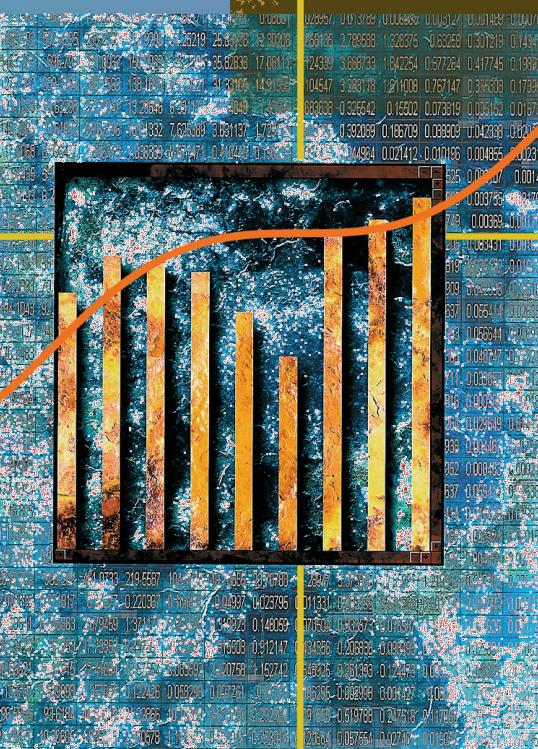


1

FUNCTIONS



*O*ne of the important themes in calculus is the analysis of relationships between physical or mathematical quantities. Such relationships can be described in terms of graphs, formulas, numerical data, or words. In this chapter we will develop the concept of a *function*, which is the basic idea that underlies almost all mathematical and physical relationships, regardless of the form in which they are expressed. We will study properties of some of the most basic functions that occur in calculus, and we will examine some familiar ideas involving lines, polynomials, and trigonometric functions from viewpoints that may be new. We will also discuss ideas relating to the use of graphing utilities such as graphing calculators and graphing software for computers. Before you start reading, you may want to scan through the appendices, since they contain various kinds of precalculus material that may be helpful if you need to review some of those ideas.

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1.1 FUNCTIONS AND THE ANALYSIS OF GRAPHICAL INFORMATION

In this section we will define and develop the concept of a function. Functions are used by mathematicians and scientists to describe the relationships between variable quantities and hence play a central role in calculus and its applications.

SCATTER PLOTS AND TABULAR DATA

Table 1.1.1

INDIANAPOLIS 500
QUALIFYING SPEEDS

YEAR t	SPEED S (mi/h)
1980	192.256
1981	200.546
1982	207.004
1983	207.395
1984	210.029
1985	212.583
1986	216.828
1987	215.390
1988	219.198
1989	223.885
1990	225.301
1991	224.113
1992	232.482
1993	223.967
1994	228.011
1995	231.604
1996	233.100
1997	218.263
1998	223.503
1999	225.179

Many scientific laws are discovered by collecting, organizing, and analyzing experimental data. Since graphs play a major role in studying data, we will begin by discussing the kinds of information that a graph can convey.

To start, we will focus on paired data. For example, Table 1.1.1 shows the top qualifying speed by year in the Indianapolis 500 auto race from 1980 to 1999. This table pairs up each year t between 1980 and 1999 with the top qualifying speed S for that year. These paired data can be represented graphically in a number of ways:

- One possibility is to plot the paired data points in a rectangular tS -coordinate system (t horizontal and S vertical), in which case we obtain a **scatter plot** of S versus t (Figure 1.1.1a).
- A second possibility is to enhance the scatter plot visually by joining successive points with straight-line segments, in which case we obtain a **line graph** (Figure 1.1.1b).
- A third possibility is to represent the paired data by a **bar graph** (Figure 1.1.1c).

All three graphical representations reveal an upward trend in the data, as one would expect with improvements in automotive technology.

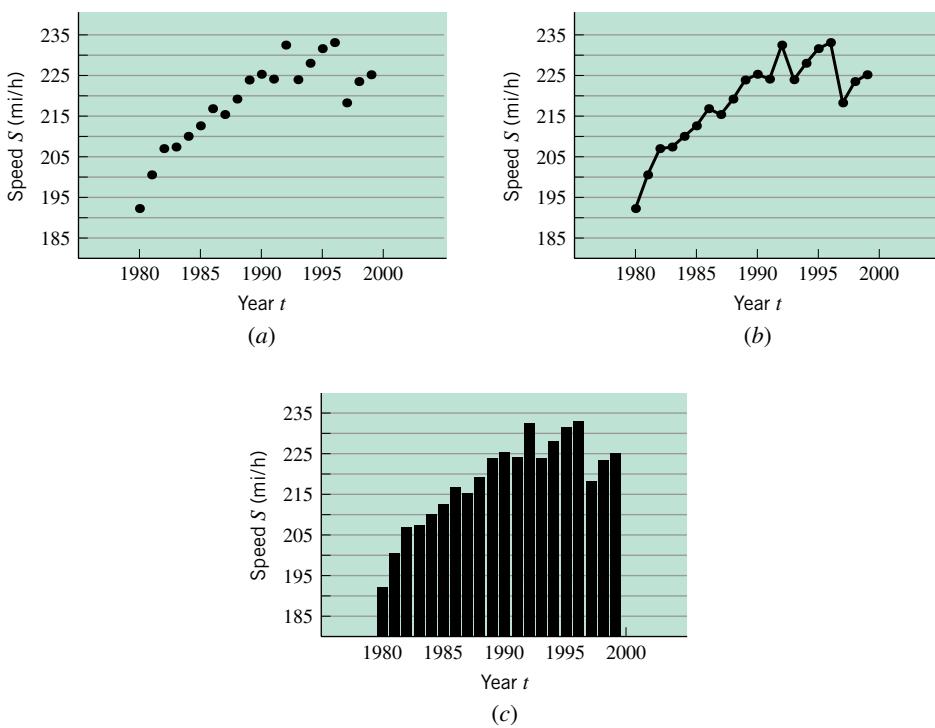


Figure 1.1.1

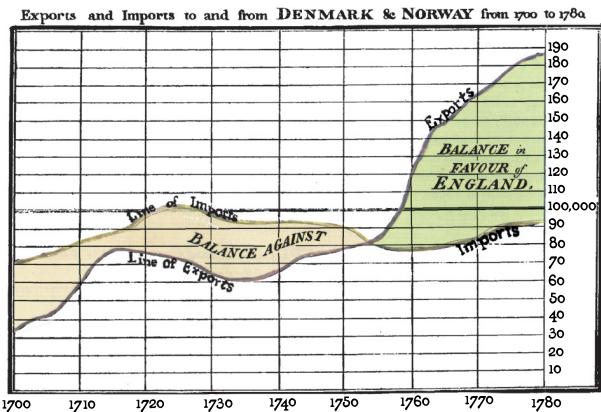
EXTRACTING INFORMATION FROM GRAPHS

One of the first books to use graphs for representing numerical data was *The Commercial and Political Atlas*, published in 1786 by the Scottish political economist William Playfair (1759–1823). Figure 1.1.2a shows an engraving from that work that compares exports and imports by England to Denmark and Norway (combined). In spite of its antiquity, the

1.1 Functions and the Analysis of Graphical Information 9

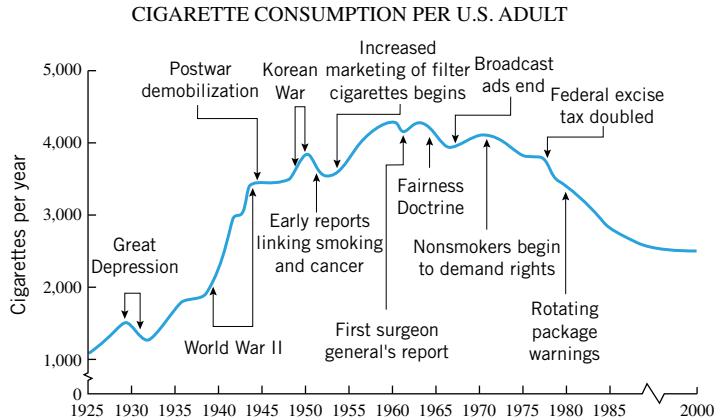
engraving is modern in spirit and provides a wealth of information. You should be able to extract the following information from Playfair's graphs:

- In the year 1700 imports were valued at about 70,000 pounds and exports at about 35,000 pounds.
- During the period from 1700 to about 1754 imports exceeded exports (a trade deficit for England).
- In the year 1754 the imports and exports were equal (a trade balance in today's economic terminology).
- From 1754 to 1780 exports exceeded imports (a trade surplus for England). The greatest surplus occurred in 1780, at which time exports exceeded imports by about 95,000 pounds.
- During the period from 1700 to 1725 imports were rising. They peaked in 1725, and then slowly fell until about 1760, at which time they bottomed out and began to rise again slowly until 1780.
- During the period from 1760 to 1780 exports and imports were both rising, but exports were rising more rapidly than imports, resulting in an ever-widening trade surplus for England.



Playfair's Graph of 1786: The horizontal scale is in years from 1700 to 1780 and the vertical scale is in units of 1,000 pounds sterling from 0 to 200.

(a)



Source: U.S. Department of Health and Human Services.

(b)

Figure 1.1.2

Figure 1.1.2b is a more contemporary graph; it describes the per capita consumption of cigarettes in the United States between 1925 and 1995.

- **FOR THE READER.** Use the graph in Figure 1.1.2b to provide reasonable answers to the following questions:

- When did the maximum annual cigarette consumption per adult occur and how many were consumed?
- What factors are likely to cause sharp decreases in cigarette consumption?
- What factors are likely to cause sharp increases in cigarette consumption?
- What were the long- and short-term effects of the first surgeon general's report on the health risks of smoking?

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GRAPHS OF EQUATIONS

Graphs can be used to describe mathematical equations as well as physical data. For example, consider the equation

$$y = x\sqrt{9 - x^2} \quad (1)$$

For each value of x in the interval $-3 \leq x \leq 3$, this equation produces a corresponding real value of y , which is obtained by substituting the value of x into the right side of the equation. Some typical values are shown in Table 1.1.2.

Table 1.1.2

x	-3	-2	-1	0	1	2	3
y	0	$-2\sqrt{5} \approx -4.47214$	$-2\sqrt{2} \approx -2.82843$	0	$2\sqrt{2} \approx 2.82843$	$2\sqrt{5} \approx 4.47214$	0

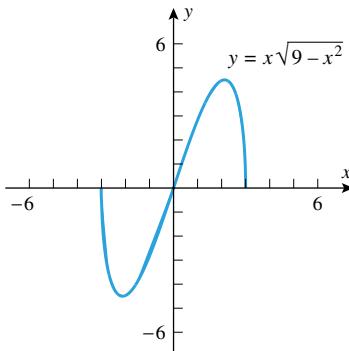


Figure 1.1.3

The set of *all* points in the xy -plane whose coordinates satisfy an equation in x and y is called the **graph** of that equation in the xy -plane. Figure 1.1.3 shows the graph of Equation (1) in the xy -plane. Notice that the graph extends only over the interval $[-3, 3]$. This is because values of x outside of this interval produce complex values of y , and in these cases the ordered pairs (x, y) do not correspond to points in the xy -plane. For example, if $x = 8$, then the corresponding value of y is $y = 8\sqrt{-55} = 8\sqrt{55}i$, and the ordered pair $(8, 8\sqrt{55}i)$ is not a point in the xy -plane.

Example 1 Figure 1.1.4 shows the graph of an unspecified equation that was used to obtain the values that appear in the shaded parts of the accompanying tables. Examine the graph and confirm that the values in the tables are reasonable approximations. 

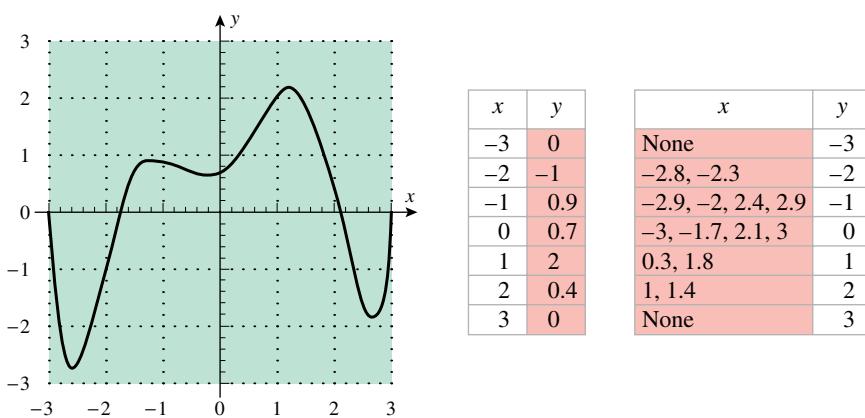


Figure 1.1.4

FUNCTIONS

Tables, graphs, and equations provide three methods for describing how one quantity depends on another—numerical, visual, and algebraic. The fundamental importance of this idea was recognized by Leibniz in 1673 when he coined the term *function* to describe the dependence of one quantity on another. The following examples illustrate how this term is used:

- The area A of a circle depends on its radius r by the equation $A = \pi r^2$, so we say that A is a function of r .

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- The velocity v of a ball falling freely in the Earth's gravitational field increases with time t until it hits the ground, so we say that v is a function of t .
- In a bacteria culture, the number n of bacteria present after 1 hour of growth depends on the number n_0 of bacteria present initially, so we say that n is a function of n_0 .

This idea is captured in the following definition.

1.1.1 DEFINITION. If a variable y depends on a variable x in such a way that each value of x determines exactly one value of y , then we say that y is a function of x .

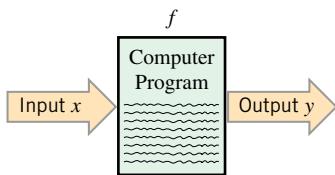


Figure 1.1.5

In the mid-eighteenth century the Swiss mathematician Leonhard Euler^{*} (pronounced "oiler") conceived the idea of denoting functions by letters of the alphabet, thereby making it possible to describe functions without stating specific formulas, graphs, or tables. To understand Euler's idea, think of a function as a computer program that takes an *input* x , operates on it in some way, and produces exactly one *output* y . The computer program is an object in its own right, so we can give it a name, say f . Thus, the function f (the computer program) associates a unique output y with each input x (Figure 1.1.5). This suggests the following definition.

1.1.2 DEFINITION. A *function* f is a rule that associates a unique output with each input. If the input is denoted by x , then the output is denoted by $f(x)$ (read " f of x ").

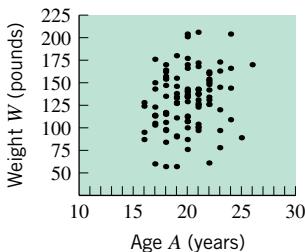


Figure 1.1.6

- REMARK.** In this definition the term *unique* means "exactly one." Thus, a function cannot assign two different outputs to the same input. For example, Figure 1.1.6 shows a scatter plot of weight versus age for a random sample of 100 college students. This scatter plot does not describe the weight W as a function of the age A because there are some values of A with more than one corresponding value of W . This is to be expected, since two people with the same age need not have the same weight. In contrast, Table 1.1.1 describes S as a function of t because there is only one top qualifying speed in a given year; similarly, Equation (1) describes y as a function of x because each input x in the interval $-3 \leq x \leq 3$ produces exactly one output $y = x\sqrt{9 - x^2}$.

* **LEONHARD EULER** (1707–1783). Euler was probably the most prolific mathematician who ever lived. It has been said that "Euler wrote mathematics as effortlessly as most men breathe." He was born in Basel, Switzerland, and was the son of a Protestant minister who had himself studied mathematics. Euler's genius developed early. He attended the University of Basel, where by age 16 he obtained both a Bachelor of Arts degree and a Master's degree in philosophy. While at Basel, Euler had the good fortune to be tutored one day a week in mathematics by a distinguished mathematician, Johann Bernoulli. At the urging of his father, Euler then began to study theology. The lure of mathematics was too great, however, and by age 18 Euler had begun to do mathematical research. Nevertheless, the influence of his father and his theological studies remained, and throughout his life Euler was a deeply religious, unaffected person. At various times Euler taught at St. Petersburg Academy of Sciences (in Russia), the University of Basel, and the Berlin Academy of Sciences. Euler's energy and capacity for work were virtually boundless. His collected works form more than 100 quarto-sized volumes and it is believed that much of his work has been lost. What is particularly astonishing is that Euler was blind for the last 17 years of his life, and this was one of his most productive periods! Euler's flawless memory was phenomenal. Early in his life he memorized the entire *Aeneid* by Virgil and at age 70 could not only recite the entire work, but could also state the first and last sentence on each page of the book from which he memorized the work. His ability to solve problems in his head was beyond belief. He worked out in his head major problems of lunar motion that baffled Isaac Newton and once did a complicated calculation in his head to settle an argument between two students whose computations differed in the fiftieth decimal place.

Following the development of calculus by Leibniz and Newton, results in mathematics developed rapidly in a disorganized way. Euler's genius gave coherence to the mathematical landscape. He was the first mathematician to bring the full power of calculus to bear on problems from physics. He made major contributions to virtually every branch of mathematics as well as to the theory of optics, planetary motion, electricity, magnetism, and general mechanics.

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WAYS TO DESCRIBE FUNCTIONS

Functions can be represented in various ways:

- Numerically by tables
- Geometrically by graphs
- Algebraically by formulas
- Verbally

The method of representation often depends on how the function arises. For example:

- Table 1.1.1 is a numerical representation of S as a function of t . This is the natural way in which data of this type are recorded.
- Figure 1.1.7 shows a record of the amount of deflection D of a seismograph needle during an earthquake. The variable D is a function of the time t that has elapsed since the shock wave left the earthquake's epicenter. In this case the function originates as a graph.
- Some of the most familiar examples of functions arise as formulas; for example, the formula $C = 2\pi r$ expresses the circumference C of a circle as a function of its radius r .
- Sometimes functions are described in words. For example, Isaac Newton's Law of Universal Gravitation is often stated as follows: The gravitational force of attraction between two bodies in the Universe is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. This is the verbal description of the formula

$$F = G \frac{m_1 m_2}{r^2} \quad (2)$$

in which F is the force of attraction, m_1 and m_2 are the masses, r is the distance between them, and G is a constant.

- We will see later that functions can also arise through limiting processes, some of which we discussed informally in the Introduction.

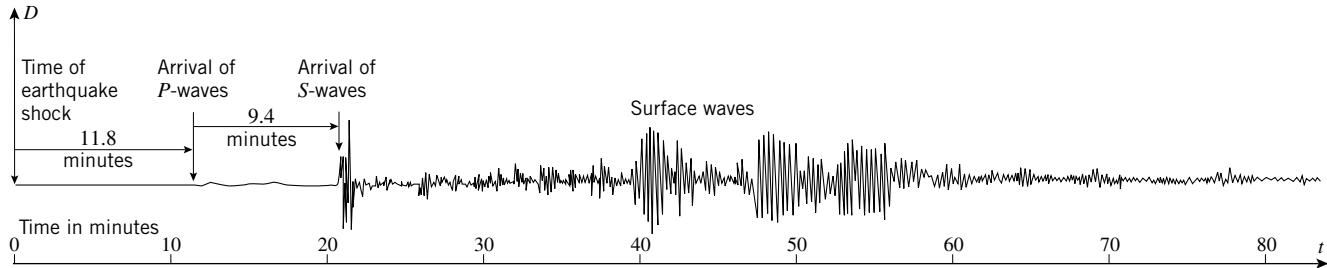


Figure 1.1.7

Sometimes it is desirable to convert one representation of a function into another. For example, in Figure 1.1.1 we converted the numerical relationship between S and t into a graphical relationship, and in writing Formula (2) we converted the verbal representation of the Law of Universal Gravitation into an algebraic relationship.

The problem of converting numerical representations of functions into algebraic formulas often requires special techniques known as **curve fitting**. For example, Table 1.1.3 gives the U.S. population at 10-year intervals from 1790 to 1850. This table is a numerical representation of the function $P = f(t)$ that relates the U.S. population P to the year t . If we plot P versus t , we obtain the scatter plot in Figure 1.1.8a, and if we use curve-fitting methods

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Table 1.1.3
U.S. POPULATION

YEAR t	POPULATION P (millions)
1790	3.9
1800	5.3
1810	7.2
1820	9.6
1830	12
1840	17
1850	23

Source: *The World Almanac*.

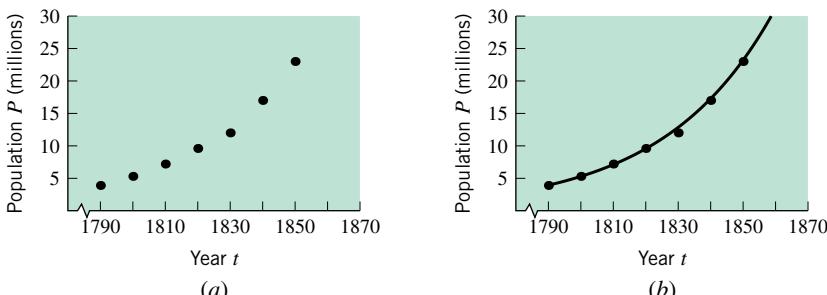


Figure 1.1.8

that will be discussed later, we can obtain the approximation

$$P \approx 3.94(1.03)^{t-1790}$$

Figure 1.1.8b shows the graph of this equation imposed on the scatter plot.

DISCRETE VERSUS CONTINUOUS DATA

Engineers and physicists distinguish between **continuous data** and **discrete data**. Continuous data have values that vary *continuously* over an interval, whereas discrete data have values that make *discrete* jumps. For example, for the seismic data in Figure 1.1.7 both the time and intensity vary continuously, whereas in Table 1.1.3 and Figure 1.1.8a both the year and population make discrete jumps. As a rule, continuous data lead to graphs that are continuous, unbroken curves, whereas discrete data lead to scatter plots consisting of isolated points. Sometimes, as in Figure 1.1.8b, it is desirable to approximate a scatter plot by a continuous curve. This is useful for making conjectures about the values of the quantities between the recorded data points.

GRAPHS AS PROBLEM-SOLVING TOOLS

Sometimes a function is buried in the statement of a problem, and it is up to the problem solver to uncover it and use it in an appropriate way to solve the problem. Here is an example that illustrates the power of graphical representations of functions as a problem-solving tool.

Example 2 Figure 1.1.9a shows an offshore oil well located at a point W that is 5 km from the closest point A on a straight shoreline. Oil is to be piped from W to a shore point B that is 8 km from A . It costs \$1,000,000/km to lay pipe under water and \$500,000/km over land. In your role as project manager you receive three proposals for piping the oil from W to B . Proposal 1 claims that it is cheapest to pipe directly from W to B , since the shortest distance between two points is a straight line. Proposal 2 claims that it is cheapest to pipe directly to point A and then along the shoreline to B , thereby using the least amount of expensive underwater pipe. Proposal 3 claims that it is cheapest to compromise by piping under water

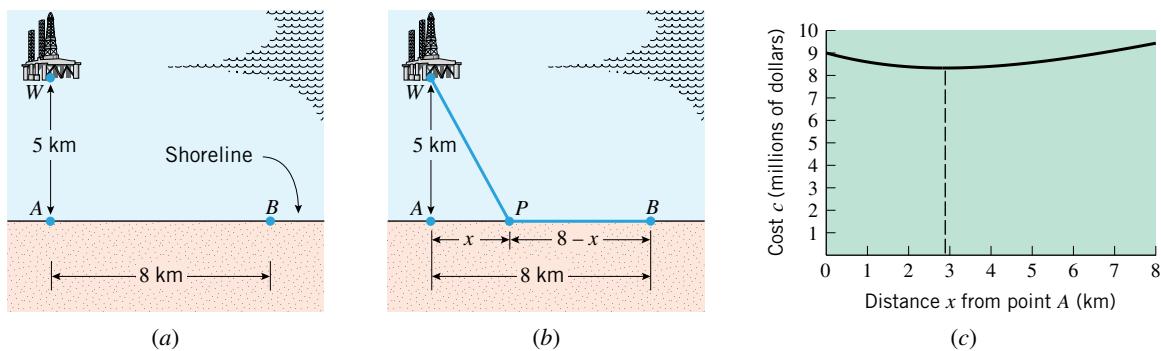


Figure 1.1.9

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to some well-chosen point between A and B , and then piping along the shoreline to B . Which proposal is correct?

Solution. Let P be any point between A and B (Figure 1.1.9b), and let

$$x = \text{distance (in kilometers) between } A \text{ and } P$$

$$c = \text{cost (in millions of dollars) for the entire pipeline}$$

Proposal 1 claims that $x = 8$ results in the least cost, Proposal 2 claims that it is $x = 0$, and Proposal 3 claims it is some value of x between 0 and 8. From Figure 1.1.9b the length of pipe along the shore is

$$8 - x \quad (3)$$

and from the Theorem of Pythagoras, the length of pipe under water is

$$\sqrt{x^2 + 25} \quad (4)$$

Thus, from (3) and (4) the total cost c (in millions of dollars) for the pipeline is

$$c = 1(\sqrt{x^2 + 25}) + 0.5(8 - x) = \sqrt{x^2 + 25} + 0.5(8 - x) \quad (5)$$

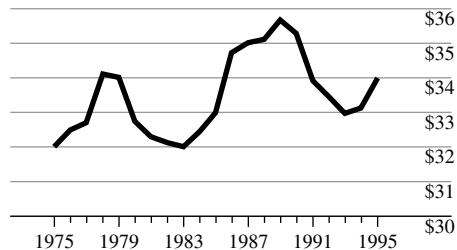
where $0 \leq x \leq 8$. The graph of Equation (5), shown in Figure 1.1.9c, makes it clear that Proposal 3 is correct—the most cost-effective strategy is to pipe to a point a little less than 3 km from point A . ◀

EXERCISE SET 1.1 Graphing Calculator

- Use the cigarette consumption graph in Figure 1.1.2b to answer the following questions, making reasonable approximations where needed.
 - When did the annual cigarette consumption reach 3000 per adult for the first time?
 - When did the annual cigarette consumption per adult reach its peak, and what was the peak value?
 - Can you tell from the graph how many cigarettes were consumed in a given year? If not, what additional information would you need to make that determination?
 - What factors are likely to cause a sharp increase in annual cigarette consumption per adult?
 - What factors are likely to cause a sharp decline in annual cigarette consumption per adult?
- The accompanying graph shows the median income in U.S. households (adjusted for inflation) between 1975 and 1995. Use the graph to answer the following questions, making reasonable approximations where needed.
 - When did the median income reach its maximum value, and what was the median income when that occurred?
 - When did the median income reach its minimum value, and what was the median income when that occurred?
 - The median income was declining during the 4-year period between 1989 and 1993. Was it declining more

rapidly during the first 2 years or the second 2 years of that period? Explain your reasoning.

MEDIAN U.S. HOUSEHOLD INCOME IN THOUSANDS OF CONSTANT 1995 DOLLARS



Source: Census Bureau, March 1996
[1996 measures 1995 income].

Figure Ex-2

- Use the accompanying graph to answer the following questions, making reasonable approximations where needed.
 - For what values of x is $y = 1$?
 - For what values of x is $y = 3$?
 - For what values of y is $x = 3$?
 - For what values of x is $y \leq 0$?
 - What are the maximum and minimum values of y and for what values of x do they occur?

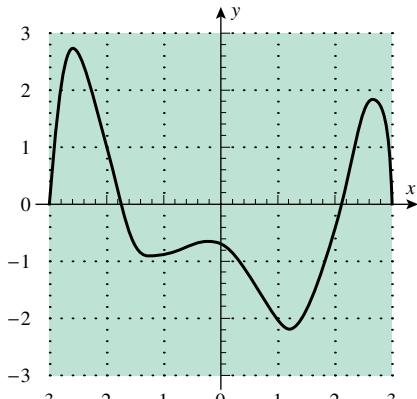
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Figure Ex-3

4. Use the accompanying table to answer the questions posed in Exercise 3.

x	-2	-1	0	2	3	4	5	6
y	5	1	-2	7	-1	1	0	9

Table Ex-4

5. Use the equation $y = x^2 - 6x + 8$ to answer the following questions.
- For what values of x is $y = 0$?
 - For what values of x is $y = -10$?
 - For what values of x is $y \geq 0$?
 - Does y have a minimum value? A maximum value? If so, find them.
6. Use the equation $y = 1 + \sqrt{x}$ to answer the following questions.
- For what values of x is $y = 4$?
 - For what values of x is $y = 0$?
 - For what values of x is $y \geq 6$?
 - Does y have a minimum value? A maximum value? If so, find them.
7. (a) If you had a device that could record the Earth's population continuously, would you expect the graph of population versus time to be a continuous (unbroken) curve? Explain what might cause breaks in the curve.
- (b) Suppose that a hospital patient receives an injection of an antibiotic every 8 hours and that between injections the concentration C of the antibiotic in the bloodstream decreases as the antibiotic is absorbed by the tissues. What might the graph of C versus the elapsed time t look like?
8. (a) If you had a device that could record the temperature of a room continuously over a 24-hour period, would you expect the graph of temperature versus time to be a continuous (unbroken) curve? Explain your reasoning.
- (b) If you had a computer that could track the number of boxes of cereal on the shelf of a market continuously

over a 1-week period, would you expect the graph of the number of boxes on the shelf versus time to be a continuous (unbroken) curve? Explain your reasoning.

9. A construction company wants to build a rectangular enclosure with an area of 1000 square feet by fencing in three sides and using its office building as the fourth side. Your objective as supervising engineer is to design the enclosure so that it uses the least amount of fencing. Proceed as follows.
- Let x and y be the dimensions of the enclosure, where x is measured parallel to the building, and let L be the length of fencing required for those dimensions. Since the area must be 1000 square feet, we must have $xy = 1000$. Find a formula for L in terms of x and y , and then express L in terms of x alone by using the area equation.
 - Are there any restrictions on the value of x ? Explain.
 - Make a graph of L versus x over a reasonable interval, and use the graph to estimate the value of x that results in the smallest value of L .
 - Estimate the smallest value of L .
10. A manufacturer constructs open boxes from sheets of cardboard that are 6 inches square by cutting small squares from the corners and folding up the sides (as shown in the accompanying figure). The Research and Development Department asks you to determine the size of the square that produces a box of greatest volume. Proceed as follows.
- Let x be the length of a side of the square to be cut, and let V be the volume of the resulting box. Show that $V = x(6 - 2x)^2$.
 - Are there any restrictions on the value of x ? Explain.
 - Make a graph of V versus x over an appropriate interval, and use the graph to estimate the value of x that results in the largest volume.
 - Estimate the largest volume.

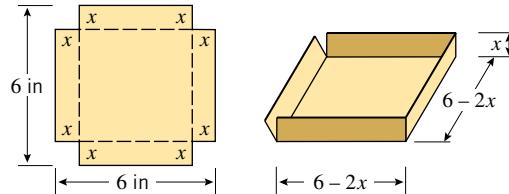


Figure Ex-10

11. A soup company wants to manufacture a can in the shape of a right circular cylinder that will hold 500 cm^3 of liquid. The material for the top and bottom costs 0.02 cent/ cm^2 , and the material for the sides costs 0.01 cent/ cm^2 .
- Use the method of Exercises 9 and 10 to estimate the radius r and height h of the can that costs the least to manufacture. [Suggestion: Express the cost C in terms of r .]
 - Suppose that the tops and bottoms of radius r are punched out from square sheets with sides of length $2r$ and the scraps are waste. If you allow for the cost of

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the waste, would you expect the can of least cost to be taller or shorter than the one in part (a)? Explain.

- (c) Estimate the radius, height, and cost of the can in part (b), and determine whether your conjecture was correct.

12. The designer of a sports facility wants to put a quarter-mile (1320 ft) running track around a football field, oriented as in the accompanying figure. The football field is 360 ft long (including the end zones) and 160 ft wide. The track consists of two straightaways and two semicircles, with the straightaways extending at least the length of the football field.
- (a) Show that it is possible to construct a quarter-mile track around the football field. [Suggestion: Find the shortest track that can be constructed around the field.]
- (b) Let L be the length of a straightaway (in feet), and let x be the distance (in feet) between a sideline of the

football field and a straightaway. Make a graph of L versus x .

- (c) Use the graph to estimate the value of x that produces the shortest straightaways, and then find this value of x exactly.
- (d) Use the graph to estimate the length of the longest possible straightaways, and then find that length exactly.

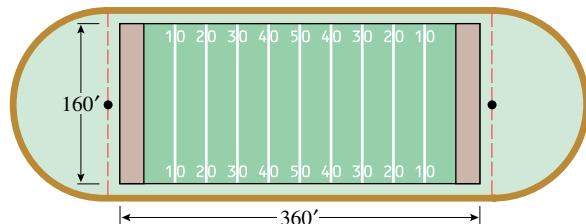


Figure Ex-12

1.2 PROPERTIES OF FUNCTIONS

In this section we will explore properties of functions in more detail. We will assume that you are familiar with the standard notation for intervals and the basic properties of absolute value. Reviews of these topics are provided in Appendices A and B.

INDEPENDENT AND DEPENDENT VARIABLES

Table 1.2.1

x	0	1	2	3
y	3	4	-1	6

Recall from the last section that a function f is a rule that associates a unique output $f(x)$ with each input x . This output is sometimes called the **value** of f at x or the **image** of x under f . Sometimes we will want to denote the output by a single letter, say y , and write

$$y = f(x)$$

This equation expresses y as a function of x ; the variable x is called the **independent variable** (or **argument**) of f , and the variable y is called the **dependent variable** of f . This terminology is intended to suggest that x is free to vary, but that once x has a specific value a corresponding value of y is determined. For now we will only consider functions in which the independent and dependent variables are real numbers, in which case we say that f is a **real-valued function of a real variable**. Later, we will consider other kinds of functions as well.

Table 1.2.1 can be viewed as a numerical representation of a function of f . For this function we have

$f(0) = 3$	f associates $y = 3$ with $x = 0$.
$f(1) = 4$	f associates $y = 4$ with $x = 1$.
$f(2) = -1$	f associates $y = -1$ with $x = 2$.
$f(3) = 6$	f associates $y = 6$ with $x = 3$.

To illustrate how functions can be defined by equations, consider

$$y = 3x^2 - 4x + 2 \tag{1}$$

This equation has the form $y = f(x)$, where

$$f(x) = 3x^2 - 4x + 2 \tag{2}$$

The outputs of f (the y -values) are obtained by substituting numerical values for x in this formula. For example,

$$f(0) = 3(0)^2 - 4(0) + 2 = 2$$

f associates $y = 2$ with $x = 0$.

$$f(-1.7) = 3(-1.7)^2 - 4(-1.7) + 2 = 17.47$$

f associates $y = 17.47$ with $x = -1.7$.

$$f(\sqrt{2}) = 3(\sqrt{2})^2 - 4\sqrt{2} + 2 = 8 - 4\sqrt{2}$$

f associates $y = 8 - 4\sqrt{2}$ with $x = \sqrt{2}$.

1.2 Properties of Functions 17

REMARK. Although f , x , and y are the most common notations for functions and variables, any letters can be used. For example, to indicate that the area A of a circle is a function of the radius r , it would be more natural to write $A = f(r)$ [where $f(r) = \pi r^2$]. Similarly, to indicate that the circumference C of a circle is a function of the radius r , we might write $C = g(r)$ [where $g(r) = 2\pi r$]. The area function and the circumference function are different, which is why we denoted them by different letters, f and g .

DOMAIN AND RANGE

If $y = f(x)$, then the set of all possible inputs (x -values) is called the **domain** of f , and the set of outputs (y -values) that result when x varies over the domain is called the **range** of f . For example, consider the equations

$$y = x^2 \quad \text{and} \quad y = x^2, \quad x \geq 2$$

In the first equation there is no restriction on x , so we may assume that any real value of x is an allowable input. Thus, the equation defines a function $f(x) = x^2$ with domain $-\infty < x < +\infty$. In the second equation, the inequality $x \geq 2$ restricts the allowable inputs to be greater than or equal to 2, so the equation defines a function $g(x) = x^2, x \geq 2$ with domain $2 \leq x < +\infty$.

As x varies over the domain of the function $f(x) = x^2$, the values of $y = x^2$ vary over the interval $0 \leq y < +\infty$, so this is the range of f . By comparison, as x varies over the domain of the function $g(x) = x^2, x \geq 2$, the values of $y = x^2$ vary over the interval $4 \leq y < +\infty$, so this is the range of g .

It is important to understand here that even though $f(x) = x^2$ and $g(x) = x^2, x \geq 2$ involve the same formula, we regard them to be different functions because they have different domains. In short, *to fully describe a function you must not only specify the rule that relates the inputs and outputs, but you must also specify the domain, that is, the set of allowable inputs.*

GRAPHS OF FUNCTIONS

If f is a real-valued function of a real variable, then the **graph** of f in the xy -plane is defined to be the graph of the equation $y = f(x)$. For example, the graph of the function $f(x) = x$ is the graph of the equation $y = x$, shown in Figure 1.2.1. That figure also shows the graphs

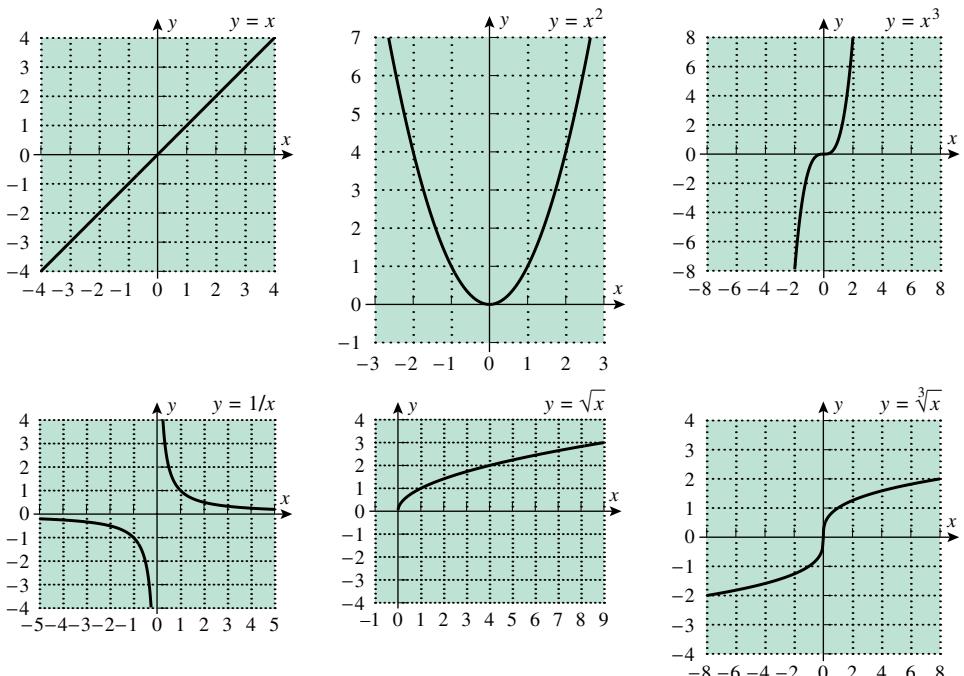


Figure 1.2.1

18 Functions

of some other basic functions that may already be familiar to you. Later in this chapter we will discuss techniques for graphing functions using graphing calculators and computers.

Graphs can provide useful visual information about a function. For example, because the graph of a function f in the xy -plane consists of all points whose coordinates satisfy the equation $y = f(x)$, the points on the graph of f are of the form $(x, f(x))$; hence each y -coordinate is the value of f at the x -coordinate (Figure 1.2.2a). Pictures of the domain and range of f can be obtained by projecting the graph of f onto the coordinate axes (Figure 1.2.2b). The values of x for which $f(x) = 0$ are the x -coordinates of the points where the graph of f intersects the x -axis (Figure 1.2.2c); these values of x are called the **zeros** of f , the **roots** of $f(x) = 0$, or the **x -intercepts** of $y = f(x)$.

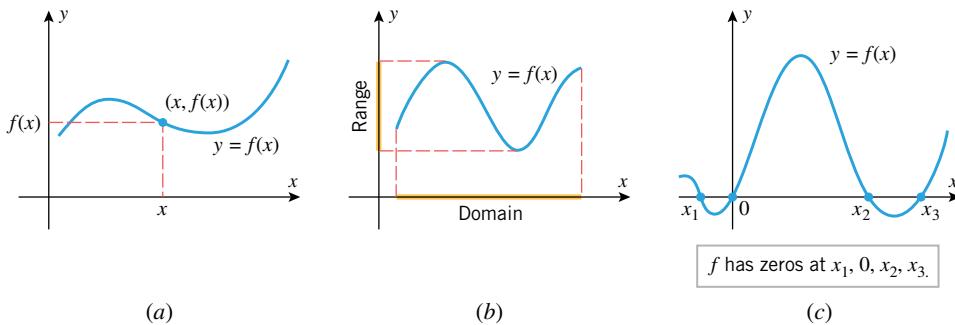


Figure 1.2.2

THE VERTICAL LINE TEST

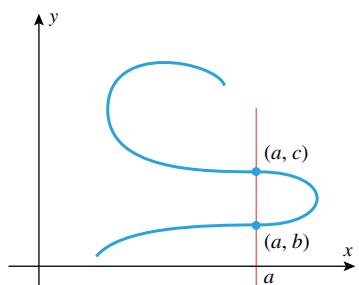


Figure 1.2.3

Not every curve in the xy -plane is the graph of a function. For example, consider the curve in Figure 1.2.3, which is cut at two distinct points, (a, b) and (a, c) , by a vertical line. This curve cannot be the graph of $y = f(x)$ for any function f ; otherwise, we would have

$$f(a) = b \quad \text{and} \quad f(a) = c$$

which is impossible, since f cannot assign two different values to a . Thus, there is no function f whose graph is the given curve. This illustrates the following general result, which we will call the **vertical line test**.

1.2.1 THE VERTICAL LINE TEST. *A curve in the xy -plane is the graph of some function f if and only if no vertical line intersects the curve more than once.*

Example 1 The graph of the equation

$$x^2 + y^2 = 25 \tag{3}$$

is a circle of radius 5, centered at the origin (see Appendix D for a review of circles), and hence there are vertical lines that cut the graph more than once. This can also be seen algebraically by solving (3) for y in terms of x :

$$y = \pm\sqrt{25 - x^2}$$

This equation does not define y as a function of x because the right side is “multiple valued” in the sense that values of x in the interval $(-5, 5)$ produce two corresponding values of y . For example, if $x = 4$, then $y = \pm 3$, and hence $(4, 3)$ and $(4, -3)$ are two points on the circle that lie on the same vertical line (Figure 1.2.4a). However, we can regard the circle as the union of two semicircles:

$$y = \sqrt{25 - x^2} \quad \text{and} \quad y = -\sqrt{25 - x^2}$$

(Figure 1.2.4b), each of which defines y as a function of x . ◀

1.2 Properties of Functions 19

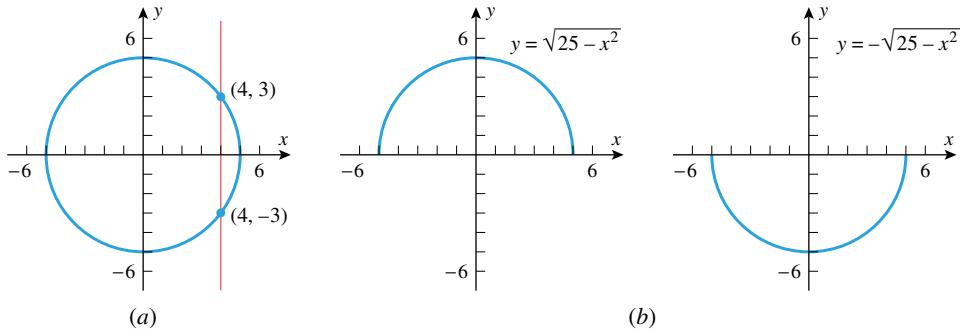


Figure 1.2.4

.....
THE ABSOLUTE VALUE FUNCTION

Recall that the **absolute value** or **magnitude** of a real number x is defined by

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

The effect of taking the absolute value of a number is to strip away the minus sign if the number is negative and to leave the number unchanged if it is nonnegative. Thus,

$$|5| = 5, \quad |-\frac{4}{7}| = \frac{4}{7}, \quad |0| = 0$$

A more detailed discussion of the properties of absolute value is given in Appendix B. However, for convenience we provide the following summary of its algebraic properties.

1.2.2 PROPERTIES OF ABSOLUTE VALUE. If a and b are real numbers, then

- (a) $|-a| = |a|$ A number and its negative have the same absolute value.
- (b) $|ab| = |a||b|$ The absolute value of a product is the product of the absolute values.
- (c) $|a/b| = |a|/|b|$ The absolute value of a ratio is the ratio of the absolute values.
- (d) $|a + b| \leq |a| + |b|$ The triangle inequality

• **REMARK.** Symbols such as $+x$ and $-x$ are deceptive, since it is tempting to conclude that $+x$ is positive and $-x$ is negative. However, this need not be so, since x itself can be positive or negative. For example, if x is negative, say $x = -3$, then $-x = 3$ is positive and $+x = -3$ is negative.

The graph of the function $f(x) = |x|$ can be obtained by graphing the two parts of the equation

$$y = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

separately. For $x \geq 0$, the graph of $y = x$ is a ray of slope 1 with its endpoint at the origin, and for $x < 0$, the graph of $y = -x$ is a ray of slope -1 with its endpoint at the origin. Combining the two parts produces the V-shaped graph in Figure 1.2.5.

Absolute values have important relationships to square roots. To see why this is so, recall from algebra that every positive real number x has two square roots, one positive and one negative. By definition, the symbol \sqrt{x} denotes the *positive* square root of x . To denote the negative square root you must write $-\sqrt{x}$. For example, the positive square root of 9 is $\sqrt{9} = 3$, and the negative square root is $-\sqrt{9} = -3$. (Do not make the mistake of writing $\sqrt{9} = \pm 3$.)

