University of Vienna

Seminar:
Mathematical/Computational Astro/Quantum Physics

Classifying Quantum Phases in 1D using MPS

Milutin Popovic

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Contents

1 Introduction

MPS or PEPS (in 2D) describe ground states of gapped local Hamiltonian of quantum many body systems. We will use this fact to generalize Landau's theory (to do: what is Landau's theory brief explanation, How do we describe these with MPS in particular). Here we consider only ground states that can be represented by MPS exactly. Two systems are in the same phase, if and only if the can be connected by a smooth path of local Hamiltonian's on the manifold of the parameters λ, where the local Hamiltonian's $h_i = h_i(\lambda)$ are all dependent on the parameters λ, intuitively this would look the figure below are all dependent on the parameters λ, intuitively this would look the figure below.

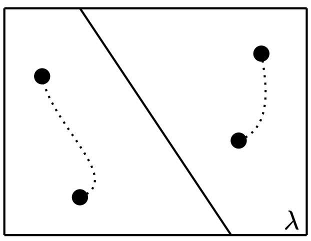


Figure 1: Two systems in the same phase are connected by a "smooth path of local Hamiltonians"

Where along such paths the physical properties of the sate smoothly change. The Hamiltonian needs to be "gapped", meaning that there is a clear separation of the ground state and the first excited state. The loss of a gap in the Hamiltonian leads mostly to discontinuous "behavior" of the ground state and affection of global properties of the system. If we introduce symmetries along the path of such Hamiltonian we can derive a refined classification of phases. Additionally if such symmetries exist we can generalize gapped quantum phases to systems with symmetry breaching!

2 Matrix Product States (MPS)

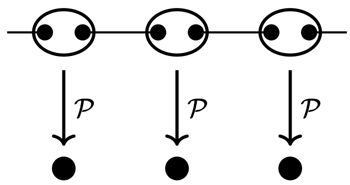
In the following we only consider translation-invariant systems on a finite chain of length N, with periodic boundary condition

Definition 1

Consider a spin chain $(\mathbb{C}^d)^{\otimes N}$. A translation-invariant MPS $|\mu[\mathcal{P}]\rangle$ of bond dimension D on $(\mathbb{C}^d)^{\otimes N}$ is constructed by placing maximally entangled pairs $|\omega_D\rangle$, as

$$|\omega_D\rangle := \sum_{n=1}^{D} |i, i\rangle \tag{1}$$

between adjacent sites and applying a linear map $\mathcal{P} : \mathbb{C}^D \otimes \mathbb{C}^D \to \mathbb{C}^d$. In graphical notation it would represent the figure below



$$|\mu[\mathcal{P}]\rangle := \mathcal{P}^{\otimes N} |\omega_D\rangle^{\otimes N} \tag{2}$$

We note that the MPS as defined above is robust under blocking sites, we are essentially blocking k-sites into one "super"-site of dimension d^k, which gives a new MPS with the same bond dimension in the lines of the projector (which is not a projection but a simple linear map)

$$\mathcal{P}' = \mathcal{P}^{\otimes k} |\omega_D\rangle^{\otimes(k-1)}. \tag{3}$$

By this blocking and using of the gauge degrees of freedom (including the variability of D) any MPS which is well defined in the Thermodynamic limit ($\beta \to 0$) can be brought into a so called **Standard form**, where the linear map $\mathcal{P}$ is supported on a block-diagonal space, i.e diagonalisation of the

$$\ker(\mathcal{P})^{\perp} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{\mathcal{A}}, \tag{4}$$

where

$$\mathcal{H}_\alpha = \operatorname{span}\{|i, j\rangle : \zeta_{\alpha-1} < i, j \leq \zeta_\alpha\}, \tag{5}$$

for $0 = \zeta_0 < \cdots < \zeta_{\mathcal{A}} = D$, and gives the partitioning $1, \ldots, D$ for $D_i = \zeta_i - \zeta_{i-1}$. The case of $\mathcal{A} = 1$ we have an injective map $\mathcal{P}$. The $\mathcal{A} > 1$ is the non-injective case.

All in all we assume that $\mathcal{P}$ is **surjective**, which is backed by the restriction of the state space $\mathbb{C}^d$ to the image of $\mathcal{P}$ (by definition).

