Notes on
Noncommutative Geometry and Particle Physics

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Contents

1 Noncommutative Geometric Spaces

1.1 Noncommutative Matrix Algebras

1.1.1 Balanced Tensor Product and Hilbert Bimodules

Definition 1. Let A be an algebra, E be a *right* A-module and F be a *left* A-module. The *balanced tensor product* of E and F forms a A-bimodule.

$$E \otimes_A F := E \otimes F / \left\{ \sum_i e_i a_i \otimes f_i - e_i \otimes a_i f_i : \quad a_i \in A,\ e_i \in E,\ f_i \in F \right\} \tag{1}$$

Note $/$ denotes the quotient space. So $\otimes_A$ takes two left/right modules and makes a bimodule with the help the tensor product of the two modules and the quotient space that takes out all the elements from the tensor product that dont preserver the left/right representation and that are duplicates.

Definition 2. Let A, B be *matrix algebras*. The *Hilbert bimodule* for (A,B) is given by

- E, an A-B-bimodule E and by
- an B-valued *inner product* $\langle\cdot,\cdot\rangle_E : E \times E \to B$

$\langle\cdot,\cdot\rangle_E$ needs to satisfy the following for $e, e_1, e_2 \in E,\ a \in A$ and $b \in B$.

$$\langle e_1, a \cdot e_2\rangle_E = \langle a^* \cdot e_1, e_2\rangle_E \qquad \text{sesquilinear in } A \tag{2}$$

$$\langle e_1, e_2 \cdot b\rangle_E = \langle e_1, e_2\rangle_E b \qquad \text{scalar in } B \tag{3}$$

$$\langle e_1, e_2\rangle_E = \langle e_2, e_1\rangle_E^* \qquad \text{hermitian} \tag{4}$$

$$\langle e, e\rangle_E \geq 0 \qquad \text{equality holds iff } e = 0 \tag{5}$$

We denote $KK_f(A,B)$ the set of all *Hilbert bimodules* of (A,B).

Exercise 1

Check that a representation $\pi : \ A \ \to L(H)$ of a matrix algebra A turns H into a Hilbert bimodule for $(A,\mathbb{C})$.

We check if the representation of $a \in A$, $\pi(a) = T \in L(H)$ fulfills the conditions on the $\mathbb{C}$-valued inner product for $h_1, h_2 \in H$:

- $\langle h_1, \pi(a)h)2\rangle_{\mathbb{C}} = \langle h_1, Th_2\rangle_{\mathbb{C}} = \langle T^* h_1, h_2\rangle_{\mathbb{C}}$, T^* given by the adjoint
- $\langle h_1, h_2\pi(a)\rangle_{\mathbb{C}} = \langle h_1, h_2 T\rangle_{\mathbb{C}} = \langle h_1, h_2\rangle_{\mathbb{C}}$, T acts from the left
- $\langle h_1, h_2\rangle^*_{\mathbb{C}} = \langle h_2, h_1\rangle_{\mathbb{C}}$, hermitian because of the $\mathbb{C}$-valued inner product
- $\langle h_1, h_2\rangle \geq 0$, $\mathbb{C}$-valued inner product.

Exercise 2

Show that the $A-A$ bimodule given by A is in $KK_f(A,A)$ by taking the following inner product $\langle\cdot,\cdot\rangle_A : A \times A \to A$:

$$\langle a, a\rangle_A = a^* a' \quad a, a' \in A \tag{6}$$

We check again the conditions on $\langle\cdot,\cdot\rangle_A$, let $a, a_1, a_2 \in A$:

- $\langle a_1, a \cdot a_2\rangle_A = a^* \ a \cdot a_2 = (a^* a_1)^* a_2 = \langle a^* a_1, a_2\rangle$
- $\langle a_1, a_2 \cdot a\rangle_A = a_1^*(a_2 \cdot a) = (a^* a_2) \cdot a = \langle a_1, a_2\rangle_A a$
- $\langle a_1, a_2\rangle^*_A = (a_1^* a_2)^* = a_2^*(a_1^*)^* = a_2^* a_1 = \langle a_2, a_1\rangle$

Example 1. Consider a $*$ homomorphism between two matrix algebras $\phi : A \to B$. From it we can construct a Hilbert bimodule $E_\phi \in KK_f(A,B)$ in the following way. We let E_ϕ be B in the vector space sense and an inner product from the above Exercise 1.1.1, with A acting on the left with ϕ.

$$a \cdot b = \phi(a)b \quad a \in A, b \in E_\phi \tag{7}$$

1.1.2 Kasparov Product and Morita Equivalence

Definition 3. Let $E \in KK_f(A,B)$ and $F \in KK_F(B,D)$ the *Kasparov product* is defined as with the balanced tensor product

$$F \circ E := E \otimes_B F \tag{8}$$

Such that $F \circ E \in KK_f(A,D)$ with a D-valued inner product.

$$\langle e_1 \otimes f_1, e_2 \otimes f_2\rangle_{E\otimes_B F} = \langle f_1, \langle e_1, e_2\rangle_E f_2\rangle_F \tag{9}$$

Question 1. How do we go from $\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle_{E \otimes_B F}$ to $\langle f_1, \langle e_1, e_2 \rangle_E f_2 \rangle_F$
This statement is still in the definition.

