16

Carl Friedrich Gauss

TOPICS IN VECTOR CALCULUS

You have reached the final chapter in this text, and in a sense you have come full circle back to the roots of calculus. The main theme of this chapter is the concept of a *flow*, and the body of mathematics that we will study here is concerned with analyzing flows of various types—the flow of a fluid or the flow of electricity, for example. Indeed, the early writings of Isaac Newton on calculus are replete with such nouns as “fluxion” and “fluent,” which are rooted in the Latin *fluens* (to flow). We will begin this chapter by introducing the concept of a *vector field*, which is the mathematical description of a flow. In subsequent sections, we will introduce two new kinds of integrals that are used in a variety of applications to analyze properties of vector fields and flows. Finally, we conclude with three major theorems, *Green’s Theorem*, the *Divergence Theorem*, and *Stokes’ Theorem*. These theorems provide a deep insight into the nature of flows and are the basis for many of the most important principles in physics and engineering.

16.1 VECTOR FIELDS

In this section we will consider functions that associate vectors with points in 2-space or 3-space. We will see that such functions play an important role in the study of fluid flow, gravitational force fields, electromagnetic force fields, and a wide range of other applied problems.

VECTOR FIELDS

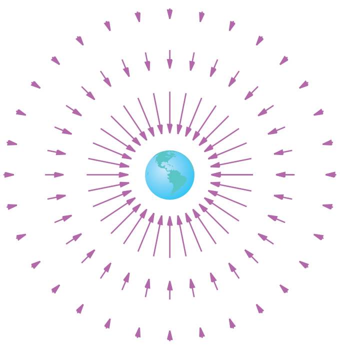


Figure 16.1.1

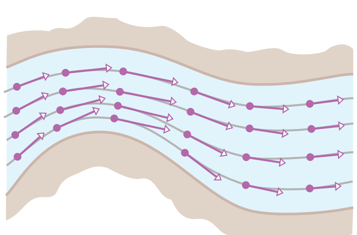


Figure 16.1.2

To motivate the mathematical ideas in this section, consider a *unit* point mass located at any point in the Universe. According to Newton's Law of Universal Gravitation, the Earth exerts an attractive force on the mass that is directed toward the center of the Earth and has a magnitude that is inversely proportional to the square of the distance from the mass to the Earth's center (Figure 16.1.1). This association of force vectors with points in space is called the Earth's *gravitational field*. A similar idea arises in fluid flow. Imagine a stream in which the water flows horizontally at every level, and consider the layer of water at a specific depth. At each point of the layer, the water has a certain velocity, which we can represent by a vector at that point (Figure 16.1.2). This association of velocity vectors with points in the two-dimensional layer is called the *velocity field* at that layer. These ideas are captured in the following definition.

16.1.1 DEFINITION. A ***vector field*** is a function that associates a unique vector $\mathbf{F}(P)$ with each point P in a region of 2-space or 3-space.

Observe that in this definition there is no reference to a coordinate system. However, for computational purposes it is usually desirable to introduce a coordinate system so that vectors can be assigned components. Specifically, if $\mathbf{F}(P)$ is a vector field in an xy-coordinate system, then the point P will have some coordinates (x, y) and the associated vector will have components that are functions of x and y. Thus, the vector field $\mathbf{F}(P)$ can be expressed as

$$\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$$

Similarly, in 3-space with an xyz-coordinate system, a vector field $\mathbf{F}(P)$ can be expressed as

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

GRAPHICAL REPRESENTATIONS OF VECTOR FIELDS

A vector field in 2-space can be pictured geometrically by drawing representative field vectors $\mathbf{F}(x, y)$ at some well-chosen points in the xy-plane. But, just as it is usually not possible to describe a plane curve completely by plotting finitely many points, so it is usually not possible to describe a vector field completely by drawing finitely many vectors. Nevertheless, such graphical representations can provide useful information about the general behavior of the field if the vectors are chosen appropriately. However, graphical representations of vector fields require a substantial amount of computation, so they are usually created using computers. Figure 16.1.3 shows four computer-generated vector fields. The vector field in part (a) might describe the velocity of the current in a stream at various depths. At the bottom of the stream the velocity is zero, but the speed of the current increases as the depth decreases. Points at the same depth have the same speed. The vector field in part (b) might describe the velocity at points on a rotating wheel. At the center of the wheel the velocity is zero, but the speed increases with the distance from the center. Points at the same distance from the center have the same speed. The vector field in part (c) might describe the repulsive force of an electrical charge—the closer to the charge, the greater the force of repulsion. Part (d) shows a vector field in 3-space. Such pictures tend to be cluttered and hence are of lesser value than graphical representations of vector fields in 2-space. Note also that the vectors in parts (b) and (c) are not to scale—their lengths have been compressed for clarity. We will follow this procedure throughout this chapter.

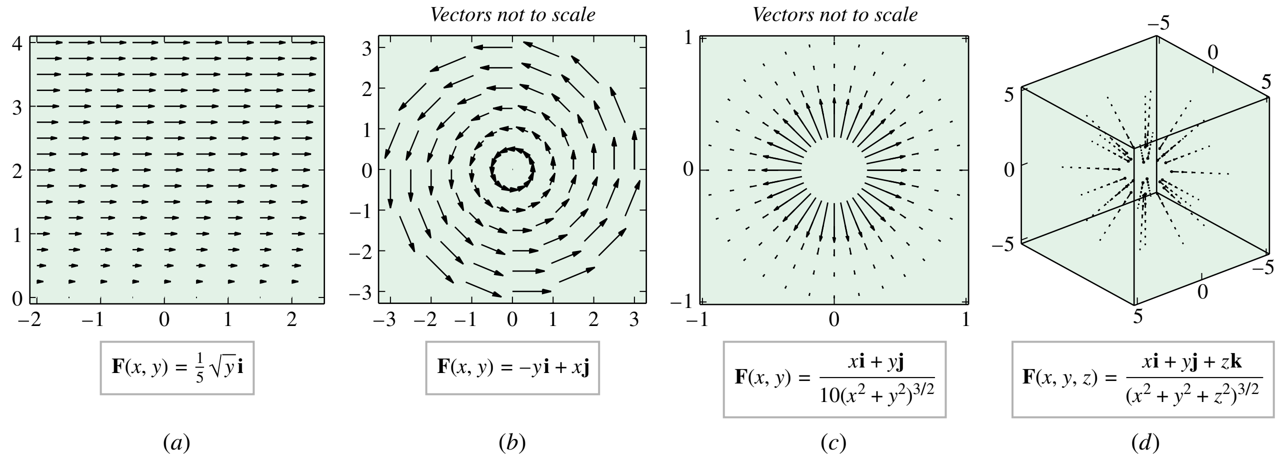


Figure 16.1.3

FOR THE READER. If you have a graphing utility that can generate vector fields, read the relevant documentation and try to make reasonable duplicates of parts (*a*) and (*b*) of Figure 16.1.3.

A COMPACT NOTATION FOR VECTOR FIELDS

Sometimes it is helpful to denote the vector fields $\mathbf{F}(x, y)$ and $\mathbf{F}(x, y, z)$ entirely in vector notation by identifying (x, y) with the radius vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ and (x, y, z) with the radius vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. With this notation a vector field in either 2-space or 3-space can be written as $\mathbf{F}(\mathbf{r})$. When no confusion is likely to arise, we will sometimes omit the $\mathbf{r}$ altogether and denote the vector field as $\mathbf{F}$.

INVERSE-SQUARE FIELDS

According to Newton's Law of Universal Gravitation, objects with masses m and M attract each other with a force $\mathbf{F}$ of magnitude

$$\|\mathbf{F}\| = \frac{GmM}{r^2} \tag{1}$$

where r is the distance between the objects (treated as point masses) and G is a constant. If we assume that the object of mass M is located at the origin of an xyz-coordinate system and $\mathbf{r}$ is the radius vector to the object of mass m, then $r = \|\mathbf{r}\|$, and the force $\mathbf{F}(\mathbf{r})$ exerted by the object of mass M on the object of mass m is in the direction of the unit vector $-\mathbf{r}/\|\mathbf{r}\|$. Thus, from (1)

$$\mathbf{F}(\mathbf{r}) = -\frac{GmM}{\|\mathbf{r}\|^2}\frac{\mathbf{r}}{\|\mathbf{r}\|} = -\frac{GmM}{\|\mathbf{r}\|^3}\mathbf{r} \tag{2}$$

If m and M are constant, and we let $c = -GmM$, then this formula can be expressed as

$$\mathbf{F}(\mathbf{r}) = \frac{c}{\|\mathbf{r}\|^3}\mathbf{r}$$

Vector fields of this form arise in electromagnetic as well as gravitational problems. Such fields are so important that they have their own terminology.

16.1.2 DEFINITION. If $\mathbf{r}$ is a radius vector in 2-space or 3-space, and if c is a constant, then a vector field of the form

$$\mathbf{F}(\mathbf{r}) = \frac{c}{\|\mathbf{r}\|^3}\mathbf{r} \tag{3}$$

is called an ***inverse-square field***.

Observe that if $c > 0$ in (3), then $\mathbf{F}(\mathbf{r})$ has the same direction as $\mathbf{r}$, so each vector in the field is directed away from the origin; and if $c < 0$, then $\mathbf{F}(\mathbf{r})$ is oppositely directed to $\mathbf{r}$, so each vector in the field is directed toward the origin. In either case the magnitude of $\mathbf{F}(\mathbf{r})$ is inversely proportional to the square of the distance from the terminal point of $\mathbf{r}$ to the origin, since

$$\|\mathbf{F}(\mathbf{r})\| = \frac{|c|}{\|\mathbf{r}\|^3}\|\mathbf{r}\| = \frac{|c|}{\|\mathbf{r}\|^2}$$

We leave it for you to verify that in 2-space Formula (3) can be written in component form as

$$\mathbf{F}(x, y) = \frac{c}{(x^2 + y^2)^{3/2}}(x\mathbf{i} + y\mathbf{j}) \tag{4}$$

and in 3-space as

$$\mathbf{F}(x, y, z) = \frac{c}{(x^2 + y^2 + z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \tag{5}$$

[see parts (*c*) and (*d*) of Figure 16.1.3].

Example 1 ***Coulomb's law*** states that *the electrostatic force exerted by one charged particle on another is directly proportional to the product of the charges and inversely proportional to the square of the distance between them*. This has the same form as Newton's Law of Universal Gravitation, so the electrostatic force field exerted by a charged particle is an inverse-square field. Specifically, if a particle of charge Q is at the origin of a coordinate system, and if $\mathbf{r}$ is the radius vector to a particle of charge q, then the force $\mathbf{F}(\mathbf{r})$ that the particle of charge Q exerts on the particle of charge q is of the form

$$\mathbf{F}(\mathbf{r}) = \frac{qQ}{4\pi\epsilon_0\|\mathbf{r}\|^3}\mathbf{r}$$

where ϵ_0 is a positive constant (called the ***permittivity constant***). This formula is of form (3) with $c = qQ/4\pi\epsilon_0$. ◀

GRADIENT FIELDS

An important class of vector fields arises from the process of finding gradients. Recall that if ϕ is a function of three variables, then the gradient of ϕ is defined as

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

This formula defines a vector field in 3-space called the ***gradient field of*** ϕ. Similarly, the gradient of a function of two variables defines a gradient field in 2-space. At each point in a gradient field where the gradient is nonzero, the vector points in the direction in which the rate of increase of ϕ is maximum.

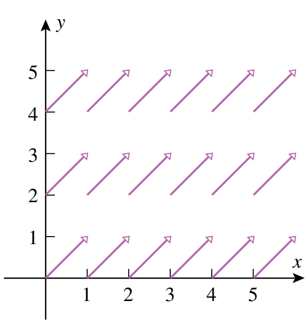


Figure 16.1.4

Example 2 Sketch the gradient field of $\phi(x, y) = x + y$.

Solution. The gradient of ϕ is

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} = \mathbf{i} + \mathbf{j}$$

which is the same at each point. A portion of the vector field is sketched in Figure 16.1.4. ◀

CONSERVATIVE FIELDS AND POTENTIAL FUNCTIONS

If $\mathbf{F}(\mathbf{r})$ is an arbitrary vector field in 2-space or 3-space, we can ask whether it is the gradient field of some function ϕ, and if so, how we can find ϕ. This is an important problem in various applications, and we will study it in more detail later. However, there is some terminology for such fields that we will introduce now.

16.1.3 DEFINITION. A vector field $\mathbf{F}$ in 2-space or 3-space is said to be ***conservative*** in a region if it is the gradient field for some function ϕ in that region. The function ϕ is called a ***potential function*** for $\mathbf{F}$ in the region.

Example 3 Inverse-square fields are conservative in any region that does not contain the origin. For example, in the two-dimensional case the function

$$\phi(x, y) = -\frac{c}{(x^2+y^2)^{1/2}} \tag{6}$$

is a potential function for (4) in any region not containing the origin, since

$$\begin{aligned}
\nabla\phi(x, y) &= \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} \\
&= \frac{cx}{(x^2+y^2)^{3/2}}\mathbf{i} + \frac{cy}{(x^2+y^2)^{3/2}}\mathbf{j} \\
&= \frac{c}{(x^2+y^2)^{3/2}}(x\mathbf{i} + y\mathbf{j}) \\
&= \mathbf{F}(x, y)
\end{aligned}$$

In a later section we will discuss methods for finding potential functions for conservative vector fields. ◀

DIVERGENCE AND CURL

We will now define two important operations on vector fields in 3-space—the *divergence* and the *curl* of the field. These names originate in the study of fluid flow, in which case the divergence relates to the way in which fluid flows toward or away from a point and the curl relates to the rotational properties of the fluid at a point. We will investigate the physical interpretations of these operations in more detail later, but for now we will focus only on their computation.

16.1.4 DEFINITION. If $\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$, then we define the ***divergence of*** $\mathbf{F}$, written div $\mathbf{F}$, by

$$\operatorname{div}\mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \tag{7}$$

16.1.5 DEFINITION. If $\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$, then we define the ***curl of*** $\mathbf{F}$, written curl $\mathbf{F}$, by

$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k} \tag{8}$$

REMARK. Observe that div $\mathbf{F}$ and curl $\mathbf{F}$ depend on the point at which they are computed, and hence are more properly written as div $\mathbf{F}(x, y, z)$ and curl $\mathbf{F}(x, y, z)$. However, even though these functions are expressed in terms of x, y, and z, it can be proved that their values at a fixed point depend on the point but not on the coordinate system selected. This is important in applications, since it allows physicists and engineers to compute the curl and divergence in any convenient coordinate system.

Before proceeding to some examples, we note that div $\mathbf{F}$ has scalar values, whereas curl $\mathbf{F}$ has vector values (i.e., curl $\mathbf{F}$ is itself a vector field). Moreover, for computational purposes it is useful to note that the formula for the curl can be expressed in the determinant form

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \tag{9}$$

You should verify that Formula (8) results if the determinant is computed by interpreting a "product" such as $(\partial/\partial x)(g)$ to mean $\partial g/\partial x$. Keep in mind, however, that (9) is just a mnemonic device and not a true determinant, since the entries in a determinant must be numbers, not vectors and partial derivative symbols.

Example 4 Find the divergence and the curl of the vector field

$$\mathbf{F}(x, y, z) = x^2y\mathbf{i} + 2y^3z\mathbf{j} + 3z\mathbf{k}$$

Solution. From (7)

$$\begin{aligned} \operatorname{div} \mathbf{F} &= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(2y^3z) + \frac{\partial}{\partial z}(3z) \\ &= 2xy + 6y^2z + 3 \end{aligned}$$

and from (9)

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & 2y^3z & 3z \end{vmatrix} \\ &= \left[\frac{\partial}{\partial y}(3z) - \frac{\partial}{\partial z}(2y^3z)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(x^2y) - \frac{\partial}{\partial x}(3z)\right]\mathbf{j} \\ &\qquad + \left[\frac{\partial}{\partial x}(2y^3z) - \frac{\partial}{\partial y}(x^2y)\right]\mathbf{k} \\ &= -2y^3\mathbf{i} - x^2\mathbf{k} \end{aligned}$$

◀

FOR THE READER. Most computer algebra systems can compute gradient fields, divergence, and curl. If you have a CAS with these capabilities, read the relevant documentation and use your CAS to check the computations in Examples 2 and 4.

Example 5 Show that the divergence of the inverse-square field

$$\mathbf{F}(x, y, z) = \frac{c}{(x^2 + y^2 + z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

is zero.

Solution. The computations can be simplified by letting $r = (x^2 + y^2 + z^2)^{1/2}$, in which case $\mathbf{F}$ can be expressed as

$$\mathbf{F}(x, y, z) = \frac{cx\mathbf{i} + cy\mathbf{j} + cz\mathbf{k}}{r^3} = \frac{cx}{r^3}\mathbf{i} + \frac{cy}{r^3}\mathbf{j} + \frac{cz}{r^3}\mathbf{k}$$

We leave it for you to show that

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Thus

$$\operatorname{div} \mathbf{F} = c\left[\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\right] \tag{10}$$

But

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{r^3 - x(3r^2)(x/r)}{(r^3)^2} = \frac{1}{r^3} - \frac{3x^2}{r^5}$$
$$\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) = \frac{1}{r^3} - \frac{3y^2}{r^5}$$
$$\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = \frac{1}{r^3} - \frac{3z^2}{r^5}$$

Substituting these expressions in (10) yields

$$\operatorname{div} \mathbf{F} = c\left[\frac{3}{r^3} - \frac{3x^2+3y^2+3z^2}{r^5}\right] = c\left[\frac{3}{r^3} - \frac{3r^2}{r^5}\right] = 0$$ ◀

THE ∇ OPERATOR

Thus far, the symbol ∇ that appears in the gradient expression $\nabla\phi$ has not been given a meaning of its own. However, it is often convenient to view ∇ as an operator

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \tag{11}$$

which when applied to $\phi(x, y, z)$ produces the gradient

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

We call (11) the ***del operator***. This is analogous to the derivative operator d/dx, which when applied to $f(x)$ produces the derivative $f'(x)$.

The del operator allows us to express the divergence of a vector field

$$\mathbf{F} = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

in dot product notation as

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \tag{12}$$

and the curl of this field in cross-product notation as

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \tag{13}$$

THE LAPLACIAN ∇^2

The operator that results by taking the dot product of the del operator with itself is denoted by ∇^2 and is called the ***Laplacian*** * ***operator***. This operator has the form

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{14}$$

When applied to $\phi(x, y, z)$ the Laplacian operator produces the function

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Note that $\nabla^2\phi$ can also be expressed as div $(\nabla\phi)$. The equation $\nabla^2\phi = 0$ or, equivalently,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

is known as ***Laplace's equation***. This partial differential equation plays an important role in a wide variety of applications, resulting from the fact that it is satisfied by the potential function for the inverse-square field.

* See biography on page 1100.

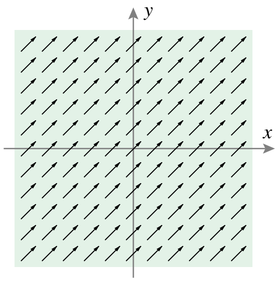
EXERCISE SET 16.1 Graphing Utility C CAS

In Exercises 1 and 2, match the vector field $\mathbf{F}(x, y)$ with one of the plots, and explain your reasoning.

1. (a) $\mathbf{F}(x, y) = x\mathbf{i}$ (b) $\mathbf{F}(x, y) = \sin x\mathbf{i} + \mathbf{j}$

2. (a) $\mathbf{F}(x, y) = \mathbf{i} + \mathbf{j}$

(b) $\mathbf{F}(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$



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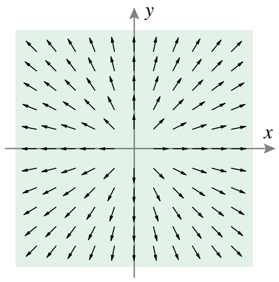

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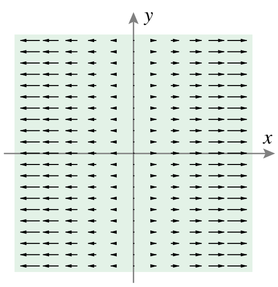

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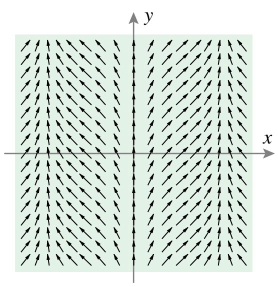

IV

In Exercises 3 and 4, determine whether the statement about the vector field $\mathbf{F}(x, y)$ is true or false. If false, explain why.

3. $\mathbf{F}(x, y) = x^2\mathbf{i} - y\mathbf{j}$.
(a) $\|\mathbf{F}(x, y)\| \to 0$ as $(x, y) \to (0, 0)$.
(b) If (x, y) is on the positive y-axis, then the vector points in the negative y-direction.
(c) If (x, y) is in the first quadrant, then the vector points down and to the right.

4. $\mathbf{F}(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}\mathbf{i} - \dfrac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$.
(a) As (x, y) moves away from the origin, the lengths of the vectors decrease.
(b) If (x, y) is a point on the positive x-axis, then the vector points up.
(c) If (x, y) is a point on the positive y-axis, the vector points to the right.

In Exercises 5–8, sketch the vector field by drawing some representative nonintersecting vectors. The vectors need not be drawn to scale, but they should be in reasonably correct proportion relative to each other.

5. $\mathbf{F}(x, y) = 2\mathbf{i} - \mathbf{j}$ **6.** $\mathbf{F}(x, y) = y\mathbf{j}, \quad y > 0$

7. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$. [*Note:* Each vector in the field is perpendicular to the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.]

8. $\mathbf{F}(x, y) = \dfrac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}$. [*Note:* Each vector in the field is a unit vector in the same direction as the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.]

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*PIERRE-SIMON DE LAPLACE (1749–1827). French mathematician and physicist. Laplace is sometimes referred to as the French Isaac Newton because of his work in celestial mechanics. In a five-volume treatise entitled *Traité de Mécanique Céleste*, he solved extremely difficult problems involving gravitational interactions between the planets. In particular, he was able to show that our solar system is stable and not prone to catastrophic collapse as a result of these interactions. This was an issue of major concern at the time because Jupiter's orbit appeared to be shrinking and Saturn's expanding; Laplace showed that these were expected periodic anomalies. In addition to his work in celestial mechanics, he founded modern probability theory, showed with Lavoisier that respiration is a form of combustion, and developed methods that fostered many new branches of pure mathematics.

Laplace was born to moderately successful parents in Normandy, his father being a farmer and cider merchant. He matriculated in the theology program at the University of Caen at age 16 but left for Paris at age 18 with a letter of introduction to the influential mathematician d'Alembert, who eventually helped him undertake a career in mathematics. Laplace was a prolific writer, and after his election to the Academy of Sciences in 1773, the secretary wrote that the Academy had never received so many important research papers by so young a person in such a short time. Laplace had little interest in pure mathematics—he regarded mathematics merely as a tool for solving applied problems. In his impatience with mathematical detail, he frequently omitted complicated arguments with the statement, "It is easy to show that. . . ." He admitted, however, that as time passed he often had trouble reconstructing the omitted details himself!

At the height of his fame, Laplace served on many government committees and held the posts of Minister of the Interior and chancellor of the Senate. He barely escaped imprisonment and execution during the period of the Revolution, probably because he was able to convince each opposing party that he sided with them. Napoleon described him as a great mathematician but a poor administrator who "sought subtleties everywhere, had only doubtful ideas, and . . . carried the spirit of the infinitely small into administration." In spite of his genius, Laplace was both egotistic and insecure, attempting to ensure his place in history by conveniently failing to credit mathematicians whose work he used—an unnecessary pettiness since his own work was so brilliant. However, on the positive side he was supportive of young mathematicians, often treating them as his own children. Laplace ranks as one of the most influential mathematicians in history.

In Exercises 9 and 10, use a graphing utility to generate a plot of the vector field.

9. $\mathbf{F}(x, y) = \mathbf{i} + \cos y\,\mathbf{j}$ **10.** $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

In Exercises 11 and 12, confirm that ϕ is a potential function for $\mathbf{F}(\mathbf{r})$ on some region, and state the region.

11. (a) $\phi(x, y) = \tan^{-1} xy$
$\mathbf{F}(x, y) = \dfrac{y}{1 + x^2y^2}\mathbf{i} + \dfrac{x}{1 + x^2y^2}\mathbf{j}$
(b) $\phi(x, y, z) = x^2 - 3y^2 + 4z^2$
$\mathbf{F}(x, y, z) = 2x\mathbf{i} - 6y\mathbf{j} + 8z\mathbf{k}$

12. (a) $\phi(x, y) = 2y^2 + 3x^2y - xy^3$
$\mathbf{F}(x, y) = (6xy - y^3)\mathbf{i} + (4y + 3x^2 - 3xy^2)\mathbf{j}$
(b) $\phi(x, y, z) = x \sin z + y \sin x + z \sin y$
$\mathbf{F}(x, y, z) = (\sin z + y \cos x)\mathbf{i} + (\sin x + z \cos y)\mathbf{j} + (\sin y + x \cos z)\mathbf{k}$

In Exercises 13–18, find div **F** and curl **F**.

13. $\mathbf{F}(x, y, z) = x^2\mathbf{i} - 2\mathbf{j} + yz\mathbf{k}$

14. $\mathbf{F}(x, y, z) = xz^3\mathbf{i} + 2y^4x^2\mathbf{j} + 5z^2y\mathbf{k}$

15. $\mathbf{F}(x, y, z) = 7y^3z^2\mathbf{i} - 8x^2z^5\mathbf{j} - 3xy^4\mathbf{k}$

16. $\mathbf{F}(x, y, z) = e^{xy}\mathbf{i} - \cos y\,\mathbf{j} + \sin^2 z\,\mathbf{k}$

17. $\mathbf{F}(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

18. $\mathbf{F}(x, y, z) = \ln x\,\mathbf{i} + e^{xyz}\mathbf{j} + \tan^{-1}(z/x)\mathbf{k}$

In Exercises 19 and 20, find $\nabla \cdot (\mathbf{F} \times \mathbf{G})$.

19. $\mathbf{F}(x, y, z) = 2x\mathbf{i} + \mathbf{j} + 4y\mathbf{k}$
$\mathbf{G}(x, y, z) = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$

20. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$
$\mathbf{G}(x, y, z) = xy\mathbf{j} + xyz\mathbf{k}$

In Exercises 21 and 22, find $\nabla \cdot (\nabla \times \mathbf{F})$.

21. $\mathbf{F}(x, y, z) = \sin x\,\mathbf{i} + \cos(x - y)\mathbf{j} + z\mathbf{k}$

22. $\mathbf{F}(x, y, z) = e^{xz}\mathbf{i} + 3xe^{y}\mathbf{j} - e^{yz}\mathbf{k}$

In Exercises 23 and 24, find $\nabla \times (\nabla \times \mathbf{F})$.

23. $\mathbf{F}(x, y, z) = xy\mathbf{j} + xyz\mathbf{k}$

24. $\mathbf{F}(x, y, z) = y^2x\mathbf{i} - 3yz\mathbf{j} + xy\mathbf{k}$

C **25.** Use a CAS to check the calculations in Exercises 19, 21, and 23.

C **26.** Use a CAS to check the calculations in Exercises 20, 22, and 24.

In Exercises 27–34, let k be a constant, and let $\mathbf{F} = \mathbf{F}(x, y, z)$, $\mathbf{G} = \mathbf{G}(x, y, z)$, and $\phi = \phi(x, y, z)$. Prove the following identities, assuming that all derivatives involved exist and are continuous.

27. $\operatorname{div}(k\mathbf{F}) = k \operatorname{div} \mathbf{F}$ **28.** $\operatorname{curl}(k\mathbf{F}) = k \operatorname{curl} \mathbf{F}$

29. $\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}$

30. $\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl} \mathbf{F} + \operatorname{curl} \mathbf{G}$

31. $\operatorname{div}(\phi\mathbf{F}) = \phi \operatorname{div} \mathbf{F} + \nabla\phi \cdot \mathbf{F}$

32. $\operatorname{curl}(\phi\mathbf{F}) = \phi \operatorname{curl} \mathbf{F} + \nabla\phi \times \mathbf{F}$

33. $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$ **34.** $\operatorname{curl}(\nabla\phi) = \mathbf{0}$

35. Rewrite the identities in Exercises 27, 29, 31, and 33 in an equivalent form using the notation $\nabla\,\cdot$ for divergence and $\nabla\,\times$ for curl.

36. Rewrite the identities in Exercises 28, 30, 32, and 34 in an equivalent form using the notation $\nabla\,\cdot$ for divergence and $\nabla\,\times$ for curl.

In Exercises 37 and 38, verify that the radius vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ has the stated property.

37. (a) $\operatorname{curl} \mathbf{r} = \mathbf{0}$ (b) $\nabla\|\mathbf{r}\| = \dfrac{\mathbf{r}}{\|\mathbf{r}\|}$

38. (a) $\operatorname{div} \mathbf{r} = 3$ (b) $\nabla\dfrac{1}{\|\mathbf{r}\|} = -\dfrac{\mathbf{r}}{\|\mathbf{r}\|^3}$

In Exercises 39 and 40, let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, let $r = \|\mathbf{r}\|$, let f be a differentiable function of one variable, and let $\mathbf{F}(\mathbf{r}) = f(r)\mathbf{r}$.

39. (a) Use the chain rule and Exercise 37(b) to show that
$$\nabla f(r) = \frac{f'(r)}{r}\mathbf{r}$$
(b) Use the result in part (a) and Exercises 31 and 38(a) to show that
$$\operatorname{div} \mathbf{F} = 3f(r) + rf'(r)$$

40. (a) Use part (a) of Exercise 39, Exercise 32, and Exercise 37(a) to show that
$$\operatorname{curl} \mathbf{F} = \mathbf{0}$$
(b) Use the result in part (a) of Exercise 39 and Exercises 31 and 38(a) to show that
$$\nabla^2 f(r) = 2\frac{f'(r)}{r} + f''(r)$$

41. Use the result in Exercise 39(b) to show that the divergence of the inverse-square field $\mathbf{F} = \mathbf{r}/\|\mathbf{r}\|^3$ is zero.

42. Use the result of Exercise 39(b) to show that if **F** is a vector field of the form $\mathbf{F} = f(\|\mathbf{r}\|)\mathbf{r}$ and if $\operatorname{div} \mathbf{F} = 0$, then **F** is an inverse-square field. [*Suggestion:* Let $r = \|\mathbf{r}\|$ and multiply $3f(r) + rf'(r) = 0$ through by r^2. Then write the result as a derivative of a product.

43. A curve C is called a ***flow line*** of a vector field **F** if **F** is a tangent vector to C at each point along C.
(a) Let C be a flow line for $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$, and let (x, y) be a point on C for which $y \neq 0$. Show that the flow lines satisfy the differential equation
$$\frac{dy}{dx} = -\frac{x}{y}$$

(b) Solve the differential equation in part (a) by separation of variables, and show that the flow lines are concentric circles centered at the origin.

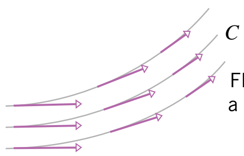


Figure Ex-43

In Exercises 44–46, find a differential equation satisfied by the flow lines of $\mathbf{F}$ (see Exercise 43), and solve it to find equations for the flow lines of $\mathbf{F}$. Sketch some typical flow lines and tangent vectors.

44. $\mathbf{F}(x, y) = \mathbf{i} + x\mathbf{j}$ **45.** $\mathbf{F}(x, y) = x\mathbf{i} + \mathbf{j}, \quad x > 0$

46. $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}, \quad x > 0$ and $y > 0$

16.2 LINE INTEGRALS

In earlier chapters we considered three kinds of integrals in rectangular coordinates: single integrals over intervals, double integrals over two-dimensional regions, and triple integrals over three-dimensional regions. In this section we will discuss integrals along curves in two- or three-dimensional space.

LINE INTEGRALS

Integrals along curves arise in a variety of problems. One such problem can be stated as follows:

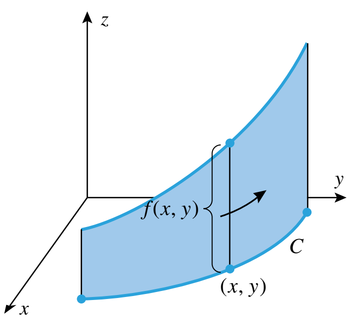


Figure 16.2.1

16.2.1 AN AREA PROBLEM. Let C be a smooth curve that extends between two points in the xy-plane, and let $f(x, y)$ be continuous and nonnegative on C. Find the area of the "sheet" that is swept out by the vertical line segment that extends upward from the point (x, y) to a height of $f(x, y)$ and moves along C from one endpoint to the other (Figure 16.2.1).

