Notes on
Noncommutative Geometry and Particle Physics

Popovic Milutin

Week 5: 12.03 - 19.03

Contents

1 Noncommutative Geometric Spaces

1.1 Exercises

Exercise 1

Make the proof of the last theorem (see week4.pdf) explicit for $N = 3$.

For the C* algebra we have $A = \mathbb{C}^3$ For H we have $H = (\mathbb{C}^2)^{\oplus 3} = H_2 \oplus H_2^1 \oplus H_2^2$.

The symmetric operator D acting on H and the representation $\pi(a)$:

$$\pi((a(1),a(2),a(3)) = \begin{pmatrix} a(1) & 0 \\ 0 & a(2) \end{pmatrix} \oplus \begin{pmatrix} a(1) & 0 \\ 0 & a(3) \end{pmatrix} \oplus \begin{pmatrix} a(2) & 0 \\ 0 & a(2) \end{pmatrix}$$

$$= \begin{pmatrix} a(1) & 0 & 0 & 0 & 0 & 0 \\ 0 & a(2) & 0 & 0 & 0 & 0 \\ 0 & 0 & a(1) & 0 & 0 & 0 \\ 0 & 0 & 0 & a(3) & 0 & 0 \\ 0 & 0 & 0 & 0 & a(2) & 0 \\ 0 & 0 & 0 & 0 & 0 & a(3) \end{pmatrix} \tag{1}$$

$$D = \begin{pmatrix} 0 & x_1 \\ x_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & x_2 \\ x_2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & x_3 \\ x_3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & x_1 & 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 & x_3 & 0 \end{pmatrix} \tag{2}$$

$$\tag{3}$$

Then the norm of the commutator would be the largest eigenvalue

$$||[D,\pi(a)]|| = ||D\pi(a) - \pi(a)D||$$

$$= \left\|\begin{pmatrix} 0 & x_1(a(2)-a(1)) & 0 & 0 & 0 & 0 \\ -x_1(a(2)-a(1)) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2(a(3)-a(1)) & 0 & 0 \\ 0 & 0 & -x_2(a(3)-a(1)) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_3(a(3)-a(2)) \\ 0 & 0 & 0 & 0 & -x_3(a(2)-a(3)) & 0 \end{pmatrix}\right. \tag{4}$$

The matrix in Equation **??** is a skew symmetric matrix its eigenvalues are $i\lambda_1, i\lambda_2, i\lambda_3, i\lambda_4$, where the λ's are on the upper and lower diagonal check `https://en.wikipedia.org/wiki/Skew-symmetric_matrix#Skew-symmetrizable_matrix`. The matrix norm of would be the maximum of the norm of the larges eigenvalues:

$$||[D,\pi(a)]|| = \max_{a\in A}\{x_1|a(2)-a(1)|, \tag{5}$$
$$x_2|(a(3)-a(1))|,$$
$$x_3|(a(3)-a(2))|,\}$$

The metric is then:

$$d = \begin{pmatrix} 0 & a(1)-a(2) & a(1)-a(3) \\ a(2)-a(1) & 0 & a(2)-a(3) \\ a(3)-a(1) & a(3)-a(2) & 0 \end{pmatrix} \tag{6}$$

Exercise 2

Compute the metric on the space of three points given by $d_{ij} = \sup_{a\in A}\{|a(i) - a(j)| : ||[D,\pi(a)]|| \leq 1\}$ for the set of data $A = \mathbb{C}^3$ acting in the defining representation $H = \mathbb{C}^3$, and

$$D = \begin{pmatrix} 0 & d^{-1} & 0 \\ d^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some $d \in \mathbb{R}$

We have $A = \mathbb{C}^3$, $H = \mathbb{C}^3$ and D from above, then

$$||[D,\pi(a)]|| = d^{-1}\left|\left|\begin{pmatrix} 0 & a(2)-a(1) & 0 \\ -(a(2)-a(1)) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right|\right| \tag{7}$$

The metric is then

$$d = \begin{pmatrix} 0 & a(1)-a(2) & a(1) \\ a(2)-a(1) & 0 & a(2) \\ -a(1) & -a(2) & 0 \end{pmatrix} \tag{8}$$

Exercise 3

Show that d_{ij} from Equation 12 is a metric on $\hat{A}$ by establishing that:

$$d_{ij} = 0 \quad \Leftrightarrow \quad i = j \tag{9}$$
$$d_{ij} = d_{ji} \tag{10}$$
$$d_{ij} \leq d_{ik} + d_{kj} \tag{11}$$

$$d_{ij} = \sup_{a\in A}\left\{|\mathbf{Tr}(a(i)) - \mathbf{Tr}((a(j))| : ||[D,a]|| \leq 1\right\} \tag{12}$$

