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# Bachelor's Thesis

Title of the Bachelor's Thesis

## **Noncommutative Geomtetry and Physics**

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## Abstract

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## 1 Introduction

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## 2 Main Section

### 2.1 Heat Kernel Expansion

#### 2.1.1 The Heat Kernel

The heat kernel  $K(t; x, y; D)$  is the fundamental solution of the heat equation. It depends on the operator  $D$  of Laplacian type.

$$(\partial_t + D_x)K(t; x, y; D) = 0 \quad (2.1)$$

For a flat manifold  $M = \mathbb{R}^n$  and  $D = D_0 := -\Delta_\mu \Delta^\mu + m^2$  the Laplacian with a mass term and the initial condition

$$K(0; x, y; D) = \delta(x, y), \quad (2.2)$$

takes to form of the standard fundamental solution

$$K(t; x, y; D_0) = (4\pi t)^{-n/2} \exp\left(-\frac{(x-y)^2}{4t} - tm^2\right). \quad (2.3)$$

Let us consider now a more general operator  $D$  with a potential term or a gauge field, the heat kernel reads then

$$K(t; x, y; D) = \langle x | e^{-tD} | y \rangle. \quad (2.4)$$

We can expand the heat kernel in  $t$ , still having a singularity from the equation 2.3 as  $t \rightarrow 0$  thus the expansion reads

$$K(t; x, y; D) = K(t; x, y; D_0) \left( 1 + tb_2(x, y) + t^2 b_4(x, y) + \dots \right), \quad (2.5)$$

where  $b_k(x, y)$  become regular as  $y \rightarrow x$ . These coefficients are called the heat kernel coefficients.

**KANN WEGELASSEN WERDEN BIS ZUM NÄCHSTEN KAPITEL** Let's turn our attention to a propagator  $D^{-1}(x, y)$  defined through the heat kernel, with an integral representation

$$D^{-1}(x, y) = \int_0^\infty dt K(t; x, y; D). \quad (2.6)$$

If we assume the heat kernel vanishes for  $t \rightarrow \infty$ , we can integrate formally to get

$$D^{-1}(x, y) \simeq 2(4\pi)^{-n/2} \sum_{j=0} \left( \frac{|x-y|}{2m} \right)^{-\frac{n}{2}+j+1} K_{-\frac{n}{2}+j+1}(|x-y|m) b_{2j}(x, y), \quad (2.7)$$

where  $b_0 = 1$  and  $K_\nu(z)$  is the Bessel function

$$K_\nu(z) = \frac{1}{\pi} \int_0^\pi \cos(\nu\tau - z \sin(\tau)) d\tau. \quad (2.8)$$

The Bessel function solves the following differential equation

$$z^2 \frac{d^2 K}{dz^2} + z \frac{dK}{dz} + (z^2 - \nu^2) = 0. \quad (2.9)$$

By looking at an integral approximation for the propagator we conclude that the singularities of  $D^{-1}$  coincide with the singularities of the heat kernel coefficients. Thus we can say, that a generating functional in terms of  $\det(D)$  is called the one-loop effective action (quantum field theory)

$$W = \frac{1}{2} \ln(\det D). \quad (2.10)$$

We have a direct relation with one-loop effective action  $W$  and the heat kernel. Furthermore notice that for each eigenvalue  $\lambda > 0$  of  $D$  we can write the identity.

$$\ln \lambda = - \int_0^\infty \frac{e^{-t\lambda}}{t} dt \quad (2.11)$$

This expression is correct up to an infinite constant which does not depend on the eigenvalue  $\lambda$ , thus we can ignore it. By substituting  $\ln(\det D) = \text{Tr}(\ln D)$  we can rewrite the one-loop effective action  $W$  into

$$W = -\frac{1}{2} \int_0^\infty dt \frac{K(t, D)}{t}, \quad (2.12)$$

where

$$K(t, D) = \text{Tr}(e^{-tD}) = \int d^n x \sqrt{g} K(t; x, x; D). \quad (2.13)$$

The problem now is that the integral of  $W$  is divergent at both limits. Yet the divergences at  $t \rightarrow \infty$  are caused by  $\lambda \leq 0$  of  $D$  (infrared divergences) and can be ignored. The divergences at  $t \rightarrow 0$  are cutoff at  $t = \Lambda^{-2}$ , simply written as

$$W_\Lambda = -\frac{1}{2} \int_{\Lambda^{-2}}^{\infty} dt \frac{K(t, D)}{t}. \quad (2.14)$$

We can calculate  $W_\Lambda$  up to an order of  $\lambda^0$

$$W_\Lambda = -(4\pi)^{-n/2} \int d^n x \sqrt{g} \left( \sum_{2(j+l) < n} \Lambda^{n-2j-2l} b_{2j}(x, x) \frac{(-m^2)^l l!}{n-2j-2l} + \right. \quad (2.15)$$

$$\left. + \sum_{2(j+l)=n} \ln(\Lambda) (-m^2)^l l! b_{2j}(x, x) \mathcal{O}(\lambda^0) \right) \quad (2.16)$$

There is an divergence at  $b_2(x, x)$  for  $k \leq n$ . Computing the limit  $\Lambda \rightarrow \infty$  we get

$$-\frac{1}{2} (4\pi)^{n/2} m^n \int d^n x \sqrt{g} \sum_{2j > n} \frac{b_{2j}(x, x)}{m^{2j}} \Gamma(2j - n), \quad (2.17)$$

where  $\Gamma$  stands for the gamma function.

### 2.1.2 Spectral Functions

Manifolds  $M$  with a disappearing boundary condition for the operator  $e^{-tD}$  for  $t > 0$  is a trace class operator on  $L^2(V)$ . Meaning for any smooth function  $f$  on  $M$  we can define

$$K(t, f, D) := \text{Tr}_{L^2}(f e^{-tD}), \quad (2.18)$$

or alternately write an integral representation

$$K(t, f, D) = \int_M d^n x \sqrt{g} \text{Tr}_V(K(t; x, x; D) f(x)), \quad (2.19)$$

in the regular limit  $y \rightarrow y$ . We can write the Heat Kernel in terms of the spectrum of  $D$ . So for an orthonormal basis  $\{\phi_\lambda\}$  of eigenfunctions for  $D$ , which corresponds to the eigenvalue  $\lambda$ , we can rewrite the heat kernel into

$$K(t; x, y; D) = \sum_\lambda \phi_\lambda^\dagger(x) \phi_\lambda(y) e^{-t\lambda}. \quad (2.20)$$

An asymptotic expansion as  $t \rightarrow 0$  for the trace is then

$$\text{Tr}_{L^2}(f e^{-tD}) \simeq \sum_{k \geq 0} t^{(k-n)/2} a_k(f, D), \quad (2.21)$$

where

$$a_k(f, D) = (4\pi)^{-n/2} \int_M d^4 x \sqrt{g} b_k(x, x) f(x). \quad (2.22)$$

### 2.1.3 General Formulae

Let us summarize what we have obtained in the last chapter, we considered a compact Riemannian manifold  $M$  without boundary condition, a vector bundle  $V$  over  $M$  to define functions which carry discrete (spin or gauge) indices, an operator  $D$  of Laplace type over  $V$  and smooth function  $f$  on  $M$ .