We use the following limit process to find the area of the sheet:

- Divide C into n arcs by inserting a succession of distinct points $P_1, P_2, \ldots, P_{n-1}$ between the initial and terminal points of C in the direction of increasing parameter. As illustrated on the left side of Figure 16.2.2, these points divide the surface into n strips. If we denote the area of the kth strip by ΔA_k, then the total area A of the sheet can be

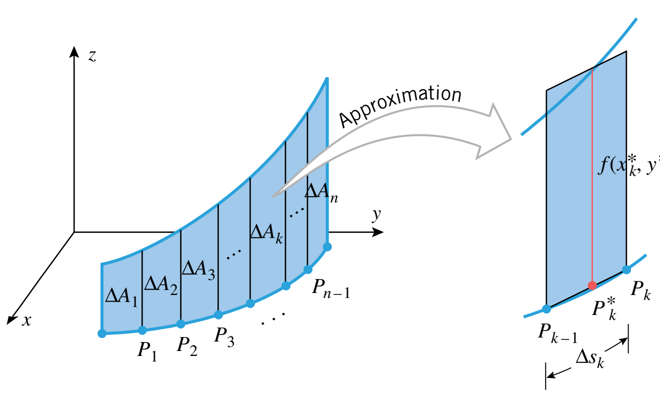


Figure 16.2.2

expressed as

$$A = \Delta A_1 + \Delta A_2 + \cdots + \Delta A_n = \sum_{k=1}^{n} \Delta A_k$$

- The next step is to approximate the area ΔA_k of the kth strip, assuming that this strip is narrow. For this purpose, let Δs_k be the length of the arc along C at the base of the kth strip, and choose an arbitrary point $P_k^*(x_k^*, y_k^*)$ on this arc. Since the strip is narrow and f is continuous, the value of f will not vary much along the kth arc, so we can assume that f has a constant value of $f(x_k^*, y_k^*)$ on this arc. Thus, the area ΔA_k of the kth strip can be closely approximated by the area of a rectangle with base Δs_k and height $f(x_k^*, y_k^*)$, as shown in the right part of Figure 16.2.2; that is,

$$\Delta A_k \approx f(x_k^*, y_k^*)\Delta s_k$$

from which it follows that

$$A \approx \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta s_k$$

- If we now increase n so that the length of each arc approaches zero, then it is plausible that the error in this approximation approaches zero, and the exact surface area is

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta s_k \tag{1}$$

In deriving Formula (1) we assumed that f is continuous and nonnegative on the curve C. If f is continuous on C and has both positive and negative values, then the limit

$$\lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta s_k$$

does not represent the area of the surface over C; rather, it represents a *difference* of areas—the area between the curve C and the graph of $f(x, y)$ above the xy-plane minus the area between C and the graph of $f(x, y)$ below the xy-plane. We call this the ***net signed area*** between the curve C and the graph of $f(x, y)$. Also, we call the limit in (1) the ***line integral of f with respect to s along C*** and denote it by

$$\int_C f(x, y)\, ds = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta s_k \tag{2}$$

With this notation, the area of the surface in Figure 16.2.1 can be expressed as

$$A = \int_C f(x, y)\, ds \tag{3}$$

REMARK. In Section 6.1 we observed that the area of a region in the xy-plane under a curve or between two curves over an interval $[a, b]$ is obtained by integrating the length of a vertical cross section of the region from a to b (see the remark preceding Example 1 in Section 6.1). Similarly, Formula (3) states that the area of a sheet along a curve C is obtained by integrating the length of a vertical cross section of the sheet along the curve C.

EVALUATING LINE INTEGRALS

Except in simple cases, it will not be feasible to evaluate a line integral directly from (2). However, we will now show that it is possible to express a line integral as an ordinary definite integral, so that no special methods of evaluation are required. To see how this can be done, suppose that the curve C is represented by the parametric equations

$$x = x(t), \quad y = y(t) \qquad (a \le t \le b)$$

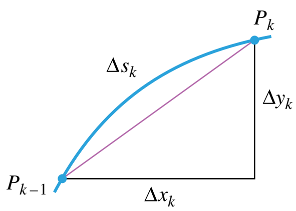


Figure 16.2.3

Moreover, suppose that the points P_{k-1} and P_k in Figure 16.2.3 correspond to parameter values of t_{k-1} and t_k, respectively, and that $P_k^*(x_k^*, y_k^*)$ corresponds to the parameter value t_k^*. If we let $\Delta t_k = t_k - t_{k-1}$, then we can approximate Δs_k as

$$\Delta s_k \approx \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{\left(\frac{\Delta x_k}{\Delta t_k}\right)^2 + \left(\frac{\Delta y_k}{\Delta t_k}\right)^2}\,\Delta t_k \tag{4}$$

from which it follows that (2) can be expressed as

$$\int_C f(x, y)\,ds = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x(t_k^*), y(t_k^*))\sqrt{\left(\frac{\Delta x_k}{\Delta t_k}\right)^2 + \left(\frac{\Delta y_k}{\Delta t_k}\right)^2}\,\Delta t_k$$

which suggests that

$$\int_C f(x, y)\,ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \tag{5}$$

In words, this formula states that *a line integral can be evaluated by expressing the integrand in terms of the parameter, multiplying the integrand by an appropriate "radical," and then integrating from the initial value of the parameter to the final value of the parameter.*

In the special case where t is an arc length parameter, say $t = s$, it follows from Formula (20) of Section 13.3 that the radical in (5) reduces to 1, so the integration formula simplifies to

$$\int_C f(x, y)\,ds = \int_a^b f(x(s), y(s))\,ds \tag{6}$$

Example 1 Evaluate the line integral $\int_C (1 + xy^2)\,ds$ from $(0, 0)$ to $(1, 2)$ along the line segment C that is represented by the parametric equations $x = t$, $y = 2t$ $(0 \le t \le 1)$.

Solution. It follows from Formula (5) that

$$\begin{aligned}\int_C (1 + xy^2)\,ds &= \int_0^1 (1 + (t)(4t^2))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\ &= \int_0^1 (1 + 4t^3)\sqrt{5}\,dt \\ &= \sqrt{5}\left[t + t^4\right]_0^1 = 2\sqrt{5}\end{aligned}$$

◀

Example 2 Find the area of the surface extending upward from the circle $x^2 + y^2 = 1$ in the xy-plane to the parabolic cylinder $z = 1 - x^2$ (Figure 16.2.4).

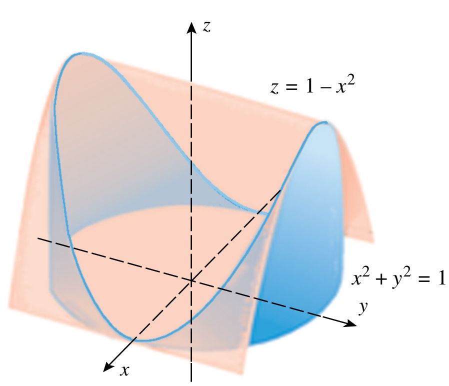


Figure 16.2.4

Solution. The area A of the surface can be expressed as the line integral

$$A = \int_C (1 - x^2)\, ds \tag{7}$$

where C is the circle $x^2 + y^2 = 1$. This circle can be parametrized in terms of arc length as

$$x = \cos s, \quad y = \sin s \quad (0 \le s \le 2\pi)$$

Thus, it follows from (6) and (7) that

$$A = \int_C (1 - x^2)\, ds = \int_0^{2\pi} (1 - \cos^2 s)\, ds$$

$$= \int_0^{2\pi} \sin^2 s\, ds = \frac{1}{2}\int_0^{2\pi} (1 - \cos 2s)\, ds = \pi \qquad \blacktriangleleft$$

REMARK. We will show later in this section that we would have obtained the same value for (7) had we used any other smooth parametrization of the circle $x^2 + y^2 = 1$ in the xy-plane.

LINE INTEGRALS IN 3-SPACE

If C is a smooth curve that extends between two points in an xyz-coordinate system in 3-space, and if $f(x, y, z)$ is continuous on C, then the ***line integral of f with respect to s along C*** is defined as

$$\int_C f(x, y, z)\, ds = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta s_k \tag{8}$$

where the sum on the right side is obtained by subdividing the curve C into n arcs, choosing an arbitrary point (x_k^*, y_k^*, z_k^*) in the kth arc, multiplying $f(x_k^*, y_k^*, z_k^*)$ by the length Δs_k of the kth arc, and summing over all n arcs. Here $n\to+\infty$ indicates the process of increasing the number of arcs on C in such a way that the lengths of the arcs approach zero. If the curve C is represented by the parametric equations

$$x = x(t), \quad y = y(t), \quad z = z(t) \quad (a \le t \le b)$$

then (8) can be evaluated from the formula

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \tag{9}$$

and if t is an arc length parameter, say $t = s$, then it follows from Formula (21) of Section 13.3 that the radical in (9) reduces to 1, so the integration formula simplifies to

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(s), y(s), z(s))\, ds \tag{10}$$

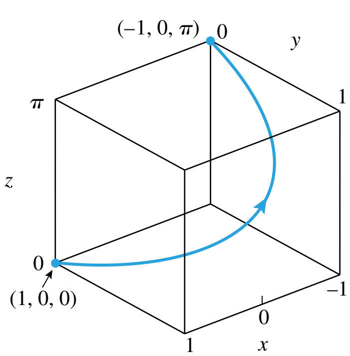


Figure 16.2.5

REMARK. Observe that Formulas (9) and (10) have the same form as (5) and (6) but with an additional z-component. In general, line integrals along curves in 3-space do not have a simple area interpretation, so there is no analog of Formula (3). However, we will see later in this section that line integrals along curves in 3-space have other important interpretations.

Example 3 Evaluate the line integral $\int_C (xy + z^3)\, ds$ from $(1, 0, 0)$ to $(-1, 0, \pi)$ along the helix C that is represented by the parametric equations

$$x = \cos t, \quad y = \sin t, \quad z = t \quad (0 \le t \le \pi)$$

(Figure 16.2.5).

Solution. From (9)

$$\int_C (xy + z^3)\, ds = \int_0^\pi (\cos t \sin t + t^3)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \int_0^\pi (\cos t \sin t + t^3)\sqrt{(-\sin t)^2 + (\cos t)^2 + 1}\, dt$$

$$= \sqrt{2}\int_0^\pi (\cos t \sin t + t^3)\, dt$$

$$= \sqrt{2}\left[\frac{\sin^2 t}{2} + \frac{t^4}{4}\right]_0^\pi = \frac{\sqrt{2}\pi^4}{4}$$ ◀

MASS OF A WIRE AS A LINE INTEGRAL

We will now show how a line integral can be used to calculate the mass of a thin wire. For this purpose consider an idealized thin wire in 2-space or 3-space that is bent in the shape of a curve C. If the composition of the wire is uniform so that its mass is distributed uniformly, then the wire is said to be ***homogeneous***, and we define the ***linear mass density*** of the wire to be the total mass divided by the total length. For example, a homogeneous wire with a mass of 2 g and a length of 8 cm would have a linear mass density of $\frac{2}{8} = 0.25$ g/cm. However, if the mass of the wire is not uniformly distributed, then the linear mass density is not a useful measure, since it does not account for variations in mass concentration. In this case we describe the mass concentration at a point by a ***mass density function*** δ, which we view as a limit; that is,

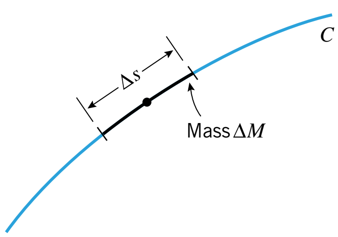


Figure 16.2.6

$$\delta = \lim_{\Delta s \to 0} \frac{\Delta M}{\Delta s} \tag{11}$$

where ΔM and Δs denote the mass and length of a small section of wire centered at the point (Figure 16.2.6). Observe that $\Delta M/\Delta s$ is the linear mass density of the small section of wire, so that the mass density function at a point can be viewed informally as *the limit of the linear mass densities of small wire sections centered at the point*.

To translate this informal idea into a useful formula, suppose that $\delta = \delta(x, y)$ is the density function for a thin smooth wire C in 2-space. Assume that the wire is subdivided into n small sections; let (x_k^*, y_k^*) be the center of the kth section, let ΔM_k be the mass of the kth section, and let Δs_k be the length of the kth section. Since we are assuming that the sections are small, it follows from (11) that the mass of the kth section can be approximated as

$$\Delta M_k \approx \delta(x_k^*, y_k^*)\Delta s_k$$

and hence the mass M of the entire wire can be approximated as

$$M = \sum_{k=1}^{n} \Delta M_k \approx \sum_{k=1}^{n} \delta(x_k^*, y_k^*)\Delta s_k \tag{12}$$

If we now increase n in such a way that the lengths of the sections approach zero, then it is plausible that the error in (12) will approach zero, and the exact value of M will be given by the line integral

$$\boxed{M = \int_C \delta(x, y)\, ds} \tag{13}$$

Similarly, the mass M of a wire C in 3-space with density function $\delta(x, y, z)$ is given by

$$\boxed{M = \int_C \delta(x, y, z)\, ds} \tag{14}$$

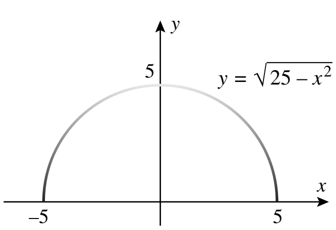


Figure 16.2.7

Example 4 Suppose that a semicircular wire has the equation $y = \sqrt{25 - x^2}$ and that its mass density is $\delta(x, y) = 15 - y$ (Figure 16.2.7). Physically, this means the wire has a maximum density of 15 units at the base ($y = 0$) and that the density of the wire decreases linearly with respect to y to a value of 10 units at the top ($y = 5$). Find the mass of the wire.

Solution. The mass M of the wire can be expressed as the line integral

$$M = \int_C \delta(x, y)\, ds = \int_C (15 - y)\, ds \tag{15}$$

along the semicircle C. To evaluate this integral we will express C parametrically as

$$x = 5\cos t, \quad y = 5\sin t \qquad (0 \le t \le \pi)$$

Thus, it follows from (5) and (15) that

$$\begin{aligned} M = \int_C (15 - y)\, ds &= \int_0^\pi (15 - 5\sin t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^\pi (15 - 5\sin t)\sqrt{(-5\sin t)^2 + (5\cos t)^2}\, dt \\ &= 5\int_0^\pi (15 - 5\sin t)\, dt \\ &= 5\big[15t + 5\cos t\big]_0^\pi \\ &\approx 75\pi - 50 \approx 185.6 \text{ units of mass} \end{aligned}$$

◀

ARC LENGTH AS A LINE INTEGRAL

In the special cases where $f(x, y)$ and $f(x, y, z)$ are 1, Formulas (5) and (9) become

$$\int_C ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\int_C ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

However, it follows from Formulas (2) and (4) of Section 13.3 that these integrals represent the arc length of C. Thus, we have established the following result.

16.2.2 THEOREM. *If C is a smooth parametric curve in 2-space or 3-space, then its arc length L can be expressed as*

$$L = \int_C ds \tag{16}$$

REMARK. This result adds nothing new computationally, since Formula (16) is just a reformulation of the arc length formulas in Section 13.3. However, the relationship between line integrals and arc length is important to know.

LINE INTEGRALS WITH RESPECT TO x, y, AND z

There are other important types of line integrals that result by replacing Δs_k in definitions (2) and (8) by $\Delta x_k = x_k - x_{k-1}$, $\Delta y_k = y_k - y_{k-1}$, or $\Delta z_k = z_k - z_{k-1}$, where (x_k, y_k, z_k) and $(x_{k-1}, y_{k-1}, z_{k-1})$ are the coordinates of the points P_k and P_{k-1} in Figure 16.2.2. For example, in 2-space we define

$$\int_C f(x, y)\, dx = \lim_{n \to +\infty} \sum_{k=1}^n f(x_k^*, y_k^*)\Delta x_k \tag{17}$$

$$\int_C f(x, y)\, dy = \lim_{n \to +\infty} \sum_{k=1}^n f(x_k^*, y_k^*)\Delta y_k \tag{18}$$

and in 3-space we define

$$\int_C f(x, y, z)\,dx = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta x_k \tag{19}$$

$$\int_C f(x, y, z)\,dy = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta y_k \tag{20}$$

$$\int_C f(x, y, z)\,dz = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta z_k \tag{21}$$

We will call these ***line integrals with respect to x, y, and z*** (as appropriate) in contrast to (2) and (8), which are line integrals with respect to s (also called ***line integrals with respect to arc length***).

The basic procedure for evaluating these line integrals is to find parametric equations for C, say

$$x = x(t), \quad y = y(t), \quad z = z(t) \qquad (a \le t \le b)$$

and then express the integrand in terms of t. For example,

$$\int_C f(x, y)\,dx = \int_a^b \left[f(x(t), y(t))\frac{dx}{dt} \right] dt = \int_a^b f(x(t), y(t))x'(t)\,dt$$

We omit the formal proof.

For reference, we list the relevant formulas.

$$\int_C f(x, y)\,dx = \int_a^b f(x(t), y(t))x'(t)\,dt \tag{22}$$

$$\int_C f(x, y)\,dy = \int_a^b f(x(t), y(t))y'(t)\,dt \tag{23}$$

$$\int_C f(x, y, z)\,dx = \int_a^b f(x(t), y(t), z(t))x'(t)\,dt \tag{24}$$

$$\int_C f(x, y, z)\,dy = \int_a^b f(x(t), y(t), z(t))y'(t)\,dt \tag{25}$$

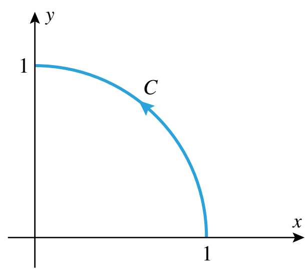
$$\int_C f(x, y, z)\,dz = \int_a^b f(x(t), y(t), z(t))z'(t)\,dt \tag{26}$$

Frequently, the line integrals with respect to x and y occur in combination, in which case we dispense with one of the integral signs and write

$$\int_C f(x, y)\,dx + g(x, y)\,dy = \int_C f(x, y)\,dx + \int_C g(x, y)\,dy \tag{27}$$

and similarly,

$$\begin{aligned}\int_C f(x, y, z)\,dx &+ g(x, y, z)\,dy + h(x, y, z)\,dz \\ &= \int_C f(x, y, z)\,dx + \int_C g(x, y, z)\,dy + \int_C h(x, y, z)\,dz\end{aligned} \tag{28}$$



$x = \cos t,\ y = \sin t\ (0 \le t \le \pi/2)$

Figure 16.2.8

Example 5 Evaluate

$$\int_C 2xy\,dx + (x^2 + y^2)\,dy$$

along the circular arc C given by $x = \cos t$, $y = \sin t$ $(0 \le t \le \pi/2)$ (Figure 16.2.8).

Solution. From (22) and (23)

$$\int_C 2xy\,dx = \int_0^{\pi/2} (2\cos t \sin t)\left[\frac{d}{dt}(\cos t)\right] dt$$

$$= -2\int_0^{\pi/2} \sin^2 t \cos t\,dt = -\frac{2}{3}\sin^3 t\Big]_0^{\pi/2} = -\frac{2}{3}$$

$$\int_C (x^2 + y^2)\,dy = \int_0^{\pi/2} (\cos^2 t + \sin^2 t)\left[\frac{d}{dt}(\sin t)\right] dt$$

$$= \int_0^{\pi/2} \cos t\,dt = \sin t\Big]_0^{\pi/2} = 1$$

Thus, from (27)

$$\int_C 2xy\,dx + (x^2 + y^2)\,dy = \int_C 2xy\,dx + \int_C (x^2 + y^2)\,dy$$

$$= -\frac{2}{3} + 1 = \frac{1}{3}$$

◀

Example 6

(a) Show that $\int_C f(x, y)\,dx = 0$ along any line segment parallel to the y-axis.

(b) Show that $\int_C f(x, y)\,dy = 0$ along any line segment parallel to the x-axis.

Solution. A line segment parallel to the y-axis can be represented parametrically by equations of the form $x = k$, $y = t$, where k is a constant. Thus, $x'(t) = 0$ in (22). Similarly, a line segment parallel to the x-axis can be represented parametrically by equations of the form $x = t$, $y = k$, where k is a constant. Thus, $y'(t) = 0$ in (23). ◀

• FOR THE READER. What is the analog of Example 6 in 3-space?

LINE INTEGRALS ALONG PIECEWISE SMOOTH CURVES

Thus far, we have only considered line integrals along smooth curves. However, the notion of a line integral can be extended to curves formed from finitely many smooth curves $C_1, C_2, \ldots, C_n$ joined end to end. Such a curve is called ***piecewise smooth*** (Figure 16.2.9). We define a line integral along a piecewise smooth curve C to be the sum of the integrals along the sections:

$$\int_C = \int_{C_1} + \int_{C_2} + \cdots + \int_{C_n}$$

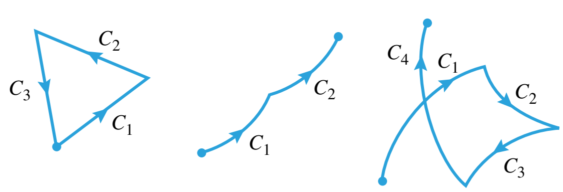


Figure 16.2.9

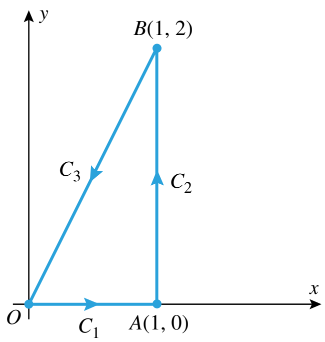


Figure 16.2.10

Example 7 Evaluate

$$\int_C x^2y\,dx + x\,dy$$

in a counterclockwise direction around the triangular path shown in Figure 16.2.10.

Solution. We will integrate over C_1, C_2, and C_3 separately and add the results. For each of the three integrals we must find parametric equations that trace the path of integration in the correct direction. For this purpose recall from Formula (7) of Section 13.1 that the graph of the vector-valued function

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 \quad (0 \le t \le 1)$$

is the line segment joining $\mathbf{r}_0$ and $\mathbf{r}_1$, oriented in the direction from $\mathbf{r}_0$ to $\mathbf{r}_1$. Thus, the line segments C_1, C_2, and C_3 can be represented in vector notation as

$$C_1\colon \mathbf{r}(t) = (1-t)\langle 0,0\rangle + t\langle 1,0\rangle = \langle t,0\rangle$$
$$C_2\colon \mathbf{r}(t) = (1-t)\langle 1,0\rangle + t\langle 1,2\rangle = \langle 1,2t\rangle$$
$$C_3\colon \mathbf{r}(t) = (1-t)\langle 1,2\rangle + t\langle 0,0\rangle = \langle 1-t,2-2t\rangle$$

where t varies from 0 to 1 in each case. From these equations and Example 6 we obtain

$$\int_{C_1} x^2y\,dx + x\,dy = \int_{C_1} x^2y\,dx = \int_0^1 (t^2)(0)\frac{d}{dt}[t]\,dt = 0$$
$$\int_{C_2} x^2y\,dx + x\,dy = \int_{C_2} x\,dy = \int_0^1 (1)\frac{d}{dt}[2t]\,dt = 2$$
$$\int_{C_3} x^2y\,dx + x\,dy = \int_0^1 (1-t)^2(2-2t)\frac{d}{dt}[1-t]\,dt + \int_0^1 (1-t)\frac{d}{dt}[2-2t]\,dt$$
$$= 2\int_0^1 (t-1)^3\,dt + 2\int_0^1 (t-1)\,dt = -\tfrac{1}{2} - 1 = -\tfrac{3}{2}$$

Thus,

$$\int_C x^2y\,dx + x\,dy = 0 + 2 + \left(-\tfrac{3}{2}\right) = \tfrac{1}{2}$$ ◀

CHANGE OF PARAMETER IN LINE INTEGRALS

Although parametric equations of a curve are used to *evaluate* line integrals along that curve, the line integrals themselves are *defined* without reference to a parametrization. It follows that the value of the line integral should be independent of any (oriented) parametrization of the curve. This is the content of following theorem, which we state without formal proof.

> **16.2.3** THEOREM (***Independence of Parametrization***). *The value of a line integral along a curve C does not depend on the parametrization of C in the sense that any two parametrizations of C with the same orientation produce the same value for the line integral.*

REMARK. This is an extremely important theorem because it allows us to choose any convenient parametrization for the path of integration without concern that the choice will affect the value of the integral. Indeed, we have tacitly used this result in all of the examples in this section where we chose the parametric equations for C.

REVERSING THE DIRECTION OF INTEGRATION

Suppose that C is a parametric curve that begins at point A and ends at point B when traced in the direction of increasing parameter. If the curve C is reparametrized so that it is traced from B to A as the parameter increases, then we denote the reparametrized curve by $-C$. Thus, C and $-C$ consist of the same points but have opposite orientations (Figure 16.2.11).

When the orientation of C is reversed, the signs of Δx_k, Δy_k, and Δz_k in (17) to (21) are reversed, so the effect is to reverse the signs of the line integrals with respect to x, y, and

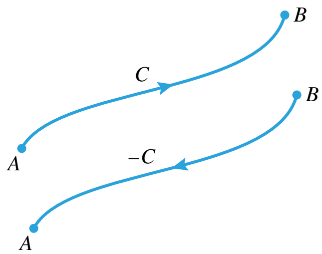


Figure 16.2.11

z. However, reversing the orientation of C has no effect on a line integral with respect to s because the quantity Δs_k in (2) and (8) denotes an arc length, which is positive regardless of the orientation. Thus, we have the following result, which we state without formal proof.

> **16.2.4 THEOREM (*Reversal of Orientation*).** *If C is a smooth parametric curve, then a smooth change of parameter that reverses the orientation of C changes the sign of a line integral along C with respect to x, y, or z, but leaves the value of a line integral along C with respect to arc length unchanged.*

It follows from this theorem that

$$\int_{-C} f(x, y)\, dx + g(x, y)\, dy = -\int_{C} f(x, y)\, dx + g(x, y)\, dy \tag{29}$$

$$\int_{-C} f(x, y)\, ds = \int_{C} f(x, y)\, ds \tag{30}$$

and similarly for line integrals in 3-space.

WORK AS A LINE INTEGRAL

In Section 6.6 we first defined the work W performed by a force of constant magnitude acting on an object in the direction of motion (Definition 6.6.1), and later in that section we extended the definition to allow for a force of variable magnitude acting in the direction of motion (Definition 6.6.3). In Section 12.3 we took the concept of work a step further by defining the work W performed by a constant force $\mathbf{F}$ acting at a fixed angle to the displacement vector $\overrightarrow{PQ}$ to be

$$W = \mathbf{F} \cdot \overrightarrow{PQ} \tag{31}$$

[Formula (14) of Section 12.3]. Our next goal is to define a more general concept of work—the work performed by a variable force acting on a particle that moves along a curved path in 2-space or 3-space.

In many applications variable forces arise from force fields (gravitational fields, electromagnetic fields, and so forth), so we will consider the problem of work in that context. More precisely, let us assume that a particle moves along a smooth parametric curve C through a continuous force field $\mathbf{F}(x, y)$ in 2-space or $\mathbf{F}(x, y, z)$ in 3-space. We will call the work done by $\mathbf{F}$ the ***work performed by the force field***. To motivate an appropriate definition for the work performed by the force field, we will use a limit process, and since the procedure is the same in 2-space and 3-space, we will discuss it for 3-space only. The idea is as follows:

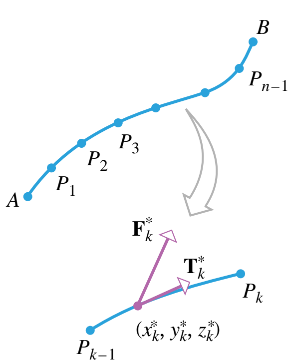


Figure 16.2.12

- Assume that the particle moves along C from a point A to a point B as the parameter increases, and divide C into n arcs by inserting a succession of distinct points $P_1, P_2, \ldots, P_{n-1}$ between A and B in the direction of increasing parameter. Denote the length of the kth arc by Δs_k. Let (x_k^*, y_k^*, z_k^*) be any point on the kth arc, and let $\mathbf{T}_k^* = \mathbf{T}(x_k^*, y_k^*, z_k^*)$ be the unit tangent vector and $\mathbf{F}_k^* = \mathbf{F}(x_k^*, y_k^*, z_k^*)$ the force vector at this point (Figure 16.2.12).
- If the kth arc is small, then the force will not vary much, so we can approximate the force by the constant value $\mathbf{F}_k^*$ on this arc. Moreover, the direction of motion will not vary much over the small arc, so we can assume that the particle moves in the direction of $\mathbf{T}_k^*$ for a distance of Δs_k; that is, the particle has a linear displacement $\Delta s_k \mathbf{T}_k^*$. Thus, it follows from (31) that the work ΔW_k performed by the vector field along the kth arc can be approximated as

  $$\Delta W_k \approx \mathbf{F}_k^* \cdot (\Delta s_k \mathbf{T}_k^*) = (\mathbf{F}_k^* \cdot \mathbf{T}_k^*)\Delta s_k$$

  and the total work W performed by the vector field as the particle moves along C from

A to B can be approximated as

$$W \approx \sum_{k=1}^{n} (\mathbf{F}_k^* \cdot \mathbf{T}_k^*)\Delta s_k$$

- If we now increase n so that the length of each arc approaches zero, then it is plausible that the error in the approximations approaches zero, and the exact work performed by the vector field is

$$W = \lim_{n\to+\infty} \sum_{k=1}^{n} (\mathbf{F}_k^* \cdot \mathbf{T}_k^*)\Delta s_k = \int_C \mathbf{F}(x, y, z) \cdot \mathbf{T}(x, y, z)\, ds$$

Thus, we are led to the following definition:

16.2.5 DEFINITION. If $\mathbf{F}$ is a continuous vector field and C is a smooth parametric curve in 2-space or 3-space with unit tangent vector $\mathbf{T}$, then the ***work performed by the vector field*** on a particle that moves along C in the direction of increasing parameter is

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds \tag{32}$$

REMARK. In words, this definition states that *the work performed by a vector field on a particle moving along a parametric curve C is obtained by integrating the scalar tangential component of force along C.*

A METHOD FOR CALCULATING WORK

Although Formula (32) can be used to calculate work, it is not usually the best choice. A more useful formula can be obtained by using Formula (6) of Section 13.4 to express $\mathbf{T}$ as

$$\mathbf{T} = \frac{d\mathbf{r}}{ds}$$

This suggests that (32) can be expressed as

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} \tag{33}$$

in which $d\mathbf{r}$ is interpreted as

$$d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} \quad \text{or} \quad d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k} \tag{34}$$

depending on whether C is in 2-space or 3-space.

Example 8 Find the work done by the force field

$$\mathbf{F}(x, y) = x^3 y\,\mathbf{i} + (x - y)\,\mathbf{j}$$

on a particle that moves along the parabola $y = x^2$ from $(-2, 4)$ to $(1, 1)$ (see Figure 16.2.13).

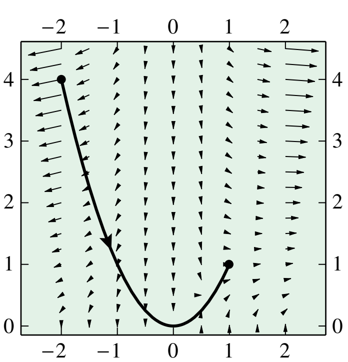


Figure 16.2.13

Solution. If we use $x = t$ as the parameter, the path C of the particle can be expressed parametrically as

$$x = t, \quad y = t^2 \qquad (-2 \le t \le 1)$$

or in vector notation as

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} \quad (-2 \le t \le 1)$$

Thus, from (33) the work W done by $\mathbf{F}$ is

$$\begin{aligned} W &= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (x^3y\mathbf{i} + (x-y)\mathbf{j})\cdot(dx\mathbf{i} + dy\mathbf{j}) \\ &= \int_C x^3y\,dx + (x-y)\,dy = \int_{-2}^{1} (t^5 + (t-t^2)(2t))\,dt \\ &= \frac{1}{6}t^6 + \frac{2}{3}t^3 - \frac{1}{2}t^4\bigg]_{-2}^{1} = 3 \end{aligned}$$

where the units for W depend on the units chosen for force and distance. ◀

REMARK. In light of Theorem 16.2.4, you might expect that reversing the orientation of C in Formula (32) would have no effect on the work W performed by the vector field. However, reversing the orientation of C reverses the orientation of $\mathbf{T}$ in the integrand and hence reverses the sign of the integral; that is,

$$\int_{-C} \mathbf{F}\cdot\mathbf{T}\,ds = -\int_C \mathbf{F}\cdot\mathbf{T}\,ds \tag{35}$$

$$\int_{-C} \mathbf{F}\cdot d\mathbf{r} = -\int_C \mathbf{F}\cdot d\mathbf{r} \tag{36}$$

Thus, in Example 8 the work performed on a particle that moves along the given parabola from (1, 1) to (−2, 4) is −3, and the work performed on a particle that moves along the parabola from (−2, 4) to (1, 1) and then back along the parabola to (−2, 4) is zero.

WORK EXPRESSED IN SCALAR FORM

We conclude this section by noting that it is sometimes useful to express Formula (33) in scalar form. For example, if $\mathbf{F} = \mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$ is a vector field in 2-space, then

$$\mathbf{F}\cdot d\mathbf{r} = f(x, y)\,dx + g(x, y)\,dy$$

so (33) can be expressed as

$$W = \int_C f(x, y)\,dx + g(x, y)\,dy \tag{37}$$

and similarly in 3-space as

$$W = \int_C f(x, y, z)\,dx + g(x, y, z)\,dy + h(x, y, z)\,dz \tag{38}$$

EXERCISE SET 16.2 C CAS

1. Let C be the line segment from (0, 0) to (0, 1). In each part, evaluate the line integral along C by inspection, and explain your reasoning.

(a) $\int_C ds$ (b) $\int_C \sin xy\,dy$

2. Let C be the line segment from (0, 2) to (0, 4). In each part, evaluate the line integral along C by inspection, and explain your reasoning.

(a) $\int_C ds$ (b) $\int_C e^{xy}\,dx$

3. Let C be the curve represented by the equations

$$x = 2t, \quad y = 3t^2 \qquad (0 \le t \le 1)$$

In each part, evaluate the line integral along C.

(a) $\displaystyle\int_C (x - y)\, ds$ (b) $\displaystyle\int_C (x - y)\, dx$

(c) $\displaystyle\int_C (x - y)\, dy$

4. Let C be the curve represented by the equations

$$x = t, \quad y = 3t^2, \quad z = 6t^3 \qquad (0 \le t \le 1)$$

In each part, evaluate the line integral along C.

(a) $\displaystyle\int_C xyz^2\, ds$ (b) $\displaystyle\int_C xyz^2\, dx$

(c) $\displaystyle\int_C xyz^2\, dy$ (d) $\displaystyle\int_C xyz^2\, dz$

5. In each part, evaluate the integral

$$\int_C (3x + 2y)\, dx + (2x - y)\, dy$$

along the stated curve.

(a) The line segment from (0, 0) to (1, 1).
(b) The parabolic arc $y = x^2$ from (0, 0) to (1, 1).
(c) The curve $y = \sin(\pi x/2)$ from (0, 0) to (1, 1).
(d) The curve $x = y^3$ from (0, 0) to (1, 1).

6. In each part, evaluate the integral

$$\int y\, dx + z\, dy - x\, dz$$

along the stated curve.