For Equation 9 set $i = j$ in 12.

$$\begin{aligned} d_{ii} &= \sup_{a\in A}\{|\mathrm{Tr}(a(i)) - \mathrm{Tr}((a(i))| : ||[D,a]|| \leq 1\} \\ &= \sup_{a\in A}\{0 : ||[D,a]|| \leq 1\} = 0 \end{aligned}$$

For Equation 10 obviously we have the commuting property of addition.
For Equation 11, for $k = j$ then $d_{kj} = 0$ and the equality holds. For $i = k$ then $d_{ik} = 0$ and equality holds. Else set $d_{ik} = 1$ and $d_{kj} = 1$ then $d_{ij} = 1 \leq d_{ik} + d_{kj} = 2$

1.2 Properties of Matrix Algebras

Lemma 1. *If A is a unital C* algebra that acts faithfully on a finite dimensional Hilbert space, then A is a matrix algebra of the Form:*

$$A \simeq \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}) \tag{13}$$

Proof. Since A acts faithfully on a Hilbert space, then A is a C* subalgebra of a matrix algebra $L(H) = M_{\dim(H)}(\mathbb{C} \Rightarrow A \simeq$ Matrix algebra. □

Question 1. What does the author mean when he sais 'acts faithfully on a Hilbertspace'? Then the representation is fully reducible, or that the presentation is irreducible?

For a *-representation 'faithful' if it is injective. For a *-homomorphism 'faithful' means one-to-one correspondance

Example 1. $A = M_n(\mathbb{C})$ and $H = \mathbb{C}^n$, A acts on H with matrix multiplication and standard inner product. D on H is a hermitian matrix $n \times n$ matrix.

D is referred to as a finite Dirac operator as in as its ∞ dimensional on Riemannian Spin manifolds coming in Chapter 4.

Now can introduce a 'differential 'geometric structure' on the finite space X with the **devided difference**

$$\frac{a(i) - a(j)}{d_{ij}} \tag{14}$$

for each pair i, $j \in X$ the finite dimensional discrete space X. This appears in the entries in the commutator $[D,a]$ in the above exercises.

Definition 1. Given an finite spectral triple (A,H,D), the A-bimodule of Connes' differential one form is:

$$\Omega^1_D(A) := \left\{ \sum_k a_k[D,b_k] : a_k, b_k \in A \right\} \tag{15}$$

Question 2. Is the Conne's differential one form the set of all '1st order differential operators' given A, that act on H?

Then there is a map $d : A \to \Omega^1_D(A)$, $d = [D,\cdot]$.

Exercise 4

Verify that 'd' is a derivation of the C* algebra

$$d(ab) = d(a)b + ad(b)$$
$$d(a^*) = -d(a)^*$$

For the record $d(\cdot) = [D, \cdot]$, then we have

1.

$$d(ab) = [D, ab] = [D, a]b + a[D, b]$$
$$= d(a)b + ad(b)$$

2.

$$d(a^*) = [D, a^*] = Da^* - a^*D$$
$$= -(D^*a - aD^*) = -[D^*, a]$$
$$= -d(a)^*$$

Exercise 5

Verify that $\Omega^1_D(A)$ is an A-bimodule by rewriting

$$a(a_k[D, b_k])b = \sum_k a'_k[D, b'_k] \quad a'_k, b'_k \in A$$

Begin

$$
\begin{aligned}
a(a_k[D, b_k])b &= a'_k(Db_k - b_kD)b = \\
&= a'_k(Db_kb - b_kDb) = a_k(Db_kb - b_kDb - Db_k + Db_kb) = \\
&= a'_k([D, b_kb] - b_kDb + Db_k + \cdots - \cdots) = \\
&= \sum_k a'_k[D, b'_k]
\end{aligned}
$$

Lemma 2. *Let $(A, H, D) = (M_n(\mathbb{C}, \mathbb{C}^n, D)$, with D a hermitian $n \times n$ matrix. If D is not a multiple of the identity then:*

$$\Omega^1_D(A) \simeq M_n(\mathbb{C}) = A \tag{16}$$

Proof. Assume $D = \sum_i \lambda_i e_{ii}$ (diagonal), $\lambda_i \in \mathbb{R}$ and $\{e_{ij}\}$ the basis of $M_n(\mathbb{C}$. For fixed i, j choose k such that $\lambda_k \neq \lambda_j$ then

$$\left(\frac{1}{\lambda_k - \lambda_j} e_{ik}\right) [D, e_{kj}] = e_{ij} \tag{17}$$