There is an asymptotic expansion where the heat kernel coefficients with an odd index  $k = 2j + 1$  vanish  $a_{2j+1}(f, D) = 0$ . On the other hand coefficients with an even index are locally computable in terms of geometric invariants

$$a_k(f, D) = \text{Tr}_V \left( \int_M d^n x \sqrt{g} (f(x) a_k(x; D)) \right) = \quad (2.23)$$

$$= \sum_I \text{Tr}_V \left( \int_M d^n x \sqrt{g} (f u^I \mathcal{A}_k^I(D)) \right). \quad (2.24)$$

We denote  $\mathcal{A}_k^I$  as all possible independent invariants of dimension  $k$ , and  $u^I$  are constants. The invariants are constructed from  $E, \Omega, R_{\mu\nu\rho\sigma}$  and their derivatives. If  $E$  has dimension two, then the derivative has dimension one. So if  $k = 2$  there are only two independent invariants,  $E$  and  $R$ . This corresponds to the statement  $a_{2j+1} = 0$ .

If we consider  $M = M_1 \times M_2$  with coordinates  $x_1$  and  $x_2$  and a decomposed Laplace style operator  $D = D_1 \otimes 1 + 1 \otimes D_2$  we can separate functions acting on operators and on coordinates linearly by the following

$$e^{-tD} = e^{-tD_1} \otimes e^{-tD_2}, \quad (2.25)$$

$$f(x_1, x_2) = f_1(x_1) f_2(x_2), \quad (2.26)$$

$$(2.27)$$

thus the heat kernel coefficients are separated by

$$a_k(x; D) = \sum_{p+q=k} a_p(x_1; D_1) a_q(x_2; D_2) \quad (2.28)$$

If we know the eigenvalues of  $D_1$  are known,  $l^2, l \in \mathbb{Z}$ , we can obtain the heat kernel asymmetries with the Poisson summation formula giving us an approximation in the order of  $e^{-1/t}$

$$K(t, D_1) = \sum_{l \in \mathbb{Z}} e^{-tl^2} = \sqrt{\frac{\pi}{t}} \sum_{l \in \mathbb{Z}} e^{-\frac{\pi^2 l^2}{t}} = \quad (2.29)$$

$$\simeq \sqrt{\frac{\pi}{t}} + \mathcal{O}(e^{-1/t}). \quad (2.30)$$

The exponentially small terms have no effect on the heat kernel coefficients and that the only nonzero coefficient is  $a_0(1, D_1) = \sqrt{\pi}$ , therefore the heat coefficients can be written as

$$a_k(f(x^2), D) = \sqrt{\pi} \int_{M_2} d^{n-1} x \sqrt{g} \sum_I \text{Tr}_V \left( f(x^2) u_{(n-1)}^I \mathcal{A}_n^I(D_2) \right). \quad (2.31)$$

Because all of the geometric invariants associated with  $D$  are in the  $D_2$  part, they are independent of  $x_1$ . Ultimately meaning we are free to choose  $M_1$ . For  $M_1 = S^1$  with

$x \in (0, 2\pi)$  and  $D_1 = -\partial_{x_1}^2$  we can rewrite the heat kernel coefficients into

$$a_k(f(x_2), D) = \int_{S^1 \times M_2} d^n x \sqrt{g} \sum_I \text{Tr}_V(f(x_2) u_{(n)}^I \mathcal{A}_k^I(D_2)) = \quad (2.32)$$

$$= 2\pi \int_{M_2} d^n x \sqrt{g} \sum_I \text{Tr}_V(f(x_2) u_{(n)}^I \mathcal{A}_k^I(D_2)). \quad (2.33)$$

Computing the two equations above we see that

$$u_{(n)}^I = \sqrt{4\pi} u_{(n+1)}^I \quad (2.34)$$

#### 2.1.4 Heat Kernel Coefficients

To calculate the heat kernel coefficients we need the following variational equations

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} a_k(1, e^{-2\varepsilon f} D) = (n-k) a_k(f, D), \quad (2.35)$$

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} a_k(1, D - \varepsilon F) = a_{k-2}(F, D), \quad (2.36)$$

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} a_k(e^{-2\varepsilon f} F, e^{-2\varepsilon f} D) = 0. \quad (2.37)$$

Let us explain the equations above. To get the first equation 2.35 we differentiate

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \text{Tr}(\exp(-e^{-2\varepsilon f} t D)) = \text{Tr}(2ft D e^{-tD}) = -2t \left. \frac{d}{dt} \right|_{t=0} \text{Tr}(f e^{-tD}) \quad (2.38)$$

then we expand both sides in  $t$  and get 2.35. Equation 2.36 is derived similarly.

For equation 2.37 we consider the following operator

$$D(\varepsilon, \delta) = e^{-2\varepsilon f} (D - \delta F) \quad (2.39)$$

for  $k = n$  we use equation 2.35 and we get

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} a_n(1, D(\varepsilon, \delta)) = 0, \quad (2.40)$$

then we take the variation in terms of  $\delta$ , evaluated at  $\delta = 0$  and swap the differentiation, allowed by theorem of Schwarz

$$0 = \left. \frac{d}{d\delta} \right|_{\delta=0} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} a_n(1, D(\varepsilon, \delta)) = \quad (2.41)$$

$$= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \left. \frac{d}{d\delta} \right|_{\delta=0} a_n(1, D(\varepsilon, \delta)) = \quad (2.42)$$

$$= a_{n-2}(e^{-2\varepsilon f} F, e^{-2\varepsilon f} D), \quad (2.43)$$

which gives us equation 2.37.

Now that we have established the ground basis, we can calculate the constants  $u^I$ , and by that the first three heat kernel coefficients read

$$a_0(f, D) = (4\pi)^{-n/2} \int_M d^n x \sqrt{g} \text{Tr}_V(a_0 f), \quad (2.44)$$

$$a_2(f, D) = (4\pi)^{-n/2} \frac{1}{6} \int_M d^n x \sqrt{g} \text{Tr}_V(f \alpha_1 E + \alpha_2 R), \quad (2.45)$$

$$a_4(f, D) = (4\pi)^{-n/2} \frac{1}{360} \int_M d^n x \sqrt{g} \text{Tr}_V(f(\alpha_3 E_{,kk} + \alpha_4 R E + \alpha_5 E^2 \alpha_6 R_{,kk} + \quad (2.46)$$

$$+ \alpha_7 R^2 + \alpha_8 R_{ij} R_{ij} + \alpha_9 R_{ijkl} R_{ijkl} + \alpha_{10} \Omega_{ij} \Omega_{ij})), \quad (2.47)$$

where the comma subscript , denotes the derivative and constants  $\alpha_I$  do not depend on the dimension of the Manifold and we can compute them with our variational identities.

