Notes on
Noncommutative Geometry and Particle Physics

Popovic Milutin

Week 2: 12.02 - 19.02

Contents

1 Noncommutative Geometric Spaces **1**

1.1 Noncommutative Matrix Algebras 1

1.1.1 Balanced Tensor Product and Hilbert Bimodules 1

1.1.2 Kasparov Product and Morita Equivalence 2

1 Noncommutative Geometric Spaces

1.1 Noncommutative Matrix Algebras

1.1.1 Balanced Tensor Product and Hilbert Bimodules

Definition 1. Let A be an algebra, E be a *right* A-module and F be a *left* A-module. The *balanced tensor product* of E and F forms a A-bimodule.

$$E \otimes_A F := E \otimes F \Big/ \left\{ \sum_i e_i a_i \otimes f_i - e_i \otimes a_i f_i : \quad a_i \in A,\ e_i \in E,\ f_i \in F \right\}$$

In other words the balanced tensor product forms only elements of

- E that preserver the *left* representation of A and
- F that preserver the *right* representation of A.

Which is the same saying:

$$E \otimes_A F = \{ea \otimes_A f = e \otimes_A af : \quad a \in A,\ e \in E,\ f \in F\}$$

Definition 2. Let A, B be *matrix algebras*. The *Hilbert bimodule* for (A,B) is given by

- E, an A-B-bimodue E and by
- an B-valued *inner product* $\langle \cdot, \cdot \rangle_E : E \times E \to B$

$\langle\cdot,\cdot\rangle_E$ needs to satisfy the following for $e, e_1, e_2 \in E,\ a \in A$ and $b \in B$.

$$\begin{aligned} \langle e_1, a\cdot e_2\rangle_E &= \langle a^*\cdot e_1, e_2\rangle_E && \text{sesquilinear in } A \\ \langle e_1, e_2\cdot b\rangle_E &= \langle e_1, e_2\rangle_E b && \text{scalar in } B \\ \langle e_1, e_2\rangle_E &= \langle e_2, e_1\rangle_E^* && \text{hermitian} \\ \langle e, e\rangle_E &\geq 0 && \text{equality holds iff } e = 0 \end{aligned}$$

We denote $KK_f(A,B)$ the set of all *Hilbert bimodules* of (A,B).

Exercise 1. *Check that a representation $\pi:\ A\ \to L(H)$ of a matrix algebra A turns H into a Hilbert bimodule for $(A,\mathbb{C})$.*

Solution 1.

Exercise 2. *Show that the $A-A$ bimodule given by A is in $KK_f(A,A)$ by taking the following inner product $\langle\cdot,\cdot\rangle_A : A\times A\to A$:*

$$\langle a, a\rangle_A = a^*a' \quad a, a' \in A$$

Solution 2.

Example 1. Consider a $*$ homomorphism between two matrix algebras $\phi : A \to B$. From it we can construct a Hilbert bimodule $E_\phi \in KK_f(A,B)$ in the following way. We let E_ϕ be B in the vector space sense and an inner product from the above Exercise 2, with A acting on the left with ϕ.

$$a\cdot b = \phi(a)b \quad a \in A, b \in E_\phi$$

1.1.2 Kasparov Product and Morita Equivalence

Definition 3. Let $E \in KK_f(A,B)$ and $F \in KK_F(B,D)$ the *Kasparov product* is defined as with the balanced tensor product

$$F\circ E := E\otimes_B F$$

Such that $F\circ E \in KK_f(A,D)$ with a D-valued inner product.

$$< e_1\otimes f_1, e_2\otimes f_2 >_{E\otimes_B F} = < f_1, < e_1, e_2 >_E f_2 >_F$$

Question 1. What is the meaning of 'associative up to isomorphism? Isomorphism of $F\circ E$ or of A, B or D?

Exercise 3. *Show that the association $\phi \rightsquigarrow E_\phi$ (from the previous Example) is natrual in the sense*

- $E_{id_A} \simeq A \in KK_f(A,A)$
- *for $*$-algebra homomorphism $\phi : A\to B$ and $\psi : B\to C$ we have an isomorphism*

$$E_\psi\circ E_\phi \ \equiv\ E_\phi\otimes_B E_\psi \ \simeq\ E_{\psi\circ\phi} \in KK_f(A,C)$$

Solution 3.

Exercise 4. *In the definition of Morita equivalence:*

- *Check that $E \otimes_B F$ is a $A - D$ bimodule*
- *Check that $< \cdot, \cdot >_{E \oplus_B F}$ defines a D valued inner product*
- *Check that $< a^*(e_1 \otimes f_1), e_2 \otimes f_2 >_{E \otimes_B F} = < e_1 \otimes f_1, a(e_2 \otimes f_2) >_{E \otimes_B F}$.*

Solution 4.

Definition 4. Let A, B be *matrix algebras*. They are called *Morita eqivalent* if there exists an $E \in KK_f(A,B)$ and an $F \in KK_f(B,A)$ such that:

$$E \otimes_B F \simeq A \quad \text{and} \quad F \otimes_A E \simeq B$$

Where $\simeq$ denotes the isomorphism between Hilbert bimodules, note that A or B is a bimodule by itself.

Question 2. Why are E and F each others inverse in the Kasparov Product?

Example 2.

