Notes on
Noncommutative Geometry and Particle Physics

Popovic Milutin

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Contents

1 Characters

Definition 1. The characters χ_D of a group representation D, are the *traces* of the linear operators of of the representation or their matrix elements.

$$\chi_D(g) \equiv \mathrm{Tr}(D(g)) = \sum_i (D(g))_{ii} \tag{1}$$

Advantages of the characters are:

- $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ so $\mathrm{Tr}(D(g^{-1}g_1g)) = \mathrm{Tr}(D(g_1))$
- equivalent representations have the *same* characters
- characters are different for inequivalent irreducible representation, say D_a and D_b then there is a orthogonality relation up to N (number of elements in the group):

$$\frac{1}{N}\sum_{g\in G} \chi_{D_a}(g)^* \chi_{D_b}(g) = \delta_{ab}$$

Furthermore characters are a *complete* basis for functions that are constant on the conjugacy class. Suppose $F(g_1)$ is such a function. We can expand this function in terms of matrix elements of irreducible representations.

$$\begin{aligned}
F(g_1) &= \sum_{a,j,l} \frac{1}{n_a} c^a_{jk} (D_a(g_1))_{jk} \\
&= \sum_{a,j,l} \frac{1}{n_a} c^a_{jk} (D_a(g^{-1} g_1 g))_{jk} \\
&= \sum_{a,j,k,g,l,m} \frac{1}{n_a} c^a_{jk} (D_a(g^{-1}))_{jl} (D_a(g_1))_{lm} (D_a(g))_{mk} \\
&= \sum_{a,j,l} \frac{1}{n_a} c^a_{jk} (D_a(g_1))_{lm} \delta_{jk} \delta_{lm} \\
&= \sum_{a,j,l} \frac{1}{n_a} c^a_{jj} (D_a(g_1))_{ll} \\
&= \sum_{a,j,l} \frac{1}{n_a} c^a_{jj} \chi_a(g_1)
\end{aligned}$$

where n_a denotes the dimension of the representation D_a.

We can use characters to find out how many irreducible representations appear in a reducible one and decompose it to it's irreducible components. We define the projection operator onto the subspace that transforms under the representation of a, where D is a arbitrary representation.

$$P_a = \frac{n_a}{N} \sum_{g \in G} \chi_{D_a}(g)^* D(g) \tag{2}$$

This gives us the projection to the original basis.

2 Spring System in a Equilateral Triangle

Consider three masses on the edges of a equilateral triangle connected by springs. The system has 6 degrees of freedom, the x, y coordinates of the three masses.

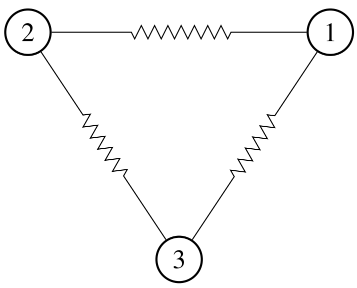


Figure 1: Spring System Equilateral Triangle (not equilateral in the picture)

First we will see what we can find just by looking at the symmetries of the system.

2.1 Group Theoretical Approach

The 6 degrees of freedom means that the system can be described with a 6 dimensional space. This is a tensor product of a 2 dimensional space of x and y coordinates and a 3 dimensional space of the masses (blocks).

$$(x_1, y_1, x_2, y_2, x_3, y_3)$$

The 3 dimensional space has S_3 symmetry (Group of all permutations of a three-element set), it can be represented with D_3 the dihedral group.

$$D_3(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad D_3(a_1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad D_3(a_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$
$$D_3(a_3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad D_3(a_4) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad D_3(a_5) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The 2 dimensional space also transforms under S_3, under a representation D_2

$$D_2(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad D_2(a_1) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad D_2(a_2) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$
$$D_2(a_3) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad D_2(a_4) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad D_2(a_5) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Then D_6 is the tensor product of D_3 and D_2:

$$(D_6(g))_{i\mu k\nu} = (D_3(g))_{ij}(D_2(g)_{\mu\nu} \tag{3}$$

For a_2 we would than have block matrices instead of 1 in D_3:

$$D_6(a_2) = \begin{pmatrix} 0 & D_2(a_2) & 0 \\ 0 & 0 & D_2(a_2) \\ D_2(a_2) & 0 & 0 \end{pmatrix}$$

All other elements follow accordingly.

