# Notes on
# Noncommutative Geometry and Particle Physics

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## Contents



# 1 Noncommutative Geometric Spaces

## 1.1 Noncommutative Matrix Algebras

### 1.1.1 Balanced Tensor Product and Hilbert Bimodules

**Definition 1.** Let $A$ be an algebra, $E$ be a *right* $A$-module and $F$ be a *left* $A$-module. The *balanced tensor product* of $E$ and $F$ forms a $A$-bimodule.

$$E \otimes_A F := E \otimes F \Big/ \left\{ \sum_i e_i a_i \otimes f_i - e_i \otimes a_i f_i : \quad a_i \in A,\ e_i \in E,\ f_i \in F \right\}$$

**Question 1.** Does $/$ denote the complement, because one usually writes $\backslash$.

In other words the balanced tensor product forms only elements of

- $E$ that preserver the *left* representation of $A$ and
- $F$ that preserver the *right* representation of $A$.

Which is the same saying:

$$E \otimes_A F = \{ ea \otimes_A f = e \otimes_A af : \quad a \in A,\ e \in E,\ f \in F \}$$

**Definition 2.** Let $A$, $B$ be *matrix algebras*. The *Hilbert bimodule* for $(A,B)$ is given by

- $E$, an $A$-$B$-bimodue $E$ and by
- an $B$-valued *inner product* $\langle \cdot, \cdot \rangle_E : E \times E \to B$

$\langle\cdot,\cdot\rangle_E$ needs to satisfy the following for $e, e_1, e_2 \in E,\ a \in A$ and $b \in B$.

$$\begin{aligned}
\langle e_1, a\cdot e_2\rangle_E &= \langle a^*\cdot e_1, e_2\rangle_E && \text{sesquilinear in } A\\
\langle e_1, e_2\cdot b\rangle_E &= \langle e_1, e_2\rangle_E b && \text{scalar in } B\\
\langle e_1, e_2\rangle_E &= \langle e_2, e_1\rangle_E^* && \text{hermitian}\\
\langle e, e\rangle_E &\geq 0 && \text{equality holds iff } e = 0
\end{aligned}$$

We denote $KK_f(A,B)$ the set of all *Hilbert bimodules* of $(A,B)$.

**Exercise 1.** *Check that a representation $\pi:\ A\ \to L(H)$ of a matrix algebra $A$ turns $H$ into a Hilbert bimodule for $(A,\mathbb{C})$.*

**Solution 1.** We check if the representation of $a \in A$, $\pi(a) = T \in L(H)$ fulfills the conditions on the $\mathbb{C}$-valued inner product for $h_1, h_2 \in H$:

- $\langle h_1, \pi(a)h)2\rangle_{\mathbb{C}} = \langle h_1, Th_2\rangle_{\mathbb{C}} = \langle T^*h_1, h_2\rangle_{\mathbb{C}}$, $T^*$ given by the adjoint
- $\langle h_1, h_2\pi(a)\rangle_{\mathbb{C}} = \langle h_1, h_2T\rangle_{\mathbb{C}} = \langle h_1, h_2\rangle_{\mathbb{C}}$, $T$ acts from the left
- $\langle h_1, h_2\rangle_{\mathbb{C}}^* = \langle h_2, h_1\rangle_{\mathbb{C}}$, hermitian because of the $\mathbb{C}$-valued inner product
- $\langle h_1, h_2\rangle \geq 0$, $\mathbb{C}$-valued inner product.

