriented and connected surface with a selected set of *special edges*. Further, we assume that some ends (at the vertices) of the special edges (not necessarily all) are assigned either the clockwise or counterclockwise boundary direction.

By *the rest of the graph* $G^\star$ we will understand the graph G with the special edges removed. The graph $G^\star$ is drawn on the same surface, even if its genus is not minimal. A *corner* of $G^\star$ is an inner angle in a vertex between two neighboring edges of $G^\star$ (or full angle for the degenerate case of no edge in a vertex). All ends of the special edges are located inside certain corners of $G^\star$.

Let $\sigma_\star$ be the permutation on the set of the corners of $G^\star$ such that each cycle of $\sigma_\star$ corresponds to the corners which belong to some face of $G^\star$, arranged in the clockwise

boundary cyclic order (i.e., by going along the boundary of the face of $G^\star$, touching it with the right hand). For the example on Figure 3d we have $\sigma_\star = (1')(1, 2, 3, 4, 5, 6)$.

(a)   (b)   (c)   (d)
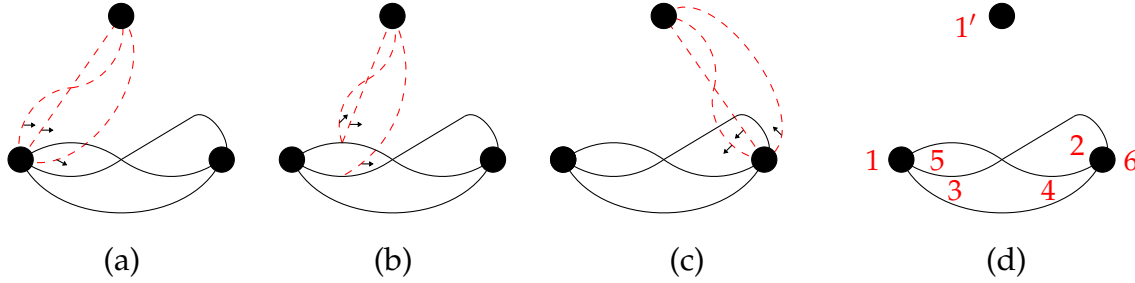
Figure 3: An example of edge sliding. (a) A graph with the special edges dashed and coloured red. The directions of the edge ends are indicated by arrows. (b) The graph during the edge sliding in the clockwise boundary direction. (c) The graph after the edge sliding. Directions have already been reversed. (d) The graph without the special edges with the numbered order of the corners.

We assume that each corner of the graph $G^\star$ contains the ends of the special edges in the following counterclockwise angular order.

- First, some number of the ends of the special edges with the clockwise boundary direction.

- Second, some number of the ends of the special edges with no assigned direction.

- Finally, some number of the ends of the special edges with the counterclockwise boundary direction.

Furthermore, we assume that there is no corner c of $G^\star$ such that the corner c contains the end of a special edge with the clockwise boundary direction and the corner $\sigma_\star(c)$ contains the end of a special edge with the counterclockwise boundary direction (see Figure 3a for an example).

The output of the edge sliding is defined as the graph G in which the ends of the special edges in a corner c which have an assigned clockwise boundary direction are slid to the next corner $\sigma_\star(c)$. However the ends of the special edges which have an assigned counterclockwise boundary direction are slid to the previous corner $\sigma_\star^{-1}(c)$. Moreover,

the output of the edge sliding has opposite assignments of directions to both ends of each special edge. Figure 3 shows an example of the edge sliding. Note that in the degenerate case of only special edges in a vertex (like at the corner $1'$ in the picture Figure 3d), sliding does never really not occur.

More precisely, after the edge sliding each corner c contains the ends of special edges in the following counterclockwise angular order.

- First, some number of the ends of special edges with the clockwise boundary direction which previously appeared in the same angular order in the corner $\sigma_\star(c)$ with the counterclockwise boundary direction.

- Second, some number of the ends of special edges which previously appeared in the same angular order in the corner c without the direction.

- Finally, some number of the ends of special edges with counterclockwise boundary direction which previously appeared in the same angular order in the corner $\sigma_\star^{-1}(c)$ with the clockwise boundary direction.

The edge sliding is an involution on the set of graphs drawn on an oriented and connected surface with a selected set of special edges which satisfy the above conditions for correct selection of directions. Edge sliding is an invertible transformation, with the inverse also given by edge sliding.

