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Applied Analysis Problems

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1 Sheet 8

1.1 Finite Discrete Fourier Transform (FDFT)

Consider the vector $\begin{pmatrix} a & b & c & d \end{pmatrix}^T \in \mathbb{C}^4$ with a FDFT $\begin{pmatrix} A & B & C & D \end{pmatrix}^T$. We can show that the vector

$$\begin{pmatrix} a & 0 & b & 0 & c & 0 & d & 0 \end{pmatrix}^T, \tag{1}$$

has the FDFT of

$$\frac{1}{2}\begin{pmatrix} A & 0 & B & 0 & C & 0 & D & 0 \end{pmatrix}^T. \tag{2}$$

For the $N = 4$, $n \in \{0, \dots, 3\}$ the coefficients a, b, c, d are denoted in $f[n]$. The FDFT is

$$\begin{aligned} \hat{f}[k] &= \frac{1}{4} * \sum_{n=0}^{3} f[n] e^{-2\pi i \frac{n}{4} k} && (3)\\ &= \frac{1}{4}\left(a + b e^{-\pi i \frac{k}{2}} + c e^{-\pi i k} + d e^{-\frac{3\pi i k}{2}}\right) = && (4)\\ (&= \begin{pmatrix} A & B & C & D \end{pmatrix}^T) && (5) \end{aligned}$$

for $k \in \{0, \dots, 3\}$ accordingly. For the $N = 8$, $\mathbb{C}^8$ case we have $f_2[n]$ for $n \in \{0, \dots 7\}$,

$$\begin{aligned} \hat{f}_2[k] &= \frac{1}{8} * \sum_{n=0}^{7} f_2[n] e^{-2\pi i \frac{n}{8} k} && (6)\\ &= \frac{1}{2}\frac{1}{4}\left(a + b e^{-\pi i \frac{k}{2}} + c e^{-\pi i k} + d e^{-\frac{3\pi i k}{2}}\right) = && (7)\\ (&= \frac{1}{2}\begin{pmatrix} A & B & C & D & A & B & C & D \end{pmatrix}^T) && (8) \end{aligned}$$

for $k \in \{0, \dots, 7\}$ accordingly. We may generalize now for $\mathbb{C}^{4N}$, and the sequence for $a, b, c, d, 0$ represented by the function $g[n]$ for $n \in \{0, \dots, 4N-1\}$,

$$g[n] = \begin{cases} f[n] & n \in \{0, N, 2N, 3N\} \\ 0 & \text{else} \end{cases} . \tag{9}$$

Now we can compute the FDFT for $k \in \{0, \dots, 4N-1\}$

$$\hat{g}[k] = \frac{1}{4N} \sum_{n=0}^{4N-1} g[n] e^{-2\pi i \frac{n}{4N} k} \tag{10}$$

$$= \frac{4}{N} \sum_{n=0} 3 f[n] e^{-2\pi i \frac{n}{4} k} \tag{11}$$

$$= \frac{1}{N} \left(\frac{1}{4} \sum_{n=0}^{3} f[n] e^{-2\pi i \frac{n}{4} k} \right) \tag{12}$$

$$= \frac{1}{N} \underbrace{\begin{pmatrix} A & B & C & D & \dots & \dots & A & B & C & D \end{pmatrix}^T}_{4N \text{ entries, } N \text{ sequences}}. \tag{13}$$

1.2 More FDFT

Consider the discrete complex exponential with frequency of $1Hz$ in $\mathbb{C}^8$, for $n \in \{0, \dots, 7\}$,

$$\exp[n] = e^{2\pi i n/8}. \tag{14}$$

The FDFT for $k \in \{0, \dots, 7\}$ is

$$\hat{\exp}[k] = \frac{1}{8} \sum_{n=0}^{7} e^{2\pi i \frac{n}{8}} e^{-2\pi i n \frac{k}{8}} \tag{15}$$