**Exercise 2.** *Show that the $A-A$ bimodule given by $A$ is in $KK_f(A,A)$ by taking the following inner product $\langle\cdot,\cdot\rangle_A : A\times A \to A$:*

$$\langle a, a\rangle_A = a^*a' \quad a, a' \in A$$

**Solution 2.** We check again the conditions on $\langle\cdot,\cdot\rangle_A$, let $a, a_1, a_2 \in A$:

- $\langle a_1, a\cdot a_2\rangle_A = a^*\ a\cdot a_2 = (a^*a_1)^*a_2 = \langle a^*a_1, a_2\rangle$
- $\langle a_1, a_2\cdot a\rangle_A = a_1^*(a_2\cdot a) = (a^*a_2)\cdot a = \langle a_1, a_2\rangle_A a$
- $\langle a_1, a_2\rangle_A^* = (a_1^*a_2)^* = a_2^*(a_1^*)^* = a_2^*a_1 = \langle a_2, a_1\rangle$

**Example 1.** Consider a $*$ homomorphism between two matrix algebras $\phi : A \to B$. From it we can construct a Hilbert bimodule $E_\phi \in KK_f(A,B)$ in the following way. We let $E_\phi$ be $B$ in the vector space sense and an inner product from the above Exercise 2, with $A$ acting on the left with $\phi$.

$$a\cdot b = \phi(a)b \quad a \in A, b \in E_\phi$$

### 1.1.2 Kasparov Product and Morita Equivalence

**Definition 3.** Let $E \in KK_f(A,B)$ and $F \in KK_F(B,D)$ the *Kasparov product* is defined as with the balanced tensor product

$$F\circ E := E\otimes_B F$$

Such that $F\circ E \in KK_f(A,D)$ with a $D$-valued inner product.

$$\langle e_1\otimes f_1, e_2\otimes f_2\rangle_{E\otimes_B F} = \langle f_1, \langle e_1, e_2\rangle_E f_2\rangle_F$$

**Question 2.** How do we go from $\langle e_1\otimes f_1, e_2\otimes f_2\rangle_{E\otimes_B F}$ to $\langle f_1, \langle e_1, e_2\rangle_E f_2\rangle_F$

**Question 3.** What is the meaning of 'associative up to isomorphism'? Isomorphism of $F \circ E$ or of $A, B$ or $D$?

**Exercise 3.** *Show that the association $\phi \rightsquigarrow E_\phi$ (from the previous Example) is natural in the sense*

1. $E_{id_A} \simeq A \in KK_f(A, A)$
2. *for $*$-algebra homomorphism $\phi : A \to B$ and $\psi : B \to C$ we have an isomorphism*

$$E_\psi \circ E_\phi \;\equiv\; E_\phi \otimes_B E_\psi \;\simeq\; E_{\psi\circ\phi} \in KK_f(A, C)$$

**Solution 3.**

1. $\mathrm{id}_A : A \to A$.
   To construct $E_\phi \in KK_f(A, A)$, we let $E_\phi$ be $A$ with a natural right representation, so $\Rightarrow E_\phi \simeq A$.
   With an inner product, acting on $A$ from the left with $\phi$, $a', a \in A$
   $a'a = (\phi(a')a) \in A$, which is satisfied by $\mathrm{id}_A$, so $\phi = \mathrm{id}_A$.
2. Not sure but: $a \cdot b \cdot c = \psi(\phi(a) \cdot b) \cdot c$ which is in a sense $\psi \circ \phi$

**Exercise 4.** *In the definition of Morita equivalence:*

1. *Check that $E \otimes_B F$ is a $A - D$ bimodule*
2. *Check that $\langle \cdot, \cdot \rangle_{E \oplus_B F}$ defines a $D$ valued inner product*
3. *Check that $\langle a^*(e_1 \otimes f_1), e_2 \otimes f_2 \rangle_{E\otimes_B F} = \langle e_1 \otimes f_1, a(e_2 \otimes f_2) \rangle_{E\otimes_B F}$.*