Now S_3 has two 1 dimensional irreducible representations which are trivial because they map to the identity and one, 2 dimensional irreducible representation D_2 bellow is a character table of these representations

Table 1: Character table of S_3

S_3	e	$\{a_1, a_2\}$	$\{a_3, a_4, a_5\}$
χ_0	1	1	1
χ_1	1	1	-1
χ_2	2	-1	0
χ_3	3	0	1
χ_6	6	0	0

To find the normal modes of the oscillation around equilibrium we project $D_6(g)$ to D_0, D_1 and D_2 which are the irreducible representatives of S_3.

We start with D_0:

$$
\begin{aligned}
P_0 &= \frac{1}{6}\sum_{g\in G}\chi_0(g)^* D_6(g) \\
&= \frac{1}{6}\begin{pmatrix} D_2(e)+D_2(a_4) & D_2(a_2)+D_2(a_3) & D_2(a_1)+D_2(a_5) \\ D_2(a_1)+D_2(a_3) & D_2(e)+D_2(a_5) & D_2(a_2)+D_2(a_4) \\ D_2(a_1)+D_2(a_5) & D_2(a_1)+D_2(a_4) & D_2(e)+D_2(a_3) \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{6} \\ -\frac{1}{2} \\ \frac{\sqrt{3}}{6} \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{6} & -\frac{1}{2} & \frac{\sqrt{3}}{6} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}
\end{aligned} \tag{4}
$$

For D_1 we get:

$$
\begin{aligned}
P_1 &= \frac{1}{6}\sum_{g\in G}\chi_1(g)^* D_6(g) \\
&= \frac{1}{6}\begin{pmatrix} D_2(e)-D_2(a_4) & D_2(a_2)-D_2(a_3) & D_2(a_1)-D_2(a_5) \\ D_2(a_1)-D_2(a_3) & D_2(e)-D_2(a_5) & D_2(a_2)-D_2(a_4) \\ D_2(a_1)-D_2(a_5) & D_2(a_1)-D_2(a_4) & D_2(e)-D_2(a_3) \end{pmatrix} \\
&= \begin{pmatrix} -\frac{\sqrt{3}}{6} \\ \frac{1}{2} \\ -\frac{\sqrt{3}}{6} \\ -\frac{1}{2} \\ \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{3}}{6} & \frac{1}{2} & -\frac{\sqrt{3}}{6} & -\frac{1}{2} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}
\end{aligned} \tag{5}
$$

And for D_2 we get:

$$
\begin{aligned}
P_2 &= \frac{2}{6}\sum_{g\in G}\chi_2(g)^* D_6(g) \\
&= \frac{2}{6}\begin{pmatrix} 2D_2(e) & -D_2(a_2) & -D_2(a_1) \\ -D_2(a_1) & 2D_2(e) & -D_2(a_2) \\ -D_2(a_1) & -D_2(a_1) & 2D_2(e) \end{pmatrix}
\end{aligned}
$$

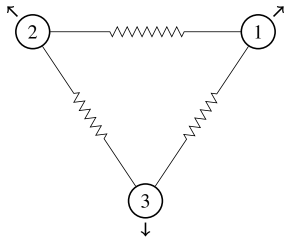
And the nontrivial modes provided by P_2 can be calculated by including translation in x and y direction T_x and T_y:

$$
T_x = \frac{1}{3}\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad T_y = \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}
$$

