

# Notes on Noncommutative Geometry and Particle Physics

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## 1 Excuse

**Manifold:** A topological space that is locally Euclidean.

**Riemannian Manifold:** A Manifold equipped with a riemannian Metric, a symmetric bilinear form on Vector Fields  $\Gamma(TM)$

$$g : \Gamma(TM) \times \Gamma(TM) \rightarrow C(M) \quad (1)$$

with

$$g(X, Y) \in \mathbb{R} \quad \text{if } X, Y \in \mathbb{R} \quad (2)$$

$$g \text{ is } C(M)\text{-bilinear } \forall f \in C(M) : g(fX, Y) = g(X, fY) = fg(X, Y) \quad (3)$$

$$g(X, X) \begin{cases} \geq 0 & \forall X \\ = 0 & \forall X = 0 \end{cases} \quad (4)$$

$g$  on  $M$  gives rise to a distance function on  $M$

$$d_g(x, y) = \inf_{\gamma} \left\{ \int_0^1 (\dot{\gamma}(t), \dot{\gamma}(t)) dt; \quad \gamma(0) = x, \gamma(1) = y \right\} \quad (5)$$

Riemannian Manifold is called  $\text{spin}^c$  if there exists a vector bundle  $S \rightarrow M$  with an algebra bundle isomorphism

$$\mathbb{C}\text{I}(TM) \simeq \text{End}(S) \quad (\dim(M) \text{ even}) \quad (6)$$

$$\mathbb{C}\text{I}(TM)^\circ \simeq \text{End}(S) \quad (\dim(M) \text{ odd}) \quad (7)$$

$$(8)$$

$(M, S)$  is called the  **$\text{spin}^c$  structure on  $M$** .

$S$  is called the **spinor Bundle**.

$\Gamma(S)$  are the **spinors**.

Riemannian  $\text{spin}^c$  Manifold is called **spin** if there exists an anti-unitary operator  $J_M : \Gamma(S) \rightarrow \Gamma(S)$  such that:

1.  $J_M$  commutes with the action of real-valued continuous functions on  $\Gamma(S)$ .
2.  $J_M$  commutes with  $\text{Cliff}^-(M)$  (even case)  
 $J_M$  commutes with  $\text{Cliff}^-(M)^\circ$  (odd case)

$(S, J_M)$  is called the **spin Structure on  $M$**

$J_M$  is called the **charge conjugation**.

## 2 Noncommutative Geomtery of Electrodynamics

### 2.1 The Two-Point Space

Consider a two point space  $X := \{x, y\}$ . This space can be described with the following spectral triple

$$F_x := (C(X) = \mathbb{C}^2, H_F, D_F; J_F, \gamma_f). \quad (9)$$

Notes on the spectral triple:

- Action of  $C(X)$  on  $H_F$  is faithful ( $\dim(H_F) \geq 2$ )  
we choose  $H_F = \mathbb{C}^2$
- $\gamma_F$  is the  $\mathbb{Z}_2$  grading, which allows us to decompose  $H_F = H_F^+ \oplus H_F^- = \mathbb{C} \oplus \mathbb{C}$   
where  $H_F^\pm = \{\psi \in H_F \mid \gamma_F \psi = \pm \psi\}$  are the two eigenspaces
- $D_F$  interchanges between  $H_F^\pm$ ,  $D_F = \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix}$  where  $t \in \mathbb{C}$

**Proposition 1.**  $F_x$  can only have a real structure if  $D_F = 0$  in that case we have  $KO - \dim = 0, 2, 6$

*Proof.* There are two diagram representations of  $F_x$  at  $\underbrace{\mathbb{C} \oplus \mathbb{C}}_{C(X)}$  on  $\underbrace{\mathbb{C} \oplus \mathbb{C}}_{H_F}$

