Applied Analysis Problems

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1 Sheet 3

1.1 Problem 8

Let us look at functions $f : \mathcal{D} \mapsto \mathbb{R}$ that show boundary layer behavior at the following manifolds.

The **first** for $\mathcal{D} = \mathbb{R}^2$ and $S = \{0\}$ we have a function e.g.

$$f_\varepsilon(x, y) = e^{-\frac{x}{\varepsilon}} + y, \tag{1}$$

with the reduced equation

$$\lim_{\varepsilon \to 0} f_\varepsilon(x, y) = \begin{cases} y & x > 0 \\ 1 + y & x = 0 \end{cases} \tag{2}$$

The **second** example is $\mathcal{D} = \mathbb{R}^n$ and $S = \{|x| = 1\}$.

$$f_\varepsilon(x_1, \dots, x_n) = \tanh\left(\frac{|x| - 1}{\varepsilon}\right), \tag{3}$$

with the reduced equation

$$\lim_{\varepsilon \to 0} f_\varepsilon(x_1, \dots, x_n) = \begin{cases} -1 & |x| < 0 \\ 1 & |x| > 0 \end{cases} \tag{4}$$

The **third** example is $\mathcal{D} = \mathbb{R}^3$ and $S = \{x_1 = 1\}$

$$f_\varepsilon(x_1, x_2, x_3) = \tanh\left(\frac{x_1 - 1}{\varepsilon}\right) + x_2 x_3 \tag{5}$$

with the reduced equation

$$\lim_{\varepsilon \to 0} f_\varepsilon(x_1, x_2, x_3) = \begin{cases} -1 + x_2 x_3 & x_1 < 0 \\ 1 + x_2 x_3 & x_1 > 0 \end{cases} \tag{6}$$

1.2 Problem 9

Consider a linear BVP

$$Lu := -\varepsilon u'' + b(x)u' + c(x)u = f(x), \tag{7}$$
$$u(0) = u(1) = 0, \tag{8}$$

for $0 < \varepsilon \ll \varepsilon_0$ and $b, c, f \in C([0,1])$ with the conditions

$$c(x) \geq 0, \qquad b(x) \geq \beta > 0 \qquad x \in [0,1] \tag{9}$$

We are to show that for all $x \in [0,1)$ the reduced solution u_0 of the above BVP satisfies

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = u_0(x)))), \tag{10}$$

where the reduced solution u_0 is the solution to the following differential equation

$$b(x)u' + c(x)u = f(x), \quad u(0) = 0. \tag{11}$$

The hint was given: Set

$$w_1(x) = e^{\beta x} \quad w_2(x) = e^{-\beta \frac{1-x}{\varepsilon}}, \tag{12}$$

such that $Lw_1 \geq \gamma > 0$ for some suitable γ and $Lw_2 \geq 0$. Then for

$$v = \pm(u_\varepsilon - u_0), \qquad w = A\varepsilon w_1 + B\varepsilon w_2, \tag{13}$$

for some suitable A, B. The following comparison principal is applicable: IF

$$Lv(x) \leq Lw(x) \quad \forall x \in (0,1) \tag{14}$$
$$v(0) \leq w(0) \tag{15}$$
$$v(1) \leq w(1) \tag{16}$$
$$\tag{17}$$

then

$$\implies v(x) \leq w(x) \quad \forall x \in (0,1) \tag{18}$$

which holds for $u, v \in C^2((0,1)) \cap C([0,1])$. Thus a boundary layer is possible only at $x = 1$. On the other hand, for $b(x) \leq \beta < 0$ it follows that the boundary layer is possible only at $x = 0$.

