Applied Analysis, WS 2021 – Problem Sheet 1

P. 1 "Fall from high"

We examine the movement of a stone of mass m (say, 20 kg) which falls down (i.e. zero initial velocity: $\dot{x}(t = 0) = 0$) from a very large height h (say, 20 km) such that the gravitational force depends on the height in a significant way. We denote the height, i.e. the distance of the stone to the surface of the earth, as $x(t)$.

The equation of motion is

$$\ddot{x}(t) = -g \frac{R^2}{(x(t) + R)^2}.$$

R is the radius of the earth (approx. 6000 km), and g the gravitational acceleration on the surface of the earth (approx. $9,81\ m/s^2$).

We want to compute, say, the time T^* at which the stone hits the ground.

(a) Determine (all) possible non-dimensionalisations of this problem. Which of these are good, which bad ?

(b) Write down the reduced problem for each choice of scaling.
If possible, calculate the time until impact by the reduced problem.

(c) In the exact problem, is the time to reach the ground longer than if calculated from the reduced problem ?
(hint: examine the maximum/ minimum of the gravitation force)

(d) Calculate the velocity at impact by the solution of the reduced problem.
Do you get the same result when calculating from energy conservation?

(e) Compare this problem and it's scaling with the "vertical throw" presented in the lecture.

(f) **(*)** Give a (complete) list of modeling assumptions and simplifications.

(g) **(*)** Is it a good approximation to replace the attractive force of the earth by the attraction of the whole mass concentrated at the center ? (hint: calculate the "sum" of the attraction of all points of the ball exacted on the test point at the surface, i.e. the 3-dimensional integral).

(h) **(*)** Derive the equation of motion from Newton's second law.

P. 2 Scale the "van der Pol equation", which is a perturbation of the "oscillation equation":

$$LC\frac{d^2I}{dt^2} + (-g_1C + 3g_3CI^2)\frac{dI}{dt} = -I\,, \tag{1}$$

with initial conditions

$$I(0) = I_0, \qquad \frac{dI}{dt}(0) = 0 \tag{2}$$

$I(t)$ ... current at time t
C ... capacitance
L ... inductance
g_1, g_3 ... parameters

As basic units we use Ampere (A), the SI unit for electric current, and seconds (s) for time. The units of the parameters are:

Quantity	SI unit
LC	s^2
g_1C	s
g_3C	$s\,A^{-2}$

(a) What is the "oscillator equation" for $I(t)$ in (1) ?
Compute it's solution(s).

(b) Determine all possible non-dimensionalisations of the problem (1) (2).

(c) Discuss different possibilities to find reduced models for the case of a small non-dimensional parameter ε.

P. 3 Scale the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi + V\psi, \quad \psi(t=0) = \psi_0$$

($i^2 = -1$ complex unit, $\hbar$ ... Planck constant, $\psi = \psi(x,t)$, m ... mass, $V = V(x)$ ... "potential"; $[\hbar] = Js = kg\ m^2 s^{-1}$, $[V] = J = kg\ m^2 s^{-2}$) in the following cases:

(a) $V \equiv 0$, $x \in \mathbb{R}^3$

(b) $V \equiv 0$, $x \in [0,L]$, $t \in [0,T]$

(c) $V = V(x) = m\omega^2x^2$, ω ... frequency, $x \in \mathbb{R}$, "harmonic oscillator potential".

(Note: $[\psi] = m^{-d/2}$, where d is the space dimension considered (d=3 or d=2,1).