

# Noncommutative Geometry

## Bachelor's seminar

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- Noncommutative geometry (NCG) brings many mathematical fields together
- Interesting physics application
- First understand duality between (classical) geometry and commutative Algebras
- Gelfand-Naimark-Theoreme in Functional Analysis in the 1940s

Introduce:

Finite topological Space  $X$  consisting of  $N$  points. (discrete topology)

$\bullet$   $\bullet$   $\cdot$   $\cdot$   $\cdot$   $\bullet$   
1 2  $\dots$   $N$

Commutative algebra of continuous functions on  $X$

$$C(X) = \{f : X \rightarrow \mathbb{C} : f \text{ is continuous}\}$$

Results of the Theorem:

- $X$  and  $C(X)$  contain the same information (duality)
- Construct  $X$ , given  $C(X)$ .
- Translate geometrical properties of  $X$  to algebraic data (metric, differential forms, vector fields, curvature, etc.)

- NCG extends this duality to noncommutative algebras
- Provides methods to deal with these algebras
- NCG is encoded in a spectral triple

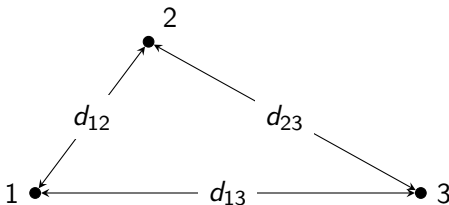
## The Finite Spectral Triple

$(A, H, D)$

- $A$  - Algebra
- $H$  - Hilbertspace
- $D$  - Symmetric operator acting on  $H$

- The metric describes distances between points on a space
- Simple example the discrete metric on a discrete space

$$d_{ij} = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$



- In NCG we can also define a metric
- replace algebra with a noncommutative matrix Algebra  $A$
- a finite-dimensional Hilbertspace  $H$
- and a hermitian matrix  $D$

A metric constructed with  $(A, H, D)$

$$d_{ij} = \sup_{a \in A} \{ |\text{Tr}(a(i)) - \text{Tr}(a(j))| : \|[D, a]\| < 1 \}$$

- Example: 1D calculus  $f(x)dx$
- In NCG defining the differential one form requires only the spectral Triple  $(A, H, D)$

## Connes' Differential One Form

$$\Omega_D^1(A) = \{\sum_k a_k [D, b_k] : a_k, b_k \in A\}$$

With a consequent derivation of a algebra  $d : A \rightarrow \Omega_D^1$ ,  $d(\cdot) = [D, \cdot]$



- Introduce a richer geometry
- From finite topological space to Manifolds generalized with noncommutativity
- From finite to general spectral triples with a self adjoint operator (Dirac Operator)

- NCG of Electrodynamics
- NCG of the Quantum Hall Effect
- NCG of the Standard Model, full Lagrangian

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