If  $F_x$  a real spectral triple then  $D_F$  can only go vertically or horizontally  $\Rightarrow D_F = 0$ . Furthermore the diagram on the left has  $KO$ -dimension 2 and 6, diagram on the right has  $KO$ -dimension 0 and 4. Yet  $KO$ -dimension 4 is not allowed because  $\dim(H_F^\pm) = 1$  (see Lemma 3.8 Book), so  $J_F^2 = -1$  is not allowed.  $\square$

$$\begin{array}{ccc}
& \mathbf{1} & \mathbf{1} \\
\mathbf{1}^\circ & & \circ \\
\mathbf{1}^\circ & \circ & 
\end{array}
\qquad
\begin{array}{ccc}
& \mathbf{1} & \mathbf{1} \\
\mathbf{1}^\circ & & \circ \\
\mathbf{1}^\circ & \circ & 
\end{array}$$

## 2.2 The product Space

Let  $M$  be a 4-dim Riemannian spin Manifold, then we have the almost commutative manifold  $M \times F_x$

$$M \times F_x = (C^\infty(M, \mathbb{C}^2, L^2(S)) \otimes \mathbb{C}^2, D_M \otimes 1; J_M \otimes J_F, \gamma_M \otimes \gamma_F) \quad (10)$$

( $J_M$  is missing need to choose)

$C^\infty(M, \mathbb{C}^2) \simeq C^\infty(M) \oplus C^\infty(M)$  (decomposition) and from Gelfand duality we we have

$$N := M \otimes X \simeq M \sqcup X \quad (11)$$

$H = L^2(S) \oplus L^2(S)$  (decomposition), such that for  $\underbrace{a, b \in C^\infty(M)}_{(a,b) \in C^\infty(N)}$  and  $\underbrace{\psi, \phi \in L^2(S)}_{(\psi, \phi) \in H}$  we have

$$(a, b)(\psi, \phi) = (a\psi, b\phi) \quad (12)$$

We can consider a distance formula on  $M \times F_x$  by

$$d_{D_F}(x, y) = \sup \{ |a(x) - a(y)| : a \in A_F, ||[D_F, a]|| \leq 1 \} \quad (13)$$

Now lets calculate the distance between two points on the two point space  $X = \{x, y\}$ , between  $x$  and  $y$ . Let  $a \in \mathbb{C}^2 = C(X)$ ,  $a$  is specified with two complex numbers  $a(x)$  and  $a(y)$

$$||[D_F, a]|| = ||(a(y) - a(x)) \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix}|| \leq 1 \quad (14)$$

$$\Rightarrow |a(y) - a(x)| \leq \frac{1}{|t|} \quad (15)$$

Therefore the distance between two points  $x$  and  $y$  is

$$d_{D_F}(x, y) = \frac{1}{|t|} \quad (16)$$

Note that if there exists  $J_M$  (real structure)  $\Rightarrow t = 0$  then  $d_{D_F}(x, y) \rightarrow \infty$ !

Now let  $p \in M$ , then take two points on  $N = M \times X$ ,  $(p, x)$  and  $(p, y)$  and  $a \in C^\infty(N)$  is determined by  $a_x(p) := a(p, x)$  and  $a_y(p) := a(p, y)$ . The distance between these two points is then

$$d_{D_F \otimes 1}(n_1, n_2) = \sup \{ |a(n_1) - a(n_2)| : a \in A, ||[D \otimes 1, a]|| \} \quad (17)$$

**Remark:** If  $n_1 = (p, x)$  and  $n_2 = (q, x)$  for  $p, q \in M$  then

$$d_{D_M \otimes 1}(n_1, n_2) = |a_x(p) - a_x(q)| \quad a_x \in C^\infty(M) \quad \text{with} \quad ||[D \otimes 1, a_x]|| \leq 1 \quad (18)$$

The distance turns to the geodesic distance formula

$$d_{D_M \otimes 1}(n_1, n_2) = d_g(p, q) \quad (19)$$

However if  $n_1 = (p, x)$  and  $n_2 = (q, y)$  then the two conditions are  $||[D_M, a_x]|| \leq 1$  and  $||[D_M, a_y]|| \leq 1$ . They have no restriction which results in the distance being infinite! And  $N = M \times X$  is given by two disjoint copies of  $M$  which are separated by infinite distance

**Note:** distance is only finite if  $[D_F, a] \neq 1$ . The commutator generates a scalar field say  $\phi$  and the finiteness of the distance is related to the existence of scalar fields.

### 2.3 $U(1)$ Gauge Group

Here we determine the Gauge theory corresponding to the almost commutative Manifold  $M \times F_X$ .

