Bachelor's Thesis

Title of the Bachelor's Thesis

Noncommutative Geomtetry and Physics

submitted by

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Abstract

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1 Introduction

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2 Main Section

2.1 Noncommutative Geometric Spaces

2.1.1 Matrix Algebras and Finite Spaces

2.1.2 $*$-Algebra

Definition 1. A *vector space* A over $\mathbb{C}$ is called a *complex, unital Algebra* if,
$\forall a,b \in A$:

1. $A \times A \to A$ *bilinear*
   $(a,b) \mapsto a \cdot b$

2. $1a = a1 = a$ *unital*

Definition 2. A $*$-algebra is an algebra A with a *conjugate linear map (involution)* $* : A \to A$, $\forall a,b \in A$ satisfying:

1. $(ab)^* = b^*a^*$ *antidistributive*

2. $(a^*)^* = a$ *closure*

In the following all unital algebras are referred to as algebras.

2.1.3 Functions on Discrete Spaces

Let X be a *discretized topological* space with N points. Consider functions of a continuous $*$-algebra $C(X)$ assigning values to $\mathbb{C}$, for $f,g \in C(X)$, $\lambda \in \mathbb{C}$ and $x \in X$ they provide the following structures:

- *pointwise linear*
  $(f+g)(x) = f(x)+g(x)$
  $(\lambda f)(x) = \lambda(f(x))$
- *pointwise multiplication*
  $fg(x) = f(x)g(x)$ same as $(fg)(x) = f(x)g(x))$?
- *pointwise involution*
  $f^*(x) = \overline{f(x)}$

Question 1. Mathematical difference between Topological Discreet Spaces and just Discreet Spaces?

The author indicates that $\mathbb{C}$-valued functions on X are automatically continuous.

Proof Idea. *CAN WE USE THE METRIC? NO! We know that X is a finite discrete space, meaning in an ε-δ approach for each $x \in X$ the only $y \in X$, that is small enough is x by itself, which implies ε is always bigger than zero, thus every function $f : X \to \mathbb{C}$ is continuous.*

2.1.4 Isomorphism Property

Furthermore $C(X)$ $*$-algebra is *isomorphic* to a $*$-algebra $\mathbb{C}^N$ with involution (N number of points in X), written as $C(X) \simeq \mathbb{C}^N$. A function $f : X \to \mathbb{C}$ can be represented with $N \times N$ diagonal matrices, where the value (ii) is the value of the function at the corresponding i-th point ($i = 1,...,N$). The structure is preserved because of the definitions of matrix multiplication and the hermitian conjugate of matrices.

Question 2. Can isomorphisms between $C(X)$ and $\mathbb{C}^N$ be shown with matrix factorization?

Isomorphisms are bijective preserve structure and don't lose physical information/

2.1.5 Mapping Finite Discrete Spaces

Definition 3. A *map* between finite discrete spaces X_1 and X_2 is a function $\phi : X_1 \to X_2$

For every map between finite discrete spaces there exists a corresponding map $\phi^* : C(X_2) \to C(X_1)$, which 'pulls back' values even if ϕ is not bijective. Note that the pullback doesn't map points back, but maps functions on an $*$-algebra $C(X)$.

This map is called a pullback (or a $*$-homomorphism or a $*$-algebra map under pointwise product). Under the pointwise product:

- $\phi^*(fg) = \phi^*(f)\phi^*(g)$
- $\phi^*(\overline{f}) = \overline{\phi^*(f)}$

- $\phi^*(\lambda f + g) = \lambda \phi^*(f) + \phi^*(g)$

Question 3. ϕ is in most cases not bijective, so how can we prove that there exists such a pullback for every map between discrete spaces which preserves information? For bijective it is given by its inverse, which by definition exists because ϕ is a map. Or I didn't understand this correctly?

Exercise 1

Show that $\phi : X_1 \rightarrow X_2$ is injective (surjective) map of finite spaces iff $\phi^* : C(X_2) \rightarrow C(X_1)$ is surjective (injective).

Consider X_1 with n points and X_2 with m points. Then there are three cases:

1. $n = m$
   Obviously ϕ is bijective and ϕ^* too.
2. $n \rangle m$
   ϕ assigns n points to m points when $n \rangle m$, which is by definition surjective.
   ϕ^* assigns m points to n points when $n \rangle m$, which is by definition injective.
3. $n \langle m$
   analogous

2.1.6 Matrix Algebras

Definition 4. A *(complex) matrix algebra* A is a direct sum, for $n_i, N \in \mathbb{N}$.

$$A = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}) \tag{2.1}$$

The involution is the hermitian conjugate, a $*$ algebra with involution is referred to as a matrix algebra

So from a topological discrete space X, we can construct a $*$-algebra $C(X)$ which is isomorphic to a matrix algebra A. The question is can we construct X given A? A is a matrix algebra, which are in most cases is not commutative, so the answer is generally no.

There are two options. We can restrict ourselves to commutative matrix algebras, which are the vast minority and not physically interesting. Or we can allow more morphisms(isomorphisms) between matrix algebras.

Question 4. Why are non-commutative algebras not physically interesting? Maybe too far fetched,but because physical observables (QM-Operators) are not commutative?

Exactly.

2.1.7 Finite Inner Product Spaces and Representations

Until now we looked at a finite topological discrete space, moreover we can consider a finite dimensional inner product space H (finite Hilbert-spaces), with inner product $(\cdot,\cdot) \to \mathbb{C}$. $L(H)$ is the $*$-algebra of operators on H with product given by composition and involution given by the adjoint, $T \mapsto T^*$. $L(H)$ is a *normed vector space* with

$$\|T\|^2 = \sup_{h\in H}\{(Th,Th) : (h,h) \leq 1\} \qquad T \in L(H) \tag{2.2}$$

$$\|T\| = \sup\{\sqrt{\lambda} : \lambda \text{ eigenvalue of } T\} \tag{2.3}$$

Definition 5. The *representation* of a finite dimensional $*$-algebra A is a pair (H,π). H is a finite, dimensional inner product space and π is a $*$-*algebra map*

$$\pi : A \ \to \ L(H) \tag{2.4}$$

Definition 6. (H,π) is called *irreducible* if:

- $H \neq \emptyset$
- only $\emptyset$ or H is invariant under the action of A on H

Examples for reducible and irreducible representations

- $A = M_n(\mathbb{C})$, representation $H = \mathbb{C}^n$, A acts as matrix multiplication
  H is irreducible.
- $A = M_n(\mathbb{C})$, representation $H = \mathbb{C}^n \oplus \mathbb{C}^n$, with $a \in A$ acting in block form
  $\pi : a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ is reducible.

Definition 7. Let (H_1,π_1) and (H_2,π_2) be representations of a $*$-algebra A. They are called *unitary equivalent* if there exists a map $U : H_1 \to H_2$ such that.

$$\pi_1(a) = U^*\pi_2(a)U \tag{2.5}$$

Question 5. In matrix representation this is diagonalization condition? (unitary diagonalization)

Yes

Definition 8. A a $*$-algebra then, $\hat{A}$ is called the structure space of all *unitary equivalence classes of irreducible representations of A*

Question 6. Gelfand duality and the spectrum of $\hat{A}$, examples Fourier-Transform and Laplace-Transform for simple spaces.

More on that in later chapters.

Exercise 2

Given (H,π) of a $*$-algebra A, the commutant $\pi(A)'$ of $\pi(A)$ is defined as a set of operators in $L(H)$ that commute with all $\pi(a)$

$$\pi(A)' = \{T \in L(H) : \pi(a)T = T\pi(a) \quad \forall a \in A\} \tag{2.6}$$

1. **Show that $\pi(A)'$ is a $*$-algebra.**
2. **Show that a representation (H,π) of A is irreducible iff the commutant**

$\pi(A)'$ consists of multiples of the identity

1. To show that $\pi(A)'$ is a $*$-algebra we have to show that it is unital, associative and involute. And note that $\pi(a) \in L(H)\ \forall a \in A$. Unitarity is given by the unital operator of the $*$-algebra of operators $L(H)$, which exists by definition because H is a inner product space. Associativity is given by $*$-algebra of $L(H)$, $L(H) \times L(H) \mapsto L(H)$, which is associative by definition. Involutnes is also given by the $*$-algebra $L(H)$ with a map $* : L(H) \mapsto L(H)$ only for T that commute with $\pi(a)$.
2.?

Exercise 3

1. **If A is a unital $*$-algebra, show that the $n \times n$ matrices $M_n(A)$ with entries in A form a unital $*$-algebra.**
2. **Let $\pi : A \to L(H)$ be a representation of a $*$-algebra A and set $H^n = H \oplus ... \oplus H$, n times. Show that $\tilde{\pi} : M_n(A) \to L(H^n)$ of $M_n(A)$ with $\tilde{\pi}((a_{ij})) = (\tilde{\pi}(a_{ij})) \in M_n(A)$.**
3. **Let $\tilde{\pi} : M_n(A) \to L(H^n)$ be a $*$ algebra representation of $M_n(A)$. Show that $\pi : A \to L(H^n)$ is a representation of A.**

1. We know A is a $*$ algebra. Unitary operaton in $M_n(A)$ is given by the identity Matrix, which has to exists because every entry in $M_n(A)$ has to behave like in A. Associativity is given by matrix multiplication. Involutnes is given by the conjugate transpose.
2. $A \simeq M_n(A)$ and $H \simeq H^n$ meaning $\tilde{\pi}$ is a valid reducible representation.
3. $\tilde{\pi}$ and π are unitary equivalent, there is a map $U : H^n \to H^n$ given by $U = \mathbb{1}_n$:
$\pi(a) = \mathbb{1}_n^* \, \tilde{\pi}((a_{ij})) \, \mathbb{1}_n = \tilde{\pi}((a_{ij})) = \pi(a_{ij}) \Rightarrow a_{ij} = a\mathbb{1}_n$.

