

# Notes on Noncommutative Geometry and Particle Physics

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## 1 Classification of Finite Real Spectral Triples

Here we classify finite real spectral triples modulo unitary equivalence with *Krajewski Diagrams*. We extend  $\Lambda$ -decorated graphs to the case of real spectral triples (grading and real structure).

**The Algebra:** Like before:

$$A \simeq \bigoplus_{i=1}^N M_{n_i}(\mathbb{C}) \quad \text{with } \hat{A} = \{\mathbf{n}_1, \dots, \mathbf{n}_N\} \quad (1)$$

Where  $\mathbf{n}_i$  are irreducible representation of  $A$  on  $\mathbb{C}^{n_i}$

**The Hilbertspace:** Faithful irreducible representation on  $A$  are the direct sum of  $\mathbb{C}^{n_i}$ 's, which act on  $A$  by left block-diagonal matrix multiplication.

$$\bigoplus_{i=1}^N \mathbb{C}^{n_i} \quad (2)$$

Furthermore we need a representation of  $A^\circ$  on  $H$  that commutes with  $A$ . That is

$$A^\circ \simeq \bigoplus_{i=1}^N M_{n_i}(\mathbb{C})^\circ \quad (3)$$

$$\text{with } \hat{A}^\circ = \{\mathbf{n}_1^\circ, \dots, \mathbf{n}_N^\circ\} \quad (4)$$

$$\text{and } \bigoplus_{i=1}^N \mathbb{C}^{n_i^\circ} \quad (5)$$

And we need the multiplicity space  $V_{ij}$  of  $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j^\circ}$ . Thus making the Hilbertspace:

$$H = \bigoplus_{i,j=1}^N \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j^\circ} \otimes V_{ij} \quad (6)$$

- $\mathbf{n}_i, \mathbf{n}_j^\circ$  form a grid
- if there is a node at  $(\mathbf{n}_i, \mathbf{n}_j^\circ)$  then  $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j^\circ}$  is nonzero in  $H$ .
- multiplicity implies multiple nodes

**Example 1.**  $A = \mathbb{C} \oplus M_2(\mathbb{C})$ , two options of the Hilbertspace.



The first diagram corresponds to  $H_1 = \mathbb{C} \oplus M_2(\mathbb{C})$ , to the second  $H_2 = \mathbb{C} \oplus \mathbb{C}^2$ .

#### Exercise 1

**Let  $J$  be an anti-unitary operator on a finite-dimensional Hilbert space. Show that  $J^2$  is a unitary operator**

Straight forward, say  $J : H \rightarrow H$ , then let  $\xi_1, \xi_2 \in H$ :

$$\langle J^2 \xi_1, J^2 \xi_2 \rangle = \langle J(J \xi_1), J(J \xi_2) \rangle = \quad (7)$$

$$= \langle J \xi_2, J \xi_1 \rangle = \langle \xi_1, \xi_2 \rangle \quad (8)$$

**The real Structure:**  $J : H \rightarrow H$ .

**Lemma 1.** *Let  $J$  be an anti-unitary operator on a finite-dimensional Hilbertspace  $H$  with  $J^2 = \pm 1$*

1. *If  $J^2 = 1 \Rightarrow \exists$  an ONB  $\{e_k\}$  of  $H$  with  $Je_k = e_k$ .*
2. *If  $J^2 = -1 \Rightarrow \exists$  an ONB  $\{e_k, f_k\}$  of  $H$  with  $Je_k = f_k$  and consequently  $Jf_k = -e_k$ .*

*Proof.* **1.**  $J^2 = 1$

$v \in H$  and set:

$$e_1 := \begin{cases} c(v + Jv) & \text{if } Jv \neq -v \\ iv & \text{if } Jv = -v \end{cases} \quad (9)$$

Where  $c$  is a normalization constant, then take  $Je_1$

$$J(v + Jv) = Jv + J^2v = v + Jv \quad \text{and} \quad (10)$$

$$J(iv) = -iJv = iv \quad (11)$$

$$\Rightarrow Je_1 = e_1 \quad (12)$$

Take  $v' \perp e_1$  making:

$$\langle e_1, Jv' \rangle = \langle J^2v', Je_1 \rangle = \langle v', Je_1 \rangle = \langle v', e_1 \rangle = 0 \quad (13)$$

Construct  $e_2 \perp e_1$  with  $v'$ :

$$e_2 := \begin{cases} c(v' + Jv') & \text{if } Jv' \neq -v' \\ iv' & \text{if } Jv' = -v' \end{cases} \quad (14)$$

Do this  $k$  times and get  $\{e_k\}$  ONB of  $H$  for  $J^2 = 1$ .

