

Notes on Noncommutative Geometry and Particle Physics

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1 Noncommutative Geometric Spaces

1.1 Exercises

Exercise 1

Make the proof of the last theorem (see week4.pdf) explicit for $N = 3$.

For the C^* algebra we have $A = \mathbb{C}^3$ For H we have $H = (\mathbb{C}^2)^{\oplus 3} = H_2 \oplus H_2^1 \oplus H_2^2$.

The symmetric operator D acting on H and the representation $\pi(a)$:

$$\begin{aligned}\pi((a(1), a(2), a(3))) &= \begin{pmatrix} a(1) & 0 \\ 0 & a(2) \end{pmatrix} \oplus \begin{pmatrix} a(1) & 0 \\ 0 & a(3) \end{pmatrix} \oplus \begin{pmatrix} a(2) & 0 \\ 0 & a(2) \end{pmatrix} \\ &= \begin{pmatrix} a(1) & 0 & 0 & 0 & 0 & 0 \\ 0 & a(2) & 0 & 0 & 0 & 0 \\ 0 & 0 & a(1) & 0 & 0 & 0 \\ 0 & 0 & 0 & a(3) & 0 & 0 \\ 0 & 0 & 0 & 0 & a(2) & 0 \\ 0 & 0 & 0 & 0 & 0 & a(3) \end{pmatrix} \quad (1)\end{aligned}$$

$$\begin{aligned}D &= \begin{pmatrix} 0 & x_1 \\ x_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & x_2 \\ x_2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & x_3 \\ x_3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & x_1 & 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 & x_3 & 0 \end{pmatrix} \quad (2)\end{aligned}$$

(3)

Then the norm of the commutator would be the largest eigenvalue

$$\begin{aligned}||[D, \pi(a)]|| &= ||D\pi(a) - \pi(a)D|| \\ &= \left\| \begin{pmatrix} 0 & x_1(a(2) - a(1)) & 0 & 0 & 0 & 0 \\ -x_1(a(2) - a(1)) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2(a(3) - a(1)) & 0 & 0 \\ 0 & 0 & -x_2(a(3) - a(1)) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_3(a(2) - a(3)) \\ 0 & 0 & 0 & 0 & -x_3(a(2) - a(3)) & 0 \end{pmatrix} \right\| \quad (4)\end{aligned}$$

The matrix in Equation ?? is a skew symmetric matrix its eigenvalues are $i\lambda_1, i\lambda_2, i\lambda_3, i\lambda_4$, where the λ 's are on the upper and lower diagonal check https://en.wikipedia.org/wiki/Skew-symmetric_matrix#Skew-symmetrizable_matrix. The matrix norm of would be the maximum of the norm of the largest eigenvalues:

$$\begin{aligned}||[D, \pi(a)]|| &= \max_{a \in A} \{x_1|a(2) - a(1)|, \\ &\quad x_2|(a(3) - a(1))|, \\ &\quad x_3|(a(3) - a(2))|, \} \quad (5)\end{aligned}$$

The metric is then:

$$d = \begin{pmatrix} 0 & a(1) - a(2) & a(1) - a(3) \\ a(2) - a(1) & 0 & a(2) - a(3) \\ a(3) - a(1) & a(3) - a(2) & 0 \end{pmatrix} \quad (6)$$

Exercise 2

Compute the metric on the space of three points given by $d_{ij} = \sup_{a \in A} \{|a(i) - a(j)| : ||[D, \pi(a)]|| \leq 1\}$ for the set of data $A = \mathbb{C}^3$ acting in the defining representation $H = \mathbb{C}^3$, and

$$D = \begin{pmatrix} 0 & d^{-1} & 0 \\ d^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some $d \in \mathbb{R}$

We have $A = \mathbb{C}^3$, $H = \mathbb{C}^3$ and D from above, then

$$||[D, \pi(a)]|| = d^{-1} \left\| \begin{pmatrix} 0 & a(2) - a(1) & 0 \\ -(a(2) - a(1)) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\| \quad (7)$$

The metric is then

$$d = \begin{pmatrix} 0 & a(1) - a(2) & a(1) \\ a(2) - a(1) & 0 & a(2) \\ -a(1) & -a(2) & 0 \end{pmatrix} \quad (8)$$

Exercise 3

Show that d_{ij} from Equation 12 is a metric on \hat{A} by establishing that:

$$d_{ij} = 0 \Leftrightarrow i = j \quad (9)$$

$$d_{ij} = d_{ji} \quad (10)$$

$$d_{ij} \leq d_{ik} + d_{kj} \quad (11)$$

$$d_{ij} = \sup_{a \in A} \{|\text{Tr}(a(i)) - \text{Tr}(a(j))| : ||[D, a]|| \leq 1\} \quad (12)$$

For Equation 9 set $i = j$ in 12.

$$\begin{aligned} d_{ii} &= \sup_{a \in A} \{|\text{Tr}(a(i)) - \text{Tr}(a(i))| : ||[D, a]|| \leq 1\} \\ &= \sup_{a \in A} \{0 : ||[D, a]|| \leq 1\} = 0 \end{aligned}$$

For Equation 10 obviously we have the commuting property of addition.