$e_{ij} \in \Omega^1_D(A)$ by the above definition. And $\Omega^1_D(A) \subset L(\mathbb{C}^n) = H \simeq M_n(\mathbb{C}) = A$ □

Exercise 6

Consider $(A = \mathbb{C}^2, H = \mathbb{C}^2, D = \begin{pmatrix} 0 & \lambda \\ \bar{\lambda} & 0 \end{pmatrix})$ **with** $\lambda \neq 0$**. Show that** $\Omega^1_D(A) \simeq M_2(\mathbb{C})$

Because of the Hilbert Basis D can be extended in terms of the basis of $M_2(\mathbb{C})$, plugging this into Equation 17 will get us the same cyclic result, thus $\Omega^1_D(A) \simeq M_2(\mathbb{C})$

1.3 Morphisms Between Finite Spectral Triples

Definition 2. two finite spectral tripes (A_1, H_1, D_1) and (A_2, H_2, D_2) are called unitarily equivalent if

- $A_1 = A_2$
- $\exists\ U : H_1 \to H_2$, unitary with
  1. $U\pi_1(a)U^* = \pi_2(a)$ with $a \in A_1$
  2. $UD_1U^* = D_2$

Some remarks

- the above is an equivalence relation
- spectral unitary equivalence is given by the unitaries of the matrix algebra itself
- for any such U then $(A, H, D) \sim (A, H, UDU^*)$
- $UDU^* = D + U[D, U^*]$ of the form of elements in $\Omega^1_D(A)$.

Exercise 7

Show that the unitary equivalence between finite spectral triples is a equivalence relation

An equivalence relation needs to satisfy reflexivity, symmetry transitivity. Let (A_1, H_1, D_1), (A_2, H_2, D_2) and (A_3, H_3, D_3) be three finite spectral triples.

For reflexivity $(A_1, H_1, D_1) \sim (A_1, H_1, D_1)$. So there exists a $U : H_1 \to H_1$ unitary, which is the identity and always exists.

For symmetry we need

$$(A_1, H_1, D_1) \sim (A_2, H_2, D_2) \Leftrightarrow (A_2, H_2, D_2) \sim (A_1, H_1, D_1)$$

because U is unitary:

$$\begin{aligned} U\pi_1(a)U^* &= \pi_2(a) \quad | \cdot U^* \boxdot U \\ U^*U\pi_1(a)U^*U &= \pi_1(a) = U^*\pi_2(a)U \end{aligned}$$

The same with the symmetric operator D.

For transitivity we need

$$\begin{gathered} (A_1,H_1,D_1) \sim (A_2,H_2,D_2) \quad \text{and} \quad (A_2,H_2,D_2) \sim (A_3,H_3,D_3) \\ \Rightarrow (A_1,H_1,D_1) \sim (A_3,H_3,D_3) \end{gathered}$$

There are two unitary maps $U_{12} : H_1 \to H_2$ and $U_{23} : H_2 \to H_3$ then

$$\begin{aligned} U_{23}U_{12}\pi_1(a)U_{12}^*U_{23}^* &= U_{23}\pi_2(a)U_23^* \\ &= \pi_3(a) \\ U_{23}U_{12}D_1U_{12}^*U_{23}^* &= U_{23}D_2U_23^* \\ &= D_3 \end{aligned}$$

Extending the this relation we look again at the notion of equivalence from Morita equivalence of Matrix Algebras.

Given a Hilbert bimodule $E \in KK_f(B,A)$ and (A,H,D) we construct a finite spectral triple on B, (B,H',D')

$$H' = E \otimes_A H \tag{18}$$

This extends the left action on B with the right action and inherits the $\mathbb{C}$ valued inner product space.

$$D'(e \otimes \xi) = e \otimes D\xi + \nabla(e)\xi \quad e \in E, a \in A \tag{19}$$

Where ∇ is called the *connection on the right A-module E* associated with the derivation $d = [D, \cdot]$ and satisfying the *Leibnitz Rule* which is

$$\nabla(ae) = \nabla(e)a + e \otimes [D,a] \quad e \in E,\ a \in A \tag{20}$$

Then the linearity of the balanced tensor product $E \otimes_A H$ is satisfied

$$\begin{aligned} D'(ea \otimes \xi - e \otimes a\xi) &= D'(ea \otimes \xi) - D'(e \otimes \xi) \\ &= ea \otimes D\xi + \nabla(ae)\xi - e \otimes D(a\xi) - \nabla(e)a\xi \\ &= 0 \end{aligned}$$