## 2. $J^2 = -1$

$v \in H$  and set  $e_1 = cv$ ,  $c$  normalization constant. Then we set  $f_1 = Je_1$  with  $f_1 \perp e_1$ , this is automatically the case because:

$$\langle f_1, e_1 \rangle = \langle Je_1, e_1 \rangle = -\langle Je_1, J^2e_1 \rangle = \quad (15)$$

$$= -\langle Je_1, e_1 \rangle = -\langle f_1, e_1 \rangle \quad (16)$$

this only holds for 0. Then take some  $v' \perp e_1, f_1$  and set

$e_2 = c'v'$  and  $f_2 = Je_2 \perp e_2, f_1, e_1$ .

$$\langle e_1, f_2 \rangle = \langle e_1, Je_2 \rangle = -\langle J^2e_1, Je_2 \rangle = -\langle e_2, Je_1 \rangle = -\langle e_2, f_1 \rangle = 0 \quad (17)$$

$$\langle f_1, f_2 \rangle = \langle Je_1, Je_2 \rangle = \langle e_2, e_1 \rangle = 0. \quad (18)$$

Do this  $k$  times and get  $\{e_k, f_k\}$  ONB of  $H$  for  $J^2 = -1$

□

Apply Lemma 1 to the real structure  $J$  on a spectral triple.  $J$  implements right action of  $A$  on  $H$  with

$$a^\circ = Ja^*J^{-1} \quad (19)$$

and satisfying  $[a, b^\circ] = 0$ . With the block form of  $A$ , this implies

$$J(a_1^* \oplus \cdots \oplus a_N^*) = (a_1^\circ \oplus \cdots \oplus a_N^\circ)J. \quad (20)$$

With this we can conclude that the Krajewski diagram for a real finite spectral triple is symmetric along the diagonal.  $J$  has then the following bilinear mapping:

$$J: \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j^\circ} \otimes V_{ij} \rightarrow \mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i^\circ} \otimes V_{ji}. \quad (21)$$

**Proposition 1.** *Let  $J$  be a real structure on a finite real spectral triple  $(A, H, D; J)$ .*

1. If  $J^2 = 1$  (KO-dimension 0, 1, 6, 7) Rightarrow  $\exists$  an ONB  $\{e_k^{(ij)}\}$  with  $e_k^{(ij)} \in \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j^\circ} \otimes V_{ij}$  such that

$$J e_k^{(ij)} = e_k^{(ij)} \quad (i, j = 1, \dots, N; k = 1, \dots, \dim(V_{ij})) \quad (22)$$

2. If  $J^2 = -1$  (KO-dimension 2, 3, 4, 5)  $\Rightarrow \exists$  ONB  $\{e_k^{(ij)}, f_k^{(ji)}\}$  with  $e_k^{(ij)} \in \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j^\circ} \otimes V_{ij}$  and  $f_k^{(ji)} \in \mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i^\circ} \otimes V_{ji}$  such that

$$J e_k^{(ij)} = f_k^{(ji)} \quad (i \leq j = 1, \dots, N; k = 1, \dots, \dim(V_{ji})). \quad (23)$$

*Proof.* Similar to Lemma 1. □

For whatever unknown reasons this implies that in the case of KO-dimension 2, 3, 4, 5, diagonals  $H_{ii}$  need to have even multiplicity.

**The finite Dirac Operator:** Is a mapping between  $H_{ij}$  to  $H_{kl}$

$$D_{ij,kl} : \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j^\circ} \otimes V_{ij} \rightarrow \mathbb{C}^{n_k} \otimes \mathbb{C}^{n_l^\circ} \otimes V_{kl} \quad (24)$$

We have  $D_{kl,ij} = D_{ij,kl}^*$ . And in the diagram we have a line between the nodes  $(\mathbf{n}_i, \mathbf{n}_j^\circ)$  and  $(\mathbf{n}_l, \mathbf{n}_k^\circ)$ . But instead of drawing directional lines draw a single undirected line that represents both  $D_{ij,kl}$  and the adjoint  $D_{kl,ij}$ .

**Lemma 2.** The conditions  $JD = \pm DJ$  and  $[[D, a], b^\circ] = 0$  imply that the connections in the diagram run only vertically or horizontally and thereby the diagonal symmetry between the nodes is preserved.