In addition, it is easy to see that the edge sliding on a graph does not change the number of faces of this graph.

3.2 The set X_k of maps

We consider any map from the set X_k. This map has one face and an odd number of edges $k \geq 5$. We denote the black vertices by b_1, b_2 and the white vertices by w_1, w_2. There is at least one edge between each pair of the vertices of different colours. Of course, $\deg(b_1) + \deg(b_2) = k$ is an odd number. Without loss of generality we may assume that $\deg(b_1) > 0$ is an odd number and $\deg(b_2) > 0$ is an even number. Let $k_1, k_2 > 0$ denote the numbers of edges which connect the vertex b_1 with the vertices w_1, w_2, respectively. As $\deg(b_1) = k_1 + k_2$ is an odd number, without loss of generality we may assume that k_1 is even and k_2 is odd. For example, the unique (up to choice of the root) map from the set X_5 is shown in Figure 4a.
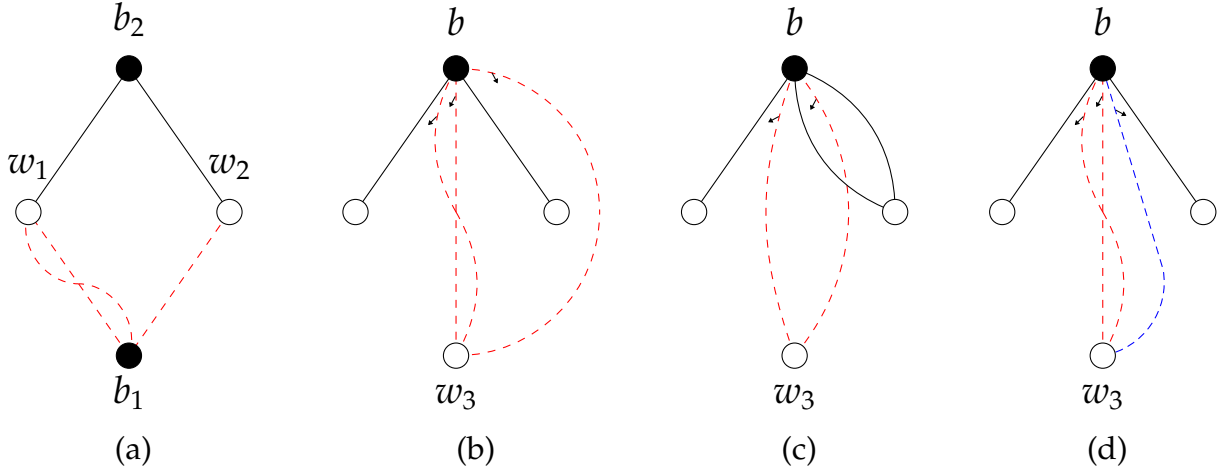
Figure 4: Examples of maps from the sets (a) X_5, (b) T_5^{odd}, (c) T_5^{even}, (d) T_5^{rest}. Special edges are dashed and coloures. The directions are indicated by arrows, and the root is not marked.

3.3 The set Y_k of maps

Let σ be the cycle that encodes the clockwise boundary cyclic order of the corners on the unique face of G (this is just $\sigma_\star$ for $G^\star$ with no special edges). We will say that *the vertex w_j is a descendant of the vertex w_i* (we denote it by $w_i \to w_j$) if using the clockwise boundary order of the corners on the unique face of the map we can move (by walking along the edges and holding them with the right hand) in two steps from a certain corner c_i of the vertex w_i to a certain corner c_j of the vertex w_j, i.e., $\sigma^2(c_i) = c_j$.

We consider any map from the set Y_k. Any such map has one face and an odd number of edges $k \geq 5$. We denote the black vertex by b and the white vertices by w_1, w_2, w_3. We will write the set Y_k as a union of three sets which will be defined below.