For Equation 11, for $k = j$ then $d_{kj} = 0$ and the equality holds. For $i = k$ then $d_{ik} = 0$ and equality holds. Else set $d_{ik} = 1$ and $d_{kj} = 1$ then $d_{ij} = 1 \leq d_{ik} + d_{kj} = 2$

1.2 Properties of Matrix Algebras

Lemma 1. *If A is a unital C^* algebra that acts faithfully on a finite dimensional Hilbert space, then A is a matrix algebra of the Form:*

$$A \simeq \bigoplus_{i=1}^N M_{n_i}(\mathbb{C}) \quad (13)$$

Proof. Since A acts faithfully on a Hilbert space, then A is a C^* subalgebra of a matrix algebra $L(H) = M_{\dim(H)}(\mathbb{C}) \Rightarrow A \simeq \text{Matrix algebra}$. \square

Question 1. What does the author mean when he says 'acts faithfully on a Hilbertspace'? Then the representation is fully reducible, or that the presentation is irreducible?

For a $*$ -representation 'faithful' if it is injective. For a $*$ -homomorphism 'faithful' means one-to-one correspondence

Example 1. $A = M_n(\mathbb{C})$ and $H = \mathbb{C}^n$, A acts on H with matrix multiplication and standard inner product. D on H is a hermitian matrix $n \times n$ matrix.

D is referred to as a finite Dirac operator as in as its ∞ dimensional on Riemannian Spin manifolds coming in Chapter 4.

Now can introduce a 'differential 'geometric structure' on the finite space X with the **devided difference**

$$\frac{a(i) - a(j)}{d_{ij}} \quad (14)$$

for each pair $i, j \in X$ the finite dimensional discrete space X . This appears in the entries in the commutator $[D, a]$ in the above exercises.

Definition 1. Given an finite spectral triple (A, H, D) , the A -bimodule of Connes' differential one form is:

$$\Omega_D^1(A) := \left\{ \sum_k a_k [D, b_k] : a_k, b_k \in A \right\} \quad (15)$$

Question 2. Is the Conne's differential one form the set of all '1st order differential operators' given A , that act on H ?

Then there is a map $d : A \rightarrow \Omega_D^1(A)$, $d = [D, \cdot]$.

Exercise 4

Verify that 'd' is a derivation of the C^* algebra

$$\begin{aligned} d(ab) &= d(a)b + ad(b) \\ d(a^*) &= -d(a)^* \end{aligned}$$

For the record $d(\cdot) = [D, \cdot]$, then we have

1.

$$\begin{aligned} d(ab) &= [D, ab] = [D, a]b + a[D, b] \\ &= d(a)b + ad(b) \end{aligned}$$

2.

$$\begin{aligned} d(a^*) &= [D, a^*] = Da^* - a^*D \\ &= -(D^*a - aD^*) = -[D^*, a] \\ &= -d(a)^* \end{aligned}$$

Exercise 5

Verify that $\Omega_D^1(A)$ is an A -bimodule by rewriting

$$a(a_k[D, b_k])b = \sum_k a'_k[D, b'_k] \quad a'_k, b'_k \in A$$

Lemma 2. Let $(A, H, D) = (M_n(\mathbb{C}, \mathbb{C}^n, D)$, with D a hermitian $n \times n$ matrix. If D is not a multiple of the identity then:

$$\Omega_D^1(A) \simeq M_n(\mathbb{C}) = A \quad (16)$$

Proof. Assume $D = \sum_i \lambda_i e_{ii}$ (diagonal), $\lambda_i \in \mathbb{R}$ and $\{e_{ij}\}$ the basis of $M_n(\mathbb{C})$. For fixed i, j choose k such that $\lambda_k \neq \lambda_j$ then

$$\left(\frac{1}{\lambda_k - \lambda_j} e_{ik} \right) [D, e_{kj}] = e_{ij} \quad (17)$$

