Introductory Seminar on Dynamical System and Nonlinear Differential Equations (University of Vienna, March–April 2023)

Due **March 14**: exercises **1–7**
Due **March 21**: exercises **8–10**
Due **March 28**: exercises **11–16**
Due **April 25**: exercises **17–20**
Due **May 2**: exercises **21–23**

1 Find the stable, center, and unstable subspaces for $\dot{x} = Ax$ with

(a) $A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$

(c) $A = \begin{bmatrix} -1 & -3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

In the two-dimensional cases, also sketch the phase portrait.

2 Take $\dot{x}(t) = Ax(t)$, $x(0) = p \in \mathbb{R}^n$ with a nonsingular $A \in \mathbb{R}^{n\times n}$. Prove the following statements.

(a) If $p \in E^s \setminus \{0\}$ then $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to-\infty} |x(t)| = \infty$.

(b) If $p \in E^u \setminus \{0\}$ then $\lim_{t\to-\infty} x(t) = 0$ and $\lim_{t\to\infty} |x(t)| = \infty$.

(c) If A is semisimple and $p \in E^c \setminus \{0\}$ then there exist $0 < m < M$ s.t. $m \le x(t) \le M$ for all $t \in \mathbb{R}$. (A matrix is *semisimple* if the multiplicity of each eigenvalue in the minimal polynomial is one.)

(d) If A is not semisimple then there exists a $p \in E^c \setminus \{0\}$ such that $\lim_{t\to\pm\infty} |x(t)| = \infty$.

(e) If $E^s \neq \{0\}$, $E^u \neq \{0\}$, and $p \in E^s \oplus E^u \setminus (E^s \cup E^u)$ then $\lim_{t\to\pm\infty} |x(t)| = \infty$.

(f) If $E^u \neq \{0\}$, $E^c \neq \{0\}$, and $p \in E^u \oplus E^c \setminus (E^u \cup E^c)$ then $\lim_{t\to\infty} |x(t)| = \infty$ and $\nexists \lim_{t\to-\infty} x(t)$.

(g) If $E^s \neq \{0\}$, $E^c \neq \{0\}$, and $p \in E^s \oplus E^c \setminus (E^s \cup E^c)$ then $\lim_{t\to-\infty} |x(t)| = \infty$ and $\nexists \lim_{t\to\infty} x(t)$.

3 Which of the following statements are true?

(a) $E^u = \{p \in \mathbb{R}^n \colon \lim_{t\to\infty} |e^{At}p| = \infty\}$

(b) $E^u = \{p \in \mathbb{R}^n \colon \lim_{t\to-\infty} e^{At}p = 0\}$

(c) For every $p \in E^u$ we have $\lim_{t\to\infty} |e^{At}p| = \infty$, but the converse is not true in general.

(d) $E^c = \{p \in \mathbb{R}^n \colon |e^{At}p| = |p| \text{ for all } t \in \mathbb{R}\}$

(e) E^c is the set of all periodic orbits

(f) Every periodic orbit lies in E^c.

(g) $E^c = \{p \in \mathbb{R}^n \colon \{e^{At}p \colon t \ge 0\} \text{ is bounded}\}$

(h) $E^c = \{p \in \mathbb{R}^n \colon \{e^{At}p \colon t \in \mathbb{R}\} \text{ is bounded}\}$

(i) $E^c = \{p \in \mathbb{R}^n \colon \text{neither } \lim_{t\to\infty} e^{At}p = 0 \text{ nor } \lim_{t\to-\infty} e^{At}p = 0\}$

4 In the four-dimensional Euclidean space of the 2×2 matrices, what is the dimension of the following manifolds? (The *algebraic multiplicity* of an eigenvalue λ is its multiplicity in the characteristic polynomial. The *geometric multiplicity* of an eigenvalue λ is the dimension of the subspace spanned by the eigenvectors associated with λ (or equivalently, the number of Jordan blocks associated with λ).)