Let $Y_k^{\text{odd}} \subseteq Y_k$ be the set of maps for which there exists an *odd degree* white vertex (let us say it is w_3) which has the other two white vertices as descendants, i.e., $w_3 \to w_1$ and $w_3 \to w_2$. Let T_k^{odd} be the set of all maps from the set Y_k^{odd} with a distinguished vertex w_3 with this property. Moreover, the set of edges of the vertex w_3 is fixed as the set of special edges. In addition, the edge ends at the vertex w_3 have no assigned direction, and the edge ends at the vertex b have assigned clockwise boundary direction. The unique

(up to choice of the root) map from the set T_5^{odd} is shown in Figure 4b. Clearly

$$|T_k^{\text{odd}}| \geq |Y_k^{\text{odd}}|. \tag{3.1}$$

Let $Y_k^{\text{even}} \subseteq Y_k$ be the set of maps such that there exists an *even degree* white vertex (let us say it is w_3) which has the other two white vertices as descendants, i.e., $w_3 \rightarrow w_1$ and $w_3 \rightarrow w_2$. Let T_k^{even} be the set of all the maps from the set Y_k^{even} with a distinguished vertex w_3 with this property. Moreover, the set of edges of the vertex w_3 is fixed as the set of special edges. In addition, the edge ends at the vertex w_3 have no assigned direction, and the edge ends at the vertex b have assigned clockwise boundary direction. The unique (up to choice of the root) map from the set T_5^{even} is shown in Figure 4c. Clearly

$$|T_k^{\text{even}}| \geq |Y_k^{\text{even}}|. \tag{3.2}$$

Let $Y_k^{\text{rest}} \subseteq Y_k$ be the set of maps not included in the sets Y_k^{odd} and Y_k^{even}, i.e.,

$$Y_k^{\text{rest}} = Y_k \setminus (Y_k^{\text{odd}} \cup Y_k^{\text{even}}). \tag{3.3}$$

Consider some map $m \in Y_k^{\text{rest}}$. Obviously $w_1 \rightarrow w_2 \rightarrow w_3 \rightarrow w_1$ or the other way around. Without loss of generality we may assume that $w_1 \rightarrow w_2 \rightarrow w_3 \rightarrow w_1$ and as a consequence $w_1 \nrightarrow w_2 \nrightarrow w_3 \nrightarrow w_1$.

Lemma. *The map m has a white vertex of odd degree, greater than* 1.

Proof. By contradiction, suppose this is not the case. The map m has at least one odd degree white vertex, because $\deg(w_1) + \deg(w_2) + \deg(w_3) = k$ is odd. Without loss of generality we may assume that $\deg(w_1)$ is odd. Since

$$\deg(w_1) + \deg(w_2) + \deg(w_3) = k > 3 = 1 + 1 + 1,$$

it follows that $\deg(w_1) = 1$ and $\deg(w_2), \deg(w_3)$ are even, because m does not have a white vertex with odd degree greater than 1. The vertex w_1 is a leaf and thus has a unique corner which we denote by c_1.

Naturally $\sigma^2(c_1)$ is a corner of the vertex w_2. Note that σ^2 is a permutation of the corners of the white vertices which has only one cycle, because the map m has only one face. The corners of the white vertices can be labelled 1, 2, 3 according to the names of the vertices they are in. If a corner c has the label a, its descendant $\sigma^2(c)$ has either the

label a or $1 + a \bmod 3$. There is only one corner which has the label 1, so the corner labels (arranged in the cyclic order according to the unique cycle of σ^2) are $(1, 2, \ldots, 2, 3, \ldots, 3)$. Since there exists only one corner c_2 of the white vertex w_2 such that $\sigma^2(c_2)$ is a corner of the vertex w_3, then there exists a unique corner $c_0 = \sigma(c_2)$ of the vertex b such that $\sigma(c_0)$ is the corner of the vertex w_3 and $\sigma^{-1}(c_0)$ is the corner of the vertex w_2. Thus the clockwise angular cyclic order of the edges around the black vertex b is as follows: one edge connected to the vertex w_1, a certain number of edges connected to the vertex w_2, a certain number of edges connected to the vertex w_3. Figure 5a visualizes this situation.
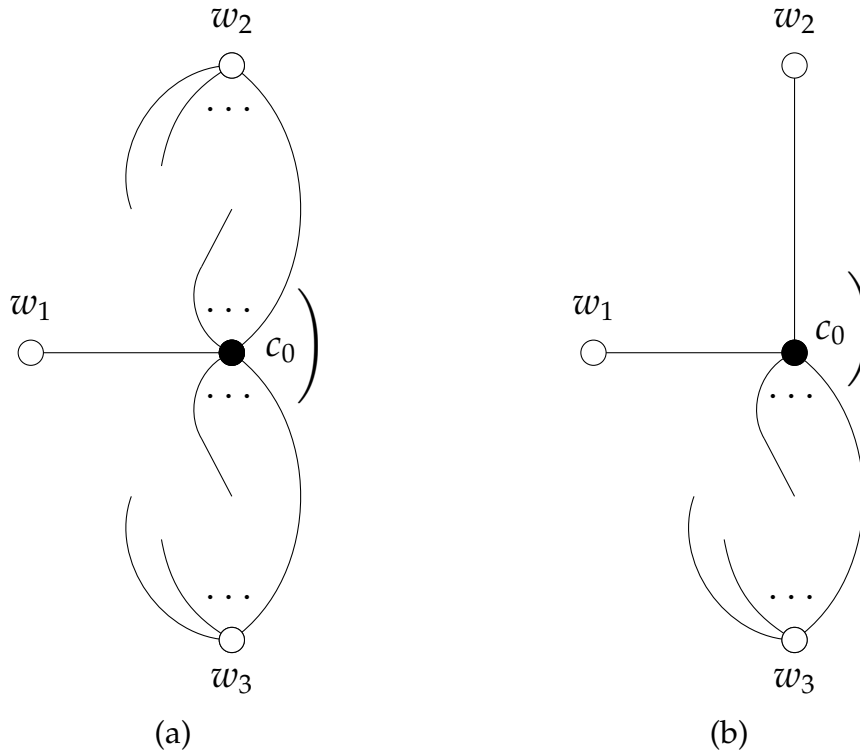