2.1.8 Commutative Matrix Algebras

- Commutative matrix algebras can be used to reconstruct a discrete space given a matrix *commutative* matrix algebra.
- The structure space $\hat{A}$ is used for this. Because $A \simeq \mathbb{C}^N$ we all any irreducible representation are of the form $\pi_i : (\lambda_1, ..., \lambda_N) \in \mathbb{C}^N \mapsto \lambda_i \in \mathbb{C}$
for $i = 1, ..., N \Rightarrow \hat{A} \simeq \{1, ..., N\}$.
- Conclusion is that there is a duality between discrete spaces and commutative matrix algebra this duality is called the *finite dimensional Gelfand duality*

2.1.9 Noncommutative Matrix Algebras

Aim is to construct duality between finite dimensional spaces and *equivalence classes* of matrix algebras, to preserve general non-commutivity of matrices.

- Equivalence classes are described by a generalized notion of ispomorphisms between matrix algebras (*Morita Equivalence*)

2.1.10 Algebraic Modules

Definition 9. Let A, B be algebras (need not be matrix algebras)

1. *left* A-module is a vector space E, that carries a left representation of A, that is $\exists$ a bilinear map $\gamma : A \times E \to E$ with

$$(a_1 a_2) \cdot e = a_1 \cdot (a_2 \cdot e); \quad a_1, a_2 \in A, e \in E \tag{2.7}$$

2. *right* B-module is a vector space F, that carries a right representation of A, that is $\exists$ a bilinear map $\gamma : F \times B \to F$ with

$$f \cdot (b_1 b_2) = (f \cdot b_1) \cdot b_2; \quad b_1, b_2 \in B, f \in F \tag{2.8}$$

3. *left* A-module and *right* B-module is a *bimodule*, a vector space E satisfying

$$a \cdot (e \cdot b) = (a \cdot e) \cdot b; \quad a \in A, b \in B, e \in E \tag{2.9}$$

Notion of A-**module homomorphism** as linear map $\phi : E \to F$ which respects the representation of A, e.g. for left module.

$$\phi(ae) = a\phi(e); \quad a \in A, e \in E. \tag{2.10}$$

Remark on the notation

- ${}_AE$ left A-module E;
- ${}_AE_B$ right B-module F;
- ${}_AE_B$ A-B-bimodule E;

Exercise 4

Check that a representation of $\pi : A \to L(H)$ of a $*$-algebra A turns H into a left module ${}_AH$.

Not quite sure but
$a \in A$, $h_1, h_2 \in H$, we know $\pi(a) = T \in L(H)$ than

$$\langle \pi(a)h_1, \pi(a)h_2 \rangle = \langle Th_1, Th_2 \rangle = \langle T^*Th_1, h_2 \rangle = \langle h_1, h_2 \rangle \tag{2.11}$$

Or maybe this
If ${}_AH$ than $(a_1 a_2)h = a_1(a_2 h)$ for $a_1, a_2 \in A$ and $h \in H$.
Then we take the representation of an $a \in A$, $\pi(a)$:

$$(\pi(a_1)\pi(a_2))h = \pi(a_1)(\pi(a_2)h) = (T_1 T_2)h = T_1(T_2 h) \tag{2.12}$$

For $T_1, T_2 \in L(H)$, which operate naturally from the left on h.

Exercise 5

Show that A is a bimodule ${}_AA_A$ with itself.

$\gamma : A \times A \times A \to A$ which is given by the inner product of the $*$-algebra.

2.2 Excurse

Manifold: A topological space that is locally Euclidean.
Riemannian Manifold:A Manifold equipped with a Riemannian Metric, a symmetric bilinear form on Vector Fields $\Gamma(TM)$

$$g : \Gamma(TM) \times \Gamma(TM) \to C(M) \tag{2.13}$$

with

$$g(X,Y) \in \mathbb{R} \quad \text{if } X,Y \in \mathbb{R} \tag{2.14}$$

$$g \text{ is } C(M)\text{-bilinear } \forall f \in C(M) : \quad g(fX,Y) = g(X,fY) = fg(X,Y) \tag{2.15}$$

$$g(X,X) \begin{cases} \geq 0 & \forall X \\ = 0 & \forall X = 0 \end{cases} \tag{2.16}$$

g on M gives rise to a distance function on M

$$d_g(x,y) = \inf_\gamma \left\{ \int_0^1 (\dot\gamma(t), \dot\gamma(t)) dt; \;\; \gamma(0) = x, \gamma(1) = y \right\} \tag{2.17}$$

Riemannian Manifold is called spin^c if there exists a vector bundle $S \to M$ with an algebra bundle isomorphism

$$\mathbb{C}\mathrm{l}(TM) \simeq \mathrm{End}(S) \qquad (dim(M) \text{ even}) \tag{2.18}$$

$$\mathbb{C}\mathrm{l}(TM)^\circ \simeq \mathrm{End}(S) \qquad (dim(M) \text{ odd}) \tag{2.19}$$

$$\tag{2.20}$$

(M,S) is called the **spin^c structure on** M.
S is called the **spinor Bundle**.
$\Gamma(S)$ are the **spinors**.

Riemannian spin^c Manifold is called spin if there exists an anti-unitary operator $J_M : \Gamma(S) \to \Gamma(S)$ such that:

1. J_M commutes with the action of real-valued continuous functions on $\Gamma(S)$.
2. J_M commutes with $\mathrm{Cliff}^-(M)$ (even case)
   J_M commutes with $\mathrm{Cliff}^-(M)^\circ$ (odd case)

(S, J_M) is called the **spin Structure on** M
J_M is called the **charge conjugation**.

2.3 Noncommutative Geometry of Electrodynamics

2.3.1 The Two-Point Space

Consider a two point space $X := \{x, y\}$. This space=an be described with the following spectral triple

$$F_x := (C(X) = \mathbb{C}^2, H_F, D_F; J_F, \gamma_f). \tag{2.21}$$

Notes on the spectral triple:

- Action of $C(X)$ on H_F is faithful ($\dim(H_F) \geq 2$)
  we choose $H_F = \mathbb{C}^2$
- γ_F is the $\mathbb{Z}_2$ grading, which allows us to decompose $H_F = H_F^+ \oplus H_F^- = \mathbb{C} \oplus \mathbb{C}$
  where $H_F^\pm = \{\psi \in H_F | \ \gamma_F \psi = \pm \psi\}$ are the two eigenspaces
- D_F interchanges between $H_F^\pm$, $D_F = \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix}$ where $t \in \mathbb{C}$

Proposition 1. *F_x can only have a real structure if $D_F = 0$ in that case we have $KO-dim = 0, 2, 6$*

Proof. There are two diagram representations of F_x at $\underbrace{\mathbb{C} \oplus \mathbb{C}}_{C(X)}$ on $\underbrace{\mathbb{C} \oplus \mathbb{C}}_{H_F}$

$$\begin{array}{ccc} & \mathbf{1} & \mathbf{1} \\ \mathbf{1}^\circ & & \circ \\ \mathbf{1}^\circ & \circ & \end{array} \qquad\qquad \begin{array}{ccc} & \mathbf{1} & \mathbf{1} \\ \mathbf{1}^\circ & \circ & \\ \mathbf{1}^\circ & & \circ \end{array}$$

If F_x a real spectral triple then D_F can only go vertically or horizontally $\Rightarrow D_F = 0$. Furthermore the diagram on the left has KO-dimension 2 and 6, diagram on the right has KO-dimension 0 and 4. Yet KO-dimension 4 is not allowed because $dim(H_F^\pm) = 1$ (see Lemma 3.8 Book), so $J_F^2 = -1$ is not allowed. □

2.3.2 The product Space

Let M be a 4-dim Riemannian spin Manifold, then we have the almost commutative manifold $M \times F_x$

$$M \times F_x = (C^\infty(M, \mathbb{C}^2, L^2(S) \otimes \mathbb{C}^2, D_M \otimes 1; J_M \otimes J_F, \gamma_M \otimes \gamma_F) \tag{2.22}$$

(J_M is missing need to choose)
$C^\infty(M, \mathbb{C}^2) \simeq C^\infty(M) \oplus C^\infty(M)$ (decomposition) and from Gelfand duality we we have

$$N := M \otimes X \simeq M \sqcup X \tag{2.23}$$

$H = L^2(S) \oplus L^2(S)$ (decomposition), such that for $\underbrace{a, b \in C^\infty(M)}_{(a,b) \in C^\infty(N)}$ and $\underbrace{\psi, \phi \in L^2(S)}_{(\psi,\phi) \in H}$ we have

$$(a, b)(\psi, \phi) = (a\psi, b\phi) \tag{2.24}$$

We can consider a distance formula on $M \times F_x$ by

$$d_{D_F}(x,y) = \sup\{|a(x) - a(y)| : a \in A_F, ||[D_F, a]|| \leq 1\} \tag{2.25}$$

Now lets calculate the distance between two points on the two point space $X = \{x, y\}$, between x and y. Let $a \in \mathbb{C}^2 = C(X)$, a is specified with two complex numbers $a(x)$ and $a(y)$

$$||[D_F, a]|| = ||(a(y) - a(x)) \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix} || \leq 1 \tag{2.26}$$

$$\Rightarrow |a(y) - a(x)| \leq \frac{1}{|t|} \tag{2.27}$$

Therefore the distance between two points x and y is

$$d_{D_F}(x,y) = \frac{1}{|t|} \tag{2.28}$$

Note that if there exists J_M (real structure) $\Rightarrow t = 0$ then $d_{D_F}(x,y) \to \infty$!

