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Abstract

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1 Introduction

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2 Main Section

2.1 Noncommutative Geometric Spaces

2.1.1 $*$-Algebra

To grasp the idea of encoding geometrical data into a spectral triple we introduce the first ingredient of a spectral triple, an unital $*$ algebra.

Definition 1

A vector space A over $\mathbb{C}$ is called a complex, unital Algebra if,
$\forall a, b \in A$:

$$A \times A \to A \tag{2.1}$$

$$(a,\ b)\ \mapsto\ a \cdot b, \tag{2.2}$$

with an identity element:

$$1a = a1 = a. \tag{2.3}$$

Extending the definition, a $*$-algebra is an algebra A with a conjugate linear map (involution) $* : A \to A$, $\forall a, b \in A$ satisfying

$$(a\,b)^* = b^* a^*, (a^*)^* \qquad = a. \tag{2.4}$$

In the following all unital algebras are referred to as algebras.

2.1.2 Finite Discrete Space

Let us consider an example of an $*$-algebra of continuous functions $C(X)$ on a discrete topological space X with N points. Functions of a continuous $*$-algebra $C(X)$ assign values to $\mathbb{C}$, thus $f,\ g \in C(X)$, $\lambda \in \mathbb{C}$ and $x \in X$ they provide the following structure:

- *pointwise linear*
  $(f+g)(x) = f(x) + g(x)$,
  $(\lambda\ f)(x) = \lambda(f(x))$,
- *pointwise multiplication*
  $f\ g\ (x) = f(x)g(x)$,
- *pointwise involution*
  $f^*(x) = \overline{f(x)}$.

The $*$-algebra $C(X)$ is *isomorphic* to a $*$-algebra $\mathbb{C}^N$ with involution (N number of points in X), we write $C(X) \simeq \mathbb{C}^N$. Isomorphisms are bijective maps that preserve structure and don't lose physical information. A function $f : X \ \rightarrow\ \mathbb{C}$ can be represented with $N \times N$ diagonal matrices, where each diagonal value represents the function value at the corresponding i-th point for $i = 1, ..., N$. Because of matrix multiplication and hermitian conjugate of matrices we have a preserving structure.

Moreover we can *map* between finite discrete spaces X_1 and X_2 with a function

$$\phi :\ X_1 \rightarrow\ X_2. \tag{2.5}$$

For every such map there exists a corresponding map

$$\phi^* : C(X_2)\ \rightarrow C(X_1), \tag{2.6}$$

which 'pulls back' values even if ϕ is not bijective. Note that the pullback doesn't map points back, but maps functions on an $*$-algebra $C(X)$. The pullback, in literature often called a $*$-homomorphism or a $*$-algebra map under pointwise product has the following properties

- $\phi^*(f\ g) = \phi^*(f)\ \phi^*(g)$,
- $\phi^*(\overline{f}) = \overline{\phi^*(f)}$,
- $\phi^*(\lambda\ f + g) = \lambda\ \phi^*(f) + \phi^*(g)$.

The map $\phi : X_1\ \rightarrow\ X_2$ is an injective (surjective) map, if only if the corresponding pullback $\phi^* : C(X_2)\ \rightarrow\ C(X_1)$ is surjective (injective). Let us say, that X_1 has n points and X_2 with m points. Then there are three different cases, first $n = m$ and obviously ϕ is bijective and ϕ^* too. Then $n > m$, in this case ϕ assigns n points to m points when $n > m$, which is by definition surjective. On the other hand ϕ^* assigns m points to n points when $n > m$, which is by definition injective. Lastly $n < m$, which is completely analogous to the case $n > m$.

2.1.3 Matrix Algebras

Definition 2

A (complex) matrix algebra A is a direct sum, for $n_i, N \in \mathbb{N}$

$$A = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}). \tag{2.7}$$

The involution is the hermitian conjugate, a $*$ algebra with involution is referred to as a matrix algebra

From a topological discrete space X, we can construct a $*$-algebra $C(X)$ which is isomorphic to a matrix algebra A. Then the question instantly arises, if we can construct X given A? For a matrix algebra A, which in most cases is not commutative, the answer is generally no.

Thus there are two options. We can restrict ourselves to commutative matrix algebras, which are the vast minority and not physically interesting. Or we can allow more morphisms (isomorphisms) between matrix algebras.

2.1.4 Finite Inner Product Spaces and Representations

Until now we looked at finite topological discrete spaces, moreover we can consider a finite dimensional inner product space H (finite Hilbert-spaces), with inner product $(\cdot,\cdot) \to \mathbb{C}$. We denote $L(H)$ as the $*$-algebra of operators on H equipped with a product given by composition and involution of the adjoint, $T \mapsto T^*$. Then $L(H)$ is a *normed vector space* with

$$\|T\|^2 = \sup_{h \in H} \big\{ (T\ h,\ T\ h) : (h,\ h) \leq 1 \big|\ T \in L(H) \big\}, \tag{2.8}$$

$$\|T\| = \sup \big\{ \sqrt{\lambda} :\ \lambda \text{ eigenvalue of } T \big\}. \tag{2.9}$$

This allows us to define representations of $*$-algebras.

Definition 3

The representation of a finite dimensional $*$-algebra A is a pair (H, π), where H is a finite dimensional inner product space and π is a $*$-algebra map

$$\pi : A \ \to \ L(H). \tag{2.10}$$

We call the representation (H, π) irreducible if

- *$H \neq \emptyset$,*
- *only $\emptyset$ or H is invariant under the action of A on H.*

Here are some examples of reducible and irreducible representations

- For $A = M_n(\mathbb{C})$ the representation $H = \mathbb{C}^n$, A acts as matrix multiplication H is irreducible.
- For $A = M_n(\mathbb{C})$ the representation $H = \mathbb{C}^n \oplus \mathbb{C}^n$, with $a \in A$ acting in block form $\pi : a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ is reducible.

Naturally there are also certain equivalences between different representations.

Definition 4

Two representations of a $*$-algebra A, (H_1, π_1) and (H_2, π_2) are called unitary equivalent if there exists a map $U : H_1 \to H_2$ such that.

$$\pi_1(a) = U^* \pi_2(a) U \tag{2.11}$$

Furthermore we define a mathematical structure called the structure space, which will later become important, when speaking of the duality between a spectral triple and a space.

Definition 5

Let A a $*$-algebra then, $\hat{A}$ is called the structure space of all unitary equivalence classes of irreducible representations of A

Given a representation (H, π) of a $*$-algebra A, the **commutant** $\pi(A)'$ of $\pi(A)$ is defined as a set of operators in $L(H)$ that commute with all $\pi(a)$

$$\pi(A)' = \{T \in L(H) : \ \pi(a)\, T = T\, \pi(a) \ \ \forall a \in A\} \tag{2.12}$$

The commutant $\pi(A)'$ is also a $*$-algebra, because it has unital, associative and involutive properties. We note that $\pi(a) \in L(H)$ $\forall a \in A$, unitary property is given by the unital operator of the $*$-algebra of operators $L(H)$, which exists by definition because H is a inner product space. Associativity is given by the $*$-algebra of $L(H)$, where $L(H) \times L(H) \mapsto L(H)$, which is associative by definition. The involutive property is also given by the $*$-algebra $L(H)$ with a map $* : L(H) \mapsto L(H)$ only for a T that commutes with $\pi(a)$.

For a unital algebra $*$-algebra A, the matrices $M_n(A)$ with entries in A form a unital $*$-algebra, because unitary operation in $M_n(A)$ is given by the identity Matrix, which has to exists in every entry in $M_n(A)$, and behaves like in A. Associativity is given by matrix multiplication. Lastly involution is given by the conjugate transpose.

A representation $\pi : A \ \to \ L(H)$ of a $*$-algebra A, for $H^n = H \oplus ... \oplus H$, n times. Then we have the following representation $\tilde{\pi} : M_n(A) \to L(H^n)$ for the Matrix Algebra with $\tilde{\pi}((a_{ij})) = (\tilde{\pi}(a_{ij})) \in M_n(A)$. We have direct isomorphisms of $A \simeq M_n(A)$ and $H \simeq H^n$ meaning $\tilde{\pi}$ is a valid reducible representation.

Let $\tilde{\pi} : M_n(A) \to L(H^n)$ be a $*$ algebra representation of $M_n(A)$, then $\pi : A \to L(H^n)$ is a representation of A. The fact that $\tilde{\pi}$ and π are unitary equivalent, there is a map $U : H^n \to H^n$ given by $U = \mathbb{1}_n$, thus

$$\pi(a) = \mathbb{1}_n^* \, \tilde{\pi}((a_{ij})), \tag{2.13}$$

$$\mathbb{1}_n = \tilde{\pi}((a_{ij})) = \pi(a_{ij}) \Rightarrow a_{ij} = a \, \mathbb{1}_n. \tag{2.14}$$

A commutative matrix algebra can be used to reconstruct a discrete space. The structure space $\hat{A}$ is used for this. Because $A \simeq \mathbb{C}^N$ all irreducible representation are of the form

$$\pi_i : (\lambda_1, ..., \lambda_N) \in \mathbb{C}^N \mapsto \lambda_i \in \mathbb{C} \tag{2.15}$$

for $i = 1, ..., N$ and thus $\hat{A} \simeq \{1, ..., N\}$. The conclusion is that, there is a duality between discrete spaces and commutative matrix algebra this duality is called the *finite dimensional Gelfand duality*

Our aim is to construct a duality between finite dimensional spaces and *equivalence classes* of matrix algebras, to preserve general non-commutativity of matrices. Equivalence classes are described by a generalized notion of isomorphisms between matrix algebras (*Morita Equivalence*)

2.1.5 Algebraic Modules

An important notion for Morita Equivalence are algebraic modules, later extended with Hilbert bimodules.

Definition 6

Let A, B be algebras (need not be matrix algebras)

1. *left A-module is a vector space E, that carries a left representation of A, that is $\exists$ a bilinear map $\gamma : A \times E \to E$ with*

$$(a_1\ a_2) \cdot e = a_1 \cdot (a_2 \cdot e); \quad a_1, a_2 \in A, e \in E. \tag{2.16}$$

2. *right B-module is a vector space F, that carries a right representation of A, that is there exists a bilinear map $\gamma : F \times B \to F$ with*

$$f \cdot (b_1\ b_2) = (f \cdot b_1) \cdot b_2; \quad b_1, b_2 \in B, f \in F \tag{2.17}$$

3. *left A-module and right B-module is a bimodule, a vector space E satisfying*

$$a \cdot (e \cdot b) = (a \cdot e) \cdot b; \quad a \in A, b \in B, e \in E \tag{2.18}$$

An A-**module homomorphism** as linear map $\phi : E \to F$ which respects the representation of A, e.g. for left module.

$$\phi(a\ e) = a\phi(e); \quad a \in A, e \in E. \tag{2.19}$$

We will use the notation

- ${}_AE$, for left A-module E;
- ${}_AE_B$, for right B-module F;
- ${}_AE_B$, for A-B-bimodule E, simply bimodule.

From a simple observation, we see that an arbitrary representation $\pi : A \to L(H)$ of a $*$-algebra A, turns H into a left module ${}_AH$. If ${}_AH$ than $(a_1\ a_2)h = a_1(a_2\ h)$ for $a_1, a_2 \in A$ and $h \in H$. We take the representation of an $a \in A$, $\pi(a)$, and write

$$\big(\pi(a_1)\ \pi(a_2)\big)h = \pi(a_1)\big(\pi(a_2)\ h\big) = \big(T_1\ T_2\big)h = T_1\big(T_2\ h\big) \tag{2.20}$$

For $T_1, T_2 \in L(H)$, which operate naturally from the left on h.

Furthermore notice that that an $*$-algebra A is a bimodule ${}_AA_A$ with itself, given by the map

$$\gamma : A \times A \times A \to A, \tag{2.21}$$

which is the inner product of a $*$-algebra.

2.1.6 Balanced Tensor Product and Hilbert Bimodules

Definition 7

Let A be an algebra, E be a right A-module and F be a left A-module. The balanced tensor product of E and F forms a A-bimodule.

$$E \otimes_A F := E \otimes F / \left\{ \sum_i e_i a_i \otimes f_i - e_i \otimes a_i f_i : \quad a_i \in A,\ e_i \in E,\ f_i \in F \right\}. \quad (2.22)$$

The $/$ denotes the quotient space. By that the operation $\otimes_A$ takes two left/right modules and makes a bimodule with the help the tensor product of the two modules and the quotient space that takes out all the elements from the tensor product that don't preserver the left/right representation and that are duplicates.

Definition 8

Let A, B be matrix algebras. The Hilbert bimodule for (A,B) is given by an A-B-bimodue E and by an B-valued inner product $\langle \cdot,\cdot \rangle_E : E \times E \to B$, which satisfies the following conditions for $e, e_1, e_2 \in E$, $a \in A$ and $b \in B$

$$\langle e_1,\ a \cdot e_2 \rangle_E = \langle a^* \cdot e_1,\ e_2 \rangle_E \qquad \textit{sesquilinear in } A, \qquad (2.23)$$
$$\langle e_1,\ e_2 \cdot b \rangle_E = \langle e_1,\ e_2 \rangle_E b \qquad \textit{scalar in } B, \qquad (2.24)$$
$$\langle e_1,\ e_2 \rangle_E = \langle e_2,\ e_1 \rangle_E^* \qquad \textit{hermitian}, \qquad (2.25)$$
$$\langle e,\ e \rangle_E \geq 0 \qquad \textit{equality holds iff } e = 0. \qquad (2.26)$$

We denote $KK_f(A,\ B)$ as the set of all Hilbert bimodules of $(A,\ B)$.

