

University of Vienna Faculty of Mathematics

Applied Analysis Problems

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1 Sheet 5

1.1 Fourier Transform

In this section we prove the linearity of the Fourier Transform \mathcal{F} on $L^1(\mathbb{R}^d)$. For $f, g \in L^1(\mathbb{R}^d)$ and $\lambda, \mu \in \mathbb{R}$ the linearity condition for \mathcal{F} is the following

$$\mathcal{F}(\lambda f + \mu g) = \lambda \mathcal{F}(f) + \mu \mathcal{F}(g). \quad (1)$$

We start by using the Fourier transform definition for $x, \xi \in \mathbb{R}^d$

$$\mathcal{F}(\lambda f + \mu g)(\xi) = \int_{\mathbb{R}^d} (\lambda f(x) + \mu g(x)) e^{-2\pi i \langle x, \xi \rangle} dx = \quad (2)$$

$$= \int_{\mathbb{R}^d} \lambda f(x) e^{-2\pi i \langle x, \xi \rangle} + \mu g(x) e^{-2\pi i \langle x, \xi \rangle} dx = \quad (3)$$

$$= \int_{\mathbb{R}^d} \lambda f(x) e^{-2\pi i \langle x, \xi \rangle} dx + \int_{\mathbb{R}^d} \mu g(x) e^{-2\pi i \langle x, \xi \rangle} dx = \quad (4)$$

$$= \lambda \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \xi \rangle} dx + \mu \int_{\mathbb{R}^d} g(x) e^{-2\pi i \langle x, \xi \rangle} dx = \quad (5)$$

$$= \lambda \mathcal{F}(f)(\xi) + \mu \mathcal{F}(g)(\xi) \quad (6)$$

1.2 Identities of the Fourier transform

The following are three identities of the Fourier transform

	$g(x)$	$\hat{g}(\xi)$
(1)	$f(x - x_0)$	$e^{-2\pi i x_0 \xi} \hat{f}(\xi)$
(2)	$e^{2\pi i \xi_0 x} f(x)$	$\hat{f}(\xi - \xi_0)$
(3)	$f(ax)$	$\frac{1}{a} \hat{f}\left(\frac{\xi}{a}\right)$

Table 1: Identities of the Fourier transform for $0 < a \in \mathbb{R}, \xi_0, x \in \mathbb{R}$

We start with (1)

$$\widehat{f(x - x_0)} = \int_{\mathbb{R}} f(x - x_0) e^{-2\pi i x \xi} dx = \quad (y = x - x_0) \quad (7)$$

$$= \int_{\mathbb{R}} f(y) e^{-2\pi i (y + x_0) \xi} dy = \quad (8)$$

$$= e^{-2\pi i x_0 \xi} \int_{\mathbb{R}} f(y) e^{-2\pi i y \xi} dy = \quad (9)$$

$$= e^{-2\pi i x_0 \xi} \hat{f}(\xi). \quad (10)$$

For (2) we have

$$\widehat{e^{2\pi i x \xi_0} f(x)} = \int_{\mathbb{R}} e^{2\pi i x \xi_0} f(x) e^{-2\pi i x \xi} dx = \quad (11)$$

$$= \int_{\mathbb{R}} f(x) e^{-2\pi i x (\xi - \xi_0)} dx = \quad (12)$$

$$= \hat{f}(\xi - \xi_0). \quad (13)$$

For (3) we have

$$\widehat{f(ax)} = \int_{\mathbb{R}} f(ax) e^{-2\pi i \xi x} dx = \quad (y = ax) \quad (14)$$

$$= \int_{\mathbb{R}} \frac{1}{a} f(y) e^{-2\pi i \frac{\xi}{a} y} dy = \quad (15)$$

$$= \frac{1}{a} \hat{f}\left(\frac{\xi}{a}\right). \quad (16)$$

1.3 The Box-Function

Consider the following Box-Function

$$\Pi(x) := \begin{cases} 1 & -\frac{3}{2} < x < \frac{1}{2} \\ 0 & \text{else} \end{cases} \quad (17)$$

The Fourier transform of this function is

$$\widehat{\Pi(x)} = \int_{\mathbb{R}} \Pi(x) e^{-2\pi i x \xi} dx = \quad (18)$$

$$= \int_{-\frac{3}{2}}^{\frac{1}{2}} e^{-2\pi i x \xi} dx = \frac{-1}{2\pi i \xi} e^{-2\pi i x \xi} \Big|_{-\frac{3}{2}}^{\frac{1}{2}} = \quad (19)$$

$$= \frac{1}{2\pi i \xi} (e^{3\pi i \xi} - e^{-\pi i \xi}) = \quad (20)$$

$$= \frac{e^{\pi i \xi} \sin(2\pi \xi)}{\pi \xi}. \quad (21)$$