(a) Eigenvalues are $\pm\omega i$ with $\omega \neq 0$.

(b) Zero is an eigenvalue with algebraic multiplicity two.

(c) Zero is an eigenvalue with algebraic multiplicity two and geometric multiplicity one.

(d) Zero is an eigenvalue with algebraic multiplicity two and geometric multiplicity two.

Sketch the phase portraits for (a), (c), (d).

5 For $A \in \mathbb{R}^{2\times 2}$, express in terms of $\det A$ and $\operatorname{tr} A$ the fact that $\sigma(A) = \{\omega i, -\omega i\}$ for some $\omega > 0$.

6 For $A \in \mathbb{R}^{3\times 3}$, express in terms of $\det A$, M, and $\operatorname{tr} A$ the fact that $\sigma(A) = \{\varrho, \omega i, -\omega i\}$ for some $\varrho < 0$ and $\omega > 0$. (Here, M is the sum of the order-two principal minors of A.)

7 Express in terms of b_0, b_1, b_2, b_3 that each root of $x^4 + b_3x^3 + b_2x^2 + b_1x + b_0$ has negative real part.

8 Compute $\int_0^\infty e^{(A+A^\top)t}\mathrm{d}t$ for $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$.

9 For $A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, find a homeomorphism $h \colon \mathbb{R}^2 \to \mathbb{R}^2$ for which $h(e^{At}p) = e^{Bt}h(p)$ for all $p \in \mathbb{R}^2$. Write up h^{-1}, too.

10 Prove the following. Let $\dot{x} = f(x)$ and $\dot{y} = g(y)$ be scalar ODEs with finite number of equilibria and solutions defined for all time. Then their flows are topologically conjugate if and only if they have the same number of equilibria, and the direction of the flow between the equilibria is the same (or opposite).

11 Consider the differential equation

$$\begin{aligned} \dot{r} &= r(1-r), \\ \dot{\varphi} &= r(1-\cos\varphi) \end{aligned}$$

given in polar coordinates ($r \geq 0$, $0 \leq \varphi < 2\pi$). Find $\dot{x}$ and $\dot{y}$, where $x = r\cos\varphi$ and $y = r\sin\varphi$. Further, sketch the phase portrait in the (x, y)-plane.

12 Find $a, b, c > 0$ such that $V(x, y, z) = ax^2 + by^2 + cz^2$ is a strict Lyapunov function for

$$\begin{aligned} \dot{x} &= -2y + yz - x^3, \\ \dot{y} &= x - xz - y^3, \\ \dot{z} &= xy - z^3. \end{aligned}$$

13 Using the Lyapunov function $V(x, y, z) = x^2 + y^2 + z^2$, show that the origin is globally asymtotically stable for

$$\begin{aligned} \dot{x} &= -y - xy^2 + z^2 - x^3, \\ \dot{y} &= x + z^3 - y^3, \\ \dot{z} &= -xz - zx^2 - yz^2 - z^5. \end{aligned}$$

Further, linearize the equation at the origin, and analyse that.

14 Use appropriate Lyapunov functions to determine the stability of the equilibrium points of the following systems:

(a) $\begin{cases} \dot{x} = -x + y + xy \\ \dot{y} = x - y - x^2 - y^3 \end{cases}$

(b) $\begin{cases} \dot{x} = -x - 2y + xy^2 \\ \dot{y} = 3x - 3y + y^3 \end{cases}$

15 Let $q\colon \mathbb{R} \to \mathbb{R}$ be a continuous function for which $xq(x) > 0$ when $x \neq 0$. Using $V(x,y) = \frac{1}{2}y^2 + \int_0^x q(\alpha)\mathrm{d}\alpha$, show that the origin of

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -q(x) \end{aligned}$$

is Lyapunov stable. How is this related to the pendulum?

