

# Notes on Noncommutative Geometry and Particle Physics

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## 1 Noncommutative Geometric Spaces

### 1.1 Exercises

#### Exercise 1

Make the proof of the last theorem (see week4.pdf) explicit for  $N = 3$ .

For the  $C^*$  algebra we have  $A = \mathbb{C}^3$  For  $H$  we have  $H = (\mathbb{C}^2)^{\oplus 3} = H_2 \oplus H_2^1 \oplus H_2^2$ .

The symmetric operator  $D$  acting on  $H$  and the representation  $\pi(a)$ :

$$\begin{aligned}\pi((a(1), a(2), a(3))) &= \begin{pmatrix} a(1) & 0 \\ 0 & a(2) \end{pmatrix} \oplus \begin{pmatrix} a(1) & 0 \\ 0 & a(3) \end{pmatrix} \oplus \begin{pmatrix} a(2) & 0 \\ 0 & a(2) \end{pmatrix} \\ &= \begin{pmatrix} a(1) & 0 & 0 & 0 & 0 & 0 \\ 0 & a(2) & 0 & 0 & 0 & 0 \\ 0 & 0 & a(1) & 0 & 0 & 0 \\ 0 & 0 & 0 & a(3) & 0 & 0 \\ 0 & 0 & 0 & 0 & a(2) & 0 \\ 0 & 0 & 0 & 0 & 0 & a(3) \end{pmatrix} \quad (1)\end{aligned}$$

$$\begin{aligned}D &= \begin{pmatrix} 0 & x_1 \\ x_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & x_2 \\ x_2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & x_3 \\ x_3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & x_1 & 0 & 0 & 0 & 0 \\ x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 & x_3 & 0 \end{pmatrix} \quad (2)\end{aligned}$$

(3)

Then the norm of the commutator would be the largest eigenvalue

$$\begin{aligned}||[D, \pi(a)]|| &= ||D\pi(a) - \pi(a)D|| \\ &= \left\| \begin{pmatrix} 0 & x_1(a(2) - a(1)) & 0 & 0 & 0 & 0 \\ -x_1(a(2) - a(1)) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2(a(3) - a(1)) & 0 & 0 \\ 0 & 0 & -x_2(a(3) - a(1)) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_3(a(3) - a(2)) \\ 0 & 0 & 0 & 0 & -x_3(a(2) - a(3)) & 0 \end{pmatrix} \right\| \quad (4)\end{aligned}$$

The matrix in Equation ?? is a skew symmetric matrix its eigenvalues are  $i\lambda_1, i\lambda_2, i\lambda_3, i\lambda_4$ , where the  $\lambda$ 's are on the upper and lower diagonal check [https://en.wikipedia.org/wiki/Skew-symmetric\\_matrix#Skew-symmetrizable\\_matrix](https://en.wikipedia.org/wiki/Skew-symmetric_matrix#Skew-symmetrizable_matrix). The matrix norm of would be the maximum of the norm of the larges eigenvalues:

$$\begin{aligned}||[D, \pi(a)]|| &= \max_{a \in A} \{x_1|a(2) - a(1)|, \\ &\quad x_2|(a(3) - a(1))|, \\ &\quad x_3|(a(3) - a(2))|, \} \quad (5)\end{aligned}$$

The metric is then:

$$d = \begin{pmatrix} 0 & a(1) - a(2) & a(1) - a(3) \\ a(2) - a(1) & 0 & a(2) - a(3) \\ a(3) - a(1) & a(3) - a(2) & 0 \end{pmatrix} \quad (6)$$

### Exercise 2

Compute the metric on the space of three points given by  $d_{ij} = \sup_{a \in A} \{|a(i) - a(j)| : ||[D, \pi(a)]|| \leq 1\}$  for the set of data  $A = \mathbb{C}^3$  acting in the defining representation  $H = \mathbb{C}^3$ , and

$$D = \begin{pmatrix} 0 & d^{-1} & 0 \\ d^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some  $d \in \mathbb{R}$

We have  $A = \mathbb{C}^3$ ,  $H = \mathbb{C}^3$  and  $D$  from above, then

$$||[D, \pi(a)]|| = d^{-1} \left\| \begin{pmatrix} 0 & a(2) - a(1) & 0 \\ -(a(2) - a(1)) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\| \quad (7)$$

