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Applied Analysis Problems

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1 Sheet 5

1.1 Fourier Transform

In this section we prove the linearity of the Fourier Transform $\mathcal{F}$ on $L^1(\mathbb{R}^d)$. For $f, g \in L^1(\mathbb{R}^d)$ and $\lambda, \mu \in \mathbb{R}$ the linearity condition for $\mathcal{F}$ is the following

$$\mathcal{F}(\lambda f + \mu g) = \lambda\mathcal{F}(f) + \mu\mathcal{F}(g). \tag{1}$$

We start by using the Fourier transform definition for $x, \xi \in \mathbb{R}^d$

$$\begin{aligned}
\mathcal{F}(\lambda f + \mu g)(\xi) &= \int_{\mathbb{R}^d} (\lambda f(x) + \mu g(x)) e^{-2\pi i\langle x,\xi\rangle}\ dx = && (2)\\
&= \int_{\mathbb{R}^d} \lambda f(x) e^{-2\pi i\langle x,\xi\rangle} + \mu g(x) e^{-2\pi i\langle x,\xi\rangle}\ dx = && (3)\\
&= \int_{\mathbb{R}^d} \lambda f(x) e^{-2\pi i\langle x,\xi\rangle}\ dx + \int_{\mathbb{R}^d} \mu g(x) e^{-2\pi i\langle x,\xi\rangle}\ dx = && (4)\\
&= \lambda \int_{\mathbb{R}^d} f(x) e^{-2\pi i\langle x,\xi\rangle}\ dx + \mu \int_{\mathbb{R}^d} g(x) e^{-2\pi i\langle x,\xi\rangle}\ dx = && (5)\\
&= \lambda\mathcal{F}(f)(\xi) + \mu\mathcal{F}(g)(\xi) && (6)
\end{aligned}$$

1.2 Identities of the Fourier transform

The following are three identities of the Fourier transform

	$g(x)$	$\hat{g}(\xi)$
(1)	$f(x - x_0)$	$e^{-2\pi i x_0 \xi}\hat{f}(\xi)$
(2)	$e^{2\pi i \xi_0 x} f(x)$	$f(\xi - \xi_0)$
(3)	$f(ax)$	$\frac{1}{a}\hat{f}(\frac{\xi}{a})$

Table 1: Identities of the Fourier transform for $a > 0, \xi_0, x \in \mathbb{R}$

We start with (1)

$$\begin{aligned}
\widehat{f(x-x_0)} &= \int_{\mathbb{R}} f(x-x_0)e^{-2\pi i x\xi}\ dx = \qquad (y = x - x_0) && (7)\\
&= \int_{\mathbb{R}} f(y)e^{-2\pi i(y+x_0)\xi}\ dy = && (8)\\
&= e^{-2\pi i x_0\xi}\int_{\mathbb{R}} f(y)e^{-2\pi i y\xi}\ dy = && (9)\\
&= e^{-2\pi i x_0\xi}\hat{f}(\xi). && (10)
\end{aligned}$$

For (2) we have

$$\begin{aligned}
\widehat{e^{2\pi i x\xi_0}f(x)} &= \int_{\mathbb{R}} e^{2\pi i x\xi_0} f(x)e^{-2\pi i x\xi}\ dx = && (11)\\
&= \int_{\mathbb{R}} f(x)e^{-2\pi i x(\xi-\xi_0)}\ dx = && (12)\\
&= \hat{f}(\xi-\xi_0). && (13)
\end{aligned}$$

For (3) we have

$$\begin{aligned}
\widehat{f(ax)} &= \int_{\mathbb{R}} f(ax)e^{-2\pi i\xi x}\ dx = \qquad \text{sub: } (y = ax) && (14)\\
&= \int_{\mathbb{R}} \frac{1}{a} f(y)e^{-2\pi i\frac{\xi}{a}y}\ dy = && (15)\\
&= \frac{1}{a}\hat{f}\left(\frac{\xi}{a}\right). && (16)
\end{aligned}$$

1.3 The Box-Function

Consider the following Box-Function

$$\Pi(x) := \begin{cases} 1 & -\frac{3}{2} < x < \frac{1}{2} \\ 0 & \text{else} \end{cases} \tag{17}$$

The Fourier transform of this function is

$$\begin{aligned}
\widehat{\Pi(x)} &= \int_{\mathbb{R}} \Pi(x)e^{-2\pi i x\xi}\ dx = && (18)\\
&= \int_{-\frac{3}{2}}^{\frac{1}{2}} e^{-2\pi i x\xi}\ dx = \frac{-1}{2\pi i\xi}e^{-2\pi i x\xi}\Big|_{-\frac{3}{2}}^{\frac{1}{2}} = && (19)\\
&= \frac{1}{2\pi i\xi}\left(e^{3\pi i\xi} - e^{-\pi i\xi}\right) = && (20)\\
&= \frac{e^{\pi i\xi}\sin(2\pi\xi)}{\pi\xi}. && (21)
\end{aligned}$$