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Applied Analysis Problems

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1 Sheet 1

1.1 Problem 1

Consider the following two matrices  $A, L_1 \in \mathbb{R}^{4 \times 4}$  defined in as

$$A := \begin{pmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{pmatrix}, \quad L_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{pmatrix}, \quad (1)$$

for  $x, y, z \in \mathbb{R}$ .

To show that  $A$  is invertible, we need to show it has maximal rank, that is  $\text{rank}(A) = 4$ . We can do this by doing Gaussian elimination steps until  $A$  is of the form of a upper triangular matrix

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} \begin{matrix} -2 \cdot I \\ -4 \cdot I \\ -3 \cdot I \end{matrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix} \begin{matrix} -3 \cdot II \\ -4 \cdot II \end{matrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{matrix} -1 \cdot III \end{matrix} \quad (2)$$

$$\rightarrow \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{\det} 8. \quad (3)$$

Next we will determine  $x, y$  and  $z$ , s.t.  $(L_1 A)_{.,1} = (2 \ 0 \ 0 \ 0)$  by solving the linear system

$$L_1 A = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 2x+4 & x+3 & x+3 & 1 \\ 2y+8 & y+7 & y+9 & 5 \\ 2z+6 & z+7 & z+9 & 8 \end{pmatrix}, \quad (4)$$

we get  $x = -2$ ,  $y = -4$  and  $z = -3$  and thereby

$$L_1 A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 3 & 5 & 8 \end{pmatrix}. \quad (5)$$

In an analogous structure we may define  $L_2, L_3 \in \mathbb{R}^{4 \times 4}$ , s.t.

$$L_3 L_2 L_1 A = U, \quad (6)$$

where  $U$  is an upper triangular matrix. We may notice that this is an LU decompositions of a matrix and can be determined by the inversion of a single step of Gaussian elimination. By that the three steps needed to achieve the upper triangular by Gaussian elimination are introduced in 2 and 3, that is also why  $-2, -4, -3$  aligns up with  $L_1$ . To summarize, by looking at 2 and 3 the matrices  $L_2, L_3$  are the following

$$L_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & -4 & 0 & 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}. \quad (7)$$

And by no calculation we know that  $U$  needs to be the upper triangular found in 3, i.e.

$$L_3 L_2 L_1 A = U = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}. \quad (8)$$

We have indeed preformed an LU decomposition of  $A$ , which is indeed useful for solving a linear system of the form

$$Ax = b \quad \text{and} \quad L_3 L_2 L_1 A = U, \quad (9)$$

$$(L_3 L_2 L_1 A)x = Ux = L_3 L_2 L_1 b = y \quad (10)$$

$$\Rightarrow Ux = y, \quad (11)$$

where the system is recursively solvable as  $U$  is the upper triangular and no additional transformation steps are required only "plug and play".

## 1.2 Problem 2

Next we consider  $A_\varepsilon \in \mathbb{R}^{2 \times 2}$  defined as

$$A_\varepsilon := \begin{pmatrix} \varepsilon & 1 \\ 1 & 1 \end{pmatrix}, \quad (12)$$

for  $\varepsilon > 0$ . The inverse of  $A_\varepsilon$  is

$$A_\varepsilon^{-1} = \frac{1}{\det(A_\varepsilon)} \text{adj}(A_\varepsilon) = \frac{1}{\varepsilon - 1} \begin{pmatrix} 1 & -1 \\ -1 & \varepsilon \end{pmatrix} \quad (13)$$

Now let  $\|x\|_\infty = \max\{|x_1|, |x_2|\}$  be the maximum norm of  $x \in \mathbb{R}^2$ , and  $\|A_\varepsilon\|_\infty$  the induced matrix norm of  $A_\varepsilon$ . We can show that

$$\lim_{\varepsilon \rightarrow 0} K(A_\varepsilon) = 4, \quad (14)$$

where  $K(A_\varepsilon) = \|A_\varepsilon\|_\infty \|A_\varepsilon^{-1}\|_\infty$  is the condition number of  $A_\varepsilon$ .

