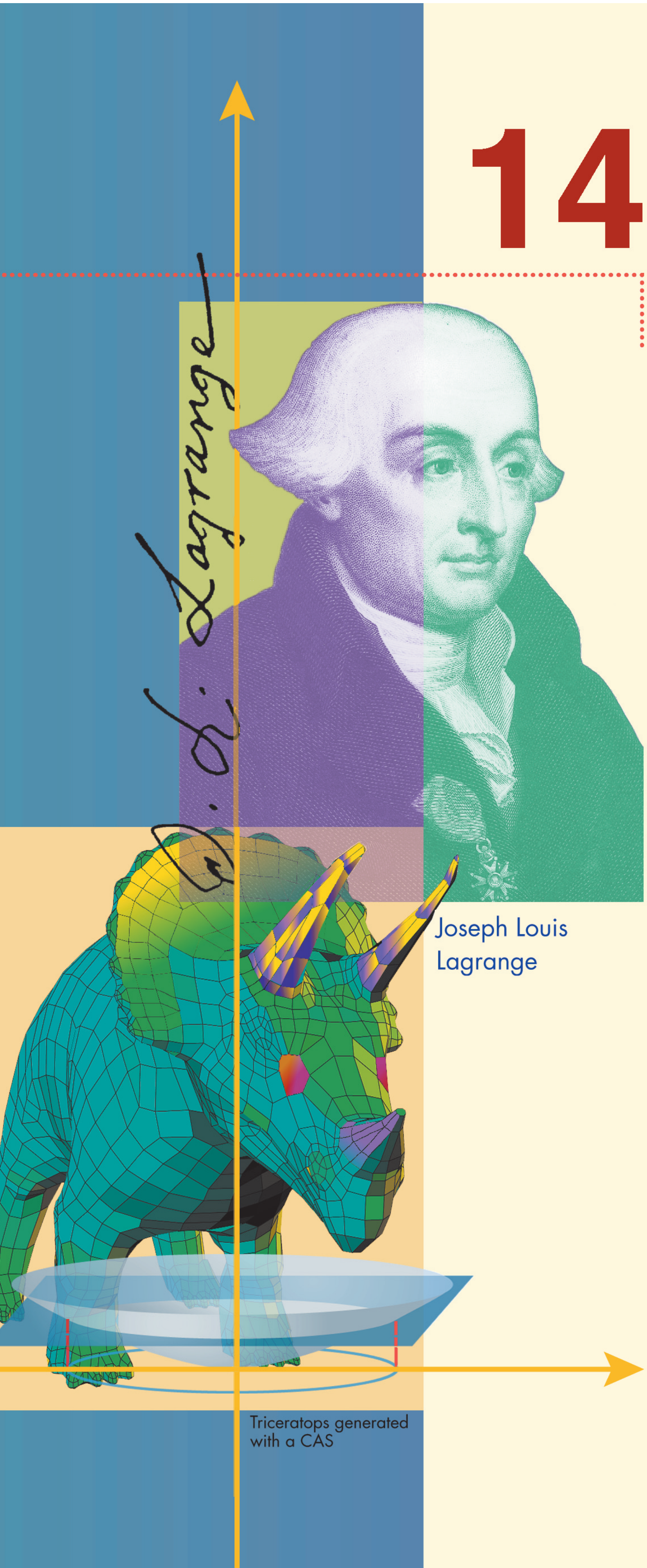
14

PARTIAL DERIVATIVES



Joseph Louis Lagrange

Triceratops generated with a CAS

In this chapter we will extend many of the basic concepts of calculus to functions of two or more variables, commonly called functions of "several variables." We will begin by discussing limits and continuity for functions of two and three variables, then we will define derivatives of such functions, and then we will use these derivatives to study tangent planes, rates of change, slopes of surfaces, and maximization and minimization problems. Although many of the basic ideas that we developed for functions of one variable will carry over in a natural way, functions of several variables are intrinsically more complicated than functions of one variable, so we will need to develop new tools and new ideas to deal with such functions.

14.1 FUNCTIONS OF TWO OR MORE VARIABLES

In previous sections we studied real-valued functions of a real variable and vector-valued functions of a real variable. In this section we will consider real-valued functions of two or more real variables.

NOTATION AND TERMINOLOGY

There are many familiar formulas in which a given variable depends on two or more other variables. For example, the area A of a triangle depends on the base length b and height h by the formula $A = \frac{1}{2}bh$; the volume V of a rectangular box depends on the length l, the width w, and the height h by the formula $V = lwh$; and the arithmetic average $\bar{x}$ of n real numbers, $x_1, x_2, \ldots, x_n$, depends on those numbers by the formula $\bar{x} = (x_1 + x_2 + \cdots + x_n)/n$. Thus, we say that

A is a function of the two variables b and h;

V is a function of the three variables l, w, and h;

$\bar{x}$ is a function of the n variables $x_1, x_2, \ldots, x_n$.

The terminology and notation for functions of two or more variables is similar to that for functions of one variable. For example, the expression

$$z = f(x, y)$$

means that z is a function of x and y in the sense that a unique value of the dependent variable z is determined by specifying values for the independent variables x and y. Similarly,

$$w = f(x, y, z)$$

expresses w as a function of x, y, and z, and

$$u = f(x_1, x_2, \ldots, x_n)$$

expresses u as a function of $x_1, x_2, \ldots, x_n$.

It is sometimes more convenient (or even necessary) to describe a function $z = f(x, y)$ using a table of values instead of an explicit formula. For example, recall from Example 3 of Section 1.2 that the windchill index is the temperature at a wind speed of 4 mi/h that would produce the same sensation on exposed skin as the current temperature and wind speed combination. Table 14.1.1 displays the windchill index (WCI) as a function of the actual air temperature T and the speed v of the wind. The entries in Table 14.1.1 were obtained by rounding the values obtained by the formula

$$\text{WCI} = 91.4 + (91.4 - T)(0.0203v - 0.304\sqrt{v} - 0.474) \tag{1}$$

to the nearest integer. Clearly, Table 14.1.1 provides us with more information "at a glance" than does Equation (1). For example, if the temperature is 30°F and the speed of the wind is 10 mi/h, then it feels as if the temperature is 16°F. We can also use Table 14.1.1 to obtain convenient estimates of windchill values that are not explicitly displayed.

Table 14.1.1

TEMPERATURE T (°F)

WIND SPEED v (mi/h)	20	25	30	35
5	16	22	27	32
10	3	10	16	22
15	–5	2	9	15
20	–11	–3	4	11

Example 1 Use Table 14.1.1 to estimate the windchill index if the air temperature is 30°F and the wind speed is 12 mi/h.

Solution. Although there is no entry in Table 14.1.1 that corresponds to $T = 30$ and $v = 12$, we can estimate the corresponding windchill by a process known as ***linear interpolation***. For $T = 30$ and $v = 10$ we have WCI = 16, and for $T = 30$ and $v = 15$ we have WCI = 9. That is, an increase of 5 mi/h in the value of v is reflected by a decrease of 7°F in WCI. If WCI were a linear function of v when T is held fixed at 30°F, then an increase of 2 mi/h in the wind speed would produce a decrease of $\frac{2}{5} \cdot 7 = \frac{14}{5} = 2.8$°F in the value of WCI. If we assume that WCI is a linear function of v for $T = 30$°F and v between 10 and 15 mi/h, then it follows that at a temperature of 30°F and a wind speed of 12 mi/h the windchill index will be $16 - 2.8 = 13.2$°F. Note that this value compares reasonably well with the value WCI = 12.5939°F obtained using Equation (1). ◀

We will find it useful to think of functions of two or three independent variables in geometric terms. For example, if $z = f(x, y)$, then we can view (x, y) as a point in the xy-plane and think of f as a rule that associates a unique numerical value z with the point (x, y); similarly, we can think of $w = f(x, y, z)$ as a rule that associates a unique numerical value w with a point (x, y, z) in an xyz-coordinate system (Figure 14.1.1).

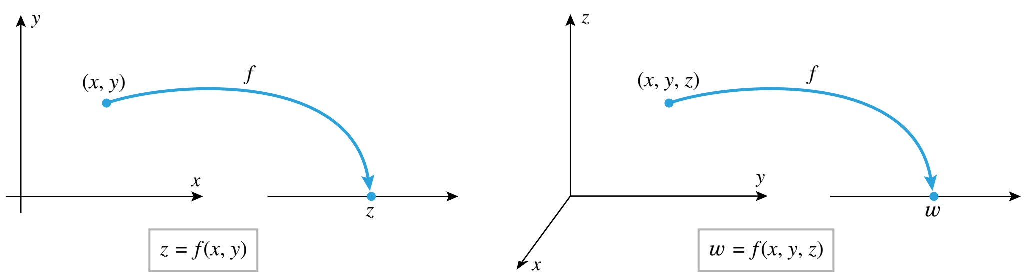


Figure 14.1.1

As with functions of one variable, the independent variables of a function of two or more variables may be restricted to lie in some set D, which we call the ***domain*** of f. Sometimes the domain will be determined by physical restrictions on the variables. If the function is defined by a formula and if there are no physical restrictions or other restrictions stated explicitly, then it is understood that the domain consists of all points for which the formula yields a real value for the dependent variable. We call this the ***natural domain*** of the function. The following definitions summarize this discussion.

14.1.1 DEFINITION. A ***function f of two variables***, x and y, is a rule that assigns a unique real number $f(x, y)$ to each point (x, y) in some set D in the xy-plane.

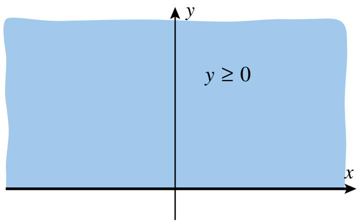
14.1.2 DEFINITION. A ***function f of three variables***, x, y, and z, is a rule that assigns a unique real number $f(x, y, z)$ to each point (x, y, z) in some set D in three-dimensional space.

REMARK. In more advanced courses the notion of "n-dimensional space" for $n > 3$ is defined, and a ***function f of n real variables***, $x_1, x_2, \ldots, x_n$, is regarded as a rule that assigns a unique real number $f(x_1, x_2, \ldots, x_n)$ to each "point" $(x_1, x_2, \ldots, x_n)$ in some set in n-dimensional space. However, we will not pursue that idea in this text.

Example 2 Let

$$f(x, y) = 3x^2\sqrt{y} - 1$$

Find $f(1, 4)$, $f(0, 9)$, $f(t^2, t)$, $f(ab, 9b)$, and the natural domain of f.



The solid boundary line is included in the domain.

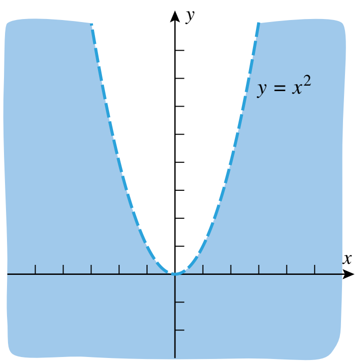
Figure 14.1.2

Solution. By substitution

$$f(1,4) = 3(1)^2\sqrt{4} - 1 = 5$$
$$f(0,9) = 3(0)^2\sqrt{9} - 1 = -1$$
$$f(t^2,t) = 3(t^2)^2\sqrt{t} - 1 = 3t^4\sqrt{t} - 1$$
$$f(ab, 9b) = 3(ab)^2\sqrt{9b} - 1 = 9a^2b^2\sqrt{b} - 1$$

Because of the radical $\sqrt{y}$ in the formula for f, we must have $y \geq 0$ to avoid imaginary values for $f(x, y)$. Thus, the natural domain of f consists of all points in the xy-plane that are on or above the x-axis. (See Figure 14.1.2.) ◀

Example 3 Sketch the natural domain of the function $f(x, y) = \ln(x^2 - y)$.



The dashed boundary does not belong to the domain.

Figure 14.1.3

Solution. $\ln(x^2 - y)$ is defined only when $0 < x^2 - y$ or $y < x^2$. We first sketch the parabola $y = x^2$ as a "dashed" curve. The region $y < x^2$ then consists of all points below this curve (Figure 14.1.3). ◀

Example 4 Let

$$f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$$

Find $f\left(0, \frac{1}{2}, -\frac{1}{2}\right)$ and the natural domain of f.

Solution. By substitution,

$$f\left(0, \tfrac{1}{2}, -\tfrac{1}{2}\right) = \sqrt{1 - (0)^2 - \left(\tfrac{1}{2}\right)^2 - \left(-\tfrac{1}{2}\right)^2} = \sqrt{\tfrac{1}{2}}$$

Because of the square root sign, we must have $0 \leq 1 - x^2 - y^2 - z^2$ in order to have a real value for $f(x, y, z)$. Rewriting this inequality in the form

$$x^2 + y^2 + z^2 \leq 1$$

we see that the natural domain of f consists of all points on or within the sphere

$$x^2 + y^2 + z^2 = 1$$ ◀

GRAPHS OF FUNCTIONS OF TWO VARIABLES

Recall that for a function f of one variable, the graph of $f(x)$ in the xy-plane was defined to be the graph of the equation $y = f(x)$. Similarly, if f is a function of two variables, we define the ***graph*** of $f(x, y)$ in xyz-space to be the graph of the equation $z = f(x, y)$. In general, such a graph will be a surface in 3-space.

Example 5 In each part, describe the graph of the function in an xyz-coordinate system.

(a) $f(x, y) = 1 - x - \frac{1}{2}y$ (b) $f(x, y) = \sqrt{1 - x^2 - y^2}$

(c) $f(x, y) = -\sqrt{x^2 + y^2}$

Solution (a). By definition, the graph of the given function is the graph of the equation

$$z = 1 - x - \tfrac{1}{2}y$$

which is a plane. A triangular portion of the plane can be sketched by plotting the intersections with the coordinate axes and joining them with line segments (Figure 14.1.4a).

Solution (b). By definition, the graph of the given function is the graph of the equation

$$z = \sqrt{1 - x^2 - y^2} \tag{2}$$

After squaring both sides, this can be rewritten as

$$x^2 + y^2 + z^2 = 1$$

which represents a sphere of radius 1, centered at the origin. Since (2) imposes the added condition that $z \geq 0$, the graph is just the upper hemisphere (Figure 14.1.4b).

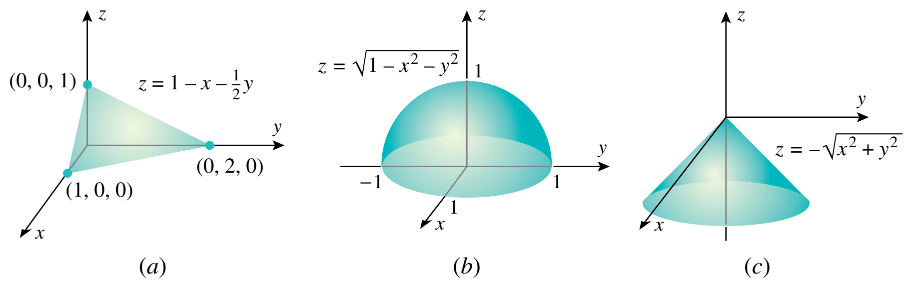


Figure 14.1.4

Solution (c). The graph of the given function is the graph of the equation

$$z = -\sqrt{x^2 + y^2} \tag{3}$$

After squaring, we obtain

$$z^2 = x^2 + y^2$$

which is the equation of a circular cone (see Table 12.7.1). Since (3) imposes the condition that $z \le 0$, the graph is just the lower nappe of the cone (Figure 14.1.4c). ◀

LEVEL CURVES

We are all familiar with the topographic (or contour) maps in which a three-dimensional landscape, such as a mountain range, is represented by two-dimensional contour lines or curves of constant elevation. Consider, for example, the model hill and its contour map shown in Figure 14.1.5. The contour map is constructed by passing planes of constant elevation through the hill, projecting the resulting contours onto a flat surface, and labeling the contours with their elevations. In Figure 14.1.5, note how the two gullies appear as indentations in the contour lines and how the curves are close together on the contour map where the hill has a steep slope and become more widely spaced where the slope is gradual.

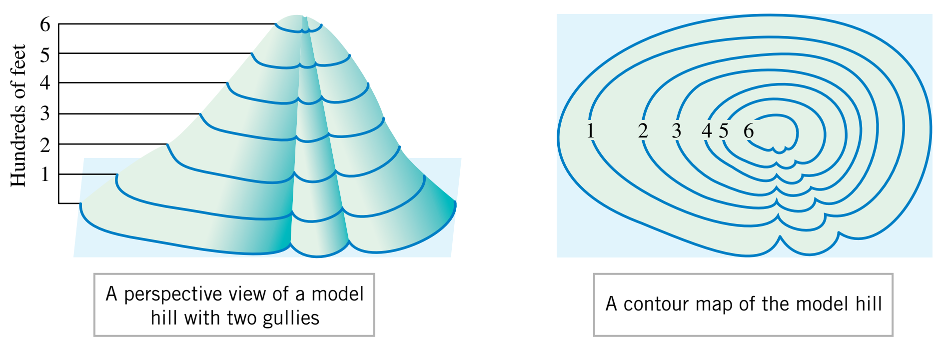


Figure 14.1.5

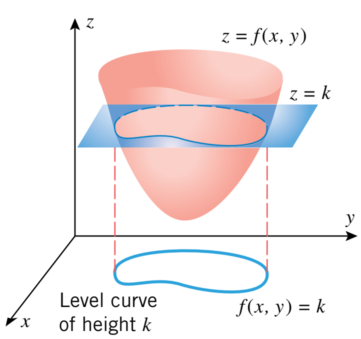
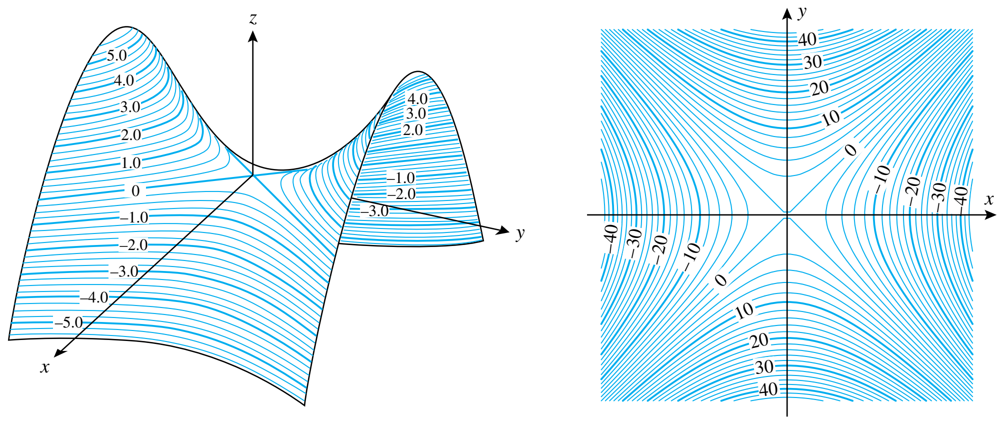


Figure 14.1.6

Contour maps are also useful for studying functions of two variables. If the surface $z = f(x, y)$ is cut by the horizontal plane $z = k$, then at all points on the intersection we have $f(x, y) = k$. The projection of this intersection onto the xy-plane is called the ***level curve of height k*** or the ***level curve with constant k*** (Figure 14.1.6). A set of level curves for $z = f(x, y)$ is called a ***contour plot*** or ***contour map*** of f.

Example 6 The graph of the function $f(x, y) = y^2 - x^2$ in xyz-space is the hyperbolic paraboloid (saddle surface) shown in Figure 14.1.7a. The level curves have equations of the form $y^2 - x^2 = k$. For $k > 0$ these curves are hyperbolas opening along lines parallel to the y-axis; for $k < 0$ they are hyperbolas opening along lines parallel to the x-axis; and for $k = 0$ the level curve consists of the intersecting lines $y + x = 0$ and $y - x = 0$ (Figure 14.1.7b). ◀



(*a*) (*b*)

Figure 14.1.7

Example 7

(a) Sketch the contour plot of $f(x, y) = 4x^2 + y^2$ using level curves of height $k = 0, 1, 2, 3, 4, 5$.

(b) Sketch the contour plot of $f(x, y) = 2 - x - y$ using level curves of height $k = -6, -4, -2, 0, 2, 4, 6$.

Solution (a). The graph of the surface $z = 4x^2 + y^2$ is the paraboloid shown in Figure 14.1.8, so we can reasonably expect the contour plot to be a family of ellipses centered at the origin. The level curve of height k has the equation $4x^2 + y^2 = k$. If $k = 0$, then the graph is the single point $(0, 0)$. For $k > 0$ we can rewrite the equation as

$$\frac{x^2}{k/4} + \frac{y^2}{k} = 1$$

which represents a family of ellipses with x-intercepts $\pm\sqrt{k}/2$ and y-intercepts $\pm\sqrt{k}$. The contour plot for the specified values of k is shown in Figure 14.1.9.

Solution (b). The graph of the surface $z = 2 - x - y$ is the plane shown in Figure 14.1.10, so we can reasonably expect the contour plot to be a family of parallel lines. The level curve of height k has the equation $2 - x - y = k$, which we can rewrite as

$$y = -x + (2 - k)$$

This represents a family of parallel lines of slope -1. The contour plot for the specified values of k is shown in Figure 14.1.11. ◀

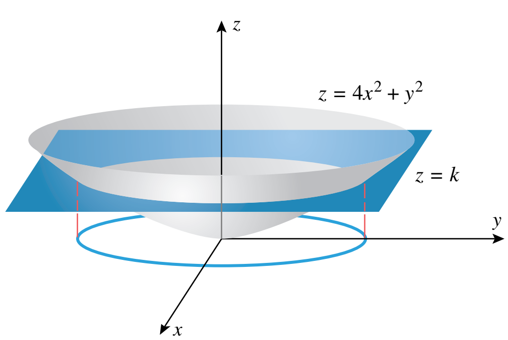


Figure 14.1.8

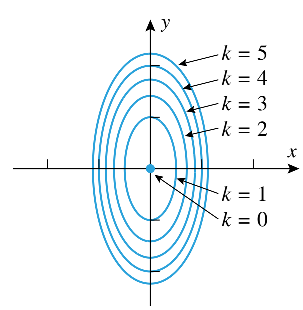


Figure 14.1.9

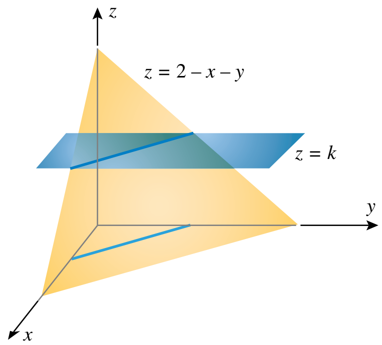


Figure 14.1.10

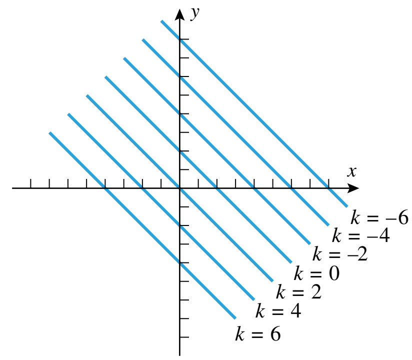


Figure 14.1.11

CONTOUR PLOTS USING TECHNOLOGY

Except in the simplest cases, contour plots can be difficult to produce without the help of a graphing utility. Figure 14.1.12 illustrates how graphing technology can be used to display level curves. The table shows two graphical representations of the level curves of the function $f(x, y) = |\sin x \sin y|$ produced with a CAS over the domain $0 \leq x \leq 2\pi$, $0 \leq y \leq 2\pi$.

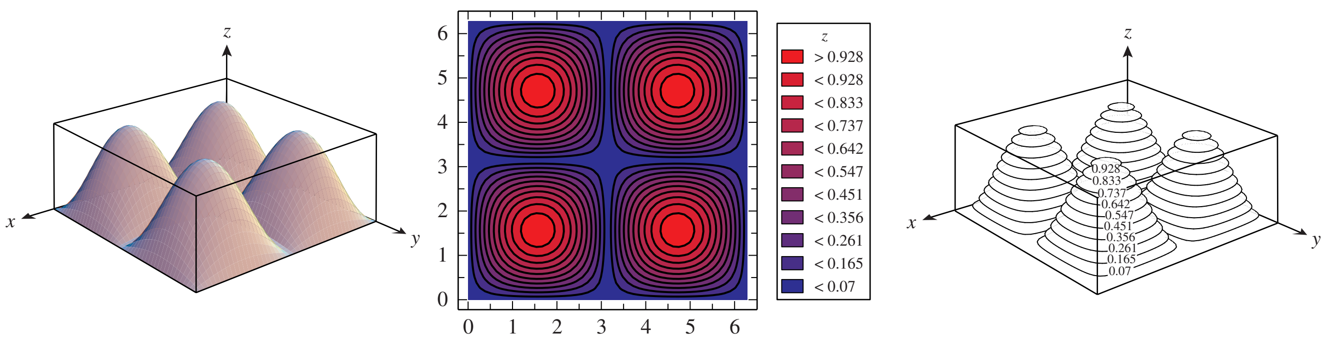


Figure 14.1.12

LEVEL SURFACES

Observe that the graph of $y = f(x)$ is a curve in 2-space, and the graph of $z = f(x, y)$ is a surface in 3-space, so the number of dimensions required for these graphs is one greater than the number of independent variables. Accordingly, there is no "direct" way to graph a function of three variables since four dimensions are required. However, if k is a constant, then the graph of the equation $f(x, y, z) = k$ will generally be a surface in 3-space (e.g., the graph of $x^2 + y^2 + z^2 = 1$ is a sphere), which we call the ***level surface with constant k***. Some geometric insight into the behavior of the function f can sometimes be obtained by graphing these level surfaces for various values of k.

REMARK. The term "level surface" is standard but confusing, since a level surface need *not* be level in the sense of being horizontal; it is simply a surface on which all values of f are the same.

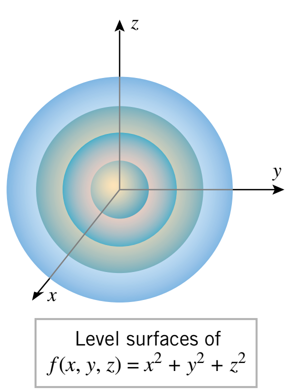


Figure 14.1.13

Example 8 Describe the level surfaces of

(a) $f(x, y, z) = x^2 + y^2 + z^2$ (b) $f(x, y, z) = z^2 - x^2 - y^2$

Solution (a). The level surfaces have equations of the form

$$x^2 + y^2 + z^2 = k$$

For $k > 0$ the graph of this equation is a sphere of radius $\sqrt{k}$, centered at the origin; for $k = 0$ the graph is the single point $(0, 0, 0)$; and for $k < 0$ there is no level surface (Figure 14.1.13).

Solution (b). The level surfaces have equations of the form

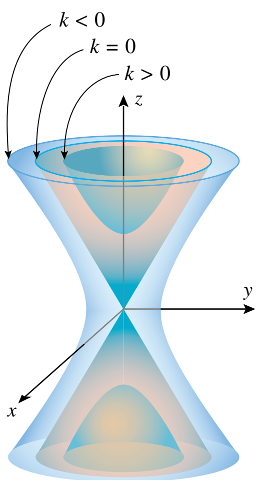
$$z^2 - x^2 - y^2 = k$$

As discussed in Section 12.7, this equation represents a cone if $k = 0$, a hyperboloid of two sheets if $k > 0$, and a hyperboloid of one sheet if $k < 0$ (Figure 14.1.14). ◀

GRAPHING FUNCTIONS OF TWO VARIABLES USING TECHNOLOGY

Generating surfaces with a graphing utility is more complicated than generating plane curves because there are more factors that must be taken into account. We can only touch on the ideas here, so if you want to use a graphing utility, its documentation will be your main source of information.

Graphing utilities can only show a portion of xyz-space in a viewing screen, so the first step in graphing a surface is to determine which portion of xyz-space you want to display. This region is called the ***viewing window*** or ***viewing box***. For example, the first row of Table 14.1.2 shows the effect of graphing the paraboloid $z = x^2 + y^2$ in three different viewing windows. However, within a fixed viewing window, the appearance of the surface is also affected by the ***viewpoint***, that is, the direction from which the surface is viewed, and the distance from the viewer to the surface. For example, the second row of Table 14.1.2 shows the graph of the paraboloid $z = x^2 + y^2$ from three different viewpoints using the viewing window in the first part of the table.



Level surfaces of
$f(x, y, z) = z^2 - x^2 - y^2$

Figure 14.1.14

FOR THE READER. If you have a graphing utility that can generate surfaces in 3-space, read the documentation and try to duplicate Table 14.1.2.

FOR THE READER. Table 14.1.3 shows six surfaces in 3-space. Examine each surface and convince yourself that the contour plot describes its level curves. This will take a little thought because the mesh lines on the surface are traces in vertical planes, whereas the level curves are traces in horizontal planes. In these contour plots the color gradation is from dark to light as z increases. If you have a graphing utility that can generate surfaces in 3-space, try to duplicate some of these figures. You need not match the colors or generate the coordinate axes.

Table 14.1.2

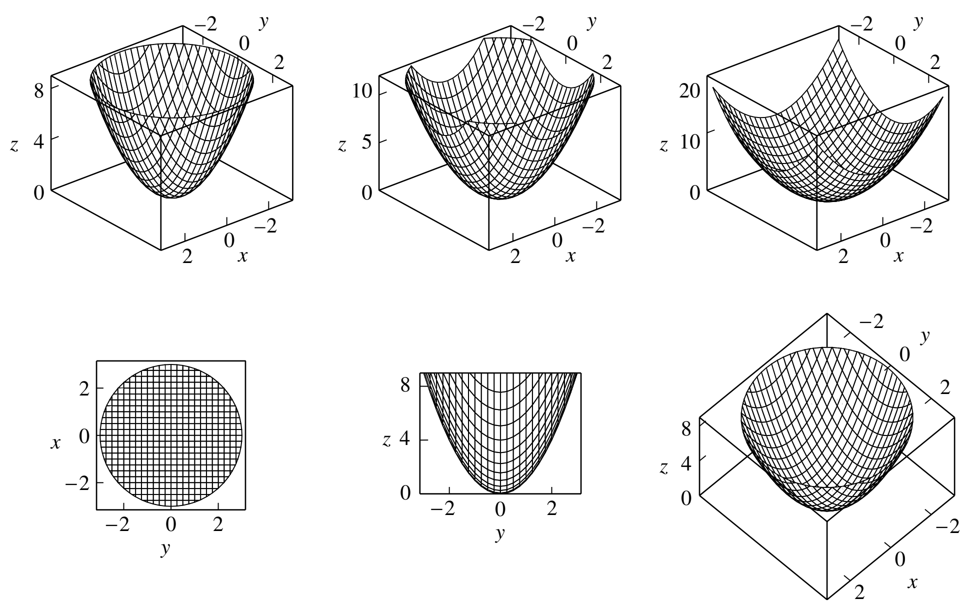
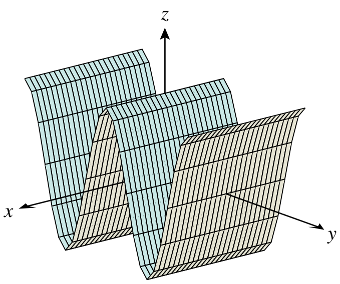
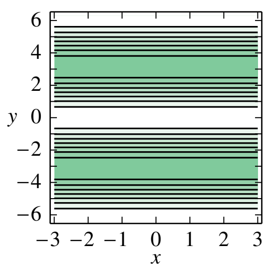
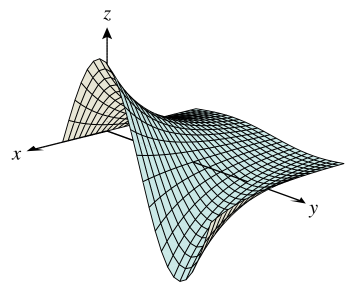
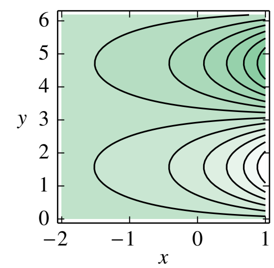
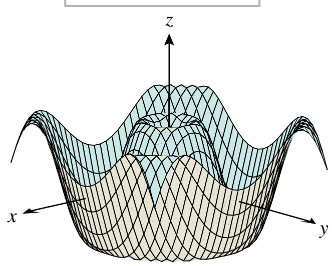
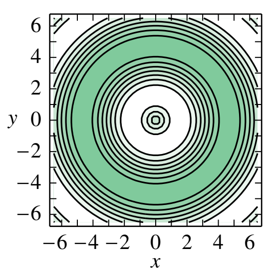
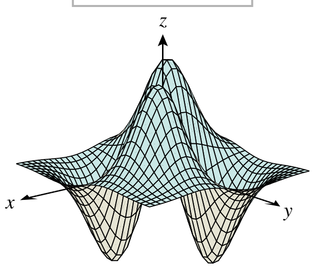
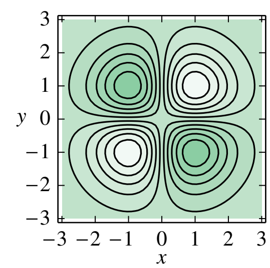
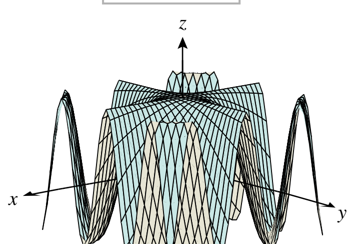
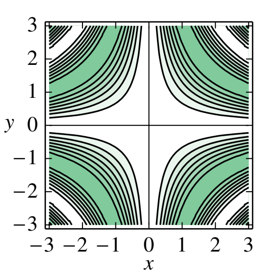
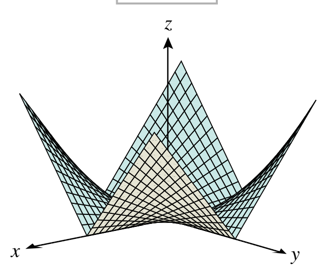
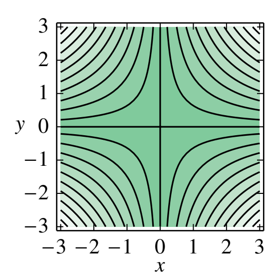


Table 14.1.3

SURFACE	CONTOUR PLOT	SURFACE	CONTOUR PLOT
$z = \cos y$ 		$z = e^x \sin y$ 	
$z = \sin(\sqrt{x^2 + y^2})$ 		$z = xye^{-\frac{1}{2}(x^2+y^2)}$ 	
$z = \cos(xy)$ 		$z = \lvert xy \rvert$ 	

EXERCISE SET 14.1 Graphing Utility C CAS

Exercises 1–8 are concerned with functions of two variables.

1. Let $f(x, y) = x^2y + 1$. Find
(a) $f(2, 1)$ (b) $f(1, 2)$ (c) $f(0, 0)$
(d) $f(1, -3)$ (e) $f(3a, a)$ (f) $f(ab, a - b)$.

2. Let $f(x, y) = x + \sqrt[3]{xy}$. Find
(a) $f(t, t^2)$ (b) $f(x, x^2)$ (c) $f(2y^2, 4y)$.

3. Let $f(x, y) = xy + 3$. Find
(a) $f(x + y, x - y)$ (b) $f(xy, 3x^2y^3)$.

4. Let $g(x) = x \sin x$. Find
(a) $g(x/y)$ (b) $g(xy)$ (c) $g(x - y)$.

5. Find $F(g(x), h(y))$ if $F(x, y) = xe^{xy}$, $g(x) = x^3$, and $h(y) = 3y + 1$.

6. Find $g(u(x, y), v(x, y))$ if $g(x, y) = y \sin(x^2y)$, $u(x, y) = x^2y^3$, and $v(x, y) = \pi xy$.

7. Let $f(x, y) = x + 3x^2y^2$, $x(t) = t^2$, and $y(t) = t^3$. Find
(a) $f(x(t), y(t))$ (b) $f(x(0), y(0))$
(c) $f(x(2), y(2))$.

8. Let $g(x, y) = ye^{-3x}$, $x(t) = \ln(t^2 + 1)$, and $y(t) = \sqrt{t}$. Find $g(x(t), y(t))$.

9. Refer to Table 14.1.1 and use the method of Example 1 to estimate the windchill index when

(a) the temperature is 25°F and the wind speed is 7 mi/h
(b) the temperature is 28°F and the wind speed is 5 mi/h.

10. Refer to Table 14.1.1 and use the method of Example 1 to estimate the windchill index when
(a) the temperature is 35°F and the wind speed is 14 mi/h
(b) the temperature is 32°F and the wind speed is 10 mi/h.

11. One method for determining relative humidity is to wet the bulb of a thermometer, whirl it through the air, and then compare the thermometer reading with the actual air temperature. If the relative humidity is less than 100%, the reading on the thermometer will be less than the temperature of the air. This difference in temperature is known as the *wet-bulb depression*. The accompanying table gives the relative humidity as a function of the air temperature and the wet-bulb depression. Use the table to complete parts (a)–(c).
(a) What is the relative humidity if the air temperature is 20°C and the wet-bulb thermometer reads 16°C?
(b) Use the method of Example 1 to estimate the relative humidity if the air temperature is 25°C and the wet-bulb depression is 3.5°C.
(c) Use the method of Example 1 to estimate the relative humidity if the air temperature is 22°C and the wet-bulb depression is 5°C.

AIR TEMPERATURE (°C)

WET-BULB DEPRESSION (°C)	15	20	25	30
3	71	74	77	79
4	62	66	70	73
5	53	59	63	67

Table Ex-11

12. Use the table in Exercise 11 to complete parts (a)–(c).
(a) What is the wet-bulb depression if the air temperature is 30°C and the relative humidity is 73%?
(b) Use the method of Example 1 to estimate the relative humidity if the air temperature is 15°C and the wet-bulb depression is 4.25°C.
(c) Use the method of Example 1 to estimate the relative humidity if the air temperature is 26°C and the wet-bulb depression is 3°C.

Exercises 13–16 involve functions of three variables.

13. Let $f(x, y, z) = xy^2z^3 + 3$. Find
(a) $f(2, 1, 2)$ (b) $f(-3, 2, 1)$
(c) $f(0, 0, 0)$ (d) $f(a, a, a)$
(e) $f(t, t^2, -t)$ (f) $f(a+b, a-b, b)$.

14. Let $f(x, y, z) = zxy + x$. Find
(a) $f(x+y, x-y, x^2)$ (b) $f(xy, y/x, xz)$.

15. Find $F(f(x), g(y), h(z))$ if $F(x, y, z) = ye^{xyz}$, $f(x) = x^2$, $g(y) = y + 1$, and $h(z) = z^2$.

16. Find $g(u(x, y, z), v(x, y, z), w(x, y, z))$ if $g(x, y, z) = z \sin xy$, $u(x, y, z) = x^2z^3$, $v(x, y, z) = \pi xyz$, and $w(x, y, z) = xy/z$.

Exercises 17 and 18 are concerned with functions of four or more variables.

17. (a) Let $f(x, y, z, t) = x^2y^3\sqrt{z+t}$. Find $f(\sqrt{5}, 2, \pi, 3\pi)$.
(b) Let $f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} kx_k$. Find $f(1, 1, \ldots, 1)$.

18. (a) Let $f(u, v, \lambda, \phi) = e^{u+v}\cos\lambda\tan\phi$. Find $f(-2, 2, 0, \pi/4)$.
(b) Let $f(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2$. Find $f(1, 2, \ldots, n)$.

In Exercises 19–22, sketch the domain of f. Use solid lines for portions of the boundary included in the domain and dashed lines for portions not included.

19. $f(x, y) = \ln(1 - x^2 - y^2)$ **20.** $f(x, y) = \sqrt{x^2 + y^2 - 4}$

21. $f(x, y) = \dfrac{1}{x - y^2}$ **22.** $f(x, y) = \ln xy$

In Exercises 23 and 24, describe the domain of f in words.

23. (a) $f(x, y) = xe^{-\sqrt{y+2}}$
(b) $f(x, y, z) = \sqrt{25 - x^2 - y^2 - z^2}$
(c) $f(x, y, z) = e^{xyz}$

24. (a) $f(x, y) = \dfrac{\sqrt{4 - x^2}}{y^2 + 3}$ (b) $f(x, y) = \ln(y - 2x)$
(c) $f(x, y, z) = \dfrac{xyz}{x + y + z}$

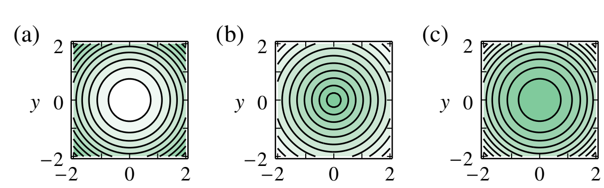
In Exercises 25–34, sketch the graph of f.

25. $f(x, y) = 3$ **26.** $f(x, y) = \sqrt{9 - x^2 - y^2}$
27. $f(x, y) = \sqrt{x^2 + y^2}$ **28.** $f(x, y) = x^2 + y^2$
29. $f(x, y) = x^2 - y^2$ **30.** $f(x, y) = 4 - x^2 - y^2$
31. $f(x, y) = \sqrt{x^2 + y^2 + 1}$ **32.** $f(x, y) = \sqrt{x^2 + y^2 - 1}$
33. $f(x, y) = y + 1$ **34.** $f(x, y) = x^2$

35. In each part, match the contour plot with one of the functions

$$f(x, y) = \sqrt{x^2 + y^2}, \quad f(x, y) = x^2 + y^2, \quad f(x, y) = 1 - x^2 - y^2$$

by inspection, and explain your reasoning. The larger the value of z, the lighter the color in the contour plot, and the contours correspond to equally spaced values of z.



36. In each part, match the contour plot with one of the surfaces in the accompanying figure by inspection, and explain your reasoning. The larger the value of z, the lighter the color in the contour plot.

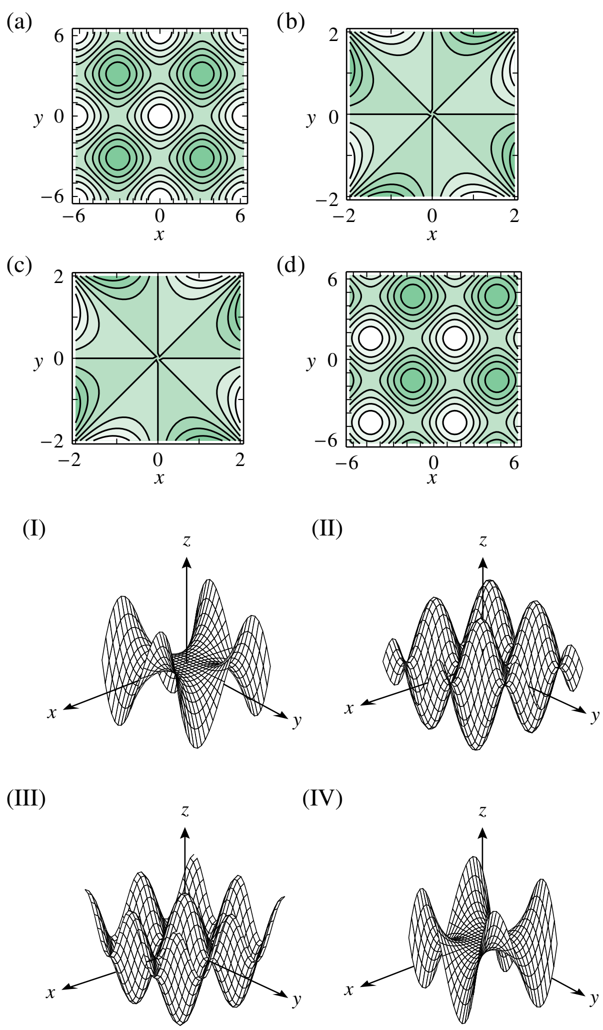


Figure Ex-36

37. In each part, the questions refer to the contour map in the accompanying figure.

(a) Is A or B the higher point? Explain your reasoning.

(b) Is A or B on the steeper slope? Explain your reasoning.

(c) Starting at A and moving so that y remains constant and x increases, will the elevation begin to increase or decrease?

(d) Starting at B and moving so that y remains constant and x increases, will the elevation begin to increase or decrease?

(e) Starting at A and moving so that x remains constant and y decreases, will the elevation begin to increase or decrease?

(f) Starting at B and moving so that x remains constant and y decreases, will the elevation begin to increase or decrease?