The metric is then

$$d = \begin{pmatrix} 0 & a(1) - a(2) & a(1) \\ a(2) - a(1) & 0 & a(2) \\ -a(1) & -a(2) & 0 \end{pmatrix} \quad (8)$$

### Exercise 3

Show that  $d_{ij}$  from Equation 12 is a metric on  $\hat{A}$  by establishing that:

$$d_{ij} = 0 \Leftrightarrow i = j \quad (9)$$

$$d_{ij} = d_{ji} \quad (10)$$

$$d_{ij} \leq d_{ik} + d_{kj} \quad (11)$$

$$d_{ij} = \sup_{a \in A} \{|\text{Tr}(a(i)) - \text{Tr}(a(j))| : ||[D, a]|| \leq 1\} \quad (12)$$

For Equation 9 set  $i = j$  in 12.

$$\begin{aligned} d_{ii} &= \sup_{a \in A} \{|\text{Tr}(a(i)) - \text{Tr}(a(i))| : ||[D, a]|| \leq 1\} \\ &= \sup_{a \in A} \{0 : ||[D, a]|| \leq 1\} = 0 \end{aligned}$$

For Equation 10 obviously we have the commuting property of addition.

For Equation 11, for  $k = j$  then  $d_{kj} = 0$  and the equality holds. For  $i = k$  then  $d_{ik} = 0$  and equality holds. Else set  $d_{ik} = 1$  and  $d_{kj} = 1$  then  $d_{ij} = 1 \leq d_{ik} + d_{kj} = 2$

## 1.2 Properties of Matrix Algebras

**Lemma 1.** *If  $A$  is a unital  $C^*$  algebra that acts faithfully on a finite dimensional Hilbert space, then  $A$  is a matrix algebra of the Form:*

$$A \simeq \bigoplus_{i=1}^N M_{n_i}(\mathbb{C}) \quad (13)$$

*Proof.* Since  $A$  acts faithfully on a Hilbert space, then  $A$  is a  $C^*$  subalgebra of a matrix algebra  $L(H) = M_{\dim(H)}(\mathbb{C}) \Rightarrow A \simeq \text{Matrix algebra}$ .  $\square$

**Question 1.** What does the author mean when he says 'acts faithfully on a Hilbertspace'? Then the representation is fully reducible, or that the presentation is irreducible?

For a  $*$ -representation 'faithful' if it is injective. For a  $*$ -homomorphism 'faithful' means one-to-one correspondence

**Example 1.**  $A = M_n(\mathbb{C})$  and  $H = \mathbb{C}^n$ ,  $A$  acts on  $H$  with matrix multiplication and standard inner product.  $D$  on  $H$  is a hermitian matrix  $n \times n$  matrix.

$D$  is referred to as a finite Dirac operator as in as its  $\infty$  dimensional on Riemannian Spin manifolds coming in Chapter 4.

Now can introduce a 'differential 'geometric structure' on the finite space  $X$  with the **devided difference**

$$\frac{a(i) - a(j)}{d_{ij}} \quad (14)$$

for each pair  $i, j \in X$  the finite dimensional discrete space  $X$ . This appears in the entries in the commutator  $[D, a]$  in the above exercises.

**Definition 1.** Given an finite spectral triple  $(A, H, D)$ , the  $A$ -bimodule of Connes' differential one form is:

$$\Omega_D^1(A) := \left\{ \sum_k a_k [D, b_k] : a_k, b_k \in A \right\} \quad (15)$$

**Question 2.** Is the Conne's differential one form the set of all '1st order differential operators' given  $A$ , that act on  $H$ ?

Then there is a map  $d : A \rightarrow \Omega_D^1(A)$ ,  $d = [D, \cdot]$ .

