University of Vienna
Faculty of Physics

Notes on
Noncommutative Geometry and Particle Phyiscs

Milutin Popovic
Supervisor: Dr. Lisa Glaser

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Contents

1 Spectral Action of the Fluctuated Dirac Operator

Proposition 1. *The spectral action of the almost commutative manifold M with $\dim(M) = 4$ with a fluctuated Dirac operator is.*

$$Tr(f\frac{D_\omega}{\Lambda}) \sim \int_M \mathscr{L}(g_{\mu\nu}, B_\mu, \Phi)\sqrt{g}\, d^4x + O(\Lambda^{-1}) \tag{1}$$

with

$$\mathscr{L}(g_{\mu\nu}, B_\mu, \Phi) = N\mathscr{L}_M(g_{\mu\nu})\mathscr{L}_B(B_\mu) + \mathscr{L}_\phi(g_{\mu\nu}, B_\mu, \Phi) \tag{2}$$

where $N = 4$ and $\mathscr{L}_M$ is the Lagrangian of the spectral triple $(C^\infty(M), L^2(S), D_M)$

$$\mathscr{L}_M(g_{\mu\nu}) := \frac{f_4\Lambda^4}{2\pi^2} - \frac{f_2\Lambda^2}{24\pi^2}s - \frac{f(0)}{320\pi^2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}. \tag{3}$$

Here $C^{\mu\nu\rho\sigma}$ is defined in terms of the Riemannian curvature tensor $R_{\mu\nu\rho\sigma}$ and the Ricci tensor $R_{\nu\sigma} = g^{\mu\rho}R_{\mu\nu\rho\sigma}$.

Furthermore $\mathscr{L}_B$ describes the kinetic term of the gauge field

$$\mathscr{L}_B(B_\mu) := \frac{f(0)}{24\pi^2}Tr(F_{\mu\nu}F^{\mu\nu}). \tag{4}$$

Last $\mathscr{L}_\phi$ is the scalar-field Lagrangian with a boundary term.

$$\mathscr{L}_\phi(g_{\mu\nu}, B_\mu, \Phi) := -\frac{2f_2\Lambda^2}{4\pi^2}Tr(\Phi^2) + \frac{f(0)}{8\pi^2}Tr(\Phi^4) + \frac{f(0)}{24\pi^2}\Delta(Tr(\Phi^2)) \tag{5}$$

$$+\frac{f(0)}{48\pi^2}sTr(\Phi^2)\frac{f(0)}{8\pi^2}Tr((D_\mu\Phi)(D^\mu\Phi)). \tag{6}$$

Proof. The dimension of our manifold M is $\dim(M) = \mathrm{Tr}(id) = 4$. Let us take a $x \in M$, we have an asymtotic expansion of $\mathrm{Tr}(f(\frac{D_\omega}{\Lambda}))$ as $\Lambda \to \infty$

$$\mathrm{Tr}(f(\frac{D_\omega}{\Lambda})) \simeq 2f_4\Lambda^4 a_0(D_\omega^2) + 2f_2\Lambda^2 a_2(D_\omega^2) \tag{7}$$

$$+ f(0)a_4(D_\omega^4) + O(\Lambda^{-1}). \tag{8}$$

Note that the heat kernel coefficients are zero for uneven k, furthermore they are dependent on the fluctuated Dirac operator D_ω. We can rewrite the heat kernel coefficients in terms of D_M, for the first two we note that $N := \mathrm{Tr}\mathbb{1}_{\mathbb{H}_\mathbb{F}})$

$$a_0(D_\omega^2) = Na_0(D_M^2) \tag{9}$$

$$a_2(D_\omega^2 = Na_2(D_M^2) - \frac{1}{4\pi^2}\int_M \mathrm{Tr}(\Phi^2)\sqrt{g}d^4x \tag{10}$$

For a_4 we need to extend in terms of coefficients of F, look week9.pdf for the standard version,

$$\frac{1}{360}\mathrm{Tr}(60sF) = -\frac{1}{6}S(Ns+4\mathrm{Tr}(\Phi^2)) \tag{11}$$

$$F^2 = \frac{1}{16}s^2\otimes 1 + 1\otimes\Phi^4 - \frac{1}{4}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma F_{\mu\nu}F^{\mu\nu} + \tag{12}$$

$$+\gamma^\mu\gamma^\nu\otimes(D_\mu\Phi)(D_\nu\Phi)+\frac{1}{2}s\otimes\Phi^2 + \text{ traceless terms} \tag{13}$$

$$\frac{1}{360}\mathrm{Tr}(180F^2) = \frac{1}{8}s^2N + 2\mathrm{Tr}(\Phi^4)+\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})+ \tag{14}$$

$$+2\mathrm{Tr}((D_\mu\Phi)(D^\mu\Phi))+s\mathrm{Tr}(\Phi^2) \tag{15}$$

$$\frac{1}{360}\mathrm{Tr}(-60\Delta F) = \frac{1}{6}\Delta(Ns+4\mathrm{Tr}(\Phi^2)). \tag{16}$$