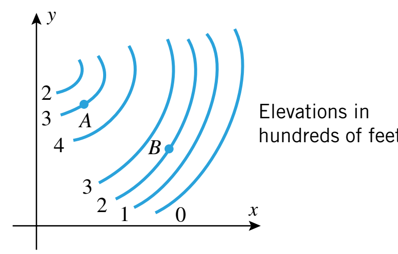


Figure Ex-37

38. A curve connecting points of equal atmospheric pressure on a weather map is called an ***isobar***. On a typical weather map the isobars refer to pressure at mean sea level and are given in units of ***millibars*** (mb). Mathematically, isobars are level curves for the pressure function $p(x, y)$ defined at the geographic points (x, y) represented on the map. Tightly packed isobars correspond to steep slopes on the graph of the pressure function, and these are usually associated with strong winds—the steeper the slope, the greater the speed of the wind.

(a) Referring to the accompanying weather map, is the wind speed greater in Medicine Hat, Alberta or in Chicago? Explain your reasoning.

(b) Estimate the average rate of change in atmospheric pressure from Medicine Hat to Chicago, given that the distance between the cities is approximately 1400 mi.

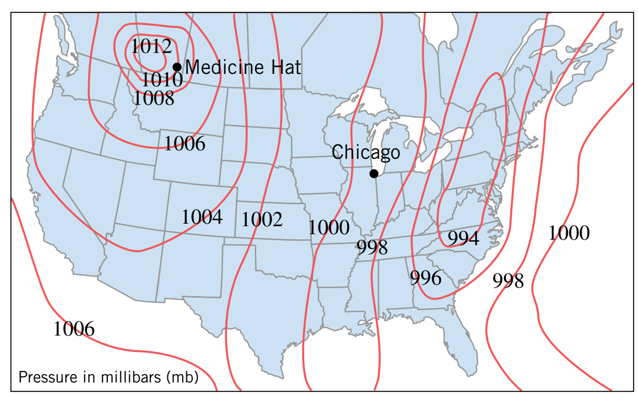


Figure Ex-38

In Exercises 39–44, sketch the level curve $z = k$ for the specified values of k.

39. $z = x^2 + y^2$; $k = 0, 1, 2, 3, 4$

40. $z = y/x$; $k = -2, -1, 0, 1, 2$

41. $z = x^2 + y$; $k = -2, -1, 0, 1, 2$

42. $z = x^2 + 9y^2$; $k = 0, 1, 2, 3, 4$

43. $z = x^2 - y^2$; $k = -2, -1, 0, 1, 2$

44. $z = y \csc x$; $k = -2, -1, 0, 1, 2$

In Exercises 45–48, sketch the level surface $f(x, y, z) = k$.

45. $f(x, y, z) = 4x^2 + y^2 + 4z^2$; $k = 16$

46. $f(x, y, z) = x^2 + y^2 - z^2$; $k = 0$

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47. $f(x, y, z) = z - x^2 - y^2 + 4;\ k = 7$

48. $f(x, y, z) = 4x - 2y + z;\ k = 1$

In Exercises 49–52, describe the level surfaces in words.

49. $f(x, y, z) = (x - 2)^2 + y^2 + z^2$

50. $f(x, y, z) = 3x - y + 2z$ **51.** $f(x, y, z) = x^2 + z^2$

52. $f(x, y, z) = z - x^2 - y^2$

53. Let $f(x, y) = x^2 - 2x^3 + 3xy$. Find an equation of the level curve that passes through the point
(a) $(-1, 1)$ (b) $(0, 0)$ (c) $(2, -1)$.

54. Let $f(x, y) = ye^x$. Find an equation of the level curve that passes through the point
(a) $(\ln 2, 1)$ (b) $(0, 3)$ (c) $(1, -2)$.

55. Let $f(x, y, z) = x^2 + y^2 - z$. Find an equation of the level surface that passes through the point
(a) $(1, -2, 0)$ (b) $(1, 0, 3)$ (c) $(0, 0, 0)$.

56. Let $f(x, y, z) = xyz + 3$. Find an equation of the level surface that passes through the point
(a) $(1, 0, 2)$ (b) $(-2, 4, 1)$ (c) $(0, 0, 0)$.

57. If $T(x, y)$ is the temperature at a point (x, y) on a thin metal plate in the xy-plane, then the level curves of T are called ***isothermal curves***. All points on such a curve are at the same temperature. Suppose that a plate occupies the first quadrant and $T(x, y) = xy$.
(a) Sketch the isothermal curves on which $T = 1$, $T = 2$, and $T = 3$.
(b) An ant, initially at $(1, 4)$, wants to walk on the plate so that the temperature along its path remains constant. What path should the ant take and what is the temperature along that path?

58. If $V(x, y)$ is the voltage or potential at a point (x, y) in the xy-plane, then the level curves of V are called ***equipotential curves***. Along such a curve, the voltage remains constant. Given that

$$V(x, y) = \frac{8}{\sqrt{16 + x^2 + y^2}}$$

sketch the equipotential curves at which $V = 2.0$, $V = 1.0$, and $V = 0.5$.

59. Let $f(x, y) = x^2 + y^3$.
(a) Use a graphing utility to generate the level curve that passes through the point $(2, -1)$.
(b) Generate the level curve of height 1.

60. Let $f(x, y) = 2\sqrt{xy}$.
(a) Use a graphing utility to generate the level curve that passes through the point $(2, 2)$.
(b) Generate the level curve of height 8.

C **61.** Let $f(x, y) = xe^{-(x^2+y^2)}$.
(a) Use a CAS to generate the graph of f for $-2 \le x \le 2$ and $-2 \le y \le 2$.
(b) Generate a contour plot for the surface, and confirm visually that it is consistent with the surface obtained in part (a).
(c) Read the appropriate documentation and explore the effect of generating the graph of f from various viewpoints.

C **62.** Let $f(x, y) = \frac{1}{10}e^x \sin y$.
(a) Use a CAS to generate the graph of f for $0 \le x \le 4$ and $0 \le y \le 2\pi$.
(b) Generate a contour plot for the surface, and confirm visually that it is consistent with the surface obtained in part (a).
(c) Read the appropriate documentation and explore the effect of generating the graph of f from various viewpoints.

63. In each part, describe in words how the graph of g is related to the graph of f.
(a) $g(x, y) = f(x - 1, y)$ (b) $g(x, y) = 1 + f(x, y)$
(c) $g(x, y) = -f(x, y + 1)$

64. (a) Sketch the graph of $f(x, y) = e^{-(x^2+y^2)}$.
(b) In this part, describe in words how the graph of the function $g(x, y) = e^{-a(x^2+y^2)}$ is related to the graph of f for positive values of a.

14.2 LIMITS AND CONTINUITY

In this section we will introduce the notions of limit and continuity for functions of two or more variables. We will not go into great detail—our objective is to develop the basic concepts accurately and to obtain results needed in later sections. A more extensive study of these topics is usually given in advanced calculus.

LIMITS ALONG CURVES

For a function of one variable there are two one-sided limits at a number x_0, namely

$$\lim_{x \to x_0^+} f(x) \quad \text{and} \quad \lim_{x \to x_0^-} f(x)$$

reflecting the fact that there are only two directions from which x can approach x_0, the right or the left. For functions of two or three variables the situation is more complicated

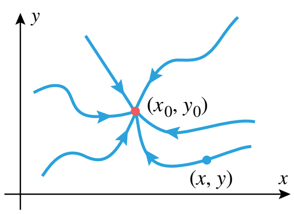

Figure 14.2.1

because there are infinitely many different curves along which one point can approach another (Figure 14.2.1). Our first objective in this section is to define the limit of $f(x, y)$ as (x, y) approaches a point (x_0, y_0) along a curve C (and similarly for functions of three variables).

If C is a smooth parametric curve in 2-space or 3-space that is represented by the equations

$$x = x(t), \quad y = y(t) \qquad \text{or} \qquad x = x(t), \quad y = y(t), \quad z = z(t)$$

and if $x_0 = x(t_0)$, $y_0 = y(t_0)$, and $z_0 = z(t_0)$, then the limits

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ (\text{along } C)}} f(x, y) \quad \text{and} \quad \lim_{\substack{(x,y,z)\to(x_0,y_0,z_0)\\ (\text{along } C)}} f(x, y, z)$$

are defined by

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ (\text{along } C)}} f(x, y) = \lim_{t\to t_0} f(x(t), y(t)) \tag{1}$$

$$\lim_{\substack{(x,y,z)\to(x_0,y_0,z_0)\\ (\text{along } C)}} f(x, y, z) = \lim_{t\to t_0} f(x(t), y(t), z(t)) \tag{2}$$

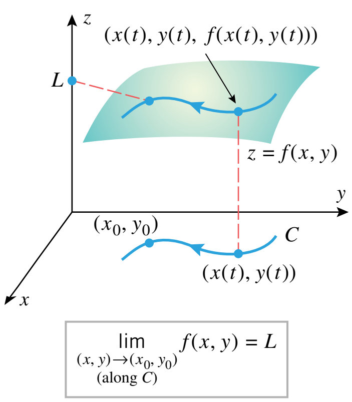

Figure 14.2.2

Simply stated, limits along parametric curves are obtained by substituting the parametric equations into the formula for the function f and computing the appropriate limit of the resulting function of one variable. A geometric interpretation of the limit along a curve for a function of two variables is shown in Figure 14.2.2: As the point $(x(t), y(t))$ moves along the curve C in the xy-plane toward (x_0, y_0), the point $(x(t), y(t), f(x(t), y(t)))$ moves directly above it along the graph of $z = f(x, y)$ with $f(x(t), y(t))$ approaching the limiting value L. In the figure we followed a common practice of omitting the zero z-coordinate for points in the xy-plane.

REMARK. In both (1) and (2), the limit of the function of t has to be treated as a one-sided limit if (x_0, y_0) or (x_0, y_0, z_0) is an endpoint of C.

Example 1 Figure 14.2.3*a* shows a computer-generated graph of the function

$$f(x, y) = -\frac{xy}{x^2 + y^2}$$

The graph reveals that the surface has a ridge above the line $y = -x$, which is to be expected since $f(x, y)$ has a constant value of $\frac{1}{2}$ for $y = -x$, except at (0, 0) where f is undefined (verify). Moreover, the graph suggests that the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ along a line through the origin varies with the direction of the line. Find this limit along

(a) the x-axis (b) the y-axis (c) the line $y = x$
(d) the line $y = -x$ (e) the parabola $y = x^2$

Solution (a). The x-axis has parametric equations $x = t$, $y = 0$, with (0, 0) corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=0)}} f(x, y) = \lim_{t\to 0} f(t, 0) = \lim_{t\to 0}\left(-\frac{0}{t^2}\right) = \lim_{t\to 0} 0 = 0$$

which is consistent with Figure 14.2.3*b*.

Solution (b). The y-axis has parametric equations $x = 0$, $y = t$, with (0, 0) corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } x=0)}} f(x, y) = \lim_{t\to 0} f(0, t) = \lim_{t\to 0}\left(-\frac{0}{t^2}\right) = \lim_{t\to 0} 0 = 0$$

which is consistent with Figure 14.2.3*b*.

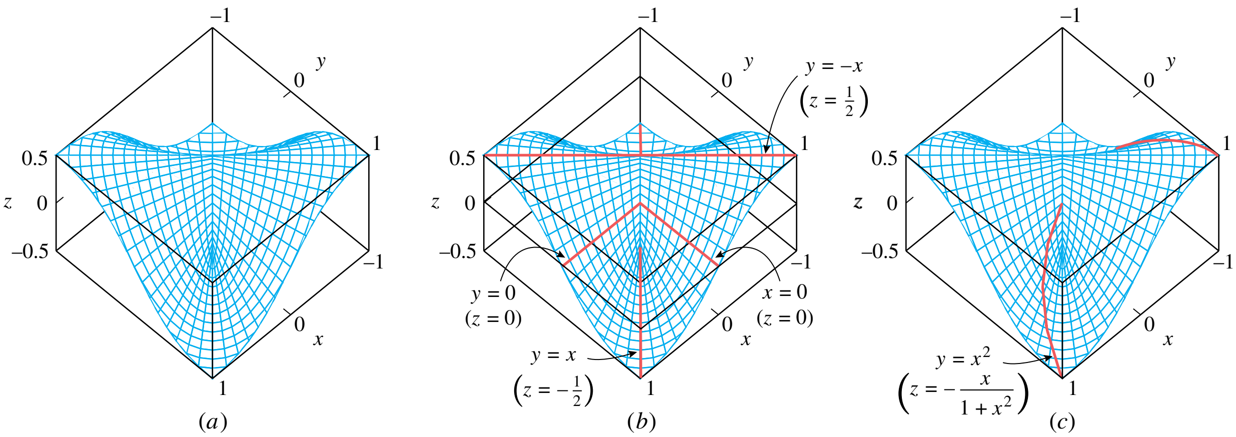

Figure 14.2.3

Solution (c). The line $y = x$ has parametric equations $x = t$, $y = t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=x)}} f(x, y) = \lim_{t\to 0} f(t, t) = \lim_{t\to 0}\left(-\frac{t^2}{2t^2}\right) = \lim_{t\to 0}\left(-\frac{1}{2}\right) = -\frac{1}{2}$$

which is consistent with Figure 14.2.3*b*.

Solution (d). The line $y = -x$ has parametric equations $x = t$, $y = -t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=-x)}} f(x, y) = \lim_{t\to 0} f(t, -t) = \lim_{t\to 0}\frac{t^2}{2t^2} = \lim_{t\to 0}\frac{1}{2} = \frac{1}{2}$$

which is consistent with Figure 14.2.3*b*.

Solution (e). The parabola $y = x^2$ has parametric equations $x = t$, $y = t^2$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=x^2)}} f(x, y) = \lim_{t\to 0} f(t, t^2) = \lim_{t\to 0}\left(-\frac{t^3}{t^2+t^4}\right) = \lim_{t\to 0}\left(-\frac{t}{1+t^2}\right) = 0$$

This is consistent with Figure 14.2.3*c*, which shows the parametric curve

$$x = t, \quad y = t^2, \quad z = -\frac{t}{1+t^2}$$

superimposed on the surface. ◀

GENERAL LIMITS OF FUNCTIONS OF TWO VARIABLES

Although limits along specific curves are useful for many purposes, they do not always tell the complete story about the limiting behavior of a function; what is required is a limit concept that accounts for the behavior of the function in an *entire vicinity* of a point, not just along smooth curves passing through the point. In our discussion of limits for functions of a single variable, this concept of "vicinity" was captured by an interval of real numbers. For functions of two variables this role is played by a *disk* in the plane. A disk is called *open* if it consists of all points inside its boundary circle, and it is called *closed* if it includes its boundary circle. (In 3-space the analogous concept is that of a *ball*. A ball is *open* if it consists of all points inside its boundary sphere, and it is *closed* if it includes its boundary sphere.) Later we will extend the notions of "open" and "closed" to more general sets.

As illustrated in Figure 14.2.4, we will want the statement

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

to mean that the value of $f(x, y)$ can be made as close as we like to L (say within ϵ units of L) by restricting (x, y) to lie within (but not at the center of) some sufficiently small disk centered at (x_0, y_0) (say a disk of radius δ). This idea is conveyed by Definition 14.2.1.

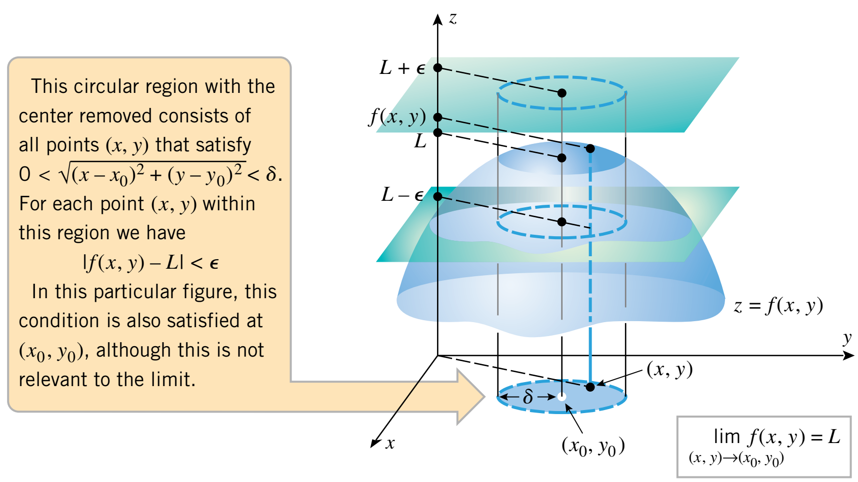

Figure 14.2.4

> **14.2.1 DEFINITION.** Let f be a function of two variables, and assume that f is defined at all points within a disk centered at (x_0, y_0), except possibly at (x_0, y_0). We will write
>
> $$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \tag{3}$$
>
> if given any number $\epsilon > 0$, we can find a number $\delta > 0$ such that $f(x, y)$ satisfies
>
> $$|f(x,y) - L| < \epsilon$$
>
> whenever the distance between (x, y) and (x_0, y_0) satisfies
>
> $$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

When convenient, (3) can also be written as

$$f(x,y) \to L \quad \text{as} \quad (x,y)\to(x_0,y_0)$$

PROPERTIES OF LIMITS

We note without proof that the standard properties of limits hold for limits along curves and for general limits of functions of two variables, so that computations involving such limits can be performed in the usual way.

Example 2

$$\begin{aligned}
\lim_{(x,y)\to(1,4)} [5x^3y^2 - 9] &= \lim_{(x,y)\to(1,4)} [5x^3y^2] - \lim_{(x,y)\to(1,4)} 9 \\
&= 5\left[\lim_{(x,y)\to(1,4)} x\right]^3 \left[\lim_{(x,y)\to(1,4)} y\right]^2 - 9 \\
&= 5(1)^3(4)^2 - 9 = 71
\end{aligned}$$

◀

RELATIONSHIPS BETWEEN GENERAL LIMITS AND LIMITS ALONG SMOOTH CURVES

The following theorem, which we state without proof, establishes an important relationship between general limits and limits along smooth curves.

14.2.2 THEOREM.

(*a*) *If $f(x, y) \to L$ as $(x, y) \to (x_0, y_0)$, then $f(x, y) \to L$ as $(x, y) \to (x_0, y_0)$ along any smooth curve.*

(*b*) *If the limit of $f(x, y)$ fails to exist as $(x, y) \to (x_0, y_0)$ along some smooth curve, or if $f(x, y)$ has different limits as $(x, y) \to (x_0, y_0)$ along two different smooth curves, then the limit of $f(x, y)$ does not exist as $(x, y) \to (x_0, y_0)$.*

Example 3 The limit

$$\lim_{(x,y)\to(0,0)} -\frac{xy}{x^2+y^2}$$

does not exist because in Example 1 we found two different smooth curves along which this limit had different values. Specifically,

$$\lim_{\substack{(x,y)\to(0,0)\\(\text{along } x=0)}} -\frac{xy}{x^2+y^2} = 0 \quad \text{and} \quad \lim_{\substack{(x,y)\to(0,0)\\(\text{along } y=x)}} -\frac{xy}{x^2+y^2} = -\frac{1}{2}$$ ◀

REMARK. One cannot prove that $f(x, y) \to L$ as $(x, y) \to (x_0, y_0)$ by showing that $f(x, y) \to L$ as $(x, y) \to (x_0, y_0)$ along a specific curve or even an entire family of curves. The problem is that there may be some curve outside of the family for which the limit does not exist or has a limit that is different from L (see Exercise 26, for example).

CONTINUITY

Stated informally, a function of one variable is continuous if its graph is an unbroken curve without jumps or holes. To extend this idea to functions of two variables, imagine that the graph of $z = f(x, y)$ is molded from a thin sheet of clay that has been hollowed or pinched into peaks and valleys. We will regard f as being continuous if the clay surface has no tears or holes. The functions graphed in Figure 14.2.5 fail to be continuous because of their behavior at (0, 0).

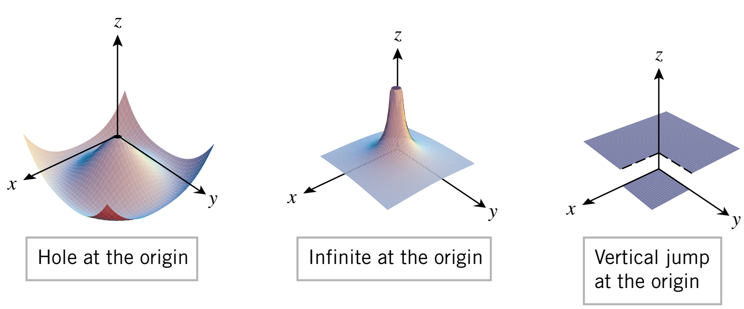

Figure 14.2.5

The precise definition of continuity at a point for functions of two variables is similar to that for functions of one variable—we require the limit of the function and the value of the function to be the same at the point.

14.2.3 DEFINITION. A function $f(x, y)$ is said to be ***continuous at*** (x_0, y_0) if $f(x_0, y_0)$ is defined and if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$

The following theorem, which we state without proof, illustrates some of the ways in which continuous functions can be combined to produce new continuous functions.

14.2.4 THEOREM.

(*a*) *If* $g(x)$ *is continuous at* x_0 *and* $h(y)$ *is continuous at* y_0, *then* $f(x, y) = g(x)h(y)$ *is continuous at* (x_0, y_0).

(*b*) *If* $h(x, y)$ *is continuous at* (x_0, y_0) *and* $g(u)$ *is continuous at* $u = h(x_0, y_0)$, *then the composition* $f(x, y) = g(h(x, y))$ *is continuous at* (x_0, y_0).

(*c*) *If* $f(x, y)$ *is continuous at* (x_0, y_0), *and if* $x(t)$ *and* $y(t)$ *are continuous at* t_0 *with* $x(t_0) = x_0$ *and* $y(t_0) = y_0$, *then the composition* $f(x(t), y(t))$ *is continuous at* t_0.

A function f of two variables that is continuous at every point (x, y) in the xy-plane is said to be ***continuous everywhere***.

Example 4 Use Theorem 14.2.4 to show that $f(x, y) = 3x^2y^5$ and $f(x, y) = \sin(3x^2y^5)$ are continuous everywhere.

Solution. The polynomials $g(x) = 3x^2$ and $h(y) = y^5$ are continuous at every real number, and therefore by part (*a*) of Theorem 14.2.4, the function $f(x, y) = 3x^2y^5$ is continuous at every point (x, y) in the xy-plane. Since $3x^2y^5$ is continuous at every point in the xy-plane and $\sin u$ is continuous at every real number u, it follows from part (*b*) of Theorem 14.2.4 that the composition $f(x, y) = \sin(3x^2y^5)$ is continuous at every point in the xy-plane. ◀

Theorem 14.2.4 is one of a whole class of theorems about continuity of functions in two or more variables. The content of these theorems can be summarized informally with three basic principles:

- A composition of continuous functions is continuous.
- A sum, difference, or product of continuous functions is continuous.
- A quotient of continuous functions is continuous, except where the denominator is zero.

By using these principles and Theorem 14.2.4, you should be able to confirm that the following functions are all continuous:

$$xe^{xy} + y^{2/3}, \quad \cosh(xy^3) - |xy|, \quad \frac{xy}{1 + x^2 + y^2}$$

Example 5 Evaluate $\displaystyle\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2 + y^2}$.

Solution. Since $f(x, y) = xy/(x^2 + y^2)$ is continuous at $(-1, 2)$ (why?), it follows from the definition of continuity for functions of two variables that

$$\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2 + y^2} = \frac{(-1)(2)}{(-1)^2 + (2)^2} = -\frac{2}{5}$$ ◀

Example 6 Since the function

$$f(x, y) = \frac{x^3y^2}{1 - xy}$$

is a quotient of continuous functions, it is continuous except where $1 - xy = 0$. Thus, $f(x, y)$ is continuous everywhere except on the hyperbola $xy = 1$. ◀

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LIMITS AT DISCONTINUITIES

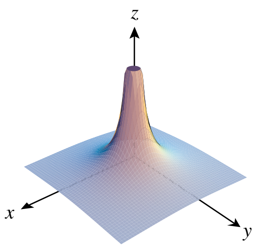


Figure 14.2.6

Sometimes it is easy to recognize when a limit does not exist. For example, it is evident that

$$\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} = +\infty$$

which implies that the values of the function approach $+\infty$ as $(x, y) \to (0, 0)$ along any smooth curve (Figure 14.2.6). However, it is not evident whether the limit

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$$

exists because it is an indeterminate form of type $0 \cdot \infty$. Although L'Hôpital's rule cannot be applied directly, the following example illustrates a method for finding this limit by converting to polar coordinates.

Example 7 Find $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$.

Solution. Let (r, θ) be polar coordinates of the point (x, y) with $r \geq 0$. Then we have

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2+y^2$$

Moreover, since $r \geq 0$ we have $r = \sqrt{x^2+y^2}$, so that $r \to 0^+$ if and only if $(x, y) \to (0, 0)$. Thus, we can rewrite the given limit as

$$\begin{aligned}\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) &= \lim_{r\to 0^+} r^2 \ln r^2 \\ &= \lim_{r\to 0^+} \frac{2\ln r}{1/r^2} && \text{This converts the limit to an indeterminate form of type } \infty/\infty. \\ &= \lim_{r\to 0^+} \frac{2/r}{-2/r^3} && \text{L'Hôpital's rule} \\ &= \lim_{r\to 0^+} (-r^2) = 0\end{aligned}$$

◀

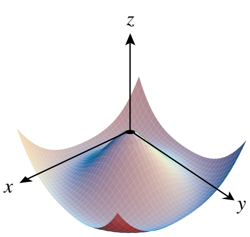


Figure 14.2.7

REMARK. Recall that for a function f of one variable, a hole occurs in the graph of f at x_0 if $f(x_0)$ is undefined but $f(x)$ has a limit as $x \to x_0$ (Figure 2.5.4*a*, for example). Similarly, a hole will occur in the graph of $f(x, y)$ at (x_0, y_0) if $f(x_0, y_0)$ is undefined but $f(x, y)$ has a limit as $(x, y) \to (x_0, y_0)$. In particular, it follows from the last example that the graph of $f(x, y) = (x^2+y^2)\ln(x^2+y^2)$ has a hole at $(0, 0)$ (Figure 14.2.7).

CONTINUITY ON A SET

Consider the function $f(x, y) = \sqrt{1-x^2-y^2}$ whose domain consists of all points (x, y) such that

$$0 \leq 1 - x^2 - y^2$$

or, equivalently, such that

$$x^2 + y^2 \leq 1$$

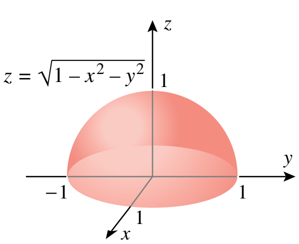


Figure 14.2.8

Thus, the domain of f is the closed unit disk of radius 1 centered at the origin. For any point (x_0, y_0) in this disk, if (x, y) is also in the disk and near (x_0, y_0), then $f(x, y)$ will be close to $f(x_0, y_0)$. Furthermore, the graph of this function is the upper hemisphere of radius 1 centered at the origin (Figure 14.2.8), and this graph displays no tears or holes. However, although this function passes our intuitive "tests" of continuity, it nonetheless fails to be continuous at every point on the boundary circle $x^2 + y^2 = 1$ of its domain. This failure is actually due to a technicality. It follows from Definitions 14.2.1 and 14.2.3 that in order for a function to be continuous at a point, it must be defined in some disk centered at that point. However, every disk centered at any point (x_0, y_0) on the circle $x^2 + y^2 = 1$ contains points outside the domain of f. Thus f is discontinuous at every point (x_0, y_0) on the boundary of its domain.

To avoid such technical failures of continuity, we need to extend the notion of continuity to the concept of ***continuity on a set***. This is analogous to the one-variable situation where in order to define the continuity of $f(x)$ on a closed interval (Definition 2.5.7), we needed to extend the notion of continuity (Definition 2.5.1) to include "continuity from the left" and "continuity from the right." Similarly, in order to define the continuity of $f(x, y)$ on a subset R of the xy-plane, we need to extend Definition 14.2.3. Suppose that R is a subset of the xy-plane that is contained within the domain of $f(x, y)$. To say that $f(x, y)$ is *continuous on* R should mean that for every point (x_0, y_0) in R, $f(x, y)$ will be close to $f(x_0, y_0)$ for all points (x, y) in R that are near (x_0, y_0). More formally, we have the following definition.

14.2.5 DEFINITION. Let R denote a subset of the xy-plane contained within the domain of a function $f(x, y)$. We say that $f(x, y)$ is ***continuous on R*** provided that for every point (x_0, y_0) in R, and for every $\epsilon > 0$, there exists a number $\delta > 0$ such that $f(x, y)$ satisfies

$$|f(x, y) - f(x_0, y_0)| < \epsilon$$

whenever (x, y) is in R and the distance between (x, y) and (x_0, y_0) satisfies

$$0 \leq \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

It follows from Definition 14.2.5 that the function $f(x, y) = \sqrt{1 - x^2 - y^2}$ is continuous on its entire domain. More generally, any function $f(x, y)$ that can be expressed by a *single formula* will *usually* be continuous on its domain.

EXTENSIONS TO THREE VARIABLES

All of the results in this section can be extended to functions of three or more variables. For example, the distance between the points (x, y, z) and (x_0, y_0, z_0) in 3-space is

$$\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

so the natural extension of Definition 14.2.1 to 3-space is as follows:

14.2.6 DEFINITION. Let f be a function of three variables, and assume that f is defined at all points within a ball centered at (x_0, y_0, z_0), except possibly at (x_0, y_0, z_0). We will write

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = L \tag{4}$$

if given any number $\epsilon > 0$, we can find a number $\delta > 0$ such that $f(x, y, z)$ satisfies

$$|f(x, y, z) - L| < \epsilon$$

whenever the distance between (x, y, z) and (x_0, y_0, z_0) satisfies

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \delta$$

As with functions of one and two variables, we define a function $f(x, y, z)$ of three variables to be continuous at a point (x_0, y_0, z_0) if the limit of the function and the value of the function are the same at this point; that is,

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = f(x_0, y_0, z_0)$$

Although we will omit the details, the properties of limits and continuity that we discussed for functions of two variables carry over to functions of three variables.

Exercise Set 14.2

In Exercises 1–6, use limit laws and continuity properties to evaluate the limit.

1. $\lim_{(x,y)\to(1,3)} (4xy^2 - x)$ **2.** $\lim_{(x,y)\to(1/2,\pi)} (xy^2 \sin xy)$

3. $\lim_{(x,y)\to(-1,2)} \dfrac{xy^3}{x+y}$ **4.** $\lim_{(x,y)\to(1,-3)} e^{2x-y^2}$

5. $\lim_{(x,y)\to(0,0)} \ln(1+x^2y^3)$ **6.** $\lim_{(x,y)\to(4,-2)} x\sqrt[3]{y^3+2x}$

In Exercises 7 and 8, show that the limit does not exist by considering the limits as $(x, y)\to(0, 0)$ along the coordinate axes.

7. (a) $\lim_{(x,y)\to(0,0)} \dfrac{3}{x^2+2y^2}$ (b) $\lim_{(x,y)\to(0,0)} \dfrac{x+y}{x+y^2}$

8. (a) $\lim_{(x,y)\to(0,0)} \dfrac{x-y}{x^2+y^2}$ (b) $\lim_{(x,y)\to(0,0)} \dfrac{\cos xy}{x+y}$

In Exercises 9–12, evaluate the limit by making the substitution $z = x^2 + y^2$ and observing that $z \to 0^+$ if and only if $(x, y)\to(0, 0)$.

9. $\lim_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2}$ **10.** $\lim_{(x,y)\to(0,0)} \dfrac{1-\cos(x^2+y^2)}{x^2+y^2}$

11. $\lim_{(x,y)\to(0,0)} e^{-1/(x^2+y^2)}$ **12.** $\lim_{(x,y)\to(0,0)} \dfrac{e^{-1/\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}$

In Exercises 13–20, determine whether the limit exists. If so, find its value.

13. $\lim_{(x,y)\to(0,0)} \dfrac{x^4-y^4}{x^2+y^2}$ **14.** $\lim_{(x,y)\to(0,0)} \dfrac{x^4-16y^4}{x^2+4y^2}$

15. $\lim_{(x,y)\to(0,0)} \dfrac{xy}{3x^2+2y^2}$ **16.** $\lim_{(x,y)\to(0,0)} \dfrac{1-x^2-y^2}{x^2+y^2}$

17. $\lim_{(x,y,z)\to(2,-1,2)} \dfrac{xz^2}{\sqrt{x^2+y^2+z^2}}$

18. $\lim_{(x,y,z)\to(2,0,-1)} \ln(2x+y-z)$

19. $\lim_{(x,y,z)\to(0,0,0)} \dfrac{\sin(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}}$

20. $\lim_{(x,y,z)\to(0,0,0)} \dfrac{\sin\sqrt{x^2+y^2+z^2}}{x^2+y^2+z^2}$

In Exercises 21 and 22, evaluate the limit, if it exists, by converting to polar coordinates, as in Example 7.

21. $\lim_{(x,y)\to(0,0)} y\ln(x^2+y^2)$ **22.** $\lim_{(x,y)\to(0,0)} \dfrac{x^2y^2}{\sqrt{x^2+y^2}}$

In Exercises 23 and 24, evaluate the limit, if it exists, by converting to spherical coordinates; that is, let $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$ and observe that $\rho\to 0^+$ if and only if $(x, y, z)\to(0, 0, 0)$, since $\rho = \sqrt{x^2+y^2+z^2}$.

23. $\lim_{(x,y,z)\to(0,0,0)} \dfrac{e^{\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}$

24. $\lim_{(x,y,z)\to(0,0,0)} \tan^{-1}\left[\dfrac{1}{x^2+y^2+z^2}\right]$

25. The accompanying figure shows a portion of the graph of

$$f(x, y) = \frac{x^2y}{x^4+y^2}$$

(a) Based on the graph in the figure, does $f(x, y)$ have a limit as $(x, y)\to(0, 0)$? Explain your reasoning.
(b) Show that $f(x, y)\to 0$ as $(x, y)\to(0, 0)$ along any line $y = mx$. Does this imply that $f(x, y)\to 0$ as $(x, y)\to(0, 0)$? Explain.
(c) Show that $f(x, y)\to\frac{1}{2}$ as $(x, y)\to(0, 0)$ along the parabola $y = x^2$, and confirm visually that this is consistent with the graph of $f(x, y)$.
(d) Based on parts (b) and (c), does $f(x, y)$ have a limit as $(x, y)\to(0, 0)$? Is this consistent with your answer to part (a)?

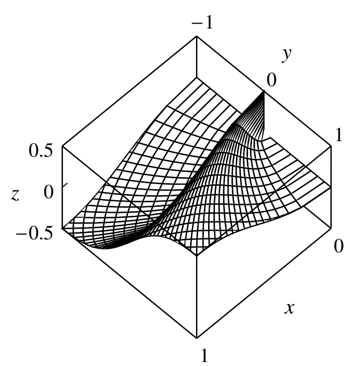


Figure Ex-25

26. (a) Show that the value of $\dfrac{x^3y}{2x^6+y^2}$ approaches 0 as $(x, y)\to(0, 0)$ along any straight line $y = mx$, or along any parabola $y = kx^2$.
(b) Show that $\lim_{(x,y)\to(0,0)} \dfrac{x^3y}{2x^6+y^2}$ does not exist by letting $(x, y)\to(0, 0)$ along the curve $y = x^3$.

27. (a) Show that the value of $\dfrac{xyz}{x^2+y^4+z^4}$ approaches 0 as $(x, y, z)\to(0, 0, 0)$ along any line $x = at$, $y = bt$, $z = ct$.
(b) Show that the limit $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{x^2+y^4+z^4}$ does not exist by letting $(x, y, z)\to(0, 0, 0)$ along the curve $x = t^2$, $y = t$, $z = t$.

28. Find
$$\lim_{(x,y)\to(0,1)} \tan^{-1}\left[\frac{x^2+1}{x^2+(y-1)^2}\right]$$

29. Find
$$\lim_{(x,y)\to(0,1)} \tan^{-1}\left[\frac{x^2-1}{x^2+(y-1)^2}\right]$$

30. Let $f(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2}, & (x,y)\neq(0,0) \\ 1, & (x,y)=(0,0). \end{cases}$

Show that f is continuous at $(0,0)$.

31. Let $f(x,y) = \dfrac{x^2}{x^2+y^2}$. Is it possible to define $f(0,0)$ so that f will be continuous at $(0,0)$?

32. Let $f(x,y) = xy\ln(x^2+y^2)$. Is it possible to define $f(0,0)$ so that f will be continuous at $(0,0)$?

In Exercises 33–40, sketch the largest region on which the function f is continuous.

33. $f(x,y) = y\ln(1+x)$

34. $f(x,y) = \sqrt{x-y}$

35. $f(x,y) = \dfrac{x^2y}{\sqrt{25-x^2-y^2}}$

36. $f(x,y) = \ln(2x-y+1)$

37. $f(x,y) = \cos\left(\dfrac{xy}{1+x^2+y^2}\right)$

38. $f(x,y) = e^{1-xy}$

39. $f(x,y) = \sin^{-1}(xy)$

40. $f(x,y) = \tan^{-1}(y-x)$

In Exercises 41–44, describe the largest region on which the function f is continuous.

41. $f(x,y,z) = 3x^2e^{yz}\cos(xyz)$

42. $f(x,y,z) = \ln(4-x^2-y^2-z^2)$

43. $f(x,y,z) = \dfrac{y+1}{x^2+z^2-1}$

44. $f(x,y,z) = \sin\sqrt{x^2+y^2+3z^2}$

14.3 PARTIAL DERIVATIVES

If $z = f(x,y)$, then one can inquire how the value of z changes if y is held fixed and x is allowed to vary, or if x is held fixed and y is allowed to vary. For example, the ideal gas law in physics states that under appropriate conditions the pressure exerted by a gas is a function of the volume of the gas and its temperature. Thus, a physicist studying gases might be interested in the rate of change of the pressure if the volume is held fixed and the temperature is allowed to vary or if the temperature is held fixed and the volume is allowed to vary. In this section we will develop the mathematical tools for studying rates of change that involve two or more independent variables.

PARTIAL DERIVATIVES OF FUNCTIONS OF TWO VARIABLES

Suppose that (x_0, y_0) is a point in the domain of a function $f(x,y)$. If we fix $y = y_0$, then $f(x, y_0)$ is a function of the variable x alone. The value of the derivative
$$\frac{d}{dx}[f(x,y_0)]$$
at x_0 then gives us a measure of the instantaneous rate of change of f with respect to x at the point (x_0, y_0). Similarly, the value of the derivative
$$\frac{d}{dy}[f(x_0,y)]$$
at y_0 gives us a measure of the instantaneous rate of change of f with respect to y at the point (x_0, y_0). These derivatives are so basic to the study of differential calculus of multivariable functions that they have their own name and notation.

14.3.1 DEFINITION. If $z = f(x, y)$ and (x_0, y_0) is a point in the domain of f, then the ***partial derivative of f with respect to x*** at (x_0, y_0) [also called the ***partial derivative of z with respect to x*** at (x_0, y_0)] is the derivative at x_0 of the function that results when $y = y_0$ is held fixed and x is allowed to vary. This partial derivative is denoted by $f_x(x_0, y_0)$ and is given by

$$f_x(x_0, y_0) = \frac{d}{dx}[f(x, y_0)]\bigg|_{x=x_0} = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} \tag{1}$$

Similarly, the ***partial derivative of f with respect to y*** at (x_0, y_0) [also called the ***partial derivative of z with respect to y*** at (x_0, y_0)] is the derivative at y_0 of the function that results when $x = x_0$ is held fixed and y is allowed to vary. This partial derivative is denoted by $f_y(x_0, y_0)$ and is given by

$$f_y(x_0, y_0) = \frac{d}{dy}[f(x_0, y)]\bigg|_{y=y_0} = \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0} \tag{2}$$

We have included the limits in Equations (1) and (2) because it is sometimes necessary (especially in proofs) to express a partial derivative in limit form. However, in many cases we can compute partial derivatives directly by applying our derivative rules for functions of a single variable.

Example 1 Determine $f_x(1, 3)$ and $f_y(1, 3)$ for the function $f(x, y) = 2x^3y^2 + 2y + 4x$.

Solution. Since

$$f_x(x, 3) = \frac{d}{dx}[f(x, 3)] = \frac{d}{dx}[18x^3 + 4x + 6] = 54x^2 + 4$$

we have $f_x(1, 3) = 54 + 4 = 58$. Also, since

$$f_y(1, y) = \frac{d}{dy}[f(1, y)] = \frac{d}{dy}[2y^2 + 2y + 4] = 4y + 2$$

we have $f_y(1, 3) = 4(3) + 2 = 14$. ◀

When computing partial derivatives, it is often more efficient to omit subscripts and to postpone substitution of specific values of the variables until after the differentiation process. For instance, with the function $f(x, y) = 2x^3y^2 + 2y + 4x$ from Example 1, we can obtain $f_x(x, y)$ by treating y as an unspecified constant and differentiating with respect to x. That is,

$$f_x(x, y) = \frac{d}{dx}[2x^3y^2 + 2y + 4x] = 6x^2y^2 + 4$$

Then $f_x(1, 3) = 6(1^2)(3^2) + 4 = 58$. Similarly,

$$f_y(x, y) = \frac{d}{dy}[2x^3y^2 + 2y + 4x] = 4x^3y + 2$$

so that $f_y(1, 3) = 4(1^3)3 + 2 = 14$.

FOR THE READER. If you have a CAS, read the relevant documentation on calculating partial derivatives, and then use the CAS to perform the computations in Example 1.

PARTIAL DERIVATIVES VIEWED AS RATES OF CHANGE AND SLOPES

Recall that if $y = f(x)$, then the value of $f'(x_0)$ can be interpreted either as the rate of change of y with respect to x at x_0 or as the slope of the graph of f at x_0. Partial derivatives have analogous interpretations. To see that this is so, suppose that C_1 is the intersection of the surface $z = f(x, y)$ with the plane $y = y_0$ and that C_2 is its intersection with the plane

$x = x_0$ (Figure 14.3.1). Thus, $f_x(x, y_0)$ can be interpreted as the rate of change of z with respect to x along the curve C_1, and $f_y(x_0, y)$ can be interpreted as the rate of change of z with respect to y along the curve C_2. In particular, $f_x(x_0, y_0)$ is the rate of change of z with respect to x along the curve C_1 at the point (x_0, y_0), and $f_y(x_0, y_0)$ is the rate of change of z with respect to y along the curve C_2 at the point (x_0, y_0). Geometrically, $f_x(x_0, y_0)$ can be viewed as the slope of the curve C_1 at the point (x_0, y_0), and $f_y(x_0, y_0)$ can be viewed as the slope of the curve C_2 at the point (x_0, y_0) (Figure 14.3.1). We will call $f_x(x_0, y_0)$ the ***slope of the surface in the x-direction*** at (x_0, y_0), and $f_y(x_0, y_0)$ the ***slope of the surface in the y-direction*** at (x_0, y_0).

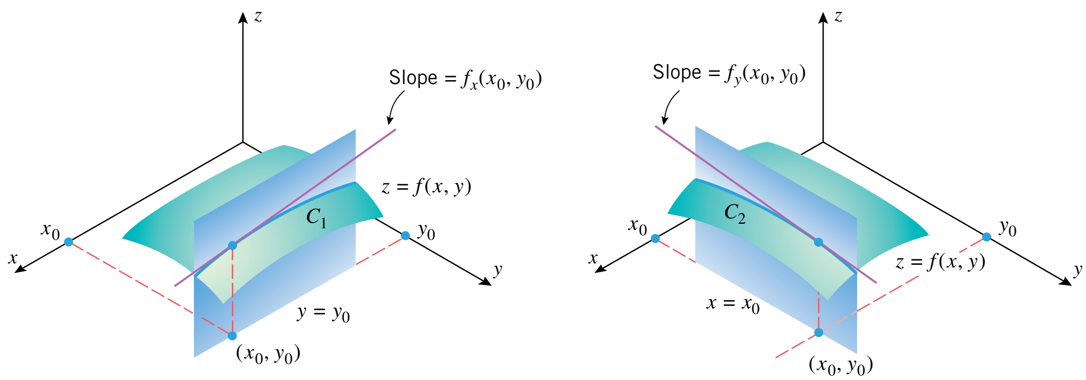


Figure 14.3.1

Example 2 Let $f(x, y) = x^2y + 5y^3$.

(a) Find the slope of the surface $z = f(x, y)$ in the x-direction at the point $(1, -2)$.

(b) Find the slope of the surface $z = f(x, y)$ in the y-direction at the point $(1, -2)$.

Solution (a). Differentiating f with respect to x with y held fixed yields

$$f_x(x, y) = 2xy$$

Thus, the slope in the x-direction is $f_x(1, -2) = -4$; that is, z is decreasing at the rate of 4 units per unit increase in x.

