Mathematical Modeling of Water-Wave Problems

Applied PDE Seminar

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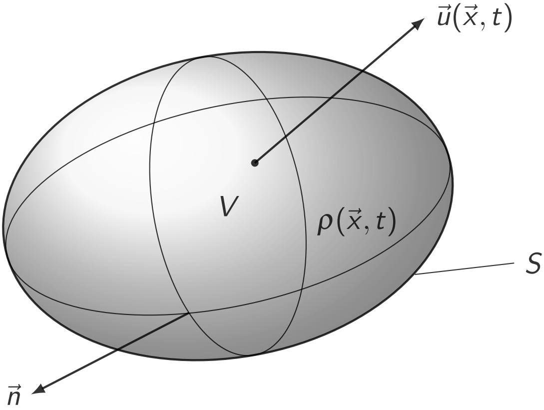
Fluid Description

The fluid is described by

- Fluid density $\rho(\vec{x}, t)$
- Velocity Field $\vec{u}(\vec{x}, t) = (u, v, w)$

Figure: Control volume of the fluid



Mass Conservation

- Mass:

$$m(t) = \int_V \rho(\vec{x}, t) \; dV$$

- Rate of change:

$$\int_V \frac{\partial \rho(\vec{x}, t)}{\partial t} \; dV = \frac{dm}{dt} = -\int_S \rho(\vec{x}, t)\vec{u} \cdot \vec{n} \; dS$$

- Use Gauss's law to get the **Equation of Mass conservation**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

$\rightarrow$ Apply Newton's second law to the Fluid

Body Force

$\vec{F} = (0, 0, -g)$

Local/Short-range Force

Stress tensor

For inviscid fluid: $P(\vec{x}, t)$

$$\Rightarrow \int_V \rho \frac{D\vec{u}}{Dt} \, dV = \int_V \left(\rho \vec{F} - \nabla P \right) \, dV$$

$\rightarrow$ Leads us to **Euler's Equation of Motion**

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho} \nabla P + \vec{F}$$

Vorticity

Vorticity	Irrotational Flow
$\vec{\omega} = \nabla \times \vec{u}$	$\vec{\omega} = 0$

$\rightarrow$ Vorticity pops up in the acceleration of the fluid particles

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + \nabla \left(\frac{1}{2} \vec{u} \cdot \vec{u} \right) - (\vec{u} \times \vec{\omega})$$

$\rightarrow$ We can incorporate vorticity into Euler's Equation of Motion

$$\frac{\partial \vec{u}}{\partial t} + \nabla \left(\frac{1}{2} \vec{u} \cdot \vec{u} + \frac{P}{\rho} + \Omega \right) = \vec{u} \times \vec{\omega}$$

Perfect Fluid

inviscid $\mu = 0$

incompressible $\rho = \text{const.}$, then $\nabla \vec{u} = 0$

Boundary Conditions for Water Waves

- **Kinematic Condition**: Fluid particles at the surface
- **Dynamic Condition**: Atmospheric Pressure on the surface
- **Bottom Condition**: Rigid and fixed bottom
- (**Integrated Mass Condition**): Combination

Nondimensionalisation

- h_0 for the typical water depth
- λ for the typical wavelength
- $\sqrt{gh_0}$ velocity scale of waves in (x, y)
- $\frac{\lambda}{\sqrt{gh_0}}$ time scale of wave propagation
- $\frac{h_0\sqrt{gh_0}}{\lambda}$ velocity scale in z

$\rightarrow$ **Shallowness parameter** $\delta = \frac{h_0}{\lambda}$

$\rightarrow$ **Amplitude parameter** $\varepsilon = \frac{a}{h_0}$

Nondimensionalisation

→ Nondimensionalisation

$$x \to \lambda x, \quad u \to \sqrt{gh_0}\,u,$$
$$y \to \lambda y, \quad v \to \sqrt{gh_0}\,v, \qquad t \to \frac{\lambda}{\sqrt{gh_0}}t,$$
$$z \to h_0 z, \quad w \to \frac{h_0\sqrt{gh_0}}{\lambda}w.$$

→ Top and Bottom conditions

$$h = h_0 + a\eta(\vec{x}_\perp, t), \qquad b \to h_0 b(\vec{x}_\perp, t)$$

→ Rewrite Pressure

$$P = P_a + \rho g(h_0 - z) + \rho g h_0 p(\vec{x})$$

Scaling

$\rightarrow$ w, p and the free surface z are $\propto \varepsilon$, leading to the scaling

$$p \rightarrow \varepsilon p, \quad w \rightarrow \varepsilon w, \quad \vec{u}_\perp \rightarrow \varepsilon \vec{u}_\perp$$

