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Abstract

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1 Introduction

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2 Main Section

2.1 Noncommutative Geometric Spaces

2.1.1 $*$-Algebra

To grasp the idea of encoding geometrical data into a spectral triple we introduce the first ingredient of a spectral triple, an unital C^* algebra.

Definition 1. A *vector space* A over $\mathbb{C}$ is called a *complex, unital Algebra* if,
$\forall a, b \in A$:

1. $A \times A \to A$, $(a,b) \;\mapsto\; a \cdot b$,
2. with an identity element $1a = a1 = a$.

Extending the definition, a $*$-algebra is an algebra A with a *conjugate linear map (involution)* $* : A \;\to A$, $\forall a, b \in A$ satisfying:

1. $(ab)^* = b^* a^*$,
2. $(a^*)^* = a$.

In the following all unital algebras are referred to as algebras.

2.1.2 Finite Discrete Space

Let us consider an example of an $*$-algebra of continuous functions $C(X)$ on a discrete topological space X with N points. Functions of a continuous $*$-algebra $C(X)$ assign values to $\mathbb{C}$, thus $f,g \in C(X)$, $\lambda \in \mathbb{C}$ and $x \in X$ they provide the following structure:

- *pointwise linear*
  $(f+g)(x) = f(x)+g(x)$,
  $(\lambda f)(x) = \lambda(f(x))$,
- *pointwise multiplication*
  $fg(x) = f(x)g(x)$,
- *pointwise involution*
  $f^*(x) = \overline{f(x)}$.

The $*$-algebra $C(X)$ is *isomorphic* to a $*$-algebra $\mathbb{C}^N$ with involution (N number of points in X), we write $C(X) \simeq \mathbb{C}^N$. Isomorphisms are bijective maps that preserve structure and don't lose physical information. A function $f : X \to \mathbb{C}$ can be represented with $N \times N$ diagonal matrices, where each diagonal value represents the function value at the corresponding i-th point for $i = 1,...,N$. Because of matrix multiplication and hermitian conjugate of matrices we have a preserving structure.

Moreover we can *map* between finite discrete spaces X_1 and X_2 with a function

$$\phi : X_1 \to X_2. \tag{2.1}$$

For every such map there exists a corresponding map

$$\phi^* : C(X_2) \to C(X_1), \tag{2.2}$$

which 'pulls back' values even if ϕ is not bijective. Note that the pullback doesn't map points back, but maps functions on an $*$-algebra $C(X)$. The pullback, in literature often called a $*$-homomorphism or a $*$-algebra map under pointwise product has the following properties

- $\phi^*(fg) = \phi^*(f)\phi^*(g)$,
- $\phi^*(\overline{f}) = \overline{\phi^*(f)}$,
- $\phi^*(\lambda f+g) = \lambda\phi^*(f)+\phi^*(g)$.

The map $\phi : X_1 \to X_2$ is an injective (surjective) map, if only if the corresponding pullback $\phi^* : C(X_2) \to C(X_1)$ is surjective (injective). Let us say, that X_1 has n points and X_2 with m points. Then there are three different cases, first $n = m$ and obviously ϕ is bijective and ϕ^* too. Then $n > m$, in this case ϕ assigns n points to m points when $n > m$, which is by definition surjective. On the other hand ϕ^* assigns m points to n points when $n > m$, which is by definition injective. Lastly $n < m$, which is completely analogous to the case $n > m$.

2.1.3 Matrix Algebras

Definition 2. A *(complex) matrix algebra* A is a direct sum, for $n_i, N \in \mathbb{N}$

$$A = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}). \tag{2.3}$$

The involution is the hermitian conjugate, a $*$ algebra with involution is referred to as a matrix algebra

From a topological discrete space X, we can construct a $*$-algebra $C(X)$ which is isomorphic to a matrix algebra A. Then the question instantly arises, if we can construct X given A? For a matrix algebra A, which in most cases is not commutative, the answer is generally no.

Thus there are two options. We can restrict ourselves to commutative matrix algebras, which are the vast minority and not physically interesting. Or we can allow more morphisms (isomorphisms) between matrix algebras.

2.1.4 Finite Inner Product Spaces and Representations

Until now we looked at finite topological discrete spaces, moreover we can consider a finite dimensional inner product space H (finite Hilbert-spaces), with inner product $(\cdot,\cdot) \to \mathbb{C}$. We denote $L(H)$ as the $*$-algebra of operators on H equipped with a product given by composition and involution of the adjoint, $T \mapsto T^*$. Then $L(H)$ is a *normed vector space* with

$$\|T\|^2 = \sup_{h \in H}\{(Th, Th) : (h,h) \leq 1\} \qquad T \in L(H) \tag{2.4}$$

$$\|T\| = \sup\{\sqrt{\lambda} : \lambda \text{ eigenvalue of } T\} \tag{2.5}$$

This allows us to define representations of $*$-algebras.

Definition 3. The *representation* of a finite dimensional $*$-algebra A is a pair (H, π), where H is a finite dimensional inner product space and π is a $*$-*algebra map*

$$\pi : A \ \to \ L(H). \tag{2.6}$$

We call the representation (H, π) *irreducible* if

- $H \neq \emptyset$,
- only $\emptyset$ or H is invariant under the action of A on H.

Here are some examples of reducible and irreducible representations

- For $A = M_n(\mathbb{C})$ the representation $H = \mathbb{C}^n$, A acts as matrix multiplication H is irreducible.
- For $A = M_n(\mathbb{C})$ the representation $H = \mathbb{C}^n \oplus \mathbb{C}^n$, with $a \in A$ acting in block form $\pi : a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ is reducible.

Naturally there are also certain equivalences between different representations.

Definition 4. Two representations of a $*$-algebra A, (H_1, π_1) and (H_2, π_2) are called *unitary equivalent* if there exists a map $U : H_1 \to H_2$ such that.

$$\pi_1(a) = U^* \pi_2(a) U \tag{2.7}$$

Furthermore we define a mathematical structure called the structure space, which will later become important, when speaking of the duality between a spectral triple and a space.

Definition 5. Let A a $*$-algebra then, $\hat{A}$ is called the structure space of all *unitary equivalence classes of irreducible representations of A*

Given a representation (H, π) of a $*$-algebra A, the **commutant** $\pi(A)'$ of $\pi(A)$ is defined as a set of operators in $L(H)$ that commute with all $\pi(a)$

$$\pi(A)' = \{T \in L(H) : \pi(a)T = T\pi(a) \quad \forall a \in A\} \tag{2.8}$$

The commutant $\pi(A)'$ is also a $*$-algebra, because it has unital, associative and involutive properties. We note that $\pi(a) \in L(H)$ $\forall a \in A$, unitary property is given by the unital operator of the $*$-algebra of operators $L(H)$, which exists by definition because H is a inner product space. Associativity is given by the $*$-algebra of $L(H)$, where $L(H) \times L(H) \mapsto L(H)$, which is associative by definition. The involutive property is also given by the $*$-algebra $L(H)$ with a map $* : L(H) \mapsto L(H)$ only for a T that commutes with $\pi(a)$.

For a unital algebra $*$-algebra A, the matrices $M_n(A)$ with entries in A form a unital $*$-algebra, because unitary operation in $M_n(A)$ is given by the identity Matrix, which has to exists in every entry in $M_n(A)$, and behaves like in A. Associativity is given by matrix multiplication. Lastly involution is given by the conjugate transpose.

