University of Vienna
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Notes on Noncommutative Geometry and Particle Phyiscs

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1 Heat Kernel Expansion

1.1 The Heat Kernel

The heat kernel $K(t;x,y;D)$ is the fundamental solution of the heat equation. It depends on the operator D of Laplacian type.

$$(\partial_t + D_x)K(t;x,y;D) = 0 \tag{1}$$

For a flat manifold $M = \mathbb{R}^n$ and $D = D_0 := -\Delta_\mu \Delta^\mu + m^2$ the Laplacian with a mass term and the initial condition

$$K(0;x,y;D) = \delta(x,y) \tag{2}$$

we have the standard fundamental solution

$$K(t;x,y;D_0) = (4\pi t)^{-n/2} \exp\left(-\frac{(x-y)^2}{4t} - tm^2\right) \tag{3}$$

Let us consider now a more general operator D with a potential term or a guage field, the heat kernel reads then

$$K(t;x,y;D) = \langle x|e^{-tD}|y\rangle. \tag{4}$$

We can expand it it in terms of D_0 and we still have the singularity from the equation 3 as $t \to 0$ thus the expansion gives

$$K(t;x,y;D) = K(t;x,y;D_0)\left(1 + tb_2(x,y) + t^2 b_4(x,y) + \dots\right) \tag{5}$$

where $b_k(x,y)$ are regular in $y \to x$. They are called the heat kernel coefficients.

1.2 Example

Now let us consider a propagator $D^{-1}(x,y)$ defined through the heat kernel in an integral representation

$$D^{-1}(x,y) = \int_0^\infty dt K(t;x,y;D). \tag{6}$$

We can integrate the expression formally if we assume the heat kernel vanishes for $t \to \infty$ we get

$$D^{-1}(x,y) \simeq 2(4\pi)^{-n/2} \sum_{j=0} \left(\frac{|x-y|}{2m}\right)^{-\frac{n}{2}+j+1} K_{-\frac{n}{2}+j+1}(|x-y|m) b_{2j}(x,y). \tag{7}$$

where $b_0 = 1$ and $K_\nu(z)$ is the Bessel function

$$K_\nu(z) = \frac{1}{\pi}\int_0^\pi cos(\nu\tau - z\sin(\tau))d\tau \tag{8}$$

it solves the differential equation

$$z^2\frac{d^2K}{dz^2} + z\frac{dK}{dz} + (z^2 - \nu^2) = 0. \tag{9}$$

By looking at integral approximation of the propagator we conclude that the singularities of D^{-1} coincide with the singularities of the heat kernel coefficients. We consider now a generating functional in terns of $\det(D)$ which is called the one-loop effective action (quantum fields theory)

$$W = \frac{1}{2}\ln(\det D) \tag{10}$$

we can relate W with the heat kernel. For each eigenvalue $\lambda > 0$ of D we can write the identity.

$$\ln\lambda = -\int_0^\infty \frac{e^{-t\lambda}}{t}dt \tag{11}$$

This expression is correct up to an infinite constant which does not depent on λ, because of this we can ignore it. Further more we use $\ln(\det D) = \mathrm{Tr}(\ln D)$ and therefor we can write for W

$$W = -\frac{1}{2}\int_0^\infty dt\frac{K(t,D)}{t} \tag{12}$$

where

$$K(t,D) = \mathrm{Tr}(e^{-tD}) = \int d^n x\sqrt{g}K(t;x,x;D). \tag{13}$$

The problem is now that the integral of W is divergent at both limits. Yet the divergences at $t \to \infty$ are caused by $\lambda \leq 0$ of D (infrared divergences) and are just ignored. The divergences at $t \to 0$ are cutoff at $t = \Lambda^{-2}$, thus we write

$$W_\Lambda = -\frac{1}{2}\int_{\Lambda^{-2}}^\infty dt\frac{K(t,D)}{t}. \tag{14}$$

