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Notes on
Noncommutative Geometry and Particle Phyiscs

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Contents

1 Classification of Finite Real Spectral Triples

Here we classify finite real spectral triples modulo unitary equivalence with *Krajewski Diagrams*. We extend Λ-decorated graphs to the case of real spectral triples (grading and real structure).

The Algebra: Like before:

$$A \simeq \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}) \qquad \text{with} \quad \hat{A} = \{\mathbf{n}_1, \dots, \mathbf{n}_N\} \tag{1}$$

Where $\mathbf{n}_i$ are irreducible representation of A on $\mathbb{C}^{n_i}$

The Hilbertspace: Faithful irreducible representation on A are the direct sum of $\mathbb{C}^{n_i}$'s, which act on A by left block-diagonal matrix multiplication.

$$\bigoplus_{i=1}^{N} \mathbb{C}^{n_i} \tag{2}$$

Furthermore we need a representation of A° on H that commutes with A. That is

$$A^\circ \simeq \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C})^\circ \tag{3}$$

$$\text{with} \quad \hat{A}^\circ = \{\mathbf{n}_1^\circ, \dots, \mathbf{n}_N^\circ\} \tag{4}$$

$$\text{and} \quad \bigoplus_{i=1}^{N} \mathbb{C}^{n_i \circ} \tag{5}$$

And we need the multiplicity space V_{ij} of $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ}$. Thus making the Hilbertspace:

$$H = \bigoplus_{i,j=1}^{N} \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ} \otimes V_{ij} \tag{6}$$

- $\mathbf{n}_i$, $\mathbf{n}_j^\circ$ form a grid
- if there is a node at $(\mathbf{n}_i, \mathbf{n}_j^\circ)$ then $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ}$ is nonzero in H.
- multiplicity implies multiple nodes

Example 1. $A = \mathbb{C} \oplus M_2(\mathbb{C})$, two options of the Hilbertspace.

$$\begin{array}{c|cc} & \mathbf{1} & \mathbf{2} \\ \hline \mathbf{1}^\circ & \circ & \\ \mathbf{2}^\circ & & \circ \end{array} \qquad\qquad \begin{array}{c|cc} & \mathbf{1} & \mathbf{2} \\ \hline \mathbf{1}^\circ & \circ & \circ \\ \mathbf{2}^\circ & & \end{array}$$

The first diagram corresponds to $H_1 = \mathbb{C} \oplus M_2(\mathbb{C})$, to the second $H_2 = \mathbb{C} \oplus \mathbb{C}^2$.

Exercise 1

Let J be an anti-unitary operator on a finite-dimensional Hilbert space. Show that J^2 is an unitary operator

Straight forward, say $J: H \to H$, then let $\xi_1, \xi_2 \in H$:

$$< J^2\xi_1, J^2\xi_2 > = < J(J\xi_1), J(J\xi_2) >= \tag{7}$$
$$= < J\xi_2, J\xi_1 > = < \xi_1, \xi_2 > \tag{8}$$

The real Structure: $J: H \to H$.

Lemma 1. *Let J be an anti-unitary operator on a finite-dimensional Hilbertspace H with $J^2 = \pm 1$*

1. *If $J^2 = 1 \Rightarrow \exists$ an ONB $\{e_k\}$ of H with $Je_k = e_k$.*
2. *If $J^2 = -1 \Rightarrow \exists$ an ONB $\{e_k, f_k\}$ of H with $Je_k = f_k$ and consequently $Jf_k = -e_k$.*

Proof. **1.** $J^2 = 1$

$v \in H$ and set:

$$e_1 := \begin{cases} c(v + Jv) & \text{if } Jv \neq -v \\ iv & \text{if } Jv = -v \end{cases} \tag{9}$$

Where c is a normalization constant, then take Je_1

$$J(v + Jv) = Jv + J^2v = v + Jv \quad \text{and} \tag{10}$$
$$J(iv) = -iJv = iv \tag{11}$$
$$\Rightarrow Je_1 = e_1 \tag{12}$$

Take $v' \perp e_1$ making:

$$< e_1, Jv' > = < J^2v', Je_1 > = < v', Je_1 > = < v', e_1 > = 0 \tag{13}$$

Construct $e_2 \perp e_1$ with v':

$$e_2 := \begin{cases} c(v' + Jv') & \text{if } Jv' \neq -v' \\ iv' & \text{if } Jv' = -v' \end{cases} \tag{14}$$

Do this k times and get $\{e_k\}$ ONB of H for $J^2 = 1$.

