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Bachelor's Thesis

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Noncommutative Geomtetry and Physics

submitted by

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Abstract

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1 Introduction

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2 Main Section

2.0.1 Metric on Finite Discrete Spaces

Let us come back to our finite discrete space X, we can describe it by a structure space $\hat{A}$ of a matrix algebra A. To describe distance between two points in X (as we would in a metric space) we use an array $\{d_{ij}\}_{i,j\in X}$ of *real non-negative* entries in X such that

- $d_{ij} = d_{ji}$ Symmetric
- $d_{ij} \leq d_{ik}d_{kj}$ Triangle Inequality
- $d_{ij} = 0$ for $i = j$ (the same element)

In the commutative case, the algebra A is commutative and can describe the metric on X in terms of algebraic data.

Theorem 1. *Let d_{ij} be a metric on X a finite discrete space with N points, $A = \mathbb{C}^N$ with elements $a = (a(i))_{i=1}^N$ such that $\hat{A} \simeq X$. Then there exists a representation π of A on a finite-dimensional inner product space H and a symmetric operator D on H such that*

$$d_{ij} = \sup_{a\in A}\{|a(i) - a(j)| : ||[D, \pi(a)]|| \leq 1\} \tag{2.1}$$

Proof. We claim that this would follow from the equality:

$$||[D, \pi(a)]|| = \max_{k\neq l}\Big\{\frac{1}{d_{kl}}|a(k) - a(l)|\Big\} \tag{2.2}$$

This can be proved with induction. Set $N = 2$ then $H = \mathbb{C}^2$, $\pi : A \to L(H)$ and a hermitian matrix D.

$$\pi(a) = \begin{pmatrix} a(1) & 0 \\ 0 & a(2) \end{pmatrix} \quad D = \begin{pmatrix} 0 & (d_{12})^{-1} \\ (d_{21})^{-1} & 0 \end{pmatrix} \tag{2.3}$$

Then we commpute the commutator

$$||[D, \pi(a)]|| = (d_{12})^{-1} |a(1) - a(2)| \tag{2.4}$$

For the case $A = \mathbb{C}^3$, we have $H = (\mathbb{C}^2)^{\oplus 3} = H_2 \oplus H_2^1 \oplus H_2^2$. The the representation $\pi(a)$ reads

$$\begin{aligned} \pi((a(1), a(2), a(3)) &= \begin{pmatrix} a(1) & 0 \\ 0 & a(2) \end{pmatrix} \oplus \begin{pmatrix} a(1) & 0 \\ 0 & a(3) \end{pmatrix} \oplus \begin{pmatrix} a(2) & 0 \\ 0 & a(2) \end{pmatrix} \\ &= \operatorname{diag}\big(a(1), a(2), a(1), a(3), a(2), a(3)\big) \end{aligned} \tag{2.5}$$

And the operator D takes the form

$$\begin{aligned} D &= \begin{pmatrix} 0 & x_1 \\ x_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & x_2 \\ x_2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & x_3 \\ x_3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & x_1 & 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 & x_3 & 0 \end{pmatrix}. \end{aligned} \tag{2.6}$$

Then the norm of the commutator would be the largest eigenvalue

$$||[D, \pi(a)]|| = ||D\pi(a) - \pi(a)D||, \tag{2.7}$$

where the this matrix in the norm from equation 2.7 is a skew symmetric matrix. Its eigenvalues are $i\lambda_1, i\lambda_2, i\lambda_3, i\lambda_4$. The λ's are on the upper and lower diagonal. The matrix norm would be the maximum of the norm with the larges eigenvalues:

$$\begin{aligned} ||[D, \pi(a)]|| = \max_{a \in A} \{ & x_1 |a(2) - a(1)|, \\ & x_2 |(a(3) - a(1))|, \\ & x_3 |(a(3) - a(2))|, \} \end{aligned} \tag{2.8}$$

Hence the metric turns out to be

$$d = \begin{pmatrix} 0 & a(1) - a(2) & a(1) - a(3) \\ a(2) - a(1) & 0 & a(2) - a(3) \\ a(3) - a(1) & a(3) - a(2) & 0 \end{pmatrix} \tag{2.9}$$