Figure 5: (a) The map m before replace including the marked corner c_0. (b) A non-existent hypothetical map with four vertices and an even number of edges.

It is easy to see that we can replace all edges of the white vertex w_2 by a single edge (see Figure 5) and get a map with 4 vertices, one face and an even number of edges. We get a contradiction, because such a map does not exist (see Equation (1.4)). Therefore, the map m has a white vertex with an odd degree greater than 1. □

We now fix all edges between the vertices b and w_3 as special. Furthermore, the ends of the special edges in the vertex w_3 have no assigned direction. To all ends of the special edges in the vertex b we assign the direction in such way that among them there is an even number with the clockwise boundary direction and an odd number with the counterclockwise boundary direction. This is always possible, e.g. for a single edge with the counterclockwise boundary direction.

Let T_k^{rest} be the set of all the maps from the set Y_k^{rest} with a distinguished white vertex denoted by w_3 with a fixed choice of the set of special edges together with the directions of their ends satisfying the conditions just mentioned above. The unique (up to choice of the root) example of the map from the set T_5^{rest} is shown in Figure 4d. Clearly

$$|T_k^{\text{rest}}| \geq |Y_k^{\text{rest}}|. \tag{3.4}$$

4   Proof of main result

4.1   Goal

In this section we will construct three bijections which show the cardinalities of the corresponding sets are equal. Using these equalities and the definitions of these sets we will prove Theorem 5.

4.2   Three bijections

The first bijection between X_k and T_k^{odd}. We start from a map $m \in X_k$. As the set of special edges we select all edges of the vertex b_1. Recall that we have assumed that $\deg(b_1)$ is odd. The ends of the special edges at the vertex b_1 have no assigned direction and the other ends have assigned the counterclockwise boundary direction. We apply the edge sliding to the map m. Then we change the colour of the black vertex b_1 to white and its name to w_3, and the name of the vertex b_2 to b. Of course, the degree of the vertex w_3 does not change and is odd. In addition, $w_3 \to w_1$ and $w_3 \to w_2$, because any map from the set X_k has at least one edge between each pair of the vertices of different colours. We obtain a map from the set T_k^{odd}. (At all times one of the edges is selected as the root.) Moreover, each map from the set T_k^{odd} can be produced in this way. Such a transformation is a bijection between the set X_k and the set T_k^{odd}, since the edge sliding