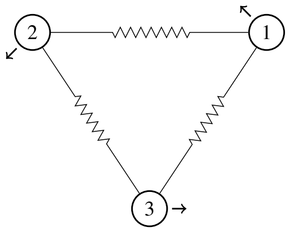
To see the modes we move mass 3 in the y direction which is the vector $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ and its mode can be get by appying it to $P_2 - T_x - T_y$:

$$
\begin{pmatrix} \frac{\sqrt{3}}{6} & -\frac{1}{6} & -\frac{\sqrt{3}}{6} & -\frac{1}{6} & 0 & \frac{1}{3} \end{pmatrix} \tag{6}
$$

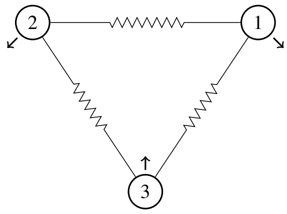
The three vectors in Equasions 4, 5 and 6 are one of the normal modes of the system, at the same time they are the eigenvectors of the motion equation of the system which will be introduced in the next chapter. Further more the corresponding modes are equaual to the following motions. Note there are three more nodes mabye I will get later into them, but they can be calculated like the mode from Eq. 6.



(a) Motion given by 4
Breathing Mode



(b) Motion given by 5
Rotation



(c) Motion given by 6

2.2 Physical Approach

The physical approach would be to construct the lagrangian $\mathfrak{L} = T - V$. Were T is simply the kinetic energy of the system in $\eta = (x_1, y_1, x_2, y_2, x_3, y_3)$ coordinates and for simplicity we set all masses to m.

$$T = \frac{m}{2}\dot{\eta}_i\dot{\eta}^i \quad \text{for}^{1} \;\; i = 1, \ldots, 6$$

For V the potential energy we have three springs and *small oscillations around equilibrium*, two of them are the offset of the one to the angle of $\theta = \pm\frac{\pi}{3}$, which is $\begin{pmatrix} cos(\theta) \\ sin(\theta) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \pm\frac{\sqrt{3}}{2} \end{pmatrix}$. Then V is:

$$\begin{aligned} V &= \frac{k}{2} U^i{}_j \eta_i \eta^j \\ &= \frac{k}{2}\Big((x_1 - x_2)^2 + (\frac{1}{2}(x_2 - x_3 + \frac{\sqrt{3}}{2}(y_2 - y_3))^2 + (\frac{1}{2}(x_1 - x_3) + \frac{\sqrt{3}}{2}(y_1 - y_3))^2 \Big) \end{aligned}$$

Where U is:

$$U = \frac{1}{4}\begin{pmatrix} 5 & \sqrt{3} & -4 & 0 & -1 & -\sqrt{3} \\ \sqrt{3} & 3 & 0 & 0 & -\sqrt{3} & -3 \\ -4 & 0 & 5 & -\sqrt{3} & -1 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 3 & \sqrt{3} & -3 \\ -1 & -\sqrt{3} & -1 & \sqrt{3} & 2 & 0 \\ -\sqrt{3} & -3 & \sqrt{3} & -3 & 0 & 6 \end{pmatrix}$$

The Euler-Lagrange equations then give us :

$$m\ddot{\eta}^i = -kU^i{}_j \eta^j \tag{7}$$

With the exponential ansatz $\eta = \eta_0 e^{i\omega t}$ we get

$$U^i{}_j \eta^j = \lambda \eta^i \qquad \text{where } \lambda = \frac{m\omega^2}{k} \tag{8}$$

The normal modes are the eigenvalues of U, which were calculated only using symmetry in Equations 4, 5 and 6. With some more character theory we can extract even more information explicitly on the eigenvalues. To sum it up there are four different eigenvalues, which means that some modes have the same frequency, to find the frequencies (eigenvalues) go through a calculation in diagonal coordinates of U and $D(g)$ (we know all traces of $D(g)$) the trace is then invariant and the sum of all eigenvectors, it will give 3 equation with four unknown but one is the trivial oscillation with $\lambda = 0$ making the system solvable.