Care must be exercised in simplifying expressions of the form $\sqrt{x^2}$, since it is *not* always true that $\sqrt{x^2} = x$. This equation is correct if x is nonnegative, but it is false for negative x . For example, if $x = -4$, then

$$\sqrt{x^2} = \sqrt{(-4)^2} = \sqrt{16} = 4 \neq x$$

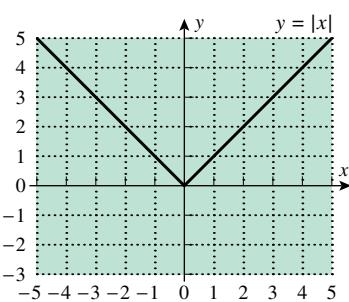


Figure 1.2.5

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A statement that is correct for all real values of x is

$$\sqrt{x^2} = |x|$$

- **FOR THE READER.** Verify this relationship by using a graphing utility to show that the equations $y = \sqrt{x^2}$ and $y = |x|$ have the same graph.

FUNCTIONS DEFINED PIECEWISE

The absolute value function $f(x) = |x|$ is an example of a function that is defined **piecewise** in the sense that the formula for f changes, depending on the value of x .

Example 2 Sketch the graph of the function defined piecewise by the formula

$$f(x) = \begin{cases} 0, & x \leq -1 \\ \sqrt{1-x^2}, & -1 < x < 1 \\ x, & x \geq 1 \end{cases}$$

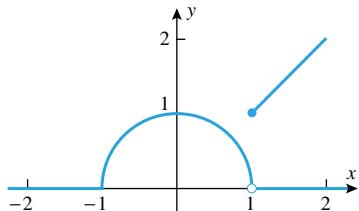


Figure 1.2.6

Solution. The formula for f changes at the points $x = -1$ and $x = 1$. (We call these the **breakpoints** for the formula.) A good procedure for graphing functions defined piecewise is to graph the function separately over the open intervals determined by the breakpoints, and then graph f at the breakpoints themselves. For the function f in this example the graph is the horizontal ray $y = 0$ on the interval $(-\infty, -1)$, it is the semicircle $y = \sqrt{1-x^2}$ on the interval $(-1, 1)$, and it is the ray $y = x$ on the interval $(1, +\infty)$. The formula for f specifies that the equation $y = 0$ applies at the breakpoint -1 [so $y = f(-1) = 0$], and it specifies that the equation $y = x$ applies at the breakpoint 1 [so $y = f(1) = 1$]. The graph of f is shown in Figure 1.2.6. ◀

- **REMARK.** In Figure 1.2.6 the solid dot and open circle at the breakpoint $x = 1$ serve to emphasize that the point on the graph lies on the ray and not the semicircle. There is no ambiguity at the breakpoint $x = -1$ because the two parts of the graph join together continuously there.

Example 3 Increasing the speed at which air moves over a person's skin increases the rate of moisture evaporation and makes the person feel cooler. (This is why we fan ourselves in hot weather.) The **windchill index** is the temperature at a wind speed of 4 mi/h that would produce the same sensation on exposed skin as the current temperature and wind speed combination. An empirical formula (i.e., a formula based on experimental data) for the windchill index W at 32°F for a wind speed of v mi/h is

$$W = \begin{cases} 32, & 0 \leq v \leq 4 \\ 91.4 + 59.4(0.0203v - 0.304\sqrt{v} - 0.474), & 4 < v < 45 \\ -3.8, & v \geq 45 \end{cases}$$

A computer-generated graph of $W(v)$ is shown in Figure 1.2.7. ◀

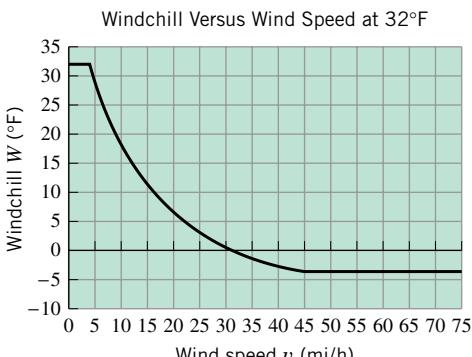


Figure 1.2.7

THE NATURAL DOMAIN

Sometimes, restrictions on the allowable values of an independent variable result from a mathematical formula that defines the function. For example, if $f(x) = 1/x$, then $x = 0$ must be excluded from the domain to avoid division by zero, and if $f(x) = \sqrt{x}$, then negative values of x must be excluded from the domain, since we are only considering real-valued functions of a real variable for now. We make the following definition.

1.2.3 DEFINITION. If a real-valued function of a real variable is defined by a formula, and if no domain is stated explicitly, then it is to be understood that the domain consists of all real numbers for which the formula yields a real value. This is called the *natural domain* of the function.

Example 4 Find the natural domain of

- (a) $f(x) = x^3$ (b) $f(x) = 1/[(x - 1)(x - 3)]$
 (c) $f(x) = \tan x$ (d) $f(x) = \sqrt{x^2 - 5x + 6}$

Solution (a). The function f has real values for all real x , so its natural domain is the interval $(-\infty, +\infty)$.

Solution (b). The function f has real values for all real x , except $x = 1$ and $x = 3$, where divisions by zero occur. Thus, the natural domain is

$$\{x : x \neq 1 \text{ and } x \neq 3\} = (-\infty, 1) \cup (1, 3) \cup (3, +\infty)$$

Solution (c). Since $f(x) = \tan x = \sin x / \cos x$, the function f has real values except where $\cos x = 0$, and this occurs when x is an odd integer multiple of $\pi/2$. Thus, the natural domain consists of all real numbers except

$$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots$$

Solution (d). The function f has real values, except when the expression inside the radical is negative. Thus the natural domain consists of all real numbers x such that

$$x^2 - 5x + 6 = (x - 3)(x - 2) \geq 0$$

This inequality is satisfied if $x \leq 2$ or $x \geq 3$ (verify), so the natural domain of f is

$$(-\infty, 2] \cup [3, +\infty)$$



REMARK. In some problems we will want to limit the domain of a function by imposing specific restrictions. For example, by writing

$$f(x) = x^2, \quad x \geq 0$$

we can limit the domain of f to the positive x -axis (Figure 1.2.8).

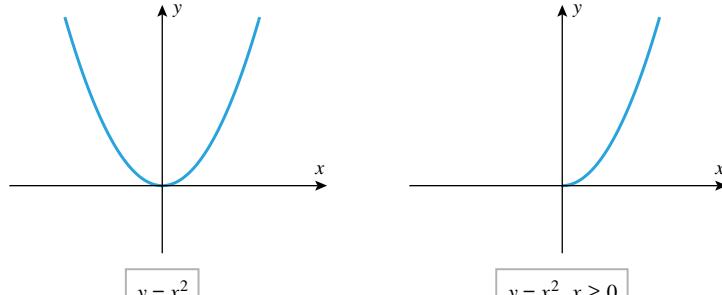


Figure 1.2.8

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THE EFFECT OF ALGEBRAIC OPERATIONS ON THE DOMAIN

Algebraic expressions are frequently simplified by canceling common factors in the numerator and denominator. However, care must be exercised when simplifying formulas for functions in this way, since this process can alter the domain.

Example 5 The natural domain of the function

$$f(x) = \frac{x^2 - 4}{x - 2}$$

consists of all real x except $x = 2$. However, if we factor the numerator and then cancel the common factor in the numerator and denominator, we obtain

$$f(x) = \frac{(x - 2)(x + 2)}{x - 2} = x + 2$$

which is defined at $x = 2$ [since $f(2) = 4$ for the altered function f]. Thus, the algebraic simplification has altered the domain of the function. Geometrically, the graph of $y = x + 2$ is a line of slope 1 and y -intercept 2, whereas the graph of $y = (x^2 - 4)/(x - 2)$ is the same line, but with a hole in it at $x = 2$, since y is undefined there (Figure 1.2.9). Thus, the geometric effect of the algebraic cancellation is to eliminate the hole in the original graph. In some situations such minor alterations in the domain are irrelevant to the problem under consideration and can be ignored. However, if we wanted to preserve the domain in this example, then we would express the simplified form of the function as

$$f(x) = x + 2, \quad x \neq 2$$

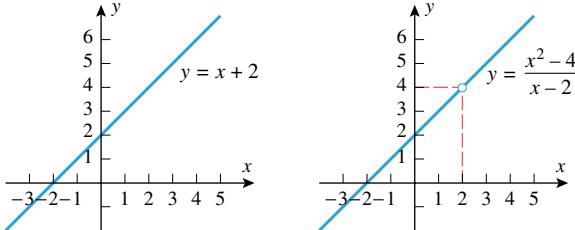


Figure 1.2.9

Example 6 Find the domain and range of

$$(a) f(x) = 2 + \sqrt{x - 1} \quad (b) f(x) = (x + 1)/(x - 1)$$

Solution (a). Since no domain is stated explicitly, the domain of f is the natural domain $[1, +\infty)$. As x varies over the interval $[1, +\infty)$, the value of $\sqrt{x - 1}$ varies over the interval $[0, +\infty)$, so the value of $f(x) = 2 + \sqrt{x - 1}$ varies over the interval $[2, +\infty)$, which is the range of f . The domain and range are shown graphically in Figure 1.2.10a.

Solution (b). The given function f is defined for all real x , except $x = 1$, so the natural domain of f is

$$\{x : x \neq 1\} = (-\infty, 1) \cup (1, +\infty)$$

To determine the range it will be convenient to introduce a dependent variable

$$y = \frac{x + 1}{x - 1} \tag{4}$$

Although the set of possible y -values is not immediately evident from this equation, the graph of (4), which is shown in Figure 1.2.10b, suggests that the range of f consists of all

1.2 Properties of Functions 23

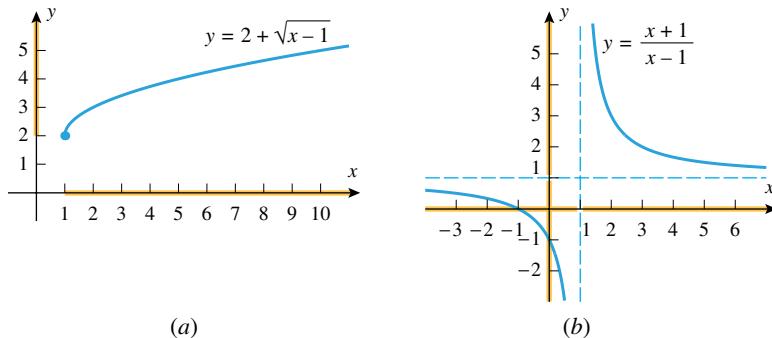


Figure 1.2.10

y , except $y = 1$. To see that this is so, we solve (4) for x in terms of y :

$$\begin{aligned} (x - 1)y &= x + 1 \\ xy - y &= x + 1 \\ xy - x &= y + 1 \\ x(y - 1) &= y + 1 \\ x &= \frac{y + 1}{y - 1} \end{aligned}$$

It is now evident from the right side of this equation that $y = 1$ is not in the range; otherwise we would have a division by zero. No other values of y are excluded by this equation, so the range of the function f is $\{y : y \neq 1\} = (-\infty, 1) \cup (1, +\infty)$, which agrees with the result obtained graphically. \blacktriangleleft

DOMAIN AND RANGE IN APPLIED PROBLEMS

In applications, physical considerations often impose restrictions on the domain and range of a function.

Example 7 An open box is to be made from a 16-inch by 30-inch piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides (Figure 1.2.11a).

- (a) Let V be the volume of the box that results when the squares have sides of length x . Find a formula for V as a function of x .
- (b) Find the domain of V .
- (c) Use the graph of V given in Figure 1.2.11c to estimate the range of V .
- (d) Describe in words what the graph tells you about the volume.

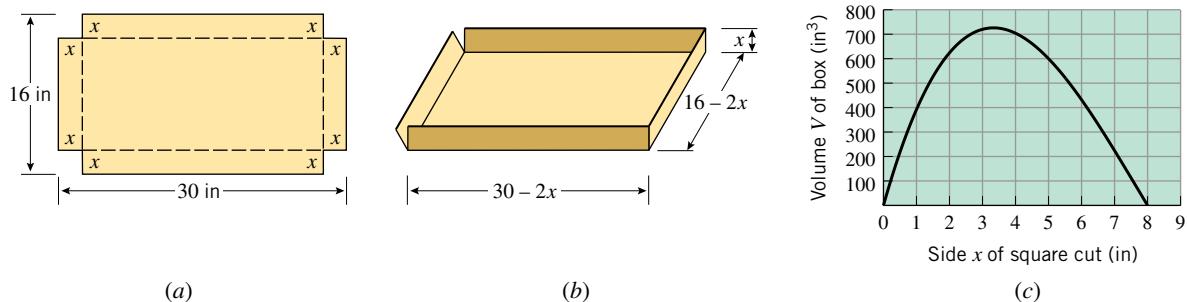


Figure 1.2.11

24 Functions

Solution (a). As shown in Figure 1.2.11b, the resulting box has dimensions $16 - 2x$ by $30 - 2x$ by x , so the volume $V(x)$ is given by

$$V(x) = (16 - 2x)(30 - 2x)x = 480x - 92x^2 + 4x^3$$

Solution (b). The domain is the set of x -values and the range is the set of V -values. Because x is a length, it must be nonnegative, and because we cannot cut out squares whose sides are more than 8 in long (why?), the x -values in the domain must satisfy

$$0 \leq x \leq 8$$

Solution (c). From the graph of V versus x in Figure 1.2.11c we estimate that the V -values in the range satisfy

$$0 \leq V \leq 725$$

Note that this is an approximation. Later we will show how to find the range exactly.

Solution (d). The graph tells us that the box of maximum volume occurs for a value of x that is between 3 and 4 and that the maximum volume is approximately 725 in³. Moreover, the volume decreases toward zero as x gets closer to 0 or 8. ◀

In applications involving time, formulas for functions are often expressed in terms of a variable t whose starting value is taken to be $t = 0$.

Example 8 At 8:05 A.M. a car is clocked at 100 ft/s by a radar detector that is positioned at the edge of a straight highway. Assuming that the car maintains a constant speed between 8:05 A.M. and 8:06 A.M., find a function $D(t)$ that expresses the distance traveled by the car during that time interval as a function of the time t .

Solution. It would be clumsy to use clock time for the variable t , so let us agree to measure the elapsed time in seconds, starting with $t = 0$ at 8:05 A.M. and ending with $t = 60$ at 8:06 A.M. At each instant, the distance traveled (in ft) is equal to the speed of the car (in ft/s) multiplied by the elapsed time (in s). Thus,

$$D(t) = 100t, \quad 0 \leq t \leq 60$$

The graph of D versus t is shown in Figure 1.2.12. ◀

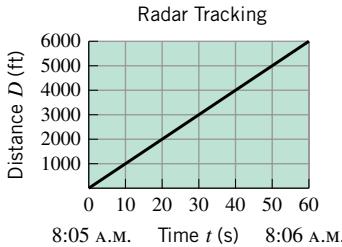
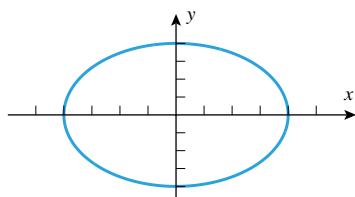


Figure 1.2.12

ISSUES OF SCALE AND UNITS



The circle is squashed because 1 unit on the y -axis has a smaller length than 1 unit on the x -axis.

Figure 1.2.13

In geometric problems where you want to preserve the “true” shape of a graph, you must use units of equal length on both axes. For example, if you graph a circle in a coordinate system in which 1 unit in the y -direction is smaller than 1 unit in the x -direction, then the circle will be squashed vertically into an elliptical shape (Figure 1.2.13). You must also use units of equal length when you want to apply the distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

to calculate the distance between two points (x_1, y_1) and (x_2, y_2) in the xy -plane.

However, sometimes it is inconvenient or impossible to display a graph using units of equal length. For example, consider the equation

$$y = x^2$$

If we want to show the portion of the graph over the interval $-3 \leq x \leq 3$, then there is no problem using units of equal length, since y only varies from 0 to 9 over that interval. However, if we want to show the portion of the graph over the interval $-10 \leq x \leq 10$, then there is a problem keeping the units equal in length, since the value of y varies between 0 and 100. In this case the only reasonable way to show all of the graph that occurs over the interval $-10 \leq x \leq 10$ is to compress the unit of length along the y -axis, as illustrated in Figure 1.2.14.

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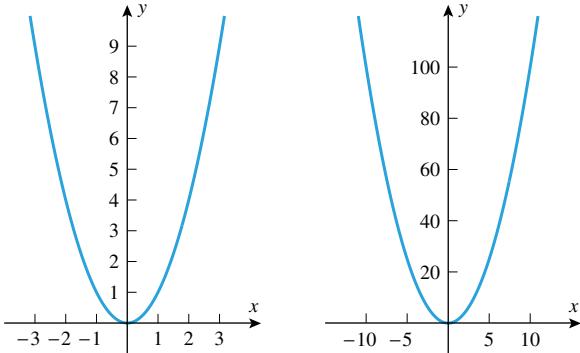


Figure 1.2.14

REMARK. In applications where the variables on the two axes have unrelated units (say, centimeters on the y -axis and seconds on the x -axis), then nothing is gained by requiring the units to have equal lengths; choose the lengths to make the graph as clear as possible.

EXERCISE SET 1.2 Graphing Calculator

1. Find $f(0)$, $f(2)$, $f(-2)$, $f(3)$, $f(\sqrt{2})$, and $f(3t)$.

(a) $f(x) = 3x^2 - 2$ (b) $f(x) = \begin{cases} \frac{1}{x}, & x > 3 \\ 2x, & x \leq 3 \end{cases}$

2. Find $g(3)$, $g(-1)$, $g(\pi)$, $g(-1.1)$, and $g(t^2 - 1)$.

(a) $g(x) = \frac{x+1}{x-1}$ (b) $g(x) = \begin{cases} \sqrt{x+1}, & x \geq 1 \\ 3, & x < 1 \end{cases}$

In Exercises 3–6, find the natural domain of the function algebraically, and confirm that your result is consistent with the graph produced by your graphing utility. [Note: Set your graphing utility to the radian mode when graphing trigonometric functions.]

3. (a) $f(x) = \frac{1}{x-3}$ (b) $g(x) = \sqrt{x^2 - 3}$

(c) $G(x) = \sqrt{x^2 - 2x + 5}$ (d) $f(x) = \frac{x}{|x|}$

(e) $h(x) = \frac{1}{1 - \sin x}$

4. (a) $f(x) = \frac{1}{5x+7}$ (b) $h(x) = \sqrt{x - 3x^2}$

(c) $G(x) = \sqrt{\frac{x^2 - 4}{x - 4}}$ (d) $f(x) = \frac{x^2 - 1}{x + 1}$

(e) $h(x) = \frac{3}{2 - \cos x}$

5. (a) $f(x) = \sqrt{3 - x}$

(c) $h(x) = 3 + \sqrt{x}$

(e) $H(x) = 3 \sin x$

(b) $g(x) = \sqrt{4 - x^2}$

(d) $G(x) = x^3 + 2$

6. (a) $f(x) = \sqrt{3x - 2}$

(c) $h(x) = \frac{1}{3 + \sqrt{x}}$

(e) $H(x) = \sin^2 \sqrt{x}$

(b) $g(x) = \sqrt{9 - 4x^2}$

(d) $G(x) = \frac{3}{x}$

7. In each part of the accompanying figure, determine whether the graph defines y as a function of x .

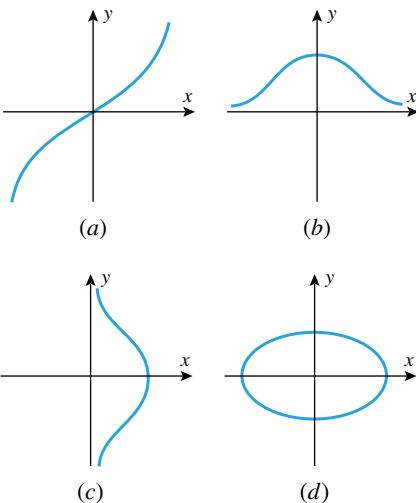


Figure Ex-7

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8. Express the length L of a chord of a circle with radius 10 cm as a function of the central angle θ (see the accompanying figure).

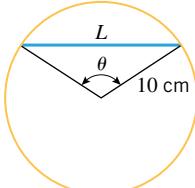


Figure Ex-8

9. As shown in the accompanying figure, a pendulum of constant length L makes an angle θ with its vertical position. Express the height h as a function of the angle θ .

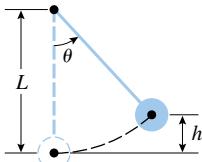


Figure Ex-9

10. A cup of hot coffee sits on a table. You pour in some cool milk and let it sit for an hour. Sketch a rough graph of the temperature of the coffee as a function of time.

11. A boat is bobbing up and down on some gentle waves. Suddenly it gets hit by a large wave and sinks. Sketch a rough graph of the height of the boat above the ocean floor as a function of time.

12. Make a rough sketch of your weight as a function of time from birth to the present.

In Exercises 13 and 14, express the function in piecewise form without using absolute values. [Suggestion: It may help to generate the graph of the function.]

13. (a) $f(x) = |x| + 3x + 1$ (b) $g(x) = |x| + |x - 1|$
 14. (a) $f(x) = 3 + |2x - 5|$ (b) $g(x) = 3|x - 2| - |x + 1|$

15. As shown in the accompanying figure, an open box is to be constructed from a rectangular sheet of metal, 8 inches by 15 inches, by cutting out squares with sides of length x from each corner and bending up the sides.

- (a) Express the volume V as a function of x .
(b) Find the natural domain and the range of the function, ignoring any physical restrictions on the values of the variables.
(c) Modify the domain and range appropriately to account for the physical restrictions on the values of V and x .
(d) In words, describe how the volume V of the box varies with x , and discuss how one might construct boxes of maximum volume and minimum volume.

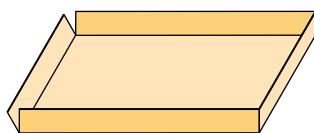
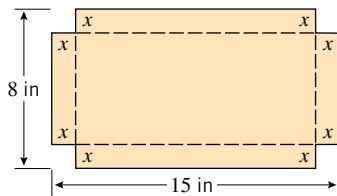


Figure Ex-15

16. As shown in the accompanying figure, a camera is mounted at a point 3000 ft from the base of a rocket launching pad. The shuttle rises vertically when launched, and the camera's elevation angle is continually adjusted to follow the bottom of the rocket.
- Choose letters to represent the height of the rocket and the elevation angle of the camera, and express the height as a function of the elevation angle.
 - Find the natural domain and the range of the function, ignoring any physical restrictions on the values of the variables.
 - Modify the domain and range appropriately to account for the physical restrictions on the values of the variables.
 - Generate the graph of height versus the elevation angle on a graphing utility, and use it to estimate the height of the rocket when the elevation angle is $\pi/4 \approx 0.7854$ radian. Compare this estimate to the exact height. [Suggestion: If you are using a graphing calculator, the trace and zoom features will be helpful here.]

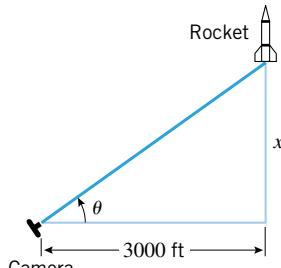


Figure Ex-16

In Exercises 17 and 18: (i) Explain why the function f has one or more holes in its graph, and state the x -values at which those holes occur. (ii) Find a function g whose graph is identical to that of f , but without the holes.

17. $f(x) = \frac{(x+2)(x^2-1)}{(x+2)(x-1)}$ 18. $f(x) = \frac{x+\sqrt{x}}{\sqrt{x}}$

19. For a given outside temperature T and wind speed v , the windchill index (WCI) is the equivalent temperature that

1.3 Graphing Functions on Calculators and Computers; Computer Algebra Systems **27**

exposed skin would feel with a wind speed of 4 mi/h. An empirical formula for the WCI (based on experience and observation) is

$$\text{WCI} = \begin{cases} T, & 0 \leq v \leq 4 \\ 91.4 + (91.4 - T)(0.0203v - 0.304\sqrt{v} - 0.474), & 4 < v < 45 \\ 1.6T - 55, & v \geq 45 \end{cases}$$

where T is the air temperature in $^{\circ}\text{F}$, v is the wind speed in mi/h, and WCI is the equivalent temperature in $^{\circ}\text{F}$. Find the WCI to the nearest degree if the air temperature is 25°F and

- (a) $v = 3$ mi/h (b) $v = 15$ mi/h
 (c) $v = 46$ mi/h.

[Adapted from UMAP Module 658, *Windchill*, W. Bosch and L. Cobb, COMAP, Arlington, MA.]

In Exercises 20–22, use the formula for the windchill index described in Exercise 19.

20. Find the air temperature to the nearest degree if the WCI is reported as -60°F with a wind speed of 48 mi/h.
21. Find the air temperature to the nearest degree if the WCI is reported as -10°F with a wind speed of 8 mi/h.
22. Find the wind speed to the nearest mile per hour if the WCI is reported as -15°F with an air temperature of 20°F .
23. At 9:23 A.M. a lunar lander that is 1000 ft above the Moon's surface begins a vertical descent, touching down at 10:13 A.M. Assuming that the lander maintains a constant speed, find a function $D(t)$ that expresses the altitude of the lander above the Moon's surface as a function of t .

1.3 GRAPHING FUNCTIONS ON CALCULATORS AND COMPUTERS; COMPUTER ALGEBRA SYSTEMS

In this section we will discuss issues that relate to generating graphs of equations and functions with graphing utilities (graphing calculators and computers). Because graphing utilities vary widely, it is difficult to make general statements about them. Therefore, at various places in this section we will ask you to refer to the documentation for your own graphing utility for specific details about the way it operates.

GRAPHING CALCULATORS AND COMPUTER ALGEBRA SYSTEMS

The development of new technology has significantly changed how and where mathematicians, engineers, and scientists perform their work, as well as their approach to problem solving. Not only have portable computers and handheld calculators with graphing capabilities become standard tools in the scientific community, but there have been major new innovations in computer software. Among the most significant of these innovations are programs called **Computer Algebra Systems** (abbreviated CAS), the most common being *Mathematica*, *Maple*, and *Derive*.* Computer algebra systems not only have powerful graphing capabilities, but, as their name suggests, they can perform many of the symbolic computations that occur in algebra, calculus, and branches of higher mathematics. For example, it is a trivial task for a CAS to perform the factorization

$$x^6 + 23x^5 + 147x^4 - 139x^3 - 3464x^2 - 2112x + 23040 = (x + 5)(x - 3)^2(x + 8)^3$$

or the exact numerical computation

$$\left(\frac{63456}{3177295} - \frac{43907}{22854377} \right)^3 = \frac{2251912457164208291259320230122866923}{382895955819369204449565945369203764688375}$$

Technology has also made it possible to generate graphs of equations and functions in seconds that in the past might have taken hours to produce. Graphing technology includes handheld graphing calculators, computer algebra systems, and software designed for that purpose. Figure 1.3.1 shows the graphs of the function $f(x) = x^4 - x^3 - 2x^2$ produced with various graphing utilities; the first two were generated with the CAS programs, *Mathematica* and *Maple*, and the third with a graphing calculator. Graphing calculators produce coarser graphs than most computer programs but have the advantage of being compact and portable.

* *Mathematica* is a product of Wolfram Research, Inc.; *Maple* is a product of Waterloo Maple Software, Inc.; and *Derive* is a product of Soft Warehouse, Inc.

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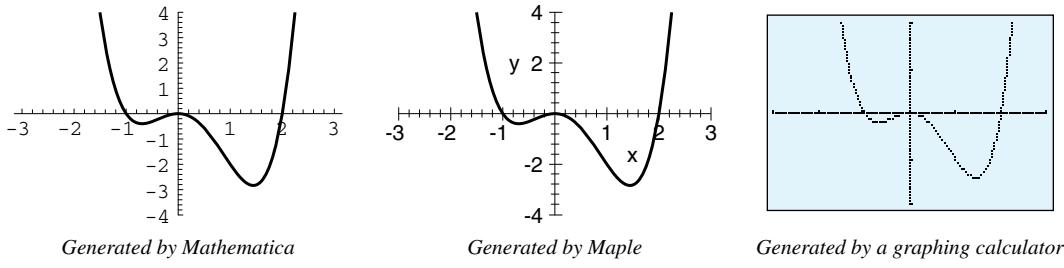


Figure 1.3.1

VIEWING WINDOWS

Graphing utilities can only show a portion of the xy -plane in the viewing screen, so the first step in graphing an equation is to determine which rectangular portion of the xy -plane you want to display. This region is called the **viewing window** (or **viewing rectangle**). For example, in Figure 1.3.1 the viewing window extends over the interval $[-3, 3]$ in the x -direction and over the interval $[-4, 4]$ in the y -direction, so we say that the viewing window is $[-3, 3] \times [-4, 4]$ (read “ $[-3, 3]$ by $[-4, 4]$ ”). In general, if the viewing window is $[a, b] \times [c, d]$, then the window extends between $x = a$ and $x = b$ in the x -direction and between $y = c$ and $y = d$ in the y -direction. We will call $[a, b]$ the **x -interval** for the window and $[c, d]$ the **y -interval** for the window (Figure 1.3.2).

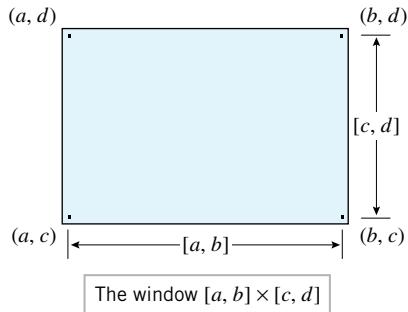


Figure 1.3.2

Different graphing utilities designate viewing windows in different ways. For example, the first two graphs in Figure 1.3.1 were produced by the commands

```
Plot[x^4 - x^3 - 2*x^2, {x, -3, 3}, PlotRange->{-4, 4}]
(Mathematica)
plot( x^4 - x^3 - 2*x^2, x = -3..3, y = -4..4);
(Maple)
```

and the last graph was produced on a graphing calculator by pressing the GRAPH button after setting the following values for the variables that determine the x -interval and y -interval:

$$x\text{Min} = -3, \quad x\text{Max} = 3, \quad y\text{Min} = -4, \quad y\text{Max} = 4$$

- **FOR THE READER.** Use your own graphing utility to generate the graph of the function $f(x) = x^4 - x^3 - 2x^2$ in the window $[-3, 3] \times [-4, 4]$.

TICK MARKS AND GRID LINES

To help locate points in a viewing window visually, graphing utilities provide methods for drawing **tick marks** (also called **scale marks**) on the coordinate axes or at other locations in the viewing window. With computer programs such as *Mathematica* and *Maple*, there are specific commands for designating the spacing between tick marks, but if the user does not

1.3 Graphing Functions on Calculators and Computers; Computer Algebra Systems **29**

specify the spacing, then the programs make certain *default* choices. For example, in the first two parts of Figure 1.3.1, the tick marks shown were the default choices.

On some graphing calculators the spacing between tick marks is determined by two *scale variables* (also called *scale factors*), which we will denote by

$$x\text{Scl} \text{ and } y\text{Scl}$$

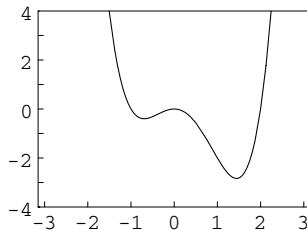
(The notation varies among calculators.) These variables specify the spacing between the tick marks in the x - and y -directions, respectively. For example, in the third part of Figure 1.3.1 the window and tick marks were designated by the settings

$$x\text{Min} = -3 \quad x\text{Max} = 3$$

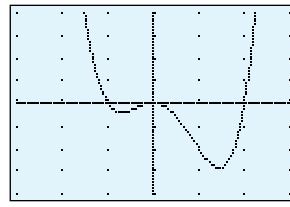
$$y\text{Min} = -4 \quad y\text{Max} = 4$$

$$x\text{Scl} = 1 \quad y\text{Scl} = 1$$

Most graphing utilities allow for variations in the design and positioning of tick marks. For example, Figure 1.3.3 shows two variations of the graphs in Figure 1.3.1; the first was generated on a computer using an option for placing the ticks and numbers on the edges of a box, and the second was generated on a graphing calculator using an option for drawing grid lines to simulate graph paper.



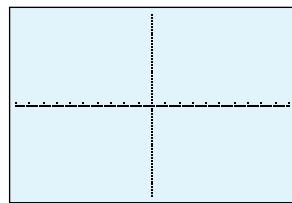
Generated by Mathematica



Generated by a graphing calculator

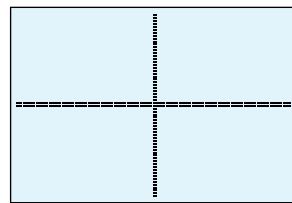
Figure 1.3.3

Example 1 Figure 1.3.4a shows the window $[-5, 5] \times [-5, 5]$ with the tick marks spaced .5 unit apart in the x -direction and 10 units apart in the y -direction. Note that no tick marks are actually visible in the y -direction because the tick mark at the origin is covered by the x -axis, and all other tick marks in the y -direction fall outside of the viewing window. ◀



$[-5, 5] \times [-5, 5]$
 $x\text{Scl} = .5, y\text{Scl} = 10$

(a)



$[-10, 10] \times [-10, 10]$
 $x\text{Scl} = .1, y\text{Scl} = .1$

(b)

Figure 1.3.4

Example 2 Figure 1.3.4b shows the window $[-10, 10] \times [-10, 10]$ with the tick marks spaced .1 unit apart in the x - and y -directions. In this case the tick marks are so close together that they create the effect of thick lines on the coordinate axes. When this occurs you will usually want to increase the scale factors to reduce the number of tick marks and make them legible. ◀

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FOR THE READER. Graphing calculators provide a way of clearing all settings and returning them to *default values*. For example, on one calculator the default window is $[-10, 10] \times [-10, 10]$ and the default scale factors are $x\text{Scl} = 1$ and $y\text{Scl} = 1$. Check your documentation to determine the default values for your calculator and how to reset the calculator to its default configuration. If you are using a computer program, check your documentation to determine the commands for specifying the spacing between tick marks.

CHOOSING A VIEWING WINDOW

When the graph of a function extends indefinitely in some direction, no single viewing window can show the entire graph. In such cases the choice of the viewing window can drastically affect one's perception of how the graph looks. For example, Figure 1.3.5 shows a computer-generated graph of $y = 9 - x^2$, and Figure 1.3.6 shows four views of this graph generated on a calculator.

- In part (a) the graph falls completely outside of the window, so the window is blank (except for the ticks and axes).
- In part (b) the graph is broken into two pieces because it passes in and out of the window.
- In part (c) the graph appears to be a straight line because we have zoomed in on such a small segment of the curve.
- In part (d) we have a more complete picture of the graph shape because the window encompasses all of the important points, namely the high point on the graph and the intersections with the x -axis.

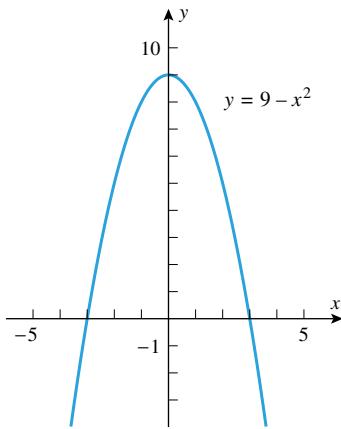


Figure 1.3.5

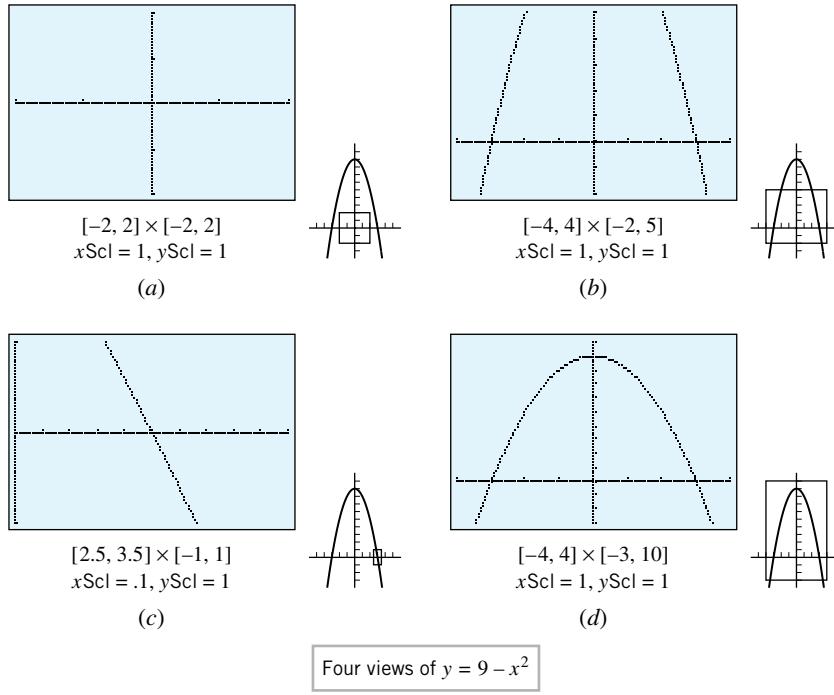


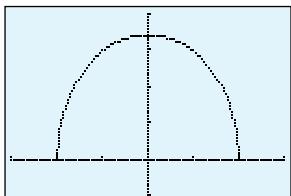
Figure 1.3.6

Four views of $y = 9 - x^2$

For a function whose graph does not extend indefinitely in either the x - or y -directions, the domain and range of the function can be used to obtain a viewing window that contains the entire graph.

1.3 Graphing Functions on Calculators and Computers; Computer Algebra Systems **31**

Example 3 Use the domain and range of the function $f(x) = \sqrt{12 - 3x^2}$ to determine a viewing window that contains the entire graph.



$[-3, 3] \times [-1, 4]$
 $xScl = 1, yScl = 1$

Figure 1.3.7

Solution. The natural domain of f is $[-2, 2]$ and the range is $[0, \sqrt{12}]$ (verify), so the entire graph will be contained in the viewing window $[-2, 2] \times [0, \sqrt{12}]$. For clarity, it is desirable to use a slightly larger window to avoid having the graph too close to the edges of the screen. For example, taking the viewing window to be $[-3, 3] \times [-1, 4]$ yields the graph in Figure 1.3.7. \blacktriangleleft

If the graph of f extends indefinitely in either the x - or y -direction, then it will not be possible to show the entire graph in any one viewing window. In such cases one tries to choose the window to show all of the important features for the problem at hand. (Of course, what is important in one problem may not be important in another, so the choice of the viewing window will often depend on the objectives in the problem.)

Example 4 Graph the equation $y = x^3 - 12x^2 + 18$ in the following windows and discuss the advantages and disadvantages of each window.

- (a) $[-10, 10] \times [-10, 10]$ with $xScl = 1, yScl = 1$
- (b) $[-20, 20] \times [-20, 20]$ with $xScl = 1, yScl = 1$
- (c) $[-20, 20] \times [-300, 20]$ with $xScl = 1, yScl = 20$
- (d) $[-5, 15] \times [-300, 20]$ with $xScl = 1, yScl = 20$
- (e) $[1, 2] \times [-1, 1]$ with $xScl = .1, yScl = .1$

Solution (a). The window in Figure 1.3.8a has chopped off the portion of the graph that intersects the y -axis, and it shows only two of three possible real roots for the given cubic polynomial. To remedy these problems we need to widen the window in both the x - and y -directions.

Solution (b). The window in Figure 1.3.8b shows the intersection of the graph with the y -axis and the three real roots, but it has chopped off the portion of the graph between the two positive roots. Moreover, the ticks in the y -direction are nearly illegible because they are so close together. We need to extend the window in the negative y -direction and increase $yScl$. We do not know how far to extend the window, so some experimentation will be required to obtain what we want.

Solution (c). The window in Figure 1.3.8c shows all of the main features of the graph. However, we have some wasted space in the x -direction. We can improve the picture by shortening the window in the x -direction appropriately.

Solution (d). The window in Figure 1.3.8d shows all of the main features of the graph without a lot of wasted space. However, the window does not provide a clear view of the roots. To get a closer view of the roots we must forget about showing all of the main features of the graph and choose windows that zoom in on the roots themselves.

Solution (e). The window in Figure 1.3.8e displays very little of the graph, but it clearly shows that the root in the interval $[1, 2]$ is slightly less than 1.3. \blacktriangleleft

• **FOR THE READER.** Sometimes you will want to determine the viewing window by choosing the x -interval for the window and allowing the graphing utility to determine a y -interval that encompasses the maximum and minimum values of the function over the x -interval. Most graphing utilities provide some method for doing this, so check your documentation to determine how to use this feature. Allowing the graphing utility to determine the y -interval of the window takes some of the guesswork out of problems like that in part (b) of the preceding example.

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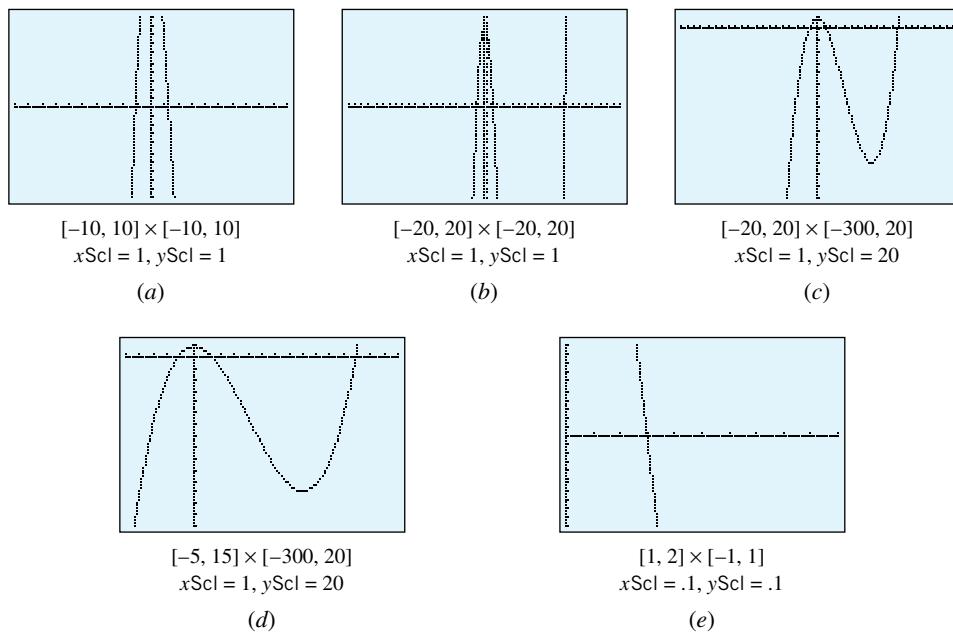


Figure 1.3.8

ZOOMING

The process of enlarging or reducing the size of a viewing window is called ***zooming***. If you reduce the size of the window, you see less of the graph as a whole, but more detail of the part shown; this is called ***zooming in***. In contrast, if you enlarge the size of the window, you see more of the graph as a whole, but less detail of the part shown; this is called ***zooming out***. Most graphing calculators provide menu items for zooming in or zooming out by fixed factors. For example, on one calculator the amount of enlargement or reduction is controlled by setting values for two ***zoom factors***, $x\text{Fact}$ and $y\text{Fact}$. If

xFact = 10 and *yFact* = 5

then each time a zoom command is executed the viewing window is enlarged or reduced by a factor of 10 in the x -direction and a factor of 5 in the y -direction. With computer programs such as *Mathematica* and *Maple*, zooming is controlled by adjusting the x -interval and y -interval directly; however, there are ways to automate this by programming.

- **FOR THE READER.** If you are using a graphing calculator, read your documentation to determine how to use the zooming feature.

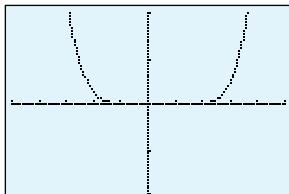
COMPRESSION

Enlarging the viewing window for a graph has the geometric effect of compressing the graph, since more of the graph is packed into the calculator screen. If the compression is sufficiently great, then some of the detail in the graph may be lost. Thus, the choice of the viewing window frequently depends on whether you want to see more of the graph or more of the detail. Figure 1.3.9 shows two views of the equation

$$y = x^5(x - 2)$$

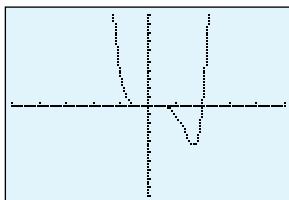
In part (a) of the figure the y -interval is very large, resulting in a vertical compression that obscures the detail in the vicinity of the x -axis. In part (b) the y -interval is smaller, and consequently we see more of the detail in the vicinity of the x -axis but less of the graph in the y -direction.

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$[-5, 5] \times [-1000, 1000]$
 $x\text{Scl} = 1, y\text{Scl} = 500$

(a)



$[-5, 5] \times [-10, 10]$
 $x\text{Scl} = 1, y\text{Scl} = 1$

(b)

Figure 1.3.9

Example 5 Describe the graph of the function $f(x) = x + 0.01 \sin 50\pi x$; then graph the function in the following windows and explain why the graphs do or do not differ from your description.

- (a) $[-10, 10] \times [-10, 10]$ (b) $[-1, 1] \times [-1, 1]$
(c) $[-.1, .1] \times [-.1, .1]$ (d) $[-.01, .01] \times [-.01, .01]$

Solution. The formula for f is the sum of the function x (whose graph is a straight line) and the function $0.01 \sin 50\pi x$ (whose graph is a sinusoidal curve with an amplitude of 0.01 and a period of $2\pi/50\pi = 0.04$). Intuitively, this suggests that the graph of f will follow the general path of the line $y = x$ but will have small bumps resulting from the contributions of the sinusoidal oscillations.