(a) The line segment from (0, 0, 0) to (1, 1, 1).
(b) The twisted cubic $x = t$, $y = t^2$, $z = t^3$ from (0, 0, 0) to (1, 1, 1).
(c) The helix $x = \cos \pi t$, $y = \sin \pi t$, $z = t$ from (1, 0, 0) to (−1, 0, 1).

In Exercises 7–10, evaluate the line integral with respect to s along the parametric curve C.

7. $\displaystyle\int_C \frac{1}{1 + x}\, ds$
C: $x = t,\ y = \frac{2}{3}t^{3/2} \quad (0 \le t \le 3)$

8. $\displaystyle\int_C \frac{x}{1 + y^2}\, ds$
C: $x = 1 + 2t,\ y = t \quad (0 \le t \le 1)$

9. $\displaystyle\int_C 3x^2yz\, ds$
C: $x = t,\ y = t^2,\ z = \frac{2}{3}t^3 \quad (0 \le t \le 1)$

10. $\displaystyle\int_C \frac{e^{-z}}{x^2 + y^2}\, ds$
C: $x = 2\cos t,\ y = 2\sin t,\ z = t \quad (0 \le t \le 2\pi)$

In Exercises 11–18, evaluate the line integral along the parametric curve C.

11. $\displaystyle\int_C (x + 2y)\, dx + (x - y)\, dy$
C: $x = 2\cos t,\ y = 4\sin t \quad (0 \le t \le \pi/4)$

12. $\displaystyle\int_C (x^2 - y^2)\, dx + x\, dy$
C: $x = t^{2/3},\ y = t \quad (-1 \le t \le 1)$

13. $\displaystyle\int_C -y\, dx + x\, dy$
C: $y^2 = 3x$ from (3, 3) to (0, 0)

14. $\displaystyle\int_C (y - x)\, dx + x^2y\, dy$
C: $y^2 = x^3$ from (1, −1) to (1, 1)

15. $\displaystyle\int_C (x^2 + y^2)\, dx - x\, dy$
C: $x^2 + y^2 = 1$, counterclockwise from (1, 0) to (0, 1)

16. $\displaystyle\int_C (y - x)\, dx + xy\, dy$
C: the line segment from (3, 4) to (2, 1)

17. $\displaystyle\int_C yz\, dx - xz\, dy + xy\, dz$
C: $x = e^t,\ y = e^{3t},\ z = e^{-t} \quad (0 \le t \le 1)$

18. $\displaystyle\int_C x^2\, dx + xy\, dy + z^2\, dz$
C: $x = \sin t,\ y = \cos t,\ z = t^2 \quad (0 \le t \le \pi/2)$

In Exercises 19 and 20, use a CAS to evaluate the line integrals along the given parametric curves.

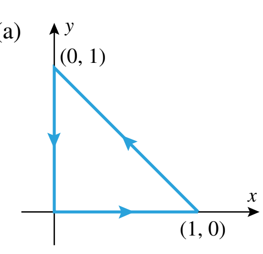
C **19.** (a) $\displaystyle\int_C (x^3 + y^3)\, ds$
C: $x = e^t,\ y = e^{-t} \quad (0 \le t \le \ln 2)$

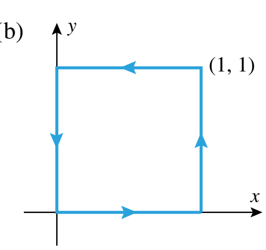
(b) $\displaystyle\int_C xe^z\, dx + (x - z)\, dy + (x^2 + y^2 + z^2)\, dz$
C: $x = \sin t,\ y = \cos t \quad (0 \le t \le \pi/2)$

C **20.** (a) $\displaystyle\int_C x^7y^3\, ds$
C: $x = \cos^3 t,\ y = \sin^3 t \quad (0 \le t \le \pi/2)$

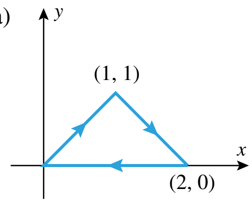
(b) $\displaystyle\int_C x^5z\, dx + 7y\, dy + y^2z\, dz$
C: $x = t,\ y = t^2,\ z = \ln t \quad (1 \le t \le e)$

In Exercises 21 and 22, evaluate $\int_C y\, dx - x\, dy$ along the curve C shown in the figure.

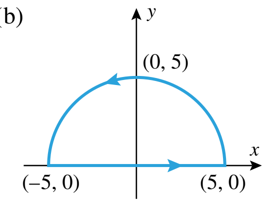
21. (a)


(b)


22. (a)

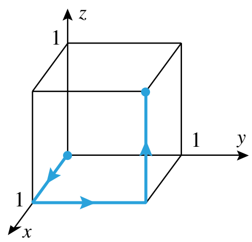


(b)

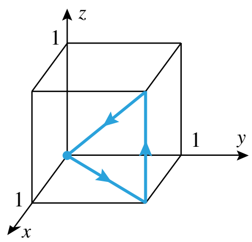


In Exercises 23 and 24, evaluate $\int_C x^2 z\,dx - yx^2\,dy + 3\,dz$ along the curve C shown in the figure.

23.



24.



In Exercises 25–28, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve C.

25. $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$
C: $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} \quad (0 \le t \le \pi)$

26. $\mathbf{F}(x, y) = x^2 y\mathbf{i} + 4\mathbf{j}$
C: $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} \quad (0 \le t \le 1)$

27. $\mathbf{F}(x, y) = (x^2 + y^2)^{-3/2}(x\mathbf{i} + y\mathbf{j})$
C: $\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j} \quad (0 \le t \le 1)$

28. $\mathbf{F}(x, y, z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$
C: $\mathbf{r}(t) = \sin t\,\mathbf{i} + 3\sin t\,\mathbf{j} + \sin^2 t\,\mathbf{k} \quad (0 \le t \le \pi/2)$

29. Find the mass of a thin wire shaped in the form of the circular arc $y = \sqrt{9 - x^2}$ $(0 \le x \le 3)$ if the density function is $\delta(x, y) = x\sqrt{y}$.

30. Find the mass of a thin wire shaped in the form of the curve $x = e^t \cos t$, $y = e^t \sin t$ $(0 \le t \le 1)$ if the density function δ is proportional to the distance from the origin.

31. Find the mass of a thin wire shaped in the form of the helix $x = 3\cos t$, $y = 3\sin t$, $z = 4t$ $(0 \le t \le \pi/2)$ if the density function is $\delta = kx/(1 + y^2)$ $(k > 0)$.

32. Find the mass of a thin wire shaped in the form of the curve $x = 2t$, $y = \ln t$, $z = 4\sqrt{t}$ $(1 \le t \le 4)$ if the density function is proportional to the distance above the xy-plane.

In Exercises 33–36, find the work done by the force field $\mathbf{F}$ on a particle that moves along the curve C.

33. $\mathbf{F}(x, y) = xy\mathbf{i} + x^2\mathbf{j}$
C: $x = y^2$ from $(0, 0)$ to $(1, 1)$

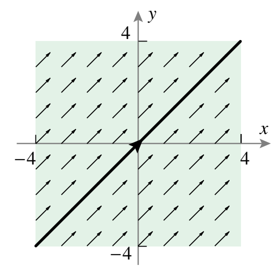
34. $\mathbf{F}(x, y) = (x^2 + xy)\mathbf{i} + (y - x^2 y)\mathbf{j}$
C: $x = t$, $y = 1/t \quad (1 \le t \le 3)$

35. $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$
C: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \quad (0 \le t \le 1)$

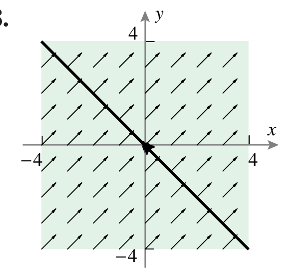
36. $\mathbf{F}(x, y, z) = (x + y)\mathbf{i} + xy\mathbf{j} - z^2\mathbf{k}$
C: along line segments from $(0, 0, 0)$ to $(1, 3, 1)$ to $(2, -1, 4)$

In Exercises 37 and 38, find $\int_C \mathbf{F} \cdot d\mathbf{r}$ by inspection for the force field $\mathbf{F}(x, y) = \mathbf{i} + \mathbf{j}$ and the curve C shown in the figure. Explain your reasoning. [For clarity, the vectors in the force field are shown at less than true scale.]

37.



38.

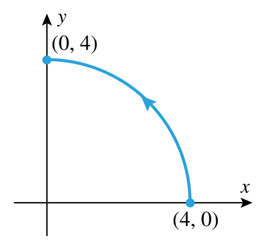


In Exercises 39 and 40, find the work done by the force field

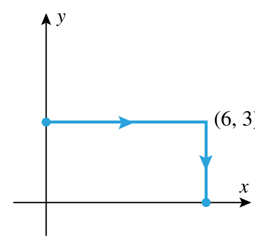
$$\mathbf{F}(x, y) = \frac{1}{x^2 + y^2}\mathbf{i} + \frac{4}{x^2 + y^2}\mathbf{j}$$

on a particle that moves along the curve C shown in the figure.

39.



40.



In Exercises 41 and 42, use a line integral to find the area of the surface.

41. The surface that extends upward from the parabola $y = x^2$ $(0 \le x \le 2)$ in the xy-plane to the plane $z = 3x$.

42. The surface that extends upward from the semicircle $y = \sqrt{4 - x^2}$ in the xy-plane to the surface $z = x^2 y$.

43. As illustrated in the accompanying figure, a sinusoidal cut is made in the top of a cylindrical tin can. Suppose that the base is modeled by the parametric equations $x = \cos t$, $y = \sin t$, $z = 0$ $(0 \le t \le 2\pi)$, and the height of the cut as a function of t is $z = 2 + 0.5\sin 3t$.
(a) Use a geometric argument to find the lateral surface area of the cut can.
(b) Write down a line integral for the surface area.
(c) Use the line integral to calculate the surface area.

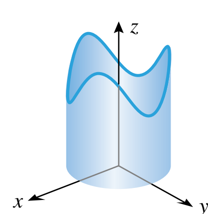


Figure Ex-43

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44. Evaluate the integral $\displaystyle\int_{-C} \frac{x\,dy - y\,dx}{x^2 + y^2}$, where C is the circle $x^2 + y^2 = a^2$ traversed counterclockwise.

45. Suppose that a particle moves through the force field $\mathbf{F}(x, y) = xy\mathbf{i} + (x - y)\mathbf{j}$ from the point $(0, 0)$ to the point $(1, 0)$ along the curve $x = t$, $y = \lambda t(1 - t)$. For what value of λ will the work done by the force field be 1?

46. A farmer weighing 150 lb carries a sack of grain weighing 20 lb up a circular helical staircase around a silo of radius 25 ft. As the farmer climbs, grain leaks from the sack at a rate of 1 lb per 10 ft of ascent. How much work is performed by the farmer in climbing through a vertical distance of 60 ft in exactly four revolutions? [*Hint:* Find a vector field that represents the force exerted by the farmer in lifting his own weight plus the weight of the sack upward at each point along his path.]

16.3 INDEPENDENCE OF PATH; CONSERVATIVE VECTOR FIELDS

In this section we will study properties of vector fields that relate to the work they perform on particles moving along various curves. In particular, we will show that for certain kinds of vector fields the work that the field performs on a particle moving along a curve depends only on the endpoints of the curve and not on the curve itself. Such vector fields are of special importance in physics and engineering.

WORK INTEGRALS

We saw in the last section that if $\mathbf{F}$ is a vector field in 2-space or 3-space, then the work performed by the field on a particle moving along a parametric curve C from an initial point A to a final point B is given by the integral

$$\int_C \mathbf{F} \cdot \mathbf{T}\,ds \quad \text{or, equivalently,} \quad \int_C \mathbf{F} \cdot d\mathbf{r}$$

Accordingly, we call an integral of this type a ***work integral***. At the end of the last section we noted that a work integral can be expressed in scalar form as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C f(x, y)\,dx + g(x, y)\,dy \qquad \boxed{\text{2-space}} \tag{1}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C f(x, y, z)\,dx + g(x, y, z)\,dy + h(x, y, z)\,dz \qquad \boxed{\text{3-space}} \tag{2}$$

where f, g, and h are the component functions of $\mathbf{F}$.

INDEPENDENCE OF PATH

The parametric curve C in a work integral is called the ***path of integration***. One of the important problems in applications is to determine how the path of integration affects the work performed by a vector field on a particle that moves from a fixed point P to a fixed point Q. We will show shortly that if the vector field $\mathbf{F}$ is conservative (i.e., is the gradient of some potential function ϕ), then the work that the field performs on a particle that moves from P to Q does not depend on the particular path C that the particle follows. This is illustrated in the following example.

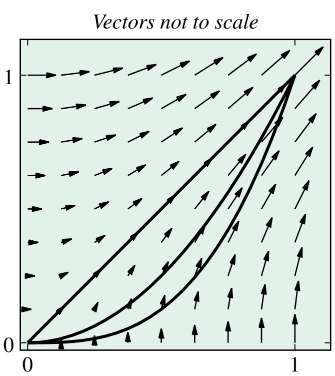


Figure 16.3.1

Example 1 The vector field $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$ is conservative since it is the gradient of $\phi(x, y) = xy$ (verify). Thus, the preceding discussion suggests that the work performed by the field on a particle that moves from the point $(0, 0)$ to the point $(1, 1)$ should be the same along different paths. Confirm that the value of the work integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

is the same along the following paths (Figure 16.3.1):

(a) The line segment $y = x$ from $(0, 0)$ to $(1, 1)$.

(b) The parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$.

(c) The cubic $y = x^3$ from $(0, 0)$ to $(1, 1)$.

Solution (a). With $x = t$ as the parameter, the path of integration is given by

$$x = t, \quad y = t \qquad (0 \le t \le 1)$$

Thus,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (y\mathbf{i} + x\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j}) = \int_C y\,dx + x\,dy$$
$$= \int_0^1 2t\,dt = 1$$

Solution (b). With $x = t$ as the parameter, the path of integration is given by

$$x = t, \quad y = t^2 \qquad (0 \le t \le 1)$$

Thus,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C y\,dx + x\,dy = \int_0^1 3t^2\,dt = 1$$

Solution (c). With $x = t$ as the parameter, the path of integration is given by

$$x = t, \quad y = t^3 \qquad (0 \le t \le 1)$$

Thus,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C y\,dx + x\,dy = \int_0^1 4t^3\,dt = 1$$

◀

THE FUNDAMENTAL THEOREM OF WORK INTEGRALS

Recall from the Fundamental Theorem of Calculus (Theorem 5.6.1) that if F is an antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The following result is the analog of that theorem for work integrals in 2-space.

16.3.1 THEOREM (*The Fundamental Theorem of Work Integrals*). *Suppose that*

$$\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$$

is a conservative vector field in some open region D containing the points (x_0, y_0) and (x_1, y_1) and that $f(x, y)$ and $g(x, y)$ are continuous in this region. If

$$\mathbf{F}(x, y) = \nabla\phi(x, y)$$

and if C is any piecewise smooth parametric curve that starts at (x_0, y_0), ends at (x_1, y_1), and lies in the region D, then

$$\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = \phi(x_1, y_1) - \phi(x_0, y_0) \tag{3}$$

or, equivalently,

$$\int_C \nabla\phi \cdot d\mathbf{r} = \phi(x_1, y_1) - \phi(x_0, y_0) \tag{4}$$

Proof. We will give the proof for a smooth curve C. The proof for a piecewise smooth curve, which is left as an exercise, can be obtained by applying the theorem to each individual smooth piece and adding the results. Suppose that C is given parametrically by $x = x(t)$, $y = y(t)$ $(a \le t \le b)$, so that the initial and final points of the curve are

$$(x_0, y_0) = (x(a), y(a)) \quad \text{and} \quad (x_1, y_1) = (x(b), y(b))$$

Since $\mathbf{F}(x, y) = \nabla\phi$, it follows that

$$\mathbf{F}(x, y) = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j}$$

so

$$\begin{aligned}
\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} &= \int_C \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy = \int_a^b \left[\frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt}\right] dt \\
&= \int_a^b \frac{d}{dt}[\phi(x(t), y(t))]\,dt = \phi(x(t), y(t))\Big]_{t=a}^{b} \\
&= \phi(x(b), y(b)) - \phi(x(a), y(a)) \\
&= \phi(x_1, y_1) - \phi(x_0, y_0)
\end{aligned}$$

Stated informally, this theorem shows that *the value of a work integral along a piecewise smooth path in a conservative vector field is* ***independent of the path***; that is, the value of the integral depends on the endpoints and not on the actual path C. Accordingly, for work integrals along paths in conservative vector fields, it is common to express (3) and (4) as

$$\int_{(x_0, y_0)}^{(x_1, y_1)} \mathbf{F} \cdot d\mathbf{r} = \int_{(x_0, y_0)}^{(x_1, y_1)} \nabla\phi \cdot d\mathbf{r} = \phi(x_1, y_1) - \phi(x_0, y_0) \tag{5}$$

Example 2

(a) Confirm that the vector field $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$ in Example 1 is conservative by showing that $\mathbf{F}(x, y)$ is the gradient of $\phi(x, y) = xy$.

(b) Use the Fundamental Theorem of Work Integrals to evaluate $\displaystyle\int_{(0,0)}^{(1,1)} \mathbf{F} \cdot d\mathbf{r}$.

Solution (a).

$$\nabla\phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} = y\mathbf{i} + x\mathbf{j}$$

Solution (b). From (5) we obtain

$$\int_{(0,0)}^{(1,1)} \mathbf{F} \cdot d\mathbf{r} = \phi(1, 1) - \phi(0, 0) = 1 - 0 = 1$$

which agrees with the results obtained in Example 1 by integrating from (0, 0) to (1, 1) along specific paths. ◀

REMARK. You can visualize the result in this example geometrically from the picture of the vector field shown in Figure 16.3.1 and the relationship

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\,ds$$

We see from this that the more closely the unit tangent vector $\mathbf{T}$ to C aligns with $\mathbf{F}$ along C, the greater the integrand and hence the greater the value of the integral. However, the length of the curve C also affects the value of the integral. Thus, in comparing the three curves in Figure 16.3.1, we see that the alignment of $\mathbf{T}$ with $\mathbf{F}$ is best for the line, but the line has the shortest length. The alignments are not as good for $y = x^2$ and $y = x^3$, but they have greater lengths to compensate. Thus, it seems plausible that the integrals have the same value.

WORK INTEGRALS ALONG CLOSED PATHS

Parametric curves that begin and end at the same point play an important role in the study of vector fields, so there is some special terminology associated with them. A parametric curve C that is represented by the vector-valued function $\mathbf{r}(t)$ for $a \leq t \leq b$ is said to be ***closed*** if the initial point $\mathbf{r}(a)$ and the terminal point $\mathbf{r}(b)$ coincide; that is, $\mathbf{r}(a) = \mathbf{r}(b)$ (Figure 16.3.2).

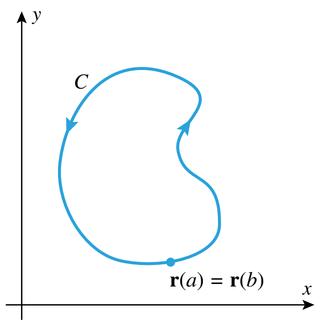


Figure 16.3.2

It follows from (5) that if a particle moving in a conservative vector field traverses a closed path C that begins and ends at (x_0, y_0), then the work performed by the field is zero. This is because the point (x_1, y_1) in (5) is the same as (x_0, y_0) and hence

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(x_1, y_1) - \phi(x_0, y_0) = 0$$

Our next objective is to show that the converse of this result is also true. That is, we want to show that under appropriate conditions a vector field in which the work is zero along *all* closed paths must be conservative. For this to be true we will need to require that the domain D of the vector field be ***connected***, by which we mean that any two points in D can be joined by some piecewise smooth curve that lies entirely in D. Stated informally, D is connected if it does not consist of two or more separate pieces (Figure 16.3.3).

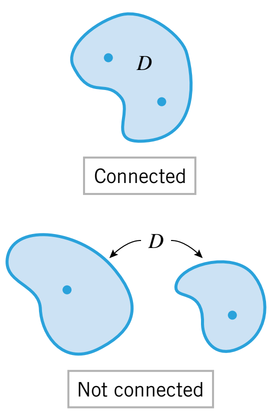


Figure 16.3.3

16.3.2 THEOREM. *If $f(x, y)$ and $g(x, y)$ are continuous on some open connected region D, then the following statements are equivalent (all true or all false):*

(*a*) $\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$ *is a conservative vector field on the region D.*

(*b*) $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ *for every piecewise smooth closed curve C in D.*

(*c*) $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ *is independent of the path from any point P in D to any point Q in D for every piecewise smooth curve C in D.*

This theorem can be established by proving three implications: $(a) \Rightarrow (b)$, $(b) \Rightarrow (c)$, and $(c) \Rightarrow (a)$. Since we showed above that $(a) \Rightarrow (b)$, we need only prove the last two implications. We will prove $(c) \Rightarrow (a)$ and leave the other implication as an exercise.

Proof. $(c) \Rightarrow (a)$. We are assuming that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the path for every piecewise smooth curve C in the region, and we want to show that there is a function $\phi = \phi(x, y)$ such that $\nabla\phi = \mathbf{F}(x, y)$ at each point of the region; that is,

$$\frac{\partial \phi}{\partial x} = f(x, y) \quad \text{and} \quad \frac{\partial \phi}{\partial y} = g(x, y) \tag{6}$$

Now choose a fixed point (a, b) in D, let (x, y) be any point in D, and define

$$\phi(x, y) = \int_{(a,b)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r} \tag{7}$$

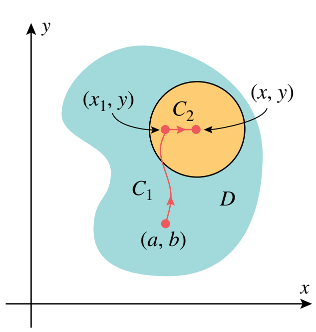


Figure 16.3.4

This is an unambiguous definition because we have assumed that the integral is independent of the path. We will show that $\nabla\phi = \mathbf{F}$. Since D is open, we can find a circular disk centered at (x, y) whose points lie entirely in D. As shown in Figure 16.3.4, choose any point (x_1, y) in this disk that lies on the same horizontal line as (x, y) such that $x_1 < x$. Because the integral in (7) is independent of path, we can evaluate it by first integrating from (a, b) to (x_1, y) along an arbitrary piecewise smooth curve C_1 in D, and then continuing along the horizontal line segment C_2 from (x_1, y) to (x, y). This yields

$$\phi(x, y) = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{(a,b)}^{(x_1,y)} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

Since the first term does not depend on x, its partial derivative with respect to x is zero and hence

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \frac{\partial}{\partial x}\int_{C_2} f(x,y)\,dx + g(x,y)\,dy$$

However, the line integral with respect to y is zero along the horizontal line segment C_2, so this equation simplifies to

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\int_{C_2} f(x,y)\,dx \tag{8}$$

To evaluate the integral in this expression, we treat y as a constant and express the line C_2 parametrically as

$$x = t, \quad y = y \qquad (x_1 \le t \le x)$$

At the risk of confusion, but to avoid complicating the notation, we have used x both as the dependent variable in the parametric equations and as the endpoint of the line segment. With the latter interpretation of x, it follows that (8) can be expressed as

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\int_{x_1}^{x} f(t,y)\,dt$$

Now we apply Part 2 of the Fundamental Theorem of Calculus (Theorem 5.6.3), treating y as constant. This yields

$$\frac{\partial \phi}{\partial x} = f(x,y)$$

which proves the first part of (6). The proof that $\partial\phi/\partial y = g(x,y)$ can be obtained in a similar manner by joining (x,y) to a point (x,y_1) with a vertical line segment (Exercise 33). ∎

A TEST FOR CONSERVATIVE VECTOR FIELDS

Although Theorem 16.3.2 is an important characterization of conservative vector fields, it is not an effective computational tool because it is usually not possible to evaluate the work integral over all possible piecewise smooth curves in D, as required in parts (b) and (c). To develop a method for determining whether a vector field is conservative, we will need to introduce some new concepts about parametric curves and connected sets. We will say that a parametric curve is ***simple*** if it does not intersect itself between its endpoints. A simple parametric curve may or may not be closed (Figure 16.3.5). In addition, we will say that a connected set D in 2-space is ***simply connected*** if no simple closed curve in D encloses points that are not in D. Stated informally, a connected set D is simply connected if it has no holes; a connected set with one or more holes is said to be ***multiply connected*** (Figure 16.3.6).

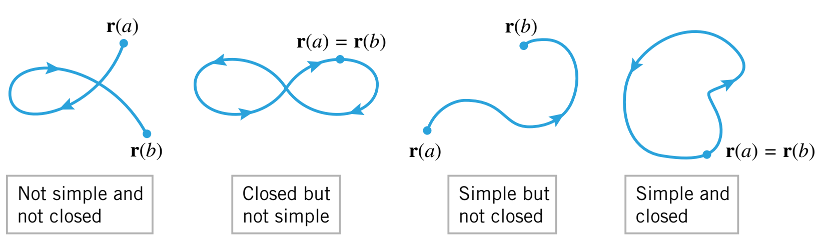


Figure 16.3.5

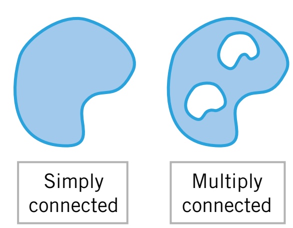


Figure 16.3.6

The following theorem is the primary tool for determining whether a vector field in 2-space is conservative.

16.3.3 THEOREM (*Conservative Field Test*). *If $f(x,y)$ and $g(x,y)$ are continuous and have continuous first partial derivatives on some open region D, and if $\mathbf{F}(x,y) = f(x,y)\mathbf{i} + g(x,y)\mathbf{j}$ is a conservative vector field on D, then*

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \tag{9}$$

at each point in D. Conversely, if D is simply connected and (9) *holds at each point in D, then $\mathbf{F}(x,y) = f(x,y)\mathbf{i} + g(x,y)\mathbf{j}$ is conservative.*

A complete proof of this theorem requires results from advanced calculus and will be omitted. However, it is not hard to see why (9) must hold if $\mathbf{F}$ is conservative. For this purpose suppose that $\mathbf{F} = \nabla\phi$, in which case we can express the functions f and g as

$$\frac{\partial \phi}{\partial x} = f \quad \text{and} \quad \frac{\partial \phi}{\partial y} = g \tag{10}$$

Thus,

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial y \partial x} \quad \text{and} \quad \frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial x \partial y}$$

But the mixed partial derivatives in these equations are equal (Theorem 14.3.2), so (9) follows.

WARNING. In (9), the $\mathbf{i}$-component of $\mathbf{F}$ is differentiated with respect to y and the $\mathbf{j}$-component with respect to x. It is easy to get this backwards by mistake.

Example 3 Use Theorem 16.3.3 to determine whether the vector field

$$\mathbf{F}(x, y) = (y + x)\mathbf{i} + (y - x)\mathbf{j}$$

is conservative on some open set.

Solution. Let $f(x, y) = y + x$ and $g(x, y) = y - x$. Then

$$\frac{\partial f}{\partial y} = 1 \quad \text{and} \quad \frac{\partial g}{\partial x} = -1$$

Thus, there are no points in the xy-plane at which condition (9) holds, and hence $\mathbf{F}$ is not conservative on any open set. ◀

REMARK. Since the vector field $\mathbf{F}$ in this example is not conservative, it follows from Theorem 16.3.2 that there must exist piecewise smooth closed curves in every open connected set in the xy-plane on which

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds \neq 0$$

One such curve is the circle shown in Figure 16.3.7. The figure suggests that $\mathbf{F} \cdot \mathbf{T} < 0$ at each point of C (why?), so $\int_C \mathbf{F} \cdot \mathbf{T}\, ds < 0$.

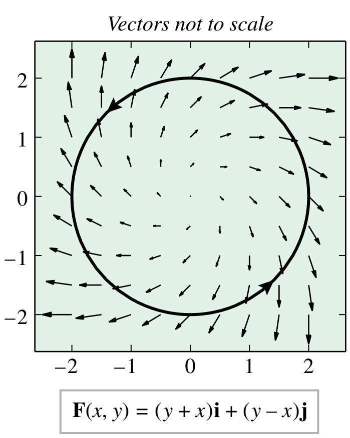


Figure 16.3.7

Once it is established that a vector field is conservative, a potential function for the field can be obtained by first integrating either of the equations in (10). This is illustrated in the following example.

Example 4 Let $\mathbf{F}(x, y) = 2xy^3\mathbf{i} + (1 + 3x^2y^2)\mathbf{j}$.

(a) Show that $\mathbf{F}$ is a conservative vector field on the entire xy-plane.

(b) Find ϕ by first integrating $\partial\phi/\partial x$.

(c) Find ϕ by first integrating $\partial\phi/\partial y$.

Solution (a). Since $f(x, y) = 2xy^3$ and $g(x, y) = 1 + 3x^2y^2$, we have

$$\frac{\partial f}{\partial y} = 6xy^2 = \frac{\partial g}{\partial x}$$

so (9) holds for all (x, y).

Solution (b). Since the field $\mathbf{F}$ is conservative, there is a potential function ϕ such that

$$\frac{\partial \phi}{\partial x} = 2xy^3 \quad \text{and} \quad \frac{\partial \phi}{\partial y} = 1 + 3x^2y^2 \tag{11}$$

Integrating the first of these equations with respect to x (and treating y as a constant) yields

$$\phi = \int 2xy^3\, dx = x^2y^3 + k(y) \tag{12}$$

where $k(y)$ represents the "constant" of integration. We are justified in treating the constant of integration as a function of y, since y is held constant in the integration process. To find $k(y)$ we differentiate (12) with respect to y and use the second equation in (11) to obtain

$$\frac{\partial\phi}{\partial y} = 3x^2y^2 + k'(y) = 1 + 3x^2y^2$$

from which it follows that $k'(y) = 1$. Thus,

$$k(y) = \int k'(y)\,dy = \int 1\,dy = y + K$$

where K is a (numerical) constant of integration. Substituting in (12) we obtain

$$\phi = x^2y^3 + y + K$$

The appearance of the arbitrary constant K tells us that ϕ is not unique. As a check on the computations, you may want to verify that $\nabla\phi = \mathbf{F}$.

Solution (c). Integrating the second equation in (11) with respect to y (and treating x as a constant) yields

$$\phi = \int (1 + 3x^2y^2)\,dy = y + x^2y^3 + k(x) \tag{13}$$

where $k(x)$ is the "constant" of integration. Differentiating (13) with respect to x and using the first equation in (11) yields

$$\frac{\partial\phi}{\partial x} = 2xy^3 + k'(x) = 2xy^3$$

from which it follows that $k'(x) = 0$ and consequently that $k(x) = K$, where K is a numerical constant of integration. Substituting this in (13) yields

$$\phi = y + x^2y^3 + K$$

which agrees with the solution in part (b). ◀

Example 5 Use the potential function obtained in Example 4 to evaluate the integral

$$\int_{(1,4)}^{(3,1)} 2xy^3\,dx + (1 + 3x^2y^2)\,dy$$

Solution. The integrand can be expressed as $\mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}$ is the vector field in Example 4. Thus, using Formula (3) and the potential function $\phi = y + x^2y^3 + K$ for $\mathbf{F}$, we obtain

$$\int_{(1,4)}^{(3,1)} 2xy^3\,dx + (1 + 3x^2y^2)\,dy = \int_{(1,4)}^{(3,1)} \mathbf{F}\cdot d\mathbf{r} = \phi(3,1) - \phi(1,4)$$
$$= (10 + K) - (68 + K) = -58$$ ◀

REMARK. Note that the constant K drops out. In future integration problems we may omit K from the computations.

Vectors not to scale

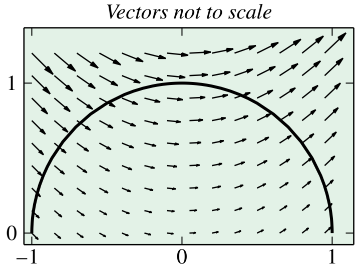


Figure 16.3.8

Example 6 Let $\mathbf{F}(x, y) = e^y\mathbf{i} + xe^y\mathbf{j}$.

(a) Verify that the vector field $\mathbf{F}$ is conservative on the entire xy-plane.

(b) Find the work done by the field on a particle that moves from $(1, 0)$ to $(-1, 0)$ along the semicircular path C shown in Figure 16.3.8.

Solution (a). For the given field we have $f(x, y) = e^y$ and $g(x, y) = xe^y$. Thus,

$$\frac{\partial}{\partial y}(e^y) = e^y = \frac{\partial}{\partial x}(xe^y)$$

so (9) holds for all (x, y) and hence $\mathbf{F}$ is conservative on the entire xy-plane.

Solution (b). From Formula (33) of Section 16.2, the work done by the field is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C e^y\, dx + xe^y\, dy \tag{14}$$

However, the calculations involved in integrating along C are tedious, so it is preferable to apply Theorem 16.3.1, taking advantage of the fact that the field is conservative and the integral is independent of path. Thus, we write (14) as

$$W = \int_{(1,0)}^{(-1,0)} e^y\, dx + xe^y\, dy = \phi(-1, 0) - \phi(1, 0) \tag{15}$$

As illustrated in Example 4, we can find ϕ by integrating either of the equations

$$\frac{\partial \phi}{\partial x} = e^y \quad \text{and} \quad \frac{\partial \phi}{\partial y} = xe^y \tag{16}$$

We will integrate the first. We obtain

$$\phi = \int e^y\, dx = xe^y + k(y) \tag{17}$$

Differentiating this equation with respect to y and using the second equation in (16) yields

$$\frac{\partial \phi}{\partial y} = xe^y + k'(y) = xe^y$$

from which it follows that $k'(y) = 0$ or $k(y) = K$. Thus, from (17)

$$\phi = xe^y + K$$

and hence from (15)

$$W = \phi(-1, 0) - \phi(1, 0) = (-1)e^0 - 1e^0 = -2$$

◀

CONSERVATIVE VECTOR FIELDS IN 3-SPACE

All of the results in this section have analogs in 3-space: Theorems 16.3.1 and 16.3.2 can be extended to vector fields in 3-space simply by adding a third variable and modifying the hypotheses appropriately. For example, in 3-space, Formula (3) becomes

$$\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = \phi(x_1, y_1, z_1) - \phi(x_0, y_0, z_0) \tag{18}$$

Theorem 16.3.3 can also be extended to vector fields in 3-space. We leave it for the exercises to show that if $\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$ is a conservative field, then

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y} \tag{19}$$

that is, curl $\mathbf{F} = \mathbf{0}$. Conversely, a vector field satisfying these conditions on a suitably restricted region is conservative on that region if f, g, and h are continuous and have continuous first partial derivatives in the region. Some problems involving Formulas (18) and (19) are given in the supplementary exercises at the end of this chapter.