We shall go through the chronological order of the conditions14, 15, 16 and check them. So for 14 we have that

$$Lw(x) = A\varepsilon Lw_1(x) + BLw_2(x) \tag{19}$$
$$\geq A\varepsilon Lw_1(x) = A\varepsilon e^{\beta x}\left(-\varepsilon\beta^2 - b(x)\beta + c(x)\right) \tag{20}$$
$$\geq A\varepsilon\beta e^{\beta x}(1 - \varepsilon) \tag{21}$$
$$\geq \varepsilon A\beta^2 e^{\beta}(1 - \varepsilon) = \gamma > 0 \tag{22}$$

And obviously

$$Lv(x) \leq 0, \tag{23}$$

by that we have that

$$Lv(x) \leq \gamma \leq Lw(x). \tag{24}$$

For the condition 15 we have

$$w(0) = A\varepsilon w_1(0)Bw_2(0) = Be^{-\frac{\beta}{\varepsilon}}, \tag{25}$$
$$v(0) = \pm\left(u_\varepsilon(0) - u_0(0)\right) = 0. \tag{26}$$

By the simple choice $B \geq 0$ we satisfy the condition

$$v(0) \leq w(0). \tag{27}$$

Now for the last condition 16 we have

$$w(1) = A\varepsilon e^{\beta} + B \geq A\varepsilon e^{\beta}, \tag{28}$$
$$v(1) = \mp u_0(1) = 0. \tag{29}$$

And choose $A = \frac{\pm u(1)}{\varepsilon} e^{-\beta}$, which satisfies the last condition

$$v(1) \leq w(1). \tag{30}$$

Thereby we have

$$v(x) \leq w(x) \tag{31}$$
$$\Rightarrow \lim_{\varepsilon \to 0} v(x) \leq \lim_{\varepsilon \to 0} w(x) = 0 \tag{32}$$
$$\Rightarrow \lim_{\varepsilon \to 0} v(x) = 0 \tag{33}$$

uniformly on $(0, 1)$.

1.3 Problem 10

Consider the following BVP

$$-\varepsilon u'' + (1 + x)u' + u = 2, \qquad u(0) = u(1) - 0, \tag{34}$$

for $0 < \varepsilon \ll 1$. **Where can this problem have a boundary layer?** To answer this question we need to look at the reduced problem

$$-(1 + x)u' + u = 2. \tag{35}$$

The solution to the equation is

$$\bar{u}(x) = 2 + A(x + 1). \tag{36}$$

According to the boundary conditions it is unclear what the value of the constant is, according to $\bar{u}(0) = 0$ we get $A = -2$ or according to $\bar{u}(1) = 0$ we get $A = -1$. Ultimately this means that there exists a boundary layer near $x = 1$ or $x = 0$. We choose $x = 0$ and according to this the local variable $\xi = x\varepsilon^{-\alpha}$ $(x = \xi\varepsilon^{-\alpha})$. The derivatives of u are calculated using the chain rule

$$\frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx} = \varepsilon^{-\alpha}\dot{u} \tag{37}$$
$$\frac{d^2u}{dx^2} = \varepsilon^{-\alpha}\frac{d^2u}{d\xi^2}\frac{d\xi}{dx} = \varepsilon^{-2\alpha}\ddot{u}. \tag{38}$$

The BVP transforms as follows

$$-\varepsilon^{1-\alpha}\ddot{u} - \dot{u} + \varepsilon(u - \xi\dot{u} - 2) = \begin{cases} -\ddot{u} - \dot{u} = 0 & \alpha = 1 \\ -\dot{u} = 0 & 0 < \alpha < 1 \end{cases} \tag{39}$$

Choosing $\alpha = 1$ for a reasonable solution

$$\hat{u}(\xi) = Be^{-\xi}, \tag{40}$$

which converges in the local limit (!). Thereby we have a asymptotic representation up to the degree of ε

$$u_{\varepsilon}(x) = \bar{u}(x) + \hat{u}(\psi) + O(\varepsilon) \tag{41}$$
$$= 2 + A(1 + x) + Be^{-\frac{x}{\varepsilon}} + O(\varepsilon) \tag{42}$$

And by the boundary conditions

$$u_\varepsilon(0) = 2 + A + B = 0, \qquad u_\varepsilon(1) = 2 + 2A + B = 0, \tag{43}$$

we get that the constants are

$$A = -4, \qquad B = 2. \tag{44}$$

The asymptotic representation is thereby

$$u_\varepsilon(x) = 2 - 4(1 + x) + 2e^{-\frac{x}{\varepsilon}} + O(\varepsilon) \tag{45}$$