2. $J^2 = -1$
$v \in H$ and set $e_1 = cv$, c normalization constant. Then we set $f_1 = Je_1$ with $f_1 \perp e_1$,

this is automatically the case because:

$$< f_1, e_1 > = < Je_1, e_1 > = - < Je_1, J^2 e_1 > = \tag{15}$$

$$= - < Je_1, e_1 > = - < f_1, e_1 > \tag{16}$$

this only holds for 0. Then take some $v' \perp e_1, f_1$ and set
$e_2 = c'v'$ and $f_2 = Je_2 \perp e_2, f_1, e_1$.

$$< e_1, f_2 > = < e_1, Je_2 > = - < J^2 e_1, Je_2 > = - < e_2, Je_1 > = - < e_2, f_1 > = 0 \tag{17}$$

$$< f_1, f_2 > = < Je_1, Je_2 > = < e_2, e_1 > = 0. \tag{18}$$

Do this k times and get $\{e_k, f_k\}$ ONB of H for $J^2 = -1$

□

Apply Lemma 1 to the real structure J on a spectral triple. J implements right action of A on H with

$$a^\circ = Ja^* J^{-1} \tag{19}$$

and satisfying $[a, b^\circ] = 0$. With the block form of A, this implies

$$J(a_1^* \oplus \cdots \oplus a_N^*) = (a_1^\circ \oplus \cdots \oplus a_N^\circ)J. \tag{20}$$

With this we can conclude that the Krajewski diagram for a real finite spectral triple is symmetric along the diagonal.J hast then the following bilinear mapping:

$$J: \ \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ} \otimes V_{ij} \to \mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i \circ} \otimes V_{ji}. \tag{21}$$

Proposition 1. *Let J be a real structure on a finite real spectral triple $(A, H, D; J)$.*

1. *If $J^2 = 1$ (K0-dimension 0, 1, 6, 7) Rightarrow $\exists$ an ONB $\{e_k^{(ij)}\}$ with $e_k^{(ij)} \in \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ} \otimes V_{ij}$ such that*

$$Je_k^{(ij)} = e_k^{(ij)} \quad (i, j = 1, \ldots, N;\ k = 1, \ldots dim(V_{ij})) \tag{22}$$

2. *If $J^2 = -1$ (KO-dimension 2, 3, 4, 5) $\Rightarrow$ $\exists$ ONB $\{e_k^{(ij)}, f_k^{(ji)}\}$ with $e_k^{(ij)} \in \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ} \otimes V_{ij}$ and $f_k^{(ji)} \in \mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i \circ} \otimes V_{ji}$ such that*

$$Je_k^{(ij)} = f_k^{(ji)} \quad (i \leq j = 1, \ldots, N;\ k = 1, \ldots, dim(V_{ji})). \tag{23}$$

Proof. Similar to Lemma 1. □

For whatever unknown reasons this implies that in the case of KO-dimension 2, 3, 4, 5, diagonals $H_i i$ need to have even multiplicity.

The finite Dirac Operator: Is a mapping between H_{ij} to H_{kl}

$$D_{ij,kl}: \ \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ} \otimes V_{ij} \to \mathbb{C}^{n_k} \otimes \mathbb{C}^{n_l \circ} \otimes V_{kl} \tag{24}$$

We have $D_{kl,ij} = D^*_{ij,kl}$. And in the diagram we have a line between the nodes $(\mathbf{n}_i, \mathbf{n}_j^\circ)$ and $(\mathbf{n}_l, \mathbf{n}_k^\circ)$. But instead of drawing directional lines draw a single undirected line that represents both $D_{ij,kl}$ and the adjoint $D_{kl,ij}$.

Lemma 2. *The conditions $JD = \pm DJ$ and $[[D,a],b^\circ] = 0$ imply that the connections in the diagram run only vertically or horizontally and thereby the diagonal symmetry between the nodes is preserved.*

Proof. The condition $JD = \pm DJ$ has the following commutative diagram.

$$\begin{array}{ccc} \mathbb{C}^{n_i \circ} \otimes \mathbb{C}^{n_j \circ} \otimes V_{ij} & \xrightarrow{D} & \mathbb{C}^{n_k \circ} \otimes \mathbb{C}^{n_l \circ} \otimes V_{kl} \\ J\big\downarrow & & \big\downarrow J \\ \mathbb{C}^{n_j \circ} \otimes \mathbb{C}^{n_i \circ} \otimes V_{ji} & \xrightarrow{\pm D} & \mathbb{C}^{n_l \circ} \otimes \mathbb{C}^{n_k \circ} \otimes V_{lk} \end{array}$$