**Solution 4.**

1. $E \otimes_B F = E \otimes F / \{\sum_i e_i b_i \otimes f_i - e_i \otimes b_i f_i; e_i \in E_i, b_i \in B, f_i \in F\}$ the last part takes out all tensor product elements of $E$ and $F$ that don't preserver the left/right representation.
2. $\langle e_1, e_2 \rangle_E \in B$ and $\langle f_1, f_2 \rangle_F \in C$ by definition. So let $\langle e_1, e_2 \rangle_E = b$.
   Then $\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle_{E\otimes_B F} = \langle f_1, \langle e_1, e_2 \rangle_E f_2 \rangle_F = \langle f_1, b f_2 \rangle_F \in C$
3. Check Question 2.
   But let $G := E \otimes_B F \in KK_f(A, C)$ then $\forall g_1, g_2 \in G$ and $a \in A$ we need by definition $\langle g_1, a g_2 \rangle_G = \langle a^* g_1, g_2 \rangle_G$ and we set $g_1 = e_1 \otimes f_1$ and $g_2 = e_2 \otimes f_2$ for some $e_1, e_2 \in E$ and $f_1, f_2 \in F$, or else $G \notin KK_f(A, C)$ which would violate the Kasparov product

**Definition 4.** Let $A$, $B$ be *matrix algebras*. They are called *Morita equivalent* if there exists an $E \in KK_f(A, B)$ and an $F \in KK_f(B, A)$ such that:

$$E \otimes_B F \simeq A \quad \text{and} \quad F \otimes_A E \simeq B$$

Where $\simeq$ denotes the isomorphism between Hilbert bimodules, note that $A$ or $B$ is a bimodule by itself.

**Question 4.** Why are $E$ and $F$ each others inverse in the Kasparov Product?

**Example 2.**

- Hilber bimodule of $(A, A)$ is $A$

- Let $E \in KK_f(A,B)$, we take $E \circ A = A \oplus_A E \simeq E$
- we conclude, that ${}_AA_A$ is the identity in the Kasparov product (up to isomorphism)

**Example 3.** Let $E = \mathbb{C}^n$, which is a $(M_n(\mathbb{C}), \mathbb{C})$ Hilbert bimodule with the standard $\mathbb{C}$ inner product.
On the other hand let $F = \mathbb{C}^n$, which is a $(\mathbb{C}, M_n(\mathbb{C}))$ Hilbert bimodule by right matrix multiplication with $M_n(\mathbb{C})$ valued inner product:

$$\langle v_1, v_2 \rangle = \bar{v_1} v_2^t \quad \in M_n(\mathbb{C})$$

Now we take the Kasparov product of $E$ and $F$:

- $F \circ E \ = \ E \otimes_{\mathbb{C}} F \qquad \simeq \ M_n(\mathbb{C})$
- $E \circ F \ = \ F \otimes_{M_n(\mathbb{C})} E \ \simeq \ \mathbb{C}$

$M_n(\mathbb{C})$ and $\mathbb{C}$ are Morita equivalent

**Theorem 1.** *Two matrix algebras are Morita Equivalent iff their their Structure spaces are isomorphic as discreet spaces (have the same cardinality / same number of elements)*

*Proof.* Let $A$, $B$ be *Morita equivalent*. So there exists ${}_AE_B$ and ${}_BF_A$ with

$$E \otimes_B F \simeq A \quad \text{and} \quad F \otimes_A E \simeq B$$

Consider $[(\pi_B, H)] \in \hat{B}$ than we construct a representation of $A$,

$$\pi_A \to L(E \otimes_B H) \quad \text{with} \quad \pi_A(a)(e \otimes v) = ae \otimes w$$

**Question 5.** Is $E \simeq H$ and $F \simeq W$?

*vice versa*, consider $[(\pi_A, W)] \in \hat{A}$ we can construct $\pi_B$

$$\pi_B : B \to L(F \otimes_A W) \quad \text{and} \quad \pi_B(b)(f \otimes w) = bf \otimes w$$