Now let $p \in M$, then take two points on $N = M \times X$, (p,x) and (p,y) and $a \in C^\infty(N)$ is determined by $a_x(p) := a(p,x)$ and $a_y(p) := a(p,y)$. The distance between these two points is then

$$d_{D_F \otimes 1}(n_1, n_2) = \sup\{|a(n_1) - a(n_2)| : a \in A, ||[D \otimes 1, a]||\} \tag{2.29}$$

Remark: If $n_1 = (p,x)$ and $n_2 = (q,x)$ for $p,q \in M$ then

$$d_{D_M \otimes 1}(n_1, n_2) = |a_x(p) - a_x(q)| \quad a_x \in C^\infty(M) \text{ with } ||[D \otimes 1, a_x]|| \leq 1 \tag{2.30}$$

The distance turns to the geodestic distance formula

$$d_{D_M \otimes 1}(n_1, n_2) = d_g(p,q) \tag{2.31}$$

However if $n_1 = (p,x)$ and $n_2 = (q,y)$ then the two conditions are $||[D_M, a_x]|| \leq 1$ and $||[D_M, a_y|| \leq 1$. They have no restriction which results in the distance being infinite! And $N = M \times X$ is given by two disjoint copies of M which are separated by infinite distance

Note: distance is only finite if $[D_F, a] \neq 1$. The commutator generates a scalar field say ϕ and the finiteness of the distance is related to the existence of scalar fields.

2.3.3 $U(1)$ Gauge Group

Here we determine the Gauge theory corresponding to the almost commutative Manifold $M \times F_x$.

Gauge Group of a Spectral Triple:

$$\mathfrak{B}(A, H; J) := \{U = uJuJ^{-1} | u \in U(A)\} \tag{2.32}$$

Definition 10. A *-automorphism of a *-algebra A is a linear invertible map

$$\alpha : A \to A \quad \text{with} \tag{2.33}$$

$$\alpha(ab) = \alpha(a)\alpha(b) \tag{2.34}$$
$$\alpha(a)^* = \alpha(a^*) \tag{2.35}$$

The **Group of automorphisms of the *-Algebra** A is (A).
The automorphism α is called **inner** if

$$\alpha(a) = uau^* \quad \text{for } U(A) \tag{2.36}$$

where $U(A)$ is

$$U(A) = \{u \in A| \ uu^* = u^*u = 1\} \quad \text{(unitary)} \tag{2.37}$$

The Gauge group is given by the quotient $U(A)/U(A_J)$. We want a nontrivial Gauge group so we need to choose $U(A_J) \neq U(A)$ which is the same as $U((A_F)_{J_F}) \neq U(A_F)$. We consider F_x to be

$$F_x := \left(\mathbb{C}^2, \mathbb{C}^2, D_F = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; J_f = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \gamma_F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right). \tag{2.38}$$

Here C is the complex conjugation, and F_X is a real even finite spectral triple (space) with $KO-dim=6$

Proposition 2. *The Gauge group $\mathfrak{B}(F)$ of the two point space is given by $U(1)$.*

Proof. Note that $U(A_F) = U(1) \times U(1)$. We need to show that $U(\mathscr{A}_F) \cap U(A_F)_{J_F}) \simeq U(1)$, such that $\mathfrak{B}(F) \simeq U(1)$.

So for $a \in \mathbb{C}^2$ to be in $(A_F)_{J_F}$ it has to satisfy $J_F a^* J_F = a$.

$$J_F a^* J^{-1} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} \begin{pmatrix} \bar{a}_1 & 0 \\ 0 & \bar{a}_2 \end{pmatrix} \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} = \begin{pmatrix} a_2 & 0 \\ 0 & a_1 \end{pmatrix} \tag{2.39}$$

Which is only the case if $a_1 = a_2$. So we have $(A_F)_{J_F} \simeq \mathbb{C}$, whose unitary elements from $U(1)$ are contained in the diagonal subgroup of $U(\mathscr{A}_F)$. ☐

Now we need to find the exact from of the field B_μ to calculate the spectral action of a spectral triple. Since $(A_F)_{J_F} \simeq \mathbb{C}$ we find that $\mathfrak{h}(F) = \mathfrak{u}((A_F)_{J_F}) \simeq i\mathbb{R}$. Where $\mathfrak{h}(F)$ is the Lie Algebra on F and $\mathfrak{u}((A_F)_{J_F})$ is the Lie algebra of the unitary group $(A_F)_{J_F}$.

An arbitrary hermitian field $A_\mu = -ia\partial_\mu b$ is given by two $U(1)$ Gauge fields $X^1_\mu, X^2_\mu \in C^\infty(M,\mathbb{R})$. However A_μ appears in combination $A_\mu - J_F A_\mu J_F^{-1}$:

$$B_\mu = A_\mu - J_F A_\mu J_F^{-1} = \begin{pmatrix} X^1_\mu & 0 \\ 0 & X^2_\mu \end{pmatrix} - \begin{pmatrix} X^2_\mu & 0 \\ 0 & X^1_\mu \end{pmatrix} =: \begin{pmatrix} Y_\mu & 0 \\ 0 & -Y_\mu \end{pmatrix} = Y_\mu \otimes \gamma_F \tag{2.40}$$

where Y_μ the $U(1)$ Gauge field is defined as

$$Y_\mu := X^1_\mu - X^2_\mu \in C^\infty(M,\mathbb{R}) = C^\infty(M, i\,u(1)). \tag{2.41}$$

Proposition 3. *The inner fluctuations of the almost-commutative manifold $M \times F_x$ described above are parametrized by a $U(1)$-gauge field Y_μ as*

$$D \mapsto D' = D + \gamma^\mu Y_\mu \otimes \gamma_F \tag{2.42}$$

The action of the gauge group $\mathfrak{B}(M \times F_X) \simeq C^\infty(M, U(1))$ on D' is implemented by

$$Y_\mu \mapsto Y_\mu - i\, u\partial_\mu u^*; \quad (u \in \mathfrak{B}(M \times F_X)). \tag{2.43}$$

2.4 Electrodynamics

Now we use the almost commutative Manifold and the abelian gauge group $U(1)$ to describe Electrodynamics. We arrive at a unified description of gravity and electrodynamics although in the classical level.

The almost commutative Manifold $M \times F_X$ describes a local gauge group $U(1)$. The inner fluctuations of the Dirac operator describe Y_μ the gauge field of $U(1)$. There arise two Problems:
(1): With F_X, D_F must vanish, however this implies that the electrons are massless (this we do not want)

(2): The Euclidean action for a free Dirac field is

$$S = -\int i\bar{\psi}(\gamma^\mu \partial_\mu - m)\psi d^4x, \tag{2.44}$$

ψ, $\bar{\psi}$ must be considered as independent variables, which means S_F need two independent Dirac Spinors. We write $\{e, \bar{e}\}$ for the ONB of H_F, where $\{e\}$ is the ONB of H_F^+ and $\{\bar{e}\}$ the ONB of H_F^- with the real structure this gives us the following relations

$$J_F e = \bar{e} \quad J_F \bar{e} = e \tag{2.45}$$

$$\gamma_F e = e \quad \gamma_F \bar{e} = \bar{e}. \tag{2.46}$$

The total Hilbertspace is $H = L^2(S) \otimes H_F$, with γ_F we can decompose $L^2(S) = L^2(S)^+ \oplus L^2(S)^-$, so with $\gamma = \gamma_M \otimes \gamma_F$ we can obtain the positive eigenspace H^+

$$H^+ = L^2(S)^+ \otimes H_F^+ \oplus L^(S)^- \otimes H_F^-. \tag{2.47}$$

For a $\xi \imath H^+$ we can write

$$\xi = \psi_L \otimes e + \psi_R \otimes \bar{e} \tag{2.48}$$

where $\psi_L \in L^2(S)^+$ and $\psi_R \in L^2(S)^-$ are the two Wheyl spinors. We denote that ξ is only determined by one Dirac spinor $\psi := \psi_L + \psi_R$, **but we require two independent spinors**. This is too much restriction for F_X.

2.4.1 The Finite Space

Here we solve the two problems by enlarging(doubling) the Hilbertspace. This is done by introducing multiplicities in Krajewski Diagrams which will also allow us to choose a nonzero Dirac operator which will connect the two vertices (next chapter).

We start of with the same algebra $C^\infty(M,\mathbb{C}^2)$, corresponding to space $N = M\times X \simeq M\sqcup M$.

