University of Vienna
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Numerical Analysis Problems

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1.1 Problem 1

Take a linear system $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. We want to solve it using the Gradient decent method, an iteration

$$x^{k+1} = x^k + \alpha_k r^k, \tag{1}$$

where $r^k = b - Ax^k$ and the residual $\alpha_k = \frac{(r^k)^T r^k}{(r^k)^T A r^k}$.

1.1.1

We compute x^1 for

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{2}$$

with an initial guess $x^0 = 0$.

$$r^0 = \begin{pmatrix}1 & 1\end{pmatrix} \qquad Ar^0 = \begin{pmatrix}1 & 1\end{pmatrix} \quad \Rightarrow \quad \alpha_0 = 1. \tag{3}$$

Then for x^1 we have

$$x^1 = \begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix} = b \tag{4}$$

1.1.2

Suppose the k-th error $e^k = x - x^k$ is an eigenvector of A to the eigenvalue λ, then

$$Ae^k = \lambda e^k = \lambda(x^k - x) = \lambda x^k - \lambda x. \tag{5}$$

For the next iteration step we need r^k which is

$$r^k = b - Ax^k + Ax - Ax = (b - Ax) - A(x^k - x) = -\lambda e^k \tag{6}$$
$$(r^k)^T r^k = \lambda^2 (e^k)^T e^k \tag{7}$$
$$(r^k)^T A r^k = \lambda^3 (e^k)^T e^k \tag{8}$$
$$\Rightarrow \alpha_k = \frac{1}{\lambda} \tag{9}$$

Then the next step x^{k+1} is

$$x^{k+1} = x^k + \alpha_k r^k = x^k - \frac{\lambda}{\lambda} e^k = x^k - e^k = x^k - x^k + x = x, \tag{10}$$

i.e. x^{k+1} is then the solution.

1.2 Exercise 2

We show the norm equivalence of the vector norms $\|\cdot\|_\infty$ and $\|\cdot\|_2$ and the optimality of the constants $C, C' > 0$and the optimality of the constants $C, C' > 0$

1.2.1

We show

$$\|v\|_\infty = \max_i |v_i| \leq \sqrt{\sum_i v_i^2} \quad \Rightarrow \quad C = 1. \tag{11}$$

Then

$$\|v\|_2 = \sqrt{\sum_i v_i^2} \leq \sqrt{\sum_i \max_i |v_i|^2} = \sqrt{\sum_i \|v\|_\infty^2} = \sqrt{n}\|v\|_\infty. \tag{12}$$

We get

$$\|v\|_\infty \leq \|v\|_2 \leq \sqrt{n}\|v\|_\infty. \tag{13}$$

For $u := \frac{v}{\|v\|_\infty} \;\Rightarrow\; \|v\|_\infty = 1$ we get

$$1 \leq \|u\|_2 \leq \sqrt{n} \tag{14}$$

which states optimality of the constants.

1.3 Problem 3

Now we do the same as in Problem 2 but with matrix norms induced by vector norms, specifically for $\|\cdot\|_2$ and $\|\cdot\|_\infty$ matrix norms induced by vector norms.

1.3.1

We know that

$$\|A\|_2 = \max_{v\neq 0} \frac{\|Ax\|_2}{\|x\|_2} \qquad \|A\|_2 = \rho(A^*A) \tag{15}$$

$$\|A\|_\infty = \max_{i,j} |a_{ij}| \tag{16}$$

Together with the norm inequalities

$$\|Av\|_\infty \leq \|A\|_\infty \|v\|_\infty \tag{17}$$

$$\|A\|_2^2 = \rho(A^TA) \leq \|A^TA\|_\infty \tag{18}$$

$$\Rightarrow \|\rho(A^TA)\|_\infty = \|A^TAx\|_\infty \leq \|A^TA\|_\infty \|x\|_\infty, \tag{19}$$

thereby

$$\|A^TA\|_\infty = \max_{i,j} \left|(A^TA)_{ij})\right| = \max_{i,j} \left|\sum_l A_{il}A_{lj}\right| \tag{20}$$