is reversible. Figure 6 shows an example of this bijection for $k = 5$. Thus

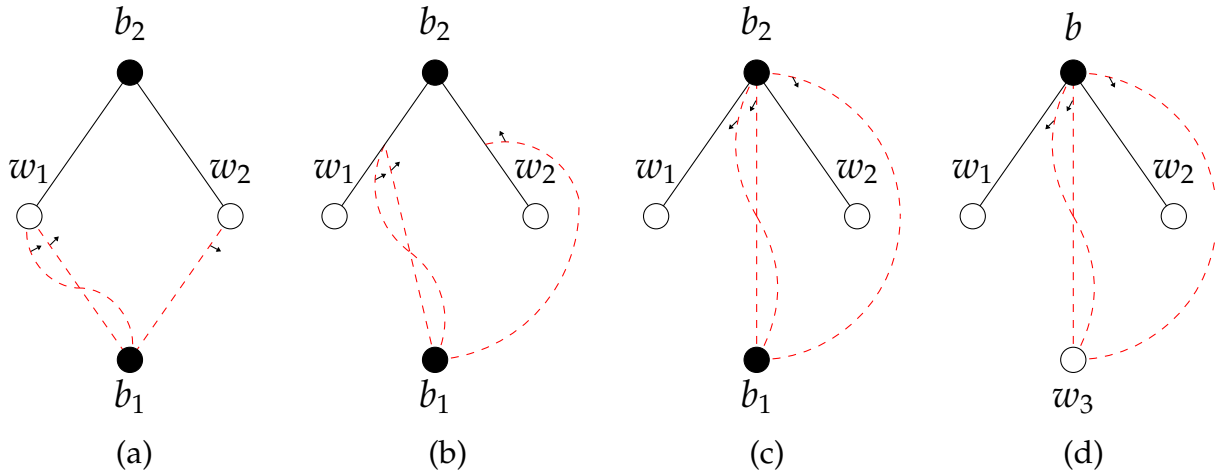$$|X_k| = |T_k^{\text{odd}}|. \tag{4.1}$$

Figure 6: The example of the first bijection for the 5-edged map. (a) The map from the set X_5. (b) The map during the edge sliding. (c) The map after the edge sliding. (d) The map from the set Y_5^{odd}.

The second bijection between X_k and T_k^{even}. We start from a map $m \in X_k$. As the set of special edges we take all edges of the vertex b_2. Recall that we have assumed that $\deg(b_2)$ is even. The ends of the special edges at the vertex b_2 have no assigned direction and the other ends have assigned the counterclockwise boundary direction. We apply the edge sliding to the map m. Then we change the colour of the black vertex b_2 to white and its name to w_3, and the name of the vertex b_1 to b. Of course, the degree of the vertex w_3 does not change and is even. In addition, $w_3 \to w_1$ and $w_3 \to w_2$, because any map from the set X_k has at least one edge between each pair of the vertices of different colours. We obtain a map from the set T_k^{even}. (At all times one of the edges is selected as the root.) Moreover, each map from the set T_k^{even} can be produced. Such a transformation is a bijection between the set X_k and the set T_k^{even}, since the edge sliding is reversible. Figure 7 shows an example of this bijection for $k = 5$. Thus

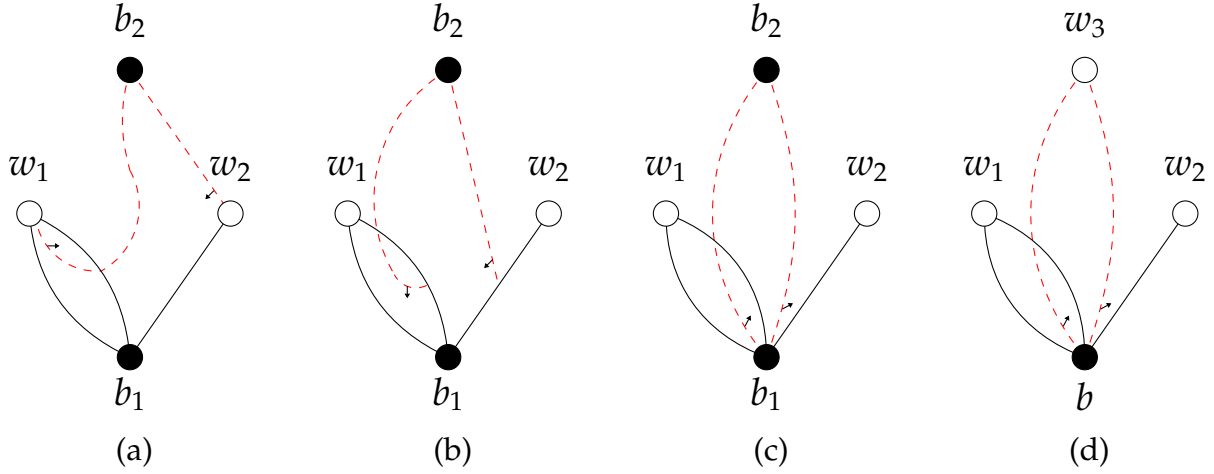$$|X_k| = |T_k^{\text{even}}|. \tag{4.2}$$

Figure 7: (a)—(d) The example of the second bijection for the 5-edged map.