The first coefficient  $\alpha_0$  can be read from the heat kernel expansion of the Laplacian on  $S^1$  (above),  $\alpha_0 = 1$ . For  $\alpha_1$  we use 2.36, the coefficient  $k = 2$  is

$$\frac{1}{6} \int_M d^n x \sqrt{g} \text{Tr}_V(\alpha_1 F) = \int_M d^n x \sqrt{g} \text{Tr}_V(F), \quad (2.48)$$

which means  $\alpha_1 = 6$ . Looking at the coefficient  $k = 4$  we have

$$\frac{1}{360} \int_M d^n x \sqrt{g} \text{Tr}_V(\alpha_4 F R + 2\alpha_5 F E) = \frac{1}{6} \int_M d^n x \sqrt{g} \text{Tr}_V(\alpha_1 F E + \alpha_2 F R), \quad (2.49)$$

thus  $\alpha_4 = 60\alpha_2$  and  $\alpha_5 = 180$ .

By applying 2.37 to  $n = 4$  we get

$$\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} a_2(e^{-2\varepsilon f} F, e^{-2\varepsilon f} D) = 0. \quad (2.50)$$

Collecting the terms with  $\text{Tr}_V(\int_M d^n x \sqrt{g} (F f_{,jj}))$  we obtain  $\alpha_1 = 6\alpha_2$ , that is  $\alpha_2 = 1$ , so  $\alpha_4 = 60$ .

Now we let  $M = M_1 \times M_2$  and split  $D = -\Delta_1 - \Delta_2$ , where  $\Delta_{1/2}$  are Laplacians for  $M_1, M_2$ . This allows us to decompose the heat kernel coefficient for  $k = 4$  into

$$a_4(1, -\Delta_1 - \Delta_2) = a_4(1, -\Delta_1) a_0(1, -\Delta_2) + a_2(1, -\Delta_1) a_2(1, -\Delta_2) \quad (2.51)$$

$$+ a_0(1, -\Delta_1) a_4(1, -\Delta_2), \quad (2.52)$$

with  $E = 0$  and  $\Omega = 0$  and by calculating the terms with  $R_1 R_2$  (scalar curvature of  $M_{1/2}$ ) we obtain  $\frac{2}{360} \alpha_7 = (\frac{\alpha_2}{6})^2$ , thus  $\alpha_7 = 5$ .

For  $n = 6$  we get

$$0 = \text{Tr}_V \left( \int_M d^n x \sqrt{g} (F(-2\alpha_3 - 10\alpha_4 + 4\alpha_5) f_{,kk} E + \quad (2.53)$$

$$+ (2\alpha_3 + 10\alpha_6) f_{,iijj} + \quad (2.54)$$

$$+ (2\alpha_4 - 2\alpha_6 - 20\alpha_7 - 2\alpha_8) f_{,ii} R \quad (2.55)$$

$$+ (-8\alpha_8 - 8\alpha_6) f_{,ij} R_{ij})), \quad (2.56)$$

we obtain  $\alpha_3 = 60$ ,  $\alpha_6 = 12$ ,  $\alpha_8 = -2$  and  $\alpha_9 = 2$

To get  $\alpha_{10}$  we use the Gauss-Bonnet theorem, ultimately giving us  $\alpha_{10} = 30$ . We leave out this lengthy calculation and refer to [1] for further reading.

Let us summarize our calculations which ultimately give us the following heat kernel coefficients

$$\alpha_0(f, D) = (4\pi)^{-n/2} \int_M d^n x \sqrt{g} \text{Tr}_V(f), \quad (2.57)$$

$$\alpha_2(f, D) = (4\pi)^{-n/2} \frac{1}{6} \int_M d^n x \sqrt{g} \text{Tr}_V(f(6E + R)), \quad (2.58)$$

$$\alpha_4(f, D) = (4\pi)^{-n/2} \frac{1}{360} \int_M d^n x \sqrt{g} \text{Tr}_V(f(60E_{,kk} + 60RE + 180E^2 + \quad (2.59)$$

$$+ 12R_{,kk} + 5R^2 - 2R_{ij}R_{ij}2R_{ijkl}R_{ijkl} + 30\Omega_{ij}\Omega_{ij})). \quad (2.60)$$

$$(2.61)$$

## 2.2 Almost-commutative Manifold

### 2.2.1 Two-Point Space

One of the basics forms of noncommutative space is the Two-Point space  $X := \{x, y\}$ . The Two-Point space can be represented by the following spectral triple

$$F_X := (C(X) = \mathbb{C}^2, H_F, D_F; J_F, \gamma_f). \quad (2.62)$$

Three properties of  $F_X$  stand out. First of all the action of  $C(X)$  on  $H_F$  is faithful for  $\dim(H_F) \geq 2$ , thus we can make a simple choice for the Hilbertspace,  $H_F = \mathbb{C}^2$ . Furthermore  $\gamma_F$  is the  $\mathbb{Z}_2$  grading, which allows us to decompose  $H_F$  into

$$H_F = H_F^+ \oplus H_F^- = \mathbb{C} \oplus \mathbb{C}, \quad (2.63)$$

where

$$H_F^\pm = \{\psi \in H_F \mid \gamma_F \psi = \pm \psi\}, \quad (2.64)$$

are two eigenspaces. And lastly the Dirac operator  $D_F$  lets us interchange between  $H_F^\pm$ ,

$$D_F = \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix}, \quad \text{with } t \in \mathbb{C}. \quad (2.65)$$

The Two-Point space  $F_X$  can only have a real structure if the Dirac operator vanishes, i.e.  $D_F = 0$ . In that case we have KO-dimension of 0, 2 or 6. To elaborate on this, we know that there are two diagram representations of  $F_X$  at  $\underbrace{\mathbb{C} \oplus \mathbb{C}}_{C(X)}$  on  $\underbrace{\mathbb{C} \oplus \mathbb{C}}_{H_F}$ , which are:

$$\begin{array}{ccc} & \mathbf{1} & \mathbf{1} \\ \mathbf{1}^\circ & & \circ \\ \mathbf{1}^\circ & \circ & \end{array} \qquad \begin{array}{ccc} & \mathbf{1} & \mathbf{1} \\ \mathbf{1}^\circ & & \circ \\ \mathbf{1}^\circ & \circ & \end{array}$$

If the Two-Point space  $F_X$  would be a real spectral triple then  $D_F$  can only go vertically or horizontally. This would mean that  $D_F$  vanishes. As for the KO-dimension The diagram on the left has KO-dimension 2 and 6, the diagram on the right 0 and 4. Yet KO-dimension 4 is ruled out because  $\dim(H_F^\pm) = 1$  (see Lemma 3.8 Book), which ultimately means  $J_F^2 = -1$  is not allowed.