$$= \max_{i,j} \sum_l |A_{il}A_{lj}| \leq n\|A\|_\infty^2 \tag{21}$$

$$\Rightarrow \|A\|_2 \leq \sqrt{n}\|A\|_\infty \tag{22}$$

1.3.2

Let $A \in \mathbb{C}^{n\times n}$, $b \in \mathbb{C}^n$ defined as

$$A = \begin{pmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \tag{23}$$

Then $Ab = \begin{pmatrix} n & 0 & \cdots & 0 \end{pmatrix}^T$ and the $\|\cdot\|_2$ of A is

$$\|A\|_2 = \max_{v\neq 0} \frac{\|Av\|_2}{\|v\|} = \frac{\|Ab\|_2}{\|b\|} = \frac{n}{\sqrt{n}} = \sqrt{n} \tag{24}$$

1.3.3

Let A be defined as above, we show that $\|A\|_\infty = \sqrt{n}\|A\|_2$, where $C = \sqrt{n}$ is optimal

$$\|A\|_\infty = \max_i \sum_j |A_{ij}| = n = \sqrt{n}\sqrt{n} = \sqrt{n}\|A\|_2, \tag{25}$$

thereby C is optimal.

1.3.4

We show that $\|A\|_2 \leq \sqrt{n}\|A\|_\infty$ for all $A \in \mathbb{C}^{n\times n}$, and equality holds for A whose columns are all zero, accept the first one filled with ones. We know the norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are induced by vector norms which are consistent. Then for all $v \in \mathbb{C}^n$ we have

$$\|v\|_\infty \leq \sqrt{n}\|v\|_2 \tag{26}$$

$$\Leftrightarrow \|A\|_\infty \leq \sqrt{n}\|A\|_2 \qquad \forall\ A \in \mathbb{C}^{n\times n}. \tag{27}$$

1.4 Problem 4

Let $\langle x, y\rangle$ be the standard Euclidean scalar product on $\mathbb{R}^n$ and $\|\cdot\|_2$ the Euclidean norm. Let $B \in \mathbb{R}^{n\times n}$ be an antisymmetric matrix, i.e. $B^T = -B$. And let $A := I - B$

1.4.1

We show that for all $x \in \mathbb{R}^n$ we have $\langle Bx, x\rangle = 0$ and $\langle Ax, x\rangle = \|x\|_2^2$.

$$\langle Bx, x\rangle = (Bx)^T x = x^T B^T x = -x^T Bx \tag{28}$$
$$(-x^T Bx)^T = x^T Bx \;\Rightarrow\; x^T Bx = jx^T Bx \tag{29}$$
$$\Rightarrow 2x^T Bx = 2\langle Bx, x\rangle = 0. \tag{30}$$

And the second statement follows from the previous equation

$$\langle Ax, x\rangle =\rangle\,(I - B)\,x, x\rangle \tag{31}$$
$$= \langle x, x\rangle - \underbrace{\langle Bx, x\rangle}_{=0} = \langle x, x\rangle = \|x\|_2^2 \tag{32}$$

1.4.2

We show that $\|Ax\|_2^2 = \|x\|_2^2 + \|Bx\|_2^2$ and that $\|A\|_2 = \sqrt{1 + \|B\|_2^2}$. We start with the vector norm

$$\|Ax\|_2^2 = \|x - Bx\|_2^2 \tag{33}$$
$$= \langle x - Bx, x - Bx\rangle \tag{34}$$
$$= (x^T - x^T B^T)(x - Bx) \tag{35}$$
$$= x^T x - x^T B^T x - x^T Bx + x^T B^T Bx \tag{36}$$
$$= x^T x + x^T B^T Bx = \|x\|_2^2 + \|Bx\|_2^2 \tag{37}$$

where i, j run from 1 to n. Then the vector induced norm follows

$$\|A\|_2^2 = \|I - B\|_2^2 = \sup_{x\neq 0} \frac{\|Ax\|_2^2}{\|x\|_2^2} \tag{38}$$
$$= \sup_{x\neq 0} \frac{\|x\|_2^2 + \|Bx\|_2^2}{\|x\|_2^2} = 1 + \sup_{x\neq 0} \frac{\|Bx\|_2^2}{\|x\|_2^2} = 1 + \|B\|_2^2, \tag{39}$$

pulling the square root on both sides gives us the answer.

