Notes on
Noncommutative Geometry and Particle Physics

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Week 2: 26.02 - 4.03

Contents

1   Excurse to Group Theory and Lie Groups

1.1   Groups and Representations

Definition 1. A Group G is a set with a binary operation on G satisfying.

1. $f, g \in G$ we have $fg = h \in G$.

2. $f(gh) = (fg)h$

3. $\exists\, e \in G \;\forall f \in G$ with $ef = fe = f$

4. $\forall f \in G \;\exists\, f^{-1} \in G$ with $ff^{-1} = f^{-1}f = e$

Definition 2. A Representation of a Group G is a mapping, D of elements of G onto a set of *linear operators* such that:

1. $D(e) = 1$, 1 is the identity operator in the space on which linear operators act

2. $D(g_1)D(g_2) = D(g_1 g_2)$, the mapping is linear in group the group operation

Just by looking at symmetries of a Group we can find a nice representation, and if the group is finite we can even find a matrix representation (Cheyley's Theorem). We all ready know a lot about linear algebra which will then allow us to study these Groups very thoroughly and derive physical properties with minimal information.

1.2   Lie Groups

Group elements now depend *smoothly* on a set *continuous parameters* $g(\alpha) \in G$. We are looking at continuous symmetries, e.g. a Sphere in $\mathbb{R}^3$ can be rotated in any direc-

tion without changing. The collection of rotations forms a Lie group because the group elements are smoothly differentiable.

1.2.1 Generators

We parameterize $g(\alpha)|_{\alpha=0} = e$ and we assume that near the identity element, the group elements can be described by a finite set of elements α_a for $a = 1,..,N$. For a representation D of this group, linear operators need to be parametrized the same way:

$$D(\alpha)|_{\alpha=0} = 1 \qquad (1)$$

Because of the smoothness and continuity we can Taylor expand a representation near the identity:

$$D(\alpha) = 1 + i d\alpha_a X_a + \cdots \qquad (2)$$

$$\text{with } X_a = -i\frac{\partial D(\alpha)}{\partial \alpha_a}\bigg|_{\alpha=0} \qquad {}^{[1]} \qquad (3)$$

We call X_a the *generators of the group*.

- If the parametrization is *parsimonious*[2] then all of X_a will be independent.

- If the representation is unitary then X_a will be *hermitian*, because of the i in the definition.

- Sophus Lie showed how to derive generators without representations.

Now let us go in some fixed infinitesimal direction from the identity.

$$D(d\alpha) = 1 + i d\alpha_a X_a \qquad (4)$$

Because of the group property of closure with respect to the group operation we can raise $D(d\alpha)$ to a large power and still get a group element.

$$D(\alpha) = \lim_{k\to\infty} (1 + i\frac{\alpha_a X_a}{k})^k = e^{i\alpha_a X_a} \qquad (5)$$

This is called the *exponential parameterization*. Looking at the expression we see that group elements can be expressed in terms of generators, and generators form a vector space. They are often referred to any element in the real linear space spanned by $X_a's$.

1.2.2 Lie Algebras

Let us consider a parameter family of group elements created by one generator X_a:

$$U(\lambda) = e^{i\lambda\alpha_a X_a} \qquad (6)$$

We know for that for the same generator the group multiplication is linear meaning:

$$U(\lambda_1)U(\lambda_2) = U(\lambda_1 + \lambda_2) \qquad (7)$$

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[2]parsimonious - All parameters are needed to distinguish between group elements

But if we multiply elements generated by two different generators the general case is

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} \neq e^{i(\alpha_a+\beta_b)X_a} \tag{8}$$

Yet because the exponentials are a representation of a group, and a group has closure under group operation we know the above needs to be true for some δ_a

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} = e^{i\delta_a X_a} \tag{9}$$

To further examine the exponent we rewrite the expression and Taylor expand $ln(1+K)$ to the second of $K = e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1$

$$i\delta_a X_a = ln(1+K) = K - \frac{K^2}{2} + \cdots$$
$$\text{and} \quad K = e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1$$
$$= (1 + i\alpha_a X_a - \frac{1}{2}(\alpha_a X_a)^2 + \cdots)$$
$$\cdot (1 + i\beta_b X_b - \frac{1}{2}(\beta_b X_b)^2 + \cdots) - 1$$
$$= i\alpha_a X_a + i\beta_b X_b - \alpha_a X_a \beta_b X_b$$
$$- \frac{1}{2}(\alpha_a X_a)^2 - \frac{1}{2}(\beta_b X_b)^2 + \cdots$$

So:

$$i\delta_a X_a = i\alpha_a X_a + i\beta_b X_b - \alpha_a X_a \beta_b X_b$$
$$- \frac{1}{2}(\alpha_a X_a)^2 - \frac{1}{2}(\beta_b X_b)^2$$
$$+ \frac{1}{2}(\alpha_a X_a + i\beta_b X_b - \alpha_a X_a \beta_b X_b)^2$$
$$= i\alpha_a X_a + i\beta_b X_b - \alpha_a X_a \beta_b X_b$$
$$- \frac{1}{2}(\alpha_a X_a)^2 - \frac{1}{2}(\beta_b X_b)^2$$
$$+ \frac{1}{2}(\alpha_a X_a)^2 + \frac{1}{2}(\beta_b X_b)^2$$
$$+ \frac{1}{2}\alpha_a X_a \beta_b X_b + \frac{1}{2}\beta_b X_b \alpha_a X_a$$

Because X's are linear operators $\alpha_a X_a \beta_b X_b \neq \beta_b X_b \alpha_a X_a$. These generators form an *algebra under commutation* and we get

$$i\delta_a X_a = i\alpha_a X_a + i\beta_b X_b - \alpha_a X_a \beta_b X_b$$
$$- \frac{1}{2}[\alpha_a X_a, \beta_b X_b] + \cdots$$

Thus rewriting the equation gives us

$$[\alpha_a X_a, \beta_b X_b] = -2i(\delta_c - \alpha_c - \beta_c)X_c \cdots \equiv i\gamma_c X_c$$

Because this is true for all α and β, and considering the group closure, there exists some *real* f_{abc} called the *structure constant* satisfying.

$$\gamma_c = \alpha_a \beta_b f_{abc} \tag{10}$$

3

Which is the same as.

$$[X_a, X_b] = i f_{abc} X_c \tag{11}$$

This is called the *Lie algebra of a group*

So f is antisymmetric because $[A, B] = -[B, A]$, which means $f_{abc} = -f_{bac}$.
And δ can now be written as

$$\delta_a = \alpha_a + \beta_a - \frac{1}{2} \gamma_a \cdots \tag{12}$$

Just by following the properties of Lie Groups (dependence on parameters and smoothness) in a fixed direction near die identity to find physical statements. E.g. $[\hat{r}_i, \hat{p}_j] = i\hbar \delta_{ij}$ tells us that we can't know the position and the momentum of a particle exactly at a given time.