Solution (b). Differentiating f with respect to y with x held fixed yields

$$f_y(x, y) = x^2 + 15y^2$$

Thus, the slope in the y-direction is $f_y(1, -2) = 61$; that is, z is increasing at the rate of 61 units per unit increase in y. ◀

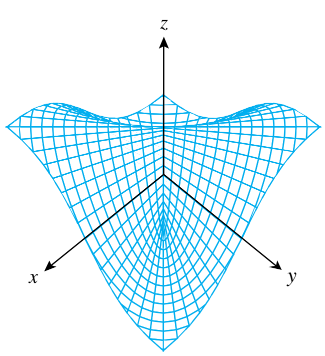


Figure 14.3.2

Example 3 Figure 14.3.2 shows the graph of the function

$$f(x, y) = \begin{cases} -\dfrac{xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases} \tag{3}$$

This is similar to the function considered in Example 1 of Section 14.2, except that here we have assigned f a value at $(0, 0)$. Except at this point, the partial derivatives of f are

$$f_x(x, y) = -\frac{(x^2 + y^2)y - xy(2x)}{(x^2 + y^2)^2} = \frac{x^2y - y^3}{(x^2 + y^2)^2} \tag{4}$$

$$f_y(x, y) = -\frac{(x^2 + y^2)x - xy(2y)}{(x^2 + y^2)^2} = \frac{xy^2 - x^3}{(x^2 + y^2)^2} \tag{5}$$

Figure 14.3.2 suggests that at each point on the x-axis [except possibly $(0, 0)$] the surface has slope 0 in the x-direction and at each point on the y-axis [except possibly $(0, 0)$] the surface has slope 0 in the y-direction. This can be confirmed by evaluating f_x at a typical point $(x, 0)$ on the x-axis and evaluating f_y at a typical point $(0, y)$ on the y-axis. Setting $y = 0$ in (4) and $x = 0$ in (5) yields

$$f_x(x, 0) = 0 \quad \text{and} \quad f_y(0, y) = 0$$

which confirms our conjecture.

It is not evident from Formula (3) whether f has partial derivatives at $(0, 0)$, and if so, what the values of those derivatives are. To answer that question we will have to use the definitions of the partial derivatives (Definition 14.3.1). Applying Formulas (1) and (2) to (3) we obtain

$$f_x(0, 0) = \lim_{x\to 0} \frac{f(x, 0) - f(0, 0)}{x - 0} = \lim_{x\to 0} \frac{0 - 0}{x} = 0$$

$$f_y(0, 0) = \lim_{y\to 0} \frac{f(0, y) - f(0, 0)}{y - 0} = \lim_{y\to 0} \frac{0 - 0}{y} = 0$$

This shows that f has partial derivatives at $(0, 0)$ and the values of both partial derivatives are 0 at that point. ◀

PARTIAL DERIVATIVE NOTATION

If $z = f(x, y)$, then the partial derivatives f_x and f_y are also denoted by the symbols*

$$\frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}, \quad \frac{\partial z}{\partial y}$$

Some typical notations for the partial derivatives of $z = f(x, y)$ at a point (x_0, y_0) are

$$\left.\frac{\partial f}{\partial x}\right|_{x=x_0,\, y=y_0}, \quad \left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)}, \quad \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)}, \quad \frac{\partial f}{\partial x}(x_0, y_0), \quad \frac{\partial z}{\partial x}(x_0, y_0)$$

Example 4 Find $\partial z/\partial x$ and $\partial z/\partial y$ if $z = x^4 \sin(xy^3)$.

Solution.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}[x^4 \sin(xy^3)] = x^4 \frac{\partial}{\partial x}[\sin(xy^3)] + \sin(xy^3) \cdot \frac{\partial}{\partial x}(x^4) \\
&= x^4 \cos(xy^3) \cdot y^3 + \sin(xy^3) \cdot 4x^3 = x^4 y^3 \cos(xy^3) + 4x^3 \sin(xy^3) \\
\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}[x^4 \sin(xy^3)] = x^4 \frac{\partial}{\partial y}[\sin(xy^3)] + \sin(xy^3) \cdot \frac{\partial}{\partial y}(x^4) \\
&= x^4 \cos(xy^3) \cdot 3xy^2 + \sin(xy^3) \cdot 0 = 3x^5 y^2 \cos(xy^3)
\end{aligned}$$ ◀

For functions that are presented in tabular form, we can estimate partial derivatives by using adjacent entries within the table.

Example 5 Recall from Table 14.1.1 that the windchill index function $\mathrm{WCI}(T, v)$ satisfies

$$\mathrm{WCI}(30, 10) = 16, \quad \mathrm{WCI}(30, 15) = 9, \quad \text{and} \quad \mathrm{WCI}(30, 20) = 4$$

Use these values to estimate the partial derivative of WCI with respect to v at the point $(T, v) = (30, 15)$. Compare this estimate with the value of the partial derivative obtained by using the formula

$$\mathrm{WCI} = 91.4 + (91.4 - T)(0.0203v - 0.304\sqrt{v} - 0.474)$$

Solution. It follows from Definition 14.3.1 that

$$\frac{\partial(\mathrm{WCI})}{\partial v}(30, 15) = \lim_{v\to 15} \frac{\mathrm{WCI}(30, v) - 9}{v - 15}$$

Consequently, to obtain an estimate for

$$\frac{\partial(\mathrm{WCI})}{\partial v}(30, 15)$$

*The symbol ∂ is called a partial derivative sign. It is derived from the Cyrillic alphabet.

we should evaluate the difference quotient

$$\frac{\mathrm{WCI}(30, v) - 9}{v - 15}$$

for a value of v close to (but not equal to) 15 mi/h. The windchill values WCI(30, 10) = 16 and WCI(30, 20) = 4 are the entries in Table 14.1.1 for temperature $T = 30°$F and wind speeds $v \neq 15$ closest to 15 mi/h. For $v = 10$ and $v = 20$ we obtain the estimates

$$\frac{\mathrm{WCI}(30, 10) - 9}{10 - 15} = \frac{16 - 9}{10 - 15} = -\frac{7}{5}$$

and

$$\frac{\mathrm{WCI}(30, 20) - 9}{20 - 15} = \frac{4 - 9}{20 - 15} = -1$$

respectively. Since $v = 15$ is equidistant from $v = 10$ and $v = 20$, a "reasonable" single estimate for

$$\frac{\partial(\mathrm{WCI})}{\partial v}(30, 15)$$

should be the average

$$\frac{\left(-\frac{7}{5}\right) + (-1)}{2} = -\frac{6}{5} = -1.2$$

of these initial estimates. That is, we make the estimate

$$\frac{\partial(\mathrm{WCI})}{\partial v}(30, 15) \approx -1.2$$

On the other hand, computing

$$\frac{\partial(\mathrm{WCI})}{\partial v}$$

from the formula for WCI yields

$$\frac{\partial(\mathrm{WCI})}{\partial v} = (91.4 - T)(0.0203 - 0.152/\sqrt{v})$$

(verify). Thus

$$\frac{\partial(\mathrm{WCI})}{\partial v}(30, 15) = (91.4 - 30)(0.0203 - 0.152/\sqrt{15}) \approx -1.1633$$ ◀

IMPLICIT PARTIAL DIFFERENTIATION

Example 6 Find the slope of the sphere $x^2 + y^2 + z^2 = 1$ in the y-direction at the points $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ and $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ (Figure 14.3.3).

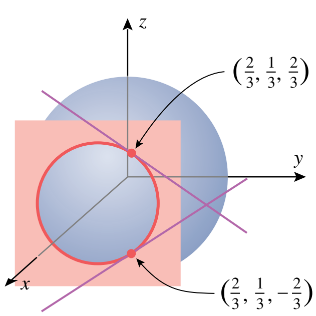


Figure 14.3.3

Solution. The point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ lies on the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$, and the point $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ lies on the lower hemisphere $z = -\sqrt{1 - x^2 - y^2}$. We could find the slopes by differentiating each expression for z separately with respect to y and then evaluating the derivatives at $x = \frac{2}{3}$ and $y = \frac{1}{3}$. However, it is more efficient to differentiate the given equation

$$x^2 + y^2 + z^2 = 1$$

implicitly with respect to y, since this will give us both slopes with one differentiation. To perform the implicit differentiation, we view z as a function of x and y and differentiate both sides with respect to y, taking x to be fixed. The computations are as follows:

$$\frac{\partial}{\partial y}[x^2 + y^2 + z^2] = \frac{\partial}{\partial y}[1]$$

$$0 + 2y + 2z\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = -\frac{y}{z}$$

Substituting the y- and z-coordinates of the points $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ and $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ in this expression, we find that the slope at the point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ is $-\frac{1}{2}$ and the slope at $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ is $\frac{1}{2}$. ◀

FOR THE READER. Check the results obtained in Example 6 by differentiating the functions $z = \sqrt{1 - x^2 - y^2}$ and $z = -\sqrt{1 - x^2 - y^2}$ directly.

Example 7 Suppose that $D = \sqrt{x^2 + y^2}$ is the length of the diagonal of a rectangle whose sides have lengths x and y that are allowed to vary. Find a formula for the rate of change of D with respect to x if x varies with y held constant, and use this formula to find the rate of change of D with respect to x at the point where $x = 3$ and $y = 4$.

Solution. Differentiating both sides of the equation $D^2 = x^2 + y^2$ with respect to x yields

$$2D\frac{\partial D}{\partial x} = 2x \quad \text{and thus} \quad D\frac{\partial D}{\partial x} = x$$

Since $D = 5$ when $x = 3$ and $y = 4$ it follows that

$$5\left.\frac{\partial D}{\partial x}\right|_{x=3,\,y=4} = 3 \quad \text{or} \quad \left.\frac{\partial D}{\partial x}\right|_{x=3,\,y=4} = \frac{3}{5}$$

Thus, D is increasing at a rate of $\frac{3}{5}$ unit per unit increase in x at the point (3, 4). ◀

PARTIAL DERIVATIVES OF FUNCTIONS WITH MORE THAN TWO VARIABLES

For a function $f(x, y, z)$ of three variables, there are three ***partial derivatives***:

$$f_x(x, y, z), \quad f_y(x, y, z), \quad f_z(x, y, z)$$

The partial derivative f_x is calculated by holding y and z constant and differentiating with respect to x. For f_y the variables x and z are held constant, and for f_z the variables x and y are held constant. If a dependent variable

$$w = f(x, y, z)$$

is used, then the three partial derivatives of f can be denoted by

$$\frac{\partial w}{\partial x}, \quad \frac{\partial w}{\partial y}, \quad \text{and} \quad \frac{\partial w}{\partial z}$$

Example 8 If $f(x, y, z) = x^3y^2z^4 + 2xy + z$, then

$$\begin{aligned}
f_x(x, y, z) &= 3x^2y^2z^4 + 2y \\
f_y(x, y, z) &= 2x^3yz^4 + 2x \\
f_z(x, y, z) &= 4x^3y^2z^3 + 1 \\
f_z(-1, 1, 2) &= 4(-1)^3(1)^2(2)^3 + 1 = -31
\end{aligned}$$ ◀

Example 9 If $f(\rho, \theta, \phi) = \rho^2 \cos\phi \sin\theta$, then

$$\begin{aligned}
f_\rho(\rho, \theta, \phi) &= 2\rho \cos\phi \sin\theta \\
f_\theta(\rho, \theta, \phi) &= \rho^2 \cos\phi \cos\theta \\
f_\phi(\rho, \theta, \phi) &= -\rho^2 \sin\phi \sin\theta
\end{aligned}$$ ◀

In general, if $f(v_1, v_2, \ldots, v_n)$ is a function of n variables, there are n partial derivatives of f, each of which is obtained by holding $n - 1$ of the variables fixed and differentiating the function f with respect to the remaining variable. If $w = f(v_1, v_2, \ldots, v_n)$, then these partial derivatives are denoted by

$$\frac{\partial w}{\partial v_1}, \frac{\partial w}{\partial v_2}, \ldots, \frac{\partial w}{\partial v_n}$$

where $\partial w/\partial v_i$ is obtained by holding all variables except v_i fixed and differentiating with respect to v_i.

Example 10 Find

$$\frac{\partial}{\partial x_i}\left[\sqrt{x_1^2+x_2^2+\cdots+x_n^2}\right]$$

for $i = 1, 2, \ldots, n$.

Solution. For each $i = 1, 2, \ldots, n$ we obtain

$$\begin{aligned}\frac{\partial}{\partial x_i}\left[\sqrt{x_1^2+x_2^2+\cdots+x_n^2}\right] &= \frac{1}{2\sqrt{x_1^2+x_2^2+\cdots+x_n^2}}\cdot\frac{\partial}{\partial x_i}[x_1^2+x_2^2+\cdots+x_n^2]\\ &= \frac{1}{2\sqrt{x_1^2+x_2^2+\cdots+x_n^2}}[2x_i] \qquad \boxed{\text{All terms except } x_i^2 \text{ are constant.}}\\ &= \frac{x_i}{\sqrt{x_1^2+x_2^2+\cdots+x_n^2}}\end{aligned}$$

◀

HIGHER-ORDER PARTIAL DERIVATIVES

Suppose that f is a function of two variables x and y. Since the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are also functions of x and y, these functions may themselves have partial derivatives. This gives rise to four possible ***second-order*** partial derivatives of f, which are defined by

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

Differentiate twice with respect to x. | Differentiate twice with respect to y.

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} \qquad \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

Differentiate first with respect to x and then with respect to y. | Differentiate first with respect to y and then with respect to x.

The last two cases are called the ***mixed second-order partial derivatives*** or the ***mixed second partials***. Also, the derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are often called the ***first-order partial derivatives*** when it is necessary to distinguish them from higher-order partial derivatives. Similar conventions apply to the second-order partial derivatives of a function of three variables.

WARNING. Observe that the two notations for the mixed second partials have opposite conventions for the order of differentiation. In the "∂" notation the derivatives are taken right to left and in the "subscript" notation they are taken left to right. However, the conventions are logical if you insert parentheses:

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Right to left. Differentiate inside the parentheses first.

$$f_{xy} = (f_x)_y$$

Left to right. Differentiate inside the parentheses first.

Example 11 Find the second-order partial derivatives of $f(x, y) = x^2y^3 + x^4y$.

Solution. We have

$$\frac{\partial f}{\partial x} = 2xy^3 + 4x^3y \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x^2y^2 + x^4$$

so that

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2xy^3 + 4x^3y) = 2y^3 + 12x^2y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x^2y^2 + x^4) = 6x^2y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3x^2y^2 + x^4) = 6xy^2 + 4x^3$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2xy^3 + 4x^3y) = 6xy^2 + 4x^3 \quad \blacktriangleleft$$

Third-order, fourth-order, and higher-order partial derivatives can be obtained by successive differentiation. Some possibilities are

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = f_{xxx} \qquad \frac{\partial^4 f}{\partial y^4} = \frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial y^3}\right) = f_{yyyy}$$

$$\frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = f_{xyy} \qquad \frac{\partial^4 f}{\partial y^2 \partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial y \partial x^2}\right) = f_{xxyy}$$

Example 12 Let $f(x, y) = y^2e^x + y$. Find f_{xyy}.

Solution.

$$f_{xyy} = \frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial^2}{\partial y^2}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial y^2}(y^2e^x) = \frac{\partial}{\partial y}(2ye^x) = 2e^x \quad \blacktriangleleft$$

EQUALITY OF MIXED PARTIALS

For a function $f(x, y)$ it might be expected that there would be four distinct second-order partial derivatives: f_{xx}, f_{xy}, f_{yx}, and f_{yy}. However, observe that the mixed second-order partial derivatives in Example 11 are equal. The following theorem (proved in advanced courses) shows that this will usually be the case for the functions that we commonly encounter.

> **14.3.2** THEOREM. *Let f be a function of two variables. If f_{xy} and f_{yx} are continuous on some open disk, then $f_{xy} = f_{yx}$ on that disk. (If f is a function of three variables, an analogous result holds for each pair of mixed second-order partial derivatives if we replace "open disk" by "open ball.")*

For example, since polynomial functions of x and y are continuous everywhere, it immediately follows from Theorem 14.3.2 that $f_{xy} = f_{yx}$ for any polynomial function f.

THE WAVE EQUATION

Consider a string of length L that is stretched taut between $x = 0$ and $x = L$ on an x-axis, and suppose that the string is set into vibratory motion by "plucking" it at time $t = 0$ (Figure 14.3.4*a*). The displacement of a point on the string depends both on its coordinate x and the elapsed time t, and hence is described by a function $u(x, t)$ of two variables. For a fixed value t, the function $u(x, t)$ depends on x alone, and the graph of u versus x describes the shape of the string—think of it as a "snapshot" of the string at time t (Figure 14.3.4*b*). It follows that at a fixed time t, the partial derivative $\partial u/\partial x$ represents the slope of the string at x, and the sign of the second partial derivative $\partial^2 u/\partial x^2$ tells us whether the string is concave up or concave down at x (Figure 14.3.4*c*).

For a fixed value of x, the function $u(x, t)$ depends on t alone, and the graph of u versus t is the position versus time curve of the point on the string with coordinate x. Thus, for a fixed value of x, the partial derivative $\partial u/\partial t$ is the velocity of the point with coordinate x, and $\partial^2 u/\partial t^2$ is the acceleration of that point.

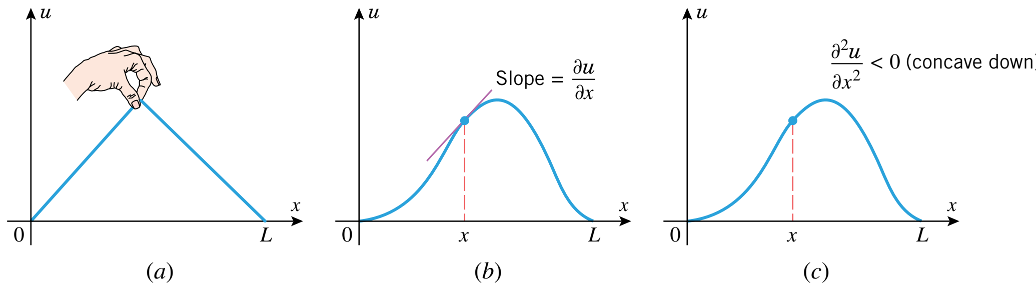


Figure 14.3.4

The vibration of a plucked string is governed by the wave equation.

It can be proved that under appropriate conditions the function $u(x, t)$ satisfies an equation of the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{6}$$

where c is a positive constant that depends on the physical characteristics of the string. This equation, which is called the ***one-dimensional wave equation***, involves partial derivatives of the unknown function $u(x, t)$ and hence is classified as a ***partial differential equation***. Techniques for solving partial differential equations are studied in advanced courses and will not be discussed in this text.

Example 13 Show that the function $u(x, t) = \sin(x - ct)$ is a solution of Equation (6).

Solution. We have

$$\frac{\partial u}{\partial x} = \cos(x - ct), \quad \frac{\partial^2 u}{\partial x^2} = -\sin(x - ct)$$

$$\frac{\partial u}{\partial t} = -c\cos(x - ct), \quad \frac{\partial^2 u}{\partial t^2} = -c^2 \sin(x - ct)$$

Thus, $u(x, t)$ satisfies (6). ◀

EXERCISE SET 14.3 Graphing Utility

1. Let $f(x, y) = 3x^3y^2$. Find
(a) $f_x(x, y)$ (b) $f_y(x, y)$ (c) $f_x(1, y)$
(d) $f_x(x, 1)$ (e) $f_y(1, y)$ (f) $f_y(x, 1)$
(g) $f_x(1, 2)$ (h) $f_y(1, 2)$.

2. Let $z = e^{2x} \sin y$. Find
(a) $\partial z/\partial x$ (b) $\partial z/\partial y$ (c) $\partial z/\partial x|_{(0,y)}$
(d) $\partial z/\partial x|_{(x,0)}$ (e) $\partial z/\partial y|_{(0,y)}$ (f) $\partial z/\partial y|_{(x,0)}$
(g) $\partial z/\partial x|_{(\ln 2,0)}$ (h) $\partial z/\partial y|_{(\ln 2,0)}$.

3. Let $f(x, y) = \sqrt{3x + 2y}$.
(a) Find the slope of the surface $z = f(x, y)$ in the x-direction at the point (4, 2).
(b) Find the slope of the surface $z = f(x, y)$ in the y-direction at the point (4, 2).

4. Let $f(x, y) = xe^{-y} + 5y$.
(a) Find the slope of the surface $z = f(x, y)$ in the x-direction at the point (3, 0).
(b) Find the slope of the surface $z = f(x, y)$ in the y-direction at the point (3, 0).

5. Let $z = \sin(y^2 - 4x)$.
(a) Find the rate of change of z with respect to x at the point (2, 1) with y held fixed.
(b) Find the rate of change of z with respect to y at the point (2, 1) with x held fixed.

6. Let $z = (x + y)^{-1}$.
(a) Find the rate of change of z with respect to x at the point $(-2, 4)$ with y held fixed.
(b) Find the rate of change of z with respect to y at the point $(-2, 4)$ with x held fixed.

7. Use the information in the accompanying figure to find the values of the first-order partial derivatives of f at the point $(1, 2)$.

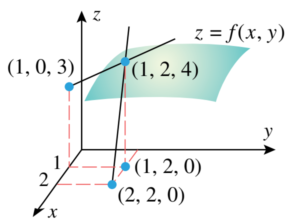


Figure Ex-7

8. The accompanying figure shows a contour plot for an unspecified function $f(x, y)$. Make a conjecture about the signs of the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$, and explain your reasoning.

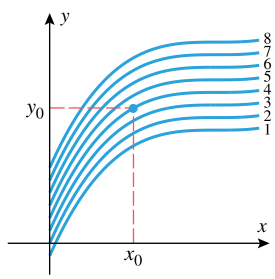


Figure Ex-8

9. Suppose that Nolan throws a baseball to Ryan and that the baseball leaves Nolan's hand at the same height at which it is caught by Ryan. It we ignore air resistance, the horizontal range r of the baseball is a function of the initial speed v of the ball when it leaves Nolan's hand and the angle θ above the horizontal at which it is thrown. Use the accompanying table and the method of Example 5 to estimate

(a) the partial derivative of r with respect to v when $v = 80$ ft/s and $\theta = 40°$

(b) the partial derivative of r with respect to θ when $v = 80$ ft/s and $\theta = 40°$.

ANGLE θ (degrees) \ SPEED v (ft/s)	75	80	85	90
35	165	188	212	238
40	173	197	222	249
45	176	200	226	253
50	173	197	222	249

Table Ex-9

10. Use the table in Exercise 9 and the method of Example 5 to estimate

(a) the partial derivative of r with respect to v when $v = 85$ ft/s and $\theta = 45°$

(b) the partial derivative of r with respect to θ when $v = 85$ ft/s and $\theta = 45°$.

In Exercises 11–16, find $\partial z/\partial x$ and $\partial z/\partial y$.

11. $z = 4e^{x^2y^3}$

12. $z = \cos(x^5y^4)$

13. $z = x^3\ln(1 + xy^{-3/5})$

14. $z = e^{xy}\sin 4y^2$

15. $z = \dfrac{xy}{x^2 + y^2}$

16. $z = \dfrac{x^2y^3}{\sqrt{x + y}}$

In Exercises 17–22, find $f_x(x, y)$ and $f_y(x, y)$.

17. $f(x, y) = \sqrt{3x^5y - 7x^3y}$

18. $f(x, y) = \dfrac{x + y}{x - y}$

19. $f(x, y) = y^{-3/2}\tan^{-1}(x/y)$

20. $f(x, y) = x^3e^{-y} + y^3\sec\sqrt{x}$

21. $f(x, y) = (y^2\tan x)^{-4/3}$

22. $f(x, y) = \cosh(\sqrt{x})\sinh^2(xy^2)$

In Exercises 23–26, evaluate the indicated partial derivatives.

23. $f(x, y) = 9 - x^2 - 7y^3$; $f_x(3, 1)$, $f_y(3, 1)$

24. $f(x, y) = x^2ye^{xy}$; $\partial f/\partial x(1, 1)$, $\partial f/\partial y(1, 1)$

25. $z = \sqrt{x^2 + 4y^2}$; $\partial z/\partial x(1, 2)$, $\partial z/\partial y(1, 2)$

26. $w = x^2\cos xy$; $\partial w/\partial x\left(\frac{1}{2}, \pi\right)$, $\partial w/\partial y\left(\frac{1}{2}, \pi\right)$

27. Let $f(x, y, z) = x^2y^4z^3 + xy + z^2 + 1$. Find

(a) $f_x(x, y, z)$ (b) $f_y(x, y, z)$ (c) $f_z(x, y, z)$

(d) $f_x(1, y, z)$ (e) $f_y(1, 2, z)$ (f) $f_z(1, 2, 3)$.

28. Let $w = x^2y\cos z$. Find

(a) $\partial w/\partial x(x, y, z)$ (b) $\partial w/\partial y(x, y, z)$

(c) $\partial w/\partial z(x, y, z)$ (d) $\partial w/\partial x(2, y, z)$

(e) $\partial w/\partial y(2, 1, z)$ (f) $\partial w/\partial z(2, 1, 0)$.

In Exercises 29–32, find f_x, f_y, and f_z.

29. $f(x, y, z) = z\ln(x^2y\cos z)$

30. $f(x, y, z) = y^{-3/2}\sec\left(\dfrac{xz}{y}\right)$

31. $f(x, y, z) = \tan^{-1}\left(\dfrac{1}{xy^2z^3}\right)$

32. $f(x, y, z) = \cosh(\sqrt{z})\sinh^2(x^2yz)$

In Exercises 33–36, find $\partial w/\partial x$, $\partial w/\partial y$, and $\partial w/\partial z$.

33. $w = ye^z\sin xz$

34. $w = \dfrac{x^2 - y^2}{y^2 + z^2}$

35. $w = \sqrt{x^2 + y^2 + z^2}$

36. $w = y^3e^{2x+3z}$

37. Let $f(x, y, z) = y^2e^{xz}$. Find

(a) $\partial f/\partial x|_{(1,1,1)}$ (b) $\partial f/\partial y|_{(1,1,1)}$ (c) $\partial f/\partial z|_{(1,1,1)}$.

38. Let $w = \sqrt{x^2 + 4y^2 - z^2}$. Find

(a) $\partial w/\partial x|_{(2,1,-1)}$ (b) $\partial w/\partial y|_{(2,1,-1)}$

(c) $\partial w/\partial z|_{(2,1,-1)}$.

39. Let $f(x, y) = e^x\cos y$. Use a graphing utility to graph the functions $f_x(0, y)$ and $f_y(x, \pi/2)$.

40. Let $f(x, y) = e^x \sin y$. Use a graphing utility to graph the functions $f_x(0, y)$ and $f_y(x, 0)$.

41. A point moves along the intersection of the elliptic paraboloid $z = x^2 + 3y^2$ and the plane $y = 1$. At what rate is z changing with x when the point is at $(2, 1, 7)$?

42. A point moves along the intersection of the elliptic paraboloid $z = x^2 + 3y^2$ and the plane $x = 2$. At what rate is z changing with y when the point is at $(2, 1, 7)$?

43. A point moves along the intersection of the plane $y = 3$ and the surface $z = \sqrt{29 - x^2 - y^2}$. At what rate is z changing with respect to x when the point is at $(4, 3, 2)$?

44. Find the slope of the tangent line at $(-1, 1, 5)$ to the curve of intersection of the surface $z = x^2 + 4y^2$ and
(a) the plane $x = -1$ (b) the plane $y = 1$.

45. The volume V of a right circular cylinder is given by the formula $V = \pi r^2 h$, where r is the radius and h is the height.
(a) Find a formula for the instantaneous rate of change of V with respect to r if r changes and h remains constant.
(b) Find a formula for the instantaneous rate of change of V with respect to h if h changes and r remains constant.
(c) Suppose that h has a constant value of 4 in, but r varies. Find the rate of change of V with respect to r at the point where $r = 6$ in.
(d) Suppose that r has a constant value of 8 in, but h varies. Find the instantaneous rate of change of V with respect to h at the point where $h = 10$ in.

46. The volume V of a right circular cone is given by
$$V = \frac{\pi}{24} d^2 \sqrt{4s^2 - d^2}$$
where s is the slant height and d is the diameter of the base.
(a) Find a formula for the instantaneous rate of change of V with respect to s if d remains constant.
(b) Find a formula for the instantaneous rate of change of V with respect to d if s remains constant.
(c) Suppose that d has a constant value of 16 cm, but s varies. Find the rate of change of V with respect to s when $s = 10$ cm.
(d) Suppose that s has a constant value of 10 cm, but d varies. Find the rate of change of V with respect to d when $d = 16$ cm.

47. According to the ideal gas law, the pressure, temperature, and volume of a gas are related by $P = kT/V$, where k is a constant of proportionality. Suppose that V is measured in cubic inches (in^3), T is measured in kelvins (K), and that for a certain gas the constant of proportionality is $k = 10$ in·lb/K.
(a) Find the instantaneous rate of change of pressure with respect to temperature if the temperature is 80 K and the volume remains fixed at 50 in^3.
(b) Find the instantaneous rate of change of volume with respect to pressure if the volume is 50 in^3 and the temperature remains fixed at 80 K.

48. The temperature at a point (x, y) on a metal plate in the xy-plane is $T(x, y) = x^3 + 2y^2 + x$ degrees centigrade. Assume that distance is measured in centimeters and find the rate at which temperature changes with respect to distance if we start at the point $(1, 2)$ and move
(a) to the right and parallel to the x-axis
(b) upward and parallel to the y-axis.

49. The length, width, and height of a rectangular box are $\ell = 5$, $w = 2$, and $h = 3$, respectively.
(a) Find the instantaneous rate of change of the volume of the box with respect to the length if w and h are held constant.
(b) Find the instantaneous rate of change of the volume of the box with respect to the width if ℓ and h are held constant.
(c) Find the instantaneous rate of change of the volume of the box with respect to the height if ℓ and w are held constant.

50. The area A of a triangle is given by $A = \frac{1}{2}ab\sin\theta$, where a and b are the lengths of two sides and θ is the angle between these sides. Suppose that $a = 5$, $b = 10$, and $\theta = \pi/3$.
(a) Find the rate at which A changes with respect to a if b and θ are held constant.
(b) Find the rate at which A changes with respect to θ if a and b are held constant.
(c) Find the rate at which b changes with respect to a if A and θ are held constant.

51. The volume of a right circular cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$. Show that if the height remains constant while the radius changes, then the volume satisfies
$$\frac{\partial V}{\partial r} = \frac{2V}{r}$$

52. Find parametric equations for the tangent line at $(1, 3, 3)$ to the curve of intersection of the surface $z = x^2 y$ and
(a) the plane $x = 1$ (b) the plane $y = 3$.

53. (a) By differentiating implicitly, find the slope of the hyperboloid $x^2 + y^2 - z^2 = 1$ in the x-direction at the points $(3, 4, 2\sqrt{6})$ and $(3, 4, -2\sqrt{6})$.
(b) Check the results in part (a) by solving for z and differentiating the resulting functions directly.

54. (a) By differentiating implicitly, find the slope of the hyperboloid $x^2 + y^2 - z^2 = 1$ in the y-direction at the points $(3, 4, 2\sqrt{6})$ and $(3, 4, -2\sqrt{6})$.
(b) Check the results in part (a) by solving for z and differentiating the resulting functions directly.

In Exercises 55–58, calculate $\partial z/\partial x$ and $\partial z/\partial y$ using implicit differentiation. Leave your answers in terms of x, y, and z.

55. $(x^2 + y^2 + z^2)^{3/2} = 1$

56. $\ln(2x^2 + y - z^3) = x$

57. $x^2 + z\sin xyz = 0$

58. $e^{xy}\sinh z - z^2 x + 1 = 0$

In Exercises 59–62, find $\partial w/\partial x$, $\partial w/\partial y$, and $\partial w/\partial z$ using implicit differentiation. Leave your answers in terms of x, y, z, and w.

59. $(x^2 + y^2 + z^2 + w^2)^{3/2} = 4$

60. $\ln(2x^2 + y - z^3 + 3w) = z$

61. $w^2 + w \sin xyz = 1$

62. $e^{xy} \sinh w - z^2 w + 1 = 0$

In Exercises 63 and 64, find f_x and f_y.

63. $f(x, y) = \displaystyle\int_y^x e^{t^2}\, dt$ **64.** $f(x, y) = \displaystyle\int_1^{xy} e^{t^2}\, dt$

65. Let $z = \sqrt{x} \cos y$. Find
(a) $\partial^2 z/\partial x^2$ (b) $\partial^2 z/\partial y^2$
(c) $\partial^2 z/\partial x \partial y$ (d) $\partial^2 z/\partial y \partial x$.

66. Let $f(x, y) = 4x^2 - 2y + 7x^4 y^5$. Find
(a) f_{xx} (b) f_{yy} (c) f_{xy} (d) f_{yx}.

In Exercises 67–74, confirm that the mixed second-order partial derivatives of f are the same.

67. $f(x, y) = 4x^2 - 8xy^4 + 7y^5 - 3$

68. $f(x, y) = \sqrt{x^2 + y^2}$ **69.** $f(x, y) = e^x \cos y$

70. $f(x, y) = e^{x - y^2}$ **71.** $f(x, y) = \ln(4x - 5y)$

72. $f(x, y) = \ln(x^2 + y^2)$

73. $f(x, y) = (x - y)/(x + y)$

74. $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$

75. The accompanying figure shows the graphs of an unspecified function $f(x, y)$ and its partial derivatives $f_x(x, y)$ and $f_y(x, y)$. Determine which is which, and explain your reasoning.

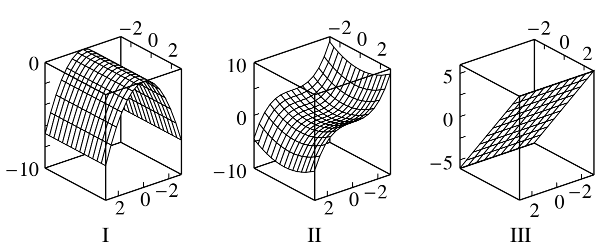


Figure Ex-75

76. What can you say about the signs of $\partial z/\partial x$, $\partial^2 z/\partial x^2$, $\partial z/\partial y$, and $\partial^2 z/\partial y^2$ at the point P in the accompanying figure? Explain your reasoning.

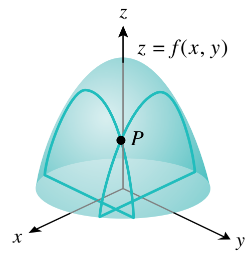


Figure Ex-76

77. Express the following derivatives in "∂" notation.
(a) f_{xxx} (b) f_{xyy} (c) f_{yyxx} (d) f_{xyyy}

78. Express the derivatives in "subscript" notation.
(a) $\dfrac{\partial^3 f}{\partial y^2 \partial x}$ (b) $\dfrac{\partial^4 f}{\partial x^4}$ (c) $\dfrac{\partial^4 f}{\partial y^2 \partial x^2}$ (d) $\dfrac{\partial^5 f}{\partial x^2 \partial y^3}$

79. Given $f(x, y) = x^3 y^5 - 2x^2 y + x$, find
(a) f_{xxy} (b) f_{yxy} (c) f_{yyy}.

80. Given $z = (2x - y)^5$, find
(a) $\dfrac{\partial^3 z}{\partial y \partial x \partial y}$ (b) $\dfrac{\partial^3 z}{\partial x^2 \partial y}$ (c) $\dfrac{\partial^4 z}{\partial x^2 \partial y^2}$.

81. Given $f(x, y) = y^3 e^{-5x}$, find
(a) $f_{xyy}(0, 1)$ (b) $f_{xxx}(0, 1)$ (c) $f_{yyxx}(0, 1)$.

82. Given $w = e^y \cos x$, find
(a) $\left.\dfrac{\partial^3 w}{\partial y^2 \partial x}\right|_{(\pi/4, 0)}$ (b) $\left.\dfrac{\partial^3 w}{\partial x^2 \partial y}\right|_{(\pi/4, 0)}$

83. Let $f(x, y, z) = x^3 y^5 z^7 + xy^2 + y^3 z$. Find
(a) f_{xy} (b) f_{yz} (c) f_{xz} (d) f_{zz}
(e) f_{zyy} (f) f_{xxy} (g) f_{zyx} (h) f_{xxyz}.

84. Let $w = (4x - 3y + 2z)^5$. Find
(a) $\dfrac{\partial^2 w}{\partial x \partial z}$ (b) $\dfrac{\partial^3 w}{\partial x \partial y \partial z}$ (c) $\dfrac{\partial^4 w}{\partial z^2 \partial y \partial x}$.

85. Show that the function satisfies ***Laplace's equation***

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

(a) $z = x^2 - y^2 + 2xy$
(b) $z = e^x \sin y + e^y \cos x$
(c) $z = \ln(x^2 + y^2) + 2 \tan^{-1}(y/x)$

86. Show that the function satisfies the ***heat equation***

$$\frac{\partial z}{\partial t} = c^2 \frac{\partial^2 z}{\partial x^2} \quad (c > 0,\ \text{constant})$$

(a) $z = e^{-t} \sin(x/c)$ (b) $z = e^{-t} \cos(x/c)$

87. Show that the function $u(x, t) = \sin c\omega t \sin \omega x$ satisfies the wave equation [Equation (6)] for all real values of ω.

88. In each part, show that $u(x, y)$ and $v(x, y)$ satisfy the ***Cauchy–Riemann equations***

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(a) $u = x^2 - y^2$, $v = 2xy$
(b) $u = e^x \cos y$, $v = e^x \sin y$
(c) $u = \ln(x^2 + y^2)$, $v = 2 \tan^{-1}(y/x)$

89. Show that if $u(x, y)$ and $v(x, y)$ each have equal mixed second partials, and if u and v satisfy the Cauchy–Riemann equations (Exercise 88), then u, v, and $u + v$ satisfy Laplace's equation (Exercise 85).

90. When two resistors having resistances R_1 ohms and R_2 ohms are connected in parallel, their combined resistance R in ohms is $R = R_1R_2/(R_1 + R_2)$. Show that

$$\frac{\partial^2 R}{\partial R_1^2}\frac{\partial^2 R}{\partial R_2^2} = \frac{4R^2}{(R_1 + R_2)^4}$$

In Exercises 91–94, find the indicated partial derivatives.

91. $f(v, w, x, y) = 4v^2w^3x^4y^5$;
$\partial f/\partial v$, $\partial f/\partial w$, $\partial f/\partial x$, $\partial f/\partial y$

92. $w = r\cos st + e^u \sin ur$;
$\partial w/\partial r$, $\partial w/\partial s$, $\partial w/\partial t$, $\partial w/\partial u$

93. $f(v_1, v_2, v_3, v_4) = \dfrac{v_1^2 - v_2^2}{v_3^2 + v_4^2}$;
$\partial f/\partial v_1$, $\partial f/\partial v_2$, $\partial f/\partial v_3$, $\partial f/\partial v_4$

94. $V = xe^{2x-y} + we^{zw} + yw$;
$\partial V/\partial x$, $\partial V/\partial y$, $\partial V/\partial z$, $\partial V/\partial w$

95. Let $u(w, x, y, z) = xe^{yw}\sin^2 z$. Find

(a) $\dfrac{\partial u}{\partial x}(0, 0, 1, \pi)$ (b) $\dfrac{\partial u}{\partial y}(0, 0, 1, \pi)$

(c) $\dfrac{\partial u}{\partial w}(0, 0, 1, \pi)$ (d) $\dfrac{\partial u}{\partial z}(0, 0, 1, \pi)$

(e) $\dfrac{\partial^4 u}{\partial x\partial y\partial w\partial z}$ (f) $\dfrac{\partial^4 u}{\partial w\partial z\partial y^2}$.

96. Let $f(v, w, x, y) = 2v^{1/2}w^4x^{1/2}y^{2/3}$. Find $f_v(1, -2, 4, 8)$, $f_w(1, -2, 4, 8)$, $f_x(1, -2, 4, 8)$, and $f_y(1, -2, 4, 8)$.

In Exercises 97 and 98, find $\partial w/\partial x_i$ for $i = 1, 2, \ldots, n$.

97. $w = \cos(x_1 + 2x_2 + \cdots + nx_n)$

98. $w = \left(\sum_{k=1}^{n} x_k\right)^{1/n}$

In Exercises 99 and 100, describe the largest set on which Theorem 14.3.2 may be used to prove that f_{xy} and f_{yx} are equal on that set. Then confirm by direct computation that $f_{xy} = f_{yx}$ on the given set.

99. (a) $f(x, y) = 4x^3y + 3x^2y$ (b) $f(x, y) = x^3/y$

100. (a) $f(x, y) = \sqrt{x^2 + y^2 - 1}$
(b) $f(x, y) = \sin(x^2 + y^3)$

101. Let $f(x, y) = 2x^2 - 3xy + y^2$. Find $f_x(2, -1)$ and $f_y(2, -1)$ by evaluating the limits in Definition 14.3.1. Then check your work by calculating the derivative in the usual way.

102. Let $f(x, y) = (x^2 + y^2)^{2/3}$. Show that

$$f_x(x, y) = \begin{cases} \dfrac{4x}{3(x^2 + y^2)^{1/3}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

[This problem, due to Don Cohen, appeared in *Mathematics and Computer Education*, Vol. 25, No. 2, 1991, p. 179.]

103. Let $f(x, y) = (x^3 + y^3)^{1/3}$.
(a) Show that $f_y(0, 0) = 1$.
(b) At what points, if any, does $f_y(x, y)$ fail to exist?

14.4 DIFFERENTIABILITY, LOCAL LINEARITY, AND DIFFERENTIALS

In this section we will extend the notion of differentiability to functions of two or three variables. Our definition of differentiability will be based on the concept of "local linearity"; that is, a function should be differentiable at a point if it can be closely approximated by a linear function near that point. In addition, we will expand the concept of a "differential" to functions of more than one variable, and we will express local linearity in terms of differentials.

DIFFERENTIABILITY

Recall that a function f of one variable is called differentiable at x_0 if it has a derivative at x_0, that is, if the limit

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{1}$$

exists. As a consequence of (1) a differentiable function enjoys a number of other important properties:

- The graph of $y = f(x)$ has a nonvertical tangent line at the point $(x_0, f(x_0))$.
- f has a local linear approximation at x_0 (Section 3.8).
- f is continuous at x_0.

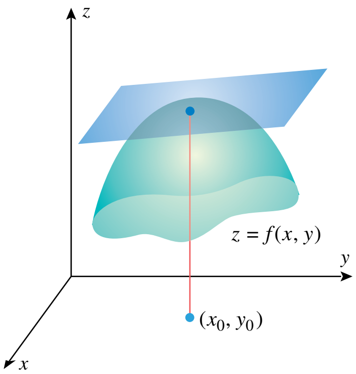


Figure 14.4.1

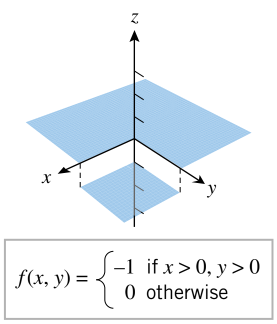


Figure 14.4.2

Our primary objective in this section is to extend the notion of differentiability to functions of two or three variables in such a way that the natural analogs of these properties hold. For example, if a function $f(x, y)$ of two variables is differentiable at a point (x_0, y_0), we want it to be the case that

- the surface $z = f(x, y)$ has a nonvertical tangent plane at the point $(x_0, y_0, f(x_0, y_0))$ (Figure 14.4.1);
- the values of f at points near (x_0, y_0) can be very closely approximated by the values of a linear function;
- f is continuous at (x_0, y_0).

It would not be unreasonable to conjecture that a function f of two or three variables should be called differentiable at a point if all the first-order partial derivatives of the function exist at that point. Unfortunately, this condition is not strong enough to guarantee that the properties above hold. For example, consider the function

$$f(x, y) = \begin{cases} -1 & \text{if } x > 0 \text{ and } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

whose graph is shown in Figure 14.4.2. This function is discontinuous at (0, 0) (why?) but does have partial derivatives at (0, 0); these derivatives are $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$ (verify).

To motivate an appropriate definition of differentiability for functions of two or three variables, it will be helpful to reexamine the definition of differentiability for a *one-variable* function f. Our goal is to interpret the differentiability of f in terms of the error involved in approximating f by a *linear function*. Suppose that f is defined on an open interval containing x_0, with $P = (x_0, f(x_0))$ the corresponding point on the graph of f. Any linear function $L(x)$ whose graph is a straight line through P can be written in the form

$$L(x) = f(x_0) + m(x - x_0)$$

where m is the slope of the line. We will refer to such a function $L(x)$ as a *linear approximation to f at x_0*. Given a linear approximation $L(x)$, define

$$E(x) = f(x) - L(x) = f(x) - f(x_0) - m(x - x_0)$$

to be the error that results if $L(x)$ is used to approximate $f(x)$. [Perhaps "remainder" would be a better name for $E(x)$ since $E(x)$ is positive when $L(x)$ is *less* than $f(x)$.] We will reformulate the differentiability of f at x_0 in terms of the limiting behavior of $E(x)$ as x approaches x_0.