To generate the four graphs, we first set the calculator to the radian mode.* Because the windows in successive parts of this example are decreasing in size by a factor of 10, it will be convenient to use the zoom in feature of the calculator with the zoom factors set to 10 in the x - and y -directions. In Figure 1.3.10a the graph appears to be a straight line because compression has hidden the small sinusoidal oscillations. (Keep in mind that the amplitude of the sinusoidal portion of the function is only 0.01.) In part (b) the oscillations have begun to appear since the y -interval has been reduced, and in part (c) the oscillations have become very clear because the vertical scale is more in keeping with the amplitude of the oscillations. In part (d) the graph appears to be a line segment because we have zoomed in on such a small portion of the curve. ◀

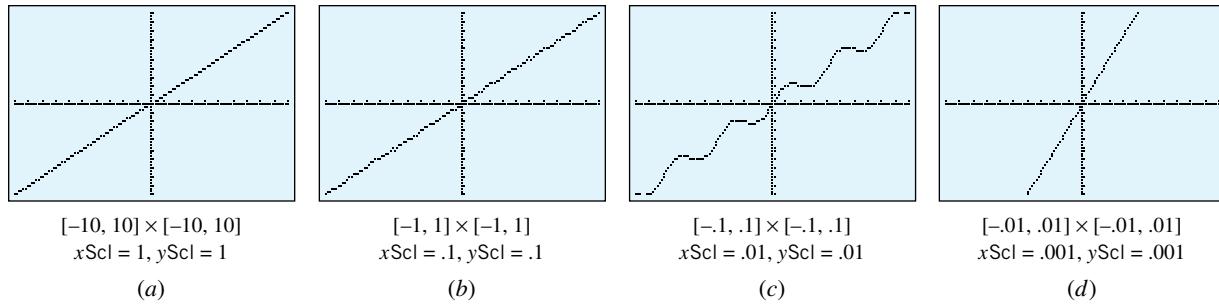


Figure 1.3.10

ASPECT RATIO DISTORTION

Figure 1.3.11a shows a circle of radius 5 and two perpendicular lines graphed in the window $[-10, 10] \times [-10, 10]$ with $x\text{Scl} = 1$ and $y\text{Scl} = 1$. However, the circle is distorted and the lines do not appear perpendicular because the calculator has not used the same length for 1 unit on the x -axis and 1 unit on the y -axis. (Compare the spacing between the ticks on the axes.) This is called **aspect ratio distortion**. Many calculators provide a menu item for automatically correcting the distortion by adjusting the viewing window appropriately. For example, one calculator makes this correction to the viewing window $[-10, 10] \times [-10, 10]$ by changing it to

$$[-16.9970674487, 16.9970674487] \times [-10, 10]$$

(Figure 1.3.11b). With computer programs such as *Mathematica* and *Maple*, aspect ratio distortion is controlled with adjustments to the physical dimensions of the viewing window on the computer screen, rather than altering the x - and y -intervals of the viewing window.

- **FOR THE READER.** Read the documentation for your graphing utility to determine how to control aspect ratio distortion.

* In this text we follow the convention that angles are measured in radians unless degree measure is specified.

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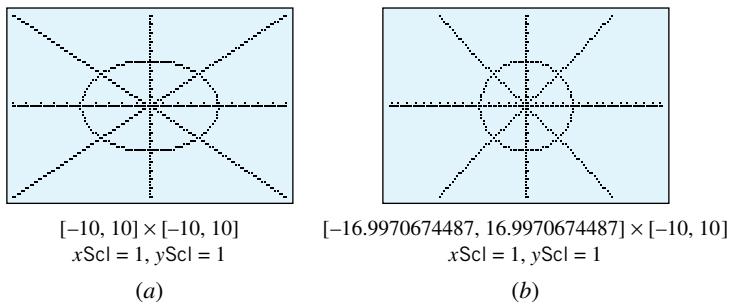


Figure 1.3.11

PIXELS AND RESOLUTION

Sometimes graphing utilities produce unexpected results. For example, Figure 1.3.12 shows the graph of $y = \cos(10\pi x)$, which was generated on a graphing calculator in four different windows. (Your own calculator may produce different results.) The first graph has the correct shape, but the remaining three do not. To explain what is happening here we need to understand more precisely how graphing utilities generate graphs.

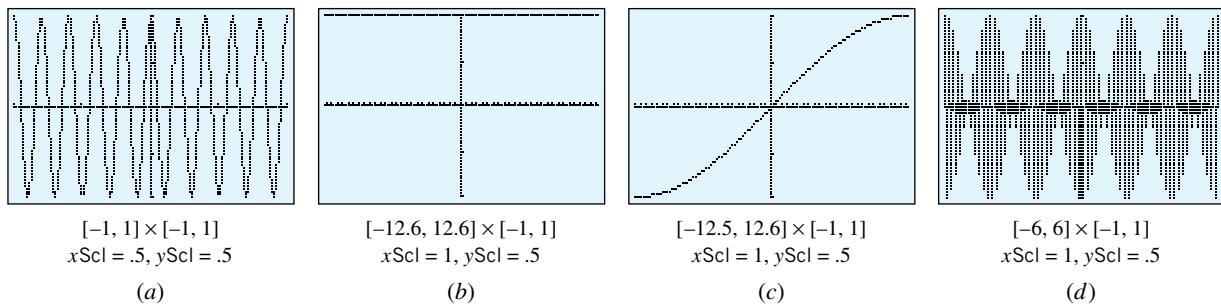


Figure 1.3.12

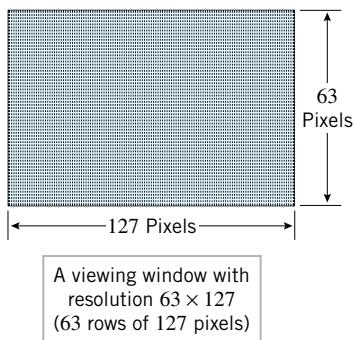


Figure 1.3.13

Screen displays for graphing utilities are divided into rows and columns of rectangular blocks, called *pixels*. For black-and-white displays each pixel has two possible states—an activated (or dark) state and a deactivated (or light) state. Since graphical elements are produced by activating pixels, the more pixels that a screen has to work with, the greater the amount of detail it can show. For example, one calculator has a *resolution* of 63×127 , meaning that there are 63 rows with 127 pixels per row (Figure 1.3.13). In contrast, a computer screen may have a resolution of 1024×1280 (1024 rows with 1280 pixels per row), so the computer screen is capable of displaying much smoother graphs than the calculator.

- **FOR THE READER.** If you are using a graphing calculator, check the documentation to determine its resolution.

SAMPLING ERROR

The procedure that a graphing utility follows to generate a graph is similar to the procedure for plotting points by hand. When a viewing window is selected and an equation is entered, the graphing utility determines the x -coordinates of certain pixels on the x -axis and computes the corresponding points (x, y) on the graph. It then activates the pixels whose coordinates most closely match those of the calculated points and uses some built-in algorithm to activate additional intermediate pixels to create the curve shape. The point to keep in mind here is that *changing the window changes the points plotted by the graphing utility*. Thus, it is possible that a particular window will produce a false impression about the graph shape because significant characteristics of the graph occur *between* the plotted pixels. This is called **sampling error**. This is exactly what occurred in Figure 1.3.12 when we graphed $y = \cos(10\pi x)$. In part (b) of the figure the plotted pixels happened to fall at

1.3 Graphing Functions on Calculators and Computers; Computer Algebra Systems **35**

the peaks of the cosine curve, giving the false impression that the graph is a horizontal line at $y = 1$. In part (c) the plotted pixels fell at successively higher points along the graph, and in part (d) the plotted pixels fell in a strange way that created yet another misleading impression of the graph shape.

- REMARK.** Figure 1.3.12 suggests that for trigonometric graphs with rapid oscillations, restricting the x -interval to a few periods is likely to produce a more accurate representation about the graph shape.

FALSE GAPS

Sometimes graphs that are continuous appear to have gaps when they are generated on a calculator. These *false gaps* typically occur where the graph rises so rapidly that vertical space is opened up between successive pixels.

Example 6 Figure 1.3.14 shows the graph of the semicircle $y = \sqrt{9 - x^2}$ in two viewing windows. Although this semicircle has x -intercepts at the points $x = \pm 3$, part (a) of the figure shows false gaps at those points because there are no pixels with x -coordinates ± 3 in the window selected. In part (b) no gaps occur because there are pixels with x -coordinates $x = \pm 3$ in the window being used. ◀

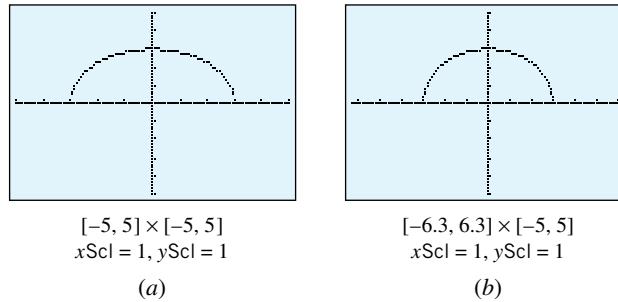


Figure 1.3.14

FALSE LINE SEGMENTS

In addition to creating false gaps in continuous graphs, calculators can err in the opposite direction by placing *false line segments* in the gaps of discontinuous curves.

Example 7 Figure 1.3.15a shows the graph of $y = 1/(x - 1)$ in the default window on a calculator. Although the graph appears to contain vertical line segments near $x = 1$, they should not be there. There is actually a gap in the curve at $x = 1$, since a division by zero occurs at that point (Figure 1.3.15b). ◀

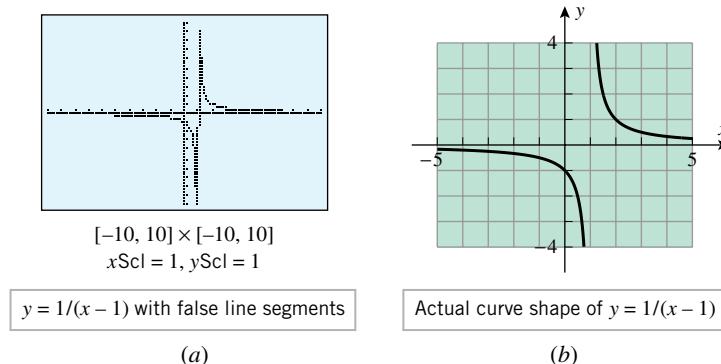


Figure 1.3.15

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ERRORS OF OMISSION

Most graphing utilities use logarithms to evaluate functions with fractional exponents such as $f(x) = x^{2/3} = \sqrt[3]{x^2}$. However, because logarithms are only defined for positive numbers, many (but not all) graphing utilities will omit portions of the graphs of functions with fractional exponents. For example, one calculator graphs $y = x^{2/3}$ as in Figure 1.3.16a, whereas the actual graph is as in Figure 1.3.16b. (See the discussion preceding Exercise 29 for a way of circumventing this problem.)

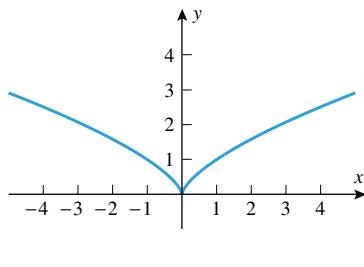
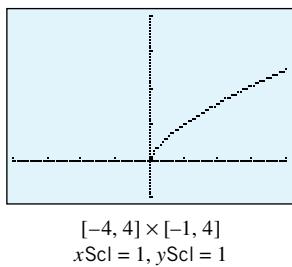


Figure 1.3.16

- **FOR THE READER.** Determine whether your graphing utility produces the graph of $y = x^{2/3}$ for both positive and negative values of x .

WHAT IS THE TRUE SHAPE OF A GRAPH?

Although graphing utilities are powerful tools for generating graphs quickly, they can produce misleading graphs as a result of compression, sampling error, false gaps, and false line segments. In short, *graphing utilities can suggest graph shapes, but they cannot establish them with certainty*. Thus, the more you know about the functions you are graphing, the easier it will be to choose good viewing windows, and the better you will be able to judge the reasonableness of the results produced by your graphing utility.

MORE INFORMATION ON GRAPHING AND CALCULATING UTILITIES

The main source of information about your graphing utility is its own documentation, and from time to time we will suggest that you refer to that documentation to learn some particular technique.

EXERCISE SET 1.3

Graphing Calculator

1. Use a graphing utility to generate the graph of the function $f(x) = x^4 - x^2$ in the given viewing windows, and specify the window that you think gives the best view of the graph.
 - $[-50, 50] \times [-50, 50]$
 - $[-5, 5] \times [-5, 5]$
 - $[-2, 2] \times [-2, 2]$
 - $[-2, 2] \times [-1, 1]$
 - $[-1.5, 1.5] \times [-0.5, 0.5]$
2. Use a graphing utility to generate the graph of the function $f(x) = x^5 - x^3$ in the given viewing windows, and specify the window that you think gives the best view of the graph.
 - $[-50, 50] \times [-50, 50]$
 - $[-5, 5] \times [-5, 5]$
 - $[-2, 2] \times [-2, 2]$
 - $[-2, 2] \times [-1, 1]$
 - $[-1.5, 1.5] \times [-0.5, 0.5]$
3. Use a graphing utility to generate the graph of the function $f(x) = x^2 + 12$ in the given viewing windows, and specify

the window that you think gives the best view of the graph.

- $[-1, 1] \times [13, 15]$
- $[-2, 2] \times [11, 15]$
- $[-4, 4] \times [10, 28]$
- A window of your choice

4. Use a graphing utility to generate the graph of the function $f(x) = -12 - x^2$ in the given viewing windows, and specify the window that you think gives the best view of the graph.
 - $[-1, 1] \times [-15, -13]$
 - $[-2, 2] \times [-15, -11]$
 - $[-4, 4] \times [-28, -10]$
 - A window of your choice

In Exercises 5 and 6, use the domain and range of f to determine a viewing window that contains the entire graph, and generate the graph in that window.

1.3 Graphing Functions on Calculators and Computers; Computer Algebra Systems **37**

5. $f(x) = \sqrt{16 - 2x^2}$ 6. $f(x) = \sqrt{3 - 2x - x^2}$
7. Graph the function $f(x) = x^3 - 15x^2 - 3x + 45$ using the stated windows and tick spacing, and discuss the advantages and disadvantages of each window.
- $[-10, 10] \times [-10, 10]$ with $xScl = 1$ and $yScl = 1$
 - $[-20, 20] \times [-20, 20]$ with $xScl = 1$ and $yScl = 1$
 - $[-5, 20] \times [-500, 50]$ with $xScl = 5$ and $yScl = 50$
 - $[-2, -1] \times [-1, 1]$ with $xScl = 0.1$ and $yScl = 0.1$
 - $[9, 11] \times [-486, -484]$ with $xScl = 0.1$ and $yScl = 0.1$
8. Graph the function $f(x) = -x^3 - 12x^2 + 4x + 48$ using the stated windows and tick spacing, and discuss the advantages and disadvantages of each window.
- $[-10, 10] \times [-10, 10]$ with $xScl = 1$ and $yScl = 1$
 - $[-20, 20] \times [-20, 20]$ with $xScl = 1$ and $yScl = 1$
 - $[-16, 4] \times [-250, 50]$ with $xScl = 2$ and $yScl = 25$
 - $[-3, -1] \times [-1, 1]$ with $xScl = 0.1$ and $yScl = 0.1$
 - $[-9, -7] \times [-241, -239]$ with $xScl = 0.1$ and $yScl = 0.1$

In Exercises 9–16, generate the graph of f in a viewing window that you think is appropriate.

9. $f(x) = x^2 - 9x - 36$ 10. $f(x) = \frac{x+7}{x-9}$
11. $f(x) = 2 \cos 80x$ 12. $f(x) = 12 \sin(x/80)$
13. $f(x) = 300 - 10x^2 + 0.01x^3$
14. $f(x) = x(30 - 2x)(25 - 2x)$
15. $f(x) = x^2 + \frac{1}{x}$ 16. $f(x) = \sqrt{11x - 18}$

In Exercises 17 and 18, generate the graph of f and determine whether your graphs contain false line segments. Sketch the actual graph and see if you can make the false line segments disappear by changing the viewing window.

17. $f(x) = \frac{x}{x^2 - 1}$ 18. $f(x) = \frac{x^2}{4 - x^2}$
19. The graph of the equation $x^2 + y^2 = 16$ is a circle of radius 4 centered at the origin.
 - Find a function whose graph is the upper semicircle and graph it.
 - Find a function whose graph is the lower semicircle and graph it.
 - Graph the upper and lower semicircles together. If the combined graphs do not appear circular, see if you can adjust the viewing window to eliminate the aspect ratio distortion.
 - Graph the portion of the circle in the first quadrant.
 - Is there a function whose graph is the right half of the circle? Explain.
20. In each part, graph the equation by solving for y in terms of x and graphing the resulting functions together.
 - $x^2/4 + y^2/9 = 1$
 - $y^2 - x^2 = 1$

21. Read the documentation for your graphing utility to determine how to graph functions involving absolute values, and graph the given equation.
- $y = |x|$
 - $y = |x - 1|$
 - $y = |x| - 1$
 - $y = |\sin x|$
 - $y = \sin |x|$
 - $y = |x| - |x + 1|$
22. Based on your knowledge of the absolute value function, sketch the graph of $f(x) = |x|/x$. Check your result using a graphing utility.
23. Make a conjecture about the relationship between the graph of $y = f(x)$ and the graph of $y = |f(x)|$; check your conjecture with some specific functions.
24. Make a conjecture about the relationship between the graph of $y = f(x)$ and the graph of $y = f(|x|)$; check your conjecture with some specific functions.
25. (a) Based on your knowledge of the absolute value function, sketch the graph of $y = |x - a|$, where a is a constant. Check your result using a graphing utility and some specific values of a .

(b) Sketch the graph of $y = |x - 1| + |x - 2|$; check your result with a graphing utility.
26. How are the graphs of $y = |x|$ and $y = \sqrt{x^2}$ related? Check your answer with a graphing utility.

Most graphing utilities provide some way of graphing functions that are defined piecewise; read the documentation for your graphing utility to find out how to do this. However, if your goal is just to find the general shape of the graph, you can graph each portion of the function separately and combine the pieces with a hand-drawn sketch. Use this method in Exercises 27 and 28.

27. Draw the graph of

$$f(x) = \begin{cases} \sqrt[3]{x-2}, & x \leq 2 \\ x^3 - 2x - 4, & x > 2 \end{cases}$$

28. Draw the graph of

$$f(x) = \begin{cases} x^3 - x^2, & x \leq 1 \\ \frac{1}{1-x}, & 1 < x < 4 \\ x^2 \cos \sqrt{x}, & 4 \leq x \end{cases}$$

We noted in the text that for functions involving fractional exponents (or radicals), graphing utilities sometimes omit portions of the graph. If $f(x) = x^{p/q}$, where p/q is a positive fraction in *lowest terms*, then you can circumvent this problem as follows:

- If p is even and q is odd, then graph $g(x) = |x|^{p/q}$ instead of $f(x)$.
- If p is odd and q is odd, then graph $g(x) = (|x|/x)|x|^{p/q}$ instead of $f(x)$.

We will explain why this works in the exercises of the next section.

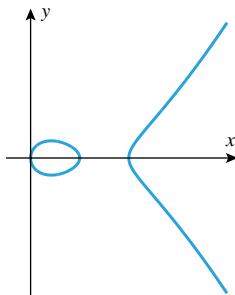
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29. (a) Generate the graphs of $f(x) = x^{2/5}$ and $g(x) = |x|^{2/5}$, and determine whether your graphing utility missed part of the graph of f .
 (b) Generate the graphs of the functions $f(x) = x^{1/5}$ and $g(x) = (|x|/x)|x|^{1/5}$, and then determine whether your graphing utility missed part of the graph of f .
 (c) Generate a graph of the function $f(x) = (x - 1)^{4/5}$ that shows all of its important features.
 (d) Generate a graph of the function $f(x) = (x + 1)^{3/4}$ that shows all of its important features.
30. The graphs of $y = (x^2 - 4)^{2/3}$ and $y = [(x^2 - 4)^2]^{1/3}$ should be the same. Does your graphing utility produce the same graph for both equations? If not, what do you think is happening?
31. In each part, graph the function for various values of c , and write a paragraph or two that describes how changes in c affect the graph in each case.
 (a) $y = cx^2$
 (b) $y = x^2 + cx$
 (c) $y = x^2 + x + c$
32. The graph of an equation of the form $y^2 = x(x - a)(x - b)$ (where $0 < a < b$) is called a **bipartite cubic**. The accompanying figure shows a typical graph of this type.
 (a) Graph the bipartite cubic $y^2 = x(x - 1)(x - 2)$ by solving for y in terms of x and graphing the two resulting functions.

- (b) Find the x -intercepts of the bipartite cubic

$$y^2 = x(x - a)(x - b)$$

and make a conjecture about how changes in the values of a and b would affect the graph. Test your conjecture by graphing the bipartite cubic for various values of a and b .



Bipartite cubic

Figure Ex-32

33. Based on your knowledge of the graphs of $y = x$ and $y = \sin x$, make a sketch of the graph of $y = x \sin x$. Check your conclusion using a graphing utility.
34. What do you think the graph of $y = \sin(1/x)$ looks like? Test your conclusion using a graphing utility. [Suggestion: Examine the graph on a succession of smaller and smaller intervals centered at $x = 0$.]

1.4 NEW FUNCTIONS FROM OLD

Just as numbers can be added, subtracted, multiplied, and divided to produce other numbers, so functions can be added, subtracted, multiplied, and divided to produce other functions. In this section we will discuss these operations and some others that have no analogs in ordinary arithmetic.

..... ARITHMETIC OPERATIONS ON FUNCTIONS

Two functions, f and g , can be added, subtracted, multiplied, and divided in a natural way to form new functions $f + g$, $f - g$, fg , and f/g . For example, $f + g$ is defined by the formula

$$(f + g)(x) = f(x) + g(x) \tag{1}$$

which states that for each input the value of $f + g$ is obtained by adding the values of f and g . For example, if

$$f(x) = x \quad \text{and} \quad g(x) = x^2$$

then

$$(f + g)(x) = f(x) + g(x) = x + x^2$$

Equation (1) provides a formula for $f + g$ but does not say anything about the domain of $f + g$. However, for the right side of this equation to be defined, x must lie in the domain of f and in the domain of g , so we define the domain of $f + g$ to be the intersection of those two domains. More generally, we make the following definition:

1.4.1 DEFINITION. Given functions f and g , we define

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)g(x)$$

$$(f/g)(x) = f(x)/g(x)$$

For the functions $f + g$, $f - g$, and fg we define the domain to be the intersection of the domains of f and g , and for the function f/g we define the domain to be the intersection of the domains of f and g but with the points where $g(x) = 0$ excluded (to avoid division by zero).

REMARK. If f is a constant function, say $f(x) = c$ for all x , then the product of f and g is cg , so multiplying a function by a constant is a special case of multiplying two functions.

Example 1 Let

$$f(x) = 1 + \sqrt{x-2} \quad \text{and} \quad g(x) = x - 3$$

Find $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$, $(f/g)(x)$, and $(7f)(x)$; state the domains of $f + g$, $f - g$, fg , f/g , and $7f$.

Solution. First, we will find formulas for the functions and then the domains. The formulas are

$$(f + g)(x) = f(x) + g(x) = (1 + \sqrt{x-2}) + (x - 3) = x - 2 + \sqrt{x-2} \quad (2)$$

$$(f - g)(x) = f(x) - g(x) = (1 + \sqrt{x-2}) - (x - 3) = 4 - x + \sqrt{x-2} \quad (3)$$

$$(fg)(x) = f(x)g(x) = (1 + \sqrt{x-2})(x - 3) \quad (4)$$

$$(f/g)(x) = f(x)/g(x) = \frac{1 + \sqrt{x-2}}{x - 3} \quad (5)$$

$$(7f)(x) = 7f(x) = 7 + 7\sqrt{x-2} \quad (6)$$

In all five cases the natural domain determined by the formula is the same as the domain specified in Definition 1.4.1, so there is no need to state the domain explicitly in any of these cases. For example, the domain of f is $[2, +\infty)$, the domain of g is $(-\infty, +\infty)$, and the natural domain for $f(x) + g(x)$ determined by Formula (2) is $[2, +\infty)$, which is precisely the intersection of the domains of f and g . ◀

REMARK. There are situations in which the natural domain associated with the formula resulting from an operation on two functions is not the correct domain for the new function. For example, if $f(x) = \sqrt{x}$ and $g(x) = \sqrt{x}$, then according to Definition 1.4.1 the domain of fg should be $[0, +\infty) \cap [0, +\infty) = [0, +\infty)$. However, $(fg)(x) = \sqrt{x}\sqrt{x} = x$, which has a natural domain of $(-\infty, +\infty)$. Thus, to be precise in describing the formula for fg , we must write $(fg)(x) = x$, $x \geq 0$.

Multiplying a function f by a *positive* constant c has the geometric effect of stretching or compressing the graph of $y = f(x)$ in the y -direction. For example, examine the graphs of $y = f(x)$, $y = 2f(x)$, and $y = \frac{1}{2}f(x)$ shown in Figure 1.4.1a. Multiplying by 2 doubles each y -coordinate, thereby stretching the graph, and multiplying by $\frac{1}{2}$ cuts each y -coordinate in half, thereby compressing the graph. In general, if $c > 0$, then the graph of $y = cf(x)$ can be obtained from the graph of $y = f(x)$ by compressing the graph of $y = f(x)$ vertically by a factor of $1/c$ if $0 < c < 1$, or stretching it by a factor of c if $c > 1$.

Analogously, multiplying x by a *positive* constant c has the geometric effect of stretching or compressing the graph of $y = f(x)$ in the x -direction. For example, examine the graphs

STRETCHES AND COMPRESSIONS

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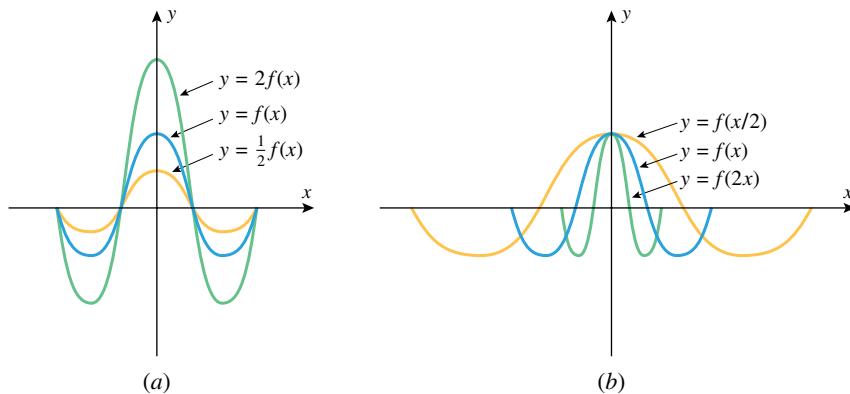


Figure 1.4.1

of $y = f(x)$, $y = f(2x)$, and $y = f(x/2)$ shown in Figure 1.4.1b. Multiplying x by 2 compresses the graph by a factor of 2 and multiplying x by $\frac{1}{2}$ stretches the graph by a factor of 2. [This is a little confusing, but think of it this way: The value of $2x$ changes twice as fast as the value of x , so a point moving along the x -axis will only have to move half as far from the origin for $y = f(2x)$ to have the same value as $y = f(x)$.] In general, if $c > 0$, then the graph of $y = f(cx)$ can be obtained from the graph of $y = f(x)$ by stretching the graph of $y = f(x)$ horizontally by a factor of $1/c$ if $0 < c < 1$, or compressing it by a factor of c if $c > 1$.

SUMS OF FUNCTIONS

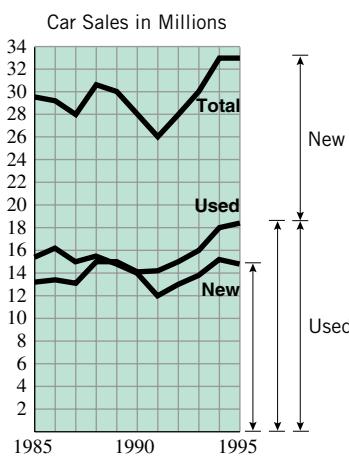


Figure 1.4.2

Adding two functions can be accomplished geometrically by adding the corresponding y -coordinates of their graphs. For example, Figure 1.4.2 shows line graphs of yearly new car sales $N(t)$ and used car sales $U(t)$ in the United States between 1985 and 1995. The sum of these functions, $T(t) = N(t) + U(t)$, represents the yearly total car sales for that period. As illustrated in the figure, the graph of $T(t)$ can be obtained by adding the values of $N(t)$ and $U(t)$ together at each time t and plotting the resulting value.

Example 2 Referring to Figure 1.2.1 for the graphs of $y = \sqrt{x}$ and $y = 1/x$, make a sketch that shows the general shape of the graph of $y = \sqrt{x} + 1/x$ for $x \geq 0$.

Solution. To add the corresponding y -values of $y = \sqrt{x}$ and $y = 1/x$ graphically, just imagine them to be “stacked” on top of one another. This yields the sketch in Figure 1.4.3.

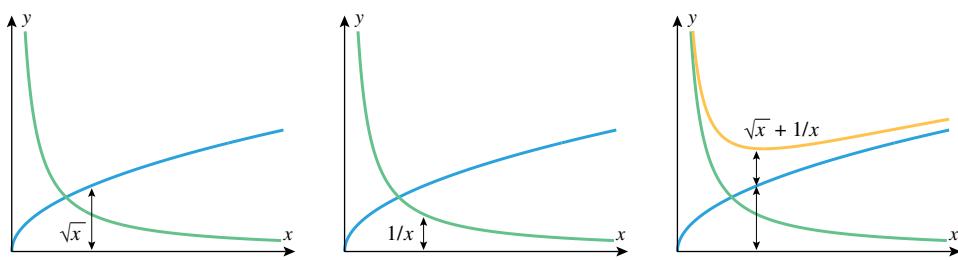


Figure 1.4.3

COMPOSITION OF FUNCTIONS

We now consider an operation on functions, called *composition*, which has no direct analog in ordinary arithmetic. Informally stated, the operation of composition is performed by substituting some function for the independent variable of another function. For example, suppose that

$$f(x) = x^2 \quad \text{and} \quad g(x) = x + 1$$

1.4 New Functions from Old **41**

If we substitute $g(x)$ for x in the formula for f , we obtain a new function

$$f(g(x)) = (g(x))^2 = (x + 1)^2$$

which we denote by $f \circ g$. Thus,

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 = (x + 1)^2$$

In general, we make the following definition.

1.4.2 DEFINITION. Given functions f and g , the **composition** of f with g , denoted by $f \circ g$, is the function defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is defined to consist of all x in the domain of g for which $g(x)$ is in the domain of f .

- **REMARK.** Although the domain of $f \circ g$ may seem complicated at first glance, it makes sense intuitively: To compute $f(g(x))$ one needs x in the domain of g to compute $g(x)$, then one needs $g(x)$ in the domain of f to compute $f(g(x))$.

.....
COMPOSITIONS VIEWED AS COMPUTER PROGRAMS

In Section 1.1 we noted that a function f can be viewed as a computer program that takes an input x , operates on it, and produces an output $f(x)$. From this viewpoint composition can be viewed as two programs, g and f , operating in succession: An input x is fed first to a program g , which produces the output $g(x)$; then this output is fed as input to a program f , which produces the output $f(g(x))$ (Figure 1.4.4). However, rather than have two separate programs operating in succession, we could create a **single** program that takes the input x and directly produces the output $f(g(x))$. This program is the composition $f \circ g$ since $(f \circ g)(x) = f(g(x))$.

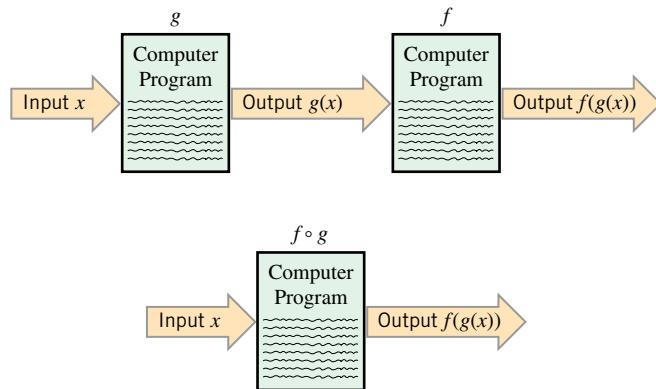


Figure 1.4.4

Example 3 Let $f(x) = x^2 + 3$ and $g(x) = \sqrt{x}$. Find

- (a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$

Solution (a). The formula for $f(g(x))$ is

$$f(g(x)) = [g(x)]^2 + 3 = (\sqrt{x})^2 + 3 = x + 3$$

Since the domain of g is $[0, +\infty)$ and the domain of f is $(-\infty, +\infty)$, the domain of $f \circ g$ consists of all x in $[0, +\infty)$ such that $g(x) = \sqrt{x}$ lies in $(-\infty, +\infty)$; thus, the domain of $f \circ g$ is $[0, +\infty)$. Therefore,

$$(f \circ g)(x) = x + 3, \quad x \geq 0$$

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Solution (b). The formula for $g(f(x))$ is

$$g(f(x)) = \sqrt{f(x)} = \sqrt{x^2 + 3}$$

Since the domain of f is $(-\infty, +\infty)$ and the domain of g is $[0, +\infty)$, the domain of $g \circ f$ consists of all x in $(-\infty, +\infty)$ such that $f(x) = x^2 + 3$ lies in $[0, +\infty)$. Thus, the domain of $g \circ f$ is $(-\infty, +\infty)$. Therefore,

$$(g \circ f)(x) = \sqrt{x^2 + 3}$$

There is no need to indicate that the domain is $(-\infty, +\infty)$, since this is the natural domain of $\sqrt{x^2 + 3}$. ◀

- **REMARK.** Note that the functions $f \circ g$ and $g \circ f$ in the preceding example are not the same. Thus, the order in which functions are composed can (and usually will) make a difference in the end result.

Compositions can also be defined for three or more functions; for example, $(f \circ g \circ h)(x)$ is computed as

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

In other words, first find $h(x)$, then find $g(h(x))$, and then find $f(g(h(x)))$.

Example 4 Find $(f \circ g \circ h)(x)$ if

$$f(x) = \sqrt{x}, \quad g(x) = 1/x, \quad h(x) = x^3$$

Solution.

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^3)) = f(1/x^3) = \sqrt{1/x^3} = 1/x^{3/2}$$



EXPRESSING A FUNCTION AS A COMPOSITION

Many problems in mathematics are attacked by “decomposing” functions into compositions of simpler functions. For example, consider the function h given by

$$h(x) = (x + 1)^2$$

To evaluate $h(x)$ for a given value of x , we would first compute $x + 1$ and then square the result. These two operations are performed by the functions

$$g(x) = x + 1 \quad \text{and} \quad f(x) = x^2$$

We can express h in terms of f and g by writing

$$h(x) = (x + 1)^2 = [g(x)]^2 = f(g(x))$$

so we have succeeded in expressing h as the composition $h = f \circ g$.

The thought process in this example suggests a general procedure for decomposing a function h into a composition $h = f \circ g$:

- Think about how you would evaluate $h(x)$ for a specific value of x , trying to break the evaluation into two steps performed in succession.
- The first operation in the evaluation will determine a function g and the second a function f .
- The formula for h can then be written as $h(x) = f(g(x))$.

For descriptive purposes, we will refer to g as the “inside function” and f as the “outside function” in the expression $f(g(x))$. The inside function performs the first operation and the outside function performs the second.

Example 5 Express $h(x) = (x - 4)^5$ as a composition of two functions.

1.4 New Functions from Old **43**

Solution. To evaluate $h(x)$ for a given value of x we would first compute $x - 4$ and then raise the result to the fifth power. Therefore, the inside function (first operation) is

$$g(x) = x - 4$$

and the outside function (second operation) is

$$f(x) = x^5$$

so $h(x) = f(g(x))$. As a check,

$$f(g(x)) = [g(x)]^5 = (x - 4)^5 = h(x)$$



Example 6 Express $\sin(x^3)$ as a composition of two functions.

Solution. To evaluate $\sin(x^3)$, we would first compute x^3 and then take the sine, so $g(x) = x^3$ is the inside function and $f(x) = \sin x$ the outside function. Therefore,

$$\sin(x^3) = f(g(x))$$

$$g(x) = x^3 \text{ and } f(x) = \sin x$$



Example 7 Table 1.4.1 gives some more examples of decomposing functions into compositions.

Table 1.4.1

FUNCTION	$g(x)$ INSIDE	$f(x)$ OUTSIDE	COMPOSITION
$(x^2 + 1)^{10}$	$x^2 + 1$	x^{10}	$(x^2 + 1)^{10} = f(g(x))$
$\sin^3 x$	$\sin x$	x^3	$\sin^3 x = f(g(x))$
$\tan(x^5)$	x^5	$\tan x$	$\tan(x^5) = f(g(x))$
$\sqrt{4 - 3x}$	$4 - 3x$	\sqrt{x}	$\sqrt{4 - 3x} = f(g(x))$
$8 + \sqrt{x}$	\sqrt{x}	$8 + x$	$8 + \sqrt{x} = f(g(x))$
$\frac{1}{x+1}$	$x + 1$	$\frac{1}{x}$	$\frac{1}{x+1} = f(g(x))$



- REMARK.** It should be noted that there is always more than one way to express a function as a composition. For example, here are two ways to express $(x^2 + 1)^{10}$ as a composition that differ from that in Table 1.4.1:

$$(x^2 + 1)^{10} = [(x^2 + 1)^2]^5 = f(g(x))$$

$$g(x) = (x^2 + 1)^2 \text{ and } f(x) = x^5$$

$$(x^2 + 1)^{10} = [(x^2 + 1)^3]^{10/3} = f(g(x))$$

$$g(x) = (x^2 + 1)^3 \text{ and } f(x) = x^{10/3}$$

.....
SYMMETRY

Figure 1.4.5 shows the graphs of three curves that have certain obvious symmetries. The graph in part (a) is **symmetric about the x -axis** in the sense that for each point (x, y) on the graph the point $(x, -y)$ is also on the graph; the graph in part (b) is **symmetric about the y -axis** in the sense that for each point (x, y) on the graph the point $(-x, y)$ is also on the graph; and the graph in part (c) is **symmetric about the origin** in the sense that for each point (x, y) on the graph the point $(-x, -y)$ is also on the graph. Geometrically, symmetry about the origin occurs if rotating the graph 180° about the origin leaves the graph unchanged.

Symmetries can often be detected from the equation of a curve. For example, the graph of

$$y = x^3 \tag{7}$$

must be symmetric about the origin because for any point (x, y) whose coordinates satisfy

44 Functions

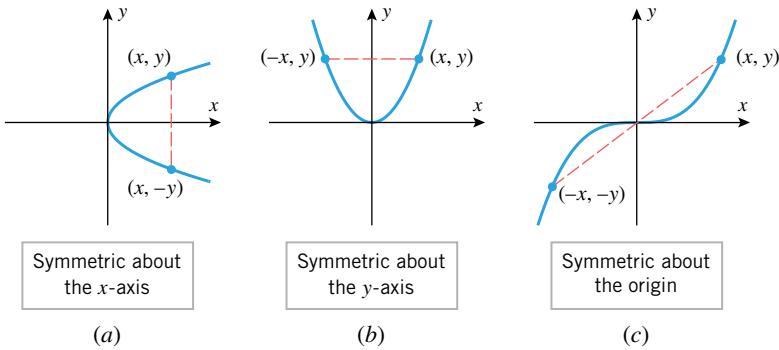


Figure 1.4.5

(7), the coordinates of the point $(-x, -y)$ also satisfy (7), since substituting these coordinates in (7) yields

$$-y = (-x)^3$$

which simplifies to (7). This suggests the following symmetry tests (Figure 1.4.6).

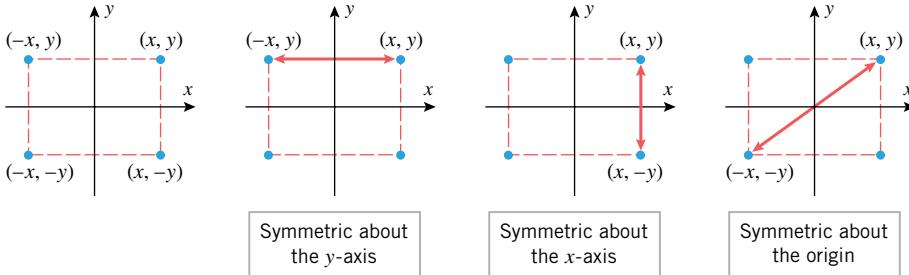


Figure 1.4.6

1.4.3 THEOREM (Symmetry Tests).

- (a) A plane curve is symmetric about the y-axis if and only if replacing x by $-x$ in its equation produces an equivalent equation.
- (b) A plane curve is symmetric about the x-axis if and only if replacing y by $-y$ in its equation produces an equivalent equation.
- (c) A plane curve is symmetric about the origin if and only if replacing both x by $-x$ and y by $-y$ in its equation produces an equivalent equation.

EVEN AND ODD FUNCTIONS

For the graph of a function f to be symmetric about the y-axis, the equations $y = f(x)$ and $y = f(-x)$ must be equivalent; for this to happen we must have

$$f(x) = f(-x)$$

A function with this property is called an **even function**. Some examples are x^2, x^4, x^6 , and $\cos x$. Similarly, for the graph of a function f to be symmetric about the origin, the equations $y = f(x)$ and $-y = f(-x)$ must be equivalent; for this to happen we must have

$$f(x) = -f(-x)$$

A function with this property is called an **odd function**. Some examples are x, x^3, x^5 , and $\sin x$.

- **FOR THE READER.** Explain why the graph of a nonzero function cannot be symmetric about the x-axis.

1.4 New Functions from Old 45

TRANSLATIONS

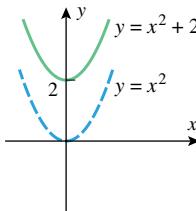
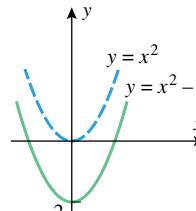
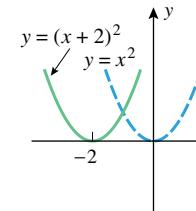
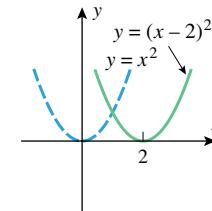
Once you know the graph of an equation $y = f(x)$, there are some techniques that can be used to help visualize the graphs of the equations

$$y = f(x) + c, \quad y = f(x) - c, \quad y = f(x + c), \quad y = f(x - c)$$

where c is any positive constant.

If a positive constant is added to or subtracted from $f(x)$, the geometric effect is to translate the graph of $y = f(x)$ parallel to the y -axis; addition translates the graph in the positive direction and subtraction translates it in the negative direction. This is illustrated in Table 1.4.2. Similarly, if a positive constant is added to or subtracted from the independent variable x , the geometric effect is to translate the graph of the function parallel to the x -axis; subtraction translates the graph in the positive direction, and addition translates it in the negative direction. This is also illustrated in Table 1.4.2.

Table 1.4.2

OPERATION ON $y = f(x)$	Add a positive constant c to $f(x)$	Subtract a positive constant c from $f(x)$	Add a positive constant c to x	Subtract a positive constant c from x
NEW EQUATION	$y = f(x) + c$	$y = f(x) - c$	$y = f(x + c)$	$y = f(x - c)$
GEOMETRIC EFFECT	Translates the graph of $y = f(x)$ up c units	Translates the graph of $y = f(x)$ down c units	Translates the graph of $y = f(x)$ left c units	Translates the graph of $y = f(x)$ right c units
EXAMPLE				

Before proceeding to the following examples, it will be helpful to review the graphs in Figures 1.2.1 and 1.2.5.

Example 8 Sketch the graph of

$$(a) y = \sqrt{x-3} \quad (b) y = \sqrt{x+3}$$

Solution. The graph of the equation $y = \sqrt{x-3}$ can be obtained by translating the graph of $y = \sqrt{x}$ right 3 units, and the graph of $y = \sqrt{x+3}$ by translating the graph of $y = \sqrt{x}$ left 3 units (Figure 1.4.7). ◀

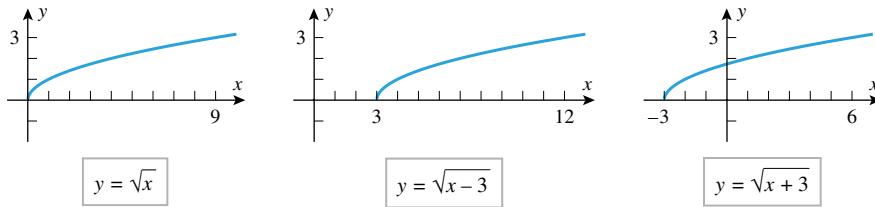


Figure 1.4.7

Example 9 Sketch the graph of $y = |x - 3| + 2$.

Solution. The graph can be obtained by two translations: first translate the graph of $y = |x|$ right 3 units to obtain the graph of $y = |x - 3|$, then translate this graph up 2 units to obtain the graph of $y = |x - 3| + 2$ (Figure 1.4.8). ◀

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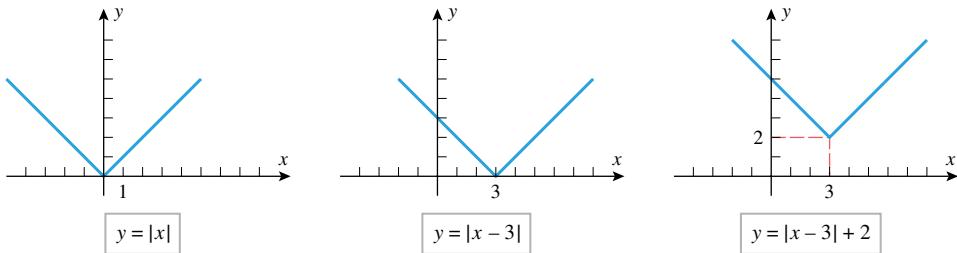


Figure 1.4.8

- REMARK. The graph in the preceding example could also have been obtained by performing the translations in the opposite order: first translating the graph of $y = |x|$ up 2 units to obtain the graph of $y = |x| + 2$, then translating this graph right 3 units to obtain the graph of $y = |x - 3| + 2$.