And indeed the Hilbert bimodule extension takes a representation $\pi : A \to L(H)$ of a matrix algebra A and turns H into a Hilbert bimodule for $(A, \mathbb{C})$, because the representation of $a \in A$, $\pi(a) = T \in L(H)$ fulfills the conditions of the $\mathbb{C}$-valued inner product for $h_1, h_2 \in H$

- $\langle h_1,\ \pi(a)\ h_2 \rangle_{\mathbb{C}} = \langle h_1,\ T\ h_2 \rangle_{\mathbb{C}} = \langle T^* h_1, h_2 \rangle_{\mathbb{C}}$, T^* given by the adjoint,
- $\langle h_1,\ h_2\ \pi(a) \rangle_{\mathbb{C}} = \langle h_1,\ h_2\ T \rangle_{\mathbb{C}} = \langle h_1,\ h_2 \rangle_{\mathbb{C}}$, T acts from the left,
- $\langle h_1,\ h_2 \rangle_{\mathbb{C}}^* = \langle h_2,\ h_1 \rangle_{\mathbb{C}}$, hermitian because of the $\mathbb{C}$-valued inner product
- $\langle h_1,\ h_2 \rangle \geq 0$, $\mathbb{C}$-valued inner product.

Take again the $A - A$ bimodule given by an $*$-algebra A, it is in $KK_f(A,\ A)$. This becomes clear by looking at the following inner product $\langle \cdot,\cdot \rangle_A : A \times A \to A$:

$$\langle a,\ a \rangle_A = a^* a' \quad a, a' \in A. \qquad (2.27)$$

Simply checking the conditions in $\langle \cdot,\cdot \rangle_A$ for $a, a_1, a_2 \in A$

$$\langle a_1,\ a \cdot a_2 \rangle_A = a^* a \cdot a_2 = (a^* a_1)^*\ a_2 = \langle a^*\ a_1,\ a_2 \rangle, \qquad (2.28)$$
$$\langle a_1,\ a_2 \cdot a \rangle_A = a_1^*\ (a_2 \cdot a) = (a^* a_2) \cdot a = \langle a_1,\ a_2 \rangle_A\ a, \qquad (2.29)$$
$$\langle a_1,\ a_2 \rangle_A^* = (a_1^*\ a_2)^* = a_2^*\ (a_1^*)^* = a_2^*\ a_1 = \langle a_2,\ a_1 \rangle. \qquad (2.30)$$

As an exemplar for overview consider a $*$ homomorphism between two matrix algebras $\phi : A \to B$, we can construct a Hilbert bimodule $E_\phi \in KK_f(A,B)$ in the following way. We let E_ϕ be B in as an vector space and an inner product from above in equation (2.27), with A acting on the left with ϕ.

$$a \cdot b = \phi(a)\, b \tag{2.31}$$

for $a \in A, b \in E_\phi$.

2.1.7 Kasparov Product and Morita Equivalence

Definition 9

Let $E \in KK_f(A,B)$ and $F \in KK_F(B,D)$ the Kasparov product is defined as with the balanced tensor product

$$F \circ E := E \otimes_B F. \tag{2.32}$$

Then $F \circ E \in KK_f(A,D)$ is equipped with a D-valued inner product

$$\langle e_1 \otimes f_1,\, e_2 \otimes f_2 \rangle_{E \otimes_B F} = \langle f_1, \langle e_1,\, e_2 \rangle_E f_2 \rangle_F \tag{2.33}$$

The Kasparov product for $*$-algebra homomorphism $\phi : A \to B$ and $\psi : B \to C$ are isomorphisms in the sense of

$$E_\psi \circ E_\phi \;\equiv\; E_\phi \otimes_B E_\psi \;\simeq\; E_{\psi \circ \phi} \in KK_f(A,C). \tag{2.34}$$

In the direct computation for elements $a \in A$, $b \in B$, and $c \in C$ which is $\psi \circ \phi$ gives us

$$a \cdot b \cdot c = \psi(\phi(a) \cdot b) \cdot c \tag{2.35}$$

An interesting case arises when looking at $E_{\mathrm{id}_A} \simeq A \in KK_f(A,A)$ for $\mathrm{id}_A : A \to A$. This is obvious when we let E_ϕ be A with a natural right representation. It follows that $E_\phi \simeq A$, thus an inner product, acting from the left on A for ϕ, $a', a \in A$ reads

$$a'\, a = (\phi(a')\, a) \in A, \tag{2.36}$$

which is satisfied by $\phi = \mathrm{id}_A$

Definition 10

Let A, B be matrix algebras. *They are called Morita equivalent if there exists an $E \in KK_f(A,B)$ and an $F \in KK_f(B,A)$ such that*

$$E \otimes_B F \simeq A \quad \text{and} \quad F \otimes_A E \simeq B, \tag{2.37}$$

where $\simeq$ denotes the isomorphism between Hilbert bimodules and note that A or B is a bimodule by itself.

The modules E and F are each others inverse in regards to the Kasparov Product, because we land in the same space as we started. To clarify, in the definition we have $E \in KK_f(A,B)$. We start from A and $E \otimes_B F$, which lands in A. Oppositely we have $F \in KK_f(B,D)$ we start from B and $F \otimes_A E$, which lands in B.

By definition $E \otimes_B F$ is a $A - D$ bimodule. Since

$$E \otimes_B F = E \otimes F \Big/ \left\{ \sum_i e_i\, b_i \otimes f_i - e_i \otimes b_i\, f_i \;\middle|\; e_i \in E_i,\, b_i \in B,\, f_i \in F \right\}, \tag{2.38}$$

the last part takes out all tensor product elements of E and F that don't preserver the left/right representation and that are duplicates.

Additionally $\langle \cdot, \cdot \rangle_{E \oplus_B F}$ defines a D valued inner product, as $\langle e_1, e_2 \rangle_E \in B$ and $\langle f_1, f_2 \rangle_F \in C$ by definition. So for $\langle e_1, e_2 \rangle_E = b$ we have

$$\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle_{E \otimes_B F} = \langle f_1, \langle e_1, e_2 \rangle_E f_2 \rangle_F = \langle f_1, b f_2 \rangle_F \in C \tag{2.39}$$

Picking up the example of (A,A), the Hilbert bimodule A, we can consider an $E \in KK_f(A,B)$ for

$$E \circ A = A \oplus_A E \simeq E. \tag{2.40}$$

We conclude, that ${}_AA_A$ is the identity element in the Kasparov product (up to isomorphism). Let us examine another example for $E = \mathbb{C}^n$, which is a $(M_n(\mathbb{C}),\mathbb{C})$ Hilbert bimodule with the standard $\mathbb{C}$ inner product. Further let $F = \mathbb{C}^n$, which is a $(\mathbb{C}, M_n(\mathbb{C}))$ Hilbert bimodule by right matrix multiplication with $M_n(\mathbb{C})$ valued inner product, we can write

$$\langle v_1, v_2 \rangle = \bar{v_1} v_2^t \in M_n(\mathbb{C}). \tag{2.41}$$

If we take the Kasparov product of E and F

$$F \circ E = E \otimes_{\mathbb{C}} F \simeq M_n(\mathbb{C}), \tag{2.42}$$
$$E \circ F = F \otimes_{M_n(\mathbb{C})} E \simeq \mathbb{C}, \tag{2.43}$$

we see that $M_n(\mathbb{C})$ and $\mathbb{C}$ are Morita equivalent!

Lemma 1

Two matrix algebras are Morita Equivalent if, and only if their their structure spaces are isomorphic as discreet spaces (have the same cardinality / same number of elements).

Proof. Let A, B be *Morita equivalent*. Then there exist ${}_AE_B$ and ${}_BF_A$ with

$$E \otimes_B F \simeq A \quad \text{and} \quad F \otimes_A E \simeq B. \tag{2.44}$$

Also consider $[(\pi_B, H)] \in \hat{B}$. We can construct a representation of A, which reads

$$\pi_A \to L(E \otimes_B H) \quad \text{with} \quad \pi_A(a)(e \otimes v) = ae \otimes w \tag{2.45}$$

Vice versa, we have $[(\pi_A, W)] \in \hat{A}$ we can construct π_B as

$$\pi_B : B \to L(F \otimes_A W) \quad \text{and} \quad \pi_B(b)(f \otimes w) = bf \otimes w. \tag{2.46}$$

Now we need to show that the representation π_A is irreducible if and only if π_B is irreducible. For (π_B, H) to be irreducible, we need $H \neq \emptyset$ and only $\emptyset$ or H to be invariant under the Action of B on H. Than $E \otimes_B H$ and $E \otimes_B H \simeq A$ cannot be empty, because E preserves left representation of A.

Lastly we need to check if the association of the class $[\pi_A]$ to $[\pi_B]$ is independent of the choice of representatives π_A and π_B. The important thing is that $[\pi_A] \in \hat{A}$ respectively $[\pi_B] \in \hat{B}$, hence any choice of representation is irreducible, because the structure space denotes all unitary equivalence classes of irreducible representations.

Note that the statements $E \simeq H$ and $F \simeq W$ are not particularly true, since all infinite dimensional Hilbert spaces are isomorphic. Here we are looking at finite dimensional Hilbert spaces. Another thing to keep in mind, is that for $[\pi_B, H] \in \hat{B}$ and looking at algebraic bimodules, we know that H is a bimodule of B, hence $E \otimes_B H \simeq A$, and for $[\pi_A, W]$, which is the same. Finally we can conclude, that these maps are each others inverses, thus $\hat{A} \simeq \hat{B}$. □

Lemma 2

The matrix algebra $M_n(\mathbb{C})$ has a unique irreducible representation (up to isomorphism) given by the defining representation on $\mathbb{C}^n$.

Proof. We know $\mathbb{C}^n$ is a irreducible representation of $A = M_n(\mathbb{C})$. Let H be irreducible and of dimension k, then we define a map

$$\phi : A \oplus ... \oplus A \to H^* \tag{2.47}$$

$$(a_1, ..., a_k) \mapsto e^1 \circ a_1^t + ... + e^k \circ a_k^t, \tag{2.48}$$

where $\{e^1, ..., e^k\}$ is the basis of the dual space H^* and $(\circ)$ being the pre-composition of elements in H^* and A acting on H. This forms a morphism of $M_n(\mathbb{C})$ modules, provided a matrix $a \in A$ acts on H^* with $v \mapsto v \circ a^t$ $(v \in H^*)$. Furthermore this morphism is surjective, thus making the pullback $\phi^* : H \mapsto (A^k)^*$ injective. Now identify $(A^k)^*$ with A^k as a A-module and note that $A = M_n(\mathbb{C}) \simeq \oplus^n \mathbb{C}^n$ as a n A module. It follows that H is a submodule of $A^k \simeq \oplus^{nk} \mathbb{C}$. By irreducibility $H \simeq \mathbb{C}$. □

Let us look at an examples for two matrix algebras A, and B.

$$A = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}), \quad B = \bigoplus_{j=1}^{M} M_{m_j}(\mathbb{C}). \tag{2.49}$$

Let $\hat{A} \simeq \hat{B}$, this implies $N = M$. Further define E with A acting by block-diagonal matrices on the first tensor and B acting in the same manner on the second tensor. Define F vice versa, ultimately reading

$$E := \bigoplus_{i=1}^{N} \mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i}, \quad F := \bigoplus_{i=1}^{N} \mathbb{C}^{m_i} \otimes \mathbb{C}^{n_i}. \tag{2.50}$$

When we calculate the Kasparov product we get the following

$$E \otimes_B F \simeq \bigoplus_{i=1}^{N} (\mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i}) \otimes_{M_{m_i}(\mathbb{C})} (\mathbb{C}^{m_i} \otimes \mathbb{C}^{n_i}) \tag{2.51}$$

$$\simeq \bigoplus_{i=1}^{N} \mathbb{C}^{n_i} \otimes \left(\mathbb{C}^{m_i} \otimes_{M_{m_i}(\mathbb{C})} \mathbb{C}^{m_i} \right) \oplus \mathbb{C}^{n_i} \tag{2.52}$$

$$\simeq \bigoplus_{i=1}^{N} \mathbb{C}^{m_i} \otimes \mathbb{C}^{n_i} \simeq A. \tag{2.53}$$

On the other hand we get

$$F \otimes_A E \simeq B. \tag{2.54}$$

To summarize, there is a duality between finite spaces and Morita equivalence classes of matrix algebras. By replacing $*$-homomorphism $A \to B$ with Hilbert bimodules (A, B) we introduce a richer structure of morphism between matrix algebras.

2.2 Finite Spectral Triples

2.2.1 Metric on Finite Discrete Spaces

Let us come back to our finite discrete space X, we can describe it by a structure space $\hat{A}$ of a matrix algebra A. To describe distance between two points in X (as we would in a metric space) we use an array $\{d_{ij}\}_{i,j\in X}$ of *real non-negative* entries in X such that

- $d_{ij} = d_{ji}$ Symmetric
- $d_{ij} \leq d_{ik}d_{kj}$ Triangle Inequality
- $d_{ij} = 0$ for $i = j$ (the same element)

In the commutative case, the algebra A is commutative and can describe the metric on X in terms of algebraic data.

