Notes on
Noncommutative Geometry and Particle Physics

Popovic Milutin

Week 6: 19.03 - 26.03

Contents

1 Finite Real Noncommutative Spaces

1.1 Finite Real Spectral Triples

Add on to finite real spectral triples a *real structure*. The requirement is that H is a A-A-bimodule (before only a A-left module).

For this we introduce a $\mathbb{Z}_2$-grading γ with

$$\gamma^* = \gamma \tag{1}$$
$$\gamma^2 = 1 \tag{2}$$
$$\gamma D = -D\gamma \tag{3}$$
$$\gamma a = a\gamma \quad a \in A \tag{4}$$

Definition 1. A *finite real spectral triple* is given by a finite spectral triple (A, H, D) and a anti-unitary operator $J : H \to H$ called the *real structure*, such that

$$a^\circ := Ja^* J^{-1} \tag{5}$$

is a right representation of A on H, that is $(ab)^\circ = b^\circ a^\circ$. With two requirements

$$[a, b^\circ] = 0 \tag{6}$$
$$[[D, a], b^\circ] = 0. \tag{7}$$

They are called the *commutant property*, and mean that the left action of an element in A and $\Omega^1_D(A)$ commutes with the right action on A.

Definition 2. The *KO*-dimension of a real spectral triple is determined by the sings $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ appearing in

$$J^2 = \varepsilon \tag{8}$$
$$JD = \varepsilon\, DJ \tag{9}$$
$$J\gamma = \varepsilon'' \gamma J. \tag{10}$$

Table 1: *KO*-dimension k modulo 8 of a real spectral triple

k	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

Definition 3. An opposite-algebra A° of a A is defined to be equal to A as a vector space with the opposite product

$$a \circ b := ba \tag{11}$$

$$\Rightarrow a^\circ = Ja^*J^{-1} \quad \text{defines the left representation of } A^\circ \text{ on } H \tag{12}$$

Example 1. Matrix algebra $M_N(\mathbb{C})$ acting on $H = M_N(\mathbb{C})$ by left matrix multiplication with the Hilbert Schmidt inner product.

$$\langle a, b \rangle = \mathrm{Tr}(a^*b) \tag{13}$$

Then we define $\gamma(a) = a$ and $J(a) = a^*$ with $a \in H$. Since D mus be odd with respect to γ it vanishes identically.

Definition 4. We call $\xi \in H$ **cyclic vector** in A if:

$$A\xi := a\xi : \ \ a \in A = H \tag{14}$$

We call $\xi \in H$ **separating vector** in A if:

$$a\xi = 0 \ \ \Rightarrow \ \ a = 0; \ \ \ a \in A \tag{15}$$

Exercise 1

In the previous example, show that the right action on $M_N(\mathbb{C})$ on $H = M_N(\mathbb{C})$ as defined by $a \mapsto a^\circ$ is given by right matrix multiplication.

$$a^\circ \xi = Ja^*J^{-1}\xi = Ja^*\xi^* = J\xi a = \xi^* a \tag{16}$$

Exercise 2

Let $A = \bigoplus_i M_{n_i}(\mathbb{C})$, represented on $H = \bigoplus_i \mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i}$, meaning that the irreducible representation $\mathbf{n}_i$ has multiplicity m_i.

1. **Show that the commutant A' of A is $A' \simeq \bigoplus_i M_{m_i}(\mathbb{C})$. As a consequence show $A'' \simeq A$.**
2. **Show that if ξ is a separating vector for A than it is cyclic for A'.**

1. We know the multiplicity space is $V_i = \mathbb{C}^{m_i}$. We know that for $T \in H$ and

$a \in A'$ to work we need $aT = Ta$ by laws of matrix multiplication we need $A' \simeq \oplus_i M_{m_i}(\mathbb{C})$ for this to work since $H = \bigoplus_i \mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i}$

2. Suppose ξ is cyclic for A then $A'\xi = \{0\}$. Under the action of A we then have $A'A\xi = AA'\xi = 0 \Rightarrow A' = 0$.
   Suppose now ξ is separating for A', we have $A'\xi = \{0\}$. We can define a projection in A', $A\xi = P'$. With this projection we have $(1-P')\xi = 0 \Rightarrow 1 - P' = 0 \Rightarrow A\xi = H$.

Exercise 3

Suppose $(A, H, D = 0)$ is a finite spectral triple such that H possesses a cyclic and separating vector for A.