These maps are each others inverses, thus $\hat{A} \simeq \hat{B}$ ☐

**Exercise 5.** *Fill in the gaps in the above proof:*

1. *show that the representation of $\pi_A$ defined is irreducible iff $\pi_B$ is.*
2. *Show that the association of the class $[\pi_A]$ to $[\pi_B]$ is independent of the choice of representatives $\pi_A$ and $\pi_B$*

**Solution 5.**

1. $(\pi_B, H)$ is irreducible means $H \neq \emptyset$ and only $\emptyset$ or $H$ is invariant under the Action of $B$ on $H$. Than $E \otimes_B H$ cannot be empty, because also $E$ preserves left representation of $A$ and also $E \otimes_B H \simeq A$.
2. The important thing is that $[\pi_A] \in \hat{A}$ respectively $[\pi_B] \in \hat{B}$, hence any choice of representation is irreducible, because the structure space denotes all unitary equivalence classes of irreducible representations.

**Lemma 1.** *The matrix algebra $M_n(\mathbb{C})$ has a unique irreducible representation (up to isomorphism) given by the defining representation on $\mathbb{C}^n$.*

*Proof.* We know $\mathbb{C}^n$ is a irreducible representation of $A = M_n(\mathbb{C})$. Let $H$ be irreducible and of dimension $k$, then we define a map

$$\begin{aligned} \phi : A \oplus \ldots \oplus A &\to H^* \\ (a_1, \ldots, a_k) &\mapsto e^1 \circ a_1^t + \ldots + e^k \circ a_k^t \end{aligned}$$

With $\{e^1, \ldots, e^k\}$ being the basis of the dual space $H^*$ and $(\circ)$ being the pre-composition of elements in $H^*$ and $A$ acting on $H$. This forms a morphism of $M_n(\mathbb{C})$ modules, provided a matrix $a \in A$ acts on $H^*$ with $v \mapsto v \circ a^t$ $(v \in H^*)$. Furthermore this morphism is surjective, thus making the pullback $\phi^* : H \mapsto (A^k)^*$ injective. Now identify $(A^k)^*$ with $A^k$ as a $A$-module and note that $A = M_n(\mathbb{C}) \simeq \oplus^n \mathbb{C}^n$ as a n A module. It follows that $H$ is a submodule of $A^k \simeq \oplus^{nk} \mathbb{C}$. By irreducibility $H \simeq \mathbb{C}$. □

**Example 4.** Consider two matrix algebras $A$, and $B$.

$$A = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}) \quad B = \bigoplus_{j=1}^{M} M_{m_j}(\mathbb{C})$$

Let $\hat{A} \simeq \hat{B}$ that implies $N = M$ and define $E$ with $A$ acting by block-diagonal matrices on the first tensor and B acting in the same way on the second tensor. Define $F$ vice versa.

$$E := \bigoplus_{i=1}^{N} \mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i} \quad F := \bigoplus_{i=1}^{N} \mathbb{C}^{m_i} \otimes \mathbb{C}^{n_i}$$

Then we calculate the Kasparov product.

$$\begin{aligned} E \otimes_B F &\simeq \bigoplus_{i=1}^{N} (\mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i}) \otimes_{M_{m_i}(\mathbb{C})} (\mathbb{C}^{m_i} \otimes \mathbb{C}^{n_i}) \\ &\simeq \bigoplus_{i=1}^{N} \mathbb{C}^{n_i} \otimes \left( \mathbb{C}^{m_i} \otimes_{M_{m_i}(\mathbb{C})} \mathbb{C}^{m_i} \right) \oplus \mathbb{C}^{n_i} \\ &\simeq \bigoplus_{i=1}^{N} \mathbb{C}^{m_i} \otimes \mathbb{C}^{n_i} \simeq A \end{aligned}$$

and from $F \otimes_A E \simeq B$.

We conclude that.

- There is a duality between finite spaces and Morita equivalence classes of matrix algebras.
- By replacing $*$-homomorphism $A \to B$ with Hilbert bimodules $(A, B)$ we introduce a richer structure of morphism between matrix algebras.