### 2.2.2 Product Space

By Extending the Two-Point space with a four dimensional Riemannian spin manifold, we get an almost commutative manifold  $M \times F_X$ , given by

$$M \times F_X = (C^\infty(M, \mathbb{C}^2), L^2(S) \otimes \mathbb{C}^2, D_M \otimes 1; J_M \otimes J_F, \gamma_M \otimes \gamma_F), \quad (2.66)$$

where

$$C^\infty(M, \mathbb{C}^2) \simeq C^\infty(M) \oplus C^\infty(M). \quad (2.67)$$

According to Gelfand duality the algebra  $C^\infty(M, \mathbb{C}^2)$  of the spectral triple corresponds to the space

$$N := M \otimes X \simeq M \sqcup X. \quad (2.68)$$

Keep in mind that we still need to find an appropriate real structure on the Riemannian spin manifold,  $J_M$ . Furthermore total Hilbertspace can be decomposed into  $H = L^2(S) \oplus L^2(S)$ , such that for  $\underbrace{a, b \in C^\infty(M)}_{(a,b) \in C^\infty(N)}$  and  $\underbrace{\psi, \phi \in L^2(S)}_{(\psi, \phi) \in H}$  we have

$$(a, b)(\psi, \phi) = (a\psi, b\phi) \quad (2.69)$$

Along with the decomposition of the total Hilbertspace we can consider a distance formula on  $M \times F_X$  with

$$d_{D_F}(x, y) = \sup \{ |a(x) - a(y)| : a \in A_F, ||[D_F, a]|| \leq 1 \}. \quad (2.70)$$

To calculate the distance between two points on the Two-Point space  $X = \{x, y\}$ , between  $x$  and  $y$ , we consider an  $a \in \mathbb{C}^2 = C(X)$ , which is specified by two complex numbers  $a(x)$  and  $a(y)$ . Then we simplify the commutator inequality in [2.70](#)

$$|[D_F, a]| = |(a(y) - a(x)) \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix}| \leq 1, \quad (2.71)$$

$$\Leftrightarrow |a(y) - a(x)| \leq \frac{1}{|t|}, \quad (2.72)$$

and the supremum gives us the distance

$$d_{D_F}(x, y) = \frac{1}{|t|}. \quad (2.73)$$

An interesting observation here is that, if the Riemannian spin manifold can be represented by a real spectral triple then a real structure  $J_M$  exists, then it follows that  $t = 0$  and the distance becomes infinite. This is a purely mathematical observation and has no physical meaning.

We can also construct a distance formula on  $N$  (in reference to a point  $p \in M$ ) between two points on  $N = M \times X$ ,  $(p, x)$  and  $(p, y)$ . Then an  $a \in C^\infty(N)$  is determined by  $a_x(p) := a(p, x)$  and  $a_y(p) := a(p, y)$ . The distance between these two points is

$$d_{D_F \otimes 1}(n_1, n_2) = \sup \{ |a(n_1) - a(n_2)| : a \in A, |[D \otimes 1, a]| \leq 1 \}. \quad (2.74)$$

On the other hand if we consider  $n_1 = (p, x)$  and  $n_2 = (q, x)$  for  $p, q \in M$  then

$$d_{D_M \otimes 1}(n_1, n_2) = |a_x(p) - a_x(q)| \quad \text{for } a_x \in C^\infty(M) \quad \text{with } \|[D \otimes 1, a_x]\| \leq 1 \quad (2.75)$$

The distance formula turns out to be the geodesic distance formula

$$d_{D_M \otimes 1}(n_1, n_2) = d_g(p, q), \quad (2.76)$$

which is to be expected since we are only looking at the manifold. However if  $n_1 = (p, x)$  and  $n_2 = (q, y)$  then the two conditions are

$$\|[D_M, a_x]\| \leq 1, \quad \text{and} \quad (2.77)$$

$$\|[D_M, a_y]\| \leq 1. \quad (2.78)$$

These conditions have no restriction which results in the distance being infinite! And  $N = M \times X$  is given by two disjoint copies of  $M$  which are separated by infinite distance

The distance is only finite if  $[D_F, a] < 1$ . In this case the commutator generates a scalar field and the finiteness of the distance is related to the existence of scalar fields.

### 2.2.3 $U(1)$ Gauge Group

To get a insight into the physical properties of the almost commutative manifold  $M \times F_X$ , that is to calculate the spectral action, we need to determine the corresponding Gauge theory. For this we set of with simple definitions and important propositions to help us break down and search for the gauge group of the Two-Point  $F_X$  space which we then extend to  $M \times F_X$ . We will only be diving superficially into this chapter, for further reading we refer to [2].

**Definition 1.** Gauge Group of a real spectral triple is given by

$$\mathfrak{B}(A, H; J) := \{U = uJuJ^{-1} | u \in U(A)\} \quad (2.79)$$

**Definition 2.** A  $*$ -automorphism of a  $*$ -algebra  $A$  is a linear invertible map

$$\alpha : A \rightarrow A \quad \text{with} \quad (2.80)$$

$$\alpha(ab) = \alpha(a)\alpha(b) \quad (2.81)$$

$$\alpha(a)^* = \alpha(a^*) \quad (2.82)$$

The **Group of automorphisms of the  $*$ -Algebra**  $A$  is denoted by  $U(A)$ .

The automorphism  $\alpha$  is called **inner** if

$$\alpha(a) = uau^* \quad \text{for } U(A) \quad (2.83)$$

where  $U(A)$  is

$$U(A) = \{u \in A \mid uu^* = u^*u = 1\} \quad (\text{unitary}) \quad (2.84)$$

The Gauge group of  $F_X$  is given by the quotient  $U(A)/U(A_J)$ . We want a nontrivial Gauge group so we need to choose a  $U(A_J) \neq U(A)$  and  $U((A_F)_{J_F}) \neq U(A_F)$ . We

consider our Two-Point space  $F_X$  to be equipped with a real structure, which means the operator vanishes, and the spectral triple representation is

$$F_X := \left( \mathbb{C}^2, \mathbb{C}^2, D_F = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; J_f = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \gamma_F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right). \quad (2.85)$$

Here  $C$  is the complex conjugation, and  $F_X$  is a real even finite spectral triple (space) of KO-dimension 6.