A representation $\pi : A \to L(H)$ of a $*$-algebra A, for $H^n = H \oplus \ldots \oplus H$, n times. Then we have the following representation $\tilde{\pi} : M_n(A) \to L(H^n)$ for the Matrix Algebra with $\tilde{\pi}((a_{ij})) = (\tilde{\pi}(a_{ij})) \in M_n(A)$. We have direct isomorphisms of $A \simeq M_n(A)$ and $H \simeq H^n$ meaning $\tilde{\pi}$ is a valid reducible representation.

Let $\tilde{\pi} : M_n(A) \to L(H^n)$ be a $*$ algebra representation of $M_n(A)$, then $\pi : A \to L(H^n)$ is a representation of A. The fact that $\tilde{\pi}$ and π are unitary equivalent, there is a map $U : H^n \to H^n$ given by $U = \mathbb{1}_n$, thus

$$\pi(a) = \mathbb{1}_n^* \, \tilde{\pi}((a_{ij})), \tag{2.9}$$

$$\mathbb{1}_n = \tilde{\pi}((a_{ij})) = \pi(a_{ij}) \Rightarrow a_{ij} = a\mathbb{1}_n. \tag{2.10}$$

A commutative matrix algebra can be used to reconstruct a discrete space. The structure space $\hat{A}$ is used for this. Because $A \simeq \mathbb{C}^N$ all irreducible representation are of the form

$$\pi_i : (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N \mapsto \lambda_i \in \mathbb{C} \tag{2.11}$$

for $i = 1, \ldots, N$ and thus $\hat{A} \simeq \{1, \ldots, N\}$. The conclusion is that, there is a duality between discrete spaces and commutative matrix algebra this duality is called the *finite dimensional Gelfand duality*

Our aim is to construct a duality between finite dimensional spaces and *equivalence classes* of matrix algebras, to preserve general non-commutativity of matrices. Equivalence classes are described by a generalized notion of isomorphisms between matrix algebras (*Morita Equivalence*)

2.1.5 Algebraic Modules

Definition 6. Let A, B be algebras (need not be matrix algebras)

1. *left* A-module is a vector space E, that carries a left representation of A, that is $\exists$ a bilinear map $\gamma : A \times E \to E$ with
$$(a_1 a_2) \cdot e = a_1 \cdot (a_2 \cdot e); \quad a_1, a_2 \in A, e \in E \tag{2.12}$$
2. *right* B-module is a vector space F, that carries a right representation of A, that is $\exists$ a bilinear map $\gamma : F \times B \to F$ with
$$f \cdot (b_1 b_2) = (f \cdot b_1) \cdot b_2; \quad b_1, b_2 \in B, f \in F \tag{2.13}$$
3. *left* A-module and *right* B-module is a *bimodule*, a vector space E satisfying
$$a \cdot (e \cdot b) = (a \cdot e) \cdot b; \quad a \in A, b \in B, e \in E \tag{2.14}$$

Notion of A-**module homomorphism** as linear map $\phi : E \to F$ which respects the representation of A, e.g. for left module.
$$\phi(ae) = a\phi(e); \quad a \in A, e \in E. \tag{2.15}$$
Remark on the notation

- ${}_AE$ left A-module E;
- ${}_AE_B$ right B-module F;
- ${}_AE_B$ A-B-bimodule E;

2.2 Heat Kernel Expansion

2.2.1 The Heat Kernel

The heat kernel $K(t;x,y;D)$ is the fundamental solution of the heat equation
$$(\partial_t + D_x)K(t;x,y;D) = 0, \tag{2.16}$$
which depends on the operator D of Laplacian type.

For a flat manifold $M = \mathbb{R}^n$ and $D = D_0 := -\Delta_\mu \Delta^\mu + m^2$ the Laplacian with a mass term and the initial condition
$$K(0;x,y;D) = \delta(x,y), \tag{2.17}$$
takes the form of the standard fundamental solution
$$K(t;x,y;D_0) = (4\pi t)^{-n/2} \exp\left(-\frac{(x-y)^2}{4t} - tm^2\right). \tag{2.18}$$
Let us consider now a more general operator D with a potential term or a gauge field, the heat kernel reads then
$$K(t;x,y;D) = \langle x|e^{-tD}|y\rangle. \tag{2.19}$$
We can expand the heat kernel in t, still having a singularity from the equation (**??**) as $t \to 0$ thus the expansion reads
$$K(t;x,y;D) = K(t;x,y;D_0)\left(1 + tb_2(x,y) + t^2 b_4(x,y) + \dots\right), \tag{2.20}$$
where $b_k(x,y)$ become regular as $y \to x$. These coefficients are called the heat kernel coefficients.

2.2.2 Spectral Functions

Manifolds M with a disappearing boundary condition for the operator e^{-tD} for $t>0$ is a trace class operator on $L^2(V)$. Meaning for any smooth function f on M we can define

$$K(t,f,D) := \mathrm{Tr}_{L^2}(fe^{-tD}), \tag{2.21}$$

or alternately write an integral representation

$$K(t,f,D) = \int_M d^n x\sqrt{g}\mathrm{Tr}_V(K(t;x,x;D)f(x)), \tag{2.22}$$

in the regular limit $y \to y$. We can write the Heat Kernel in terms of the spectrum of D. So for an orthonormal basis $\{\phi_\lambda\}$ of eigenfunctions for D, which corresponds to the eigenvalue λ, we can rewrite the heat kernel into

$$K(t;x,y;D) = \sum_\lambda \phi_\lambda^\dagger(x)\phi_\lambda(y)e^{-t\lambda}. \tag{2.23}$$

An asymptotic expansion as $t \to 0$ for the trace is then

$$\mathrm{Tr}_{L^2}(fe^{-tD}) \simeq \sum_{k\geq 0} t^{(k-n)/2}a_k(f,D), \tag{2.24}$$

where

$$a_k(f,D) = (4\pi)^{-n/2}\int_M d^4x\sqrt{g}b_k(x,x)f(x). \tag{2.25}$$

2.2.3 General Formulae

Let us summarize what we have obtained in the last chapter, we considered a compact Riemannian manifold M without boundary condition, a vector bundle V over M to define functions which carry discrete (spin or gauge) indices, an operator D of Laplace type over V and smooth function f on M.

There is an asymptotic expansion where the heat kernel coefficients with an odd index $k=2j+1$ vanish $a_{2j+1}(f,D)=0$. On the other hand coefficients with an even index are locally computable in terms of geometric invariants

$$\begin{aligned} a_k(f,D) &= \mathrm{Tr}_V\left(\int_M d^n x\sqrt{g}(f(x)a_k(x;D)\right) = \\ &= \sum_I \mathrm{Tr}_V\left(\int_M d^n x\sqrt{g}(fu^I\mathscr{A}_k^I(D))\right). \end{aligned} \tag{2.26}$$

We denote $\mathscr{A}_k^I$ as all possible independent invariants of dimension k, and u^I are constants. The invariants are constructed from $E, \Omega, R_{\mu\nu\rho\sigma}$ and their derivatives If E has dimension two, then the derivative has dimension one. So if $k=2$ there are only two independent invariants, E and R. This corresponds to the statement $a_{2j+1}=0$.

