Notes on
Noncommutative Geometry and Particle Physics

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Contents

1 Excurse

Manifold: A topological space that is locally Euclidean.
Riemannian Manifold:A Manifold equipped with a riemannian Metric, a symmetric bilinear form on Vector Fields $\Gamma(TM)$

$$g:\Gamma(TM)\times\Gamma(TM)\to C(M) \tag{1}$$

with

$$g(X,Y)\in\mathbb{R}\quad \text{if } X,Y\in\mathbb{R} \tag{2}$$

$$g \text{ is } C(M)\text{-bilinear } \forall f\in C(M):\ \ g(fX,Y)=g(X,fY)=fg(X,Y) \tag{3}$$

$$g(X,X)\begin{cases}\geq 0 & \forall X\\ =0 & \forall X=0\end{cases} \tag{4}$$

g on M gives rise to a distance function on M

$$d_g(x,y)=\inf_\gamma\left\{\int_0^1(\dot\gamma(t),\dot\gamma(t))dt;\ \ \gamma(0)=x,\gamma(1)=y\right\} \tag{5}$$

Riemannian Manifold is called spin^c if there exists a vector bundle $S \to M$ with an algebra bundle isomorphism

$$\mathbb{C}\mathrm{l}(TM) \simeq \mathrm{End}(S) \qquad (dim(M) \text{ even}) \tag{6}$$
$$\mathbb{C}\mathrm{l}(TM)^\circ \simeq \mathrm{End}(S) \qquad (dim(M) \text{ odd}) \tag{7}$$
$$\tag{8}$$

(M,S) is called the **spin^c structure on** M.
S is called the **spinor Bundle**.
$\Gamma(S)$ are the **spinors**.

Riemannian spin^c Manifold is called spin if there exists an anti-unitary operator J_M : $\Gamma(S) \to \Gamma(S)$ such that:

1. J_M commutes with the action of real-valued continuous functions on $\Gamma(S)$.
2. J_M commutes with $\mathrm{Cliff}^-(M)$ (even case)
   J_M commutes with $\mathrm{Cliff}^-(M)^\circ$ (odd case)

(S, J_M) is called the **spin Structure on** M
J_M is called the **charge conjugation**.

2 Noncommutative Geomtery of Electrodynamics

2.1 The Two-Point Space

Consider a two point space $X := \{x, y\}$. This space=an be described with the following spectral triple

$$F_x := (C(X) = \mathbb{C}^2, H_F, D_F; J_F, \gamma_f). \tag{9}$$

Notes on the spectral triple:

- Action of $C(X)$ on H_F is faithful $(\dim(H_F) \geq 2)$
  we choose $H_F = \mathbb{C}^2$
- γ_F is the $\mathbb{Z}_2$ grading, which allows us to decompose $H_F = H_F^+ \oplus H_F^- = \mathbb{C} \oplus \mathbb{C}$ where $H_F^\pm = \{\psi \in H_F | \ \gamma_F \psi = \pm\psi\}$ are the two eigenspaces
- D_F interchanges between $H_F^\pm$, $D_F = \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix}$ where $t \in \mathbb{C}$

Proposition 1. *F_x can only have a real structure if $D_F = 0$ in that case we have $KO - dim = 0, 2, 6$*

Proof. There are two diagram representations of F_x at $\underbrace{\mathbb{C} \oplus \mathbb{C}}_{C(X)}$ on $\underbrace{\mathbb{C} \oplus \mathbb{C}}_{H_F}$

If F_x a real spectral triple then D_F can only go vertically or horizontally $\Rightarrow D_F = 0$. Furthermore the diagram on the left has KO-dimension 2 and 6, diagram on the right has KO-dimension 0 and 4. Yet KO-dimension 4 is not allowed because $dim(H_F^\pm) = 1$ (see Lemma 3.8 Book), so $J_F^2 = -1$ is not allowed. $\square$

$$\begin{array}{c|cc} & \mathbf{1} & \mathbf{1} \\ \hline \mathbf{1}^\circ & & \circ \\ \mathbf{1}^\circ & \circ & \end{array} \qquad\qquad \begin{array}{c|cc} & \mathbf{1} & \mathbf{1} \\ \hline \mathbf{1}^\circ & \circ & \\ \mathbf{1}^\circ & & \circ \end{array}$$