Relating $D_{ij,kl}$ to $D_{ji,lk}$ and maintaining diagonal symmetry. Wit the condition $[[D,a],b^\circ] = 0$ for the diagonal elements $a = \lambda_1 \mathbb{I}_{n_1} \oplus \cdots \oplus \lambda_N \mathbb{I}_{n_N} \in A$ and $b = \mu_1 \mathbb{I}_{n_1} \oplus \cdots \oplus \mu_N \mathbb{I}_{n_N} \in A$, with some $\lambda_i, \mu_i \in \mathbb{C}$, we can commute:

$$D_{ij,kl}(\lambda_i - \lambda_k)(\bar{\mu}_j - \bar{\mu}_l) = 0 \tag{25}$$

$\forall \lambda_i, \mu_j \in \mathbb{C}$, thus $D_i j, kl = 0$ for $i \neq j$ or $j \neq i$. □

The Grading: $\gamma\colon\ H \to H$ each node gets labeled by a $+$ or a $-$ sign.

- D only connects nodes with different signs
- If $(\mathbf{n}_i, \mathbf{n}_j^\circ)$ has a $\pm$ sing then $(\mathbf{n}_j, \mathbf{n}_i^\circ)$ has a $\mp$, ε'' sign according to $J\gamma = \varepsilon'' \gamma J$

Definition 1. A Krajewski Diagram of KO-dimension k is an ordered pair (Γ, Λ) where Γ is a finite graph and Λ is a set of positive integers with a labeling:

- of $v \in \Gamma^{(0)}$ of vertices by elements $\iota(v) = (n(v), m(v)) \in \Lambda \times \Lambda$, an edge from v to v' implies that either $n(v) = n(v')$ or $m(v) = m('v)$ or both
- of $e = (v_1, v_2) \in \Gamma^{(1)}$ edges with non-zero operators D_e and their adjoints D_e^*:

$$D_e : \mathbb{C}^{n(v_1)} \to \mathbb{C}^{n(v_2)} \qquad \text{if} \quad m(v_1) = m(v_2) \tag{26}$$

$$D_e : \mathbb{C}^{m(v_1)} \to \mathbb{C}^{m(v_2)} \qquad \text{if} \quad n(v_1) = n(v_2) \tag{27}$$

Together with an involutive graph automorphism $j : \Gamma \Rightarrow \Gamma$ such that the following conditions hold:

1. every row or column in $\Gamma \times \Gamma$ has non-empty intersection with $\iota(\Gamma)$
2. for each vertex v we have $n(j(v)) = m(v)$
3. for each edge e we have $D_e = \varepsilon' D_{j(e)}$
4. if KO dimension k is even, then the vertices are labeled by ± 1 and the edges only connect opposite signs. The signs at v and $j(v)$ differ by a factor of ε
5. if the K0-dimension is 2, 3, 4, 5 then the inverse image of ι of the diagonal elements in $\Lambda \times \Lambda$ contains an even number of vertices of Γ

With this definition we can label different vertices by the same element in $\Lambda \times \Lambda$ (accounting for the multiplicities in V_{ij})

Diagram: To sum it up we have the following diagram

- Node at $(\mathbf{n}_i, \mathbf{n}_j^\circ)$ for each vertex with that label
- Operators D_e add up to $D_{ij,kl}$ connecting nodes $(\mathbf{n}_i, \mathbf{n}_j^\circ)$ with $(\mathbf{n}_k, \mathbf{n}_l^\circ)$

$$D_{ij,kl} = \sum_{\substack{e=(v_1,v_2)\in\Gamma^{(1)} \\ \iota(v_1)=(\mathbf{n}_i,\mathbf{n}_j) \\ \iota(v_2)=(\mathbf{n}_k,\mathbf{n}_l)}} D_e \tag{28}$$

- only vertical or horizontal connections

Theorem 1. *There is a one-to-one correspondence between finite real spectral triples $(A,H,D;J,\gamma)$ of K0-dimension k modulo unitary equivalence and Krajewski diagrams of KO-dimension k in the following way:*

$$A = \bigoplus_{n\in\Lambda} M_n(\mathbb{C}) \tag{29}$$

$$H = \bigoplus_{v\in\Gamma^{(0)}} \mathbb{C}^{n(v)} \otimes \mathbb{C}^{m(v)\circ} \tag{30}$$

$$D = \sum_{e\in\Gamma^{(1)}} D_e + D_e^* \tag{31}$$

The real structure $J : H \to H$ is given as as in Proposition 1 with a basis dictated by a graph automorphism $j : \Gamma \to \Gamma$. The grading γ is difened by setting $\gamma = \pm 1$ on $\mathbb{C}^{n(v)} \otimes \mathbb{C}^{m(v)\circ} \subset H$ according to the labeling $\pm$ of the vertex v.