Theorem 1

Let d_{ij} be a metric on X a finite discrete space with N points, $A = \mathbb{C}^N$ with elements $a = (a(i))_{i=1}^N$ such that $\hat{A} \simeq X$. Then there exists a representation π of A on a finite-dimensional inner product space H and a symmetric operator D on H such that

$$d_{ij} = \sup_{a\in A}\left\{\big|a(i)-a(j)\big| : \big|\big|\big[D,\pi(a)\big]\big|\big| \leq 1\right\} \tag{2.55}$$

Proof. We claim that this would follow from the equality:

$$\big|\big|[D,\pi(a)]\big|\big| = \max_{k\neq l}\left\{\frac{1}{d_{kl}}\big|a(k)-a(l)\big|\right\} \tag{2.56}$$

This can be proved with induction. Set $N = 2$ then $H = \mathbb{C}^2$, $\pi : A \to L(H)$ and a hermitian matrix D.

$$\pi(a) = \begin{pmatrix} a(1) & 0 \\ 0 & a(2) \end{pmatrix} \quad D = \begin{pmatrix} 0 & (d_{12})^{-1} \\ (d_{21})^{-1} & 0 \end{pmatrix} \tag{2.57}$$

Then we compute the commutator

$$\big|\big|[D,\pi(a)]\big|\big| = (d_{12})^{-1}\big|a(1)-a(2)\big| \tag{2.58}$$

For the case $A = \mathbb{C}^3$, we have $H = (\mathbb{C}^2)^{\oplus 3} = H_2 \oplus H_2^1 \oplus H_2^2$. The representation $\pi(a)$ reads

$$\begin{aligned}\pi((a(1),a(2),a(3)) &= \begin{pmatrix} a(1) & 0 \\ 0 & a(2) \end{pmatrix} \oplus \begin{pmatrix} a(1) & 0 \\ 0 & a(3) \end{pmatrix} \oplus \begin{pmatrix} a(2) & 0 \\ 0 & a(2) \end{pmatrix} \\ &= \mathrm{diag}\big(a(1),a(2),a(1),a(3),a(2),a(3)\big) \end{aligned} \tag{2.59}$$

And the operator D takes the form

$$\begin{aligned} D &= \begin{pmatrix} 0 & x_1 \\ x_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & x_2 \\ x_2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & x_3 \\ x_3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & x_1 & 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 & x_3 & 0 \end{pmatrix}. \end{aligned} \tag{2.60}$$

Then the norm of the commutator would be the largest eigenvalue

$$||[D,\pi(a)]|| = ||D\pi(a) - \pi(a)D||, \tag{2.61}$$

where the matrix in the norm from equation (2.61) is a skew symmetric matrix. Its eigenvalues are $i\lambda_1, i\lambda_2, i\lambda_3, i\lambda_4$. The λ's are on the upper and lower diagonal. The matrix norm would be the maximum of the norm with the larges eigenvalues:

$$\begin{aligned} ||[D,\pi(a)]|| = \max_{a\in A} \Big\{ & x_1 |a(2)-a(1)|, \\ & x_2 |(a(3)-a(1))|, \\ & x_3 |(a(3)-a(2))| \Big\}. \end{aligned} \tag{2.62}$$

Hence the metric turns out to be

$$d = \begin{pmatrix} 0 & a(1)-a(2) & a(1)-a(3) \\ a(2)-a(1) & 0 & a(2)-a(3) \\ a(3)-a(1) & a(3)-a(2) & 0 \end{pmatrix} \tag{2.63}$$

Suppose this holds for N with π_N, $H_N = \mathbb{C}^N$ and D_N. Then it has to holds for $N+1$ with $H_{N+1} = H_N \oplus \bigoplus_{i=1}^N H_N^i$, since the representation reads

$$\begin{aligned} \pi_{N+1}(a(1),\dots,a(N+1)) = \pi_N(a(1),\dots,a(N)) \oplus \begin{pmatrix} a(1) & 0 \\ 0 & a(N+1) \end{pmatrix} \oplus \\ \oplus \dots \oplus \begin{pmatrix} a(N) & 0 \\ 01 & a(N+1) \end{pmatrix} \end{aligned} \tag{2.64}$$

And the operator D_{N+1} is

$$\begin{aligned} D_{N+1} = D_N \oplus \begin{pmatrix} 0 & (d_{1(N+1)})^{-1} \\ (d_{1(N+1)})^{-1} & 0 \end{pmatrix} \oplus \\ \oplus \dots \oplus \begin{pmatrix} 0 & (d_{N(N+1)})^{-1} \\ (d_{N(N+1)})^{-1} & 0 \end{pmatrix} \end{aligned} \tag{2.65}$$

From this follows equation (2.56). Thus we can continue the proof by setting for fixed i, j, $a(k) = d_{ik}$, which then gives $|a(i) - a(j)| = d_{ij}$ and thereby it follows that

$$\frac{1}{d_{kl}} |a(k) - a(l)| = \frac{1}{d_{kl}} |d_{ik} - d_{il}| \leq 1. \tag{2.66}$$

□

To get a better understanding of the results of the theorem let us compute a metric on the space of three points given by $d_{ij} = \sup_{a\in A}\{|a(i) - a(j)| : ||[D,\pi(a)]|| \leq 1\}$ for the set of data $A = \mathbb{C}^3$ acting in the defining representation $H = \mathbb{C}^3$, and

$$D = \begin{pmatrix} 0 & d^{-1} & 0 \\ d^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.67}$$

for some $d \in \mathbb{R}$. From the data $A = \mathbb{C}^3$, $H = \mathbb{C}^3$ and D we compute the commutator

$$||[D,\pi(a)]|| = d^{-1}\left\|\begin{pmatrix} 0 & a(2)-a(1) & 0 \\ -(a(2)-a(1)) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right\|. \tag{2.68}$$

Hence the metric is

$$d = \begin{pmatrix} 0 & a(1)-a(2) & a(1) \\ a(2)-a(1) & 0 & a(2) \\ -a(1) & -a(2) & 0 \end{pmatrix}. \tag{2.69}$$

The translation of the metric on X into algebraic data assumes commutativity in A, this can be extended to a noncommutative matrix algebra, by the following metric on a structure space $\hat{A}$ of a matrix algebra $M_{n_i}(\mathbb{C}$

$$d_{ij} = \sup_{a\in A}\left\{|\mathrm{Tr}(a(i)) - \mathrm{Tr}((a(j))| : ||[D,a]|| \leq 1\right\}. \tag{2.70}$$

Equation (2.70) is special case of the Connes' distance formula on a structure space of A.

Finally we have all three ingredients to define a finite spectral triple, an mathematical structure which encodes finite discrete geometry into algebraic data.

Definition 11

A finite spectral triple is a tripe (A,H,D), where A is a unital $*$-algebra, faithfully represented on a finite-dimensional Hilbert space H, with a symmetric operator $D : H \to H$. (Note that A is automatically a matrix algebra.)

2.2.2 Properties of Matrix Algebras

Lemma 3

If A is a unital C algebra acting faithfully on a finite dimensional Hilbert space, then A is a matrix algebra of the Form:*

$$A \simeq \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}) \tag{2.71}$$

Proof. The wording 'acting faithfully on a Hilbertspace' means that the $*$-representation is injective, or for a $*$-homomorphism that means one-to-one correspondence. And since A acts faithfully on a Hilbert space, this means that A is a $*$ subalgebra of a matrix algebra $L(H) = M_{\dim(H)}(\mathbb{C}$. Hence it follows, that A is isomorphic to a matrix algebra. □

A simple illustration would be for an algebra $A = M_n(\mathbb{C})$ and $H = \mathbb{C}^n$. Since A acts on H with matrix multiplication and standard inner product and D on H is a hermitian matrix $n \times n$ matrix.

Definition 12

Given an finite spectral triple (A,H,D), the A-bimodule of Connes' differential *one-forms is*

$$\Omega_D^1(A) := \left\{ \sum_k a_k[D,b_k] : a_k, b_k \in A \right\}. \tag{2.72}$$

Then there is a map $d : A \to \Omega_D^1(A)$, $d = [D,\cdot]$. Where d is a derivation of the $*$-algebra in the sense that

$$d(a\,b) = d(a)b + ad(b), \tag{2.73}$$
$$d(a^*) = -d(a)^*. \tag{2.74}$$

Since we have $d(\cdot) = [D,\cdot]$, we can easily check the above equations

$$\begin{aligned} d(a\,b) &= [D, a\,b] = [D,a]b + a[D,b] \\ &= d(a)\,b + a\,d(b). \end{aligned} \tag{2.75}$$

And

$$\begin{aligned} d(a^*) &= [D,a^*] = Da^* - a^*D \\ &= -(D^*\,a - a\,D^*) = -[D^*, a] \\ &= -d(a)^*. \end{aligned} \tag{2.76}$$

Furthermore $\Omega_D^1(A)$ is an A-bimodule, which can be seen by rewriting the defining equation (2.72) into

$$\begin{aligned} a\,(a_k[D,b_k])\,b &= a\,a_k(D\,b_k - b_k\,D)\,b = \\ &= a\,a_k(D\,b_k\,b - b_k\,D\,b) = \\ &= a\,a_k(D\,b_k\,b - b_k\,D\,b - b_k\,b\,D + b_k\,b\,D) = \\ &= a\,a_k(D\,b_k\,b - b_k\,b\,D + b_k\,b\,D - b_k\,D\,b) = \\ &= a\,a_k[D, b_k\,b] + a\,a_k\,b[D,b] = \\ &= \sum_k a'_k\,[D, b'_k] \end{aligned} \tag{2.77}$$

Lemma 4

Let $(A,H,D) = (M_n(\mathbb{C}), \mathbb{C}^n, D)$, where D is a hermitian $n \times n$ matrix. If D is not a multiple of the identity then

$$\Omega_D^1(A) \simeq M_n(\mathbb{C}) = A \tag{2.78}$$

Proof. Assume $D = \sum_i \lambda_i e_{ii}$ is diagonal, $\lambda_i \in \mathbb{R}$ and $\{e_{ij}\}$ is the basis of $M_n(\mathbb{C})$. Then for fixed i, j choose k such that $\lambda_k \neq \lambda_j$, hence we have

$$\left(\frac{1}{\lambda_k - \lambda_j} e_{ik}\right)[D, e_{kj}] = e_{ij}, \tag{2.79}$$

for $e_{ij} \in \Omega_D^1(A)$ by the above definition (2.72). Ultimately we have

$$\Omega_D^1(A) \subset L(\mathbb{C}^n) = H \simeq M_n(\mathbb{C}) = A \tag{2.80}$$

$\square$

Consider an example

$$\left(A = \mathbb{C}^2, H = \mathbb{C}^2, D = \begin{pmatrix} 0 & \lambda \\ \bar{\lambda} & 0 \end{pmatrix}\right) \tag{2.81}$$

with $\lambda \neq 0$. We can show that $\Omega_D^1(A) \simeq M_2(\mathbb{C})$. The Hilbert Basis D can be extended in terms of the basis of $M_2(\mathbb{C})$, plugging this into Equation (2.79) will get us the same cyclic result, thus $\Omega_D^1(A) \simeq M_2(\mathbb{C})$.

2.2.3 Morphisms Between Finite Spectral Triples

Next we will define an equivalence relation between finite spectral triples, called spectral unitary equivalence, which is given by the unitarity of the two matrix algebras themselves, and an additional map U which allows us to associate a one operator to another second operator.

Definition 13

Two finite spectral tripes (A_1, H_1, D_1) and (A_2, H_2, D_2) are called unitary equivalent if $A_1 = A_2$ and there exists a map $U: H_1 \to H_2$ that satisfies

$$U\ \pi_1(a)\ U^* = \pi_2(a) \quad \text{with} \quad a \in A_1, \tag{2.82}$$

$$U\ D_1\ U^* = D_2. \tag{2.83}$$

Notice that for any such U we have the relation $(A, H, D) \sim (A, H, UDU^*)$. And hence $U\ D\ U^* = D + U[D, U^*]$ are of the form of elements in $\Omega_D^1(A)$.

To make it clear that the above definition is an equivalence relation between finite spectral triples, we need to see if the relation satisfies reflexivity, symmetry and transitivity. Let us look then at three spectral triples (A_1, H_1, D_1), (A_2, H_2, D_2) and (A_3, H_3, D_3). For reflexivity $(A_1, H_1, D_1) \sim (A_1, H_1, D_1)$. So there exists the unitary map $U: H_1 \to H_1$, which is the identity and always exists. On the other hand the symmetry condition requires

$$(A_1, H_1, D_1) \sim (A_2, H_2, D_2) \Leftrightarrow (A_2, H_2, D_2) \sim (A_1, H_1, D_1). \tag{2.84}$$

Because U is unitary we can rewrite for the representation for A_1

$$\begin{aligned} U\pi_1(a)U^* &= \pi_2(a) \quad | \cdot U^* \boxdot U \\ U^*U\pi_1(a)U^*U &= \pi_1(a) = U^*\pi_2(a)U. \end{aligned} \tag{2.85}$$

The same relation applies for the symmetric operator D. Lastly for transitivity the condition is

$$\begin{aligned} (A_1, H_1, D_1) \sim (A_2, H_2, D_2) \quad &\text{and} \quad (A_2, H_2, D_2) \sim (A_3, H_3, D_3) \\ &\Rightarrow (A_1, H_1, D_1) \sim (A_3, H_3, D_3). \end{aligned} \tag{2.86}$$

Therefore the two unitary maps $U_{12}: H_1 \to H_2$ and $U_{23}: H_2 \to H_3$ are

$$\begin{aligned} U_{23}\ U_{12}\ \pi_1(a)\ U_{12}^*\ U_{23}^* &= U_{23}\ \pi_2(a)\ U_{23}^* \\ &= \pi_3(a), \end{aligned} \tag{2.87}$$

$$\begin{aligned} U_{23}\ U_{12}\ D_1\ U_{12}^*\ U_{23}^* &= U_{23}\ D_2 U_{23}^* \\ &= D_3. \end{aligned} \tag{2.88}$$

In order to extend this relation we take a look at Morita equivalence of Matrix Algebras.