Let $L(x) = f(x_0) + m(x - x_0)$ denote a linear approximation to f at x_0. The number m is equal to $f'(x_0)$ if and only if

$$m = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

or equivalently, if and only if

$$\begin{aligned} 0 &= \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} - m \right] = \lim_{x \to x_0} \frac{f(x) - f(x_0) - m(x - x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{E(x)}{x - x_0} \end{aligned}$$

We conclude that f is differentiable at x_0 if and only if there exists a linear function

$$L(x) = f(x_0) + m(x - x_0)$$

such that

$$\lim_{x \to x_0} \frac{E(x)}{x - x_0} = 0 \tag{2}$$

Furthermore, when such a linear function $L(x)$ exists, it is given uniquely as the local linear approximation $L(x) = f(x_0) + f'(x_0)(x - x_0)$ to f at x_0 (Section 3.8). In order to formulate this description in such a way that it extends naturally to functions of two or three variables, we need to replace the difference $x - x_0$ in Equation (2) by $|x - x_0|$. This is permissible since it can be shown that given any two functions $f(x)$ and $g(x)$,

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0 \quad \text{if and only if} \quad \lim_{x \to x_0} \frac{f(x)}{|g(x)|} = 0$$

(Exercise 61). Therefore,

$$\lim_{x \to x_0} \frac{E(x)}{x - x_0} = 0 \quad \text{if and only if} \quad \lim_{x \to x_0} \frac{E(x)}{|x - x_0|} = 0$$

and we have the following alternative definition of differentiability for functions of one variable.

14.4.1 DEFINITION. A function f of one variable is said to be ***differentiable*** at x_0 provided there exists a linear approximation $L(x) = f(x_0) + m(x - x_0)$ to f at x_0 for which the error $E(x) = f(x) - L(x)$ satisfies

$$\lim_{x \to x_0} \frac{E(x)}{|x - x_0|} = 0 \tag{3}$$

When f is differentiable at x_0, we denote the number m by $f'(x_0)$ and refer to it as the ***derivative of f at*** $\boldsymbol{x_0}$.

We can interpret Equation (3) to mean that when x is very close to x_0, the magnitude of the error in approximating $f(x)$ by $L(x) = f(x_0) + f'(x_0)(x - x_0)$ is *much* smaller than the distance $|x - x_0|$ between x and x_0.

Although the definition of differentiability in Definition 14.4.1 is more complicated than that given earlier in the text, it has a natural extension to functions of two or more variables. Let us consider first the case of a function $f(x, y)$ of two variables. Suppose that f is defined in an open disk containing the point (x_0, y_0) with $P = (x_0, y_0, f(x_0, y_0))$ the corresponding point on the graph of f. Any linear function $L(x, y)$ whose graph is a plane through P can be written in the form

$$L(x, y) = f(x_0, y_0) + m_1(x - x_0) + m_2(y - y_0)$$

for some choice of the constants m_1 and m_2. We will refer to such a function $L(x, y)$ as a *linear approximation to* f *at* (x_0, y_0). Given a linear approximation $L(x, y)$, define

$$E(x, y) = f(x, y) - L(x, y) = f(x, y) - f(x_0, y_0) - m_1(x - x_0) - m_2(y - y_0)$$

to be the error that results if $L(x, y)$ is used to approximate $f(x, y)$.

By analogy to the one-variable case, suppose that there exists a linear approximation $L(x, y)$ to f at (x_0, y_0) such that when (x, y) is very close to (x_0, y_0), the size of the error $E(x, y)$ is much smaller than the distance $\sqrt{(x - x_0)^2 + (y - y_0)^2}$ between (x, y) and (x_0, y_0). More precisely, suppose that

$$\lim_{(x,y) \to (x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0 \tag{4}$$

We will argue that in this case $m_1 = f_x(x_0, y_0)$ and $m_2 = f_y(x_0, y_0)$.

On the line $y = y_0$ we have

$$\frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = \frac{E(x, y_0)}{\sqrt{(x - x_0)^2}} = \frac{f(x, y_0) - f(x_0, y_0) - m_1(x - x_0)}{|x - x_0|}$$

and $(x, y) = (x, y_0)$ approaches (x_0, y_0) if and only if x approaches x_0. Therefore, on the line $y = y_0$, Equation (4) becomes

$$\lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0) - m_1(x - x_0)}{|x - x_0|} = 0$$

and it follows from Definition 14.4.1 [with $f(x)$ replaced by $f(x, y_0)$] that

$$m_1 = \frac{d}{dx}[f(x, y_0)]_{x=x_0} = f_x(x_0, y_0)$$

Similarly, if we allow (x, y) to approach (x_0, y_0) along the line $x = x_0$, then we conclude from Equation (4) that $m_2 = f_y(x_0, y_0)$. The linear function $L(x, y)$ must therefore have the form

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Based on this analysis, we can now give our two-variable version of Definition 14.4.1.

14.4.2 DEFINITION. A function f of two variables is said to be ***differentiable*** at (x_0, y_0) provided $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist and

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0 \tag{5}$$

where $E(x, y) = f(x, y) - L(x, y)$ denotes the error in the linear approximation

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{6}$$

to f at (x_0, y_0). When f is differentiable at (x_0, y_0), we will refer to (6) as the ***local linear approximation to f at*** (x_0, y_0).

As with the one-variable case, verification of differentiability using this definition involves the computation of a limit.

Example 1 Let $f(x, y) = x^2 + y^2$.

(a) Use Definition 14.4.2 to prove that f is differentiable at $(0, 0)$.

(b) Let $E(x, y)$ denote the error in the local linear approximation to f at $(0, 0)$. Determine all points $(x, y) \neq (0, 0)$ such that

$$\left| \frac{E(x, y)}{\sqrt{x^2 + y^2}} \right| < 10^{-6}$$

Solution (a). We have $f_x(x, y) = 2x$ and $f_y(x, y) = 2y$ so that

$$f_x(0, 0) = f_y(0, 0) = f(0, 0) = 0$$

and thus $L(x, y) = 0$. It follows that

$$E(x, y) = f(x, y) - L(x, y) = f(x, y) = x^2 + y^2$$

and

$$\lim_{(x,y)\to(0,0)} \frac{E(x, y)}{\sqrt{x^2 + y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2}} = \lim_{(x,y)\to(0,0)} \sqrt{x^2 + y^2} = 0$$

which shows that f is differentiable at $(0, 0)$.

Solution (b). It follows immediately from the solution to part (a) that

$$\left| \frac{E(x, y)}{\sqrt{x^2 + y^2}} \right| = \sqrt{x^2 + y^2} < 10^{-6}$$

if and only if $(x, y) \neq (0, 0)$ belongs to the open disk of radius 10^{-6} centered at the origin. ◀

For functions of three variables we have an analogous definition of differentiability.

14.4.3 DEFINITION. A function f of three variables is said to be ***differentiable*** at (x_0, y_0, z_0) provided $f_x(x_0, y_0, z_0)$, $f_y(x_0, y_0, z_0)$, and $f_z(x_0, y_0, z_0)$ exist and

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{E(x,y,z)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} = 0 \tag{7}$$

where $E(x,y,z) = f(x,y,z) - L(x,y,z)$ denotes the error in the linear approximation

$$\begin{aligned} L(x,y,z) = f(x_0,y_0,z_0) &+ f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) \\ &+ f_z(x_0,y_0,z_0)(z-z_0) \end{aligned} \tag{8}$$

to f at (x_0, y_0, z_0). When f is differentiable at (x_0, y_0, z_0), we will refer to (8) as the ***local linear approximation to f at*** (x_0, y_0, z_0).

If a function f of two variables is differentiable at each point of a region R in the xy-plane, then we say that f is ***differentiable on R***; and if f is differentiable at every point in the xy-plane, then we say that f is ***differentiable everywhere***. For a function f of three variables we have corresponding conventions.

DIFFERENTIABILITY AND CONTINUITY

Recall that we want a function to be continuous at every point at which it is differentiable. The following theorem shows this to be the case.

14.4.4 THEOREM. *If a function is differentiable at a point, then it is continuous at that point.*

Proof. We will give the proof for a function of two variables $f(x, y)$ since that will reveal the essential ideas. Assume that f is differentiable at (x_0, y_0) and let $L(x, y)$ and $E(x, y)$ be the functions in Definition 14.4.2. Then

$$\begin{aligned} \lim_{(x,y)\to(x_0,y_0)} L(x,y) &= \lim_{(x,y)\to(x_0,y_0)} [f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)] \\ &= f(x_0,y_0) + 0 + 0 = f(x_0,y_0) \end{aligned}$$

and

$$\begin{aligned} \lim_{(x,y)\to(x_0,y_0)} E(x,y) &= \lim_{(x,y)\to(x_0,y_0)} \left[\frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\right]\sqrt{(x-x_0)^2+(y-y_0)^2} \\ &= (0)(0) = 0 \end{aligned}$$

Since $f(x,y) = E(x,y) + L(x,y)$, we conclude that

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \lim_{(x,y)\to(x_0,y_0)} [E(x,y) + L(x,y)] = 0 + f(x_0,y_0) = f(x_0,y_0)$$

and thus f is continuous at (x_0, y_0). ∎

At the beginning of this section we gave an example of a function $f(x, y)$ that was discontinuous at (0, 0) but for which $f_x(0, 0)$ and $f_y(0, 0)$ both exist. This example, together with Theorem 14.4.4, shows that the mere *existence* of first-order partial derivatives is not enough to ensure differentiability. However, note that although both $f_x(0, 0)$ and $f_y(0, 0)$ are defined for this function, $f_y(x, 0)$ is undefined for $x > 0$ and $f_x(0, y)$ is undefined for $y > 0$ (Exercise 65). In particular, neither f_x nor f_y is *continuous* at (0, 0). The following theorem, whose proof we omit, states that the existence of continuous first-order partial derivatives at a point *is* sufficient to imply differentiability.

14.4.5 THEOREM. *If all first-order partial derivatives of f exist and are continuous at a point, then f is differentiable at that point.*

For example, consider the function $f(x, y, z) = x + yz$. Since

$$f_x(x, y, z) = 1, \quad f_y(x, y, z) = z, \quad \text{and} \quad f_z(x, y, z) = y$$

are defined and continuous everywhere, we conclude from Theorem 14.4.5 that f is differentiable everywhere.

For most of the functions that we will encounter, the verification of differentiability using Theorem 14.4.5 will be a relatively straightforward process. For this reason, we will not be concerned with such verification in the remainder of this chapter.

LOCAL LINEARITY; DIFFERENTIALS

Our definitions of differentiability assure us that if a function f is differentiable at some point, then it can be very closely approximated by a linear function near that point. For example, if a function $f(x, y)$ is differentiable at (x_0, y_0), then Equation (5) tells us that the error in the local linear approximation to f at (x_0, y_0) approaches 0 much more quickly than $\sqrt{(x-x_0)^2 + (y-y_0)^2}$. Thus, if (x, y) is very close to (x_0, y_0), then we can expect the size of the error in the local linear approximation of $f(x, y)$ by $L(x, y)$ to be much smaller than the distance $\sqrt{(x-x_0)^2 + (y-y_0)^2}$ between the points (x, y) and (x_0, y_0). Comparable results hold for functions of three variables.

Example 2 Let $L(x, y)$ denote the local linear approximation to $f(x, y) = \sqrt{x^2 + y^2}$ at the point $(3, 4)$. Compare the error in approximating

$$f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2}$$

by $L(3.04, 3.98)$ with the distance between the points $(3, 4)$ and $(3.04, 3.98)$.

Solution. We have

$$f_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad f_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}}$$

with $f_x(3, 4) = \frac{3}{5}$ and $f_y(3, 4) = \frac{4}{5}$. Therefore, the local linear approximation to f at $(3, 4)$ is given by

$$L(x, y) = 5 + \tfrac{3}{5}(x - 3) + \tfrac{4}{5}(y - 4)$$

Consequently,

$$f(3.04, 3.98) \approx L(3.04, 3.98) = 5 + \tfrac{3}{5}(0.04) + \tfrac{4}{5}(-0.02) = 5.008$$

Since

$$f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2} \approx 5.00819$$

the error in the approximation is about $5.00819 - 5.008 = 0.00019$. This is less than $\frac{1}{200}$ of the distance

$$\sqrt{(3.04 - 3)^2 + (3.98 - 4)^2} \approx 0.045$$

between the points $(3, 4)$ and $(3.04, 3.98)$. ◀

It is possible to interpret the "smallness" of the error in the local linear approximation in another way. Assume that f is differentiable at (x_0, y_0) and define the function

$$\epsilon(x, y) = \begin{cases} \dfrac{E(x, y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}, & (x, y) \neq (x_0, y_0) \\ 0, & (x, y) = (x_0, y_0) \end{cases}$$

where $E(x, y)$ is the error function from Definition 14.4.2. Then

$$E(x, y) = \epsilon(x, y)\sqrt{(x-x_0)^2 + (y-y_0)^2} \tag{9}$$

and it follows from Equation (5) that

$$\lim_{(x,y)\to(x_0,y_0)} \epsilon(x, y) = \lim_{(x,y)\to(x_0,y_0)} \frac{E(x, y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0 = \epsilon(x_0, y_0)$$

In other words, $\epsilon(x, y)$ is continuous at (x_0, y_0). We conclude that $E(x, y)$ can be expressed as the product of $\sqrt{(x-x_0)^2+(y-y_0)^2}$ and a second function $\epsilon(x, y)$ that is both continuous at (x_0, y_0) and vanishes there. [As an illustration, recall from Example 1 that for

$$f(x, y) = x^2 + y^2 \quad \text{and} \quad (x_0, y_0) = (0, 0)$$

we had $E(x, y) = x^2 + y^2$, which can be written in the form $E(x, y) = \sqrt{x^2+y^2}\sqrt{x^2+y^2}$.] Conversely, if $E(x, y)$ can be written in this form, then $f(x, y)$ must be differentiable at (x_0, y_0) (Exercise 66).

As with the one-variable case, local linear approximations can be interpreted using the language of differentials. If $z = f(x, y)$ is differentiable at a point (x_0, y_0), we let

$$dz = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy \tag{10}$$

denote a new function with dependent variable dz and independent variables dx and dy. We refer to this function (also denoted df) as the ***total differential of z*** at (x_0, y_0) or as the ***total differential of f*** at (x_0, y_0). Similarly, for a function $w = f(x, y, z)$ of three variables we have the ***total differential of w*** at (x_0, y_0, z_0),

$$dw = f_x(x_0, y_0, z_0)\,dx + f_y(x_0, y_0, z_0)\,dy + f_z(x_0, y_0, z_0)\,dz \tag{11}$$

which is also referred to as the ***total differential of f*** at (x_0, y_0, z_0). It is common practice to omit the subscripts and write Equations (10) and (11) as

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy \tag{12}$$

and

$$dw = f_x(x, y, z)\,dx + f_y(x, y, z)\,dy + f_z(x, y, z)\,dz \tag{13}$$

In the two-variable case, the approximation

$$f(x, y) \approx L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

can be written in the form

$$\Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y \tag{14}$$

where $\Delta f = f(x, y) - f(x_0, y_0)$ denotes the change in the values of f between the points (x_0, y_0) and (x, y), and $\Delta x = x - x_0$ and $\Delta y = y - y_0$ denote the changes in x and y, respectively. Note that the right side of (14) is the value of dz for $dx = \Delta x$ and $dy = \Delta y$. Letting $\Delta z = \Delta f$, we can write (14) in the form

$$\Delta z \approx dz \tag{15}$$

That is, we can estimate the change Δz in z by the value of the differential dz provided dx is the change in x and dy is the change in y.

Example 3 Use (15) to estimate the change in $z = xy^2$ from its value at (0.5, 1.0) to its value at (0.503, 1.004). Compare the error in this estimate with the distance between the points (0.5, 1.0) and (0.503, 1.004).

Solution. For $z = xy^2$ we have $dz = y^2\,dx + 2xy\,dy$. Evaluating this differential at $(x, y) = (0.5, 1.0)$ with

$$dx = \Delta x = 0.503 - 0.5 = 0.003 \quad \text{and} \quad dy = \Delta y = 1.004 - 1.0 = 0.004$$

yields the approximation

$$\Delta z \approx 1.0^2(0.003) + 2(0.5)(1.0)(0.004) = 0.007$$

Since $z = 0.5$ at (0.5, 1.0) and $z \approx 0.50703$ at (0.503, 1.004), we have

$$\Delta z \approx 0.50703 - 0.5 = 0.00703$$

and the error in approximating Δz by dz is

$$\Delta z - dz \approx 0.00703 - 0.007 = 0.00003$$

Since the distance between the two points is

$$\sqrt{(0.503-0.5)^2+(1.004-1.0)^2} = \sqrt{0.000025} = 0.005$$

we see that the error in our approximation is less than $\frac{1}{150}$ of the distance between the two points. ◀

Note that for approximation (15)

$$\begin{aligned}\Delta z - dz &= [f(x,y) - f(x_0,y_0)] - [f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)] \\ &= f(x,y) - L(x,y) = E(x,y)\end{aligned}$$

That is, the error $\Delta z - dz$ in approximation (15) is equal to the error $E(x,y)$ in the corresponding local linear approximation and we have

$$\Delta z = dz + E(x,y) \tag{16}$$

Using Equation (9) we can write (16) as

$$\Delta z = dz + \epsilon(x,y)\sqrt{(x-x_0)^2+(y-y_0)^2}$$

where $\epsilon(x,y)$ is both continuous at (x_0,y_0) and vanishes there. Equivalently,

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \epsilon(x,y)\sqrt{(\Delta x)^2+(\Delta y)^2} \tag{17}$$

We will use Equation (17) in the next section to prove the chain rule for functions of two variables.

With the appropriate changes in notation, the preceding analysis is also valid for a function f of three or more variables.

Example 4 The length, width, and height of a rectangular box are measured with an error of at most 5%. Use a total differential to estimate the maximum percentage error that results if these quantities are used to calculate the diagonal of the box.

Solution. The diagonal D of a box with length x, width y, and height z is given by

$$D = \sqrt{x^2+y^2+z^2}$$

Let x_0, y_0, z_0, and $D_0 = \sqrt{x_0^2+y_0^2+z_0^2}$ denote the actual values of the length, width, height, and diagonal of the box. The total differential dD of D at (x_0,y_0,z_0) is given by

$$dD = \frac{x_0}{\sqrt{x_0^2+y_0^2+z_0^2}}dx + \frac{y_0}{\sqrt{x_0^2+y_0^2+z_0^2}}dy + \frac{z_0}{\sqrt{x_0^2+y_0^2+z_0^2}}dz$$

If x, y, z, and $D = \sqrt{x^2+y^2+z^2}$ are the measured and computed values of the length, width, height, and diagonal, respectively, then $\Delta x = x - x_0$, $\Delta y = y - y_0$, $\Delta z = z - z_0$, and

$$\left|\frac{\Delta x}{x_0}\right| \le 0.05, \quad \left|\frac{\Delta y}{y_0}\right| \le 0.05, \quad \left|\frac{\Delta z}{z_0}\right| \le 0.05$$

We are seeking an estimate for the maximum size of $\Delta D/D_0$. With the aid of Equation (13) we have

$$\begin{aligned}\frac{\Delta D}{D_0} \approx \frac{dD}{D_0} &= \frac{1}{x_0^2+y_0^2+z_0^2}[x_0\Delta x + y_0\Delta y + z_0\Delta z] \\ &= \frac{1}{x_0^2+y_0^2+z_0^2}\left[x_0^2\frac{\Delta x}{x_0} + y_0^2\frac{\Delta y}{y_0} + z_0^2\frac{\Delta z}{z_0}\right]\end{aligned}$$

Since

$$\begin{aligned}\left|\frac{dD}{D_0}\right| &= \frac{1}{x_0^2+y_0^2+z_0^2}\left|x_0^2\frac{\Delta x}{x_0}+y_0^2\frac{\Delta y}{y_0}+z_0^2\frac{\Delta z}{z_0}\right|\\ &\le \frac{1}{x_0^2+y_0^2+z_0^2}\left(x_0^2\left|\frac{\Delta x}{x_0}\right|+y_0^2\left|\frac{\Delta y}{y_0}\right|+z_0^2\left|\frac{\Delta z}{z_0}\right|\right)\\ &\le \frac{1}{x_0^2+y_0^2+z_0^2}\left(x_0^2(0.05)+y_0^2(0.05)+z_0^2(0.05)\right)=0.05\end{aligned}$$

we estimate the maximum percentage error in D to be 5%. ◀

We have formulated our definitions in this section in such a way that continuity and local linearity are consequences of differentiability. It remains to be shown that if a function $f(x, y)$ is differentiable at a point (x_0, y_0), then the graph of f has a nonvertical tangent plane at the point $(x_0, y_0, f(x_0, y_0))$. This we will do in Section 14.7.

EXERCISE SET 14.4

1. In each part, confirm that the stated formula is the local linear approximation at $(0, 0)$.
(a) $e^x \sin y \approx y$ (b) $\dfrac{2x+1}{y+1} \approx 1+2x-y$

2. Show that if $\alpha \neq 1$ and $\beta \neq 1$, then the local linear approximation of the function $f(x, y) = x^\alpha y^\beta$ at $(1, 1)$ is
$$x^\alpha y^\beta \approx 1+\alpha(x-1)+\beta(y-1)$$

3. In each part, confirm that the stated formula is the local linear approximation at $(1, 1, 1)$.
(a) $xyz+2 \approx x+y+z$ (b) $\dfrac{4x}{y+z} \approx 2x-y-z+2$

4. Based on Exercise 2, what would you conjecture is the local linear approximation to $x^\alpha y^\beta z^\gamma$ at $(1, 1, 1)$, provided none of the exponents α, β, or γ are equal to 1? Verify your conjecture by finding this local linear approximation.

5. Suppose that a function $f(x, y)$ is differentiable at the point $(3, 4)$ with $f_x(3, 4) = 2$ and $f_y(3, 4) = -1$. If $f(3, 4) = 5$, estimate the value of $f(3.01, 3.98)$.

6. Suppose that a function $f(x, y)$ is differentiable at the point $(-1, 2)$ with $f_x(-1, 2) = 1$ and $f_y(-1, 2) = 3$. If $f(-1, 2) = 2$, estimate the value of $f(-0.99, 2.02)$.

7. Suppose that a function $f(x, y)$ is differentiable at the point $(1, 1)$ with $f_x(1, 1) = 2$ and $f(1, 1) = 3$. Let $L(x, y)$ denote the local linear approximation of f at $(1, 1)$. If $L(1.1, 0.9) = 3.15$, find the value of $f_y(1, 1)$.

8. Suppose that a function $f(x, y)$ is differentiable at the point $(0, -1)$ with $f_y(0, -1) = -2$ and $f(0, -1) = 3$. Let $L(x, y)$ denote the local linear approximation of f at $(0, -1)$. If $L(0.1, -1.1) = 3.3$, find the value of $f_x(0, -1)$.

9. Suppose that a function $f(x, y, z)$ is differentiable at the point $(1, 2, 3)$ with $f_x(1, 2, 3) = 1$, $f_y(1, 2, 3) = 2$, and $f_z(1, 2, 3) = 3$. If $f(1, 2, 3) = 4$, estimate the value of $f(1.01, 2.02, 3.03)$.

10. Suppose that a function $f(x, y, z)$ is differentiable at the point $(2, 1, -2)$ with $f_x(2, 1, -2) = -1$, $f_y(2, 1, -2) = 1$, and $f_z(2, 1, -2) = -2$. If $f(2, 1, -2) = 0$, estimate the value of $f(1.98, 0.99, -1.97)$.

11. Suppose that a function $f(x, y, z)$ is differentiable at the point $(3, 2, 1)$ and $L(x, y, z) = x-y+2z-2$ is the local linear approximation to f at $(3, 2, 1)$. Find $f(3, 2, 1)$, $f_x(3, 2, 1)$, $f_y(3, 2, 1)$, and $f_z(3, 2, 1)$.

12. Suppose that a function $f(x, y, z)$ is differentiable at the point $(0, -1, -2)$ and $L(x, y, z) = x+2y+3z+4$ is the local linear approximation to f at $(0, -1, -2)$. Find $f(0, -1, -2)$, $f_x(0, -1, -2)$, $f_y(0, -1, -2)$, and $f_z(0, -1, -2)$.

In Exercises 13–16, a function f is given along with a local linear approximation L to f at a point P. Use the information given to determine point P.

13. $f(x, y) = x^2+y^2$; $L(x, y) = 2y-2x-2$

14. $f(x, y) = x^2y$; $L(x, y) = 4y-4x+8$

15. $f(x, y, z) = xy+z^2$; $L(x, y, z) = y+2z-1$

16. $f(x, y, z) = xyz$; $L(x, y, z) = x-y-z-2$

In Exercises 17–24: (a) Find the local linear approximation L to the specified function f at the designated point P. (b) Compare the error in approximating f by L at the specified point Q with the distance between P and Q.

17. $f(x, y) = \dfrac{1}{\sqrt{x^2+y^2}}$; $P(4, 3)$, $Q(3.92, 3.01)$

18. $f(x, y) = x^{0.5}y^{0.3}$; $P(1, 1)$, $Q(1.05, 0.97)$

19. $f(x, y) = x\sin y$; $P(0, 0)$, $Q(0.003, 0.004)$

20. $f(x, y) = \ln xy$; $P(1, 2)$, $Q(1.01, 2.02)$

21. $f(x, y, z) = xyz$; $P(1, 2, 3)$, $Q(1.001, 2.002, 3.003)$

22. $f(x, y, z) = \dfrac{x + y}{y + z}$; $P(-1, 1, 1)$, $Q(-0.99, 0.99, 1.01)$

23. $f(x, y, z) = xe^{yz}$; $P(1, -1, -1)$, $Q(0.99, -1.01, -0.99)$

24. $f(x, y, z) = \ln(x + yz)$; $P(2, 1, -1)$, $Q(2.02, 0.97, -1.01)$

In Exercises 25–36, compute the differential dz or dw of the specified function.

25. $z = 7x - 2y$ **26.** $z = e^{xy}$ **27.** $z = x^3y^2$

28. $z = 5x^2y^5 - 2x + 4y + 7$

29. $z = \tan^{-1} xy$ **30.** $z = \sec^2(x - 3y)$

31. $w = 8x - 3y + 4z$ **32.** $w = e^{xyz}$

33. $w = x^3y^2z$

34. $w = 4x^2y^3z^7 - 3xy + z + 5$

35. $w = \tan^{-1}(xyz)$ **36.** $w = \sqrt{x} + \sqrt{y} + \sqrt{z}$

In Exercises 37–42, use a total differential to approximate the change in $f(x, y)$ as (x, y) varies from P to Q. Compare your estimate with the actual change in $f(x, y)$.

37. $f(x, y) = x^2 + 2xy - 4x$; $P(1, 2)$, $Q(1.01, 2.04)$

38. $f(x, y) = x^{1/3}y^{1/2}$; $P(8, 9)$, $Q(7.78, 9.03)$

39. $f(x, y) = \dfrac{x + y}{xy}$; $P(-1, -2)$, $Q(-1.02, -2.04)$

40. $f(x, y) = \ln\sqrt{1 + xy}$; $P(0, 2)$, $Q(-0.09, 1.98)$

41. $f(x, y, z) = 2xy^2z^3$; $P(1, -1, 2)$, $Q(0.99, -1.02, 2.02)$

42. $f(x, y, z) = \dfrac{xyz}{x + y + z}$; $P(-1, -2, 4)$, $Q(-1.04, -1.98, 3.97)$

43. In the accompanying figure a rectangle with initial length x_0 and initial width y_0 has been increased, resulting in a larger rectangle with length x and width y. What portion of the figure represents the increase in the area of the rectangle? What portion of the figure represents an approximation of the increase in area by a total differential?

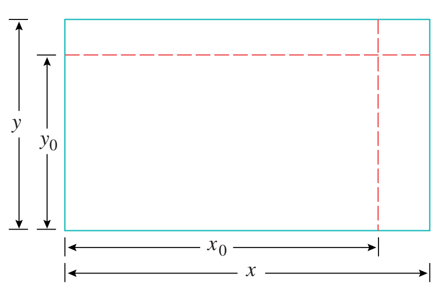


Figure Ex-43

44. The volume V of a right circular cone of radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$. Suppose that the height decreases from 20 in to 19.95 in and the radius increases from 4 in to 4.05 in. Compare the change in volume of the cone with an approximation of this change using a total differential.

45. The length and width of a rectangle are measured with errors of at most 3% and 5%, respectively. Use differentials to approximate the maximum percentage error in the calculated area.

46. The radius and height of a right circular cone are measured with errors of at most 1% and 4%, respectively. Use differentials to approximate the maximum percentage error in the calculated volume.

47. The length and width of a rectangle are measured with errors of at most r%, where r is small. Use differentials to approximate the maximum percentage error in the calculated length of the diagonal.

48. The legs of a right triangle are measured to be 3 cm and 4 cm, with a maximum error of 0.05 cm in each measurement. Use differentials to approximate the maximum possible error in the calculated value of (a) the hypotenuse and (b) the area of the triangle.

49. The period T of a simple pendulum with small oscillations is calculated from the formula $T = 2\pi\sqrt{L/g}$, where L is the length of the pendulum and g is the acceleration due to gravity. Suppose that measured values of L and g have errors of at most 0.5% and 0.1%, respectively. Use differentials to approximate the maximum percentage error in the calculated value of T.

50. According to the ideal gas law, the pressure, temperature, and volume of a confined gas are related by $P = kT/V$, where k is a constant. Use differentials to approximate the percentage change in pressure if the temperature of a gas is increased 3% and the volume is increased 5%.

51. Suppose that certain measured quantities x and y have errors of at most r% and s%, respectively. For each of the following formulas in x and y, use differentials to approximate the maximum possible error in the calculated result.
(a) xy (b) x/y (c) x^2y^3 (d) $x^3\sqrt{y}$

52. The total resistance R of three resistances R_1, R_2, and R_3, connected in parallel, is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Suppose that R_1, R_2, and R_3 are measured to be 100 ohms, 200 ohms, and 500 ohms, respectively, with a maximum error of 10% in each. Use differentials to approximate the maximum percentage error in the calculated value of R.

53. The area of a triangle is to be computed from the formula $A = \frac{1}{2}ab\sin\theta$, where a and b are the lengths of two sides and θ is the included angle. Suppose that a, b, and θ are measured to be 40 ft, 50 ft, and 30°, respectively. Use differentials to approximate the maximum error in the calculated value of A if the maximum errors in a, b, and θ are $\frac{1}{2}$ ft, $\frac{1}{4}$ ft, and 2°, respectively.

54. The length, width, and height of a rectangular box are measured with errors of at most r% (where r is small). Use differentials to approximate the maximum percentage error in the computed value of the volume.

55. Use Definitions 14.4.2 and 14.4.3 to prove that a constant function of two or three variables is differentiable everywhere.

56. Use Definitions 14.4.2 and 14.4.3 to prove that a linear function of two or three variables is differentiable everywhere.

57. Use Theorem 14.4.5 to prove that $f(x, y) = x^2 \sin y$ is differentiable everywhere.

58. Use Theorem 14.4.5 to prove that $f(x, y, z) = xy \sin z$ is differentiable everywhere.

59. Use Definition 14.4.3 to prove that

$$f(x, y, z) = x^2 + y^2 + z^2$$

is differentiable at (0, 0, 0).

60. Use Definition 14.4.3 to determine all values of r such that $f(x, y, z) = (x^2 + y^2 + z^2)^r$ is differentiable at (0, 0, 0).

61. Prove that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0 \quad \text{if and only if} \quad \lim_{x \to x_0} \frac{f(x)}{|g(x)|} = 0$$

62. Let

$$M(x, y) = f(x_0, y_0) + m_1(x - x_0) + m_2(y - y_0)$$

denote a linear approximation to $f(x, y)$ at (x_0, y_0), with

$$E_M(x, y) = f(x, y) - M(x, y)$$

the corresponding error term. Prove that

$$\lim_{(x,y) \to (x_0, y_0)} E_M(x, y) = 0$$

if and only if $f(x, y)$ is continuous at (x_0, y_0).

63. Suppose that $f(x, y)$ is differentiable at (x_0, y_0), with $E(x, y)$ the error function in Definition 14.4.2. Prove that $E(x, y)$ is differentiable at (x_0, y_0).

64. Suppose that $f(x, y)$ is differentiable at the point (x_0, y_0) and let $z_0 = f(x_0, y_0)$. Prove that the function $g(x, y, z) = z - f(x, y)$ is differentiable at (x_0, y_0, z_0).

65. Let

$$f(x, y) = \begin{cases} -1 & \text{if } x > 0 \text{ and } y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Explain why $f_y(x, 0)$ is undefined for $x > 0$ and $f_x(0, y)$ is undefined for $y > 0$.

66. Suppose that the error term $E(x, y) = f(x, y) - L(x, y)$ in Definition 14.4.2 can be written in the form

$$E(x, y) = \epsilon(x, y)\sqrt{(x - x_0)^2 + (y - y_0)^2}$$

where $\epsilon(x, y)$ is continuous at (x_0, y_0) with $\epsilon(x_0, y_0) = 0$. Prove that f is differentiable at (x_0, y_0).

14.5 THE CHAIN RULE

In this section we will derive versions of the chain rule for functions of two or three variables. These new versions will allow us to generate useful relationships among the derivatives and partial derivatives of various functions.

THE CHAIN RULE FOR DERIVATIVES

If y is a differentiable function of x and x is a differentiable function of t, then the chain rule for functions of one variable states that, under composition, y becomes a differentiable function of t with

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

We will now derive a version of the chain rule for functions of two variables.

Assume that $z = f(x, y)$ is a function of x and y, and suppose that x and y are in turn functions of a single variable t, say

$$x = x(t), \quad y = y(t)$$

The composition $z = f(x(t), y(t))$ then expresses z as a function of the single variable t. Thus, we can ask for the derivative dz/dt and we can inquire about its relationship to the derivatives $\partial z/\partial x$, $\partial z/\partial y$, dx/dt, and dy/dt.

14.5.1 THEOREM (***Two-Variable Chain Rule***). *If $x = x(t)$ and $y = y(t)$ are differentiable at t, and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(t), y(t))$, then $z = f(x(t), y(t))$ is differentiable at t and*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \tag{1}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y).

Proof. In order to use Equation (17) of Section 14.4, we will be a bit more specific and assume that $x = x(t)$ and $y = y(t)$ are differentiable at $t = t_0$ and that $z = f(x, y)$ is differentiable at the point $(x_0, y_0) = (x(t_0), y(t_0))$. We then must prove that $z = f(x(t), y(t))$ is differentiable at t_0 with

$$\frac{dz}{dt}(t_0) = \frac{\partial z}{\partial x}(x_0, y_0)\frac{dx}{dt}(t_0) + \frac{\partial z}{\partial y}(x_0, y_0)\frac{dy}{dt}(t_0) \tag{2}$$

To simplify the notation we will let

$$\begin{aligned}
\Delta x &= x(t) - x(t_0)\\
\Delta y &= y(t) - y(t_0)\\
\Delta z &= f(x(t), y(t)) - f(x(t_0), y(t_0))\\
\Delta t &= t - t_0
\end{aligned}$$

In terms of this notation

$$\frac{dz}{dt}(t_0) = \lim_{t\to t_0}\frac{\Delta z}{\Delta t}, \quad \frac{dx}{dt}(t_0) = \lim_{t\to t_0}\frac{\Delta x}{\Delta t}, \quad \text{and} \quad \frac{dy}{dt}(t_0) = \lim_{t\to t_0}\frac{\Delta y}{\Delta t}$$

Since $f(x, y)$ is differentiable at (x_0, y_0), it follows from Equation (17) of Section 14.4 that

$$\Delta z = \frac{\partial z}{\partial x}(x_0, y_0)\Delta x + \frac{\partial z}{\partial y}(x_0, y_0)\Delta y + \epsilon(x(t), y(t))\sqrt{(\Delta x)^2 + (\Delta y)^2} \tag{3}$$

where $\epsilon(x, y)$ is continuous at (x_0, y_0) with $\epsilon(x_0, y_0) = 0$. The functions $x(t)$ and $y(t)$ are differentiable at t_0, and thus also continuous at t_0, with

$$\lim_{t\to t_0} x(t) = x(t_0) = x_0 \quad \text{and} \quad \lim_{t\to t_0} y(t) = y(t_0) = y_0$$

Therefore,

$$\lim_{t\to t_0} \epsilon(x(t), y(t)) = \epsilon(x_0, y_0) = 0$$

by part (*c*) of Theorem 14.2.4. Since

$$\begin{aligned}
\lim_{t\to t_0}\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{|\Delta t|} &= \lim_{t\to t_0}\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} = \sqrt{\left(\lim_{t\to t_0}\frac{\Delta x}{\Delta t}\right)^2 + \left(\lim_{t\to t_0}\frac{\Delta y}{\Delta t}\right)^2}\\
&= \sqrt{\left(\frac{dx}{dt}(t_0)\right)^2 + \left(\frac{dy}{dt}(t_0)\right)^2}
\end{aligned}$$

we have

$$\begin{aligned}
\lim_{t\to t_0}\frac{\epsilon(x(t), y(t))\sqrt{(\Delta x)^2 + (\Delta y)^2}}{|\Delta t|} &= \lim_{t\to t_0}\epsilon(x(t), y(t))\lim_{t\to t_0}\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{|\Delta t|}\\
&= 0\sqrt{\left(\frac{dx}{dt}(t_0)\right)^2 + \left(\frac{dy}{dt}(t_0)\right)^2} = 0
\end{aligned}$$

Thus

$$\lim_{t\to t_0}\frac{\epsilon(x(t), y(t))\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta t} = 0$$

With the aid of Equation (3) we now have

$$\begin{aligned}
\frac{dz}{dt}(t_0) &= \lim_{t\to t_0}\frac{\Delta z}{\Delta t}\\
&= \lim_{t\to t_0}\left[\frac{\partial z}{\partial x}(x_0, y_0)\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}(x_0, y_0)\frac{\Delta y}{\Delta t} + \frac{\epsilon(x(t), y(t))\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta t}\right]\\
&= \frac{\partial z}{\partial x}(x_0, y_0)\frac{dx}{dt}(t_0) + \frac{\partial z}{\partial y}(x_0, y_0)\frac{dy}{dt}(t_0) + 0\\
&= \frac{\partial z}{\partial x}(x_0, y_0)\frac{dx}{dt}(t_0) + \frac{\partial z}{\partial y}(x_0, y_0)\frac{dy}{dt}(t_0)
\end{aligned}$$

which is Equation (2). ∎

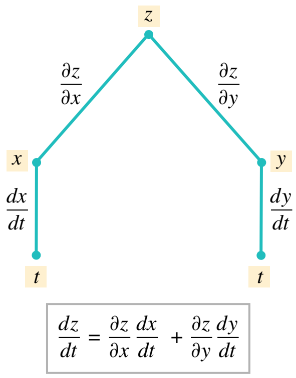


Figure 14.5.1

Formula (1) can be represented schematically by a "tree diagram" that is constructed as follows (Figure 14.5.1). Starting with z at the top of the tree and moving downward, join each variable by lines (or branches) to those variables on which it depends *directly*. Thus, z is joined to x and y and these in turn are joined to t. Next, label each branch with a derivative whose "numerator" contains the variable at the top end of that branch and whose "denominator" contains the variable at the bottom end of that branch. This completes the "tree." To find the formula for dz/dt, follow the two paths through the tree that start with z and end with t. Each such path corresponds to a term in Formula (1).

Example 1 Suppose that

$$z = x^2y, \quad x = t^2, \quad y = t^3$$

Use the chain rule to find dz/dt, and check the result by expressing z as a function of t and differentiating directly.

Solution. By the chain rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2xy)(2t) + (x^2)(3t^2)$$
$$= (2t^5)(2t) + (t^4)(3t^2) = 7t^6$$

Alternatively, we can express z directly as a function of t,

$$z = x^2y = (t^2)^2(t^3) = t^7$$

and then differentiate to obtain $dz/dt = 7t^6$. However, this procedure may not always be convenient. ◀

Example 2 Suppose that

$$z = \sqrt{xy + y}, \quad x = \cos\theta, \quad y = \sin\theta$$

Use the chain rule to find $dz/d\theta$ when $\theta = \pi/2$.

Solution. From the chain rule with θ in place of t,

$$\frac{dz}{d\theta} = \frac{\partial z}{\partial x}\frac{dx}{d\theta} + \frac{\partial z}{\partial y}\frac{dy}{d\theta}$$

we obtain

$$\frac{dz}{d\theta} = \frac{1}{2}(xy + y)^{-1/2}(y)(-\sin\theta) + \frac{1}{2}(xy + y)^{-1/2}(x + 1)(\cos\theta)$$

When $\theta = \pi/2$, we have

$$x = \cos\frac{\pi}{2} = 0, \quad y = \sin\frac{\pi}{2} = 1$$

Substituting $x = 0$, $y = 1$, $\theta = \pi/2$ in the formula for $dz/d\theta$ yields

$$\left.\frac{dz}{d\theta}\right|_{\theta=\pi/2} = \frac{1}{2}(1)(1)(-1) + \frac{1}{2}(1)(1)(0) = -\frac{1}{2}$$ ◀

REMARK. There are many variations in derivative notations, each of which gives the chain rule a different look. If $z = f(x, y)$, where x and y are functions of t, then some possibilities are

$$\frac{dz}{dt} = f_x\frac{dx}{dt} + f_y\frac{dy}{dt}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\frac{df}{dt} = f_x x'(t) + f_y y'(t)$$

Theorem 14.5.1 has a natural extension to functions $w = f(x, y, z)$ of three variables, which we state without proof.

14.5.2 THEOREM (*Three-Variable Chain Rule*). *If $x = x(t)$, $y = y(t)$, and $z = z(t)$ are differentiable at t, and if $w = f(x, y, z)$ is differentiable at the point $(x, y, z) = (x(t), y(t), z(t))$, then $w = f(x(t), y(t), z(t))$ is differentiable at t and*

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \tag{4}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y, z).

One of the principal uses of the chain rule for functions of a *single* variable was to compute formulas for the derivatives of compositions of functions. Theorems 14.5.1 and 14.5.2 are important not so much for the computation of formulas but because they allow us to express *relationships* among various derivatives. As illustrations, we revisit the topics of implicit differentiation and related rates problems.

IMPLICIT DIFFERENTIATION

Consider the special case where $z = f(x, y)$ is a function of x and y and y is a differentiable function of x. Equation (1) then becomes

$$\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \tag{5}$$

This result can be used to find derivatives of functions that are defined implicitly. For example, suppose that the equation

$$f(x, y) = c \tag{6}$$

defines y implicitly as a differentiable function of x and we are interested in finding dy/dx. Differentiating both sides of (6) with respect to x and applying (5) yields

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

Thus, if $\partial f/\partial y \neq 0$, we obtain

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

In summary, we have the following result.

14.5.3 THEOREM. *If the equation $f(x, y) = c$ defines y implicitly as a differentiable function of x, and if $\partial f/\partial y \neq 0$, then*

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} \tag{7}$$

Example 3 Given that

$$x^3 + y^2 x - 3 = 0$$

find dy/dx using (7), and check the result using implicit differentiation.

Solution. By (7) with $f(x, y) = x^3 + y^2 x - 3$,

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{3x^2 + y^2}{2yx}$$

Alternatively, differentiating the given equation implicitly yields

$$3x^2 + y^2 + x\left(2y\frac{dy}{dx}\right) - 0 = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{3x^2 + y^2}{2yx}$$

which agrees with the result obtained by (7). ◀

RELATED RATES PROBLEMS

Theorems 14.5.1 and 14.5.2 provide us with additional perspective on related rates problems such as those in Section 3.7.

Example 4 At what rate is the volume of a box changing if its length is 8 ft and increasing at 3 ft/s, its width is 6 ft and increasing at 2 ft/s, and its height is 4 ft and increasing at 1 ft/s?

Solution. Let x, y, and z denote the length, width, and height of the box, respectively, and let t denote time in seconds. We can interpret the given rates to mean that

$$\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 2, \quad \text{and} \quad \frac{dz}{dt} = 1 \tag{8}$$

at the instant when

$$x = 8, \quad y = 6, \quad \text{and} \quad z = 4 \tag{9}$$

We want to find dV/dt at that instant. For this purpose we use the volume formula $V = xyz$ to obtain

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt} = yz\frac{dx}{dt} + xz\frac{dy}{dt} + xy\frac{dz}{dt}$$

Substituting (8) and (9) into this equation yields

$$\frac{dV}{dt} = (6)(4)(3) + (8)(4)(2) + (8)(6)(1) = 184$$

Thus, the volume is increasing at a rate of 184 ft^3/s at the given instant. ◀

THE CHAIN RULE FOR PARTIAL DERIVATIVES

In Theorem 14.5.1 the variables x and y are each functions of a single variable t. We now consider the case where x and y are each functions of two variables. Let

$$z = f(x, y) \tag{10}$$

and suppose that x and y are functions of u and v, say

$$x = x(u, v), \quad y = y(u, v)$$

On substituting these functions of u and v into (10), we obtain the relationship

$$z = f(x(u, v), y(u, v))$$

which expresses z as a function of the two variables u and v. Thus, we can ask for the partial derivatives $\partial z/\partial u$ and $\partial z/\partial v$; and we can inquire about the relationship between these derivatives and the derivatives $\partial z/\partial x$, $\partial z/\partial y$, $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, and $\partial y/\partial v$.