Example 2. $A = M_n(\mathbb{C})$ with $\hat{A} = \mathbf{n}$. We have the following Krajewski diagram.

$$\begin{array}{cc} & \mathbf{n} \\ \mathbf{n}^\circ & \circ \end{array}$$

- We can label the node either with a $+$ or a $-$ sign, the choice being irrelevant
- $H = \mathbb{C}^n \otimes \mathbb{C}^{n\circ} \simeq M_n(\mathbb{C})$
- γ trivial grading $(+1)$
- J is a combination of complex conjugation and the flip $n \otimes n^\circ$ ($\Rightarrow M_n(\mathbb{C})$ as matrix adjoint)
- Because node label is $\pm$ there is no non-zero Dirac operator
- $\Rightarrow (A = M_n(\mathbb{C}), H = M_n(\mathbb{C}), D = 0; J = (\cdot)^*, \gamma = 1)$

2 Real Algebras and Krajewski Diagrams

Definition 2. A real Algebra is a Vector space A over $\mathbb{R}$ with $A \times A \to A$, $(a,b) \mapsto ab$ and $1a = a1 = a \ \forall a \in A$

A real *-algebra is a real algebra with a bilinear map $* : A \to A$ such that $(ab)^* = b^* a^*$ and $(a^*)^* \quad \forall a, b \in A$

Example 3. Real *-algebra of quaternions $\mathbb{H}$ subalgebra of $M_2(\mathbb{C})$.

$$\mathbb{H} = \{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C} \} \tag{32}$$

$\mathbb{H}$ consists of matrices that commute in $M_2(\mathbb{C})$ with the operator I defined by:

$$I \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\bar{v}_2 \\ \bar{v}_1 \end{pmatrix} \tag{33}$$

The involution is the hermitian conjugation of $M_2(\mathbb{C})$.

Exercise 2

1. **Show that $\mathbb{H}$ is a real *-algebra which contains a real subalgebra isomorphic to $\mathbb{C}$.**
2. **Show that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq M_2(\mathbb{C})$ as complex *-algebras.**
3. **Show that $M_k(\mathbb{H})$ is areal *-algebra for any k**
4. **Show that $M_k(\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq M_{2k}(\mathbb{C})$ as complex *algebras.**

1). Let us take some $a, b \in \mathbb{H}$ with

$$a = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \quad b = \begin{pmatrix} \gamma & \delta \\ -\bar{\delta} & \bar{\gamma} \end{pmatrix} \tag{34}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Since $\mathbb{H}$ is represented in standard $2x2$ matrices, the involution is just subsequent from there, the only thing left to show is the closure $ab \in \mathbb{H}$.

$$ab = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} \gamma & \delta \\ -\bar{\delta} & \bar{\gamma} \end{pmatrix} = \tag{35}$$

$$= \begin{pmatrix} \alpha\beta - \beta\bar{\delta} & \alpha\delta + \beta\bar{\gamma} \\ -(\bar{\alpha}\bar{\delta} + \bar{\beta}\gamma) & \bar{\alpha}\gamma - \bar{\beta}\delta \end{pmatrix} = \begin{pmatrix} \xi & \psi \\ -\bar{\psi} & \bar{\xi} \end{pmatrix} \in \mathbb{H} \tag{36}$$

where $\xi, \psi \in \mathbb{C}$ because of closure of complex numbers in regards to multiplication and addition, which is $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}$, e.g. $\beta \cdot c \in \mathbb{C}$ with $c \in \mathbb{C}$.

2)For $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ we have for some $h \in \mathbb{H}$ and $c \in mathbbC$

$$h \otimes c = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \otimes c = \tag{37}$$

$$= \begin{pmatrix} \alpha c & \beta c \\ -\bar{\beta} c & \bar{\alpha} c \end{pmatrix} \simeq M_2(\mathbb{C}) \tag{38}$$

because again of $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}$.
3)We know that $\mathbb{H}$ is a real subalgebra of $M_2(\mathbb{C})$, so $M_k(\mathbb{H})$ is just an extension and an real subalgebra of $M_{2k}(\mathbb{C})$.
4) Here we use what we have learned

$$M_k(\mathbb{H}) \otimes_{\mathbb{R}} \mathbb{C} \simeq M_k(M_2(\mathbb{C})) = M_{2k}(\mathbb{C}) \tag{39}$$

Definition 3. A representation of a finite-dimensional real * algebra A is a pair (π, H), H- Hilbertspace, $\pi : A \to L(H)$

Exercise 3

Show that there is a one-to-one correspondence between Hilbertspace representations of real *-algebras A and complex representations of its complexification $A \otimes_{\mathbb{R}} \mathbb{C}$. Conclude that the unique irreducible Hilbertspace representation of $M_k(\mathbb{H})$ is $\mathbb{C}^{2k}$

Lemma 3. *Real *-algebra A represented faithfully on a finite dimensional Hilbertspace H through a real linear *-algebra map $\pi : A \to L(H)$ hen A is a matrix algebra.*

$$A \simeq \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{F}_i) \tag{40}$$

Where $\mathbb{F}_i = \mathbb{R}, \mathbb{C}, \mathbb{H}$ depending on i.