### Exercise 4

**Verify that 'd' is a derivation of the  $C^*$  algebra**

$$\begin{aligned} d(ab) &= d(a)b + ad(b) \\ d(a^*) &= -d(a)^* \end{aligned}$$

For the record  $d(\cdot) = [D, \cdot]$ , then we have

1.

$$\begin{aligned} d(ab) &= [D, ab] = [D, a]b + a[D, b] \\ &= d(a)b + ad(b) \end{aligned}$$

2.

$$\begin{aligned} d(a^*) &= [D, a^*] = Da^* - a^*D \\ &= -(D^*a - aD^*) = -[D^*, a] \\ &= -d(a)^* \end{aligned}$$

### Exercise 5

Verify that  $\Omega_D^1(A)$  is an  $A$ -bimodule by rewriting

$$a(a_k[D, b_k])b = \sum_k a'_k[D, b'_k] \quad a'_k, b'_k \in A$$

Begin

$$\begin{aligned} a(a_k[D, b_k])b &= aa_k(Db_k - b_kD)b = \\ &= aa_k(Db_kb - b_kDb) = aa_k(Db_kb - b_kDb - b_kbD + b_kbD) = \\ &= aa_k(Db_kb - b_kbD + b_kbD - b_kDb) = \\ &= aa_k[D, b_kb] + aa_kb[D, b] = \\ &= \sum_k a'_k[D, b'_k] \end{aligned}$$

**Lemma 2.** Let  $(A, H, D) = (M_n(\mathbb{C}, \mathbb{C}^n, D)$ , with  $D$  a hermitian  $n \times n$  matrix. If  $D$  is not a multiple of the identity then:

$$\Omega_D^1(A) \simeq M_n(\mathbb{C}) = A \quad (16)$$

*Proof.* Assume  $D = \sum_i \lambda_i e_{ii}$  (diagonal),  $\lambda_i \in \mathbb{R}$  and  $\{e_{ij}\}$  the basis of  $M_n(\mathbb{C})$ . For fixed  $i, j$  choose  $k$  such that  $\lambda_k \neq \lambda_j$  then

$$\left( \frac{1}{\lambda_k - \lambda_j} e_{ik} \right) [D, e_{kj}] = e_{ij} \quad (17)$$

$e_{ij} \in \Omega_D^1(A)$  by the above definition. And  $\Omega_D^1(A) \subset L(\mathbb{C}^n) = H \simeq M_n(\mathbb{C}) = A \quad \square$

### Exercise 6

**Consider**  $(A = \mathbb{C}^2, H = \mathbb{C}^2, D = \begin{pmatrix} 0 & \lambda \\ \bar{\lambda} & 0 \end{pmatrix})$  with  $\lambda \neq 0$ . **Show that**  $\Omega_D^1(A) \simeq M_2(\mathbb{C})$

Because of the Hilbert Basis  $D$  can be extended in terms of the basis of  $M_2(\mathbb{C})$ , plugging this into Equation 17 will get us the same cyclic result, thus  $\Omega_D^1(A) \simeq M_2(\mathbb{C})$

## 1.3 Morphisms Between Finite Spectral Triples

**Definition 2.** two finite spectral triples  $(A_1, H_1, D_1)$  and  $(A_2, H_2, D_2)$  are called unitarily equivalent if

- $A_1 = A_2$
- $\exists U : H_1 \rightarrow H_2$ , unitary with
  1.  $U\pi_1(a)U^* = \pi_2(a)$  with  $a \in A_1$
  2.  $UD_1U^* = D_2$

Some remarks

- the above is an equivalence relation
- spectral unitary equivalence is given by the unitaries of the matrix algebra itself
- for any such  $U$  then  $(A, H, D) \sim (A, H, UDU^*)$
- $UDU^* = D + U[D, U^*]$  of the form of elements in  $\Omega_D^1(A)$ .

### Exercise 7

**Show that the unitary equivalence between finite spectral triples is a equivalence relation**

An equivalence relation needs to satisfy reflexivity, symmetry transitivity. Let  $(A_1, H_1, D_1)$ ,  $(A_2, H_2, D_2)$  and  $(A_3, H_3, D_3)$  be three finite spectral triples.

For reflexivity  $(A_1, H_1, D_1) \sim (A_1, H_1, D_1)$ . So there exists a  $U : H_1 \rightarrow H_1$  unitary, which is the identity and always exists.