The Hilbertspace will describe four particles,

- left handed electrons
- right handed positrons

Thus we have $\{ \underbrace{e_R, e_L}_{\text{left-handed}}, \underbrace{\bar{e}_R, \bar{e}_L}_{\text{right-handed}} \}$ the ONB for $H_F\mathbb{C}^4$.
Then with J_F we interchange particles with antiparticles we have the following properties

$$J_F e_R = \bar{e}_R \qquad J_F e_L = \bar{e_L} \tag{2.49}$$
$$\gamma_F e_R = -e_R \qquad \gamma_F e_L = e_L \tag{2.50}$$
and
$$J_F^2 = 1 \qquad J_F\gamma_F = -\gamma_F J_F \tag{2.51}$$

This corresponds to KO-dim= 6. Then γ_F allows us to can decompose H

$$H_F = \underbrace{H_F^+}_{\text{ONB }\{e_L,\bar{e}_L\}} \oplus \underbrace{H_F^-}_{\text{ONB }\{e_R,\bar{e}_R\}} . \tag{2.52}$$

Alternatively we can decompose H into the eigenspace of particles and their antiparticles (electrons and positrons) which we will use going further.

$$H_F = \underbrace{H_e}_{\text{ONB }\{e_L,e_R\}} \oplus \underbrace{H_{\bar{e}}}_{\text{ONB }\{\bar{e}_L,\bar{e}_R\}} \tag{2.53}$$

Now the action of $a \in A = \mathbb{C}^2$ on H with respect to the ONB $\{e_L, e_R, \bar{e}_L, \bar{e}_R\}$ is represented by

$$a = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \mapsto \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix} \tag{2.54}$$

Do note that this action commutes wit the grading and that $[a,b^\circ] = 0$ with $b := J_F b^* J_F$ because both the left and the right action is given by diagonal matrices.

Proposition 4. *The data*

$$\left(\mathbb{C}^2, \mathbb{C}^2, D_F = 0; J_F = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \gamma_F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \tag{2.55}$$

defines a real even spectral triple of KO-dimension 6.

This spectral triple can be represented in the following Krajewski diagram, with two nodes of multiplicity two

$$\begin{array}{ccc} & \mathbf{1} & \mathbf{1} \\ \mathbf{1}^\circ & & \circledcirc \\ \mathbf{1}^\circ & \circledcirc & \end{array}$$

2.4.2 A noncommutative Finite Dirac Operator

Add a non-zero Dirac Operator to F_{ED}. From the Krajewski Diagram, we see that edges only exist between the multiple vertices. So we construct a Dirac operator mapping between the two vertices.

$$D_F = \begin{pmatrix} 0 & d & 0 & 0 \\ \bar{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{d} \\ 0 & 0 & d & 0 \end{pmatrix} \tag{2.56}$$

We can now consider the finite space F_{ED}.

$$F_{ED} := (\mathbb{C}^2, \mathbb{C}^4, D_F; J_F, \gamma_F) \tag{2.57}$$

where J_F and γ_F like before, D_F like above.

2.4.3 The almost-commutative Manifold

The almost commutative manifold $M \times F_{ED}$ has KO-dim$= 2$, it is the following spectral triple

$$M \times F_{ED} := \Big(C^\infty(M, \mathbb{C}^2),\ L^2(S) \otimes \mathbb{C}^4,\ D_M \otimes 1 + \gamma_M \otimes D_F;\ J_M \otimes J_F,\ \gamma_M \otimes \gamma_F\Big) \tag{2.58}$$

The algebra decomposition is like before

$$C^\infty(M, \mathbb{C}^2) = C^\infty(M) \oplus C^\infty(M) \tag{2.59}$$

The Hilbertspace decomposition is

$$H = (L^2(S) \otimes H_e) \oplus (L^2(S) \otimes H_{\bar{e}}). \tag{2.60}$$

Here we have the one component of the algebra acting on $L^2(S) \otimes H_e$, and the other one acting on $L^2(S) \otimes H_{\bar{e}}$

The derivation of the gauge theory is the same for F_{ED} as for F_X, we have $\mathfrak{B}(F) \simeq U(1)$ and for $B_\mu = A_\mu - J_F A_\mu J_F^{-1}$

$$B_\mu = \begin{pmatrix} Y_\mu & 0 & 0 & 0 \\ 0 & Y_\mu & 0 & 0 \\ 0 & 0 & Y_\mu & 0 \\ 0 & 0 & 0 & Y_\mu \end{pmatrix} \quad \text{for} \quad Y_\mu(x) \in \mathbb{R}. \tag{2.61}$$

We have one single $U(1)$ gauge field Y_μ, carrying the action of the gauge group

$$\mathfrak{B}(M \times F_{ED}) \simeq C^\infty(M, U(1)) \tag{2.62}$$

Our space $N = M \times X \simeq M \sqcup M$ consists of two compies of M. If $D_F = 0$ we have infinite distance between the two copies. Now we have D_F nonzero but $[D_F, a] = 0$ $\forall a \in A$ which still yields infinite distance.

Question 7. What does this imply (physically, mathematically)? Why can we continue even thought we have infinite distance between the same manifold? What do we get if we fix this?

2.4.4 The Spectral Action

Here we calculate the Lagrangian of the almost commutative Manifold $M \times F_{ED}$, which corresponds to the Lagrangian of Electrodynamics on a curved background Manifold (+ gravitational Lagrangian). It consists of the spectral action S_b (bosonic) and of the fermionic action S_f.

The simples spectral action of a spectral triple (A,H,D) is given by the trace of some function of D, we also allow inner fluctuations of the Dirac operator $D_\omega = D + \omega + \varepsilon' J\omega J^{-1}$ where $\omega = \omega^* \in \Omega^1_D(A)$.

Definition 11. Let $f : \mathbb{R} \to \mathbb{R}$ be a suitable function **positive and even**. The spectral action is then

$$S_b[\omega] := \mathrm{Tr} f(\frac{D_\omega}{\Lambda}) \tag{2.63}$$

where Λ is a real cutoff parameter. The minimal condition on f is that $f(\frac{D_\omega}{\Lambda})$ is a traclass operator, which mean that it should be compact operator with well defined finite trace independent of the basis. The subscript b of S_b refers to bosonic, because in physical applications ω will describe bosonic fields.

Furthermore there is a topological spectral action, defined with the grading γ

$$S_{\mathrm{top}}[\omega] := \mathrm{Tr}(\gamma f(\frac{D_\omega}{\Lambda})). \tag{2.64}$$

Definition 12. The fermionic action is defined by

$$S_f[\omega,\psi] = (J\tilde{\psi}, D_\omega \tilde{\psi}) \tag{2.65}$$

with $\tilde{\psi} \in H^+_{cl} := \{\tilde{\psi} : \psi \in H^+\}$. H^+_{cl} is the set of Grassmann variables in H in the $+1$-eigenspace of the grading γ.

The grasmann variables are a set of Basis vectors of a vector space, they form a unital algebra over a vector field say V where the generators are anti commuting, that is for θ_i, θ_j some Grassmann variables we have

$$\theta_i\theta_j = -\theta_j\theta_i \tag{2.66}$$

$$\theta_i x = x\theta_j \quad x \in V \tag{2.67}$$

$$(\theta_i)^2 = 0 \quad (\theta_i\theta_i = -\theta_i\theta_i) \tag{2.68}$$

Proposition 5. *The spectral action of the almost commutative manifold M with* $\dim(M) = 4$ *with a fluctuated Dirac operator is.*

$$Tr(f\frac{D_\omega}{\Lambda}) \sim \int_M \mathscr{L}(g_{\mu\nu}, B_\mu, \Phi)\sqrt{g}\, d^4x + O(\Lambda^{-1}) \tag{2.69}$$

with

$$\mathscr{L}(g_{\mu\nu}, B_\mu, \Phi) = N\mathscr{L}_M(g_{\mu\nu})\mathscr{L}_B(B_\mu) + \mathscr{L}_\phi(g_{\mu\nu}, B_\mu, \Phi) \tag{2.70}$$

where $N = 4$ and $\mathscr{L}_M$ is the Lagrangian of the spectral triple $(C^\infty(M), L^2(S), D_M)$

$$\mathscr{L}_M(g_{\mu\nu}) := \frac{f_4\Lambda^4}{2\pi^2} - \frac{f_2\Lambda^2}{24\pi^2}s - \frac{f(0)}{320\pi^2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}. \tag{2.71}$$

Here $C^{\mu\nu\rho\sigma}$ is defined in terms of the Riemannian curvature tensor $R_{\mu\nu\rho\sigma}$ and the Ricci tensor $R_{\nu\sigma} = g^{\mu\rho}R_{\mu\nu\rho\sigma}$.