Now for the cross terms of $\Omega^E_{\mu\nu}\Omega^{E\mu\nu}$ the trace vanishes because of the anti-symmetric properties of the Riemannian curvature Tensor

$$\Omega^E_{\mu\nu}\Omega^{E\mu\nu} = \Omega^S_{\mu\nu}\Omega^{S\mu\nu}\otimes 1 - 1\otimes F_{\mu\nu}F^{\mu\nu} + 2i\Omega^S_{\mu\nu}\otimes F^{\mu\nu} \tag{17}$$

the trace of the cross term vanishes because

$$\mathrm{Tr}(\Omega^S_{\mu\nu} = \frac{1}{4}R_{\mu\nu\rho\sigma}\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \frac{1}{4}R_{\mu\nu\rho\sigma}g^{\mu\nu} = 0 \tag{18}$$

and the trace of the whole term is

$$\frac{1}{360}\mathrm{Tr}(30\Omega^E_{\mu\nu}\Omega^{E\mu\nu}) = \frac{N}{24}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{1}{3}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}). \tag{19}$$

Plugging the results into a_4 and simplifying we can write

$$a_4(x,D^4_\omega) = Na_4(x,D^2_M)+\frac{1}{4\pi^2}\Bigg(\frac{1}{12}s\mathrm{Tr}(\Phi^2)+\frac{1}{2}\mathrm{Tr}(\Phi^4) \tag{20}$$

$$+\frac{1}{4}\mathrm{Tr}((D_\mu\Phi)(D^\mu\Phi))+\frac{1}{6}\Delta\mathrm{Tr}(\Phi^2)+\frac{1}{6}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\Bigg) \tag{21}$$

The only thing left is to plug in the heat kernel coefficients into the heat kernel expansion above. □

2 Fermionic Action

A quick reminder with what we are dealing with, the fermionic action is defined in the following way.

Definition 1. The fermionic action is defined by

$$S_f[\omega,\psi] = (J\tilde{\psi}, D_\omega\tilde{\psi}) \tag{22}$$

with $\tilde{\psi}\in H^+_{cl} := \{\tilde{\psi} : \psi\in H^+\}$. H^+_{cl} is the set of Grassmann variables in H in the +1-eigenspace of the grading γ.

The almostcommutative Manifold we are dealing with is the following

$$M \times F_{ED} := \big(C^\infty(M,\mathbb{C}^2),\ L^2(S)\otimes\mathbb{C}^4,\ D_M\otimes 1+\gamma_M\otimes D_F;\ J_M\otimes J_F,\ \gamma_M\otimes\gamma_F\big). \quad (23)$$

where:

$$C^\infty(M,\mathbb{C}^2) = C^\infty(M)\otimes C^\infty(M) \qquad \mathscr{H} = \mathscr{H}^+\otimes\mathscr{H}^- \quad (24)$$

$$\mathscr{H} = L^2(S)^+\otimes H_F^+ \oplus L^2(S)^-\otimes H_F^-. \quad (25)$$

Where H_F is separated into the particle-anitparticle states with ONB $\{e_R, e_L, \bar{e}_R, \bar{e}_L\}$. The ONB of H_F^+ is $\{e_L, \bar{e}_R\}$ and for H_F^- we have $\{e_R, \bar{e}_L\}$. Furthermore we can decompose a spinor $\psi \in L^2(S)$ for each of the eigenspaces $H_F^\pm$, $\psi = \psi_R\psi_L$. Thus we can write for an arbitrary $\psi \in \mathscr{H}^+$

$$\psi = \chi_R\otimes e_R + \chi_L\otimes e_L + \psi_L\otimes\bar{e}_R\psi_R\otimes\bar{e}_L \quad (26)$$

for $\chi_L, \psi_L \in L^2(S)^+$ and $\chi_R, \psi_R \in L^2(S)^-$.