**Proposition 1.** *The Gauge group of the Two-Point space  $\mathfrak{B}(F_X)$  is  $U(1)$ .*

*Proof.* Note that  $U(A_F) = U(1) \times U(1)$ . We need to show that  $U(A_F) \cap U(A_F)_{J_F} \simeq U(1)$ , such that  $\mathfrak{B}(F) \simeq U(1)$ . So for an element  $a \in \mathbb{C}^2$  to be in  $(A_F)_{J_F}$ , it has to satisfy  $J_F a^* J_F = a$ ,

$$J_F a^* J_F^{-1} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} \begin{pmatrix} \bar{a}_1 & 0 \\ 0 & \bar{a}_2 \end{pmatrix} \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} = \begin{pmatrix} a_2 & 0 \\ 0 & a_1 \end{pmatrix}. \quad (2.86)$$

This can only be the case if  $a_1 = a_2$ . So we have  $(A_F)_{J_F} \simeq \mathbb{C}$ , whose unitary elements from  $U(1)$  are contained in the diagonal subgroup of  $U(A_F)$ .  $\square$

An arbitrary hermitian field  $A_\mu = -ia\partial_\mu b$  is given by two  $U(1)$  Gauge fields  $X_\mu^1, X_\mu^2 \in C^\infty(M, \mathbb{R})$ . However  $A_\mu$  appears in combination  $A_\mu - J_F A_\mu J_F^{-1}$ :

$$A_\mu - J_F A_\mu J_F^{-1} = \begin{pmatrix} X_\mu^1 & 0 \\ 0 & X_\mu^2 \end{pmatrix} - \begin{pmatrix} X_\mu^2 & 0 \\ 0 & X_\mu^1 \end{pmatrix} =: \begin{pmatrix} Y_\mu & 0 \\ 0 & -Y_\mu \end{pmatrix} = Y_\mu \otimes \gamma_F, \quad (2.87)$$

where  $Y_\mu$  the  $U(1)$  Gauge field is defined as

$$Y_\mu := X_\mu^1 - X_\mu^2 \in C^\infty(M, \mathbb{R}) = C^\infty(M, i u(1)). \quad (2.88)$$

**Proposition 2.** *The inner fluctuations of the almost-commutative manifold  $M \times F_X$  are parameterized by a  $U(1)$ -gauge field  $Y_\mu$  as*

$$D \mapsto D' = D + \gamma^\mu Y_\mu \otimes \gamma_F \quad (2.89)$$

*The action of the gauge group  $\mathfrak{B}(M \times F_X) \simeq C^\infty(M, U(1))$  on  $D'$  is implemented by*

$$Y_\mu \mapsto Y_\mu - i u \partial_\mu u^*; \quad (u \in \mathfrak{B}(M \times F_X)). \quad (2.90)$$

### 2.3 Noncommutative Geometry of Electrodynamics

In this chapter we describe Electrodynamics with the almost commutative manifold  $M \times F_X$  and the abelian gauge group  $U(1)$ . We arrive at a unified description of gravity and electrodynamics although in the classical level.

The almost commutative Manifold  $M \times F_X$  describes a local gauge group  $U(1)$ . The inner fluctuations of the Dirac operator relate to  $Y_\mu$  the gauge field of  $U(1)$ . According to the setup we ultimately arrive at two serious problems.

First of all in the Two-Point space  $F_X$ , the operator  $D_F$  must vanish for us to have a real structure. However this implies that the electrons are massless, which would be absurd.

The second problem arises when looking at the Euclidean action for a free Dirac field

$$S = - \int i\bar{\psi}(\gamma^\mu \partial_\mu - m)\psi d^4x, \quad (2.91)$$

where  $\psi$ ,  $\bar{\psi}$  must be considered as independent variables, which means that the fermionic action  $S_f$  needs two independent Dirac spinors. Let us try and construct two independent Dirac spinors with our data. To do this we take a look at the decomposition of the basis and of the total Hilbertspace  $H = L^2(S) \otimes H_F$ . For the orthonormal basis of  $H_F$  we can write  $\{e, \bar{e}\}$ , where  $\{e\}$  is the orthonormal basis of  $H_F^+$  and  $\{\bar{e}\}$  the orthonormal basis of  $H_F^-$ . Accompanied with the real structure we arrive at the following relations

$$J_F e = \bar{e} \quad J_F \bar{e} = e, \quad (2.92)$$

$$\gamma_F e = e \quad \gamma_F \bar{e} = \bar{e}. \quad (2.93)$$

Along with the decomposition of  $L^2(S) = L^2(S)^+ \oplus L^2(S)^-$  and  $\gamma = \gamma_M \otimes \gamma_F$  we can obtain the positive eigenspace

$$H^+ = L^2(S)^+ \otimes H_F^+ \oplus L^2(S)^- \otimes H_F^-. \quad (2.94)$$

So, for a  $\xi \in H^+$  we can write

$$\xi = \psi_L \otimes e + \psi_R \otimes \bar{e} \quad (2.95)$$

where  $\psi_L \in L^2(S)^+$  and  $\psi_R \in L^2(S)^-$  are the two Weyl spinors. We denote that  $\xi$  is only determined by one Dirac spinor  $\psi := \psi_L + \psi_R$ , **but we require two independent spinors**. Our conclusion is that the definition of the fermionic action gives too much restrictions to the Two-Point space  $F_X$ .

### 2.3.1 The Finite Space

To solve the two problems we simply enlarge (double) the Hilbertspace. This is visualized by introducing multiplicities in Krajewski Diagrams which will also allow us to choose a nonzero Dirac operator that will connect the two vertices and preserve real structure making our particles massive and bringing anti-particles into the mix.

We start of with the same algebra  $C^\infty(M, \mathbb{C}^2)$ , corresponding to space  $N = M \times X$ . The Hilbertspace describes four particles, meaning it has four orthonormal basis elements. It describes **left handed electrons** and **right handed positrons**. Pointing this out, we have  $\{ \underbrace{e_R, e_L}_{\text{left-handed}}, \underbrace{\bar{e}_R, \bar{e}_L}_{\text{right-handed}} \}$  the orthonormal basis for  $H_F = \mathbb{C}^4$ . Accompanied

with the real structure  $J_F$ , which allows us to interchange particles with antiparticles by the following equations

$$J_F e_R = \bar{e}_R, \quad (2.96)$$

$$J_F e_L = \bar{e}_L, \quad (2.97)$$

$$\gamma_F e_R = -e_R, \quad (2.98)$$

$$\gamma_F e_L = e_L \quad (2.99)$$

$$(2.100)$$

where  $J_F$  and  $\gamma_F$  have to following properties

$$J_F^2 = 1, \quad (2.101)$$

$$J_F \gamma_F = -\gamma_F J_F. \quad (2.102)$$

By means of  $\gamma_F$  we have two options to decompose the total Hilbertspace  $H$ , firstly into

$$H_F = \underbrace{H_F^+}_{\text{ONB } \{e_L, \bar{e}_L\}} \oplus \underbrace{H_F^-}_{\text{ONB } \{e_R, \bar{e}_R\}}, \quad (2.103)$$

or alternatively into the eigenspace of particles and their antiparticles (electrons and positrons) which is preferred in literature and which we will use going further

$$H_F = \underbrace{H_e}_{\text{ONB } \{e_L, e_R\}} \oplus \underbrace{H_{\bar{e}}}_{\text{ONB } \{\bar{e}_L, \bar{e}_R\}}. \quad (2.104)$$

Here ONB means orthonormal basis.