2.2 The product Space

Let M be a 4-dim Riemannian spin Manifold, then we have the almost commutative manifold $M \times F_x$

$$M \times F_x = (C^\infty(M, \mathbb{C}^2, L^2(S) \otimes \mathbb{C}^2, D_M \otimes 1; J_M \otimes J_F, \gamma_M \otimes \gamma_F) \tag{10}$$

(J_M is missing need to choose)
$C^\infty(M, \mathbb{C}^2) \simeq C^\infty(M) \oplus C^\infty(M)$ (decomposition) and from Gelfand duality we we have

$$N := M \otimes X \simeq M \sqcup X \tag{11}$$

$H = L^2(S) \oplus L^2(S)$ (decomposition), such that for $\underbrace{a, b \in C^\infty(M)}_{(a,b) \in C^\infty(N)}$ and $\underbrace{\psi, \phi \in L^2(S)}_{(\psi,\phi) \in H}$ we have

$$(a,b)(\psi, \phi) = (a\psi, b\phi) \tag{12}$$

We can consider a distance formula on $M \times F_x$ by

$$d_{D_F}(x,y) = \sup\{|a(x) - a(y)| : a \in A_F, ||[D_F, a]|| \leq 1\} \tag{13}$$

Now lets calculate the distance between two points on the two point space $X = \{x, y\}$, between x and y. Let $a \in \mathbb{C}^2 = C(X)$, a is specified with two complex numbers $a(x)$ and $a(y)$

$$||[D_F, a]|| = ||(a(y) - a(x)) \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix} || \leq 1 \tag{14}$$

$$\Rightarrow |a(y) - a(x)| \leq \frac{1}{|t|} \tag{15}$$

Therefore the distance between two points x and y is

$$d_{D_F}(x,y) = \frac{1}{|t|} \tag{16}$$

Note that if there exists J_M (real structure) $\Rightarrow t = 0$ then $d_{D_F}(x,y) \to \infty$!

Now let $p \in M$, then take two points on $N = M \times X$, (p,x) and (p,y) and $a \in C^\infty(N)$ is determined by $a_x(p) := a(p,x)$ and $a_y(p) := a(p,y)$. The distance between these two points is then

$$d_{D_F \otimes 1}(n_1, n_2) = \sup\{|a(n_1) - a(n_2)| : a \in A, ||[D \otimes 1, a]||\} \tag{17}$$

Remark: If $n_1 = (p,x)$ and $n_2 = (q,x)$ for $p, q \in M$ then

$$d_{D_M \otimes 1}(n_1, n_2) = |a_x(p) - a_x(q)| \quad a_x \in C^\infty(M) \text{ with } ||[D \otimes 1, a_x]|| \leq 1 \tag{18}$$

The distance turns to the geodestic distance formula

$$d_{D_M \otimes 1}(n_1, n_2) = d_g(p, q) \tag{19}$$

However if $n_1 = (p, x)$ and $n_2 = (q, y)$ then the two conditions are $||[D_M, a_x]|| \leq 1$ and $||[D_M, a_y|| \leq 1$. They have no restriction which results in the distance being infinite! And $N = M \times X$ is given by two disjoint copies of M which are separated by infinite distance

Note: distance is only finite if $[D_F, a] \neq 1$. The commutator generates a scalar field say ϕ and the finiteness of the distance is related to the existence of scalar fields.

2.3 $U(1)$ Gauge Group

Here we determine the Gauge theory corresponding to the almost commutative Manifold $M \times F_x$.