16 Classify the equilibria (as sinks, sources, or saddles) for the systems below.

(a) $\begin{cases} \dot{x} = x - xy \\ \dot{y} = y - x^2 \end{cases}$

(b) $\begin{cases} \dot{x} = -4y + 2xy - 8 \\ \dot{y} = 4y^2 - x^2 \end{cases}$

(c) $\begin{cases} \dot{x} = 2x - 2xy \\ \dot{y} = 2y - x^2 + y^2 \end{cases}$

(d) $\begin{cases} \dot{x} = -x \\ \dot{y} = -y + x^2 \\ \dot{z} = z + x^2 \end{cases}$

(e) $\begin{cases} \dot{x} = y - x \\ \dot{y} = \alpha x - y - xz \\ \dot{z} = xy - z \end{cases}$

17 Show that

$$\begin{aligned} \dot{r} &= r(1 - r^2) + \mu r \cos\varphi, \\ \dot{\varphi} &= 1 \end{aligned}$$

(given in polar coordinates) has a limit cycle, where $0 < \mu < 1$ is a parameter.

18 Find $a, b > 0$ such that

$$\begin{aligned} \dot{x} &= -x + ay + x^2 y, \\ \dot{y} &= b - ay - x^2 y \end{aligned}$$

has a limit cycle in the nonnegative quadrant.

19 Show that

$$\begin{aligned} \dot{x} &= x - y - x^3, \\ \dot{y} &= x + y - y^3 \end{aligned}$$

has a limit cycle.

20 In each of the following cases, show that there is no periodic solution.

(a) $\begin{cases} \dot{x} = y \\ \dot{y} = -\sin x - \delta y \end{cases}$ with $\delta > 0$

(b) $\begin{cases} \dot{x} = x(1-x)(x-y) \\ \dot{y} = y(1-y)(2x-1) \end{cases}$ in $(0,1)^2$

(c) $\begin{cases} \dot{x} = x(r - ax + by) \\ \dot{y} = y(s + cx - dy) \end{cases}$ in $\mathbb{R}^2_+$ with $a, d > 0$

(d) $\ddot{x} + p(x)\dot{x} + q(x) = 0$ with $p(x) > 0$ for all $x \in \mathbb{R}$

21 Consider the equation

$$\begin{aligned} \dot{x} &= -x + x^2 + 3xy^2, \\ \dot{y} &= y - 2xy - y^3. \end{aligned}$$

Is it reversible (w.r.t. some line)? Is it Hamiltonian? If so, find the Hamiltonian function H. Find all equilibria. Does any of the equilibria a center?

22 Consider the equation

$$\dot{x} = y - x \sin y,$$
$$\dot{y} = -\cos y.$$

Is it reversible (w.r.t. some line)? Is it Hamiltonian? If so, find the Hamiltonian function H. Find all equilibria. Does any of the equilibria a center

23 Find the (global) stable manifold at the origin for

$$\dot{x} = -x,$$
$$\dot{y} = y - x^3.$$

Introductory Seminar on Dynamical System and Nonlinear Differential Equations
(first test, April 18, 2023)

1 Find the stable, center, and unstable subspaces for the linear equation

$$\begin{aligned}\dot{x} &= x + z,\\ \dot{y} &= by,\\ \dot{z} &= -2x - z,\end{aligned}$$

where $b \in \mathbb{R}$ is a parameter.

2 For which $r, a \in \mathbb{R}$ does the Lotka–Volterra equation

$$\begin{aligned}\dot{x} &= x(r + ax - y),\\ \dot{y} &= y(x - 1)\end{aligned}$$

have a positive equilibrium? Can a positive equilibrium of this system be a saddle? Can a positive equilibrium of this system be asymptotically stable?

3 Find $b > 0$ such that $V(x, y, z) = x^2 + by^2 + z^2$ is a strict Lyapunov function for the origin for the equation

$$\begin{aligned}\dot{x} &= -2y + yz - x^3,\\ \dot{y} &= x - xz - y^3,\\ \dot{z} &= xy - z^3.\end{aligned}$$