Notes on
Noncommutative Geometry and Particle Physics

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1 Noncommutative Geometric Spaces

1.1 Noncommutative Matrix Algebras

1.1.1 Balanced Tensor Product and Hilbert Bimodules

Definition 1. Let A be an algebra, E be a *right* A-module and F be a *left* A-module. The *balanced tensor product* of E and F forms a A-bimodule.

$$E \otimes_A F := E \otimes F \Big/ \left\{ \sum_i e_i a_i \otimes f_i - e_i \otimes a_i f_i : \quad a_i \in A,\ e_i \in E,\ f_i \in F \right\}$$

In other words the balanced tensor product forms only elements of

- E that preserver the *left* representation of A and
- F that preserver the *right* representation of A.

Which is the same saying:

$$E \otimes_A F = \{ ea \otimes_A f = e \otimes_A af : \quad a \in A,\ e \in E,\ f \in F \}$$

Definition 2. Let A, B be *matrix algebras*. The *Hilbert bimodule* for (A,B) is given by

- E, an A-B-bimodue E and by
- an B-valued *inner product* $\langle \cdot, \cdot \rangle_E : E \times E \to B$

$\langle \cdot, \cdot \rangle_E$ needs to satisfy the following for $e, e_1, e_2 \in E,\ a \in A$ and $b \in B$.

$$\begin{aligned} \langle e_1, a \cdot e_2 \rangle_E &= \langle a^* \cdot e_1, e_2 \rangle_E && \text{sesquilinear in } A \\ \langle e_1, e_2 \cdot b \rangle_E &= \langle e_1, e_2 \rangle_E b && \text{scalar in } B \\ \langle e_1, e_2 \rangle_E &= \langle e_2, e_1 \rangle_E^* && \text{hermitian} \\ \langle e, e \rangle_E &\geq 0 && \text{equality holds iff } e = 0 \end{aligned}$$

We denote $KK_f(A,B)$ the set of all *Hilbert bimodules* of (A,B).

Exercise 1. *Check that a representation $\pi :\ A\ \to L(H)$ of a matrix algebra A turns H into a Hilbert bimodule for $(A, \mathbb{C})$.*

Solution 1.

Exercise 2. *Show that the $A - A$ bimodule given by A is in $KK_f(A,A)$ by taking the following inner product $\langle \cdot, \cdot \rangle_A : A \times A \to A$:*

$$\langle a, a \rangle_A = a^* a' \quad a, a' \in A$$

Solution 2.

1.1.2 Kasparov Product and Morita Equivalence

Definition 3. Let $E \in KK_f(A,B)$ and $F \in KK_F(B,D)$ the *Kasparov product* is defined as with the balanced tensor product

$$F \circ E := E \otimes_B F$$

Such that $F \circ E \in KK_f(A,D)$ with a D-valued inner product.

$$< e_1 \otimes f_1, e_2 \otimes f_2 >_{E \otimes_B F} = < f_1, < e_1, e_2 >_E f_2 >_F$$

Question 1. What is the meaning of 'associative up to isomorphism? Isomorphism of $F \circ E$ or of A, B or D?

Exercise 3. *Show that the association $\phi \rightsquigarrow E_\phi$ (from the previous Example) is natrual in the sense*

- $E_{id_A} \simeq A \in KK_f(A,A)$
- *for $*$-algebra homomorphism $\phi : A \to B$ and $\psi : B \to C$ we have an isomorphism*

$$E_\psi \circ E_\phi \ \equiv \ E_\phi \otimes_B E_\psi \ \simeq \ E_{\psi \circ \phi} \in KK_f(A,C)$$

Solution 3.

Exercise 4. *In the definition of Morita equivalence:*

- *Check that $E \otimes_B F$ is a $A - D$ bimodule*
- *Check that $< \cdot, \cdot >_{E \oplus_B F}$ defines a D valued inner product*
- *Check that $< a^*(e_1 \otimes f_1), e_2 \otimes f_2 >_{E \otimes_B F} = < e_1 \otimes f_1, a(e_2 \otimes f_2) >_{E \otimes_B F}$.*

Solution 4.

Definition 4. Let A, B be *matrix algebras*. They are called *Morita eqivalent* if there exists an $E \in KK_f(A,B)$ and an $F \in KK_f(B,A)$ such that:

$$E \otimes_B F \simeq A \quad \text{and} \quad F \otimes_A E \simeq B$$

Where $\simeq$ denotes the isomorphism between Hilbert bimodules, note that A or B is a bimodule by itself.

Question 2. Why are E and F each others inverse in the Kasparov Product?

Theorem 1. *Two matrix algebras are Morita Equivalent iff their their Structure spaces are isomorphic as discreet spaces (have the same cardinality / same number of elements)*

Proof. Let A, B be *Morita equivalent*. So there exists ${}_AE_B$ and ${}_BF_A$ with

$$E \otimes_B F \simeq A \quad \text{and} \quad F \otimes_A E \simeq B$$

Consider $[(\pi_B, H)] \in \hat{B}$ than we construct a representation of A, $\pi_A \to L(E \otimes_B H)$ with $\pi_A(a)(e \otimes v) = ae \otimes w$

Question 3. Is $E \simeq H$ and $F \simeq W$?

vice versa, consider $[(\pi_A, W)] \in \hat{A} \Rightarrow \pi_B : B \to L(F \otimes_A W)$ and $\pi_B(b)(f \otimes w) = bf \otimes w$
These maps are each others inverses, thus $\hat{A} \simeq \hat{B}$ $\square$