Example 10 Sketch the graph of $y = x^2 - 4x + 5$.

Solution. Completing the square on the first two terms yields

$$y = (x^2 - 4x + 4) - 4 + 5 = (x - 2)^2 + 1$$

(see Appendix D for a review of this technique). In this form we see that the graph can be obtained by translating the graph of $y = x^2$ right 2 units because of the $x - 2$, and up 1 unit because of the $+1$ (Figure 1.4.9). ◀

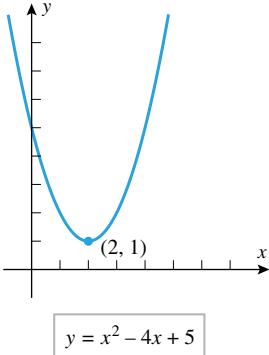


Figure 1.4.9

Example 11 By completing the square, an equation of the form $y = ax^2 + bx + c$ with $a \neq 0$ can be expressed as

$$y = a(x - h)^2 + k \quad (8)$$

Sketch the graph of this equation.

Solution. We can build up Equation (8) in three steps from the equation $y = x^2$. First, we can multiply by a to obtain $y = ax^2$. If $a > 0$, this operation has the geometric effect of stretching or compressing the graph of $y = x^2$; and if $a < 0$, it has the geometric effect of reflecting the graph about the x -axis, in addition to stretching or compressing it. Since stretching or compressing does not alter the general parabolic shape of the original curve, the graph of $y = ax^2$ looks roughly like one of those in Figure 1.4.10a. Next, we can subtract h from x to obtain the equation $y = a(x - h)^2$, and then we can add k to obtain $y = a(x - h)^2 + k$. Subtracting h causes a horizontal translation (right or left, depending on the sign of h), and adding k causes a vertical translation (up or down, depending on the sign of k). Thus, the graph of (8) looks roughly like one of those in Figure 1.4.10b, which are shown with $h > 0$ and $k > 0$ for simplicity. ◀

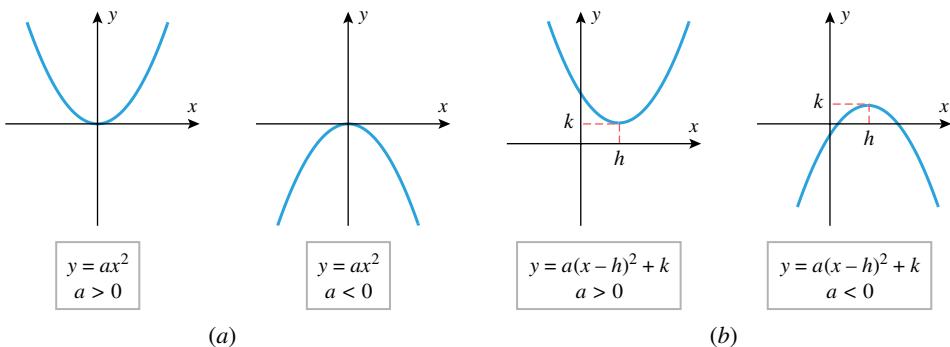


Figure 1.4.10

1.4 New Functions from Old 47

REFLECTIONS

The graph of $y = f(-x)$ is the reflection of the graph of $y = f(x)$ about the y -axis, and the graph of $y = -f(x)$ [or equivalently, $-y = f(x)$] is the reflection of the graph of $y = f(x)$ about the x -axis. Thus, if you know what the graph of $y = f(x)$ looks like, you can obtain the graphs of $y = f(-x)$ and $y = -f(x)$ by making appropriate reflections. This is illustrated in Table 1.4.3.

Table 1.4.3

OPERATION ON $y = f(x)$	Replace x by $-x$	Multiply $f(x)$ by -1
NEW EQUATION	$y = f(-x)$	$y = -f(x)$
GEOMETRIC EFFECT	Reflects the graph of $y = f(x)$ about the y -axis	Reflects the graph of $y = f(x)$ about the x -axis
EXAMPLE	<p>The figure shows two Cartesian coordinate systems. The left system has a horizontal axis labeled x with tick marks at -6, -3, 0, 3, and 6. The vertical axis is labeled y with tick marks at -3, 0, and 3. A green curve labeled $y = \sqrt{-x}$ starts at the origin (0,0) and curves downwards towards the negative x-axis. The right system has a horizontal axis labeled x with tick marks at -6, -3, 0, 3, and 6. The vertical axis is labeled y with tick marks at -3, 0, and 3. A blue curve labeled $y = \sqrt{x}$ starts at the origin (0,0) and curves upwards towards the positive x-axis.</p>	<p>The figure shows two Cartesian coordinate systems. The left system has a horizontal axis labeled x with tick marks at -6, -3, 0, 3, and 6. The vertical axis is labeled y with tick marks at -3, 0, and 3. A blue curve labeled $y = \sqrt{x}$ starts at the origin (0,0) and curves upwards towards the positive x-axis. The right system has a horizontal axis labeled x with tick marks at -6, -3, 0, 3, and 6. The vertical axis is labeled y with tick marks at -3, 0, and 3. A green curve labeled $y = -\sqrt{x}$ starts at the origin (0,0) and curves downwards towards the negative x-axis.</p>

- FOR THE READER. Describe the geometric effect of multiplying a function f by a *negative* constant in terms of reflection and stretching or compressing. What is the geometric effect of multiplying the independent variable of a function f by a *negative* constant?

Example 12 Sketch the graph of $y = \sqrt[3]{2-x}$.

Solution. The graph can be obtained by a reflection and a translation: first reflect the graph of $y = \sqrt[3]{x}$ about the y -axis to obtain the graph of $y = \sqrt[3]{-x}$, then translate this graph right 2 units to obtain the graph of the equation $y = \sqrt[3]{-(x-2)} = \sqrt[3]{2-x}$ (Figure 1.4.11). ◀

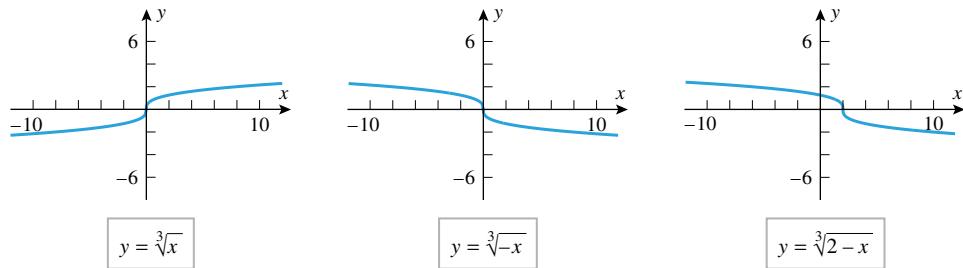


Figure 1.4.11

Example 13 Sketch the graph of $y = 4 - |x - 2|$.

Solution. The graph can be obtained by a reflection and two translations: first translate the graph of $y = |x|$ right 2 units to obtain the graph of $y = |x - 2|$; then reflect this graph about the x -axis to obtain the graph of $y = -|x - 2|$; and then translate this graph up 4 units to obtain the graph of the equation $y = -|x - 2| + 4 = 4 - |x - 2|$ (Figure 1.4.12). ◀

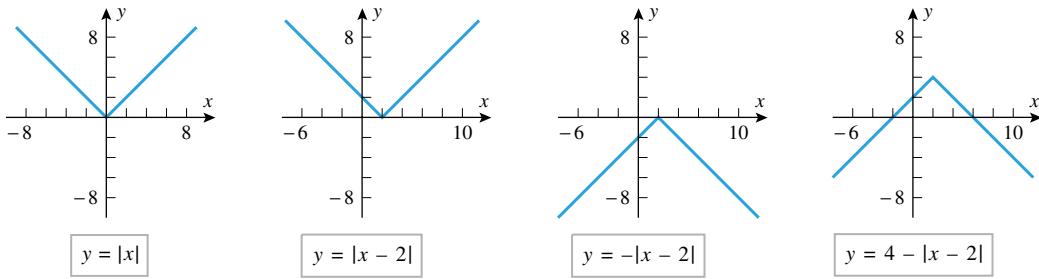
48 Functions

Figure 1.4.12

EXERCISE SET 1.4 Graphing Calculator

1. The graph of a function f is shown in the accompanying figure. Sketch the graphs of the following equations.

(a) $y = f(x) - 1$ (b) $y = f(x - 1)$
 (c) $y = \frac{1}{2}f(x)$ (d) $y = f\left(-\frac{1}{2}x\right)$

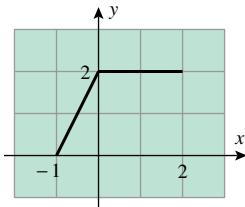


Figure Ex-1

2. Use the graph in Exercise 1 to sketch the graphs of the following equations.

(a) $y = -f(-x)$ (b) $y = f(2 - x)$
 (c) $y = 1 - f(2 - x)$ (d) $y = \frac{1}{2}f(2x)$

3. The graph of a function f is shown in the accompanying figure. Sketch the graphs of the following equations.

(a) $y = f(x + 1)$ (b) $y = f(2x)$
 (c) $y = |f(x)|$ (d) $y = 1 - |f(x)|$

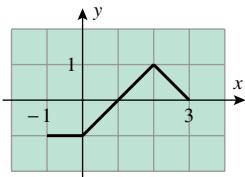


Figure Ex-3

4. Use the graph in Exercise 3 to sketch the graph of the equation $y = f(|x|)$.

In Exercises 5–12, sketch the graph of the equation by translating, reflecting, compressing, and stretching the graph of $y = x^2$ appropriately, and then use a graphing utility to confirm that your sketch is correct.

5. $y = 1 + (x - 2)^2$ 6. $y = 2 - (x + 1)^2$
 7. $y = -2(x + 1)^2 - 3$ 8. $y = \frac{1}{2}(x - 3)^2 + 2$
 9. $y = x^2 + 6x$ 10. $y = x^2 + 6x - 10$

In Exercises 13–16, sketch the graph of the equation by translating, reflecting, compressing, and stretching the graph of $y = \sqrt{x}$ appropriately, and then use a graphing utility to confirm that your sketch is correct.

13. $y = 3 - \sqrt{x + 1}$ 14. $y = 1 + \sqrt{x - 4}$
 15. $y = \frac{1}{2}\sqrt{x} + 1$ 16. $y = -\sqrt{3x}$

In Exercises 17–20, sketch the graph of the equation by translating, reflecting, compressing, and stretching the graph of $y = 1/x$ appropriately, and then use a graphing utility to confirm that your sketch is correct.

17. $y = \frac{1}{x - 3}$ 18. $y = \frac{1}{1 - x}$
 19. $y = 2 - \frac{1}{x + 1}$ 20. $y = \frac{x - 1}{x}$

In Exercises 21–24, sketch the graph of the equation by translating, reflecting, compressing, and stretching the graph of $y = |x|$ appropriately, and then use a graphing utility to confirm that your sketch is correct.

21. $y = |x + 2| - 2$ 22. $y = 1 - |x - 3|$
 23. $y = |2x - 1| + 1$ 24. $y = \sqrt{x^2 - 4x + 4}$

In Exercises 25–28, sketch the graph of the equation by translating, reflecting, compressing, and stretching the graph of $y = \sqrt[3]{x}$ appropriately, and then use a graphing utility to confirm that your sketch is correct.

25. $y = 1 - 2\sqrt[3]{x}$ 26. $y = \sqrt[3]{x - 2} - 3$
 27. $y = 2 + \sqrt[3]{x + 1}$ 28. $y = \sqrt[3]{x - 2} = 0$
 29. (a) Sketch the graph of $y = x + |x|$ by adding the corresponding y -coordinates on the graphs of $y = x$ and $y = |x|$.

- (b) Express the equation $y = x + |x|$ in piecewise form with no absolute values, and confirm that the graph you obtained in part (a) is consistent with this equation.

30. Sketch the graph of $y = x + (1/x)$ by adding corresponding y -coordinates on the graphs of $y = x$ and $y = 1/x$. Use a graphing utility to confirm that your sketch is correct.

In Exercises 31–34, find formulas for $f + g$, $f - g$, fg , and f/g , and state the domains of the functions.

In Exercises 39–44, find formulas for $f \circ g$ and $g \circ f$, and state the domains of the functions.

39. $f(x) = 2x + 1$, $g(x) = x^2 - x$
 40. $f(x) = 2 - x^2$, $g(x) = x^3$
 41. $f(x) = x^2$, $g(x) = \sqrt{1 - x}$
 42. $f(x) = \sqrt{x - 3}$, $g(x) = \sqrt{x^2 + 3}$
 43. $f(x) = \frac{1+x}{1-x}$, $g(x) = \frac{x}{1-x}$
 44. $f(x) = \frac{x}{1+x^2}$, $g(x) = \frac{1}{x}$

In Exercises 45 and 46, find a formula for $f \circ g \circ h$.

- 45.** $f(x) = x^2 + 1$, $g(x) = \frac{1}{x}$, $h(x) = x^3$

46. $f(x) = \frac{1}{1+x}$, $g(x) = \sqrt[3]{x}$, $h(x) = \frac{1}{x^3}$

In Exercises 47–50, express f as a composition of two functions; that is, find g and h such that $f = g \circ h$. [Note: Each exercise has more than one solution.]

- 47.** (a) $f(x) = \sqrt{x+2}$ (b) $f(x) = |x^2 - 3x + 5|$

48. (a) $f(x) = x^2 + 1$ (b) $f(x) = \frac{1}{x-3}$

49. (a) $f(x) = \sin^2 x$ (b) $f(x) = \frac{3}{5 + \cos x}$

50. (a) $f(x) = 3 \sin(x^2)$ (b) $f(x) = 3 \sin^2 x + 4 \sin x$

In Exercises 51 and 52, express F as a composition of three functions; that is, find f , g , and h such that $F = f \circ g \circ h$. [Note: Each exercise has more than one solution.]

- 51.** (a) $F(x) = (1 + \sin(x^2))^3$ (b) $F(x) = \sqrt{1 - \sqrt[3]{x}}$

52. (a) $F(x) = \frac{1}{1 - x^2}$ (b) $F(x) = |5 + 2x|$

53. Use the accompanying table to make a scatter plot of $y = f(g(x))$.

x	-3	-2	-1	0	1	2	3
$f(x)$	-4	-3	-2	-1	0	1	2
$g(x)$	-1	0	1	2	3	-2	-3

Table Ex-53

- 54.** Find the domain of $g \circ f$ for the functions f and g in Exercise 53.

55. Sketch the graph of $y = f(g(x))$ for the functions graphed in the accompanying figure.

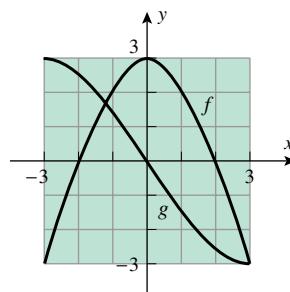


Figure Ex-55

- 56.** Sketch the graph of $y = g(f(x))$ for the functions graphed in Exercise 55.

57. Use the graphs of f and g in Exercise 55 to estimate the solutions of the equations $f(g(x)) = 0$ and $g(f(x)) = 0$.

58. Use the table in Exercise 53 to solve the equations $f(g(x)) = 0$ and $g(f(x)) = 0$.

50 Functions

In Exercises 59–62, find

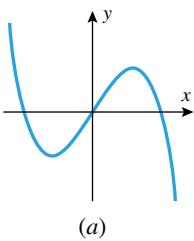
$$\frac{f(x+h) - f(x)}{h} \quad \text{and} \quad \frac{f(w) - f(x)}{w-x}$$

Simplify as much as possible.

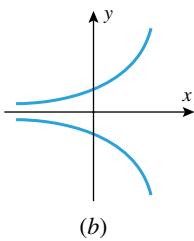
59. $f(x) = 3x^2 - 5$

61. $f(x) = 1/x$

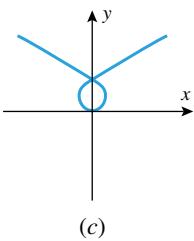
63. In each part of the accompanying figure determine whether the graph is symmetric about the x -axis, the y -axis, the origin, or none of the preceding.



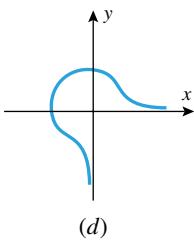
(a)



(b)



(c)



(d)

Figure Ex-63

64. The accompanying figure shows a portion of a graph. Complete the graph so that the entire graph is symmetric about
 (a) the x -axis (b) the y -axis (c) the origin.

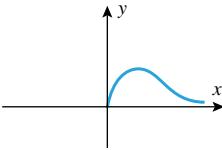


Figure Ex-64

65. Complete the accompanying table so that the graph of $y = f(x)$ (which is a scatter plot) is symmetric about
 (a) the y -axis (b) the origin.

x	-3	-2	-1	0	1	2	3
$f(x)$	1		-1	0		-5	

Table Ex-65

66. The accompanying figure shows a portion of the graph of a function f . Complete the graph assuming that
 (a) f is an even function (b) f is an odd function.

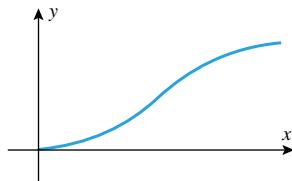
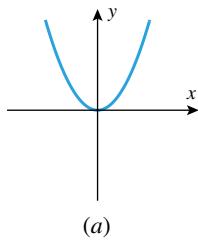
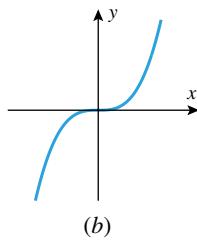


Figure Ex-66

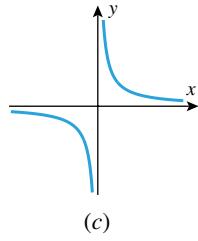
67. Classify the functions graphed in the accompanying figure as even, odd, or neither.



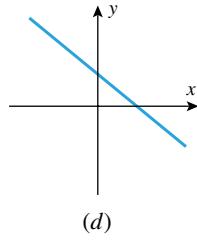
(a)



(b)



(c)



(d)

Figure Ex-67

68. Classify the functions whose values are given in the following table as even, odd, or neither.

x	-3	-2	-1	0	1	2	3
$f(x)$	5	3	2	3	1	-3	5
$g(x)$	4	1	-2	0	2	-1	-4
$h(x)$	2	-5	8	-2	8	-5	2

69. In each part, classify the function as even, odd, or neither.

$$\begin{array}{ll} (\text{a}) f(x) = x^2 & (\text{b}) f(x) = x^3 \\ (\text{c}) f(x) = |x| & (\text{d}) f(x) = x + 1 \\ (\text{e}) f(x) = \frac{x^5 - x}{1 + x^2} & (\text{f}) f(x) = 2 \end{array}$$

In Exercises 70 and 71, use Theorem 1.4.3 to determine whether the graph has symmetries about the x -axis, the y -axis, or the origin.

70. (a) $x = 5y^2 + 9$ (b) $x^2 - 2y^2 = 3$
 (c) $xy = 5$
71. (a) $x^4 = 2y^3 + y$ (b) $y = \frac{x}{3 + x^2}$
 (c) $y^2 = |x| - 5$

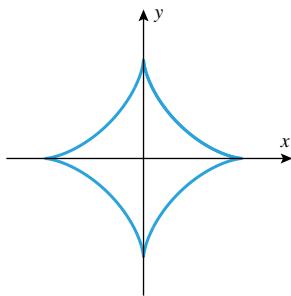
In Exercises 72 and 73: (i) Use a graphing utility to graph the equation in the first quadrant. [Note: To do this you will have to solve the equation for y in terms of x .] (ii) Use symmetry to make a hand-drawn sketch of the entire graph. (iii) Confirm your work by generating the graph of the equation in the remaining three quadrants.

72. $9x^2 + 4y^2 = 36$

73. $4x^2 + 16y^2 = 16$

74. The graph of the equation $x^{2/3} + y^{2/3} = 1$, which is shown in the accompanying figure, is called a **four-cusped hypocycloid**.

- Use Theorem 1.4.3 to confirm that this graph is symmetric about the x -axis, the y -axis, and the origin.
- Find a function f whose graph in the first quadrant coincides with the four-cusped hypocycloid, and use a graphing utility to confirm your work.
- Repeat part (b) for the remaining three quadrants.



Four-cusped hypocycloid

Figure Ex-74

75. The equation $y = |f(x)|$ can be written as

$$y = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases}$$

which shows that the graph of $y = |f(x)|$ can be obtained from the graph of $y = f(x)$ by retaining the portion that lies on or above the x -axis and reflecting about the x -axis the portion that lies below the x -axis. Use this method to obtain the graph of $y = |2x - 3|$ from the graph of $y = 2x - 3$.

In Exercises 76 and 77, use the method described in Exercise 75.

76. Sketch the graph of $y = |1 - x^2|$.

77. Sketch the graph of

(a) $f(x) = |\cos x|$ (b) $f(x) = \cos x + |\cos x|$.

78. The **greatest integer function**, $[x]$, is defined to be the greatest integer that is less than or equal to x . For example, $[2.7] = 2$, $[-2.3] = -3$, and $[4] = 4$. Sketch the graph of

(a) $f(x) = [x]$	(b) $f(x) = [x]^2$
(c) $f(x) = [x]^2$	(d) $f(x) = [\sin x]$

79. Is it ever true that $f \circ g = g \circ f$ if f and g are nonconstant functions? If not, prove it; if so, give some examples for which it is true.

80. In the discussion preceding Exercise 29 of Section 1.3, we gave a procedure for generating a complete graph of $f(x) = x^{p/q}$ in which we suggested graphing the function $g(x) = |x|^{p/q}$ instead of $f(x)$ when p is even and q is odd and graphing $g(x) = (|x|/x)|x|^{p/q}$ if p is odd and q is odd. Show that in both cases $f(x) = g(x)$ if $x > 0$ or $x < 0$. [Hint: Show that $f(x)$ is an even function if p is even and q is odd and is an odd function if p is odd and q is odd.]

1.5 LINES

This section includes a quick review of precalculus material on lines. Readers who want to review this material in more depth are referred to Appendix C.

EQUATIONS OF LINES

An equation that is expressible in the form

$$Ax + By + C = 0 \tag{1}$$

where A and B are not both zero, is called a **first-degree equation** or a **linear equation** in x and y . It is shown in precalculus that every first-degree equation in x and y has a straight line as its graph and, conversely, every straight line can be represented by a first-degree equation in x and y . For this reason (1) is sometimes called the **general equation** of a line. Recall that equations of lines may be written in several different forms:

$$y = mx + b \quad \text{Slope-intercept form} \tag{2}$$

$$y - y_1 = m(x - x_1) \quad \text{Point-slope form} \tag{3}$$

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{Double-intercept form} \tag{4}$$

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In these equations m is the slope of the line, a is the x -intercept, b is the y -intercept, and (x_1, y_1) is any point on the line (Figure 1.5.1). Keep in mind that these equations do not apply to vertical lines. For vertical lines the slope is *undefined*, or stated informally, a vertical line has infinite slope. Vertical and horizontal lines have particularly simple equations:

$$x = a \quad \boxed{\text{The vertical line with } x\text{-intercept } a} \quad (5)$$

$$y = b \quad \boxed{\text{The horizontal line with } y\text{-intercept } b} \quad (6)$$

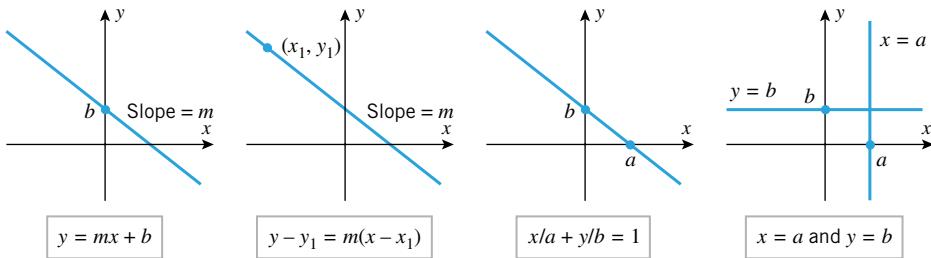


Figure 1.5.1

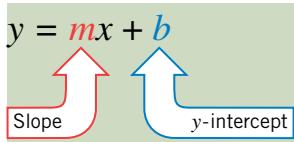


Figure 1.5.2

INTERPRETATIONS OF SLOPE

Equation (2) is especially useful because the slope and the y -intercept of the line can be determined by inspection: the slope is the coefficient of x , and the y -intercept is the constant term (Figure 1.5.2). This equation expresses y as a function of x , the function being $f(x) = mx + b$. A function of this form is called a *linear function* of x .

- m is a measure of the *steepness* of the line.
- m is the rate of change of y with respect to x .

The steepness interpretation has an analog in surveying. Surveyors measure the grade or slope of a hill as the ratio of its rise over its run (Figure 1.5.3a). The same idea applies to lines. Consider a particle that moves left to right along a nonvertical line from a point $P_1(x_1, y_1)$ to a point $P_2(x_2, y_2)$. In the course of its travel the point moves $y_2 - y_1$ units vertically as it travels $x_2 - x_1$ units horizontally (Figure 1.5.3b). The vertical change, which is denoted by $\Delta y = y_2 - y_1$, is called the *rise*, and the horizontal change, which is denoted by $\Delta x = x_2 - x_1$, is called the *run*. The ratio of the rise over the run is always equal to the

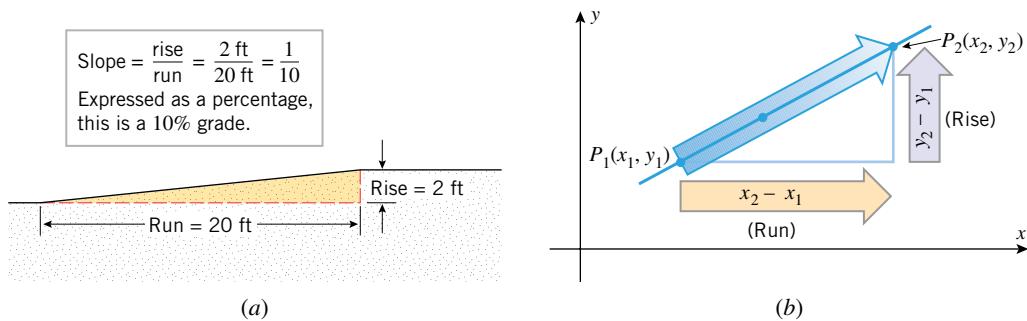


Figure 1.5.3

slope, regardless of where the points P_1 and P_2 are located on the line; that is,

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \quad (7)$$

REMARK. The symbols Δx and Δy should not be interpreted as products; rather, Δx should be viewed as a single entity representing the *change* in the value of x , and Δy as a single entity representing the *change* in the value of y . In general, if v is any variable whose value changes from an initial value of v_1 to a final value of v_2 , then we call $\Delta v = v_2 - v_1$ (final value minus initial value) an **increment** in v . Increments can be positive or negative, depending on whether the final value is larger or smaller than the initial value.

ANGLE OF INCLINATION

The slope of a nonvertical line L is related to the angle that L makes with the positive x -axis. If ϕ is the smallest positive angle measured counterclockwise from the x -axis to L , then the slope of the line can be expressed as

$$m = \tan \phi \quad (8)$$

(Figure 1.5.4a). The angle ϕ , which is called the **angle of inclination** of the line, satisfies $0^\circ \leq \phi < 180^\circ$ in degree measure (or, equivalently, $0 \leq \phi < \pi$ in radian measure). If ϕ is an acute angle, then $m = \tan \phi$ is positive and the line slopes up to the right, and if ϕ is an obtuse angle, then $m = \tan \phi$ is negative and the line slopes down to the right. For example, a line whose angle of inclination is 45° has slope $m = \tan 45^\circ = 1$, and a line whose angle of inclination is 135° has a slope of $m = \tan 135^\circ = -1$ (Figure 1.5.4b). Figure 1.5.5 shows a convenient way of using the line $x = 1$ as a “ruler” for visualizing the relationship between lines of various slopes.

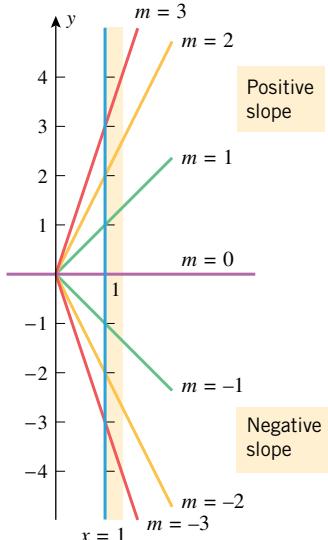


Figure 1.5.5

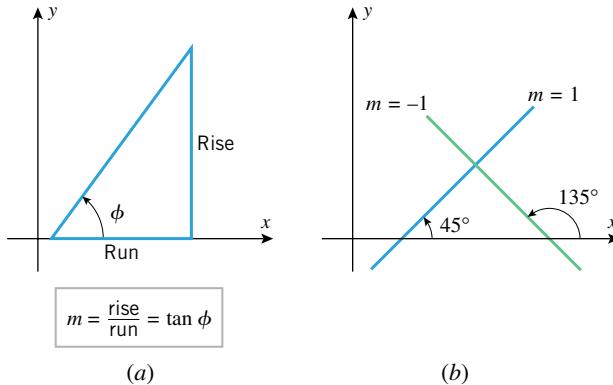


Figure 1.5.4

SLOPES OF LINES IN APPLIED PROBLEMS

In applied problems, changing the units of measurement can change the slope of a line, so it is essential to include the units when calculating the slope. The following example illustrates this.

Example 1 Suppose that a uniform rod of length 40 cm ($= 0.4$ m) is thermally insulated around the lateral surface and that the exposed ends of the rod are held at constant temperatures of 25°C and 5°C , respectively (Figure 1.5.6a). It is shown in physics that under appropriate conditions the graph of the temperature T versus the distance x from the left-hand end of the rod will be a straight line. Parts (b) and (c) of Figure 1.5.6 show two such graphs: one in which x is measured in centimeters and one in which it is measured in

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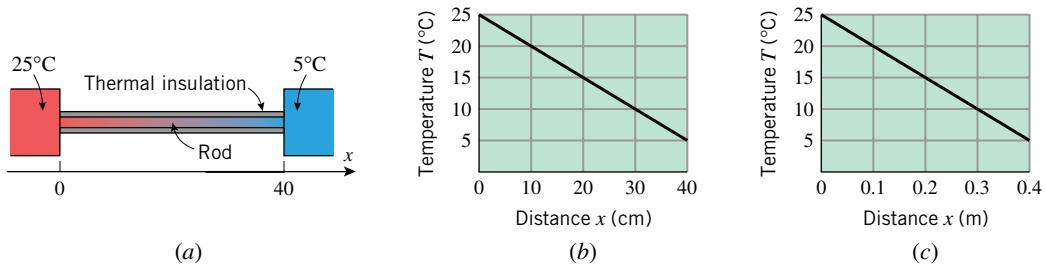


Figure 1.5.6

meters. The slopes in the two cases are

$$m = \frac{5 - 25}{40 - 0} = \frac{-20}{40} = -0.5 \quad (9)$$

$$m = \frac{5 - 25}{0.4 - 0} = \frac{-20}{0.4} = -50 \quad (10)$$

The slope in (9) implies that the temperature *decreases* at a rate of 0.5°C per centimeter of distance from the left end of the rod, and the slope in (10) implies that the temperature decreases at a rate of 50°C per meter of distance from the left end of the rod. The two statements are equivalent physically, even though the slopes differ. ◀

Example 2 Find the slope-intercept form of the equation of the temperature distribution in the preceding example if the temperature T is measured in degrees Celsius ($^{\circ}\text{C}$) and the distance x is measured in (a) centimeters and (b) meters.

Solution (a). The slope is $m = -0.5$ and the intercept on the T -axis is 25, so

$$T = -0.5x + 25, \quad 0 \leq x \leq 40$$

where the restriction on x is required because the rod is 40 cm in length. The graph of this equation with the restriction is a line segment rather than a line.

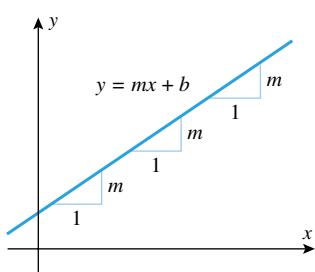
Solution (b). The slope is $m = -50$, the intercept on the T -axis is 25, and the restriction on x is $0 \leq x \leq 0.4$. Thus, the equation is

$$T = -50x + 25, \quad 0 \leq x \leq 0.4$$

If y is a linear function of x , say $y = mx + b$, then it follows from (7) that

$$\Delta y = m \Delta x$$

Thus, a 1-unit increase in x ($\Delta x = 1$) produces an m -unit change in y ($\Delta y = m$). Moreover, this is true at every point on the line (Figure 1.5.7), so we say that y changes at a *constant rate* with respect to x , and we call m the **rate of change of y with respect to x** . This idea can be summarized as follows.



A 1-unit increase in x always produces an m -unit change in y .

1.5.1 CONSTANT RATE OF CHANGE. If a variable y is related to a variable x in such a way that the rate of change of y with respect to x is constant, say m , then y is a linear function of x of the form

$$v \equiv mx + b$$

Conversely, if y is a linear function of x whose graph has slope m , then the rate of change of y with respect to x is constant and equal to m .

Figure 1.5.7

It follows from this that linear functions are appropriate whenever experimentation or theory suggests that the rate of change of y with respect to x is constant.

UNIFORM RECTILINEAR MOTION

One of the important themes in calculus is the study of motion. To describe the motion of an object completely, one must specify its *speed* (how fast it is going) and the direction in which it is moving. The speed and the direction of motion together comprise what is called the *velocity* of the object. For example, knowing that the speed of an aircraft is 500 mi/h tells us how fast it is going, but not which way it is moving. In contrast, knowing that the velocity of the aircraft is 500 mi/h *due south* pins down the speed and the direction of motion.

Later, we will study the motion of particles that move along curves in two- or three-dimensional space, but for now we will focus on motion along a line; this is called **rectilinear motion**. In general rectilinear motion, a particle can move back and forth along the line (as with a piston moving up and down in a cylinder); however, for now we will only consider the simple case in which the particle moves in just *one direction* along a line (as with a car traveling on a straight road).

For simplicity, we will assume that the motion is along a coordinate line, such as an *x*-axis or *y*-axis, and that the particle is moving in the positive direction. In general discussions we will usually name the coordinate line the *s*-axis to avoid being specific. A graphical description of rectilinear motion along an *s*-axis can be obtained by making a plot of the *s*-coordinate of the particle versus the elapsed time *t*. This is called the **position versus time curve** for the particle. Figure 1.5.8a shows a typical position versus time curve for a car moving in the positive direction along an *s*-axis.

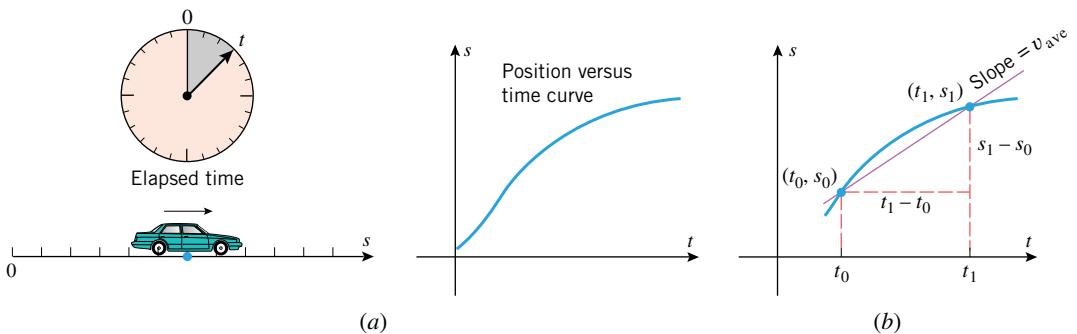


Figure 1.5.8

- **FOR THE READER.** How can you tell from the position versus time curve in Figure 1.5.8a that the car does not reverse direction?

Because we are assuming that the particle is moving in the positive direction of the *s*-axis, there is no ambiguity about the direction of motion, and hence the terms “speed” and “velocity” can be used interchangeably. However, later, when we consider general rectilinear motion or motion along a curved path, it will be necessary to distinguish between these terms, since the direction of motion may vary.

For a particle in rectilinear motion along a coordinate axis, we define the **average velocity** v_{ave} of the particle during the time interval from t_0 to t_1 to be

$$v_{\text{ave}} = \frac{s_1 - s_0}{t_1 - t_0} = \frac{\Delta s}{\Delta t} \quad (11)$$

where s_0 and s_1 are the *s*-coordinates of the particle at times t_0 and t_1 , respectively. Geometrically, this is the slope of the secant line connecting the points (t_0, s_0) and (t_1, s_1) on the position versus time curve (Figure 1.5.8b). The quantity $\Delta s = s_1 - s_0$ is called the **displacement** or **change in position** of the particle during the time interval from t_0 to t_1 . With this terminology, Formula (11) states that for a particle in rectilinear motion *the average velocity over a time interval is the displacement during the time interval divided by the length of the time interval*. For example, if a car moving in one direction along a straight road travels 75 miles in 3 hours, then its average velocity is $75/3 = 25$ mi/h.

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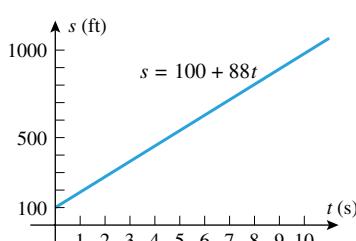


Figure 1.5.9

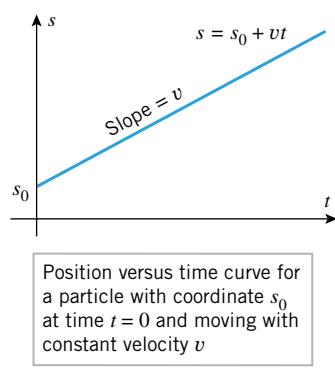


Figure 1.5.10

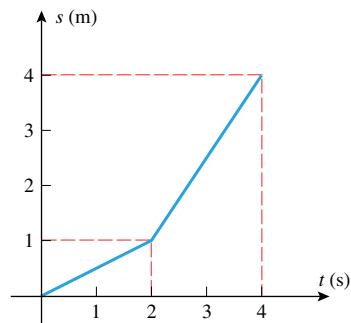


Figure 1.5.11

CONSTANT ACCELERATION

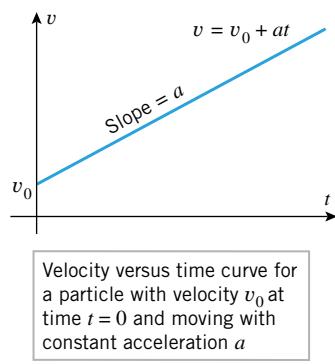


Figure 1.5.12

In the special case where the average velocity of a particle in rectilinear motion is the same over every time interval, the particle is said to have **constant velocity** or **uniform rectilinear motion**. If the average velocity over every time interval is v , then we will refer to v as the **velocity** of the particle (dropping the adjective “average”).

For a particle with uniform rectilinear motion the displacement over *any* time interval is given by the formula

$$\text{displacement} = \text{velocity} \times \text{elapsed time} \quad (12)$$

Example 3 Suppose that a car moves with a constant velocity of 88 ft/s in the positive direction of an s -axis. Given that the s -coordinate of the car at time $t = 0$ is $s = 100$, find an equation for s as a function of t , and graph the position versus time curve.

Solution. It follows from (12) that in a period of t seconds, the car will move $88t$ feet from its starting point, so its coordinate s at time t will be

$$s = 100 + 88t$$

The graph of this equation is the line in Figure 1.5.9. ◀

It is not accidental that the position versus time curve turned out to be a line in the last example; this will always be the case for uniform rectilinear motion. To see why this is so, suppose that a particle moves with constant velocity v in the positive direction along an s -axis, starting at the point s_0 at time $t = 0$. It follows from (12) that in t units of time the particle will move vt units from its starting point s_0 , so its coordinate s at time t will be

$$s = s_0 + vt$$

which is a line with s -intercept s_0 and slope v (Figure 1.5.10). It follows from this equation and 1.5.1 that we can view the velocity v as the rate of change of s with respect to t , that is, the rate of change of position with respect to time.

Example 4 Figure 1.5.11 shows the position versus time curve for a particle moving along an s -axis. Describe the motion of the particle in words.

Solution. At time $t = 0$ the particle is at the origin. From time $t = 0$ to $t = 2$ the slope of the line segment is $\frac{1}{2}$, so the particle is moving with a constant velocity of $\frac{1}{2} = 0.5$ m/s. At time $t = 2$ the particle is at the point $s = 1$ (i.e., 1 meter from the origin). From time $t = 2$ to $t = 4$ the slope of the line segment is $\frac{3}{2}$, so the particle is moving with a constant velocity of $\frac{3}{2} = 1.5$ m/s. At time $t = 4$ it is at the point $s = 4$. ◀

In everyday language we say that an object is “accelerating” if it is speeding up and “decelerating” if it is slowing down. Mathematically, the **acceleration** of a particle in rectilinear motion is defined to be the *rate of change of velocity with respect to time*, where the acceleration is positive if the velocity is increasing and negative if it is decreasing. Thus, for a particle that moves in the positive direction of an s -axis, negative acceleration means the particle is “decelerating” in everyday language. Acceleration, like velocity, can be variable or constant. For example, by pressing the gas pedal of a car toward the floor smoothly, the driver can make the car’s velocity increase at a constant rate (a constant acceleration); however, if the driver suddenly slams the pedal to the floor, the car will lurch forward, reflecting a nonconstant acceleration. Later in the text we will study acceleration in more depth, but for now we will only consider the case in which acceleration is constant.

• **REMARK.** The units of acceleration are units of velocity divided by units of time. For example, if the velocity of a particle is increasing at a rate of 3 feet per second each second, then its acceleration is 3 ft/s/s (velocity in ft/s divided by time in s); this is usually written as 3 ft/s^2 (read “3 feet per second per second” or “3 feet per second squared”). Similarly, if the velocity of a particle is decreasing at a rate of 3 feet per second each second, then it has an acceleration of -3 ft/s^2 .

Graphical information about the acceleration of a particle can be obtained from the graph of velocity versus time; this is called the **velocity versus time curve**. In the case where the particle has constant acceleration, the velocity versus time curve will be linear, and its slope, which is the rate of change of velocity with time, will be the acceleration (Figure 1.5.12).

Example 5 Suppose that a car moves in the positive direction of an s -axis in such a way that its velocity v increases at a constant rate of 2 ft/s^2 .

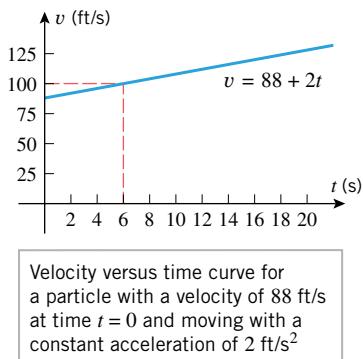


Figure 1.5.13

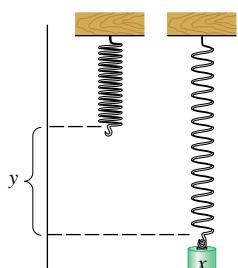
DIRECT PROPORTION

Figure 1.5.14

Recall that a variable y is said to be **directly proportional** to a variable x if there is a positive constant k , called the **constant of proportionality**, such that

$$y = kx \quad (14)$$

The graph of this equation is a line through the origin whose slope k is the constant of proportionality. Thus, linear functions are appropriate in physical problems where one variable is directly proportional to another.

Hooke's law* in physics provides a nice example of direct proportion. It follows from this law that if a weight of x units is suspended from a spring, then the spring will be stretched by an amount y that is directly proportional to x , that is, $y = kx$ (Figure 1.5.14). The constant k depends on the stiffness of the spring: the stiffer the spring, the smaller the value of k (why?).

Example 6 Figure 1.5.15 shows an old-fashioned spring scale that is calibrated in pounds.

- (a) Given that the pound scale marks are 0.5 in apart, find an equation that expresses the length y that the spring is stretched (in inches) in terms of the suspended weight x (in pounds).
- (b) Graph the equation obtained in part (a).

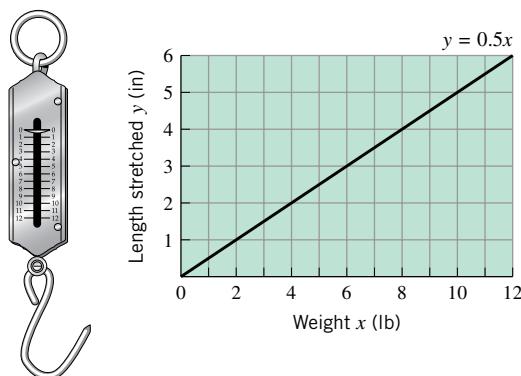


Figure 1.5.15

* Hooke's law, named for the English physicist Robert Hooke (1635–1703), applies only for small displacements that do not stretch the spring to the point of permanently distorting it.

