Notes on
Noncommutative Geometry and Particle Physics

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1 Noncommutative Geometric Spaces

1.1 Matrix Algebras and Finite Spaces

1.1.1 $*$-Algebra

Definition 1. A *vector space* A over $\mathbb{C}$ is called a *complex, unital Algebra* if,
$\forall a,b \in A$:

1. $A \times A \to A$ *bilinear*
   $(a,b) \mapsto a \cdot b$
2. $1a = a1 = a$ *unitary*

Definition 2. A $*$-algebra is an algebra A with a *conjugate linear map (involution)* $* : A \to A$, $\forall a,b \in A$ satisfying:

1. $(ab)^* = b^* a^*$ *antidistributive*
2. $(a^*)^* = a$ *closure*

In the following all unitary algebras are referred to as algebras.

1.1.2 Functions on Discrete Spaces

Let X be a *discretized topological* space with N points. Consider functions of a continuous $*$-algebra $C(X)$ assigning values to $\mathbb{C}$, for $f,g \in C(X)$, $\lambda \in \mathbb{C}$ and $x \in X$ they provide the following structures:

- *pointwise linear*
  $(f+g)(x) = f(x) + g(x)$
  $(\lambda f)(x) = \lambda(f(x))$
- *pointwise multiplication*
  $fg(x) = f(x)g(x)$ same as $(fg)(x) = f(x)g(x))$?

- *pointwise involution*
  $f^*(x) = \overline{f(x)}$

Question 1. Mathematical difference between Topological Discreet Spaces and just Discreet Spaces?

The author indicates that $\mathbb{C}$-valued functions on X are automatically continuous.

Proof Idea. CAN WE USE THE METRIC? » NO! We know that X is a *finite discrete space*, meaning in an ε-δ approach for each $x \in X$ the only $y \in X$, that is small enough is x by itself, which implies ε is always bigger than zero, thus every function $f : X \to \mathbb{C}$ is continuous.

1.1.3 Isomorphism Property

Furthermore $C(X)$ $*$-algebra is *isomorphic* to a $*$-algebra $\mathbb{C}^N$ with involution (N number of points in X), written as $C(X) \simeq \mathbb{C}^N$. A function $f : X \to \mathbb{C}$ can be represented with $N \times N$ diagonal matrices, where the value (ii) is the value of the function at the corresponding i-th point ($i = 1, ..., N$). The structure is preserved because of the definitions of matrix multiplication and the hermitian conjugate of matrices.

Question 2. Can isomorphisms between $C(X)$ and $\mathbb{C}^N$ be shown with matrix factorization?

Isomorphisms are bijective preserve structure and don't lose physical information/

1.1.4 Mapping Finite Discrete Spaces

Definition 3. A *map* between finite discrete spaces X_1 and X_2 is a function $\phi : X_1 \to X_2$

For every map between finite discrete spaces there exists a corresponding map $\phi^* : C(X_2) \to C(X_1)$, which 'pulls back' values even if ϕ is not bijective. Note that the pullback doesn't map points back, but maps functions on an $*$-algebra $C(X)$.

This map is called a pullback (or a $*$-homomorphism or a $*$-algebra map under pointwise product). Under the pointwise product:

- $\phi^*(fg) = \phi^*(f)\phi^*(g)$
- $\phi^*(\overline{f}) = \overline{\phi^*(f)}$
- $\phi^*(\lambda f + g) = \lambda\phi^*(f) + \phi^*(g)$

Question 3. ϕ is in most cases not bijective, so how can we prove that there exists such a pullback for every map between discreet spaces which preserves information? For bijective it is given by its inverse, which by definition exists because ϕ is a map. Or I didn't understand this correctly?

Exercise 1. *Show that $\phi : X_1 \to X_2$ is injective (surjective) map of finite spaces iff $\phi^* : C(X_2) \to C(X_1)$ is surjective (injective).*

Solution 1. Consider X_1 with n points and X_2 with m points. Then there are three cases:

1. $n = m$
   Obviously ϕ is bijective and ϕ^* too.

2. $n > m$
   ϕ assigns n points to m points when $n > m$, which is by definition surjective.
   ϕ^* assigns m points to n points when $n > m$, which is by definition injective.