3 Noncommutative geometric Finite Spaces

3.1 Metric on Finite Discrete Spaces

Let X be a *finite discrete space*, described by a structure space $\hat{A}$ of a matrix algebra A. To describe distance between two points in X (as we would in a metric space) we use an array $\{d_{ij}\}_{i,j\in X}$ of *real nonnegative* entries on X such that

- $d_{ij} = d_{ji}$ Symmetric
- $d_{ij} \leq d_{ik}d_{kj}$ Triangle Inequality
- $d_{ij} = 0$ for $i = j$ (the same element)

Example 1. The *discrete metric* on a discrete space X is $d_{ij} = 1$ for $i \neq j$ and $d_{ij} = 0$ for $i = j$
Properties of the discrete metric https://en.wikipedia.org/wiki/Discrete_space#Properties

The commutative case, where A is assumed commutative can desrcibe the metric on X in terms of algebraic data. The result is the following theorem can be proved.

Theorem 1. *Let d_{ij} be a metric on X a finite discrete space with N points, $A = \mathbb{C}^N$ with elements $a = (a(i))_{i=1}^N$ such that $\hat{A} \simeq X$. Then there exists a representation π of A on a finite-dimensional inner product space H and a symmetric operator D on H such that*

$$d_{ij} = \sup_{a\in A}\{|a(i) - a(j)| : ||[D, \pi(a)]|| \leq 1\} \tag{9}$$

Proof. Claim that this would follow from the equality:

$$||[D, \pi(a)]|| = \max_{k\neq l}\{\frac{1}{d_{kl}}|a(k) - a(l)|\} \tag{10}$$

This can be proved with induction. Set $N = 2$ then $H = \mathbb{C}^2$, $\pi : A \to L(H)$ and a hermitian matrix D.

$$\pi(a) = \begin{pmatrix} a(1) & 0 \\ 0 & a(2) \end{pmatrix} \quad D = \begin{pmatrix} 0 & (d_{12})^{-1} \\ (d_{21})^{-1} & 0 \end{pmatrix} \tag{11}$$

Then:

$$||[D, \pi(a)]|| = (d_{12})^{-1}|a(1) - a(2)| \tag{12}$$

Suppose this holds for N with π_N, $H_N = \mathbb{C}^N$ and D_N. Then it holds for $N+1$ with $H_{N+1} = H_N \oplus \bigoplus_{i=1}^N H_N^i$ and

$$\pi_{N+1}(a(1),\ldots,a(N+1)) = \pi_N(a(1),\ldots,a(N)) \oplus \begin{pmatrix} a(1) & 0 \\ 0 & a(N+1) \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} a(N) & 0 \\ 01 & a(N+1) \end{pmatrix} \tag{13}$$

And D_{N+1}:

$$D_{N+1} = D_N \oplus \begin{pmatrix} 0 & (d_{1(N+1)})^-1 \\ (d_{1(N+1)})^-1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & (d_{N(N+1)})^-1 \\ (d_{N(N+1)})^-1 & 0 \end{pmatrix} \tag{14}$$

From this follows 10. Then we can continue the proof, we set for fixed i, j, $a(k) = d_{ik}$, which gives $|a(i) - a(j)| = d_{ij}$

$$\Rightarrow \frac{1}{d_{kl}} |a(k) - a(l)| = \frac{1}{d_{kl}} |d_{ik} - d_{il}| \leq 1 \tag{15}$$

□

The translation of the metric on X into algebraic data assumes commutativity in A, but this can be extended to noncommutative matrix algebra, with the following metric on a structure space $\hat{A}$ of a matrix algebra $M_{n_i}(\mathbb{C}$

$$d_{ij} = \sup_{a \in A} \left\{ |\mathrm{Tr}(a(i)) - \mathrm{Tr}((a(j))| : ||[D,a]|| \leq 1 \right\} \tag{16}$$

This is special case of the Connes' distance formula on a structure space of A.

Definition 2. A *finite spectral triple* is a tripe (A,H,D), where A is a unital $*$-algebra, faithfully represented on a finite-dimensional Hilbert space H, with a symmetric operator $D : H \to H$.

A is automatically a matrix algebra.