14.5.4 THEOREM (*Two-Variable Chain Rule*). *If $x = x(u, v)$ and $y = y(u, v)$ have first-order partial derivatives at the point (u, v), and if $z = f(x, y)$ is differentiable at the point $(x(u, v), y(u, v))$, then $z = f(x(u, v), y(u, v))$ has first-order partial derivatives at (u, v) given by*

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Proof. If v is held fixed, then $x = x(u, v)$ and $y = y(u, v)$ become functions of u alone. Thus, we are back to the case of Theorem 14.5.1. If we apply that theorem with u in place

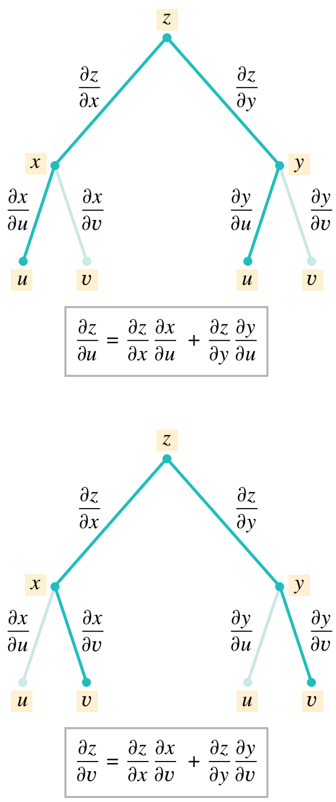


Figure 14.5.2

of t, and if we use ∂ rather than d to indicate that the variable v is fixed, we obtain

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

The formula for $\partial z/\partial v$ is derived similarly.

Figure 14.5.2 shows tree diagrams for the formulas in Theorem 14.5.4. The formula for $\partial z/\partial u$ can be obtained by tracing all paths through the tree that start with z and end with u, and the formula for $\partial z/\partial v$ can be obtained by tracing all paths through the tree that start with z and end with v.

Example 5 Given that

$$z = e^{xy}, \quad x = 2u + v, \quad y = u/v$$

find $\partial z/\partial u$ and $\partial z/\partial v$ using the chain rule.

Solution.

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (ye^{xy})(2) + (xe^{xy})\left(\frac{1}{v}\right) = \left[2y + \frac{x}{v}\right]e^{xy} \\ &= \left[\frac{2u}{v} + \frac{2u+v}{v}\right]e^{(2u+v)(u/v)} = \left[\frac{4u}{v} + 1\right]e^{(2u+v)(u/v)} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (ye^{xy})(1) + (xe^{xy})\left(-\frac{u}{v^2}\right) \\ &= \left[y - x\left(\frac{u}{v^2}\right)\right]e^{xy} = \left[\frac{u}{v} - (2u+v)\left(\frac{u}{v^2}\right)\right]e^{(2u+v)(u/v)} \\ &= -\frac{2u^2}{v^2}e^{(2u+v)(u/v)}\end{aligned}$$

Theorem 14.5.4 has a natural extension to functions $w = f(x, y, z)$ of three variables, which we state without proof.

14.5.5 THEOREM (*Three-Variable Chain Rule*). *If the functions $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ have first-order partial derivatives at the point (u, v), and if the function $w = f(x, y, z)$ is differentiable at the point $(x(u, v), y(u, v), z(u, v))$, then the function $w = f(x(u, v), y(u, v), z(u, v))$ has first-order partial derivatives at (u, v) given by*

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} \quad \text{and} \quad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

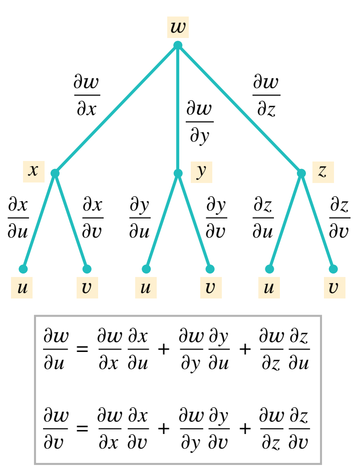


Figure 14.5.3

Example 6 Suppose that

$$w = e^{xyz}, \quad x = 3u + v, \quad y = 3u - v, \quad z = u^2v$$

Use appropriate forms of the chain rule to find $\partial w/\partial u$ and $\partial w/\partial v$.

Solution. From the tree diagram and corresponding formulas in Figure 14.5.3 we obtain

$$\frac{\partial w}{\partial u} = yze^{xyz}(3) + xze^{xyz}(3) + xye^{xyz}(2uv) = e^{xyz}(3yz + 3xz + 2xyuv)$$

and

$$\frac{\partial w}{\partial v} = yze^{xyz}(1) + xze^{xyz}(-1) + xye^{xyz}(u^2) = e^{xyz}(yz - xz + xyu^2)$$

If desired, we can express $\partial w/\partial u$ and $\partial w/\partial v$ in terms of u and v alone by replacing x, y, and z by their expressions in terms of u and v. ◀

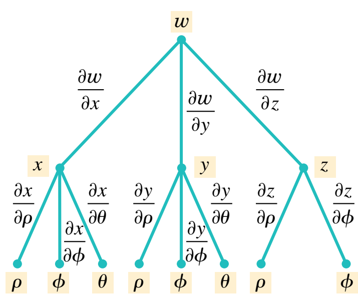
OTHER VERSIONS OF THE CHAIN RULE

Although we will not develop the underlying theory, the chain rule extends to functions $w = f(v_1, v_2, \ldots, v_n)$ of n variables. For example, if each v_i is a function of t, $i = 1, 2, \ldots, n$, the relevant formula is

$$\frac{dw}{dt} = \frac{\partial w}{\partial v_1}\frac{dv_1}{dt} + \frac{\partial w}{\partial v_2}\frac{dv_2}{dt} + \cdots + \frac{\partial w}{\partial v_n}\frac{dv_n}{dt} \tag{11}$$

Note that (11) is a natural extension of Formula (1) in Theorem 14.5.1 and Formula (4) in Theorem 14.5.2.

There are infinitely many variations of the chain rule, depending on the number of variables and the choice of independent and dependent variables. A good working procedure is to use tree diagrams to derive new versions of the chain rule as needed. This approach will give correct results for the functions that we will usually encounter.



$$\frac{\partial w}{\partial \rho} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \rho}$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta}$$

Figure 14.5.4

Example 7 Suppose that $w = x^2 + y^2 - z^2$ and

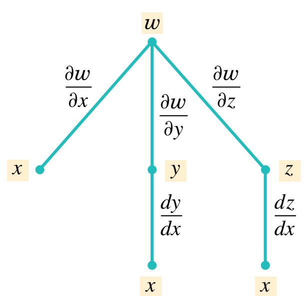
$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

Use appropriate forms of the chain rule to find $\partial w/\partial \rho$ and $\partial w/\partial \theta$.

Solution. From the tree diagram and corresponding formulas in Figure 14.5.4 we obtain

$$\begin{aligned}
\frac{\partial w}{\partial \rho} &= 2x \sin\phi\cos\theta + 2y\sin\phi\sin\theta - 2z\cos\phi \\
&= 2\rho \sin^2\phi\cos^2\theta + 2\rho\sin^2\phi\sin^2\theta - 2\rho\cos^2\phi \\
&= 2\rho\sin^2\phi(\cos^2\theta + \sin^2\theta) - 2\rho\cos^2\phi \\
&= 2\rho(\sin^2\phi - \cos^2\phi) \\
&= -2\rho\cos 2\phi \\
\frac{\partial w}{\partial \theta} &= (2x)(-\rho\sin\phi\sin\theta) + (2y)\rho\sin\phi\cos\theta \\
&= -2\rho^2\sin^2\phi\sin\theta\cos\theta + 2\rho^2\sin^2\phi\sin\theta\cos\theta \\
&= 0
\end{aligned}$$

This result is explained by the fact that w does not vary with θ. You can see this directly by expressing the variables x, y, and z in terms of ρ, ϕ, and θ in the formula for w. (Verify that $w = -\rho^2\cos 2\phi$.) ◀



$$\frac{dw}{dx} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\frac{dy}{dx} + \frac{\partial w}{\partial z}\frac{dz}{dx}$$

Figure 14.5.5

Example 8 Suppose that

$$w = xy + yz, \quad y = \sin x, \quad z = e^x$$

Use an appropriate form of the chain rule to find dw/dx.

Solution. From the tree diagram and corresponding formulas in Figure 14.5.5 we obtain

$$\begin{aligned}
\frac{dw}{dx} &= y + (x+z)\cos x + ye^x \\
&= \sin x + (x + e^x)\cos x + e^x \sin x
\end{aligned}$$

This result can also be obtained by first expressing w explicitly in terms of x as

$$w = x\sin x + e^x \sin x$$

and then differentiating with respect to x; however, such direct substitution is not always possible. ◀

WARNING. The symbol ∂z, unlike the differential dz, has no meaning of its own. For example, if we were to "cancel" partial symbols in the chain-rule formula

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

we would obtain

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial u}$$

which is false in cases where $\partial z/\partial u \neq 0$.

In each of the expressions

$$z = \sin xy, \quad z = \frac{xy}{1+xy}, \quad z = e^{xy}$$

the independent variables occur only in the combination xy, so the substitution $t = xy$ reduces the expression to a function of one variable:

$$z = \sin t, \quad z = \frac{t}{1+t}, \quad z = e^t$$

Conversely, if we begin with a function of one variable $z = f(t)$ and substitute $t = xy$, we obtain a function $z = f(xy)$ in which the variables appear only in the combination xy. Functions whose variables occur in fixed combinations arise frequently in applications.

Example 9 Show that when f is differentiable, a function of the form $z = f(xy)$ satisfies the equation

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 0$$

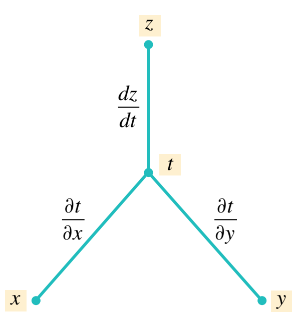


Figure 14.5.6

Solution. Let $t = xy$, so that $z = f(t)$. From the tree diagram in Figure 14.5.6 we obtain the formulas

$$\frac{\partial z}{\partial x} = \frac{dz}{dt}\frac{\partial t}{\partial x} = y\frac{dz}{dt} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{dz}{dt}\frac{\partial t}{\partial y} = x\frac{dz}{dt}$$

from which it follows that

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = xy\frac{dz}{dt} - yx\frac{dz}{dt} = 0$$ ◀

EXERCISE SET 14.5

In Exercises 1–6, use an appropriate form of the chain rule to find dz/dt.

1. $z = 3x^2y^3$; $x = t^4$, $y = t^2$
2. $z = \ln(2x^2 + y)$; $x = \sqrt{t}$, $y = t^{2/3}$
3. $z = 3\cos x - \sin xy$; $x = 1/t$, $y = 3t$
4. $z = \sqrt{1 + x - 2xy^4}$; $x = \ln t$, $y = t$
5. $z = e^{1-xy}$; $x = t^{1/3}$, $y = t^3$
6. $z = \cosh^2 xy$; $x = t/2$, $y = e^t$

In Exercises 7–10, use an appropriate form of the chain rule to find dw/dt.

7. $w = 5x^2y^3z^4$; $x = t^2$, $y = t^3$, $z = t^5$
8. $w = \ln(3x^2 - 2y + 4z^3)$; $x = t^{1/2}$, $y = t^{2/3}$, $z = t^{-2}$
9. $w = 5\cos xy - \sin xz$; $x = 1/t$, $y = t$, $z = t^3$
10. $w = \sqrt{1 + x - 2yz^4x}$; $x = \ln t$, $y = t$, $z = 4t$
11. Suppose that
$$w = x^3y^2z^4; \; x = t^2, \; y = t + 2, \; z = 2t^4$$
Find the rate of change of w with respect to t at $t = 1$ by using the chain rule, and then check your work by expressing w as a function of t and differentiating.
12. Suppose that
$$w = x \sin yz^2; \; x = \cos t, \; y = t^2, \; z = e^t$$

Find the rate of change of w with respect to t at $t = 0$ by using the chain rule, and then check your work by expressing w as a function of t and differentiating.

In Exercises 13–18, use appropriate forms of the chain rule to find $\partial z/\partial u$ and $\partial z/\partial v$.

13. $z = 8x^2y - 2x + 3y;\ x = uv,\ y = u - v$

14. $z = x^2 - y\tan x;\ x = u/v,\ y = u^2v^2$

15. $z = x/y;\ x = 2\cos u,\ y = 3\sin v$

16. $z = 3x - 2y;\ x = u + v\ln u,\ y = u^2 - v\ln v$

17. $z = e^{x^2y};\ x = \sqrt{uv},\ y = 1/v$

18. $z = \cos x \sin y;\ x = u - v,\ y = u^2 + v^2$

In Exercises 19–26, use appropriate forms of the chain rule to find the derivatives.

19. Let $T = x^2y - xy^3 + 2;\ x = r\cos\theta,\ y = r\sin\theta$. Find $\partial T/\partial r$ and $\partial T/\partial\theta$.

20. Let $R = e^{2s-t^2};\ s = 3\phi,\ t = \phi^{1/2}$. Find $dR/d\phi$.

21. Let $t = u/v;\ u = x^2 - y^2,\ v = 4xy^3$. Find $\partial t/\partial x$ and $\partial t/\partial y$.

22. Let $w = rs/(r^2 + s^2);\ r = uv,\ s = u - 2v$. Find $\partial w/\partial u$ and $\partial w/\partial v$.

23. Let $z = \ln(x^2 + 1)$, where $x = r\cos\theta$. Find $\partial z/\partial r$ and $\partial z/\partial\theta$.

24. Let $u = rs^2\ln t,\ r = x^2,\ s = 4y + 1,\ t = xy^3$. Find $\partial u/\partial x$ and $\partial u/\partial y$.

25. Let $w = 4x^2 + 4y^2 + z^2,\ x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$. Find $\partial w/\partial\rho$, $\partial w/\partial\phi$, and $\partial w/\partial\theta$.

26. Let $w = 3xy^2z^3,\ y = 3x^2 + 2,\ z = \sqrt{x - 1}$. Find dw/dx.

27. Use a chain rule to find the value of $\left.\dfrac{dw}{ds}\right|_{s=1/4}$ if $w = r^2 - r\tan\theta;\ r = \sqrt{s},\ \theta = \pi s$.

28. Use a chain rule to find the values of

$$\left.\frac{\partial f}{\partial u}\right|_{u=1,v=-2} \quad\text{and}\quad \left.\frac{\partial f}{\partial v}\right|_{u=1,v=-2}$$

if $f(x, y) = x^2y^2 - x + 2y;\ x = \sqrt{u},\ y = uv^3$.

29. Use a chain rule to find the values of

$$\left.\frac{\partial z}{\partial r}\right|_{r=2,\theta=\pi/6} \quad\text{and}\quad \left.\frac{\partial z}{\partial\theta}\right|_{r=2,\theta=\pi/6}$$

if $z = xye^{x/y};\ x = r\cos\theta,\ y = r\sin\theta$.

30. Use a chain rule to find $\left.\dfrac{dz}{dt}\right|_{t=3}$ if $z = x^2y;\ x = t^2,\ y = t + 7$.

In Exercises 31–34, use Theorem 14.5.3 to find dy/dx and check your result using implicit differentiation.

31. $x^2y^3 + \cos y = 0$

32. $x^3 - 3xy^2 + y^3 = 5$

33. $e^{xy} + ye^y = 1$

34. $x - \sqrt{xy} + 3y = 4$

35. Assume that $F(x, y, z) = 0$ defines z implicitly as a function of x and y. Show that if $\partial F/\partial z \neq 0$, then

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}$$

36. Assume that $F(x, y, z) = 0$ defines z implicitly as a function of x and y. Show that if $\partial F/\partial z \neq 0$, then

$$\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}$$

In Exercises 37–40, find $\partial z/\partial x$ and $\partial z/\partial y$ by implicit differentiation, and confirm that the results obtained agree with those predicted by the formulas in Exercises 35 and 36.

37. $x^2 - 3yz^2 + xyz - 2 = 0$

38. $\ln(1 + z) + xy^2 + z = 1$

39. $ye^x - 5\sin 3z = 3z$

40. $e^{xy}\cos yz - e^{yz}\sin xz + 2 = 0$

41. Two straight roads intersect at right angles. Car A, moving on one of the roads, approaches the intersection at 25 mi/h and car B, moving on the other road, approaches the intersection at 30 mi/h. At what rate is the distance between the cars changing when A is 0.3 mile from the intersection and B is 0.4 mile from the intersection?

42. Use the ideal gas law $P = kT/V$ with V in cubic inches (in^3), T in kelvins (K), and $k = 10$ in·lb/K to find the rate at which the temperature of a gas is changing when the volume is 200 in^3 and increasing at the rate of 4 in^3/s, while the pressure is 5 lb/in^2 and decreasing at the rate of 1 lb/in^2/s.

43. Two sides of a triangle have lengths $a = 4$ cm and $b = 3$ cm, but are increasing at the rate of 1 cm/s. If the area of the triangle remains constant, at what rate is the angle θ between a and b changing when $\theta = \pi/6$?

44. Two sides of a triangle have lengths $a = 5$ cm and $b = 10$ cm, and the included angle is $\theta = \pi/3$. If a is increasing at a rate of 2 cm/s, b is increasing at a rate of 1 cm/s, and θ remains constant, at what rate is the third side changing? Is it increasing or decreasing? [*Hint:* Use the law of cosines.]

45. Suppose that the portion of a tree that is usable for lumber is a right circular cylinder. If the usable height of a tree increases 2 ft per year and the usable diameter increases 3 in per year, how fast is the volume of usable lumber increasing when the usable height of the tree is 20 ft and the usable diameter is 30 in?

46. Suppose that a particle moving along a metal plate in the xy-plane has velocity $\mathbf{v} = \mathbf{i} - 4\mathbf{j}$ (cm/s) at the point (3, 2). Given that the temperature of the plate at points in the xy-plane is $T(x, y) = y^2\ln x,\ x \geq 1$, in degrees Celsius, find dT/dt at the point (3, 2).

47. The length, width, and height of a rectangular box are increasing at rates of 1 in/s, 2 in/s, and 3 in/s, respectively.

(a) At what rate is the volume increasing when the length is 2 in, the width is 3 in, and the height is 6 in?

(b) At what rate is the length of the diagonal increasing at that instant?

48. Consider the box in Exercise 47. At what rate is the surface area of the box increasing at the given instant?

A function $f(x, y)$ is said to be ***homogeneous of degree n*** if $f(tx, ty) = t^n f(x, y)$ for $t > 0$. This terminology is needed in Exercises 49 and 50.

49. In each part, show that the function is homogeneous, and find its degree.

(a) $f(x, y) = 3x^2 + y^2$ (b) $f(x, y) = \sqrt{x^2 + y^2}$

(c) $f(x, y) = x^2y - 2y^3$ (d) $f(x, y) = \dfrac{5}{(x^2 + 2y^2)^2}$

50. (a) Show that if $f(x, y)$ is a homogeneous function of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

[*Hint:* Let $u = tx$ and $v = ty$ in $f(tx, ty)$, and differentiate both sides of $f(u, v) = t^n f(x, y)$ with respect to t.]

(b) Confirm that the functions in Exercise 49 satisfy the equation in part (a).

51. (a) Suppose that $z = f(u)$ and $u = g(x, y)$. Draw a tree diagram, and use it to construct chain rules that express $\partial z/\partial x$ and $\partial z/\partial y$ in terms of dz/du, $\partial u/\partial x$, and $\partial u/\partial y$.

(b) Show that

$$\frac{\partial^2 z}{\partial x^2} = \frac{dz}{du}\frac{\partial^2 u}{\partial x^2} + \frac{d^2 z}{du^2}\left(\frac{\partial u}{\partial x}\right)^2$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{dz}{du}\frac{\partial^2 u}{\partial y^2} + \frac{d^2 z}{du^2}\left(\frac{\partial u}{\partial y}\right)^2$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{dz}{du}\frac{\partial^2 u}{\partial y \partial x} + \frac{d^2 z}{du^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$$

52. (a) Let $z = f(x^2 - y^2)$. Use the result in Exercise 51(a) to show that

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$$

(b) Let $z = f(xy)$. Use the result in Exercise 51(a) to show that

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 0$$

(c) Confirm the result in part (a) in the case where $z = \sin(x^2 - y^2)$.

(d) Confirm the result in part (b) in the case where $z = e^{xy}$.

53. Let f be a differentiable function of one variable, and let $z = f(x + 2y)$. Show that

$$2\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$$

54. Let f be a differentiable function of one variable, and let $z = f(x^2 + y^2)$. Show that

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$$

55. Let f be a differentiable function of one variable, and let $w = f(u)$, where $u = x + 2y + 3z$. Show that

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 6\frac{dw}{du}$$

56. Let f be a differentiable function of one variable, and let $w = f(\rho)$, where $\rho = (x^2 + y^2 + z^2)^{1/2}$. Show that

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 = \left(\frac{dw}{d\rho}\right)^2$$

57. Let $z = f(x - y, y - x)$. Show that $\partial z/\partial x + \partial z/\partial y = 0$.

58. Let f be a differentiable function of three variables and suppose that $w = f(x - y, y - z, z - x)$. Show that

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 0$$

59. Suppose that the equation $z = f(x, y)$ is expressed in the polar form $z = g(r, \theta)$ by making the substitution $x = r\cos\theta$ and $y = r\sin\theta$.

(a) View r and θ as functions of x and y and use implicit differentiation to show that

$$\frac{\partial r}{\partial x} = \cos\theta \quad \text{and} \quad \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$$

(b) View r and θ as functions of x and y and use implicit differentiation to show that

$$\frac{\partial r}{\partial y} = \sin\theta \quad \text{and} \quad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

(c) Use the results in parts (a) and (b) to show that

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial z}{\partial \theta}\sin\theta$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial z}{\partial \theta}\cos\theta$$

(d) Use the result in part (c) to show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

(e) Use the result in part (c) to show that if $z = f(x, y)$ satisfies Laplace's equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

then $z = g(r, \theta)$ satisfies the equation

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r} = 0$$

and conversely. The latter equation is called the ***polar form of Laplace's equation***.

60. Show that the function

$$z = \tan^{-1}\frac{2xy}{x^2 - y^2}$$

satisfies Laplace's equation; then make the substitution $x = r\cos\theta$, $y = r\sin\theta$, and show that the resulting function of r and θ satisfies the polar form of Laplace's equation given in part (e) of Exercise 59.

61. (a) Show that if $u(x, y)$ and $v(x, y)$ satisfy the Cauchy–Riemann equations (Exercise 88, Section 14.3), and if $x = r\cos\theta$ and $y = r\sin\theta$, then

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

This is called the ***polar form of the Cauchy–Riemann equations***.

(b) Show that the functions

$$u = \ln(x^2 + y^2), \quad v = 2\tan^{-1}(y/x)$$

satisfy the Cauchy–Riemann equations; then make the substitution $x = r\cos\theta$, $y = r\sin\theta$, and show that the resulting functions of r and θ satisfy the polar form of the Cauchy–Riemann equations.

62. Recall from Formula (6) of Section 14.3 that under appropriate conditions a plucked string satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

where c is a positive constant.

(a) Show that a function of the form $u(x, t) = f(x + ct)$ satisfies the wave equation.

(b) Show that a function of the form $u(x, t) = g(x - ct)$ satisfies the wave equation.

(c) Show that a function of the form

$$u(x, t) = f(x + ct) + g(x - ct)$$

satisfies the wave equation.

(d) It can be proved that every solution of the wave equation is expressible in the form stated in part (c). Confirm that $u(x, t) = \sin t \sin x$ satisfies the wave equation in which $c = 1$, and then use appropriate trigonometric identities to express this function in the form $f(x+t)+g(x-t)$.

63. Let f be a differentiable function of three variables, and let $w = f(x, y, z)$, $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$. Express $\partial w/\partial\rho$, $\partial w/\partial\phi$, and $\partial w/\partial\theta$ in terms of $\partial w/\partial x$, $\partial w/\partial y$, and $\partial w/\partial z$.

64. Let $w = f(x, y, z)$, where $z = g(x, y)$. Taking x and y as the independent variables, express each of the following in terms of $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$, $\partial z/\partial x$, and $\partial z/\partial y$.

(a) $\partial w/\partial x$ (b) $\partial w/\partial y$

65. Let $w = \ln(e^r + e^s + e^t + e^u)$. Show that

$$w_{rstu} = -6e^{r+s+t+u-4w}$$

[*Hint:* Take advantage of the relationship $e^w = e^r + e^s + e^t + e^u$.]

66. Suppose that w is a differentiable function of x_1, x_2, and x_3, and

$$x_1 = a_1y_1 + b_1y_2$$
$$x_2 = a_2y_1 + b_2y_2$$
$$x_3 = a_3y_1 + b_3y_2$$

where the a's and b's are constants. Express $\partial w/\partial y_1$ and $\partial w/\partial y_2$ in terms of $\partial w/\partial x_1$, $\partial w/\partial x_2$, and $\partial w/\partial x_3$.

67. (a) Let w be a differentiable function of x_1, x_2, x_3, and x_4, and let each x_i be a function of t. Find a chain-rule formula for dw/dt.

(b) Let w be a differentiable function of x_1, x_2, x_3, and x_4, and let each x_i be a differentiable function of v_1, v_2, and v_3. Find chain-rule formulas for $\partial w/\partial v_1$, $\partial w/\partial v_2$, and $\partial w/\partial v_3$.

68. Let $w = (x_1^2 + x_2^2 + \cdots + x_n^2)^k$, where $n > 2$. For what values of k does

$$\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} + \cdots + \frac{\partial^2 w}{\partial x_n^2} = 0$$

hold?

69. We showed in Exercise 24 of Section 7.5 that

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f(g(x))g'(x) - f(h(x))h'(x)$$

Derive this same result by letting $u = g(x)$ and $v = h(x)$ and then differentiating the function

$$F(u, v) = \int_v^u f(t)\,dt$$

with respect to x.

70. Prove: If f, f_x, and f_y are continuous on a circular region containing $A(x_0, y_0)$ and $B(x_1, y_1)$, then there is a point (x^*, y^*) on the line segment joining A and B such that

$$f(x_1, y_1) - f(x_0, y_0) = f_x(x^*, y^*)(x_1 - x_0) + f_y(x^*, y^*)(y_1 - y_0)$$

This result is the two-dimensional version of the Mean-Value Theorem. [*Hint:* Express the line segment joining A and B in parametric form and use the Mean-Value Theorem for functions of one variable.]

71. Prove: If $f_x(x, y) = 0$ and $f_y(x, y) = 0$ throughout a circular region, then $f(x, y)$ is constant on that region. [*Hint:* Use the result of Exercise 70.]

14.6 DIRECTIONAL DERIVATIVES AND GRADIENTS

The partial derivatives $f_x(x, y)$ and $f_y(x, y)$ represent the rates of change of $f(x, y)$ in directions parallel to the x- and y-axes. In this section we will investigate rates of change of $f(x, y)$ in other directions.

DIRECTIONAL DERIVATIVES

Now that we have defined *differentiability* for a function of two or three variables, it is natural to ask what is meant by the *derivative* of such a function. Before we answer this question, it will be helpful to extend the concept of a *partial* derivative to the more general notion of a *directional* derivative. We have seen that the partial derivatives of a function give the instantaneous rates of change of that function in directions parallel to the coordinate axes. Directional derivatives allow us to compute the rates of change of a function with respect to distance in *any* direction.

Suppose that we wish to compute the instantaneous rate of change of a function $f(x, y)$ with respect to distance from a point (x_0, y_0) in some direction. Since there are infinitely many different directions from (x_0, y_0) in which we could move, we need a convenient method for describing a specific direction starting at (x_0, y_0). One way to do this is to use a unit vector

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$$

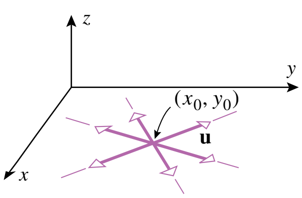


Figure 14.6.1

that has its initial point at (x_0, y_0) and points in the desired direction (Figure 14.6.1). This vector determines a line ℓ in the xy-plane that can be expressed parametrically as

$$x = x_0 + su_1, \quad y = y_0 + su_2 \tag{1}$$

where s is the arc length parameter that has its reference point at (x_0, y_0) and has positive values in the direction of $\mathbf{u}$. For $s = 0$, the point (x, y) is at the reference point (x_0, y_0), and as s increases, the point (x, y) moves along ℓ in the direction of $\mathbf{u}$. On the line ℓ the variable $z = f(x_0+su_1, y_0+su_2)$ is a function of the parameter s. The value of the derivative dz/ds at $s = 0$ then gives an instantaneous rate of change of $f(x, y)$ with respect to distance from (x_0, y_0) in the direction of $\mathbf{u}$.

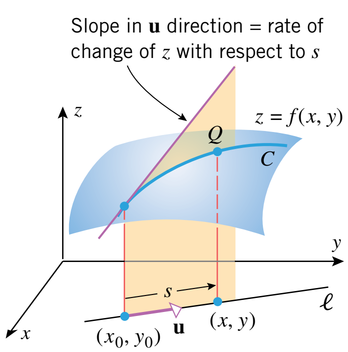


Figure 14.6.2

14.6.1 DEFINITION. If $f(x, y)$ is a function of x and y, and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector, then the ***directional derivative of f in the direction of* u** at (x_0, y_0) is denoted by $D_{\mathbf{u}}f(x_0, y_0)$ and is defined by

$$D_{\mathbf{u}}f(x_0, y_0) = \frac{d}{ds}[f(x_0 + su_1, y_0 + su_2)]_{s=0} \tag{2}$$

provided this derivative exists.

Geometrically, $D_{\mathbf{u}}f(x_0, y_0)$ can be interpreted as the ***slope of the surface z = f(x, y) in the direction of* u** at the point $(x_0, y_0, f(x_0, y_0))$ (Figure 14.6.2). Usually the value of $D_{\mathbf{u}}f(x_0, y_0)$ will depend on both the point (x_0, y_0) and the direction $\mathbf{u}$. Thus, at a fixed point the slope of the surface may vary with the direction (Figure 14.6.3). Analytically, the directional derivative represents the ***instantaneous rate of change of f(x, y) with respect to distance in the direction of* u** at the point (x_0, y_0).

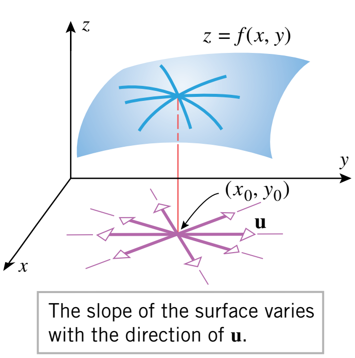


Figure 14.6.3

Example 1 Let $f(x, y) = xy$ and find $D_{\mathbf{u}}f(1, 2)$, where $\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$.

Solution. It follows from Equation (2) that

$$D_{\mathbf{u}}f(1, 2) = \frac{d}{ds}\left[f\left(1 + \frac{\sqrt{3}s}{2}, 2 + \frac{s}{2}\right)\right]_{s=0}$$

Since

$$f\left(1+\frac{\sqrt{3}s}{2}, 2+\frac{s}{2}\right) = \left(1+\frac{\sqrt{3}s}{2}\right)\left(2+\frac{s}{2}\right) = \frac{\sqrt{3}}{4}s^2 + \left(\frac{1}{2}+\sqrt{3}\right)s + 2$$

we have

$$\frac{d}{ds}\left[f\left(1+\frac{\sqrt{3}s}{2}, 2+\frac{s}{2}\right)\right] = \frac{\sqrt{3}}{2}s + \frac{1}{2} + \sqrt{3}$$

and thus

$$D_{\mathbf{u}}f(1,2) = \frac{d}{ds}\left[f\left(1+\frac{\sqrt{3}s}{2}, 2+\frac{s}{2}\right)\right]_{s=0} = \frac{1}{2} + \sqrt{3}$$

Since $\frac{1}{2}+\sqrt{3} \approx 2.23$, we conclude that if we move a small distance from the point $(1, 2)$ in the direction of $\mathbf{u}$, the function $f(x, y) = xy$ will increase by about 2.23 times the distance moved. ◀

The definition of a directional derivative for a function $f(x, y, z)$ of three variables is similar to Definition 14.6.1.

14.6.2 DEFINITION. If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ is a unit vector, and if $f(x, y, z)$ is a function of x, y, and z, then the ***directional derivative of f in the direction of*** $\mathbf{u}$ at (x_0, y_0, z_0) is denoted by $D_{\mathbf{u}}f(x_0, y_0, z_0)$ and is defined by

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = \frac{d}{ds}[f(x_0+su_1, y_0+su_2, z_0+su_3)]_{s=0} \tag{3}$$

provided this derivative exists.

Although Equation (3) does not have a convenient geometric interpretation, we can still interpret directional derivatives for functions of three variables in terms of instantaneous rates of change in a specified direction.

For a function that is differentiable at a point, directional derivatives exist in every direction from the point and can be computed directly in terms of the first-order partial derivatives of the function.

14.6.3 THEOREM.

(*a*) *If $f(x, y)$ is differentiable at (x_0, y_0), and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector, then the directional derivative $D_{\mathbf{u}}f(x_0, y_0)$ exists and is given by*

$$D_{\mathbf{u}}f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 \tag{4}$$

(*b*) *If $f(x, y, z)$ is differentiable at (x_0, y_0, z_0), and if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ is a unit vector, then the directional derivative $D_{\mathbf{u}}f(x_0, y_0, z_0)$ exists and is given by*

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)u_1 + f_y(x_0, y_0, z_0)u_2 + f_z(x_0, y_0, z_0)u_3 \tag{5}$$

Proof. We will give the proof of part (*a*); the proof of part (*b*) is similar and will be omitted. The function $z = f(x_0 + su_1, y_0 + su_2)$ is the composition of the function $z = f(x, y)$ with the functions

$$x = x(s) = x_0 + su_1 \quad \text{and} \quad y = y(s) = y_0 + su_2$$

As such, the chain rule in Theorem 14.5.1 immediately gives

$$\begin{aligned} D_{\mathbf{u}} f(x_0, y_0) &= \frac{d}{ds}[f(x_0 + su_1, y_0 + su_2)]_{s=0} \\ &= \frac{dz}{ds}(0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 \end{aligned}$$

REMARK. For conciseness, we used the chain rule in the proof of Theorem 14.6.3. However, a more "elementary" proof follows directly from Definitions 14.4.1 and 14.4.2 (Exercises 86 and 87).

We can use Theorem 14.6.3 to confirm the result of Example 1. For $f(x, y) = xy$ we have $f_x(1, 2) = 2$ and $f_y(1, 2) = 1$ (verify). With

$$\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$$

Equation (4) becomes

$$D_{\mathbf{u}} f(1, 2) = 2\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2} = \sqrt{3} + \frac{1}{2}$$

which agrees with our solution in Example 1.

Recall from Formula (13) of Section 12.2 that a unit vector $\mathbf{u}$ in the xy-plane can be expressed as

$$\mathbf{u} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j} \tag{6}$$

where ϕ is the angle from the positive x-axis to $\mathbf{u}$. Thus, Formula (4) can also be expressed as

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)\cos\phi + f_y(x_0, y_0)\sin\phi \tag{7}$$

Example 2 Find the directional derivative of $f(x, y) = e^{xy}$ at $(-2, 0)$ in the direction of the unit vector that makes an angle of $\pi/3$ with the positive x-axis.

Solution. The partial derivatives of f are

$$\begin{aligned} &f_x(x, y) = ye^{xy}, \quad &&f_y(x, y) = xe^{xy} \\ &f_x(-2, 0) = 0, &&f_y(-2, 0) = -2 \end{aligned}$$

The unit vector $\mathbf{u}$ that makes an angle of $\pi/3$ with the positive x-axis is

$$\mathbf{u} = \cos(\pi/3)\mathbf{i} + \sin(\pi/3)\mathbf{j} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

Thus, from (7)

$$\begin{aligned} D_{\mathbf{u}} f(-2, 0) &= f_x(-2, 0)\cos(\pi/3) + f_y(-2, 0)\sin(\pi/3) \\ &= 0(1/2) + (-2)(\sqrt{3}/2) = -\sqrt{3} \end{aligned}$$

◀

It is important that the direction of a directional derivative be specified by a *unit vector* when applying either Equation (4) or Equation (5).

Example 3 Find the directional derivative of $f(x, y, z) = x^2y - yz^3 + z$ at the point $(1, -2, 0)$ in the direction of the vector $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Solution. The partial derivatives of f are

$$\begin{aligned} &f_x(x, y, z) = 2xy, \quad &&f_y(x, y, z) = x^2 - z^3, \quad &&f_z(x, y, z) = -3yz^2 + 1 \\ &f_x(1, -2, 0) = -4, &&f_y(1, -2, 0) = 1, &&f_z(1, -2, 0) = 1 \end{aligned}$$

Since **a** is not a unit vector, we normalize it, getting

$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{\sqrt{9}}(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

Formula (5) then yields

$$D_{\mathbf{u}} f(1, -2, 0) = (-4)\left(\frac{2}{3}\right) + \frac{1}{3} - \frac{2}{3} = -3$$ ◀

THE GRADIENT

Formula (4) can be expressed in the form of a dot product as

$$\begin{aligned} D_{\mathbf{u}} f(x_0, y_0) &= (f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j}) \\ &= (f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}) \cdot \mathbf{u} \end{aligned}$$

Similarly, Formula (5) can be expressed as

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = (f_x(x_0, y_0, z_0)\mathbf{i} + f_y(x_0, y_0, z_0)\mathbf{j} + f_z(x_0, y_0, z_0)\mathbf{k}) \cdot \mathbf{u}$$

In both cases the directional derivative is obtained by dotting the direction vector **u** with a new vector constructed from the first-order partial derivatives of f.

14.6.4 DEFINITION.

(a) If f is a function of x and y, then the ***gradient of f*** is defined by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} \tag{8}$$

(b) If f is a function of x, y, and z, then the ***gradient of f*** is defined by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} \tag{9}$$

The symbol ∇ (read "del") is an inverted delta. (It is sometimes called a "nabla" because of its similarity in form to an ancient Hebrew ten-stringed harp of that name.)

Formulas (4) and (5) can now be written as

$$D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u} \tag{10}$$

and

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \mathbf{u} \tag{11}$$

respectively. For example, using Formula (11) our solution to Example 3 would take the form

$$\begin{aligned} D_{\mathbf{u}} f(1, -2, 0) &= \nabla f(1, -2, 0) \cdot \mathbf{u} = (-4\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot \left(\tfrac{2}{3}\mathbf{i} + \tfrac{1}{3}\mathbf{j} - \tfrac{2}{3}\mathbf{k}\right) \\ &= (-4)\left(\tfrac{2}{3}\right) + \tfrac{1}{3} - \tfrac{2}{3} = -3 \end{aligned}$$

Formula (10) can be interpreted to mean that the slope of the surface $z = f(x, y)$ at the point (x_0, y_0) in the direction of **u** is the dot product of the gradient with **u** (Figure 14.6.4).

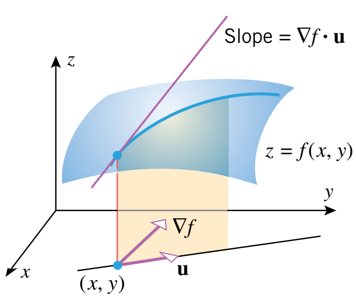


Figure 14.6.4

REMARK. It is important to keep in mind that ∇f is not the product of ∇ and f. The symbol ∇ does not have a value in and of itself; rather, you should think of it as an operator that acts on the function f to produce the gradient ∇f in the same sense that d/dx is an operator that acts on a function f of a single variable to produce its derivative f'.

PROPERTIES OF THE GRADIENT

The gradient is not merely a notational device to simplify the formula for the directional derivative: we will see that the length and direction of the gradient ∇f provide important information about the function f and the surface $z = f(x, y)$. For example, suppose that $\nabla f(x, y) \neq \mathbf{0}$, and let us use Formula (4) of Section 12.3 to rewrite (10) as

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = \|\nabla f(x, y)\|\|\mathbf{u}\| \cos\theta = \|\nabla f(x, y)\| \cos\theta \tag{12}$$

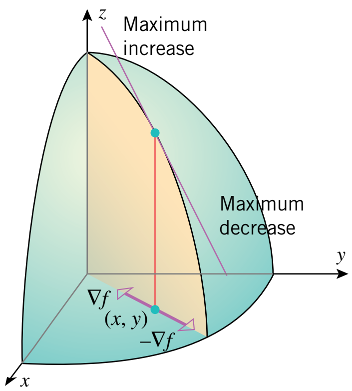


Figure 14.6.5

where θ is the angle between $\nabla f(x, y)$ and $\mathbf{u}$. This equation tells us that the maximum value of $D_{\mathbf{u}} f(x, y)$ is $\|\nabla f(x, y)\|$, and this maximum occurs when $\theta = 0$, that is, when $\mathbf{u}$ is in the direction of $\nabla f(x, y)$. Geometrically, this means that *the surface $z = f(x, y)$ has its maximum slope at a point (x, y) in the direction of the gradient, and the maximum slope is* $\|\nabla f(x, y)\|$ (Figure 14.6.5). Similarly, (12) tells us that the minimum value of $D_{\mathbf{u}} f(x, y)$ is $-\|\nabla f(x, y)\|$, and this minimum occurs when $\theta = \pi$, that is, when $\mathbf{u}$ is oppositely directed to $\nabla f(x, y)$. Geometrically, this means that *the surface $z = f(x, y)$ has its minimum slope at a point (x, y) in the direction that is opposite to the gradient, and the minimum slope is* $-\|\nabla f(x, y)\|$ (Figure 14.6.5).

Finally, in the case where $\nabla f(x, y) = \mathbf{0}$, it follows from (12) that $D_{\mathbf{u}} f(x, y) = 0$ in all directions at the point (x, y). This typically occurs where the surface $z = f(x, y)$ has a "relative maximum," a "relative minimum," or a saddle point.

A similar analysis applies to functions of three variables. As a consequence, we have the following result.

14.6.5 THEOREM. *Let f be a function of either two variables or three variables, and let P denote the point $P(x_0, y_0)$ or $P(x_0, y_0, z_0)$, respectively. Assume that f is differentiable at P.*

(*a*) *If $\nabla f = \mathbf{0}$ at P, then all directional derivatives of f at P are zero.*

(*b*) *If $\nabla f \neq \mathbf{0}$ at P, then among all possible directional derivatives of f at P, the derivative in the direction of ∇f at P has the largest value. The value of this largest directional derivative is $\|\nabla f\|$ at P.*

(*c*) *If $\nabla f \neq \mathbf{0}$ at P, then among all possible directional derivatives of f at P, the derivative in the direction opposite to that of ∇f at P has the smallest value. The value of this smallest directional derivative is $-\|\nabla f\|$ at P.*

Example 4 Let $f(x, y) = x^2 e^y$. Find the maximum value of a directional derivative at $(-2, 0)$, and find the unit vector in the direction in which the maximum value occurs.