CONSERVATION OF ENERGY

If $\mathbf{F}(x, y, z)$ is a conservative force field with a potential function $\phi(x, y, z)$, then we call $V(x, y, z) = -\phi(x, y, z)$ the ***potential energy*** of the field at the point (x, y, z). Thus, it follows from the 3-space version of Theorem 16.3.1 that the work W done by $\mathbf{F}$ on a particle that moves along any path C from a point (x_0, y_0, z_0) to a point (x_1, y_1, z_1) is related to the potential energy by the equation

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \phi(x_1, y_1, z_1) - \phi(x_0, y_0, z_0) = -[V(x_1, y_1, z_1) - V(x_0, y_0, z_0)] \tag{20}$$

That is, the work done by the field is the negative of the change in potential energy. In particular, it follows from the 3-space analog of Theorem 16.3.2 that if a particle traverses a piecewise smooth closed path in a conservative vector field, then the work done by the field is zero, and there is no change in potential energy. To take this a step further, suppose that a particle of mass m moves along any piecewise smooth curve (not necessarily closed) in a conservative vector field, starting at (x_0, y_0, z_0) with velocity v_i and ending at (x_1, y_1, z_1) with velocity v_f. If we let V_i denote the potential energy at the starting point and V_f the potential energy at the final point, then it follows from the work–energy relationship [Equation (5), Section 6.6] that

$$\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2 = -[V_f - V_i]$$

which we can rewrite as

$$\tfrac{1}{2}mv_f^2 + V_f = \tfrac{1}{2}mv_i^2 + V_i$$

This equation states that the total energy of the particle (kinetic energy + potential energy) does not change as the particle moves along a path in a conservative vector field. This result, called the ***conservation of energy principle***, explains the origin of the term "conservative vector field."

EXERCISE SET 16.3 C CAS

In Exercises 1–6, determine whether **F** is a conservative vector field. If so, find a potential function for it.

1. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$ **2.** $\mathbf{F}(x, y) = 3y^2\mathbf{i} + 6xy\mathbf{j}$

3. $\mathbf{F}(x, y) = x^2y\mathbf{i} + 5xy^2\mathbf{j}$

4. $\mathbf{F}(x, y) = e^x \cos y\mathbf{i} - e^x \sin y\mathbf{j}$

5. $\mathbf{F}(x, y) = (\cos y + y\cos x)\mathbf{i} + (\sin x - x\sin y)\mathbf{j}$

6. $\mathbf{F}(x, y) = x\ln y\mathbf{i} + y\ln x\mathbf{j}$

7. (a) Show that the line integral $\int_C y^2\,dx + 2xy\,dy$ is independent of the path.
(b) Evaluate the integral in part (a) along the line segment from $(-1, 2)$ to $(1, 3)$.
(c) Evaluate the integral $\int_{(-1,2)}^{(1,3)} y^2\,dx + 2xy\,dy$ using Theorem 16.3.1, and confirm that the value is the same as that obtained in part (b).

8. (a) Show that the line integral $\int_C y\sin x\,dx - \cos x\,dy$ is independent of the path.
(b) Evaluate the integral in part (a) along the line segment from $(0, 1)$ to $(\pi, -1)$.
(c) Evaluate the integral $\int_{(0,1)}^{(\pi,-1)} y\sin x\,dx - \cos x\,dy$ using Theorem 16.3.1, and confirm that the value is the same as that obtained in part (b).

In Exercises 9–14, show that the integral is independent of the path, and use Theorem 16.3.1 to find its value.

9. $\displaystyle\int_{(1,2)}^{(4,0)} 3y\,dx + 3x\,dy$

10. $\displaystyle\int_{(0,0)}^{(1,\pi/2)} e^x \sin y\,dx + e^x \cos y\,dy$

11. $\displaystyle\int_{(0,0)}^{(3,2)} 2xe^y\,dx + x^2e^y\,dy$

12. $\displaystyle\int_{(-1,2)}^{(0,1)} (3x - y + 1)\,dx - (x + 4y + 2)\,dy$

13. $\displaystyle\int_{(2,-2)}^{(-1,0)} 2xy^3\,dx + 3y^2x^2\,dy$

14. $\displaystyle\int_{(1,1)}^{(3,3)} \left(e^x \ln y - \frac{e^y}{x}\right)dx + \left(\frac{e^x}{y} - e^y \ln x\right)dy$, where x and y are positive.

In Exercises 15–18, confirm that the force field **F** is conservative in some open connected region containing the points P and Q, and then find the work done by the force field on a particle moving along an arbitrary smooth curve in the region from P to Q.

15. $\mathbf{F}(x, y) = xy^2\mathbf{i} + x^2y\mathbf{j}$; $P(1, 1)$, $Q(0, 0)$

16. $\mathbf{F}(x, y) = 2xy^3\mathbf{i} + 3x^2y^2\mathbf{j}$; $P(-3, 0)$, $Q(4, 1)$

17. $\mathbf{F}(x, y) = ye^{xy}\mathbf{i} + xe^{xy}\mathbf{j}$; $P(-1, 1)$, $Q(2, 0)$

18. $\mathbf{F}(x, y) = e^{-y}\cos x\mathbf{i} - e^{-y}\sin x\mathbf{j}$; $P(\pi/2, 1)$, $Q(-\pi/2, 0)$

In Exercises 19 and 20, find the exact value of $\int_C \mathbf{F}\cdot d\mathbf{r}$ using any method.

19. $\mathbf{F}(x, y) = (e^y + ye^x)\mathbf{i} + (xe^y + e^x)\mathbf{j}$
$C: \mathbf{r}(t) = \sin(\pi t/2)\mathbf{i} + \ln t\,\mathbf{j} \quad (1 \le t \le 2)$

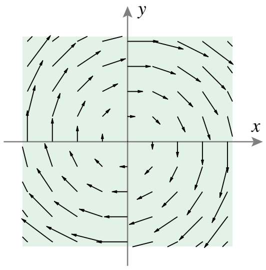
20. $\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 + \cos y)\mathbf{j}$
$C: \mathbf{r}(t) = t\mathbf{i} + t\cos(t/3)\mathbf{j} \quad (0 \le t \le \pi)$

C **21.** Use the numerical integration capability of a CAS or other calculating utility to approximate the value of the integral in Exercise 19 by direct integration. Confirm that the numerical approximation is consistent with the exact value.

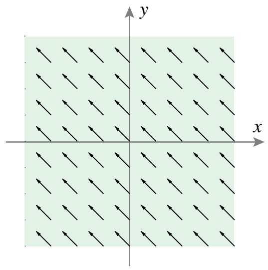
C **22.** Use the numerical integration capability of a CAS or other calculating utility to approximate the value of the integral in Exercise 20 by direct integration. Confirm that the numerical approximation is consistent with the exact value.

In Exercises 23 and 24, is the vector field conservative? Explain your reasoning.

23.



24.



25. Prove: If $\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$ is a conservative field and f, g, and h are continuous and have continuous first partial derivatives in a region, then

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}$$

in the region.

26. Use the result in Exercise 25 to show that the integral

$$\int_C yz\,dx + xz\,dy + yx^2\,dz$$

is not independent of the path.

27. Find a nonzero function h for which

$$\mathbf{F}(x, y) = h(x)[x \sin y + y \cos y]\mathbf{i} + h(x)[x \cos y - y \sin y]\mathbf{j}$$

is conservative.

28. (a) In Example 3 of Section 16.1 we showed that

$$\phi(x, y) = -\frac{c}{(x^2 + y^2)^{1/2}}$$

is a potential function for the two-dimensional inverse-square field

$$\mathbf{F}(x, y) = \frac{c}{(x^2 + y^2)^{3/2}}(x\mathbf{i} + y\mathbf{j})$$

but we did not explain how the potential function $\phi(x, y)$ was obtained. Use Theorem 16.3.3 to show that the two-dimensional inverse-square field is conservative everywhere except at the origin, and then use the method of Example 4 to derive the formula for $\phi(x, y)$.

(b) Use an appropriate generalization of the method of Example 4 to derive the potential function

$$\phi(x, y, z) = -\frac{c}{(x^2 + y^2 + z^2)^{1/2}}$$

for the three-dimensional inverse-square field given by Formula (5) of Section 16.1.

In Exercises 29 and 30, use the result in Exercise 28(b).

29. In each part, find the work done by the three-dimensional inverse-square field

$$\mathbf{F}(\mathbf{r}) = \frac{1}{\|\mathbf{r}\|^3}\mathbf{r}$$

on a particle that moves along the curve C.

(a) C is the line segment from $P(1, 1, 2)$ to $Q(3, 2, 1)$.

(b) C is the curve $\mathbf{r}(t) = (2t^2+1)\mathbf{i}+(t^3+1)\mathbf{j}+(2-\sqrt{t})\mathbf{k}$, where $0 \le t \le 1$.

(c) C is the circle in the xy-plane of radius 1 centered at $(2, 0, 0)$ traversed counterclockwise.

30. Let $\mathbf{F}(x, y) = \dfrac{y}{x^2 + y^2}\mathbf{i} - \dfrac{x}{x^2 + y^2}\mathbf{j}$.

(a) Show that

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \ne \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

if C_1 and C_2 are the semicircular paths from $(1, 0)$ to $(-1, 0)$ given by

$$C_1\colon x = \cos t,\quad y = \sin t \qquad (0 \le t \le \pi)$$
$$C_2\colon x = \cos t,\quad y = -\sin t \qquad (0 \le t \le \pi)$$

(b) Show that the components of $\mathbf{F}$ satisfy Formula (9).

(c) Do the results in parts (a) and (b) violate Theorem 16.3.3? Explain.

31. Prove Theorem 16.3.1 if C is a piecewise smooth curve composed of smooth curves $C_1, C_2, \ldots, C_n$.

32. Prove that (*b*) implies (*c*) in Theorem 16.3.2. [*Hint:* Consider any two piecewise smooth oriented curves C_1 and C_2 in the region from a point P to a point Q, and integrate around the closed curve consisting of C_1 and $-C_2$.]

33. Complete the proof of Theorem 16.3.2 by showing that $\partial\phi/\partial y = g(x, y)$, where $\phi(x, y)$ is the function in (7).

16.4 GREEN'S THEOREM

In this section we will discuss a remarkable and beautiful theorem that expresses a double integral over a plane region in terms of a line integral around its boundary.

GREEN'S THEOREM

16.4.1 THEOREM (*Green's*[*] *Theorem*). *Let R be a simply connected plane region whose boundary is a simple, closed, piecewise smooth curve C oriented counterclockwise. If $f(x, y)$ and $g(x, y)$ are continuous and have continuous first partial derivatives on some open set containing R, then*

$$\int_C f(x, y)\,dx + g(x, y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA \qquad (1)$$

Proof. For simplicity, we will prove the theorem for regions that are simultaneously type I and type II (see Definition 15.2.1). Such a region is shown in Figure 16.4.1. The crux of the proof is to show that

$$\int_C f(x, y)\,dx = -\iint_R \frac{\partial f}{\partial y}\,dA \quad \text{and} \quad \int_C g(x, y)\,dy = \iint_R \frac{\partial g}{\partial x}\,dA \qquad (2\text{–}3)$$

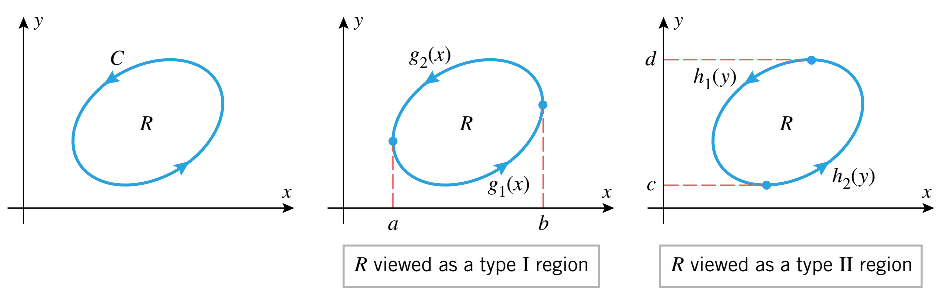


Figure 16.4.1

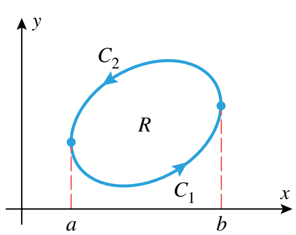


Figure 16.4.2

To prove (2), view R as a type I region and let C_1 and C_2 be the lower and upper boundary curves, oriented as in Figure 16.4.2. Then

$$\int_C f(x, y)\,dx = \int_{C_1} f(x, y)\,dx + \int_{C_2} f(x, y)\,dx$$

or, equivalently,

$$\int_C f(x, y)\,dx = \int_{C_1} f(x, y)\,dx - \int_{-C_2} f(x, y)\,dx \qquad (4)$$

[*] GEORGE GREEN (1793–1841). English mathematician and physicist. Green left school at an early age to work in his father's bakery and consequently had little early formal education. When his father opened a mill, the boy used the top room as a study in which he taught himself physics and mathematics from library books. In 1828 Green published his most important work, *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*. Although Green's Theorem appeared in that paper, the result went virtually unnoticed because of the small pressrun and local distribution. Following the death of his father in 1829, Green was urged by friends to seek a college education. In 1833, after four years of self-study to close the gaps in his elementary education, Green was admitted to Caius College, Cambridge. He graduated four years later, but with a disappointing performance on his final examinations—possibly because he was more interested in his own research. After a succession of works on light and sound, he was named to be Perse Fellow at Caius College. Two years later he died. In 1845, four years after his death, his paper of 1828 was published and the theories developed therein by this obscure, self-taught baker's son helped pave the way to the modern theories of electricity and magnetism.

(This step will help simplify our calculations since C_1 and $-C_2$ are then both oriented left to right.) The curves C_1 and $-C_2$ can be expressed parametrically as

$$\begin{aligned} C_1\colon x = t, \quad & y = g_1(t) \qquad (a \le t \le b) \\ -C_2\colon x = t, \quad & y = g_2(t) \qquad (a \le t \le b) \end{aligned}$$

Thus, we can rewrite (4) as

$$\begin{aligned} \int_C f(x, y)\, dx &= \int_a^b f(t, g_1(t))x'(t)\, dt - \int_a^b f(t, g_2(t))x'(t)\, dt \\ &= \int_a^b f(t, g_1(t))\, dt - \int_a^b f(t, g_2(t))\, dt \\ &= -\int_a^b [f(t, g_2(t)) - f(t, g_1(t))]\, dt \\ &= -\int_a^b \Big[f(t, y) \Big]_{y=g_1(t)}^{y=g_2(t)} dt = -\int_a^b \left[\int_{g_1(t)}^{g_2(t)} \frac{\partial f}{\partial y}\, dy \right] dt \\ &= -\int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial f}{\partial y}\, dy\, dx = -\iint_R \frac{\partial f}{\partial y}\, dA \end{aligned}$$

Since $x = t$

The proof of (3) is obtained similarly by treating R as a type II region. We omit the details. ■

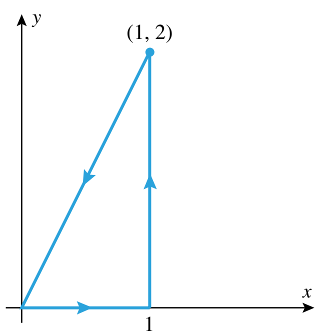


Figure 16.4.3

Example 1 Use Green's Theorem to evaluate

$$\int_C x^2 y\, dx + x\, dy$$

along the triangular path shown in Figure 16.4.3.

Solution. Since $f(x, y) = x^2 y$ and $g(x, y) = x$, it follows from (1) that

$$\begin{aligned} \int_C x^2 y\, dx + x\, dy &= \iint_R \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(x^2 y) \right] dA = \int_0^1 \int_0^{2x} (1 - x^2)\, dy\, dx \\ &= \int_0^1 (2x - 2x^3)\, dx = \left[x^2 - \frac{x^4}{2} \right]_0^1 = \frac{1}{2} \end{aligned}$$

This agrees with the result obtained in Example 7 of Section 16.2, where we evaluated the line integral directly. Note how much simpler this solution is. ◀

A NOTATION FOR LINE INTEGRALS AROUND SIMPLE CLOSED CURVES

It is common practice to denote a line integral around a simple closed curve by an integral sign with a superimposed circle. With this notation Formula (1) would be written as

$$\oint_C f(x, y)\, dx + g(x, y)\, dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

Sometimes a direction arrow is added to the circle to indicate whether the integration is clockwise or counterclockwise. Thus, if we wanted to emphasize the counterclockwise direction of integration required by Theorem 16.4.1, we could express (1) as

$$\ointctrclockwise_C f(x, y)\, dx + g(x, y)\, dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA \tag{5}$$

FINDING WORK USING GREEN'S THEOREM

It follows from Formula (37) of Section 16.2 that the integral on the left side of (5) is the work performed by the vector field $\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$ on a particle moving counterclockwise around the simple closed curve C. In the case where this vector field is conservative, it follows from Theorem 16.3.2 that the integrand in the double integral on the right side of (5) is zero, so the work performed by the field is zero, as expected. For vector fields that are not conservative, it is often more efficient to calculate the work around simple closed curves by using Green's Theorem than by parametrizing the curve.

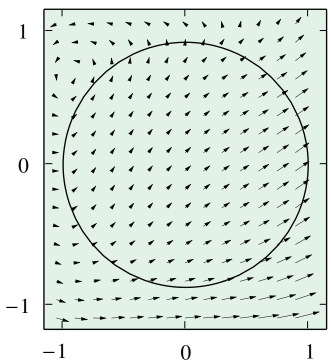


Figure 16.4.4

Example 2 Find the work done by the force field

$$\mathbf{F}(x, y) = (e^x - y^3)\mathbf{i} + (\cos y + x^3)\mathbf{j}$$

on a particle that travels once around the unit circle $x^2 + y^2 = 1$ in the counterclockwise direction (Figure 16.4.4).

Solution. The work W performed by the field is

$$\begin{aligned} W = \oint_C \mathbf{F}\cdot d\mathbf{r} &= \oint_C (e^x - y^3)\,dx + (\cos y + x^3)\,dy \\ &= \iint_R \left[\frac{\partial}{\partial x}(\cos y + x^3) - \frac{\partial}{\partial y}(e^x - y^3)\right] dA \qquad \boxed{\text{Green's Theorem}} \\ &= \iint_R (3x^2 + 3y^2)\,dA = 3\iint_R (x^2 + y^2)\,dA \\ &= 3\int_0^{2\pi}\int_0^1 (r^2)r\,dr\,d\theta = \frac{3}{4}\int_0^{2\pi} d\theta = \frac{3\pi}{2} \end{aligned}$$

We converted to polar coordinates.

◀

FINDING AREAS USING GREEN'S THEOREM

Green's Theorem leads to some useful new formulas for the area A of a region R that satisfies the conditions of the theorem. Two such formulas can be obtained as follows:

$$A = \iint_R dA = \oint_C x\,dy \quad \text{and} \quad A = \iint_R dA = \oint_C (-y)\,dx$$

Set $f(x, y) = 0$ and $g(x, y) = x$ in (1).

Set $f(x, y) = -y$ and $g(x, y) = 0$ in (1).

A third formula can be obtained by adding these two equations together. Thus, we have the following three formulas that express the area A of a region R in terms of line integrals around the boundary:

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C -y\,dx + x\,dy \tag{6}$$

REMARK. Although the third formula in (6) looks more complicated than the other two, it often leads to simpler integrations; but each has advantages in certain situations.

Example 3 Use a line integral to find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution. The ellipse, with counterclockwise orientation, can be represented parametrically by

$$x = a\cos t, \quad y = b\sin t \qquad (0 \le t \le 2\pi)$$

If we denote this curve by C, then from the third formula in (6) the area A enclosed by the ellipse is

$$A = \frac{1}{2}\oint_C -y\,dx + x\,dy$$
$$= \frac{1}{2}\int_0^{2\pi} [(-b\sin t)(-a\sin t) + (a\cos t)(b\cos t)]\,dt$$
$$= \frac{1}{2}ab\int_0^{2\pi}(\sin^2 t + \cos^2 t)\,dt = \frac{1}{2}ab\int_0^{2\pi} dt = \pi ab \qquad \blacktriangleleft$$

GREEN'S THEOREM FOR MULTIPLY CONNECTED REGIONS

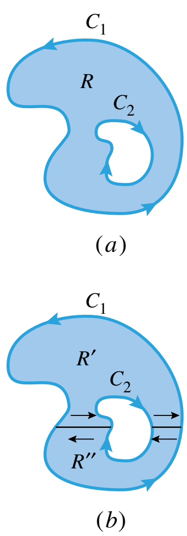


Figure 16.4.5

Recall that a plane region is said to be simply connected if it has no holes and is said to be multiply connected if it has one or more holes (see Figure 16.3.6). At the beginning of this section we stated Green's Theorem for a counterclockwise integration around the boundary of a simply connected region R (Theorem 16.4.1). Our next goal is to extend this theorem to multiply connected regions. To make this extension we will need to assume that *the region lies on the left when any portion of the boundary is traversed in the direction of its orientation*. This implies that the outer boundary curve of the region is oriented counterclockwise and the boundary curves that enclose holes have clockwise orientation (Figure 16.4.5*a*). If all portions of the boundary of a multiply connected region R are oriented in this way, then we say that the boundary of R has ***positive orientation***.

We will now derive a version of Green's Theorem that applies to multiply connected regions with positively oriented boundaries. For simplicity, we will consider a multiply connected region R with one hole, and we will assume that $f(x, y)$ and $g(x, y)$ have continuous first partial derivatives on some open set containing R. As shown in Figure 16.4.5*b*, let us divide R into two regions R' and R'' by introducing two "cuts" in R. The cuts are shown as line segments, but any piecewise smooth curves will suffice. If we assume that f and g satisfy the hypotheses of Green's Theorem on R (and hence on R' and R''), then we can apply this theorem to both R' and R'' to obtain

$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \iint_{R'} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA + \iint_{R''} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$
$$= \oint_{\substack{\text{Boundary}\\ \text{of } R'}} f(x, y)\,dx + g(x, y)\,dy + \oint_{\substack{\text{Boundary}\\ \text{of } R''}} f(x, y)\,dx + g(x, y)\,dy$$

However, the two line integrals are taken in opposite directions along the cuts, and hence cancel there, leaving only the contributions along C_1 and C_2. Thus,

$$\iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = \oint_{C_1} f(x, y)\,dx + g(x, y)\,dy + \oint_{C_2} f(x, y)\,dx + g(x, y)\,dy \tag{7}$$

which is an extension of Green's Theorem to a multiply connected region with one hole. Observe that the integral around the outer boundary is taken counterclockwise and the integral around the hole is taken clockwise. More generally, if R is a multiply connected region with n holes, then the analog of (7) involves a sum of $n+1$ integrals, one taken counterclockwise around the outer boundary of R and the rest taken clockwise around the holes.

Example 4 Evaluate the integral

$$\oint_C \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

if C is a piecewise smooth simple closed curve oriented counterclockwise such that (a) C does not enclose the origin and (b) C encloses the origin.

Solution (a). Let

$$f(x, y) = -\frac{y}{x^2 + y^2}, \quad g(x, y) = \frac{x}{x^2 + y^2} \tag{8}$$

so that

$$\frac{\partial g}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial f}{\partial y}$$

if x and y are not both zero. Thus, if C does not enclose the origin, we have

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0 \tag{9}$$

on the simply connected region enclosed by C, and hence the given integral is zero by Green's Theorem.

Solution (b). Unlike the situation in part (a), we cannot apply Green's Theorem directly because the functions $f(x, y)$ and $g(x, y)$ in (8) are discontinuous at the origin. Our problems are further compounded by the fact that we do not have a specific curve C that we can parametrize to evaluate the integral. Our strategy circumventing these problems will be to replace C with a specific curve that produces the same value for the integral and then use that curve for the evaluation. To obtain such a curve, we will apply Green's Theorem for multiply connected regions to a region that does not contain the origin. For this purpose we construct a circle C_a with *clockwise* orientation, centered at the origin, and with sufficiently small radius a that it lies inside the region enclosed by C (Figure 16.4.6). This creates a multiply connected region R whose boundary curves C and C_a have the orientations required by Formula (7) and such that within R the functions $f(x, y)$ and $g(x, y)$ in (8) satisfy the hypotheses of Green's Theorem (the origin is outside of R). Thus, it follows from (7) and (9) that

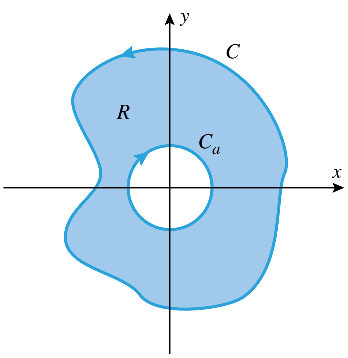


Figure 16.4.6

$$\oint_C \frac{-y\,dx + x\,dy}{x^2 + y^2} + \oint_{C_a} \frac{-y\,dx + x\,dy}{x^2 + y^2} = \iint_R 0\,dA = 0$$

It follows from this equation that

$$\oint_C \frac{-y\,dx + x\,dy}{x^2 + y^2} = -\oint_{C_a} \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

which we can rewrite as

$$\oint_C \frac{-y\,dx + x\,dy}{x^2 + y^2} = \oint_{-C_a} \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

Reversing the orientation of C_a reverses the sign of the integral.

But C_a has clockwise orientation, so $-C_a$ has counterclockwise orientation. Thus, we have shown that the original integral can be evaluated by integrating counterclockwise around a circle of radius a that is centered at the origin and lies within the region enclosed by C. Such a circle can be expressed parametrically as $x = a\cos t$, $y = a\sin t$ $(0 \le t \le 2\pi)$; and hence

$$\oint_C \frac{-y\,dx + x\,dy}{x^2 + y^2} = \int_0^{2\pi} \frac{(-a\sin t)(-a\sin t)\,dt + (a\cos t)(a\cos t)\,dt}{(a\cos t)^2 + (a\sin t)^2}$$

$$= \int_0^{2\pi} 1\,dt = 2\pi$$

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EXERCISE SET 16.4 C CAS

In Exercises 1 and 2, evaluate the line integral using Green's Theorem and check the answer by evaluating it directly.

1. $\oint_C y^2\,dx + x^2\,dy$, where C is the square with vertices (0, 0), (1, 0), (1, 1), and (0, 1) oriented counterclockwise.

2. $\oint_C y\,dx + x\,dy$, where C is the unit circle oriented counterclockwise.

In Exercises 3–13, use Green's Theorem to evaluate the integral. In each exercise, assume that the curve C is oriented counterclockwise.

3. $\oint_C 3xy\,dx + 2xy\,dy$, where C is the rectangle bounded by $x = -2$, $x = 4$, $y = 1$, and $y = 2$.

4. $\oint_C (x^2 - y^2)\,dx + x\,dy$, where C is the circle $x^2 + y^2 = 9$.

5. $\oint_C x\cos y\,dx - y\sin x\,dy$, where C is the square with vertices $(0, 0)$, $(\pi/2, 0)$, $(\pi/2, \pi/2)$, and $(0, \pi/2)$.

6. $\oint_C y\tan^2 x\,dx + \tan x\,dy$, where C is the circle $x^2 + (y + 1)^2 = 1$.

7. $\oint_C (x^2 - y)\,dx + x\,dy$, where C is the circle $x^2 + y^2 = 4$.

8. $\oint_C (e^x + y^2)\,dx + (e^y + x^2)\,dy$, where C is the boundary of the region between $y = x^2$ and $y = x$.

9. $\oint_C \ln(1 + y)\,dx - \dfrac{xy}{1 + y}\,dy$, where C is the triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 4)$.

10. $\oint_C x^2 y\,dx - y^2 x\,dy$, where C is the boundary of the region in the first quadrant, enclosed between the coordinate axes and the circle $x^2 + y^2 = 16$.

11. $\oint_C \tan^{-1} y\,dx - \dfrac{y^2 x}{1 + y^2}\,dy$, where C is the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$.

12. $\oint_C \cos x\sin y\,dx + \sin x\cos y\,dy$, where C is the triangle with vertices $(0, 0)$, $(3, 3)$, and $(0, 3)$.

13. $\oint_C x^2 y\,dx + (y + xy^2)\,dy$, where C is the boundary of the region enclosed by $y = x^2$ and $x = y^2$.

14. Let C be the boundary of the region enclosed between $y = x^2$ and $y = 2x$. Assuming that C is oriented counterclockwise, evaluate the following integrals by Green's Theorem:

(a) $\oint_C (6xy - y^2)\,dx$ (b) $\oint_C (6xy - y^2)\,dy$

C **15.** Use a CAS to check Green's Theorem by evaluating both integrals in the equation

$$\oint_C e^y\,dx + ye^x\,dy = \iint_R \left[\frac{\partial}{\partial x}(ye^x) - \frac{\partial}{\partial y}(e^y)\right] dA$$

where

(a) C is the circle $x^2 + y^2 = 1$

(b) C is the boundary of the region enclosed by $y = x^2$ and $x = y^2$.

16. In Example 3, we used Green's Theorem to obtain the area of an ellipse. Obtain this area using the first and then the second formula in (6).

17. Use a line integral to find the area of the region enclosed by the astroid

$$x = a\cos^3\phi, \quad y = a\sin^3\phi \qquad (0 \le \phi \le 2\pi)$$

[See Exercise 14 of Section 6.4.]

18. Use a line integral to find the area of the triangle with vertices $(0, 0)$, $(a, 0)$, and $(0, b)$, where $a > 0$ and $b > 0$.

19. Use the formula

$$A = \frac{1}{2}\oint_C -y\,dx + x\,dy$$

to find the area of the region swept out by the line from the origin to the ellipse $x = a\cos t$, $y = b\sin t$ if t varies from $t = 0$ to $t = t_0$ $(0 \le t_0 \le 2\pi)$.

20. Use the formula

$$A = \frac{1}{2}\oint_C -y\,dx + x\,dy$$

to find the area of the region swept out by the line from the origin to the hyperbola $x = a\cosh t$, $y = b\sinh t$ if t varies from $t = 0$ to $t = t_0$ $(t_0 \ge 0)$.

In Exercises 21 and 22, use Green's Theorem to find the work done by the force field $\mathbf{F}$ on a particle that moves along the stated path.

21. $\mathbf{F}(x, y) = xy\mathbf{i} + \left(\frac{1}{2}x^2 + xy\right)\mathbf{j}$; the particle starts at $(5, 0)$, traverses the upper semicircle $x^2 + y^2 = 25$, and returns to its starting point along the x-axis.

22. $\mathbf{F}(x, y) = \sqrt{y}\,\mathbf{i} + \sqrt{x}\,\mathbf{j}$; the particle moves counterclockwise one time around the closed curve given by the equations $y = 0$, $x = 2$, and $y = x^3/4$.

23. Evaluate $\oint_C y\,dx - x\,dy$, where C is the cardioid

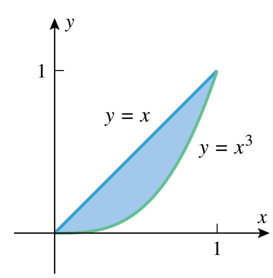
$$r = a(1 + \cos\theta) \qquad (0 \le \theta \le 2\pi)$$

24. Let R be a plane region with area A whose boundary is a piecewise smooth simple closed curve C. Use Green's Theorem to prove that the centroid $(\bar{x}, \bar{y})$ of R is given by

$$\bar{x} = \frac{1}{2A}\oint_C x^2\,dy, \qquad \bar{y} = -\frac{1}{2A}\oint_C y^2\,dx$$

In Exercises 25–28, use the result in Exercise 24 to find the centroid of the region.

25.



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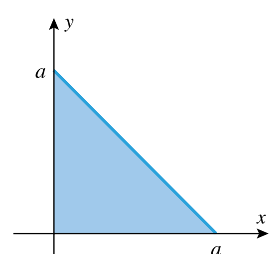

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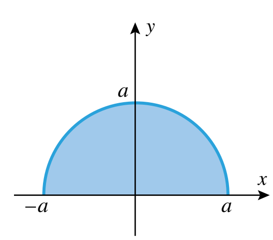

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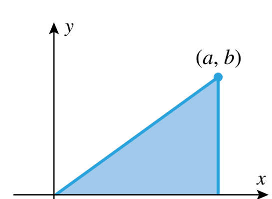

29. Find a simple closed curve C with counterclockwise orientation that maximizes the value of

$$\oint_C \tfrac{1}{3}y^3\,dx + \left(x - \tfrac{1}{3}x^3\right)dy$$

and explain your reasoning.

30. (a) Let C be the line segment from a point (a, b) to a point (c, d). Show that

$$\int_C -y\,dx + x\,dy = ad - bc$$

(b) Use the result in part (a) to show that the area A of a triangle with successive vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) going counterclockwise is

$$A = \tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]$$

(c) Find a formula for the area of a polygon with successive vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ going counterclockwise.

(d) Use the result in part (c) to find the area of a quadrilateral with vertices $(0, 0)$, $(3, 4)$, $(-2, 2)$, $(-1, 0)$.

In Exercises 31 and 32, evaluate the integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the boundary of the region R and C is oriented so that the region is on the left when the boundary is traversed in the direction of its orientation.

31. $\mathbf{F}(x, y) = (x^2 + y)\mathbf{i} + (4x - \cos y)\mathbf{j}$; C is the boundary of the region R that is inside the square with vertices $(0, 0)$, $(5, 0)$, $(5, 5)$, $(0, 5)$ but is outside the rectangle with vertices $(1, 1)$, $(3, 1)$, $(3, 2)$, $(1, 2)$.