$$= \frac{1}{8} \sum_{n=0}^{7} e^{-2\pi i (k-1) \frac{n}{8}} \tag{16}$$

$$= \begin{cases} 1 & k = 1 \\ 0 & k \neq 1 \end{cases} . \tag{17}$$

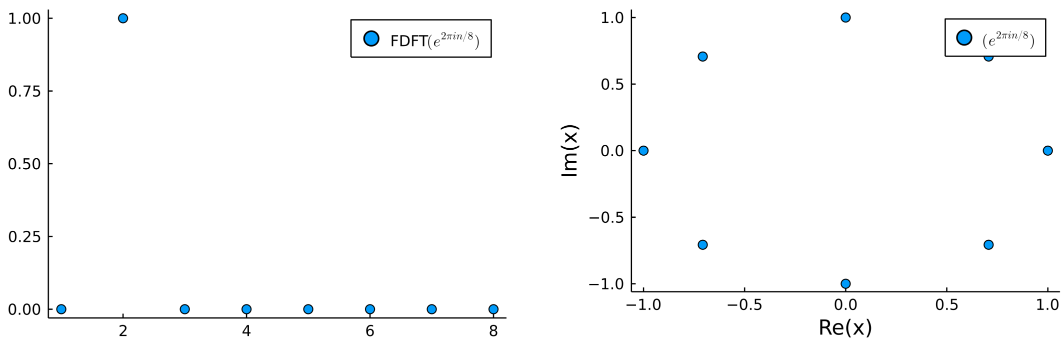


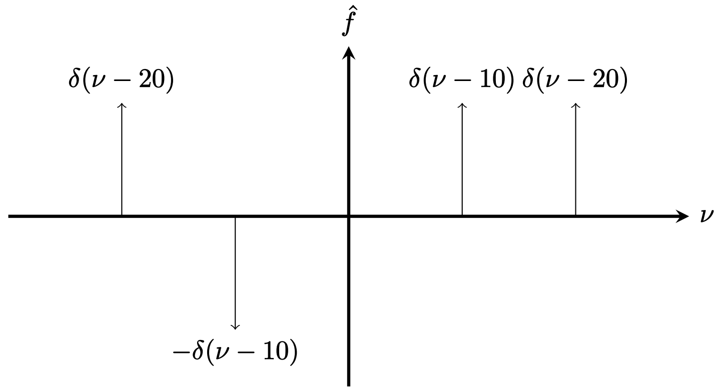
Figure 1: Test in Julia

1.3 Sampling Sinusoids

Consider the following continuous signal

$$f(t) = sin(20\pi t) + sin(40\pi t) \tag{18}$$

with frequencies $\omega = 2\pi\nu$, $\nu_1 = 10$ Hz and $\nu_2 = 20$ Hz. Sketching its Fourier transform would be something like this



The Nyquist frequency for sampling would be

$$\nu_{\text{Nyquist}} = 2\nu_{\max} = 2\nu_2 = 40 \text{ Hz}, \tag{19}$$

If we choose 50 Hz for sampling we would get aliasing with the following frequencies

$$n \cdot 50 \text{ Hz} - 20 \text{ Hz} = 30 \text{ Hz}, 80 \text{ Hz}, 130 \text{ Hz}, \dots \tag{20}$$

1.4 Short-Time Fourier Transform (STFT)

The Definition of the STFT is

$$\begin{aligned} \text{STFT}\{f\} = S_\varphi f(\tau, \omega) &= \int_{\mathbb{R}} f(t)\overline{\mathrm{M}_\omega \mathrm{T}_\tau \varphi} dt && (21) \\ &= \int_{\mathbb{R}} f(t)\bar{\varphi}(t-\tau)e^{-2\pi i \omega t}\ dt && (22) \\ & && (23) \end{aligned}$$

Then we have the following identity

$$\begin{aligned} S_\varphi(\mathrm{T}_u \mathrm{M}_\eta f)(x, \omega) &= \int_{\mathbb{R}} (\mathrm{T}_u \mathrm{M}_\eta f(t))\, \bar{\varphi}(t-x)e^{-2\pi i \omega t}\ dt && (24) \\ &= \int_{\mathbb{R}} e^{2\pi i \eta(t-u)} f(t-u)e^{-2\pi i \omega t}\bar{\varphi}(t-x)\ dt \qquad (\text{sub: } s = t-u) && (25) \\ &= \int_{\mathbb{R}} f(s)\bar{\varphi}(s-(x-u))e^{2\pi i \eta s}e^{-2\pi i \omega s}e^{-2\pi i \omega u}\ ds && (26) \\ &= e^{-2\pi i \omega u} \int_{\mathbb{R}} f(s)\bar{\varphi}(s-(x-u))e^{-2\pi i(\omega-\eta)s}\ ds && (27) \\ &= e^{-2\pi i \omega u} \int_{\mathbb{R}} f(s)\overline{\mathrm{M}_{(\omega-\eta)}\mathrm{T}_{(x-u)}\varphi(s)}\ ds && (28) \\ &= e^{-2\pi i \omega u} S_\varphi f\left(x-u,\ \omega-\eta\right). && (29) \end{aligned}$$

The second identity we can show

$$
\begin{aligned}
S_\varphi f(x,\omega) &= \langle f, \overline{\mathrm{M}_\omega \mathrm{T}_x \varphi} \rangle && (30)\\
&= \langle \mathcal{F} f, \mathcal{F}\overline{\mathrm{M}_\omega \mathrm{T}_x \varphi} \rangle && (31)\\
&= \int_\xi \hat{f}(\xi) \int_t \overline{\mathrm{M}_\omega \mathrm{T}_x \varphi}(t) e^{-2\pi i \xi t}\ dt\ d\xi && (32)\\
&= \int_\xi \hat{f}(\xi) \int_t \hat{\hat{\varphi}}(t-x) e^{2\pi i \omega t} e^{-2\pi i \xi t}\ dt\ d\xi && (33)\\
&= \int_\xi \hat{f}(\xi) \int_t \hat{\hat{\varphi}}(t-x) e^{-2\pi i (\xi-\omega) t}\ dt\ d\xi \qquad \text{sub } u = t - x && (34)\\
&= \int_\xi \hat{f}(\xi) \int_t \hat{\hat{\varphi}}(u) e^{-2\pi i (\xi-\omega) u} e^{-2\pi i (\xi-\omega) x}\ dt\ d\xi && (35)\\
&= \int_\xi \hat{f}(\xi) e^{-2\pi i (\xi-\omega) x} \int_t \hat{\varphi}(u) e^{-2\pi i (\xi-\omega) u}\ dt\ d\xi && (36)\\
&= e^{2\pi i \omega x} \int_\xi \hat{f}(\xi) \hat{\hat{\varphi}}(\xi-\omega) e^{-2\pi i \xi x} d\xi && (37)\\
&= e^{2\pi i \omega x} S_{\hat{\varphi}} \hat{f}(\omega, -x). && (38)
\end{aligned}
$$