Solution. Since

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = 2xe^y\mathbf{i} + x^2e^y\mathbf{j}$$

the gradient of f at $(-2, 0)$ is

$$\nabla f(-2, 0) = -4\mathbf{i} + 4\mathbf{j}$$

By Theorem 14.6.5, the maximum value of the directional derivative is

$$\|\nabla f(-2, 0)\| = \sqrt{(-4)^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$$

This maximum occurs in the direction of $\nabla f(-2, 0)$. The unit vector in this direction is

$$\mathbf{u} = \frac{\nabla f(-2, 0)}{\|\nabla f(-2, 0)\|} = \frac{1}{4\sqrt{2}}(-4\mathbf{i} + 4\mathbf{j}) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \qquad \blacktriangleleft$$

GRADIENTS ARE NORMAL TO LEVEL CURVES

We have seen that the gradient points in the direction in which a function increases most rapidly. For a function $f(x, y)$ of two variables, we will now consider how this direction of maximum rate of increase can be determined from a contour map of the function. Suppose that (x_0, y_0) is a point on a level curve $f(x, y) = c$ of f, and assume that this curve can be smoothly parametrized as

$$x = x(s), \quad y = y(s) \tag{13}$$

where s is an arc length parameter. Recall from Formula (6) of Section 13.4 that the unit tangent vector to (13) is

$$\mathbf{T} = \mathbf{T}(s) = \left(\frac{dx}{ds}\right)\mathbf{i} + \left(\frac{dy}{ds}\right)\mathbf{j}$$

Since $\mathbf{T}$ gives a direction along which f is nearly constant, we would expect the instantaneous rate of change of f with respect to distance in the direction of $\mathbf{T}$ to be 0. That is, we would expect that

$$D_{\mathbf{T}} f(x, y) = \nabla f(x, y) \cdot \mathbf{T}(s) = 0$$

To show this to be the case, we differentiate both sides of the equation $f(x, y) = c$ with respect to s. Assuming that f is differentiable at (x, y), we can use the chain rule to obtain

$$\frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} = 0$$

which we can rewrite as

$$\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right) \cdot \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}\right) = 0$$

or alternatively, as

$$\nabla f(x, y) \cdot \mathbf{T} = 0$$

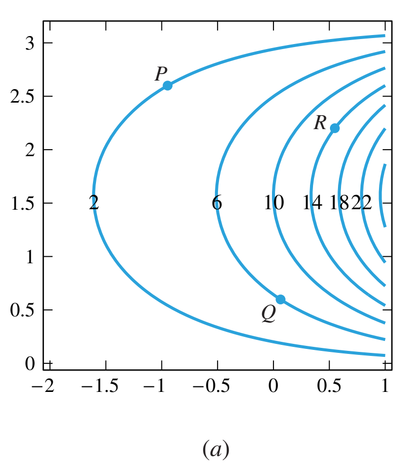
Therefore, if $\nabla f(x, y) \neq \mathbf{0}$, then $\nabla f(x, y)$ should be normal to the level curve $f(x, y) = c$ at any point (x, y) on the curve.

It is proved in advanced courses that if $f(x, y)$ has continuous first-order partial derivatives, and if $\nabla f(x_0, y_0) \neq \mathbf{0}$, then near (x_0, y_0) the graph of $f(x, y) = c$ is indeed a smooth curve through (x_0, y_0). Furthermore, we also know from Theorem 14.4.5 that f will be differentiable at (x_0, y_0). We therefore have the following result.

14.6.6 THEOREM. *Assume that $f(x, y)$ has continuous first-order partial derivatives and that $\nabla f(x_0, y_0) \neq \mathbf{0}$. Then $\nabla f(x_0, y_0)$ is normal to the level curve of f through (x_0, y_0).*

When we examine a contour map, we instinctively regard the distance between adjacent contours to be measured in a normal direction. If the contours correspond to equally spaced values of f, then the closer together the contours appear to be, the more rapidly the values of f will be changing in that normal direction. It follows from Theorems 14.6.5 and 14.6.6 that this rate of change of f is given by $\|\nabla f(x, y)\|$. Thus, the closer together the contours appear to be, the greater the length of the gradient of f.

Example 5 A contour plot of a function f is given in Figure 14.6.6*a*. Sketch the directions of the gradient of f at the points P, Q, and R. At which of these three points does the gradient vector have maximum length? Minimum length?



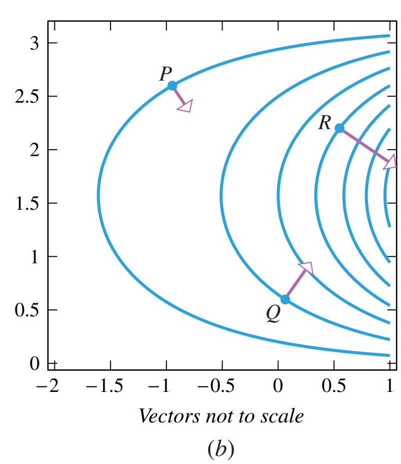


Figure 14.6.6

Solution. It follows from Theorems 14.6.5 and 14.6.6 that the directions of the gradient vectors will be as given in Figure 14.6.6*b*. Based on the density of the contour lines, we would guess that the gradient of f has maximum length at R and minimum length at P, with the length at Q somewhere in between. ◀

REMARK. If (x_0, y_0) is a point on the level curve $f(x, y) = c$, then the slope of the surface $z = f(x, y)$ at that point in the direction of $\mathbf{u}$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u}$$

If $\mathbf{u}$ is tangent to the level curve at (x_0, y_0), then $f(x, y)$ is neither increasing nor decreasing in that direction, so $D_{\mathbf{u}} f(x_0, y_0) = 0$. Thus, $\nabla f(x_0, y_0)$, $-\nabla f(x_0, y_0)$, and the tangent vector $\mathbf{u}$ mark the directions of maximum slope, minimum slope, and zero slope at a point (x_0, y_0) on a level curve (Figure 14.6.7). Good skiers use these facts intuitively to control their speed by zigzagging down ski slopes—they ski across the slope with their skis tangential to a level curve to stop their downhill motion, and they point their skis down the slope and normal to the level curve to obtain the most rapid descent.

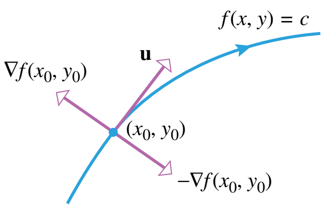


Figure 14.6.7

AN APPLICATION OF GRADIENTS

There are numerous applications in which the motion of an object must be controlled so that it moves toward a heat source. For example, in medical applications the operation of certain diagnostic equipment is designed to locate heat sources generated by tumors or infections, and in military applications the trajectories of heat-seeking missiles are controlled to seek and destroy enemy aircraft. The following example illustrates how gradients are used to solve such problems.

Example 6 A heat-seeking particle is located at the point (2, 3) on a flat metal plate whose temperature at a point (x, y) is

$$T(x, y) = 10 - 8x^2 - 2y^2$$

Find an equation for the trajectory of the particle if it moves continuously in the direction of maximum temperature increase.

Solution. Assume that the trajectory is represented parametrically by the equations

$$x = x(t), \quad y = y(t)$$

where the particle is at the point (2, 3) at time $t = 0$. Because the particle moves in the direction of maximum temperature increase, its direction of motion at time t is in the direction of the gradient of $T(x, y)$, and hence its velocity vector $\mathbf{v}(t)$ at time t points in the direction of the gradient. Thus, there is a scalar k that depends on t such that

$$\mathbf{v}(t) = k\nabla T(x, y)$$

from which we obtain

$$\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} = k(-16x\mathbf{i} - 4y\mathbf{j})$$

Equating components yields

$$\frac{dx}{dt} = -16kx, \quad \frac{dy}{dt} = -4ky$$

and dividing to eliminate k yields

$$\frac{dy}{dx} = \frac{-4ky}{-16kx} = \frac{y}{4x}$$

Thus, we can obtain the trajectory by solving the initial-value problem

$$\frac{dy}{dx} - \frac{y}{4x} = 0, \quad y(2) = 3$$

The differential equation is a separable first-order linear equation and hence can be solved by separating the variables or by the method of integrating factors discussed in Section 9.1. We leave it for you to show that the solution of the initial-value problem is

$$y = \frac{3}{\sqrt[4]{2}}x^{1/4}$$

The graph of the trajectory and a contour plot of the temperature function are shown in Figure 14.6.8. ◀

THE DERIVATIVE OF A FUNCTION OF TWO OR THREE VARIABLES

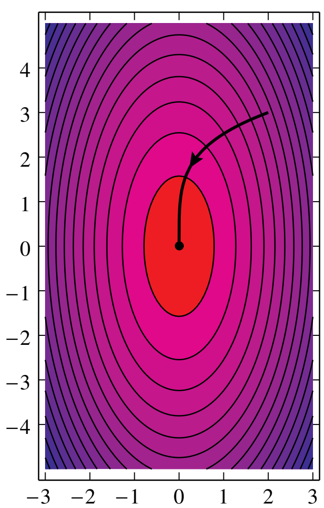


Figure 14.6.8

We now return to the question of what is meant by the *derivative* of a differentiable function of two or three variables. This question can actually be answered in several ways: some authors identify the derivative with the total differential of the function, and others define the derivative to be the gradient. We will choose the latter approach, justifying it with a purely symbolic argument.

We will restrict our attention to functions $f(x, y)$ of two variables and write the operator ∇ as $d/d\mathbf{r}$ and the gradient ∇f as $df/d\mathbf{r}$. (This notation is meant to be suggestive only and will not be used elsewhere.) We can identify any point (x, y) in the xy-plane with the vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$. Under this identification we can write $f(\mathbf{r})$ instead of $f(x, y)$. Any point (x_0, y_0) can be identified with the vector $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j}$. We then have

$$\mathbf{r} - \mathbf{r}_0 = (x - x_0)\,\mathbf{i} + (y - y_0)\,\mathbf{j}$$

and the local linear approximation

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

can be written as

$$f(\mathbf{r}) \approx f(\mathbf{r}_0) + \frac{df}{d\mathbf{r}}(\mathbf{r}_0) \cdot (\mathbf{r} - \mathbf{r}_0)$$

which has the same form as the one-variable local linear approximation

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

from Section 3.8. Note that the gradient $df/d\mathbf{r}$ plays the same role in the approximation for $f(\mathbf{r})$ that the derivative f' plays in the approximation for $f(x)$.

For a second illustration, consider the version of the chain rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

that appears in Theorem 14.5.1. Using our vector notation, we can write the composition $z = f(x(t), y(t))$ as $z = f(\mathbf{r}(t))$, where $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. With the notation

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$$

and

$$\frac{dz}{d\mathbf{r}} = \frac{df}{d\mathbf{r}} = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

the chain rule above takes the form

$$\frac{dz}{dt} = \frac{dz}{d\mathbf{r}} \cdot \frac{d\mathbf{r}}{dt}$$

This has the same form as the one-variable chain rule

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt}$$

and again we see that the gradient $dz/d\mathbf{r}$ plays the role of the derivative.

EXERCISE SET 14.6 Graphing Utility C CAS

In Exercises 1–8, find $D_{\mathbf{u}} f$ at P.

1. $f(x, y) = (1+xy)^{3/2}$; $P(3, 1)$; $\mathbf{u} = \dfrac{1}{\sqrt{2}}\mathbf{i} + \dfrac{1}{\sqrt{2}}\mathbf{j}$
2. $f(x, y) = e^{2xy}$; $P(4, 0)$; $\mathbf{u} = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$
3. $f(x, y) = \ln(1+x^2+y)$; $P(0, 0)$; $\mathbf{u} = -\dfrac{1}{\sqrt{10}}\mathbf{i} - \dfrac{3}{\sqrt{10}}\mathbf{j}$
4. $f(x, y) = \dfrac{cx+dy}{x-y}$; $P(3, 4)$; $\mathbf{u} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$
5. $f(x, y, z) = 4x^5y^2z^3$; $P(2, -1, 1)$; $\mathbf{u} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$
6. $f(x, y, z) = ye^{xz} + z^2$; $P(0, 2, 3)$; $\mathbf{u} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$
7. $f(x, y, z) = \ln(x^2+2y^2+3z^2)$; $P(-1, 2, 4)$; $\mathbf{u} = -\frac{3}{13}\mathbf{i} - \frac{4}{13}\mathbf{j} - \frac{12}{13}\mathbf{k}$
8. $f(x, y, z) = \sin xyz$; $P\left(\frac{1}{2}, \frac{1}{3}, \pi\right)$; $\mathbf{u} = \dfrac{1}{\sqrt{3}}\mathbf{i} - \dfrac{1}{\sqrt{3}}\mathbf{j} + \dfrac{1}{\sqrt{3}}\mathbf{k}$

In Exercises 9–18, find the directional derivative of f at P in the direction of $\mathbf{a}$.

9. $f(x, y) = 4x^3y^2$; $P(2, 1)$; $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j}$
10. $f(x, y) = x^2 - 3xy + 4y^3$; $P(-2, 0)$; $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$
11. $f(x, y) = y^2 \ln x$; $P(1, 4)$; $\mathbf{a} = -3\mathbf{i} + 3\mathbf{j}$
12. $f(x, y) = e^x \cos y$; $P(0, \pi/4)$; $\mathbf{a} = 5\mathbf{i} - 2\mathbf{j}$
13. $f(x, y) = \tan^{-1}(y/x)$; $P(-2, 2)$; $\mathbf{a} = -\mathbf{i} - \mathbf{j}$
14. $f(x, y) = xe^y - ye^x$; $P(0, 0)$; $\mathbf{a} = 5\mathbf{i} - 2\mathbf{j}$
15. $f(x, y, z) = x^3z - yx^2 + z^2$; $P(2, -1, 1)$; $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$
16. $f(x, y, z) = y - \sqrt{x^2+z^2}$; $P(-3, 1, 4)$; $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$
17. $f(x, y, z) = \dfrac{z-x}{z+y}$; $P(1, 0, -3)$; $\mathbf{a} = -6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$
18. $f(x, y, z) = e^{x+y+3z}$; $P(-2, 2, -1)$; $\mathbf{a} = 20\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$

In Exercises 19–22, find the directional derivative of f at P in the direction of a vector making the counterclockwise angle θ with the positive x-axis.

19. $f(x, y) = \sqrt{xy}$; $P(1, 4)$; $\theta = \pi/3$
20. $f(x, y) = \dfrac{x-y}{x+y}$; $P(-1, -2)$; $\theta = \pi/2$
21. $f(x, y) = \tan(2x+y)$; $P(\pi/6, \pi/3)$; $\theta = 7\pi/4$
22. $f(x, y) = \sinh x \cosh y$; $P(0, 0)$; $\theta = \pi$
23. Find the directional derivative of
$$f(x, y) = \frac{x}{x+y}$$
at $P(1, 0)$ in the direction of $Q(-1, -1)$.
24. Find the directional derivative of $f(x, y) = e^{-x}\sec y$ at $P(0, \pi/4)$ in the direction of the origin.
25. Find the directional derivative of $f(x, y) = \sqrt{xy}e^y$ at $P(1, 1)$ in the direction of the negative y-axis.
26. Let
$$f(x, y) = \frac{y}{x+y}$$
Find a unit vector $\mathbf{u}$ for which $D_{\mathbf{u}} f(2, 3) = 0$.
27. Find the directional derivative of
$$f(x, y, z) = \frac{y}{x+z}$$
at $P(2, 1, -1)$ in the direction from P to $Q(-1, 2, 0)$.
28. Find the directional derivative of the function
$$f(x, y, z) = x^3y^2z^5 - 2xz + yz + 3x$$
at $P(-1, -2, 1)$ in the direction of the negative z-axis.
29. Suppose that $D_{\mathbf{u}} f(1, 2) = -5$ and $D_{\mathbf{v}} f(1, 2) = 10$, where $\mathbf{u} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$ and $\mathbf{v} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$. Find
(a) $f_x(1, 2)$ (b) $f_y(1, 2)$
(c) the directional derivative of f at $(1, 2)$ in the direction of the origin.
30. Given that $f_x(-5, 1) = -3$ and $f_y(-5, 1) = 2$, find the directional derivative of f at $P(-5, 1)$ in the direction of the vector from P to $Q(-4, 3)$.
31. The accompanying figure shows some level curves of an unspecified function $f(x, y)$. Which of the three vectors shown in the figure is most likely to be ∇f? Explain.
32. The accompanying figure shows some level curves of an unspecified function $f(x, y)$. Of the gradients at P and Q, which probably has the greater length? Explain.

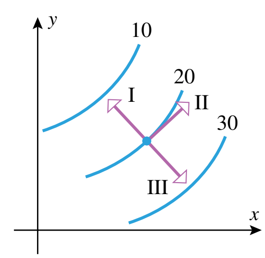


Figure Ex-31

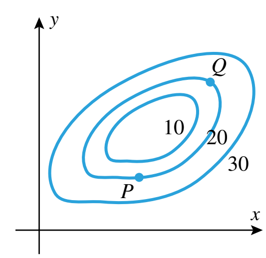


Figure Ex-32

In Exercises 33–36, find ∇z or ∇w.

33. $z = 4x - 8y$
34. $z = e^{-3y}\cos 4x$
35. $w = \ln\sqrt{x^2+y^2+z^2}$
36. $w = e^{-5x}\sec x^2yz$

In Exercises 37–40, find the gradient of f at the indicated point.

37. $f(x, y) = (x^2+xy)^3$; $(-1, -1)$
38. $f(x, y) = (x^2+y^2)^{-1/2}$; $(3, 4)$
39. $f(x, y, z) = y\ln(x+y+z)$; $(-3, 4, 0)$

40. $f(x, y, z) = y^2 z \tan^3 x$; $(\pi/4, -3, 1)$

In Exercises 41–44, sketch the level curve of $f(x, y)$ that passes through P and draw the gradient vector at P.

41. $f(x, y) = 4x - 2y + 3$; $P(1, 2)$

42. $f(x, y) = y/x^2$; $P(-2, 2)$

43. $f(x, y) = x^2 + 4y^2$; $P(-2, 0)$

44. $f(x, y) = x^2 - y^2$; $P(2, -1)$

45. Find a unit vector $\mathbf{u}$ that is normal at $P(1, -2)$ to the level curve of $f(x, y) = 4x^2 y$ through P.

46. Find a unit vector $\mathbf{u}$ that is normal at $P(2, -3)$ to the level curve of $f(x, y) = 3x^2 y - xy$ through P.

In Exercises 47–54, find a unit vector in the direction in which f increases most rapidly at P; and find the rate of change of f at P in that direction.

47. $f(x, y) = 4x^3 y^2$; $P(-1, 1)$

48. $f(x, y) = 3x - \ln y$; $P(2, 4)$

49. $f(x, y) = \sqrt{x^2 + y^2}$; $P(4, -3)$

50. $f(x, y) = \dfrac{x}{x + y}$; $P(0, 2)$

51. $f(x, y, z) = x^3 z^2 + y^3 z + z - 1$; $P(1, 1, -1)$

52. $f(x, y, z) = \sqrt{x - 3y + 4z}$; $P(0, -3, 0)$

53. $f(x, y, z) = \dfrac{x}{z} + \dfrac{z}{y^2}$; $P(1, 2, -2)$

54. $f(x, y, z) = \tan^{-1}\left(\dfrac{x}{y + z}\right)$; $P(4, 2, 2)$

In Exercises 55–60, find a unit vector in the direction in which f decreases most rapidly at P; and find the rate of change of f at P in that direction.

55. $f(x, y) = 20 - x^2 - y^2$; $P(-1, -3)$

56. $f(x, y) = e^{xy}$; $P(2, 3)$

57. $f(x, y) = \cos(3x - y)$; $P(\pi/6, \pi/4)$

58. $f(x, y) = \sqrt{\dfrac{x - y}{x + y}}$; $P(3, 1)$

59. $f(x, y, z) = \dfrac{x + z}{z - y}$; $P(5, 7, 6)$

60. $f(x, y, z) = 4e^{xy} \cos z$; $P(0, 1, \pi/4)$

61. Given that $\nabla f(4, -5) = 2\mathbf{i} - \mathbf{j}$, find the directional derivative of the function f at the point $(4, -5)$ in the direction of $\mathbf{a} = 5\mathbf{i} + 2\mathbf{j}$.

62. Given that $\nabla f(x_0, y_0) = \mathbf{i} - 2\mathbf{j}$ and $D_{\mathbf{u}} f(x_0, y_0) = -2$, find $\mathbf{u}$ (two answers).

63. The accompanying figure shows some level curves of an unspecified function $f(x, y)$.

(a) Use the available information to approximate the length of the vector $\nabla f(1, 2)$, and sketch the approximation. Explain how you approximated the length and determined the direction of the vector.

(b) Sketch an approximation of the vector $-\nabla f(4, 4)$.

64. (a) The accompanying figure shows a topographic map of a hill and a point P at the bottom of the hill. Suppose that you want to climb from the point P toward the top of the hill in such a way that you are always ascending in the direction of steepest slope. Sketch the projection of your path on the contour map. This is called the ***path of steepest ascent***. Explain how you determined the path.

(b) Suppose that when you are at the top you want to climb down the hill in such a way that you are always descending in the direction of steepest slope. Sketch the projection of your path on the contour map. This is called the ***path of steepest descent***. Explain how you determined the path.

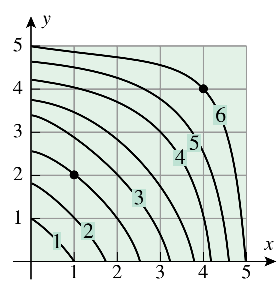


Figure Ex-63

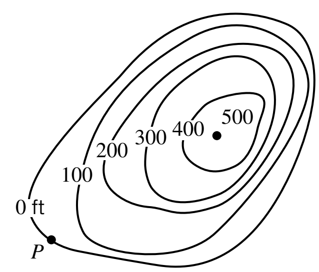


Figure Ex-64

65. Let $z = 3x^2 - y^2$. Find all points at which $\|\nabla z\| = 6$.

66. Given that $z = 3x + y^2$, find $\nabla \|\nabla z\|$ at the point $(5, 2)$.

67. A particle moves along a path C given by the equations $x = t$ and $y = -t^2$. If $z = x^2 + y^2$, find dz/ds along C at the instant when the particle is at the point $(2, -4)$.

68. The temperature (in degrees Celsius) at a point (x, y) on a metal plate in the xy-plane is

$$T(x, y) = \frac{xy}{1 + x^2 + y^2}$$

(a) Find the rate of change of temperature at $(1, 1)$ in the direction of $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$.

(b) An ant at $(1, 1)$ wants to walk in the direction in which the temperature drops most rapidly. Find a unit vector in that direction.

69. If the electric potential at a point (x, y) in the xy-plane is $V(x, y)$, then the ***electric intensity vector*** at the point (x, y) is $\mathbf{E} = -\nabla V(x, y)$. Suppose that $V(x, y) = e^{-2x} \cos 2y$.

(a) Find the electric intensity vector at $(\pi/4, 0)$.

(b) Show that at each point in the plane, the electric potential decreases most rapidly in the direction of the vector $\mathbf{E}$.

70. On a certain mountain, the elevation z above a point (x, y) in an xy-plane at sea level is $z = 2000 - 0.02x^2 - 0.04y^2$, where x, y, and z are in meters. The positive x-axis points

east, and the positive y-axis north. A climber is at the point $(-20, 5, 1991)$.
(a) If the climber uses a compass reading to walk due west, will she begin to ascend or descend?
(b) If the climber uses a compass reading to walk northeast, will she ascend or descend? At what rate?
(c) In what compass direction should the climber begin walking to travel a level path (two answers)?

71. Given that the directional derivative of $f(x, y, z)$ at the point $(3, -2, 1)$ in the direction of $\mathbf{a} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ is -5 and that $\|\nabla f(3, -2, 1)\| = 5$, find $\nabla f(3, -2, 1)$.

72. The temperature (in degrees Celsius) at a point (x, y, z) in a metal solid is

$$T(x, y, z) = \frac{xyz}{1 + x^2 + y^2 + z^2}$$

(a) Find the rate of change of temperature with respect to distance at $(1, 1, 1)$ in the direction of the origin.
(b) Find the direction in which the temperature rises most rapidly at the point $(1, 1, 1)$. (Express your answer as a unit vector.)
(c) Find the rate at which the temperature rises moving from $(1, 1, 1)$ in the direction obtained in part (b).

73. Let $r = \sqrt{x^2 + y^2}$.
(a) Show that $\nabla r = \dfrac{\mathbf{r}}{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.
(b) Show that $\nabla f(r) = f'(r)\nabla r = \dfrac{f'(r)}{r}\mathbf{r}$.

74. Use the formula in part (b) of Exercise 73 to find
(a) $\nabla f(r)$ if $f(r) = re^{-3r}$
(b) $f(r)$ if $\nabla f(r) = 3r^2\mathbf{r}$ and $f(2) = 1$.

75. Let $\mathbf{u}_r$ be a unit vector whose counterclockwise angle from the positive x-axis is θ, and let $\mathbf{u}_\theta$ be a unit vector $90°$ counterclockwise from $\mathbf{u}_r$. Show that if $z = f(x, y)$, $x = r\cos\theta$, and $y = r\sin\theta$, then

$$\nabla z = \frac{\partial z}{\partial r}\mathbf{u}_r + \frac{1}{r}\frac{\partial z}{\partial \theta}\mathbf{u}_\theta$$

[*Hint:* Use part (c) of Exercise 59, Section 14.5.]

76. Prove: If f and g are differentiable, then
(a) $\nabla(f + g) = \nabla f + \nabla g$
(b) $\nabla(cf) = c\nabla f$ (c constant)
(c) $\nabla(fg) = f\nabla g + g\nabla f$
(d) $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$
(e) $\nabla(f^n) = nf^{n-1}\nabla f$.

In Exercises 77 and 78, a heat-seeking particle is located at the point P on a flat metal plate whose temperature at a point (x, y) is $T(x, y)$. Find parametric equations for the trajectory of the particle if it moves continuously in the direction of maximum temperature increase.

77. $T(x, y) = 5 - 4x^2 - y^2$; $P(1, 4)$

78. $T(x, y) = 100 - x^2 - 2y^2$; $P(5, 3)$

79. Use a graphing utility to generate the trajectory of the particle together with some representative level curves of the temperature function in Exercise 77.

80. Use a graphing utility to generate the trajectory of the particle together with some representative level curves of the temperature function in Exercise 78.

C **81.** (a) Use a CAS to graph $f(x, y) = (x^2 + 3y^2)e^{-(x^2+y^2)}$.
(b) At how many points do you think it is true that $D_{\mathbf{u}}f(x, y) = 0$ for all unit vectors $\mathbf{u}$?
(c) Use a CAS to find ∇f.
(d) Use a CAS to solve the equation $\nabla f(x, y) = 0$ for x and y.
(e) Use the result in part (d) together with Theorem 14.6.5 to check your conjecture in part (b).

82. Prove: If $x = x(t)$ and $y = y(t)$ are differentiable at t, and if $z = f(x, y)$ is differentiable at the point $(x(t), y(t))$, then

$$\frac{dz}{dt} = \nabla z \cdot \mathbf{r}'(t)$$

where $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$.

83. Prove: If f, f_x, and f_y are continuous on a circular region, and if $\nabla f(x, y) = \mathbf{0}$ throughout the region, then $f(x, y)$ is constant on the region. [*Hint:* See Exercise 71, Section 14.5.]

84. Prove: If the function f is differentiable at the point (x, y) and if $D_{\mathbf{u}}f(x, y) = 0$ in two nonparallel directions, then $D_{\mathbf{u}}f(x, y) = 0$ in all directions.

85. Given that the functions $u = u(x, y, z)$, $v = v(x, y, z)$, and $w = w(x, y, z)$, and $f(u, v, w)$ are all differentiable, show that

$$\nabla f(u, v, w) = \frac{\partial f}{\partial u}\nabla u + \frac{\partial f}{\partial v}\nabla v + \frac{\partial f}{\partial w}\nabla w$$

86. (a) Let $g(s) = f(x_0 + su_1, y_0 + su_2)$ where $u_1^2 + u_2^2 = 1$. Use Definition 14.4.1 to show that if you want to prove that $g'(0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$ then it suffices to prove that

$$\lim_{s\to 0}\frac{E(s)}{|s|} = 0$$

where

$$E(s) = f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0) - f_x(x_0, y_0)su_1 - f_y(x_0, y_0)su_2$$

(b) Let ℓ denote the line whose parametric equations are $x = x_0 + su_1$, $y = y_0 + su_2$. Prove that if $(x, y) \to (x_0, y_0)$ on line ℓ, then Equation (5) in Definition 14.4.2 reduces to the limit equation from part (a).
(c) Use parts (a) and (b) to give another proof of Equation (4) in Theorem 14.6.3.

87. (a) Explain why Equation (4) of Theorem 14.6.3 may be written in the form

$$\lim_{s\to 0}\frac{f(x_0+su_1, y_0+su_2)-f(x_0, y_0)}{s} = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

(b) Use Definition 2.4.1 to express the limit from part (a) in a statement involving the ϵ-δ notation.

(c) Use Definition 14.2.1 to express Equation (5) of Definition 14.4.2 in a statement involving the ϵ-δ notation.

(d) Rewrite your answer in part (c) using the substitutions

$$s=\sqrt{(x-x_0)^2+(y-y_0)^2},\quad u_1=\frac{x-x_0}{s},\quad u_2=\frac{y-y_0}{s}$$

(e) Use parts (a)–(d) to give another proof of Equation (4) in Theorem 14.6.3.

14.7 TANGENT PLANES AND NORMAL VECTORS

In this section we will discuss tangent planes to surfaces in three-dimensional space. We will be concerned with three main questions: What is a tangent plane? When do tangent planes exist? How do we find equations of tangent planes?

TANGENT PLANES

Recall from Section 14.4 that if a function $f(x, y)$ is differentiable at a point (x_0, y_0), then we want it to be the case that the surface $z = f(x, y)$ has a nonvertical tangent plane at the point $P_0(x_0, y_0, f(x_0, y_0))$. We also saw in Section 14.4 that the linear function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

approximates $f(x, y)$ very closely near (x_0, y_0) and has a nonvertical tangent plane through the point P_0. This suggests that the graph of L is the tangent plane we seek. We can now provide some *geometric* justification for this conclusion.

We will base our concept of a tangent plane to a surface $S : z = f(x, y)$ on the more elementary notion of a tangent line to a curve C in 3-space (Figure 14.7.1). Intuitively, we would expect a tangent plane to S at a point P_0 to be composed of the tangent lines at P_0 of all curves on S that pass through P_0 (Figure 14.7.2). The following theorem shows that the graph of the local linear approximation is indeed tangent to the surface $z = f(x, y)$ in this geometric sense.

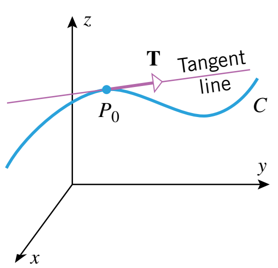


Figure 14.7.1

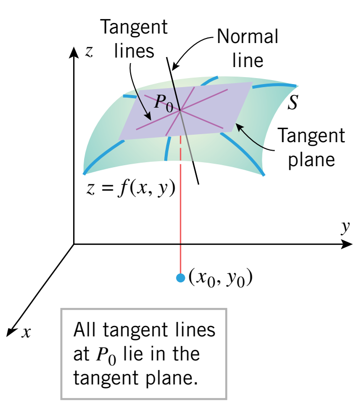


Figure 14.7.2

14.7.1 THEOREM. *Assume that the function $f(x, y)$ is differentiable at (x_0, y_0) and let $P_0(x_0, y_0, f(x_0, y_0))$ denote the corresponding point on the graph of f. Let T denote the graph of the local linear approximation*

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{1}$$

to f at (x_0, y_0). Then a line is tangent at P_0 to a curve C on the surface $z = f(x, y)$ if and only if the line is contained in T.

Proof. The graph T of (1) is the plane

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

for which

$$\mathbf{n} = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k}$$

is a normal vector (verify). Let C denote a curve on the surface $z = f(x, y)$ through P_0 and assume that C is parametrized by

$$x = x(t),\quad y = y(t),\quad z = z(t)$$

with

$$x_0 = x(t_0), \quad y_0 = y(t_0), \quad f(x_0, y_0) = z(t_0)$$

The tangent line ℓ to C through P_0 is then parallel to the vector

$$\mathbf{r}' = x'(t_0)\mathbf{i} + y'(t_0)\mathbf{j} + z'(t_0)\mathbf{k}$$

where we assume that $\mathbf{r}' \neq \mathbf{0}$ (Definition 13.2.8). To prove that ℓ is contained in T, it suffices to prove that $\mathbf{n} \cdot \mathbf{r}' = 0$. Since C lies on the graph of f, we have

$$z(t) = f(x(t), y(t))$$

Using the chain rule to compute the derivative of $z(t)$ at t_0 yields

$$z'(t_0) = f_x(x_0, y_0)x'(t_0) + f_y(x_0, y_0)y'(t_0)$$

or, equivalently, that

$$(f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k}) \cdot (x'(t_0)\mathbf{i} + y'(t_0)\mathbf{j} + z'(t_0)\mathbf{k}) = 0$$

But this is just the equation $\mathbf{n} \cdot \mathbf{r}' = 0$, which completes the proof that ℓ is contained in T.

Conversely, let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ denote the direction vector for a line ℓ through P_0 contained in T. Then

$$0 = \mathbf{n} \cdot \mathbf{a} = a_1 f_x(x_0, y_0) + a_2 f_y(x_0, y_0) - a_3$$

which implies that

$$a_3 = a_1 f_x(x_0, y_0) + a_2 f_y(x_0, y_0)$$

Let C denote the curve with parametric equations

$$x = x(t) = x_0 + a_1 t, \quad y = y(t) = y_0 + a_2 t, \quad z = z(t) = f(x(t), y(t))$$

The curve C passes through P_0 when $t = 0$ and the tangent line to C at P_0 has direction vector

$$\mathbf{r}' = x'(0)\mathbf{i} + y'(0)\mathbf{j} + z'(0)\mathbf{k} = a_1\mathbf{i} + a_2\mathbf{j} + z'(0)\mathbf{k}$$

It follows from the chain rule that

$$z'(0) = a_1 f_x(x_0, y_0) + a_2 f_y(x_0, y_0) = a_3$$

and therefore

$$\mathbf{r}' = x'(0)\mathbf{i} + y'(0)\mathbf{j} + z'(0)\mathbf{k} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = \mathbf{a}$$

Thus, the vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is the direction vector $\mathbf{r}'$ for the line through P_0 tangent to C. Therefore, this line is ℓ, which completes the proof that ℓ is tangent at P_0 to a curve C on the surface $z = f(x, y)$. ■

Based on Theorem 14.7.1 we make the following definition.

14.7.2 DEFINITION. If $f(x, y)$ is differentiable at the point (x_0, y_0), then the ***tangent plane*** to the surface $z = f(x, y)$ at the point $P_0(x_0, y_0, f(x_0, y_0))$ [or (x_0, y_0)] is the plane

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{2}$$

The line through the point P_0 parallel to the vector $\mathbf{n}$ is perpendicular to the tangent plane (2). We will refer to this line as the ***normal line*** to the surface $z = f(x, y)$ at P_0. It follows that this normal line can be expressed parametrically as

$$x = x_0 + f_x(x_0, y_0)t, \quad y = y_0 + f_y(x_0, y_0)t, \quad z = f(x_0, y_0) - t \tag{3}$$

Example 1 Find an equation for the tangent plane and parametric equations for the normal line to the surface $z = x^2y$ at the point (2, 1, 4).

Solution. The partial derivatives of f are

$$f_x(x, y) = 2xy, \quad f_y(x, y) = x^2$$
$$f_x(2, 1) = 4, \quad f_y(2, 1) = 4$$

Therefore, the tangent plane has equation

$$z = 4 + 4(x - 2) + 4(y - 1) = 4x + 4y - 8$$

and the normal line has equations

$$x = 2 + 4t, \quad y = 1 + 4t, \quad z = 4 - t$$ ◀

TANGENT PLANES AND TOTAL DIFFERENTIALS

Recall that for a function $f(x, y)$ of two variables, approximation by differentials geometrically:

$$\Delta z = \Delta f = f(x, y) - f(x_0, y_0) \approx dz = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The tangent plane provides a geometric interpretation of this approximation. We see in Figure 14.7.3 that Δz is the change in z *along the surface* $z = f(x, y)$ from the point $P_0(x_0, y_0, f(x_0, y_0))$ to the point $P(x, y, f(x, y))$, and dz is the change in z *along the tangent plane* from P_0 to $Q(x, y, L(x, y))$. The small vertical displacement at (x, y) between the surface and the plane represents the error $E(x, y)$ in the local linear approximation to f at (x_0, y_0). We have seen that near (x_0, y_0) this error term has magnitude much smaller than the distance between (x, y) and (x_0, y_0).

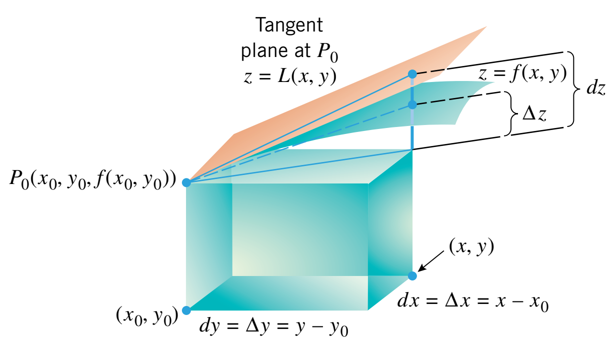


Figure 14.7.3

TANGENT PLANES TO LEVEL SURFACES

We now consider the problem of finding tangent planes to surfaces that can be represented implicitly by equations of the form $F(x, y, z) = c$. We will assume that F has continuous first-order partial derivatives. This assumption poses no real restriction on the functions we will routinely encounter and has an important geometric consequence. In advanced courses it is proved that if F has continuous first-order partial derivatives, and if $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then near $P_0(x_0, y_0, z_0)$ the graph of $F(x, y, z) = c$ is actually the graph of an implicitly defined differentiable function of (at least) one of the following forms:

$$z = f(x, y), \quad y = g(x, z), \quad x = h(y, z) \tag{4}$$

This guarantees that near P_0 the graph of $F(x, y, z) = c$ is indeed a "surface" (rather than some possibly exotic-looking set of points in 3-space), and it follows from Theorem 14.7.1 that there is a tangent plane to the surface at the point P_0.

Fortunately, we do not need to solve the equation $F(x, y, z) = c$ for one of the functions in (4) in order to find the tangent plane at P_0. (In practice, this may be impossible.) We know from Theorem 14.7.1 that a line through P_0 will belong to this tangent plane if and only if it is a tangent line at P_0 of a curve C on the surface $F(x, y, z) = c$. Suppose that C

is parametrized by

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

with

$$x_0 = x(t_0), \quad y_0 = y(t_0), \quad z_0 = z(t_0)$$

The tangent line ℓ to C through P_0 is then parallel to the vector

$$\mathbf{r}' = x'(t_0)\mathbf{i} + y'(t_0)\mathbf{j} + z'(t_0)\mathbf{k}$$

where we assume that $\mathbf{r}' \neq \mathbf{0}$ (Definition 13.2.8). Since C is on the surface $F(x, y, z) = c$, we have

$$c = F(x(t), y(t), z(t)) \tag{5}$$

Computing the derivative at t_0 of both sides of (5), we have by the chain rule that

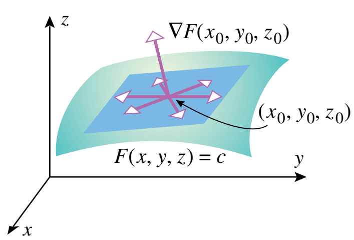
$$0 = F_x(x_0, y_0, z_0)x'(t_0) + F_y(x_0, y_0, z_0)y'(t_0) + F_z(x_0, y_0, z_0)z'(t_0)$$

We can write this equation in vector form as

$$0 = (F_x(x_0, y_0, z_0)\mathbf{i} + F_y(x_0, y_0, z_0)\mathbf{j} + F_z(x_0, y_0, z_0)\mathbf{k}) \cdot (x'(t_0)\mathbf{i} + y'(t_0)\mathbf{j} + z'(t_0)\mathbf{k})$$

or

$$0 = \nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'$$



The gradient is normal to the level surface.

Figure 14.7.4

It follows that if $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then $\nabla F(x_0, y_0, z_0)$ is normal to line ℓ. We conclude that if $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then $\nabla F(x_0, y_0, z_0)$ is normal to any line through P_0 that is contained in the tangent plane to the surface $F(x, y, z) = c$ at P_0. It follows that $\nabla F(x_0, y_0, z_0)$ is a normal vector to this plane and hence is normal to the level surface (Figure 14.7.4).

We can now express the equation of the tangent plane to the level surface at P_0 in point-normal form as

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

[see Formula (3) of Section 12.6]. Based on this analysis we have the following theorem.

14.7.3 THEOREM. *Assume that $F(x, y, z)$ has continuous first-order partial derivatives and let $c = F(x_0, y_0, z_0)$. If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then $\nabla F(x_0, y_0, z_0)$ is a* ***normal vector*** *to the surface $F(x, y, z) = c$ at the point $P_0(x_0, y_0, z_0)$, and the* ***tangent plane*** *to this surface at P_0 is the plane with equation*

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \tag{6}$$

REMARK. Theorem 14.7.3 can be viewed as an extension of Theorem 14.6.6 from curves to surfaces.

Example 2 Find an equation of the tangent plane to the ellipsoid $x^2 + 4y^2 + z^2 = 18$ at the point $(1, 2, 1)$, and determine the acute angle that this plane makes with the xy-plane.

Solution. The ellipsoid is a level surface of the function $F(x, y, z) = x^2 + 4y^2 + z^2$, so we begin by finding the gradient of this function at the point $(1, 2, 1)$. The computations are

$$\nabla F(x, y, z) = \frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k} = 2x\mathbf{i} + 8y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla F(1, 2, 1) = 2\mathbf{i} + 16\mathbf{j} + 2\mathbf{k}$$

Thus,

$$F_x(1, 2, 1) = 2, \quad F_y(1, 2, 1) = 16, \quad F_z(1, 2, 1) = 2$$

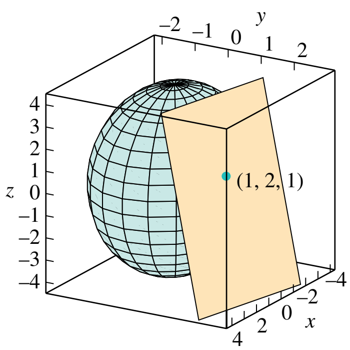


Figure 14.7.5

and hence from (6) the equation of the tangent plane is

$$2(x-1)+16(y-2)+2(z-1)=0 \quad \text{or} \quad x+8y+z=18$$

To find the acute angle θ between the tangent plane and the xy-plane, we will apply Formula (9) of Section 12.6 with $\mathbf{n}_1 = \nabla F(1,2,1) = 2\mathbf{i}+16\mathbf{j}+2\mathbf{k}$ and $\mathbf{n}_2 = \mathbf{k}$. This yields

$$\cos\theta = \frac{|\nabla F(1,2,1)\cdot\mathbf{k}|}{\|\nabla F(1,2,1)\|\|\mathbf{k}\|} = \frac{2}{(2\sqrt{66})(1)} = \frac{1}{\sqrt{66}}$$

Thus,

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{66}}\right) \approx 83^\circ$$

(Figure 14.7.5). ◀

USING GRADIENTS TO FIND TANGENT LINES TO INTERSECTIONS OF SURFACES

In general, the intersection of two surfaces $F(x,y,z)=0$ and $G(x,y,z)=0$ will be a curve in 3-space. If (x_0,y_0,z_0) is a point on this curve, then $\nabla F(x_0,y_0,z_0)$ will be normal to the surface $F(x,y,z)=0$ at (x_0,y_0,z_0) and $\nabla G(x_0,y_0,z_0)$ will be normal to the surface $G(x,y,z)=0$ at (x_0,y_0,z_0). Thus, if the curve of intersection can be smoothly parametrized, then its unit tangent vector $\mathbf{T}$ at (x_0,y_0,z_0) will be orthogonal to both $\nabla F(x_0,y_0,z_0)$ and $\nabla G(x_0,y_0,z_0)$ (Figure 14.7.6). Consequently, if

$$\nabla F(x_0,y_0,z_0)\times\nabla G(x_0,y_0,z_0)\neq\mathbf{0}$$

then this cross product will be parallel to $\mathbf{T}$ and hence will be tangent to the curve of intersection. This tangent vector can be used to determine the direction of the tangent line to the curve of intersection at the point (x_0,y_0,z_0).

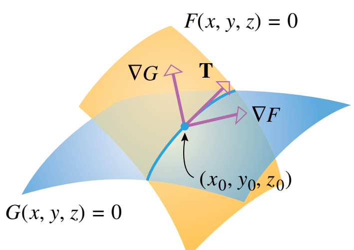


Figure 14.7.6

Example 3 Find parametric equations of the tangent line to the curve of intersection of the paraboloid $z=x^2+y^2$ and the ellipsoid $3x^2+2y^2+z^2=9$ at the point $(1,1,2)$ (Figure 14.7.7).