We can calculate W_Λ at up to an order of λ^0

$$W_\Lambda = -(4\pi)^{-n/2}\int d^n x\sqrt{g}\Bigg(\sum_{2(j+l)<n}\Lambda^{n-2j-2l}b_{2j}(x,x)\frac{(-m^2)^l l!}{n-2j-2l}+ \tag{15}$$

$$+\sum_{2(j+l)=n}\ln(\Lambda)(-m^2)^l l! b_{2j}(x,x)\mathscr{O}(\lambda^0)\Bigg) \tag{16}$$

There is an divergence at $b_2(x,x)$ with $k \leq n$. Now we compute the limit $\Lambda \to \infty$

$$-\frac{1}{2}(4\pi)^{n/2}m^n\int d^n x\sqrt{g}\sum_{2j>n}\frac{b_{2j}(x,x)}{m^{2j}}\Gamma(2j-n) \tag{17}$$

here Γ is the gamma function.

1.3 Differential Geometry and Operators of Laplace Type

Let M be a n dimensional compact Riemannian manifold with $\partial M = 0$. Then consider a vector bundle V over M (i.e. there is a vector space to each point on M), so we can define smooth functions. We want to look at arbitrary differential operators D of Laplace type on V, they have the general from

$$D = -(g^{\mu\nu}\partial_\mu\partial_\nu + a^\sigma\partial_\sigma + b) \tag{18}$$

where a^σ, b are matrix valued functions on M and $g^{\mu\nu}$ is the inverse metric on M. There is a unique connection on V and a unique endomorphism (matrix valued function) E on V, then we can rewrite D in terms of E and covariant derivatives

$$D = -(g^{\mu\nu}\nabla_\mu\nabla_\nu + E) \tag{19}$$

Where the covariant derivative consists of $\nabla = \nabla^{[R]} + \omega$ the standard Riemannian covariant derivative $\nabla^{[R]}$ and a "gauge" bundle ω (fluctuations). WE can write E and ω in terms of geometrical identities

$$\omega_\delta = \frac{1}{2}g_{\nu\delta}(a^\nu + g^{\mu\sigma}\Gamma^\nu_{\mu\sigma}I_V) \tag{20}$$

$$E = b - g^{\nu\mu}(\partial_\mu\omega_\nu + \omega_\nu\omega_\mu - \omega_\sigma\Gamma^\sigma_{\nu\mu}) \tag{21}$$

where I_V is the identity in V and the Christoffel symbol

$$\Gamma^\sigma_{\mu\nu} = g^{\sigma\rho}\frac{1}{2}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \tag{22}$$

Furthermore we remind ourselves of the Riemmanian curvature tensor, Ricci Tensor and the Scalar curavture.

$$R^\mu_{\nu\rho\sigma} = \partial_\sigma\Gamma^\mu_{\nu\rho} - \partial_\rho\Gamma^\mu_{\nu\sigma}\Gamma^\lambda_{\nu\rho}\Gamma^\mu_{\lambda\sigma}\Gamma^\lambda_{\nu\sigma}\Gamma^\mu_{\lambda\rho} \tag{23}$$

$$R_{\mu\nu} := R^\sigma_{\mu\nu\sigma} \tag{24}$$

$$R := R^\mu_\mu \tag{25}$$

The we let $\{e_1,\dots,e_n\}$ be the local orthonormal frame of TM(tangent bundle M), which will be noted with flat indices $i,j,k,l \in \{1,\dots,n\}$, we use e^k_μ, e^ν_j to transform between flat indices and curved indices μ,ν,ρ.

$$e^\mu_j e^\nu_k g_{\mu\nu} = \delta_{jk} \tag{26}$$

$$e^\mu_j e^\nu_k \delta^{jk} = g^{\mu\nu} \tag{27}$$

$$e^j_\mu e^\mu_k = \delta^j_k \tag{28}$$

The Riemannian part of the covariant derivative contains the standard Levi-Civita connection, so that for a v_ν we write

$$\nabla^{[R]}_\mu v_\nu = \partial_\mu v_\nu - \Gamma^\rho_{\mu\nu}v_\rho. \tag{29}$$

The extended covariant derivative reads then

$$\nabla_\mu v^j = \partial_\mu v^j + \sigma^{jk}_\mu v_k. \tag{30}$$

the condition $\nabla_\mu e^k_\nu = 0$ gives us the general connection

$$\sigma^{kl}_\mu = e^\nu_l\Gamma^\rho_{\mu\nu}e^k_\rho - e^\nu_l\partial_\mu e^k_\nu \tag{31}$$