Furthermore $\mathscr{L}_B$ describes the kinetic term of the gauge field

$$\mathscr{L}_B(B_\mu) := \frac{f(0)}{24\pi^2}Tr(F_{\mu\nu}F^{\mu\nu}). \tag{2.72}$$

Last $\mathscr{L}_\phi$ is the scalar-field Lagrangian with a boundary term.

$$\mathscr{L}_\phi(g_{\mu\nu}, B_\mu, \Phi) := -\frac{2f_2\Lambda^2}{4\pi^2}Tr(\Phi^2) + \frac{f(0)}{8\pi^2}Tr(\Phi^4) + \frac{f(0)}{24\pi^2}\Delta(Tr(\Phi^2)) \tag{2.73}$$

$$+\frac{f(0)}{48\pi^2}sTr(\Phi^2)\frac{f(0)}{8\pi^2}Tr((D_\mu\Phi)(D^\mu\Phi)). \tag{2.74}$$

Proof. Will maybe be filled in if I go through the last two chapters in the book and understand the proof. **PROOF: in week10.pdf** □

Here on we go and calculate the spectral action of $M \times F_{ED}$

Proposition 6. *The Spectral action of $M \times F_{ED}$ is*

$$Tr(f\frac{D_\omega}{\Lambda}) \sim \int_M \mathscr{L}(g_{\mu\nu}, Y_\mu)\sqrt{g}\, d^4x + O(\Lambda^{-1}) \tag{2.75}$$

where the Lagrangian is

$$\mathscr{L}(g_{\mu\nu}, Y_\mu) = 4\mathscr{L}_M(g_{\mu\nu}) + \mathscr{L}_Y(Y_\mu) + \mathscr{L}_\phi(g_{\mu\nu}, d) \tag{2.76}$$

here the d in $\mathscr{L}_\phi$ is from D_F in equation 2.56. The Lagrangian $\mathscr{L}_M$ is like in equation 2.156. The Lagrangian $\mathscr{L}_Y$ is the kinetic term of the $U(1)$ gauge field Y_μ

$$\mathscr{L}_Y(Y_\mu) := \frac{f(0)}{6\pi^2}Y_{\mu\nu}Y^{\mu\nu} \qquad with \quad Y_{\mu\nu} = \partial_\mu Y_\nu - \partial_\nu Y_\mu. \tag{2.77}$$

Then there is $\mathscr{L}_\phi$, which has two constant terms (disregarding the boundary term) that add up to the Cosmological Constant and a term that for the Einstein-Hilbert action

$$\mathscr{L}_\phi(g_{\mu\nu}, d) := \frac{2f_2\Lambda^2}{\pi^2}|d|^2 + \frac{f(0)}{2\pi^2}|d|^4 + \frac{f(0)}{12\pi^2}s|d|^2. \tag{2.78}$$

Proof. The Trace of $\mathbb{C}^4$ (the Hilbertspace) gives $N = 4$. With B_μ like in equation 2.61 we have $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = 4Y_{\mu\nu}Y^{\mu\nu}$. This provides $\mathscr{L}_Y$. Furthermore we have $\Phi^2 = D_F^2 = |d|^2$ and $\mathscr{L}_\phi$ only give numerical contributions to the cosmological constant and the Einstein-Hilbert action.

The proof is relying itself on just plugging the terms into the previous proposition, for which I didn't write the proof for. □

2.5 Heat Kernel Expansion

2.5.1 The Heat Kernel

The heat kernel $K(t;x,y;D)$ is the fundamental solution of the heat equation. It depends on the operator D of Laplacian type.

$$(\partial_t + D_x)K(t;x,y;D) = 0 \tag{2.79}$$

For a flat manifold $M = \mathbb{R}^n$ and $D = D_0 := -\Delta_\mu \Delta^\mu + m^2$ the Laplacian with a mass term and the initial condition

$$K(0;x,y;D) = \delta(x,y) \tag{2.80}$$

we have the standard fundamental solution

$$K(t;x,y;D_0) = (4\pi t)^{-n/2} \exp\left(-\frac{(x-y)^2}{4t} - tm^2\right) \tag{2.81}$$

Let us consider now a more general operator D with a potential term or a guage field, the heat kernel reads then

$$K(t;x,y;D) = \langle x|e^{-tD}|y\rangle. \tag{2.82}$$

We can expand it it in terms of D_0 and we still have the singularity from the equation 2.81 as $t \to 0$ thus the expansion gives

$$K(t;x,y;D) = K(t;x,y;D_0)\left(1 + tb_2(x,y) + t^2 b_4(x,y) + \dots\right) \tag{2.83}$$

where $b_k(x,y)$ are regular in $y \to x$. They are called the heat kernel coefficients.

2.5.2 Example

Now let us consider a propagator $D^{-1}(x,y)$ defined through the heat kernel in an integral representation

$$D^{-1}(x,y) = \int_0^\infty dt K(t;x,y;D). \tag{2.84}$$

We can integrate the expression formally if we assume the heat kernel vanishes for $t \to \infty$ we get

$$D^{-1}(x,y) \simeq 2(4\pi)^{-n/2} \sum_{j=0} \left(\frac{|x-y|}{2m}\right)^{-\frac{n}{2}+j+1} K_{-\frac{n}{2}+j+1}(|x-y|m) b_{2j}(x,y). \tag{2.85}$$

where $b_0 = 1$ and $K_\nu(z)$ is the Bessel function

$$K_\nu(z) = \frac{1}{\pi}\int_0^\pi cos(\nu\tau - z\sin(\tau))d\tau \tag{2.86}$$

it solves the differential equation

$$z^2 \frac{d^2K}{dz^2} + z\frac{dK}{dz} + (z^2 - \nu^2) = 0. \tag{2.87}$$

By looking at integral approximation of the propagator we conclude that the singularities of D^{-1} coincide with the singularities of the heat kernel coefficients. We consider now a generating functional in terns of $\det(D)$ which is called the one-loop effective action (quantum fields theory)

$$W = \frac{1}{2}\ln(\det D) \tag{2.88}$$

we can relate W with the heat kernel. For each eigenvalue $\lambda > 0$ of D we can write the identity.

$$\ln\lambda = -\int_0^\infty \frac{e^{-t\lambda}}{t}dt \tag{2.89}$$

This expression is correct up to an infinite constant which does not depent on λ, because of this we can ignore it. Further more we use $\ln(\det D) = \mathrm{Tr}(\ln D)$ and therefor we can write for W

$$W = -\frac{1}{2}\int_0^\infty dt\frac{K(t,D)}{t} \tag{2.90}$$

where

$$K(t,D) = \mathrm{Tr}(e^{-tD}) = \int d^n x\sqrt{g}K(t;x,x;D). \tag{2.91}$$

The problem is now that the integral of W is divergent at both limits. Yet the divergences at $t \to \infty$ are caused by $\lambda \leq 0$ of D (infrared divergences) and are just ignored. The divergences at $t \to 0$ are cutoff at $t = \Lambda^{-2}$, thus we write

$$W_\Lambda = -\frac{1}{2}\int_{\Lambda^{-2}}^\infty dt\frac{K(t,D)}{t}. \tag{2.92}$$

We can calculate W_Λ at up to an order of λ^0

$$W_\Lambda = -(4\pi)^{-n/2}\int d^n x\sqrt{g}\Big(\sum_{2(j+l)<n}\Lambda^{n-2j-2l}b_{2j}(x,x)\frac{(-m^2)^l l!}{n-2j-2l}+ \tag{2.93}$$

$$+\sum_{2(j+l)=n}\ln(\Lambda)(-m^2)^l l! b_{2j}(x,x)\mathscr{O}(\lambda^0)\Big) \tag{2.94}$$

There is an divergence at $b_2(x,x)$ with $k \leq n$. Now we compute the limit $\Lambda \to \infty$

$$-\frac{1}{2}(4\pi)^{n/2}m^n\int d^n x\sqrt{g}\sum_{2j>n}\frac{b_{2j}(x,x)}{m^{2j}}\Gamma(2j-n) \tag{2.95}$$

here Γ is the gamma function.

2.5.3 Differential Geometry and Operators of Laplace Type

Let M be a n dimensional compact Riemannian manifold with $\partial M = 0$. Then consider a vector bundle V over M (i.e. there is a vector space to each point on M), so we can define smooth functions. We want to look at arbitrary differential operators D of Laplace type on V, they have the general from

$$D = -(g^{\mu\nu}\partial_\mu\partial_\nu + a^\sigma\partial_\sigma + b) \tag{2.96}$$

where a^σ, b are matrix valued functions on M and $g^{\mu\nu}$ is the inverse metric on M. There is a unique connection on V and a unique endomorphism (matrix valued function) E on V, then we can rewrite D in terms of E and covariant derivatives

$$D = -(g^{\mu\nu}\nabla_\mu\nabla_\nu + E) \tag{2.97}$$

Where the covariant derivative consists of $\nabla = \nabla^{[R]} + \omega$ the standard Riemannian covariant derivative $\nabla^{[R]}$ and a "gauge" bundle ω (fluctuations). WE can write E and ω in terms of geometrical identities

$$\omega_\delta = \frac{1}{2}g_{\nu\delta}(a^\nu + g^{\mu\sigma}\Gamma^\nu_{\mu\sigma}I_V) \tag{2.98}$$

$$E = b - g^{\nu\mu}(\partial_\mu\omega_\nu + \omega_\nu\omega_\mu - \omega_\sigma\Gamma^\sigma_{\nu\mu}) \tag{2.99}$$

where I_V is the identity in V and the Christoffel symbol

$$\Gamma^\sigma_{\mu\nu} = g^{\sigma\rho}\frac{1}{2}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \tag{2.100}$$

Furthermore we remind ourselves of the Riemmanian curvature tensor, Ricci Tensor and the Scalar curavture.

