

University of Vienna
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Notes on
Noncommutative Geometry and Particle Physics

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Contents

1 Finite Real Noncommutative Spaces

1.1 Finite Real Spectral Triples

Add on to finite real spectral triples a *real structure*. The requirement is that H is a A - A -bimodule (before only a A -left module).

For this we introduce a \mathbb{Z}_2 -grading γ with

$$\gamma^* = \gamma \quad (1)$$

$$\gamma^2 = 1 \quad (2)$$

$$\gamma D = -D\gamma \quad (3)$$

$$\gamma a = a\gamma \quad a \in A \quad (4)$$

Definition 1. A *finite real spectral triple* is given by a finite spectral triple (A, H, D) and a anti-unitary operator $J : H \rightarrow H$ called the *real structure*, such that

$$a^\circ := Ja^*J^{-1} \quad (5)$$

is a right representation of A on H , that is $(ab)^\circ = b^\circ a^\circ$. With two requirements

$$[a, b^\circ] = 0 \quad (6)$$

$$[[D, a], b^\circ] = 0. \quad (7)$$

They are called the *commutant property*, and mean that the left action of an element in A and $\Omega_D^1(A)$ commutes with the right action on A .

Definition 2. The *KO-dimension* of a real spectral triple is determined by the sings $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ appearing in

$$J^2 = \varepsilon \quad (8)$$

$$JD = \varepsilon DJ \quad (9)$$

$$J\gamma = \varepsilon''\gamma J. \quad (10)$$

Table 1: *KO-dimension* k modulo 8 of a real spectral triple

k	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

Definition 3. An opposite-algebra A° of a A is defined to be equal to A as a vector space with the opposite product

$$a \circ b := ba \quad (11)$$

$$\Rightarrow a^\circ = Ja^*J^{-1} \quad \text{defines the left representation of } A^\circ \text{ on } H \quad (12)$$

Example 1. Matrix algebra $M_N(\mathbb{C})$ acting on $H = M_N(\mathbb{C})$ by left matrix multiplication with the Hilbert Schmidt inner product.

$$\langle a, b \rangle = \text{Tr}(a^*b) \quad (13)$$

Then we define $\gamma(a) = a$ and $J(a) = a^*$ with $a \in H$. Since D must be odd with respect to γ it vanishes identically.

Definition 4. We call $\xi \in H$ **cyclic vector** in A if:

$$A\xi := a\xi : a \in A = H \quad (14)$$

We call $\xi \in H$ **separating vector** in A if:

$$a\xi = 0 \Rightarrow a = 0; a \in A \quad (15)$$

Exercise 1

In the previous example, show that the right action on $M_N(\mathbb{C})$ on $H = M_N(\mathbb{C})$ as defined by $a \mapsto a^\circ$ is given by right matrix multiplication.

$$a^\circ \xi = Ja^*J^{-1}\xi = Ja^*\xi^* = J\xi a = \xi^*a \quad (16)$$

Exercise 2

Let $A = \bigoplus_i M_{n_i}(\mathbb{C})$, represented on $H = \bigoplus_i \mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i}$, meaning that the irreducible representation n_i has multiplicity m_i .

- 1. Show that the commutant A' of A is $A' \simeq \bigoplus_i M_{m_i}(\mathbb{C})$. As a consequence show $A'' \simeq A$.**
- 2. Show that if ξ is a separating vector for A then it is cyclic for A' .**
 1. We know the multiplicity space is $V_i = \mathbb{C}^{m_i}$. We know that for $T \in H$ and $a \in A'$ to work we need $aT = Ta$ by laws of matrix multiplication we need $A' \simeq \bigoplus_i M_{m_i}(\mathbb{C})$ for this to work since $H = \bigoplus_i \mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i}$
 2. Suppose ξ is cyclic for A then $A'\xi = \{0\}$. Under the action of A we then have $A'A\xi = AA'\xi = 0 \Rightarrow A' = 0$.
Suppose now ξ is separating for A' , we have $A'\xi = \{0\}$. We can define a projection in A' , $A\xi = P'\xi$. With this projection we have $(1 - P')\xi = 0 \Rightarrow 1 - P' = 0 \Rightarrow A\xi = H$.

Exercise 3

Suppose $(A, H, D = 0)$ is a finite spectral triple such that H possesses a cyclic and separating vector for A .