For symmetry we need

$$(A_1, H_1, D_1) \sim (A_2, H_2, D_2) \Leftrightarrow (A_2, H_2, D_2) \sim (A_1, H_1, D_1)$$

because  $U$  is unitary:

$$\begin{aligned} U\pi_1(a)U^* &= \pi_2(a) \quad | \cdot U^* \sqcup U \\ U^*U\pi_1(a)U^*U &= \pi_1(a) = U^*\pi_2(a)U \end{aligned}$$

The same with the symmetric operator  $D$ .

For transitivity we need

$$\begin{aligned} (A_1, H_1, D_1) &\sim (A_2, H_2, D_2) \quad \text{and} \quad (A_2, H_2, D_2) \sim (A_3, H_3, D_3) \\ \Rightarrow (A_1, H_1, D_1) &\sim (A_3, H_3, D_3) \end{aligned}$$

There are two unitary maps  $U_{12} : H_1 \rightarrow H_2$  and  $U_{23} : H_2 \rightarrow H_3$  then

$$\begin{aligned} U_{23}U_{12}\pi_1(a)U_{12}^*U_{23}^* &= U_{23}\pi_2(a)U_{23}^* \\ &= \pi_3(a) \\ U_{23}U_{12}D_1U_{12}^*U_{23}^* &= U_{23}D_2U_{23}^* \\ &= D_3 \end{aligned}$$

Extending the this relation we look again at the notion of equivalence from Morita equivalence of Matrix Algebras.

Given a Hilbert bimodule  $E \in KK_f(B, A)$  and  $(A, H, D)$  we construct a finite spectral triple on  $B$ ,  $(B, H', D')$

$$H' = E \otimes_A H \tag{18}$$

This extends the left action on  $B$  with the right action and inherits the  $\mathbb{C}$  valued inner product space.

$$D'(e \otimes \xi) = e \otimes D\xi + \nabla(e)\xi \quad e \in E, a \in A \tag{19}$$

Where  $\nabla$  is called the *connection on the right  $A$ -module  $E$*  associated with the derivation  $d = [D, \cdot]$  and satisfying the *Leibnitz Rule* which is

$$\nabla(ae) = \nabla(e)a + e \otimes [D, a] \quad e \in E, a \in A \tag{20}$$

Then the linearity of the balanced tensor product  $E \otimes_A H$  is satisfied

$$\begin{aligned} D'(ea \otimes \xi - e \otimes a\xi) &= D'(ea \otimes \xi) - D'(e \otimes a\xi) \\ &= ea \otimes D\xi + \nabla(ea)\xi - e \otimes D(a\xi) - \nabla(e)a\xi \\ &= 0 \end{aligned}$$

With the information thus far we can prove the following theorem

**Theorem 1.** *If  $(A, H, D)$  a finite spectral triple,  $E \in KK_f(B, A)$ . Then  $(V, E \otimes_A H, D')$  is a finite spectral triple, provided that  $\nabla$  satisfies the compatibility condition*

$$\langle e_1, \nabla e_2 \rangle_E - \langle \nabla e_1, e_2 \rangle_E = d\langle e_1, e_2 \rangle_E \quad e_1, e_2 \in E \tag{21}$$

*Proof.*  $E \otimes_A H$  was shown in the previous section (text before the theorem). The only thing left is to show that  $D'$  is a symmetric operator, this we can just compute. Let  $e_1, e_2 \in E$  and  $\xi_1, \xi_2 \in H$  then

$$\begin{aligned} \langle e_1 \otimes \xi_1, D'(e_2 \otimes \xi_2) \rangle_{E \otimes_A H} &= \langle \xi_1, \langle e_1, \nabla e_2 \rangle_E \xi_2 \rangle_H + \langle \xi_1, \langle e_1, e_2 \rangle_E D \xi_2 \rangle_H \\ &= \langle \xi_1, \langle \nabla e_1, e_2 \rangle_E \xi_2 \rangle_H + \langle \xi_1, d \langle e_1, e_2 \rangle_E \xi_2 \rangle_H \\ &\quad + \langle D \xi_1, \langle e_1, e_2 \rangle_E \xi_2 \rangle_H - \langle \xi_1, [D, \langle e_1, e_2 \rangle_E] \xi_2 \rangle_H \\ &= \langle D'(e_1 \otimes \xi_1), e_2 \otimes \xi_2 \rangle_{E \otimes_A H} \end{aligned}$$

