Noncommutative Geometry

Bachelor's seminar

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Introduction

- Noncommutative geometry (NCG) brings many mathematical fields together
- Interesting physics application
- First understand duality between (classical) geometry and commutative Algebras
- Gelfand-Naimark-Theoreme in Functional Analysis in the 1940s

Introduce:

Finite topological Space X consisting of N points. (discrete topology)

$$\begin{array}{ccccc} \bullet & \bullet & \cdots & \bullet \\ 1 & 2 & & N \end{array}$$

Commutative algebra of continuous functions on X

$$C(X) = \{f : X \to \mathbb{C} : \ f \text{ is continuous}\}$$

Spaces and commutative Algebras

Results of the Theorem:

- X and $C(X)$ contain the same information (duality)
- Construct X, given $C(X)$.
- Translate geometrical properties of X to algebraic data (metric, differential forms, vector fields, curvature, etc.)

The Finite Spectral Triple

- NCG extends this duality to noncommutative algebras
- Provides methods to deal with these algebras
- NCG is encoded in a spectral triple

The Finite Spectral Triple

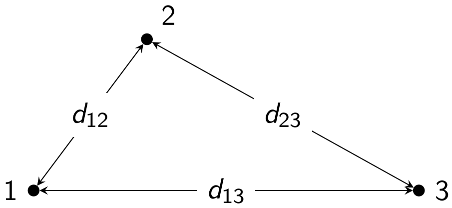
$$(A, H, D)$$

- A - Algebra
- H - Hilbertspace
- D - Symmetric operator acting on H

Introducing the Metric

- The metric describes distances between points on a space
- Simple example the discrete metric on a discrete space

$$d_{ij} = \begin{cases} 1 & \text{if} \quad i \neq j \\ 0 & \text{if} \quad i = j \end{cases}$$



Metric on NCG

- In NCG we can also define a metric
- replace algebra with a noncommutative matrix Algebra A
- a finite-dimensional Hilbertspace H
- and a hermitian matrix D

A metric constructed with (A, H, D)

$$d_{ij} = \sup_{a \in A}\{|\mathrm{Tr}(a(i)) - \mathrm{Tr}(a(j)) : ||[D, a]|| < 1\}$$

Differential One Forms

- Example: 1D calculus $f(x)dx$
- In NCG defining the differential one form requires only the spectral Triple (A, H, D)

Connes' Differential One Form

$$\Omega^1_D(A) = \{\sum_k a_k[D, b_k] : a_k, b_k \in A\}$$

With a consequent derivation of a algebra $d : A \to \Omega^1_D$, $d(\cdot) = [D, \cdot]$

Generalization to the Continuum

- Introduce a richer geometry
- From finite topological space to Manifolds generalized with noncommutativity
- From finite to general spectral triples with a self adjoint operator (Dirac Operator)

Applications In Physics

- NCG of Electrodynamics
- NCG of the Quantum Hall Effect
- NCG of the Standard Model, full Lagrangian

Bibliography

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