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Notes on
Noncommutative Geometry and Particle Phyiscs

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Contents

1 Excurse to Group Theory and Lie Groups

1.1 Groups and Representations

Definition 1. A Group G is a set with a binary operation on G satisfying.

1. $f, g \in G$ we have $fg = h \in G$.
2. $f(gh) = (fg)h$
3. $\exists\, e \in G\, \forall f \in G$ with $ef = fe = f$
4. $\forall f \in G\, \exists\, f^{-1} \in G$ with $ff^{-1} = f^{-1}f = e$

Definition 2. A Representation of a Group G is a mapping, D of elements of G onto a set of *linear operators* such that:

1. $D(e) = 1$, 1 is the identity operator in the space on which linear operators act
2. $D(g_1)D(g_2) = D(g_1 g_2)$, the mapping is linear in group the group operation

Just by looking at symmetries of a Group we can find a nice representation, and if the group is finite we can even find a matrix representation (Cheyley's Theorem). We all ready know a lot about linear algebra which will then allow us to study these Groups very thoroughly and derive physical properties with minimal information.

1.2 Lie Groups

Group elements now depend *smoothly* on a set *continuous parameters* $g(\alpha) \in G$. We are looking at continuous symmetries, e.g. a Sphere in $\mathbb{R}^3$ can be rotated in any direction without changing. The collection of rotations forms a Lie group because the group elements are smoothly differentiable.

1.2.1 Generators

We parameterize $g(\alpha)|_{\alpha=0} = e$ and we assume that near the identity element, the group elements can be described by a finite set of elements α_a for $a = 1, .., N$. For a representation D of this group, linear operators need to be parametrized the same way:

$$D(\alpha)|_{\alpha=0} = 1 \tag{1}$$

Because of the smoothness and continuity we can Taylor expand a representation near the identity:

$$D(\alpha) = 1 + id\alpha_a X_a + \cdots \tag{2}$$

$$\text{with } X_a = -i \frac{\partial D(\alpha)}{\partial \alpha_a}\bigg|_{\alpha=0} \quad [1] \tag{3}$$

We call X_a the *generators of the group*.

- If the parametrization is *parsimonious*[2] then all of X_a will be independent.
- If the representation is unitary then X_a will be *hermitian*, because of the i in the definition.

[2]parsimonious - All parameters are needed to distinguish between group elements

- Sophus Lie showed how to derive generators without representations.

Now let us go in some fixed infinitesimal direction from the identity.

$$D(d\alpha) = 1 + id\alpha_a X_a \tag{4}$$

Because of the group property of closure with respect to the group operation we can raise $D(d\alpha)$ to a large power and still get a group element.

$$D(\alpha) = \lim_{k\to\infty}(1 + i\frac{\alpha_a X_a}{k})^k = e^{i\alpha_a X_a} \tag{5}$$

This is called the *exponential parameterization*. Looking at the expression we see that group elements can be expressed in terms of generators, and generators form a vector space. They are often referred to any element in the real linear space spanned by $X_a's$.

1.2.2 Lie Algebras

Let us consider a parameter family of group elements created by one generator X_a:

$$U(\lambda) = e^{i\lambda\alpha_a X_a} \tag{6}$$

We know for that for the same generator the group multiplication is linear meaning:

$$U(\lambda_1)U(\lambda_2) = U(\lambda_1 + \lambda_2) \tag{7}$$

But if we multiply elements generated by two different generators the general case is

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} \neq e^{i(\alpha_a+\beta_b)X_a} \tag{8}$$

Yet because the exponentials are a representation of a group, and a group has closure under group operation we know the above needs to be true for some δ_a

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} = e^{i\delta_a X_a} \tag{9}$$

To further examine the exponent we rewrite the expression and Taylor expand $ln(1+K)$ to the second of $K = e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1$

$$\begin{aligned}
i\delta_a X_a =& ln(1+K) = K - \frac{K^2}{2} + \cdots && (10)\\
\text{and} \quad K =& e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1 && (11)\\
=& (1 + i\alpha_a X_a - \frac{1}{2}(\alpha_a X_a)^2 + \cdots) && (12)\\
&\cdot (1 + i\beta_b X_b - \frac{1}{2}(\beta_b X_b)^2 + \cdots) - 1 && (13)\\
=& i\alpha_a X_a + i\beta_b X_b - \alpha_a X_a \beta_b X_b && (14)\\
&- \frac{1}{2}(\alpha_a X_a)^2 - \frac{1}{2}(\beta_b X_b)^2 + \cdots && (15)
\end{aligned}$$

So:

$$
\begin{aligned}
i\delta_a X_a = & \; i\alpha_a X_a + i\beta_b X_b - \alpha_a X_a \beta_b X_b & (16)\\
& - \frac{1}{2}(\alpha_a X_a)^2 - \frac{1}{2}(\beta_b X_b)^2 & (17)\\
& + \frac{1}{2}(\alpha_a X_a + i\beta_b X_b - \alpha_a X_a \beta_b X_b)^2 & (18)\\
= & \; i\alpha_a X_a + i\beta_b X_b - \alpha_a X_a \beta_b X_b & (19)\\
& - \frac{1}{2}(\alpha_a X_a)^2 - \frac{1}{2}(\beta_b X_b)^2 & (20)\\
& + \frac{1}{2}(\alpha_a X_a)^2 + \frac{1}{2}(\beta_b X_b)^2 & (21)\\
& + \frac{1}{2}\alpha_a X_a \beta_b X_b + \frac{1}{2}\beta_b X_b \alpha_a X_a & (22)
\end{aligned}
$$

Because X's are linear operators $\alpha_a X_a \beta_b X_b \neq \beta_b X_b \alpha_a X_a$. These generators form an *algebra under commutation* and we get

$$
\begin{aligned}
i\delta_a X_a = & \; i\alpha_a X_a + i\beta_b X_b - \alpha_a X_a \beta_b X_b & (23)\\
& - \frac{1}{2}[\alpha_a X_a, \beta_b X_b] + \cdots & (24)
\end{aligned}
$$

Thus rewriting the equation gives us

$$[\alpha_a X_a, \beta_b X_b] = -2i(\delta_c - \alpha_c - \beta_c)X_c \cdots \equiv i\gamma_c X_c \tag{25}$$

Because this is true for all α and β, and considering the group closure, there exists some *real* f_{abc} called the *structure constant* satisfying.

$$\gamma_c = \alpha_a \beta_b f_{abc} \tag{26}$$

Which is the same as.

$$[X_a, X_b] = i f_{abc} X_c \tag{27}$$

This is called the *Lie algebra of a group*

So f is antisymmetric because $[A,B] = -[B,A]$, which means $f_{abc} = -f_{bac}$.
And δ can now be written as

$$\delta_a = \alpha_a + \beta_a - \frac{1}{2}\gamma_a \cdots \tag{28}$$

Just by following the properties of Lie Groups (dependence on parameters and smoothness) in a fixed direction near die identity to find physical statements. E.g. $[\hat{r}_i, \hat{p}_j] = i\hbar\delta_{ij}$ tells us that we can't know the position and the momentum of a particle exactly at a given time.