□

### Exercise 8

Let  $\nabla$  and  $\nabla'$  be two connections on a right  $A$ -module  $E$ . Show that  $\nabla - \nabla'$  is a right  $A$ -linear map  $E \rightarrow E \otimes_A \Omega_D^1(A)$

Both  $\nabla$  and  $\nabla'$  need to satisfy the Leibnitz rule, so let's see if  $\nabla - \nabla'$  does.

$$\begin{aligned} \nabla(ea) - \nabla'(ea) &= \nabla(e) + e \otimes [D, a] \\ &\quad - (\nabla'(e)a + e \otimes [D', a]) \\ &= \bar{\nabla}a + e \otimes (Da - aD - D'a + aD') \\ &= \bar{\nabla}a + e \otimes ((D - D')a - a(D - D')) \\ &= \bar{\nabla}a + e \otimes [D', a] \\ &= \bar{\nabla}(ea) \end{aligned}$$

Therefore  $\nabla - \nabla'$  is a linear map.

### Exercise 9

Construct a finite spectral triple  $(A, H', D')$  from  $(A, H, D)$

1. show that the derivation  $d(\cdot) : A \rightarrow A \otimes_A \Omega_D^1(A) = \Omega_D^1(A)$  is a connection on  $A$  considered a right  $A$ -module
2. Upon identifying  $A \otimes_A H \simeq H$ , what is  $D'$  when the connection is  $d(\cdot)$ .
3. Use 1) and 2) to show that any connection  $\nabla : A \rightarrow A \otimes_A \Omega_D^1(A)$  is given by

$$\nabla = d + \omega$$

where  $\omega \in \Omega_D^1(A)$

4. Upon identifying  $A \otimes_A H \simeq H$ , what is the difference operator  $D'$  with the connection on  $A$  given by  $\nabla = d + \omega$



1.  $\nabla(e \cdot a) = d(a)$
2.  $D'(a\xi) = a(D\xi) + (\nabla a)\xi = D(a\xi)$
3. Use the identity element  $e \in A$   
 $\nabla(e \cdot a) = \nabla(e)a + 1 \otimes d(a) = d(a)\nabla(e)a$
4.  $D'(a \otimes \xi) = D'(a\xi) = a(D\xi) + (\nabla a)\xi = a(D\xi) + \nabla(e \cdot a)\xi$   
 $= D(a\xi) + \nabla(e)(a\xi)$

## 1.4 Graphing Finite Spectral Triples

**Definition 3.** A *graph* is a ordered pair  $(\Gamma^{(0)}, \Gamma^{(1)})$ . Where  $\Gamma^{(0)}$  is the set of vertices (nodes) and  $\Gamma^{(1)}$  a set of pairs of vertices (edges)

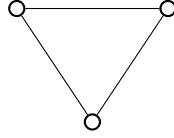


Figure 1: A simple graph with three vertices and three edges

### Exercise 10

**Show that any finite-dimensional faithful representation  $H$  of a matrix algebra  $A$  is completely reducible. To do that show that the complement  $W^\perp$  of an  $A$ -submodule  $W \subset H$  is also an  $A$ -submodule of  $H$ .**

$A \simeq \bigoplus_{i=1}^N M_{n_i}(\mathbb{C})$  is the matrix algebra then  $H$  is a Hilbert  $A$ -bimodule and  $W$  a submodule of  $A$ . Because we have  $H = W \cup W^\perp$ , then  $W^\perp$  is naturally a  $A$ -submodule, because elements in  $W^\perp$  need to satisfy the bimodularity.

**Definition 4.** A  $\Lambda$ -decorated graph is given by an ordered pair  $(\Gamma, \Lambda)$  of a finite graph  $\Gamma$  and a set of positive integers  $\Lambda$  with the labeling

- of the vetices  $v \in \Gamma^{(0)}$  given by  $n(v) \in \Lambda$
- of the edges  $e = (v_1, v_2) \in \Gamma^{(1)}$  by operators
  - $D_e : \mathbb{C}^{n(v_1)} \rightarrow \mathbb{C}^{n(v_2)}$
  - and  $D_e^* : \mathbb{C}^{n(v_2)} \rightarrow \mathbb{C}^{n(v_1)}$  its conjugate traspose (pullback?)

such that

$$n(\Gamma^{(0)}) = \Lambda \quad (22)$$

**Question 3.** Would then  $D_e$  be the pullback?

**Question 4.** These graphs are important in the next chapter I should look into it more, I don't understand much here, specific how to construct them with the abstraction of a spectral triple...

