University of Vienna
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# Notes on
# Noncommutative Geometry and Particle Phyiscs

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## Contents

# 1 Finite Real Noncommutative Spaces

## 1.1 Finite Real Spectral Triples

Add on to finite real spectral triples a *real structure*. The requirement is that $H$ is a $A$-$A$-bimodule (before only a $A$-left module).

For this we introduce a $\mathbb{Z}_2$-grading $\gamma$ with

$$\gamma^* = \gamma \tag{1}$$
$$\gamma^2 = 1 \tag{2}$$
$$\gamma D = -D\gamma \tag{3}$$
$$\gamma a = a\gamma \quad a \in A \tag{4}$$

**Definition 1.** A *finite real spectral triple* is given by a finite spectral triple $(A,H,D)$ and a anti-unitary operator $J : H \to H$ called the *real structure*, such that

$$a^\circ := Ja^* J^{-1} \tag{5}$$

is a right representation of $A$ on $H$, that is $(ab)^\circ = b^\circ a^\circ$. With two requirements

$$[a, b^\circ] = 0 \tag{6}$$
$$[[D,a], b^\circ] = 0. \tag{7}$$

They are called the *commutant property*, and mean that the left action of an element in $A$ and $\Omega^1_D(A)$ commutes with the right action on $A$.

**Definition 2.** The *KO*-dimension of a real spectral triple is determined by the sings $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ appearing in

$$J^2 = \varepsilon \tag{8}$$
$$JD = \varepsilon\, DJ \tag{9}$$
$$J\gamma = \varepsilon'' \gamma J. \tag{10}$$

Table 1: *KO*-dimension $k$ modulo 8 of a real spectral triple

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon$ | 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 |
| $\varepsilon'$ | 1 | -1 | 1 | 1 | 1 | -1 | 1 | 1 |
| $\varepsilon''$ | 1 | | -1 | | 1 | | -1 | |

**Definition 3.** An opposite-algebra $A^\circ$ of a $A$ is defined to be equal to $A$ as a vector space with the opposite product

$$a \circ b := ba \tag{11}$$
$$\Rightarrow a^\circ = Ja^* J^{-1} \quad \text{defines the left representation of } A^\circ \text{ on } H \tag{12}$$

**Example 1.** Matrix algebra $M_N(\mathbb{C})$ acting on $H = M_N(\mathbb{C})$ by left matrix multiplication with the Hilbert Schmidt inner product.

$$\langle a, b \rangle = \mathrm{Tr}(a^* b) \tag{13}$$

Then we define $\gamma(a) = a$ and $J(a) = a^*$ with $a \in H$. Since $D$ mus be odd with respect to $\gamma$ it vanishes identically.

**Definition 4.** We call $\xi \in H$ **cyclic vector** in $A$ if:

$$A\xi := a\xi : \ \ a \in A = H \tag{14}$$

We call $\xi \in H$ **separating vector** in $A$ if:

$$a\xi = 0 \ \ \Rightarrow \ \ a = 0; \ \ \ a \in A \tag{15}$$

**Exercise 1**

**In the previous example, show that the right action on $M_N(\mathbb{C})$ on $H = M_N(\mathbb{C})$ as defined by $a \mapsto a^\circ$ is given by right matrix multiplication.**

$$a^\circ \xi = Ja^* J^{-1} \xi = Ja^* \xi^* = J\xi a = \xi^* a \tag{16}$$

**Exercise 2**

**Let $A = \bigoplus_i M_{n_i}(\mathbb{C})$, represented on $H = \bigoplus_i \mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i}$, meaning that the irreducible representation $\mathbf{n}_i$ has multiplicity $m_i$.**

1. **Show that the commutant $A'$ of $A$ is $A' \simeq \bigoplus_i M_{m_i}(\mathbb{C})$. As a consequence show $A'' \simeq A$.**
2. **Show that if $\xi$ is a separating vector for $A$ than it is cyclic for $A'$.**

1. We know the multiplicity space is $V_i = \mathbb{C}^{m_i}$. We know that for $T \in H$ and $a \in A'$ to work we need $aT = Ta$ by laws of matrix multiplication we need $A' \simeq \oplus_i M_{m_i}(\mathbb{C})$ for this to work since $H = \bigoplus_i \mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i}$
2. Suppose $\xi$ is cyclic for $A$ then $A'\xi = \{0\}$. Under the action of $A$ we then have $A'A\xi = AA'\xi = 0 \Rightarrow A' = 0$.
   Suppose now $\xi$ is separating for $A'$, we have $A'\xi = \{0\}$. We can define a projection in $A'$, $A\xi = P'$. With this projection we have $(1-P')\xi = 0 \Rightarrow 1 - P' = 0 \Rightarrow A\xi = H$.