Gauge Group of a Spectral Triple:

$$\mathfrak{B}(A, H; J) := \{U = uJuJ^{-1} | u \in U(A)\} \tag{20}$$

Definition 1. A *-automorphism of a *-algebra A is a linear invertible map

$$\alpha : A \to A \quad \text{with} \tag{21}$$

$$\alpha(ab) = \alpha(a)\alpha(b) \tag{22}$$

$$\alpha(a)^* = \alpha(a^*) \tag{23}$$

The **Group of automorphisms of the *-Algebra** A is (A).
The automorphism α is called **inner** if

$$\alpha(a) = uau^* \quad \text{for } U(A) \tag{24}$$

where $U(\mathfrak{A})$ is

$$U(A) = \{u \in A | \ uu^* = u^*u = 1\} \quad \text{(unitary)} \tag{25}$$

The Gauge group is given by the quotient $U(A)/U(A_J)$. We want a nontrivial Gauge group so we need to choose $U(A_J) \neq U(A)$ which is the same as $U((A_F)_{J_F}) \neq U(A_F)$. We consider F_x to be

$$F_x := \left(\mathbb{C}^2, \mathbb{C}^2, D_F = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; J_f = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \gamma_F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right). \tag{26}$$

Here C is the complex conjugation, and F_X is a real even finite spectral triple (space) with $KO - dim = 6$

Proposition 2. *The Gauge group $\mathfrak{B}(F)$ of the two point space is given by $U(1)$.*

Proof. Note that $U(A_F) = U(1) \times U(1)$. We need to show that $U(\mathscr{A}_F) \cap U(A_F)_{J_F}) \simeq U(1)$, such that $\mathfrak{B}(F) \simeq U(1)$.

So for $a \in \mathbb{C}^2$ to be in $(A_F)_{J_F}$ it has to satisfy $J_F a^* J_F = a$.

$$J_F a^* J^{-1} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} \begin{pmatrix} \bar{a}_1 & 0 \\ 0 & \bar{a}_2 \end{pmatrix} \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} = \begin{pmatrix} a_2 & 0 \\ 0 & a_1 \end{pmatrix} \tag{27}$$

Which is only the case if $a_1 = a_2$. So we have $(A_F)_{J_F} \simeq \mathbb{C}$, whose unitary elements from $U(1)$ are contained in the diagonal subgroup of $U(\mathscr{A}_F)$. $\square$

Now we need to find the exact from of the field B_μ to calculate the spectral action of a spectral triple. Since $(A_F)_{J_F} \simeq \mathbb{C}$ we find that $\mathfrak{h}(F) = \mathfrak{u}((A_F)_{J_F}) \simeq i\mathbb{R}$.

An arbitrary hermitian field $A_\mu = -ia\partial_\mu b$ is given by two $U(1)$ Gauge fields $X_\mu^1, X_\mu^2 \in C^\infty(M, \mathbb{R})$. However A_μ appears in combination $A_\mu - J_F A_\mu J_F^{-1}$:

$$B_\mu = A_\mu - J_F A_\mu J_F^{-1} = \begin{pmatrix} X_\mu^1 & 0 \\ 0 & X_\mu^2 \end{pmatrix} - \begin{pmatrix} X_\mu^2 & 0 \\ 0 & X_\mu^1 \end{pmatrix} =: \begin{pmatrix} Y_\mu & 0 \\ 0 & -Y_\mu \end{pmatrix} = Y_\mu \otimes \gamma_F \tag{28}$$

where Y_μ the $U(1)$ Gauge field is defined as

$$Y_\mu := X_\mu^1 - X_\mu^2 \in C^\infty(M, \mathbb{R}) = C^\infty(M, i\, u(1)). \tag{29}$$

Proposition 3. *The inner fluctuations of the almost-commutative manifold $M \times F_x$ described above are parametrized by a $U(1)$-gauge field Y_μ as*

$$D \mapsto D' = D + \gamma^\mu Y_\mu \otimes \gamma_F \tag{30}$$

The action of the gauge group $\mathfrak{B}(M \times F_X) \simeq C^\infty(M, U(1))$ on D' is implemented by

$$Y_\mu \mapsto Y_\mu - i\, u \partial_\mu u^*; \quad (u \in \mathfrak{B}(M \times F_X)). \tag{31}$$

3 Electrodynamics

Now we use the almost commutative Manifold and the abelian gauge group $U(1)$ to describe Electrodynamics. We arrive at a unified description of gravity and electrodynamics although in the classical level.