If we consider $M = M_1 \times M_2$ with coordinates x_1 and x_2 and a decomposed Laplace style operator $D = D_1 \otimes 1 + 1 \otimes D_2$ we can separate functions acting on operators and on coordinates linearly by the following

$$e^{-tD} = e^{-tD_1} \otimes e^{-tD_2}, \tag{2.27}$$

$$f(x_1,x_2) = f_1(x_1)f_2(x_2), \tag{2.28}$$

thus the heat kernel coefficients are separated by

$$a_k(x;D) = \sum_{p+q=k} a_p(x_1;D_1)a_q(x_2;D_2) \tag{2.29}$$

If we know the eigenvalues of D_1 are known, $l^2, l \in \mathbb{Z}$, we can obtain the heat kernel asymmetries with the Poisson summation formula giving us an approximation in the order of $e^{-1/t}$

$$\begin{aligned} K(t,D_1) = \sum_{l\in\mathbb{Z}} e^{-tl^2} &= \sqrt{\frac{\pi}{t}} \sum_{l\in\mathbb{Z}} e^{-\frac{\pi^2 l^2}{t}} = \\ &\simeq \sqrt{\frac{\pi}{t}} + \mathscr{O}(e^{-1/t}). \end{aligned} \tag{2.30}$$

The exponentially small terms have no effect on the heat kernel coefficients and that the only nonzero coefficient is $a_0(1,D_1) = \sqrt{\pi}$, therefore the heat coefficients can be written as

$$a_k(f(x^2),D) = \sqrt{\pi}\int_{M_2} d^{n-1}x\sqrt{g}\sum_I \mathrm{Tr}_V\left(f(x^2)u^I_{(n-1)}\mathscr{A}^I_n(D_2)\right). \tag{2.31}$$

Because all of the geometric invariants associated with D are in the D_2 part, they are independent of x_1. Ultimately meaning we are free to choose M_1. For $M_1 = S^1$ with $x \in (0,2\pi)$ and $D_1 = -\partial^2_{x_1}$ we can rewrite the heat kernel coefficients into

$$\begin{aligned} a_k(f(x_2),D) &= \int_{S^1\times M_2} d^n x\sqrt{g}\sum_I \mathrm{Tr}_V(f(x_2)u^I_{(n)}\mathscr{A}^I_k(D_2)) = \\ &= 2\pi\int_{M_2} d^n x\sqrt{g}\sum_I \mathrm{Tr}_V(f(x_2)u^I_{(n)}\mathscr{A}^I_k(D_2)). \end{aligned} \tag{2.32}$$

Computing the two equations above we see that

$$u^I_{(n)} = \sqrt{4\pi}u^I_{(n+1)} \tag{2.33}$$

2.2.4 Heat Kernel Coefficients

To calculate the heat kernel coefficients we need the following variational equations

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} a_k(1,e^{-2\varepsilon f}D) = (n-k)a_k(f,D), \tag{2.34}$$

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} a_k(1,D-\varepsilon F) = a_{k-2}(F,D), \tag{2.35}$$

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} a_k(e^{-2\varepsilon f}F,e^{-2\varepsilon f}D) = 0. \tag{2.36}$$

Let us explain the equations above. To get the first equation (**??**) we differentiate

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathrm{Tr}(\exp(-e^{-2\varepsilon f}tD) = \mathrm{Tr}(2ftDe^{-tD}) = -2t\frac{d}{dt}\mathrm{Tr}(fe^{-tD})) \tag{2.37}$$

then we expand both sides in t and get (**??**). Equation (**??**) is derived similarly.

For equation (??) we consider the following operator

$$D(\varepsilon,\delta)=e^{-2\varepsilon f}(D-\delta F) \tag{2.38}$$

for $k=n$ we use equation (??) and we get

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} a_n(1,D(\varepsilon,\delta))=0, \tag{2.39}$$

then we take the variation in terms of δ, evaluated at $\delta=0$ and swap the differentiation, allowed by theorem of Schwarz

$$\begin{aligned} 0 &= \left.\frac{d}{d\delta}\right|_{\delta=0}\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} a_n(1,D(\varepsilon,\delta))= \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\left.\frac{d}{d\delta}\right|_{\delta=0} a_n(1,D(\varepsilon,\delta))= \\ &= a_{n-2}(e^{-2\varepsilon f}F,e^{-2\varepsilon f}D), \end{aligned} \tag{2.40}$$

which gives us equation (??).

Now that we have established the ground basis, we can calculate the constants u^I, and by that the first three heat kernel coefficients read

$$a_0(f,D)=(4\pi)^{-n/2}\int_M d^nx\sqrt{g}\mathrm{Tr}_V(a_0f), \tag{2.41}$$

$$a_2(f,D)=(4\pi)^{-n/2}\frac{1}{6}\int_M d^nx\sqrt{g}\mathrm{Tr}_V)(f\alpha_1E+\alpha_2R), \tag{2.42}$$

$$\begin{aligned} a_4(f,D)=(4\pi)^{-n/2}\frac{1}{360}\int_M d^nx\sqrt{g}\mathrm{Tr}_V(f(\alpha_3E_{,kk}+\alpha_4\,R\,E+\alpha_5E^2\alpha_6R_{,kk}+ \\ +\alpha_7R^2+\alpha_8R_{ij}R_{ij}+\alpha_9R_{ijkl}R_{ijkl}+\alpha_{10}\Omega_{ij}\Omega ij)), \end{aligned} \tag{2.43}$$

where the comma subscript , denotes the derivative and constants α_I do not depend on the dimension of the Manifold and we can compute them with our variational identities.

The first coefficient α_0 can be read from the heat kernel expansion of the Laplacian on S^1 (above), $\alpha_0=1$. For α_1 we use (??), the coefficient $k=2$ is

$$\frac{1}{6}\int_M d^nx\sqrt{g}\mathrm{Tr}_V(\alpha_1F)=\int_M d^nx\sqrt{g}\mathrm{Tr}_V(F), \tag{2.44}$$

which means $\alpha_1=6$. Looking at the coefficient $k=4$ we have

$$\frac{1}{360}\int_M d^nx\sqrt{g}\mathrm{Tr}_V(\alpha_4\,F\,R+2\alpha_5\,F\,E)=\frac{1}{6}\int_M d^nx\sqrt{g}\mathrm{Tr}_V(\alpha_1\,F\,E+\alpha_2\,F\,R), \tag{2.45}$$

thus $\alpha_4=60\alpha_2$ and $\alpha_5=180$.

By applying (??) to $n=4$ we get

$$\frac{d}{d\varepsilon}|_{\varepsilon=0}a_2(e^{-2\varepsilon f}F,e^{-2\varepsilon f}D)=0. \tag{2.46}$$

Collecting the terms with $\mathrm{Tr}_V(\int_M d^nx\sqrt{g}(Ff_{,jj}))$ we obtain $\alpha_1=6\alpha_2$, that is $\alpha_2=1$, so $\alpha_4=60$.

Now we let $M = M_1 \times M_2$ and split $D = -\Delta_1 - \Delta_2$, where $\Delta_{1/2}$ are Laplacians for M_1, M_2. This allows us to decompose the heat kernel coefficient for $k = 4$ into

$$\begin{aligned} a_4(1, -\Delta_1 - \Delta_2) =& a_4(1, -\Delta_1)a_0(1, -\Delta_2) + \\ &+ a_2(1, -\Delta_1)a_2(1, -\Delta_2) \\ &+ a_0(1, -\Delta_1)a_4(1, -\Delta_2), \end{aligned} \tag{2.47}$$

with $E = 0$ and $\Omega = 0$ and by calculating the terms with R_1R_2 (scalar curvature of $M_{1/2}$) we obtain $\frac{2}{360}\alpha_7 = (\frac{\alpha_2}{6})^2$, thus $\alpha_7 = 5$.