**Exercise 3**

**Suppose $(A, H, D = 0)$ is a finite spectral triple such that $H$ possesses a cyclic and separating vector for $A$.**

1. **Show that the formula $S(a\xi) = a * \xi$ defines a anti-linear operator $S : H \to H$.**
2. **Show that $S$ is invertible**

3. **Let $J : H \to H$ be the operator in $S = J\Delta^{1/2}$ with $\Delta = S * S$. Show that $J$ is anti-unitary**

1. By composition $S(a\xi) = a * \xi$ this is literally anti-linearity. Does this mean $S\xi = \xi$?
2. Let $\xi \in H$ be cyclic then: $S(A\xi) = A * \xi = A\xi = H$. The same has to work for $S^{-1}$ if not then $\xi$ wouldn't exist. $S^{-1}(A * \xi) = S^{-1}(H) = H$.
3. Since $S$ is bijective then $\Delta^{1/2}$ and $J$ need to be bijective.
   Now let $\xi_1, \xi_2 \in H$.

$$
\begin{aligned}
< J\xi_1, J\xi_2 > &=< J^* J\xi_1, \xi_2 >^* = && (17)\\
&=< (\Delta^{1/2})^* S^* S\Delta^{1/2}\xi_1, \xi_2 >^* = && (18)\\
&=< (SS^*)^{1/2} S^* S(SS^*)\xi_1, \xi_2 >^* = && (19)\\
&=< (SS^* SS^*)^{1/2}\xi_1, \xi_2 >^* = && (20)\\
&=< \xi_1, \xi_2 >^* =< \xi_2, \xi_1 > . && (21)
\end{aligned}
$$

## 1.2 Morphisms Between Finite Real Spectral Triples

Extend unitary equivalence of finite spectral triples to real ones (with $J$ and $\gamma$)

**Definition 5.** We call two finite real spectral triples $(A_1, H_1, D_1; J_1, \gamma_1)$ and $(A_2, H_2, D_2; J_2, \gamma_2)$ unitarily equivalent if $A_1 = A_2$ and if there exists a unitary operator $U : H_1 \to H_2$ such that

$$U\pi_1(a)U^* = \pi_2(a) \tag{22}$$
$$UD_1U^* = D_2 \tag{23}$$
$$U\gamma_1U^* = \gamma_2 \tag{24}$$
$$UJ_1U^* = J_2 \tag{25}$$

**Definition 6.** Let $E$ be a $B$-$A$ bimodule. The *conjugate Module* $E^\circ$ is given by the $A$-$B$-bimodule.

$$E^\circ = \{\bar{e} : e \in E\} \tag{26}$$

with

$$a \cdot \bar{e} \cdot b = b^* \bar{e} a^* \quad \forall a \in A, b \in B \tag{27}$$

$E^\circ$ is not a Hilbert bimodule for $(A, B)$ because it doesn't have a natural $B$-valued inner product. But there is a $A$-valued inner product on the left $A$-module $E^\circ$ with

$$\langle \bar{e}_1, \bar{e}_2 \rangle = \langle e_2, e_1 \rangle \quad e_1, e_2 \in E \tag{28}$$

and linearity in $A$:

$$\langle a\bar{e}_1, \bar{e}_2 \rangle = a\langle \bar{e}_1, \bar{e}_2 \rangle \quad \forall a \in A. \tag{29}$$

**Exercise 4**

**Show that $E^\circ$ is a Hilbert bimodule** $(B^\circ, A^\circ)$

Straightforward show properties of the Hilbert bimodule and its $B^\circ$ valued inner product. Let $\bar{e}_1, \bar{e}_2 \in E^\circ$ and $a^\circ \in A, b^\circ \in B$.

$$< \bar{e}_1, a^\circ \bar{e}_2 > = < \bar{e}_1, Ja^* J^{-1} \bar{e}_2 > = \tag{30}$$
$$= < \bar{e}_1, Ja^* e_2 > = \tag{31}$$
$$= < J^{-1} e_1, a^* e_2 > = \tag{32}$$
$$= < a^* e_1, e_2 > = < J^{-1} (a^\circ)^* J e_1, e_2 > = \tag{33}$$
$$= < J^{-1} (a^\circ)^* \bar{e}_1, e_2 > = \tag{34}$$
$$= < (a^\circ)^* \bar{e}_1, \bar{e}_2 > . \tag{35}$$

Next $< \bar{e}_1, \bar{e}_2 b^\circ > = < \bar{e}_1, \bar{e}_2 > b^\circ$.

$$< \bar{e}_1, \bar{e}_2 b^\circ > = < \bar{e}_1, \bar{e}_2 J b^* J^{-1} > = \tag{36}$$
$$= < \bar{e}_1, \bar{e}_2 > J b^* J^{-1} = \tag{37}$$
$$= < \bar{e}_1, \bar{e}_2 > b^\circ . \tag{38}$$

Then:

$$(< \bar{e}_1, \bar{e}_2) >_{E^\circ})^* = (< e_2, e_1 >_E)^* = \tag{39}$$
$$= < e_1, e_2 >_E^* = < \bar{e}_2, \bar{e}_2 >_{E^\circ} \tag{40}$$