*Proof.* The condition  $JD = \pm DJ$  has the following commutative diagram.

$$\begin{array}{ccc} \mathbb{C}^{n_i^\circ} \otimes \mathbb{C}^{n_j^\circ} \otimes V_{ij} & \xrightarrow{D} & \mathbb{C}^{n_k^\circ} \otimes \mathbb{C}^{n_l^\circ} \otimes V_{kl} \\ J \downarrow & & \downarrow J \\ \mathbb{C}^{n_j^\circ} \otimes \mathbb{C}^{n_i^\circ} \otimes V_{ji} & \xrightarrow{\pm D} & \mathbb{C}^{n_l^\circ} \otimes \mathbb{C}^{n_k^\circ} \otimes V_{lk} \end{array}$$

Relating  $D_{ij,kl}$  to  $D_{ji,lk}$  and maintaining diagonal symmetry. Wit the condition  $[[D, a], b^\circ] = 0$  for the diagonal elements  $a = \lambda_1 \mathbb{I}_{n_1} \oplus \dots \oplus \lambda_N \mathbb{I}_{n_N} \in A$  and  $b = \mu_1 \mathbb{I}_{n_1} \oplus \dots \oplus \mu_N \mathbb{I}_{n_N} \in A$ , with some  $\lambda_i, \mu_i \in \mathbb{C}$ , we can commute:

$$D_{ij,kl}(\lambda_i - \lambda_k)(\bar{\mu}_j - \bar{\mu}_l) = 0 \quad (25)$$

$\forall \lambda_i, \mu_j \in \mathbb{C}$ , thus  $D_{ij,kl} = 0$  for  $i \neq j$  or  $j \neq l$ . □

**The Grading:**  $\gamma : H \rightarrow H$  each node gets labeled by a + or a - sign.

- D only connects nodes with different signs
- If  $(\mathbf{n}_i, \mathbf{n}_j^\circ)$  has a  $\pm$  sign then  $(\mathbf{n}_j, \mathbf{n}_i^\circ)$  has a  $\mp$ ,  $\varepsilon''$  sign according to  $J\gamma = \varepsilon''\gamma J$

**Definition 1.** A Krajewski Diagram of KO-dimension  $k$  is an ordered pair  $(\Gamma, \Lambda)$  where  $\Gamma$  is a finite graph and  $\Lambda$  is a set of positive integers with a labeling:

- of  $v \in \Gamma^{(0)}$  of vertices by elements  $\iota(v) = (n(v), m(v)) \in \Lambda \times \Lambda$ , an edge from  $v$  to  $v'$  implies that either  $n(v) = n(v')$  or  $m(v) = m(v')$  or both
- of  $e = (v_1, v_2) \in \Gamma^{(1)}$  edges with non-zero operators  $D_e$  and their adjoints  $D_e^*$ :

$$\begin{aligned} D_e : \mathbb{C}^{n(v_1)} &\rightarrow \mathbb{C}^{n(v_2)} && \text{if } m(v_1) = m(v_2) \\ D_e : \mathbb{C}^{m(v_1)} &\rightarrow \mathbb{C}^{m(v_2)} && \text{if } n(v_1) = n(v_2) \end{aligned}$$

Together with an involutive graph automorphism  $j : \Gamma \Rightarrow \Gamma$  such that the following conditions hold:

1. every row or column in  $\Gamma \times \Gamma$  has non-empty intersection with  $\iota(\Gamma)$
2. for each vertex  $v$  we have  $n(j(v)) = m(v)$
3. for each edge  $e$  we have  $D_e = \varepsilon' D_{j(e)}$
4. if KO dimension  $k$  is even, then the vertices are labeled by  $\pm 1$  and the edges only connect opposite signs. The signs at  $v$  and  $j(v)$  differ by a factor of  $\varepsilon$
5. if the KO-dimension is 2, 3, 4, 5 then the inverse image of  $\iota$  of the diagonal elements in  $\Lambda \times \Lambda$  contains an even number of vertices of  $\Gamma$

With this definition we can label different vertices by the same element in  $\Lambda \times \Lambda$  (accounting for the multiplicities in  $V_{ij}$ )

**Diagram:** To sum it up we have the following diagram

- Node at  $(\mathbf{n}_i, \mathbf{n}_j^\circ)$  for each vertex with that label
- Operators  $D_e$  add up to  $D_{ij,kl}$  connecting nodes  $(\mathbf{n}_i, \mathbf{n}_j^\circ)$  with  $(\mathbf{n}_k, \mathbf{n}_l^\circ)$

$$D_{ij,kl} = \sum_{\substack{e=(v_1, v_2) \in \Gamma^{(1)} \\ \iota(v_1) = (\mathbf{n}_i, \mathbf{n}_j) \\ \iota(v_2) = (\mathbf{n}_k, \mathbf{n}_l)}} D_e \quad (26)$$