32. $\mathbf{F}(x, y) = (e^{-x} + 3y)\mathbf{i} + x\mathbf{j}$; C is the boundary of the region R between the circles $x^2 + y^2 = 16$ and $x^2 - 2x + y^2 = 3$.

16.5 SURFACE INTEGRALS

In previous sections we considered four kinds of integrals—integrals over intervals, double integrals over two-dimensional regions, triple integrals over three-dimensional solids, and line integrals along curves in two- or three-dimensional space. In this section we will discuss integrals over surfaces in three-dimensional space. Such integrals occur in problems involving fluid and heat flow, electricity, magnetism, mass, and center of gravity.

DEFINITION OF A SURFACE INTEGRAL

Recall that if C is a smooth parametric curve in 3-space, and $f(x, y, z)$ is continuous on C, then the line integral of f along C with respect to arc length is defined by subdividing C into n arcs and defining the line integral as the limit

$$\int_C f(x, y, z)\,ds = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta s_k$$

where (x_k^*, y_k^*, z_k^*) is a point on the kth arc and Δs_k is the length of the kth arc. We will define *surface integrals* in an analogous manner.

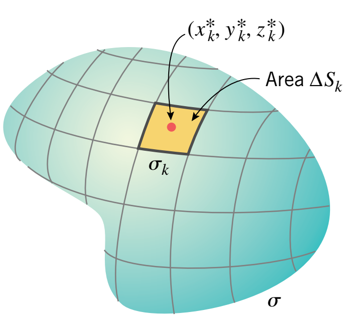

Figure 16.5.1

Let σ be a surface in 3-space with finite surface area, and let $f(x, y, z)$ be a continuous function defined on σ. As shown in Figure 16.5.1, subdivide σ into patches, $\sigma_1, \sigma_2, \ldots, \sigma_n$ with areas $\Delta S_1, \Delta S_2, \ldots, \Delta S_n$, and form the sum

$$\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta S_k \tag{1}$$

where (x_k^*, y_k^*, z_k^*) is an arbitrary point on σ_k. Now repeat the subdivision process, dividing σ into more and more patches in such a way that the maximum dimension of each patch approaches zero as $n\to+\infty$. If (1) approaches a limit that does not depend on the way the subdivisions are made or how the points (x_k^*, y_k^*, z_k^*) are chosen, then this limit is called the ***surface integral*** of $f(x, y, z)$ over σ and is denoted by

$$\iint_\sigma f(x, y, z)\,dS = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta S_k \tag{2}$$

EVALUATING SURFACE INTEGRALS

There are various procedures for evaluating surface integrals that depend on how the surface σ is represented. The following theorem provides a method for evaluating a surface integral when σ is represented parametrically.

16.5.1 THEOREM. *Let σ be a smooth parametric surface whose vector equation is*
$\mathbf{r} = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$
where (u, v) varies over a region R in the uv-plane. If $f(x, y, z)$ is continuous on σ, then

$$\iint_{\sigma} f(x, y, z)\, dS = \iint_{R} f(x(u, v), y(u, v), z(u, v)) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA \tag{3}$$

To motivate this result, suppose that the parameter domain R is subdivided as in Figure 15.4.10, and suppose that the point (x_k^*, y_k^*, z_k^*) in (2) corresponds to parameter values of u_k^* and v_k^*. If we use Formula (9) of Section 15.4 to approximate ΔS_k, and if we assume that the errors in the approximations approach zero as $n \to +\infty$, then it follows from (2) that

$$\iint_{\sigma} f(x, y, z)\, dS = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x(u_k^*, v_k^*), y(u_k^*, v_k^*), z(u_k^*, v_k^*)) \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \Delta A_k$$

which suggests Formula (3).

We will discuss various applications and interpretations of surface integrals later in this section and in subsequent sections, but for now we will focus on techniques for evaluating such integrals.

Example 1 Evaluate the surface integral $\iint_{\sigma} x^2\, dS$ over the sphere $x^2 + y^2 + z^2 = 1$.

Solution. As in Example 9 of Section 15.4 (with $a = 1$), the sphere is the graph of the vector-valued function

$$\mathbf{r}(\phi, \theta) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k} \quad (0 \le \phi \le \pi,\ \ 0 \le \theta \le 2\pi) \tag{4}$$

and

$$\left\| \frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\| = \sin\phi$$

From the **i**-component of **r**, the integrand in the surface integral can be expressed in terms of ϕ and θ as $x^2 = \sin^2\phi\cos^2\theta$. Thus, it follows from (3) with ϕ and θ in place of u and v and R as the rectangular region in the $\phi\theta$-plane determined by the inequalities in (4) that

$$\begin{aligned}
\iint_{\sigma} x^2\, dS &= \iint_{R} (\sin^2\phi\cos^2\theta) \left\| \frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\| dA \\
&= \int_0^{2\pi}\!\!\int_0^{\pi} \sin^3\phi\cos^2\theta\, d\phi\, d\theta \\
&= \int_0^{2\pi} \left[\int_0^{\pi} \sin^3\phi\, d\phi\right] \cos^2\theta\, d\theta \\
&= \int_0^{2\pi} \left[\frac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\pi} \cos^2\theta\, d\theta && \text{Formula (11), Section 8.3} \\
&= \frac{4}{3}\int_0^{2\pi} \cos^2\theta\, d\theta \\
&= \frac{4}{3}\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{4\pi}{3} && \text{Formula (8), Section 8.3}
\end{aligned}$$

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SURFACE INTEGRALS OVER $z = g(x, y)$, $y = g(x, z)$, AND $x = g(y, z)$

In the case where σ is a surface of the form $z = g(x, y)$, we can take $x = u$ and $y = v$ as parameters and express the equation of the surface as

$$\mathbf{r} = u\mathbf{i} + v\mathbf{j} + g(u, v)\mathbf{k}$$

in which case we obtain

$$\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

[see the derivation of Formula (11) in Section 15.4]. Thus, it follows from (3) that

$$\iint_\sigma f(x, y, z)\, dS = \iint_R f(x, y, g(x, y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

Note that in this formula the region R lies in the xy-plane because the parameters are x and y. Geometrically, this region is the projection of σ on the xy-plane. The following theorem summarizes this result and gives analogous formulas for surface integrals over surfaces of the form $y = g(x, z)$ and $x = g(y, z)$.

16.5.2 THEOREM.

(*a*) *Let σ be a surface with equation $z = g(x, y)$ and let R be its projection on the xy-plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ, then*

$$\iint_\sigma f(x, y, z)\, dS = \iint_R f(x, y, g(x, y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA \tag{5}$$

(*b*) *Let σ be a surface with equation $y = g(x, z)$ and let R be its projection on the xz-plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ, then*

$$\iint_\sigma f(x, y, z)\, dS = \iint_R f(x, g(x, z), z)\sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\, dA \tag{6}$$

(*c*) *Let σ be a surface with equation $x = g(y, z)$ and let R be its projection on the yz-plane. If g has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on σ, then*

$$\iint_\sigma f(x, y, z)\, dS = \iint_R f(g(y, z), y, z)\sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1}\, dA \tag{7}$$

Example 2 Evaluate the surface integral

$$\iint_\sigma xz\, dS$$

where σ is the part of the plane $x + y + z = 1$ that lies in the first octant.

Solution. The equation of the plane can be written as

$$z = 1 - x - y$$

which is of the form $z = g(x, y)$. Consequently, we can apply Formula (5) with

$z = g(x, y) = 1 - x - y$ and $f(x, y, z) = xz$. We have

$$\frac{\partial z}{\partial x} = -1 \quad \text{and} \quad \frac{\partial z}{\partial y} = -1$$

so (5) becomes

$$\iint_{\sigma} xz\,dS = \iint_{R} x(1 - x - y)\sqrt{(-1)^2 + (-1)^2 + 1}\,dA \tag{8}$$

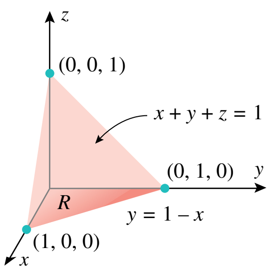

Figure 16.5.2

where R is the projection of σ on the xy-plane (Figure 16.5.2). Rewriting the double integral in (8) as an iterated integral yields

$$\begin{aligned}
\iint_{\sigma} xz\,dS &= \sqrt{3}\int_0^1 \int_0^{1-x} (x - x^2 - xy)\,dy\,dx \\
&= \sqrt{3}\int_0^1 \left[xy - x^2y - \frac{xy^2}{2}\right]_{y=0}^{1-x} dx \\
&= \sqrt{3}\int_0^1 \left(\frac{x}{2} - x^2 + \frac{x^3}{2}\right) dx \\
&= \sqrt{3}\left[\frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8}\right]_0^1 = \frac{\sqrt{3}}{24}
\end{aligned}$$

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Example 3 Evaluate the surface integral

$$\iint_{\sigma} y^2z^2\,dS$$

where σ is the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the planes $z = 1$ and $z = 2$ (Figure 16.5.3).

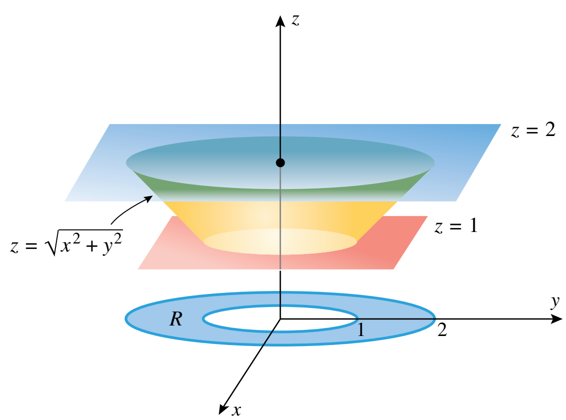


Figure 16.5.3

Solution. We will apply Formula (5) with

$$z = g(x, y) = \sqrt{x^2 + y^2} \quad \text{and} \quad f(x, y, z) = y^2z^2$$

Thus,

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

so

$$\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} = \sqrt{2}$$

(verify), and (5) yields

$$\iint_{\sigma} y^2 z^2\, dS = \iint_{R} y^2 \left(\sqrt{x^2+y^2}\right)^2 \sqrt{2}\, dA = \sqrt{2} \iint_{R} y^2(x^2+y^2)\, dA$$

where R is the annulus enclosed between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ (Figure 16.5.3). Using polar coordinates to evaluate this double integral over the annulus R yields

$$\begin{aligned}\iint_{\sigma} y^2 z^2\, dS &= \sqrt{2} \int_0^{2\pi} \int_1^2 (r \sin\theta)^2 (r^2) r\, dr\, d\theta \\ &= \sqrt{2} \int_0^{2\pi} \int_1^2 r^5 \sin^2\theta\, dr\, d\theta \\ &= \sqrt{2} \int_0^{2\pi} \left[\frac{r^6}{6} \sin^2\theta\right]_{r=1}^{2} d\theta = \frac{21}{\sqrt{2}} \int_0^{2\pi} \sin^2\theta\, d\theta \\ &= \frac{21}{\sqrt{2}} \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{21\pi}{\sqrt{2}} \qquad \boxed{\text{Formula (7), Section 8.3}}\end{aligned}$$

◀

MASS OF A CURVED LAMINA AS A SURFACE INTEGRAL

In Section 15.6 we defined a *lamina* to be an idealized flat object that is thin enough to be viewed as a plane region. Analogously, a ***curved lamina*** is an idealized object that is thin enough to be viewed as a surface in 3-space. A curved lamina may look like a bent plate, as in Figure 16.5.4, or it may enclose a region in 3-space, like the shell of an egg. If the composition of a curved lamina is uniform so that its mass is distributed uniformly, then it is said to be ***homogeneous***, and we define its ***mass density*** to be the total mass divided by the total surface area. However, if the mass of the lamina is not uniformly distributed, then this is not a useful measure, since it does not account for the variations in mass concentration. In this case we describe the mass concentration at a point by a ***mass density function*** δ, which we view as a limit; that is,

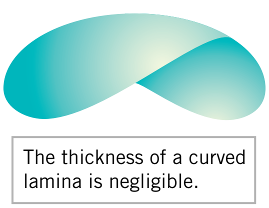


Figure 16.5.4

$$\delta = \lim_{\Delta S \to 0} \frac{\Delta M}{\Delta S} \tag{9}$$

where ΔM and ΔS denote the mass and surface area of a small section of lamina containing the point (Figure 16.5.5).

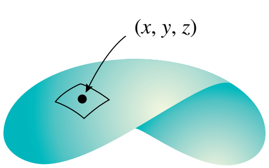


Figure 16.5.5

To translate this informal idea into a useful formula, suppose that $\delta = \delta(x, y, z)$ is the density function of a smooth curved lamina σ. Assume that the lamina is subdivided into n small sections; let (x_k^*, y_k^*, z_k^*) be a point in the kth section, let ΔM_k be the mass of the kth section, and let ΔS_k be the surface area of the kth section. Since we are assuming that the sections are small, it follows from (9) that the mass of the kth section can be approximated as

$$\Delta M_k \approx \delta(x_k^*, y_k^*, z_k^*)\Delta S_k$$

and hence the mass M of the entire lamina can be approximated as

$$M = \sum_{k=1}^{n} \Delta M_k \approx \sum_{k=1}^{n} \delta(x_k^*, y_k^*, z_k^*)\Delta S_k \tag{10}$$

If we now increase n in such a way that the dimensions of the sections approach zero, then it is plausible that the error in (10) will approach zero, and the exact value of M will be given by the surface integral

$$M = \iint_{\sigma} \delta(x, y, z)\, dS \tag{11}$$

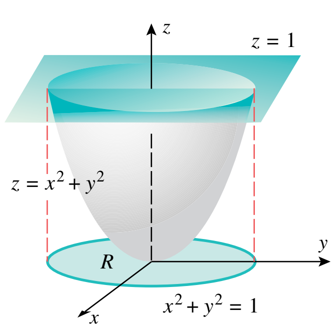


Figure 16.5.6

Example 4 Suppose that a curved lamina σ with constant density $\delta(x, y, z) = \delta_0$ is the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$ (Figure 16.5.6). Find the mass of the lamina.

Solution. Since $z = g(x, y) = x^2 + y^2$, it follows that

$$\frac{\partial z}{\partial x} = 2x \quad \text{and} \quad \frac{\partial z}{\partial y} = 2y$$

Substituting these expressions and $\delta(x, y, z) = \delta(x, y, g(x, y)) = \delta_0$ into (11) yields

$$M = \iint_{\sigma} \delta_0 \, dS = \iint_{R} \delta_0 \sqrt{(2x)^2 + (2y)^2 + 1}\, dA = \delta_0 \iint_{R} \sqrt{4x^2 + 4y^2 + 1}\, dA \tag{12}$$

where R is the circular region enclosed by $x^2 + y^2 = 1$. To evaluate (12) we use polar coordinates:

$$M = \delta_0 \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1}\, r \, dr \, d\theta = \frac{\delta_0}{12} \int_0^{2\pi} (4r^2 + 1)^{3/2} \Big]_{r=0}^{1} d\theta$$

$$= \frac{\delta_0}{12} \int_0^{2\pi} (5^{3/2} - 1)\, d\theta = \frac{\pi \delta_0}{6}(5\sqrt{5} - 1)$$ ◀

SURFACE AREA AS A SURFACE INTEGRAL

In the special case where $f(x, y, z)$ is 1, Formula (3) becomes

$$\iint_{\sigma} dS = \iint_{R} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA$$

However, it follows from Formula (10) of Section 15.4 that this integral represents the surface area of σ. Thus, we have established the following result.

16.5.3 THEOREM. *If σ is a smooth parametric surface in 3-space, then its surface area S can be expressed as*

$$S = \iint_{\sigma} dS \tag{13}$$

REMARK. This result adds nothing new computationally, since Formula (13) is just a reformulation of Formula (10) in Section 15.4. However, the relationship between surface integrals and surface area is important to understand.

EXERCISE SET 16.5 C CAS

In Exercises 1–10, evaluate the surface integral

$$\iint_{\sigma} f(x, y, z)\, dS$$

1. $f(x, y, z) = z^2$; σ is the portion of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$.

2. $f(x, y, z) = xy$; σ is the portion of the plane $x + y + z = 1$ lying in the first octant.

3. $f(x, y, z) = x^2 y$; σ is the portion of the cylinder $x^2 + z^2 = 1$ between the planes $y = 0$, $y = 1$, and above the xy-plane.

4. $f(x, y, z) = (x^2 + y^2)z$; σ is the portion of the sphere $x^2 + y^2 + z^2 = 4$ above the plane $z = 1$.

5. $f(x, y, z) = x - y - z$; σ is the portion of the plane $x + y = 1$ in the first octant between $z = 0$ and $z = 1$.

6. $f(x, y, z) = x + y$; σ is the portion of the plane $z = 6 - 2x - 3y$ in the first octant.

7. $f(x, y, z) = x + y + z$; σ is the surface of the cube defined by the inequalities $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$. [*Hint:* Integrate over each face separately.]

8. $f(x, y, z) = z + 1$; σ is the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$.

9. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$; σ is the portion of the cone $z = \sqrt{x^2 + y^2}$ below the plane $z = 1$.

10. $f(x, y, z) = x^2 + y^2$; σ is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

In Exercises 11 and 12, set up, but do not evaluate, an iterated integral equal to the given surface integral by projecting σ on (a) the xy-plane, (b) the yz-plane, and (c) the xz-plane.

11. $\iint_\sigma xyz\, dS$, where σ is the portion of the plane $2x + 3y + 4z = 12$ in the first octant.

12. $\iint_\sigma xz\, dS$, where σ is the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant.

C **13.** Use a CAS to confirm that the three integrals you obtained in Exercise 11 are equal, and find the exact value of the surface integral.

C **14.** Try to confirm with a CAS that the three integrals you obtained in Exercise 12 are equal. If you did not succeed, what was the difficulty?

In Exercises 15 and 16, set up, but do not evaluate, two different iterated integrals equal to the given integral.

15. $\iint_\sigma xyz\, dS$, where σ is the portion of the surface $y^2 = x$ between the planes $z = 0$, $z = 4$, $y = 1$, and $y = 2$.

16. $\iint_\sigma x^2 y\, dS$, where σ is the portion of the cylinder $y^2 + z^2 = a^2$ in the first octant between the planes $x = 0$, $x = 9$, $z = y$, and $z = 2y$.

C **17.** Use a CAS to confirm that the two integrals you obtained in Exercise 15 are equal, and find the exact value of the surface integral.

C **18.** Use a CAS to find the value of the surface integral

$$\iint_\sigma x^2 yz\, dS$$

over the portion of the elliptic paraboloid $z = 5 - 3x^2 - 2y^2$ that lies above the xy-plane.

In Exercises 19 and 20, find the mass of the lamina with constant density δ_0.

19. The lamina that is the portion of the circular cylinder $x^2 + z^2 = 4$ that lies directly above the rectangle $R = \{(x, y) : 0 \le x \le 1, 0 \le y \le 4\}$ in the xy-plane.

20. The lamina that is the portion of the paraboloid $2z = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 8$.

21. Find the mass of the lamina that is the portion of the surface $y^2 = 4 - z$ between the planes $x = 0$, $x = 3$, $y = 0$, and $y = 3$ if the density is $\delta(x, y, z) = y$.

22. Find the mass of the lamina that is the portion of the cone $z = \sqrt{x^2 + y^2}$ between $z = 1$ and $z = 4$ if the density is $\delta(x, y, z) = x^2 z$.

23. If a curved lamina has constant density δ_0, what relationship must exist between its mass and surface area? Explain your reasoning.

24. Show that if the density of the lamina $x^2 + y^2 + z^2 = a^2$ at each point is equal to the distance between that point and the xy-plane, then the mass of the lamina is $2\pi a^3$.

The centroid of a surface σ is defined by

$$\bar{x} = \frac{\iint_\sigma x\, dS}{\text{area of } \sigma}, \quad \bar{y} = \frac{\iint_\sigma y\, dS}{\text{area of } \sigma}, \quad \bar{z} = \frac{\iint_\sigma z\, dS}{\text{area of } \sigma}$$

In Exercises 25 and 26, find the centroid of the surface.

25. The portion of the paraboloid $z = \frac{1}{2}(x^2 + y^2)$ below the plane $z = 4$.

26. The portion of the sphere $x^2 + y^2 + z^2 = 4$ above the plane $z = 1$.

In Exercises 27–30, evaluate the integral $\iint_\sigma f(x, y, z)\, dS$ over the surface σ represented by the vector-valued function $\mathbf{r}(u, v)$.

27. $f(x, y, z) = xyz$; $\mathbf{r}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + 3u\mathbf{k}$ $(1 \le u \le 2,\ 0 \le v \le \pi/2)$

28. $f(x, y, z) = \dfrac{x^2 + z^2}{y}$; $\mathbf{r}(u, v) = 2\cos v\mathbf{i} + u\mathbf{j} + 2\sin v\mathbf{k}$ $(1 \le u \le 3,\ 0 \le v \le 2\pi)$

29. $f(x, y, z) = \dfrac{1}{\sqrt{1 + 4x^2 + 4y^2}}$;
$\mathbf{r}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + u^2\mathbf{k}$
$(0 \le u \le \sin v,\ 0 \le v \le \pi)$

30. $f(x, y, z) = e^{-z}$;
$\mathbf{r}(u, v) = 2\sin u\cos v\mathbf{i} + 2\sin u\sin v\mathbf{j} + 2\cos u\mathbf{k}$
$(0 \le u \le \pi/2, 0 \le v \le 2\pi)$

C **31.** Use a CAS to approximate the mass of the curved lamina $z = e^{-x^2-y^2}$ that lies above the region in the xy-plane enclosed by $x^2 + y^2 = 9$ given that the density function is $\delta(x, y, z) = \sqrt{x^2 + y^2}$.

C **32.** The surface σ shown in the accompanying figure, called a ***Möbius strip***, is represented by the parametric equations

$$x = (5 + u\cos(v/2))\cos v$$
$$y = (5 + u\cos(v/2))\sin v \quad (-1 \le u \le 1, 0 \le v \le 2\pi)$$
$$z = u\sin(v/2)$$

(a) Use a CAS to generate a reasonable facsimile of this surface.

(b) Use a CAS to approximate the location of the centroid of σ (see the definition preceding Exercise 25).

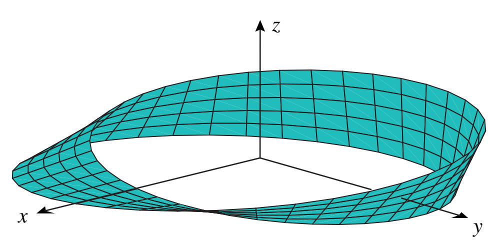


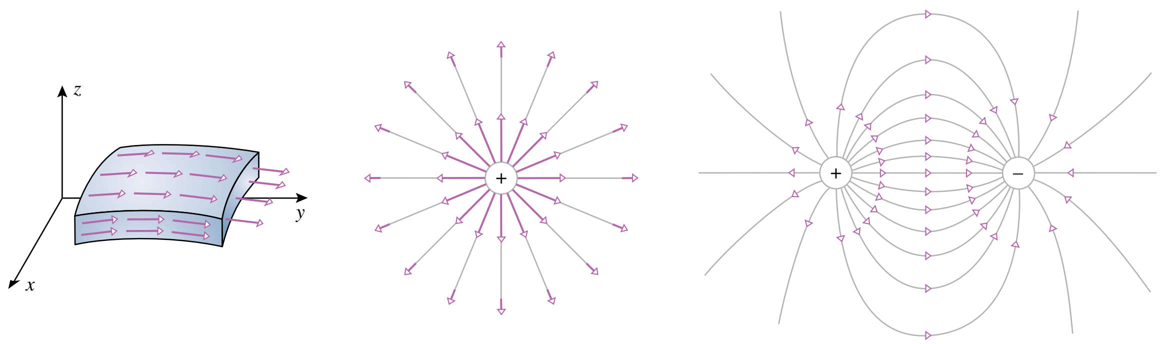
Figure Ex-32

16.6 APPLICATIONS OF SURFACE INTEGRALS; FLUX

In this section we will discuss applications of surface integrals in vector fields associated with fluid flow and electrostatic forces. However, the ideas that we will develop will be general in nature and applicable to other kinds of vector fields as well.

FLOW FIELDS

We will be concerned in this section with vector fields in 3-space that involve some type of "flow"—the flow of a fluid or the flow of charged particles in an electrostatic field, for example. In the case of fluid flow, the vector field $\mathbf{F}(x, y, z)$ represents the velocity of a fluid particle at the point (x, y, z), and the fluid particles flow along "streamlines" that are tangential to the velocity vectors (Figure 16.6.1*a*). In the case of an electrostatic field, $\mathbf{F}(x, y, z)$ is the force that the field exerts on a small unit of positive charge at the point (x, y, z), and such charges have acceleration in the directions of "electric lines" that are tangential to the force vectors (Figures 16.6.1*b* and 16.6.1*c*).



The velocity vectors of the fluid particles are tangent to the streamlines.

By Coulomb's law the electrostatic field resulting from a single positive charge is an inverse-square field in which **F** is the repulsive force on a small unit positive charge.

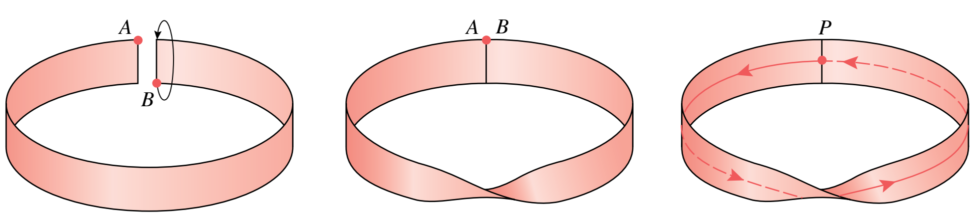
The electrostatic field **F** that results from two charges of equal strength but opposite polarity.

(*a*) (*b*) (*c*)

Figure 16.6.1

ORIENTED SURFACES

Our main goal in this section is to study flows of vector fields through permeable surfaces placed in the field. For this purpose we will need to consider some basic ideas about surfaces. Most surfaces that we encounter in applications have two sides—a sphere has an inside and an outside, and an infinite horizontal plane has a top side and a bottom side, for example. However, there exist mathematical surfaces with only one side. For example, Figure 16.6.2*a* shows the construction of a surface called a ***Möbius strip*** [in honor of the German mathematician August Möbius (1790–1868)]. The Möbius strip has only one side in the sense that a bug can traverse the *entire* surface without crossing an edge (Figure 16.6.2*b*). In contrast, a sphere is two-sided in the sense that a bug walking on the sphere can traverse



If an ant starts at P with its back facing you and makes one circuit around the strip, then its back will face away from you when it returns to P. Thus, the Möbius strip has only one side.

(*a*) (*b*)

Figure 16.6.2

the inside surface or the outside surface but cannot traverse both without somehow passing through the sphere. A two-sided surface is said to be ***orientable***, and a one-sided surface is said to be ***nonorientable***. In the rest of this text we will only be concerned with orientable surfaces.

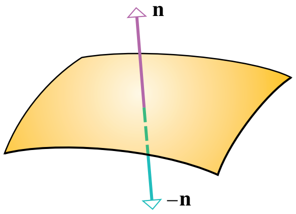


Figure 16.6.3

In applications, it is important to have some way of distinguishing between the two sides of an orientable surface. For this purpose let us suppose that σ is an orientable surface that has a unit normal vector $\mathbf{n}$ at each point. As illustrated in Figure 16.6.3, the vectors $\mathbf{n}$ and $-\mathbf{n}$ point to opposite sides of the surface and hence serve to distinguish between the two sides. It can be proved that if σ is a smooth orientable surface, then it is always possible to choose the direction of $\mathbf{n}$ at each point so that $\mathbf{n} = \mathbf{n}(x, y, z)$ varies continuously over the surface. These unit vectors are then said to form an ***orientation*** of the surface. It can also be proved that a smooth orientable surface has only two possible orientations. For example, the surface in Figure 16.6.4 is oriented up by the purple vectors and down by the green vectors. However, we cannot create a third orientation by mixing the two since this produces points on the surface at which there is an abrupt change in direction (across the black curve in the figure, for example).

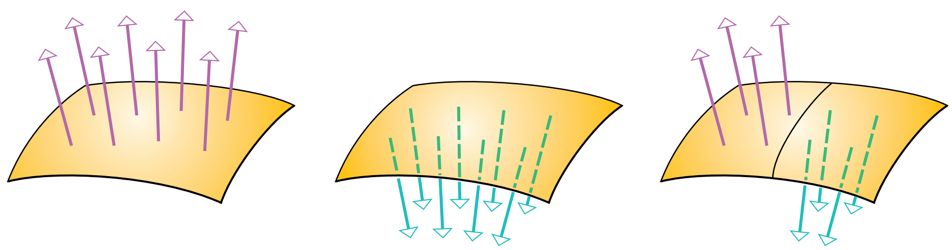


Figure 16.6.4

ORIENTATION OF A SMOOTH PARAMETRIC SURFACE

When a surface is expressed parametrically, the parametric equations create a natural orientation of the surface. To see why this is so, recall from Section 15.4 that if a smooth parametric surface σ is given by the vector equation

$$\mathbf{r} = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

then the unit normal

$$\mathbf{n} = \mathbf{n}(u, v) = \frac{\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}}{\left\| \dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} \right\|} \tag{1}$$

is a continuous vector-valued function of u and v. Thus, Formula (1) defines an orientation of the surface; we call this the ***positive orientation*** of the parametric surface and we say that $\mathbf{n}$ points in the ***positive direction*** from the surface. The orientation determined by $-\mathbf{n}$ is called the ***negative orientation*** of the surface and we say that $-\mathbf{n}$ points in the ***negative direction*** from the surface. For example, consider the sphere that is represented parametrically by the vector equation

$$\mathbf{r}(\phi, \theta) = a \sin\phi \cos\theta\,\mathbf{i} + a \sin\phi \sin\theta\,\mathbf{j} + a\cos\phi\,\mathbf{k} \quad (0 \le \phi \le \pi,\;\; 0 \le \theta \le 2\pi)$$

We showed in Example 9 of Section 15.4 that

$$\mathbf{n} = \frac{1}{a}\mathbf{r}$$

This vector points in the same direction as the radius vector $\mathbf{r}$ (outward from the center). Thus, for the given parametrization, the positive orientation of the sphere is *outward* and the negative orientation is *inward* (Figure 16.6.5).

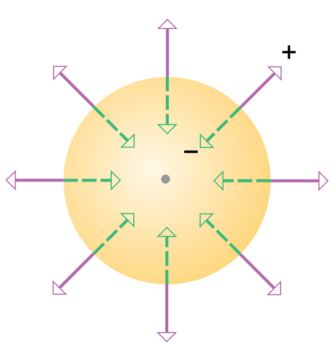


Figure 16.6.5

FOR THE READER. See if you can find a parametrization of the sphere in which the positive direction is inward.

FLUX

In physics, the term *fluid* is used to describe both liquids and gases. Liquids are usually regarded to be ***incompressible***, meaning that the liquid has a uniform density (mass per unit volume) that cannot be altered by compressive forces. Gases are regarded to be ***compressible***, meaning that the density may vary from point to point and can be altered by compressive forces. In this text we will be concerned primarily with incompressible fluids. Moreover, we will assume that the velocity of the fluid at a fixed point does not vary with time. Fluid flows with this property are said to be in a ***steady state***.

Our next goal in this section is to define a fundamental concept of physics known as *flux* (from the Latin word *fluxus*, meaning "flow"). This concept is applicable in any vector field, but we will motivate it in the context of steady-state flow of an incompressible fluid. We consider the following problem:

16.6.1 PROBLEM. Suppose that an oriented surface σ is immersed in an incompressible, steady-state fluid flow, and assume further that the surface is permeable so that the fluid can flow through it freely in either direction. Find the net volume of fluid Φ that passes through the surface per unit of time, where the net volume is interpreted to mean the volume that passes through the surface in the positive direction minus the volume that passes through the surface in the negative direction.

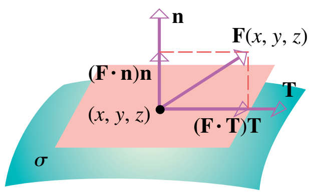


Figure 16.6.6

To solve this problem, suppose that the velocity of the fluid at a point (x, y, z) on the surface σ is given by

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

Let $\mathbf{n}$ be the unit normal toward the positive side of σ at the point (x, y, z), and let $\mathbf{T}$ be a unit vector that is orthogonal to $\mathbf{n}$ and lies in the plane of $\mathbf{F}$ and $\mathbf{n}$. As illustrated in Figure 16.6.6, the velocity vector $\mathbf{F}$ can be resolved into two orthogonal components—a component $(\mathbf{F}\cdot\mathbf{T})\mathbf{T}$ along the "face" of the surface σ and a component $(\mathbf{F}\cdot\mathbf{n})\mathbf{n}$ that is perpendicular to σ. The component of velocity along the face of the surface does not contribute to the flow through σ and hence can be ignored in our computations. Moreover, observe that the sign of $\mathbf{F}\cdot\mathbf{n}$ determines the direction of flow—a positive value means the flow is in the direction of $\mathbf{n}$ and a negative value means that it is opposite to $\mathbf{n}$.

