Noncommutative Geometry

Bachelor's seminar

Popovic Milutin
Supervisor: Dr. Lisa Glaser

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Introduction

- Noncommutative geometry (NCG) brings many mathematical fields together (e.g. K-Theory, Differential Geometry)
- Physics application (spectral Standard Model)
- Gelfand-Naimark-Theorem in Functional Analysis in the 1940s duality between (classical) geometry and Algebra

Introduce:

Algebra

Vectorspace with a multiplication operation
(associative and possesses an identity element)

Finite topological Space X consisting of N points. (discrete topology)

$$\underset{1}{\bullet} \qquad \underset{2}{\bullet} \qquad \cdots \qquad \underset{N}{\bullet}$$

Commutative algebra of continuous functions on X

$$C(X) = \{f : X \to \mathbb{C} : \ f \text{ is continuous}\}$$

Results of the Theorem:

- X and $C(X)$ contain the same information (duality)
- Construct X, given $C(X)$.
- Translate geometrical properties of X to algebraic data (metric, differential forms, vector fields, curvature, etc.)

The Spectral Triple

$$(A, \ H, \ D)$$

- A - Algebra
- H - Hilbertspace
- D - self adjoint Operator acting on H

The Spectral Triple of a Circle $\mathbb{S}^1$

$$\left(\; C^\infty(\mathbb{S}^1), \;\; L^2(\mathbb{S}^1), \;\; -i\,\tfrac{d}{dt} \; \right)$$

Introducing the Metric

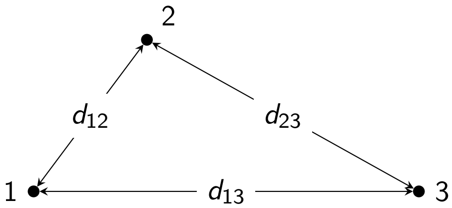
- The metric describes distances between points on a space

Discrete Metric

$$d_{ij} = \begin{cases} 0 & \text{if} \quad i = j \\ 1 & \text{if} \quad i \neq j \end{cases}$$

Minkowski Metric

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Algebraic Formulation of the Metric

- Utilize results of the Gelfand-Naimark Theorem
- Characterize the Metric with
  - commutative Algebra
  - finite-dimensional Hilbertspace H
  - symmetric operator D

Metric with (A, H, D) on finite Space (commutative case)

$$d_{ij} = \sup_{a \in A}\{|a(i) - a(j)| : ||[D, a]|| \leq 1\}$$

Algebraic Formulation of the Metric

In the noncommutative Case:

- replace Algebra with matrix Algebra (noncommutative)
- define in terms of invariants

Metric with (A, H, D) on finite Space (noncommutative case)

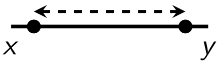
$$d_{ij} = \sup_{a \in A}\{|\mathrm{Tr}(a(i)) - \mathrm{Tr}(a(j))| : ||[D, a]|| \leq 1\}$$

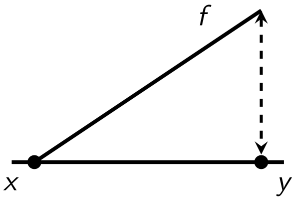
Algebraic Formulation of the Metric

- describe the Metric on a Manifold M
- We need
  - $C^\infty(M)$ - Algebra
  - $H^2(S)$ - Hilbertspace
  - D - Dirac Operator

Metric with $(C^\infty(M),\ H^2(S),\ D)$ on a Manifod

$$d(x,y) = \sup_{f \in C^\infty(M)} \{|f(x) - f(y)| : ||[D,f]|| \leq 1\}$$





Applications In Physics

- NCG of the Quantum Hall Effect
- NCG of the Standard Model
  - going to noncommutative Manifolds
  - obtain Standard Model gauge fields (scalar Higgs filed)
  - construct the Full Lagrangian
  - minimal coupling to gravity

Bibliography

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[3] Thijs van den Broek. *NCG in 4 pages: A very brief introduction into noncommutative geometry*. 2013. URL: `http://www.noncommutativegeometry.nl/wp-content/uploads/2013/09/NCGin4pages1.pdf` (visited on 04/13/2021).

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