$$\|A_\varepsilon\|_\infty = \|(\varepsilon + 1 \quad 1 + 1)\|_\infty = 2 \quad (15)$$

$$\|A_\varepsilon^{-1}\|_\infty = \|(-\frac{2}{\varepsilon - 1} \quad 1)\|_\infty = \frac{2}{1 - \varepsilon}, \quad (16)$$

and thereby

$$\lim_{\varepsilon \rightarrow 0} K(A_\varepsilon) = \lim_{\varepsilon \rightarrow 0} 2 \cdot \frac{2}{1 - \varepsilon} = 4 \quad (17)$$

If we performed an LU decomposition of  $A_\varepsilon$  like in the first problem to get an upper diagonal the decomposition would be

$$LA_\varepsilon = \begin{pmatrix} 1 & 0 \\ -\frac{1}{\varepsilon} & 1 \end{pmatrix} \begin{pmatrix} \varepsilon & 1 \\ 1 & 1 \end{pmatrix} \quad (18)$$

$$= \begin{pmatrix} \varepsilon & 1 \\ 0 & 1 - \frac{1}{\varepsilon} \end{pmatrix} = U_\varepsilon, \quad (19)$$

with the inverse

$$U_\varepsilon^{-1} = \frac{1}{\varepsilon - 1} \begin{pmatrix} 1 - \frac{1}{\varepsilon} & -1 \\ 0 & \varepsilon \end{pmatrix}. \quad (20)$$

The condition number of the resulting upper triangular matrix  $U_\varepsilon$ ,  $K(U_\varepsilon)$  as  $\varepsilon \rightarrow 0$  is

$$\|U_\varepsilon\|_\infty = \left\| \begin{pmatrix} \varepsilon + 1 & | & 1 - \frac{1}{\varepsilon} \end{pmatrix} \right\|_\infty = \frac{1}{\varepsilon} - 1 \quad (21)$$

$$\|U_\varepsilon^{-1}\|_\infty = \left\| \begin{pmatrix} | & \frac{1-\frac{1}{\varepsilon}}{\varepsilon-1} & | & \frac{\varepsilon}{\varepsilon-1} \end{pmatrix} \right\|_\infty = \frac{1}{\varepsilon(\varepsilon - 1)} \quad (22)$$

$$\implies \lim_{\varepsilon \rightarrow 0} K(U_\varepsilon) = \lim_{\varepsilon \rightarrow 0} \frac{1 - \varepsilon}{\varepsilon} \frac{1}{\varepsilon(1 - \varepsilon)} = \infty. \quad (23)$$

But if we on the other hand considered a pivoting step in which we exchange the rows of  $A_\varepsilon$

$$PA_\varepsilon = A'_\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \varepsilon & 1 \end{pmatrix} \quad (24)$$

Then the P-LU decomposition is

$$L'A'_\varepsilon = \begin{pmatrix} 1 & 0 \\ -\varepsilon & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \varepsilon & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 - \varepsilon \end{pmatrix} = U'_\varepsilon, \quad (25)$$

with the inverse

$$(U'_\varepsilon)^{-1} = \frac{1}{1 - \varepsilon} \begin{pmatrix} 1 - \varepsilon & -1 \\ 0 & 1 \end{pmatrix}. \quad (26)$$

Then the condition number as  $\varepsilon \rightarrow 0$

$$\|U'_\varepsilon\|_\infty = \left\| \begin{pmatrix} 2 & 1 - \varepsilon \end{pmatrix} \right\|_\infty = 2 \quad (27)$$

$$\|(U'_\varepsilon)^{-1}\|_\infty = \left\| \begin{pmatrix} \frac{1-\varepsilon+1}{1-\varepsilon} & \frac{1}{1-\varepsilon} \end{pmatrix} \right\|_\infty = \frac{2 - \varepsilon}{1 - \varepsilon} \quad (28)$$

$$\implies \lim_{\varepsilon \rightarrow 0} K(U'_\varepsilon) = \lim_{\varepsilon \rightarrow 0} 2 \cdot \frac{2 - \varepsilon}{1 - \varepsilon} = 2 \cdot 2 = 4 \quad (29)$$