Proof. π allows A to be considered as a real *-subalgebra of $M_{dim(H)}(\mathbb{C}) \Rightarrow A + iA$ complex *-subalgebra of $M_{dim(H)}(\mathbb{C})$. Then $A + iA$ is a matrix algebra and $A + iA = M_k(\mathbb{C})$ for $k \geq 1$. Thus we have

$$A \cap iA = \begin{cases} \{0\} & \text{if } A = M_k(\mathbb{C}) \\ A + iA = M_k(\mathbb{C}) \end{cases} \tag{41}$$

Furthermore A is a fixed point algebra of an anti-linear automorphism α of $M_k(\mathbb{C})$ with $\alpha(a + ib) = a - ib$ for $a, b \in A$. Implement α by an anti-linear isometry I on $\mathbb{C}^n$ such that $\alpha(x) = I \times I^{-1} \quad \forall x \in M_k(\mathbb{C})$. Now since $\alpha^2 = 1$, I^2 commutes with $M_k(\mathbb{C})$ and is proportional to a complex scalar $I^2 = \pm 1$ and A is the commutant of I

- if $I^2 = 1 \quad \Rightarrow \quad \exists \quad \{e_i\}$ ONB of $\mathbb{C}^k$ with $Ie_i = e_i$, then $A = M_k(\mathbb{R})$
- if $I^2 = -1 \quad \Rightarrow \quad \exists \quad \{e_i, f_i\}$ ONB of $\mathbb{C}^k$ with $Ie_i = f_i$, (k even)
  Therefor I must be a $k/2 \times k/2$ matrix because of commutation with $M_k(\mathbb{C})$, then $A = M_{k/2}(\mathbb{H})$

$\square$

The Krajewski diagrams can also classify real algebras, as long as we take $\mathbb{F}_i$ for each i into account. That is we enhance the set Λ to be

$$\Lambda = \{\mathbf{n}_1\mathbb{F}_1, \ldots, \mathbf{n}_N\mathbb{F}_N\} \tag{42}$$

Reducing in to the previous Λ if all $\mathbb{F}_i = \mathbb{C}$.

3 Classification of Irreducible Geometries

Classify irreducible real spectral triples based on $M_N(\mathbb{C} \oplus M_N(\mathbb{C})$ for some N

Definition 4. A finite real spectral triple $(A, H, D; J, \gamma)$ is called irreducible if the triple (A, H, J) is irreducible, that is when

1. The representation of A and J on H are irreducible
2. The action of A on H has a separating vector

Theorem 2. *Let $(A, H, D; J, \gamma)$ be an irreducible finite real spectral triple of KO-dimension 6. Then exists a positive integer N such that $A \simeq M_N(\mathbb{C}) \oplus M_N(\mathbb{C})$.*

Proof. Let $(A, H, D; J, \gamma)$ be an arbitrary finite real spectral triple, corresponding to

$$A = \bigoplus_{i}^{N} M_{n_i}(\mathbb{C}) \tag{43}$$

$$H = \bigoplus_{i,j=1}^{N} \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ} \otimes V_{ij} \tag{44}$$

Remember that each $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j}$ is a irreducible representation of A. In order for H to support the real structure J we need both $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j}$ and $\mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i}$. With Lemma 1 with $J^2 = 1$ with multiplicity $dim(V_{ij}) = 1$ we have such a structure. Hence

$$H = \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j} \oplus \mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i} \tag{45}$$

For $i, j \in \{1, \ldots, N\}$

For the second condition (existence of the separating vector). The representations of A in H are only faithful if $A = M_{n_i}(\mathbb{C}) \oplus M_{n_j}(\mathbb{C})$. The stronger condition applies $n_i = n_j$ then we have $A'\xi = H$ with the commutant of A and $\xi \in H$ the separating vector. Normally since $A' = M_{n_j}(\mathbb{C}) \oplus M_{n_i}(\mathbb{C})$ with $dim(A') = n_i^2 + n_j^2$ and $dim(H) = 2n_in_j$ we have a equality $n_i = n_j$. $\square$