**Gauge Group of a Spectral Triple:**

$$\mathfrak{B}(A, H; J) := \{U = uJuJ^{-1} | u \in U(A)\} \quad (20)$$

**Definition 1.** A  $*$ -automorphism of a  $*$ -algebra  $A$  is a linear invertible map

$$\alpha : A \rightarrow A \quad \text{with} \quad (21)$$

$$\alpha(ab) = \alpha(a)\alpha(b) \quad (22)$$

$$\alpha(a)^* = \alpha(a^*) \quad (23)$$

The **Group of automorphisms of the  $*$ -Algebra  $A$**  is  $(A)$ .

The automorphism  $\alpha$  is called **inner** if

$$\alpha(a) = uau^* \quad \text{for } U(A) \quad (24)$$

where  $U(\mathfrak{A})$  is

$$U(A) = \{u \in A \mid uu^* = u^*u = 1\} \quad (\text{unitary}) \quad (25)$$

The Gauge group is given by the quotient  $U(A)/U(A_J)$ . We want a nontrivial Gauge group so we need to choose  $U(A_J) \neq U(A)$  which is the same as  $U((A_F)_{J_F}) \neq U(A_F)$ . We consider  $F_X$  to be

$$F_X := \left( \mathbb{C}^2, \mathbb{C}^2, D_F = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; J_F = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \gamma_F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right). \quad (26)$$

Here  $C$  is the complex conjugation, and  $F_X$  is a real even finite spectral triple (space) with  $KO - \dim = 6$

**Proposition 2.** The Gauge group  $\mathfrak{B}(F)$  of the two point space is given by  $U(1)$ .

*Proof.* Note that  $U(A_F) = U(1) \times U(1)$ . We need to show that  $U(\mathcal{A}_F) \cap U(A_F)_{J_F} \simeq U(1)$ , such that  $\mathfrak{B}(F) \simeq U(1)$ .

So for  $a \in \mathbb{C}^2$  to be in  $(A_F)_{J_F}$  it has to satisfy  $J_F a^* J_F = a$ .

$$J_F a^* J_F^{-1} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} \begin{pmatrix} \bar{a}_1 & 0 \\ 0 & \bar{a}_2 \end{pmatrix} \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} = \begin{pmatrix} a_2 & 0 \\ 0 & a_1 \end{pmatrix} \quad (27)$$

Which is only the case if  $a_1 = a_2$ . So we have  $(A_F)_{J_F} \simeq \mathbb{C}$ , whose unitary elements from  $U(1)$  are contained in the diagonal subgroup of  $U(\mathcal{A}_F)$ .  $\square$

Now we need to find the exact form of the field  $B_\mu$  to calculate the spectral action of a spectral triple. Since  $(A_F)_{J_F} \simeq \mathbb{C}$  we find that  $\mathfrak{h}(F) = \mathfrak{u}((A_F)_{J_F}) \simeq i\mathbb{R}$ .

An arbitrary hermitian field  $A_\mu = -ia\partial_\mu b$  is given by two  $U(1)$  Gauge fields  $X_\mu^1, X_\mu^2 \in C^\infty(M, \mathbb{R})$ . However  $A_\mu$  appears in combination  $A_\mu - J_F A_\mu J_F^{-1}$ :

$$B_\mu = A_\mu - J_F A_\mu J_F^{-1} = \begin{pmatrix} X_\mu^1 & 0 \\ 0 & X_\mu^2 \end{pmatrix} - \begin{pmatrix} X_\mu^2 & 0 \\ 0 & X_\mu^1 \end{pmatrix} =: \begin{pmatrix} Y_\mu & 0 \\ 0 & -Y_\mu \end{pmatrix} = Y_\mu \otimes \gamma_F \quad (28)$$

where  $Y_\mu$  the  $U(1)$  Gauge field is defined as

$$Y_\mu := X_\mu^1 - X_\mu^2 \in C^\infty(M, \mathbb{R}) = C^\infty(M, i\mathfrak{u}(1)). \quad (29)$$