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Solution (a). It follows from Hooke's law that y is related to x by an equation of the form $y = kx$. To find k we rewrite this equation as $k = y/x$ and use the fact that a weight of $x = 1$ lb stretches the spring $y = 0.5$ in. Thus,

$$k = \frac{y}{x} = \frac{0.5}{1} = 0.5 \quad \text{and hence} \quad y = 0.5x$$

Solution (b). The graph of the equation $y = 0.5x$ is shown in Figure 1.5.15. 

LINEAR DATA

One method for determining whether n points

$$(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$$

lie on a line is to compare the slopes of the line segments joining successive points. The points lie on a line if and only if those slopes are equal (Figure 1.5.16).

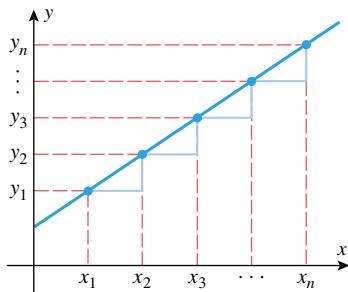


Figure 1.5.16

Example 7 Consider the data in Table 1.5.1.

- Explain why a linear function is appropriate for the data in the table.
- Find a linear equation that relates x and y , and graph the equation and the data together.

Solution (a). The five data points lie on a line, since each 1-unit increase in x produces a corresponding 0.8-unit increase in y . Thus, the slope of the line segment joining any two successive data points is

$$m = \frac{\Delta y}{\Delta x} = \frac{0.8}{1} = 0.8$$

Table 1.5.1

x	y
1.5	0.3
2.5	1.1
3.5	1.9
5.5	3.5
9.5	6.7

Figure 1.5.17

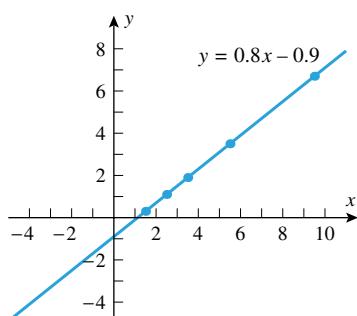
Solution (b). A linear equation relating x and y can be obtained from the point-slope form of the line using the slope $m = 0.8$ calculated in part (a) and any one of the five data points. If we use the first data point, $(1.5, 0.3)$, we obtain

$$y - 0.3 = 0.8(x - 1.5)$$

or in slope-intercept form,

$$y = 0.8x - 0.9$$

The graph of this equation together with the given data are shown in Figure 1.5.17. 

**OTHER APPLICATIONS OF LINEAR FUNCTIONS**

Linear functions arise in a variety of practical problems. Here is a typical example.

Example 8 A university parking lot charges \$3.00 per day but offers a \$40.00 monthly sticker with which the student pays only \$0.25 per day.

- **REMARK.** Sometimes, data points that should theoretically lie on a line do not because of experimental error and other factors. In such cases curve-fitting techniques are used to find a line that most closely fits the data. Such techniques will be discussed later in the text.

- (a) Find equations for the cost C of parking for x days per month under both payment methods, and graph the equations for $0 \leq x \leq 30$. (Treat C as a continuous function of x , even though x only assumes integer values.)
- (b) Find the value of x for which the graphs intersect, and discuss the significance of this value.

Solution (a). The cost in dollars of parking for x days at \$3.00 per day is $C = 3x$, and the cost for the \$40.00 sticker plus x days at \$0.25 per day is $C = 40 + 0.25x$ (Figure 1.5.18).

Solution (b). The graphs intersect at the point where

$$3x = 40 + 0.25x$$

which is $x = 40/2.75 \approx 14.5$. This value of x is not an option for the student, since x must be an integer. However, it is the dividing point at which the monthly sticker method becomes less expensive than the daily payment method; that is, for $x \geq 15$ it is cheaper to buy the monthly sticker and for $x \leq 14$ it is cheaper to pay the daily rate. \blacktriangleleft

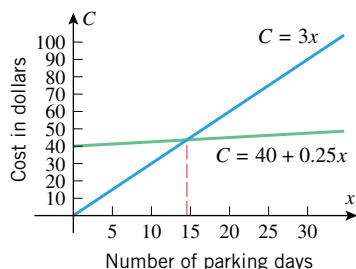


Figure 1.5.18

EXERCISE SET 1.5 Graphing Calculator

Exercises 1–26 involve the basic properties of lines and slope. In some of these exercises you will need to use slopes to determine whether two lines are parallel or perpendicular. If you have forgotten how to do this, review Appendix C.

1. (a) Find the slopes of the sides of the triangle with vertices $(0, 3)$, $(2, 0)$, and $(6, \frac{8}{3})$.
- (b) Is this a right triangle? Explain.
2. (a) Find the slopes of the sides of the quadrilateral with vertices $(-3, -1)$, $(5, -1)$, $(7, 3)$, and $(-1, 3)$.
- (b) Is this a parallelogram? Explain.
3. List the lines in the accompanying figure in the order of increasing slope.

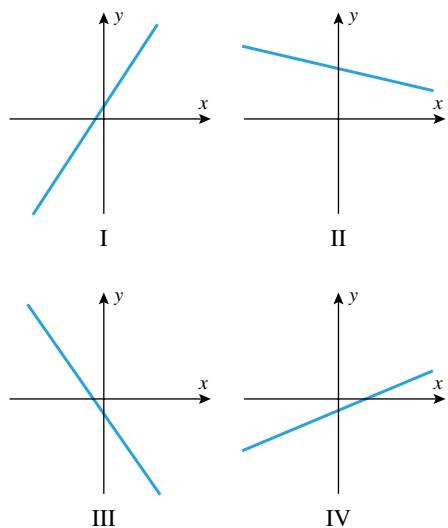


Figure Ex-3

4. List the lines in the accompanying figure in the order of increasing slope.

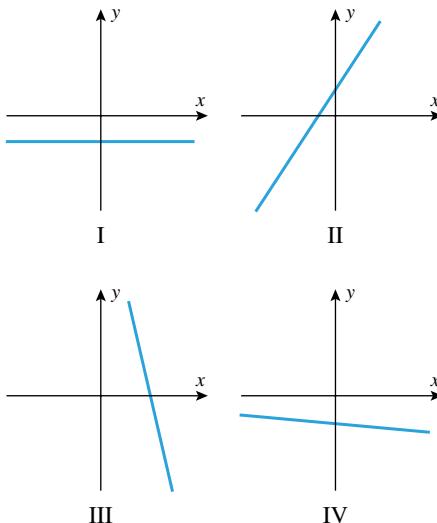


Figure Ex-4

5. Use slopes to determine whether the given points lie on the same line.
 - (a) $(1, 1)$, $(-2, -5)$, and $(0, -1)$
 - (b) $(-2, 4)$, $(0, 2)$, and $(1, 5)$
6. A particle, initially at $(7, 5)$, moves along a line of slope $m = -2$ to a new position (x, y) .
 - (a) Find y if $x = 9$.
 - (b) Find x if $y = 12$.
7. A particle, initially at $(1, 2)$, moves along a line of slope $m = 3$ to a new position (x, y) .
 - (a) Find y if $x = 5$.
 - (b) Find x if $y = -2$.

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8. Find x and y if the line through $(0, 0)$ and (x, y) has slope $\frac{1}{2}$, and the line through (x, y) and $(7, 5)$ has slope 2.
9. Find x if the slope of the line through $(1, 2)$ and $(x, 0)$ is the negative of the slope of the line through $(4, 5)$ and $(x, 0)$.

In Exercises 10 and 11, find the angle of inclination of the line with slope m to the nearest degree. Use a calculating utility, where needed.

10. (a) $m = \frac{1}{2}$ (b) $m = -1$
 (c) $m = 2$ (d) $m = -57$
11. (a) $m = -\frac{1}{2}$ (b) $m = 1$
 (c) $m = -2$ (d) $m = 57$

In Exercises 12 and 13, find the angle of inclination of the line to the nearest degree. Use a calculating utility, where needed.

12. (a) $3y = 2 - \sqrt{3}x$ (b) $y - 4x + 7 = 0$
 13. (a) $y = \sqrt{3}x + 2$ (b) $y + 2x + 5 = 0$
14. Find equations for the x - and y -axes.

In Exercises 15–22, find the slope-intercept form of the equation of the line satisfying the stated conditions, and check your answer using a graphing utility.

15. Slope = -2 , y -intercept = 4
16. $m = 5$, $b = -3$
17. The line is parallel to $y = 4x - 2$ and its y -intercept is 7.
18. The line is parallel to $3x + 2y = 5$ and passes through $(-1, 2)$.
19. The line is perpendicular to $y = 5x + 9$ and has y -intercept 6.
20. The line is perpendicular to $x - 4y = 7$ and passes through $(3, -4)$.
21. The line passes through $(2, 4)$ and $(1, -7)$.
22. The line passes through $(-3, 6)$ and $(-2, 1)$.
23. In each part, classify the lines as parallel, perpendicular, or neither.
 (a) $y = 4x - 7$ and $y = 4x + 9$
 (b) $y = 2x - 3$ and $y = 7 - \frac{1}{2}x$
 (c) $5x - 3y + 6 = 0$ and $10x - 6y + 7 = 0$
 (d) $Ax + By + C = 0$ and $Bx - Ay + D = 0$
 (e) $y - 2 = 4(x - 3)$ and $y - 7 = \frac{1}{4}(x - 3)$

24. In each part, classify the lines as parallel, perpendicular, or neither.
 (a) $y = -5x + 1$ and $y = 3 - 5x$
 (b) $y - 1 = 2(x - 3)$ and $y - 4 = -\frac{1}{2}(x + 7)$
 (c) $4x + 5y + 7 = 0$ and $5x - 4y + 9 = 0$
 (d) $Ax + By + C = 0$ and $Ax + By + D = 0$
 (e) $y = \frac{1}{2}x$ and $x = \frac{1}{2}y$

In Exercises 25 and 26, use the graph to find the equation of the line in slope-intercept form, and then check your result by using a graphing utility to graph the equation.

25.

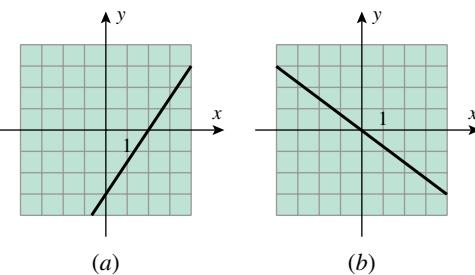


Figure Ex-25

26.

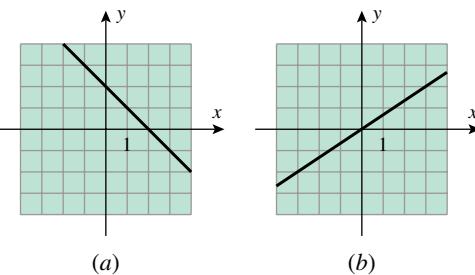


Figure Ex-26

27. The accompanying figure shows the position versus time curve for a particle moving along an x -axis.
 (a) What is the velocity of the particle?
 (b) What is the x -coordinate of the particle at time $t = 0$?
 (c) What is the x -coordinate of the particle at time $t = 2$?
 (d) At what time does the particle have an x -coordinate of $x = 4$?

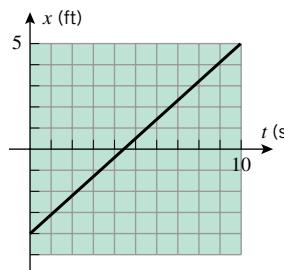


Figure Ex-27

28. A particle moving along an x -axis with constant velocity is at the point $x = 1$ when $t = 2$ and is at the point $x = 5$ when $t = 4$.
 (a) Find the velocity of the particle if x is in meters and t is in seconds.
 (b) Find an equation that expresses x as a function of t .
 (c) What is the coordinate of the particle at time $t = 0$?

29. A particle moving along an x -axis with constant acceleration has velocity $v = 3$ ft/s at time $t = 1$ and velocity $v = -1$ ft/s at time $t = 4$.
- Find the acceleration of the particle.
 - Find an equation that expresses v as a function of t .
 - What is the velocity of the particle at time $t = 0$?
30. The accompanying figure shows the velocity versus time curve for a particle moving along the x -axis.
- What is the acceleration of the particle?
 - What is the velocity of the particle at time $t = 0$?
 - What is the velocity of the particle at time $t = 2$?
 - At what time does the particle have a velocity of $v = 3$ ft/s?

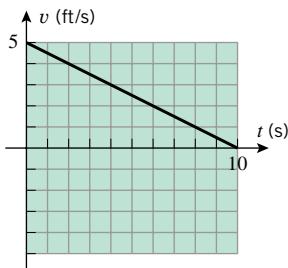


Figure Ex-30

31. The accompanying figure shows the position versus time curve for a particle moving along an x -axis.
- Describe the motion of the particle in words.
 - Find the average velocity of the particle from $t = 0$ to $t = 10$.
 - Find the average speed of the particle from $t = 0$ to $t = 10$.

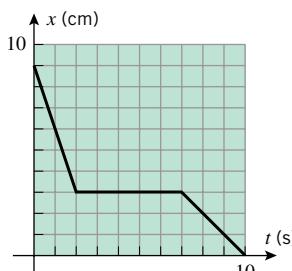


Figure Ex-31

32. The accompanying figure shows the velocity versus time curve for a particle moving along an x -axis. Describe the motion of the particle in words.

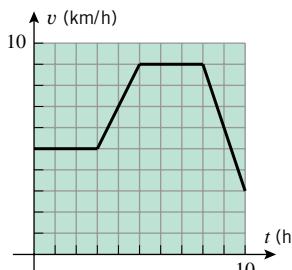


Figure Ex-32

33. A locomotive travels on a straight track at a constant speed of 40 mi/h, then reverses direction and returns to its starting point, traveling at a constant speed of 60 mi/h.
- What is the average velocity for the round-trip?
 - What is the average speed for the round-trip?
 - What is the total distance traveled by the train if the total trip took 5 h?
34. A ball is tossed straight up at time $t = 0$ with an initial velocity of 64 ft/s. We will show later using basic principles of physics that the velocity of the ball as a function of time is $v = 64 - 32t$.
- What direction is the ball traveling 3 s after it is released? Explain your reasoning.
 - At what time does the ball reach its maximum height above the ground? Explain your reasoning.
 - What can you say about the acceleration of the ball?
35. A car is stopped at a toll booth on a straight highway. Starting at time $t = 0$ it accelerates at a constant rate of 10 ft/s^2 for 10 s. It then travels at a constant speed of 100 ft/s for 90 s. At that time it begins to decelerate at a constant rate of 5 ft/s^2 for 20 s, at which point in time it reaches a full stop at a traffic light.
- Sketch the velocity versus time curve.
 - Express v as a piecewise function of t .
36. Make a reasonable sketch of a position versus time curve for a particle that moves in the positive x -direction with positive constant acceleration.
37. A spring with a natural length of 15 in stretches to a length of 20 in when a 45-lb object is suspended from it.
- Use Hooke's law to find an equation that expresses the amount y by which the spring is stretched (in inches) in terms of the suspended weight x (in pounds).
 - Graph the equation obtained in part (a).
 - Find the length of the spring when a 100-lb object is suspended from it.
 - What is the largest weight that can be suspended from the spring if the spring cannot be stretched to more than twice its natural length?
38. The spring in a heavy-duty shock absorber has a natural length of 3 ft and is compressed 0.2 ft by a load of 1 ton. An additional load of 5 tons compresses the spring an additional 1 ft.
- Assuming that Hooke's law applies to compression as well as extension, find an equation that expresses the length y that the spring is compressed from its natural length (in feet) in terms of the load x (in tons).
 - Graph the equation obtained in part (a).
 - Find the amount that the spring is compressed from its natural length by a load of 3 tons.
 - Find the maximum load that can be applied if safety regulations prohibit compressing the spring to less than half its natural length.

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In Exercises 39 and 40, confirm that a linear function is appropriate for the relationship between x and y . Find a linear equation relating x and y , and verify that the data points lie on the graph of your equation.

39.

x	0	1	2	4	6
y	2	3.2	4.4	6.8	9.2

40.

x	-1	0	2	5	8
y	12.6	10.5	6.3	0	-6.3

41. There are two common systems for measuring temperature, Celsius and Fahrenheit. Water freezes at 0°C (0°C) and 32°F (32°F); it boils at 100°C and 212°F .

- (a) Assuming that the Celsius temperature T_C and the Fahrenheit temperature T_F are related by a linear equation, find the equation.
- (b) What is the slope of the line relating T_F and T_C if T_F is plotted on the horizontal axis?
- (c) At what temperature is the Fahrenheit reading equal to the Celsius reading?
- (d) Normal body temperature is 98.6°F . What is it in $^{\circ}\text{C}$?

42. Thermometers are calibrated using the so-called “triple point” of water, which is 273.16 K on the Kelvin scale and 0.01°C on the Celsius scale. A one-degree difference on the Celsius scale is the same as a one-degree difference on the Kelvin scale, so there is a linear relationship between the temperature T_C in degrees Celsius and the temperature T_K in kelvins.

- (a) Find an equation that relates T_C and T_K .
- (b) Absolute zero (0 K on the Kelvin scale) is the temperature below which a body’s temperature cannot be lowered. Express absolute zero in $^{\circ}\text{C}$.

43. To the extent that water can be assumed to be incompressible, the pressure p in a body of water varies linearly with the distance h below the surface.

- (a) Given that the pressure is 1 atm at the surface and 5.9 atm at a depth of 50 m , find an equation that relates pressure to depth.
- (b) At what depth is the pressure twice that at the surface?

44. A resistance thermometer is a device that determines temperature by measuring the resistance of a fine wire whose resistance varies with temperature. Suppose that the resistance R in ohms (Ω) varies linearly with the temperature T in $^{\circ}\text{C}$ and that $R = 123.4\ \Omega$ when $T = 20^{\circ}\text{C}$ and that $R = 133.9\ \Omega$ when $T = 45^{\circ}\text{C}$.

- (a) Find an equation for R in terms of T .
- (b) If R is measured experimentally as $128.6\ \Omega$, what is the temperature?

45. Suppose that the mass of a spherical mothball decreases with time, due to evaporation, at a rate that is proportional to its surface area. Assuming that it always retains the shape of a sphere, it can be shown that the radius r of the sphere decreases linearly with the time t .

- (a) If, at a certain instant, the radius is 0.80 mm and 4 days later it is 0.75 mm , find an equation for r (in millimeters) in terms of the elapsed time t (in days).
- (b) How long will it take for the mothball to completely evaporate?

46. The accompanying figure shows three masses suspended from a spring: a mass of 11 g , a mass of 24 g , and an unknown mass of $W\text{ g}$.

- (a) What will the pointer indicate on the scale if no mass is suspended?
- (b) Find W .

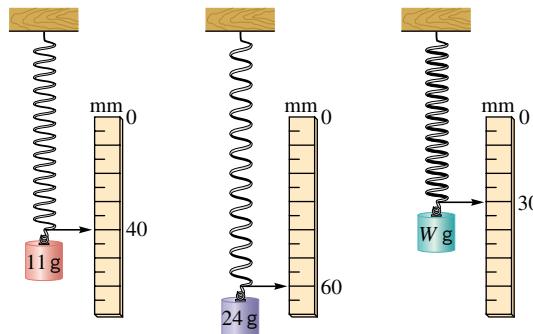


Figure Ex-46

47. The price for a round-trip bus ride from a university to center city is $\$2.00$, but it is possible to purchase a monthly commuter pass for $\$25.00$ with which each round-trip ride costs an additional $\$0.25$.

- (a) Find equations for the cost C of making x round-trips per month under both payment plans, and graph the equations for $0 \leq x \leq 30$ (treating C as a continuous function of x , even though x assumes only integer values).
- (b) How many round-trips per month would a student have to make for the commuter pass to be worthwhile?

48. A student must decide between buying one of two used cars: car A for $\$4000$ or car B for $\$5500$. Car A gets $20\text{ miles per gallon}$ of gas, and car B gets $30\text{ miles per gallon}$. The student estimates that gas will run $\$1.25$ per gallon. Both cars are in excellent condition, so the student feels that repair costs should be negligible for the foreseeable future. How many miles would the student have to drive before car B becomes the better buy?

1.6 FAMILIES OF FUNCTIONS

Functions are often grouped into families according to the form of their defining formulas or other common characteristics. In this section we will discuss some of the most basic families of functions.

This section includes quick reviews of precalculus material on polynomials and trigonometry. Readers who want to review this material in more depth are referred to Appendices E and F. Instructors who want to spend some additional time on precalculus review can divide this section into two parts, covering the trigonometry material in a second lecture.

FAMILIES OF LINES

A function f whose values are all the same is called a **constant function**. For example, the formula $f(x) = c$ defines the constant function whose value is c for all x . The graph of the constant function $f(x) = c$ is the horizontal line $y = c$ (Figure 1.6.1a). If we vary c , then we obtain a set or **family** of horizontal lines (Figure 1.6.1b).

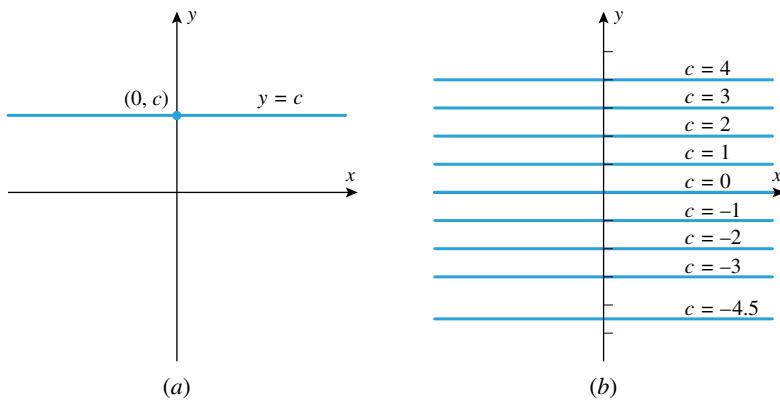


Figure 1.6.1

• **REMARK.** The expression $f(x) = c$ can be confusing because it can be interpreted either as an equation that is satisfied for certain x (as in $x^2 = c$) or as an identity that is satisfied for all x ; it is the latter interpretation that defines a constant function. Thus, when you see an expression of the form $f(x) = c$, you will have to determine from its context whether it is intended as an equation or a constant function.

The quantities m and b in the equation $y = mx + b$ can be viewed as unspecified constants whose values may change from one application to another; such changeable constants are called **parameters**.

If we keep b fixed and vary the parameter m in the equation $y = mx + b$, then we obtain a family of lines whose members all have y -intercept b (Figure 1.6.2a); and if we keep m fixed and vary the parameter b , then we obtain a family of parallel lines whose members all have slope m (Figure 1.6.2b).

Example 1

- Find an equation for the family of lines with slope $\frac{1}{2}$.
- Find the member of the family in part (a) that passes through the point $(4, 1)$.
- Find an equation for the family of lines whose members are perpendicular to the lines in part (a).

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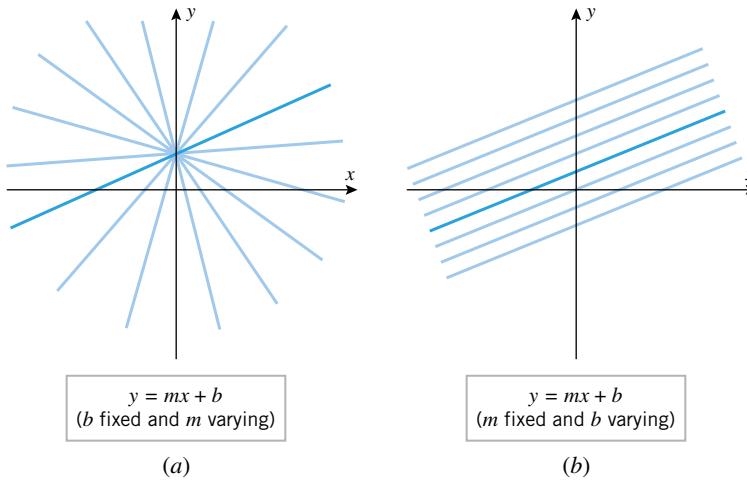


Figure 1.6.2

Solution (a). The lines of slope $\frac{1}{2}$ are of the form

$$y = \frac{1}{2}x + b \quad (1)$$

where the parameter b can have any real value.

Solution (b). To find the line in the family that passes through the point $(4, 1)$, we must find the value of b for which the coordinates $x = 4$ and $y = 1$ satisfy (1). Substituting these coordinates into (1) and solving for b yields $b = -1$, and hence the equation of the line is

$$y = \frac{1}{2}x - 1 \quad (2)$$

(Figure 1.6.3a).

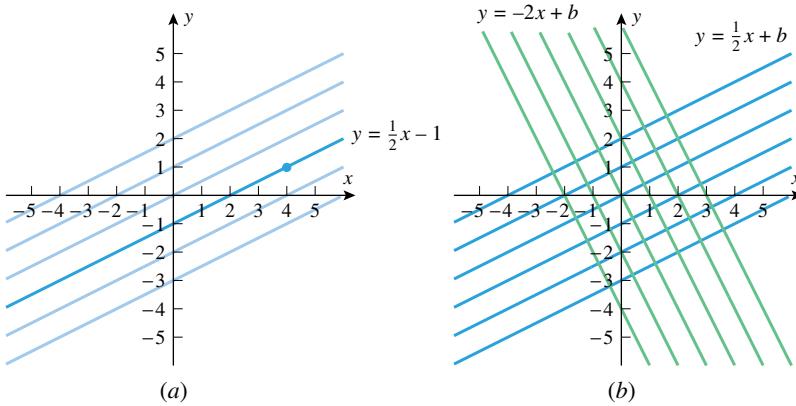


Figure 1.6.3

Solution (c). Since the slopes of perpendicular lines are negative reciprocals, it follows that the lines perpendicular to those in part (a) have slope -2 and hence are of the form

$$y = -2x + b$$

Some typical lines in families (1) and (2) are graphed in Figure 1.6.3b. ◀

.....
THE FAMILY $y = x^n$

A function of the form $f(x) = x^p$, where p is constant, is called a **power function**. Consider the case where p is a positive integer, say $p = n$. The graphs of the curves $y = x^n$ for $n = 1, 2, 3, 4$, and 5 are shown in Figure 1.6.4. The first graph is the line $y = x$ with slope 1 that passes through the origin, and the second is a parabola that opens up and has its vertex at the origin (see Appendix D).

1.6 Families of Functions 65

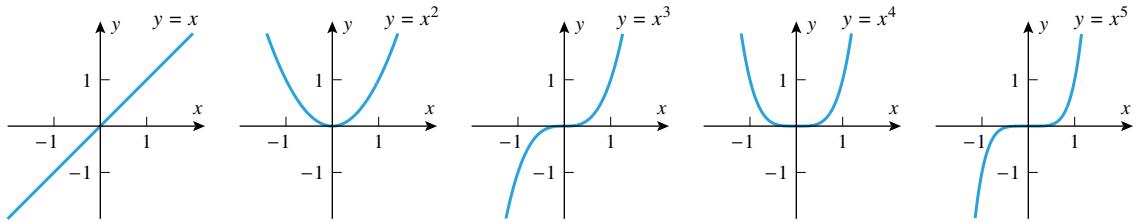


Figure 1.6.4

For $n > 2$ the shape of the graph of $y = x^n$ depends on whether n is even or odd (Figure 1.6.5). For even values of n the graphs have the same general shape as the parabola $y = x^2$ (though they are not actually parabolas if $n > 2$), and for odd values of n greater than 1 they have the same general shape as $y = x^3$. The graphs in the family $y = x^n$ share a number of important characteristics:

- For even values of n the functions $f(x) = x^n$ are even, and their graphs are symmetric about the y -axis; for odd values of n the functions $f(x) = x^n$ are odd, and their graphs are symmetric about the origin.
- For all values of n the graphs pass through the origin and the point $(1, 1)$. For even values of n the graphs pass through $(-1, 1)$, and for odd values of n they pass through $(-1, -1)$.
- Increasing n causes the graph to become flatter over the interval $-1 < x < 1$ and steeper over the intervals $x > 1$ and $x < -1$.

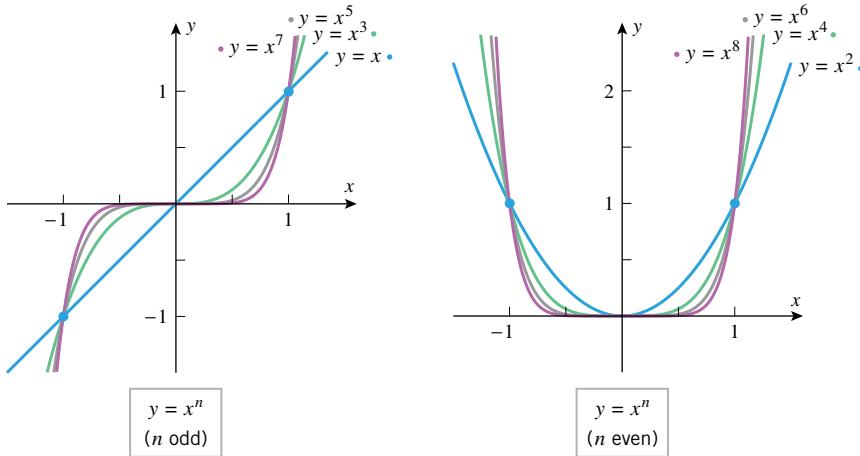


Figure 1.6.5

- **REMARK.** The last characteristic can be explained numerically by considering the effect of raising a real number x to successively higher powers. If x is a fraction, that is, $-1 < x < 1$, then the absolute value of x^n decreases as n increases (try raising $\frac{1}{2}$ or $-\frac{1}{2}$ to higher and higher powers, for example). This explains why successive graphs in Figure 1.6.5 become flatter over the interval $-1 < x < 1$. On the other hand, if $x > 1$ or $x < -1$, then the absolute value of x^n increases as n increases (try raising 2 or -2 to higher and higher powers). This explains why successive graphs become steeper if $x > 1$ or $x < -1$.

.....
THE FAMILY $y = x^{-n}$

If p is a negative integer, say $p = -n$, then the power functions $f(x) = x^p$ have the form $f(x) = x^{-n} = 1/x^n$. Figure 1.6.6a shows the graphs of $y = 1/x$ and $y = 1/x^2$, and Figure 1.6.6b shows how these graphs relate to other members of the family. The graph of $y = 1/x$ is called an **equilateral hyperbola** (for reasons to be discussed later).

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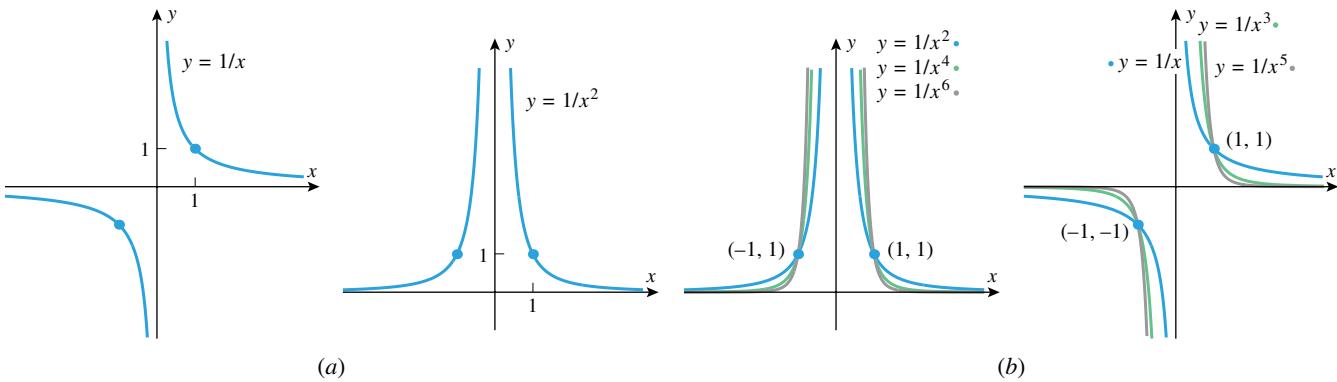


Figure 1.6.6

For odd values of n the graphs have the same general shape as $y = 1/x$, and for even values of n they have the same general shape as $y = 1/x^2$. The graphs in the family $y = 1/x^n$ share a number of important characteristics:

- For even values of n the functions $f(x) = 1/x^n$ are even, and their graphs are symmetric about the y -axis; for odd values of n the functions $f(x) = x^n$ are odd, and their graphs are symmetric about the origin.
 - For all values of n the graphs pass through the point $(1, 1)$ and have a break (called a *discontinuity*) at $x = 0$. This is caused by the division by zero that occurs when $x = 0$. For even values of n the graphs pass through $(-1, 1)$, and for odd values of n they pass through $(-1, -1)$.
 - Increasing n causes the graph to become steeper over the intervals $-1 < x < 0$ and $0 < x < 1$, and flatter over the intervals $x > 1$ and $x < -1$.

REMARK. The last characteristic can be explained numerically by considering the effect of raising the reciprocal of a number x to successively higher powers. If x is a nonzero fraction, then it lies in the interval $-1 < x < 1$, and its reciprocal satisfies $1/x > 1$ or $1/x < -1$. Thus, as n increases the absolute value of $1/x^n$ also increases. This explains why successive graphs in Figure 1.6.6 become successively steeper over the interval $-1 < x < 1$. On the other hand, if $x > 1$ or $x < -1$, then $-1 < 1/x < 1$. Thus, as n increases the absolute value of $1/x^n$ decreases. This explains why successive graphs in Figure 1.6.6 get successively flatter if $x > 1$ or $x < -1$.

THE FAMILY $y = x^{1/n}$

If $p = 1/n$, where n is a positive integer, then the power functions $f(x) = x^p$ have the form $f(x) = x^{1/n} = \sqrt[n]{x}$. In particular, if $n = 2$, then $f(x) = \sqrt{x}$, and if $n = 3$, then $f(x) = \sqrt[3]{x}$. The graphs of these functions are shown in parts (a) and (b) of Figure 1.6.7.

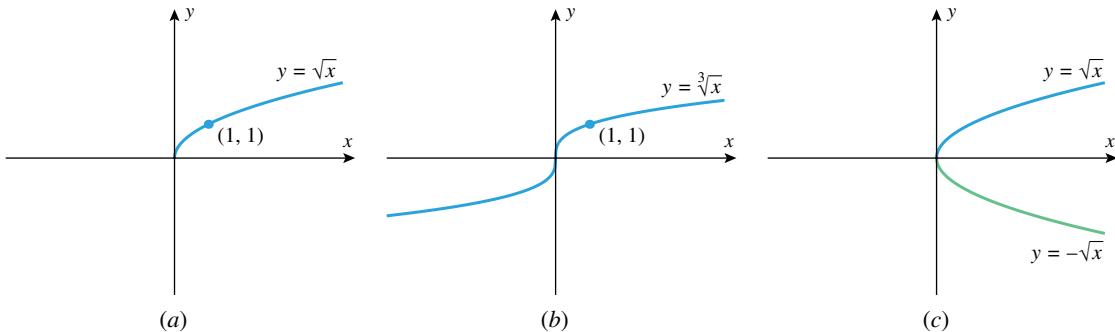


Figure 1.6.7

1.6 Families of Functions **67**

Observe that the graph of $y = \sqrt[3]{x}$ extends over the entire x -axis because $f(x) = \sqrt[3]{x}$ is defined for all real values of x (every real number has a cube root); in contrast, the graph of $y = \sqrt{x}$ only extends over the nonnegative x -axis (negative numbers have imaginary square roots). Observe also that the graph of $y = \sqrt{x}$ is the upper half of the parabola $x = y^2$ (Figure 1.6.7c).

For even values of n the graphs of $y = \sqrt[n]{x}$ have the same general shape as $y = \sqrt{x}$, and for odd values of n they have the same general shape as $y = \sqrt[3]{x}$.

- **FOR THE READER.** Sketch the graphs of $y = \sqrt[n]{x}$ for $n = 2, 4, 6$ on one set of axes and for $n = 3, 5, 7$ on another set. Use a graphing utility to check your work.

POWER FUNCTIONS WITH FRACTIONAL AND IRRATIONAL EXPONENTS

Power functions can also have fractional or irrational exponents. For example,

$$f(x) = x^{2/3}, \quad f(x) = \sqrt[5]{x^3}, \quad f(x) = x^{-7/8}, \quad \text{and} \quad f(x) = x^{\sqrt{2}} \quad (3)$$

are all power functions of this type; we will discuss power functions of these forms in later sections.

- **FOR THE READER.** The graph of $f(x) = x^{2/3}$ is given in Figure 1.3.16b. Read the note preceding Exercise 29 of Section 1.3, and use a graphing utility to generate graphs of $f(x) = \sqrt[5]{x}$ and $f(x) = x^{-7/8}$ that show all of their significant features.

Read the note preceding Exercise 29 of Section 1.3, and use a graphing utility to generate complete graphs of the functions in (3).

INVERSE PROPORTIONS

Recall that a variable y is said to be *inversely proportional to a variable x* if there is a positive constant k , called the *constant of proportionality*, such that

$$y = \frac{k}{x} \quad (4)$$

Since k is assumed to be positive, the graph of this equation has the same basic shape as $y = 1/x$ but is compressed or stretched in the y -direction.

Observe that in Formula (4) doubling x decreases y by a factor of $1/2$, tripling x decreases y by a factor of $1/3$, and, more generally, increasing x by a factor of r decreases y by a factor of $1/r$.

Functions involving inverse proportion arise in various laws of physics. For example, *Boyle's law* in physics states that at a constant temperature the pressure P exerted by a fixed quantity of an ideal gas is inversely proportional to the volume V occupied by the gas, that is,

$$P = \frac{k}{V}$$

(Figure 1.6.8).

If y is inversely proportional to x , then it follows from (4) that the product of y and x is constant, since $yx = k$. This provides a useful way of identifying inverse proportionality in experimental data.

Example 2 Table 1.6.1 shows some experimental data.

- (a) Explain why the data suggest that y is inversely proportional to x .
- (b) Express y as a function of x .
- (c) Graph your function and the data together for $x \geq 0$.

Solution. For every data point we have $xy = 5$, so y is inversely proportional to x and $y = 5/x$. The graph of this equation with the data points is shown in Figure 1.6.9. ◀

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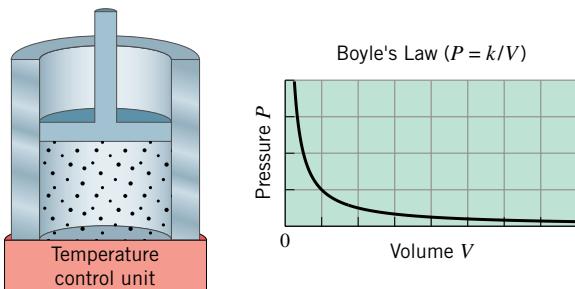
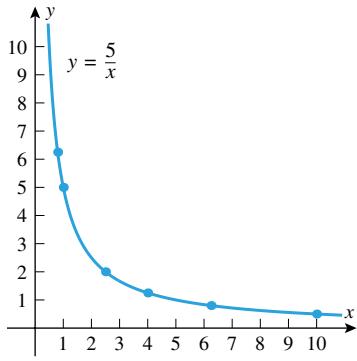
Boyle's Law ($P = k/V$)

Figure 1.6.9

As the volume of the gas changes, the temperature control unit adds or removes heat to maintain a constant temperature.

Figure 1.6.8

Table 1.6.1

EXPERIMENTAL DATA

x	0.8	1	2.5	4	6.25	10
y	6.25	5	2	1.25	0.8	0.5

A QUICK REVIEW OF POLYNOMIALS

A detailed review of polynomials is given in Appendix F, but for convenience we will review some of the terminology here.

A **polynomial in x** is a function that is expressible as a sum of finitely many terms of the form cx^n , where c is a constant and n is a nonnegative integer. Some examples of polynomials are

$$2x + 1, \quad 3x^2 + 5x - \sqrt{2}, \quad x^3, \quad 4 (= 4x^0), \quad 5x^7 - x^4 + 3$$

The function $(x^2 - 4)^3$ is also a polynomial because it can be expanded by the binomial formula (see the inside front cover) and expressed as a sum of terms of the form cx^n :

$$(x^2 - 4)^3 = (x^2)^3 - 3(x^2)^2(4) + 3(x^2)(4^2) - (4^3) = x^6 - 12x^4 + 48x^2 - 64 \quad (5)$$

A general polynomial can be written in either of the following forms, depending on whether one wants the powers of x in ascending or descending order:

$$c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$$

$$c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$$

The constants c_0, c_1, \dots, c_n are called the **coefficients** of the polynomial. When a polynomial is expressed in one of these forms, the highest power of x that occurs with a nonzero coefficient is called the **degree** of the polynomial. Nonzero constant polynomials are considered to have degree 0, since we can write $c = cx^0$. Polynomials of degree 1, 2, 3, 4, and 5 are described as **linear**, **quadratic**, **cubic**, **quartic**, and **quintic**, respectively. For example,

$$3 + 5x \quad \text{Has degree 1 (linear)}$$

$$x^2 - 3x + 1 \quad \text{Has degree 2 (quadratic)}$$

$$2x^3 - 7 \quad \text{Has degree 3 (cubic)}$$

$$8x^4 - 9x^3 + 5x - 3 \quad \text{Has degree 4 (quartic)}$$

$$\sqrt{3} + x^3 + x^5 \quad \text{Has degree 5 (quintic)}$$

$$(x^2 - 4)^3 \quad \text{Has degree 6 [see (5)]}$$

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The natural domain of a polynomial in x is $(-\infty, +\infty)$, since the only operations involved are multiplication and addition; the range depends on the particular polynomial. We already know that the graphs of polynomials of degree 0 and 1 are lines and that the graphs of polynomials of degree 2 are parabolas. Figure 1.6.10 shows the graphs of some typical polynomials of higher degree. Later, we will discuss polynomial graphs in detail, but for now it suffices to observe that graphs of polynomials are very well behaved in the sense that they have no discontinuities or sharp corners. As illustrated in Figure 1.6.10, the graphs of polynomials wander up and down for awhile in a roller-coaster fashion, but eventually that behavior stops and the graphs steadily rise or fall indefinitely as one travels along the curve in either the positive or negative direction. We will see later that the number of peaks and valleys is less than the degree of the polynomial.

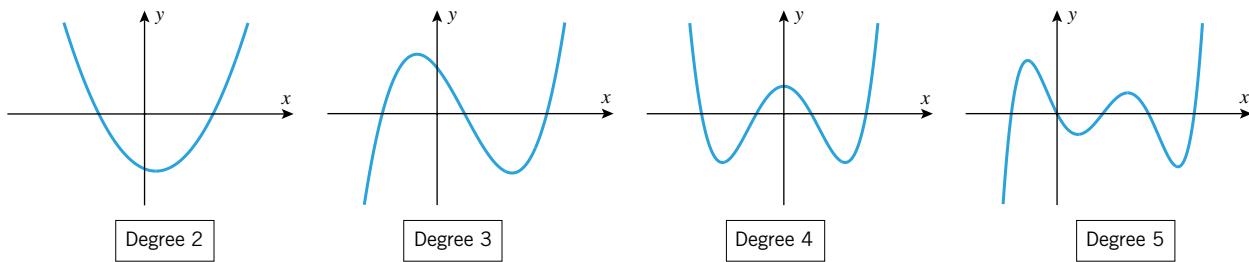


Figure 1.6.10

RATIONAL FUNCTIONS

A function that can be expressed as a ratio of two polynomials is called a *rational function*. If $P(x)$ and $Q(x)$ are polynomials, then the domain of the rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

consists of all values of x such that $Q(x) \neq 0$. For example, the domain of the rational function

$$f(x) = \frac{x^2 + 2x}{x^2 - 1}$$

consists of all values of x , except $x = 1$ and $x = -1$. Its graph is shown in Figure 1.6.11 along with the graphs of two other typical rational functions.

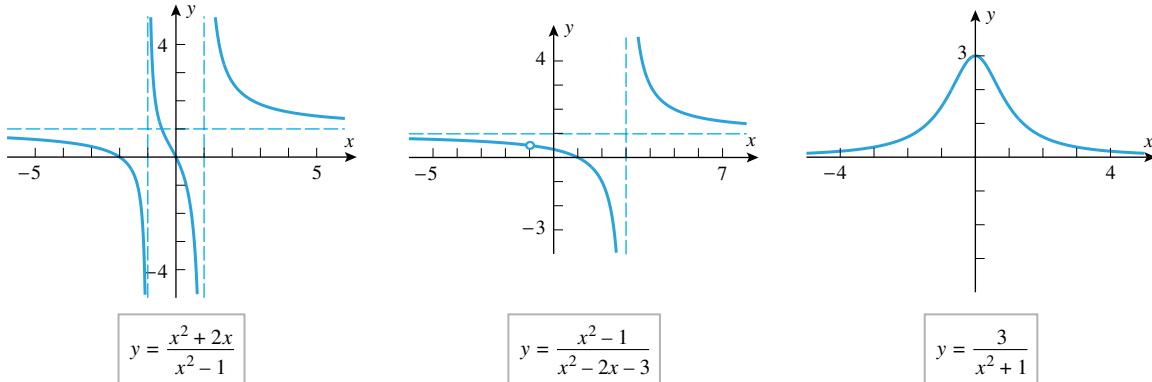


Figure 1.6.11

The graphs of rational functions with nonconstant denominators differ from the graphs of polynomials in some essential ways:

- Unlike polynomials whose graphs are continuous (unbroken) curves, the graphs of rational functions have discontinuities at the points where the denominator is zero.

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- Unlike polynomials, rational functions may have numbers at which they are not defined. Near such points, many (but not all) rational functions have graphs that approximate a vertical line, called a *vertical asymptote*. These are represented by the dashed vertical lines in Figure 1.6.11.
- Unlike the graphs of polynomials, which eventually rise or fall indefinitely, the graphs of many (but not all) rational functions eventually get closer and closer to some horizontal line, called a *horizontal asymptote*, as one travels along the curve in either the positive or negative direction. The horizontal asymptotes are represented by the dashed horizontal lines in the first two parts of Figure 1.6.11; in the third part of the figure the x -axis is a horizontal asymptote.