Proposition 2. *We can define the action of the fermionic art of $M\times F_{ED}$ in the following way*

$$S_f = -i\big(J_M\tilde{\chi}, \gamma(\nabla_\mu^S - i\Gamma_\mu)\tilde{\Psi}\big) + \big(S_M\tilde{\chi}_L, \bar{d}\tilde{\psi}_L\big) - \big(J_M\tilde{\chi}_R, d\tilde{\psi}_R\big) \quad (27)$$

Proof. We take the fluctuated Dirac operator

$$D_\omega = D_M\otimes i + \gamma^\mu\otimes B_\mu + \gamma_M\otimes D_F \quad (28)$$

□

The Fermionic Action is $S_F = (J\tilde{\xi}, D_\omega\tilde{\xi})$ for a $\xi\in\mathscr{H}^+$, we can begin to calculate (note that we add the constant $\frac{1}{2}$ to the action)

$$\begin{aligned}\frac{1}{2}(J\tilde{\xi}, D_\omega\tilde{\xi}) = & && (29)\\ & +\frac{1}{2}(J\tilde{\xi}, (D_M\otimes i)\tilde{\xi}) && (30)\\ & +\frac{1}{2}(J\tilde{\xi}, (\gamma^\mu\otimes B_\mu)\tilde{\xi}) && (31)\\ & +\frac{1}{2}(J\tilde{\xi}, (\gamma_M\otimes D_F)\tilde{\xi}). && (32)\end{aligned}$$

For equation 30 we calculate

$$\begin{aligned}\frac{1}{2}(J\tilde{\xi}, (D_M\otimes 1)\tilde{\xi}) &= \frac{1}{2}(J_M\tilde{\chi}_R, D_M\tilde{\psi}_L) + \frac{1}{2}(J_M\tilde{\chi}_L, D_M\tilde{\psi}_R) + && (33)\\ &+\frac{1}{2}(J_M\tilde{\psi}_L, D_M\tilde{\psi}_R) + \frac{1}{2}(J_M\tilde{\chi}_R, D_M\tilde{\chi}_L) && (34)\\ &= (J_M\tilde{\chi}, D_M\tilde{\chi}). && (35)\end{aligned}$$

For equation 31 we have

$$\frac{1}{2}(J\tilde{\xi},(\gamma^{\mu}\otimes B_{\mu})\tilde{\xi}) = -\frac{1}{2}(J_M\tilde{\chi}_R,\gamma^{\mu}Y_{\mu}\tilde{\psi}_R) - \frac{1}{2}(J_M\tilde{\chi}_L,\gamma^{\mu}Y_{\mu}\tilde{\psi}_R)+ \tag{36}$$

$$+\frac{1}{2}(J_M\tilde{\psi}_L,\gamma^{\mu}Y_{\mu}\tilde{\chi}_R) + \frac{1}{2}(J_M\tilde{\psi}_R,\gamma^{\mu}Y_{\mu}\tilde{\chi}_L) = \tag{37}$$

$$= -(J_M\tilde{\chi},\gamma^{\mu}Y_{\mu}\tilde{\psi}). \tag{38}$$

For equation 32 we have

$$\frac{1}{2}(J\tilde{\xi},(\gamma_M\otimes D_F)\tilde{\xi}) = +\frac{1}{2}(J_M\tilde{\chi}_R, d\gamma_M\tilde{\chi}_R) + \frac{1}{2}(J_M\tilde{\chi}_L,\bar{d}\gamma_M\tilde{\chi}_L)+ \tag{39}$$

$$+\frac{1}{2}(J_M\tilde{\chi}_L,\bar{d}\gamma_M\tilde{\chi}_L) + \frac{1}{2}(J_M\tilde{\chi}_R, d\gamma_M\tilde{\chi}_R) = \tag{40}$$

$$= i(J_M\tilde{\chi}, m\tilde{\psi}) \tag{41}$$

Note that we obtain a complex mass parameter d, so we write $d := im$ for $m \in \mathbb{R}$, which stands for the real mass and we obtain a nice result

Theorem 1. *The full Lagrangian of $M \times F_{ED}$ is the sum of purely gravitational Lagrangian*

$$\mathscr{L}_{grav}(g_{\mu\nu}) = 4\mathscr{L}_M(g_{\mu\nu})\mathscr{L}_{\phi}(g_{\mu\nu}) \tag{42}$$

and the Lagrangian of electrodynamics

$$\mathscr{L}_{ED} = -i\left\langle J_M\tilde{\chi}, \left(\gamma^{\mu}(\nabla^S_{\mu} - iY_{\mu}) - m\right)\tilde{\psi}\right\rangle + \frac{f(0)}{6\pi^2}Y_{\mu\nu}Y^{\mu\nu}. \tag{43}$$