2.1 Parent Hamiltonian

Given an MPS in the standard form, we can construct local, translation-invariant **parent Hamiltonians**, which have the given MPS as the **ground state**

$$H = \sum_{i=1}^{N} h(i, i+1). \tag{6}$$

The local terms $h(i, i+1) \geq 0$ act on one MPS object $(i, i+1)$ mapped by $\mathcal{P}$. The kernels of these local terms support the reduced density operator of the corresponding MPS, that is the kernel can be written as

$$\ker(h(i, i+1)) = (\mathcal{P} \otimes \mathcal{P})(\mathbb{C}^D \otimes |\omega\rangle \otimes \mathbb{C}^D). \tag{7}$$

Note that by the definition we have first that $H \geq 0$, and that $H\,|\mu[\mathcal{P}]\rangle = 0$, because the system $|\mu[\mathcal{P}]\rangle$ is the ground state of H.

To summarize, given a matrix product state (MPS) there exists a unique gapped local parent Hamiltonian, where the given MPS is in the groundstate (Perez-Garcia et al. 2007). Also, backed up by the fact that the groundstate of any one dimensional, gapped Hamiltonian can be well approximated by an MPS (proven by Hastings 2007).

2.2 Definition of quantum phases

We arrive at the definition of quantum phases, where we initially pose a question whether two systems are in the same phase. Two systems are in the same phase if they can be connected by a continuous path of gapped local Hamiltonians

2.2.1 Phases without symmetries

Let H_1, H_2 be a family of translation-invariant gapped local Hamiltonians on a ring (i.e. periodic boundary conditions). We say that H_1 and H_2 are in the same phase, if and only if exists an finite k, when blocking k sites both H_1 and H_2 are two local and can be written as

$$H_p = \sum_{i=1}^{N} h_p(i, i+1) \qquad p = 0, 1. \tag{8}$$

Additionally to this there exists a translation-invariant path of local gapped Hamiltonians

$$H_\gamma = \sum_{i=1}^{N} h_\gamma(i, i+1) \qquad \gamma \in [0, 1], \tag{9}$$

where h_γ is acting locally with the following properties

- $h_0 = h_{\gamma=0}$; $h_1 = h_{\gamma=1}$
- $\|h\|_{op} \leq 1$
- h_γ is continuous w.r.t. $\gamma \in [0, 1]$
- H_γ has a spectral gap above the ground state manifold, bounded below by some constant $\Delta > 0$ independent of N and γ.

We can say that H_0 and H_1 are in the same phase if they are connected by a local, bound-strength, continuous and gapped path, which applies to both Hamiltonians with unique and degenerate ground states.

2.2.2 Phases with symmetries

Let H_p, with $p \in \{0, 1\}$ be a Hamiltonian acting on the space $\mathcal{H}_p^{\otimes N}$ where $\mathcal{H}_p = \mathbb{C}^{d_p}$ and U_g^p be a linear unitary representation of some group $G \ni g$ of $\mathcal{H}_p$. Now, U_g is a symmetry of a family of local gapped Hamiltonians H_p, if

$$[H_p, (U_g^p)^{\otimes N}] = 0 \qquad \forall g \in G, \tag{10}$$

where U_g^p is a strictly one dimensional representation of the group G as

$$U_g^p \leftrightarrow e^{i\phi_g^p} U_g^p. \tag{11}$$

We can say that H_1 and H_2 are in the same phase under symmetry G, if there exists a phase gauge of U_g^0 and U_g^1 and a representation

$$U = U_g^0 \oplus U_g^1 \oplus U_g^\alpha \qquad \alpha \in (0,1) \tag{12}$$

on the Hilbertspace $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_\alpha$ with the properties of the previous section, such that

$$[H_\gamma, U_g^{\otimes N}] = 0 \tag{13}$$

and H_p is supported on $\mathcal{H}_p$ for $p = 0, 1$ respectively.

2.2.3 Robust definition of Phases

It is usually required for a phase to be **robust**, meaning that the phase is an open set in the space of allowed Hamiltonians. For all Hamiltonians 6 there exists and $\varepsilon > 0$, such that

$$H = \sum_{i=1}^{N} (h(i, i+1) + \varepsilon k(i, i+1)) \tag{14}$$

is in the same phase for any bound-strength $k(i, i+1)$ which obeys the symmetries of the system.