3. $n < m$
   analogous

1.1.5 Matrix Algebras

Definition 4. A *(complex) matrix algebra* A is a direct sum, for $n_i, N \in \mathbb{N}$.

$$A = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C})$$

The involution is the hermitian conjugate, a $*$ algebra with involution is referred to as a matrix algebra

So from a topological discreet space X, we can construct a $*$-algebra $C(X)$ which is isomorphic to a matrix algebra A. The question is can we construct X given A? A is a matrix algebra, which are in most cases is not commutative, so the answer is generally no.

There are two options. We can restrict ourselves to commutative matrix algebras, which are the vast minority and not physically interesting. Or we can allow more morphisms(isomorphisms) between matrix algebras.

Question 4. Why are non-commutative algebras not physically interesting? Maybe too far fetched,but because physical observables (QM-Operators) are not commutative?

Exactly.

1.1.6 Finite Inner Product Spaces and Representations

Until now we looked at a finite topological discreet space, moreover we can consider a finite dimensional inner product space H (finite Hilbert-spaces), with inner product $(\cdot,\cdot) \to \mathbb{C}$. $L(H)$ is the $*$-algebra of operators on H with product given by composition and involution given by the adjoint, $T \mapsto T^*$. $L(H)$ is a *normed vector space* with

$$\|T\|^2 = \sup_{h\in H}\{(Th,Th) : (h,h) \leq 1\} \qquad T \in L(H)$$
$$\|T\| = \sup\{\sqrt{\lambda} : \lambda \text{ eigenvalue of } T\}$$

Definition 5. The *representation* of a finite dimensional $*$-algebra A is a pair (H,π). H is a finite, dimensional inner product space and π is a $*$-*algebra map*

$$\pi : A \to L(H)$$

Definition 6. (H,π) is called *irreducible* if:

- $H \neq \emptyset$
- only $\emptyset$ or H is invariant under the action of A on H

Examples for reducible and irreducible representations

- $A = M_n(\mathbb{C})$, representation $H = \mathbb{C}^n$, A acts as matrix multiplication
  H is irreducible.
- $A = M_n(\mathbb{C})$, representation $H = \mathbb{C}^n \oplus \mathbb{C}^n$, with $a \in A$ acting in block form
  $\pi : a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ is reducible.

Definition 7. Let (H_1, π_1) and (H_2, π_2) be representations of a $*$-algebra A. They are called *unitarily equivalent* if there exists a map $U : H_1 \to H_2$ such that.

$$\pi_1(a) = U^* \pi_2(a) U$$

Question 5. In matrix representation this is diagonalization condition? (unitary diagonalization)

Yes

Definition 8. A a $*$-algebra then, $\hat{A}$ is called the structure space of all *unitary equivalence classes of irreducible representations of A*

Question 6. Gelfand duality and the spectrum of $\hat{A}$, examples Fourier-Transform and Laplace-Transform for simple spaces.

More on that in later chapters.

Exercise 2. *Given (H, π) of a $*$-algebra A, the* ***commutant*** *$\pi(A)'$ of $\pi(A)$ is defined as a set of operators in $L(H)$ that commute with all $\pi(a)$*

$$\pi(A)' = \{T \in L(H) : \pi(a)T = T\pi(a) \quad \forall a \in A\}$$

1. *Show that $\pi(A)'$ is a $*$-algebra.*
2. *Show that a representation (H, π) of A is irreducible iff the commutant $\pi(A)'$ consists of multiples of the identity*

Solution 2. 1. To show that $\pi(A)'$ is a $*$-algebra we have to show that it is unital, associative and involute. And note that $\pi(a) \in L(H)\ \forall a \in A$. Unity is given by the unitary operator of the $*$-algebra of operators $L(H)$, which exists by definition because H is a inner product space. Associativity is given by $*$-algebra of $L(H)$, $L(H) \times L(H) \mapsto L(H)$, which is associative by definition. Involutnes is also given by the $*$-algebra $L(H)$ with a map $* : L(H) \mapsto L(H)$ only for T that commute with $\pi(a)$.
2.?