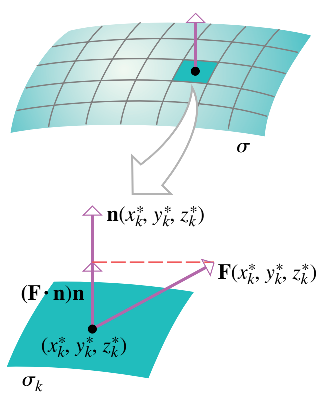


Figure 16.6.7

To solve Problem 16.6.1, we subdivide σ into n patches $\sigma_1, \sigma_2, \ldots, \sigma_n$ with areas

$$\Delta S_1, \Delta S_2, \ldots, \Delta S_n$$

If the patches are small and the flow is not too erratic, it is reasonable to assume that the velocity does not vary much on each patch. Thus, if (x_k^*, y_k^*, z_k^*) is any point in the kth patch, we can assume that $\mathbf{F}(x, y, z)$ is constant and equal to $\mathbf{F}(x_k^*, y_k^*, z_k^*)$ throughout the patch and that the component of velocity across the surface σ_k is

$$\mathbf{F}(x_k^*, y_k^*, z_k^*)\cdot\mathbf{n}(x_k^*, y_k^*, z_k^*) \tag{2}$$

(Figure 16.6.7). Thus, we can interpret

$$\mathbf{F}(x_k^*, y_k^*, z_k^*)\cdot\mathbf{n}(x_k^*, y_k^*, z_k^*)\Delta S_k$$

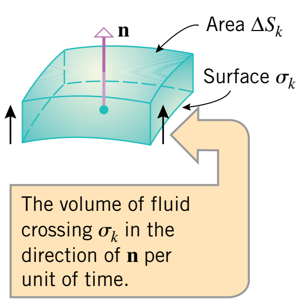


Figure 16.6.8

as the approximate volume of fluid crossing the patch σ_k in the direction of $\mathbf{n}$ per unit of time (Figure 16.6.8). For example, if the component of velocity in the direction of $\mathbf{n}$ is $\mathbf{F}(x_k^*, y_k^*, z_k^*)\cdot\mathbf{n} = 25$ cm/s, and the area of the patch is $\Delta S_k = 2$ cm^2, then the volume of fluid ΔV_k crossing the patch in the direction of $\mathbf{n}$ per unit of time is approximately

$$\Delta V_k \approx \mathbf{F}(x_k^*, y_k^*, z_k^*)\cdot\mathbf{n}(x_k^*, y_k^*, z_k^*)\Delta S_k = 25\text{ cm/s}\cdot 2\text{ cm}^2 = 50\text{ cm}^3/\text{s}$$

In the case where the velocity component $\mathbf{F}(x_k^*, y_k^*, z_k^*)\cdot\mathbf{n}(x_k^*, y_k^*, z_k^*)$ is negative, the flow is in the direction opposite to $\mathbf{n}$, so that $-\Delta V_k$ is the approximate volume of fluid crossing the patch σ_k in the direction opposite to $\mathbf{n}$ per unit time. Thus, the sum

$$\sum_{k=1}^{n}\mathbf{F}(x_k^*, y_k^*, z_k^*)\cdot\mathbf{n}(x_k^*, y_k^*, z_k^*)\Delta S_k$$

measures the approximate net volume of fluid that crosses the surface σ in the direction of its orientation $\mathbf{n}$ per unit of time.

If we now increase n in such a way that the maximum dimension of each patch approaches zero, then it is plausible that the errors in the approximations approach zero, and the limit

$$\Phi = \lim_{n \to +\infty} \sum_{k=1}^{n} \mathbf{F}(x_k^*, y_k^*, z_k^*) \cdot \mathbf{n}(x_k^*, y_k^*, z_k^*)\Delta S_k \tag{3}$$

represents the exact net volume of fluid that crosses the surface σ in the direction of its orientation $\mathbf{n}$ per unit of time. The quantity Φ defined by Equation (3) is called the ***flux of F across σ***. The flux can also be expressed as the surface integral

$$\Phi = \iint_{\sigma} \mathbf{F}(x, y, z) \cdot \mathbf{n}(x, y, z)\, dS \tag{4}$$

A positive flux means that in one unit of time a greater volume of fluid passes through σ in the positive direction than in the negative direction, a negative flux means that a greater volume passes through the surface in the negative direction than in the positive direction, and a zero flux means that the same volume passes through the surface in each direction. Integrals of form (4) arise in other contexts as well and are called ***flux integrals***.

REMARK. If the fluid has mass density δ, then $\Phi\delta$ (volume $\times$ density) represents the net mass of fluid that passes through σ per unit of time.

EVALUATING FLUX INTEGRALS

An effective formula for evaluating flux integrals can be obtained by applying Theorem 16.5.1 and using Formula (1) for $\mathbf{n}$. This yields

$$\begin{aligned}
\iint_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_{R} \mathbf{F} \cdot \mathbf{n} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA \\
&= \iint_{R} \mathbf{F} \cdot \frac{\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}}{\left\| \dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v} \right\|} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA \\
&= \iint_{R} \mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) dA
\end{aligned}$$

In summary, we have the following result.

16.6.2 THEOREM. *Let σ be a smooth parametric surface represented by the vector equation $\mathbf{r} = \mathbf{r}(u, v)$ in which (u, v) varies over a region R in the uv-plane. If the component functions of the vector field $\mathbf{F}$ are continuous on σ, and if $\mathbf{n}$ determines the positive orientation of σ, then*

$$\Phi = \iint_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{R} \mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) dA \tag{5}$$

where it is understood that the integrand on the right side of the equation is expressed in terms of u and v.

Example 1 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{k}$ across the downward-oriented sphere $x^2 + y^2 + z^2 = a^2$.

Solution. The sphere with outward positive orientation can be represented by the vector-valued function

$$\mathbf{r}(\phi, \theta) = a \sin\phi \cos\theta \mathbf{i} + a \sin\phi \sin\theta \mathbf{j} + a \cos\phi \mathbf{k} \quad (0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi)$$

From this formula we obtain (see Example 9 of Section 15.4 for the computations)

$$\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} = a^2 \sin^2\phi \cos\theta \mathbf{i} + a^2 \sin^2\phi \sin\theta \mathbf{j} + a^2 \sin\phi \cos\phi \mathbf{k}$$

Moreover, for points on the sphere we have $\mathbf{F} = z\mathbf{k} = a\cos\phi\mathbf{k}$; hence,

$$\mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta}\right) = a^3 \sin\phi \cos^2\phi$$

Thus, it follows from (5) with the parameters u and v replaced by ϕ and θ that

$$\begin{aligned}
\Phi &= \iint_\sigma \mathbf{F} \cdot \mathbf{n}\, dS \\
&= \iint_R \mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta}\right) dA \\
&= \int_0^{2\pi} \int_0^{\pi} a^3 \sin\phi \cos^2\phi \, d\phi\, d\theta \\
&= a^3 \int_0^{2\pi} \left[-\frac{\cos^3\phi}{3}\right]_0^{\pi} d\theta \\
&= \frac{2a^3}{3} \int_0^{2\pi} d\theta = \frac{4\pi a^3}{3}
\end{aligned}$$ ◀

REMARK. Although the computations in this example give a correct result, they are technically flawed in that the parametric representation used for the sphere is not smooth at $\phi = 0$ or $\phi = \pi$ (see Example 9 of Section 15.4). However, this difficulty can be circumvented by cutting holes with a small radius in the sphere around the z-axis (to avoid the problem areas), performing the required computations on the cut surface, and then taking the limit as the radius approaches zero. It can be shown that this leads to the same result that we obtained in our formal computations. In general, no problems occur when Formula (5) is applied directly to spheres that are parametrized as in this example.

REMARK. Reversing the orientation of the surface σ in (5) reverses the sign $\mathbf{n}$, hence the sign of $\mathbf{F} \cdot \mathbf{n}$, and hence reverses the sign of Φ. This can also be seen physically by interpreting the flux integral as the volume of fluid per unit time that crosses σ in the positive direction minus the volume per unit time that crosses in the negative direction—reversing the orientation of σ changes the sign of the difference. Thus, in Example 1 an inward orientation of the sphere would produce a flux of $-4\pi a^3/3$.

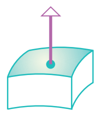
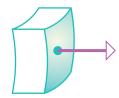
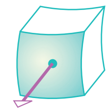
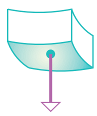
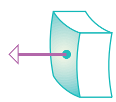
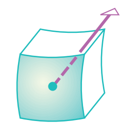
ORIENTATION OF NONPARAMETRIC SURFACES

Nonparametric surfaces of the form $z = g(x, y)$, $y = g(z, x)$, and $x = g(y, z)$ can be expressed parametrically using the independent variables as parameters. More precisely, these surfaces can be represented by the vector equations

$$\underbrace{\mathbf{r} = u\mathbf{i} + v\mathbf{j} + g(u, v)\mathbf{k}}_{\boxed{z = g(x, y)}}, \quad \underbrace{\mathbf{r} = v\mathbf{i} + g(u, v)\mathbf{j} + u\mathbf{k}}_{\boxed{y = g(z, x)}}, \quad \underbrace{\mathbf{r} = g(u, v)\mathbf{i} + u\mathbf{j} + v\mathbf{k}}_{\boxed{x = g(y, z)}} \qquad (6\text{–}8)$$

These representations impose positive and negative orientations on the surfaces in accordance with Formula (1). We leave it as an exercise to calculate $\mathbf{n}$ and $-\mathbf{n}$ in each case and to show that the positive and negative orientations are as shown in Table 16.6.1.

Table 16.6.1

$z = g(x, y)$	$y = g(z, x)$	$x = g(y, z)$
 $\mathbf{n} = \dfrac{-\dfrac{\partial z}{\partial x}\mathbf{i} - \dfrac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1}}$ Positive **k**-component; Positive orientation	 $\mathbf{n} = \dfrac{-\dfrac{\partial y}{\partial x}\mathbf{i} + \mathbf{j} - \dfrac{\partial y}{\partial z}\mathbf{k}}{\sqrt{\left(\dfrac{\partial y}{\partial x}\right)^2 + \left(\dfrac{\partial y}{\partial z}\right)^2 + 1}}$ Positive **j**-component; Positive orientation	 $\mathbf{n} = \dfrac{\mathbf{i} - \dfrac{\partial x}{\partial y}\mathbf{j} - \dfrac{\partial x}{\partial z}\mathbf{k}}{\sqrt{\left(\dfrac{\partial x}{\partial y}\right)^2 + \left(\dfrac{\partial x}{\partial z}\right)^2 + 1}}$ Positive **i**-component; Positive orientation
 $-\mathbf{n} = \dfrac{\dfrac{\partial z}{\partial x}\mathbf{i} + \dfrac{\partial z}{\partial y}\mathbf{j} - \mathbf{k}}{\sqrt{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1}}$ Negative **k**-component; Negative orientation	 $-\mathbf{n} = \dfrac{\dfrac{\partial y}{\partial x}\mathbf{i} - \mathbf{j} + \dfrac{\partial y}{\partial z}\mathbf{k}}{\sqrt{\left(\dfrac{\partial y}{\partial x}\right)^2 + \left(\dfrac{\partial y}{\partial z}\right)^2 + 1}}$ Negative **j**-component; Negative orientation	 $-\mathbf{n} = \dfrac{-\mathbf{i} + \dfrac{\partial x}{\partial y}\mathbf{j} + \dfrac{\partial x}{\partial z}\mathbf{k}}{\sqrt{\left(\dfrac{\partial x}{\partial y}\right)^2 + \left(\dfrac{\partial x}{\partial z}\right)^2 + 1}}$ Negative **i**-component; Negative orientation

The results in Table 16.6.1 can also be obtained using gradients. To see how this can be done, rewrite the equations of the surfaces as

$$z - g(x, y) = 0, \quad y - g(z, x) = 0, \quad x - g(y, z) = 0$$

Each of these equations has the form $G(x, y, z) = 0$ and hence can be viewed as a level surface of a function $G(x, y, z)$. Since the gradient of G is normal to the level surface, it follows that the unit normal $\mathbf{n}$ is either $\nabla G/\|\nabla G\|$ or $-\nabla G/\|\nabla G\|$. However, if $G(x, y, z) = z - g(x, y)$, then ∇G has a **k**-component of 1; if $G(x, y, z) = y - g(z, x)$, then ∇G has a **j**-component of 1; and if $G(x, y, z) = x - g(y, z)$, then ∇G has an **i**-component of 1. Thus, it is evident from Table 16.6.1 that in all three cases we have

$$\mathbf{n} = \frac{\nabla G}{\|\nabla G\|} \tag{9}$$

Moreover, we leave it as an exercise to show that if the surfaces $z = g(x, y)$, $y = g(z, x)$, and $x = g(y, z)$ are expressed in vector forms (6), (7), and (8), then

$$\nabla G = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \tag{10}$$

[compare (1) and (9)]. Thus, we are led to the following version of Theorem 16.6.2 for nonparametric surfaces.

16.6.3 THEOREM. *Let σ be a smooth surface of the form $z = g(x, y)$, $y = g(z, x)$, or $x = g(y, z)$, and suppose that the component functions of the vector field* $\mathbf{F}$ *are continuous on σ. Suppose also that the equation for σ is rewritten as $G(x, y, z) = 0$ by taking g to the left side of the equation, and let R be the projection of σ on the coordinate plane determined by the independent variables of g. If σ has positive orientation, then*

$$\Phi = \iint_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{R} \mathbf{F} \cdot \nabla G\, dA \tag{11}$$

Formula (11) can either be used directly for computations or to derive some more specific formulas for each of the three surface types. For example, if $z = g(x, y)$, then we have $G(x, y, z) = z - g(x, y)$, so

$$\nabla G = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k} = -\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}$$

Substituting this expression for ∇G in (11) and taking R to be the projection of the surface $z = g(x, y)$ on the xy-plane yields

$$\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \mathbf{F}\cdot\left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}\right) dA \qquad (12)$$

σ of the form $z = f(x, y)$ and oriented up

$$\iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \mathbf{F}\cdot\left(\frac{\partial z}{\partial x}\mathbf{i} + \frac{\partial z}{\partial y}\mathbf{j} - \mathbf{k}\right) dA \qquad (13)$$

σ of the form $z = f(x, y)$ and oriented down

The derivations of the corresponding formulas when $y = g(z, x)$ and $x = g(y, z)$ are left as exercises.

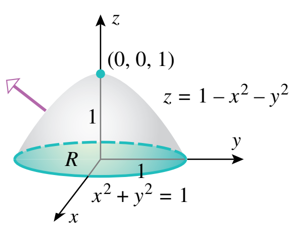


Figure 16.6.9

Example 2 Let σ be the portion of the surface $z = 1 - x^2 - y^2$ that lies above the xy-plane, and suppose that σ is oriented up, as shown in Figure 16.6.9. Find the flux of the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across σ.

Solution. From (12) the flux Φ is given by

$$\begin{aligned}\Phi = \iint_\sigma \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_R \mathbf{F}\cdot\left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}\right) dA\\ &= \iint_R (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\cdot(2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\,dA\\ &= \iint_R (x^2 + y^2 + 1)\,dA\\ &= \int_0^{2\pi}\int_0^1 (r^2+1)r\,dr\,d\theta\\ &= \int_0^{2\pi}\left(\frac{3}{4}\right)d\theta = \frac{3\pi}{2}\end{aligned}$$

Since $z = 1 - x^2 - y^2$ on the surface

Using polar coordinates to evaluate the integral

◀

EXERCISE SET 16.6

1. Suppose that the surface σ of the unit cube in the accompanying figure has an outward orientation. In each part, determine whether the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{j}$ across the specified face is positive, negative, or zero.
   (a) The face $x = 1$ (b) The face $x = 0$
   (c) The face $y = 1$ (d) The face $y = 0$
   (e) The face $z = 1$ (f) The face $z = 0$

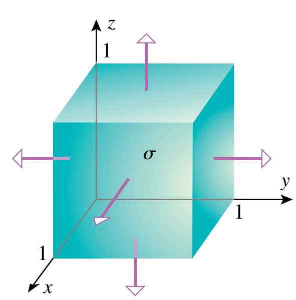


Figure Ex-1

2. Answer the questions posed in Exercise 1 for the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{k}$.

3. Answer the questions posed in Exercise 1 for the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

4. What is the flux of the constant vector field $\mathbf{F}(x, y, z) = \mathbf{i}$ across the entire surface σ in Figure Ex-1? Explain your reasoning.

5. Let σ be the cylindrical surface that is represented by the vector-valued function $\mathbf{r}(u, v) = \cos v\mathbf{i} + \sin v\mathbf{j} + u\mathbf{k}$ with $0 \le u \le 1$ and $0 \le v \le 2\pi$.
(a) Find the unit normal $\mathbf{n} = \mathbf{n}(u, v)$ that defines the positive orientation of σ.
(b) Is the positive orientation inward or outward? Justify your answer.

6. Let σ be the conical surface that is represented by the parametric equations $x = r\cos\theta$, $y = r\sin\theta$, $z = r$ with $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.
(a) Find the unit normal $\mathbf{n} = \mathbf{n}(r, \theta)$ that defines the positive orientation of σ.
(b) Is the positive orientation inward or outward? Justify your answer.

In Exercises 7–12, find the flux of the vector field $\mathbf{F}$ across σ.

7. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$; σ is the portion of the surface $z = 1 - x^2 - y^2$ above the xy-plane, oriented by upward normals.

8. $\mathbf{F}(x, y, z) = (x + y)\mathbf{i} + (y + z)\mathbf{j} + (z + x)\mathbf{k}$; σ is the portion of the plane $x + y + z = 1$ in the first octant, oriented by unit normals with positive components.

9. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$; σ is the portion of the cone $z^2 = x^2 + y^2$ between the planes $z = 1$ and $z = 2$, oriented by upward unit normals.

10. $\mathbf{F}(x, y, z) = y\mathbf{j} + \mathbf{k}$; σ is the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 4$, oriented by downward unit normals.

11. $\mathbf{F}(x, y, z) = x\mathbf{k}$; the surface σ is the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = y$, oriented by downward unit normals.

12. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + yx\mathbf{j} + zx\mathbf{k}$; σ is the portion of the plane $6x + 3y + 2z = 6$ in the first octant, oriented by unit normals with positive components.

In Exercises 13–16, find the flux of the vector field $\mathbf{F}$ across σ in the direction of positive orientation.

13. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$; σ is the portion of the paraboloid

$$\mathbf{r}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + (1 - u^2)\mathbf{k}$$

with $1 \le u \le 2$, $0 \le v \le 2\pi$.

14. $\mathbf{F}(x, y, z) = e^{-y}\mathbf{i} - y\mathbf{j} + x\sin z\mathbf{k}$; σ is the portion of the elliptic cylinder

$$\mathbf{r}(u, v) = 2\cos v\mathbf{i} + \sin v\mathbf{j} + u\mathbf{k}$$

with $0 \le u \le 5$, $0 \le v \le 2\pi$.

15. $\mathbf{F}(x, y, z) = \sqrt{x^2 + y^2}\,\mathbf{k}$; σ is the portion of the cone

$$\mathbf{r}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + 2u\mathbf{k}$$

with $0 \le u \le \sin v$, $0 \le v \le \pi$.

16. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; σ is the portion of the sphere

$$\mathbf{r}(u, v) = 2\sin u\cos v\mathbf{i} + 2\sin u\sin v\mathbf{j} + 2\cos u\mathbf{k}$$

with $0 \le u \le \pi/3$, $0 \le v \le 2\pi$.

17. Let σ be the surface of the cube bounded by the planes $x = \pm 1$, $y = \pm 1$, $z = \pm 1$, oriented by outward unit normals. In each part, find the flux of $\mathbf{F}$ across σ.
(a) $\mathbf{F}(x, y, z) = x\mathbf{i}$
(b) $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
(c) $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

18. Let σ be the closed surface consisting of the portion of the paraboloid $z = x^2 + y^2$ for which $0 \le z \le 1$ and capped by the disk $x^2 + y^2 \le 1$ in the plane $z = 1$. Find the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{j} - y\mathbf{k}$ in the outward direction across σ.

In Exercises 19 and 20, find the flux of $\mathbf{F}$ across σ by expressing σ parametrically.

19. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$; the surface σ is the portion of the cone $z = \sqrt{x^2 + y^2}$ below the plane $z = 1$, oriented by downward unit normals.

20. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; σ is the portion of the cylinder $x^2 + z^2 = 1$ between the planes $y = 1$ and $y = -2$, oriented by outward unit normals.

21. Let x, y, and z be measured in meters, and suppose that $\mathbf{F}(x, y, z) = 2x\mathbf{i} - 3y\mathbf{j} + z\mathbf{k}$ be the velocity vector (in m/s) of a fluid particle at the point (x, y, z) in a steady-state fluid flow.
(a) Find the net volume of fluid that passes in the upward direction through the portion of the plane $x + y + z = 1$ in the first octant in 1 s.
(b) Assuming that the fluid has a mass density of 806 kg/m^3, find the net mass of fluid that passes in the upward direction through the surface in part (a) in 1 s.

22. Let x, y, and z be measured in meters, and suppose that $\mathbf{F}(x, y, z) = -y\mathbf{i} + z\mathbf{j} + 3x\mathbf{k}$ is the velocity vector (in m/s) of a fluid particle at the point (x, y, z) in a steady-state incompressible fluid flow.
(a) Find the net volume of fluid that passes in the upward direction through the hemisphere $z = \sqrt{9 - x^2 - y^2}$ in 1 s.
(b) Assuming that the fluid has a mass density of 1060 kg/m^3, find the net mass of fluid that passes in the upward direction through the surface in part (a) in 1 s.

23. (a) Derive the analogs of Formulas (12) and (13) for surfaces of the form $x = g(y, z)$.
(b) Let σ be the portion of the paraboloid $x = y^2 + z^2$ for $x \le 1$ and $z \ge 0$ oriented by unit normals with negative

x-components. Use the result in part (a) to find the flux of

$$\mathbf{F}(x, y, z) = y\mathbf{i} - z\mathbf{j} + 8\mathbf{k}$$

across σ.

24. (a) Derive the analogs of Formulas (12) and (13) for surfaces of the form $y = g(z, x)$.
(b) Let σ be the portion of the paraboloid $y = z^2 + x^2$ for $y \leq 1$ and $z \geq 0$ oriented by unit normals with positive y-components. Use the result in part (a) to find the flux of

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

across σ.

25. Let $\mathbf{F} = \|\mathbf{r}\|^k\mathbf{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and k is a constant. (Note that if $k = -3$, this is an inverse-square field.) Let σ be the sphere of radius a centered at the origin and oriented by the outward normal $\mathbf{n} = \mathbf{r}/\|\mathbf{r}\| = \mathbf{r}/a$.
(a) Find the flux of $\mathbf{F}$ across σ without performing any integrations. [*Hint:* The surface area of a sphere of radius a is $4\pi a^2$.]
(b) For what value of k is the flux independent of the radius of the sphere?

26. Let

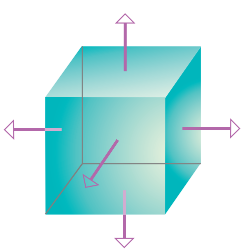
$$\mathbf{F}(x, y, z) = a^2x\mathbf{i} + (y/a)\mathbf{j} + az^2\mathbf{k}$$

and let σ be the sphere of radius 1, centered at the origin and oriented outward. Approximate all values of a such that the flux of $\mathbf{F}$ across σ is 10.

16.7 THE DIVERGENCE THEOREM

In this section we will be concerned with flux across surfaces, such as spheres, that "enclose" a region of space. We will show that the flux across such surfaces can be expressed in terms of the divergence of the vector field, and we will use this result to give a physical interpretation of the concept of divergence.

ORIENTATION OF PIECEWISE SMOOTH CLOSED SURFACES



Box with outward orientation

Figure 16.7.1

In the last section we studied flux across general surfaces. Here we will be concerned exclusively with surfaces that are boundaries of finite solids—the surface of a solid sphere, the surface of a solid box, or the surface of a solid cylinder, for example. Such surfaces are said to be ***closed***. A closed surface may or may not be smooth, but most of the surfaces that arise in applications are ***piecewise smooth***; that is, they consist of finitely many smooth surfaces joined together at the edges (a box, for example). We will limit our discussion to piecewise smooth surfaces that can be assigned an ***inward orientation*** (toward the interior of the solid) and an ***outward orientation*** (away from the interior). It is very difficult to make this concept mathematically precise, but the basic idea is that each piece of the surface is orientable, and oriented pieces fit together in such a way that the entire surface can be assigned an orientation (Figure 16.7.1).

THE DIVERGENCE THEOREM

In Section 16.1 we defined the divergence of a vector field

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

as

$$\operatorname{div} \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

but we did not attempt to give a physical explanation of its meaning at that time. The following result, known as the ***Divergence Theorem*** or ***Gauss's*** * ***Theorem***, will provide us with a physical interpretation of divergence in the context of fluid flow.

*See biography on page 1148.

16.7.1 THEOREM (*The Divergence Theorem*). *Let G be a solid whose surface σ is oriented outward. If*

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

where f, g, and h have continuous first partial derivatives on some open set containing G, and if $\mathbf{n}$ is the outward unit normal on σ, then

$$\iint_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_{G} \operatorname{div} \mathbf{F}\, dV \tag{1}$$

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*CARL FRIEDRICH GAUSS (1777–1855). German mathematician and scientist. Sometimes called the "prince of mathematicians," Gauss ranks with Newton and Archimedes as one of the three greatest mathematicians who ever lived. His father, a laborer, was an uncouth but honest man who would have liked Gauss to take up a trade such as gardening or bricklaying; but the boy's genius for mathematics was not to be denied. In the entire history of mathematics there may never have been a child so precocious as Gauss—by his own account he worked out the rudiments of arithmetic before he could talk. One day, before he was even three years old, his genius became apparent to his parents in a very dramatic way. His father was preparing the weekly payroll for the laborers under his charge while the boy watched quietly from a corner. At the end of the long and tedious calculation, Gauss informed his father that there was an error in the result and stated the answer, which he had worked out in his head. To the astonishment of his parents, a check of the computations showed Gauss to be correct!

For his elementary education Gauss was enrolled in a squalid school run by a man named Büttner whose main teaching technique was thrashing. Büttner was in the habit of assigning long addition problems which, unknown to his students, were arithmetic progressions that he could sum up using formulas. On the first day that Gauss entered the arithmetic class, the students were asked to sum the numbers from 1 to 100. But no sooner had Büttner stated the problem than Gauss turned over his slate and exclaimed in his peasant dialect, "Ligget se'." (Here it lies.) For nearly an hour Büttner glared at Gauss, who sat with folded hands while his classmates toiled away. When Büttner examined the slates at the end of the period, Gauss's slate contained a single number, 5050—the only correct solution in the class. To his credit, Büttner recognized the genius of Gauss and with the help of his assistant, John Bartels, had him brought to the attention of Karl Wilhelm Ferdinand, Duke of Brunswick. The shy and awkward boy, who was then fourteen, so captivated the Duke that he subsidized him through preparatory school, college, and the early part of his career.

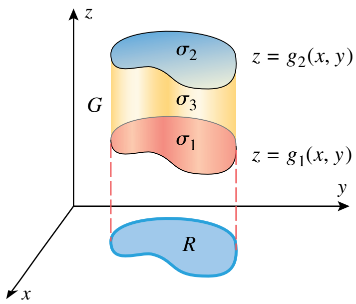
From 1795 to 1798 Gauss studied mathematics at the University of Göttingen, receiving his degree in absentia from the University of Helmstadt. For his dissertation, he gave the first complete proof of the fundamental theorem of algebra, which states that every polynomial equation has as many solutions as its degree. At age 19 he solved a problem that baffled Euclid, inscribing a regular polygon of 17 sides in a circle using straightedge and compass; and in 1801, at age 24, he published his first masterpiece, *Disquisitiones Arithmeticae*, considered by many to be one of the most brilliant achievements in mathematics. In that book Gauss systematized the study of number theory (properties of the integers) and formulated the basic concepts that form the foundation of that subject.

In the same year that the *Disquisitiones* was published, Gauss again applied his phenomenal computational skills in a dramatic way. The astronomer Giuseppi Piazzi had observed the asteroid Ceres for $\frac{1}{40}$ of its orbit, but lost it in the Sun. Using only three observations and the "method of least squares" that he had developed in 1795, Gauss computed the orbit with such accuracy that astronomers had no trouble relocating it the following year. This achievement brought him instant recognition as the premier mathematician in Europe, and in 1807 he was made Professor of Astronomy and head of the astronomical observatory at Göttingen.

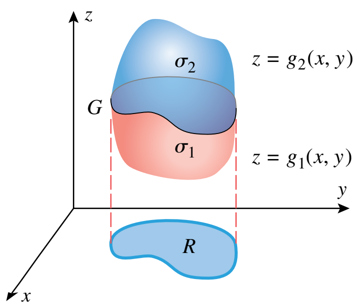
In the years that followed, Gauss revolutionized mathematics by bringing to it standards of precision and rigor undreamed of by his predecessors. He had a passion for perfection that drove him to polish and rework his papers rather than publish less finished work in greater numbers—his favorite saying was "Pauca, sed matura" (Few, but ripe). As a result, many of his important discoveries were squirreled away in diaries that remained unpublished until years after his death.

Among his myriad achievements, Gauss discovered the Gaussian or "bell-shaped" error curve fundamental in probability, gave the first geometric interpretation of complex numbers and established their fundamental role in mathematics, developed methods of characterizing surfaces intrinsically by means of the curves that they contain, developed the theory of conformal (angle-preserving) maps, and discovered non-Euclidean geometry 30 years before the ideas were published by others. In physics he made major contributions to the theory of lenses and capillary action, and with Wilhelm Weber he did fundamental work in electromagnetism. Gauss invented the heliotrope, bifilar magnetometer, and an electrotelegraph.

Gauss was deeply religious and aristocratic in demeanor. He mastered foreign languages with ease, read extensively, and enjoyed mineralogy and botany as hobbies. He disliked teaching and was usually cool and discouraging to other mathematicians, possibly because he had already anticipated their work. It has been said that if Gauss had published all of his discoveries, the current state of mathematics would be advanced by 50 years. He was without a doubt the greatest mathematician of the modern era.



(*a*)



(*b*)

Figure 16.7.2

The proof of this theorem for a general solid G is too difficult to present here. However, we can give a proof for the special case where G is simultaneously a simple xy-solid, a simple yz-solid, and a simple zx-solid (see Figure 15.5.3 and the related discussion for terminology).

Proof. Suppose that G has upper surface $z = g_2(x, y)$, lower surface $z = g_1(x, y)$, and projection R on the xy-plane. Let σ_1 denote the lower surface, σ_2 the upper surface, and σ_3 the lateral surface (Figure 16.7.2*a*). If the upper surface and lower surface meet as in Figure 16.7.2*b*, then there is no lateral surface σ_3. Our proof will allow for both cases shown in those figures.

Formula (1) can be expressed as

$$\iint_{\sigma} [f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}] \cdot \mathbf{n}\, dS = \iiint_{G} \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dV$$

so it suffices to prove the three equalities

$$\iint_{\sigma} f(x, y, z)\mathbf{i} \cdot \mathbf{n}\, dS = \iiint_{G} \frac{\partial f}{\partial x}\, dV \tag{2a}$$

$$\iint_{\sigma} g(x, y, z)\mathbf{j} \cdot \mathbf{n}\, dS = \iiint_{G} \frac{\partial g}{\partial y}\, dV \tag{2b}$$

$$\iint_{\sigma} h(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS = \iiint_{G} \frac{\partial h}{\partial z}\, dV \tag{2c}$$

Since the proofs of all three equalities are similar, we will prove only the third.

It follows from Theorem 15.5.2 that

$$\iiint_{G} \frac{\partial h}{\partial z}\, dV = \iint_{R} \left[\int_{g_1(x,y)}^{g_2(x,y)} \frac{\partial h}{\partial z} dz\right] dA = \iint_{R} \Big[h(x, y, z)\Big]_{z=g_1(x,y)}^{g_2(x,y)} dA$$

so

$$\iiint_{G} \frac{\partial h}{\partial z}\, dV = \iint_{R} [h(x, y, g_2(x, y)) - h(x, y, g_1(x, y))]\, dA \tag{3}$$

Next we will evaluate the surface integral in (2c) by integrating over each surface of G separately. If there is a lateral surface σ_3, then at each point of this surface $\mathbf{k} \cdot \mathbf{n} = 0$ since $\mathbf{n}$ is horizontal and $\mathbf{k}$ is vertical. Thus,

$$\iint_{\sigma_3} h(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS = 0$$

Therefore, regardless of whether G has a lateral surface, we can write

$$\iint_{\sigma} h(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS = \iint_{\sigma_1} h(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS + \iint_{\sigma_2} h(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS \tag{4}$$

On the upper surface σ_2, the outer normal is an upward normal, and on the lower surface σ_1, the outer normal is a downward normal. Thus, Formulas (12) and (13) of Section 16.6 imply that

$$\begin{aligned}\iint_{\sigma_2} h(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS &= \iint_{R} h(x, y, g_2(x, y))\mathbf{k} \cdot \left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}\right) dA \\ &= \iint_{R} h(x, y, g_2(x, y))\, dA \end{aligned} \tag{5}$$

and

$$\iint_{\sigma_1} h(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS = \iint_R h(x, y, g_1(x, y))\mathbf{k} \cdot \left(\frac{\partial z}{\partial x}\mathbf{i} + \frac{\partial z}{\partial y}\mathbf{j} - \mathbf{k}\right) dA$$

$$= -\iint_R h(x, y, g_1(x, y))\, dA \tag{6}$$

Substituting (5) and (6) into (4) and combining the terms into a single integral yields

$$\iint_\sigma h(x, y, z)\mathbf{k} \cdot \mathbf{n}\, dS = \iint_R [h(x, y, g_2(x, y)) - h(x, y, g_1(x, y))]\, dA \tag{7}$$

Equation (2c) now follows from (3) and (7).

In words, the Divergence Theorem states:

The flux of a vector field across a closed surface with outward orientation is equal to the triple integral of the divergence over the region enclosed by the surface.

This is sometimes called the ***outward flux*** across the surface.

USING THE DIVERGENCE THEOREM TO FIND FLUX

Sometimes it is easier to find the flux across a closed surface by using the Divergence Theorem than by evaluating the flux integral directly. This is illustrated in the following example.

Example 1 Use the Divergence Theorem to find the outward flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{k}$ across the sphere $x^2 + y^2 + z^2 = a^2$.