Definition 14

Let A be an algebra. We say that $I \subset A$, as a vector space, is a right(left) ideal if $a\ b \in I$ for $a \in A$ and $b \in I$ (or $b\ a \in I$, $b \in I$, $a \in A$). We call a left-right ideal simply an ideal.

Given a Hilbert bimodule $E \in KK_f(B,A)$ and (A,H,D) we construct a finite spectral triple on B, (B,H',D')

$$H' = E \otimes_A H. \tag{2.89}$$

We might define D' with $D'(e \otimes \xi) = e \otimes D\xi$, thought this would not satisfy the ideal defining the balanced tensor product over A, which is generated by elements of the form

$$e\ a \otimes \xi - e \otimes a\ \xi, \quad e \in E, a \in A, \xi \in H. \tag{2.90}$$

This inherits the left action on B from E and has a $\mathbb{C}$ valued inner product space. B also satisfies the ideal

$$D'(e \otimes \xi) = e \otimes D\ \xi + \nabla(e)\ \xi, \quad e \in E, a \in A, \tag{2.91}$$

where ∇ is called the *connection on the right A-module E* associated with the derivation $d = [D,\cdot]$. The connection needs to satisfy the *Leibnitz Rule*

$$\nabla(ae) = \nabla(e)a + e \otimes [D,a], \quad e \in E,\ a \in A. \tag{2.92}$$

Hence D' is well defined on $E \otimes_A H$

$$\begin{aligned} D'(e\ a \otimes \xi - e \otimes a\ \xi) &= D'(e\ a \otimes \xi) - D'(e \otimes \xi) \\ &= e\ a \otimes D\ \xi + \nabla(a\ e)\ \xi - e \otimes D(a\ \xi) - \nabla(e)\ a\ \xi \\ &= 0. \end{aligned} \tag{2.93}$$

With the information thus far we can prove the following theorem

Theorem 2

If (A,H,D) a finite spectral triple, $E \in KK_f(B,A)$. Then $(V, E \otimes_A H, D')$ is a finite spectral triple, provided that ∇ satisfies the compatibility condition

$$\langle e_1, \nabla e_2 \rangle_E - \langle \nabla e_1, e_2 \rangle_E = d\langle e_1, e_2 \rangle_E \quad e_1, e_2 \in E \tag{2.94}$$

Proof. $E \otimes_A H$ was previously. The only thing left is to show that D' is a symmetric operator, this we can just compute. Let $e_1, e_2 \in E$ and $\xi_1, \xi_2 \in H$ then

$$\begin{aligned} \langle e_1 \otimes \xi_1, D'(e_2 \otimes \xi_2)\rangle_{E\otimes_A H} &= \langle \xi_1, \langle e_1, \nabla e_2 \rangle_E\ \xi_2 \rangle \langle \xi_1, \langle e_1, e_2 \rangle_E\ D\ \xi_2 \rangle_H \\ &= \langle \xi_1, \langle \nabla e_1, e_2 \rangle_E\ \xi_2 \rangle_H + \langle \xi_1, d\langle e_1, e_2 \rangle_E\ \xi_2 \rangle_H \\ &+ \langle D\ \xi_1, \langle e_1, e_2 \rangle_E\ \xi_2 \rangle_H - \langle \xi_1, [D, \langle e_1, e_2 \rangle_E]\ \xi_2 \rangle_H \\ &= \langle D'(e_1 \otimes \xi_1), e_2 \otimes \xi_2 \rangle_{E\otimes_A H} \end{aligned} \tag{2.95}$$

□

Let us examin what happens if we look at difference of connectoins ∇ and ∇' on a right A-module E. Since both connections need to satisfy the Leiblitz rule, the difference should also

$$\begin{aligned}\nabla(ea) - \nabla'(ea) &= \nabla(e) + e \otimes [D,a] \\ &\quad - (\nabla'(e)a + e \otimes [D',a]) \\ &= \bar{\nabla}a + e \otimes (Da - aD - D'a + aD') \\ &= \bar{\nabla}a + e \otimes ((D-D')a - a(D-D')) \\ &= \bar{\nabla}a + e \otimes [D',a] \\ &= \bar{\nabla}(ea). \end{aligned} \tag{2.96}$$

Therefore $\nabla - \nabla'$ is a right A-linear map $E \to E \otimes_A \Omega_D^1(A)$.

To get a better grasp of the results let us construct a finite spectral triple (A,H',D') from (A,H,D). The derivation $d(\cdot): A \to A \otimes_A \Omega_D^1(A) = \Omega_D^1(A)$ is a connection on A considered a right A-module

$$\nabla(e \cdot a) = d(a), \tag{2.97}$$

hence $A \otimes_A H \simeq H$. Next we can construct the operator D' for the connection $d(\cdot)$,

$$D'(a\xi) = a(D\xi) + (\nabla a)\xi = D(a\xi). \tag{2.98}$$

By using the identity element in the connection relation

$$\nabla(e \cdot a) = \nabla(e)a + 1 \otimes d(a) = d(a)\nabla(e)a, \tag{2.99}$$

we see that any connection $\nabla : A \to A \otimes_A \Omega_D^1(A)$ is given by

$$\nabla = d + \omega, \tag{2.100}$$

where $\omega \in \Omega_D^1(A)$. Ultimately the the difference operator D' with the connection on A is given by

$$\begin{aligned} D'(a \otimes \xi) = D'(a\xi) &= a(D\xi) + (\nabla a)\xi \\ &= a(D\xi) + \nabla(e \cdot a)\xi \\ &= D(a\xi) + \nabla(e)(a\xi). \end{aligned} \tag{2.101}$$

So any such connection is of the form

$$\nabla = d + \omega. \tag{2.102}$$

2.3 Finite Real Noncommutative Spaces

2.3.1 Finite Real Spectral Triples

In this chapter we supplement the finite spectral triples with a *real structure*. We additionally require a symmetry condition that that H is a A-A-bimodule rather than only a A-left module. This ansatz has tight bounds with physical properties such as charge conjugation, which we will dive in deeper in later chapters. For this we will need to set a basis of definitions to get an overview. First we introduce a $\mathbb{Z}_2$-grading γ with the following properties

$$\gamma^* = \gamma, \tag{2.103}$$
$$\gamma^2 = 1, \tag{2.104}$$
$$\gamma D = -D\gamma, \tag{2.105}$$
$$\gamma a = a\gamma, \quad a \in A. \tag{2.106}$$

Then we can define a finite real spectral triple.

Definition 15

A finite real spectral triple is given by a finite spectral triple (A, H, D) and a anti-unitary operator $J : H \to H$ called the real structure, such that

$$a^\circ := J\, a^*\, J^{-1}, \tag{2.107}$$

is a right representation of A on H, that is $(ab)^\circ = b^\circ a^\circ$. With two requirements

$$[a, b^\circ] = 0, \tag{2.108}$$
$$[[D, a],\, b^\circ] = 0. \tag{2.109}$$

The two properties are called the commutant property, they require that the left action of an element in A and $\Omega^1_D(A)$ commutes with the right action on A.

Definition 16

The KO-dimension of a real spectral triple is determined by the sings $\varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\}$ appearing in

$$J^2 = \varepsilon, \tag{2.110}$$
$$J\, D = \varepsilon\, D\, J, \tag{2.111}$$
$$J\, \gamma = \varepsilon''\, \gamma\, J. \tag{2.112}$$

Table 1: *KO*-dimension k modulo 8 of a real spectral triple

k	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

Even thought the KO-dimension of a real spectral triple is important, we will not be doing in-depth introduction of the KO-dimension, for this we reference to [1].

Definition 17

An opposite-algebra A° of a A is defined to be equal to A as a vector space with the opposite product

$$a \circ b := ba \tag{2.113}$$
$$\Rightarrow a^\circ = Ja^*J^{-1}, \tag{2.114}$$

which defines the left representation of A° on H

Let us examine an example of a matrix algebra $M_N(\mathbb{C})$ acting on $H = M_N(\mathbb{C})$ by left matrix multiplication with the Hilbert Schmidt inner product.

$$\langle a,b\rangle = \mathrm{Tr}(a^*b). \tag{2.115}$$

We can define $\gamma(a) = a$ and $J(a) = a^*$ with $a \in H$. Since D must be odd with respect to γ it vanishes identically. Furthermore we know the multiplicity space is $V_i = \mathbb{C}^{m_i}$, and also we know that for $T \in H$ and$a \in A'$ to work we need $a\ T = T\ a$. Thus by laws of matrix multiplication we need $A' \simeq \bigoplus_i M_{m_i}(\mathbb{C})$. For this to work we naturally need $H = \bigoplus_i \mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i}$. Hence the right action of $M_N(\mathbb{C})$ on $H = M_N(\mathbb{C})$ as defined by $a \mapsto a^\circ$ is given by right matrix multiplication

$$a^\circ\xi = Ja^*J^{-1}\xi = Ja^*\xi^* = J\xi a = \xi^* a \tag{2.116}$$

Definition 18

*We call $\xi \in H$ **cyclic vector** in A if:*

$$A\xi := a\xi : \ \ a \in A = H \tag{2.117}$$

*We call $\xi \in H$ **separating vector** in A if:*

$$a\xi = 0 \ \ \Rightarrow \ \ a = 0; \ \ a \in A \tag{2.118}$$

Suppose $(A,H,D = 0)$ is a finite spectral triple such that H possesses a cyclic and separating vector for A and let $J : H \to H$ be the operator in $S = J\Delta^{1/2}$ with $\Delta = S^*S$. By composition $S(a\xi) = a * \xi$ this is literally anti-linearity, then $S(a\xi) = a * \xi$ defines a anti-linear operator. Furthermore the operator S is invertible because, if a $\xi \in H$ is cyclic then we have $S(A\xi) = A^*\xi = A\xi = H$. Vice versa the same has to work for S^{-1}, otherwise ξ wouldn't exist. And hence $S^{-1}(A^*\xi) = S^{-1}(H) = H$. Additionally J is anti-unitary because firstly, S is bijective thus $\Delta^{1/2}$ and J need to be bijective. Also have $J = S\Delta^{-1/2}$ and $\Delta^* = \Delta$, so for a $\xi_1, \xi_2 \in H$ we can write

$$\begin{aligned} < J\xi_1, J\xi_2 > =& < J^*J\xi_1, \xi_2 >^* = \\ &=< (\Delta^{-1/2})^* S^*S\Delta^{-1/2}\xi_1, \xi_2 >^* = \\ &=< (\Delta^{-1/2})^*\Delta\Delta^{-1/2}\xi_1, \xi_2 >^* = \\ &=< \Delta^{-1/2}\Delta^{1/2}\Delta^{1/2}\Delta^{-1/2}\xi_1, \xi_2 >^* = \\ &=< \xi_1, \xi_2 >^* = < \xi_2, \xi_1 >, \end{aligned} \tag{2.119}$$

which concludes the anti-unitarity by definition.

2.3.2 Morphisms Between Finite Real Spectral Triples

Like the unitary equivalence relation for finite spectral triples, we can it to finite real spectral triples.