Exercise 3

Show that the association $\phi \rightsquigarrow E_\phi$ (from the previous Example) is natural in the sense

1. $E_{\mathbf{id}_A} \simeq A \in KK_f(A, A)$
2. **for $*$-algebra homomorphism $\phi : A \to B$ and $\psi : B \to C$ we have an isomorphism**

$$E_\psi \circ E_\phi \;\equiv\; E_\phi \otimes_B E_\psi \;\simeq\; E_{\psi \circ \phi} \in KK_f(A, C) \tag{10}$$

1. $\mathrm{id}_A : A \to A$.
To construct $E_\phi \in KK_f(A, A)$, we let E_ϕ be A with a natural right representation, so $\Rightarrow E_\phi \simeq A$.
With an inner product, acting on A from the left with ϕ, $a', a \in A$
$a'a = (\phi(a')a) \in A$, which is satisfied by id_A, so $\phi = \mathrm{id}_A$.
2. $a \cdot b \cdot c = \psi(\phi(a) \cdot b) \cdot c$ for $a \in A$, $b \in B$, and $c \in C$ which is $\psi \circ \phi$

Exercise 4

In the definition of Morita equivalence:

1. **Check that $E \otimes_B F$ is a $A - D$ bimodule**
2. **Check that $\langle \cdot, \cdot \rangle_{E \oplus_B F}$ defines a D valued inner product**
3. **Check that $\langle a^*(e_1 \otimes f_1), e_2 \otimes f_2 \rangle_{E \otimes_B F} = \langle e_1 \otimes f_1, a(e_2 \otimes f_2) \rangle_{E \otimes_B F}$.**

1. $E \otimes_B F = E \otimes F / \{\sum_i e_i b_i \otimes f_i - e_i \otimes b_i f_i ; e_i \in E_i, b_i \in B, f_i \in F\}$ the last part takes out all tensor product elements of E and F that don't preserver the left/right representation and that are duplicates.
2. $\langle e_1, e_2 \rangle_E \in B$ and $\langle f_1, f_2 \rangle_F \in C$ by definition. So let $\langle e_1, e_2 \rangle_E = b$.
Then $\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle_{E \otimes_B F} = \langle f_1, \langle e_1, e_2 \rangle_E f_2 \rangle_F = \langle f_1, b f_2 \rangle_F \in C$
3. Check Question 1.
But let $G := E \otimes_B F \in KK_f(A, C)$ then $\forall g_1, g_2 \in G$ and $a \in A$ we need by definition $\langle g_1, a g_2 \rangle_G = \langle a^* g_1, g_2 \rangle_G$ and we set $g_1 = e_1 \otimes f_1$ and $g_2 = e_2 \otimes f_2$ for some $e_1, e_2 \in E$ and $f_1, f_2 \in F$, or else $G \notin KK_f(A, C)$ which would violate the Kasparov product

Definition 4. Let A, B be *matrix algebras*. They are called *Morita equivalent* if there exists an $E \in KK_f(A, B)$ and an $F \in KK_f(B, A)$ such that:

$$E \otimes_B F \simeq A \quad \text{and} \quad F \otimes_A E \simeq B \tag{11}$$

Where $\simeq$ denotes the isomorphism between Hilbert bimodules, note that A or B is a bimodule by itself.

Question 2. Why are E and F each others inverse in the Kasparov Product?
They are each others inverse with respect to the Kasparov Product because we land in the same space as we started. In the definition we have $E \in KK_f(A,B)$ we start from A and $E \otimes_B F$ lands in A.
On the other hand we have $F \in KK_f(B,D)$ we start from B and $F \otimes_A E$ lands in B.

Example 2.

- Hilber bimodule of (A,A) is A
- Let $E \in KK_f(A,B)$, we take $E \circ A = A \oplus_A E \simeq E$
- we conclude, that ${}_A A_A$ is the identity in the Kasparov product (up to isomorphism)

Example 3. Let $E = \mathbb{C}^n$, which is a $(M_n(\mathbb{C}), \mathbb{C})$ Hilbert bimodule with the standard $\mathbb{C}$ inner product.
On the other hand let $F = \mathbb{C}^n$, which is a $(\mathbb{C}, M_n(\mathbb{C}))$ Hilbert bimodule by right matrix multiplication with $M_n(\mathbb{C})$ valued inner product:

$$\langle v_1, v_2 \rangle = \bar{v_1} v_2^t \quad \in M_n(\mathbb{C}) \tag{12}$$

Now we take the Kasparov product of E and F:

- $F \circ E \;=\; E \otimes_{\mathbb{C}} F \qquad \simeq \; M_n(\mathbb{C})$
- $E \circ F \;=\; F \otimes_{M_n(\mathbb{C})} E \;\simeq\; \mathbb{C}$

$M_n(\mathbb{C})$ and $\mathbb{C}$ are Morita equivalent

Theorem 1. *Two matrix algebras are Morita Equivalent iff their their Structure spaces are isomorphic as discreet spaces (have the same cardinality / same number of elements)*