- 1. Show that the formula $S(a\xi) = a^*\xi$ defines a anti-linear operator $S : H \rightarrow H$.**
- 2. Show that S is invertible**

3. Let $J : H \rightarrow H$ be the operator in $S = J\Delta^{1/2}$ with $\Delta = S^*S$. Show that J is anti-unitary

1. By composition $S(a\xi) = a * \xi$ this is literally anti-linearity. Does this mean $S\xi = \xi$?
2. Let $\xi \in H$ be cyclic then: $S(A\xi) = A * \xi = A\xi = H$. The same has to work for S^{-1} if not then ξ wouldn't exist. $S^{-1}(A * \xi) = S^{-1}(H) = H$.
3. Since S is bijective then $\Delta^{1/2}$ and J need to be bijective.
Now let $\xi_1, \xi_2 \in H$.

$$\langle J\xi_1, J\xi_2 \rangle = \langle J^*J\xi_1, \xi_2 \rangle^* = \quad (17)$$

$$= \langle (\Delta^{1/2})^* S^* S \Delta^{1/2} \xi_1, \xi_2 \rangle^* = \quad (18)$$

$$= \langle (SS^*)^{1/2} S^* S (SS^*) \xi_1, \xi_2 \rangle^* = \quad (19)$$

$$= \langle (SS^* SS^*)^{1/2} \xi_1, \xi_2 \rangle^* = \quad (20)$$

$$= \langle \xi_1, \xi_2 \rangle^* = \langle \xi_2, \xi_1 \rangle. \quad (21)$$

1.2 Morphisms Between Finite Real Spectral Triples

Extend unitary equivalence of finite spectral triples to real ones (with J and γ)

Definition 5. We call two finite real spectral triples $(A_1, H_1, D_1; J_1, \gamma_1)$ and $(A_2, H_2, D_2; J_2, \gamma_2)$ unitarily equivalent if $A_1 = A_2$ and if there exists a unitary operator $U : H_1 \rightarrow H_2$ such that

$$U\pi_1(a)U^* = \pi_2(a) \quad (22)$$

$$UD_1U^* = D_2 \quad (23)$$

$$U\gamma_1U^* = \gamma_2 \quad (24)$$

$$UJ_1U^* = J_2 \quad (25)$$

Definition 6. Let E be a B - A bimodule. The *conjugate Module* E° is given by the A - B -bimodule.

$$E^\circ = \{\bar{e} : e \in E\} \quad (26)$$

with

$$a \cdot \bar{e} \cdot b = b^* \bar{e} a^* \quad \forall a \in A, b \in B \quad (27)$$

E° is not a Hilbert bimodule for (A, B) because it doesn't have a natural B -valued inner product. But there is a A -valued inner product on the left A -module E° with

$$\langle \bar{e}_1, \bar{e}_2 \rangle = \langle e_2, e_1 \rangle \quad e_1, e_2 \in E \quad (28)$$

and linearity in A :

$$\langle a\bar{e}_1, \bar{e}_2 \rangle = a \langle \bar{e}_1, \bar{e}_2 \rangle \quad \forall a \in A. \quad (29)$$

Exercise 4

Show that E° is a Hilbert bimodule (B°, A°)

Straightforward show properties of the Hilbert bimodule and its B° valued inner product. Let $\bar{e}_1, \bar{e}_2 \in E^\circ$ and $a^\circ \in A, b^\circ \in B$.

$$\langle \bar{e}_1, a^\circ \bar{e}_2 \rangle = \langle \bar{e}_1, J a^* J^{-1} \bar{e}_2 \rangle = \quad (30)$$

$$= \langle \bar{e}_1, J a^* e_2 \rangle = \quad (31)$$

$$= \langle J^{-1} e_1, a^* e_2 \rangle = \quad (32)$$

$$= \langle a^* e_1, e_2 \rangle = \langle J^{-1} (a^\circ)^* J e_1, e_2 \rangle = \quad (33)$$

$$= \langle J^{-1} (a^\circ)^* \bar{e}_1, e_2 \rangle = \quad (34)$$

$$= \langle (a^\circ)^* \bar{e}_1, \bar{e}_2 \rangle. \quad (35)$$

Next $\langle \bar{e}_1, \bar{e}_2 b^\circ \rangle = \langle \bar{e}_1, \bar{e}_2 \rangle b^\circ$.

$$\langle \bar{e}_1, \bar{e}_2 b^\circ \rangle = \langle \bar{e}_1, \bar{e}_2 J b^* J^{-1} \rangle = \quad (36)$$

$$= \langle \bar{e}_1, \bar{e}_2 \rangle J b^* J^{-1} = \quad (37)$$

$$= \langle \bar{e}_1, \bar{e}_2 \rangle b^\circ. \quad (38)$$

Then:

$$(\langle \bar{e}_1, \bar{e}_2 \rangle_{E^\circ})^* = (\langle e_2, e_1 \rangle_E)^* = \quad (39)$$

$$= \langle e_1, e_2 \rangle_E^* = \langle \bar{e}_2, \bar{e}_1 \rangle_{E^\circ} \quad (40)$$