Definition 19

We call two finite real spectral triples $(A_1, H_1, D_1; J_1, \gamma_1)$ and $(A_2, H_2, D_2; J_2, \gamma_2)$ unitarily equivalent if $A_1 = A_2$ and if there exists a unitary operator $U : H_1 \to H_2$

such that

$$U\,\pi_1(a)\,U^* = \pi_2(a), \tag{2.120}$$
$$U\,D_1\,U^* = D_2, \tag{2.121}$$
$$U\gamma_1\,U^* = \gamma_2, \tag{2.122}$$
$$U\,J_1\,U^* = J_2. \tag{2.123}$$

Definition 20

Let E be a B-A bimodule. The conjugate Module E° is given by the A-B-bimodule.

$$E^\circ = \{\bar{e} : e \in E\}, \tag{2.124}$$

with

$$a \cdot \bar{e} \cdot b = b^*\ \bar{e}\ a^*, \quad \forall a \in A, b \in B. \tag{2.125}$$

We bear in mind that E° is not a Hilbert bimodule for (A,B) because it doesn't have a natural B-valued inner product. But there is a A-valued inner product on the left A-module E° with

$$\langle \bar{e}_1, \bar{e}_2 \rangle = \langle e_2, e_1 \rangle, \quad e_1, e_2 \in E. \tag{2.126}$$

And linearity in A by the terms

$$\langle a\,\bar{e}_1, \bar{e}_2 \rangle = a\langle \bar{e}_1, \bar{e}_2 \rangle, \quad \forall a \in A. \tag{2.127}$$

With this it becomes obvious that E° is a Hilbert bimodule of (B°, A°). A straightforward calculation of the properties of the Hilbert bimodule and its B° valued inner product gives the results. So for $\bar{e}_1, \bar{e}_2 \in E^\circ$ and $a^\circ \in A, b^\circ \in B$ we write

$$\begin{aligned}
\langle \bar{e}_1, a^\circ \bar{e}_2 \rangle &= \langle \bar{e}_1, Ja^* J^{-1} \bar{e}_2 \rangle = \\
&= \langle \bar{e}_1, Ja^* e_2 \rangle \\
&= \langle J^{-1} e_1, a^* e_2 \rangle \\
&= \langle a^* e_1, e_2 \rangle = \langle J^{-1}(a^\circ)^* J e_1, e_2 \rangle \\
&= \langle J^{-1}(a^\circ)^* \bar{e}_1, e_2 \rangle \\
&= \langle (a^\circ)^* \bar{e}_1, \bar{e}_2 \rangle.
\end{aligned} \tag{2.128}$$

Next for $\langle \bar{e}_1, \bar{e}_2 b^\circ \rangle = \langle \bar{e}_1, \bar{e}_2 \rangle b^\circ$ we write

$$\begin{aligned}
\langle \bar{e}_1, \bar{e}_2 b^\circ \rangle &= \langle \bar{e}_1, \bar{e}_2 J b^* J^{-1} \rangle \\
&= \langle \bar{e}_1, \bar{e}_2 \rangle J b^* J^{-1} \\
&= \langle \bar{e}_1, \bar{e}_2 \rangle b^\circ.
\end{aligned} \tag{2.129}$$

Additionally we have

$$\begin{aligned}
(\langle \bar{e}_1, \bar{e}_2)\rangle_{E^\circ})^* &= (\langle e_2, e_1 \rangle_E)^* \\
&= \langle e_1, e_2 \rangle_E^* \\
&= \langle \bar{e}_2, \bar{e}_2 \rangle_{E^\circ}.
\end{aligned} \tag{2.130}$$

And finally of course we have

$$\langle \bar{e}, \bar{e} \rangle = \langle e, e \rangle \geq 0 \tag{2.131}$$

Given the results thus far with a Hilbert bimodule E for (B,A), we construct a spectral triple $(B,H',D';J',\gamma')$ from $(A,H,D;J,\gamma)$. For H' we make a $\mathbb{C}$-valued inner product on H' by combining the A valued inner product on E and E° with the $\mathbb{C}$-valued inner product on H by defining

$$H' := E \otimes_A H \otimes_A E^\circ. \tag{2.132}$$

Then the action of B on H' takes the following form

$$b(e_2 \otimes \xi \otimes \bar{e}_2) = (be_1) \otimes \xi \otimes \bar{e}_2. \tag{2.133}$$

The right action of B on H' defined by action on the right component E° is

$$J'(e_1 \otimes \xi \otimes \bar{e}_2) = e_2 \otimes J\xi \otimes \bar{e}_1, \tag{2.134}$$

where $b^\circ = J'b^*(J')^{-1}$, $b^* \in B$ is the action on H'. Hence the connection reads

$$\nabla : E \to E \otimes_A \Omega^1_D(A) \tag{2.135}$$

$$\bar{\nabla} : E^\circ \to \Omega^1_D(A) \otimes_A E^\circ, \tag{2.136}$$

which gives us the Dirac operator on $H' = E \otimes_A H \otimes_A E^\circ$ as

$$D'(e_1 \otimes \xi \otimes \bar{e}_2) = (\nabla e_1)\xi \otimes \bar{e}_2 + e_1 \otimes D\xi \otimes \bar{e}_2 + e_1 \otimes \xi(\bar{\nabla}\bar{e}_2). \tag{2.137}$$

And the right action of $\omega \in \Omega^1_D(A)$ on $\xi \in H$ is defined by

$$\xi \mapsto \varepsilon' J\omega^* J^{-1}\xi. \tag{2.138}$$

Finally for the grading we have

$$\gamma' = 1 \otimes \gamma \otimes 1. \tag{2.139}$$

Summarizing we can write down the following theorem

Theorem 3

Suppose $(A,H,D;J,\gamma)$ is a finite spectral triple of KO-dimension k, let ∇ be a connection satisfying the compatibility condition (same as with finite spectral triples). Then $(B,H',D';J',\gamma')$ is a finite spectral triple of KO-Dimension k. (H',D',J',γ')

Proof. The only thing left is to check if the KO-dimension is preserved, for this we check if the ε's are the same.

$$(J')^2 = 1 \otimes J^2 \otimes 1 = \varepsilon, \tag{2.140}$$

$$J'\gamma' = \varepsilon''\gamma' J'. \tag{2.141}$$

Lastly for ε' we have

$$\begin{aligned} J'D'(e_1 \otimes \xi \otimes \bar{e}_2) &= J'\big((\nabla e_1)\xi \otimes \bar{e}_2 + e_1 \otimes D\xi \otimes \bar{e}_2 + e_1 \otimes \xi(\tau\nabla e_2)\big) \\ &= \varepsilon' D'(e_2 \otimes J\xi \otimes \bar{e}_2) \\ &= \varepsilon' D'J'(e_1 \otimes \xi \bar{e}_2) \end{aligned} \tag{2.142}$$

□

Let us take a look at $\nabla : E \Rightarrow E \otimes_A \Omega^1_d(A)$ right connection on E and consider the following anti-linear map

$$\tau : E \otimes_A \Omega^1_D(A) \to \Omega^1_D(A) \otimes_A E^\circ \tag{2.143}$$

$$e \otimes \omega \mapsto -\omega^* \otimes \bar{e}. \tag{2.144}$$

Interestingly the map $\bar{\nabla} : E^\circ \to \Omega^1_D(A) \otimes E^\circ$ with $\bar{\nabla}(\bar{e}) = \tau \circ \nabla(e)$ is a left connection, that means show that it satisfied the left Leibniz rule, for one

$$\tau \circ \nabla(ae) = \bar{\nabla}(a\bar{e}) = \bar{\nabla}(a^*\bar{e}). \tag{2.145}$$

And for two

$$\begin{aligned} \tau \circ \nabla(ae) &= \tau(\nabla(e)a) + \tau \circ (e \otimes d(a)) \\ &= a^*\bar{\nabla}(\bar{e}) - d(a)^* \otimes \bar{e}. \\ &= a^*\bar{\nabla}(\bar{e}) + d(a^*) \otimes \bar{e}. \end{aligned} \tag{2.146}$$

2.4 Heat Kernel Expansion

2.4.1 The Heat Kernel

The heat kernel $K(t;x,y;D)$ is the fundamental solution of the heat equation

$$(\partial_t + D_x)K(t;x,y;D) = 0, \tag{2.147}$$

which depends on the operator D of Laplacian type.

For a flat manifold $M = \mathbb{R}^n$ and $D = D_0 := -\Delta_\mu \Delta^\mu + m^2$ the Laplacian with a mass term and the initial condition

$$K(0;x,y;D) = \delta(x,y), \tag{2.148}$$

takes the form of the standard fundamental solution

$$K(t;x,y;D_0) = (4\pi t)^{-n/2} \exp\left(-\frac{(x-y)^2}{4t} - tm^2\right). \tag{2.149}$$

Let us consider now a more general operator D with a potential term or a gauge field, the heat kernel reads then

$$K(t;x,y;D) = \langle x|e^{-tD}|y\rangle. \tag{2.150}$$

We can expand the heat kernel in t, still having a singularity from the equation (2.149) as $t \to 0$ thus the expansion reads

$$K(t;x,y;D) = K(t;x,y;D_0)\left(1 + tb_2(x,y) + t^2 b_4(x,y) + \dots\right), \tag{2.151}$$

where $b_k(x,y)$ become regular as $y \to x$. These coefficients are called the heat kernel coefficients.

2.4.2 Spectral Functions

Manifolds M with a disappearing boundary condition for the operator e^{-tD} for $t > 0$ is a trace class operator on $L^2(V)$. Meaning for any smooth function f on M we can define

$$K(t,f,D) := \mathrm{Tr}_{L^2}(fe^{-tD}), \tag{2.152}$$

or alternately write an integral representation

$$K(t,f,D) = \int_M d^n x\sqrt{g}\mathrm{Tr}_V(K(t;x,x;D)f(x)), \tag{2.153}$$

in the regular limit $y \to y$. We can write the Heat Kernel in terms of the spectrum of D. So for an orthonormal basis $\{\phi_\lambda\}$ of eigenfunctions for D, which corresponds to the eigenvalue λ, we can rewrite the heat kernel into

$$K(t;x,y;D) = \sum_\lambda \phi_\lambda^\dagger(x)\phi_\lambda(y)e^{-t\lambda}. \tag{2.154}$$

An asymptotic expansion as $t \to 0$ for the trace is then

$$\mathrm{Tr}_{L^2}(fe^{-tD}) \simeq \sum_{k\geq 0} t^{(k-n)/2}a_k(f,D), \tag{2.155}$$

where

$$a_k(f,D) = (4\pi)^{-n/2}\int_M d^4x\sqrt{g}b_k(x,x)f(x). \tag{2.156}$$

2.4.3 General Formulae

Let us summarize what we have obtained in the last chapter, we considered a compact Riemannian manifold M without boundary condition, a vector bundle V over M to define functions which carry discrete (spin or gauge) indices, an operator D of Laplace type over V and smooth function f on M.

There is an asymptotic expansion where the heat kernel coefficients with an odd index $k = 2j+1$ vanish $a_{2j+1}(f,D) = 0$. On the other hand coefficients with an even index are locally computable in terms of geometric invariants

$$\begin{aligned} a_k(f,D) &= \mathrm{Tr}_V\left(\int_M d^n x\sqrt{g}(f(x)a_k(x;D)\right) = \\ &= \sum_I \mathrm{Tr}_V\left(\int_M d^n x\sqrt{g}(fu^I\mathscr{A}_k^I(D))\right). \end{aligned} \tag{2.157}$$

We denote $\mathscr{A}_k^I$ as all possible independent invariants of dimension k, and u^I are constants. The invariants are constructed from $E, \Omega, R_{\mu\nu\rho\sigma}$ and their derivatives If E has dimension two, then the derivative has dimension one. So if $k = 2$ there are only two independent invariants, E and R. This corresponds to the statement $a_{2j+1} = 0$.

If we consider $M = M_1 \times M_2$ with coordinates x_1 and x_2 and a decomposed Laplace style operator $D = D_1 \otimes 1 + 1 \otimes D_2$ we can separate functions acting on operators and on coordinates linearly by the following

$$e^{-tD} = e^{-tD_1} \otimes e^{-tD_2}, \tag{2.158}$$

$$f(x_1,x_2) = f_1(x_1)f_2(x_2), \tag{2.159}$$

thus the heat kernel coefficients are separated by

$$a_k(x;D) = \sum_{p+q=k} a_p(x_1;D_1) a_q(x_2;D_2) \tag{2.160}$$

If we know the eigenvalues of D_1 are known, $l^2, l \in \mathbb{Z}$, we can obtain the heat kernel asymmetries with the Poisson summation formula giving us an approximation in the order of $e^{-1/t}$

$$\begin{aligned} K(t,D_1) = \sum_{l\in\mathbb{Z}} e^{-tl^2} = \sqrt{\frac{\pi}{t}} \sum_{l\in\mathbb{Z}} e^{-\frac{\pi^2 l^2}{t}} = \\ \simeq \sqrt{\frac{\pi}{t}} + \mathscr{O}(e^{-1/t}). \end{aligned} \tag{2.161}$$

The exponentially small terms have no effect on the heat kernel coefficients and that the only nonzero coefficient is $a_0(1,D_1) = \sqrt{\pi}$, therefore the heat coefficients can be written as

$$a_k(f(x^2),D) = \sqrt{\pi} \int_{M_2} d^{n-1}x\sqrt{g} \sum_I \mathrm{Tr}_V \left(f(x^2) u^I_{(n-1)} \mathscr{A}^I_n (D_2) \right). \tag{2.162}$$

Because all of the geometric invariants associated with D are in the D_2 part, they are independent of x_1. Ultimately meaning we are free to choose M_1. For $M_1 = S^1$ with $x \in (0,2\pi)$ and $D_1 = -\partial^2_{x_1}$ we can rewrite the heat kernel coefficients into

$$\begin{aligned} a_k(f(x_2),D) = \int_{S^1\times M_2} d^n x\sqrt{g} \sum_I \mathrm{Tr}_V (f(x_2) u^I_{(n)} \mathscr{A}^I_k (D_2)) = \\ = 2\pi \int_{M_2} d^n x\sqrt{g} \sum_I \mathrm{Tr}_V (f(x_2) u^I_{(n)} \mathscr{A}^I_k (D_2)). \end{aligned} \tag{2.163}$$

Computing the two equations above we see that

$$u^I_{(n)} = \sqrt{4\pi} u^I_{(n+1)} \tag{2.164}$$

2.4.4 Heat Kernel Coefficients

To calculate the heat kernel coefficients we need the following variational equations

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} a_k(1, e^{-2\varepsilon f} D) = (n-k) a_k(f,D), \tag{2.165}$$

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} a_k(1, D - \varepsilon F) = a_{k-2}(F,D), \tag{2.166}$$

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} a_k(e^{-2\varepsilon f} F, e^{-2\varepsilon f} D) = 0. \tag{2.167}$$

Let us explain the equations above. To get the first equation (2.165) we differentiate

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathrm{Tr}(\exp(-e^{-2\varepsilon f} tD) = \mathrm{Tr}(2ftDe^{-tD}) = -2t \frac{d}{dt} \mathrm{Tr}(fe^{-tD})) \tag{2.168}$$

then we expand both sides in t and get (2.165). Equation (2.166) is derived similarly.