With the information thus far we can prove the following theorem

Theorem 1. *If (A,H,D) a finite spectral triple, $E \in KK_f(B,A)$. Then $(V, E \otimes_A H, D')$ is a finite spectral triple, provided that ∇ satisfies the compatibility condition*

$$\langle e_1, \nabla e_2 \rangle_E - \langle \nabla e_1, e_2 \rangle_E = d\langle e_1, e_2 \rangle_E \quad e_1, e_2 \in E \tag{21}$$

Proof. $E \otimes_A H$ was shown in the previous section (text before the theorem). The only thing left is to show that D' is a symmetric operator, this we can just compute. Let $e_1, e_2 \in E$ and $\xi_1, \xi_2 \in H$ then

$$\begin{aligned}
\langle e_1 \otimes \xi_1, D'(e_2 \otimes \xi_2) \rangle_{E \otimes_A H} &= \langle \xi_1, \langle e_1, \nabla e_2 \rangle_E \xi_2 \rangle + \langle \xi_1, \langle e_1, e_2 \rangle_E D \xi_2 \rangle_H \\
&= \langle \xi_1, \langle \nabla e_1, e_2 \rangle_E \xi_2 \rangle_H + \langle \xi_1, d\langle e_1, e_2 \rangle_E \xi_2 \rangle_H \\
&+ \langle D\xi_1, \langle e_1, e_2 \rangle_E \xi_2 \rangle_H - \langle \xi_1, [D, \langle e_1, e_2 \rangle_E] \xi_2 \rangle_H \\
&= \langle D'(e_1 \otimes \xi_1), e_2 \otimes \xi_2 \rangle_{E \otimes_A H}
\end{aligned}$$

□

Exercise 8

Let ∇ and ∇' be two connections on a right A-module E. Show that $\nabla - \nabla'$ is a right A-linear map $E \to E \otimes_A \Omega^1_D(A)$

Both ∇ and ∇' need to satisfy the Leiblitz rule, so let's see if $\nabla - \nabla'$ does.

$$\begin{aligned}
\nabla(ea) - \nabla'(ea) &= \nabla(e) + e \otimes [D, a] \\
&- (\nabla'(e)a + e \otimes [D', a]) \\
&= \bar{\nabla} a + e \otimes (Da - aD - D'a + aD') \\
&= \bar{\nabla} a + e \otimes ((D - D')a - a(D - D')) \\
&= \bar{\nabla} a + e \otimes [D', a] \\
&= \bar{\nabla}(ea)
\end{aligned}$$

Therefore $\nabla - \nabla'$ is a linear map.

Exercise 9

Construct a finite spectral triple (A, H', D') from (A, H, D)

1. **show that the derivation $d(\cdot) : A \to A \otimes_A \Omega^1_D(A) = \Omega^1_D(A)$ is a connection on A considered a right A-module**
2. **Upon identifying $A \otimes_A H \simeq H$, what is D' when the connection is $d(\cdot)$.**
3. **Use 1) and 2) to show that any connection $\nabla : A \to A \otimes_A \Omega^1_D(A)$ is given by**

$$\nabla = d + \omega$$

where $\omega \in \Omega^1_D(A)$

4. **Upon identifying $A \otimes_A H \simeq H$, what is the difference operator D' with the connection on A given by $\nabla = d + \omega$**

1. $\nabla(e \cdot a) = d(a)$
2. $D'(a\xi) = a(D\xi) + (\nabla a)\xi = D(a\xi)$
3. Use the identity element $e \in A$
   $\nabla(e \cdot a) = \nabla(e)a + 1 \otimes d(a) = d(a)\nabla(e)a$
4. $D'(a \otimes \xi) = D'(a\xi) = a(D\xi) + (\nabla a)\xi = a(D\xi) + \nabla(e \cdot a)\xi$
   $= D(a\xi) + \nabla(e)(a\xi)$

1.4 Graphing Finite Spectral Triples

Definition 3. A *graph* is a ordered pair $(\Gamma^{(0)}, \Gamma^{(1)})$. Where $\Gamma^{(0)}$ is the set of vertices (nodes) and $\Gamma^{(1)}$ a set of pairs of vertices (edges)

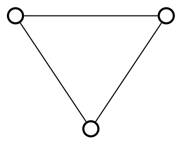


Figure 1: A simple graph with three vertices and three edges

Exercise 10

Show that any finite-dimensional faithful representation H of a matrix algebra A is completely reducible. To do that show that the complement $W^{\perp}$ of an A-submodule $W \subset H$ is also an A-submodule of H.