ALGEBRAIC FUNCTIONS

Functions that can be constructed from polynomials by applying finitely many algebraic operations (addition, subtraction, division, and root extraction) are called *algebraic functions*. Some examples are

$$f(x) = \sqrt{x^2 - 4}, \quad f(x) = 3\sqrt[3]{x}(2+x), \quad f(x) = x^{2/3}(x+2)^2$$

As illustrated in Figure 1.6.12, the graphs of algebraic functions vary widely, so it is difficult to make general statements about them. Later in this text we will develop general calculus methods for analyzing such functions.

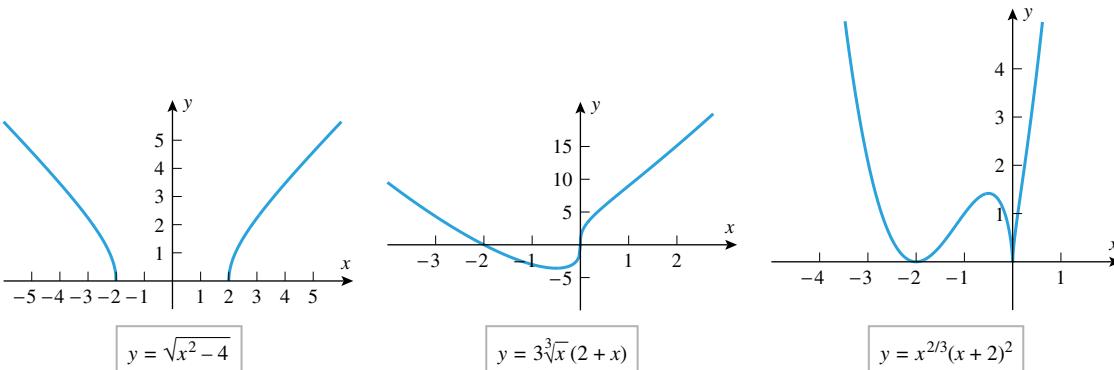


Figure 1.6.12

A QUICK REVIEW OF TRIGONOMETRIC FUNCTIONS

A detailed review of trigonometric functions is given in Appendix E, but for convenience we will summarize some of the main ideas here.

It is often convenient to think of the trigonometric functions in terms of circles rather than triangles. For this purpose, consider a point that moves either clockwise or counterclockwise along the *unit circle* $u^2 + v^2 = 1$ in the uv -plane, starting at $(1, 0)$ and stopping at a point P (Figure 1.6.13a). Let x denote the *signed* arc length traveled by the moving point, taking x to be positive for counterclockwise motion and negative for clockwise motion. (We allow for the possibility that the point may traverse the circle more than once.) When convenient, the variable x can also be interpreted as the angle in radians that is swept out by the radial line from the origin to P , with the usual convention that angles are positive if generated by counterclockwise rotations and negative if generated by clockwise rotations. We can *define* $\cos x$ to be the u -coordinate of P and $\sin x$ to be the v -coordinate of P (Figure 1.6.13b).

The remaining trigonometric functions can be defined in terms of the functions $\sin x$ and $\cos x$:

$$\begin{aligned} \tan x &= \frac{\sin x}{\cos x} & \cot x &= \frac{\cos x}{\sin x} \\ \sec x &= \frac{1}{\cos x} & \csc x &= \frac{1}{\sin x} \end{aligned}$$

The graphs of the six trigonometric functions in Figure 1.6.14 should already be familiar

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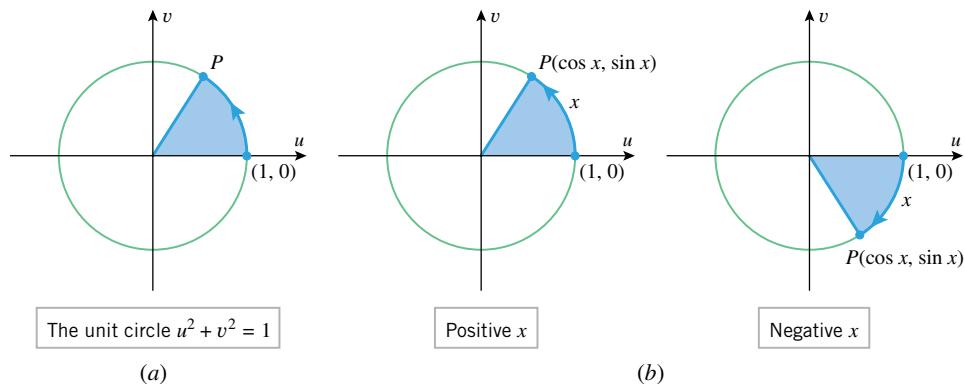


Figure 1.6.13

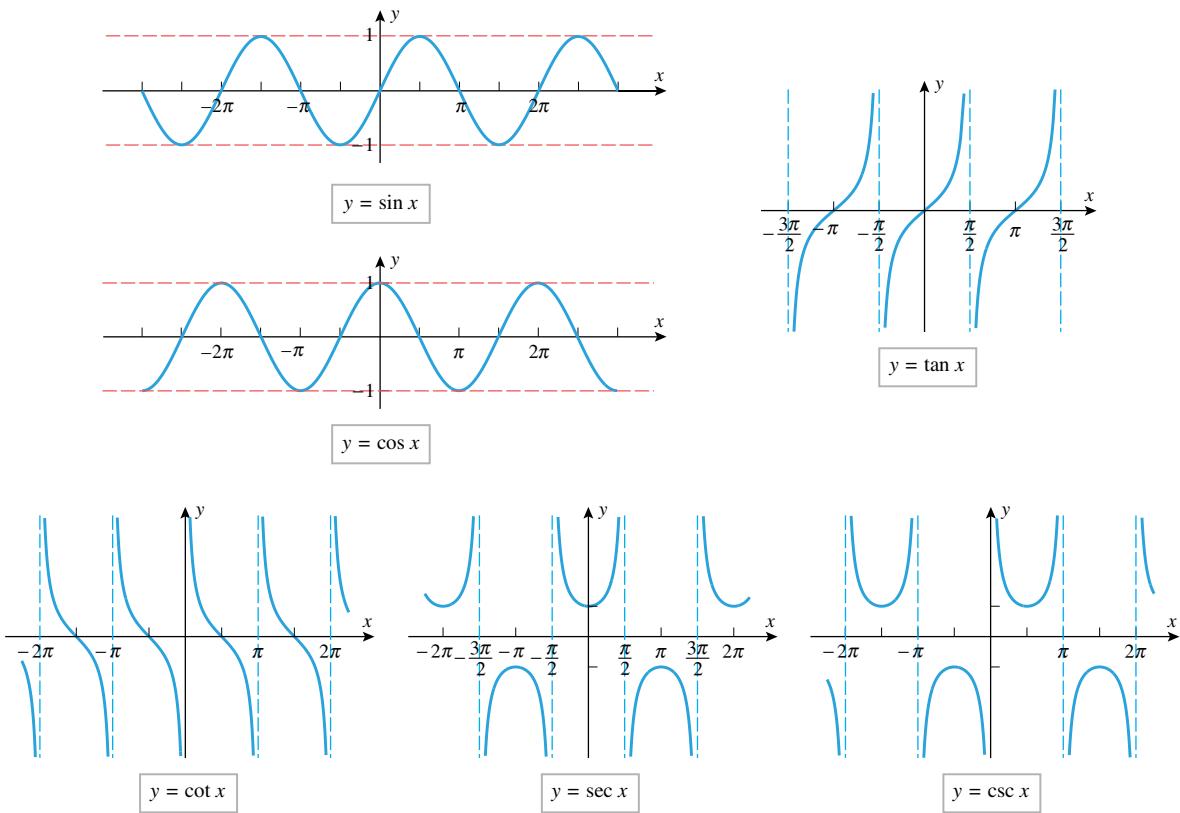


Figure 1.6.14

to you, but try generating them using a graphing utility, making sure to use radian measure for x .

- REMARK. In this text we will always assume that the independent variable in a trigonometric function is in radians unless specifically stated otherwise.

.....
PROPERTIES OF $\sin x$, $\cos x$, AND $\tan x$

Many of the basic properties of $\sin x$ and $\cos x$ can be deduced from the circle definitions of these functions. For example:

- As the point $P(\cos x, \sin x)$ moves around the unit circle, its coordinates vary between -1 and 1 , and hence

$$-1 \leq \sin x \leq 1 \quad \text{and} \quad -1 \leq \cos x \leq 1$$

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- If x increases or decreases by 2π radians, then the point $P(\cos x, \sin x)$ makes one complete revolution around the unit circle, and the coordinates return to their starting values. Thus, $\sin x$ and $\cos x$ have period 2π ; that is,

$$\begin{aligned}\sin(x \pm 2\pi) &= \sin x \\ \cos(x \pm 2\pi) &= \cos x\end{aligned}$$

- As $P(\cos x, \sin x)$ moves around the unit circle, $\sin x$ is zero when P is on the horizontal axis (which occurs when x is an integer multiple of π), and $\cos x$ is zero when P is on the vertical axis (which occurs when x is an odd multiple of $\pi/2$). Thus,

$$\begin{aligned}\sin x = 0 &\quad \text{if and only if } x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \dots \\ \cos x = 0 &\quad \text{if and only if } x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \dots\end{aligned}$$

- As $P(\cos x, \sin x)$ moves around the unit circle $u^2 + v^2 = 1$, its coordinates satisfy this equation for all x , which produces the fundamental trigonometric identity

$$\cos^2 x + \sin^2 x = 1$$

Observe that the graph of $y = \tan x$ has vertical asymptotes at the points $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \dots$. This is to be expected since $\tan x = \sin x / \cos x$, and these are the values of x at which $\cos x$ is zero. What is less obvious, however, is the fact that $\tan x$ repeats every π radians (i.e., has period π), even though $\sin x$ and $\cos x$ have period 2π . This can be explained by interpreting

$$\tan x = \frac{\sin x}{\cos x}$$

as the slope of the line L that passes through the origin and the point $P(\cos x, \sin x)$ on the unit circle in the uv -plane (Figure 1.6.15). Each time x increases or decreases by π radians, the point P traverses half the circumference, and the line L rotates π radians, so its starting and ending slope are the same.

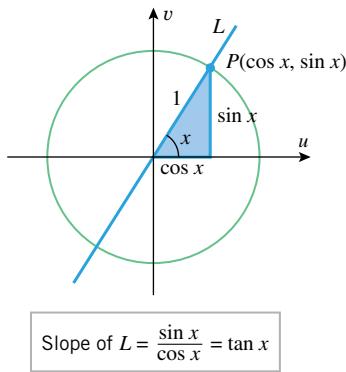


Figure 1.6.15

RADIANS AS A DIMENSIONLESS UNIT

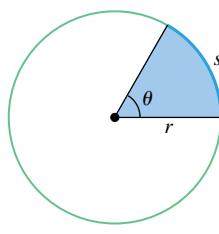


Figure 1.6.16

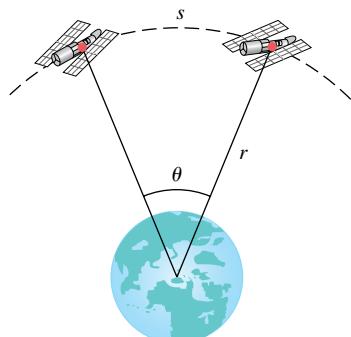


Figure 1.6.17

The choice of radian measure as opposed to degree measure depends on the nature of the problem being considered; degree measure is usually chosen in engineering problems involving measurements of angles, and radian measure is usually chosen when the function properties of $\sin x$, $\cos x$, $\tan x$, ... are the primary focus. Radian measure is also usually chosen in problems involving arc lengths on circles because of the basic result in trigonometry which states that the arc length s of a sector with radius r and a central angle of θ (radians) is given by

$$s = r\theta \tag{6}$$

(Figure 1.6.16).

In applications involving angles, radians require special treatment to ensure that quantities are assigned proper units. To see why this is so, let us rewrite (6) as

$$\theta = \frac{s}{r}$$

The left side of this equation is in radians, and the right side is the ratio of two lengths, say meters/meters or feet/feet. However, because these units of length cancel, the right side of this equation is actually *dimensionless* (has no units). Thus, to ensure consistency between the two sides of the equation, we would have to omit the units of radians on the left side to make it dimensionless as well. In practical terms this means that units of radians can be used in intermediate computations, when convenient, but they need to be omitted in the end result to ensure consistency of units. This is confusing, to say the least, but the following example should clarify the idea.

Example 3 Suppose that two satellites circle the equator in an orbit whose radius is $r = 4.23 \times 10^7$ m (Figure 1.6.17). Find the arc length s that separates the satellites if they have an angular separation of $\theta = 2.00^\circ$.

Solution. To apply Formula (6), we must convert the angular separation to radians:

$$2.00^\circ = \frac{\pi}{180}(2.00) \approx 0.0349 \text{ rad}$$

Thus, from (6)

$$s = r\theta \approx (4.23 \times 10^7 \text{ m})(0.0349 \text{ rad}) \approx 1.48 \times 10^6 \text{ m}$$

In this computation the product $r\theta$ produces units of meters \times radians, but if we treat radians as dimensionless, we have meters \times radians = meters, which correctly produces units of meters (m) for the arc length s . \blacktriangleleft

THE FAMILIES $y = A \sin Bx$
AND $y = A \cos Bx$

Many important applications lead to trigonometric functions of the form

$$f(x) = A \sin(Bx - C) \quad \text{and} \quad g(x) = A \cos(Bx - C) \quad (7)$$

where A , B , and C are nonzero constants. The graphs of such functions can be obtained by stretching, compressing, translating, and reflecting the graphs of $y = \sin x$ and $y = \cos x$ appropriately. To see why this is so, let us start with the case where $C = 0$ and consider how the graphs of the equations

$$y = A \sin Bx \quad \text{and} \quad y = A \cos Bx$$

relate to the graphs of $y = \sin x$ and $y = \cos x$. If A and B are positive, then the effect of the constant A is to stretch or compress the graphs of $y = \sin x$ and $y = \cos x$ vertically by a factor of A , and the effect of the constant B is to compress or stretch the graphs of $\sin x$ and $\cos x$ horizontally by a factor of B . For example, the graph of $y = 2 \sin 4x$ can be obtained by stretching the graph of $y = \sin x$ vertically by a factor of 2 and compressing it horizontally by a factor of 4. (Recall from Section 1.4 that the multiplier of x stretches when it is less than 1 and compresses when it is greater than 1.) Thus, as shown in Figure 1.6.18, the graph of $y = 2 \sin 4x$ varies between -2 and 2 , and repeats every $2\pi/4 = \pi/2$ units.

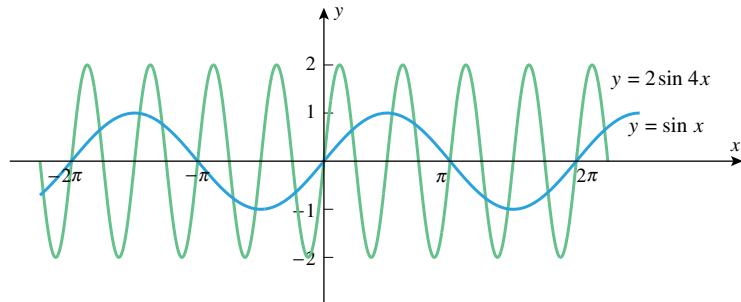


Figure 1.6.18

In general, if A and B are positive numbers, then the graphs of

$$y = A \sin Bx \quad \text{and} \quad y = A \cos Bx$$

oscillate between $-A$ and A and repeat every $2\pi/B$ units, so we say that these functions have **amplitude** A and **period** $2\pi/B$. In addition, we define the **frequency** of these functions to be the reciprocal of the period, that is, the frequency is $B/2\pi$. If A or B is negative, then these constants cause reflections of the graphs about the axes as well as compressing or stretching them; and in this case the amplitude, period, and frequency are given by $|A|$, $2\pi/|B|$, and $|B|/2\pi$, respectively.

Example 4 Make sketches of the following graphs that show the period and amplitude.

- (a) $y = 3 \sin 2\pi x$ (b) $y = -3 \cos 0.5x$ (c) $y = 1 + \sin x$

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Solution (a). The equation is of the form $y = A \sin Bx$ with $A = 3$ and $B = 2\pi$, so the graph has the shape of a sine function, but with amplitude $A = 3$ and period $2\pi/B = 2\pi/2\pi = 1$ (Figure 1.6.19a).

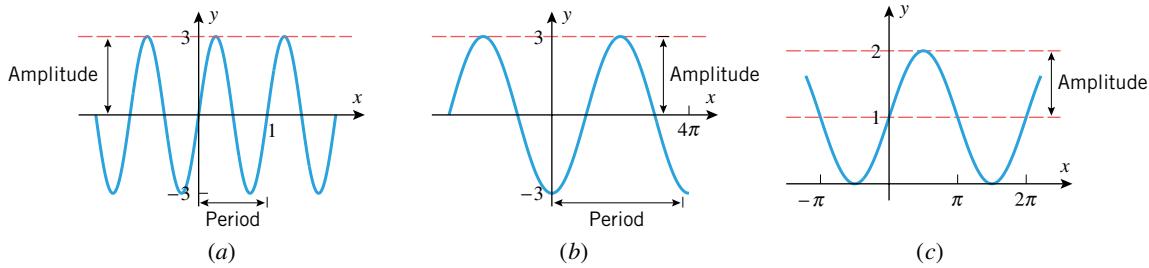


Figure 1.6.19

Solution (b). The equation is of the form $y = A \cos Bx$ with $A = -3$ and $B = 0.5$, so the graph has the shape of a cosine function that has been reflected about the x -axis (because $A = -3$ is negative), but with amplitude $|A| = 3$ and period $2\pi/B = 2\pi/0.5 = 4\pi$ (Figure 1.6.19b).

Solution (c). The graph has the shape of a sine function that has been translated up 1 unit (Figure 1.6.19c). ◀

**THE FAMILIES $y = A \sin(Bx - C)$
AND $y = A \cos(Bx - C)$**

To investigate the graphs of the more general families

$$y = A \sin(Bx - C) \quad \text{and} \quad y = A \cos(Bx - C)$$

it will be helpful to rewrite these equations as

$$y = A \sin\left[B\left(x - \frac{C}{B}\right)\right] \quad \text{and} \quad y = A \cos\left[B\left(x - \frac{C}{B}\right)\right]$$

In this form we see that the graphs of these equations can be obtained by translating the graphs of $y = A \sin Bx$ and $y = A \cos Bx$ to the left or right, depending on the sign of C/B . For example, if $C/B > 0$, then the graph of

$$y = A \sin[B(x - C/B)] = A \sin(Bx - C)$$

can be obtained by translating the graph of $y = A \sin Bx$ to the right by C/B units (Figure 1.6.20). If $C/B < 0$, the graph of $y = A \sin(Bx - C)$ is obtained by translating the graph of $y = A \sin Bx$ to the left by $|C/B|$ units

Example 5 Find the amplitude and period of

$$y = 3 \cos\left(2x + \frac{\pi}{2}\right)$$

and determine how the graph of $y = 3 \cos 2x$ should be translated to produce the graph of this equation. Confirm your results by graphing on a calculator or computer.

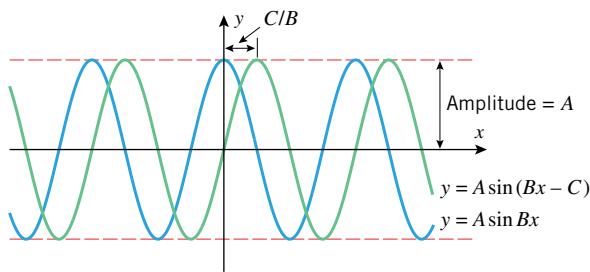


Figure 1.6.20

Solution. The equation can be rewritten as

$$y = 3 \cos \left[2x - \left(-\frac{\pi}{2} \right) \right] = 3 \cos \left[2 \left(x - \left(-\frac{\pi}{4} \right) \right) \right]$$

which is of the form

$$y = A \cos \left[B \left(x - \frac{C}{B} \right) \right]$$

with $A = 3$, $B = 2$, and $C/B = -\pi/4$. Thus, the amplitude is $A = 3$, the period is $2\pi/B = \pi$, and the graph is obtained by translating the graph of $y = 3 \cos 2x$ left by $|C/B| = \pi/4$ units (Figure 1.6.21). ◀

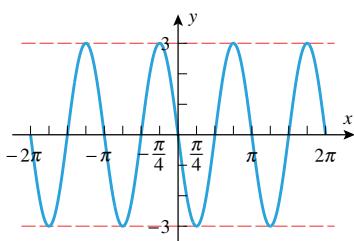


Figure 1.6.21

OTHER FAMILIES

In addition to the functions mentioned in this section, there are exponential and logarithmic functions, which we will study later, and various special functions that arise in physics and engineering. There are also many kinds of functions that have no names; indeed, one of the important themes of calculus is to provide methods for analyzing new types of functions.

EXERCISE SET 1.6

Graphing Calculator

1. (a) Find an equation for the family of lines whose members have slope $m = 3$.
 (b) Find an equation for the member of the family that passes through $(-1, 3)$.
 (c) Sketch some members of the family, and label them with their equations. Include the line in part (b).
2. Find an equation for the family of lines whose members are perpendicular to those in Exercise 1.
3. (a) Find an equation for the family of lines with y -intercept $b = 2$.
 (b) Find an equation for the member of the family whose angle of inclination is 135° .
 (c) Sketch some members of the family, and label them with their equations. Include the line in part (b).
4. Find an equation for
 - (a) the family of lines that pass through the origin
 - (b) the family of lines with x -intercept $a = 1$
 - (c) the family of lines that pass through the point $(1, -2)$
 - (d) the family of lines parallel to $2x + 4y = 1$.

In Exercises 5 and 6, state a geometric property common to all lines in the family, and sketch five of the lines.

5. (a) The family $y = -x + b$
 (b) The family $y = mx - 1$
 (c) The family $y = m(x + 4) + 2$
 (d) The family $x - ky = 1$
6. (a) The family $y = b$
 (b) The family $Ax + 2y + 1 = 0$
 (c) The family $2x + By + 1 = 0$
 (d) The family $y - 1 = m(x + 1)$
7. Find an equation for the family of lines tangent to the circle with center at the origin and radius 3.

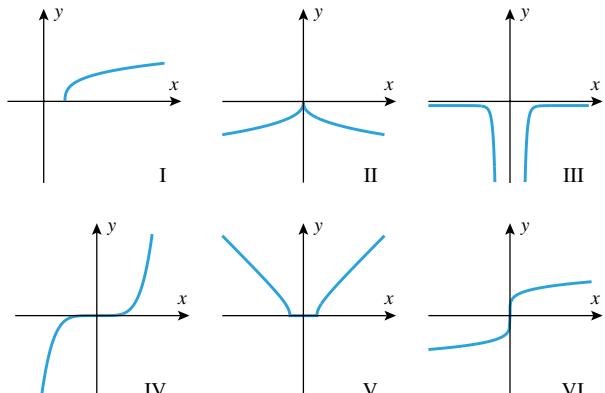


Figure Ex-11

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12. The accompanying table gives approximate values of three functions: one of the form kx^2 , one of the form kx^{-3} , and one of the form $kx^{3/2}$. Identify which is which, and estimate k in each case.

x	0.25	0.37	2.1	4.0	5.8	6.2	7.9	9.3
$f(x)$	640	197	1.08	0.156	0.0513	0.0420	0.0203	0.0124
$g(x)$	0.0312	0.0684	2.20	8.00	16.8	19.2	31.2	43.2
$h(x)$	0.250	0.450	6.09	16.0	27.9	30.9	44.4	56.7

Table Ex-12

In Exercises 13 and 14, sketch the graph of the equation for $n = 1, 3$, and 5 in one coordinate system and for $n = 2, 4$, and 6 in another coordinate system. Check your work with a graphing utility.

13. (a) $y = -x^n$ (b) $y = 2x^{-n}$ (c) $y = (x - 1)^{1/n}$
 14. (a) $y = 2x^n$ (b) $y = -x^{-n}$
 (c) $y = -3(x + 2)^{1/n}$
 15. (a) Sketch the graph of $y = ax^2$ for $a = \pm 1, \pm 2$, and ± 3 in a single coordinate system.
 (b) Sketch the graph of $y = x^2 + b$ for $b = \pm 1, \pm 2$, and ± 3 in a single coordinate system.
 (c) Sketch some typical members of the family of curves $y = ax^2 + b$.
 16. (a) Sketch the graph of $y = a\sqrt{x}$ for $a = \pm 1, \pm 2$, and ± 3 in a single coordinate system.
 (b) Sketch the graph of $y = \sqrt{x} + b$ for $b = \pm 1, \pm 2$, and ± 3 in a single coordinate system.
 (c) Sketch some typical members of the family of curves $y = a\sqrt{x} + b$.

In Exercises 17–20, sketch the graph of the equation by making appropriate transformations to the graph of a basic power function. Check your work with a graphing utility.

17. (a) $y = 2(x + 1)^2$ (b) $y = -3(x - 2)^3$
 (c) $y = \frac{-3}{(x + 1)^2}$ (d) $y = \frac{1}{(x - 3)^5}$
 18. (a) $y = 1 - \sqrt{x + 2}$ (b) $y = 1 - \sqrt[3]{x + 2}$
 (c) $y = \frac{5}{(1 - x)^3}$ (d) $y = \frac{2}{(4 + x)^4}$
 19. (a) $y = \sqrt[3]{x + 1}$ (b) $y = 1 - \sqrt{x - 2}$
 (c) $y = (x - 1)^5 + 2$ (d) $y = \frac{x + 1}{x}$
 20. (a) $y = 1 + \frac{1}{x - 2}$ (b) $y = \frac{1}{1 + 2x - x^2}$
 (c) $y = -\frac{2}{x^7}$ (d) $y = x^2 + 2x$

21. Sketch the graph of $y = x^2 + 2x$ by completing the square and making appropriate transformations to the graph of $y = x^2$.

22. (a) Use the graph of $y = \sqrt{x}$ to help sketch the graph of $y = \sqrt{|x|}$.

- (b) Use the graph of $y = \sqrt[3]{x}$ to help sketch the graph of $y = \sqrt[3]{|x|}$.

23. As discussed in this section, Boyle's law states that at a constant temperature the pressure P exerted by a gas is related to the volume V by the equation $P = k/V$.

- (a) Find the appropriate units for the constant k if pressure (which is force per unit area) is in newtons per square meter (N/m^2) and volume is in cubic meters (m^3).

- (b) Find k if the gas exerts a pressure of $20,000 N/m^2$ when the volume is 1 liter ($0.001 m^3$).

- (c) Make a table that shows the pressures for volumes of 0.25, 0.5, 1.0, 1.5, and 2.0 liters.

- (d) Make a graph of P versus V .

24. A manufacturer of cardboard drink containers wants to construct a closed rectangular container that has a square base and will hold $\frac{1}{10}$ liter (100 cm^3). Estimate the dimension of the container that will require the least amount of material for its manufacture.

A variable y is said to be **inversely proportional to the square of a variable x** if y is related to x by an equation of the form $y = k/x^2$, where k is a nonzero constant, called the **constant of proportionality**. This terminology is used in Exercises 25 and 26.

25. According to **Coulomb's law**, the force F of attraction between positive and negative point charges is inversely proportional to the square of the distance x between them.

- (a) Assuming that the force of attraction between two point charges is 0.0005 newton when the distance between them is 0.3 meter, find the constant of proportionality (with proper units).

- (b) Find the force of attraction between the point charges when they are 3 meters apart.

- (c) Make a graph of force versus distance for the two charges.

- (d) What happens to the force as the particles get closer and closer together? What happens as they get farther and farther apart?

26. It follows from Newton's Law of Universal Gravitation that the weight W of an object (relative to the Earth) is inversely proportional to the square of the distance x between the object and the center of the Earth, that is, $W = C/x^2$.

- (a) Assuming that a weather satellite weighs 2000 pounds on the surface of the Earth and that the Earth is a sphere of radius 4000 miles, find the constant C .

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- (b) Find the weight of the satellite when it is 1000 miles above the surface of the Earth.
 (c) Make a graph of the satellite's weight versus its distance from the center of the Earth.
 (d) Is there any distance from the center of the Earth at which the weight of the satellite is zero? Explain your reasoning.
27. In each part, match the equation with one of the accompanying graphs, and give the equations for the horizontal and vertical asymptotes.

(a) $y = \frac{x^2}{x^2 - x - 2}$

(b) $y = \frac{x - 1}{x^2 - x - 6}$

(c) $y = \frac{2x^4}{x^4 + 1}$

(d) $y = \frac{4}{(x + 2)^2}$

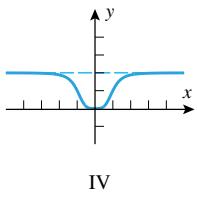
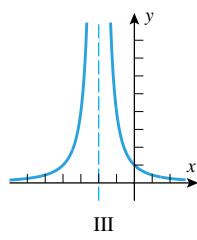
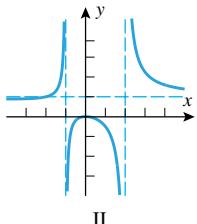
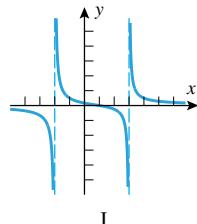


Figure Ex-27

28. Find an equation of the form $y = k/(x^2 + bx + c)$ whose graph is a reasonable match to that in the accompanying figure. Check your work with a graphing utility.

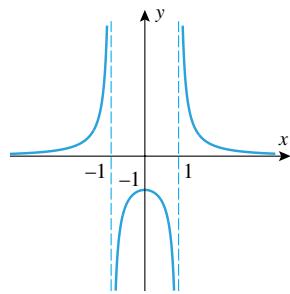


Figure Ex-28

In Exercises 29 and 30, draw a radial line from the origin with the given angle, and determine whether the six trigonometric functions are positive, negative, or undefined for that angle.

29. (a) $\frac{\pi}{3}$ (b) $-\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$
 (d) -1 (e) $\frac{5\pi}{4}$ (f) $\frac{11\pi}{6}$
30. (a) $\frac{3\pi}{2}$ (b) $-\frac{5\pi}{4}$ (c) π
 (d) $\frac{5\pi}{2}$ (e) 4 (f) $-\frac{33\pi}{7}$

In Exercises 31 and 32, use a calculating utility set to the radian mode to confirm the approximations $\sin(\pi/5) \approx 0.588$ and $\cos(\pi/8) \approx 0.924$, and then use these values to approximate the given expressions by hand calculation. Check your answers using the trigonometric function operations of your calculating utility.

31. (a) $\sin \frac{4\pi}{5}$ (b) $\cos(-\frac{\pi}{8})$ (c) $\sin \frac{11\pi}{5}$
 (d) $\cos \frac{7\pi}{8}$ (e) $\cos^2 \frac{\pi}{5}$ (f) $\sin^2 \frac{2\pi}{5}$
32. (a) $\sin \frac{16\pi}{5}$ (b) $\cos(-\frac{17\pi}{8})$ (c) $\sin \frac{41\pi}{5}$
 (d) $\sin(-\frac{\pi}{16})$ (e) $\cos \frac{27\pi}{8}$ (f) $\tan^2 \frac{\pi}{8}$
33. Assuming that $\sin \alpha = a$, $\cos \beta = b$, and $\tan \gamma = c$, express the stated quantities in terms of a , b , and c .
 (a) $\sin(-\alpha)$ (b) $\cos(-\beta)$ (c) $\tan(-\gamma)$
 (d) $\sin(\frac{\pi}{2} - \alpha)$ (e) $\cos(\pi - \beta)$ (f) $\sin(\alpha + \pi)$
 (g) $\sin(2\beta)$ (h) $\cos(2\beta)$ (i) $\sec(\beta + 2\pi)$
 (j) $\csc(\alpha + \pi)$ (k) $\cot(\gamma + 5\pi)$ (l) $\sin^2 \left(\frac{\beta}{2}\right)$

34. A ship travels from a point near Hawaii at 20° N latitude directly north to a point near Alaska at 56° N latitude.
 (a) Assuming the Earth to be a sphere of radius 4000 mi, find the actual distance traveled by the ship.
 (b) What fraction of the Earth's circumference did the ship travel?
35. The Moon completes one revolution around the Earth in approximately 27.3 days. Assuming that the Moon's orbit is a circle with a radius of 0.38×10^9 m from the center of the Earth, find the arc length traveled by the Moon in 1 day.
36. A spoked wheel with a diameter of 3 ft rolls along a flat road without slipping. How far along the road does the wheel roll if the spokes turn through 225° ?
37. As illustrated in the accompanying figure, suppose that you hold one quarter flat against a table while you rotate a second quarter around it without slippage. Through what angle will the second quarter have turned about its own center when it returns to its original location?

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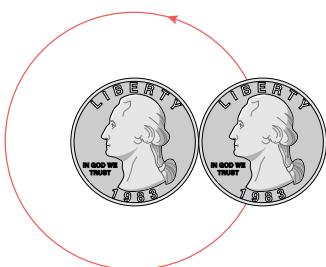
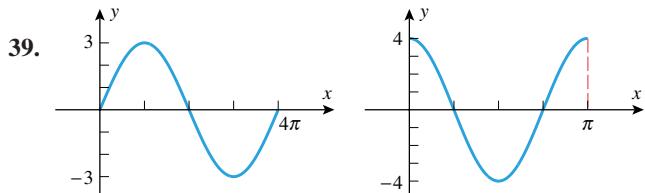


Figure Ex-37

38. Suppose that you begin cutting wedge-shaped pieces from a pie so that the arc length along the outer crust of each piece is equal to the radius. What fraction of the pie will remain after all pieces that can be cut in this way are eaten?

In Exercises 39 and 40, find an equation of the form $y = D + A \sin Bx$ or $y = D + A \cos Bx$ for each graph.

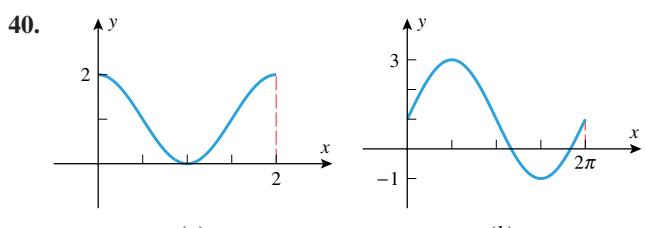


(a)

(b)

(c)

Figure Ex-39



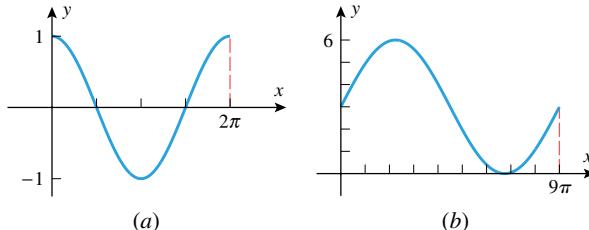
(a)

(b)

(c)

Figure Ex-40

41. In each part, find an equation for the graph that has the form $y = y_0 + A \sin(Bx - C)$.



(a)

(b)

(c)

Figure Ex-41

42. In the United States, a standard electrical outlet supplies sinusoidal electrical current with a maximum voltage of $V = 120\sqrt{2}$ volts (V) at a frequency of 60 Hertz (Hz). Write an equation that expresses V as a function of the time t , assuming that $V = 0$ if $t = 0$. [Note: 1 Hz = 1 cycle per second.]

In Exercises 43 and 44, find the amplitude, period, and phase shift, and sketch at least two periods of the graph by hand. Check your work with a graphing utility.

43. (a) $y = 3 \sin 4x$ (b) $y = -2 \cos \pi x$
 (c) $y = 2 + \cos \left(\frac{x}{2}\right)$

44. (a) $y = -1 - 4 \sin 2x$ (b) $y = \frac{1}{2} \cos(3x - \pi)$
 (c) $y = -4 \sin \left(\frac{x}{3} + 2\pi\right)$

45. Equations of the form

$$x = A_1 \sin \omega t + A_2 \cos \omega t$$

arise in the study of vibrations and other periodic motion.

- (a) Use the trigonometric identity for $\sin(\alpha + \beta)$ to show that this equation can be expressed in the form

$$x = A \sin(\omega t + \theta)$$

- (b) State formulas that express A and θ in terms of the constants A_1 , A_2 , and ω .

- (c) Express the equation

$$x = 5\sqrt{3} \sin 2\pi t + \frac{5}{2} \cos 2\pi t$$

in the form $x = A \sin(\omega t + \theta)$, and use a graphing utility to confirm that both equations have the same graph.

46. Determine the number of solutions of $x = 2 \sin x$, and use a graphing or calculating utility to estimate them.

1.7 MATHEMATICAL MODELS

In this section we will introduce some simple mathematical models that are based on linear, quadratic, and trigonometric functions. The method of “least squares” will be discussed briefly within the context of linear and quadratic regression.

MATHEMATICAL MODELS

A **mathematical model** of a physical law or phenomenon is a description of that law or phenomenon in the language of mathematics. The modern scientific view about mathematical models was first expressed in the seventeenth century by the Italian mathematician and scientist Galileo Galilei (1564–1642) who wrote, “The book of nature is written in mathematics.” This seemingly innocuous statement represented a major departure from the approach of the ancient Greek scientists and philosophers, who generally described scientific laws in words. For example, the Greek philosopher Aristotle (384 B.C.–322 B.C.) would have described the motion of a falling body by saying that such bodies “seek their natural position”, whereas today we would use functions, equations, or other mathematical means to describe the motion precisely.

Mathematical models allow us to use mathematical models to deduce results about the physical world that are not evident or have never been observed. For example, the possibility of placing a satellite in orbit around the Earth was deduced mathematically from Isaac Newton’s model of mechanics nearly 200 years before the launching of *Sputnik*, and Albert Einstein (1879–1955) gave a relativistic model of mechanics in 1915 that explained a precession (position shift) in the perihelion of the planet Mercury that was not confirmed by physical measurement until 1967.

One of the most important steps in creating a mathematical model of a physical phenomenon is deciding which factors to consider and which to ignore—the more factors one takes into account the more complicated the formulas and equations of the model tend to become, so there is always a balance to be struck between keeping a model mathematically simple and considering enough factors to make the model useful. For example, if a meteorologist were trying to model the relationship between the speed of a raindrop when it hits the ground and the height of the cloud in which it was formed, then he or she would certainly have to take air resistance into account, but with equal certainty he or she would ignore the infinitesimal effect that the Pluto’s gravitational pull has on the raindrop. The danger is that in trying to keep a mathematical model from becoming too complicated one might oversimplify to the point where the results it produces do not agree with reality. We are reminded of this by Einstein’s admonition: “Everything must be as simple as possible, but not simpler.” A good mathematical model is one that produces results that are consistent with the physical world. If a time comes when the mathematical results produced by the model do not agree with real-world observations, then the model must be abandoned or modified in favor of a new model that does. This is the nature of the **scientific method**—old models constantly being replaced by new models that more accurately describe the real world.

FUNCTIONS AS MODELS

In this section we will consider some simple models that involve only two variables. In our general discussion we will refer to these variables as x and y , but in specific examples other letters will be more appropriate. We will assume that the data for the phenomenon being modeled consists of a collection of ordered pairs of measurements

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots, (x_n, y_n)$$

that relate corresponding values of the variables x and y . We distinguish between two types of phenomena—**deterministic phenomena** in which each value of x determines one value of y and **probabilistic phenomena** in which the value of y associated with a specific x is not uniquely determined, but rather depends on probabilities in some way. For example, if y is the amount that a spring is stretched by a force x , then for a given spring the value of y is uniquely determined by the value of x , so this is a deterministic phenomenon. In contrast, if y is the weight of person whose height is x , then the value of y is not uniquely

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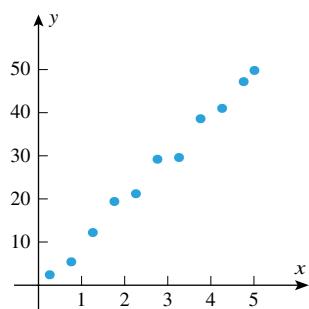


Figure 1.7.1

LINEAR FUNCTIONS AS MODELS

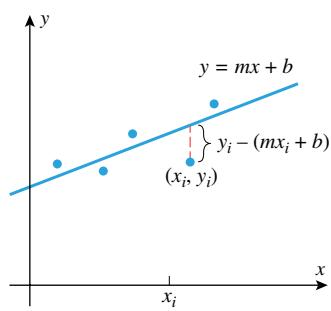


Figure 1.7.2

determined by x , but there is a **correlation** between weight and height that makes it more likely for a taller person to weigh more, so this is a probabilistic phenomenon.

In a deterministic model the variable y is a function of x , so the goal is to use the data in some reasonable way to find a formula $y = f(x)$ for this function. For example, Figure 1.7.1 strongly suggests that the relationship between x and y is linear, so in absence of additional information it would be natural to look for a linear function $y = mx + b$ as a model. In a probabilistic model the variable y need not be a function of x , so the goal is to find an equation $y = f(x)$ that specifies the *average value* of y that can be expected to occur for a given x . A more precise explanation of what is meant by “average value” and “expected to occur” requires ideas from probability and statistics, so we will depend on your intuition here.

Let us suppose that we have decided to model a particular phenomenon with a (yet to be determined) linear function $y = mx + b$. Ideally, we would then like to choose the parameters m and b such that the line $y = mx + b$ passes through all our data points. In practice, this may be impossible, either because of errors in our measurements or because there is not a strictly linear relationship between the variables x and y . We are then faced with the problem of finding the line $y = mx + b$ that “best fits” our set of data points. The key to determining this line is based on the following idea: For any proposed linear function $y = mx + b$, we draw a vertical connector from each data point (x_i, y_i) to the point $(x_i, mx_i + b)$ on the line and consider the differences $y_i - (mx_i + b)$ (Figure 1.7.2). These differences, which are called **residuals**, may be viewed as “errors” that result when the line is used to model the data. Data points above the line have positive errors, those below the line have negative errors, and those on the line have no error. One of the most common procedures is to look for a line such that *the sum of the squares* of the residuals is as small as possible. This line, known as a **least squares line** or **linear regression line**, is one choice for a line that “best fits” a given set of data. Most graphing calculators, spreadsheets, and CAS programs provide methods for finding regression lines. We will assume that you have access to some such method in this section.

It is possible to compute a regression line, even in cases where the data have no apparent linear pattern. Thus, it is important to have some quantitative method of determining whether a linear model is appropriate for the data. The most common measure of linearity in data is called the **correlation coefficient**. Following convention, we denote the correlation coefficient by the letter r . Although a detailed discussion of correlation coefficients is beyond the scope of this text, here are some of the basic facts:

- The values of r are in the interval $-1 \leq r \leq 1$, where r has the same sign as the slope of the regression line.
- If r is equal to 1 or -1 , then the data points all lie on a line, so a linear model is a perfect fit for the data.
- If $r = 0$, then the data points exhibit no linear tendency, so a linear model is inappropriate for the data.

The closer r is to 1 or -1 , the more tightly the data points hug the regression line and the more appropriate the regression line is as a model; the closer r is to 0, the more scattered the points and the less appropriate the regression line is as a model (Figure 1.7.3).

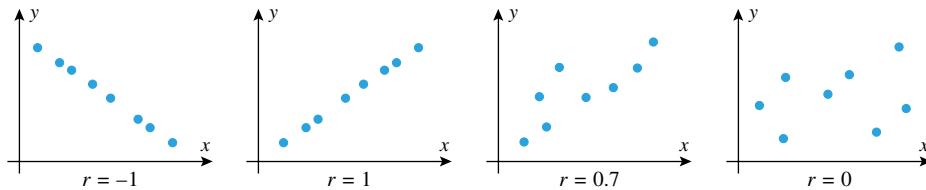


Figure 1.7.3

Roughly stated, the value of r^2 is a measure of the percentage of data points that fall in a “tight linear band.” Thus $r = 0.5$ means that 25% of the points fall in a tight linear band, and $r = 0.9$ means that 81% of the points fall in a tight linear band. (A precise explanation of what is meant by a “tight linear band” requires ideas from statistics.)

Table 1.7.1

TEMPERATURE T (°C)	PRESSURE p (atm)
0	2.54
50	3.06
100	3.46
150	4.00
200	4.41

Example 1 Table 1.7.1 gives a set of data points relating the pressure p in atmospheres (atm) and the temperature T (in °C) of a fixed quantity of carbon dioxide in a closed cylinder. The associated scatter plot in Figure 1.7.4a suggests that there is a linear relationship between the pressure and the temperature.

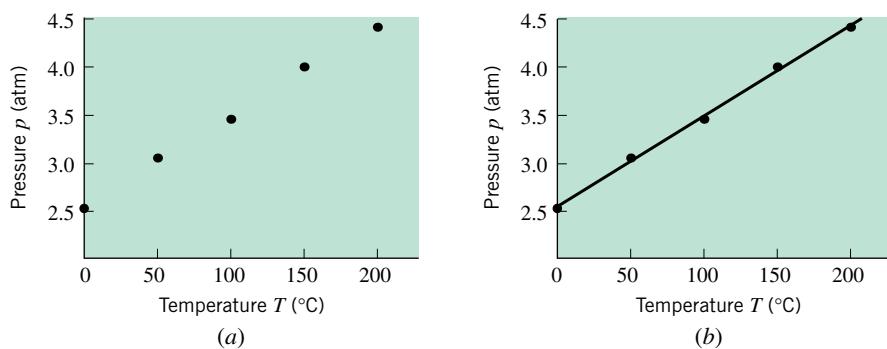


Figure 1.7.4

- (a) Use your calculating utility to find the least squares line for the data. If your utility can produce the correlation coefficient, then find it.
- (b) Use the model obtained in part (a) to predict the pressure when the temperature is 250°C.
- (c) Use the model obtained in part (a) to predict a temperature at which the pressure of the gas will be zero.