- only vertical or horizontal connections

**Theorem 1.** *There is a one-to-one correspondence between finite real spectral triples  $(A, H, D; J, \gamma)$  of KO-dimension  $k$  modulo unitary equivalence and Krajewski diagrams of KO-dimension  $k$  in the following way:*

$$A = \bigoplus_{n \in \Lambda} M_n(\mathbb{C}) \quad (27)$$

$$H = \bigoplus_{v \in \Gamma^{(0)}} \mathbb{C}^{n(v)} \otimes \mathbb{C}^{m(v)^\circ} \quad (28)$$

$$D = \sum_{e \in \Gamma^{(1)}} D_e + D_e^* \quad (29)$$

The real structure  $J : H \rightarrow H$  is given as as in Proposition 1 with a basis dictated by a graph automorphism  $j : \Gamma \rightarrow \Gamma$ . The grading  $\gamma$  is difened by setting  $\gamma = \pm 1$  on  $\mathbb{C}^{n(v)} \otimes \mathbb{C}^{m(v)^\circ} \subset H$  according to the labeling  $\pm$  of the vertex  $v$ .

**Example 2.**  $A = M_n(\mathbb{C})$  with  $\hat{A} = \mathbf{n}$ . We have the following Krajewski diagram.

$$\begin{array}{c} \mathbf{n} \\ \mathbf{n}^\circ \quad \circ \end{array}$$

- We can label the node either with a  $+$  or a  $-$  sign, the choice being irrelevant
- $H = \mathbb{C}^n \otimes \mathbb{C}^{n^\circ} \simeq M_n(\mathbb{C})$
- $\gamma$  trivial grading ( $+1$ )
- $J$  is a combination of complex conjugation and the flip  $n \otimes n^\circ (\Rightarrow M_n(\mathbb{C})$  as matrix adjoint)
- Because node label is  $\pm$  there is no non-zero Dirac operator
- $\Rightarrow (A = M_n(\mathbb{C}), H = M_n(\mathbb{C}), D = 0; J = (\cdot)^*, \gamma = 1)$

## 2 Real Algebras and Krajewski Diagrams

**Definition 2.** A real Algebra is a Vector space  $A$  over  $\mathbb{R}$  with  $A \times A \rightarrow A$ ,  $(a, b) \mapsto ab$  and  $1a = a1 = a \quad \forall a \in A$

A real  $*$ -algebra is a real algebra with a bilinear map  $*$  :  $A \rightarrow A$  such that  $(ab)^* = b^*a^*$  and  $(a^*)^* = a \quad \forall a, b \in A$

**Example 3.** Real  $*$ -algebra of quaternions  $\mathbb{H}$  subalgebra of  $M_2(\mathbb{C})$ .

$$\mathbb{H} = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\} \quad (30)$$

$\mathbb{H}$  consists of matrices that commute in  $M_2(\mathbb{C})$  with the operator  $I$  defined by:

$$I \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\bar{v}_2 \\ \bar{v}_1 \end{pmatrix} \quad (31)$$

The involution is the hermitian conjugation of  $M_2(\mathbb{C})$ .

### Exercise 2

1. Show that  $\mathbb{H}$  is a real  $*$ -algebra which contains a real subalgebra isomorphic to  $\mathbb{C}$ .
2. Show that  $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq M_2(\mathbb{C})$  as complex  $*$ -algebras.
3. Show that  $M_k(\mathbb{H})$  is areal  $*$ -algebra for any  $k$
4. Show that  $M_k(\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}) \simeq M_{2k}(\mathbb{C})$  as complex  $*$ -algebras.

**Definition 3.** A representation of a finite-dimensional real  $*$  algebra  $A$  is a pair  $(\pi, H)$ ,  $H$ - Hilbertspace,  $\pi : A \rightarrow L(H)$

### Exercise 3

Show that there is a one-to-one correspondence between Hilbertspace representations of real  $*$ -algebras  $A$  and complex representations of its complexification  $A \otimes_{\mathbb{R}} \mathbb{C}$ . Conclude that the unique irreducible Hilbertspace representation of  $M_k(\mathbb{H})$  is  $\mathbb{C}^{2k}$

**Lemma 3.** Real  $*$ -algebra  $A$  represented faithfully on a finite dimensional Hilbertspace  $H$  through a real linear  $*$ -algebra map  $\pi : A \rightarrow L(H)$  then  $A$  is a matrix algebra.

$$A \simeq \bigoplus_{i=1}^N M_{n_i}(\mathbb{F}_i) \quad (32)$$

Where  $\mathbb{F}_i = \mathbb{R}, \mathbb{C}, \mathbb{H}$  depending on  $i$ .