The we may define the field strength $\Omega_{\mu\nu}$ of the connection ω

$$\Omega_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + \omega_\mu\omega_\nu - \omega_\nu\omega_\mu. \tag{32}$$

If we apply the covariant derivative on Ω we get

$$\nabla_\rho\Omega_{\mu\nu} = \partial_\rho\Omega_{\mu\nu} - \Gamma^\sigma_{\rho\mu}\Omega_{\sigma\mu} + [\omega_\rho, \Omega_{\mu\nu}] \tag{33}$$

1.4 Spectral Functions

Manifolds without M boundary condition for the operator e^{-tD} for $t > 0$ is a trace class operator on $L^2(V)$, this means that for any smooth function f on M we can define

$$K(t,f,D) = \mathrm{Tr}_{L^2}(fe^{-tD}) \tag{34}$$

and we can rewrite

$$K(t,f,D) = \int_M d^n x\sqrt{g}\mathrm{Tr}_V(K(t;x,x;D)f(x)). \tag{35}$$

in terms of the Heat kernel $K(t;x,y;D)$ in the regular limit $y \to y$. We can write the Heat Kernel in terms of the spectrum of D. Say $\{\phi_\lambda\}$ is a ONB of eigenfunctions of D corresponding to the eigenvalue λ

$$K(t;x,y;D) = \sum_\lambda \phi_\lambda^\dagger(x)\phi_\lambda(y)e^{-t\lambda}. \tag{36}$$

We have an asymtotic expansion at $t \to 0$ for the trace

$$Tr_{L^2}(fe^{-tD}) \simeq \sum_{k\geq 0} t^{(k-n)/2} a_k(f,D). \tag{37}$$

where

$$a_k(f,D) = (4\pi)^{-n/2}\int_M d^4x\sqrt{g}b_k(x,x)f(x) \tag{38}$$

1.5 General Formulae

We consider a compact Riemmanian Manifold M without boundary condition, a vector bundle V over M to define functions which carry discrete (spin or gauge) indices. An Laplace style operator D over V and smooth function f on M. There is an asymtotic expansion where the heat kernel coefficients

1. with odd index $k = 2j+1$ vanish $a_{2j+1}(f,D) = 0$
2. with even index are locally computable in terms of geometric invariants

$$a_k(f,D) = \mathrm{Tr}_V\left(\int_M d^n x\sqrt{g}(f(x)a_k(x;D)\right) = \tag{39}$$

$$= \sum_I \mathrm{Tr}_V\left(\int_M d^n x\sqrt{g}(fu^I\mathscr{A}_k^I(D))\right) \tag{40}$$

here $\mathscr{K}_k^I$ are all possible independent invariants of dimension k, constructed from $E,\Omega,R_{\mu\nu\rho\sigma}$ and their derivatives, u^I are some constants.

If E has dimension two, then the derivative has dimension one. So if $k = 2$ there are only two independent invariants, E and R. This corresponds to the statement $a_{2j+1} = 0$.

If we consider $M = M_1 \times M_2$ with coordinates x_1 and x_2 and a decomposed Laplace style operator $D = D_1 \otimes 1 + 1 \otimes D_2$ we can separate everything, i.e.

$$e^{-tD} = e^{-tD_1} \otimes e^{-tD_2} \tag{41}$$

$$f(x_1,x_2) = f_1(x_1)f_2(x_2) \tag{42}$$

$$a_k(x;D) = \sum_{p+q=k} a_p(x_1;D_1)a_q(x_2;D_2) \tag{43}$$

Say the spectrum of D_1 is known, $l^2, l \in \mathbb{Z}$. We obtain the heat kernel asymmetries with the Poisson Summation formula

$$K(t,D_1) = \sum_{l\in\mathbb{Z}} e^{-tl^2} = \sqrt{\frac{\pi}{t}} \sum_{l\in\mathbb{Z}} e^{-\frac{\pi^2 l^2}{t}} = \tag{44}$$

$$\simeq \sqrt{\frac{\pi}{t}} + \mathscr{O}(e^{-1/t}). \tag{45}$$

Note that the exponentially small terms have no effect on the heat kernel coefficients and that the only nonzero coefficient is $a_0(1,D_1) = \sqrt{\pi}$. Therefore we can write

$$a_k(f(x^2),D) = \sqrt{\pi} \int_{M_2} d^{n-1}x\sqrt{g} \sum_I \mathrm{Tr}_V \left(f(x^2) u^I_{(n-1)} \mathscr{A}^I_n(D_2) \right). \tag{46}$$