The action of  $a \in A = \mathbb{C}^2$  on  $H$  with respect to the ONB  $\{e_L, e_R, \bar{e}_L, \bar{e}_R\}$  is represented by

$$a = (a_1, a_2) \mapsto \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix} \quad (2.105)$$

Do note that this action commutes with the grading and that  $[a, b^\circ] = 0$  with  $b := J_F b^* J_F$  because both the left and the right action is given by diagonal matrices by equation 2.105. Note that we are still left with  $D_F = 0$  and the following spectral triple

$$\left( \mathbb{C}^2, \mathbb{C}^2, D_F = 0; J_F = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \gamma_F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right). \quad (2.106)$$

It can be represented in the following Krajewski diagram, with two nodes of multiplicity two

$$\begin{array}{cc} \mathbf{1} & \mathbf{1} \\ \mathbf{1}^\circ & \odot \\ \mathbf{1}^\circ & \odot \end{array}$$

### 2.3.2 A noncommutative Finite Dirac Operator

To extend our spectral triple with a non-zero Operator, we need to take a closer look at the Krajewski diagram above. Notice that edges only exist between multiple vertices, meaning we can construct a Dirac operator mapping between the two vertices. The operator can be represented by the following matrix

$$D_F = \begin{pmatrix} 0 & d & 0 & 0 \\ \bar{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{d} \\ 0 & 0 & d & 0 \end{pmatrix} \quad (2.107)$$

We can now define the finite space  $F_{ED}$ .

$$F_{ED} := (\mathbb{C}^2, \mathbb{C}^4, D_F; J_F, \gamma_F) \quad (2.108)$$

where  $J_F$  and  $\gamma_F$  are like in equation 2.106 and  $D_F$  from equation 2.107.

### 2.3.3 Almost commutative Manifold of Electrodynamics

The almost commutative manifold  $M \times F_{ED}$  has KO-dimension 2, and is represented by the following spectral triple

$$M \times F_{ED} := (C^\infty(M, \mathbb{C}^2), L^2(S) \otimes \mathbb{C}^4, D_M \otimes 1 + \gamma_M \otimes D_F; J_M \otimes J_F, \gamma_M \otimes \gamma_F) \quad (2.109)$$

The algebra didn't change, thus we can decompose it like before

$$C^\infty(M, \mathbb{C}^2) = C^\infty(M) \oplus C^\infty(M) \quad (2.110)$$

As for the Hilbertspace, we can decompose it in the following way

$$H = (L^2(S) \otimes H_e) \oplus (L^2(S) \otimes H_{\bar{e}}). \quad (2.111)$$

Note that the one component of the algebra is acting on  $L^2(S) \otimes H_e$ , and the other one acting on  $L^2(S) \otimes H_{\bar{e}}$ . In other words the components of the decomposition of both the algebra and the Hilbertspace match by the action of the algebra.

The derivation of the gauge theory is the same for  $F_{ED}$  as for the Two-Point space  $F_X$ . We have  $\mathfrak{B}(F) \simeq U(1)$  and for an arbitrary gauge field  $B_\mu = A_\mu - J_F A_\mu J_F^{-1}$  we can write

$$B_\mu = \begin{pmatrix} Y_\mu & 0 & 0 & 0 \\ 0 & Y_\mu & 0 & 0 \\ 0 & 0 & Y_\mu & 0 \\ 0 & 0 & 0 & Y_\mu \end{pmatrix} \quad \text{for } Y_\mu(x) \in \mathbb{R}. \quad (2.112)$$

We have one single  $U(1)$  gauge field  $Y_\mu$ , carrying the action of the gauge group

$$\mathfrak{B}(M \times F_{ED}) \simeq C^\infty(M, U(1)) \quad (2.113)$$

The space  $N = M \times X$  consists of two copies of  $M$ . If  $D_F = 0$  we have infinite distance between the two copies. Now have hacked the spectral triple to have nonzero Dirac operator  $D_F$ . The new Dirac operator still has a commuting relation with the algebra  $[D_F, a] = 0 \forall a \in A$ , and we should note that the distance between the two copies of  $M$  is still infinite. This is purely an mathematically abstract observation and doesn't affect physical results.

### 2.3.4 Spectral Action

In this chapter we bring all our results together to establish an Action functional to describe a physical system. It turns out that the Lagrangian of the almost commutative manifold  $M \times F_{ED}$  corresponds to the Lagrangian of Electrodynamics on a curved background manifold (+ gravitational Lagrangian), consisting of the spectral action  $S_b$  (bosonic) and of the fermionic action  $S_f$ .

The simplest spectral action of a spectral triple  $(A, H, D)$  is given by the trace of a function of  $D$ . We also consider inner fluctuations of the Dirac operator  $D_\omega = D + \omega + \varepsilon' J \omega J^{-1}$  where  $\omega = \omega^* \in \Omega_D^1(A)$ .

**Definition 3.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a suitable function **positive and even**. The spectral action is then

$$S_b[\omega] := \text{Tr}(f(\frac{D_\omega}{\Lambda})) \quad (2.114)$$

where  $\Lambda$  is a real cutoff parameter. The minimal condition on  $f$  is that  $f(\frac{D_\omega}{\Lambda})$  is a trace class operator. A trace class operator is a compact operator with a well defined finite trace independent of the basis. The subscript  $b$  in  $S_b$  stands for bosonic, because in physical applications  $\omega$  will describe bosonic fields.

In addition to the bosonic action  $S_b$  we can define a topological spectral action  $S_{top}$ . Leaning on the grading  $\gamma$  the topological spectral action is

$$S_{top}[\omega] := \text{Tr}(\gamma f(\frac{D_\omega}{\Lambda})). \quad (2.115)$$

**Definition 4.** The fermionic action is defined by

$$S_f[\omega, \psi] = (J\tilde{\psi}, D_\omega \tilde{\psi}) \quad (2.116)$$

with  $\tilde{\psi} \in H_{cl}^+ := \{\tilde{\psi} : \psi \in H^+\}$ , where  $H_{cl}^+$  is a set of Grassmann variables in  $H$  in the +1-eigenspace of the grading  $\gamma$ .

**APPENDIX??** Grassmann variables are a set of Basis vectors of a vector space, they form a unital algebra over a vector field  $V$ , where the generators are anti commuting, that is for Grassmann variables  $\theta_i, \theta_j$  we have

$$\theta_i \theta_j = -\theta_j \theta_i \quad (2.117)$$

$$\theta_i x = x \theta_j \quad x \in V \quad (2.118)$$

$$(\theta_i)^2 = 0 \quad (\theta_i \theta_i = -\theta_i \theta_i) \quad (2.119)$$

**Proposition 3.** *The spectral action of the almost commutative manifold  $M$  with  $\dim(M) = 4$  with a fluctuated Dirac operator is*

$$\text{Tr}(f \frac{D_\omega}{\Lambda}) \sim \int_M \mathcal{L}(g_{\mu\nu}, B_\mu, \Phi) \sqrt{g} d^4x + O(\Lambda^{-1}), \quad (2.120)$$

where

$$\mathcal{L}(g_{\mu\nu}, B_\mu, \Phi) = N \mathcal{L}_M(g_{\mu\nu}) \mathcal{L}_B(B_\mu) + \mathcal{L}_\phi(g_{\mu\nu}, B_\mu, \Phi). \quad (2.121)$$