The operator  $D_e$  between  $\mathbf{n}_i$  and  $\mathbf{n}_j$  add up to  $D_{ij}$

$$D_{ij} = \sum_{\substack{e=(v_1, v_2) \\ n(v_1)=\mathbf{n}_i \\ n(v_2)=\mathbf{n}_j}} D_e$$

**Theorem 2.** *There is a one to one correspondence between finite spectral triples modulo unitary equivalence and  $\Lambda$ -decorated graphs, given by associating a finite spectral triple  $(A, H, D)$  to a  $\Lambda$  decorated graph  $(\Gamma, \Lambda)$  in the following way:*

$$A = \bigoplus_{n \in \Lambda} M_n(\mathbb{C}); \quad H = \bigoplus_{v \in \Gamma^{(0)}} \mathbb{C}^{n(v)}; \quad D = \sum_{e \in \Gamma^{(1)}} D_e + D_e^* \quad (23)$$

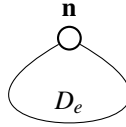


Figure 2: A  $\Lambda$ -decorated Graph of  $(M_n(\mathbb{C}), \mathbb{C}^n, D = D_e + D_e^*)$

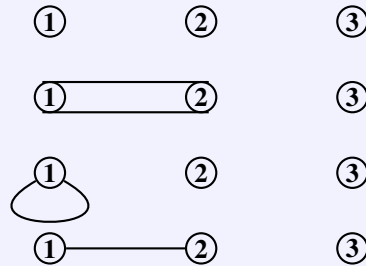
#### Exercise 11

Draw a  $\Lambda$  decorated graph corresponding to the spectral triple  $(A = \mathbb{C}^3, H = \mathbb{C}^3, D = \begin{pmatrix} 0 & \lambda & 0 \\ \bar{\lambda} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix})$



#### Exercise 12

Use  $\Lambda$ -decorated graphs to classify all finite spectral triples (modulo unitary equivalence) on the matrix algebra  $A = \mathbb{C} \oplus M_2(\mathbb{C})$



### 1.4.1 Graph Construction of Finite Spectral Triples

**Algebra:** We know if  $A$  acts on a finite dimensional Hilbert space then this  $C^*$  algebra is isomorphic to a matrix algebra so  $A \simeq \bigoplus_{i=1}^N M_{n_i}(\mathbb{C})$ . Where  $i \in \hat{A}$  represents an equivalence class and runs from 1 to  $N$ , thus  $\hat{A} \simeq \{1, \dots, N\}$ . We label equivalence classes by  $\mathbf{n}_i$ , then  $\hat{A} \simeq \{\mathbf{n}_1, \dots, \mathbf{n}_N\}$ .

**Hilbert Space:** Since every Hilbert space that acts faithfully on a  $C^*$  algebra is completely reducible, it is isomorphic to the composition of irreducible representations.  $H \simeq \bigoplus_{i=1}^N \mathbb{C}^{n_i} \otimes V_i$ . Where all  $V_i$ 's are Vector spaces, their dimension is the multiplicity of the representation landed by  $\mathbf{n}_i$  to  $V_i$  itself by the multiplicity space.

**Finite Dirac Operator:**  $D_{ij}$  is connecting nodes  $\mathbf{n}_i$  and  $\mathbf{n}_j$ , with a symmetric map  $D_{ij} : \mathbb{C}^{n_i} \otimes V_i \rightarrow \mathbb{C}^{n_j} \otimes V_j$

To draw a graph, draw nodes in position  $\mathbf{n}_i \in \hat{A}$ . Multiple nodes at the same position represent multiplicities in  $H$ . Draw lines between nodes to represent  $D_{ij}$ .



Figure 3: Example