$e_{ij} \in \Omega_D^1(A)$ by the above definition. And $\Omega_D^1(A) \subset L(\mathbb{C}^n) = H \simeq M_n(\mathbb{C}) = A \quad \square$

Exercise 6

Consider $(A = \mathbb{C}^2, H = \mathbb{C}^2, D = \begin{pmatrix} 0 & \lambda \\ \bar{\lambda} & 0 \end{pmatrix})$ with $\lambda \neq 0$. **Show that** $\Omega_D^1(A) \simeq M_2(\mathbb{C})$

Because of the Hilbert Basis D can be extended in terms of the basis of $M_2(\mathbb{C})$, plugging this into Equation 17 will get us the same cyclic result, thus $\Omega_D^1(A) \simeq M_2(\mathbb{C})$

1.3 Morphisms Between Finite Spectral Triples

Definition 2. two finite spectral triples (A_1, H_1, D_1) and (A_2, H_2, D_2) are called unitarily equivalent if

- $A_1 = A_2$
- $\exists U : H_1 \rightarrow H_2$, unitary with
 1. $U \pi_1(a) U^* = \pi_2(a)$ with $a \in A_1$
 2. $U D_1 U^* = D_2$

Some remarks

- the above is an equivalence relation
- spectral unitary equivalence is given by the unitaries of the matrix algebra itself
- for any such U then $(A, H, D) \sim (A, H, U D U^*)$
- $U D U^* = D + U[D, U^*]$ of the form of elements in $\Omega_D^1(A)$.

Exercise 7

Show that the unitary equivalence between finite spectral triples is a equivalence relation

An equivalence relation needs to satisfy reflexivity, symmetry transitivity. Let (A_1, H_1, D_1) , (A_2, H_2, D_2) and (A_3, H_3, D_3) be three finite spectral triples.

For reflexivity $(A_1, H_1, D_1) \sim (A_1, H_1, D_1)$. So there exists a $U : H_1 \rightarrow H_1$ unitary, which is the identity and always exists.

For symmetry we need

$$(A_1, H_1, D_1) \sim (A_2, H_2, D_2) \Leftrightarrow (A_2, H_2, D_2) \sim (A_1, H_1, D_1)$$

because U is unitary:

$$\begin{aligned} U \pi_1(a) U^* &= \pi_2(a) \quad | \cdot U^* \sqcup U \\ U^* U \pi_1(a) U^* U &= \pi_1(a) = U^* \pi_2(a) U \end{aligned}$$

The same with the symmetric operator D .

For transitivity we need

$$\begin{aligned} (A_1, H_1, D_1) \sim (A_2, H_2, D_2) \quad \text{and} \quad (A_2, H_2, D_2) \sim (A_3, H_3, D_3) \\ \Rightarrow (A_1, H_1, D_1) \sim (A_3, H_3, D_3) \end{aligned}$$

There are two unitary maps $U_{12} : H_1 \rightarrow H_2$ and $U_{23} : H_2 \rightarrow H_3$ then

$$\begin{aligned} U_{23}U_{12}\pi_1(a)U_{12}^*U_{23}^* &= U_{23}\pi_2(a)U_{23}^* \\ &= \pi_3(a) \\ U_{23}U_{12}D_1U_{12}^*U_{23}^* &= U_{23}D_2U_{23}^* \\ &= D_3 \end{aligned}$$

Extending this relation we look again at the notion of equivalence from Morita equivalence of Matrix Algebras.

Given a Hilbert bimodule $E \in KK_f(B, A)$ and (A, H, D) we construct a finite spectral triple on B , (B, H', D')

$$H' = E \otimes_A H \quad (18)$$

This extends the left action on B with the right action and inherits the \mathbb{C} valued inner product space.

$$D'(e \otimes \xi) = e \otimes D\xi + \nabla(e)\xi \quad e \in E, a \in A \quad (19)$$

Where ∇ is called the *connection on the right A -module E* associated with the derivation $d = [D, \cdot]$ and satisfying the *Leibnitz Rule* which is

$$\nabla(ae) = \nabla(e)a + e \otimes [D, a] \quad e \in E, a \in A \quad (20)$$

Then the linearity of the balanced tensor product $E \otimes_A H$ is satisfied

$$\begin{aligned} D'(ea \otimes \xi - e \otimes a\xi) &= D'(ea \otimes \xi) - D'(e \otimes a\xi) \\ &= ea \otimes D\xi + \nabla(ea)\xi - e \otimes D(a\xi) - \nabla(e)a\xi \\ &= 0 \end{aligned}$$