2.2.4 Restriction to parent Hamiltonians

Indeed we want a classification of phases of gapped local Hamiltonians with an exact MPS ground state. We are in luck because for every MPS we can find such a Hamiltonian, the parent Hamiltonian which is sufficient enough to classify the phases.

For two gapped Hamiltonians H, H' with some ground state subspace, the interpolating path

$$\gamma H + (1-\gamma) H' \tag{15}$$

has all the desired properties and it is gapped. Indeed all parent Hamiltonians for a given MPS are interchangeable!

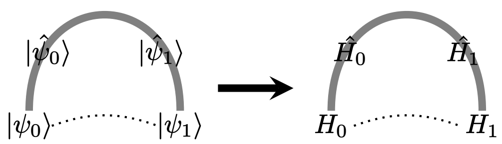


Figure 2: Interchangeability of MPS and parent Hamiltonians

2.3 The isometric Form

2.3.1 Reduction to a standard form

Given two MPS $|\mu[\mathcal{P}_p]\rangle$ with $p = 0, 1$ together with their nearest neighbor parent Hamiltonians H_p. Our goal is to see weather H_1 and H_2 are in the same phase. This is achieved by interpolating $\mathcal{P}_0$ and $\mathcal{P}_1$ along $\mathcal{P}_\gamma$, such that the result is the path H_γ in the space of parent Hamiltonians satisfying all the requirements (continuity and gap).

2.3.2 The isometric form

The isometric form of a MPS captures the essential entanglement, long range properties of the sate and forms a fixed point of a renormalization procedure. Given an MPS state $|\mu[\mathcal{P}]\rangle$ we decompose $\mathcal{P}$ by the **Polar-decomposition** of $\mathcal{P}|_{(\ker \mathcal{P})^\perp}$ as

$$\mathcal{P} = QW, \tag{16}$$

where $WW^\dagger = \mathbb{1}$ and $Q > 0$. And w.l.o.g. we assume $0 < Q \leq \mathbb{1}$ which can be achieved by rescaling of $\mathcal{P}$. The isometry form of $|\mu[\mathcal{P}]\rangle$ is $|\mu[W]\rangle$, where the MPS described by W is the isometric part of the tensor $\mathcal{P}$. To see that $|\mu[\mathcal{P}]\rangle$ and $|\mu[W]\rangle$ are in the same phase, we essentially define an interpolating path in terns of Q_γ

$$\mathcal{P}_\gamma = Q_\gamma W \quad \text{where} \quad Q_\gamma = \gamma Q + (1-\gamma)\mathbb{1}, \tag{17}$$

for $\gamma \in [0,1]$. No consider the parent Hamiltonian of $|\mu[\mathcal{P}_0]\rangle$

$$H_0 = \sum_{i=1}^{N} h_0(i, i+1) \tag{18}$$

where h_0 is a projector and we define $\Lambda_\gamma = (Q_\gamma^{-1})^{\otimes 2}$ for a γ-deformed Hamiltonian

$$H_\gamma = \sum_{i=1}^{N} h_\gamma(i, i+1) \quad \text{where} \quad h_\gamma = \Lambda_\gamma h_0 \Lambda_\gamma \geq 0. \tag{19}$$

Now we have that $|\mu[\mathcal{P}_0]\rangle = 0$ is equivalent to $|\mu[\mathcal{P}_\gamma]\rangle = 0$, i.e. H_γ is a parent Hamiltonian of $|\mu[\mathcal{P}_\gamma]\rangle$. All we need to show now is that H_γ is uniformly gapped, that there exists a constant $\Delta > 0$ which H_γ by bellow independent of γ and N. By this we would have that the whole set of $|\mu[\mathcal{P}_\gamma]\rangle$ for $\gamma \in [0,1]$ are indeed in the same phase.

Additional observation is that the lower bound of the gapped parent Hamiltonians is bound by correlation length ξ of the gap H_γ, restricted to ξ sites and since both depend smoothly, positive definite on γ and $\xi \to 0$ as $\gamma \to 0$ we have a uniform lower bound on the gap.