Proof. Let A, B be *Morita equivalent*. So there exists ${}_A E_B$ and ${}_B F_A$ with

$$E \otimes_B F \simeq A \quad \text{and} \quad F \otimes_A E \simeq B \tag{13}$$

Consider $[(\pi_B, H)] \in \hat{B}$ than we construct a representation of A,

$$\pi_A \to L(E \otimes_B H) \quad \text{with} \quad \pi_A(a)(e \otimes v) = ae \otimes w \tag{14}$$

Question 3. Is $E \simeq H$ and $F \simeq W$?
Not in particular, there is a theorem that all infinite dimensional Hilbert spaces are isomorphic. Here we are looking at finite dimensional Hilbert spaces.
Another thing to is that $[\pi_B, H] \in \hat{B}$ and looking at Exercise 1.1.1 we know that H is a bimodule of B, hence $E \otimes_B H \simeq A$, and for $[\pi_A, W]$ the same.

vice versa, consider $[(\pi_A, W)] \in \hat{A}$ we can construct π_B

$$\pi_B : B \to L(F \otimes_A W) \quad \text{and} \quad \pi_B(b)(f \otimes w) = bf \otimes w \tag{15}$$

These maps are each others inverses, thus $\hat{A} \simeq \hat{B}$ $\square$

Exercise 5

Fill in the gaps in the above proof:

1. **show that the representation of π_A defined is irreducible iff π_B is.**
2. **Show that the association of the class $[\pi_A]$ to $[\pi_B]$ is independent of the choice of representatives π_A and π_B**

1. (π_B, H) is irreducible means $H \neq \emptyset$ and only $\emptyset$ or H is invariant under the Action of B on H. Than $E \otimes_B H$ cannot be empty, because also E preserves left representation of A and also $E \otimes_B H \simeq A$.
2. The important thing is that $[\pi_A] \in \hat{A}$ respectively $[\pi_B] \in \hat{B}$, hence any choice of representation is irreducible, because the structure space denotes all unitary equivalence classes of irreducible representations.

Lemma 1. *The matrix algebra $M_n(\mathbb{C})$ has a unique irreducible representation (up to isomorphism) given by the defining representation on $\mathbb{C}^n$.*

Proof. We know $\mathbb{C}^n$ is a irreducible representation of $A = M_n(\mathbb{C})$. Let H be irreducible and of dimension k, then we define a map

$$\phi : A \oplus ... \oplus A \to H^* \tag{16}$$

$$(a_1, ..., a_k) \mapsto e^1 \circ a_1^t + ... + e^k \circ a_k^t \tag{17}$$

With $\{e^1, ..., e^k\}$ being the basis of the dual space H^* and $(\circ)$ being the pre-composition of elements in H^* and A acting on H. This forms a morphism of $M_n(\mathbb{C})$ modules, provided a matrix $a \in A$ acts on H^* with $v \mapsto v \circ a^t$ $(v \in H^*)$. Furthermore this morphism is surjective, thus making the pullback $\phi^* : H \mapsto (A^k)^*$ injective. Now identify $(A^k)^*$ with A^k as a A-module and note that $A = M_n(\mathbb{C}) \simeq \oplus^n \mathbb{C}^n$ as a n A module. It follows that H is a submodule of $A^k \simeq \oplus^{nk} \mathbb{C}$. By irreducibility $H \simeq \mathbb{C}$. □

Example 4. Consider two matrix algebras A, and B.

$$A = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}) \quad B = \bigoplus_{j=1}^{M} M_{m_j}(\mathbb{C}) \tag{18}$$

Let $\hat{A} \simeq \hat{B}$ that implies $N = M$ and define E with A acting by block-diagonal matrices on the first tensor and B acting in the same way on the second tensor. Define F vice versa.

$$E := \bigoplus_{i=1}^{N} \mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i} \quad F := \bigoplus_{i=1}^{N} \mathbb{C}^{m_i} \otimes \mathbb{C}^{n_i} \tag{19}$$

Then we calculate the Kasparov product.

$$E \otimes_B F \simeq \bigoplus_{i=1}^{N} (\mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i}) \otimes_{M_{m_i}(\mathbb{C})} (\mathbb{C}^{m_i} \otimes \mathbb{C}^{n_i}) \tag{20}$$

$$\simeq \bigoplus_{i=1}^{N} \mathbb{C}^{n_i} \otimes \left(\mathbb{C}^{m_i} \otimes_{M_{m_i}(\mathbb{C})} \mathbb{C}^{m_i} \right) \oplus \mathbb{C}^{n_i} \tag{21}$$

$$\simeq \bigoplus_{i=1}^{N} \mathbb{C}^{m_i} \otimes \mathbb{C}^{n_i} \simeq A \tag{22}$$

and from $F \otimes_A E \simeq B$.

We conclude that.

- There is a duality between finite spaces and Morita equivalence classes of matrix algebras.
- By replacing $*$-homomorphism $A \to B$ with Hilbert bimodules (A, B) we introduce a richer structure of morphism between matrix algebras.