The Lagrangian  $\mathcal{L}_M$  is of the spectral triple, represented by the following term  $(C^\infty(M), L^2(S), D_M)$

$$\mathcal{L}_M(g_{\mu\nu}) := \frac{f_4 \Lambda^4}{2\pi^2} - \frac{f_2 \Lambda^2}{24\pi^2} s - \frac{f(0)}{320\pi^2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}, \quad (2.122)$$

here  $C^{\mu\nu\rho\sigma}$  is defined in terms of the Riemannian curvature tensor  $R_{\mu\nu\rho\sigma}$  and the Ricci tensor  $R_{\nu\sigma} = g^{\mu\rho} R_{\mu\nu\rho\sigma}$ . The kinetic term of the gauge field is described by the Lagrangian  $\mathcal{L}_B$ , which takes the following shape

$$\mathcal{L}_B(B_\mu) := \frac{f(0)}{24\pi^2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}). \quad (2.123)$$

Lastly  $\mathcal{L}_\phi$  is the scalar-field Lagrangian with a boundary term, given by

$$\mathcal{L}_\phi(g_{\mu\nu}, B_\mu, \Phi) := -\frac{2f_2 \Lambda^2}{4\pi^2} \text{Tr}(\Phi^2) + \frac{f(0)}{8\pi^2} \text{Tr}(\Phi^4) + \frac{f(0)}{24\pi^2} \Delta(\text{Tr}(\Phi^2)) \quad (2.124)$$

$$+ \frac{f(0)}{48\pi^2} s \text{Tr}(\Phi^2) + \frac{f(0)}{8\pi^2} \text{Tr}((D_\mu \Phi)(D^\mu \Phi)). \quad (2.125)$$

*Proof.* The dimension of the manifold  $M$  is  $\dim(M) = \text{Tr}(id) = 4$ . For an  $x \in M$ , we have an asymptotic expansion of the term  $\text{Tr}(f(\frac{D_\omega}{\Lambda}))$  as  $\Lambda$  goes to infinity, which can be written as

$$\text{Tr}(f(\frac{D_\omega}{\Lambda})) \simeq 2f_4\Lambda^4 a_0(D_\omega^2) + 2f_2\Lambda^2 a_2(D_\omega^2) \quad (2.126)$$

$$+ f(0)a_4(D_\omega^4) + O(\Lambda^{-1}). \quad (2.127)$$

We have to note here that the heat kernel coefficients are zero for uneven  $k$ , and they are dependent on the fluctuated Dirac operator  $D_\omega$ . We can rewrite the heat kernel coefficients in terms of  $D_M$ , for the first two terms  $a_0$  and  $a_2$  we use  $N := \text{Tr}\mathbb{1}_{\mathbb{H}_\mathbb{F}}$  and write

$$a_0(D_\omega^2) = Na_0(D_M^2), \quad (2.128)$$

$$a_2(D_\omega^2) = Na_2(D_M^2) - \frac{1}{4\pi^2} \int_M \text{Tr}(\Phi^2) \sqrt{g} d^4x \quad (2.129)$$

For  $a_4$  we extend in terms of coefficients of  $F$ , **REWRITE: look week9.pdf for the standard version**

$$\frac{1}{360} \text{Tr}(60sF) = -\frac{1}{6} S(Ns + 4\text{Tr}(\Phi^2)) \quad (2.130)$$

$$F^2 = \frac{1}{16} s^2 \otimes 1 + 1 \otimes \Phi^4 - \frac{1}{4} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma F_{\mu\nu} F^{\mu\nu} + \quad (2.131)$$

$$+ \gamma^\mu \gamma^\nu \otimes (D_\mu \Phi)(D_\nu \Phi) + \frac{1}{2} s \otimes \Phi^2 + \text{traceless terms} \quad (2.132)$$

$$\frac{1}{360} \text{Tr}(180F^2) = \frac{1}{8} s^2 N + 2\text{Tr}(\Phi^4) + \text{Tr}(F_{\mu\nu} F^{\mu\nu}) + \quad (2.133)$$

$$+ 2\text{Tr}((D_\mu \Phi)(D^\mu \Phi)) + s\text{Tr}(\Phi^2) \quad (2.134)$$

$$\frac{1}{360} \text{Tr}(-60\Delta F) = \frac{1}{6} \Delta(Ns + 4\text{Tr}(\Phi^2)). \quad (2.135)$$

The cross terms of the trace in  $\Omega_{\mu\nu}^E \Omega^{E\mu\nu}$  vanishes because of the antisymmetric property of the Riemannian curvature tensor, thus we can write

$$\Omega_{\mu\nu}^E \Omega^{E\mu\nu} = \Omega_{\mu\nu}^S \Omega^{S\mu\nu} \otimes 1 - 1 \otimes F_{\mu\nu} F^{\mu\nu} + 2i\Omega_{\mu\nu}^S \otimes F^{\mu\nu}. \quad (2.136)$$

The trace of the cross term  $\Omega_{\mu\nu}^S$  vanishes because

$$\text{Tr}(\Omega_{\mu\nu}^S) = \frac{1}{4} R_{\mu\nu\rho\sigma} \text{Tr}(\gamma^\mu \gamma^\nu) = \frac{1}{4} R_{\mu\nu\rho\sigma} g^{\mu\nu} = 0, \quad (2.137)$$

then the trace of the whole term is given by

$$\frac{1}{360} \text{Tr}(30\Omega_{\mu\nu}^E \Omega^{E\mu\nu}) = \frac{N}{24} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{1}{3} \text{Tr}(F_{\mu\nu} F^{\mu\nu}). \quad (2.138)$$



Finally plugging the results into the coefficient  $a_4$  and simplifying we get

$$a_4(x, D_\omega^4) = Na_4(x, D_M^2) + \frac{1}{4\pi^2} \left( \frac{1}{12} s \text{Tr}(\Phi^2) + \frac{1}{2} \text{Tr}(\Phi^4) \right) \quad (2.139)$$

$$+ \frac{1}{4} \text{Tr}((D_\mu \Phi)(D^\mu \Phi)) + \frac{1}{6} \Delta \text{Tr}(\Phi^2) + \frac{1}{6} \text{Tr}(F_{\mu\nu} F^{\mu\nu}). \quad (2.140)$$

The only thing left is to substitute the heat kernel coefficients into the heat kernel expansion in equation 2.126.  $\square$

### 2.3.5 Fermionic Action

We remind ourselves the definition of the fermionic action in 4 and the manifold we are dealing with in equation 2.109. The Hilbertspace  $H_F$  is separated into the particle-antiparticle states with ONB  $\{e_R, e_L, \bar{e}_R, \bar{e}_L\}$ . The orthonormal basis of  $H_F^+$  is  $\{e_L, \bar{e}_R\}$  and consequently for  $H_F^-$ ,  $\{e_R, \bar{e}_L\}$ . We can decompose a spinor  $\psi \in L^2(S)$  in each of the eigenspaces  $H_F^\pm$ ,  $\psi = \psi_R + \psi_L$ . That means for an arbitrary  $\psi \in H^+$  we can write

$$\psi = \chi_R \otimes e_R + \chi_L \otimes e_L + \psi_L \otimes \bar{e}_R + \psi_R \otimes \bar{e}_L, \quad (2.141)$$

where  $\chi_L, \psi_L \in L^2(S)^+$  and  $\chi_R, \psi_R \in L^2(S)^-$ .