The almost commutative Manifold $M \times F_X$ describes a local gauge group $U(1)$. The inner fluctuations of the Dirac operator describe Y_μ the gauge field of $U(1)$. There arise two Problems:
(1): With F_X, D_F must vanish, however this implies that the electrons are massless (this we do not want)

(2): The Euclidean action for a free Dirac field is

$$S = -\int i\bar{\psi}(\gamma^\mu \partial_\mu - m)\psi d^4x, \tag{32}$$

ψ, $\bar{\psi}$ must be considered as independent variables, which means S_F need two independent Dirac Spinors. We write $\{e, \bar{e}\}$ for the ONB of H_F, where $\{e\}$ is the ONB of

H_F^+ and $\{\bar{e}\}$ the ONB of H_F^- with the real structure this gives us the following relations

$$J_F e = \bar{e} \qquad J_F \bar{e} = e \tag{33}$$
$$\gamma_F e = e \qquad \gamma_F \bar{e} = \bar{e}. \tag{34}$$

The total Hilbertspace is $H = L^2(S) \otimes H_F$, with γ_F we can decompose $L^2(S) = L^2(S)^+ \oplus L^2(S)^-$, so with $\gamma = \gamma_M \otimes \gamma_F$ we can obtain the positive eigenspace H^+

$$H^+ = L^2(S)^+ \otimes H_F^+ \oplus L^(S)^- \otimes H_F^-. \tag{35}$$

For a $\xi ß H^+$ we can write

$$\xi = \psi_L \otimes e + \psi_R \otimes \bar{e} \tag{36}$$

where $\psi_L \in L^2(S)^+$ and $\psi_R \in L^2(S)^-$ are the two Wheyl spinors. We denote that ξ is only determined by one Dirac spinor $\psi := \psi_L + \psi_R$, **but we require two independent spinors**. This is too much restriction for F_X.

3.1 The Finite Space

Here we solve the two problems by enlarging(doubling) the Hilbertspace. This is done by introducing multiplicities in Krajewski Diagrams which will also allow us to choose a nonzero Dirac operator which will connect the two vertices (next chapter).

We start of with the same algebra $C^\infty(M, \mathbb{C}^2)$, corresponding to space $N = M \times X \simeq M \sqcup M$.

The Hilbertspace will describe four particles,

- left handed electrons
- right handed positrons

Thus we have $\{ \underbrace{e_R, e_L}_{\text{left-handed}} , \underbrace{\bar{e}_R, \bar{e}_L}_{\text{right-handed}} \}$ the ONB for $H_F \mathbb{C}^4$.

Then with J_F we interchange particles with antiparticles we have the following properties

$$J_F e_R = \bar{e}_R \qquad J_F e_L = \bar{e}_L \tag{37}$$
$$\gamma_F e_R = -e_R \qquad \gamma_F e_L = e_L \tag{38}$$

and

$$J_F^2 = 1 \qquad J_F \gamma_F = -\gamma_F J_F \tag{39}$$

This corresponds to KO-dim$= 6$. Then γ_F allows us to can decompose H

$$H_F = \underbrace{H_F^+}_{\text{ONB } \{e_L, \bar{e}_L\}} \oplus \underbrace{H_F^-}_{\text{ONB } \{e_R, \bar{e}_R\}} . \tag{40}$$

Alternatively we can decompose H into the eigenspace of particles and their antiparticles (electrons and positrons) which we will use going further.