Exercise 3. 1. *If A is a unital $*$-algebra, show that the $n \times n$ matrices $M_n(A)$ with entries in A form a unital $*$-algebra.*

2. *Let $\pi : A \to L(H)$ be a representation of a $*$-algebra A and set $H^n = H \oplus \ldots \oplus H$, n times. Show that $\tilde{\pi} : M_n(A) \to L(H^n)$ of $M_n(A)$ with $\tilde{\pi}((a_{ij})) = (\tilde{\pi}(a_{ij})) \in M_n(A)$.*
3. *Let $\tilde{\pi} : M_n(A) \to L(H^n)$ be a $*$ algebra representation of $M_n(A)$. Show that $\pi : A \to L(H^n)$ is a representation of A.*

Solution 3. 1. We know A is a $*$ algebra. Unitary operaton in $M_n(A)$ is given by the identity Matrix, which has to exists because every entry in $M_n(A)$ has to behave like in A. Associativity is given by matrix multiplication. Involutnes is given by the conjugate

transpose.
2. $A \simeq M_n(A)$ and $H \simeq H^n$ meaning $\tilde{\pi}$ is a valid reducible representation.
3. $\tilde{\pi}$ and π are unitary equivalent, there is a map $U : H^n \to H^n$ given by $U = \mathbb{1}_n$:
$\pi(a) = \mathbb{1}_n^* \, \tilde{\pi}((a_{ij})) \, \mathbb{1}_n = \tilde{\pi}((a_{ij})) = \pi(a_{ij}) \Rightarrow a_{ij} = a\mathbb{1}_n$.

1.2 Commutative Matrix Algebras

- Commutative matrix algebras can be used to reconstruct a discrete space given a matrix *commutative* matrix algebra.
- The structure space $\hat{A}$ is used for this. Because $A \simeq \mathbb{C}^N$ we all any irreducible representation are of the form $\pi_i : (\lambda_1, ..., \lambda_N) \in \mathbb{C}^N \mapsto \lambda_i \in \mathbb{C}$
for $i = 1, ..., N \Rightarrow \hat{A} \simeq \{1, ..., N\}$.
- Conclusion is that there is a duality between discreet spaces and commutative matrix algebra this duality is called the *finite dimensional Gelfand duality*

1.3 Noncommutative Matrix Algebras

Aim is to construct duality between finite dimensional spaces and *equivalence classes* of matrix algebras, to preserve general non-commutivity of matrices.

- Equivalence classes are described by a generalized notion of ispomorphisms between matrix algebras (*Morita Equivalence*)

1.3.1 Algebraic Modules

Definition 9. Let A, B be algebras (need not be matrix algebras)

1. *left* A-module is a vector space E, that carries a left representation of A, that is $\exists$ a bilinear map $\gamma : A \times E \to E$ with

$$(a_1 a_2) \cdot e = a_1 \cdot (a_2 \cdot e); \quad a_1, a_2 \in A, e \in E$$

2. *right* B-module is a vector space F, that carries a right representation of A, that is $\exists$ a bilinear map $\gamma : F \times B \to F$ with

$$f \cdot (b_1 b_2) = (f \cdot b_1) \cdot b_2; \quad b_1, b_2 \in B, f \in F$$

3. *left* A-module and *right* B-module is a *bimodule*, a vector space E satisfying

$$a \cdot (e \cdot b) = (a \cdot e) \cdot b; \quad a \in A, b \in B, e \in E$$

Notion of A-**module homomorphism** as linear map $\phi : E \to F$ which respects the representation of A, e.g. for left module.

$$\phi(ae) = a\phi(e); \quad a \in A, e \in E.$$

Remark on the notation

- ${}_A E$ left A-module E;
- ${}_A E_B$ right B-module F;

- ${}_A E_B$ A-B-bimodule E;

Exercise 4. *Check that a representation of $\pi : A \to L(H)$ of a $*$-algebra A turns H into a left module ${}_A H$.*

Solution 4. Not quite sure but
$a \in A$, $h_1, h_2 \in H$, we know $\pi(a) = T \in L(H)$ than

$$< \pi(a)h_1, \pi(a)h_2 >=< Th_1, Th_2 >=< T^* Th_1, h_2 >=< h_1, h_2 >$$

Exercise 5. *Show that A is a bimodule ${}_A A_A$ with itself.*

Solution 5. $\gamma : A \times A \times A \to A$ which is given by the inner product of the $*$-algebra.