On the other had all geometric invariants associated with D are in the D_2 part. Thus all invariants are independent of x_1, so we can choose for M_1. Say $M_1 = S^1$ with $x \in (0,2\pi)$ and $D_1 = -\partial^2_{x_1}$ we may rewrite the heat kernel coefficients in

$$a_k(f(x_2),D) = \int_{S^1\times M_2} d^n x\sqrt{g} \sum_I \mathrm{Tr}_V (f(x_2) u^I_{(n)} \mathscr{A}^I_k(D_2)) = \tag{47}$$

$$= 2\pi \int_{M_2} d^n x\sqrt{g} \sum_I \mathrm{Tr}_V (f(x_2) u^I_{(n)} \mathscr{A}^I_k(D_2)). \tag{48}$$

Computing the two equations above we see that

$$u^I_{(n)} = \sqrt{4\pi} u^I_{(n+1)} \tag{49}$$

1.6 Heat Kernel Coefficients

To calculate the heat kernel coefficients we need the following variational equations

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} a_k(1, e^{-2\varepsilon f} D) = (n-k) a_k(f,D), \tag{50}$$

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} a_k(1, D - \varepsilon F) = a_{k-2}(F,D), \tag{51}$$

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} a_k(e^{-2\varepsilon f} F, e^{-2\varepsilon f} D) = 0. \tag{52}$$

To prove the equation 50 we differentiate

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} \mathrm{Tr}(\exp(-e^{-2\varepsilon f} tD) = \mathrm{Tr}(2ftDe^{-tD}) = -2t\frac{d}{dt}\mathrm{Tr}(fe^{-tD})) \tag{53}$$

then we expand both sides in t and get 50. Equation 51 is derived similarly. For equation 52 we consider the following operator

$$D(\varepsilon,\delta)=e^{-2\varepsilon f}(D-\delta F) \tag{54}$$

for $k=n$ we use equation 50 and we get

$$\frac{d}{d\varepsilon}|_{\varepsilon=0}a_n(1,D(\varepsilon,\delta))=0 \tag{55}$$

then we take the variation in terms of δ, evaluated at $\delta=0$ and swap the differentiation, allowed by theorem of Schwarz

$$0=\frac{d}{d\delta}|_{\delta=0}\frac{d}{d\varepsilon}|_{\varepsilon=0}a_n(1,D(\varepsilon,\delta))=\frac{d}{d\varepsilon}|_{\varepsilon=0}\frac{d}{d\delta}|_{\delta=0}a_n(1,D(\varepsilon,\delta))= \tag{56}$$
$$=a_{n-2}(e^{-2\varepsilon f}F,e^{-2\varepsilon f}D) \tag{57}$$

which proves equation 52. With this we calculate the constants u^I and we can write the first three heat kernel coefficients as

$$a_0(f,D)=(4\pi)^{-n/2}\int_M d^nx\sqrt{g}\mathrm{Tr}_V(a_0 f) \tag{58}$$
$$a_2(f,D)=(4\pi)^{-n/2}\frac{1}{6}\int_M d^nx\sqrt{g}\mathrm{Tr}_V)(f\alpha_1 E+\alpha_2 R) \tag{59}$$
$$a_4(f,D)=(4\pi)^{-n/2}\frac{1}{360}\int_M d^nx\sqrt{g}\mathrm{Tr}_V(f(\alpha_3 E_{,kk}+\alpha_4 RE+\alpha_5 E^2\alpha_6 R_{,kk}+ \tag{60}$$
$$+\alpha_7 R^2+\alpha_8 R_{ij}R_{ij}+\alpha_9 R_{ijkl}R_{ijkl}+\alpha_{10}\Omega_{ij}\Omega ij)). \tag{61}$$

The constants α_I do not depend on the dimension n of the Manifold and we can compute them with our variational identities.