For $n = 6$ we get

$$\begin{aligned} 0 = \mathrm{Tr}_V(\int_M d^n x\sqrt{g}(F(-2\alpha_3 - 10\alpha_4 + 4\alpha_5)f_{,kk}E + \\ + (2\alpha_3 + 10\alpha_6)f_{,iijj} + \\ + (2\alpha_4 - 2\alpha_6 - 20\alpha_7 - 2\alpha_8)f_{,ii}R \\ + (-8\alpha_8 - 8\alpha_6)f_{,ij}R_{ij})) \end{aligned} \tag{2.48}$$

we obtain $\alpha_3 = 60$, $\alpha_6 = 12$, $\alpha_8 = -2$ and $\alpha_9 = 2$

To get α_{10} we use the Gauss-Bonnet theorem, ultimately giving us $\alpha_{10} = 30$. We leave out this lengthy calculation and refer to [**heatkernel**] for further reading.

Let us summarize our calculations which ultimately give us the following heat kernel coefficients

$$\alpha_0(f, D) = (4\pi)^{-n/2}\int_M d^n x\sqrt{g}\mathrm{Tr}_V(f), \tag{2.49}$$

$$\alpha_2(f, D) = (4\pi)^{-n/2}\frac{1}{6}\int_M d^n x\sqrt{g}\mathrm{Tr}_V(f(6E + R)), \tag{2.50}$$

$$\alpha_4(f, D) = (4\pi)^{-n/2}\frac{1}{360}\int_M d^n x\sqrt{g}\mathrm{Tr}_V(f(60E_{,kk} + 60RE + 180E^2 + \tag{2.51}$$

$$+ 12R_{,kk} + 5R^2 - 2R_{ij}R_{ij}2R_{ijkl}R_{ijkl} + 30\Omega_{ij}\Omega_{ij})). \tag{2.52}$$

2.3 Almost-commutative Manifold

2.3.1 Two-Point Space

One of the basics forms of noncommutative space is the Two-Point space $X := \{x, y\}$. The Two-Point space can be represented by the following spectral triple

$$F_X := (C(X) = \mathbb{C}^2, H_F, D_F; J_F, \gamma_f). \tag{2.53}$$

Three properties of F_X stand out. First of all the action of $C(X)$ on H_F is faithful for $dim(H_F) \geq 2$, thus we can make a simple choice for the Hilbertspace, $H_F = \mathbb{C}^2$. Furthermore γ_F is the $\mathbb{Z}_2$ grading, which allows us to decompose H_F into

$$H_F = H_F^+ \otimes H_F^- = \mathbb{C} \otimes \mathbb{C}, \tag{2.54}$$

where

$$H_F^{\pm} = \{\psi \in H_F | \ \gamma_F \psi = \pm\psi\}, \tag{2.55}$$

are two eigenspaces. And lastly the Dirac operator D_F lets us interchange between $H_F^{\pm}$,

$$D_F = \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix}, \quad \text{with } t \in \mathbb{C}. \tag{2.56}$$

The Two-Point space F_X can only have a real structure if the Dirac operator vanishes, i.e. $D_F = 0$. In that case we have KO-dimension of 0, 2 or 6. To elaborate on this, we know that there are two diagram representations of F_X at $\underbrace{\mathbb{C} \oplus \mathbb{C}}_{C(X)}$ on $\underbrace{\mathbb{C} \oplus \mathbb{C}}_{H_F}$, which are:

$$\begin{array}{ccc} & \mathbf{1} & \mathbf{1} \\ \mathbf{1}^{\circ} & & \circ \\ \mathbf{1}^{\circ} & \circ & \end{array} \qquad\qquad \begin{array}{ccc} & \mathbf{1} & \mathbf{1} \\ \mathbf{1}^{\circ} & \circ & \\ \mathbf{1}^{\circ} & & \circ \end{array}$$

If the Two-Point space F_X would be a real spectral triple then D_F can only go vertically or horizontally. This would mean that D_F vanishes. As for the KO-dimension The diagram on the left has KO-dimension 2 and 6, the diagram on the right 0 and 4. Yet KO-dimension 4 is ruled out because $dim(H_F^{\pm}) = 1$ (see Lemma 3.8 Book), which ultimately means $J_F^2 = -1$ is not allowed.

2.3.2 Product Space

By Extending the Two-Point space with a four dimensional Riemannian spin manifold, we get an almost commutative manifold $M \times F_X$, given by

$$M \times F_X = \left(C^{\infty}(M, \mathbb{C}^2), L^2(S) \otimes \mathbb{C}^2, D_M \otimes 1; J_M \otimes J_F, \gamma_M \otimes \gamma_F\right), \tag{2.57}$$

where

$$C^{\infty}(M, \mathbb{C}^2) \simeq C^{\infty}(M) \oplus C^{\infty}(M). \tag{2.58}$$

According to Gelfand duality the algebra $C^{\infty}(M, \mathbb{C}^2)$ of the spectral triple corresponds to the space

$$N := M \otimes X \simeq M \sqcup X. \tag{2.59}$$

Keep in mind that we still need to find an appropriate real structure on the Riemannian spin manifold, J_M. Furthermore total Hilbertspace can be decomposed into $H = L^2(S) \oplus L^2(S)$, such that for $\underbrace{a, b \in C^{\infty}(M)}_{(a,b) \in C^{\infty}(N)}$ and $\underbrace{\psi, \phi \in L^2(S)}_{(\psi,\phi) \in H}$ we have

$$(a, b)(\psi, \phi) = (a\psi, b\phi) \tag{2.60}$$

Along with the decomposition of the total Hilbertspace we can consider a distance formula on $M \times F_X$ with

$$d_{D_F}(x, y) = \sup\{|a(x) - a(y)| : a \in A_F, ||[D_F, a]|| \leq 1\}. \tag{2.61}$$

To calculate the distance between two points on the Two-Point space $X = \{x,y\}$, between x and y, we consider an $a \in \mathbb{C}^2 = C(X)$, which is specified by two complex numbers $a(x)$ and $a(y)$. Then we simplify the commutator inequality in (**??**)

$$||[D_F,a]|| = ||(a(y)-a(x))\begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix}|| \leq 1, \tag{2.62}$$

$$\Leftrightarrow |a(y)-a(x)| \leq \frac{1}{|t|}, \tag{2.63}$$

and the supremum gives us the distance

$$d_{D_F}(x,y) = \frac{1}{|t|}. \tag{2.64}$$

An interesting observation here is that, if the Riemannian spin manifold can be represented by a real spectral triple then a real structure J_M exists, then it follows that $t = 0$ and the distance becomes infinite. This is a purely mathematical observation and has no physical meaning.