And of course $\langle \bar{e}, \bar{e} \rangle = \langle e, e \rangle \geq 0$

1.2.1 Construction of a Finite Real Spectral Triple from a Finite Real Spectral Triple

Given a Hilbert bimodule E for (B, A) we construct a spectral triple $(B, H', D'; J', \gamma')$ from $(A, H, D; J, \gamma)$

For the H' we make a \mathbb{C} -valued inner product on H' by combining the A valued inner product on E and E° with the \mathbb{C} -valued inner product on H .

$$H' := E \otimes_A H \otimes_A E^\circ \quad (41)$$

Then the action of B on H' is:

$$b(e_2 \otimes \xi \otimes \bar{e}_2) = (b e_1) \otimes \xi \otimes \bar{e}_2 \quad (42)$$

The right action of B on H' defined by action on the right component E°

$$J'(e_1 \otimes \xi \otimes \bar{e}_2) = e_2 \otimes J \xi \otimes \bar{e}_1 \quad (43)$$

with $b^\circ = J' b^* (J')^{-1}$, $b^* \in B$ action on H' .

Exercise 5

Let $\nabla : E \Rightarrow E \otimes_A \Omega_D^1(A)$ be a right connection on E consider the following anti-linear map:

$$\tau : E \otimes_A \Omega_D^1(A) \rightarrow \Omega_D^1(A) \otimes_A E^\circ \quad (44)$$

$$e \otimes \omega \mapsto -\omega^* \otimes \bar{e} \quad (45)$$

Show that the map $\bar{\nabla} : E^\circ \otimes_A \Omega_D^1(A) \rightarrow E^\circ$ with $\bar{\nabla}(\bar{e}) = \tau \circ \nabla(e)$ is a left connection, that means show that it satisfied the left Leibniz rule:

$$\bar{\nabla}(a\bar{e}) = [D, a] \otimes \bar{e} + a\bar{\nabla}(\bar{e}) \quad (46)$$

Hagime:

$$\text{For one:} \quad (47)$$

$$\tau \circ \nabla(ae) = \bar{\nabla}(a\bar{e}) = \bar{\nabla}(a^* \bar{e}) \quad (48)$$

$$\text{For two:} \quad (49)$$

$$\tau \circ \nabla(ae) = \tau(\nabla(e)a) + \tau \circ (e \otimes d(a)) = \quad (50)$$

$$= a^* \bar{\nabla}(\bar{e}) - d(a)^* \otimes \bar{e}. \quad (51)$$

$$= a^* \bar{\nabla}(\bar{e}) + d(a^*) \otimes \bar{e}. \quad (52)$$

Then the connections

$$\nabla : E \rightarrow E \otimes_A \Omega_D^1(A) \quad (53)$$

$$\bar{\nabla} : E^\circ \rightarrow \Omega_D^1(A) \otimes_A E^\circ \quad (54)$$

give us the Dirac operator on $H' = E \otimes_A H \otimes_A E^\circ$

$$D'(e_1 \otimes \xi \otimes \bar{e}_2) = (\nabla e_1) \xi \otimes \bar{e}_2 + e_1 \otimes D\xi \otimes \bar{e}_2 + e_1 \otimes \xi \otimes (\bar{\nabla} \bar{e}_2) \quad (55)$$

And the right action of $\omega \in \Omega_D^1(A)$ on $\xi \in H$ is defined by

$$\xi \mapsto \varepsilon' J \omega^* J^{-1} \xi \quad (56)$$

Finally for the grading

$$\gamma' = 1 \otimes \gamma \otimes 1 \quad (57)$$

Theorem 1. Suppose $(A, H, D; J, \gamma)$ is a finite spectral triple of KO-dimension k , let ∇ be like above satisfying the compatibility condition (like with finite spectral triples).

Then $(B, H', D'; J', \gamma')$ is a finite spectral triple of KO-Dimension k . (H', D', J', γ' like above)

Proof. The only thing left is to check if the KO -dimension is preserved, for this we check if the ε 's are the same.

$$(J')^2 = 1 \otimes J^2 \otimes 1 = \varepsilon \quad (58)$$

$$J'\gamma' = \varepsilon''\gamma'J' \quad (59)$$

and for ε'

$$J'D'(e_1 \otimes \xi \otimes \bar{e}_2) = J'((\nabla e_1)\xi \otimes \bar{e}_2 + e_1 \otimes D\xi \otimes \bar{e}_2 + e_1 \otimes \xi(\tau\nabla e_2)) \quad (60)$$

$$= \varepsilon'D'(e_2 \otimes J\xi \otimes \bar{e}_2) \quad (61)$$

$$= \varepsilon'D'J'(e_1 \otimes \xi \otimes \bar{e}_2) \quad (62)$$

□