Solution. Let σ denote the outward-oriented spherical surface and G the region that it encloses. The divergence of the vector field is

$$\text{div}\, \mathbf{F} = \frac{\partial z}{\partial z} = 1$$

so from (1) the flux across σ is

$$\Phi = \iint_\sigma \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_G dV = \text{volume of } G = \frac{4\pi a^3}{3}$$

Note how much simpler this calculation is than that in Example 1 of Section 16.6. ◀

The Divergence Theorem is usually the method of choice for finding the flux across closed piecewise smooth surfaces with multiple sections, since it eliminates the need for a separate integral evaluation over each section. This is illustrated in the next three examples.

Example 2 Use the Divergence Theorem to find the outward flux of the vector field

$$\mathbf{F}(x, y, z) = 2x\mathbf{i} + 3y\mathbf{j} + z^2\mathbf{k}$$

across the unit cube in Figure 16.7.3.

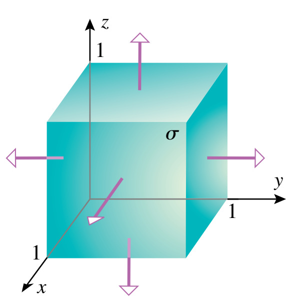


Figure 16.7.3

Solution. Let σ denote the outward-oriented surface of the cube and G the region that it encloses. The divergence of the vector field is

$$\text{div}\, \mathbf{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(z^2) = 5 + 2z$$

so from (1) the flux across σ is

$$\Phi = \iint_{\sigma} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_{G} (5+2z)\,dV = \int_0^1\int_0^1\int_0^1 (5+2z)\,dz\,dy\,dx$$

$$= \int_0^1\int_0^1 \left[5z+z^2\right]_{z=0}^{1} dy\,dx = \int_0^1\int_0^1 6\,dy\,dx = 6 \quad ◀$$

Example 3 Use the Divergence Theorem to find the outward flux of the vector field

$$\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^2\mathbf{k}$$

across the surface of the region that is enclosed by the circular cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 2$ (Figure 16.7.4).

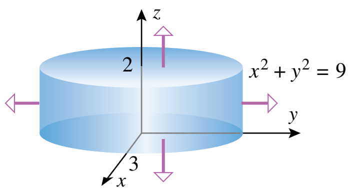


Figure 16.7.4

Solution. Let σ denote the outward-oriented surface and G the region that it encloses. The divergence of the vector field is

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^2) = 3x^2 + 3y^2 + 2z$$

so from (1) the flux across σ is

$$\Phi = \iint_{\sigma} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_{G} (3x^2+3y^2+2z)\,dV$$

$$= \int_0^{2\pi}\int_0^3\int_0^2 (3r^2+2z)r\,dz\,dr\,d\theta$$

Using cylindrical coordinates

$$= \int_0^{2\pi}\int_0^3 \left[3r^3z + z^2r\right]_{z=0}^{2} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 (6r^3+4r)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{3r^4}{2} + 2r^2\right]_0^3 d\theta$$

$$= \int_0^{2\pi} \frac{279}{2}\,d\theta = 279\pi \quad ◀$$

Example 4 Use the Divergence Theorem to find the outward flux of the vector field

$$\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$$

across the surface of the region that is enclosed by the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ and the plane $z = 0$ (Figure 16.7.5).

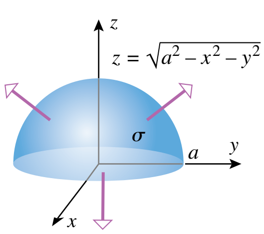


Figure 16.7.5

Solution. Let σ denote the outward-oriented surface and G the region that it encloses. The divergence of the vector field is

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3x^2 + 3y^2 + 3z^2$$

so from (1) the flux across σ is

$$
\begin{aligned}
\Phi = \iint_{\sigma} \mathbf{F}\cdot\mathbf{n}\,dS &= \iiint_G (3x^2+3y^2+3z^2)\,dV \\
&= \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (3\rho^2)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \qquad \boxed{\text{Using spherical coordinates}} \\
&= 3\int_0^{2\pi}\int_0^{\pi/2}\int_0^a \rho^4\sin\phi\,d\rho\,d\phi\,d\theta \\
&= 3\int_0^{2\pi}\int_0^{\pi/2}\left[\frac{\rho^5}{5}\sin\phi\right]_{\rho=0}^{a} d\phi\,d\theta \\
&= \frac{3a^5}{5}\int_0^{2\pi}\int_0^{\pi/2}\sin\phi\,d\phi\,d\theta \\
&= \frac{3a^5}{5}\int_0^{2\pi}[-\cos\phi]_0^{\pi/2}\,d\theta \\
&= \frac{3a^5}{5}\int_0^{2\pi} d\theta = \frac{6\pi a^5}{5}
\end{aligned}
$$

◀

DIVERGENCE VIEWED AS FLUX DENSITY

The Divergence Theorem provides a way of interpreting the divergence of a vector field $\mathbf{F}$. Suppose that G is a *small* spherical region centered at the point P_0 and that its surface, denoted by $\sigma(G)$, is oriented outward. Denote the volume of the region by $\text{vol}(G)$ and the flux of $\mathbf{F}$ across $\sigma(G)$ by $\Phi(G)$. If div $\mathbf{F}$ is continuous on G, then across the small region G the value of div $\mathbf{F}$ will not vary much from its value div $\mathbf{F}(P_0)$ at the center, and we can reasonably approximate div $\mathbf{F}$ by the constant div $\mathbf{F}(P_0)$ on G. Thus, the Divergence Theorem implies that the flux $\Phi(G)$ of $\mathbf{F}$ across $\sigma(G)$ can be approximated as

$$
\Phi(G) = \iint_{\sigma(G)} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_G \operatorname{div}\mathbf{F}\,dV \approx \operatorname{div}\mathbf{F}(P_0)\iiint_G dV = \operatorname{div}\mathbf{F}(P_0)\,\text{vol}(G)
$$

from which we obtain the approximation

$$
\operatorname{div}\mathbf{F}(P_0) \approx \frac{\Phi(G)}{\text{vol}(G)} \tag{8}
$$

The expression on the right side of (8) is called the ***outward flux density of* F *across G***. If we now let the radius of the sphere approach zero [so that $\text{vol}(G)$ approaches zero], then it is plausible that the error in this approximation will approach zero, and the divergence of $\mathbf{F}$ at the point P_0 will be given exactly by

$$
\operatorname{div}\mathbf{F}(P_0) = \lim_{\text{vol}(G)\to 0}\frac{\Phi(G)}{\text{vol}(G)}
$$

which we can express as

$$
\operatorname{div}\mathbf{F}(P_0) = \lim_{\text{vol}(G)\to 0}\frac{1}{\text{vol}(G)}\iint_{\sigma(G)}\mathbf{F}\cdot\mathbf{n}\,dS \tag{9}
$$

This limit, which is called the ***outward flux density of* F *at* P_0**, tells us that *in a steady-state fluid flow*, div $\mathbf{F}$ *can be interpreted as the limiting flux per unit volume at a point*. Moreover, it follows from (8) that for a small spherical region G centered at a point P_0 in the flow, the outward flux across the surface of G can be approximated as

$$
\Phi(G) \approx (\operatorname{div}\mathbf{F}(P_0))(\text{vol}(G)) \tag{10}
$$

REMARK. Formula (9) is sometimes taken as the definition of divergence. This is a useful alternative to Definition 16.1.4 because it does not require a coordinate system.

SOURCES AND SINKS

If P_0 is a point in an incompressible fluid at which div $\mathbf{F}(P_0) > 0$, then it follows from (8) that $\Phi(G) > 0$ for a sufficiently small sphere G centered at P_0. Thus, there is a greater volume of fluid going out through the surface of G than coming in. But this can only happen if there is some point *inside* the sphere at which fluid is entering the flow (say by condensation, melting of a solid, or a chemical reaction); otherwise the net outward flow through the surface would result in a decrease in density within the sphere, contradicting the incompressibility assumption. Similarly, if div $\mathbf{F}(P_0) < 0$, there would have to be a point *inside* the sphere at which fluid is leaving the flow (say by evaporation); otherwise the net inward flow through the surface would result in an increase in density within the sphere. In an incompressible fluid, points at which div $\mathbf{F}(P_0) > 0$ are called ***sources*** and points at which div $\mathbf{F}(P_0) < 0$ are called ***sinks***. Fluid enters the flow at a source and drains out at a sink. In an incompressible fluid without sources or sinks we must have

$$\operatorname{div} \mathbf{F}(P) = 0$$

at every point P. In hydrodynamics this is called the ***continuity equation for incompressible fluids*** and is sometimes taken as the defining characteristic of an incompressible fluid.

GAUSS'S LAW FOR INVERSE-SQUARE FIELDS

The Divergence Theorem applied to inverse-square fields (see Definition 16.1.2) produces a result called ***Gauss's Law for Inverse-Square Fields***. This result is the basis for many important principles in physics.

16.7.2 GAUSS'S LAW FOR INVERSE-SQUARE FIELDS. If

$$\mathbf{F}(\mathbf{r}) = \frac{c}{\|\mathbf{r}\|^3}\mathbf{r}$$

is an inverse-square field in 3-space, and if σ is a closed orientable surface that surrounds the origin, then the outward flux of $\mathbf{F}$ across σ is

$$\Phi = \iint_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS = 4\pi c \tag{11}$$

Recall from Formula (5) of Section 16.1 that $\mathbf{F}$ can be expressed in component form as

$$\mathbf{F}(x, y, z) = \frac{c}{(x^2 + y^2 + z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \tag{12}$$

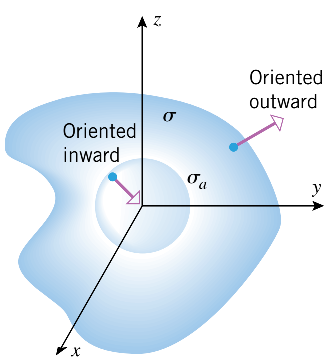


Figure 16.7.6

Since the components of $\mathbf{F}$ are not continuous at the origin, we cannot apply the Divergence Theorem across the solid enclosed by σ. However, we can circumvent this difficulty by constructing a sphere of radius a centered at the origin, where the radius is sufficiently small that the sphere lies entirely within the region enclosed by σ (Figure 16.7.6). We will denote the surface of this sphere by σ_a. The solid G enclosed between σ_a and σ is an example of a three-dimensional solid with an internal "cavity." Just as we were able to extend Green's Theorem to multiply connected regions in the plane (regions with holes), so it is possible to extend the Divergence Theorem to solids in 3-space with internal cavities, provided the surface integral in the theorem is taken over the *entire* boundary with the outside boundary of the solid oriented outward and the boundaries of the cavities oriented inward. Thus, if $\mathbf{F}$ is the inverse-square field in (12), and if σ_a is oriented inward, then the Divergence Theorem yields

$$\iiint_G \operatorname{div} \mathbf{F}\, dV = \iint_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS + \iint_{\sigma_a} \mathbf{F} \cdot \mathbf{n}\, dS \tag{13}$$

But we showed in Example 5 of Section 16.1 that div $\mathbf{F} = 0$, so (13) yields

$$\iint_{\sigma} \mathbf{F} \cdot \mathbf{n}\, dS = -\iint_{\sigma_a} \mathbf{F} \cdot \mathbf{n}\, dS \tag{14}$$

We can evaluate the surface integral over σ_a by expressing the integrand in terms of components; however, it is easier to leave it in vector form. At each point on the sphere the unit normal $\mathbf{n}$ points inward along a radius from the origin, and hence $\mathbf{n} = -\mathbf{r}/\|\mathbf{r}\|$. Thus, (14) yields

$$\begin{aligned}
\iint_{\sigma} \mathbf{F}\cdot\mathbf{n}\,dS &= -\iint_{\sigma_a} \frac{c}{\|\mathbf{r}\|^3}\mathbf{r}\cdot\left(-\frac{\mathbf{r}}{\|\mathbf{r}\|}\right) dS \\
&= \iint_{\sigma_a} \frac{c}{\|\mathbf{r}\|^4}(\mathbf{r}\cdot\mathbf{r})\,dS \\
&= \iint_{\sigma_a} \frac{c}{\|\mathbf{r}\|^2}\,dS \\
&= \frac{c}{a^2}\iint_{\sigma_a} dS && \boxed{\|\mathbf{r}\| = a \text{ on } \sigma_a} \\
&= \frac{c}{a^2}(4\pi a^2) && \boxed{\text{The integral is the surface area of the sphere.}} \\
&= 4\pi c
\end{aligned}$$

which establishes (11).

GAUSS'S LAW IN ELECTROSTATICS

It follows from Example 1 of Section 16.1 with $q = 1$ that a single charged particle of charge Q located at the origin creates an inverse-square field

$$\mathbf{F}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0\|\mathbf{r}\|^3}\mathbf{r}$$

in which $\mathbf{F}(\mathbf{r})$ is the electrical force exerted by Q on a unit positive charge ($q = 1$) located at the point with position vector $\mathbf{r}$. In this case Gauss's law (16.7.2) states that the outward flux Φ across any closed orientable surface σ that surrounds Q is

$$\Phi = \iint_{\sigma} \mathbf{F}\cdot\mathbf{n}\,dS = 4\pi\left(\frac{Q}{4\pi\epsilon_0}\right) = \frac{Q}{\epsilon_0}$$

This result, which is called ***Gauss's Law for Electric Fields***, can be extended to more than one charge. It is one of the fundamental laws in electricity and magnetism.

EXERCISE SET 16.7 C CAS

In Exercises 1–4, verify Formula (1) in the Divergence Theorem by evaluating the surface integral and the triple integral.

1. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; σ is the surface of the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

2. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; σ is the spherical surface $x^2 + y^2 + z^2 = 1$.

3. $\mathbf{F}(x, y, z) = 2x\mathbf{i} - yz\mathbf{j} + z^2\mathbf{k}$; the surface σ is the paraboloid $z = x^2 + y^2$ capped by the disk $x^2 + y^2 \le 1$ in the plane $z = 1$.

4. $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$; σ is the surface of the cube bounded by the planes $x = 0$, $x = 2$, $y = 0$, $y = 2$, $z = 0$, $z = 2$.

In Exercises 5–15, use the Divergence Theorem to find the flux of $\mathbf{F}$ across the surface σ with outward orientation.

5. $\mathbf{F}(x, y, z) = (x^2 + y)\mathbf{i} + z^2\mathbf{j} + (e^y - z)\mathbf{k}$; σ is the surface of the rectangular solid bounded by the coordinate planes and the planes $x = 3$, $y = 1$, and $z = 2$.

6. $\mathbf{F}(x, y, z) = z^3\mathbf{i} - x^3\mathbf{j} + y^3\mathbf{k}$, where σ is the sphere $x^2 + y^2 + z^2 = a^2$.

7. $\mathbf{F}(x, y, z) = (x - z)\mathbf{i} + (y - x)\mathbf{j} + (z - y)\mathbf{k}$; σ is the surface of the cylindrical solid bounded by $x^2 + y^2 = a^2$, $z = 0$, and $z = 1$.

8. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; σ is the surface of the solid bounded by the paraboloid $z = 1 - x^2 - y^2$ and the xy-plane.

9. $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$; σ is the surface of the cylindrical solid bounded by $x^2 + y^2 = 4$, $z = 0$, and $z = 3$.

10. $\mathbf{F}(x, y, z) = (x^2 + y)\mathbf{i} + xy\mathbf{j} - (2xz + y)\mathbf{k}$; σ is the surface of the tetrahedron in the first octant bounded by $x+y+z = 1$ and the coordinate planes.

11. $\mathbf{F}(x, y, z) = (x^3 - e^y)\mathbf{i} + (y^3 + \sin z)\mathbf{j} + (z^3 - xy)\mathbf{k}$, where σ is the surface of the solid bounded by $z = \sqrt{4 - x^2 - y^2}$ and the xy-plane. [*Hint:* Use spherical coordinates.]

12. $\mathbf{F}(x, y, z) = 2xz\mathbf{i} + yz\mathbf{j} + z^2\mathbf{k}$, where σ is the surface of the hemispherical solid bounded above by $z = \sqrt{a^2 - x^2 - y^2}$ and below by the xy-plane.

13. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$; σ is the surface of the conical solid bounded by $z = \sqrt{x^2 + y^2}$ and $z = 1$.

14. $\mathbf{F}(x, y, z) = x^2y\mathbf{i} - xy^2\mathbf{j} + (z+2)\mathbf{k}$; σ is the surface of the solid bounded above by the plane $z = 2x$ and below by the paraboloid $z = x^2 + y^2$.

15. $\mathbf{F}(x, y, z) = x^3\mathbf{i} + x^2y\mathbf{j} + xy\mathbf{k}$; σ is the surface of the solid bounded by $z = 4 - x^2$, $y + z = 5$, $z = 0$, and $y = 0$.

16. Let $\mathbf{F}(x, y, z) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ be a constant vector field and let σ be the surface a solid G. Use the Divergence Theorem to show that the flux of $\mathbf{F}$ across σ is zero. Give an informal physical explanation of this result.

17. Prove that if $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and σ is the surface of a solid G oriented by outward unit normals, then

$$\operatorname{vol}(G) = \frac{1}{3}\iint_{\sigma} \mathbf{r} \cdot \mathbf{n}\, dS$$

where vol(G) is the volume of G.

18. Use the result in Exercise 17 to find the outward flux of the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across the surface σ of the cylindrical solid bounded by $x^2 + 4x + y^2 = 5$, $z = -1$, and $z = 4$.

In Exercises 19–23, prove the identity, assuming that $\mathbf{F}$, σ, and G satisfy the hypotheses of the Divergence Theorem and that all necessary differentiability requirements for the functions $f(x, y, z)$ and $g(x, y, z)$ are met.

19. $\displaystyle\iint_{\sigma} \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS = 0$ [*Hint:* See Exercise 33, Section 16.1.]

20. $\displaystyle\iint_{\sigma} \nabla f \cdot \mathbf{n}\, dS = \iiint_{G} \nabla^2 f\, dV$

$$\left(\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right)$$

21. $\displaystyle\iint_{\sigma} (f\nabla g) \cdot \mathbf{n}\, dS = \iiint_{G} (f\nabla^2 g + \nabla f \cdot \nabla g)\, dV$

22. $\displaystyle\iint_{\sigma} (f\nabla g - g\nabla f) \cdot \mathbf{n}\, dS = \iiint_{G} (f\nabla^2 g - g\nabla^2 f)\, dV$

[*Hint:* Interchange f and g in 21.]

23. $\displaystyle\iint_{\sigma} (f\mathbf{n}) \cdot \mathbf{v}\, dS = \iiint_{G} \nabla f \cdot \mathbf{v}\, dV$ ($\mathbf{v}$ a fixed vector)

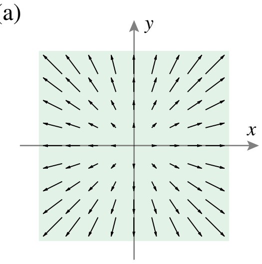
24. Find all positive values of k such that

$$\mathbf{F}(\mathbf{r}) = \frac{\mathbf{r}}{\|\mathbf{r}\|^k}$$

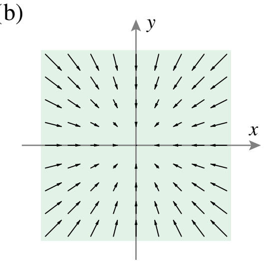
satisfies the condition div $\mathbf{F} = 0$ when $\mathbf{r} \neq \mathbf{0}$.

25. In each part, the figure shows a horizontal layer of the vector field of a fluid flow in which the flow is parallel to the xy-plane at every point and is identical in each layer (i.e., is independent of z). For each flow, what can you say about the sign of the divergence at the origin? Explain your reasoning.

(a)



(b)



26. Find a vector field $\mathbf{F}(x, y, z)$ that has
(a) positive divergence everywhere
(b) negative divergence everywhere.

In Exercises 27–30, determine whether the vector field $\mathbf{F}(x, y, z)$ is free of sources and sinks. If it is not, locate them.

27. $\mathbf{F}(x, y, z) = (y + z)\mathbf{i} - xz^3\mathbf{j} + (x^2 \sin y)\mathbf{k}$

28. $\mathbf{F}(x, y, z) = xy\mathbf{i} - xy\mathbf{j} + y^2\mathbf{k}$

29. $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$

30. $\mathbf{F}(x, y, z) = (x^3 - x)\mathbf{i} + (y^3 - y)\mathbf{j} + (z^3 - z)\mathbf{k}$

C **31.** Let σ be the surface of the solid G that is enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$. Use a CAS to verify Formula (1) in the Divergence Theorem for the vector field

$$\mathbf{F} = (x^2y - z^2)\mathbf{i} + (y^3 - x)\mathbf{j} + (2x + 3z - 1)\mathbf{k}$$

by evaluating the surface integral and the triple integral.

16.8 STOKES' THEOREM

In this section we will discuss a generalization of Green's Theorem to three dimensions that has important applications in the study of vector fields, particularly in the analysis of rotational motion of fluids. This theorem will also provide us with a physical interpretation of the curl of a vector field.

RELATIVE ORIENTATION OF CURVES AND SURFACES

We will be concerned in this section with oriented surfaces in 3-space that are bounded by simple closed parametric curves (Figure 16.8.1*a*). If σ is an oriented surface bounded by a simple closed parametric curve C, then there are two possible relationships between the orientations of σ and C, which can be described as follows. Imagine a person walking along the curve C with his or her head in the direction of the orientation of σ. The person is said to be walking in the ***positive direction*** of C relative to the orientation of σ if the surface is on the person's left (Figure 16.8.1*b*), and the person is said to be walking in the ***negative direction*** of C relative to the orientation of σ if the surface is on the person's right (Figure 16.8.1*c*). The positive direction of C establishes a right-hand relationship between the orientations of σ and C in the sense that if the fingers of the right hand are cupped in the positive direction of C, then the thumb points (roughly) in the direction of the orientation of σ.

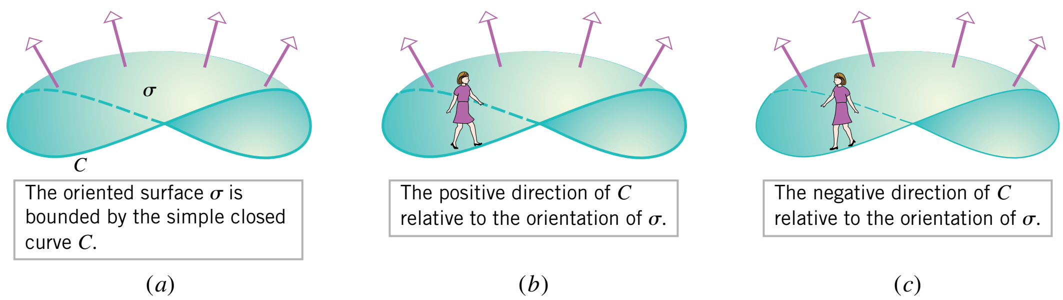


Figure 16.8.1

STOKES' THEOREM

In Section 16.1 we defined the curl of a vector field

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

as

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f & g & h \end{vmatrix} \tag{1}$$

but we did not attempt to give a physical explanation of its meaning at that time. The following result, known as ***Stokes'*** [*] ***Theorem***, (see biography on p. 1157) will provide us with a physical interpretation of the curl in the context of fluid flow.

16.8.1 THEOREM (*Stokes' Theorem*). *Let σ be a piecewise smooth oriented surface that is bounded by a simple, closed, piecewise smooth curve C with positive orientation. If the components of the vector field*

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

are continuous and have continuous first partial derivatives on some open set containing σ, and if $\mathbf{T}$ is the unit tangent vector to C, then

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_\sigma (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dS \tag{2}$$

The proof of this theorem is beyond the scope of this text, so we will focus on its applications.

Recall from Formula (32) of Section 16.2 that the integral on the left side of (2) represents the work performed by the vector field **F** on a particle that traverses the curve C. Thus, loosely phrased, Stokes' Theorem states:

The work performed by a vector field on a particle that traverses a simple, closed, piecewise smooth curve C in the positive direction can be obtained by integrating the normal component of the curl over an oriented surface σ bounded by C.

USING STOKES' THEOREM TO CALCULATE WORK

For computational purposes it is usually preferable to use Formula (33) of Section 16.2 to rewrite the formula in Stokes' Theorem as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS \tag{3}$$

Stokes' Theorem is usually the method of choice for calculating work around piecewise smooth curves with multiple sections, since it eliminates the need for a separate integral evaluation over each section. This is illustrated in the following example.

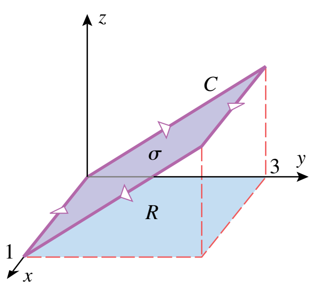


Figure 16.8.2

Example 1 Find the work performed by the vector field

$$\mathbf{F}(x, y, z) = x^2\mathbf{i} + 4xy^3\mathbf{j} + y^2x\mathbf{k}$$

on a particle that traverses the rectangle C in the plane $z = y$ shown in Figure 16.8.2.

Solution. The work performed by the field is

$$W = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

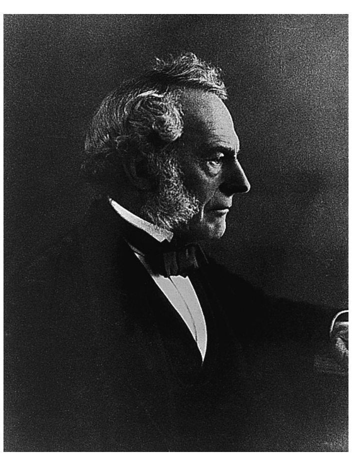
However, to evaluate this integral directly would require four separate integrations, one over each side of the rectangle. Instead, we will use Formula (3) to express the work as the surface integral

$$W = \iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$$

in which the plane surface σ enclosed by C is assigned a *downward* orientation to make the orientation of C positive, as required by Stokes' Theorem.

Since the surface σ has equation $z = y$ and

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & 4xy^3 & xy^2 \end{vmatrix} = 2xy\mathbf{i} - y^2\mathbf{j} + 4y^3\mathbf{k}$$



*GEORGE GABRIEL STOKES (1819–1903). Irish mathematician and physicist. Born in Skreen, Ireland, Stokes came from a family deeply rooted in the Church of Ireland. His father was a rector, his mother the daughter of a rector, and three of his brothers took holy orders. He received his early education from his father and a local parish clerk. In 1837, he entered Pembroke College and after graduating with top honors accepted a fellowship at the college. In 1847 he was appointed Lucasian professor of mathematics at Cambridge, a position once held by Isaac Newton, but one that had lost its esteem through the years. By virtue of his accomplishments, Stokes ultimately restored the position to the eminence it once held. Unfortunately, the position paid very little and Stokes was forced to teach at the Government School of Mines during the 1850s to supplement his income.

Stokes was one of several outstanding nineteenth century scientists who helped turn the physical sciences in a more empirical direction. He systematically studied hydrodynamics, elasticity of solids, behavior of waves in elastic solids, and diffraction of light. For Stokes, mathematics was a tool for his physical studies. He wrote classic papers on the motion of viscous fluids that laid the foundation for modern hydrodynamics; he elaborated on the wave theory of light; and he wrote papers on gravitational variation that established him as a founder of the modern science of geodesy.

Stokes was honored in his later years with degrees, medals, and memberships in foreign societies. He was knighted in 1889. Throughout his life, Stokes gave generously of his time to learned societies and readily assisted those who sought his help in solving problems. He was deeply religious and vitally concerned with the relationship between science and religion.

it follows from Formula (13) of Section 16.6 with curl **F** replacing **F** that

$$W = \iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_R (\text{curl } \mathbf{F}) \cdot \left(\frac{\partial z}{\partial x}\mathbf{i} + \frac{\partial z}{\partial y}\mathbf{j} - \mathbf{k}\right) dA$$

$$= \iint_R (2xy\mathbf{i} - y^2\mathbf{j} + 4y^3\mathbf{k}) \cdot (0\mathbf{i} + \mathbf{j} - \mathbf{k})\, dA$$

$$= \int_0^1 \int_0^3 (-y^2 - 4y^3)\, dy\, dx$$

$$= -\int_0^1 \left[\frac{y^3}{3} + y^4\right]_{y=0}^3 dx$$

$$= -\int_0^1 90\, dx = -90 \qquad \blacktriangleleft$$

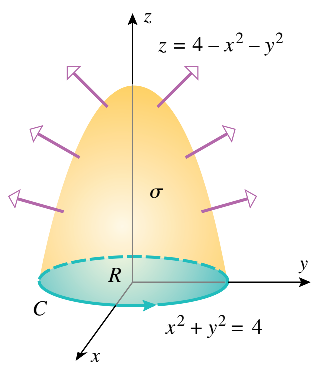


Figure 16.8.3

Example 2 Verify Stokes' Theorem for the vector field $\mathbf{F}(x, y, z) = 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$, taking σ to be the portion of the paraboloid $z = 4 - x^2 - y^2$ for which $z \geq 0$ with upward orientation, and C to be the positively oriented circle $x^2 + y^2 = 4$ that forms the boundary of σ in the xy-plane (Figure 16.8.3).

Solution. We will verify Formula (3). Since σ is oriented up, the positive orientation of C is counterclockwise looking down the positive z-axis. Thus, C can be represented parametrically (with positive orientation) by

$$x = 2\cos t, \quad y = 2\sin t, \quad z = 0 \quad (0 \leq t \leq 2\pi) \tag{4}$$

Therefore,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C 2z\, dx + 3x\, dy + 5y\, dz$$

$$= \int_0^{2\pi} [0 + (6\cos t)(2\cos t) + 0]\, dt$$

$$= \int_0^{2\pi} 12\cos^2 t\, dt = 12\left[\frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{2\pi} = 12\pi$$

To evaluate the right side of (3), we start by finding curl **F**. We obtain

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2z & 3x & 5y \end{vmatrix} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$

Since σ is oriented up and is expressed in the form $z = g(x, y) = 4 - x^2 - y^2$, it follows from Formula (12) of Section 16.6 with curl **F** replacing **F** that

$$\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_R (\text{curl } \mathbf{F}) \cdot \left(-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}\right) dA$$

$$= \iint_R (5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\, dA$$

$$= \iint_R (10x + 4y + 3)\, dA$$

$$= \int_0^{2\pi} \int_0^2 (10r\cos\theta + 4r\sin\theta + 3)r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{10r^3}{3} \cos\theta + \frac{4r^3}{3} \sin\theta + \frac{3r^2}{2} \right]_{r=0}^{2} d\theta$$

$$= \int_0^{2\pi} \left(\frac{80}{3} \cos\theta + \frac{32}{3} \sin\theta + 6 \right) d\theta$$

$$= \left[\frac{80}{3} \sin\theta - \frac{32}{3} \cos\theta + 6\theta \right]_0^{2\pi} = 12\pi$$

As guaranteed by Stokes' Theorem, the value of this surface integral is the same as the value of the line integral obtained above. Note, however, that the line integral was simpler to evaluate and hence would be the method of choice in this case. ◀

REMARK. Observe that in Formula (3) the only relationship required between σ and C is that C be the boundary of σ and that C be positively oriented relative to the orientation of σ. Thus, if σ_1 and σ_2 are *different* oriented surfaces but have the *same* positively oriented boundary curve C, then it follows from (3) that

$$\iint_{\sigma_1} \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{\sigma_2} \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS$$

For example, the parabolic surface in Example 2 has the same positively oriented boundary C as the disk R in Figure 16.8.3 with upper orientation. Thus, the value of the surface integral in that example would not change if σ is replaced by R (or by any other oriented surface that has the positively oriented circle C as its boundary). This can be useful in computations because it is sometimes possible to circumvent a difficult integration by changing the surface of integration.

RELATIONSHIP BETWEEN GREEN'S THEOREM AND STOKES' THEOREM

It is sometimes convenient to regard a vector field

$$\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$$

in 2-space as a vector field in 3-space by expressing it as

$$\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j} + 0\mathbf{k} \tag{5}$$

If R is a region in the xy-plane enclosed by a simple, closed, piecewise smooth curve C, then we can treat R as a *flat* surface, and we can treat a surface integral over R as an ordinary double integral over R. Thus, if we orient R up and C counterclockwise looking down the positive z-axis, then Formula (3) applied to (5) yields

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dA \tag{6}$$

But

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f & g & 0 \end{vmatrix} = -\frac{\partial g}{\partial z}\mathbf{i} + \frac{\partial f}{\partial z}\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)\mathbf{k} = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)\mathbf{k}$$

since $\partial g/\partial z = \partial f/\partial z = 0$. Substituting this expression in (6) and expressing the integrals in terms of components yields

$$\oint_C f\, dx + g\, dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

which is Green's Theorem [Formula (1) of Section 16.4]. Thus, we have shown that Green's Theorem can be viewed as a special case of Stokes' Theorem.

CURL VIEWED AS CIRCULATION

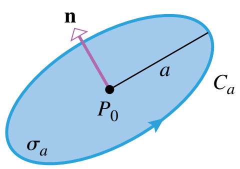


Figure 16.8.4

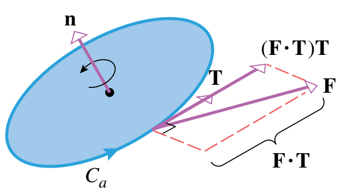


Figure 16.8.5

Stokes' Theorem provides a way of interpreting the curl of a vector field $\mathbf{F}$ in the context of fluid flow. For this purpose let σ_a be a small oriented disk of radius a centered at a point P_0 in a steady-state fluid flow, and let $\mathbf{n}$ be a unit normal vector at the center of the disk that points in the direction of orientation. Let us assume that the flow of liquid past the disk causes it to spin around the axis through $\mathbf{n}$, and let us try to find the direction of $\mathbf{n}$ that will produce the maximum rotation rate in the positive direction of the boundary curve C_a (Figure 16.8.4). For convenience, we will denote the area of the disk σ_a by $A(\sigma_a)$; that is, $A(\sigma_a) = \pi a^2$.