→ Nondimensionalized Euler's Equation of motion

$$\frac{Du}{Dt} = -p_x \quad \frac{Dv}{Dt} = -p_y \quad \delta^2 \frac{Dw}{Dt} = -p_z$$

$$\nabla \cdot \vec{u} = 0$$

→ With boundary conditions

$$\left. \begin{aligned} p &= \eta - \frac{\delta^2 \varepsilon h_0}{\lambda^2} \frac{W_e}{R} \\ w &= \frac{1}{\varepsilon} \eta_t + (\mathrm{u}_\perp \nabla_\perp) \eta \end{aligned} \right\} \quad \text{on} \;\; z = 1 + \varepsilon \eta \tag{1}$$

$$w = \frac{1}{\varepsilon} b_t + (\mathrm{u}_\perp \nabla_\perp) b \quad \text{on} \;\; z = b \tag{2}$$

History of the Soliton

- John Scott Russell discovered the solitary wave in 1834, firstly calling it the **wave of translation**
- a **soliton** is a solitary wave that resists dispersion, maintaining its shape while it propagates at constant velocity

Korteweg-de Vries equation: nonlinear PDE

$$\eta_t + 6K\eta\eta_x + \eta_{xxx} = 0$$

With Solution

$$\eta(x,t) = 2c^2 \operatorname{sech}^2\left(c\left(x - 4c^2 t\right)\right)$$

KdV Regime

1) The KdV equation arises in the $\varepsilon = O(\delta^2)$
2) by rescaling δ in favor of ε in Euler's Equations of motion
3) going into the frame of the moving wave ($\xi = x - t, \tau = \varepsilon t$)
4) conducting an Asymptotic expansion of u, w, p and η.
5) KdV equation is present in the ε^1 term

2004 Tsunami: Description

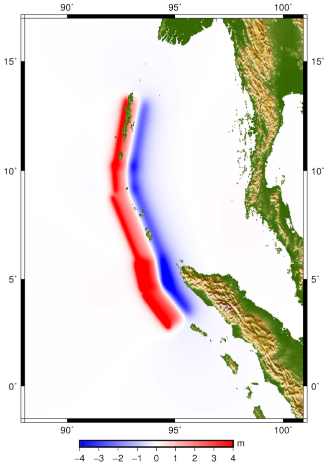


Figure: Earthquake generating a tsunami with $\lambda = 100$ km, $a = 1$ m (found in [6])

2004 Tsunami: $\varepsilon = O\left(\delta^2\right)$

- $\varepsilon = \frac{a}{h_0}$ and $\delta = \frac{h_0}{\lambda}$ need to enter the regime $\varepsilon = O(\delta^2)$ for the KdV equation to become relevant
- But also the geophysical scales need to be $\xi = O(1)$ and $\tau = O(1)$ for the KdV dynamics to become relevant, that is

$$x = O\left(\varepsilon^{-1}\lambda\right) \tag{3}$$

- KdV dynamics is when the waves are ordered with the highest in front following an oscillatory tale
- This happens because wave amplitude is proportional to wave speed

2004 Tsunami: Regime of Validity

$$\lambda = 100 \text{ km} \qquad a = 1 \text{ m}$$

Waves propagating westwards to India/Sri Lanka

$$h_0 = 4 \text{ km} \Rightarrow \begin{cases} \varepsilon \simeq 25 \cdot 10^{-5} \\ \delta \simeq 4 \cdot 10^{-2} \end{cases}$$

Waves propagating eastwards to Thailand

$$h_0 = 1 \text{ km} \Rightarrow \begin{cases} \varepsilon \simeq 10^{-3} \\ \delta \simeq 10^{-2} \end{cases}$$

$\Rightarrow$ Both enter the regime $\varepsilon = O(\delta^2)$

2004 Tsunami: Regime of Validity

Waves propagating westwards to India/Sri Lanka ($\simeq$ 1600 km)

$$\left.\begin{array}{l}\varepsilon \simeq 25 \cdot 10^{-5} \\ \lambda = 100\text{km}\end{array}\right\} \Rightarrow \; x \simeq 4 \cdot 10^{5} \text{ km}$$

Waves propagating eastwards to Thailand ($\simeq$ 700 km)

$$\left.\begin{array}{l}\varepsilon \simeq 10^{-3} \\ \lambda = 100\text{km}\end{array}\right\} \Rightarrow \; x \simeq 10^{5} \text{ km}$$

$\Rightarrow$ The propagation distance of the tsunami waves in both directions is not enough for KdV dynamics to take place.

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Thank you for listening!