Solution. We begin by rewriting the equations of the surfaces as

$$x^2+y^2-z=0 \quad \text{and} \quad 3x^2+2y^2+z^2-9=0$$

and we take

$$F(x,y,z)=x^2+y^2-z \quad \text{and} \quad G(x,y,z)=3x^2+2y^2+z^2-9$$

We will need the gradients of these functions at the point $(1,1,2)$. The computations are

$$\nabla F(x,y,z)=2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}, \quad \nabla G(x,y,z)=6x\mathbf{i}+4y\mathbf{j}+2z\mathbf{k}$$
$$\nabla F(1,1,2)=2\mathbf{i}+2\mathbf{j}-\mathbf{k}, \quad \nabla G(1,1,2)=6\mathbf{i}+4\mathbf{j}+4\mathbf{k}$$

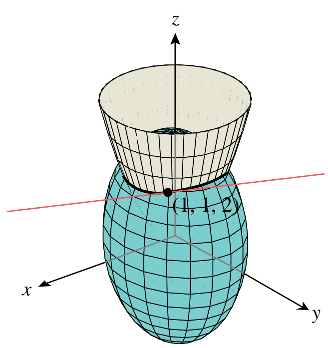


Figure 14.7.7

Thus, a tangent vector at $(1,1,2)$ to the curve of intersection is

$$\nabla F(1,1,2)\times\nabla G(1,1,2)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 2 & -1\\ 6 & 4 & 4\end{vmatrix}=12\mathbf{i}-14\mathbf{j}-4\mathbf{k}$$

Since any scalar multiple of this vector will do just as well, we can multiply by $\frac{1}{2}$ to reduce the size of the coefficients and use the vector of $6\mathbf{i}-7\mathbf{j}-2\mathbf{k}$ to determine the direction of the tangent line. This vector and the point $(1,1,2)$ yield the parametric equations

$$x=1+6t, \quad y=1-7t, \quad z=2-2t$$ ◀

EXERCISE SET 14.7 C CAS

In Exercises 1–8, find an equation for the tangent plane and parametric equations for the normal line to the surface at the point P.

1. $z = 4x^3y^2 + 2y$; $P(1, -2, 12)$

2. $z = \frac{1}{2}x^7y^{-2}$; $P(2, 4, 4)$

3. $z = xe^{-y}$; $P(1, 0, 1)$

4. $z = \ln\sqrt{x^2 + y^2}$; $P(-1, 0, 0)$

5. $z = e^{3y}\sin 3x$; $P(\pi/6, 0, 1)$

6. $z = x^{1/2} + y^{1/2}$; $P(4, 9, 5)$

7. $x^2 + y^2 + z^2 = 25$; $P(-3, 0, 4)$

8. $x^2y - 4z^2 = -7$; $P(-3, 1, -2)$

9. Find all points on the surface at which the tangent plane is horizontal.
(a) $z = x^3y^2$
(b) $z = x^2 - xy + y^2 - 2x + 4y$

10. Find a point on the surface $z = 3x^2 - y^2$ at which the tangent plane is parallel to the plane $6x + 4y - z = 5$.

11. Find a point on the surface $z = 8 - 3x^2 - 2y^2$ at which the tangent plane is perpendicular to the line $x = 2 - 3t$, $y = 7 + 8t$, $z = 5 - t$.

12. Show that the surfaces

$$z = \sqrt{x^2 + y^2} \quad \text{and} \quad z = \tfrac{1}{10}(x^2 + y^2) + \tfrac{5}{2}$$

intersect at $(3, 4, 5)$ and have a common tangent plane at that point.

13. (a) Find all points of intersection of the line $x = -1 + t$, $y = 2 + t$, $z = 2t + 7$ and the surface $z = x^2 + y^2$.
(b) At each point of intersection, find the cosine of the acute angle between the given line and the line normal to the surface.

14. Show that if f is differentiable and $z = xf(x/y)$, then all tangent planes to the graph of this equation pass through the origin.

15. Consider the ellipsoid $x^2 + y^2 + 4z^2 = 12$.
(a) Use the method of Example 2 to find an equation of the tangent plane to the ellipsoid at the point $(2, 2, 1)$.
(b) Find parametric equations of the line that is normal to the ellipsoid at the point $(2, 2, 1)$.
(c) Find the acute angle that the tangent plane at the point $(2, 2, 1)$ makes with the xy-plane.

16. Consider the surface $xz - yz^3 + yz^2 = 2$.
(a) Use the method of Example 2 to find an equation of the tangent plane to the surface at the point $(2, -1, 1)$.
(b) Find parametric equations of the line that is normal to the surface at the point $(2, -1, 1)$.
(c) Find the acute angle that the tangent plane at the point $(2, -1, 1)$ makes with the xy-plane.

In Exercises 17 and 18, find two unit vectors that are normal to the given surface at the point P.

17. $\sqrt{\dfrac{z + x}{y - 1}} = z^2$; $P(3, 5, 1)$

18. $\sin xz - 4\cos yz = 4$; $P(\pi, \pi, 1)$

19. Show that every line that is normal to the sphere

$$x^2 + y^2 + z^2 = 1$$

passes through the origin.

20. Find all points on the ellipsoid $2x^2 + 3y^2 + 4z^2 = 9$ at which the tangent plane is parallel to the plane $x - 2y + 3z = 5$.

21. Find all points on the surface $x^2 + y^2 - z^2 = 1$ at which the normal line is parallel to the line through $P(1, -2, 1)$ and $Q(4, 0, -1)$.

22. Show that the ellipsoid $2x^2 + 3y^2 + z^2 = 9$ and the sphere

$$x^2 + y^2 + z^2 - 6x - 8y - 8z + 24 = 0$$

have a common tangent plane at the point $(1, 1, 2)$.

23. Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $x^2 + 4y^2 + z^2 = 9$ at the point $(1, -1, 2)$.

24. Find parametric equations for the tangent line to the curve of intersection of the cone $z = \sqrt{x^2 + y^2}$ and the plane $x + 2y + 2z = 20$ at the point $(4, 3, 5)$.

25. Find parametric equations for the tangent line to the curve of intersection of the cylinders $x^2 + z^2 = 25$ and $y^2 + z^2 = 25$ at the point $(3, -3, 4)$.

C **26.** The accompanying figure shows the intersection of the surfaces $z = 8 - x^2 - y^2$ and $4x + 2y - z = 0$.
(a) Find parametric equations for the tangent line to the curve of intersection at the point $(0, 2, 4)$.
(b) Use a CAS to generate a reasonable facsimile of the figure. You need not generate the colors, but try to obtain a similar viewpoint.

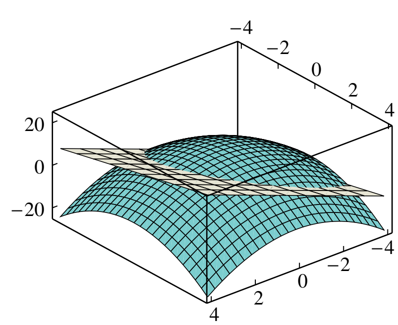


Figure Ex-26

27. Show that the equation of the plane that is tangent to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at (x_0, y_0, z_0) can be written in the form

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = 1$$

28. Show that the equation of the plane that is tangent to the paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

at (x_0, y_0, z_0) can be written in the form

$$z + z_0 = \frac{2x_0 x}{a^2} + \frac{2y_0 y}{b^2}$$

29. Prove: If the surfaces $z = f(x, y)$ and $z = g(x, y)$ intersect at $P(x_0, y_0, z_0)$, and if f and g are differentiable at (x_0, y_0), then the normal lines at P are perpendicular if and only if

$$f_x(x_0, y_0)g_x(x_0, y_0) + f_y(x_0, y_0)g_y(x_0, y_0) = -1$$

30. Use the result in Exercise 29 to show that the normal lines to the cones $z = \sqrt{x^2 + y^2}$ and $z = -\sqrt{x^2 + y^2}$ are perpendicular to the normal lines to the sphere $x^2 + y^2 + z^2 = a^2$ at every point of intersection (see Figure Ex-32).

31. Two surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$ are said to be ***orthogonal*** at a point P of intersection if ∇f and ∇g are nonzero at P and the normal lines to the surfaces are perpendicular at P. Show that if $\nabla f(x_0, y_0, z_0) \neq \mathbf{0}$ and $\nabla g(x_0, y_0, z_0) \neq \mathbf{0}$, then the surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$ are orthogonal at the point (x_0, y_0, z_0) if and only if

$$f_x g_x + f_y g_y + f_z g_z = 0$$

at this point. [*Note:* This is a more general version of the result in Exercise 29.]

32. Use the result of Exercise 31 to show that the sphere $x^2 + y^2 + z^2 = a^2$ and the cone $z^2 = x^2 + y^2$ are orthogonal at every point of intersection (see the accompanying figure).

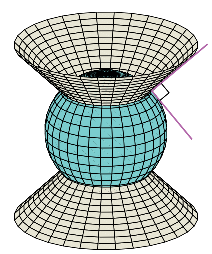


Figure Ex-32

33. Show that the volume of the solid bounded by the coordinate planes and the plane tangent to the portion of the surface $xyz = k$, $k > 0$, in the first octant does not depend on the point of tangency.

14.8 MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

Earlier in this text we learned how to find maximum and minimum values of a function of one variable. In this section we will develop similar techniques for functions of two variables.

EXTREMA

If we imagine the graph of a function f of two variables to be a mountain range (Figure 14.8.1), then the mountaintops, which are the high points in their immediate vicinity, are called *relative maxima* of f, and the valley bottoms, which are the low points in their immediate vicinity, are called *relative minima* of f.

Just as a geologist might be interested in finding the highest mountain and deepest valley in an entire mountain range, so a mathematician might be interested in finding the largest and smallest values of $f(x, y)$ over the *entire* domain of f. These are called the *absolute maximum* and *absolute minimum values* of f. The following definitions make these informal ideas precise.

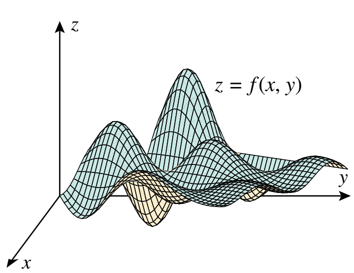


Figure 14.8.1

14.8.1 DEFINITION. A function f of two variables is said to have a ***relative maximum*** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an ***absolute maximum*** at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) in the domain of f.

14.8.2 DEFINITION. A function f of two variables is said to have a ***relative minimum*** at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an ***absolute minimum*** at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) in the domain of f.

If f has a relative maximum or a relative minimum at (x_0, y_0), then we say that f has a ***relative extremum*** at (x_0, y_0), and if f has an absolute maximum or absolute minimum at (x_0, y_0), then we say that f has an ***absolute extremum*** at (x_0, y_0).

Figure 14.8.2 shows the graph of a function f whose domain is the square region in the xy-plane whose points satisfy the inequalities $0 \le x \le 1, 0 \le y \le 1$. The function f has relative minima at the points A and C and a relative maximum at B. There is an absolute minimum at A and an absolute maximum at D.

For functions of two variables we will be concerned with two important questions:

- Are there any relative or absolute extrema?
- If so, where are they located?

OPEN AND CLOSED SETS

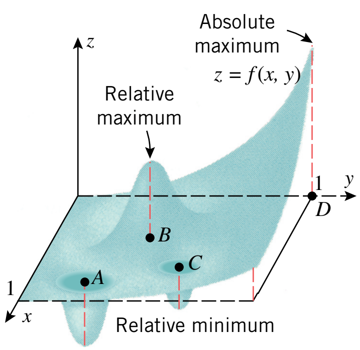


Figure 14.8.2

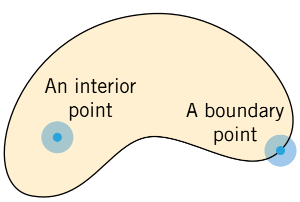


Figure 14.8.3

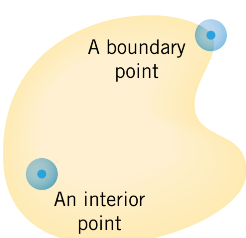


Figure 14.8.4

In our study of extrema for functions of one variable, the domains of the functions we encountered were generally intervals. For functions of two or three variables the situation is more complicated, so we will need to discuss some terminology about sets in 2-space and 3-space that will be helpful when we want to accurately describe the domain of a function of two or three variables.

If D is a set of points in 2-space, then a point (x_0, y_0) is called an ***interior point*** of D if there is *some* circular disk with positive radius, centered at (x_0, y_0), and containing only points in D (Figure 14.8.3). A point (x_0, y_0) is called a ***boundary point*** of D if *every* circular disk with positive radius and centered at (x_0, y_0) contains both points in D and points not in D (Figure 14.8.3). Similarly, if D is a set of points in 3-space, then a point (x_0, y_0, z_0) is called an ***interior point*** of D if there is *some* spherical ball with positive radius, centered at (x_0, y_0, z_0), and containing only points in D (Figure 14.8.4). A point (x_0, y_0, z_0) is called a ***boundary point*** of D if *every* spherical ball with positive radius and centered at (x_0, y_0, z_0) contains both points in D and points not in D (Figure 14.8.4).

For a set D in either 2-space or 3-space, the set of all boundary points of D is called the ***boundary*** of D and the set of all interior points of D is called the ***interior*** of D.

Recall that an open interval (a, b) on a coordinate line contains *neither* of its endpoints and a closed interval $[a, b]$ contains *both* of its endpoints. Analogously, a set D in 2-space or 3-space is called ***open*** if it contains *none* of its boundary points and ***closed*** if it contains *all* of its boundary points. The set D of all points in 2-space has no boundary points; it is regarded as both open and closed. Similarly, the set D of all points in 3-space is both open and closed.

Example 1 Let D be the set containing points in the xy-plane that are inside or on the circle of radius 1 centered at the origin. The set D, its interior I, and its boundary B can be expressed in set notation as

$$D = \{(x, y) : x^2 + y^2 \le 1\}, \quad I = \{(x, y) : x^2 + y^2 < 1\}, \quad B = \{(x, y) : x^2 + y^2 = 1\}$$

respectively (Figure 14.8.5). The sets B and D are closed and the set I is open. ◀

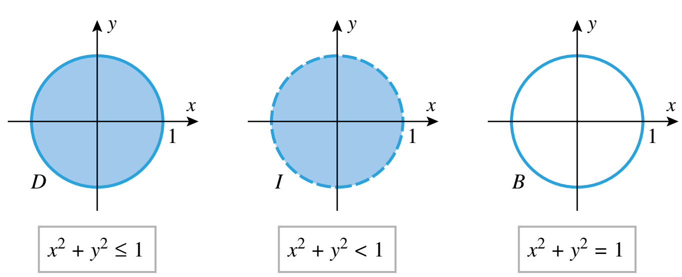


Figure 14.8.5

BOUNDED SETS

Just as we distinguished between finite intervals and infinite intervals on the real line, so we will want to distinguish between regions of "finite extent" and regions of "infinite extent" in 2-space and 3-space. A set of points in 2-space is called ***bounded*** if the entire set can be contained within some rectangle, and is called ***unbounded*** if there is no rectangle that contains all the points of the set. Similarly, a set of points in 3-space is ***bounded*** if the entire set can be contained within some box, and is unbounded otherwise (Figure 14.8.6).

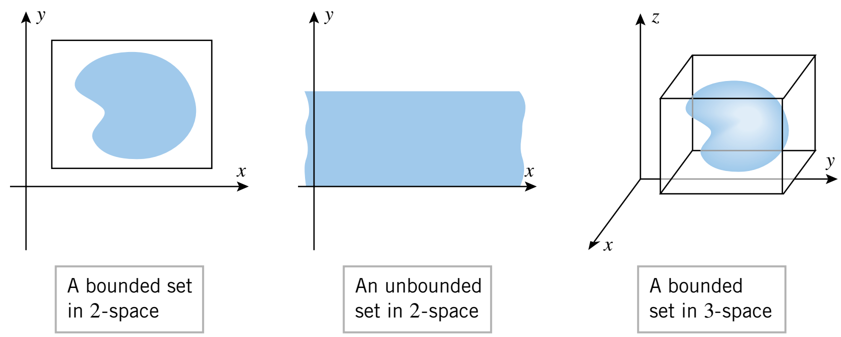


Figure 14.8.6

THE EXTREME-VALUE THEOREM

For functions of one variable that are continuous on a closed interval, the Extreme-Value Theorem (Theorem 4.5.3) answered the existence question for absolute extrema. The following theorem, which we state without proof, is the corresponding result for functions of two variables.

> **14.8.3** THEOREM (***Extreme-Value Theorem***). *If $f(x, y)$ is continuous on a closed and bounded set R, then f has both an absolute maximum and an absolute minimum on R.*

Example 2 The square region R whose points satisfy the inequalities

$$0 \leq x \leq 1 \quad \text{and} \quad 0 \leq y \leq 1$$

is a closed and bounded set in the xy-plane. The function f whose graph is shown in Figure 14.8.2 is continuous on R; thus, it is guaranteed to have an absolute maximum and minimum on R by the last theorem. These occur at points D and A that are shown in the figure. ◀

REMARK. If any of the conditions in the Extreme-Value Theorem fail to hold, then there is no guarantee that an absolute maximum or absolute minimum exists on the region R. Thus, a discontinuous function on a closed and bounded set need not have any absolute extrema, and a continuous function on a set that is not closed and bounded also need not have any absolute extrema.

FINDING RELATIVE EXTREMA

Recall that if a function g of one variable has a relative extremum at a number x_0 where g is differentiable, then $g'(x_0) = 0$. To obtain the analog of this result for functions of two variables, suppose that $f(x, y)$ has a relative maximum at a point (x_0, y_0) and that the partial derivatives of f exist at (x_0, y_0). It seems plausible geometrically that the traces of the surface $z = f(x, y)$ on the planes $x = x_0$ and $y = y_0$ have horizontal tangent lines at (x_0, y_0) (Figure 14.8.7), so

$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0$$

The same conclusion holds if f has a relative minimum at (x_0, y_0), all of which suggests the following result, which we state without formal proof.

> **14.8.4** THEOREM. *If f has a relative extremum at a point (x_0, y_0), and if the first-order partial derivatives of f exist at this point, then*
>
> $$f_x(x_0, y_0) = 0 \quad \textit{and} \quad f_y(x_0, y_0) = 0$$

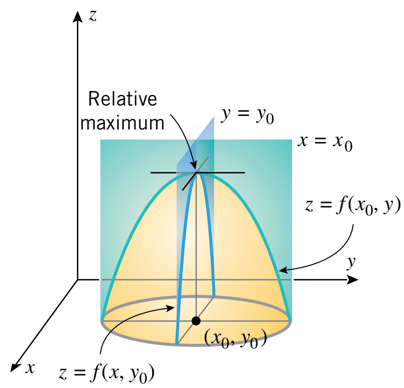


Figure 14.8.7

Recall that the *critical numbers* of a function f of one variable are those values of x in the domain of f at which $f'(x) = 0$ or f is not differentiable. The following definition is the analog for functions of two variables.

14.8.5 DEFINITION. A point (x_0, y_0) in the domain of a function $f(x, y)$ is called a ***critical point*** of the function if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ or if one or both partial derivatives do not exist at (x_0, y_0).

It follows from this definition and Theorem 14.8.4 that relative extrema occur at critical points, just as for a function of one variable. However, recall that for a function of one variable a relative extremum need not occur at *every* critical number. For example, the function might have an inflection point with a horizontal tangent line at the critical number (see Figure 4.2.4). Similarly, a function of two variables need not have a relative extremum at every critical point. For example, consider the function

$$f(x, y) = y^2 - x^2$$

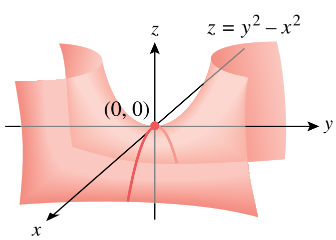
This function, whose graph is the hyperbolic paraboloid shown in Figure 14.8.8, has a critical point at (0, 0), since

$$f_x(x, y) = -2x \quad \text{and} \quad f_y(x, y) = 2y$$

from which it follows that

$$f_x(0, 0) = 0 \quad \text{and} \quad f_y(0, 0) = 0$$

However, the function f has neither a relative maximum nor a relative minimum at (0, 0). For obvious reasons, the point (0, 0) is called a *saddle point* of f. In general, we will say that a surface $z = f(x, y)$ has a ***saddle point*** at (x_0, y_0) if there are two distinct vertical planes through this point such that the trace of the surface in one of the planes has a relative maximum at (x_0, y_0) and the trace in the other has a relative minimum at (x_0, y_0).



The function $f(x, y) = y^2 - x^2$ has neither a relative maximum nor a relative minimum at the critical point (0, 0).

Figure 14.8.8

Example 3 The three functions graphed in Figure 14.8.9 all have critical points at (0, 0). For the paraboloids, the partial derivatives at the origin are zero. You can check this algebraically by evaluating the partial derivatives at (0, 0), but you can see it geometrically by observing that the traces in the xz-plane and yz-plane have horizontal tangent lines at (0, 0). For the cone neither partial derivative exists at the origin because the traces in the xz-plane and the yz-plane have corners there. The paraboloid in part (a) and the cone in part (c) have a relative minimum and absolute minimum at the origin, and the paraboloid in part (b) has a relative maximum and an absolute maximum at the origin. ◀

THE SECOND PARTIALS TEST

For functions of one variable the second derivative test (Theorem 4.2.4) was used to determine the behavior of a function at a critical number. The following theorem, which is

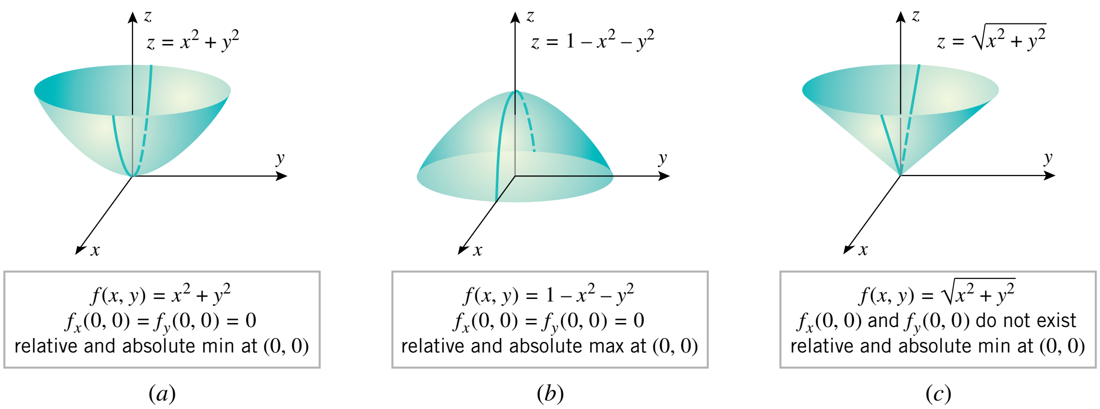


Figure 14.8.9

usually proved in advanced calculus, is the analog of that theorem for functions of two variables.

14.8.6 THEOREM (*The Second Partials Test*). *Let f be a function of two variables with continuous second-order partial derivatives in some disk centered at a critical point (x_0, y_0), and let*

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$$

(*a*) *If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a relative minimum at (x_0, y_0).*
(*b*) *If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a relative maximum at (x_0, y_0).*
(*c*) *If $D < 0$, then f has a saddle point at (x_0, y_0).*
(*d*) *If $D = 0$, then no conclusion can be drawn.*

Example 4 Locate all relative extrema and saddle points of

$$f(x, y) = 3x^2 - 2xy + y^2 - 8y$$

Solution. Since $f_x(x, y) = 6x - 2y$ and $f_y(x, y) = -2x + 2y - 8$, the critical points of f satisfy the equations

$$\begin{aligned} 6x - 2y &= 0 \\ -2x + 2y - 8 &= 0 \end{aligned}$$

Solving these for x and y yields $x = 2$, $y = 6$ (verify), so (2, 6) is the only critical point. To apply Theorem 14.8.6 we need the second-order partial derivatives

$$f_{xx}(x, y) = 6, \quad f_{yy}(x, y) = 2, \quad f_{xy}(x, y) = -2$$

At the point (2, 6) we have

$$D = f_{xx}(2, 6) f_{yy}(2, 6) - f_{xy}^2(2, 6) = (6)(2) - (-2)^2 = 8 > 0$$

and

$$f_{xx}(2, 6) = 6 > 0$$

so f has a relative minimum at (2, 6) by part (*a*) of the second partials test. Figure 14.8.10 shows a graph of f in the vicinity of the relative minimum. ◀

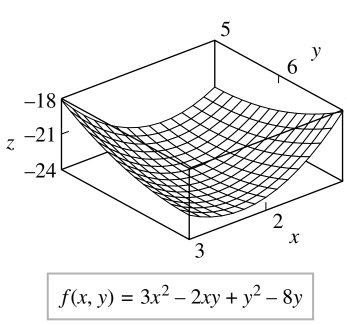


Figure 14.8.10

Example 5 Locate all relative extrema and saddle points of

$$f(x, y) = 4xy - x^4 - y^4$$

Solution. Since

$$\begin{aligned} f_x(x, y) &= 4y - 4x^3 \\ f_y(x, y) &= 4x - 4y^3 \end{aligned} \qquad (1)$$

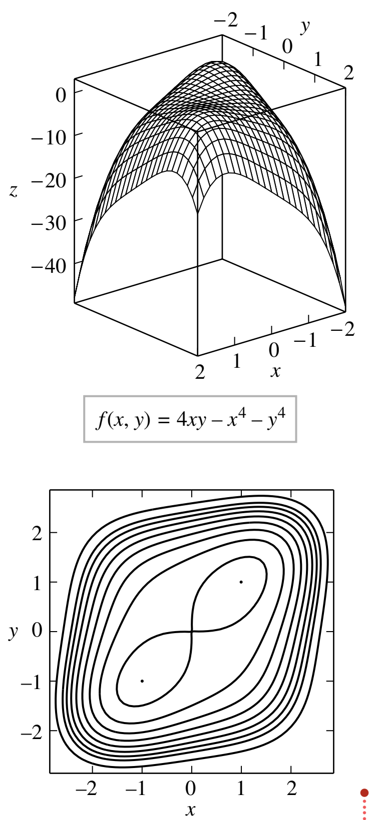


Figure 14.8.11

the critical points of f have coordinates satisfying the equations

$$\begin{aligned} 4y - 4x^3 &= 0 \\ 4x - 4y^3 &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} y &= x^3 \\ x &= y^3 \end{aligned} \tag{2}$$

Substituting the top equation in the bottom yields $x = (x^3)^3$ or $x^9 - x = 0$ or $x(x^8 - 1) = 0$, which has solutions $x = 0$, $x = 1$, $x = -1$. Substituting these values in the top equation of (2), we obtain the corresponding y-values $y = 0$, $y = 1$, $y = -1$. Thus, the critical points of f are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

From (1),

$$f_{xx}(x, y) = -12x^2, \quad f_{yy}(x, y) = -12y^2, \quad f_{xy}(x, y) = 4$$

which yields the following table:

CRITICAL POINT (x_0, y_0)	$f_{xx}(x_0, y_0)$	$f_{yy}(x_0, y_0)$	$f_{xy}(x_0, y_0)$	$D = f_{xx}f_{yy} - f_{xy}^2$
$(0, 0)$	0	0	4	−16
$(1, 1)$	−12	−12	4	128
$(-1, -1)$	−12	−12	4	128

At the points $(1, 1)$ and $(-1, -1)$, we have $D > 0$ and $f_{xx} < 0$, so relative maxima occur at these critical points. At $(0, 0)$ there is a saddle point since $D < 0$. The surface and a contour plot are shown in Figure 14.8.11. ◀

FOR THE READER. The "figure eight" pattern at $(0, 0)$ in the contour plot for the surface in Figure 14.8.11 is typical for level curves that pass through a saddle point. If a bug starts at the point $(0, 0, 0)$ on the surface, in how many directions can it walk and remain in the xy-plane?

The following theorem, which is the analog for functions of two variables of Theorem 4.5.4, will lead to an important method for finding absolute extrema.

> **14.8.7 THEOREM.** *If a function f of two variables has an absolute extremum (either an absolute maximum or an absolute minimum) at an interior point of its domain, then this extremum occurs at a critical point.*

Proof. If f has an absolute maximum at the point (x_0, y_0) in the interior of the domain of f, then f has a relative maximum at (x_0, y_0). If both partial derivatives exist at (x_0, y_0), then

$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0$$

by Theorem 14.8.4, so (x_0, y_0) is a critical point of f. If either partial derivative does not exist, then again (x_0, y_0) is a critical point, so (x_0, y_0) is a critical point in all cases. The proof for an absolute minimum is similar. ■

FINDING ABSOLUTE EXTREMA ON CLOSED AND BOUNDED SETS

If $f(x, y)$ is continuous on a closed and bounded set R, then the Extreme-Value Theorem (Theorem 14.8.3) guarantees the existence of an absolute maximum and an absolute minimum of f on R. These absolute extrema can occur either on the boundary of R or in the interior of R, but if an absolute extremum occurs in the interior, then it occurs at a critical point by Theorem 14.8.7. Thus, we are led to the following procedure for finding absolute extrema:

How to Find the Absolute Extrema of a Continuous Function f of Two Variables on a Closed and Bounded Set R

Step 1. Find the critical points of f that lie in the interior of R.

Step 2. Find all boundary points at which the absolute extrema can occur.

Step 3. Evaluate $f(x, y)$ at the points obtained in the preceding steps. The largest of these values is the absolute maximum and the smallest the absolute minimum.

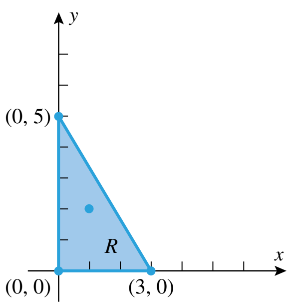


Figure 14.8.12

Example 6 Find the absolute maximum and minimum values of

$$f(x, y) = 3xy - 6x - 3y + 7 \tag{3}$$

on the closed triangular region R with vertices (0, 0), (3, 0), and (0, 5).

Solution. The region R is shown in Figure 14.8.12. We have

$$\frac{\partial f}{\partial x} = 3y - 6 \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x - 3$$

so all critical points occur where

$$3y - 6 = 0 \quad \text{and} \quad 3x - 3 = 0$$

Solving these equations yields $x = 1$ and $y = 2$, so (1, 2) is the only critical point. As shown in Figure 14.8.12, this critical point is in the interior of R.

Next, we want to determine the locations of the points on the boundary of R at which the absolute extrema might occur. The boundary of R consists of three line segments, each of which we will treat separately:

The line segment between (0, 0) *and* (3, 0): On this line segment we have $y = 0$, so (3) simplifies to a function of the single variable x,

$$u(x) = f(x, 0) = -6x + 7, \quad 0 \le x \le 3$$

This function has no critical numbers because $u'(x) = -6$ is nonzero for all x. Thus the extreme values of $u(x)$ occur at the endpoints $x = 0$ and $x = 3$, which correspond to the points (0, 0) and (3, 0) of R.

The line segment between (0, 0) *and* (0, 5): On this line segment we have $x = 0$, so (3) simplifies to a function of the single variable y,

$$v(y) = f(0, y) = -3y + 7, \quad 0 \le y \le 5$$

This function has no critical numbers because $v'(y) = -3$ is nonzero for all y. Thus, the extreme values of $v(y)$ occur at the endpoints $y = 0$ and $y = 5$, which correspond to the points (0, 0) and (0, 5) of R.

The line segment between (3, 0) *and* (0, 5): In the xy-plane, an equation for this line segment is

$$y = -\tfrac{5}{3}x + 5, \quad 0 \le x \le 3 \tag{4}$$

so (3) simplifies to a function of the single variable x,

$$\begin{aligned} w(x) &= f\left(x, -\tfrac{5}{3}x + 5\right) = 3x\left(-\tfrac{5}{3}x + 5\right) - 6x - 3\left(-\tfrac{5}{3}x + 5\right) + 7 \\ &= -5x^2 + 14x - 8, \qquad 0 \le x \le 3 \end{aligned}$$

Since $w'(x) = -10x + 14$, the equation $w'(x) = 0$ yields $x = \frac{7}{5}$ as the only critical number of w. Thus, the extreme values of w occur either at the critical number $x = \frac{7}{5}$ or at the endpoints $x = 0$ and $x = 3$. The endpoints correspond to the points (0, 5) and (3, 0) of R, and from (4) the critical number corresponds to $\left(\frac{7}{5}, \frac{8}{3}\right)$.

Finally, Table 14.8.1 lists the values of $f(x, y)$ at the interior critical point and at the points on the boundary where an absolute extremum can occur. From the table we conclude that the absolute maximum value of f is $f(0, 0) = 7$ and the absolute minimum value is $f(3, 0) = -11$. ◀

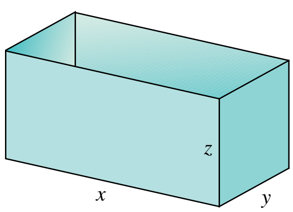
Table 14.8.1

(x, y)	$(0, 0)$	$(3, 0)$	$(0, 5)$	$\left(\frac{7}{5}, \frac{8}{3}\right)$	$(1, 2)$
$f(x, y)$	7	–11	–8	$\frac{9}{5}$	1

Example 7 Determine the dimensions of a rectangular box, open at the top, having a volume of 32 ft^3, and requiring the least amount of material for its construction.

Solution. Let

x = length of the box (in feet)
y = width of the box (in feet)
z = height of the box (in feet)
S = surface area of the box (in square feet)



Two sides each have area xz.
Two sides each have area yz.
The base has area xy.

Figure 14.8.13

We may reasonably assume that the box with least surface area requires the least amount of material, so our objective is to minimize the surface area

$$S = xy + 2xz + 2yz \tag{5}$$

(Figure 14.8.13) subject to the volume requirement

$$xyz = 32 \tag{6}$$

From (6) we obtain $z = 32/xy$, so (5) can be rewritten as

$$S = xy + \frac{64}{y} + \frac{64}{x} \tag{7}$$

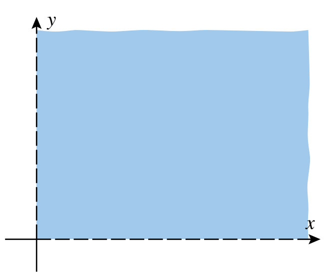


Figure 14.8.14

which expresses S as a function of two variables. The dimensions x and y in this formula must be positive, but otherwise have no limitation, so our problem reduces to finding the absolute minimum value of S over the first quadrant: $x > 0$, $y > 0$ (Figure 14.8.14). Because this region is neither closed nor bounded we have no mathematical guarantee at this stage that an absolute minimum exists. However, note that S will have a large value at any point (x, y) in the first quadrant for which the product xy is large or for which either x or y is close to 0. We can use this observation to prove the existence of an absolute minimum value of S.

Let R denote the region in the first quadrant defined by the inequalities

$$1/2 \le x, \quad 1/2 \le y, \quad \text{and} \quad xy \le 128$$

This region is both closed and bounded (verify) and the function S is continuous on R. It follows from Theorem 14.8.3 that S has an absolute minimum on R. Furthermore, note that $S > 128$ at any point (x, y) not in R and that the point $(8, 8)$ belongs to R with $S = 80 < 128$ at this point. We conclude that the minimum value of S on R is also the minimum value of S on the entire first quadrant.

Since S has an absolute minimum value in the first quadrant, it must occur at a critical point of S. Differentiating (7) we obtain

$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2}, \quad \frac{\partial S}{\partial y} = x - \frac{64}{y^2} \tag{8}$$

so the coordinates of the critical points of S satisfy

$$y - \frac{64}{x^2} = 0, \quad x - \frac{64}{y^2} = 0$$

Solving the first equation for y yields

$$y = \frac{64}{x^2} \tag{9}$$

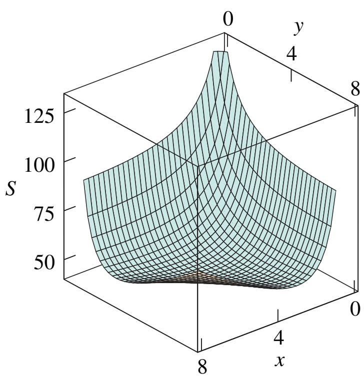
and substituting this expression in the second equation yields

$$x - \frac{64}{(64/x^2)^2} = 0$$

which can be rewritten as

$$x\left(1 - \frac{x^3}{64}\right) = 0$$

The solutions of this equation are $x = 0$ and $x = 4$. Since we require $x > 0$, the only solution of significance is $x = 4$. Substituting this value into (9) yields $y = 4$. Substituting $x = 4$ and $y = 4$ into (6) yields $z = 2$, so the box using least material has a height of 2 ft and a square base whose edges are 4 ft long. ◀



$S = xy + \dfrac{64}{y} + \dfrac{64}{x}$

Figure 14.8.15

REMARK. Fortunately, in our solution to Example 7 we were able to prove the existence of an absolute minimum of S on the first quadrant. The general problem of finding the absolute extrema of a function on an unbounded region, or on a region that is not closed, can be difficult and will not be considered in this text. However, in applied problems we can sometimes use physical considerations to deduce that an absolute extremum has been found. For example, the graph of Equation (7) in Figure 14.8.15 strongly suggests that the relative minimum at $x = 4$ and $y = 4$ is also an absolute minimum.

EXERCISE SET 14.8 Graphing Utility C CAS

In Exercises 1 and 2, locate all absolute maxima and minima, if any, by inspection. Then check your answers using calculus.

1. (a) $f(x, y) = (x - 2)^2 + (y + 1)^2$
(b) $f(x, y) = 1 - x^2 - y^2$ (c) $f(x, y) = x + 2y - 5$

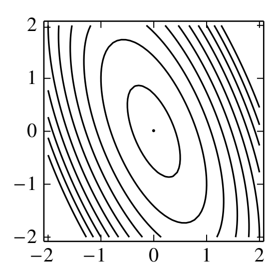
2. (a) $f(x, y) = 1 - (x + 1)^2 - (y - 5)^2$
(b) $f(x, y) = e^{xy}$ (c) $f(x, y) = x^2 - y^2$

In Exercises 3 and 4, complete the squares and locate all absolute maxima and minima, if any, by inspection. Then check your answers using calculus.

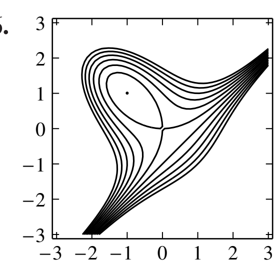
3. $f(x, y) = 13 - 6x + x^2 + 4y + y^2$

4. $f(x, y) = 1 - 2x - x^2 + 4y - 2y^2$

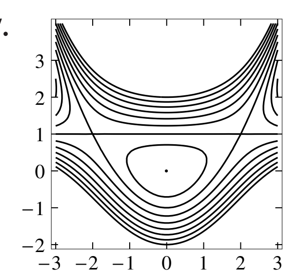
In Exercises 5–8, the contour plots show all significant features of the function. Make a conjecture about the number and the location of all relative extrema and saddle points, and then use calculus to check your conjecture.

5.


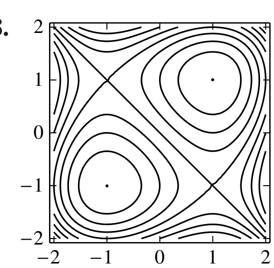
$f(x, y) = 3x^2 + 2xy + y^2$

6.


$f(x, y) = x^3 - 3xy - y^3$

7.


$f(x, y) = x^2 + 2y^2 - x^2y$

8.


$f(x, y) = x^3 + y^3 - 3x - 3y$

In Exercises 9–20, locate all relative maxima, relative minima, and saddle points, if any.

9. $f(x, y) = y^2 + xy + 3y + 2x + 3$

10. $f(x, y) = x^2 + xy - 2y - 2x + 1$

11. $f(x, y) = x^2 + xy + y^2 - 3x$

12. $f(x, y) = xy - x^3 - y^2$ **13.** $f(x, y) = x^2 + y^2 + \dfrac{2}{xy}$

14. $f(x, y) = xe^y$ **15.** $f(x, y) = x^2 + y - e^y$

16. $f(x, y) = xy + \dfrac{2}{x} + \dfrac{4}{y}$ **17.** $f(x, y) = e^x \sin y$

18. $f(x, y) = y \sin x$ **19.** $f(x, y) = e^{-(x^2+y^2+2x)}$

20. $f(x, y) = xy + \dfrac{a^3}{x} + \dfrac{b^3}{y} \quad (a \neq 0, b \neq 0)$

C **21.** Use a CAS to generate a contour plot of

$$f(x, y) = 2x^2 - 4xy + y^4 + 2$$

for $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$, and use the plot to approximate the locations of all relative extrema and saddle points in the region. Check your answer using calculus, and identify the relative extrema as relative maxima or minima.

C **22.** Use a CAS to generate a contour plot of

$$f(x, y) = 2y^2x - yx^2 + 4xy$$

for $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$, and use the plot to approximate the locations of all relative extrema and saddle points in the region. Check your answer using calculus, and identify the relative extrema as relative maxima or minima.

23. (a) Show that the second partials test provides no information about the critical points of $f(x, y) = x^4 + y^4$.
(b) Classify all critical points of f as relative maxima, relative minima, or saddle points.

24. (a) Show that the second partials test provides no information about the critical points of $f(x, y) = x^4 - y^4$.
(b) Classify all critical points of f as relative maxima, relative minima, or saddle points.

25. Recall from Theorem 4.5.5 that if a continuous function of one variable has exactly one relative extremum on an interval, then that relative extremum is an absolute extremum on the interval. This exercise shows that this result does not extend to functions of two variables.
(a) Show that $f(x, y) = 3xe^y - x^3 - e^{3y}$ has only one critical point and that a relative maximum occurs there. (See the accompanying figure.)
(b) Show that f does not have an absolute maximum.

[This exercise is based on the article "The Only Critical Point in Town Test" by Ira Rosenholtz and Lowell Smylie, *Mathematics Magazine*, Vol. 58, No. 3, May 1985, pp. 149–150.]

26. If f is a continuous function of one variable with two relative maxima on an interval, then there must be a relative minimum between the relative maxima. (Convince yourself of this by drawing some pictures.) The purpose of this exercise is to show that this result does not extend to functions of two variables. Show that $f(x, y) = 4x^2e^y - 2x^4 - e^{4y}$ has two relative maxima but no other critical points (see the accompanying figure).

[This exercise is based on the problem "Two Mountains Without a Valley" proposed and solved by Ira Rosenholtz, *Mathematics Magazine*, Vol. 60, No. 1, February 1987, p. 48.]

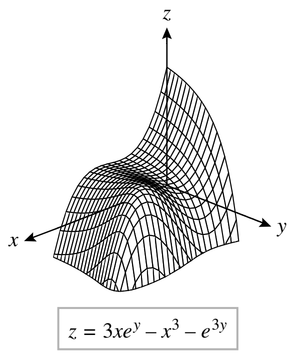


Figure Ex-25

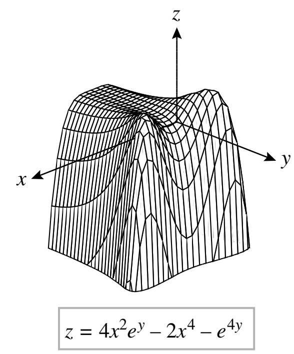


Figure Ex-26

In Exercises 27–32, find the absolute extrema of the given function on the indicated closed and bounded set R.

27. $f(x, y) = xy - x - 3y$; R is the triangular region with vertices $(0, 0)$, $(0, 4)$, and $(5, 0)$.

28. $f(x, y) = xy - 2x$; R is the triangular region with vertices $(0, 0)$, $(0, 4)$, and $(4, 0)$.

29. $f(x, y) = x^2 - 3y^2 - 2x + 6y$; R is the square region with vertices $(0, 0)$, $(0, 2)$, $(2, 2)$, and $(2, 0)$.

30. $f(x, y) = xe^y - x^2 - e^y$; R is the rectangular region with vertices $(0, 0)$, $(0, 1)$, $(2, 1)$, and $(2, 0)$.

31. $f(x, y) = x^2+2y^2-x$; R is the circular region $x^2+y^2 \leq 4$.

32. $f(x, y) = xy^2$; R is the region that satisfies the inequalities $x \geq 0$, $y \geq 0$, and $x^2 + y^2 \leq 1$.

33. Find three positive numbers whose sum is 48 and such that their product is as large as possible.

34. Find three positive numbers whose sum is 27 and such that the sum of their squares is as small as possible.

35. Find all points on the portion of the plane $x + y + z = 5$ in the first octant at which $f(x, y, z) = xy^2z^2$ has a maximum value.

36. Find the points on the surface $x^2 - yz = 5$ that are closest to the origin.

37. Find the dimensions of the rectangular box of maximum volume that can be inscribed in a sphere of radius a.

38. Find the maximum volume of a rectangular box with three faces in the coordinate planes and a vertex in the first octant on the plane $x + y + z = 1$.

39. A closed rectangular box with a volume of 16 ft^3 is made from two kinds of materials. The top and bottom are made of material costing 10¢ per square foot and the sides from material costing 5¢ per square foot. Find the dimensions of the box so that the cost of materials is minimized.