$$H_F = \underbrace{H_e}_{\text{ONB } \{e_L, e_R\}} \oplus \underbrace{H_{\bar{e}}}_{\text{ONB } \{\bar{e}_L, \bar{e}_R\}} \tag{41}$$

Now the action of $a \in A = \mathbb{C}^2$ on H with respect to the ONB $\{e_L, e_R, \bar{e}_L, \bar{e}_R\}$ is represented by

$$a = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \mapsto \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix} \tag{42}$$

Do note that this action commutes wit the grading and that $[a, b^\circ] = 0$ with $b := J_F b^* J_F$ because both the left and the right action is given by diagonal matrices.

Proposition 4. *The data*

$$\left(\mathbb{C}^2, \mathbb{C}^2, D_F = 0; J_F = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \gamma_F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \tag{43}$$

defines a real even spectral triple of KO-dimension 6.

This spectral triple can be represented in the following Krajewski diagram, with two nodes of multiplicity two

$$\begin{array}{ccc} & \mathbf{1} & \mathbf{1} \\ \mathbf{1}^\circ & & \circledcirc \\ \mathbf{1}^\circ & \circledcirc & \end{array}$$

3.2 A noncommutative Finite Dirac Operator

Add a non-zero Dirac Operator to F_{ED}. From the Krajewski Diagram, we see that edges only exist between the multiple vertices. So we construct a Dirac operator mapping between the two vertices.

$$D_F = \begin{pmatrix} 0 & d & 0 & 0 \\ \bar{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{d} \\ 0 & 0 & d & 0 \end{pmatrix} \tag{44}$$

We can now consider the finite space F_{ED}.

$$F_{ED} := (\mathbb{C}^2, \mathbb{C}^4, D_F; J_F, \gamma_F) \tag{45}$$

where J_F and γ_F like before, D_F like above.

3.3 The almost-commutative Manifold

The almost commutative manifold $M \times F_{ED}$ has KO-dim$= 2$, it is the following spectral triple

$$M \times F_{ED} := \left(C^\infty(M, \mathbb{C}^2, L^2(S) \otimes \mathbb{C}^4, D_M \otimes 1 + \gamma_M \otimes D_F; J_M \otimes J_F, \gamma_M \otimes \gamma_F\right) \qquad (46)$$

The algebra decomposition is like before

$$C^\infty(M, \mathbb{C}^2) = C^\infty(M) \oplus C^\infty(M) \qquad (47)$$

The Hilbertspace decomposition is

$$H = (L^2(S) \otimes H_e) \oplus (L^2(S) \otimes H_{\bar{e}}). \qquad (48)$$

Here we have the one component of the algebra acting on $L^2(S) \otimes H_e$, and the other one acting on $L^2(S) \otimes H_{\bar{e}}$

The derivation of the gauge theory is the same for F_{ED} as for F_X, we have $\mathfrak{B}(F) \simeq U(1)$ and for $B_\mu = A_\mu - J_F A_\mu J_F^{-1}$

$$B_\mu = \begin{pmatrix} Y_\mu & 0 & 0 & 0 \\ 0 & Y_\mu & 0 & 0 \\ 0 & 0 & Y_\mu & 0 \\ 0 & 0 & 0 & Y_\mu \end{pmatrix} \quad \text{for} \quad Y_\mu(x) \in \mathbb{R}. \qquad (49)$$

We have one single $U(1)$ gauge field Y_μ, carrying the action of the gauge group

$$\mathfrak{B}(M \times F_{ED}) \simeq C^\infty(M, U(1)) \qquad (50)$$

Our space $N = M \times X \simeq M \sqcup M$ consists of two compies of M. If $D_F = 0$ we have infinite distance of the two copies. Now we have D_F nonzero but the $[D_F, a] = 0$ $\forall a \in A$ which still yields infinite distance.

Question 1. What does this imply (physically, mathematically)? Why can we continue even thought we have infinite distance between the same manifold? What do we get if we fix this?