We can also construct a distance formula on N (in reference to a point $p \in M$) between two points on $N = M \times X$, (p,x) and (p,y). Then an $a \in C^\infty(N)$ is determined by $a_x(p) := a(p,x)$ and $a_y(p) := a(p,y)$. The distance between these two points is

$$d_{D_F\otimes 1}(n_1,n_2) = \sup\left\{|a(n_1)-a(n_2)| : a \in A, ||[D\otimes 1,a]||\right\}. \tag{2.65}$$

On the other hand if we consider $n_1 = (p,x)$ and $n_2 = (q,x)$ for $p,q \in M$ then

$$d_{D_M\otimes 1}(n_1,n_2) = |a_x(p)-a_x(q)| \quad \text{for } a_x \in C^\infty(M) \text{ with } ||[D\otimes 1,a_x]|| \leq 1 \tag{2.66}$$

The distance formula turns to out to be the geodesic distance formula

$$d_{D_M\otimes 1}(n_1,n_2) = d_g(p,q), \tag{2.67}$$

which is to be expected since we are only looking at the manifold. However if $n_1 = (p,x)$ and $n_2 = (q,y)$ then the two conditions are

$$||[D_M,a_x]|| \leq 1, \quad \text{and} \tag{2.68}$$

$$||[D_M,a_y|| \leq 1. \tag{2.69}$$

These conditions have no restriction which results in the distance being infinite! And $N = M \times X$ is given by two disjoint copies of M which are separated by infinite distance

The distance is only finite if $[D_F,a] < 1$. In this case the commutator generates a scalar field and the finiteness of the distance is related to the existence of scalar fields.

2.3.3 $U(1)$ Gauge Group

To get a insight into the physical properties of the almost commutative manifold $M \times F_X$, that is to calculate the spectral action, we need to determine the corresponding Gauge theory. For this we set of with simple definitions and important propositions to help us break down and search for the gauge group of the Two-Point F_X space which we then extend to $M \times F_X$. We will only be diving superficially into this chapter, for further reading we refer to [**ncgwalter**].

Definition 7. Gauge Group of a real spectral triple is given by

$$\mathfrak{B}(A,H;J) := \{U = uJuJ^{-1} | u \in U(A)\} \tag{2.70}$$

Definition 8. A *-automorphism of a *-algebra A is a linear invertible map

$$\alpha : A \to A \quad \text{with} \tag{2.71}$$

$$\alpha(ab) = \alpha(a)\alpha(b) \tag{2.72}$$

$$\alpha(a)^* = \alpha(a^*) \tag{2.73}$$

The **Group of automorphisms of the *-Algebra** A is denoted by (A).
The automorphism α is called **inner** if

$$\alpha(a) = uau^* \quad \text{for } U(A) \tag{2.74}$$

where $U(A)$ is

$$U(A) = \{u \in A| \ uu^* = u^*u = 1\} \quad \text{(unitary)} \tag{2.75}$$

The Gauge group of F_X is given by the quotient $U(A)/U(A_J)$. We want a nontrivial Gauge group so we need to choose a $U(A_J) \neq U(A)$ and $U((A_F)_{J_F}) \neq U(A_F)$. We consider our Two-Point space F_X to be equipped with a real structure, which means the operator vanishes, and the spectral triple representation is

$$F_X := \left(\mathbb{C}^2, \mathbb{C}^2, D_F = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; J_f = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \gamma_F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right). \tag{2.76}$$

Here C is the complex conjugation, and F_X is a real even finite spectral triple (space) of KO-dimension 6.

Proposition 1. *The Gauge group of the Two-Point space $\mathfrak{B}(F_X)$ is $U(1)$.*

Proof. Note that $U(A_F) = U(1) \times U(1)$. We need to show that $U(A_F) \cap U(A_F)_{J_F} \simeq U(1)$, such that $\mathfrak{B}(F) \simeq U(1)$. So for an element $a \in \mathbb{C}^2$ to be in $(A_F)_{J_F}$, it has to satisfy $J_F a^* J_F = a$,

$$J_F a^* J^{-1} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} \begin{pmatrix} \bar{a}_1 & 0 \\ 0 & \bar{a}_2 \end{pmatrix} \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} = \begin{pmatrix} a_2 & 0 \\ 0 & a_1 \end{pmatrix}. \tag{2.77}$$

This can only be the case if $a_1 = a_2$. So we have $(A_F)_{J_F} \simeq \mathbb{C}$, whose unitary elements from $U(1)$ are contained in the diagonal subgroup of $U(A_F)$. □

An arbitrary hermitian field $A_\mu = -ia\partial_\mu b$ is given by two $U(1)$ Gauge fields $X^1_\mu, X^2_\mu \in C^\infty(M,\mathbb{R})$. However A_μ appears in combination $A_\mu - J_F A_\mu J_F^{-1}$:

$$A_\mu - J_F A_\mu J_F^{-1} = \begin{pmatrix} X^1_\mu & 0 \\ 0 & X^2_\mu \end{pmatrix} - \begin{pmatrix} X^2_\mu & 0 \\ 0 & X^1_\mu \end{pmatrix} =: \begin{pmatrix} Y_\mu & 0 \\ 0 & -Y_\mu \end{pmatrix} = Y_\mu \otimes \gamma_F, \tag{2.78}$$

where Y_μ the $U(1)$ Gauge field is defined as

$$Y_\mu := X^1_\mu - X^2_\mu \in C^\infty(M,\mathbb{R}) = C^\infty(M, i\, u(1)). \tag{2.79}$$

Proposition 2. *The inner fluctuations of the almost-commutative manifold $M \times F_X$ are parameterized by a $U(1)$-gauge field Y_μ as*

$$D \mapsto D' = D + \gamma^\mu Y_\mu \otimes \gamma_F \tag{2.80}$$

The action of the gauge group $\mathfrak{B}(M \times F_X) \simeq C^\infty(M, U(1))$ on D' is implemented by

$$Y_\mu \mapsto Y_\mu - i\, u\partial_\mu u^*; \quad (u \in \mathfrak{B}(M \times F_X)). \tag{2.81}$$

2.4 Noncommutative Geometry of Electrodynamics

In this chapter we describe Electrodynamics with the almost commutative manifold $M \times F_X$ and the abelian gauge group $U(1)$. We arrive at a unified description of gravity and electrodynamics although in the classical level.

The almost commutative Manifold $M \times F_X$ describes a local gauge group $U(1)$. The inner fluctuations of the Dirac operator relate to Y_μ the gauge field of $U(1)$. According to the setup we ultimately arrive at two serious problems.

First of all in the Two-Point space F_X, the operator D_F must vanish for us to have a real structure. However this implies that the electrons are massless, which would be absurd.