Suppose this holds for N with π_N, $H_N = \mathbb{C}^N$ and D_N. Then it has to holds for $N+1$ with $H_{N+1} = H_N \oplus \bigoplus_{i=1}^N H_N^i$, since the representation reads

$$\begin{aligned} \pi_{N+1}(a(1), \dots, a(N+1)) = \pi_N(a(1), \dots, a(N)) \oplus \begin{pmatrix} a(1) & 0 \\ 0 & a(N+1) \end{pmatrix} \oplus \\ \oplus \cdots \oplus \begin{pmatrix} a(N) & 0 \\ 01 & a(N+1) \end{pmatrix} \end{aligned} \tag{2.10}$$

And the operator D_{N+1} is

$$D_{N+1} = D_N \oplus \begin{pmatrix} 0 & (d_{1(N+1)})^{-1} \\ (d_{1(N+1)})^{-1} & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & (d_{N(N+1)})^{-1} \\ (d_{N(N+1)})^{-1} & 0 \end{pmatrix} \tag{2.11}$$

From this follows equation 2.2. Thus we can continue the proof by setting for fixed i, j, $a(k) = d_{ik}$, which then gives $|a(i) - a(j)| = d_{ij}$ and thereby

$$\Rightarrow \frac{1}{d_{kl}}|a(k) - a(l)| = \frac{1}{d_{kl}}|d_{ik} - d_{il}| \leq 1 \tag{2.12}$$

□

The translation of the metric on X into algebraic data assumes commutativity in A, this can be extended to a noncommutative matrix algebra, by the following metric on a structure space $\hat{A}$ of a matrix algebra $M_{n_i}(\mathbb{C}$

$$d_{ij} = \sup_{a \in A} \left\{ |\mathrm{Tr}(a(i)) - \mathrm{Tr}((a(j))| : ||[D, a]|| \leq 1 \right\}. \tag{2.13}$$

Equation 2.13 is special case of the Connes' distance formula on a structure space of A.

Finally we have all three ingredients to define a finite spectral triple, an mathematical structure which encodes finite discrete geometry into algebraic data.

Definition 1. A *finite spectral triple* is a tripe (A, H, D), where A is a unital $*$-algebra, faithfully represented on a finite-dimensional Hilbert space H, with a symmetric operator $D : H \to H$.

Note that A is automatically a matrix algebra.

2.0.2 Exercises

Exercise 1

Compute the metric on the space of three points given by $d_{ij} = \sup_{a \in A}\{|a(i) - a(j)| : ||[D, \pi(a)]|| \leq 1\}$ **for the set of data** $A = \mathbb{C}^3$ **acting in the defining representation** $H = \mathbb{C}^3$**, and**

$$D = \begin{pmatrix} 0 & d^{-1} & 0 \\ d^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.14}$$

for some $d \in \mathbb{R}$

We have $A = \mathbb{C}^3$, $H = \mathbb{C}^3$ and D from above, then

$$||[D, \pi(a)]|| = d^{-1} \left\| \begin{pmatrix} 0 & a(2) - a(1) & 0 \\ -(a(2) - a(1)) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\| \tag{2.15}$$

The metric is then

$$d = \begin{pmatrix} 0 & a(1)-a(2) & a(1) \\ a(2)-a(1) & 0 & a(2) \\ -a(1) & -a(2) & 0 \end{pmatrix} \tag{2.16}$$

Exercise 2

Show that d_{ij} from Equation 2.20 is a metric on $\hat{A}$ by establishing that:

$$d_{ij} = 0 \quad \Leftrightarrow \quad i = j \tag{2.17}$$
$$d_{ij} = d_{ji} \tag{2.18}$$
$$d_{ij} \leq d_{ik} + d_{kj} \tag{2.19}$$

$$d_{ij} = \sup_{a \in A} \left\{ |\mathbf{Tr}(a(i)) - \mathbf{Tr}((a(j))| : ||[D,a]|| \leq 1 \right\} \tag{2.20}$$

For Equation 2.17 set $i = j$ in 2.20.

$$d_{ii} = \sup_{a \in A} \{ |\mathrm{Tr}(a(i)) - \mathrm{Tr}((a(i))| : ||[D,a]|| \leq 1 \} \tag{2.21}$$
$$= \sup_{a \in A} \{ 0 : ||[D,a]|| \leq 1 \} = 0 \tag{2.22}$$

For Equation 2.18 obviously we have the commuting property of addition.
For Equation 2.19, for $k = j$ then $d_{kj} = 0$ and the equality holds. For $i = k$ then $d_{ik} = 0$ and equality holds. Else set $d_{ik} = 1$ and $d_{kj} = 1$ then $d_{ij} = 1 \leq d_{ik} + d_{kj} = 2$

3 Conclusion

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4 Acknowledgment

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