$$R^\mu_{\nu\rho\sigma} = \partial_\sigma\Gamma^\mu_{\nu\rho} - \partial_\rho\Gamma^\mu_{\nu\sigma}\Gamma^\lambda_{\nu\rho}\Gamma^\mu_{\lambda\sigma}\Gamma^\lambda_{\nu\sigma}\Gamma^\mu_{\lambda\rho} \tag{2.101}$$

$$R_{\mu\nu} := R^\sigma_{\mu\nu\sigma} \tag{2.102}$$

$$R := R^\mu_\mu \tag{2.103}$$

The we let $\{e_1, \ldots, e_n\}$ be the local orthonormal frame of TM(tangent bundle M), which will be noted with flat indices $i, j, k, l \in \{1, \ldots, n\}$, we use e^k_μ, e^ν_j to transform between flat indices and curved indices μ, ν, ρ.

$$e^\mu_j e^\nu_k g_{\mu\nu} = \delta_{jk} \tag{2.104}$$

$$e^\mu_j e^\nu_k \delta^{jk} = g^{\mu\nu} \tag{2.105}$$

$$e^j_\mu e^\mu_k = \delta^j_k \tag{2.106}$$

The Riemannian part of the covariant derivative contains the standard Levi-Civita connection, so that for a v_ν we write

$$\nabla^{[R]}_\mu v_\nu = \partial_\mu v_\nu - \Gamma^\rho_{\mu\nu}v_\rho. \tag{2.107}$$

The extended covariant derivative reads then

$$\nabla_\mu v^j = \partial_\mu v^j + \sigma^{jk}_\mu v_k. \tag{2.108}$$

the condition $\nabla_\mu e^k_\nu = 0$ gives us the general connection

$$\sigma^{kl}_\mu = e^\nu_l\Gamma^\rho_{\mu\nu}e^k_\rho - e^\nu_l\partial_\mu e^k_\nu \tag{2.109}$$

The we may define the field strength $\Omega_{\mu\nu}$ of the connection ω

$$\Omega_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + \omega_\mu\omega_\nu - \omega_\nu\omega_\mu. \tag{2.110}$$

If we apply the covariant derivative on Ω we get

$$\nabla_\rho\Omega_{\mu\nu} = \partial_\rho\Omega_{\mu\nu} - \Gamma^\sigma_{\rho\mu}\Omega_{\sigma\mu} + [\omega_\rho, \Omega_{\mu\nu}] \tag{2.111}$$

2.5.4 Spectral Functions

Manifolds without M boundary condition for the operator e^{-tD} for $t > 0$ is a trace class operator on $L^2(V)$, this means that for any smooth function f on M we can define

$$K(t, f, D) = \mathrm{Tr}_{L^2}(fe^{-tD}) \tag{2.112}$$

and we can rewrite

$$K(t, f, D) = \int_M d^n x \sqrt{g} \mathrm{Tr}_V (K(t; x, x; D) f(x)). \tag{2.113}$$

in terms of the Heat kernel $K(t;x,y;D)$ in the regular limit $y \to y$. We can write the Heat Kernel in terms of the spectrum of D. Say $\{\phi_\lambda\}$ is a ONB of eigenfunctions of D corresponding to the eigenvalue λ

$$K(t;x,y;D) = \sum_\lambda \phi_\lambda^\dagger(x)\phi_\lambda(y)e^{-t\lambda}. \tag{2.114}$$

We have an asymtotic expansion at $t \to 0$ for the trace

$$Tr_{L^2}(fe^{-tD}) \simeq \sum_{k \geq 0} t^{(k-n)/2} a_k(f, D). \tag{2.115}$$

where

$$a_k(f, D) = (4\pi)^{-n/2} \int_M d^4 x \sqrt{g} b_k(x, x) f(x) \tag{2.116}$$

2.5.5 General Formulae

We consider a compact Riemmanian Manifold M without boundary condition, a vector bundle V over M to define functions which carry discrete (spin or gauge) indices. An Laplace style operator D over V and smooth function f on M. There is an asymtotic expansion where the heat kernel coefficients

1. with odd index $k = 2j + 1$ vanish $a_{2j+1}(f, D) = 0$
2. with even index are locally computable in terms of geometric invariants

$$a_k(f, D) = \mathrm{Tr}_V \left(\int_M d^n x \sqrt{g} (f(x) a_k(x; D) \right) = \tag{2.117}$$

$$= \sum_I \mathrm{Tr}_V \left(\int_M d^n x \sqrt{g} (f u^I \mathscr{A}_k^I(D)) \right) \tag{2.118}$$

here $\mathscr{K}_k^I$ are all possible independent invariants of dimension k, constructed from $E, \Omega, R_{\mu\nu\rho\sigma}$ and their derivatives, u^I are some constants.

If E has dimension two, then the derivative has dimension one. So if $k = 2$ there are only two independent invariants, E and R. This corresponds to the statement $a_{2j+1} = 0$.

If we consider $M = M_1 \times M_2$ with coordinates x_1 and x_2 and a decomposed Laplace style operator $D = D_1 \otimes 1 + 1 \otimes D_2$ we can separate everything, i.e.

$$e^{-tD} = e^{-tD_1} \otimes e^{-tD_2} \tag{2.119}$$

$$f(x_1, x_2) = f_1(x_1) f_2(x_2) \tag{2.120}$$

$$a_k(x; D) = \sum_{p+q=k} a_p(x_1; D_1) a_q(x_2; D_2) \tag{2.121}$$

Say the spectrum of D_1 is known, $l^2, l \in \mathbb{Z}$. We obtain the heat kernel asymmetries with the Poisson Summation formula

$$K(t,D_1) = \sum_{l\in\mathbb{Z}} e^{-tl^2} = \sqrt{\frac{\pi}{t}} \sum_{l\in\mathbb{Z}} e^{-\frac{\pi^2 l^2}{t}} = \tag{2.122}$$

$$\simeq \sqrt{\frac{\pi}{t}} + \mathscr{O}(e^{-1/t}). \tag{2.123}$$

Note that the exponentially small terms have no effect on the heat kernel coefficients and that the only nonzero coefficient is $a_0(1,D_1) = \sqrt{\pi}$. Therefore we can write

$$a_k(f(x^2),D) = \sqrt{\pi} \int_{M_2} d^{n-1}x\sqrt{g} \sum_I \mathrm{Tr}_V \left(f(x^2) u^I_{(n-1)} \mathscr{A}^I_n(D_2) \right). \tag{2.124}$$

On the other had all geometric invariants associated with D are in the D_2 part. Thus all invariants are independent of x_1, so we can choose for M_1. Say $M_1 = S^1$ with $x \in (0,2\pi)$ and $D_1 = -\partial^2_{x_1}$ we may rewrite the heat kernel coefficients in

$$a_k(f(x_2),D) = \int_{S^1\times M_2} d^n x\sqrt{g} \sum_I \mathrm{Tr}_V (f(x_2) u^I_{(n)} \mathscr{A}^I_k(D_2)) = \tag{2.125}$$

$$= 2\pi \int_{M_2} d^n x\sqrt{g} \sum_I \mathrm{Tr}_V (f(x_2) u^I_{(n)} \mathscr{A}^I_k(D_2)). \tag{2.126}$$

Computing the two equations above we see that

$$u^I_{(n)} = \sqrt{4\pi} u^I_{(n+1)} \tag{2.127}$$

2.5.6 Heat Kernel Coefficients

To calculate the heat kernel coefficients we need the following variational equations

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} a_k(1, e^{-2\varepsilon f} D) = (n-k) a_k(f,D), \tag{2.128}$$

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} a_k(1, D-\varepsilon F) = a_{k-2}(F,D), \tag{2.129}$$

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} a_k(e^{-2\varepsilon f} F, e^{-2\varepsilon f} D) = 0. \tag{2.130}$$

To prove the equation 2.128 we differentiate

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} \mathrm{Tr}(\exp(-e^{-2\varepsilon f} tD) = \mathrm{Tr}(2ftDe^{-tD}) = -2t\frac{d}{dt}\mathrm{Tr}(fe^{-tD})) \tag{2.131}$$

then we expand both sides in t and get 2.128. Equation 2.129 is derived similarly. For equation 2.130 we consider the following operator

$$D(\varepsilon,\delta) = e^{-2\varepsilon f}(D-\delta F) \tag{2.132}$$

for $k=n$ we use equation 2.128 and we get

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} a_n(1, D(\varepsilon,\delta)) = 0 \tag{2.133}$$

then we take the variation in terms of δ, evaluated at $\delta = 0$ and swap the differentiation, allowed by theorem of Schwarz

$$0 = \frac{d}{d\delta}|_{\delta=0}\frac{d}{d\varepsilon}|_{\varepsilon=0}a_n(1, D(\varepsilon,\delta)) = \frac{d}{d\varepsilon}|_{\varepsilon=0}\frac{d}{d\delta}|_{\delta=0}a_n(1, D(\varepsilon,\delta)) = \tag{2.134}$$
$$= a_{n-2}(e^{-2\varepsilon f}F, e^{-2\varepsilon f}D) \tag{2.135}$$

which proves equation 2.130. With this we calculate the constants u^I and we can write the first three heat kernel coefficients as

$$a_0(f,D) = (4\pi)^{-n/2}\int_M d^n x\sqrt{g}\mathrm{Tr}_V(a_0 f) \tag{2.136}$$
$$a_2(f,D) = (4\pi)^{-n/2}\frac{1}{6}\int_M d^n x\sqrt{g}\mathrm{Tr}_V)(f\alpha_1 E + \alpha_2 R) \tag{2.137}$$
$$a_4(f,D) = (4\pi)^{-n/2}\frac{1}{360}\int_M d^n x\sqrt{g}\mathrm{Tr}_V(f(\alpha_3 E_{,kk} + \alpha_4 RE + \alpha_5 E^2\alpha_6 R_{,kk} + \tag{2.138}$$
$$+ \alpha_7 R^2 + \alpha_8 R_{ij}R_{ij} + \alpha_9 R_{ijkl}R_{ijkl} + \alpha_{10}\Omega_{ij}\Omega ij)). \tag{2.139}$$

The constants α_I do not depend on the dimension n of the Manifold and we can compute them with our variational identities.