**Proposition 3.** *The inner fluctuations of the almost-commutative manifold  $M \times F_X$  described above are parametrized by a  $U(1)$ -gauge field  $Y_\mu$  as*

$$D \mapsto D' = D + \gamma^\mu Y_\mu \otimes \gamma_F \quad (30)$$

The action of the gauge group  $\mathfrak{B}(M \times F_X) \simeq C^\infty(M, U(1))$  on  $D'$  is implemented by

$$Y_\mu \mapsto Y_\mu - i u \partial_\mu u^*; \quad (u \in \mathfrak{B}(M \times F_X)). \quad (31)$$

### 3 Electrodynamics

Now we use the almost commutative Manifold and the abelian gauge group  $U(1)$  to describe Electrodynamics. We arrive at a unified description of gravity and electrodynamics although in the classical level.

The almost commutative Manifold  $M \times F_X$  describes a local gauge group  $U(1)$ . The inner fluctuations of the Dirac operator describe  $Y_\mu$  the gauge field of  $U(1)$ . There arise two Problems:

(1): With  $F_X$ ,  $D_F$  must vanish, however this implies that the electrons are massless (this we do not want)

(2): The Euclidean action for a free Dirac field is

$$S = - \int i \bar{\psi} (\gamma^\mu \partial_\mu - m) \psi d^4 x, \quad (32)$$

$\psi$ ,  $\bar{\psi}$  must be considered as independent variables, which means  $S_F$  need two independent Dirac Spinors. We write  $\{e, \bar{e}\}$  for the ONB of  $H_F$ , where  $\{e\}$  is the ONB of

$H_F^+$  and  $\{\bar{e}\}$  the ONB of  $H_F^-$  with the real structure this gives us the following relations

$$J_F e = \bar{e} \quad J_F \bar{e} = e \quad (33)$$

$$\gamma_F e = e \quad \gamma_F \bar{e} = \bar{e}. \quad (34)$$

The total Hilbertspace is  $H = L^2(S) \otimes H_F$ , with  $\gamma_F$  we can decompose  $L^2(S) = L^2(S)^+ \oplus L^2(S)^-$ , so with  $\gamma = \gamma_M \otimes \gamma_F$  we can obtain the positive eigenspace  $H^+$

$$H^+ = L^2(S)^+ \otimes H_F^+ \oplus L^2(S)^- \otimes H_F^-. \quad (35)$$

For a  $\xi \in H^+$  we can write

$$\xi = \psi_L \otimes e + \psi_R \otimes \bar{e} \quad (36)$$

where  $\psi_L \in L^2(S)^+$  and  $\psi_R \in L^2(S)^-$  are the two Weyl spinors. We denote that  $\xi$  is only determined by one Dirac spinor  $\psi := \psi_L + \psi_R$ , **but we require two independent spinors**. This is too much restriction for  $F_X$ .

### 3.1 The Finite Space

Here we solve the two problems by enlarging(doubling) the Hilbertspace. This is done by introducing multiplicities in Krajewski Diagrams which will also allow us to choose a nonzero Dirac operator which will connect the two vertices (next chapter).

We start of with the same algebra  $C^\infty(M, \mathbb{C}^2)$ , corresponding to space  $N = M \times X \simeq M \sqcup M$ .

The Hilbertspace will describe four particles,

- left handed electrons
- right handed positrons

Thus we have  $\{ \underbrace{e_R, e_L}_{\text{left-handed}}, \underbrace{\bar{e}_R, \bar{e}_L}_{\text{right-handed}} \}$  the ONB for  $H_F \mathbb{C}^4$ .

Then with  $J_F$  we interchange particles with antiparticles we have the following properties

$$J_F e_R = \bar{e}_R \quad J_F e_L = \bar{e}_L \quad (37)$$

$$\gamma_F e_R = -e_R \quad \gamma_F e_L = e_L \quad (38)$$

and

$$J_F^2 = 1 \quad J_F \gamma_F = -\gamma_F J_F \quad (39)$$

This corresponds to KO-dim= 6. Then  $\gamma_F$  allows us to can decompose  $H$

$$H_F = \underbrace{H_F^+}_{\text{ONB } \{e_L, \bar{e}_L\}} \oplus \underbrace{H_F^-}_{\text{ONB } \{e_R, \bar{e}_R\}}. \quad (40)$$