And of course $< \bar{e}, \bar{e} > = < e, e > \geq 0$

### 1.2.1 Construction of a Finite Real Spectral Triple from a Finite Real Spectral Triple

Given a Hilbert bimodule $E$ for $(B, A)$ we construct a spectral triple $(B, H', D'; J', \gamma')$ from $(A, H, D; J, \gamma)$

For the $H'$ we make a $\mathbb{C}$-valued inner product on $H'$ by combining the $A$ valued inner product on $E$ and $E^\circ$ with the $\mathbb{C}$-valued inner product on $H$.

$$H' := E \otimes_A H \otimes_A E^\circ \tag{41}$$

Then the action of $B$ on $H'$ is:

$$b(e_2 \otimes \xi \otimes \bar{e}_2) = (be_1) \otimes \xi \otimes \bar{e}_2 \tag{42}$$

The right action of $B$ on $H'$ defined by action on the right component $E^\circ$

$$J'(e_1 \otimes \xi \otimes \bar{e}_2) = e_2 \otimes J\xi \otimes \bar{e}_1 \tag{43}$$

with $b^\circ = J' b^* (J')^{-1}$, $b^* \in B$ action on $H'$.

**Exercise 5**

**Let $\nabla : E \Rightarrow E \otimes_A \Omega^1_d(A)$ be a right connection on $E$ consider the following anti-linear map:**

$$\tau : E \otimes_A \Omega^1_D(A) \to \Omega^1_D(A) \otimes_A E^\circ \tag{44}$$

$$e \otimes \omega \mapsto -\omega^* \otimes \bar{e} \tag{45}$$

**Show that the map $\bar{\nabla} : E^\circ \Omega^1_D(A) \otimes E^\circ$ with $\bar{\nabla}(\bar{e}) = \tau \circ \nabla(e)$ is a left connection, that means show that it satisfied the left Leibniz rule:**

$$\bar{\nabla}(a\bar{e}) = [D,a] \otimes \bar{e} + a\bar{\nabla}(\bar{e}) \tag{46}$$

Hagime:

$$\text{For one:} \tag{47}$$

$$\tau \circ \nabla(ae) = \bar{\nabla}(a\bar{e}) = \bar{\nabla}(a^*\bar{e}) \tag{48}$$

$$\text{For two:} \tag{49}$$

$$\tau \circ \nabla(ae) = \tau(\nabla(e)a) + \tau \circ (e \otimes d(a)) = \tag{50}$$

$$= a^*\bar{\nabla}(\bar{e}) - d(a)^* \otimes \bar{e}. \tag{51}$$

$$= a^*\bar{\nabla}(\bar{e}) + d(a^*) \otimes \bar{e}. \tag{52}$$

Then the connections

$$\nabla : E \to E \otimes_A \Omega^1_D(A) \tag{53}$$

$$\bar{\nabla} : E^\circ \to \Omega^1_D(A) \otimes_A E^\circ \tag{54}$$

give us the Dirac operator on $H' = E \otimes_A H \otimes_A E^\circ$

$$D'(e_1 \otimes \xi \otimes \bar{e}_2) = (\nabla e_1)\xi \otimes \bar{e}_2 + e_1 \otimes D\xi \otimes \bar{e}_2 + e_1 \otimes \xi(\bar{\nabla}\bar{e}_2) \tag{55}$$

And the right action of $\omega \in \Omega^1_D(A)$ on $\xi \in H$ is defined by

$$\xi \mapsto \varepsilon' J \omega^* J^{-1} \xi \tag{56}$$

Finally for the grading

$$\gamma' = 1 \otimes \gamma \otimes 1 \tag{57}$$

**Theorem 1.** *Suppose $(A,H,D;J,\gamma)$ is a finite spectral triple of KO-dimension k, let $\nabla$ be like above satisfying the compatibility condition (like with finite spectral triples).*

*Then $(B,H',D';J',\gamma')$ is a finite spectral triple of KO-Dimension k. ($H',D',J',\gamma'$ like above)*

*Proof.* The only thing left is to check if the *KO*-dimension is preserved, for this we check if the $\varepsilon$'s are the same.

$$(J')^2 = 1 \otimes J^2 \otimes 1 = \varepsilon \tag{58}$$

$$J'\gamma' = \varepsilon''\gamma' J' \tag{59}$$

and for $\varepsilon'$

$$\begin{aligned} J'D'(e_1 \otimes \xi \otimes \bar{e}_2) &= J'((\nabla e_1)\xi \otimes \bar{e_2} + e_1 \otimes D\xi \otimes \bar{e}_2 + e_1 \otimes \xi(\tau\nabla e_2)) && (60) \\ &= \varepsilon' D'(e_2 \otimes J\xi \otimes \bar{e}_2) && (61) \\ &= \varepsilon' D' J'(e_1 \otimes \xi \bar{e}_2) && (62) \end{aligned}$$

□