$A \simeq \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C})$ is the matrix algebra then H is a Hilbert A-bimodule and W a submodule of A. Because we have $H = W \cup W^{\perp}$, then $W^{\perp}$ is naturally a A-submodule, because elements in $W^{\perp}$ need to satisfy the bimodularity.

Definition 4. A Λ-decorated graph is given by an ordered pair (Γ, Λ) of a finite graph Γ and a set of positive integers Λ with the labeling

- of the vetices $v \in \Gamma^{(0)}$ given by $n(v) \in \Lambda$
- of the edges $e = (v_1, v_2) \in \Gamma^{(1)}$ by operators
  - $D_e : \mathbb{C}^{n(v_1)} \to \mathbb{C}^{n(v_2)}$
  - and $D_e^* : \mathbb{C}^{n(v_2)} \to \mathbb{C}^{n(v_1)}$ its conjugate traspose (pullback?)

such that

$$n(\Gamma^{(0)}) = \Lambda \tag{22}$$

Question 3. Would then D_e be the pullback?

Question 4. These graphs are important in the next chapter I should look into it more, I don't understand much here, specific how to construct them with the abstraction of a spectral triple...

The operator D_e between $\mathbf{n}_i$ and $\mathbf{n}_j$ add up to D_{ij}

$$D_{ij} = \sum_{\substack{e=(v_1,v_2)\\ n(v_1)=\mathbf{n}_i\\ n(v_2)=\mathbf{n}_j}} D_e$$

Theorem 2. *There is a on to one correspondence between finite spectral triples modulo unitary equivalence and Λ-decorated graphs, given by associating a finite spectral triples (A,H,D) to a Λ decorated graph (Γ,Λ) in the following way:*

$$A = \bigoplus_{n\in\Lambda} M_n(\mathbb{C}); \quad H = \bigoplus_{v\in\Gamma^{(0)}} \mathbb{C}^{n(v)}; \quad D = \sum_{e\in\Gamma^{(1)}} D_e + D_e^* \tag{23}$$

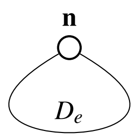


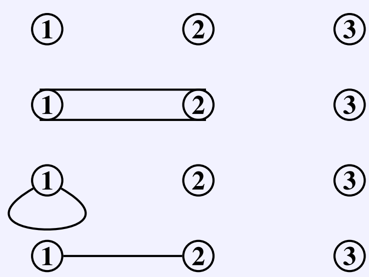
Figure 2: A Λ-decorated Graph of $(M_n(\mathbb{C}), \mathbb{C}^n, D = D_e + D_e^*)$

Exercise 11

Draw a Λ decorated graph corresponding to the spectral triple $(A = \mathbb{C}^3, H = \mathbb{C}^3, D = \begin{pmatrix} 0 & \lambda & 0 \\ \bar{\lambda} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix})$

Exercise 12

Use Λ-decorated graphs to classify all finite spectral triples (modulo unitary equivalence) on the matrix algebra $A = \mathbb{C} \oplus M_2(\mathbb{C})$



1.4.1 Graph Construction of Finite Spectral Triples

Algebra: We know if a acts on a finite dimensional Hilbert space then this C* algebra is isomorphic to a matrix algebra so $A \simeq \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C})$. Where $i \in \hat{A}$ represents an equivalence class and runs from 1 to N, thus $\hat{A} \simeq \{1,\ldots,N\}$. We label equivalence classes by $\mathbf{n}_i$, then $\hat{A} \simeq \{\mathbf{n}_1,\ldots,\mathbf{n}_N\}$.

Hilbert Space: Since every Hilbert space that acts faithfully on a C* algebra is completely reducible, it is isomorphic to the composition of irreducible representations. $H \simeq \bigoplus_{i=1}^{N} \mathbb{C}^{n_i} \otimes V_i$. Where all V_i's are Vector spaces, their dimension is the multiplicity of the representation landed by $\mathbf{n}_i$ to V_i itself by the multiplicity space.

Finite Dirac Operator: D_{ij} is connecting nodes $\mathbf{n}_i$ and $\mathbf{n}_j$, with a symmetric map $D_{ij} : \mathbb{C}^{n_i} \otimes V_i \to \mathbb{C}^{n_j} \otimes V_j$

To draw a graph, draw nodes in position $\mathbf{n}_i \in \hat{A}$. Multiple nodes at the same position represent multiplicities in H. Draw lines between nodes to represent D_{ij}.

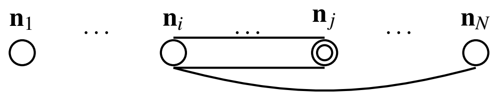


Figure 3: Example