For equation (2.167) we consider the following operator

$$D(\varepsilon,\delta) = e^{-2\varepsilon f}(D-\delta F) \tag{2.169}$$

for $k=n$ we use equation (2.165) and we get

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} a_n(1, D(\varepsilon,\delta)) = 0, \tag{2.170}$$

then we take the variation in terms of δ, evaluated at $\delta = 0$ and swap the differentiation, allowed by theorem of Schwarz

$$\begin{aligned} 0 &= \frac{d}{d\delta}\bigg|_{\delta=0}\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} a_n(1, D(\varepsilon,\delta)) = \\ &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\frac{d}{d\delta}\bigg|_{\delta=0} a_n(1, D(\varepsilon,\delta)) = \\ &= a_{n-2}(e^{-2\varepsilon f}F, e^{-2\varepsilon f}D), \end{aligned} \tag{2.171}$$

which gives us equation (2.167).

Now that we have established the ground basis, we can calculate the constants u^I, and by that the first three heat kernel coefficients read

$$a_0(f,D) = (4\pi)^{-n/2}\int_M d^n x\sqrt{g}\mathrm{Tr}_V(a_0 f), \tag{2.172}$$

$$a_2(f,D) = (4\pi)^{-n/2}\frac{1}{6}\int_M d^n x\sqrt{g}\mathrm{Tr}_V)(f\alpha_1 E + \alpha_2 R), \tag{2.173}$$

$$\begin{aligned} a_4(f,D) = (4\pi)^{-n/2}\frac{1}{360}\int_M d^n x\sqrt{g}\mathrm{Tr}_V(f(\alpha_3 E_{,kk} + \alpha_4\, R\, E + \alpha_5 E^2\alpha_6 R_{,kk} + \\ + \alpha_7 R^2 + \alpha_8 R_{ij}R_{ij} + \alpha_9 R_{ijkl}R_{ijkl} + \alpha_{10}\Omega_{ij}\Omega ij)), \end{aligned} \tag{2.174}$$

where the comma subscript , denotes the derivative and constants α_I do not depend on the dimension of the Manifold and we can compute them with our variational identities.

The first coefficient α_0 can be read from the heat kernel expansion of the Laplacian on S^1 (above), $\alpha_0 = 1$. For α_1 we use (2.166), the coefficient $k=2$ is

$$\frac{1}{6}\int_M d^n x\sqrt{g}\mathrm{Tr}_V(\alpha_1 F) = \int_M d^n x\sqrt{g}\mathrm{Tr}_V(F), \tag{2.175}$$

which means $\alpha_1 = 6$. Looking at the coefficient $k=4$ we have

$$\frac{1}{360}\int_M d^n x\sqrt{g}\mathrm{Tr}_V(\alpha_4\, F\, R + 2\alpha_5\, F\, E) = \frac{1}{6}\int_M d^n x\sqrt{g}\mathrm{Tr}_V(\alpha_1\, F\, E + \alpha_2\, F\, R), \tag{2.176}$$

thus $\alpha_4 = 60\alpha_2$ and $\alpha_5 = 180$.

By applying (2.167) to $n=4$ we get

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} a_2(e^{-2\varepsilon f}F, e^{-2\varepsilon f}D) = 0. \tag{2.177}$$

Collecting the terms with $\mathrm{Tr}_V(\int_M d^n x\sqrt{g}(Ff_{,jj}))$ we obtain $\alpha_1 = 6\alpha_2$, that is $\alpha_2 = 1$, so $\alpha_4 = 60$.

Now we let $M = M_1 \times M_2$ and split $D = -\Delta_1 - \Delta_2$, where $\Delta_{1/2}$ are Laplacians for M_1, M_2. This allows us to decompose the heat kernel coefficient for $k = 4$ into

$$\begin{aligned} a_4(1,-\Delta_1-\Delta_2) =& a_4(1,-\Delta_1)a_0(1,-\Delta_2)+ \\ &+a_2(1,-\Delta_1)a_2(1,-\Delta_2) \\ &+a_0(1,-\Delta_1)a_4(1,-\Delta_2), \end{aligned} \tag{2.178}$$

with $E = 0$ and $\Omega = 0$ and by calculating the terms with R_1R_2 (scalar curvature of $M_{1/2}$) we obtain $\frac{2}{360}\alpha_7 = (\frac{\alpha_2}{6})^2$, thus $\alpha_7 = 5$.

For $n = 6$ we get

$$\begin{aligned} 0 = \mathrm{Tr}_V(\int_M d^n x\sqrt{g}(F(&-2\alpha_3 - 10\alpha_4 + 4\alpha_5)f_{,kk}E+ \\ &+(2\alpha_3+10\alpha_6)f_{,iijj}+ \\ &+(2\alpha_4-2\alpha_6-20\alpha_7-2\alpha_8)f_{,ii}R \\ &+(-8\alpha_8-8\alpha_6)f_{,ij}R_{ij})) \end{aligned} \tag{2.179}$$

we obtain $\alpha_3 = 60$, $\alpha_6 = 12$, $\alpha_8 = -2$ and $\alpha_9 = 2$

To get α_{10} we use the Gauss-Bonnet theorem, ultimately giving us $\alpha_{10} = 30$. We leave out this lengthy calculation and refer to [2] for further reading.

Let us summarize our calculations which ultimately give us the following heat kernel coefficients

$$\alpha_0(f,D) = (4\pi)^{-n/2}\int_M d^n x\sqrt{g}\mathrm{Tr}_V(f), \tag{2.180}$$

$$\alpha_2(f,D) = (4\pi)^{-n/2}\frac{1}{6}\int_M d^n x\sqrt{g}\mathrm{Tr}_V(f(6E+R)), \tag{2.181}$$

$$\alpha_4(f,D) = (4\pi)^{-n/2}\frac{1}{360}\int_M d^n x\sqrt{g}\mathrm{Tr}_V(f(60E_{,kk}+60RE+180E^2+ \tag{2.182}$$

$$+12R_{,kk}+5R^2-2R_{ij}R_{ij}2R_{ijkl}R_{ijkl}+30\Omega_{ij}\Omega_{ij})). \tag{2.183}$$

2.5 Almost-commutative Manifold

2.5.1 Two-Point Space

One of the basics forms of noncommutative space is the Two-Point space $X := \{x, y\}$. The Two-Point space can be represented by the following spectral triple

$$F_X := (C(X) = \mathbb{C}^2, H_F, D_F; J_F, \gamma_f). \tag{2.184}$$

Three properties of F_X stand out. First of all the action of $C(X)$ on H_F is faithful for $dim(H_F) \geq 2$, thus we can make a simple choice for the Hilbertspace, $H_F = \mathbb{C}^2$. Furthermore γ_F is the $\mathbb{Z}_2$ grading, which allows us to decompose H_F into

$$H_F = H_F^+ \otimes H_F^- = \mathbb{C} \otimes \mathbb{C}, \tag{2.185}$$

where

$$H_F^\pm = \{\psi \in H_F |\ \gamma_F \psi = \pm\psi\}, \tag{2.186}$$

are two eigenspaces. And lastly the Dirac operator D_F lets us interchange between $H_F^{\pm}$,

$$D_F = \begin{pmatrix} 0 & t \\ \bar{t} & 0 \end{pmatrix}, \quad \text{with } t \in \mathbb{C}. \tag{2.187}$$

The Two-Point space F_X can only have a real structure if the Dirac operator vanishes, i.e. $D_F = 0$. In that case we have KO-dimension of 0, 2 or 6. To elaborate on this, we know that there are two diagram representations of F_X at $\underbrace{\mathbb{C}\oplus\mathbb{C}}_{C(X)}$ on $\underbrace{\mathbb{C}\oplus\mathbb{C}}_{H_F}$, which are:

$$\begin{array}{ccc} & \mathbf{1} & \mathbf{1} \\ \mathbf{1}^{\circ} & & \circ \\ \mathbf{1}^{\circ} & \circ & \end{array} \qquad\qquad \begin{array}{ccc} & \mathbf{1} & \mathbf{1} \\ \mathbf{1}^{\circ} & \circ & \\ \mathbf{1}^{\circ} & & \circ \end{array}$$

If the Two-Point space F_X would be a real spectral triple then D_F can only go vertically or horizontally. This would mean that D_F vanishes. As for the KO-dimension The diagram on the left has KO-dimension 2 and 6, the diagram on the right 0 and 4. Yet KO-dimension 4 is ruled out because $dim(H_F^{\pm}) = 1$ (see Lemma 3.8 Book), which ultimately means $J_F^2 = -1$ is not allowed.

2.5.2 Product Space

By Extending the Two-Point space with a four dimensional Riemannian spin manifold, we get an almost commutative manifold $M \times F_X$, given by

$$M \times F_X = \left(C^{\infty}(M, \mathbb{C}^2), L^2(S) \otimes \mathbb{C}^2, D_M \otimes 1; J_M \otimes J_F, \gamma_M \otimes \gamma_F\right), \tag{2.188}$$

where

$$C^{\infty}(M, \mathbb{C}^2) \simeq C^{\infty}(M) \oplus C^{\infty}(M). \tag{2.189}$$

According to Gelfand duality the algebra $C^{\infty}(M, \mathbb{C}^2)$ of the spectral triple corresponds to the space

$$N := M \otimes X \simeq M \sqcup X. \tag{2.190}$$

Keep in mind that we still need to find an appropriate real structure on the Riemannian spin manifold, J_M. Furthermore total Hilbertspace can be decomposed into $H = L^2(S) \oplus L^2(S)$, such that for $\underbrace{a, b \in C^{\infty}(M)}_{(a,b) \in C^{\infty}(N)}$ and $\underbrace{\psi, \phi \in L^2(S)}_{(\psi,\phi) \in H}$ we have

$$(a,b)(\psi,\phi) = (a\psi, b\phi) \tag{2.191}$$

Along with the decomposition of the total Hilbertspace we can consider a distance formula on $M \times F_X$ with

$$d_{D_F}(x,y) = \sup\{|a(x) - a(y)| : a \in A_F, ||[D_F, a]|| \leq 1\}. \tag{2.192}$$

To calculate the distance between two points on the Two-Point space $X = \{x,y\}$, between x and y, we consider an $a \in \mathbb{C}^2 = C(X)$, which is specified by two complex numbers $a(x)$ and $a(y)$. Then we simplify the commutator inequality in (2.192)

$$||[D_F,a]|| = ||(a(y)-a(x))\begin{pmatrix}0 & t\\ \bar{t} & 0\end{pmatrix}|| \le 1, \tag{2.193}$$

$$\Leftrightarrow |a(y)-a(x)| \le \frac{1}{|t|}, \tag{2.194}$$

and the supremum gives us the distance

$$d_{D_F}(x,y) = \frac{1}{|t|}. \tag{2.195}$$

An interesting observation here is that, if the Riemannian spin manifold can be represented by a real spectral triple then a real structure J_M exists, then it follows that $t = 0$ and the distance becomes infinite. This is a purely mathematical observation and has no physical meaning.

We can also construct a distance formula on N (in reference to a point $p \in M$) between two points on $N = M \times X$, (p,x) and (p,y). Then an $a \in C^\infty(N)$ is determined by $a_x(p) := a(p,x)$ and $a_y(p) := a(p,y)$. The distance between these two points is

$$d_{D_F\otimes 1}(n_1,n_2) = \sup\{|a(n_1)-a(n_2)| : a \in A, ||[D\otimes 1,a]||\}. \tag{2.196}$$

On the other hand if we consider $n_1 = (p,x)$ and $n_2 = (q,x)$ for $p,q \in M$ then

$$d_{D_M\otimes 1}(n_1,n_2) = |a_x(p)-a_x(q)| \quad \text{for } a_x \in C^\infty(M) \text{ with } ||[D\otimes 1,a_x]|| \le 1 \tag{2.197}$$

The distance formula turns to out to be the geodesic distance formula

$$d_{D_M\otimes 1}(n_1,n_2) = d_g(p,q), \tag{2.198}$$

which is to be expected since we are only looking at the manifold. However if $n_1 = (p,x)$ and $n_2 = (q,y)$ then the two conditions are

$$||[D_M,a_x]|| \le 1, \quad \text{and} \tag{2.199}$$

$$||[D_M,a_y|| \le 1. \tag{2.200}$$

These conditions have no restriction which results in the distance being infinite! And $N = M \times X$ is given by two disjoint copies of M which are separated by infinite distance

The distance is only finite if $[D_F,a] < 1$. In this case the commutator generates a scalar field and the finiteness of the distance is related to the existence of scalar fields.

2.5.3 $U(1)$ Gauge Group

To get a insight into the physical properties of the almost commutative manifold $M \times F_X$, that is to calculate the spectral action, we need to determine the corresponding Gauge theory. For this we set of with simple definitions and important propositions to help us break down and search for the gauge group of the Two-Point F_X space which we then extend to $M \times F_X$. We will only be diving superficially into this chapter, for further reading we refer to [1].