The second problem arises when looking at the Euclidean action for a free Dirac field

$$S = -\int i\bar{\psi}(\gamma^\mu \partial_\mu - m)\psi d^4x, \tag{2.82}$$

where $\psi,\ \bar{\psi}$ must be considered as independent variables, which means that the fermionic action S_f needs two independent Dirac spinors. Let us try and construct two independent Dirac spinors with our data. To do this we take a look at the decomposition of the basis and of the total Hilbertspace $H = L^2(S) \otimes H_F$. For the orthonormal basis of H_F we can write $\{e, \bar{e}\}$, where $\{e\}$ is the orthonormal basis of H_F^+ and $\{\bar{e}\}$ the orthonormal basis of H_F^-. Accompanied with the real structure we arrive at the following relations

$$J_F e = \bar{e} \qquad J_F \bar{e} = e, \tag{2.83}$$

$$\gamma_F e = e \qquad \gamma_F \bar{e} = \bar{e}. \tag{2.84}$$

Along with the decomposition of $L^2(S) = L^2(S)^+ \oplus L^2(S)^-$ and $\gamma = \gamma_M \otimes \gamma_F$ we can obtain the positive eigenspace

$$H^+ = L^2(S)^+ \otimes H_F^+ \oplus L^(S)^- \otimes H_F^-. \tag{2.85}$$

So, for a $\xi \in H^+$ we can write

$$\xi = \psi_L \otimes e + \psi_R \otimes \bar{e} \tag{2.86}$$

where $\psi_L \in L^2(S)^+$ and $\psi_R \in L^2(S)^-$ are the two Wheyl spinors. We denote that ξ is only determined by one Dirac spinor $\psi := \psi_L + \psi_R$, **but we require two independent spinors**. Our conclusion is that the definition of the fermionic action gives too much restrictions to the Two-Point space F_X.

2.4.1 The Finite Space

To solve the two problems we simply enlarge (double) the Hilbertspace. This is visualized by introducing multiplicities in Krajewski Diagrams which will also allow us to choose a nonzero Dirac operator that will connect the two vertices and preserve real structure making our particles massive and bringing anti-particles into the mix.

We start of with the same algebra $C^\infty(M, \mathbb{C}^2)$, corresponding to space $N = M \times X$. The Hilbertspace describes four particles, meaning it has four orthonormal basis elements. It describes **left handed electrons** and **right handed positrons**. Pointing this out, we have $\{\ \underbrace{e_R, e_L}_{\text{left-handed}}\ ,\ \underbrace{\bar{e}_R, \bar{e}_L}_{\text{right-handed}}\ \}$ the orthonormal basis for $H_F = \mathbb{C}^4$. Accompanied with the real structure J_F, which allows us to interchange particles with antiparticles by the following equations

$$J_F e_R = \bar{e}_R, \tag{2.87}$$
$$J_F e_L = \bar{e}_L, \tag{2.88}$$

$$\gamma_F e_R = -e_R, \tag{2.89}$$
$$\gamma_F e_L = e_L, \tag{2.90}$$

where J_F and γ_F have to following properties

$$J_F^2 = 1, \tag{2.91}$$
$$J_F \gamma_F = -\gamma_F J_F. \tag{2.92}$$

By means of γ_F we have two options to decompose the total Hilbertspace H, firstly into

$$H_F = \underbrace{H_F^+}_{\text{ONB } \{e_L, \bar{e}_L\}} \oplus \underbrace{H_F^-}_{\text{ONB } \{e_R, \bar{e}_R\}}, \tag{2.93}$$

or alternatively into the eigenspace of particles and their antiparticles (electrons and positrons) which is preferred in literature and which we will use going further

$$H_F = \underbrace{H_e}_{\text{ONB } \{e_L, e_R\}} \oplus \underbrace{H_{\bar{e}}}_{\text{ONB } \{\bar{e}_L, \bar{e}_R\}}. \tag{2.94}$$

Here ONB means orthonormal basis.

The action of $a \in A = \mathbb{C}^2$ on H with respect to the ONB $\{e_L, e_R, \bar{e}_L, \bar{e}_R\}$ is represented by

$$a = (a_1, a_2) \mapsto \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix} \tag{2.95}$$

Do note that this action commutes wit the grading and that $[a, b^\circ] = 0$ with $b := J_F b^* J_F$ because both the left and the right action is given by diagonal matrices by equation (**??**). Note that we are still left with $D_F = 0$ and the following spectral triple

$$\left(\mathbb{C}^2, \mathbb{C}^2, D_F = 0; J_F = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \gamma_F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right). \tag{2.96}$$

It can be represented in the following Krajewski diagram, with two nodes of multiplicity two

$$\begin{array}{ccc} & \mathbf{1} & \mathbf{1} \\ \mathbf{1}^\circ & & \circledcirc \\ \mathbf{1}^\circ & \circledcirc & \end{array}$$

2.4.2 A noncommutative Finite Dirac Operator

To extend our spectral triple with a non-zero Operator, we need to take a closer look at the Krajewski diagram above. Notice that edges only exist between multiple vertices, meaning we can construct a Dirac operator mapping between the two vertices. The operator can be represented by the following matrix

$$D_F = \begin{pmatrix} 0 & d & 0 & 0 \\ \bar{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{d} \\ 0 & 0 & d & 0 \end{pmatrix} \tag{2.97}$$

We can now define the finite space F_{ED}.

$$F_{ED} := (\mathbb{C}^2, \mathbb{C}^4, D_F; J_F, \gamma_F) \tag{2.98}$$

where J_F and γ_F are like in equation (**??**) and D_F from equation (**??**).

2.4.3 Almost commutative Manifold of Electrodynamics

The almost commutative manifold $M \times F_{ED}$ has KO-dimension 2, and is represented by the following spectral triple

$$M \times F_{ED} := \big(C^\infty(M,\mathbb{C}^2),\ L^2(S)\otimes\mathbb{C}^4,\ D_M\otimes 1 + \gamma_M \otimes D_F;\ J_M\otimes J_F,\ \gamma_M\otimes\gamma_F\big) \tag{2.99}$$

The algebra didn't change, thus we can decompose it like before

$$C^\infty(M,\mathbb{C}^2) = C^\infty(M) \oplus C^\infty(M) \tag{2.100}$$

As for the Hilbertspace, we can decomposition it in the following way

$$H = (L^2(S)\otimes H_e) \oplus (L^2(S)\otimes H_{\bar{e}}). \tag{2.101}$$

Note that the one component of the algebra is acting on $L^2(S)\otimes H_e$, and the other one acting on $L^2(S)\otimes H_{\bar{e}}$. In other words the components of the decomposition of both the algebra and the Hilbertspace match by the action of the algebra.

The derivation of the gauge theory is the same for F_{ED} as for the Two-Point space F_X. We have $\mathfrak{B}(F) \simeq U(1)$ and for an arbitrary gauge field $B_\mu = A_\mu - J_F A_\mu J_F^{-1}$ we can write

$$B_\mu = \begin{pmatrix} Y_\mu & 0 & 0 & 0 \\ 0 & Y_\mu & 0 & 0 \\ 0 & 0 & Y_\mu & 0 \\ 0 & 0 & 0 & Y_\mu \end{pmatrix} \quad \text{for} \quad Y_\mu(x) \in \mathbb{R}. \tag{2.102}$$

We have one single $U(1)$ gauge field Y_μ, carrying the action of the gauge group

$$\mathfrak{B}(M \times F_{ED}) \simeq C^\infty(M, U(1)) \tag{2.103}$$

The space $N = M \times X$ consists of two copies of M. If $D_F = 0$ we have infinite distance between the two copies. Now have hacked the spectral triple to have nonzero Dirac operator D_F. The new Dirac operator still has a commuting relation with the algebra $[D_F, a] = 0\ \forall a \in A$, and we should note that the distance between the two copies of M is still infinite. This is purely an mathematically abstract observation and doesn't affect physical results.

2.4.4 Spectral Action

In this chapter we bring all our results together to establish an Action functional to describe a physical system. It turns out that the Lagrangian of the almost commutative manifold $M \times F_{ED}$ corresponds to the Lagrangian of Electrodynamics on a curved background manifold (+ gravitational Lagrangian), consisting of the spectral action S_b (bosonic) and of the fermionic action S_f.