- Hilber bimodule of (A,A) is A
- Let $E \in KK_f(A,B)$, we take $E \circ A = A \oplus_A E \simeq E$
- we conclude, that ${}_AA_A$ is the identity in the Kasparov product (up to isomorphism)

Example 3. Let $E = \mathbb{C}^n$, which is a $(M_n(\mathbb{C}), \mathbb{C})$ Hilbert bimodule with the standard $\mathbb{C}$ inner product.
On the other hand let $F = \mathbb{C}^n$, which is a $(\mathbb{C}, M_n(\mathbb{C}))$ Hilbert bimodule by right matrix multiplication with $M_n(\mathbb{C})$ valued inner product:

$$< v_1, v_2 >= \bar{v_1} v_2^t \quad \in M_n(\mathbb{C})$$

Now we take the Kasparov product of E and F:

- $F \circ E \ = \ E \otimes_{\mathbb{C}} F \qquad \simeq \ M_n(\mathbb{C})$
- $E \circ F \ = \ F \otimes_{M_n(\mathbb{C})} E \ \simeq \ \mathbb{C}$

$M_n(\mathbb{C})$ and $\mathbb{C}$ are Morita equivalent

Theorem 1. *Two matrix algebras are Morita Equivalent iff their their Structure spaces are isomorphic as discreet spaces (have the same cardinality / same number of elements)*

Proof. Let A, B be *Morita equivalent*. So there exists ${}_AE_B$ and ${}_BF_A$ with

$$E \otimes_B F \simeq A \quad \text{and} \quad F \otimes_A E \simeq B$$

Consider $[(\pi_B, H)] \in \hat{B}$ than we construct a representation of A,

$$\pi_A \to L(E \otimes_B H) \quad \text{with} \quad \pi_A(a)(e \otimes v) = ae \otimes w$$

Question 3. Is $E \simeq H$ and $F \simeq W$?

vice versa, consider $[(\pi_A, W)] \in \hat{A}$ we can construct π_B

$$\pi_B : B \to L(F \otimes_A W) \quad \text{and} \quad \pi_B(b)(f \otimes w) = bf \otimes w$$

These maps are each others inverses, thus $\hat{A} \simeq \hat{B}$ $\square$

Exercise 5. *Fill in the gaps in the above proof:*

- *show that the representation of π_A defined is irreducible iff π_B is.*
- *Show that the association of the class $[\pi_A]$ to $[\pi_B]$ is independent of the choice of representatives π_A and π_B*

Solution 5.

Lemma 1. *The matrix algebra $M_n(\mathbb{C})$ has a unique inrreducible representation (up to isomorphism) given by the defining representation on $\mathbb{C}^n$.*

Proof. We know $\mathbb{C}^n$ is a irreducible representation of $A = M_n(\mathbb{C})$. Let H be irreducible and of dimension k, then we define a map

$$\begin{aligned} \phi : A \oplus \ldots \oplus A &\to H^* \\ (a_1, \ldots, a_k) &\mapsto e^1 \circ a_1^t + \ldots + e^k \circ a_k^t \end{aligned}$$

With $\{e^1, \ldots, e^k\}$ being the basis of the dual space H^* and $(\circ)$ being the pre-composition of elements in H^* and A acting on H. This forms a morphism of $M_n(\mathbb{C})$ modules, provided a matrix $a \in A$ acts on H^* with $v \mapsto v \circ a^t$ $(v \in H^*)$. Furthermore this morphism is surjective, thus making the pullback $\phi^* : H \mapsto (A^k)^*$ injective. Now identify $(A^k)^*$ with A^k as a A-module and note that $A = M_n(\mathbb{C}) \simeq \oplus^n \mathbb{C}^n$ as a n A module. It follows that H is a submodule of $A^k \simeq \oplus^{nk}\mathbb{C}$. By irreducibility $H \simeq \mathbb{C}$. $\square$

Example 4. Consider two matrix algebras A, and B.

$$A = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}) \quad B = \bigoplus_{j=1}^{M} M_{m_j}(\mathbb{C})$$

Let $\hat{A} \simeq \hat{B}$ that implies $N = M$ and define E with A acting by block-diagonal matrices on the first tensor and B acting in the same way on the second tensor. Define F vice versa.

$$E := \bigoplus_{i=1}^{N} \mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i} \quad F := \bigoplus_{i=1}^{N} \mathbb{C}^{m_i} \otimes \mathbb{C}^{n_i}$$

Then we calculate the Kasparov product.

$$\begin{aligned} E \otimes_B F &\simeq \bigoplus_{i=1}^{N} (\mathbb{C}^{n_i} \otimes \mathbb{C}^{m_i}) \otimes_{M_{m_i}(\mathbb{C})} (\mathbb{C}^{m_i} \otimes \mathbb{C}^{n_i}) \\ &\simeq \bigoplus_{i=1}^{N} \mathbb{C}^{n_i} \otimes \left(\mathbb{C}^{m_i} \otimes_{M_{m_i}(\mathbb{C})} \mathbb{C}^{m_i}\right) \oplus \mathbb{C}^{n_i} \\ &\simeq \bigoplus_{i=1}^{N} \mathbb{C}^{m_i} \otimes \mathbb{C}^{n_i} \simeq A \end{aligned}$$

and from $F \otimes_A E \simeq B$.

We conclude that.

- There is a duality between finite spaces and Morita equivalence classes of matrix algebras.
- By replacing $*$-homomorphism $A \to B$ with Hilbert bimodules (A,B) we introduce a richer structure of morphism between matrix algebras.