The first coefficient α_0 can be seen from the heat kernel expanion of the Laplacian on S^1 (above), $\alpha_0 = 1$. For α_1 we use 2.129, for $k = 2$

$$\frac{1}{6}\int_M d^n x\sqrt{g}\mathrm{Tr}_V(\alpha_1 F) = \int_M d^n x\sqrt{g}\mathrm{Tr}_V(F), \tag{2.140}$$

thus we conclude that $\alpha_1 = 6$. Now we take $k = 4$

$$\frac{1}{360}\int_M d^n x\sqrt{g}\mathrm{Tr}_V(\alpha_4 FR + 2\alpha_5 FE) = \frac{1}{6}\int_M d^n x\sqrt{g}\mathrm{Tr}_V(\alpha_1 FE + \alpha_2 FR), \tag{2.141}$$

thus $\alpha_4 = 60\alpha_2$ and $\alpha_5 = 180$.

Furthermore we apply 2.130 to $n = 4$

$$\frac{d}{d\varepsilon}|_{\varepsilon=0}a_2(e^{-2\varepsilon f}F, e^{-2\varepsilon f}D) = 0. \tag{2.142}$$

By collecting the terms with $\mathrm{Tr}_V(\int_M d^n x\sqrt{g}(Ff_{,jj}))$ we obtain $\alpha_1 = 6\alpha_2$, that is $\alpha_2 = 1$, so $\alpha_4 = 60$.

Now we let $M = M_1 \times M_2$ and split $D = -\Delta_1 - \Delta_2$, where $\Delta_{1/2}$ are Laplacians for M_1, M_2, then we can decompose the heat kernel coefficients for $k = 4$

$$a_4(1, -\Delta_1 - \Delta_2) = a_4(1, -\Delta_1)a_0(1, -\Delta_2) + a_2(1, -\Delta_1)a_2(1, -\Delta_2) \tag{2.143}$$
$$+ a_0(1, -\Delta_1)a_4(1, -\Delta_2) \tag{2.144}$$

with $E = 0$ and $\Omega = 0$ and by calculating the terms with $R_1 R_2$ (scalar curvature of $M_{1/2}$) we obtain $\frac{2}{360}\alpha_7 = (\frac{\alpha_2}{6})^2$, thus $\alpha_7 = 5$.

For $n = 6$ we get

$$0 = \mathrm{Tr}_V(\int_M d^n x\sqrt{g}(F(-2\alpha_3 - 10\alpha_4 + 4\alpha_5)f_{,kk}E + \tag{2.145}$$
$$+ (2\alpha_3 + 10\alpha_6)f_{,iijj} + \tag{2.146}$$
$$+ (2\alpha_4 - 2\alpha_6 - 20\alpha_7 - 2\alpha_8)f_{,ii}R \tag{2.147}$$
$$+ (-8\alpha_8 - 8\alpha_6)f_{,ij}R_{ij})) \tag{2.148}$$

we obtain $\alpha_3 = 60$, $\alpha_6 = 12$, $\alpha_8 = -2$ and $\alpha_9 = 2$

For α_{10} we use the Gauss-Bonnet theorem to get $\alpha_{10} = 30$, which is left out because it is a lengthy computation.

Summarizing we get for the heat kernel coefficients

$$\alpha_0(f,D) = (4\pi)^{-n/2} \int_M d^n x \sqrt{g} \mathrm{Tr}_V(f) \tag{2.149}$$

$$\alpha_2(f,D) = (4\pi)^{-n/2} \frac{1}{6} \int_M d^n x \sqrt{g} \mathrm{Tr}_V(f(6E+R)) \tag{2.150}$$

$$\alpha_4(f,D) = (4\pi)^{-n/2} \frac{1}{360} \int_M d^n x \sqrt{g} \mathrm{Tr}_V(f(60E_{,kk} + 60RE + 180E^2 + \tag{2.151}$$

$$+ 12R_{,kk} + 5R^2 - 2R_{ij}R_{ij} 2R_{ijkl}R_{ijkl} + 30\Omega_{ij}\Omega_{ij})) \tag{2.152}$$

$$\tag{2.153}$$

2.6 Spectral Action of the Fluctuated Dirac Operator

Proposition 7. *The spectral action of the almost commutative manifold M with* $\dim(M) = 4$ *with a fluctuated Dirac operator is.*

$$Tr(f\frac{D_\omega}{\Lambda}) \sim \int_M \mathscr{L}(g_{\mu\nu}, B_\mu, \Phi)\sqrt{g}\, d^4x + O(\Lambda^{-1}) \tag{2.154}$$

with

$$\mathscr{L}(g_{\mu\nu}, B_\mu, \Phi) = N\mathscr{L}_M(g_{\mu\nu})\mathscr{L}_B(B_\mu) + \mathscr{L}_\phi(g_{\mu\nu}, B_\mu, \Phi) \tag{2.155}$$

where $N = 4$ *and* $\mathscr{L}_M$ *is the Lagrangian of the spectral triple* $(C^\infty(M), L^2(S), D_M)$

$$\mathscr{L}_M(g_{\mu\nu}) := \frac{f_4\Lambda^4}{2\pi^2} - \frac{f_2\Lambda^2}{24\pi^2}s - \frac{f(0)}{320\pi^2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}. \tag{2.156}$$

Here $C^{\mu\nu\rho\sigma}$ *is defined in terms of the Riemannian curvature tensor* $R_{\mu\nu\rho\sigma}$ *and the Ricci tensor* $R_{\nu\sigma} = g^{\mu\rho}R_{\mu\nu\rho\sigma}$.

Furthermore $\mathscr{L}_B$ *describes the kinetic term of the gauge field*

$$\mathscr{L}_B(B_\mu) := \frac{f(0)}{24\pi^2}Tr(F_{\mu\nu}F^{\mu\nu}). \tag{2.157}$$

Last $\mathscr{L}_\phi$ *is the scalar-field Lagrangian with a boundary term.*

$$\mathscr{L}_\phi(g_{\mu\nu}, B_\mu, \Phi) := -\frac{2f_2\Lambda^2}{4\pi^2}Tr(\Phi^2) + \frac{f(0)}{8\pi^2}Tr(\Phi^4) + \frac{f(0)}{24\pi^2}\Delta(Tr(\Phi^2)) \tag{2.158}$$

$$+ \frac{f(0)}{48\pi^2}sTr(\Phi^2)\frac{f(0)}{8\pi^2}Tr((D_\mu\Phi)(D^\mu\Phi)). \tag{2.159}$$

Proof. The dimension of our manifold M is $\dim(M) = \mathrm{Tr}(id) = 4$. Let us take a $x \in M$, we have an asymtotic expansion of $\mathrm{Tr}(f(\frac{D_\omega}{\Lambda}))$ as $\Lambda \to \infty$

$$\mathrm{Tr}(f(\frac{D_\omega}{\Lambda})) \simeq 2f_4\Lambda^4 a_0(D_\omega^2) + 2f_2\Lambda^2 a_2(D_\omega^2) \tag{2.160}$$

$$+ f(0)a_4(D_\omega^4) + O(\Lambda^{-1}). \tag{2.161}$$

Note that the heat kernel coefficients are zero for uneven k, furthermore they are dependent on the fluctuated Dirac operator D_ω. We can rewrite the heat kernel coefficients in terms of D_M, for the first two we note that $N := \mathrm{Tr}\mathbb{1}_{\mathbb{H}_\mathbb{F}})$

$$a_0(D_\omega^2) = Na_0(D_M^2) \tag{2.162}$$

$$a_2(D_\omega^2 = Na_2(D_M^2) - \frac{1}{4\pi^2}\int_M \mathrm{Tr}(\Phi^2)\sqrt{g}d^4x \tag{2.163}$$