The simplest spectral action of a spectral triple (A, H, D) is given by the trace of a function of D. We also consider inner fluctuations of the Dirac operator $D_\omega = D + \omega + \varepsilon' J \omega J^{-1}$ where $\omega = \omega^* \in \Omega^1_D(A)$.

Definition 9. Let $f : \mathbb{R} \to \mathbb{R}$ be a suitable function **positive and even**. The spectral action is then

$$S_b[\omega] := \mathrm{Tr}\big(f(\frac{D_\omega}{\Lambda})\big) \tag{2.104}$$

where Λ is a real cutoff parameter. The minimal condition on f is that $f(\frac{D_\omega}{\Lambda})$ is a trace class operator. A trace class operator is a compact operator with a well defined finite trace independent of the basis. The subscript b in S_b stands for bosonic, because in physical applications ω will describe bosonic fields.

In addition to the bosonic action S_b we can define a topological spectral action S_{top}. Leaning on the grading γ the topological spectral action is

$$S_{\mathrm{top}}[\omega] := \mathrm{Tr}(\gamma f(\frac{D_\omega}{\Lambda})). \tag{2.105}$$

Definition 10. The fermionic action is defined by

$$S_f[\omega, \psi] = (J\tilde{\psi}, D_\omega \tilde{\psi}) \tag{2.106}$$

with $\tilde{\psi} \in H^+_{cl} := \{\tilde{\psi} : \psi \in H^+\}$, where H^+_{cl} is a set of Grassmann variables in H in the $+1$-eigenspace of the grading γ.

APPENDIX?? Grassmann variables are a set of Basis vectors of a vector space, they form a unital algebra over a vector field V, where the generators are anti commuting, that is for Grassmann variables θ_i, θ_j we have

$$\theta_i \theta_j = -\theta_j \theta_i \tag{2.107}$$

$$\theta_i x = x \theta_j \quad x \in V \tag{2.108}$$

$$(\theta_i)^2 = 0 \quad (\theta_i \theta_i = -\theta_i \theta_i) \tag{2.109}$$

Proposition 3. *The spectral action of the almost commutative manifold M with* $\dim(M) = 4$ *with a fluctuated Dirac operator is*

$$Tr(f\frac{D_\omega}{\Lambda}) \sim \int_M \mathscr{L}(g_{\mu\nu}, B_\mu, \Phi)\sqrt{g}\, d^4x + O(\Lambda^{-1}), \tag{2.110}$$

where

$$\mathscr{L}(g_{\mu\nu}, B_\mu, \Phi) = N\mathscr{L}_M(g_{\mu\nu})\mathscr{L}_B(B_\mu) + \mathscr{L}_\phi(g_{\mu\nu}, B_\mu, \Phi). \tag{2.111}$$

The Lagrangian $\mathcal{L}_M$ is of the spectral triple, represented by the following term $(C^\infty(M), L^2(S), D_M)$

$$\mathcal{L}_M(g_{\mu\nu}) := \frac{f_4\Lambda^4}{2\pi^2} - \frac{f_2\Lambda^2}{24\pi^2}s - \frac{f(0)}{320\pi^2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}, \tag{2.112}$$

here $C^{\mu\nu\rho\sigma}$ is defined in terms of the Riemannian curvature tensor $R_{\mu\nu\rho\sigma}$ and the Ricci tensor $R_{\nu\sigma} = g^{\mu\rho}R_{\mu\nu\rho\sigma}$. The kinetic term of the gauge field is described by the Lagrangian $\mathcal{L}_B$, which takes the following shape

$$\mathcal{L}_B(B_\mu) := \frac{f(0)}{24\pi^2}Tr(F_{\mu\nu}F^{\mu\nu}). \tag{2.113}$$

Lastly $\mathcal{L}_\phi$ is the scalar-field Lagrangian with a boundary term, given by

$$\begin{aligned}\mathcal{L}_\phi(g_{\mu\nu}, B_\mu, \Phi) :=& -\frac{2f_2\Lambda^2}{4\pi^2}Tr(\Phi^2) + \frac{f(0)}{8\pi^2}Tr(\Phi^4) + \frac{f(0)}{24\pi^2}\Delta(Tr(\Phi^2)) \\ &+ \frac{f(0)}{48\pi^2}sTr(\Phi^2)\frac{f(0)}{8\pi^2}Tr((D_\mu\Phi)(D^\mu\Phi)).\end{aligned} \tag{2.114}$$

Proof. The dimension of the manifold M is $\dim(M) = \mathrm{Tr}(id) = 4$. For an $x \in M$, we have an asymptotic expansion of the term $\mathrm{Tr}(f(\frac{D_\omega}{\Lambda}))$ as Λ goes to infinity, which can be written as

$$\begin{aligned}\mathrm{Tr}(f(\frac{D_\omega}{\Lambda})) \simeq& 2f_4\Lambda^4 a_0(D_\omega^2) + 2f_2\Lambda^2 a_2(D_\omega^2) \\ &+ f(0)a_4(D_\omega^4) + O(\Lambda^{-1}).\end{aligned} \tag{2.115}$$

We have to note here that the heat kernel coefficients are zero for uneven k, and they are dependent on the fluctuated Dirac operator D_ω. We can rewrite the heat kernel coefficients in terms of D_M, for the first two terms a_0 and a_2 we use $N := \mathrm{Tr}\mathbb{1}_{\mathbb{H}_\mathbb{F}})$ and write

$$a_0(D_\omega^2) = Na_0(D_M^2), \tag{2.116}$$

$$a_2(D_\omega^2 = Na_2(D_M^2) - \frac{1}{4\pi^2}\int_M \mathrm{Tr}(\Phi^2)\sqrt{g}d^4x. \tag{2.117}$$

For a_4 we extend in terms of coefficients of F, **REWRITE: look week9.pdf for the standard version**

$$\frac{1}{360}\mathrm{Tr}(60sF) = -\frac{1}{6}S(Ns + 4\mathrm{Tr}(\Phi^2)) \tag{2.118}$$

$$F^2 = \frac{1}{16}s^2 \otimes 1 + 1 \otimes \Phi^4 - \frac{1}{4}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma F_{\mu\nu}F^{\mu\nu} + \tag{2.119}$$

$$+\gamma^\mu\gamma^\nu \otimes (D_\mu\Phi)(D_\nu\Phi) + \frac{1}{2}s \otimes \Phi^2 + \text{ traceless terms} \tag{2.120}$$

$$\frac{1}{360}\mathrm{Tr}(180F^2) = \frac{1}{8}s^2N + 2\mathrm{Tr}(\Phi^4) + \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \tag{2.121}$$

$$+2\mathrm{Tr}((D_\mu\Phi)(D^\mu\Phi)) + s\mathrm{Tr}(\Phi^2) \tag{2.122}$$

$$\frac{1}{360}\mathrm{Tr}(-60\Delta F) = \frac{1}{6}\Delta(Ns + 4\mathrm{Tr}(\Phi^2)). \tag{2.123}$$

The cross terms of the trace in $\Omega^E_{\mu\nu}\Omega^{E\mu\nu}$ vanishes because of the antisymmetric property of the Riemannian curvature tensor, thus we can write

$$\Omega^E_{\mu\nu}\Omega^{E\mu\nu} = \Omega^S_{\mu\nu}\Omega^{S\mu\nu}\otimes 1 - 1\otimes F_{\mu\nu}F^{\mu\nu} + 2i\Omega^S_{\mu\nu}\otimes F^{\mu\nu}. \tag{2.124}$$