Solution (a). The least squares line is given by $p = 0.00936T + 2.558$ (Figure 1.7.4b) with correlation coefficient $r = 0.999475$.

Solution (b). If $T = 250$, then $p = (0.0094)(250) + 2.54 = 4.898$ (atm).

Solution (c). Solving the equation $0 = p = 0.00936T + 2.558$ yields $T \approx -273.291$ °C. ◀

It is not always convenient (or necessary) to obtain the least squares line for a linear phenomenon in order to create a model. In some cases, more elementary methods suffice. Here is an example.

Example 2 Figure 1.7.5a shows a graph of temperature versus altitude that was transmitted by the *Magellan* spacecraft when it entered the atmosphere of Venus in October 1991. The graph strongly suggests that there is a linear relationship between temperature and altitude for altitudes between 35 km and 60 km.

- (a) Use the graph transmitted by the *Magellan* spacecraft to find a linear model of temperature versus altitude in the Venusian atmosphere that is valid for altitudes between 35 km and 60 km.
- (b) Use the model obtained in part (a) to estimate the temperature at the surface of Venus, and discuss the assumptions you are making in obtaining the estimate.

Solution (a). Let T be the temperature in kelvins and h the altitude in kilometers. We will first estimate the slope m of the linear portion of the graph, then estimate the coordinates

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of a data point (h_1, T_1) on that portion of the graph, and then use the point-slope form of a line

$$T - T_1 = m(h - h_1) \quad (1)$$

The graph nearly passes through the point $(60, 250)$, so we will take $h_1 \approx 60$ and $T_1 \approx 250$. In Figure 1.7.5b we have sketched a line that closely approximates the linear portion of the data. Using the intersections of that line with the edges of the grid box, we estimate the slope to be

$$m \approx \frac{100 - 490}{78 - 30} = -\frac{390}{48} = -8.125 \text{ K/km}$$

Substituting our estimates of h_1 , T_1 , and m into (1) yields the equation

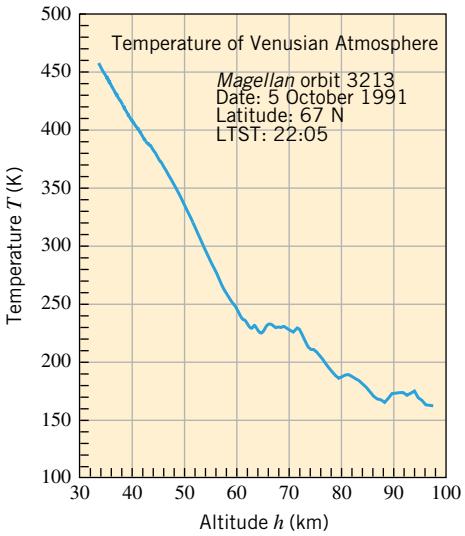
$$T - 250 = -8.125(h - 60)$$

or equivalently,

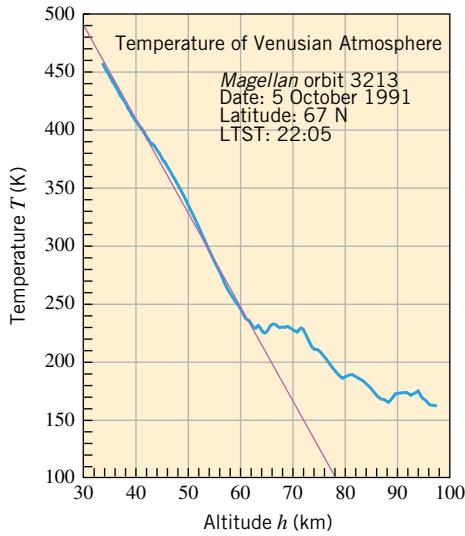
$$T = -8.125h + 737.5 \quad (2)$$

Solution (b). The *Magellan* spacecraft stopped transmitting data at an altitude of approximately 35 km, so we cannot be certain that the linear model applies at lower altitudes. However, if we *assume* that the model is valid at all lower altitudes, then we can approximate the temperature at the surface of Venus by setting $h = 0$ in (2). We obtain $T \approx 737.5$ K. ◀

- **REMARK.** The method of the preceding example is crude, at best, since it relies on extracting rough estimates of numerical data from a graph. Nevertheless, the final result is quite good, since the most recent information from NASA places the surface temperature of Venus at about 740 K (hot enough to melt lead).



(a)



(b)

Figure 1.7.5

QUADRATIC AND TRIGONOMETRIC FUNCTIONS AS MODELS

Although models based on linear functions $y = mx + b$ are simple, the relationship between the variables x and y associated with a particular phenomenon may be nonlinear, in which case replacing the function $y = mx + b$ by the quadratic function $y = ax^2 + bx + c$ may provide a better model. Most calculators, spreadsheets, and CAS programs will perform a least squares quadratic regression in a manner that is similar to linear regression.

Table 1.7.2

TIME t (s)	HEIGHT h (cm)
0.008333	98.4
0.025	96.9
0.04167	95.1
0.05833	92.9
0.075	90.8
0.09167	88.1
0.10833	85.3
0.125	82.1
0.14167	78.6
0.15833	74.9

Example 3 A student in a physics lab is studying the equations of motion of a falling body. She collects the data displayed in Table 1.7.2, which gives the height of the object at a number of times over a 0.15-s period of time. She knows that if air resistance is negligible and if the acceleration of the object due to gravity is assumed to be constant, then the height h of the object should be a quadratic function of time t . A scatter plot of the data is provided in Figure 1.7.6a, which suggests a portion of an inverted parabola.

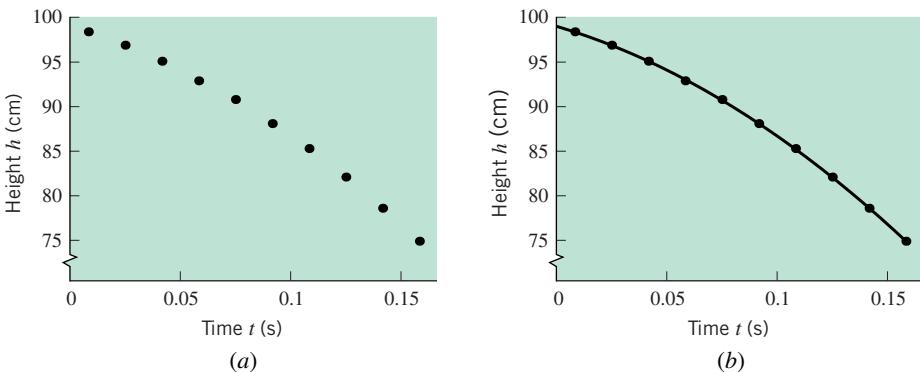


Figure 1.7.6

- Determine the quadratic regression curve for the data in Table 1.7.2.
- According to the model obtained in part (a), when will the object strike the ground?

Solution (a). Using the quadratic regression routine on a calculator, we find that the quadratic curve that best fits the data in Table 1.7.2 has equation

$$h = 99.02 - 73.21t - 499.13t^2$$

Figure 1.7.6b shows the data points and the graph of this quadratic function on the same set of axes. It appears that we have excellent agreement between our curve and the data.

Solution (b). Solving the equation $0 = h = 99.02 - 73.21t - 499.13t^2$, we find that the object will strike the ground at $t \approx 0.38$ s. ◀

The trigonometric functions $y = A \sin(Bx - C)$ and $y = A \cos(Bx - C)$ are particularly useful for modeling periodic phenomena.

Example 4 Figure 1.7.7a shows a table and scatter plot of temperature data recorded over a 24-hour period in the city of Philadelphia.* Find a function that models the data, and graph your function and data together.

Solution. The pattern of the data suggests that the relationship between the temperature T and the time t can be modeled by a sinusoidal function that has been translated both horizontally and vertically, so we will look for an equation of the form

$$T = D + A \sin[Bt - C] = D + A \sin\left[B\left(t - \frac{C}{B}\right)\right] \quad (3)$$

Since the highest temperature is 95°F and the lowest temperature is 75°F , we take $2A = 20$ or $A = 10$. The midpoint between the high and low is 85°F , so we have a vertical shift of $D = 85$. The period seems to be about 24, so $2\pi/B = 24$ or $B = \pi/12$. The horizontal shift appears to be about 10 (verify), so $C/B = 10$. Substituting these values in (3) yields

*This example is based on the article “Everybody Talks About It!—Weather Investigations,” by Gloria S. Dion and Iris Brann Fetta, *The Mathematics Teacher*, Vol. 89, No. 2, February 1996, pp. 160–165.

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PHILADELPHIA TEMPERATURES
FROM 1:00 A.M. TO 12:00 MIDNIGHT ON 27 AUGUST 1993
(t = HOURS AFTER MIDNIGHT AND T = DEGREES FAHRENHEIT)

	A.M.		P.M.	
	t	T	t	T
1:00	1	78°	13	91°
2:00	2	77°	14	93°
3:00	3	77°	15	94°
4:00	4	76°	16	95°
5:00	5	76°	17	93°
6:00	6	75°	18	92°
7:00	7	75°	19	89°
8:00	8	77°	20	86°
9:00	9	79°	21	84°
10:00	10	83°	22	83°
11:00	11	87°	23	81°
12:00	12	90°	24	79°

Source: *Philadelphia Inquirer*, 28 August 1993.

Figure 1.7.7

the equation

$$T = 85 + 10 \sin \left[\frac{\pi}{12}(t - 10) \right]$$

(Figure 1.7.7b). ◀

Note that in Example 4 we did not use a regression routine to fit the curve to the data. Some calculators may not be equipped to compute regression for trigonometric functions. In this case, we can use the calculator's graphing capability to see that a proposed model gives a *reasonable* fit to the data points, though it may not be the *best* fit.

• **FOR THE READER.** Using regression, a best fit to the data in Example 4 is

$$y = 84.203713 + 9.5964 \sin(0.2849t - 2.9300)$$

How does the graph of this best-fit curve compare with that found in Example 4?

EXERCISE SET 1.7 Graphing Calculator

- One of the lines in the accompanying figure is the regression line. Which one is it?
- Conjecture an appropriate model (linear, quadratic, or trigonometric), if any, for each of the data sets shown in the scatter plots.

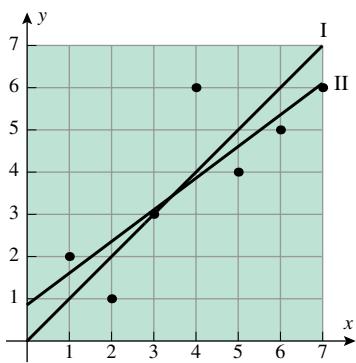
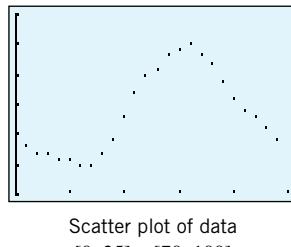


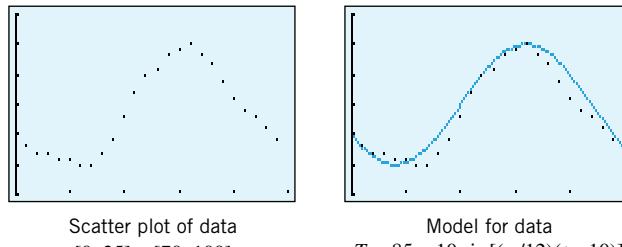
Figure Ex-1



Scatter plot of data

$[0, 25] \times [70, 100]$

t



Model for data

$T = 85 + 10 \sin[(\pi/12)(t - 10)]$

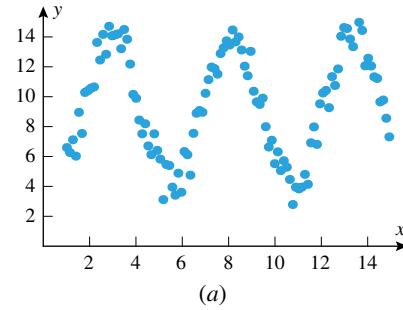
$[0, 25] \times [70, 100]$

t

T

(a)

(b)



(a)

1.7 Mathematical Models 85

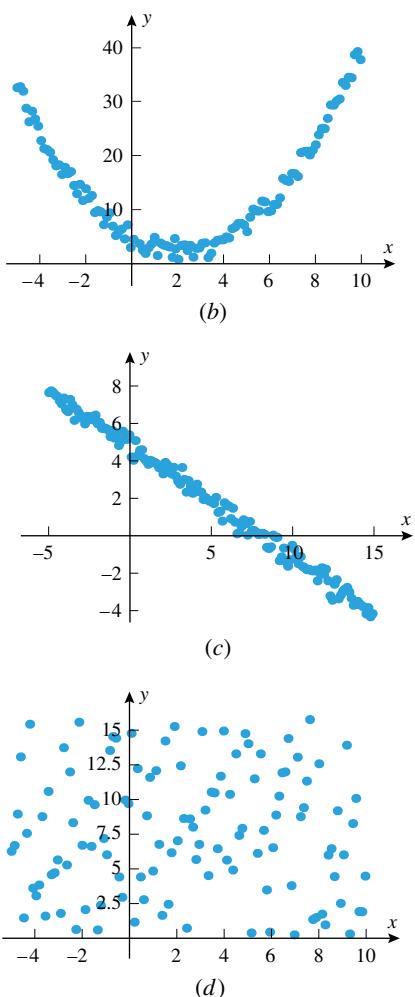


Figure Ex-2

5. A 20-liter container holds 100 g of N₂. The pressure p of this gas is measured at various temperatures T (see the accompanying table).

- Find the least squares line for this collection of data points. If your calculating utility can produce the correlation coefficient, then find it.
- Use the model obtained in part (a) to predict the pressure of the gas at a temperature of -50°C .
- Use the model obtained in part (a) to predict a temperature at which the pressure of the gas will be zero.

TEMPERATURE T ($^{\circ}\text{C}$)	PRESSURE p (atm)
0	3.99
25	4.34
50	4.70
75	5.08
100	5.45

Table Ex-5

6. A 40-liter container holds 20 g of H₂. The pressure p of this gas is measured at various temperatures T (see the accompanying table).

- Find the least squares line for this collection of data points. If your calculating utility can produce the correlation coefficient, then find it.
- Use the model obtained in part (a) to predict a temperature at which the pressure of the gas will be zero.
- At approximately what temperature of the gas will a 10°C increase in temperature result in a 5% increase in pressure?

TEMPERATURE T ($^{\circ}\text{C}$)	PRESSURE p (atm)
0	5.55
30	6.13
60	6.75
90	7.35
120	7.98

Table Ex-6

3. Table 1.1.1 provides data for the top qualifying speeds at the Indianapolis 500 from 1980 to 1999. Find the least squares line for these data. What is the correlation coefficient? Sketch the least squares line on a scatter plot of the data points.

4. A 25-liter container holds 150 g of O₂. The pressure p of the gas is measured at various temperatures T (see the accompanying table).

- Determine the least squares line for the data given in the table.
- Use the model obtained in part (a) to estimate the pressure of the gas at a temperature of -50°C .

TEMPERATURE T ($^{\circ}\text{C}$)	PRESSURE p (atm)
0	4.18
50	4.96
100	5.74
150	6.49
200	7.26

Table Ex-4

7. The **resistivity** of a metal is a measure of the extent to which a wire made from the metal will resist the flow of electrical current. (The actual **resistance** of the wire will depend on both the resistivity of the metal and the dimensions of the wire.) A common unit for resistivity is the ohm-meter ($\Omega \cdot \text{m}$). Experiments show that lowering the temperature of a metal also lowers its resistivity. The accompanying table gives the resistivity of copper at various temperatures.

- Find the least squares line for this collection of data points.
- Using the model obtained in part (a), at what temperature will copper have a resistivity of zero?

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TEMPERATURE ($^{\circ}\text{C}$)	RESISTIVITY ($10^{-8} \Omega \cdot \text{m}$)
-100	0.82
-50	1.19
0	1.54
50	1.91
100	2.27
150	2.63

Table Ex-7

8. The accompanying table gives the resistivity of tungsten at various temperatures.
- Find the least squares line for this collection of data points.
 - Using the model obtained in part (a), at what temperature will tungsten have a resistivity of zero?

TEMPERATURE ($^{\circ}\text{C}$)	RESISTIVITY ($10^{-8} \Omega \cdot \text{m}$)
-100	2.43
-50	3.61
0	4.78
50	5.96
100	7.16
150	8.32

Table Ex-8

9. The accompanying table gives the number of inches that a spring is stretched by various attached weights.
- Use linear regression to express the amount of stretch of the spring as a function of the weight attached.
 - Use the model obtained in part (a) to determine the weight required to stretch the spring 8 in.

WEIGHT (lb)	STRETCH (in)
0	0
2	0.99
4	2.01
6	2.99
8	4.00
10	5.03
12	6.01

Table Ex-9

10. The accompanying table gives the number of inches that a spring is stretched by various attached weights.
- Use linear regression to express the amount of stretch of the spring as a function of the weight attached.
 - Suppose that the spring has been stretched a certain amount by a weight and that adding another 5 lb to the weight doubles the stretch of the spring. Use the model obtained in part (a) to determine the original amount that the spring was stretched.
11. The accompanying table provides the heights and rebounds per minute for players on the 1998–1999 Davidson Col-

lege women's basketball team who played more than 100 minutes during the season.

- Find the least squares line for these data. If your calculating utility can produce the correlation coefficient, then find it.
- Sketch the least squares line on a scatter plot of the data points.
- Is the least squares line a good model for these data? Explain.

WEIGHT (lb)	STRETCH (in)
0	0
1	0.73
2	1.50
3	2.24
4	3.02
5	3.77

Table Ex-10

HEIGHT	REBOUNDS PER MINUTE
5'11"	0.25
6'2"	0.176
5'6"	0.141
5'11"	0.162
6'1"	0.167
5'8"	0.091
5'11"	0.278
6'3"	0.167
6'0"	0.214

Table Ex-11

12. The accompanying table provides the heights and weights for players on the 1999–2000 Davidson College men's basketball team.
- Find the least squares line for these data. If your calculating utility can produce the correlation coefficient, then find it.
 - Sketch the least squares line on a scatter plot of the data points.
 - Use this model to predict the weight of the team's new 7-ft recruit.

HEIGHT	WEIGHT (lb)
6'0"	165
6'0"	180
6'4"	195
6'3"	185
6'7"	210
6'4"	190
6'3"	190
6'9"	240
7'2"	280
5'10"	175
6'7"	215
6'7"	235
6'8"	225

Table Ex-12

13. (The Age of the Universe) In the early 1900s the astronomer Edwin P. Hubble (1889–1953) noted an unexpected relationship between the radial velocity of a galaxy and its distance d from any reference point (Earth, for example). That relationship, now known as **Hubble's law**, states that the galaxies are receding with a velocity v that is directly proportional to the distance d . This is usually expressed as $v = Hd$, where

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H (the constant of proportionality) is called **Hubble's constant**. When applying this formula it is usual to express v in kilometers per second (km/s) and d in millions of light-years (Mly), in which case H has units of km/s/Mly. The accompanying figure shows an original plot and trend line of the velocity-distance relationship obtained by Hubble and a collaborator Milton L. Humason (1891–1972).

- Use the trend line in the figure to estimate Hubble's constant.
- An estimate of the age of the universe can be obtained by assuming that the galaxies move with constant velocity v , in which case v and d are related by $d = vt$. Assuming that the Universe began with a “big bang” that initiated its expansion, show that the Universe is roughly 1.5×10^{10} years old. [Use the conversion 1 Mly $\approx 9.048 \times 10^{18}$ and take $H = 20$ km/s/Mly, which is in keeping with current estimates that place H between 15 and 27 km/s/Mly. (Note that the current estimates are significantly less than that resulting from Hubble's data.)]
- In a more realistic model of the Universe, the velocity v would decrease with time. What effect would that have on your estimate in part (b)?

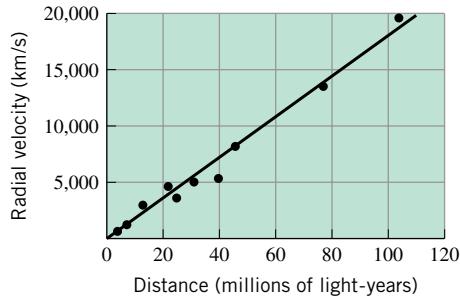


Figure Ex-13

14. A professor wishes to use midterm grades as a predictor of final grades in a small seminar that he teaches once a year. The midterm grades and final grades for last year's seminar are listed in the accompanying table.
- Find the linear regression model that expresses the final grade in terms of the midterm grade.
 - Suppose that a student in this year's seminar earned a midterm grade of 88. Use the model obtained in part (a) to predict the student's final grade in the seminar.

MIDTERM GRADE	FINAL GRADE
78	78
94	91
78	76
84	82
95	92
96	93
77	75

Table Ex-14

15. A student is studying the equations of motion for an object moving along a number line with constant acceleration. The accompanying table gives the position in meters of the object at various times.
- Use quadratic regression to model the position of the object as a function of time.
 - Based on the model obtained in part (a), what will be the position of the object after 2 s?

TIME (s)	POSITION (m)
0.2537	0.045
0.4064	0.09
0.5981	0.165
0.75	0.24
0.8781	0.315
1.032	0.42
1.1846	0.54
1.3208	0.66

Table Ex-15

16. Table 1.1.3 gives data for the U.S. population at 10-year intervals from 1790 to 1850. Use quadratic regression to model the U.S. population as a function of time since 1790. What does your model predict as the population of the United States in the year 2000? How accurate is this prediction?
17. The accompanying table gives the minutes of daylight predicted for Davidson, North Carolina, in 10-day increments during the year 2000. Find a function that models the data in this table, and graph your function on a scatter plot of the data.

DAY	DAYLIGHT (min)	DAY	DAYLIGHT (min)
10	716	190	986
20	727	200	975
30	744	210	961
40	762	220	944
50	783	230	926
60	804	240	905
70	826	250	883
80	848	260	861
90	872	270	839
100	894	280	817
110	915	290	795
120	935	300	774
130	954	310	755
140	969	320	738
150	982	330	723
160	990	340	712
170	993	350	706
180	992	360	706

Table Ex-17

18. The accompanying table gives the fraction of the Moon that is illuminated at midnight and visible from Earth (eastern

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standard time) in 2-day intervals for the first 60 days of 1999. Find a function that models the data in this table, and graph your function on a scatter plot of the data.

DAY	ILLUMINATION	DAY	ILLUMINATION
2	1	32	1
4	0.94	34	0.93
6	0.81	36	0.79
8	0.63	38	0.62
10	0.44	40	0.43
12	0.26	42	0.25
14	0.12	44	0.10
16	0.02	46	0.01
18	0	48	0.01
20	0.07	50	0.11
22	0.22	52	0.29
24	0.43	54	0.51
26	0.66	56	0.73
28	0.85	58	0.90
30	0.97	60	0.99

Table Ex-18

d (meters)	0	2.5	5	10	15	20	25
t (seconds)	0	0.7	1.0	1.4	1.7	2	2.3

Table Ex-19

20. (a) The accompanying table below provides data on five moons of the planet Saturn. In this table r is the *orbital radius* (the average distance between the moon and Saturn) and t is the time in days required for the moon to complete one orbit around Saturn. For each data pair calculate $tr^{-3/2}$, and use your results to find a reasonable model for r as a function of t .
- (b) Use the model obtained in part (a) to estimate the orbital radius of the moon Enceladus, given that its orbit time is $t \approx 1.370$ days.
- (c) Use the model obtained in part (a) to estimate the orbit time of the moon Tethys, given that its orbital radius is $r \approx 2.9467 \times 10^5$ km.

MOON	RADIUS (100,000 km)	ORBIT TIME (days)
1980S28	1.3767	0.602
1980S27	1.3935	0.613
1980S26	1.4170	0.629
1980S3	1.5142	0.694
1980S1	1.5147	0.695

Table Ex-20

1.8 PARAMETRIC EQUATIONS

Thus far, our study of graphs has focused on graphs of functions. However, because such graphs must pass the vertical line test, this limitation precludes curves with self-intersections or even such basic curves as circles. In this section we will study an alternative method for describing curves algebraically that is not subject to the severe restriction of the vertical line test.

This material is placed here to provide an early parametric option. However, it can be deferred until Chapter 11, if preferred.

PARAMETRIC EQUATIONS

Suppose that a particle moves along a curve C in the xy -plane in such a way that its x - and y -coordinates, as functions of time, are

$$x = f(t), \quad y = g(t)$$

We call these the *parametric equations* of motion for the particle and refer to C as the *trajectory* of the particle or the *graph* of the equations (Figure 1.8.1). The variable t is called the *parameter* for the equations.

Example 1 Sketch the trajectory over the time interval $0 \leq t \leq 10$ of the particle whose parametric equations of motion are

$$x = t - 3 \sin t, \quad y = 4 - 3 \cos t \tag{1}$$

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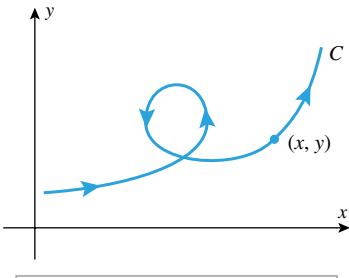


Figure 1.8.1

Solution. One way to sketch the trajectory is to choose a representative succession of times, plot the (x, y) coordinates of points on the trajectory at those times, and connect the points with a smooth curve. The trajectory in Figure 1.8.2 was obtained in this way from Table 1.8.1 in which the approximate coordinates of the particle are given at time increments of 1 unit. Observe that there is no t -axis in the picture; the values of t appear only as labels on the plotted points, and even these are usually omitted unless it is important to emphasize the location of the particle at specific times. ◀

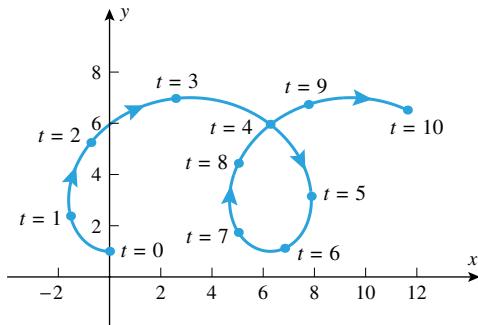


Figure 1.8.2

Table 1.8.1

t	x	y
0	0.0	1.0
1	-1.5	2.4
2	-0.7	5.2
3	2.6	7.0
4	6.3	6.0
5	7.9	3.1
6	6.8	1.1
7	5.0	1.7
8	5.0	4.4
9	7.8	6.7
10	11.6	6.5

- **FOR THE READER.** Read the documentation for your graphing utility to learn how to graph parametric equations, and then generate the trajectory in Example 1. Explore the behavior of the particle beyond time $t = 10$.

Although parametric equations commonly arise in problems of motion with time as the parameter, they arise in other contexts as well. Thus, unless the problem dictates that the parameter t in the equations

$$x = f(t), \quad y = g(t)$$

represents time, it should be viewed simply as an independent variable that varies over some interval of real numbers. (In fact, there is no need to use the letter t for the parameter; any letter not reserved for another purpose can be used.) If no restrictions on the parameter are stated explicitly or implied by the equations, then it is understood that it varies from $-\infty$ to $+\infty$. To indicate that a parameter t is restricted to an interval $[a, b]$, we will write

$$x = f(t), \quad y = g(t) \quad (a \leq t \leq b)$$

Example 2 Find the graph of the parametric equations

$$x = \cos t, \quad y = \sin t \quad (0 \leq t \leq 2\pi) \quad (2)$$

Solution. One way to find the graph is to eliminate the parameter t by noting that

$$x^2 + y^2 = \sin^2 t + \cos^2 t = 1$$

Thus, the graph is contained in the unit circle $x^2 + y^2 = 1$. Geometrically, t can be interpreted as the angle swept out by the radial line from the origin to the point $(x, y) = (\cos t, \sin t)$ on the unit circle (Figure 1.8.3). As t increases from 0 to 2π , the point traces the circle counterclockwise, starting at $(1, 0)$ when $t = 0$ and completing one full revolution when $t = 2\pi$. One can obtain different portions of the circle by varying the interval over which the parameter varies. For example,

$$x = \cos t, \quad y = \sin t \quad (0 \leq t \leq \pi) \quad (3)$$

represents just the upper semicircle in Figure 1.8.3. ◀

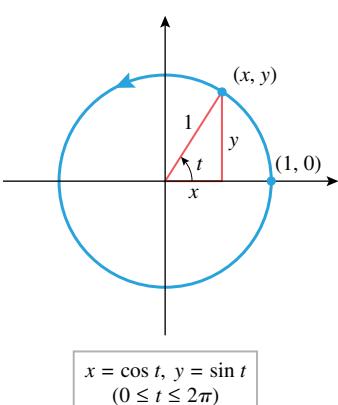


Figure 1.8.3

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..... ORIENTATION

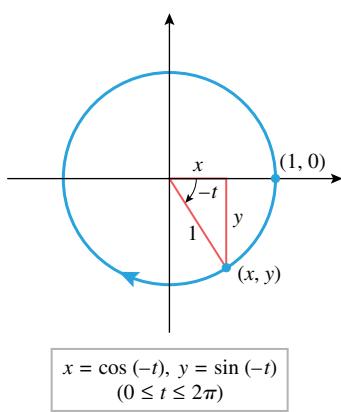


Figure 1.8.4

The direction in which the graph of a pair of parametric equations is traced as the parameter increases is called the **direction of increasing parameter** or sometimes the **orientation** imposed on the curve by the equations. Thus, we make a distinction between a **curve**, which is a set of points, and a **parametric curve**, which is a curve with an orientation imposed on it by a set of parametric equations. For example, we saw in Example 2 that the circle represented parametrically by (2) is traced counterclockwise as t increases and hence has **counterclockwise orientation**. As shown in Figures 1.8.2 and 1.8.3, the orientation of a parametric curve can be indicated by arrowheads.

To obtain parametric equations for the unit circle with **clockwise orientation**, we can replace t by $-t$ in (2), and use the identities $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$. This yields

$$x = \cos t, \quad y = -\sin t \quad (0 \leq t \leq 2\pi)$$

Here, the circle is traced clockwise by a point that starts at $(1, 0)$ when $t = 0$ and completes one full revolution when $t = 2\pi$ (Figure 1.8.4).

- **FOR THE READER.** When parametric equations are graphed using a calculator, the orientation can often be determined by watching the direction in which the graph is traced on the screen. However, many computers graph so fast that it is often hard to discern the orientation. See if you can use your graphing utility to confirm that (3) has a counterclockwise orientation.

Example 3 Graph the parametric curve

$$x = 2t - 3, \quad y = 6t - 7$$

by eliminating the parameter, and indicate the orientation on the graph.

Solution. To eliminate the parameter we will solve the first equation for t as a function of x , and then substitute this expression for t into the second equation:

$$t = \left(\frac{1}{2}\right)(x + 3)$$

$$y = 6\left(\frac{1}{2}\right)(x + 3) - 7$$

$$y = 3x + 2$$

Thus, the graph is a line of slope 3 and y -intercept 2. To find the orientation we must look to the original equations; the direction of increasing t can be deduced by observing that x increases as t increases or by observing that y increases as t increases. Either piece of information tells us that the line is traced left to right as shown in Figure 1.8.5. ◀

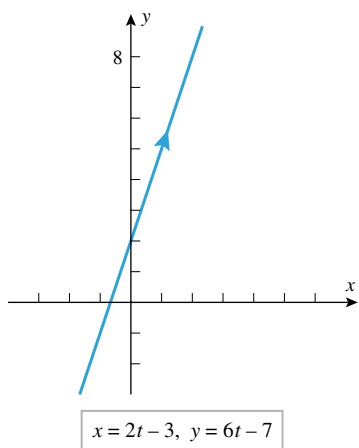


Figure 1.8.5

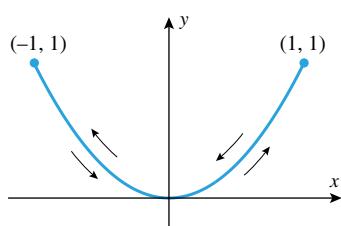


Figure 1.8.6

- **REMARK.** Not all parametric equations produce curves with definite orientations; if the equations are badly behaved, then the point tracing the curve may leap around sporadically or move back and forth, failing to determine a definite direction. For example, if

$$x = \sin t, \quad y = \sin^2 t$$

then the point (x, y) moves along the parabola $y = x^2$. However, the value of x varies periodically between -1 and 1 , so the point (x, y) moves periodically back and forth along the parabola between the points $(-1, 1)$ and $(1, 1)$ (as shown in Figure 1.8.6). Later in the text we will discuss restrictions that eliminate such erratic behavior, but for now we will just avoid such complications.

..... EXPRESSING ORDINARY FUNCTIONS PARAMETRICALLY

An equation $y = f(x)$ can be expressed in parametric form by introducing the parameter $t = x$; this yields the parametric equations $x = t$, $y = f(t)$. For example, the portion of the curve $y = \cos x$ over the interval $[-2\pi, 2\pi]$ can be expressed parametrically as

$$x = t, \quad y = \cos t \quad (-2\pi \leq t \leq 2\pi)$$

(Figure 1.8.7).

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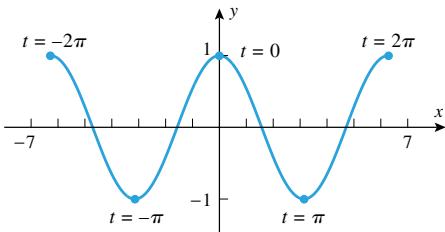


Figure 1.8.7

GENERATING PARAMETRIC CURVES WITH GRAPHING UTILITIES

Many graphing utilities allow you to graph equations of the form $y = f(x)$ but not equations of the form $x = g(y)$. Sometimes you will be able to rewrite $x = g(y)$ in the form $y = f(x)$; however, if this is inconvenient or impossible, then you can graph $x = g(y)$ by introducing a parameter $t = y$ and expressing the equation in the parametric form $x = g(t)$, $y = t$. (You may have to experiment with various intervals for t to produce a complete graph.)

Example 4 Use a graphing utility to graph the equation $x = 3y^5 - 5y^3 + 1$.

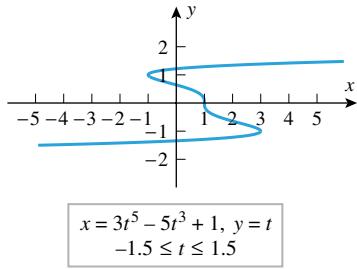


Figure 1.8.8

Solution. If we let $t = y$ be the parameter, then the equation can be written in parametric form as

$$x = 3t^5 - 5t^3 + 1, \quad y = t$$

Figure 1.8.8 shows the graph of these equations for $-1.5 \leq t \leq 1.5$. ◀

Some parametric curves are so complex that it is virtually impossible to visualize them without using some kind of graphing utility. Figure 1.8.9 shows three such curves.

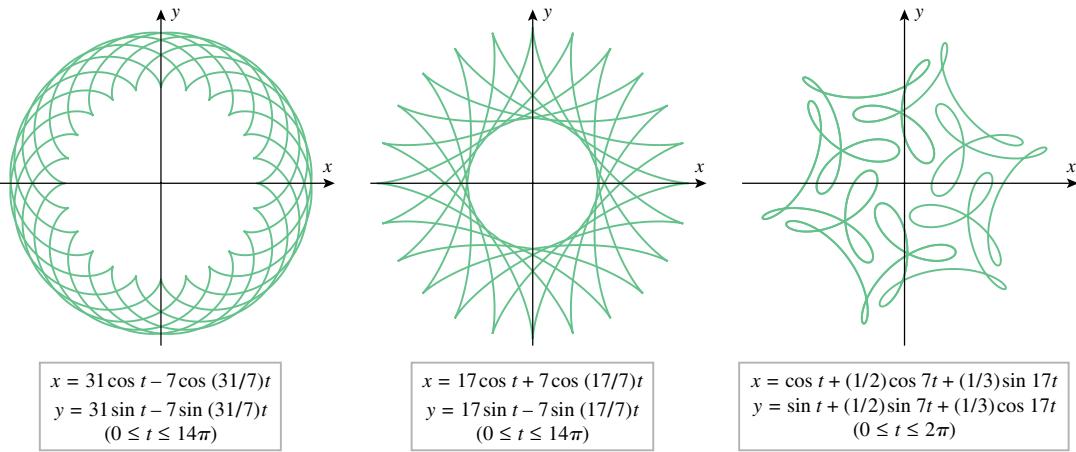


Figure 1.8.9

FOR THE READER. Without spending too much time, try your hand at generating some parametric curves with a graphing utility that you think are interesting or beautiful.

TRANSLATION

If a parametric curve C is given by the equations $x = f(t)$, $y = g(t)$, then adding a constant to $f(t)$ translates the curve C in the x -direction, and adding a constant to $g(t)$ translates it in the y -direction. Thus, a circle of radius r , centered at (x_0, y_0) can be represented parametrically as

$$x = x_0 + r \cos t, \quad y = y_0 + r \sin t \quad (0 \leq t \leq 2\pi) \quad (4)$$

(Figure 1.8.10). If desired, we can eliminate the parameter from these equations by noting

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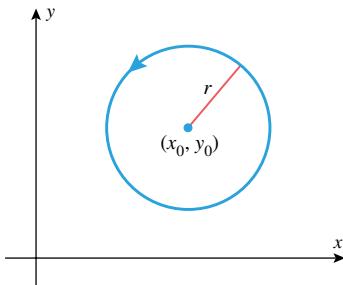


Figure 1.8.10

$$\begin{aligned}x &= x_0 + r \cos t \\y &= y_0 + r \sin t \\(0 \leq t \leq 2\pi)\end{aligned}$$

that

$$(x - x_0)^2 + (y - y_0)^2 = (r \cos t)^2 + (r \sin t)^2 = r^2$$

Thus, we have obtained the familiar equation in rectangular coordinates for a circle of radius r , centered at (x_0, y_0) :

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \quad (5)$$

- FOR THE READER.** Use the parametric capability of your graphing utility to generate a circle of radius 5 that is centered at $(3, -2)$.

SCALING

If a parametric curve C is given by the equations $x = f(t)$, $y = g(t)$, then multiplying $f(t)$ by a constant stretches or compresses C in the x -direction, and multiplying $g(t)$ by a constant stretches or compresses C in the y -direction. For example, we would expect the parametric equations

$$x = 3 \cos t, \quad y = 2 \sin t \quad (0 \leq t \leq 2\pi)$$

to represent an ellipse, centered at the origin, since the graph of these equations results from stretching the unit circle

$$x = \cos t, \quad y = \sin t \quad (0 \leq t \leq 2\pi)$$

by a factor of 3 in the x -direction and a factor of 2 in the y -direction. In general, if a and b are positive constants, then the parametric equations

$$x = a \cos t, \quad y = b \sin t \quad (0 \leq t \leq 2\pi) \quad (6)$$

represent an ellipse, centered at the origin, and extending between $-a$ and a on the x -axis and between $-b$ and b on the y -axis (Figure 1.8.11). The numbers a and b are called the **semiaxes** of the ellipse. If desired, we can eliminate the parameter t in (6) and rewrite the equations in rectangular coordinates as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (7)$$

- FOR THE READER.** Use the parametric capability of your graphing utility to generate an ellipse that is centered at the origin and that extends between -4 and 4 in the x -direction and between -3 and 3 in the y -direction. Generate an ellipse with the same dimensions, but translated so that its center is at $(2, 3)$.

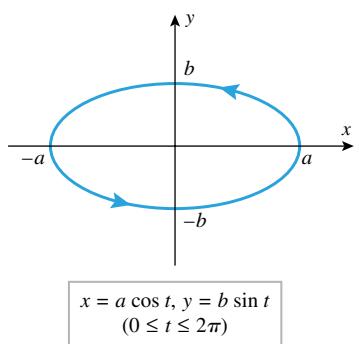
Figure 1.8.11

LISSAJOUS CURVES

In the mid-1850s the French physicist Jules Antoine Lissajous (1822–1880) became interested in parametric equations of the form

$$x = \sin at, \quad y = \sin bt \quad (8)$$

in the course of studying vibrations that combine two perpendicular sinusoidal motions.



$$\begin{aligned}x &= a \cos t, y = b \sin t \\(0 \leq t \leq 2\pi)\end{aligned}$$

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The first equation in (8) describes a sinusoidal oscillation in the x -direction with frequency $a/2\pi$, and the second describes a sinusoidal oscillation in the y -direction with frequency $b/2\pi$. If a/b is a rational number, then the combined effect of the oscillations is a periodic motion along a path called a **Lissajous curve**. Figure 1.8.12 shows some typical Lissajous curves.

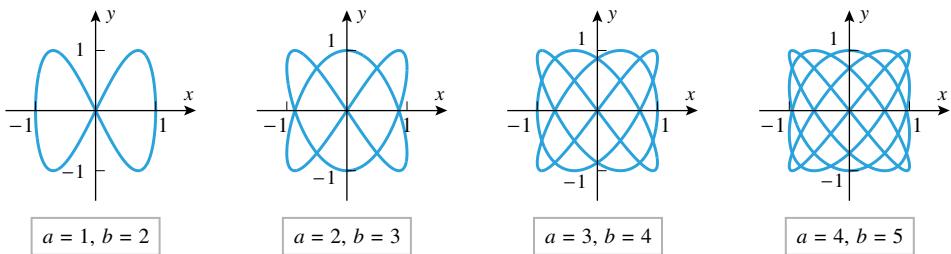


Figure 1.8.12

- FOR THE READER.** Generate some Lissajous curves on your graphing utility, and also see if you can figure out when each of the curves in Figure 1.8.12 begins to repeat.

CYCLOIDS

If a wheel rolls in a straight line along a flat road, then a point on the rim of the wheel will trace a curve called a **cycloid** (Figure 1.8.13). This curve has a fascinating history, which we will discuss shortly; but first we will show how to obtain parametric equations for it. For this purpose, let us assume that the wheel has radius a and rolls along the positive x -axis of a rectangular coordinate system. Let $P(x, y)$ be the point on the rim that traces the cycloid, and assume that P is initially at the origin. We will take as our parameter the angle θ that is swept out by the radial line to P as the wheel rolls (Figure 1.8.13). It is standard here to regard θ to be positive, even though it is generated by a clockwise rotation.

The motion of P is a combination of the movement of the wheel's center parallel to the x -axis and the rotation of P around the center. As the radial line sweeps out an angle θ , the point P traverses an arc of length $a\theta$, and the wheel moves a distance $a\theta$ along the x -axis (why?). Thus, as suggested by Figure 1.8.14, the center moves to the point $(a\theta, a)$, and the coordinates of $P(x, y)$ are

$$x = a\theta - a \sin \theta, \quad y = a - a \cos \theta \quad (9)$$

These are the equations of the cycloid in terms of the parameter θ .

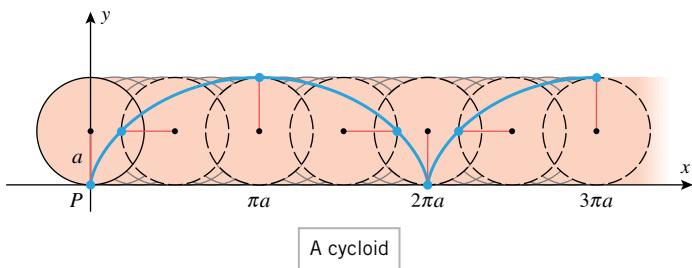


Figure 1.8.13

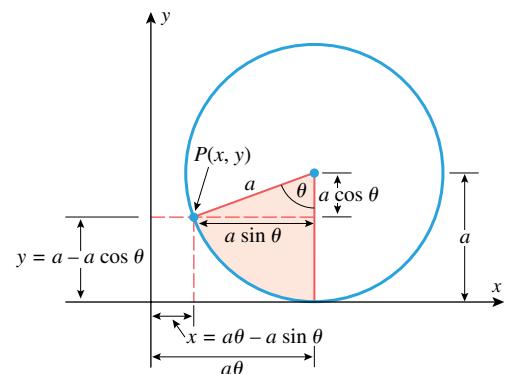


Figure 1.8.14

- FOR THE READER.** Use your graphing utility to generate two “arches” of the cycloid produced by a point on the rim of a wheel of radius 1.

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THE ROLE OF THE CYCLOID IN MATHEMATICS HISTORY

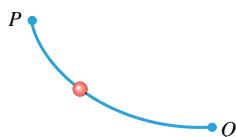


Figure 1.8.15

The cycloid is of interest because it provides the solution to two famous mathematical problems—the **brachistochrone problem** (from Greek words meaning “shortest time”) and the **tautochrone problem** (from Greek words meaning “equal time”). The brachistochrone problem is to determine the shape of a wire along which a bead might slide from a point P to another point Q , not directly below, in the *shortest time*. The tautochrone problem is to find the shape of a wire from P to Q such that two beads started at any points on the wire between P and Q reach Q in the same amount of time (Figure 1.8.15). The solution to both problems turns out to be an inverted cycloid.