The third bijection. We start from a map $m \in X_k$. As the set of special edges we select all edges of the vertex b_1. Recall that we have assumed that $\deg(b_1)$ is odd. The ends of the special edges at the vertex b_1 have no assigned direction, the ends at the vertex w_1 have assigned the counterclockwise boundary direction and at the vertex w_2 have the clockwise boundary direction. We apply the edge sliding to the map m. Then we change the colour of the black vertex b_1 to white and its name to w_3, and the name of the vertex b_2 to b. Of course, the degree of the vertex w_3 does not change and is odd. In addition, $w_3 \to w_1$ and $w_2 \to w_3$ (and $w_1 \to w_2$), because any map from the set X_k has at least one edge between each pair of the vertices of different colours. We do not necessarily obtain a map from the set T_k^{rest} (it may be that we obtain a map from set T_k^{odd}), but it can be seen that each map from the set T_k^{rest} can be produced. (At all times one of the edges is selected as the root.) Such a transformation is a bijection between the set X_k and some superset of the set T_k^{odd}, since the edge sliding is reversible. Figure 8 shows an example of this bijection for $k = 5$. Thus

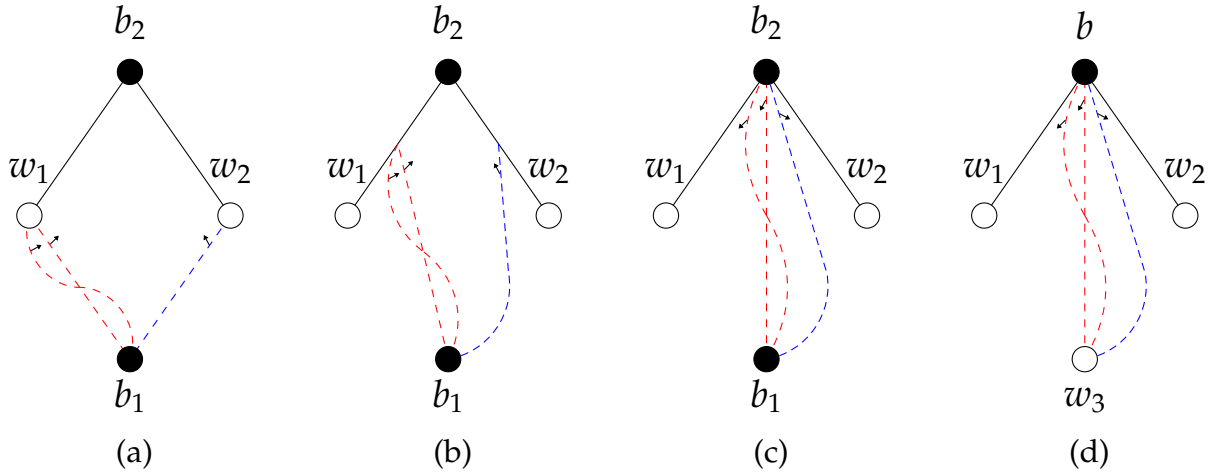$$|X_k| \geq |T_k^{\mathrm{rest}}|. \tag{4.3}$$

Figure 8: (a) - (d) The example of the third bijection for the 5-edged map.

4.3   The conclusion of the proof

We can now proceed to the proof of Theorem 5, we have:

$$
\begin{aligned}
3[C_2^2]L_k = 3[R_2^2]K_k - [R_4]K_k && \text{by (1.5)} \\
= 3|X_k| - |Y_k| && \text{by (2.1), (2.2)} \\
\geq |T_k^{\text{odd}}| + |T_k^{\text{even}}| + |T_k^{\text{rest}}| - |Y_k| && \text{by (4.1), (4.2), (4.3)} \\
\geq |Y_k^{\text{odd}}| + |Y_k^{\text{even}}| + |Y_k^{\text{rest}}| - |Y_k| && \text{by (3.1), (3.2), (3.4)} \\
= |Y_k^{\text{odd}} \cap Y_k^{\text{even}}| && \text{by (3.3)} \\
\geq 0.
\end{aligned}
$$

$\square$

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