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Introductory Seminar Advanced Numerical Analysis

Exercise sheet 3, due date: 21.03.2022

Exercise 1 (Gradient decent and CG).
We wish to solve the linear system $Ax = b, A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n$ using the Gradient decent method, i.e. the iteration

$$x^{k+1} = x^k + \alpha_k r^k$$

where $r^k = b - Ax^k$ is the residual and $\alpha_k = \frac{(r^k)^T r^k}{(r^k)^T A r^k}$.

(1) Compute x^1 if $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, b = (1,1)^T$ and the initial guess is given by $x^0 = 0$.
(2) Suppose the k-th error $e^k = x - x^k$ is an eigenvector of A. What can you say about x^{k+1}?

Exercise 2 (Norm equivalence: optimality, part 1).
Recall that two norms $\|\cdot\|_*$ and $\|\cdot\|_\star$ on a vector space V are called equivalent if there exist positive constants $C, C' > 0$ such that

$$\textbf{(E)} \qquad \|v\|_* \leq C\|v\|_\star, \quad \|v\|_\star \leq C'\|v\|_*$$

for every $v \in V$. Each of these constants is called *sharp* or *optimal* if the inequalities in **(E)** not hold with any smaller constant. If V is finite-dimensional then any two norms $\|\cdot\|_*$ and $\|\cdot\|_\star$ on V are equivalent and there exist $\hat{v}, \hat{v}' \in V \setminus \{0\}$ at which the inequalities in **(E)** with sharp constants become equalities. In this case, the sharp constants C, C' are given by

$$C = \max_{v \in V \setminus \{0\}} \frac{\|v\|_*}{\|v\|_\star} = \frac{\|\hat{v}\|_*}{\|\hat{v}\|_\star}, \quad C' = \max_{v \in V \setminus \{0\}} \frac{\|v\|_\star}{\|v\|_*} = \frac{\|\hat{v}'\|_\star}{\|\hat{v}'\|_*}.$$

In the following we determine the optimal constants C, C' in the case where $V = \mathbb{C}^n$, $\|\cdot\|_* = \|\cdot\|_\infty$ is the maximum norm and $\|\cdot\|_\star = \|\cdot\|_2$ is the Euclidean norm.

(1) Show that $\|v\|_\infty \leq C\|v\|_2$ for every $v \in \mathbb{C}^n$ with $C = 1$. Further show that $C = 1$ is optimal.
(2) Show that $\|v\|_2 \leq C'\|v\|_\infty$ for every $v \in \mathbb{C}^n$ with $C' = \sqrt{n}$. Further show that $C' = \sqrt{n}$ is optimal.

Exercise 3 (Norm equivalence: optimality, part 2).
This is a continuation of exercise 2. We wish to determine the optimal constants for the matrix norms $\|\cdot\|_\infty$ and $\|\cdot\|_2$ (the matrix norms induced by the maximum resp. Euclidean norm on $\mathbb{C}^n$) on $V = \mathbb{C}^{n \times n}$.

(1) Show that $\|A\|_\infty \leq C\|A\|_2$ for every $A \in \mathbb{C}^{n \times n}$ where $C = \sqrt{n}$.
(2) Let A be the matrix all whose rows but the first row are zero and the first row is a row of ones. Further let $b = (1, 1, \ldots, 1)^T \in \mathbb{C}^n$. Use these definitions of A and b to show that $\|A\|_2 = \sqrt{n}$.
(3) Let A be defined as in (4). Show that $\|A\|_\infty = \sqrt{n}\|A\|_2$ and conclude that $C = \sqrt{n}$ is optimal.

(4) Show that $\|A\|_2 \leq \sqrt{n}\|A\|_\infty$ for every $A \in \mathbb{C}^{n\times n}$. Further prove that equality in the previous inequality is attained for the matrix A whose columns but the first are zero and whose first column is a column of ones.

Exercise 4 (Antisymmetric systems, part 1).
Let $\langle x, y\rangle = \sum_{j=1}^n x_j y_j$ denote the Euclidean scalar product on $\mathbb{R}^n$ and let $\|\cdot\|_2$ be the Euclidean norm. Let $B \in \mathbb{R}^{n\times n}$ be an antisymmetric matrix, i.e. $B^T = -B$ and let $A \coloneqq \mathrm{Id} - B$ where Id is the identity matrix in $\mathbb{R}^{n\times n}$.

(1) Show that for every $x \in \mathbb{R}^n$ we have $\langle Bx, x\rangle = 0$ and $\langle Ax, x\rangle = \|x\|_2^2$.
(2) Show that $\|Ax\|_2^2 = \|x\|_2^2 + \|Bx\|_2^2$ and that $\|A\|_2 = \sqrt{1 + \|B\|_2^2}$.
(3) Show that A is invertible and the matrix-norm of its inverse is given by
$$\|A^{-1}\|_2 = \max_{x\neq 0} \frac{\|x\|_2}{\|Ax\|_2}.$$
(4) Show that $\|A\|_2 \leq 1$.

Exercise 5 (Antisymmetric systems, part 2).
Let the matrices A and B be given as in Exercise 4, i.e. $B \in \mathbb{R}^{n\times n}$ is antisymmetric and $A \coloneqq \mathrm{Id} - B$.

(1) Let $k \in \{1, \ldots, n\}$ and let $\mathcal{W} \subset \mathbb{R}^n$ be a k-dimensional subspace of $\mathbb{R}^n$ spanned by the vectors $w_1, \ldots, w_k \in \mathbb{R}^n$. Show that if $x \in \mathcal{W}$ is such that
$$\textbf{(P)} \qquad \langle Ax, w\rangle = \langle b, w\rangle \quad \forall w \in \mathcal{W}$$
then $\|x\|_2 \leq \|b\|_2$.
(2) With $x \coloneqq \sum_{j=1}^k c_j w_j$ the problem **(P)** is equivalent to finding real numbers $c_1, \ldots, c_k$ solving the linear system
$$\sum_{j=1}^k c_j \langle Aw_j, w_i\rangle = \langle b, w_i\rangle, \quad i = 1, \ldots, k.$$
Show that **(P)** has a unique solution $x \in \mathcal{W}$.
(3) Let $x^* \coloneqq A^{-1}b$. Show that
$$\|x^* - x\|_2 \leq \|A\|_2 \min_{w\in\mathcal{W}} \|x^* - w\|_2.$$