The first coefficient α_0 can be seen from the heat kernel expanion of the Laplacian on S^1 (above), $\alpha_0=1$. For α_1 we use 51, for $k=2$

$$\frac{1}{6}\int_M d^nx\sqrt{g}\mathrm{Tr}_V(\alpha_1 F)=\int_M d^nx\sqrt{g}\mathrm{Tr}_V(F), \tag{62}$$

thus we conclude that $\alpha_1=6$. Now we take $k=4$

$$\frac{1}{360}\int_M d^nx\sqrt{g}\mathrm{Tr}_V(\alpha_4 FR+2\alpha_5 FE)=\frac{1}{6}\int_M d^nx\sqrt{g}\mathrm{Tr}_V(\alpha_1 FE+\alpha_2 FR), \tag{63}$$

thus $\alpha_4=60\alpha_2$ and $\alpha_5=180$.

Furthermore we apply 52 to $n=4$

$$\frac{d}{d\varepsilon}|_{\varepsilon=0}a_2(e^{-2\varepsilon f}F,e^{-2\varepsilon f}D)=0. \tag{64}$$

By collecting the terms with $\mathrm{Tr}_V(\int_M d^nx\sqrt{g}(Ff_{,jj}))$ we obtain $\alpha_1=6\alpha_2$, that is $\alpha_2=1$, so $\alpha_4=60$.

Now we let $M=M_1\times M_2$ and split $D=-\Delta_1-\Delta_2$, where $\Delta_{1/2}$ are Laplacians for M_1,M_2, then we can decompose the heat kernel coefficients for $k=4$

$$a_4(1,-\Delta_1-\Delta_2)=a_4(1,-\Delta_1)a_0(1,-\Delta_2)+a_2(1,-\Delta_1)a_2(1,-\Delta_2) \tag{65}$$
$$+a_0(1,-\Delta_1)a_4(1,-\Delta_2) \tag{66}$$

with $E = 0$ and $\Omega = 0$ and by calculating the terms with $R_1 R_2$ (scalar curvature of $M_{1/2}$) we obtain $\frac{2}{360}\alpha_7 = (\frac{\alpha_2}{6})^2$, thus $\alpha_7 = 5$.

For $n = 6$ we get

$$0 = \mathrm{Tr}_V(\int_M d^n x \sqrt{g}(F(-2\alpha_3 - 10\alpha_4 + 4\alpha_5) f_{,kk} E + \quad (67)$$

$$+ (2\alpha_3 + 10\alpha_6) f_{,iijj} + \quad (68)$$

$$+ (2\alpha_4 - 2\alpha_6 - 20\alpha_7 - 2\alpha_8) f_{,ii} R \quad (69)$$

$$+ (-8\alpha_8 - 8\alpha_6) f_{,ij} R_{ij})) \quad (70)$$

we obtain $\alpha_3 = 60$, $\alpha_6 = 12$, $\alpha_8 = -2$ and $\alpha_9 = 2$

For α_{10} we use the Gauss-Bonnet theorem to get $\alpha_{10} = 30$, which is left out because it is a lengthy computation.

Summarizing we get for the heat kernel coefficients

$$\alpha_0(f, D) = (4\pi)^{-n/2} \int_M d^n x \sqrt{g} \mathrm{Tr}_V(f) \quad (71)$$

$$\alpha_2(f, D) = (4\pi)^{-n/2} \frac{1}{6} \int_M d^n x \sqrt{g} \mathrm{Tr}_V(f(6E + R)) \quad (72)$$

$$\alpha_4(f, D) = (4\pi)^{-n/2} \frac{1}{360} \int_M d^n x \sqrt{g} \mathrm{Tr}_V(f(60E_{,kk} + 60RE + 180E^2 + \quad (73)$$

$$+ 12R_{,kk} + 5R^2 - 2R_{ij}R_{ij} 2R_{ijkl}R_{ijkl} + 30\Omega_{ij}\Omega_{ij})) \quad (74)$$

$$(75)$$