Alternatively we can decompose  $H$  into the eigenspace of particles and their antiparticles (electrons and positrons) which we will use going further.

$$H_F = \underbrace{H_e}_{\text{ONB } \{e_L, e_R\}} \oplus \underbrace{H_{\bar{e}}}_{\text{ONB } \{\bar{e}_L, \bar{e}_R\}} \quad (41)$$

Now the action of  $a \in A = \mathbb{C}^2$  on  $H$  with respect to the ONB  $\{e_L, e_R, \bar{e}_L, \bar{e}_R\}$  is represented by

$$a = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \mapsto \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix} \quad (42)$$

Do note that this action commutes with the grading and that  $[a, b^\circ] = 0$  with  $b := J_F b^* J_F$  because both the left and the right action is given by diagonal matrices.

**Proposition 4.** *The data*

$$\left( \mathbb{C}^2, \mathbb{C}^2, D_F = 0; J_F = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \gamma_F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \quad (43)$$

*defines a real even spectral triple of KO-dimension 6.*

This spectral triple can be represented in the following Krajewski diagram, with two nodes of multiplicity two

$$\begin{array}{cc} \mathbf{1} & \mathbf{1} \\ \mathbf{1}^\circ & \odot \\ \mathbf{1}^\circ & \odot \end{array}$$

### 3.2 A noncommutative Finite Dirac Operator

Add a non-zero Dirac Operator to  $F_{ED}$ . From the Krajewski Diagram, we see that edges only exist between the multiple vertices. So we construct a Dirac operator mapping between the two vertices.

$$D_F = \begin{pmatrix} 0 & d & 0 & 0 \\ \bar{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{d} \\ 0 & 0 & d & 0 \end{pmatrix} \quad (44)$$

We can now consider the finite space  $F_{ED}$ .

$$F_{ED} := (\mathbb{C}^2, \mathbb{C}^4, D_F; J_F, \gamma_F) \quad (45)$$

where  $J_F$  and  $\gamma_F$  like before,  $D_F$  like above.

### 3.3 The almost-commutative Manifold

The almost commutative manifold  $M \times F_{ED}$  has  $\text{KO-dim}=2$ , it is the following spectral triple

$$M \times F_{ED} := (C^\infty(M, \mathbb{C}^2), L^2(S) \otimes \mathbb{C}^4, D_M \otimes 1 + \gamma_M \otimes D_F; J_M \otimes J_F, \gamma_M \otimes \gamma_F) \quad (46)$$

The algebra decomposition is like before

$$C^\infty(M, \mathbb{C}^2) = C^\infty(M) \oplus C^\infty(M) \quad (47)$$

The Hilbertspace decomposition is

$$H = (L^2(S) \otimes H_e) \oplus (L^2(S) \otimes H_{\bar{e}}). \quad (48)$$

Here we have the one component of the algebra acting on  $L^2(S) \otimes H_e$ , and the other one acting on  $L^2(S) \otimes H_{\bar{e}}$

The derivation of the gauge theory is the same for  $F_{ED}$  as for  $F_X$ , we have  $\mathfrak{B}(F) \simeq U(1)$  and for  $B_\mu = A_\mu - J_F A_\mu J_F^{-1}$

$$B_\mu = \begin{pmatrix} Y_\mu & 0 & 0 & 0 \\ 0 & Y_\mu & 0 & 0 \\ 0 & 0 & Y_\mu & 0 \\ 0 & 0 & 0 & Y_\mu \end{pmatrix} \quad \text{for } Y_\mu(x) \in \mathbb{R}. \quad (49)$$

We have one single  $U(1)$  gauge field  $Y_\mu$ , carrying the action of the gauge group

$$\mathfrak{B}(M \times F_{ED}) \simeq C^\infty(M, U(1)) \quad (50)$$

Our space  $N = M \times X \simeq M \sqcup M$  consists of two copies of  $M$ . If  $D_F = 0$  we have infinite distance between the two copies. Now we have  $D_F$  nonzero but  $[D_F, a] = 0 \quad \forall a \in A$  which still yields infinite distance.

**Question 1.** What does this imply (physically, mathematically)? Why can we continue even though we have infinite distance between the same manifold? What do we get if we fix this?