40. A manufacturer makes two models of an item, standard and deluxe. It costs \$40 to manufacture the standard model and \$60 for the deluxe. A market research firm estimates that if the standard model is priced at x dollars and the deluxe at y dollars, then the manufacturer will sell $500(y-x)$ of the standard items and $45{,}000+500(x-2y)$ of the deluxe each year. How should the items be priced to maximize the profit?

41. Consider the function

$$f(x, y) = 4x^2 - 3y^2 + 2xy$$

over the unit square $0 \le x \le 1, 0 \le y \le 1$.
(a) Find the maximum and minimum values of f on each edge of the square.
(b) Find the maximum and minimum values of f on each diagonal of the square.
(c) Find the maximum and minimum values of f on the entire square.

42. Show that among all parallelograms with perimeter l, a square with sides of length $l/4$ has maximum area. [*Hint:* The area of a parallelogram is given by the formula $A = ab\sin\alpha$, where a and b are the lengths of two adjacent sides and α is the angle between them.]

43. Determine the dimensions of a rectangular box, open at the top, having volume V, and requiring the least amount of material for its construction.

44. A length of sheet metal 27 inches wide is to be made into a water trough by bending up two sides as shown in the accompanying figure. Find x and ϕ so that the trapezoid-shaped cross section has a maximum area.

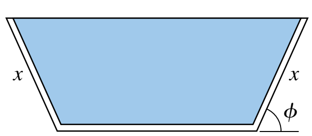


Figure Ex-44

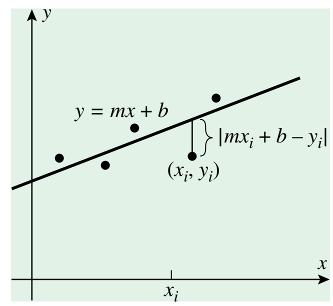
A common problem in experimental work is to obtain a mathematical relationship $y = f(x)$ between two variables x and y by "fitting" a curve to points in the plane that correspond to experimentally determined values of x and y, say

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

The curve $y = f(x)$ is called a ***mathematical model*** of the data. The general form of the function f is commonly determined by some underlying physical principle, but sometimes it is just determined by the pattern of the data. We are concerned with fitting a straight line $y = mx + b$ to data. Usually, the data will not lie on a line (possibly due to experimental error or variations in experimental conditions), so the problem is to find a line that fits the data "best" according to some criterion. One criterion for selecting the line of best fit is to choose m and b to minimize the function

$$g(m, b) = \sum_{i=1}^{n} (mx_i + b - y_i)^2$$

This is called the ***method of least squares***, and the resulting line is called the ***regression line*** or the ***least squares line of best fit***. Geometrically, $|mx_i + b - y_i|$ is the vertical distance between the data point (x_i, y_i) and the line $y = mx + b$.



These vertical distances are called the ***residuals*** of the data points, so the effect of minimizing $g(m, b)$ is to minimize the sum of the squares of the residuals. In Exercises 45 and 46, we will derive a formula for the regression line. More on this topic can be found in Section 1.7.

45. The purpose of this exercise is to find the values of m and b that produce the regression line.
(a) To minimize $g(m, b)$, we start by finding values of m and b such that $\partial g/\partial m = 0$ and $\partial g/\partial b = 0$. Show that these equations are satisfied if m and b satisfy the conditions

$$\left(\sum_{i=1}^{n} x_i^2\right) m + \left(\sum_{i=1}^{n} x_i\right) b = \sum_{i=1}^{n} x_i y_i$$

$$\left(\sum_{i=1}^{n} x_i\right) m + nb = \sum_{i=1}^{n} y_i$$

(b) Let $\bar{x} = \dfrac{1}{n}\sum_{i=1}^{n} x_i$ denote the arithmetic average of $x_1, x_2, \ldots, x_n$. Use the fact that $\sum_{i=1}^{n}(x_i - \bar{x})^2 \ge 0$ to show that

$$n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2 \ge 0$$

with equality if and only if all the x_i's are the same.

(c) Assuming that not all the x_i's are the same, prove that the equations in (a) have the unique solution

$$m = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - m\sum_{i=1}^{n} x_i\right)$$

[*Note:* We have shown that g has a critical point at these values of m and b. In the next exercise we will show that g has an absolute minimum at this critical point. Accepting this to be so, we have shown that the line $y = mx+b$ is the regression line for these values of m and b.]

46. Assume that not all the x_i's are the same, so that $g(m, b)$ has a unique critical point at the values of m and b obtained in Exercise 45(c). The purpose of this exercise is to show that g has an absolute minimum value at this point.

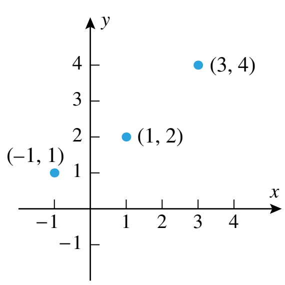
(a) Find the partial derivatives $g_{mm}(m, b)$, $g_{bb}(m, b)$, and $g_{mb}(m, b)$, and then apply the second partials test to show that g has a relative minimum at the critical point obtained in Exercise 45.

(b) Show that the graph of the equation $z = g(m, b)$ is a quadric surface. [*Hint:* See Formula (4) of Section 12.7.]

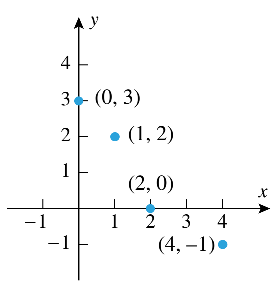
(c) It can be proved that the graph of $z = g(m, b)$ is an elliptic paraboloid. Accepting this to be so, show that this paraboloid opens in the positive z-direction, and explain how this shows that g has an absolute minimum at the critical point obtained in Exercise 45.

In Exercises 47–50, use the formulas obtained in Exercise 45 to find and draw the regression line. If you have a calculating utility that can calculate regression lines, use it to check your work.

47.



48.



49.

x	1	2	3	4
y	1.5	1.6	2.1	3.0

50.

x	1	2	3	4	5
y	4.2	3.5	3.0	2.4	2.0

51. The following table shows the life expectancy by year of birth of females in the United States:

YEAR OF BIRTH	1930	1940	1950	1960	1970	1980	1990
LIFE EXPECTANCY	61.6	65.2	71.1	73.1	74.7	77.5	78.8

(a) Take $t = 0$ to be the year 1930, and let y be the life expectancy for birth year t. Use the regression capability of a calculating utility to find the regression line of y as a function of t.

(b) Use a graphing utility to make a graph that shows the data points and the regression line.

(c) Use the regression line to make a conjecture about the life expectancy of females born in the year 2000.

52. A company manager wants to establish a relationship between the sales of a certain product and the price. The company research department provides the following data:

PRICE (x) IN DOLLARS	\$35.00	\$40.00	\$45.00	\$48.00	\$50.00
DAILY SALES VOLUME (y) IN UNITS	80	75	68	66	63

(a) Use a calculating utility to find the regression line of y as a function of x.

(b) Use a graphing utility to make a graph that shows the data points and the regression line.

(c) Use the regression line to make a conjecture about the number of units that would be sold at a price of \$60.00.

53. If a gas is cooled with its volume held constant, then it follows from the ***ideal gas law*** in physics that its pressure drops proportionally to the drop in temperature. The temperature that, in theory, corresponds to a pressure of zero is called ***absolute zero***. Suppose that an experiment produces the following data for pressure P versus temperature T with the volume held constant:

P (KILOPASCALS)	134	142	155	160	171	184
T (°CELSIUS)	0	20	40	60	80	100

(a) Use a calculating utility to find the regression line of P as a function of T.

(b) Use a graphing utility to make a graph that shows the data points and the regression line.

(c) Use the regression line to estimate the value of absolute zero in degrees Celsius.

54. Find:

(a) a continuous function $f(x, y)$ that is defined on the entire xy-plane and has no absolute extrema on the xy-plane;

(b) a function $f(x, y)$ that is defined everywhere on the rectangle $0 \le x \le 1$, $0 \le y \le 1$ and has no absolute extrema on the rectangle.

55. Show that if f has a relative maximum at (x_0, y_0), then $G(x) = f(x, y_0)$ has a relative maximum at $x = x_0$ and $H(y) = f(x_0, y)$ has a relative maximum at $y = y_0$.

14.9 LAGRANGE MULTIPLIERS

In this section we will study a powerful new method for maximizing or minimizing a function subject to constraints on the variables. This method will help us to solve certain optimization problems that are difficult or impossible to solve using the methods studied in the last section.

EXTREMUM PROBLEMS WITH CONSTRAINTS

In Example 7 of the last section, we solved the problem of minimizing

$$S = xy + 2xz + 2yz \tag{1}$$

subject to the constraint

$$xyz - 32 = 0 \tag{2}$$

This is a special case of the following general problem:

14.9.1 ***Three-Variable Extremum Problem with One Constraint***
Maximize or minimize the function $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$.

We will also be interested in the following two-variable version of this problem:

14.9.2 ***Two-Variable Extremum Problem with One Constraint***
Maximize or minimize the function $f(x, y)$ subject to the constraint $g(x, y) = 0$.

LAGRANGE MULTIPLIERS

One way to attack problems of these types is to solve the constraint equation for one of the variables in terms of the others and substitute the result into f. This produces a new function of one or two variables that incorporates the constraint and can be maximized or minimized by applying standard methods. For example, to solve the problem in Example 7 of the last section we substituted (2) into (1) to obtain

$$S = xy + \frac{64}{y} + \frac{64}{x}$$

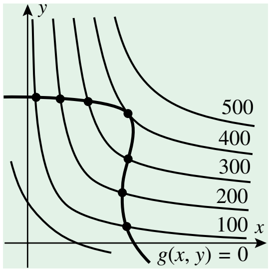
which we then minimized by finding the critical points and applying the second partials test. However, this approach hinges on our ability to solve the constraint equation for one of the variables in terms of the others. If this cannot be done, then other methods must be used. One such method, called the *method of Lagrange* multipliers*, will be discussed in this section.

*JOSEPH LOUIS LAGRANGE (1736–1813). French–Italian mathematician and astronomer. Lagrange, the son of a public official, was born in Turin, Italy. (Baptismal records list his name as Giuseppe Lodovico Lagrangia.) Although his father wanted him to be a lawyer, Lagrange was attracted to mathematics and astronomy after reading a memoir by the astronomer Halley. At age 16 he began to study mathematics on his own and by age 19 was appointed to a professorship at the Royal Artillery School in Turin. The following year Lagrange sent Euler solutions to some famous problems using new methods that eventually blossomed into a branch of mathematics called calculus of variations. These methods and Lagrange's applications of them to problems in celestial mechanics were so monumental that by age 25 he was regarded by many of his contemporaries as the greatest living mathematician.

In 1776, on the recommendations of Euler, he was chosen to succeed Euler as the director of the Berlin Academy. During his stay in Berlin, Lagrange distinguished himself not only in celestial mechanics, but also in algebraic equations and the theory of numbers. After twenty years in Berlin, he moved to Paris at the invitation of Louis XVI. He was given apartments in the Louvre and treated with great honor, even during the revolution.

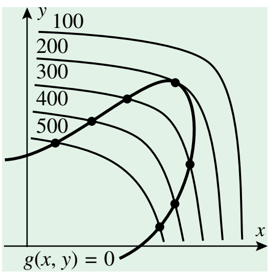
Napoleon was a great admirer of Lagrange and showered him with honors—count, senator, and Legion of Honor. The years Lagrange spent in Paris were devoted primarily to didactic treatises summarizing his mathematical conceptions. One of Lagrange's most famous works is a memoir, *Mécanique Analytique*, in which he reduced the theory of mechanics to a few general formulas from which all other necessary equations could be derived.

It is an interesting historical fact that Lagrange's father speculated unsuccessfully in several financial ventures, so his family was forced to live quite modestly. Lagrange himself stated that if his family had money, he would not have made mathematics his vocation. In spite of his fame, Lagrange was always a shy and modest man. On his death, he was buried with honor in the Pantheon.



Maximum of $f(x, y)$ is 400

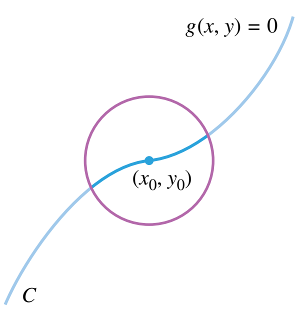
(*a*)



Minimum of $f(x, y)$ is 200

(*b*)

Figure 14.9.1



A constrained relative maximum occurs at (x_0, y_0) if $f(x_0, y_0) \ge f(x, y)$ on some segment of C that extends on both sides of (x_0, y_0).

Figure 14.9.2

To motivate the method of Lagrange multipliers, suppose that we are trying to maximize a function $f(x, y)$ subject to the constraint $g(x, y) = 0$. Geometrically, this means that we are looking for a point (x_0, y_0) on the graph of the constraint curve at which $f(x, y)$ is as large as possible. To help locate such a point, let us construct a contour plot of $f(x, y)$ in the same coordinate system as the graph of $g(x, y) = 0$. For example, Figure 14.9.1*a* shows some typical level curves of $f(x, y) = c$, which we have labeled $c = 100, 200, 300, 400$, and 500 for purposes of illustration. In this figure, each point of intersection of $g(x, y) = 0$ with a level curve is a candidate for a solution, since these points lie on the constraint curve. Among the seven such intersections shown in the figure, the maximum value of $f(x, y)$ occurs at the intersection where $f(x, y)$ has a value of 400, which is the point where the constraint curve and the level curve just touch. Observe that at this point the level curve and the constraint curve have a common normal line. This suggests that the maximum of $f(x, y)$, if it exists, occurs at a point (x_0, y_0) on the constraint curve at which the gradient vectors ∇f and ∇g are scalar multiples of one another; that is,

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \tag{3}$$

for some scalar λ. The same condition holds at points on the constraint curve where $f(x, y)$ has a minimum. For example, if the level curves are as shown in Figure 14.9.1*b*, then the minimum value of $f(x, y)$ occurs where the constraint curve just touches a level curve. Thus, to find the maximum or minimum of $f(x, y)$ subject to the constraint $g(x, y) = 0$, we look for points at which (3) holds—this is the method of Lagrange multipliers.

Our next objective in this section is to make the preceding intuitive argument more precise. For this purpose it will help to begin with some terminology about the problem of maximizing or minimizing a function $f(x, y)$ subject to a constraint $g(x, y) = 0$. As with other kinds of maximization and minimization problems, we need to distinguish between relative and absolute extrema. We will say that f has a ***constrained absolute maximum* (*minimum*)** at (x_0, y_0) if $f(x_0, y_0)$ is the largest (smallest) value of f on the constraint curve, and we will say that f has a ***constrained relative maximum* (*minimum*)** at (x_0, y_0) if $f(x_0, y_0)$ is the largest (smallest) value of f on some segment of the constraint curve that extends on both sides of the point (x_0, y_0) (Figure 14.9.2).

Let us assume that a constrained relative maximum or minimum occurs at the point (x_0, y_0) and for simplicity, let us further assume that the equation $g(x, y) = 0$ can be smoothly parametrized as

$$x = x(s), \quad y = y(s)$$

where s is an arc length parameter with reference point (x_0, y_0) at $s = 0$. Thus, the quantity

$$z = f(x(s), y(s))$$

has a relative maximum or minimum at $s = 0$, and this implies that $dz/ds = 0$ at that point. From the chain rule, this equation can be expressed as

$$\frac{dz}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} = \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right) \cdot \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}\right) = 0$$

where the derivatives are all evaluated at $s = 0$. However, the first factor in the dot product is the gradient of f, and the second factor is the unit tangent vector to the constraint curve. Since the point (x_0, y_0) corresponds to $s = 0$, it follows from this equation that

$$\nabla f(x_0, y_0) \cdot \mathbf{T}(0) = 0$$

which implies that the gradient is either $\mathbf{0}$ or is normal to the constraint curve at a constrained relative extremum. However, the constraint curve $g(x, y) = 0$ is a level curve for the function $g(x, y)$, so that if $\nabla g(x_0, y_0) \neq \mathbf{0}$, then $\nabla g(x_0, y_0)$ is normal to this curve at (x_0, y_0). It then follows that there is some scalar λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \tag{4}$$

This scalar is called a ***Lagrange multiplier***. Thus, the ***method of Lagrange multipliers*** for finding constrained relative extrema is to look for points on the constraint curve $g(x, y) = 0$ at which Equation (4) is satisfied for some scalar λ.

14.9.3 THEOREM (*Constrained-Extremum Principle for Two Variables and One Constraint*). *Let f and g be functions of two variables with continuous first partial derivatives on some open set containing the constraint curve $g(x, y) = 0$, and assume that $\nabla g \neq \mathbf{0}$ at any point on this curve. If f has a constrained relative extremum, then this extremum occurs at a point (x_0, y_0) on the constraint curve at which the gradient vectors $\nabla f(x_0, y_0)$ and $\nabla g(x_0, y_0)$ are parallel; that is, there is some number λ such that*

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

Example 1 At what point or points on the circle $x^2 + y^2 = 1$ does $f(x, y) = xy$ have an absolute maximum, and what is that maximum?

Solution. Since the circle $x^2+y^2 = 1$ is a closed and bounded set, and since $f(x, y) = xy$ is a continuous function, it follows from the Extreme-Value Theorem (Theorem 14.8.3) that f has an absolute maximum and an absolute minimum on the circle. To find these extrema, we will use Lagrange multipliers to find the constrained relative extrema, and then we will evaluate f at those relative extrema to find the absolute extrema.

We want to maximize $f(x, y) = xy$ subject to the constraint

$$g(x, y) = x^2 + y^2 - 1 = 0 \tag{5}$$

First we will look for constrained *relative* extrema. For this purpose we will need the gradients

$$\nabla f = y\mathbf{i} + x\mathbf{j} \quad \text{and} \quad \nabla g = 2x\mathbf{i} + 2y\mathbf{j}$$

From the formula for ∇g we see that $\nabla g = \mathbf{0}$ if and only if $x = 0$ and $y = 0$, so $\nabla g \neq \mathbf{0}$ at any point on the circle $x^2 + y^2 = 1$. Thus, at a constrained relative extremum we must have

$$\nabla f = \lambda \nabla g \quad \text{or} \quad y\mathbf{i} + x\mathbf{j} = \lambda(2x\mathbf{i} + 2y\mathbf{j})$$

which is equivalent to the pair of equations

$$y = 2x\lambda \quad \text{and} \quad x = 2y\lambda$$

It follows from these equations that if $x = 0$, then $y = 0$, and if $y = 0$, then $x = 0$. In either case we have $x^2 + y^2 = 0$, so the constraint equation $x^2 + y^2 = 1$ is not satisfied. Thus, we can assume that x and y are nonzero, and we can rewrite the equations as

$$\lambda = \frac{y}{2x} \quad \text{and} \quad \lambda = \frac{x}{2y}$$

from which we obtain

$$\frac{y}{2x} = \frac{x}{2y}$$

or

$$y^2 = x^2 \tag{6}$$

Substituting this in (5) yields

$$2x^2 - 1 = 0$$

from which we obtain $x = \pm 1/\sqrt{2}$. Each of these values, when substituted in Equation (6), produces y-values of $y = \pm 1/\sqrt{2}$. Thus, constrained relative extrema occur at the points $(1/\sqrt{2}, 1/\sqrt{2})$, $(1/\sqrt{2}, -1/\sqrt{2})$, $(-1/\sqrt{2}, 1/\sqrt{2})$, and $(-1/\sqrt{2}, -1/\sqrt{2})$. The values of xy at these points are as follows:

(x, y)	$(1/\sqrt{2}, 1/\sqrt{2})$	$(1/\sqrt{2}, -1/\sqrt{2})$	$(-1/\sqrt{2}, 1/\sqrt{2})$	$(-1/\sqrt{2}, -1/\sqrt{2})$
xy	1/2	–1/2	–1/2	1/2

Thus, the function $f(x, y) = xy$ has an absolute maximum of $\frac{1}{2}$ occurring at the two points $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$. Although it was not asked for, we can also see that f has an absolute minimum of $-\frac{1}{2}$ occurring at the points $(1/\sqrt{2}, -1/\sqrt{2})$ and

$(-1/\sqrt{2}, 1/\sqrt{2})$. Figure 14.9.3 shows some level curves $xy = c$ and the constraint curve in the vicinity of the maxima. A similar figure for the minima can be obtained using negative values of c for the level curves $xy = c$. ◀

REMARK. If c is a constant, then the functions $g(x, y)$ and $g(x, y) - c$ have the same gradient since the constant c drops out when we differentiate. Consequently, it is *not* essential to rewrite a constraint of the form $g(x, y) = c$ as $g(x, y) - c = 0$ in order to apply the constrained-extremum principle. Thus, in the last example, we could have kept the constraint in the form $x^2 + y^2 = 1$ and then taken $g(x, y) = x^2 + y^2$ rather than $g(x, y) = x^2 + y^2 - 1$.

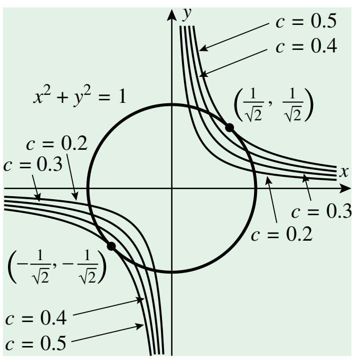


Figure 14.9.3

Example 2 Use the method of Lagrange multipliers to find the dimensions of a rectangle with perimeter p and maximum area.

Solution. Let

x = length of the rectangle

y = width of the rectangle

A = area of the rectangle

We want to maximize $A = xy$ on the line segment

$$2x + 2y = p, \quad 0 \le x, y \tag{7}$$

that corresponds to the perimeter constraint. This segment is a closed and bounded set, and since $f(x, y) = xy$ is a continuous function, it follows from the Extreme-Value Theorem (Theorem 14.8.3) that f has an absolute maximum on this segment. This absolute maximum must also be a constrained relative maximum since f is 0 at the endpoints of the segment and positive elsewhere on the segment. If $g(x, y) = 2x + 2y$, then we have

$$\nabla f = y\mathbf{i} + x\mathbf{j} \quad \text{and} \quad \nabla g = 2\mathbf{i} + 2\mathbf{j}$$

Noting that $\nabla g \neq \mathbf{0}$, it follows from (4) that

$$y\mathbf{i} + x\mathbf{j} = \lambda(2\mathbf{i} + 2\mathbf{j})$$

at a constrained relative maximum. This is equivalent to the two equations

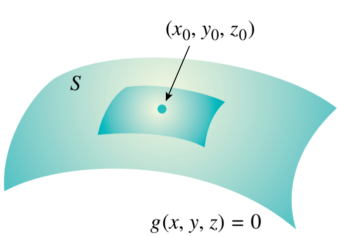
$$y = 2\lambda \quad \text{and} \quad x = 2\lambda$$

Eliminating λ from these equations we obtain $x = y$, which shows that the rectangle is actually a square. Using this condition and constraint (7), we obtain $x = p/4$, $y = p/4$. ◀

THREE VARIABLES AND ONE CONSTRAINT

The method of Lagrange multipliers can also be used to maximize or minimize a function of three variables $f(x, y, z)$ subject to a constraint $g(x, y, z) = 0$. As a rule, the graph of $g(x, y, z) = 0$ will be some surface S in 3-space. Thus, from a geometric viewpoint, the problem is to maximize or minimize $f(x, y, z)$ as (x, y, z) varies over the surface S (Figure 14.9.4). As usual, we distinguish between relative and absolute extrema. We will say that f has a ***constrained absolute maximum (minimum)*** at (x_0, y_0, z_0) if $f(x_0, y_0, z_0)$ is the largest (smallest) value of $f(x, y, z)$ on S, and we will say that f has a ***constrained relative maximum (minimum)*** at (x_0, y_0, z_0) if $f(x_0, y_0, z_0)$ is the largest (smallest) value of $f(x, y, z)$ at all points of S "near" (x_0, y_0, z_0).

The following theorem, which we state without proof, is the three-variable analog of Theorem 14.9.3.



A constrained relative maximum occurs at (x_0, y_0, z_0) if $f(x_0, y_0, z_0) \ge f(x, y, z)$ at all points of S near (x_0, y_0, z_0).

Figure 14.9.4

14.9.4 THEOREM (*Constrained-Extremum Principle for Three Variables and One Constraint*). *Let f and g be functions of three variables with continuous first partial derivatives on some open set containing the constraint surface $g(x, y, z) = 0$, and assume that $\nabla g \neq \mathbf{0}$ at any point on this surface. If f has a constrained relative extremum, then this extremum occurs at a point (x_0, y_0, z_0) on the constraint surface at which the gradient vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ are parallel; that is, there is some number λ such that*

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

Example 3 Find the points on the sphere $x^2 + y^2 + z^2 = 36$ that are closest to and farthest from the point (1, 2, 2).

Solution. To avoid radicals, we will find points on the sphere that minimize and maximize the *square* of the distance to (1, 2, 2). Thus, we want to find the relative extrema of

$$f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-2)^2$$

subject to the constraint

$$x^2 + y^2 + z^2 = 36 \tag{8}$$

If we let $g(x, y, z) = x^2 + y^2 + z^2$, then $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$. Thus, $\nabla g = \mathbf{0}$ if and only if $x = y = z = 0$. It follows that $\nabla g \neq \mathbf{0}$ at any point of the sphere (8), and hence the constrained relative extrema must occur at points where

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

That is,

$$2(x-1)\mathbf{i} + 2(y-2)\mathbf{j} + 2(z-2)\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k})$$

which leads to the equations

$$2(x-1) = 2x\lambda, \quad 2(y-2) = 2y\lambda, \quad 2(z-2) = 2z\lambda \tag{9}$$

We may assume that x, y, and z are nonzero since $x = 0$ does not satisfy the first equation, $y = 0$ does not satisfy the second, and $z = 0$ does not satisfy the third. Thus, we can rewrite (9) as

$$\frac{x-1}{x} = \lambda, \quad \frac{y-2}{y} = \lambda, \quad \frac{z-2}{z} = \lambda$$

The first two equations imply that

$$\frac{x-1}{x} = \frac{y-2}{y}$$

from which it follows that

$$y = 2x \tag{10}$$

Similarly, the first and third equations imply that

$$z = 2x \tag{11}$$

Substituting (10) and (11) in the constraint equation (8), we obtain

$$9x^2 = 36 \quad \text{or} \quad x = \pm 2$$

Substituting these values in (10) and (11) yields two points:

$$(2, 4, 4) \quad \text{and} \quad (-2, -4, -4)$$

Since $f(2, 4, 4) = 9$ and $f(-2, -4, -4) = 81$, it follows that (2, 4, 4) is the point on the sphere closest to (1, 2, 2), and $(-2, -4, -4)$ is the point that is farthest (Figure 14.9.5). ◀

Next we will use Lagrange multipliers to solve the problem of Example 7 in the last section.

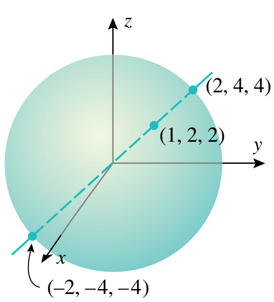


Figure 14.9.5

Example 4 Use Lagrange multipliers to determine the dimensions of a rectangular box, open at the top, having a volume of 32 ft^3, and requiring the least amount of material for its construction.

Solution. With the notation introduced in Example 7 of the last section, the problem is to minimize the surface area

$$S = xy + 2xz + 2yz$$

subject to the volume constraint

$$xyz = 32 \tag{12}$$

If we let $f(x, y, z) = xy + 2xz + 2yz$ and $g(x, y, z) = xyz$, then

$$\nabla f = (y + 2z)\mathbf{i} + (x + 2z)\mathbf{j} + (2x + 2y)\mathbf{k} \quad \text{and} \quad \nabla g = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$

It follows that $\nabla g \neq \mathbf{0}$ at any point on the surface $xyz = 32$, since x, y, and z are all nonzero on this surface. Thus, at a constrained relative extremum we must have $\nabla f = \lambda \nabla g$, that is,

$$(y + 2z)\mathbf{i} + (x + 2z)\mathbf{j} + (2x + 2y)\mathbf{k} = \lambda(yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k})$$

This condition yields the three equations

$$y + 2z = \lambda yz, \quad x + 2z = \lambda xz, \quad 2x + 2y = \lambda xy$$

Because x, y, and z are nonzero, these equations can be rewritten as

$$\frac{1}{z} + \frac{2}{y} = \lambda, \quad \frac{1}{z} + \frac{2}{x} = \lambda, \quad \frac{2}{y} + \frac{2}{x} = \lambda$$

From the first two equations,

$$y = x \tag{13}$$

and from the first and third equations,

$$z = \tfrac{1}{2}x \tag{14}$$

Substituting (13) and (14) in the volume constraint (12) yields

$$\tfrac{1}{2}x^3 = 32$$

This equation, together with (13) and (14), yields

$$x = 4, \quad y = 4, \quad z = 2$$

which agrees with the result that was obtained in Example 7 of the last section. ◀

There are variations in the method of Lagrange multipliers that can be used to solve problems with two or more constraints. However, we will not discuss that topic here.

EXERCISE SET 14.9 Graphing Utility C CAS

1. The accompanying figure shows graphs of the line $x + y = 4$ and the level curves of height $c = 2, 4, 6$, and 8 for the function $f(x, y) = xy$.
(a) Use the figure to find the maximum value of the function $f(x, y) = xy$ subject to the constraint $x + y = 4$, and explain your reasoning.
(b) How can you tell from the figure that you have not obtained the minimum value of f subject to the constraint?
(c) Use Lagrange multipliers to check your work.

2. The accompanying figure shows the graphs of the line $3x + 4y = 25$ and the level curves of height $c = 9, 16, 25, 36$, and 49 for the function $f(x, y) = x^2 + y^2$.
(a) Use the accompanying figure to find the minimum value of the function $f(x, y) = x^2 + y^2$ subject to the constraint $3x + 4y = 25$, and explain your reasoning.
(b) How can you tell from the accompanying figure that you have not obtained the maximum value of f subject to the constraint?
(c) Use Lagrange multipliers to check your work.

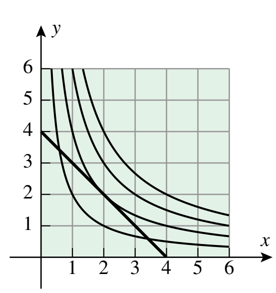


Figure Ex-1

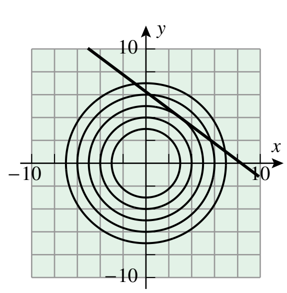


Figure Ex-2

3. (a) Use a graphing utility to graph the circle $x^2 + y^2 = 25$ and two distinct level curves of $f(x, y) = x^2 - y$ that just touch the circle.
(b) Use the results you obtained in part (a) to approximate the maximum and minimum values of f subject to the constraint $x^2 + y^2 = 25$.
(c) Check your approximations in part (b) using Lagrange multipliers.

C **4.** (a) If you have a CAS that can generate implicit curves, use it to graph the circle $(x - 4)^2 + (y - 4)^2 = 4$ and two level curves of the function $f(x, y) = x^3 + y^3 - 3xy$ that just touch the circle.
(b) Use the result you obtained in part (a) to approximate the minimum value of f subject to the constraint $(x - 4)^2 + (y - 4)^2 = 4$.
(c) Confirm graphically that you have found a minimum and not a maximum.
(d) Check your approximation using Lagrange multipliers and solving the required equations numerically.

In Exercises 5–12, use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraint. Also, find the points at which these extreme values occur.

5. $f(x, y) = xy$; $4x^2 + 8y^2 = 16$
6. $f(x, y) = x^2 - y^2$; $x^2 + y^2 = 25$
7. $f(x, y) = 4x^3 + y^2$; $2x^2 + y^2 = 1$
8. $f(x, y) = x - 3y - 1$; $x^2 + 3y^2 = 16$
9. $f(x, y, z) = 2x + y - 2z$; $x^2 + y^2 + z^2 = 4$

10. $f(x, y, z) = 3x + 6y + 2z;\ 2x^2 + 4y^2 + z^2 = 70$

11. $f(x, y, z) = xyz;\ x^2 + y^2 + z^2 = 1$

12. $f(x, y, z) = x^4 + y^4 + z^4;\ x^2 + y^2 + z^2 = 1$

In Exercises 13–20, solve using Lagrange multipliers.

13. Find the point on the line $2x - 4y = 3$ that is closest to the origin.

14. Find the point on the line $y = 2x + 3$ that is closest to $(4, 2)$.

15. Find the point on the plane $x + 2y + z = 1$ that is closest to the origin.

16. Find the point on the plane $4x + 3y + z = 2$ that is closest to $(1, -1, 1)$.

17. Find the points on the circle $x^2 + y^2 = 45$ that are closest to and farthest from $(1, 2)$.

18. Find the points on the surface $xy - z^2 = 1$ that are closest to the origin.

19. Find a vector in 3-space whose length is 5 and whose components have the largest possible sum.

20. Suppose that the temperature at a point (x, y) on a metal plate is $T(x, y) = 4x^2 - 4xy + y^2$. An ant, walking on the plate, traverses a circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the ant?

In Exercises 21–28, use Lagrange multipliers to solve the indicated problems from Section 14.8.

21. Exercise 34

22. Exercise 35

23. Exercise 36

24. Exercise 37

25. Exercise 39

26. Exercises 41(a) and (b)

27. Exercise 42

28. Exercise 43

C **29.** Let α, β, and γ be the angles of a triangle.
(a) Use Lagrange multipliers to find the maximum value of $f(\alpha, \beta, \gamma) = \cos\alpha \cos\beta \cos\gamma$, and determine the angles for which the maximum occurs.
(b) Express $f(\alpha, \beta, \gamma)$ as a function of α and β alone, and use a CAS to graph this function of two variables. Confirm that the result obtained in part (a) is consistent with the graph.

30. The accompanying figure shows the intersection of the elliptic paraboloid $z = x^2 + 4y^2$ and the right circular cylinder $x^2 + y^2 = 1$. Use Lagrange multipliers to find the highest and lowest points on the curve of intersection.

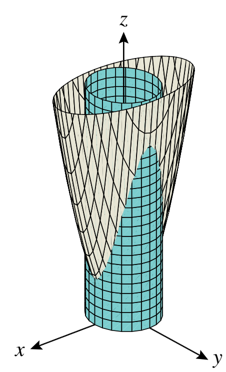


Figure Ex-30

SUPPLEMENTARY EXERCISES

Graphing Utility C CAS

1. (a) A company manufactures two types of computer monitors: standard and high resolution. Suppose that $P(x, y)$ is the profit that results from producing and selling x standard monitors and y high-resolution monitors. What do the two partial derivatives $\partial P/\partial x$ and $\partial P/\partial y$ represent?
(b) Suppose that the temperature at time t at a point (x, y) on the surface of a lake is $T(x, y, t)$. What do the partial derivatives $\partial T/\partial x$, $\partial T/\partial y$, and $\partial T/\partial t$ represent?

2. Let $z = f(x, y)$.
(a) Express $\partial z/\partial x$ and $\partial z/\partial y$ as limits.
(b) In words, what do the derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ tell you about the surface $z = f(x, y)$?
(c) In words, what do the derivatives $\partial z/\partial x(x_0, y_0)$ and $\partial z/\partial y(x_0, y_0)$ tell you about the rates of change of z with respect to x and y?
(d) In words, what does the derivative $D_{\mathbf{u}} f(x_0, y_0)$ tell you about the surface $z = f(x, y)$?

3. Show that the level curves of the cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = x^2 + y^2$ are circles, and make a sketch that illustrates the difference between the contour plots of the two functions.

4. (a) How are the directional derivative and the gradient of a function related?
(b) Under what conditions is the directional derivative of a differentiable function 0?
(c) In what direction does the directional derivative of a differentiable function have its maximum value? Its minimum value?

5. (a) In words, describe the level surfaces of the function $f(x, y, z) = a^2x^2 + a^2y^2 + z^2$, where $a > 0$.
(b) Find a function $f(x, y, z)$ whose level surfaces form a family of circular paraboloids that open in the positive z-direction.

6. What do Δf and df represent, and how are they related?

7. Let $f(x, y) = e^x \ln y$. Find
(a) $f(\ln y, e^x)$ (b) $f(r + s, rs)$.

8. Sketch the domain of f using solid lines for portions of the boundary included in the domain and dashed lines for portions not included.
(a) $f(x, y) = \ln(xy - 1)$ (b) $f(x, y) = (\sin^{-1} x)/e^y$

In Exercises 9–12, verify the assertion.

9. If $w = \tan(x^2 + y^2) + x\sqrt{y}$, then $w_{xy} = w_{yx}$.

10. If $w = \ln(3x - 3y) + \cos(x + y)$, then $\partial^2 w/\partial x^2 = \partial^2 w/\partial y^2$.

11. If $F(x, y, z) = 2z^3 - 3(x^2 + y^2)z$, then $F_{xx} + F_{yy} + F_{zz} = 0$.

12. If $f(x, y, z) = xyz + x^2 + \ln(y/z)$, then $f_{xyzx} = f_{zxxy}$.

13. The pressure in N/m^2 of a gas in a cylinder is given by $P = 10T/V$ with T in kelvins (K) and V in m^3.
(a) If T is increasing at a rate of 3 K/min with V held fixed at 2.5 m^3, find the rate at which the pressure is changing when $T = 50$ K.
(b) If T is held fixed at 50 K while V is decreasing at the rate of 3 m^3/min, find the rate at which the pressure is changing when $V = 2.5$ m^3.

14. Find the slope of the tangent line at the point $(1, -2, -3)$ on the curve of intersection of the surface $z = 5 - 4x^2 - y^2$ with
(a) the plane $x = 1$ (b) the plane $y = -2$.

In Exercises 15 and 16, (a) find the limit of the function $f(x, y)$ as $(x, y) \to (0, 0)$ if it exists, and (b) determine whether f is continuous at $(0, 0)$.

15. $f(x, y) = \dfrac{x^4 - x + y - x^3 y}{x - y}$

16. $f(x, y) = \begin{cases} \dfrac{x^4 - y^4}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

17. At the point $(1, 2)$, the directional derivative $D_{\mathbf{u}} f$ is $2\sqrt{2}$ toward $P_1(2, 3)$ and -3 toward $P_2(1, 0)$. Find $D_{\mathbf{u}} f(1, 2)$ toward the origin.

18. Find equations for the tangent plane and normal line to the given surface at P_0.
(a) $z = x^2 e^{2y}$; $P_0(1, \ln 2, 4)$
(b) $x^2 y^3 z^4 + xyz = 2$; $P_0(2, 1, -1)$

19. Find all points P_0 on the surface $z = 2 - xy$ at which the normal line passes through the origin.

20. Show that for all tangent planes to the surface

$$x^{2/3} + y^{2/3} + z^{2/3} = 1$$

the sum of the squares of the x-, y-, and z-intercepts is 1.

21. Find all points on the paraboloid $z = 9x^2 + 4y^2$ at which the normal line is parallel to the line through the points $P(4, -2, 5)$ and $Q(-2, -6, 4)$.

22. If $w = x^2 y - 2xy + y^2 x$, find the increment Δw and the differential dw if (x, y) varies from $(1, 0)$ to $(1.1, -0.1)$.

23. Use differentials to estimate the change in the volume $V = \frac{1}{3}x^2 h$ of a pyramid with a square base when its height h is increased from 2 to 2.2 m and its base dimension x is decreased from 1 to 0.9 m. Compare this to ΔV.

In Exercises 24–26, locate all relative minima, relative maxima, and saddle points.

24. $f(x, y) = x^2 + 3xy + 3y^2 - 6x + 3y$

25. $f(x, y) = x^2 y - 6y^2 - 3x^2$

26. $f(x, y) = x^3 - 3xy + \frac{1}{2}y^2$

In economics, a ***production model*** is a mathematical relationship between the output of a company or a country and the labor and capital equipment required to produce that output. Much of the pioneering work in the field of production models occurred in the 1920s when Paul Douglas of the University of Chicago and his collaborator Charles Cobb proposed that the output P can be expressed in terms of the labor L and the capital equipment K by an equation of the form

$$P = cL^\alpha K^\beta$$

where c is a constant of proportionality and α and β are constants such that $0 < \alpha < 1$ and $0 < \beta < 1$. This is called the ***Cobb–Douglas production model***. Typically, P, L, and K are all expressed in terms of their equivalent monetary values. Exercises 27–29 explore properties of this model.

27. (a) Consider the Cobb–Douglas production model given by the formula $P = L^{0.75}K^{0.25}$. Sketch the level curves $P(L, K) = 1$, $P(L, K) = 2$, and $P(L, K) = 3$ in an LK-coordinate system (L horizontal and K vertical). Your sketch need not be accurate numerically, but it should show the general shape of the curves and their relative positions.
(b) Use a graphing utility to make a more extensive contour plot of the model.

28. (a) Find $\partial P/\partial L$ and $\partial P/\partial K$ for the Cobb–Douglas production model $P = cL^\alpha K^\beta$.
(b) The derivative $\partial P/\partial L$ is called the ***marginal productivity of labor***, and the derivative $\partial P/\partial K$ is called the ***marginal productivity of capital***. Explain what these quantities mean in practical terms.
(c) Show that if $\beta = 1 - \alpha$, then P satisfies the partial differential equation

$$K\frac{\partial P}{\partial K} + L\frac{\partial P}{\partial L} = P$$

29. Consider the Cobb–Douglas production model

$$P = 1000L^{0.6}K^{0.4}$$

(a) Find the maximum output value of P if labor costs \$50.00 per unit, capital costs \$100.00 per unit, and the total cost of labor and capital is set at \$200,000.

(b) How should the \$200,000 be allocated between labor and capital to achieve the maximum?

Solve Exercises 30 and 31 two ways:
(a) Use the constraint to eliminate a variable.
(b) Use Lagrange multipliers.

30. Find all relative extrema of x^2y^2 subject to the constraint $4x^2 + y^2 = 8$.

31. Find the dimensions of the rectangular box of maximum volume that can be inscribed in the ellipsoid

$$(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$$

32. In each part, use Theorem 14.5.3 to find dy/dx.
(a) $3x^2 - 5xy + \tan xy = 0$
(b) $x \ln y + \sin(x - y) = \pi$

33. Given that $f(x, y) = 0$, use Theorem 14.5.3 to express d^2y/dx^2 in terms of partial derivatives of f.

34. As illustrated in the accompanying figure, suppose that a current I branches into currents I_1, I_2, and I_3 through resistors R_1, R_2, and R_3 in such a way that the total energy to the three resistors is a minimum. Find the ratios $I_1 : I_2 : I_3$ if the energy delivered to R_i is $I_i^2 R_i (i = 1, 2, 3)$ and $I_1 + I_2 + I_3 = I$.

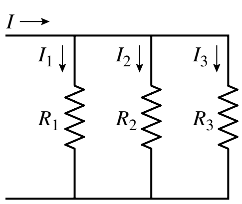


Figure Ex-34

35. Suppose the equations of motion of a particle are $x = t - 1$, $y = 4e^{-t}$, $z = 2 - \sqrt{t}$, where $t > 0$. Find, to the nearest tenth of a degree, the acute angle between the velocity vector and the normal line to the surface $(x^2/4) + y^2 + z^2 = 1$ at the points where the particle collides with the surface. Use a calculating utility with a root-finding capability where needed.

C **36.** Let

$$F(x) = \int_c^d f(x, y)\, dy, \quad a \le x \le b$$

It can be shown that if $f(x, y)$ and $\partial f/\partial x$ are continuous for $a \le x \le b$ and $c \le y \le d$, then

$$F'(x) = \int_c^d \frac{\partial f}{\partial x} dy$$

(a) Use this result to find $F'(x)$ if

$$F(x) = \int_0^1 \sin(xe^y)\, dy$$

(b) Use a CAS and the result in part (a) to find the maximum value of $F(x)$ for $0 \le x \le 2$. Express your answer to six decimal places.

37. Angle A of triangle ABC is increasing at a rate of $\pi/60$ rad/s, side AB is increasing at a rate of 2 cm/s, and side AC is increasing at a rate of 4 cm/s. At what rate is the length of BC changing when angle A is $\pi/3$ rad, $AB = 20$ cm, and $AC = 10$ cm? Is the length of BC increasing or decreasing? [*Hint:* Use the law of cosines.]

38. Let $z = f(x, y)$, where $x = g(t)$ and $y = h(t)$.
(a) Show that

$$\frac{d}{dt}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 z}{\partial y \partial x}\frac{dy}{dt}$$

and

$$\frac{d}{dt}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}\frac{dx}{dt} + \frac{\partial^2 z}{\partial y^2}\frac{dy}{dt}$$

(b) Use the formulas in part (a) to help find a formula for d^2z/dt^2.