The trace of the cross term $\Omega^S_{\mu\nu}$ vanishes because

$$\mathrm{Tr}(\Omega^S_{\mu\nu}) = \frac{1}{4}R_{\mu\nu\rho\sigma}\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \frac{1}{4}R_{\mu\nu\rho\sigma}g^{\mu\nu} = 0, \tag{2.125}$$

then the trace of the whole term is given by

$$\frac{1}{360}\mathrm{Tr}(30\Omega^E_{\mu\nu}\Omega^{E\mu\nu}) = \frac{N}{24}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{1}{3}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}). \tag{2.126}$$

Finally plugging the results into the coefficient a_4 and simplifying we get

$$\begin{aligned} a_4(x, D^4_\omega) = Na_4(x, D^2_M) + \frac{1}{4\pi^2}\Big(&\frac{1}{12}s\mathrm{Tr}(\Phi^2) + \frac{1}{2}\mathrm{Tr}(\Phi^4) \\ &+ \frac{1}{4}\mathrm{Tr}((D_\mu\Phi)(D^\mu\Phi)) + \frac{1}{6}\Delta\mathrm{Tr}(\Phi^2) + \frac{1}{6}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\Big). \end{aligned} \tag{2.127}$$

The only thing left is to substitute the heat kernel coefficients into the heat kernel expansion in equation (**??**). □

2.4.5 Fermionic Action

We remind ourselves the definition of the fermionic action in definition **??** and the manifold we are dealing with in equation (**??**). The Hilbertspace H_F is separated into the particle-antiparticle states with ONB $\{e_R, e_L, \bar{e}_R, \bar{e}_L\}$. The orthonormal basis of H_F^+ is $\{e_L, \bar{e}_R\}$ and consequently for H_F^-, $\{e_R, \bar{e}_L\}$. We can decompose a spinor $\psi \in L^2(S)$ in each of the eigenspaces $H_F^\pm$, $\psi = \psi_R + \psi_L$. That means for an arbitrary $\psi \in H^+$ we can write

$$\psi = \chi_R \otimes e_R + \chi_L \otimes e_L + \psi_L \otimes \bar{e}_R + \psi_R \otimes \bar{e}_L, \tag{2.128}$$

where $\chi_L, \psi_L \in L^2(S)^+$ and $\chi_R, \psi_R \in L^2(S)^-$.

Since the fermionic action yields too much restriction on F_{ED} (modified Two-Point space F_X) we redefine it by taking account the fluctuated Dirac operator

$$D_\omega = D_M \otimes i + \gamma^\mu \otimes B_\mu + \gamma_M \otimes D_F. \tag{2.129}$$

The Fermionic Action is

$$S_F = (J\tilde{\xi}, D_\omega\tilde{\xi}) \tag{2.130}$$

for a $\xi \in H^+$. Then the straight forward calculation gives

$$\frac{1}{2}(J\tilde{\xi}, D_\omega\tilde{\xi}) = \frac{1}{2}(J\tilde{\xi}, (D_M \otimes i)\tilde{\xi}) \tag{2.131}$$

$$+ \frac{1}{2}(J\tilde{\xi}, (\gamma^\mu \otimes B_\mu)\tilde{\xi}) \tag{2.132}$$

$$+ \frac{1}{2}(J\tilde{\xi}, (\gamma_M \otimes D_F)\tilde{\xi}), \tag{2.133}$$

(note that we add the constant $\frac{1}{2}$ to the action). For the term in (**??**) we calculate

$$\begin{aligned}\frac{1}{2}(J\tilde{\xi},(D_M\otimes 1)\tilde{\xi}) &= \frac{1}{2}(J_M\tilde{\chi}_R,D_M\tilde{\psi}_L)+\frac{1}{2}(J_M\tilde{\chi}_L,D_M\tilde{\psi}_R)+\\ &+\frac{1}{2}(J_M\tilde{\psi}_L,D_M\tilde{\psi}_R)+\frac{1}{2}(J_M\tilde{\chi}_R,D_M\tilde{\chi}_L)\\ &=(J_M\tilde{\chi},D_M\tilde{\chi}).\end{aligned}\tag{2.134}$$

For the term in (**??**) we have

$$\begin{aligned}\frac{1}{2}(J\tilde{\xi},(\gamma^\mu\otimes B_\mu)\tilde{\xi}) &= -\frac{1}{2}(J_M\tilde{\chi}_R,\gamma^\mu Y_\mu\tilde{\psi}_R)-\frac{1}{2}(J_M\tilde{\chi}_L,\gamma^\mu Y_\mu\tilde{\psi}_R)+\\ &+\frac{1}{2}(J_M\tilde{\psi}_L,\gamma^\mu Y_\mu\tilde{\chi}_R)+\frac{1}{2}(J_M\tilde{\psi}_R,\gamma^\mu Y_\mu\tilde{\chi}_L)=\\ &=-(J_M\tilde{\chi},\gamma^\mu Y_\mu\tilde{\psi}).\end{aligned}\tag{2.135}$$

And for (**??**) we can write

$$\begin{aligned}\frac{1}{2}(J\tilde{\xi},(\gamma_M\otimes D_F)\tilde{\xi}) &= +\frac{1}{2}(J_M\tilde{\chi}_R,d\gamma_M\tilde{\chi}_R)+\frac{1}{2}(J_M\tilde{\chi}_L,\bar{d}\gamma_M\tilde{\chi}_L)+\\ &+\frac{1}{2}(J_M\tilde{\chi}_L,\bar{d}\gamma_M\tilde{\chi}_L)+\frac{1}{2}(J_M\tilde{\chi}_R,d\gamma_M\tilde{\chi}_R)=\\ &=i(J_M\tilde{\chi},m\tilde{\psi}).\end{aligned}\tag{2.136}$$

A small problem arises, we obtain a complex mass parameter d, but we can write $d := im$ for $m\in\mathbb{R}$, which stands for the real mass.

Finally the fermionic action of $M\times F_{ED}$ takes the form

$$S_f=-i\big(J_M\tilde{\chi},\gamma(\nabla^S_\mu-i\Gamma_\mu)\tilde{\Psi}\big)+\big(S_M\tilde{\chi}_L,\bar{d}\tilde{\psi}_L\big)-\big(J_M\tilde{\chi}_R,d\tilde{\psi}_R\big).\tag{2.137}$$

Ultimately we arrive at the full Lagrangian of $M\times F_{ED}$, which is the sum of purely gravitational Lagrangian

$$\mathcal{L}_{grav}(g_{\mu\nu})=4\mathcal{L}_M(g_{\mu\nu})+\mathcal{L}_\phi(g_{\mu\nu}),\tag{2.138}$$

and the Lagrangian of electrodynamics

$$\mathcal{L}_{ED}=-i\Big\langle J_M\tilde{\chi},\big(\gamma^\mu(\nabla^S_\mu-iY_\mu)-m\big)\tilde{\psi})\Big\rangle+\frac{f(0)}{6\pi^2}Y_{\mu\nu}Y^{\mu\nu}.\tag{2.139}$$

3 Conclusion

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4 Acknowledgment

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