1.4.3

We show that A is invertible and the inverse matrix norm is given

$$\|A^{-1}\|^2 = \max_{x\neq} \frac{\|x\|^2}{\|Ax\|^2} \tag{40}$$

To show that A is invertible we use the Rank-Nullity Theorem

$$n = \operatorname{rg}(A) + \dim \operatorname{Im}(A) \tag{41}$$
$$\Rightarrow n - \dim \operatorname{Im}(A) = rg(A). \tag{42}$$

If A is invertible it has maximal rank that means $\dim \operatorname{Im}(A) = 0$. Let's look what the image of A is

$$Ax = 0 \Rightarrow \|Ax\|_2^2 = \|x\|_2^2 + \|Bx\|_2^2 = 0 \tag{43}$$

holds only for $x = 0$, thereby $\dim \operatorname{Im}(A) = 0$ and the rank is maximal. Now let $A^{-1}y = x$ then

$$\frac{\|A^{-1}y\|_2}{\|y\|_2} = \frac{\|x\|_2}{\|Ax\|_2} \tag{44}$$
$$\Rightarrow \|A^{-1}\|_2 = \max_{x\neq 0} \frac{\|x\|_2}{\|Ax\|_2}. \tag{45}$$

1.4.4

Next we show that $\|A^{-1}\|_2 \leq 1$.

$$\|A^{-1}\|_2 = \max_{x\neq 0} \frac{\|x\|_2}{\|Ax\|_2} = \frac{1}{\sqrt{1+\|B\|_2^2}} \leq 1 \tag{46}$$

1.5 Problem 5

We take A and B from last exercise.

1.5.1

We show that for $k \in \{1, \ldots, m\}$ and $\mathcal{W} \subset \mathbb{R}^n$ with $\dim \mathcal{W} = k$ spanned by $w_1, \ldots, w_k \in \mathbb{R}^n$. We show that if $x \in \mathcal{W}$ is such that

$$\langle Ax, w\rangle = \langle b, w\rangle \qquad \forall w \in \mathcal{W}. \tag{47}$$

then $\|x\|_2 \leq \|b\|_2$. We can choose $b = x$ then

$$\langle Ax, x\rangle = \|x\|_2^2 = \langle b, x\rangle \leq \|b\|_2\|x\|_2. \tag{48}$$

$$\Rightarrow \|x\|_2 \leq \|b\|_2 \tag{49}$$

1.5.2

Next we show that for $x := \sum_j c_j w_j$ and

$$\sum_j c_j \langle Aw_j, w_i\rangle = \langle b, w_i\rangle \tag{50}$$

for $i = 1, \ldots, k$, we have a unique solution for $x \in \mathcal{W}$. We do this by showing that every solution is the 0 solution $b = 0$

$$\langle 0, w\rangle = 0 = \langle Ax, x\rangle = \|x\|_2^2. \tag{51}$$

1.5.3

Lastly for $x^* := A^{-1}b$ we show and inequality relation

$$\|x^* - x\|_2 \leq \|A\|_2 \min_{x\in\mathcal{W}} \|x^* - w\|_2. \tag{52}$$

(We take the minimum $w = x$ and calculate)

$$\begin{aligned}
\|x^* - x\|_2^2 &= \langle A(x^* - x), x^* - x\rangle && (53)\\
&= \langle A(x^* - x), x^* - x + w - w\rangle && (54)\\
&= \langle A(x^* - x), x^* - w\rangle + \langle A(x^* - x), w - x\rangle && (55)\\
&= \langle A(x^* - x), x^* - w\rangle + \underbrace{\langle A(x^* - x), w - x\rangle}_{=0} && (56)\\
\leq \|A\|_2\|x^* - x\|_2\|x^* - w\|. & && (57)
\end{aligned}$$

When we take the minimum and divide by $\|x^* - x\|_2$ we get

$$\|x^* - x\| \leq \|A\|_2 \min_{x\in\mathcal{W}} \|x^* - w\|_2. \tag{58}$$