*Proof.*  $\pi$  allows  $A$  to be considered as a real  $*$ -subalgebra of  $M_{\dim(H)}(\mathbb{C}) \Rightarrow A + iA$  complex  $*$ -subalgebra of  $M_{\dim(H)}(\mathbb{C})$ . Then  $A + iA$  is a matrix algebra and  $A + iA = M_k(\mathbb{C})$  for  $k \geq 1$ . Thus we have

$$A \cap iA = \begin{cases} \{0\} & \text{if } A = M_k(\mathbb{C}) \\ A + iA = M_k(\mathbb{C}) & \end{cases} \quad (33)$$

Furthermore  $A$  is a fixed point algebra of an anti-linear automorphism  $\alpha$  of  $M_k(\mathbb{C})$  with  $\alpha(a + ib) = a - ib$  for  $a, b \in A$ . Implement  $\alpha$  by an anti-linear isometry  $I$  on  $\mathbb{C}^n$  such that  $\alpha(x) = I \times I^{-1} \quad \forall x \in M_k(\mathbb{C})$ . Now since  $\alpha^2 = 1$ ,  $I^2$  commutes with  $M_k(\mathbb{C})$  and is proportional to a complex scalar  $I^2 = \pm 1$  and  $A$  is the commutant of  $I$

- if  $I^2 = 1 \Rightarrow \exists \{e_i\}$  ONB of  $\mathbb{C}^k$  with  $Ie_i = e_i$ , then  $A = M_k(\mathbb{R})$
- if  $I^2 = -1 \Rightarrow \exists \{e_i, f_i\}$  ONB of  $\mathbb{C}^k$  with  $Ie_i = f_i$ , ( $k$  even)  
Therefor  $I$  must be a  $k/2 \times k/2$  matrix because of commutation with  $M_k(\mathbb{C})$ , then  $A = M_{k/2}(\mathbb{H})$

□

The Krajewski diagrams can also classify real algebras, as long as we take  $\mathbb{F}_i$  for each  $i$  into account. That is we enhance the set  $\Lambda$  to be

$$\Lambda = \{\mathbf{n}_1 \mathbb{F}_1, \dots, \mathbf{n}_N \mathbb{F}_N\} \quad (34)$$

Reducing in to the previous  $\Lambda$  if all  $\mathbb{F}_i = \mathbb{C}$ .

## 3 Classification of Irreducible Geometries

Classify irreducible real spectral triples based on  $M_N(\mathbb{C} \oplus M_N(\mathbb{C}))$  for some  $N$

**Definition 4.** A finite real spectral triple  $(A, H, D; J, \gamma)$  is called irreducible if the triple  $(A, H, J)$  is irreducible, that is when

1. The representation of  $A$  and  $J$  on  $H$  are irreducible
2. The action of  $A$  on  $H$  has a separating vector

**Theorem 2.** Let  $(A, H, D; J, \gamma)$  be an irreducible finite real spectral triple of KO-dimension 6. Then exists a positive integer  $N$  such that  $A \simeq M_N(\mathbb{C}) \oplus M_N(\mathbb{C})$ .

*Proof.* Let  $(A, H, D; J, \gamma)$  be an arbitrary finite real spectral triple, corresponding to

$$A = \bigoplus_i^N M_{n_i}(\mathbb{C}) \quad (35)$$

$$H = \bigoplus_{i,j=1}^N \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j^\circ} \otimes V_{ij} \quad (36)$$

Remember that each  $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j}$  is a irreducible representation of  $A$ . In order for  $H$  to support the real structure  $J$  we need both  $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j}$  and  $\mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i}$ . With Lemma 1 with  $J^2 = 1$  with multiplicity  $\dim(V_{ij}) = 1$  we have such a structure. Hence

$$H = \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j} \oplus \mathbb{C}^{n_j} \otimes \mathbb{C}^{n_i} \quad (37)$$

For  $i, j \in \{1, \dots, N\}$

For the second condition (existence of the separating vector). The representations of  $A$  in  $H$  are only faithful if  $A = M_{n_i}(\mathbb{C}) \oplus M_{n_j}(\mathbb{C})$ . The stronger condition applies  $n_i = n_j$  then we have  $A'\xi = H$  with the commutant of  $A$  and  $\xi \in H$  the separating vector. Normally since  $A' = M_{n_j}(\mathbb{C}) \oplus M_{n_i}(\mathbb{C})$  with  $\dim(A') = n_i^2 + n_j^2$  and  $\dim(H) = 2n_i n_j$  we have a equality  $n_i = n_j$ .  $\square$