Definition 21

Gauge Group of a real spectral triple is given by

$$\mathfrak{B}(A,H;J) := \{U = uJuJ^{-1} | u \in U(A)\} \tag{2.201}$$

Definition 22

*A *-automorphism of a *-algebra A is a linear invertible map*

$$\alpha : A \to A \quad \textit{with} \tag{2.202}$$

$$\alpha(ab) = \alpha(a)\alpha(b) \tag{2.203}$$

$$\alpha(a)^* = \alpha(a^*) \tag{2.204}$$

The ***Group of automorphisms of the *-Algebra*** *A is denoted by (A).*
The automorphism α is called ***inner*** *if*

$$\alpha(a) = uau^* \quad \text{for } U(A) \tag{2.205}$$

where $U(A)$ is

$$U(A) = \{u \in A| \ uu^* = u^*u = 1\} \quad \textit{(unitary)} \tag{2.206}$$

The Gauge group of F_X is given by the quotient $U(A)/U(A_J)$. We want a nontrivial Gauge group so we need to choose a $U(A_J) \neq U(A)$ and $U((A_F)_{J_F}) \neq U(A_F)$. We consider our Two-Point space F_X to be equipped with a real structure, which means the operator vanishes, and the spectral triple representation is

$$F_X := \left(\mathbb{C}^2, \mathbb{C}^2, D_F = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; J_f = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \gamma_F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right). \tag{2.207}$$

Here C is the complex conjugation, and F_X is a real even finite spectral triple (space) of KO-dimension 6.

Proposition 1

The Gauge group of the Two-Point space $\mathfrak{B}(F_X)$ is $U(1)$.

Proof. Note that $U(A_F) = U(1) \times U(1)$. We need to show that $U(A_F) \cap U(A_F)_{J_F}) \simeq U(1)$, such that $\mathfrak{B}(F) \simeq U(1)$. So for an element $a \in \mathbb{C}^2$ to be in $(A_F)_{J_F}$, it has to satisfy $J_F a^* J_F = a$,

$$J_F a^* J^{-1} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} \begin{pmatrix} \bar{a}_1 & 0 \\ 0 & \bar{a}_2 \end{pmatrix} \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix} = \begin{pmatrix} a_2 & 0 \\ 0 & a_1 \end{pmatrix}. \tag{2.208}$$

This can only be the case if $a_1 = a_2$. So we have $(A_F)_{J_F} \simeq \mathbb{C}$, whose unitary elements from $U(1)$ are contained in the diagonal subgroup of $U(A_F)$. □

An arbitrary hermitian field $A_\mu = -ia\partial_\mu b$ is given by two $U(1)$ Gauge fields $X^1_\mu, X^2_\mu \in C^\infty(M,\mathbb{R})$. However A_μ appears in combination $A_\mu - J_F A_\mu J_F^{-1}$:

$$A_\mu - J_F A_\mu J_F^{-1} = \begin{pmatrix} X^1_\mu & 0 \\ 0 & X^2_\mu \end{pmatrix} - \begin{pmatrix} X^2_\mu & 0 \\ 0 & X^1_\mu \end{pmatrix} =: \begin{pmatrix} Y_\mu & 0 \\ 0 & -Y_\mu \end{pmatrix} = Y_\mu \otimes \gamma_F, \tag{2.209}$$

where Y_μ the $U(1)$ Gauge field is defined as

$$Y_\mu := X_\mu^1 - X_\mu^2 \in C^\infty(M, \mathbb{R}) = C^\infty(M, i\, u(1)). \tag{2.210}$$

Proposition 2

The inner fluctuations of the almost-commutative manifold $M \times F_X$ are parameterized by a $U(1)$-gauge field Y_μ as

$$D \mapsto D' = D + \gamma^\mu Y_\mu \otimes \gamma_F \tag{2.211}$$

The action of the gauge group $\mathfrak{B}(M \times F_X) \simeq C^\infty(M, U(1))$ on D' is implemented by

$$Y_\mu \mapsto Y_\mu - i\, u \partial_\mu u^*; \quad (u \in \mathfrak{B}(M \times F_X)). \tag{2.212}$$

2.6 Noncommutative Geometry of Electrodynamics

In this chapter we describe Electrodynamics with the almost commutative manifold $M \times F_X$ and the abelian gauge group $U(1)$. We arrive at a unified description of gravity and electrodynamics although in the classical level.

The almost commutative Manifold $M \times F_X$ describes a local gauge group $U(1)$. The inner fluctuations of the Dirac operator relate to Y_μ the gauge field of $U(1)$. According to the setup we ultimately arrive at two serious problems.

First of all in the Two-Point space F_X, the operator D_F must vanish for us to have a real structure. However this implies that the electrons are massless, which would be absurd.

The second problem arises when looking at the Euclidean action for a free Dirac field

$$S = -\int i\bar{\psi}(\gamma^\mu \partial_\mu - m)\psi d^4x, \tag{2.213}$$

where ψ, $\bar{\psi}$ must be considered as independent variables, which means that the fermionic action S_f needs two independent Dirac spinors. Let us try and construct two independent Dirac spinors with our data. To do this we take a look at the decomposition of the basis and of the total Hilbertspace $H = L^2(S) \otimes H_F$. For the orthonormal basis of H_F we can write $\{e, \bar{e}\}$, where $\{e\}$ is the orthonormal basis of H_F^+ and $\{\bar{e}\}$ the orthonormal basis of H_F^-. Accompanied with the real structure we arrive at the following relations

$$J_F e = \bar{e} \quad J_F \bar{e} = e, \tag{2.214}$$

$$\gamma_F e = e \quad \gamma_F \bar{e} = \bar{e}. \tag{2.215}$$

Along with the decomposition of $L^2(S) = L^2(S)^+ \oplus L^2(S)^-$ and $\gamma = \gamma_M \otimes \gamma_F$ we can obtain the positive eigenspace

$$H^+ = L^2(S)^+ \otimes H_F^+ \oplus L^(S)^- \otimes H_F^-. \tag{2.216}$$

So, for a $\xi \in H^+$ we can write

$$\xi = \psi_L \otimes e + \psi_R \otimes \bar{e} \tag{2.217}$$

where $\psi_L \in L^2(S)^+$ and $\psi_R \in L^2(S)^-$ are the two Wheyl spinors. We denote that ξ is only determined by one Dirac spinor $\psi := \psi_L + \psi_R$, **but we require two independent spinors**. Our conclusion is that the definition of the fermionic action gives too much restrictions to the Two-Point space F_X.

2.6.1 The Finite Space

To solve the two problems we simply enlarge (double) the Hilbertspace. This is visualized by introducing multiplicities in Krajewski Diagrams which will also allow us to choose a nonzero Dirac operator that will connect the two vertices and preserve real structure making our particles massive and bringing anti-particles into the mix.

We start of with the same algebra $C^\infty(M,\mathbb{C}^2)$, corresponding to space $N = M \times X$. The Hilbertspace describes four particles, meaning it has four orthonormal basis elements. It describes **left handed electrons** and **right handed positrons**. Pointing this out, we have $\{\underbrace{e_R, e_L}_{\text{left-handed}}, \underbrace{\bar{e}_R, \bar{e}_L}_{\text{right-handed}}\}$ the orthonormal basis for $H_F = \mathbb{C}^4$. Accompanied with the real structure J_F, which allows us to interchange particles with antiparticles by the following equations

$$J_F e_R = \bar{e}_R, \tag{2.218}$$
$$J_F e_L = \bar{e_L}, \tag{2.219}$$

$$\gamma_F e_R = -e_R, \tag{2.220}$$
$$\gamma_F e_L = e_L, \tag{2.221}$$

where J_F and γ_F have to following properties

$$J_F^2 = 1, \tag{2.222}$$
$$J_F \gamma_F = -\gamma_F J_F. \tag{2.223}$$

By means of γ_F we have two options to decompose the total Hilbertspace H, firstly into

$$H_F = \underbrace{H_F^+}_{\text{ONB } \{e_L, \bar{e}_L\}} \oplus \underbrace{H_F^-}_{\text{ONB } \{e_R, \bar{e}_R\}}, \tag{2.224}$$

or alternatively into the eigenspace of particles and their antiparticles (electrons and positrons) which is preferred in literature and which we will use going further

$$H_F = \underbrace{H_e}_{\text{ONB } \{e_L, e_R\}} \oplus \underbrace{H_{\bar{e}}}_{\text{ONB } \{\bar{e}_L, \bar{e}_R\}}. \tag{2.225}$$

Here ONB means orthonormal basis.

The action of $a \in A = \mathbb{C}^2$ on H with respect to the ONB $\{e_L, e_R, \bar{e}_L, \bar{e}_R\}$ is represented by

$$a = (a_1, a_2) \mapsto \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix} \tag{2.226}$$

Do note that this action commutes wit the grading and that $[a, b^\circ] = 0$ with $b := J_F b^* J_F$ because both the left and the right action is given by diagonal matrices by equation (2.226). Note that we are still left with $D_F = 0$ and the following spectral triple

$$\left(\mathbb{C}^2, \mathbb{C}^2, D_F = 0; J_F = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \gamma_F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right). \tag{2.227}$$

It can be represented in the following Krajewski diagram, with two nodes of multiplicity two

$$\begin{array}{ccc} & \mathbf{1} & \mathbf{1} \\ \mathbf{1}^\circ & & \circledcirc \\ \mathbf{1}^\circ & \circledcirc & \end{array}$$

2.6.2 A noncommutative Finite Dirac Operator

To extend our spectral triple with a non-zero Operator, we need to take a closer look at the Krajewski diagram above. Notice that edges only exist between multiple vertices, meaning we can construct a Dirac operator mapping between the two vertices. The operator can be represented by the following matrix

$$D_F = \begin{pmatrix} 0 & d & 0 & 0 \\ \bar{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{d} \\ 0 & 0 & d & 0 \end{pmatrix} \tag{2.228}$$

We can now define the finite space F_{ED}.

$$F_{ED} := (\mathbb{C}^2, \mathbb{C}^4, D_F; J_F, \gamma_F) \tag{2.229}$$

where J_F and γ_F are like in equation (2.227) and D_F from equation (2.228).

2.6.3 Almost commutative Manifold of Electrodynamics

The almost commutative manifold $M \times F_{ED}$ has KO-dimension 2, and is represented by the following spectral triple

$$M \times F_{ED} := \left(C^\infty(M, \mathbb{C}^2),\ L^2(S) \otimes \mathbb{C}^4,\ D_M \otimes 1 + \gamma_M \otimes D_F;\ J_M \otimes J_F,\ \gamma_M \otimes \gamma_F\right) \tag{2.230}$$

The algebra didn't change, thus we can decompose it like before

$$C^\infty(M, \mathbb{C}^2) = C^\infty(M) \oplus C^\infty(M) \tag{2.231}$$

As for the Hilbertspace, we can decomposition it in the following way

$$H = (L^2(S) \otimes H_e) \oplus (L^2(S) \otimes H_{\bar{e}}). \tag{2.232}$$

Note that the one component of the algebra is acting on $L^2(S) \otimes H_e$, and the other one acting on $L^2(S) \otimes H_{\bar{e}}$. In other words the components of the decomposition of both the algebra and the Hilbertspace match by the action of the algebra.

The derivation of the gauge theory is the same for F_{ED} as for the Two-Point space F_X. We have $\mathfrak{B}(F) \simeq U(1)$ and for an arbitrary gauge field $B_\mu = A_\mu - J_F A_\mu J_F^{-1}$ we can write

$$B_\mu = \begin{pmatrix} Y_\mu & 0 & 0 & 0 \\ 0 & Y_\mu & 0 & 0 \\ 0 & 0 & Y_\mu & 0 \\ 0 & 0 & 0 & Y_\mu \end{pmatrix} \quad \text{for} \quad Y_\mu(x) \in \mathbb{R}. \tag{2.233}$$

We have one single $U(1)$ gauge field Y_μ, carrying the action of the gauge group

$$\mathfrak{B}(M \times F_{ED}) \simeq C^\infty(M, U(1)) \tag{2.234}$$

The space $N = M \times X$ consists of two copies of M. If $D_F = 0$ we have infinite distance between the two copies. Now have hacked the spectral triple to have nonzero Dirac operator D_F. The new Dirac operator still has a commuting relation with the algebra $[D_F, a] = 0\, \forall a \in A$, and we should note that the distance between the two copies of M is still infinite. This is purely an mathematically abstract observation and doesn't affect physical results.

2.6.4 Spectral Action

In this chapter we bring all our results together to establish an Action functional to describe a physical system. It turns out that the Lagrangian of the almost commutative manifold $M \times F_{ED}$ corresponds to the Lagrangian of Electrodynamics on a curved background manifold (+ gravitational Lagrangian), consisting of the spectral action S_b (bosonic) and of the fermionic action S_f.

The simplest spectral action of a spectral triple (A, H, D) is given by the trace of a function of D. We also consider inner fluctuations of the Dirac operator $D_\omega = D + \omega + \varepsilon' J \omega J^{-1}$ where $\omega = \omega^* \in \Omega^1_D(A)$.

Definition 23

Let $f : \mathbb{R} \to \mathbb{R}$ be a suitable function ***positive and even****. The spectral action is then*

$$S_b[\omega] := Tr\big(f(\frac{D_\omega}{\Lambda})\big) \tag{2.235}$$

where Λ is a real cutoff parameter. The minimal condition on f is that $f(\frac{D_\omega}{\Lambda})$ is a trace class operator. A trace class operator is a compact operator with a well defined finite trace independent of the basis. The subscript b in S_b stands for bosonic, because in physical applications ω will describe bosonic fields.