Since the fermionic action yields too much restriction on  $F_{ED}$  (modified Two-Point space  $F_X$ ) we redefine it by taking account the fluctuated Dirac operator

$$D_\omega = D_M \otimes i + \gamma^\mu \otimes B_\mu + \gamma_M \otimes D_F. \quad (2.142)$$

The Fermionic Action is

$$S_F = (J\tilde{\xi}, D_\omega \tilde{\xi}) \quad (2.143)$$

for a  $\xi \in H^+$ . Then the straight forward calculation gives

$$\frac{1}{2} (J\tilde{\xi}, D_\omega \tilde{\xi}) = \frac{1}{2} (J\tilde{\xi}, (D_M \otimes i) \tilde{\xi}) \quad (2.144)$$

$$+ \frac{1}{2} (J\tilde{\xi}, (\gamma^\mu \otimes B_\mu) \tilde{\xi}) \quad (2.145)$$

$$+ \frac{1}{2} (J\tilde{\xi}, (\gamma_M \otimes D_F) \tilde{\xi}), \quad (2.146)$$

(note that we add the constant  $\frac{1}{2}$  to the action). For the term in ?? we calculate

$$\frac{1}{2} (J\tilde{\xi}, (D_M \otimes 1) \tilde{\xi}) = \frac{1}{2} (J_M \tilde{\chi}_R, D_M \tilde{\psi}_L) + \frac{1}{2} (J_M \tilde{\chi}_L, D_M \tilde{\psi}_R) + \quad (2.147)$$

$$+ \frac{1}{2} (J_M \tilde{\psi}_L, D_M \tilde{\psi}_R) + \frac{1}{2} (J_M \tilde{\chi}_R, D_M \tilde{\chi}_L) \quad (2.148)$$

$$= (J_M \tilde{\chi}, D_M \tilde{\chi}). \quad (2.149)$$

For the term in 2.145 we have

$$\frac{1}{2} (J\tilde{\xi}, (\gamma^\mu \otimes B_\mu) \tilde{\xi}) = -\frac{1}{2} (J_M \tilde{\chi}_R, \gamma^\mu Y_\mu \tilde{\psi}_R) - \frac{1}{2} (J_M \tilde{\chi}_L, \gamma^\mu Y_\mu \tilde{\psi}_R) + \quad (2.150)$$

$$+ \frac{1}{2} (J_M \tilde{\psi}_L, \gamma^\mu Y_\mu \tilde{\chi}_R) + \frac{1}{2} (J_M \tilde{\psi}_R, \gamma^\mu Y_\mu \tilde{\chi}_L) = \quad (2.151)$$

$$= -(J_M \tilde{\chi}, \gamma^\mu Y_\mu \tilde{\psi}). \quad (2.152)$$

And for 2.146 we can write

$$\frac{1}{2}(J\tilde{\xi}, (\gamma_M \otimes D_F)\tilde{\xi}) = +\frac{1}{2}(J_M\tilde{\chi}_R, d\gamma_M\tilde{\chi}_R) + \frac{1}{2}(J_M\tilde{\chi}_L, \bar{d}\gamma_M\tilde{\chi}_L) + \quad (2.153)$$

$$+ \frac{1}{2}(J_M\tilde{\chi}_L, \bar{d}\gamma_M\tilde{\chi}_L) + \frac{1}{2}(J_M\tilde{\chi}_R, d\gamma_M\tilde{\chi}_R) = \quad (2.154)$$

$$= i(J_M\tilde{\chi}, m\tilde{\psi}). \quad (2.155)$$

A small problem arises, we obtain a complex mass parameter  $d$ , but we can write  $d := im$  for  $m \in \mathbb{R}$ , which stands for the real mass.

Finally the fermionic action of  $M \times F_{ED}$  takes the form

$$S_f = -i(J_M\tilde{\chi}, \gamma(\nabla_\mu^S - i\Gamma_\mu)\tilde{\Psi}) + (S_M\tilde{\chi}_L, \bar{d}\tilde{\psi}_L) - (J_M\tilde{\chi}_R, d\tilde{\psi}_R). \quad (2.156)$$

Ultimately we arrive at the full Lagrangian of  $M \times F_{ED}$ , which is the sum of purely gravitational Lagrangian

$$\mathcal{L}_{grav}(g_{\mu\nu}) = 4\mathcal{L}_M(g_{\mu\nu}) + \mathcal{L}_\phi(g_{\mu\nu}), \quad (2.157)$$

and the Lagrangian of electrodynamics

$$\mathcal{L}_{ED} = -i\left\langle J_M\tilde{\chi}, (\gamma^\mu(\nabla_\mu^S - iY_\mu) - m)\tilde{\psi} \right\rangle + \frac{f(0)}{6\pi^2}Y_{\mu\nu}Y^{\mu\nu}. \quad (2.158)$$

### 3 Conclusion

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### 4 Acknowledgment

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## References

- [1] D.V. Vassilevich. “Heat kernel expansion: user’s manual”. In: *Physics Reports* 388.5-6 (Dec. 2003), pp. 279–360. ISSN: 0370-1573. DOI: [10.1016/j.physrep.2003.09.002](https://doi.org/10.1016/j.physrep.2003.09.002). URL: <http://dx.doi.org/10.1016/j.physrep.2003.09.002>.
- [2] Walter D. van Suijlekom. *Noncommutative Geometry and Particle Physics*. eng. Springer Netherlands, 2015.
- [3] Koen van den Dungen and Walter van Suijlekom. “Electrodynamics from non-commutative geometry”. In: *Journal of Noncommutative Geometry* 7.2 (2013), pp. 433–456. ISSN: 1661-6952. DOI: [10.4171/jncg/122](https://doi.org/10.4171/jncg/122). URL: <http://dx.doi.org/10.4171/JNCG/122>.
- [4] Howard Georgi. *Lie Algebras in Particle Physics From Isospin to Unified Theories*. eng. 2. ed. Westview Press, 1999.
- [5] Thijs van den Broek. *NCG in 4 pages: A very brief introduction into noncommutative geometry*. 2013. URL: <http://www.noncommutativegeometry.nl/wp-content/uploads/2013/09/NCGin4pages1.pdf> (visited on 04/13/2021).
- [6] Peter Bongaarts. *A short introduction to noncommutative geometry*. 2004. URL: <http://www.lorentz.leidenuniv.nl/modphys/nCG-lecture.pdf> (visited on 04/13/2021).