1. **Show that the formula $S(a\xi) = a * \xi$ defines a anti-linear operator $S : H \to H$.**
2. **Show that S is invertible**
3. **Let $J : H \to H$ be the operator in $S = J\Delta^{1/2}$ with $\Delta = S * S$. Show that J is anti-unitary**

1. By composition $S(a\xi) = a * \xi$ this is literally anti-linearity. Does this mean $S\xi = \xi$?
2. Let $\xi \in H$ be cyclic then: $S(A\xi) = A * \xi = A\xi = H$. The same has to work for S^{-1} if not then ξ wouldn't exist. $S^{-1}(A * \xi) = S^{-1}(H) = H$.
3. Since S is bijective then $\Delta^{1/2}$ and J need to be bijective.
   Now let $\xi_1, \xi_2 \in H$.

$$< J\xi_1, J\xi_2 > = < J^* J\xi_1, \xi_2 >^* = \tag{17}$$

$$= < (\Delta^{1/2})^* S^* S \Delta^{1/2} \xi_1, \xi_2 >^* = \tag{18}$$

$$= < (SS^*)^{1/2} S^* S (SS^*) \xi_1, \xi_2 >^* = \tag{19}$$

$$= < (SS^* SS^*)^{1/2} \xi_1, \xi_2 >^* = \tag{20}$$

$$= < \xi_1, \xi_2 >^* = < \xi_2, \xi_1 > . \tag{21}$$

1.2 Morphisms Between Finite Real Spectral Triples

Extend unitary equivalence of finite spectral triples to real ones (with J and γ)

Definition 5. We call two finite real spectral triples $(A_1, H_1, D_1; J_1, \gamma_1)$ and $(A_2, H_2, D_2; J_2, \gamma_2)$ unitarily equivalent if $A_1 = A_2$ and if there exists a unitary operator $U : H_1 \to H_2$ such that

$$
\begin{aligned}
U\pi_1(a)U^* &= \pi_2(a) && (22)\\
UD_1U^* &= D_2 && (23)\\
U\gamma_1U^* &= \gamma_2 && (24)\\
UJ_1U^* &= J_2 && (25)
\end{aligned}
$$

Definition 6. Let E be a B-A bimodule. The *conjugate Module* E° is given by the A-B-bimodule.

$$E^\circ = \{\bar{e} : e \in E\} \tag{26}$$

with

$$a \cdot \bar{e} \cdot b = b^* \bar{e} a^* \quad \forall a \in A, b \in B \tag{27}$$

E° is not a Hilbert bimodule for (A,B) because it doesn't have a natural B-valued inner product. But there is a A-valued inner product on the left A-module E° with

$$\langle \bar{e}_1, \bar{e}_2 \rangle = \langle e_2, e_1 \rangle \quad e_1, e_2 \in E \tag{28}$$

and linearity in A:

$$\langle a\bar{e}_1, \bar{e}_2 \rangle = a \langle \bar{e}_1, \bar{e}_2 \rangle \quad \forall a \in A. \tag{29}$$

Exercise 4

Show that E° is a Hilbert bimodule (B°, A°)

Straightforward show properties of the Hilbert bimodule and its B° valued inner product. Let $\bar{e}_1, \bar{e}_2 \in E^\circ$ and $a^\circ \in A, b^\circ \in B$.

$$
\begin{aligned}
< \bar{e}_1, a^\circ \bar{e}_2 > &= < \bar{e}_1, Ja^* J^{-1} \bar{e}_2 >= && (30)\\
&= < \bar{e}_1, Ja^* e_2 >= && (31)\\
&= < J^{-1} e_1, a^* e_2 >= && (32)\\
&= < a^* e_1, e_2 >=< J^{-1}(a^\circ)^* J e_1, e_2 >= && (33)\\
&= < J^{-1}(a^\circ)^* \bar{e}_1, e_2 >= && (34)\\
&= < (a^\circ)^* \bar{e}_1, \bar{e}_2 > . && (35)
\end{aligned}
$$

Next $< \bar{e}_1, \bar{e}_2 b^\circ >=< \bar{e}_1, \bar{e}_2 > b^\circ$.

$$
\begin{aligned}
< \bar{e}_1, \bar{e}_2 b^\circ > &= < \bar{e}_1, \bar{e}_2 J b^* J^{-1} >= && (36)\\
&= < \bar{e}_1, \bar{e}_2 > J b^* J^{-1} = && (37)\\
&= < \bar{e}_1, \bar{e}_2 > b^\circ . && (38)
\end{aligned}
$$

Then:

$$(<\bar{e}_1,\bar{e}_2)>_{E^\circ})^* = (<e_2,e_1>_E)^* = \tag{39}$$
$$=<e_1,e_2>^*_E=<\bar{e}_2,\bar{e}_2>_{E^\circ} \tag{40}$$

