Notes on
Noncommutative Geometry and Particle Physics

Popovic Milutin

Week 6: 19.03 - 26.03

Contents

1 Finite Real Noncommutative Spaces

1.1 Finite Real Spectral Triples

Add on to finite real spectral triples a *real structure*. The requirement is that H is a A-A-bimodule (before only a A-left module).

For this we introduce a $\mathbb{Z}_2$-grading γ with

$$\gamma^* = \gamma \tag{1}$$
$$\gamma^2 = 1 \tag{2}$$
$$\gamma D = -D\gamma \tag{3}$$
$$\gamma a = a\gamma \quad a \in A \tag{4}$$

Definition 1. A *finite real spectral triple* is given by a finite spectral triple (A,H,D) and a anti-unitary operator $J : H \to H$ called the *real structure*, such that

$$a^\circ := Ja^*J^{-1} \tag{5}$$

is a right representation of A on H, that is $(ab)^\circ = b^\circ a^\circ$. With two requirements

$$[a,b^\circ] = 0 \tag{6}$$
$$[[D,a],b^\circ] = 0. \tag{7}$$

They are called the *commutant property*, and mean that the left action of an element in A and $\Omega^1_D(A)$ commutes with the right action on A.

Definition 2. The *KO*-dimension of a real spectral triple is determined by the sings $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ appearing in

$$J^2 = \varepsilon \tag{8}$$

$$JD = \varepsilon\, DJ \tag{9}$$

$$J\gamma = \varepsilon'' \gamma J. \tag{10}$$

Table 1: *KO*-dimension k modulo 8 of a real spectral triple

k	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

Definition 3. An opposite-algebra A° of a A is defined to be equal to A as a vector space with the opposite product

$$a \circ b := ba \tag{11}$$

$$\Rightarrow a^\circ = Ja^* J^{-1} \quad \text{defines the left representation of } A^\circ \text{ on } H \tag{12}$$

Example 1. Matrix algebra $M_N(\mathbb{C})$ acting on $H = M_N(\mathbb{C})$ by left matrix multiplication with the Hilbert Schmidt inner product.

$$\langle a, b \rangle = \mathrm{Tr}(a^* b) \tag{13}$$

Then we define $\gamma(a) = a$ and $J(a) = a^*$ with $a \in H$. Since D mus be odd with respect to γ it vanishes identically.

Exercise 1

In the previous example, show that the right action on $M_N(\mathbb{C})$ on $H = M_N(\mathbb{C})$ as defined by $a \mapsto a^\circ$ is given by right matrix multiplication.

$$a^\circ \xi = Ja^* J^{-1} = Ja^* \xi^* = J\xi a = \xi^* a$$

1.2 Morphisms Between Finite Real Spectral Triples

Extend unitary equivalence of finite spectral triples to real ones (with J and γ)

Definition 4. We call two finite real spectral triples $(A_1, H_1, D_1; J_1, \gamma_1)$ and $(A_2, H_2, D_2; J_2, \gamma_2)$ unitarily equivalent if $A_1 = A_2$ and if there exists a unitary operator $U : H_1 \to H_2$ such that

$$U\pi_1(a)U^* = \pi_2(a) \tag{14}$$

$$UD_1U^* = D_2 \tag{15}$$

$$U\gamma_1 U^* = \gamma_2 \tag{16}$$

$$UJ_1U^* = J_2 \tag{17}$$

Definition 5. Let E be a B-A bimodule. The *conjugate Module* E° is given by the A-B-bimodule.

$$E^\circ = \{\bar{e} : e \in E\} \tag{18}$$

with

$$a \cdot \bar{e} \cdot b = b^* \bar{e} a^* \quad \forall a \in A, b \in B \tag{19}$$

E° is not a Hilbert bimodule for (A,B) because it doesn't have a natural B-valued inner product. But there is a A-valued inner product on the left A-module E° with

$$\langle \bar{e}_1, \bar{e}_2 \rangle = \langle e_2, e_1 \rangle \quad e_1, e_2 \in E \tag{20}$$

and linearity in A:

$$\langle a\bar{e}_1, \bar{e}_2 \rangle = a \langle \bar{e}_1, \bar{e}_2 \rangle \quad \forall a \in A. \tag{21}$$

1.2.1 Construction of a Finite Real Spectral Triple from a Finite Real Spectral Triple

Given a Hilbert bimodule E for (B,A) we construct a spectral triple $(B, H', D'; J', \gamma')$ from $(A, H, D; J, \gamma)$

For the H' we make a $\mathbb{C}$-valued inner product on H' by combining the A valued inner product on E and E° with the $\mathbb{C}$-valued inner product on H.

$$H' := E \otimes_A H \otimes_A E^\circ \tag{22}$$

Then the action of B on H' is:

$$b(e_2 \otimes \xi \otimes \bar{e}_2) = (be_1) \otimes \xi \otimes \bar{e}_2 \tag{23}$$

The right action of B on H' defined by action on the right component E°

$$J'(e_1 \otimes \xi \otimes \bar{e}_2) = e_2 \otimes J\xi \otimes \bar{e}_1 \tag{24}$$

with $b^\circ = J' b^* (J')^{-1}$, $b^* \in B$ action on H'.

Then the connections

$$\nabla : E \to E \otimes_A \Omega^1_D(A) \tag{25}$$
$$\bar{\nabla} : E^\circ \to \Omega^1_D(A) \otimes_A E^\circ \tag{26}$$

give us the Dirac operator on $H' = E \otimes_A H \otimes_A E^\circ$

$$D'(e_1 \otimes \xi \otimes \bar{e}_2) = (\nabla e_1)\xi \otimes \bar{e}_2 + e_1 \otimes D\xi \otimes \bar{e}_2 + e_1 \otimes \xi(\bar{\nabla}\bar{e}_2) \tag{27}$$

And the right action of $\omega \in \Omega^1_D(A)$ on $\xi \in H$ is defined by

$$\xi \mapsto \varepsilon' J \omega^* J^{-1} \xi \tag{28}$$

Finally for the grading

$$\gamma' = 1 \otimes \gamma \otimes 1 \tag{29}$$

Theorem 1. *Suppose $(A, H, D; J, \gamma)$ is a finite spectral triple of KO-dimension k, let ∇ be like above satisfying the compatibility condition (like with finite spectral triples).*

Then $(B, H', D'; J', \gamma')$ is a finite spectral triple of KO-Dimension k. (H', D', J', γ' like above)

Proof. The only thing left is to check if the *KO*-dimension is preserved, for this we check if the ε's are the same.

$$(J')^2 = 1 \otimes J^2 \otimes 1 = \varepsilon$$
$$J'\gamma' = \varepsilon''\gamma' J'$$

and for ε'

$$\begin{aligned} J'D'(e_1 \otimes \xi \otimes \bar{e}_2) &= J'((\nabla e_1)\xi \otimes \bar{e}_2 + e_1 \otimes D\xi \otimes \bar{e}_2 + e_1 \otimes \xi(\tau \nabla e_2)) \\ &= \varepsilon' D'(e_2 \otimes J\xi \otimes \bar{e}_2) \\ &= \varepsilon' D' J'(e_1 \otimes \xi \bar{e}_2) \end{aligned}$$

$\square$