### 1.3 Problem 3

Let  $v \in \mathbb{R}^n$ ,  $n \in \mathbb{N}$  and  $v \neq 0$ . We define the Housholder matrix

$$H = \text{Id} - \frac{2}{\langle v, v \rangle} vv^T. \quad (30)$$

Indeed  $H$  is an orthogonal matrix, it satisfies  $HH^T = H^TH = \text{Id}$ .

$$HH^T = \left( \text{Id} - \frac{2}{\langle v, v \rangle} vv^T \right) \left( \text{Id} - \frac{2}{\langle v, v \rangle} vv^T \right)^T \quad (31)$$

$$= \left( \text{Id} - \frac{2}{\langle v, v \rangle} vv^T \right) \left( \text{Id} - \frac{2}{\langle v, v \rangle} (vv^T)^T \right) \quad (32)$$

$$= \left( \text{Id} - \frac{2}{\langle v, v \rangle} vv^T \right) \left( \text{Id} - \frac{2}{\langle v, v \rangle} vv^T \right) \quad (33)$$

$$= \text{Id} - \frac{4}{\langle v, v \rangle} vv^T + \frac{4}{\langle v, v \rangle^2} (vv^T)(vv^T) \quad (34)$$

$$= \text{Id} - \frac{4}{\langle v, v \rangle} vv^T + \frac{4}{\langle v, v \rangle} (vv^T) = \text{Id} \quad (35)$$

$$H^TH = \left( \text{Id} - \frac{2}{\langle v, v \rangle} vv^T \right) \left( \text{Id} - \frac{2}{\langle v, v \rangle} vv^T \right) \quad (36)$$

$$= \text{Id} \quad (37)$$

Let us look at the projection of some  $x \in \mathbb{R}^n$  in the  $v$  direction is given by

$$\frac{\langle v, x \rangle}{\langle v, v \rangle} v, \quad (38)$$

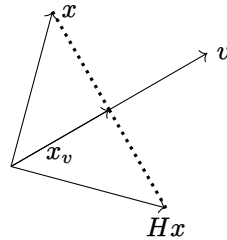
The projection of  $x$  in the orthogonal direction is

$$x - \frac{\langle v, x \rangle}{\langle v, v \rangle} v, \quad (39)$$

A reflection of  $x$  in  $v$  has  $-1$  times the projection onto  $v$ , that  $x$  has onto  $v$ , so the orthogonal projection of the reflection onto  $v$  is the orthogonal projection of  $x$  onto  $v$ , therefor the reflection in  $v$  of  $x$  is

$$x - 2 \frac{\langle v, x \rangle}{\langle v, v \rangle} v, \quad (40)$$

if we wanted to reflect  $x$  on the line spanned by  $v$  we would have to subtract the vector  $\frac{\langle v, x \rangle}{\langle v, v \rangle} v$  twice, graphically it would look like this



The Householder matrix acting on a vector  $x$ ,  $Hx$  is exactly the above case since vector multiplication is associative we have

$$Hx = x - \frac{2}{\langle v, v \rangle} vv^T x \quad (41)$$

$$= x - 2 \frac{\langle v, x \rangle}{\langle v, v \rangle} v \quad (42)$$

The condition number of an orthogonal matrix  $A$  in the  $\|\cdot\|_2$  induced norm is

$$K(A) = \|A\|_2 \|A^{-1}\|_2 = 1, \quad (43)$$

because the orthogonal matrix preserves distance, i.e.  $\|Ax\|_2 = \|x\|_2$  for all  $x$ . Also  $A^{-1} = A^T$  is orthogonal as well

$$\|A\|_2 = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sup_{x \neq 0} \frac{\|x\|_2}{\|x\|_2} = 1 \quad (44)$$