And of course $<\bar{e},\bar{e}>=<e,e>\geq 0$

1.2.1 Construction of a Finite Real Spectral Triple from a Finite Real Spectral Triple

Given a Hilbert bimodule E for (B,A) we construct a spectral triple $(B,H',D';J',\gamma')$ from $(A,H,D;J,\gamma)$

For the H' we make a $\mathbb{C}$-valued inner product on H' by combining the A valued inner product on E and E° with the $\mathbb{C}$-valued inner product on H.

$$H' := E\otimes_A H\otimes_A E^\circ \tag{41}$$

Then the action of B on H' is:

$$b(e_2\otimes\xi\otimes\bar{e}_2) = (be_1)\otimes\xi\otimes\bar{e}_2 \tag{42}$$

The right action of B on H' defined by action on the right component E°

$$J'(e_1\otimes\xi\otimes\bar{e}_2) = e_2\otimes J\xi\otimes\bar{e}_1 \tag{43}$$

with $b^\circ = J'b^*(J')^{-1}$, $b^*\in B$ action on H'.

Exercise 5

Let $\nabla : E \Rightarrow E \otimes_A \Omega^1_d(A)$ be a right connection on E consider the following anti-linear map:

$$\tau : E \otimes_A \Omega^1_D(A) \to \Omega^1_D(A) \otimes_A E^\circ \tag{44}$$
$$e \otimes \omega \mapsto -\omega^* \otimes \bar{e} \tag{45}$$

Show that the map $\bar{\nabla} : E^\circ \Omega^1_D(A) \otimes E^\circ$ with $\bar{\nabla}(\bar{e}) = \tau \circ \nabla(e)$ is a left connection, that means show that it satisfied the left Leibniz rule:

$$\bar{\nabla}(a\bar{e}) = [D,a] \otimes \bar{e} + a\bar{\nabla}(\bar{e}) \tag{46}$$

Hagime:

$$\text{For one:} \tag{47}$$
$$\tau \circ \nabla(ae) = \bar{\nabla}(a\bar{e}) = \bar{\nabla}(a^*\bar{e}) \tag{48}$$
$$\text{For two:} \tag{49}$$
$$\tau \circ \nabla(ae) = \tau(\nabla(e)a) + \tau \circ (e \otimes d(a)) = \tag{50}$$
$$= a^*\bar{\nabla}(\bar{e}) - d(a)^* \otimes \bar{e}. \tag{51}$$
$$= a^*\bar{\nabla}(\bar{e}) + d(a^*) \otimes \bar{e}. \tag{52}$$

Then the connections

$$\nabla : E \to E \otimes_A \Omega^1_D(A) \tag{53}$$
$$\bar{\nabla} : E^\circ \to \Omega^1_D(A) \otimes_A E^\circ \tag{54}$$

give us the Dirac operator on $H' = E \otimes_A H \otimes_A E^\circ$

$$D'(e_1 \otimes \xi \otimes \bar{e}_2) = (\nabla e_1)\xi \otimes \bar{e}_2 + e_1 \otimes D\xi \otimes \bar{e}_2 + e_1 \otimes \xi(\bar{\nabla}\bar{e}_2) \tag{55}$$

And the right action of $\omega \in \Omega^1_D(A)$ on $\xi \in H$ is defined by

$$\xi \mapsto \varepsilon' J\omega^* J^{-1}\xi \tag{56}$$

Finally for the grading

$$\gamma' = 1 \otimes \gamma \otimes 1 \tag{57}$$

Theorem 1. *Suppose $(A,H,D;J,\gamma)$ is a finite spectral triple of KO-dimension k, let ∇ be like above satisfying the compatibility condition (like with finite spectral triples).*

Then $(B,H',D';J',\gamma')$ is a finite spectral triple of KO-Dimension k. (H',D',J',γ' like above)

Proof. The only thing left is to check if the *KO*-dimension is preserved, for this we check if the ε's are the same.

$$(J')^2 = 1 \otimes J^2 \otimes 1 = \varepsilon \tag{58}$$
$$J'\gamma' = \varepsilon''\gamma' J' \tag{59}$$

and for ε'

$$\begin{aligned} J'D'(e_1 \otimes \xi \otimes \bar{e}_2) &= J'((\nabla e_1)\xi \otimes \bar{e_2} + e_1 \otimes D\xi \otimes \bar{e}_2 + e_1 \otimes \xi(\tau\nabla e_2)) &(60)\\ &= \varepsilon' D'(e_2 \otimes J\xi \otimes \bar{e}_2) &(61)\\ &= \varepsilon' D'J'(e_1 \otimes \xi\bar{e}_2) &(62) \end{aligned}$$

□