University of Vienna
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Notes on
Noncommutative Geometry and Particle Phyiscs

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Contents

1 Spectral Action of the Fluctuated Dirac Operator

Proposition 1. *The spectral action of the almost commutative manifold* M *with* $\dim(M) = 4$ *with a fluctuated Dirac operator is.*

$$Tr(f\frac{D_\omega}{\Lambda}) \sim \int_M \mathscr{L}(g_{\mu\nu}, B_\mu, \Phi)\sqrt{g}\, d^4x + O(\Lambda^{-1}) \tag{1}$$

with

$$\mathscr{L}(g_{\mu\nu}, B_\mu, \Phi) = N\mathscr{L}_M(g_{\mu\nu})\mathscr{L}_B(B_\mu) + \mathscr{L}_\phi(g_{\mu\nu}, B_\mu, \Phi) \tag{2}$$

where $N = 4$ *and* $\mathscr{L}_M$ *is the Lagrangian of the spectral triple* $(C^\infty(M), L^2(S), D_M)$

$$\mathscr{L}_M(g_{\mu\nu}) := \frac{f_4\Lambda^4}{2\pi^2} - \frac{f_2\Lambda^2}{24\pi^2}s - \frac{f(0)}{320\pi^2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}. \tag{3}$$

Here $C^{\mu\nu\rho\sigma}$ *is defined in terms of the Riemannian curvature tensor* $R_{\mu\nu\rho\sigma}$ *and the Ricci tensor* $R_{\nu\sigma} = g^{\mu\rho}R_{\mu\nu\rho\sigma}$.

Furthermore $\mathscr{L}_B$ *describes the kinetic term of the gauge field*

$$\mathscr{L}_B(B_\mu) := \frac{f(0)}{24\pi^2}Tr(F_{\mu\nu}F^{\mu\nu}). \tag{4}$$

Last $\mathscr{L}_\phi$ *is the scalar-field Lagrangian with a boundary term.*

$$\mathscr{L}_\phi(g_{\mu\nu}, B_\mu, \Phi) := -\frac{2f_2\Lambda^2}{4\pi^2}Tr(\Phi^2) + \frac{f(0)}{8\pi^2}Tr(\Phi^4) + \frac{f(0)}{24\pi^2}\Delta(Tr(\Phi^2)) \tag{5}$$

$$+\frac{f(0)}{48\pi^2}sTr(\Phi^2)\frac{f(0)}{8\pi^2}Tr((D_\mu\Phi)(D^\mu\Phi)). \tag{6}$$

Proof. The dimension of our manifold M is $\dim(M) = \mathrm{Tr}(id) = 4$. Let us take a $x \in M$, we have an asymtotic expansion of $\mathrm{Tr}(f(\frac{D_\omega}{\Lambda}))$ as $\Lambda \to \infty$

$$\mathrm{Tr}(f(\frac{D_\omega}{\Lambda})) \simeq 2f_4\Lambda^4 a_0(D_\omega^2) + 2f_2\Lambda^2 a_2(D_\omega^2) \tag{7}$$

$$+ f(0)a_4(D_\omega^4) + O(\Lambda^{-1}). \tag{8}$$

Note that the heat kernel coefficients are zero for uneven k, furthermore they are dependent on the fluctuated Dirac operator D_ω. We can rewrite the heat kernel coefficients in terms of D_M, for the first two we note that $N := \mathrm{Tr}\mathbb{1}_{\mathbb{H}_\mathbb{F}})$

$$a_0(D_\omega^2) = Na_0(D_M^2) \tag{9}$$

$$a_2(D_\omega^2 = Na_2(D_M^2) - \frac{1}{4\pi^2}\int_M \mathrm{Tr}(\Phi^2)\sqrt{g}d^4x \tag{10}$$

For a_4 we need to extend in terms of coefficients of F, look week9.pdf for the standard version,

$$\frac{1}{360}\mathrm{Tr}(60sF) = -\frac{1}{6}S(Ns + 4\mathrm{Tr}(\Phi^2)) \tag{11}$$

$$F^2 = \frac{1}{16}s^2 \otimes 1 + 1 \otimes \Phi^4 - \frac{1}{4}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma F_{\mu\nu}F^{\mu\nu} + \tag{12}$$

$$+ \gamma^\mu\gamma^\nu \otimes (D_\mu\Phi)(D_\nu\Phi) + \frac{1}{2}s \otimes \Phi^2 + \text{ traceless terms} \tag{13}$$

$$\frac{1}{360}\mathrm{Tr}(180F^2) = \frac{1}{8}s^2N + 2\mathrm{Tr}(\Phi^4) + \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \tag{14}$$

$$+ 2\mathrm{Tr}((D_\mu\Phi)(D^\mu\Phi)) + s\mathrm{Tr}(\Phi^2) \tag{15}$$

□

2 Fermionic Action

Definition 1. The fermionic action is defined by

$$S_f[\omega,\psi] = (J\tilde{\psi}, D_\omega\tilde{\psi}) \tag{16}$$

with $\tilde{\psi} \in H^+_{cl} := \{\tilde{\psi} : \psi \in H^+\}$. H^+_{cl} is the set of Grassmann variables in H in the +1-eigenspace of the grading γ.

$$M \times F_{ED} := \left(C^\infty(M,\mathbb{C}^2),\ L^2(S) \otimes \mathbb{C}^4,\ D_M \otimes 1 + \gamma_M \otimes D_F;\ J_M \otimes J_F,\ \gamma_M \otimes \gamma_F\right) \tag{17}$$