With the information thus far we can prove the following theorem

Theorem 1. *If (A, H, D) a finite spectral triple, $E \in KK_f(B, A)$. Then $(B, E \otimes_A H, D')$ is a finite spectral triple, provided that ∇ satisfies the compatibility condition*

$$\langle e_1, \nabla e_2 \rangle_E - \langle \nabla e_1, e_2 \rangle_E = d\langle e_1, e_2 \rangle_E \quad e_1, e_2 \in E \quad (21)$$

Proof. $E \otimes_A H$ was shown in the previous section (text before the theorem). The only thing left is to show that D' is a symmetric operator, this we can just compute. Let $e_1, e_2 \in E$ and $\xi_1, \xi_2 \in H$ then

$$\begin{aligned} \langle e_1 \otimes \xi_1, D'(e_2 \otimes \xi_2) \rangle_{E \otimes_A H} &= \langle \xi_1, \langle e_1, \nabla e_2 \rangle_E \xi_2 \rangle + \langle \xi_1, \langle e_1, e_2 \rangle_E D\xi_2 \rangle_H \\ &= \langle \xi_1, \langle \nabla e_1, e_2 \rangle_E \xi_2 \rangle_H + \langle \xi_1, d\langle e_1, e_2 \rangle_E \xi_2 \rangle_H \\ &\quad + \langle D\xi_1, \langle e_1, e_2 \rangle_E \xi_2 \rangle_H - \langle \xi_1, [D, \langle e_1, e_2 \rangle_E] \xi_2 \rangle_H \\ &= \langle D'(e_1 \otimes \xi_1), e_2 \otimes \xi_2 \rangle_{E \otimes_A H} \end{aligned}$$

□

Exercise 8

Let ∇ and ∇' be two connections on a right A -module E . Show that $\nabla - \nabla'$ is a right A -linear map $E \rightarrow E \otimes_A \Omega_D^1(A)$

Both ∇ and ∇' need to satisfy the Leibnitz rule, so let's see if $\nabla - \nabla'$ does.

$$\begin{aligned}
 \nabla(ea) - \nabla'(ea) &= \nabla(e) + e \otimes [D, a] \\
 &\quad - (\nabla'(e)a + e \otimes [D', a]) \\
 &= \bar{\nabla}a + e \otimes (Da - aD - D'a + aD') \\
 &= \bar{\nabla}a + e \otimes ((D - D')a - a(D - D')) \\
 &= \bar{\nabla}a + e \otimes [D', a] \\
 &= \bar{\nabla}(ea)
 \end{aligned}$$

Therefore $\nabla - \nabla'$ is a linear map.

Exercise 9

Construct a finite spectral triple (A, H', D') from (A, H, D)

1. show that the derivation $d(\cdot) : A \rightarrow A \otimes_A \Omega_D^1(A) = \Omega_D^1(A)$ is a connection on A considered a right A -module
2. Upon identifying $A \otimes_A H \simeq H$, what is D' when the connection is $d(\cdot)$.
3. Use 1) and 2) to show that any connection $\nabla : A \rightarrow A \otimes_A \Omega_D^1(A)$ is given by

$$\nabla = d + \omega$$

where $\omega \in \Omega_D^1(A)$

4. Upon identifying $A \otimes_A H \simeq H$, what is the difference operator D' with the connection on A given by $\nabla = d + \omega$
 1. $\nabla(e \cdot a) = d(a)$
 2. $D'(a\xi) = a(D\xi) + (\nabla a)\xi = D(a\xi)$
 3. Use the identity element $e \in A$
 $\nabla(e \cdot a) = \nabla(e)a + 1 \otimes d(a) = d(a)\nabla(e)a$
 4. $D'(a \otimes \xi) = D'(a\xi) = a(D\xi) + (\nabla a)\xi = a(D\xi) + \nabla(e \cdot a)\xi$
 $= D(a\xi) + \nabla(e)(a\xi)$

1.4 Graphing Finite Spectral Triples

Definition 3. A *graph* is a ordered pair $(\Gamma^{(0)}, \Gamma^{(1)})$. Where $\Gamma^{(0)}$ is the set of vertices (nodes) and $\Gamma^{(1)}$ a set of pairs of vertices (edges)

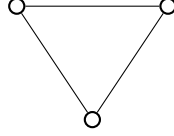


Figure 1: A simple graph with three vertices and three edges

Exercise 10

Show that any finite-dimensional faithful representation H of a matrix algebra A is completely reducible. To do that show that the complement W^\perp of an A -submodule $W \subset H$ is also an A -submodule of H .