If the direction of $\mathbf{n}$ is fixed, then at each point of C_a the only component of $\mathbf{F}$ that contributes to the rotation of the disk about $\mathbf{n}$ is the component $\mathbf{F}\cdot\mathbf{T}$ tangent to C_a (Figure 16.8.5). Thus, for a fixed $\mathbf{n}$ the integral

$$\oint_{C_a} \mathbf{F}\cdot\mathbf{T}\,ds \tag{7}$$

can be viewed as a measure of the tendency for the fluid to flow in the positive direction around C_a. Accordingly, (7) is called the ***circulation of*** $\mathbf{F}$ ***around*** $\boldsymbol{C_a}$. For example, in the extreme case where the flow is normal to the circle at each point, the circulation around C_a is zero, since $\mathbf{F}\cdot\mathbf{T} = 0$ at each point. The more closely that $\mathbf{F}$ aligns with $\mathbf{T}$ along the circle, the larger the value of $\mathbf{F}\cdot\mathbf{T}$ and the larger the value of the circulation.

To see the relationship between circulation and curl, suppose that curl $\mathbf{F}$ is continuous on σ_a, so that when σ_a is small the value of curl $\mathbf{F}$ at any point of σ_a will not vary much from the value of curl $\mathbf{F}(P_0)$ at the center. Thus, for a small disk σ_a we can reasonably approximate curl $\mathbf{F}$ by the constant value curl $\mathbf{F}(P_0)$ on σ_a. Moreover, because the surface σ_a is flat, the unit normal vectors that orient σ_a are all equal. Thus, the vector quantity $\mathbf{n}$ in Formula (3) can be treated as a constant, and we can write

$$\oint_{C_a} \mathbf{F}\cdot\mathbf{T}\,ds = \iint_{\sigma_a} (\text{curl}\,\mathbf{F})\cdot\mathbf{n}\,dS \approx \text{curl}\,\mathbf{F}(P_0)\cdot\mathbf{n}\iint_{\sigma_a} dS$$

where the line integral is taken in the positive direction of C_a. But the double integral in this equation represents the surface area of σ_a, so

$$\oint_{C_a} \mathbf{F}\cdot\mathbf{T}\,ds \approx [\text{curl}\,\mathbf{F}(P_0)\cdot\mathbf{n}]A(\sigma_a)$$

from which we obtain

$$\text{curl}\,\mathbf{F}(P_0)\cdot\mathbf{n} \approx \frac{1}{A(\sigma_a)}\oint_{C_a} \mathbf{F}\cdot\mathbf{T}\,ds \tag{8}$$

The quantity on the right side of (8) is called the ***circulation density of*** $\mathbf{F}$ ***around*** $\boldsymbol{C_a}$. If we now let the radius a of the disk approach zero (with $\mathbf{n}$ fixed), then it is plausible that the error in this approximation will approach zero and the exact value of curl $\mathbf{F}(P_0)\cdot\mathbf{n}$ will be given by

$$\text{curl}\,\mathbf{F}(P_0)\cdot\mathbf{n} = \lim_{a\to 0}\frac{1}{A(\sigma_a)}\oint_{C_a} \mathbf{F}\cdot\mathbf{T}\,ds \tag{9}$$

We call curl $\mathbf{F}(P_0)\cdot\mathbf{n}$ the ***circulation density of*** $\mathbf{F}$ ***at*** $\boldsymbol{P_0}$ ***in the direction of*** $\mathbf{n}$. This quantity has its maximum value when $\mathbf{n}$ is in the same direction as curl $\mathbf{F}(P_0)$; this tells us that *at each point in a steady-state fluid flow the maximum circulation density occurs in the direction of the curl*. Physically, this means that if a small paddle wheel is immersed in the fluid so that the pivot point is at P_0, then the paddles will turn most rapidly when the spindle is aligned with curl $\mathbf{F}(P_0)$ (Figure 16.8.6). If curl $\mathbf{F} = \mathbf{0}$ at each point of a region, then $\mathbf{F}$ is said to be ***irrotational*** in that region, since no circulation occurs about any point of the region.

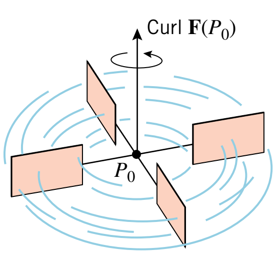


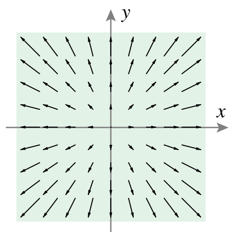
Figure 16.8.6

REMARK. Formula (9) is sometimes taken as a definition of curl. This is a useful alternative to Definition 16.1.5 because it does not require a coordinate system.

EXERCISE SET 16.8 C CAS

The figures in Exercises 1 and 2 show a horizontal layer of the vector field of a fluid flow in which the flow is parallel to the xy-plane at every point and is identical in each layer (i.e., is independent of z). For each flow, state whether you believe that the curl is nonzero at the origin, and explain your reasoning. If you believe that it is nonzero, then state whether it points in the positive or negative z-direction.

1. (a)



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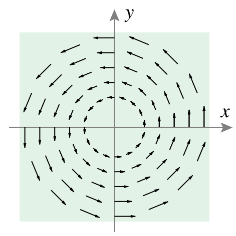
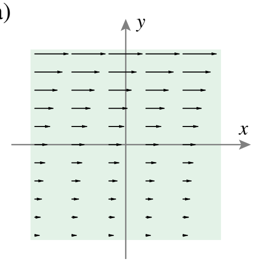
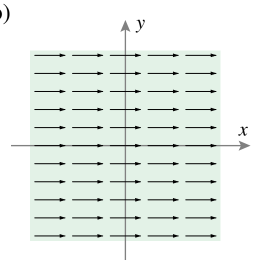


2. (a)



(b)



In Exercises 3–6, verify Formula (2) in Stokes' Theorem by evaluating the line integral and the double integral. Assume that the surface has an upward orientation.

3. $\mathbf{F}(x, y, z) = (x-y)\mathbf{i}+(y-z)\mathbf{j}+(z-x)\mathbf{k}$; σ is the portion of the plane $x+y+z=1$ in the first octant.

4. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$; σ is the portion of the cone $z = \sqrt{x^2+y^2}$ below the plane $z = 1$.

5. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; σ is the upper hemisphere $z = \sqrt{a^2-x^2-y^2}$.

6. $\mathbf{F}(x, y, z) = (z-y)\mathbf{i}+(z+x)\mathbf{j}-(x+y)\mathbf{k}$; σ is the portion of the paraboloid $z = 9-x^2-y^2$ above the xy-plane.

In Exercises 7–14, use Stokes' Theorem to evaluate the integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$.

7. $\mathbf{F}(x, y, z) = z^2\mathbf{i} + 2x\mathbf{j} - y^3\mathbf{k}$; C is the circle $x^2+y^2 = 1$ in the xy-plane with counterclockwise orientation looking down the positive z-axis.

8. $\mathbf{F}(x, y, z) = xz\mathbf{i} + 3x^2y^2\mathbf{j} + yx\mathbf{k}$; C is the rectangle in the plane $z = y$ shown in Figure 16.8.2.

9. $\mathbf{F}(x, y, z) = 3z\mathbf{i} + 4x\mathbf{j} + 2y\mathbf{k}$; C is the boundary of the paraboloid shown in Figure 16.8.3.

10. $\mathbf{F}(x, y, z) = -3y^2\mathbf{i} + 4z\mathbf{j} + 6x\mathbf{k}$; C is the triangle in the plane $z = \frac{1}{2}y$ with vertices $(2, 0, 0)$, $(0, 2, 1)$, and $(0, 0, 0)$ with a counterclockwise orientation looking down the positive z-axis.

11. $\mathbf{F}(x, y, z) = xy\mathbf{i} + x^2\mathbf{j} + z^2\mathbf{k}$; C is the intersection of the paraboloid $z = x^2+y^2$ and the plane $z = y$ with a counterclockwise orientation looking down the positive z-axis.

12. $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$; C is the triangle in the plane $x+y+z = 1$ with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ with a counterclockwise orientation looking from the first octant toward the origin.

13. $\mathbf{F}(x, y, z) = (x-y)\mathbf{i} + (y-z)\mathbf{j} + (z-x)\mathbf{k}$; C is the circle $x^2+y^2 = a^2$ in the xy-plane with counterclockwise orientation looking down the positive z-axis.

14. $\mathbf{F}(x, y, z) = (z+\sin x)\mathbf{i} + (x+y^2)\mathbf{j} + (y+e^z)\mathbf{k}$; C is the intersection of the sphere $x^2+y^2+z^2 = 1$ and the cone $z = \sqrt{x^2+y^2}$ with counterclockwise orientation looking down the positive z-axis.

15. Consider the vector field given by the formula

$$\mathbf{F}(x, y, z) = (x-z)\mathbf{i} + (y-x)\mathbf{j} + (z-xy)\mathbf{k}$$

(a) Use Stokes' Theorem to find the circulation around the triangle with vertices $A(1, 0, 0)$, $B(0, 2, 0)$, and $C(0, 0, 1)$ oriented counterclockwise looking from the origin toward the first octant.

(b) Find the circulation density of $\mathbf{F}$ at the origin in the direction of $\mathbf{k}$.

(c) Find the unit vector $\mathbf{n}$ such that the circulation density of $\mathbf{F}$ at the origin is maximum in the direction of $\mathbf{n}$.

16. (a) Let σ denote the surface of a solid G with $\mathbf{n}$ the outward unit normal vector field to σ. Assume that $\mathbf{F}$ is a vector field with continuous first-order partial derivatives on σ. Prove that

$$\iint_\sigma (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = 0$$

(*Hint:* Let C denote a simple closed curve on σ that separates the surface into two (sub)surfaces σ_1 and σ_2 that share C as their common boundary. Apply Stokes' Theorem to σ_1 and σ_2 and add the results.)

(b) The vector field curl($\mathbf{F}$) is called the ***curl field*** of $\mathbf{F}$. In words, interpret the formula in part (b) as a statement about the flux of the curl field.

17. In 1831 the physicist Michael Faraday discovered that an electric current can be produced by varying the magnetic flux through a conducting loop. His experiments showed that the electromotive force $\mathbf{E}$ is related to the magnetic induction **B** by the equation

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = -\iint_\sigma \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n}\, dS$$

Use this result to make a conjecture about the relationship between curl **E** and **B**, and explain your reasoning.

C **18.** Let σ be the portion of the paraboloid $z = 1 - x^2 - y^2$ for which $z \geq 0$, and let C be the circle $x^2 + y^2 = 1$ that forms the boundary of σ in the xy-plane. Assuming that σ is oriented up, use a CAS to verify Formula (2) in Stokes' Theorem for the vector field

$$\mathbf{F} = (x^2 y - z^2)\mathbf{i} + (y^3 - x)\mathbf{j} + (2x + 3z - 1)\mathbf{k}$$

by evaluating the line integral and the surface integral.

SUPPLEMENTARY EXERCISES

1. In words, what is a vector field? Give some physical examples of vector fields.

2. (a) Give a physical example of an inverse-square field $\mathbf{F}(\mathbf{r})$ in 3-space.
(b) Write a formula for a general inverse-square field $\mathbf{F}(\mathbf{r})$ in terms of the radius vector $\mathbf{r}$.
(c) Write a formula for a general inverse-square field $\mathbf{F}(x, y, z)$ in 3-space using rectangular coordinates.

3. Assume that C is the parametric curve $x = x(t)$, $y = y(t)$, where t varies from a to b. In each part, express the line integral as a definite integral with variable of integration t.

(a) $\displaystyle\int_C f(x, y)\, dx + g(x, y)\, dy$ (b) $\displaystyle\int_C f(x, y)\, ds$

4. (a) Express the mass M of a thin wire in 3-space as a line integral.
(b) Express the length of a curve as a line integral.
(c) Express the area of a surface as a surface integral.
(d) Express the area of a plane region as a line integral.

5. In each part, give a physical interpretation of the integral.

(a) $\displaystyle\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ (b) $\displaystyle\iint_\sigma \mathbf{F} \cdot \mathbf{n}\, dS$

6. State some alternative notations for $\displaystyle\int_C \mathbf{F} \cdot \mathbf{T}\, ds$.

7. (a) State the Fundamental Theorem of Work Integrals, including all required hypotheses.
(b) State Green's Theorem, including all of the required hypotheses.

8. What conditions must C, D, and $\mathbf{F}$ satisfy to be assured that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

around every piecewise smooth curve C in the region D in 2-space?

9. How can you tell whether the vector field

$$\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$$

is conservative on a simply connected open region D?

10. Make a sketch of a vector field that is not conservative, and give an argument in support of your answer.

11. Assume that σ is the parametric surface

$$\mathbf{r} = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

where (u, v) varies over a region R. Express the surface integral

$$\iint_\sigma f(x, y, z)\, dS$$

as a double integral with variables of integration u and v.

12. State the Divergence Theorem and Stokes' Theorem, including all required hypotheses.

13. Let α and β denote angles that satisfy $0 < \beta - \alpha \leq 2\pi$ and assume that $r = f(\theta)$ is a smooth polar curve with $f(\theta) > 0$ on the interval $[\alpha, \beta]$. Use the formula

$$A = \frac{1}{2}\int_C -y\, dx + x\, dy$$

to find the area of the region R enclosed by the curve $r = f(\theta)$ and the rays $\theta = \alpha$ and $\theta = \beta$.

14. As discussed in Example 1 of Section 16.1, *Coulomb's law* states that the electrostatic force $\mathbf{F}(\mathbf{r})$ that a particle of charge Q exerts on a particle of charge q is given by the formula

$$\mathbf{F}(\mathbf{r}) = \frac{qQ}{4\pi\epsilon_0 \|\mathbf{r}\|^3}\mathbf{r}$$

where $\mathbf{r}$ is the radius vector from Q to q and ϵ_0 is the permittivity constant.
(a) Express the vector field $\mathbf{F}(\mathbf{r})$ in coordinate form $\mathbf{F}(x, y, z)$ with Q at the origin.
(b) Find the work performed by the vector field $\mathbf{F}$ on a charge q that moves along a straight line from $(3, 0, 0)$ to $(3, 1, 5)$.

15. As discussed in Section 16.1, it follows from *Newton's Law of Universal Gravitation* that the gravitational force $\mathbf{F}(\mathbf{r})$ exerted by an object of mass M on an object of mass m is given by the formula

$$\mathbf{F}(\mathbf{r}) = -\frac{GmM}{\|\mathbf{r}\|^3}\mathbf{r}$$

where $\mathbf{r}$ is the radius vector from M to m and G is the universal gravitational constant.

(a) Show that the work W done by the gravitational field $\mathbf{F}(\mathbf{r})$ when the mass m moves from a distance of r_1 from M to a distance of r_2 from M is

$$W = GmM\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

(b) The value of the constant GM for the Earth is approximately $3.99 \times 10^5\ \text{km}^3/\text{s}^2$. Find the work done by the Earth's gravitational field on a satellite with a mass of 1000 kg that moves from a perigee of 600 km above the surface of the Earth to an apogee of 800 km above the surface of the Earth, assuming the Earth to be a sphere of radius 6370 km.

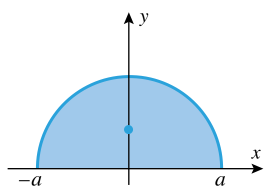
16. Let

$$\mathbf{F}(x, y, z) = \frac{x}{x^2 + y^2}\mathbf{i} + \frac{y}{x^2 + y^2}\mathbf{j} + \frac{z}{x^2 + y^2}\mathbf{k}$$

Sketch the level surface div $\mathbf{F} = 1$.

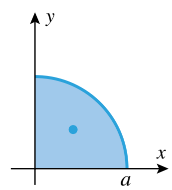
In Exercises 17–20, use the result in Exercise 16 to confirm that the centroid of the region is as shown in the figure.

17.



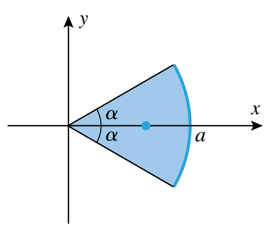
$\bar{x} = 0, \bar{y} = \dfrac{4a}{3\pi}$

18.



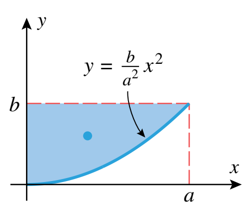
$\bar{x} = \bar{y} = \dfrac{4a}{3\pi}$

19.



$\bar{x} = \dfrac{2a}{3}\dfrac{\sin\alpha}{\alpha}, \bar{y} = 0$

20.



$\bar{x} = \dfrac{3a}{8}, \bar{y} = \dfrac{3b}{5}$

21. (a) Use Green's Theorem to prove that

$$\int_C f(x)\,dx + g(y)\,dy = 0$$

if f and g are differentiable functions and C is a simple, closed, piecewise smooth curve.

(b) What does this tell you about the vector field $\mathbf{F}(x, y) = f(x)\mathbf{i} + g(y)\mathbf{j}$?

22. The purpose of this exercise is to establish the role of the curl in describing the rotation of a rigid body. As illustrated in the accompanying figure, consider a rigid body rotating about the z-axis of an xyz-coordinate system at a constant ***angular speed*** of ω rad/s. Let P be a point on the body, and let $\mathbf{r}$ be the position vector of P. Thus, the velocity of P is $\mathbf{v} = d\mathbf{r}/dt$, where $\mathbf{v}$ is tangent to the circle of rotation of P. Let θ and ϕ be the angles shown in the figure, and define the ***angular velocity*** of the point P to be $\boldsymbol{\omega} = \omega\mathbf{k}$.

(a) Show that $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$.
(b) Show that $\mathbf{v} = -\omega y\mathbf{i} + \omega x\mathbf{j}$.
(c) Show that curl $\mathbf{v} = 2\boldsymbol{\omega}$.
(d) Is the velocity field $\mathbf{v}$ conservative? Justify your answer.

23. Do you think that the surface in the accompanying figure is orientable? Explain your reasoning.

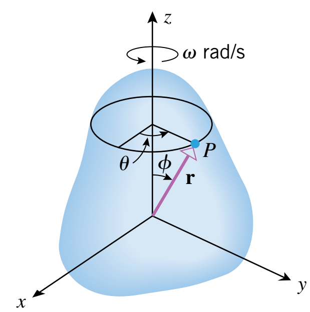


Figure Ex-22

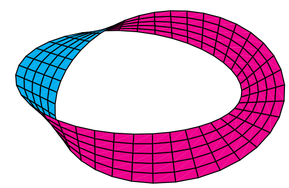


Figure Ex-23

24. Let G be a solid with the surface σ oriented by outward unit normals, suppose that ϕ has continuous first and second partial derivatives in some open set containing G, and let $D_{\mathbf{n}}\phi$ be the directional derivative of ϕ, where $\mathbf{n}$ is an outward unit normal to σ. Show that

$$\iint_\sigma D_{\mathbf{n}}\phi\,dS = \iiint_G \left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right] dV$$

25. Let σ be the sphere $x^2 + y^2 + z^2 = 1$, let $\mathbf{n}$ be an inward unit normal, and let $D_{\mathbf{n}}f$ be the directional derivative of $f(x, y, z) = x^2 + y^2 + z^2$. Use the result in Exercise 24 to evaluate the surface integral

$$\iint_\sigma D_{\mathbf{n}}f\,dS$$

26. Let $\mathbf{F}(x, y) = (ye^{xy} - 1)\mathbf{i} + xe^{xy}\mathbf{j}$.

(a) Show that $\mathbf{F}$ is a conservative vector field.
(b) Find a potential function for $\mathbf{F}$.
(c) Find the work performed by the vector field on a particle that moves along the sawtooth curve represented by the parametric equations

$$x = t + \sin^{-1}(\sin t), \quad y = (2/\pi)\sin^{-1}(\sin t) \qquad (0 \le t \le 8\pi)$$

(see accompanying figure).

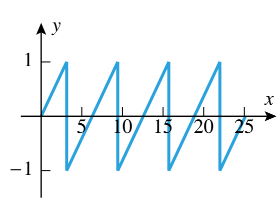


Figure Ex-26

27. Let $\mathbf{F}(x, y) = y\mathbf{i} - 2x\mathbf{j}$.
(a) Find a nonzero function $h(x)$ such that $h(x)\mathbf{F}(x, y)$ is a conservative vector field.
(b) Find a nonzero function $g(y)$ such that $g(y)\mathbf{F}(x, y)$ is a conservative vector field.

28. Let $\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$ and suppose that f, g, and h are continuous and have continuous first partial derivatives in a region. It was shown in Exercise 25 of Section 16.3 that if $\mathbf{F}$ is conservative in the region, then

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}, \quad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}$$

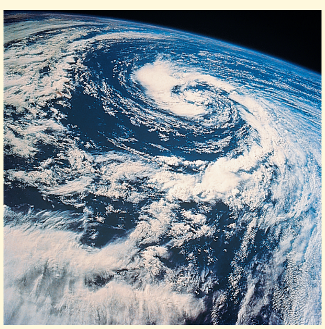
there. Use this result and Stokes' Theorem to help show that $\mathbf{F}$ is conservative in an open spherical region if and only if curl $\mathbf{F} = \mathbf{0}$ in that region.

In Exercises 29 and 30, use the result in Exercise 28 to determine whether $\mathbf{F}$ is conservative. If so, find a potential function for it.

29. (a) $\mathbf{F}(x, y, z) = z^2\mathbf{i} + e^{-y}\mathbf{j} + 2xz\mathbf{k}$
(b) $\mathbf{F}(x, y, z) = xy\mathbf{i} + x^2\mathbf{j} + \sin z\mathbf{k}$

30. (a) $\mathbf{F}(x, y, z) = \sin x\mathbf{i} + z\mathbf{j} + y\mathbf{k}$
(b) $\mathbf{F}(x, y, z) = z\mathbf{i} + 2yz\mathbf{j} + y^2\mathbf{k}$

EXPANDING THE CALCULUS HORIZON



Hurricane Modeling

Each year population centers throughout the world are ravaged by hurricanes, and it is the mission of the National Hurricane Center to minimize the damage and loss of life by issuing warnings and forecasts of hurricanes developing in the Caribbean, Atlantic, Gulf of Mexico, and Eastern Pacific regions. Your assignment as a trainee at the Center is to construct a simple mathematical model of a hurricane using basic principles of fluid flow and properties of vector fields.

Modeling Assumptions

You have been notified of a developing hurricane in the Bahamas (designated hurricane *Isaac*) and have been asked to construct a model of its velocity field. Because hurricanes are complicated three-dimensional fluid flows, you will have to make many simplifying assumptions about the structure of a hurricane and the properties of the fluid flow. Accordingly, you decide to model the moisture in Isaac as an ***ideal fluid***, meaning that it is ***incompressible*** and its ***viscosity*** can be ignored. An incompressible fluid is one in which the density of the fluid is the same at all points and cannot be altered by compressive forces. Experience has shown that water can be regarded as incompressible but water vapor cannot. However, incompressibility is a reasonable assumption for a basic hurricane model because a hurricane is not restricted to a closed container that would produce compressive forces.

All fluids have a certain amount of viscosity, which is a resistance to flow—oil and molasses have a high viscosity, whereas water has almost none at subsonic speeds. Thus, it is reasonable to ignore viscosity in a basic model. Next, you decide to assume that the flow is in a ***steady state***, meaning that the velocity of the fluid at any point does not vary with time. This is reasonable over very short time periods for hurricanes that move and change slowly. Finally, although hurricanes are three-dimensional flows, you decide to model a two-dimensional horizontal cross section, so you make the simplifying assumption that the fluid in the cross section flows horizontally.

The photograph of Isaac shown at the beginning of this module reveals a typical pattern of a Caribbean hurricane—a counterclockwise swirl of fluid around the ***eye*** through which the fluid exits the flow in the form of rain. The lower pressure in the eye causes an inward-rushing air mass, and circular winds around the eye contribute to the swirling effect.

Your first objective is to find an explicit formula for Isaac's velocity field $\mathbf{F}(x, y)$, so you begin by introducing a rectangular coordinate system with its origin at the eye and its y-axis pointing north. Moreover, based on the hurricane picture and your knowledge of meteorological theory, you decide to build up the velocity field for Isaac from the velocity fields of simpler flows—a counterclockwise "vortex flow" $\mathbf{F}_1(x, y)$ in which fluid flows counterclockwise in concentric circles around the eye and a "sink flow" $\mathbf{F}_2(x, y)$ in which the fluid flows in straight lines toward a sink at the eye. Once you find explicit formulas for $\mathbf{F}_1(x, y)$ and $\mathbf{F}_2(x, y)$, your plan is to use the ***superposition principle*** from fluid dynamics to express the velocity field for Isaac as $\mathbf{F}(x, y) = \mathbf{F}_1(x, y) + \mathbf{F}_2(x, y)$.

Modeling a Vortex Flow

A ***counterclockwise vortex flow*** of an ideal fluid around the origin has four defining characteristics (Figure 1a on the following page):

- The velocity vector at a point (x, y) is tangent to the circle that is centered at the origin and passes through the point (x, y).
- The direction of the velocity vector at a point (x, y) indicates a counterclockwise motion.
- The speed of the fluid is constant on circles centered at the origin.
- The speed of the fluid along a circle is inversely proportional to the radius of the circle (and hence the speed approaches $+\infty$ as the radius of the circle approaches 0).

In fluid dynamics, the ***strength*** k of a vortex flow is defined to be 2π times the speed of the fluid along the unit circle. If the strength of a vortex flow is known, then the speed of the fluid along any other circle can be found from the fact that speed is inversely proportional to the radius of the circle. Thus, your first objective is to find a formula for a vortex flow $\mathbf{F}_1(x, y)$ with a specified strength k.

Exercise 1 Show that

$$\mathbf{F}_1(x, y) = -\frac{k}{2\pi(x^2 + y^2)}(y\mathbf{i} - x\mathbf{j})$$

is a model for the velocity field of a counterclockwise vortex flow around the origin of strength k by confirming that

(a) $\mathbf{F}_1(x, y)$ has the four properties required of a counterclockwise vortex flow around the origin;

(b) k is 2π times the speed of the fluid along the unit circle.

Exercise 2 Use a graphing utility that can generate vector fields to generate a vortex flow of strength 2π.

Modeling a Sink Flow

A ***uniform sink flow*** of an ideal fluid toward the origin has four defining characteristics (Figure 1b):

- The velocity vector at every point (x, y) is directed toward the origin.
- The speed of the fluid is the same at all points on a circle centered at the origin.
- The speed of the fluid at a point is inversely proportional to its distance from the origin (from which it follows that the speed approaches $+\infty$ as the distance from the origin approaches 0).
- There is a sink at the origin at which fluid leaves the flow.

As with a vortex flow, the ***strength*** q of a uniform sink flow is defined to be 2π times the speed of the fluid at points on the unit circle. If the strength of a sink flow is known, then the speed of the fluid at any point in the flow can be found using the fact that the speed is inversely proportional to the distance from the origin. Thus, your next objective is to find a formula for a uniform sink flow $\mathbf{F}_2(x, y)$ with a specified strength q.

Exercise 3 Show that

$$\mathbf{F}_2(x, y) = -\frac{q}{2\pi(x^2 + y^2)}(x\mathbf{i} + y\mathbf{j})$$

is a model for the velocity field of a uniform sink flow toward the origin of strength q by confirming the following facts:

(a) $\mathbf{F}_2(x, y)$ has the four properties required of a uniform sink flow toward the origin. [A reasonable physical argument to confirm the existence of the sink will suffice.]

(b) q is 2π times the speed of the fluid at points on the unit circle.

Exercise 4 Use a graphing utility that can generate vector fields to generate a uniform sink flow of strength 2π.

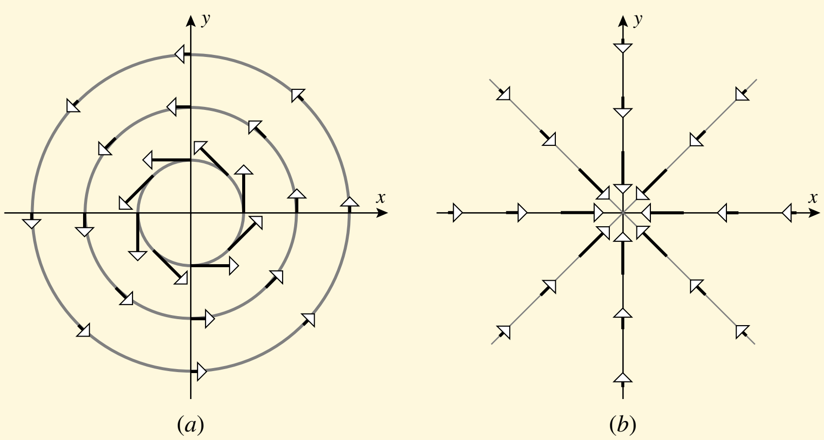


Figure 1

A Basic Hurricane Model

It now follows from Exercises 1 and 3 that the vector field $\mathbf{F}(x, y)$ for a hurricane model that combines a vortex flow around the origin of strength k and a uniform sink flow toward the origin of strength q is

$$\mathbf{F}(x, y) = -\frac{1}{2\pi(x^2 + y^2)}[(qx + ky)\mathbf{i} + (qy - kx)\mathbf{j}] \tag{1}$$

Exercise 5

(a) Figure 2 shows a vector field for a hurricane with vortex strength $k = 2\pi$ and sink strength $q = 2\pi$. Use a graphing utility that can generate vector fields to produce a reasonable facsimile of this figure.

(b) Make a conjecture about the effect of increasing k and keeping q fixed, and check your conjecture using a graphing utility.

(c) Make a conjecture about the effect of increasing q and keeping k fixed, and check your conjecture using a graphing utility.

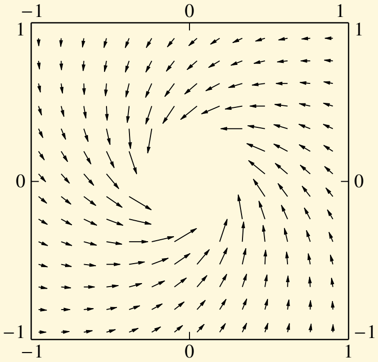


Figure 2

Modeling Hurricane Isaac

You are now ready to apply Formula (1) to obtain a model of the vector field $\mathbf{F}(x, y)$ of hurricane Isaac. You need some observational data to determine the constants k and q, so you call the Technical Support Branch of the Center for the latest information on hurricane Isaac. They report that 20 km from the eye the wind velocity has a component of 15 km/h toward the eye and a counterclockwise tangential component of 45 km/h.

Exercise 6

(a) Find the strengths k and q of the vortex and sink for hurricane Isaac.

(b) Find the vector field $\mathbf{F}(x, y)$ for hurricane Isaac.

(c) Estimate the size of hurricane Isaac by finding a radius beyond which the wind speed is less than 5 km/h.

Streamlines for the Basic Hurricane Model

The paths followed by the fluid particles in a fluid flow are called the ***streamlines*** of the flow. Thus, the vectors $\mathbf{F}(x, y)$ in the velocity field of a fluid flow are tangent to the streamlines. If the streamlines can be represented as the level curves of some function $\psi(x, y)$, then the function ψ is called a ***stream function*** for the flow. Since $\nabla\psi$ is normal to the level curves $\psi(x, y) = c$, it follows that $\nabla\psi$ is normal to the streamlines; and this in turn implies that

$$\nabla\psi \cdot \mathbf{F} = 0 \tag{2}$$

Your plan is to use this equation to find the stream function and then the streamlines of the basic hurricane model.

Since the vortex and sink flows that produce the basic hurricane model have a central symmetry, intuition suggests that polar coordinates may lead to simpler equations for the streamlines than rectangular coordinates. Thus, you decide to express the velocity vector $\mathbf{F}$ at a point (r, θ) in terms of the orthogonal unit vectors

$$\mathbf{u}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} \quad \text{and} \quad \mathbf{u}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$$

The vector $\mathbf{u}_r$, called the ***radial unit vector***, points away from the origin, and the vector $\mathbf{u}_\theta$, called the ***transverse unit vector***, is obtained by rotating $\mathbf{u}_r$ counterclockwise 90° (Figure 3).

Exercise 7 Show that the vector field for the basic hurricane model given in (1) can be expressed in terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$ as

$$\mathbf{F} = -\frac{1}{2\pi r}(q\mathbf{u}_r - k\mathbf{u}_\theta)$$

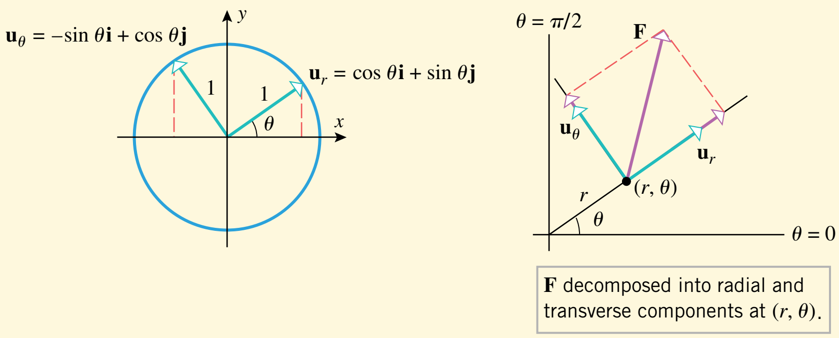


Figure 3

It follows from Exercise 75 of Section 14.6 that the gradient of the stream function can be expressed in terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$ as

$$\nabla\psi = \frac{\partial\psi}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{u}_\theta$$

Exercise 8 Confirm that for the basic hurricane model the orthogonality condition in (2) is satisfied if

$$\frac{\partial\psi}{\partial r} = \frac{k}{r} \quad \text{and} \quad \frac{\partial\psi}{\partial\theta} = q$$

Exercise 9 By integrating the equations in Exercise 8, show that

$$\psi = k\ln r + q\theta$$

is a stream function for the basic hurricane model.

Exercise 10 Show that the streamlines for the basic hurricane model are logarithmic spirals of the form

$$r = Ke^{-q\theta/k} \quad (K > 0)$$

Exercise 11 Use a graphing utility to generate some typical streamlines for the basic hurricane model with vortex strength 2π and sink strength 2π.

Streamlines for Hurricane Isaac

Exercise 12 In Exercise 6 you found the strengths k and q of the vortex and sink for hurricane Isaac. Use that information to find a formula for the family of streamlines for Isaac; and then use a graphing utility to graph the streamline that passes through the point that is 20 km from the eye in the direction that is 45°NE from the eye.

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