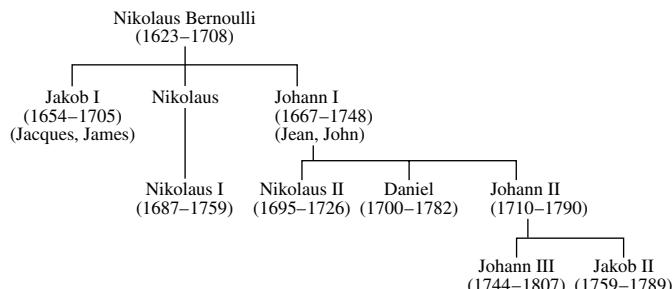
In June of 1696, Johann Bernoulli* posed the brachistochrone problem in the form of a challenge to other mathematicians. At first, one might conjecture that the wire should form a straight line, since that shape results in the shortest distance from P to Q . However, the inverted cycloid allows the bead to fall more rapidly at first, building up sufficient initial speed to reach Q in the shortest time, even though it travels a longer distance. The problem was solved by Newton and Leibniz as well as by Johann Bernoulli and his older brother

* **BERNOULLI.** An amazing Swiss family that included several generations of outstanding mathematicians and scientists. Nikolaus Bernoulli (1623–1708), a druggist, fled from Antwerp to escape religious persecution and ultimately settled in Basel, Switzerland. There he had three sons, Jakob I (also called Jacques or James), Nikolaus, and Johann I (also called Jean or John). The Roman numerals are used to distinguish family members with identical names (see the family tree below). Following Newton and Leibniz, the Bernoulli brothers, Jakob I and Johann I, are considered by some to be the two most important founders of calculus. Jakob I was self-taught in mathematics. His father wanted him to study for the ministry, but he turned to mathematics and in 1686 became a professor at the University of Basel. When he started working in mathematics, he knew nothing of Newton’s and Leibniz’ work. He eventually became familiar with Newton’s results, but because so little of Leibniz’ work was published, Jakob duplicated many of Leibniz’ results.

Jakob’s younger brother Johann I was urged to enter into business by his father. Instead, he turned to medicine and studied mathematics under the guidance of his older brother. He eventually became a mathematics professor at Groningen in Holland, and then, when Jakob died in 1705, Johann succeeded him as mathematics professor at Basel. Throughout their lives, Jakob I and Johann I had a mutual passion for criticizing each other’s work, which frequently erupted into ugly confrontations. Leibniz tried to mediate the disputes, but Jakob, who resented Leibniz’ superior intellect, accused him of siding with Johann, and thus Leibniz became entangled in the arguments. The brothers often worked on common problems that they posed as challenges to one another. Johann, interested in gaining fame, often used unscrupulous means to make himself appear the originator of his brother’s results; Jakob occasionally retaliated. Thus, it is often difficult to determine who deserves credit for many results. However, both men made major contributions to the development of calculus. In addition to his work on calculus, Jakob helped establish fundamental principles in probability, including the Law of Large Numbers, which is a cornerstone of modern probability theory.

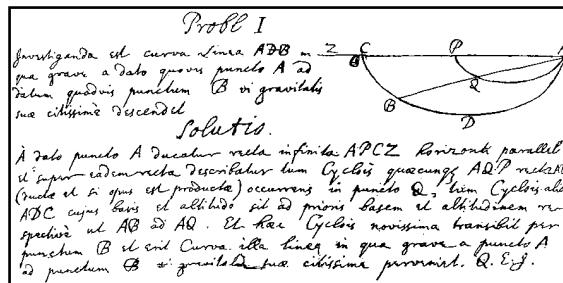
Among the other members of the Bernoulli family, Daniel, son of Johann I, is the most famous. He was a professor of mathematics at St. Petersburg Academy in Russia and subsequently a professor of anatomy and then physics at Basel. He did work in calculus and probability, but is best known for his work in physics. A basic law of fluid flow, called Bernoulli’s principle, is named in his honor. He won the annual prize of the French Academy 10 times for work on vibrating strings, tides of the sea, and kinetic theory of gases.

Johann II succeeded his father as professor of mathematics at Basel. His research was on the theory of heat and sound. Nikolaus I was a mathematician and law scholar who worked on probability and series. On the recommendation of Leibniz, he was appointed professor of mathematics at Padua and then went to Basel as a professor of logic and then law. Nikolaus II was professor of jurisprudence in Switzerland and then professor of mathematics at St. Petersburg Academy. Johann III was a professor of mathematics and astronomy in Berlin and Jakob II succeeded his uncle Daniel as professor of mathematics at St. Petersburg Academy in Russia. Truly an incredible family!



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Jakob; it was formulated and solved *incorrectly* years earlier by Galileo, who thought the answer was a circular arc.



Newton's solution of the brachistochrone problem in his own handwriting

EXERCISE SET 1.8

1. (a) By eliminating the parameter, sketch the trajectory over the time interval $0 \leq t \leq 5$ of the particle whose parametric equations of motion are

$$x = t - 1, \quad y = t + 1$$

- (b) Indicate the direction of motion on your sketch.
 (c) Make a table of x - and y -coordinates of the particle at times $t = 0, 1, 2, 3, 4, 5$.
 (d) Mark the position of the particle on the curve at the times in part (c), and label those positions with the values of t .
2. (a) By eliminating the parameter, sketch the trajectory over the time interval $0 \leq t \leq 1$ of the particle whose parametric equations of motion are

$$x = \cos(\pi t), \quad y = \sin(\pi t)$$

- (b) Indicate the direction of motion on your sketch.
 (c) Make a table of x - and y -coordinates of the particle at times $t = 0, 0.25, 0.5, 0.75, 1$.
 (d) Mark the position of the particle on the curve at the times in part (c), and label those positions with the values of t .

In Exercises 3–12, sketch the curve by eliminating the parameter, and indicate the direction of increasing t .

3. $x = 3t - 4, \quad y = 6t + 2$
 4. $x = t - 3, \quad y = 3t - 7 \quad (0 \leq t \leq 3)$
 5. $x = 2 \cos t, \quad y = 5 \sin t \quad (0 \leq t \leq 2\pi)$
 6. $x = \sqrt{t}, \quad y = 2t + 4$
 7. $x = 3 + 2 \cos t, \quad y = 2 + 4 \sin t \quad (0 \leq t \leq 2\pi)$
 8. $x = \sec t, \quad y = \tan t \quad (\pi \leq t < 3\pi/2)$
 9. $x = \cos 2t, \quad y = \sin t \quad (-\pi/2 \leq t \leq \pi/2)$
 10. $x = 4t + 3, \quad y = 16t^2 - 9$
 11. $x = 2 \sin^2 t, \quad y = 3 \cos^2 t$
 12. $x = \sec^2 t, \quad y = \tan^2 t$

In Exercises 13–18, find parametric equations for the curve, and check your work by generating the curve with a graphing utility.

13. A circle of radius 5, centered at the origin, oriented clockwise.
 14. The portion of the circle $x^2 + y^2 = 1$ that lies in the third quadrant, oriented counterclockwise.
 15. A vertical line intersecting the x -axis at $x = 2$, oriented upward.
 16. The ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, oriented counterclockwise.
 17. The portion of the parabola $x = y^2$ joining $(1, -1)$ and $(1, 1)$, oriented down to up.
 18. The circle of radius 4, centered at $(1, -3)$, oriented counterclockwise.
 19. In each part, match the parametric equation with one of the curves labeled (I)–(VI), and explain your reasoning.
 (a) $x = \sqrt{t}, \quad y = \sin 3t$ (b) $x = 2 \cos t, \quad y = 3 \sin t$
 (c) $x = t \cos t, \quad y = t \sin t$ (d) $x = \frac{3t}{1+t^3}, \quad y = \frac{3t^2}{1+t^3}$
 (e) $x = \frac{t^3}{1+t^2}, \quad y = \frac{2t^2}{1+t^2}$ (f) $x = 2 \cos t, \quad y = \sin 2t$

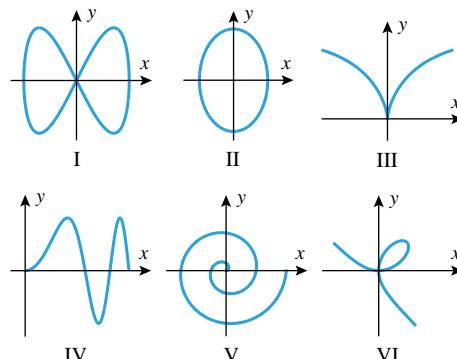


Figure Ex-19

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20. Use a graphing utility to generate the curves in Exercise 19, and in each case identify the orientation.
21. (a) Use a graphing utility to generate the trajectory of a particle whose equations of motion over the time interval $0 \leq t \leq 5$ are

$$x = 6t - \frac{1}{2}t^3, \quad y = 1 + \frac{1}{2}t^2$$
(b) Make a table of x - and y -coordinates of the particle at times $t = 0, 1, 2, 3, 4, 5$.
(c) At what times is the particle on the y -axis?
(d) During what time interval is $y < 5$?
(e) At what time is the x -coordinate of the particle maximum?
22. (a) Use a graphing utility to generate the trajectory of a paper airplane whose equations of motion for $t \geq 0$ are

$$x = t - 2 \sin t, \quad y = 3 - 2 \cos t$$
(b) Assuming that the plane flies in a room in which the floor is at $y = 0$, explain why the plane will not crash into the floor. [For simplicity, ignore the physical size of the plane by treating it as a particle.]
(c) How high must the ceiling be to ensure that the plane does not touch or crash into it?

In Exercises 23 and 24, graph the equation using a graphing utility.

23. (a) $x = y^2 + 2y + 1$
(b) $x = \sin y, -2\pi \leq y \leq 2\pi$
24. (a) $x = y + 2y^3 - y^5$
(b) $x = \tan y, -\pi/2 < y < \pi/2$
25. (a) By eliminating the parameter, show that the equations

$$x = x_0 + (x_1 - x_0)t, \quad y = y_0 + (y_1 - y_0)t$$
represent the line passing through the points (x_0, y_0) and (x_1, y_1) .
- (b) Show that if $0 \leq t \leq 1$, then the equations in part (a) represent the line segment joining (x_0, y_0) and (x_1, y_1) , oriented in the direction from (x_0, y_0) to (x_1, y_1) .
- (c) Use the result in part (b) to find parametric equations for the line segment joining the points $(1, -2)$ and $(2, 4)$, oriented in the direction from $(1, -2)$ to $(2, 4)$.
- (d) Use the result in part (b) to find parametric equations for the line segment in part (c), but oriented in the direction from $(2, 4)$ to $(1, -2)$.

26. Use the result in Exercise 25 to find
- (a) parametric equations for the line segment joining the points $(-3, -4)$ and $(-5, 1)$, oriented from $(-3, -4)$ to $(-5, 1)$
(b) parametric equations for the line segment traced from $(0, b)$ to $(a, 0)$, oriented from $(0, b)$ to $(a, 0)$.

27. (a) Suppose that the line segment from the point $P(x_0, y_0)$ to $Q(x_1, y_1)$ is represented parametrically by

$$\begin{aligned} x &= x_0 + (x_1 - x_0)t, \\ y &= y_0 + (y_1 - y_0)t \end{aligned} \quad (0 \leq t \leq 1)$$

and that $R(x, y)$ is the point on the line segment corresponding to a specified value of t (see the accompanying figure). Show that $t = r/q$, where r is the distance from P to R and q is the distance from P to Q .

- (b) What value of t produces the midpoint between points P and Q ?
(c) What value of t produces the point that is three-fourths of the way from P to Q ?

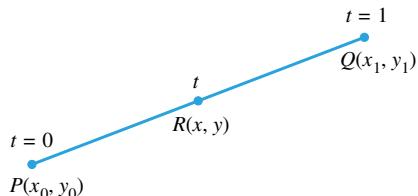


Figure Ex-27

28. Find parametric equations for the line segment joining $P(2, -1)$ and $Q(3, 1)$, and use the result in Exercise 27 to find
- (a) the midpoint between P and Q
(b) the point that is one-fourth of the way from P to Q
(c) the point that is three-fourths of the way from P to Q .

29. Explain why the parametric curve

$$x = t^2, \quad y = t^4 \quad (-1 \leq t \leq 1)$$

does not have a definite orientation.

30. (a) In parts (a) and (b) of Exercise 25 we obtained parametric equations for a line segment in which the parameter varied from $t = 0$ to $t = 1$. Sometimes it is desirable to have parametric equations for a line segment in which the parameter varies over some other interval, say $t_0 \leq t \leq t_1$. Use the ideas in Exercise 25 to show that the line segment joining the points (x_0, y_0) and (x_1, y_1) can be represented parametrically as

$$\begin{aligned} x &= x_0 + (x_1 - x_0) \frac{t - t_0}{t_1 - t_0}, \\ y &= y_0 + (y_1 - y_0) \frac{t - t_0}{t_1 - t_0} \end{aligned} \quad (t_0 \leq t \leq t_1)$$

- (b) Which way is the line segment oriented?
(c) Find parametric equations for the line segment traced from $(3, -1)$ to $(1, 4)$ as t varies from 1 to 2, and check your result with a graphing utility.

31. (a) By eliminating the parameter, show that if a and c are not both zero, then the graph of the parametric equations

$$x = at + b, \quad y = ct + d \quad (t_0 \leq t \leq t_1)$$

is a line segment.

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- (b) Sketch the parametric curve

$$x = 2t - 1, \quad y = t + 1 \quad (1 \leq t \leq 2)$$

and indicate its orientation.

32. (a) What can you say about the line in Exercise 31 if a or c (but not both) is zero?
 (b) What do the equations represent if a and c are both zero?
 33. Parametric curves can be defined piecewise by using different formulas for different values of the parameter. Sketch the curve that is represented piecewise by the parametric equations

$$x = 2t, \quad y = 4t^2 \quad (0 \leq t \leq \frac{1}{2})$$

$$x = 2 - 2t, \quad y = 2t \quad (\frac{1}{2} \leq t \leq 1)$$

34. Find parametric equations for the rectangle in the accompanying figure, assuming that the rectangle is traced counterclockwise as t varies from 0 to 1, starting at $(\frac{1}{2}, \frac{1}{2})$ when $t = 0$. [Hint: Represent the rectangle piecewise, letting t vary from 0 to $\frac{1}{4}$ for the first edge, from $\frac{1}{4}$ to $\frac{1}{2}$ for the second edge, and so forth.]

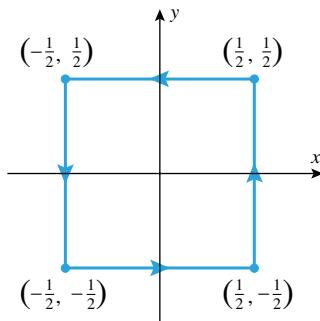


Figure Ex-34

35. (a) Find parametric equations for the ellipse that is centered at the origin and has intercepts $(4, 0)$, $(-4, 0)$, $(0, 3)$, and $(0, -3)$.
 (b) Find parametric equations for the ellipse that results by translating the ellipse in part (a) so that its center is at $(-1, 2)$.
 (c) Confirm your results in parts (a) and (b) using a graphing utility.

36. We will show later in the text that if a projectile is fired from ground level with an initial speed of v_0 meters per second at an angle α with the horizontal, and if air resistance is neglected, then its position after t seconds, relative to the coordinate system in the accompanying figure is

$$x = (v_0 \cos \alpha)t, \quad y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$$

where $g \approx 9.8 \text{ m/s}^2$.

- (a) By eliminating the parameter, show that the trajectory is a parabola.
 (b) Sketch the trajectory if $\alpha = 30^\circ$ and $v_0 = 1000 \text{ m/s}$.

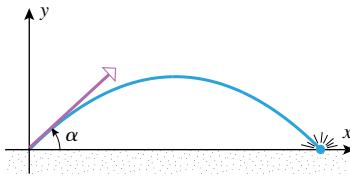


Figure Ex-36

37. A shell is fired from a cannon at an angle of $\alpha = 45^\circ$ with an initial speed of $v_0 = 800 \text{ m/s}$.
 (a) Find parametric equations for the shell's trajectory relative to the coordinate system in Figure Ex-36.
 (b) How high does the shell rise?
 (c) How far does the shell travel horizontally?
38. A robot arm, designed to buff flat surfaces on an automobile, consists of two attached rods, one that moves back and forth horizontally, and a second, with the buffering pad at the end, that moves up and down (see the accompanying figure).
 (a) Suppose that the horizontal arm of the robot moves so that the x -coordinate of the buffer's center at time t is $x = 25 \sin \pi t$ and the vertical arm moves so that the y -coordinate of the buffer's center at time t is $y = 12.5 \sin \pi t$. Graph the trajectory of the center of the buffering pad.
 (b) Suppose that the x - and y -coordinates in part (a) are $x = 25 \sin \pi at$ and $y = 12.5 \sin \pi bt$, where the constants a and b can be controlled by programming the robot arm. Graph the trajectory of the center of the pad if $a = 4$ and $b = 5$.
 (c) Investigate the trajectories that result in part (b) for various choices of a and b .

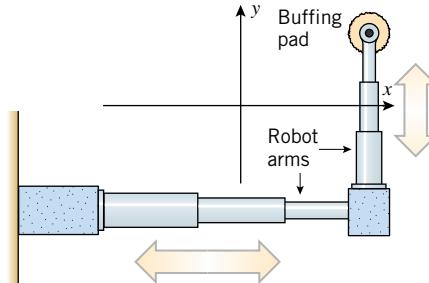


Figure Ex-38

39. Describe the family of curves described by the parametric equations
- $$x = a \cos t + h, \quad y = b \sin t + k \quad (0 \leq t \leq 2\pi)$$
- if
- (a) h and k are fixed but a and b can vary
 (b) a and b are fixed but h and k can vary
 (c) $a = 1$ and $b = 1$, but h and k vary so that $h = k + 1$.
40. A **hypocycloid** is a curve traced by a point P on the circumference of a circle that rolls inside a larger fixed circle. Suppose that the fixed circle has radius a , the rolling circle has radius b , and the fixed circle is centered at the origin. Let ϕ be the angle shown in the following figure, and assume that the point P is at $(a, 0)$ when $\phi = 0$. Show that the

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hypocycloid generated is given by the parametric equations

$$x = (a - b) \cos \phi + b \cos \left(\frac{a - b}{b} \phi \right)$$

$$y = (a - b) \sin \phi - b \sin \left(\frac{a - b}{b} \phi \right)$$

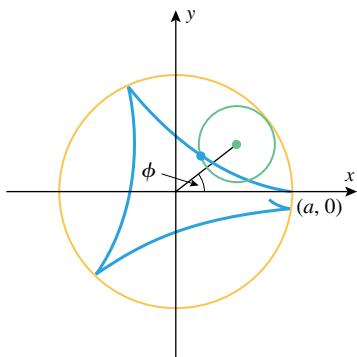


Figure Ex-40

41. If $b = \frac{1}{4}a$ in Exercise 40, then the resulting curve is called a four-cusped hypocycloid.

(a) Sketch this curve.

(b) Show that the curve is given by the parametric equations $x = a \cos^3 \phi$, $y = a \sin^3 \phi$.

(c) Show that the curve is given by the equation

$$x^{2/3} + y^{2/3} = a^{2/3}$$

in rectangular coordinates.

42. (a) Use a graphing utility to study how the curves in the family

$$x = 2a \cos^2 t, \quad y = 2a \cos t \sin t \quad (-2\pi < t < 2\pi)$$

change as a varies from 0 to 5.

(b) Confirm your conclusion algebraically.

(c) Write a brief paragraph that describes your findings.

SUPPLEMENTARY EXERCISES

1. Referring to the cigarette consumption graph in Figure 1.1.2b, during what 5-year period was the annual cigarette consumption per adult increasing most rapidly on average? Explain your reasoning.

2. Use the graphs of the functions f and g in the accompanying figure to solve the following problems.

- (a) Find the values of $f(-1)$ and $g(3)$.
- (b) For what values of x is $f(x) = g(x)$?
- (c) For what values of x is $f(x) < 2$?
- (d) What are the domain and range of f ?
- (e) What are the domain and range of g ?
- (f) Find the zeros of f and g .

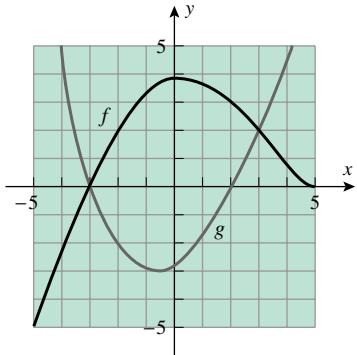


Figure Ex-2

3. A glass filled with water that has a temperature of 40°F is placed in a room in which the temperature is a constant 70°F . Sketch a rough graph that reasonably describes the temperature of the water in the glass as a function of the elapsed time.

4. A student begins driving toward school but 5 minutes into the trip remembers that he forgot his homework. He drives home hurriedly, retrieves his notes, and then drives at great speed toward school, hitting a tree 5 minutes after leaving home. Sketch a rough graph that reasonably describes the student's distance from home as a function of the elapsed time.

5. A rectangular storage container with an open top and a square base has a volume of 8 cubic meters. Material for the base costs \$5 per square meter, and material for the sides \$2 per square meter. Express the total cost of the materials as a function of the length of a side of the base.

6. You want to paint the top of a circular table. Find a formula that expresses the amount of paint required as a function of the radius, and discuss all of the assumptions you have made in finding the formula.

7. Sketch the graph of the function

$$f(x) = \begin{cases} -1, & x \leq -5 \\ \sqrt{25 - x^2}, & -5 < x < 5 \\ x - 5, & x \geq 5 \end{cases}$$

8. A ball of radius 3 inches is coated uniformly with plastic. Express the volume of the plastic as a function of its thickness.

9. A box with a closed top is to be made from a 6-ft by 10-ft piece of cardboard by cutting out four squares of equal size (see the accompanying figure), folding along the dashed lines, and tucking the two extra flaps inside.

- (a) Find a formula that expresses the volume of the box as a function of the length of the sides of the cut-out squares.

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- (b) Find an inequality that specifies the domain of the function in part (a).
 (c) Estimate the dimensions of the box of largest volume.

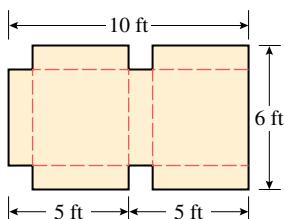


Figure Ex-9

10. Let $f(x) = -x^2$ and $g(x) = 1/\sqrt{x}$. Find the natural domains of $f \circ g$ and $g \circ f$.
11. Given that $f(x) = x^2 + 1$ and $g(x) = 3x + 2$, find all values of x such that $f(g(x)) = g(f(x))$.
12. Let $f(x) = (2x - 1)/(x + 1)$ and $g(x) = 1/(x - 1)$.
- Find $f(g(x))$.
 - Is the natural domain of the function $f(g(x))$ obtained in part (a) the same as the domain of $f \circ g$? Explain.
13. Find $f(g(h(x)))$, given that
- $$f(x) = \frac{x}{x - 1}, \quad g(x) = \frac{1}{x}, \quad h(x) = x^2 - 1$$
14. Given that $f(x) = 2x + 1$ and $h(x) = 2x^2 + 4x + 1$, find a function g such that $f(g(x)) = h(x)$.
15. Complete the following table.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	0	-1	2	1	3	-2	-3	4	-4
$g(x)$	3	2	1	-3	-1	-4	4	-2	0
$(f \circ g)(x)$									
$(g \circ f)(x)$									

16. (a) Write an equation for the graph that is obtained by reflecting the graph of $y = |x - 1|$ about the y -axis, then stretching that graph vertically by a factor of 2, then translating that graph down 3 units, and then reflecting that graph about the x -axis.
 (b) Sketch the original graph and the final graph.
17. In each part, classify the function as even, odd, or neither.
- $x^2 \sin x$
 - $\sin^2 x$
 - $x + x^2$
 - $\sin x \tan x$
18. (a) Find exact values for all x -intercepts of

$$y = \cos x - \sin 2x$$

in the interval $-2\pi \leq x \leq 2\pi$.

- (b) Find the coordinates of all intersections of the graphs of $y = \cos x$ and $y = \sin 2x$ if $-2\pi \leq x \leq 2\pi$, and use a graphing utility to check your answer.

19. (a) A surveyor measures the angle of elevation α of a tower from a point A due south of the tower and also measures the angle of elevation β from a point B that is d feet due east of the point A (see the accompanying figure). Show that the height h of the tower in feet is given by

$$h = \frac{d \tan \alpha \tan \beta}{\sqrt{\tan^2 \alpha - \tan^2 \beta}}$$

- (b) Use a calculating utility to approximate the height of the tower to the nearest tenth of a foot if $\alpha = 17^\circ$, $\beta = 12^\circ$, and $d = 1000$ ft.

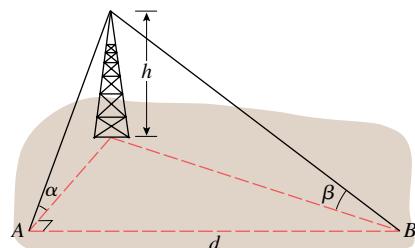


Figure Ex-19

20. Suppose that the expected low temperature in Anchorage, Alaska (in $^\circ\text{F}$), is modeled by the equation

$$T = 50 \sin \frac{2\pi}{365}(t - 101) + 25$$

where t is in days and $t = 0$ corresponds to January 1.

- Sketch the graph of T versus t for $0 \leq t \leq 365$.
- Use the model to predict when the coldest day of the year will occur.
- Based on this model, how many days during the year would you expect the temperature to be below 0°F ?

21. The accompanying figure shows the graph of the equations $y = 1 + 2 \sin x$ and $y = 2 \sin(x/2) + 2 \cos(x/2)$ for $-2\pi \leq x \leq 2\pi$. Without the aid of a calculator, label each curve by its equation, and find the coordinates of the points A , B , C , and D . Explain your reasoning.

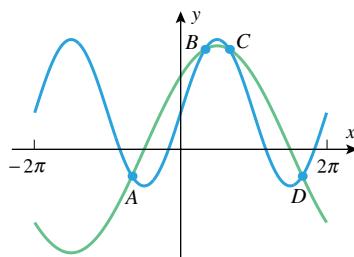


Figure Ex-21

22. The accompanying figure shows a model for the tide variation in an inlet to San Francisco Bay during a 24-hour period. Find an equation of the form $y = y_0 + y_1 \sin(at + b)$ for the model, assuming that $t = 0$ corresponds to midnight.

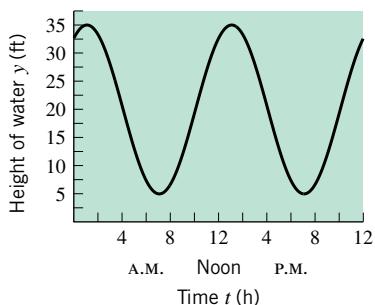
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Figure Ex-22

- 23.** In each part describe the family of curves.

(a) $(x - a)^2 + (y - a^2)^2 = 1$
 (b) $y = a + (x - 2a)^2$

- 24.** (a) Suppose that the equations $x = f(t)$, $y = g(t)$ describe a curve C as t increases from 0 to 1. Find parametric equations that describe the same curve C but traced in the opposite direction as t increases from 0 to 1.
 (b) Check your work using the parametric graphing feature of a graphing utility by generating the line segment between $(1, 2)$ and $(4, 0)$ in both possible directions as t increases from 0 to 1.

- 25.** Sketch the graph of the equation $x^2 - 4y^2 = 0$.
26. Find an equation for a parabola that passes through the points $(2, 0)$, $(8, 18)$, and $(-8, 18)$.

- 27.** Sketch the curve described by the parametric equations
 $x = t \cos(2\pi t)$, $y = t \sin(2\pi t)$

and check your result with a graphing utility.

- 28.** The electrical resistance R in ohms (Ω) for a pure metal wire is related to its temperature T in $^{\circ}\text{C}$ by the formula
 $R = R_0(1 + kT)$

in which R_0 and k are positive constants.

- (a) Make a hand-drawn sketch of the graph of R versus T , and explain the geometric significance of R_0 and k for your graph.
 (b) In theory, the resistance R of a pure metal wire drops to zero when the temperature reaches absolute zero ($T = -273^{\circ}\text{C}$). What information does this give you about k ?
 (c) A tungsten bulb filament has a resistance of 1.1Ω at a temperature of 20°C . What information does this give you about R_0 for the filament?
 (d) At what temperature will a tungsten filament have a resistance of 1.5Ω ?

Most of the following exercises require access to graphing and calculating utilities. When you are asked to *find* an answer or to *solve* an equation, you may choose to find either an exact result or a numerical approximation, depending on the particular technology you are using and your own imagination.

- 29.** Find the distance between the point $P(1, 2)$ and an arbitrary point (x, \sqrt{x}) on the curve $y = \sqrt{x}$. Graph this distance versus x , and use the graph to find the x -coordinate of the point on the curve that is closest to the point P .
- 30.** Find the distance between the point $P(1, 0)$ and an arbitrary point $(x, 1/x)$ on the curve $y = 1/x$, where $x > 0$. Graph this distance versus x , and use the graph to find the x -coordinate of the point on the curve that is closest to the point P .

In Exercises 31 and 32, use **Archimedes' principle**: *A body wholly or partially immersed in a fluid is buoyed up by a force equal to the weight of the fluid that it displaces.*

- 31.** A hollow metal sphere of diameter 5 feet weighs 108 pounds and floats partially submerged in seawater. Assuming that seawater weighs 63.9 pounds per cubic foot, how far below the surface is the bottom of the sphere? [Hint: If a sphere of radius r is submerged to a depth h , then the volume V of the submerged portion is given by the formula $V = \pi h^2(r - h/3)$.]
32. Suppose that a hollow metal sphere of diameter 5 feet and weight w pounds floats in seawater. (See Exercise 31.)
 (a) Graph w versus h for $0 \leq h \leq 5$.
 (b) Find the weight of the sphere if exactly half of the sphere is submerged.
33. A breeding group of 20 bighorn sheep is released in a protected area in Colorado. It is expected that with careful management the number of sheep, N , after t years will be given by the formula

$$N = \frac{220}{1 + 10(0.83)^t}$$
 and that the sheep population will be able to maintain itself without further supervision once the population reaches a size of 80.
 (a) Graph N versus t .
 (b) How many years must the state of Colorado maintain a program to care for the sheep?
 (c) How many bighorn sheep can the environment in the protected area support? [Hint: Examine the graph of N versus t for large values of t .]

In Exercises 34 and 35, use the following empirical formula for the windchill index (WCI) [see Example 3 of Section 1.2]:

$$\text{WCI} = \begin{cases} T, & 0 \leq v \leq 4 \\ 91.4 + (91.4 - T)(0.0203v - 0.304\sqrt{v} - 0.474), & 4 < v < 45 \\ 1.6T - 55, & v \geq 45 \end{cases}$$

where T is the air temperature in $^{\circ}\text{F}$, v is the wind speed in mi/h, and WCI is the equivalent temperature in $^{\circ}\text{F}$.

34. (a) Graph T versus v over the interval $4 \leq v \leq 45$ for $\text{WCI} = 0$.

(b) Use your graph to estimate the values of T for $\text{WCI} = 0$ corresponding to $v = 10, 20, 30$, to the nearest degree.

35. (a) Graph WCI versus v over the interval $0 \leq v \leq 50$ for $T = 20$.

(b) Use your graph to estimate the values of the WCI corresponding to $v = 10, 20, 30, 40$, to the nearest degree.

(c) Use your graph to estimate the values of v corresponding to $\text{WCI} = -20, -10, 0, 10$, to the nearest mile per hour.

36. Find the domain and range of the function

$$f(x) = \frac{\sin x}{x^4 + x^3 + 5}$$

37. Find the domain and range of the function

$$f(x) = x^2 - \sqrt{1 + x - x^4}$$

38. An oven is preheated and then remains at a constant temperature. A potato is placed in the oven to bake. Suppose that the temperature T (in $^{\circ}\text{F}$) of the potato t minutes later is given by $T = 400 - 325(0.97^t)$. The potato will be considered done when its temperature is anywhere between 260°F and 280°F .

- (a) During what interval of time would the potato be considered done?
 (b) How long does it take for the temperature of the potato to reach 95% of the oven temperature?

39. Suppose that a package of medical supplies is dropped from a helicopter straight down by parachute into a remote area. The velocity v (in feet per second) of the package t seconds after it is released is given by $v = 24.61(1 - 0.273^t)$.

- (a) Graph v versus t .
 (b) Show that the graph has a horizontal asymptote $v = c$.
 (c) The constant c is called the **terminal velocity**. Explain what the terminal velocity means in practical terms.
 (d) Can the package actually reach its terminal velocity? Explain.
 (e) How long does it take for the package to reach 98% of its terminal velocity?

40. An ancient Babylonian tablet known as Plimpton 322 contains a sequence of numbers that appear to be the squares of secants of various angles ranging from about 45° to 31° . The secants of these angles are listed in the accompanying table.

- (a) Using linear regression, find a function that (approximately) expresses these secants in terms of their position number within the table.
 (b) Do you see any connection between your linear function from part (a) and the fact that the base for the Babylonian number system was 60?

ENTRY NUMBER	FUNCTION VALUE
1	1.4083
2	1.3961
3	1.3852
4	1.3734
5	1.3472
6	1.3361
7	1.3110
8	1.3010
9	1.2817
10	1.2594
11	1.2500
12	1.2204
13	1.2042
14	1.1959
15	1.1778

Table Ex-40

41. An important problem addressed by calculus is that of finding a good linear approximation to the function $f(x)$ near a particular x -value. One possible approach (not the best) is to sample values of the function near the specified x -value, find the least squares line for this sample, and translate the least squares line so that it passes through the point on the graph of $y = f(x)$ corresponding to the given x -value. Let $f(x) = x^2 \sin x$.

- (a) Make a table of $(x, f(x))$ values for $x = 1.9, 1.92, 1.94, \dots, 2.1$.
 (b) Find a least squares line for the data in part (a).
 (c) Find the equation of the line passing through the point $(2, f(2))$ and parallel to the least squares line.
 (d) Using a graphing utility with a graphing window containing $(2, f(2))$, graph $y = f(x)$ and the line you found in part (c). How do the graphs compare as you zoom closer to the point $(2, f(2))$?

[Note: The best linear approximation to $y = x^2 \sin x$ near $x = 2$ is given by $y \approx 1.9726x - 0.308015$. In Chapter 3, we will see how to use the tools of calculus to find this answer.]

42. An extension of the linear approximation problem is finding a good polynomial approximation to the function $f(x)$ near a particular x -value. One possible approach (not the best) is to sample values of the function near the specified x -value, apply polynomial regression to this sample, and translate the regression curve so that it passes through the point on the graph of $y = f(x)$ corresponding to the given x -value. Let $f(x) = \cos x$.

- (a) Make a table of $(x, f(x))$ values for $x = -0.1, -0.08, -0.06, \dots, 0.1$.
 (b) Use quadratic regression to model the data in part (a) with a quadratic polynomial.
 (c) Translate your quadratic modeling function from part (b) to obtain a quadratic function that passes through the point $(0, f(0))$.

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- (d) Using a graphing utility with a graphing window containing $(0, f(0))$, graph $y = f(x)$ and the polynomial you found in part (c). How do the graphs compare as you zoom closer to the point $(0, f(0))$?

[Note: The best quadratic approximation to $y = \cos x$ near $x = 0$ is given by $y \approx -0.5x^2 + 1$.]

-  **43.** The accompanying table gives the water level (in meters above the mean low-water mark) at a Cape Hatteras, North Carolina, fishing pier, recorded in 2-hour increments starting from midnight, July 1, 1999. Why should we expect that a trigonometric function should fit these data? Find a function that models the data, and graph your function on a scatter plot.

HOUR	WATER LEVEL (m)	HOUR	WATER LEVEL (m)
0	0.526	36	0.534
2	0.157	38	0.192
4	0.161	40	0.141
6	0.486	42	0.426
8	0.779	44	0.849
10	0.740	46	1.032
12	0.412	48	0.765
14	0.141	50	0.281
16	0.260	52	0.042
18	0.633	54	0.157
20	1.015	56	0.587
22	1.021	58	0.777
24	0.670	60	0.620
26	0.231	62	0.241
28	0.128	64	0.045
30	0.345	66	0.195
32	0.697	68	0.613
34	0.821	70	0.945

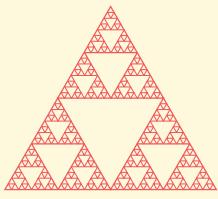
Table Ex-43

EXPANDING THE CALCULUS HORIZON**Iteration and Dynamical Systems**

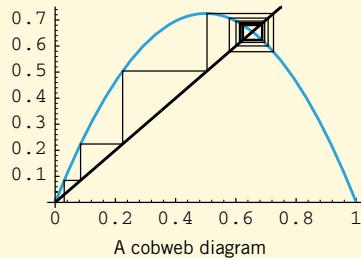
What do the four figures below have in common? The answer is that all of them are of interest in contemporary research and all involve a mathematical process called **iteration**. In this module we will introduce this concept and touch on some of the fascinating ideas to which it leads.



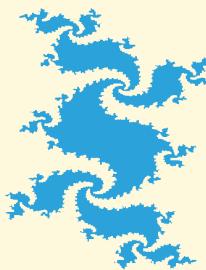
Barnsley's fern



The Sierpinski triangle



A cobweb diagram



A Julia set

Iterative Processes

Recall that in the notation $y = f(x)$, the variable x is called an **input** of the function f , and the variable y is called the corresponding **output**. Suppose that we start with some input, say $x = c$, and each time we compute an output we feed it back into f as an input. This generates the following sequence of numbers:

$$f(c), \quad f(f(c)), \quad f(f(f(c))), \quad f(f(f(f(c)))), \dots$$

This is called an **iterated function sequence** for f (from the Latin word *iteratus*, meaning “to repeat”). The number c is called the **seed value** for the sequence, the terms in the sequence are called **iterates**, and each time f is applied we say that we have performed an **iteration**. Iterated function sequences arise in a wide variety of physical processes that are collectively called **dynamical systems**.

Exercise 1 Let $f(x) = x^2$.

- Calculate the first 10 iterates in the iterated function sequence for f , starting with seed values of $c = 0.5, 1$, and 2 . In each case make a conjecture about the **long-term behavior** of the iterates, that is, the behavior of the iterates as more and more iterations are performed.
- Try your own seed values, and make a conjecture about the effect of a seed value on the long-term behavior of the iterates.

Recursion Formulas

The proliferation of parentheses in an iterated function sequence can become confusing, so for simplicity let us introduce the following notation for the successive iterates

$$y_0 = c, \quad y_1 = f(c), \quad y_2 = f(f(c)), \quad y_3 = f(f(f(c))), \quad y_4 = f(f(f(f(c)))), \dots$$

or expressed more simply,

$$y_0 = c, \quad y_1 = f(y_0), \quad y_2 = f(y_1), \quad y_3 = f(y_2), \quad y_4 = f(y_3), \dots$$

Thus, successive terms in the sequence are related by the formulas

$$y_0 = c, \quad y_{n+1} = f(y_n) \quad (n = 0, 1, 2, 3, \dots)$$

These two formulas, taken together, comprise what is called a *recursion formula* for the iterated function sequence. In general, a **recursion formula** is any formula or set of formulas that provides a method for generating the terms of a sequence from the preceding terms and a seed value. For example, the recursion formula for the iterated function sequence of $f(x) = x^2$ with seed value c is

$$y_0 = c, \quad y_{n+1} = y_n^2$$

As another example, the recursion formula

$$y_0 = 1, \quad y_{n+1} = \frac{1}{2} \left(y_n + \frac{p}{y_n} \right) \quad (1)$$

produces an iterated function sequence whose iterates can be used to approximate \sqrt{p} to any degree of accuracy.

Exercise 2 Use (1) to approximate $\sqrt{5}$ by generating successive iterates on a calculator until you encounter two successive iterates that are the same. Compare this approximation of $\sqrt{5}$ to that produced directly by your calculator.

Exercise 3

- Find iterates y_1 up to y_6 of the sequence that is generated by the recursion formula

$$y_0 = 1, \quad y_{n+1} = \frac{1}{2} y_n$$

- By examining the terms generated in part (a), find a formula that expresses y_n as a function of n .

Exercise 4 Suppose that you deposit \$1000 in a bank at 5% interest per year and allow it to accumulate value without making withdrawals.

- If y_n denotes the value of the account at the end of the n th year, how could you find the value of y_{n+1} if you knew the value of y_n ?
- Starting with $y_0 = 1000$ (dollars), use the result in part (a) to calculate y_1, y_2, y_3, y_4 , and y_5 .
- Find a recursion formula for the sequence of yearly account values assuming that $y_0 = 1000$.
- Find a formula that expresses y_n as a function of n , and use that formula to calculate the value of the account at the end of the 15th year.

104 Functions**Exploring Iterated Function Sequences**

Iterated function sequences for a function f can be explored in various ways. Here are three possibilities:

- Choose a specific seed value, and investigate the long-term behavior of the iterates (as in Exercise 1).
- Let the seed value be a variable x (in which case the iterates become functions of x), and investigate what happens to the graphs of the iterates as more and more iterations are performed.
- Choose a specific iterate, say the 10th, and investigate how the value of this iterate varies with different seed values.

Exercise 5 Let $f(x) = \sqrt{x}$.

- (a) Find formulas for the first five iterates in the iterated function sequence for f , taking the seed value to be x .
- (b) Graph the iterates in part (a) in the same coordinate system, and make a conjecture about the behavior of the graphs as more and more iterations are performed.

Continued Fractions and Fibonacci Sequences

If $f(x) = 1/x$, and the seed value is x , then the iterated function sequence for f flip-flops between x and $1/x$:

$$y_1 = \frac{1}{x}, \quad y_2 = \frac{1}{1/x} = x, \quad y_3 = \frac{1}{x}, \quad y_4 = \frac{1}{1/x} = x, \dots$$

However, if $f(x) = 1/(x+1)$, then the iterated function sequence becomes a sequence of fractions that, if continued indefinitely, is an example of a *continued fraction*:

$$\frac{1}{1+x}, \quad \frac{1}{1 + \frac{1}{1+x}}, \quad \frac{1}{1 + \frac{1}{1 + \frac{1}{1+x}}}, \quad \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1+x}}}}, \dots$$

Exercise 6 Let $f(x) = 1/(x+1)$ and $c = 1$.

- (a) Find *exact values* for the first 10 terms in the iterated function sequence for f ; that is, express each term as a fraction p/q with no common factors in the numerator and denominator.
- (b) Write down the numerators from part (a) in sequence, and see if you can discover how each term after the first two is related to its predecessors. The sequence of numerators is called a **Fibonacci sequence** [in honor of its medieval discoverer Leonardo ("Fibonacci") da Pisa]. Do some research on Fibonacci and his sequence, and write a paper on the subject.
- (c) Use the pattern you discovered in part (b) to write down the exact values of the second 10 terms in the iterated function sequence.
- (d) Find a recursion formula that will generate all the terms in the Fibonacci sequence after the first two.
- (e) It can be proved that the terms in the iterated function sequence for f get closer and closer to one of the two solutions of the equation $q = 1/(1+q)$. Which solution is it? This solution is a number known as the **golden ratio**. Do some research on the golden ratio, and write a paper on the subject.

Applications to Ecology

There are numerous models for predicting the growth and decline of populations (flowers, plants, people, animals, etc.). One way to model populations is to give a recursion formula that describes how the number of individuals in each generation relates to the number of individuals in the

preceding generation. One of the simplest such models, called the *exponential model*, assumes that the number of individuals in each generation is a fixed percentage of the number of individuals in the preceding generation. Thus, if there are c individuals initially and if the number of individuals in any generation is r times the number of individuals in the preceding generation, then the growth through successive generations is given by the recursion formula

$$y_0 = c, \quad y_{n+1} = ry_n \quad (n = 0, 1, 2, 3, \dots)$$

Exercise 7 Suppose that a population with an exponential growth model has c individuals initially.

- Express the iterates y_1 , y_2 , y_3 , and y_4 in terms of c and r .
- Find a formula for y_{n+1} in terms of c and r .
- Describe the eventual fate of the population if $r = 1$, $r < 1$, and $r > 1$.

There is a more sophisticated model of population growth, called the *logistic model*, that takes environmental constraints into account. In this model, it is assumed that there is some maximum population that can be supported by the environment, and the population is expressed as a fraction of the maximum. Thus, in each generation the population is represented as a number in the interval $0 \leq y_n \leq 1$. When y_n is near 0 the population has lots of room to grow, but when y_n is near 1 the population is close to the maximum and the environmental factors tend to inhibit further growth. Models of this type are given by recursion formulas of the form

$$y_0 = c, \quad y_{n+1} = ky_n(1 - y_n) \quad (2)$$

in which k is a positive constant that depends on the ecological conditions.

Figure 1 illustrates a graphical method for tracking the growth of a population described by (2). That figure, which is called a *cobweb diagram*, shows graphs of the line $y = x$ and the curve $y = kx(1 - x)$.

Exercise 8 Explain why the values y_1 , y_2 , and y_3 are the populations for the first three generations of the logistic growth model given by (2).

Exercise 9 The cobweb diagram in Figure 2 tracks the growth of a population with a logistical growth model given by the recursion formula

$$y_0 = 0.1, \quad y_{n+1} = 2.9y_n(1 - y_n)$$

- Find the populations y_1, y_2, \dots, y_5 of the first five generations.
- What happens to the population over the long term?

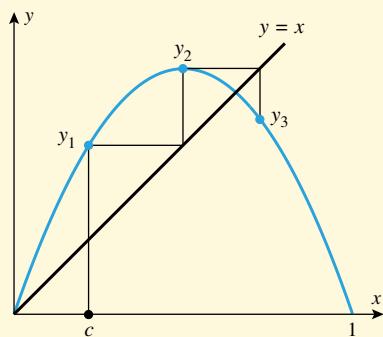


Figure 1

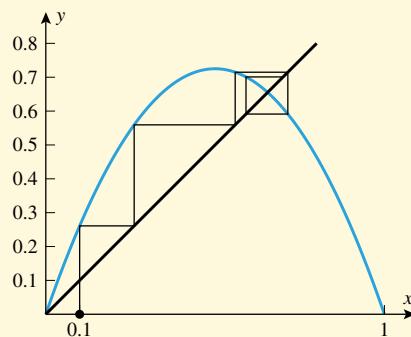


Figure 2

106 Functions**Chaos and Fractals**

Observe that (2) is a recursion formula for the iterated function sequence of $f(x) = kx(1 - x)$. Iterated function sequences of this form are called *iterated quadratic systems*. These are important not only in modeling populations but also in the study of *chaos* and *fractals*—two important fields of contemporary research.

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Howard Anton, Drexel University*