$A \simeq \bigoplus_{i=1}^N M_{n_i}(\mathbb{C})$ is the matrix algebra then H is a Hilbert A -bimodule and W a submodule of A . Because we have $H = W \cup W^\perp$, then W^\perp is naturally a A -submodule, because elements in W^\perp need to satisfy the bimodularity.

Definition 4. A Λ -decorated graph is given by an ordered pair (Γ, Λ) of a finite graph Γ and a set of positive integers Λ with the labeling

- of the vertices $v \in \Gamma^{(0)}$ given by $n(v) \in \Lambda$
- of the edges $e = (v_1, v_2) \in \Gamma^{(1)}$ by operators
 - $D_e : \mathbb{C}^{n(v_1)} \rightarrow \mathbb{C}^{n(v_2)}$
 - and $D_e^* : \mathbb{C}^{n(v_2)} \rightarrow \mathbb{C}^{n(v_1)}$ its conjugate traspose (pullback?)

such that

$$n(\Gamma^{(0)}) = \Lambda \quad (22)$$

Question 3. Would then D_e be the pullback?

Question 4. These graphs are important in the next chapter I should look into it more, I don't understand much here, specific how to construct them with the abstraction of a spectral triple...

The operator D_e between \mathbf{n}_i and \mathbf{n}_j add up to D_{ij}

$$D_{ij} = \sum_{\substack{e=(v_1, v_2) \\ n(v_1)=\mathbf{n}_i \\ n(v_2)=\mathbf{n}_j}} D_e$$

Theorem 2. There is a on to one correspondence between finite spectral triples modulo unitary equivalence and Λ -decorated graphs, given by associating a finite spectral

triples (A, H, D) to a Λ decorated graph (Γ, Λ) in the following way:

$$A = \bigoplus_{n \in \Lambda} M_n(\mathbb{C}); \quad H = \bigoplus_{v \in \Gamma^{(0)}} \mathbb{C}^{n(v)}; \quad D = \sum_{e \in \Gamma^{(1)}} D_e + D_e^* \quad (23)$$

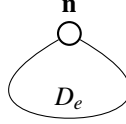
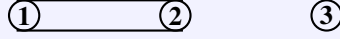


Figure 2: A Λ -decorated Graph of $(M_n(\mathbb{C}), \mathbb{C}^n, D = D_e + D_e^*)$

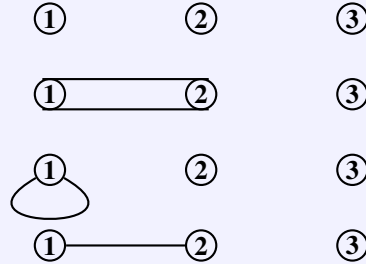
Exercise 11

Draw a Λ decorated graph corresponding to the spectral triple $(A = \mathbb{C}^3, H = \mathbb{C}^3, D = \begin{pmatrix} 0 & \lambda & 0 \\ \bar{\lambda} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix})$



Exercise 12

Use Λ -decorated graphs to classify all finite spectral triples (modulo unitary equivalence) on the matrix algebra $A = \mathbb{C} \oplus M_2(\mathbb{C})$



1.4.1 Graph Construction of Finite Spectral Triples

Algebra: We know if A acts on a finite dimensional Hilbert space then this C^* algebra is isomorphic to a matrix algebra so $A \simeq \bigoplus_{i=1}^N M_{n_i}(\mathbb{C})$. Where $i \in \hat{A}$ represents an equivalence class and runs from 1 to N , thus $\hat{A} \simeq \{1, \dots, N\}$. We label equivalence classes by \mathbf{n}_i , then $\hat{A} \simeq \{\mathbf{n}_1, \dots, \mathbf{n}_N\}$.

Hilbert Space: Since every Hilbert space that acts faithfully on a C^* algebra is completely reducible, it is isomorphic to the composition of irreducible representations.

$H \simeq \bigoplus_{i=1}^N \mathbb{C}^{n_i} \otimes V_i$. Where all V_i 's are Vector spaces, their dimension is the multiplicity of the representation landed by \mathbf{n}_i to V_i itself by the multiplicity space.

Finite Dirac Operator: D_{ij} is connecting nodes \mathbf{n}_i and \mathbf{n}_j , with a symmetric map $D_{ij} : \mathbb{C}^{n_i} \otimes V_i \rightarrow \mathbb{C}^{n_j} \otimes V_j$

To draw a graph, draw nodes in position $\mathbf{n}_i \in \hat{A}$. Multiple nodes at the same position represent multiplicities in H . Draw lines between nodes to represent D_{ij} .



Figure 3: Example