In addition to the bosonic action S_b we can define a topological spectral action S_{top}. Leaning on the grading γ the topological spectral action is

$$S_{top}[\omega] := Tr(\gamma f(\frac{D_\omega}{\Lambda})). \tag{2.236}$$

Definition 24

The fermionic action is defined by

$$S_f[\omega, \psi] = (J\tilde{\psi}, D_\omega \tilde{\psi}) \tag{2.237}$$

with $\tilde{\psi} \in H^+_{cl} := \{\tilde{\psi} : \psi \in H^+\}$, where H^+_{cl} is a set of Grassmann variables in H in the +1-eigenspace of the grading γ.

APPENDIX?? Grassmann variables are a set of Basis vectors of a vector space, they form a unital algebra over a vector field V, where the generators are anti commuting, that is for Grassmann variables θ_i, θ_j we have

$$\theta_i\theta_j = -\theta_j\theta_i \tag{2.238}$$
$$\theta_i x = x\theta_j \quad x \in V \tag{2.239}$$
$$(\theta_i)^2 = 0 \;\; (\theta_i\theta_i = -\theta_i\theta_i) \tag{2.240}$$

Proposition 3

The spectral action of the almost commutative manifold M with $\dim(M) = 4$ with a fluctuated Dirac operator is

$$Tr(f\frac{D_\omega}{\Lambda}) \sim \int_M \mathscr{L}(g_{\mu\nu}, B_\mu, \Phi)\sqrt{g}\, d^4x + O(\Lambda^{-1}), \tag{2.241}$$

where

$$\mathscr{L}(g_{\mu\nu}, B_\mu, \Phi) = N\mathscr{L}_M(g_{\mu\nu})\mathscr{L}_B(B_\mu) + \mathscr{L}_\phi(g_{\mu\nu}, B_\mu, \Phi). \tag{2.242}$$

The Lagrangian $\mathscr{L}_M$ is of the spectral triple, represented by the following term $(C^\infty(M), L^2(S), D_M)$

$$\mathscr{L}_M(g_{\mu\nu}) := \frac{f_4\Lambda^4}{2\pi^2} - \frac{f_2\Lambda^2}{24\pi^2}s - \frac{f(0)}{320\pi^2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}, \tag{2.243}$$

here $C^{\mu\nu\rho\sigma}$ is defined in terms of the Riemannian curvature tensor $R_{\mu\nu\rho\sigma}$ and the Ricci tensor $R_{\nu\sigma} = g^{\mu\rho}R_{\mu\nu\rho\sigma}$. The kinetic term of the gauge field is described by the Lagrangian $\mathscr{L}_B$, which takes the following shape

$$\mathscr{L}_B(B_\mu) := \frac{f(0)}{24\pi^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}). \tag{2.244}$$

Lastly $\mathscr{L}_\phi$ is the scalar-field Lagrangian with a boundary term, given by

$$\begin{aligned}\mathscr{L}_\phi(g_{\mu\nu}, B_\mu, \Phi) :=& -\frac{2f_2\Lambda^2}{4\pi^2}Tr(\Phi^2) + \frac{f(0)}{8\pi^2}Tr(\Phi^4) + \frac{f(0)}{24\pi^2}\Delta(Tr(\Phi^2)) \\ &+ \frac{f(0)}{48\pi^2}sTr(\Phi^2)\frac{f(0)}{8\pi^2}Tr((D_\mu\Phi)(D^\mu\Phi)).\end{aligned} \tag{2.245}$$

Proof. The dimension of the manifold M is $\dim(M) = \mathrm{Tr}(id) = 4$. For an $x \in M$, we have an asymptotic expansion of the term $\mathrm{Tr}(f(\frac{D_\omega}{\Lambda}))$ as Λ goes to infinity, which can be written as

$$\begin{aligned}\mathrm{Tr}(f(\frac{D_\omega}{\Lambda})) \simeq& \, 2f_4\Lambda^4 a_0(D^2_\omega) + 2f_2\Lambda^2 a_2(D^2_\omega) \\ &+ f(0)a_4(D^4_\omega) + O(\Lambda^{-1}).\end{aligned} \tag{2.246}$$

We have to note here that the heat kernel coefficients are zero for uneven k, and they are dependent on the fluctuated Dirac operator D_ω. We can rewrite the heat kernel coefficients in terms of D_M, for the first two terms a_0 and a_2 we use $N := \mathrm{Tr}\mathbb{1}_{\mathbb{H}_\mathbb{F}})$ and write

$$a_0(D_\omega^2) = Na_0(D_M^2), \tag{2.247}$$

$$a_2(D_\omega^2 = Na_2(D_M^2) - \frac{1}{4\pi^2}\int_M \mathrm{Tr}(\Phi^2)\sqrt{g}d^4x. \tag{2.248}$$

For a_4 we extend in terms of coefficients of F, **REWRITE: look week9.pdf for the standard version**

$$\frac{1}{360}\mathrm{Tr}(60sF) = -\frac{1}{6}S(Ns + 4\mathrm{Tr}(\Phi^2)) \tag{2.249}$$

$$F^2 = \frac{1}{16}s^2 \otimes 1 + 1 \otimes \Phi^4 - \frac{1}{4}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma F_{\mu\nu}F^{\mu\nu} + \tag{2.250}$$

$$+\gamma^\mu\gamma^\nu \otimes (D_\mu\Phi)(D_\nu\Phi) + \frac{1}{2}s \otimes \Phi^2 + \text{ traceless terms} \tag{2.251}$$

$$\frac{1}{360}\mathrm{Tr}(180F^2) = \frac{1}{8}s^2N + 2\mathrm{Tr}(\Phi^4) + \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \tag{2.252}$$

$$+2\mathrm{Tr}((D_\mu\Phi)(D^\mu\Phi)) + s\mathrm{Tr}(\Phi^2) \tag{2.253}$$

$$\frac{1}{360}\mathrm{Tr}(-60\Delta F) = \frac{1}{6}\Delta(Ns + 4\mathrm{Tr}(\Phi^2)). \tag{2.254}$$

The cross terms of the trace in $\Omega^E_{\mu\nu}\Omega^{E\mu\nu}$ vanishes because of the antisymmetric property of the Riemannian curvature tensor, thus we can write

$$\Omega^E_{\mu\nu}\Omega^{E\mu\nu} = \Omega^S_{\mu\nu}\Omega^{S\mu\nu} \otimes 1 - 1 \otimes F_{\mu\nu}F^{\mu\nu} + 2i\Omega^S_{\mu\nu} \otimes F^{\mu\nu}. \tag{2.255}$$

The trace of the cross term $\Omega^S_{\mu\nu}$ vanishes because

$$\mathrm{Tr}(\Omega^S_{\mu\nu}) = \frac{1}{4}R_{\mu\nu\rho\sigma}\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \frac{1}{4}R_{\mu\nu\rho\sigma}g^{\mu\nu} = 0, \tag{2.256}$$

then the trace of the whole term is given by

$$\frac{1}{360}\mathrm{Tr}(30\Omega^E_{\mu\nu}\Omega^{E\mu\nu}) = \frac{N}{24}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{1}{3}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}). \tag{2.257}$$

Finally plugging the results into the coefficient a_4 and simplifying we get

$$\begin{aligned} a_4(x, D_\omega^4) = Na_4(x, D_M^2) + \frac{1}{4\pi^2}\Big(\frac{1}{12}s\mathrm{Tr}(\Phi^2) + \frac{1}{2}\mathrm{Tr}(\Phi^4) \\ + \frac{1}{4}\mathrm{Tr}((D_\mu\Phi)(D^\mu\Phi)) + \frac{1}{6}\Delta\mathrm{Tr}(\Phi^2) + \frac{1}{6}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\Big). \end{aligned} \tag{2.258}$$

The only thing left is to substitute the heat kernel coefficients into the heat kernel expansion in equation (2.246). □

2.6.5 Fermionic Action

We remind ourselves the definition of the fermionic action in definition 2.6.4 and the manifold we are dealing with in equation (2.230). The Hilbertspace H_F is separated into the particle-antiparticle states with ONB $\{e_R, e_L, \bar{e}_R, \bar{e}_L\}$. The orthonormal basis of H_F^+ is $\{e_L, \bar{e}_R\}$ and consequently for H_F^-, $\{e_R, \bar{e}_L\}$. We can decompose a spinor $\psi \in L^2(S)$ in each of the eigenspaces $H_F^{\pm}$, $\psi = \psi_R + \psi_L$. That means for an arbitrary $\psi \in H^+$ we can write

$$\psi = \chi_R \otimes e_R + \chi_L \otimes e_L + \psi_L \otimes \bar{e}_R + \psi_R \otimes \bar{e}_L, \tag{2.259}$$

where $\chi_L, \psi_L \in L^2(S)^+$ and $\chi_R, \psi_R \in L^2(S)^-$.

Since the fermionic action yields too much restriction on F_{ED} (modified Two-Point space F_X) we redefine it by taking account the fluctuated Dirac operator

$$D_\omega = D_M \otimes i + \gamma^\mu \otimes B_\mu + \gamma_M \otimes D_F. \tag{2.260}$$

The Fermionic Action is

$$S_F = (J\tilde{\xi}, D_\omega \tilde{\xi}) \tag{2.261}$$

for a $\xi \in H^+$. Then the straight forward calculation gives

$$\frac{1}{2}(J\tilde{\xi}, D_\omega \tilde{\xi}) = \frac{1}{2}(J\tilde{\xi}, (D_M \otimes i)\tilde{\xi}) \tag{2.262}$$

$$+\frac{1}{2}(J\tilde{\xi}, (\gamma^\mu \otimes B_\mu)\tilde{\xi}) \tag{2.263}$$

$$+\frac{1}{2}(J\tilde{\xi}, (\gamma_M \otimes D_F)\tilde{\xi}), \tag{2.264}$$

(note that we add the constant $\frac{1}{2}$ to the action). For the term in (2.262) we calculate

$$\begin{aligned}\frac{1}{2}(J\tilde{\xi}, (D_M \otimes 1)\tilde{\xi}) &= \frac{1}{2}(J_M\tilde{\chi}_R, D_M\tilde{\psi}_L) + \frac{1}{2}(J_M\tilde{\chi}_L, D_M\tilde{\psi}_R) + \\ &+ \frac{1}{2}(J_M\tilde{\psi}_L, D_M\tilde{\psi}_R) + \frac{1}{2}(J_M\tilde{\chi}_R, D_M\tilde{\chi}_L) \\ &= (J_M\tilde{\chi}, D_M\tilde{\chi}).\end{aligned} \tag{2.265}$$

For the term in (2.263) we have

$$\begin{aligned}\frac{1}{2}(J\tilde{\xi}, (\gamma^\mu \otimes B_\mu)\tilde{\xi}) &= -\frac{1}{2}(J_M\tilde{\chi}_R, \gamma^\mu Y_\mu \tilde{\psi}_R) - \frac{1}{2}(J_M\tilde{\chi}_L, \gamma^\mu Y_\mu \tilde{\psi}_R) + \\ &+ \frac{1}{2}(J_M\tilde{\psi}_L, \gamma^\mu Y_\mu \tilde{\chi}_R) + \frac{1}{2}(J_M\tilde{\psi}_R, \gamma^\mu Y_\mu \tilde{\chi}_L) = \\ &= -(J_M\tilde{\chi}, \gamma^\mu Y_\mu \tilde{\psi}).\end{aligned} \tag{2.266}$$

And for (2.264) we can write

$$\begin{aligned}\frac{1}{2}(J\tilde{\xi}, (\gamma_M \otimes D_F)\tilde{\xi}) &= +\frac{1}{2}(J_M\tilde{\chi}_R, d\gamma_M\tilde{\chi}_R) + \frac{1}{2}(J_M\tilde{\chi}_L, \bar{d}\gamma_M\tilde{\chi}_L) + \\ &+ \frac{1}{2}(J_M\tilde{\chi}_L, \bar{d}\gamma_M\tilde{\chi}_L) + \frac{1}{2}(J_M\tilde{\chi}_R, d\gamma_M\tilde{\chi}_R) = \\ &= i(J_M\tilde{\chi}, m\tilde{\psi}).\end{aligned} \tag{2.267}$$

A small problem arises, we obtain a complex mass parameter d, but we can write $d := im$ for $m \in \mathbb{R}$, which stands for the real mass.

Finally the fermionic action of $M \times F_{ED}$ takes the form

$$S_f = -i\big(J_M\tilde{\chi}, \gamma(\nabla^S_\mu - i\Gamma_\mu)\tilde{\Psi}\big) + \big(S_M\tilde{\chi}_L, \bar{d}\tilde{\psi}_L\big) - \big(J_M\tilde{\chi}_R, d\tilde{\psi}_R\big). \tag{2.268}$$

Ultimately we arrive at the full Lagrangian of $M \times F_{ED}$, which is the sum of purely gravitational Lagrangian

$$\mathscr{L}_{grav}(g_{\mu\nu}) = 4\mathscr{L}_M(g_{\mu\nu}) + \mathscr{L}_\phi(g_{\mu\nu}), \tag{2.269}$$

and the Lagrangian of electrodynamics

$$\mathscr{L}_{ED} = -i\Big\langle J_M\tilde{\chi}, \big(\gamma^\mu(\nabla^S_\mu - iY_\mu) - m\big)\tilde{\psi}\big)\Big\rangle + \frac{f(0)}{6\pi^2}Y_{\mu\nu}Y^{\mu\nu}. \tag{2.270}$$

3 Conclusion

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4 Acknowledgment

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