For a_4 we need to extend in terms of coefficients of F, look week9.pdf for the standard version,

$$\frac{1}{360}\mathrm{Tr}(60sF) = -\frac{1}{6}S(Ns + 4\mathrm{Tr}(\Phi^2)) \tag{2.164}$$

$$F^2 = \frac{1}{16}s^2 \otimes 1 + 1 \otimes \Phi^4 - \frac{1}{4}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma F_{\mu\nu}F^{\mu\nu} + \tag{2.165}$$

$$+\gamma^\mu\gamma^\nu \otimes (D_\mu\Phi)(D_\nu\Phi) + \frac{1}{2}s \otimes \Phi^2 + \text{ traceless terms} \tag{2.166}$$

$$\frac{1}{360}\mathrm{Tr}(180F^2) = \frac{1}{8}s^2N + 2\mathrm{Tr}(\Phi^4) + \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \tag{2.167}$$

$$+2\mathrm{Tr}((D_\mu\Phi)(D^\mu\Phi)) + s\mathrm{Tr}(\Phi^2) \tag{2.168}$$

$$\frac{1}{360}\mathrm{Tr}(-60\Delta F) = \frac{1}{6}\Delta(Ns + 4\mathrm{Tr}(\Phi^2)). \tag{2.169}$$

Now for the cross terms of $\Omega^E_{\mu\nu}\Omega^{E\mu\nu}$ the trace vanishes because of the anti-symmetric properties of the Riemannian curvature Tensor

$$\Omega^E_{\mu\nu}\Omega^{E\mu\nu} = \Omega^S_{\mu\nu}\Omega^{S\mu\nu} \otimes 1 - 1 \otimes F_{\mu\nu}F^{\mu\nu} + 2i\Omega^S_{\mu\nu} \otimes F^{\mu\nu} \tag{2.170}$$

the trace of the cross term vanishes because

$$\mathrm{Tr}(\Omega^S_{\mu\nu} = \frac{1}{4}R_{\mu\nu\rho\sigma}\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \frac{1}{4}R_{\mu\nu\rho\sigma}g^{\mu\nu} = 0 \tag{2.171}$$

and the trace of the whole term is

$$\frac{1}{360}\mathrm{Tr}(30\Omega^E_{\mu\nu}\Omega^{E\mu\nu}) = \frac{N}{24}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{1}{3}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}). \tag{2.172}$$

Plugging the results into a_4 and simplifying we can write

$$a_4(x, D_\omega^4) = Na_4(x, D_M^2) + \frac{1}{4\pi^2}\bigg(\frac{1}{12}s\mathrm{Tr}(\Phi^2) + \frac{1}{2}\mathrm{Tr}(\Phi^4) \tag{2.173}$$

$$+\frac{1}{4}\mathrm{Tr}((D_\mu\Phi)(D^\mu\Phi)) + \frac{1}{6}\Delta\mathrm{Tr}(\Phi^2) + \frac{1}{6}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\bigg) \tag{2.174}$$

The only thing left is to plug in the heat kernel coefficients into the heat kernel expansion above. $\square$

2.7 Fermionic Action

A quick reminder with what we are dealing with, the fermionic action is defined in the following way.

Definition 13. The fermionic action is defined by

$$S_f[\omega,\psi] = (J\tilde{\psi}, D_\omega\tilde{\psi}) \tag{2.175}$$

with $\tilde{\psi} \in H^+_{cl} := \{\tilde{\psi} : \psi \in H^+\}$. H^+_{cl} is the set of Grassmann variables in H in the +1-eigenspace of the grading γ.

The almostcommutative Manifold we are dealing with is the following

$$M \times F_{ED} := \left(C^\infty(M,\mathbb{C}^2),\ L^2(S)\otimes\mathbb{C}^4,\ D_M\otimes 1 + \gamma_M\otimes D_F;\ J_M\otimes J_F,\ \gamma_M\otimes\gamma_F \right). \tag{2.176}$$

where:

$$C^\infty(M,\mathbb{C}^2) = C^\infty(M)\otimes C^\infty(M) \qquad \mathscr{H} = \mathscr{H}^+\otimes\mathscr{H}^- \tag{2.177}$$

$$\mathscr{H} = L^2(S)^+\otimes H_F^+ \oplus L^2(S)^-\otimes H_F^-. \tag{2.178}$$

Where H_F is separated into the particle-anitparticle states with ONB $\{e_R, e_L, \bar{e}_R, \bar{e}_L\}$. The ONB of H_F^+ is $\{e_L, \bar{e}_R\}$ and for H_F^- we have $\{e_R, \bar{e}_L\}$. Furthermore we can decompose a spinor $\psi \in L^2(S)$ for each of the eigenspaces $H_F^\pm$, $\psi = \psi_R\psi_L$. Thus we can write for an arbitrary $\psi \in \mathscr{H}^+$

$$\psi = \chi_R\otimes e_R + \chi_L\otimes e_L + \psi_L\otimes\bar{e}_R\psi_R\otimes\bar{e}_L \tag{2.179}$$

for $\chi_L, \psi_L \in L^2(S)^+$ and $\chi_R, \psi_R \in L^2(S)^-$.

Proposition 8. *We can define the action of the fermionic art of $M\times F_{ED}$ in the following way*

$$S_f = -i\big(J_M\tilde{\chi}, \gamma(\nabla^S_\mu - i\Gamma_\mu)\tilde{\Psi}\big) + \big(S_M\tilde{\chi}_L, \bar{d}\tilde{\psi}_L\big) - \big(J_M\tilde{\chi}_R, d\tilde{\psi}_R\big) \tag{2.180}$$

Proof. We take the fluctuated Dirac operator

$$D_\omega = D_M\otimes i + \gamma^\mu\otimes B_\mu + \gamma_M\otimes D_F \tag{2.181}$$

□

The Fermionic Action is $S_F = (J\tilde{\xi}, D_\omega\tilde{\xi})$ for a $\xi \in \mathscr{H}^+$, we can begin to calculate (note that we add the constant $\frac{1}{2}$ to the action)

$$\frac{1}{2}(J\tilde{\xi}, D_\omega\tilde{\xi}) = \tag{2.182}$$

$$+\frac{1}{2}(J\tilde{\xi}, (D_M\otimes i)\tilde{\xi}) \tag{2.183}$$

$$+\frac{1}{2}(J\tilde{\xi}, (\gamma^\mu\otimes B_\mu)\tilde{\xi}) \tag{2.184}$$

$$+\frac{1}{2}(J\tilde{\xi}, (\gamma_M\otimes D_F)\tilde{\xi}). \tag{2.185}$$

For equation 2.183 we calculate

$$\frac{1}{2}(J\tilde{\xi},(D_M\otimes 1)\tilde{\xi}) = \frac{1}{2}(J_M\tilde{\chi}_R, D_M\tilde{\psi}_L) + \frac{1}{2}(J_M\tilde{\chi}_L, D_M\tilde{\psi}_R) + \tag{2.186}$$

$$+\frac{1}{2}(J_M\tilde{\psi}_L, D_M\tilde{\psi}_R) + \frac{1}{2}(J_M\tilde{\chi}_R, D_M\tilde{\chi}_L) \tag{2.187}$$

$$= (J_M\tilde{\chi}, D_M\tilde{\chi}). \tag{2.188}$$

For equation 2.184 we have

$$\frac{1}{2}(J\tilde{\xi},(\gamma^\mu\otimes B_\mu)\tilde{\xi}) = -\frac{1}{2}(J_M\tilde{\chi}_R, \gamma^\mu Y_\mu\tilde{\psi}_R) - \frac{1}{2}(J_M\tilde{\chi}_L, \gamma^\mu Y_\mu\tilde{\psi}_R) + \tag{2.189}$$

$$+\frac{1}{2}(J_M\tilde{\psi}_L, \gamma^\mu Y_\mu\tilde{\chi}_R) + \frac{1}{2}(J_M\tilde{\psi}_R, \gamma^\mu Y_\mu\tilde{\chi}_L) = \tag{2.190}$$

$$= -(J_M\tilde{\chi}, \gamma^\mu Y_\mu\tilde{\psi}). \tag{2.191}$$

For equation 2.185 we have

$$\frac{1}{2}(J\tilde{\xi},(\gamma_M\otimes D_F)\tilde{\xi}) = +\frac{1}{2}(J_M\tilde{\chi}_R, d\gamma_M\tilde{\chi}_R) + \frac{1}{2}(J_M\tilde{\chi}_L, \bar{d}\gamma_M\tilde{\chi}_L) + \tag{2.192}$$

$$+\frac{1}{2}(J_M\tilde{\chi}_L, \bar{d}\gamma_M\tilde{\chi}_L) + \frac{1}{2}(J_M\tilde{\chi}_R, d\gamma_M\tilde{\chi}_R) = \tag{2.193}$$

$$= i(J_M\tilde{\chi}, m\tilde{\psi}) \tag{2.194}$$

Note that we obtain a complex mass parameter d, so we write $d := im$ for $m \in \mathbb{R}$, which stands for the real mass and we obtain a nice result

Theorem 1. *The full Lagrangian of $M \times F_{ED}$ is the sum of purely gravitational Lagrangian*

$$\mathscr{L}_{grav}(g_{\mu\nu}) = 4\mathscr{L}_M(g_{\mu\nu})\mathscr{L}_\phi(g_{\mu\nu}) \tag{2.195}$$

and the Lagrangian of electrodynamics

$$\mathscr{L}_{ED} = -i\left\langle J_M\tilde{\chi}, \left(\gamma^\mu(\nabla^S_\mu - iY_\mu) - m\right)\tilde{\psi}\right\rangle + \frac{f(0)}{6\pi^2}Y_{\mu\nu}Y^{\mu\nu}. \tag{2.196}$$

3 Conclusion

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4 Acknowledgment

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