University of Vienna
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Introductory Seminar Advanced Numerical Analysis

Exercise sheet 1, due date: 07.03.2022

Exercise 1 (Gauss Algorithm and LU-decomposition).
Let matrices $A, L_1 \in \mathbb{R}^{4\times 4}$ be defined as

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbb{R}$.

(a) Show that A is invertible.

(b) Determine $x, y, z \in \mathbb{R}$ such that

$$L_1 A = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}$$

where "$*$" stand for some real number. Further, calculate the inverse of L_1.

(c) Define analogous matrices $L_2, L_3 \in \mathbb{R}^{4\times 4}$ in such a way that

$$L_3 L_2 L_1 A = U$$

and U is upper triangular.

(d) A consequence of exercise (c) is that we have performed the LU-decomposition of the matrix A. Describe how to solve the linear system $Ax = b$ with a given right-hand side $b \in \mathbb{R}^4$ based on the LU-decomposition of A.

Exercise 2 (Pivoting and condition number).
Let $\varepsilon > 0$ and let $A_\varepsilon \in \mathbb{R}^{2\times 2}$ be the matrix

$$A_\varepsilon := \begin{pmatrix} \varepsilon & 1 \\ 1 & 1 \end{pmatrix}$$

(a) Calculate the inverse A_ε^{-1} of A_ε.

(b) Let $\|x\|_\infty = \max\{|x_1|, |x_2|\}$ be the maximum norm of a vector $x = (x_1, x_2) \in \mathbb{R}^2$. Further, let $\|A_\varepsilon\|_\infty$ be the induced matrix norm of A_ε. Show that

$$\lim_{\varepsilon \to 0} K(A_\varepsilon) = 4$$

where $K(A_\varepsilon) = \|A_\varepsilon\|_\infty \|A_\varepsilon^{-1}\|_\infty$ is the *condition number* of A_ε.

(c) Suppose we perform a standard LU-decomposition of A (as described in Exercise 1), i.e. we multiply A from the left by a matrix L to obtain an upper triangular matrix U_ε of the form

$$LA_\varepsilon = U_\varepsilon = \begin{pmatrix} \varepsilon & 1 \\ 0 & * \end{pmatrix}$$

and "$*$" is a real number (depending on ε). Show that the condition number of the resulting matrix U_ε satisfies

$$\lim_{\varepsilon \to 0} K(U_\varepsilon) = \infty.$$

(d) Suppose we perform the LU-decomposition of A_ε with a *pivoting step*: we interchange the first with the second row of A_ε to obtain the matrix

$$A'_\varepsilon := \begin{pmatrix} 1 & 1 \\ \varepsilon & 1 \end{pmatrix}.$$

As in part (c), perform the standard LU-decomposition with the new matrix A'_ε to obtain an upper triangular matrix U'_ε. Show that in this case we have

$$\lim_{\varepsilon \to 0} K(U'_\varepsilon) = 4$$

Exercise 3 (Orthogonality).
Let $v \in \mathbb{R}^n, n \in \mathbb{N}$, and $v \neq 0$. Further, define the Householder matrix associated with v by

$$H = \mathrm{Id} - \frac{2}{\langle v, v \rangle} v v^T.$$

(a) Show that H is an orthogonal matrix.
(b) Show that for a vector $x \in \mathbb{R}^n$, the vector Hx is the reflection of x at the line spanned by v.
(c) Let $\|x\|_2$ be the Euclidean norm of a vector $x \in \mathbb{R}^n$ and let $\|A\|_2$ be the induced matrix norm ($A \in \mathbb{R}^{n \times n}$). Show that if A is orthogonal then $K(A) = 1$ where $K(A) = \|A\|_2 \|A^{-1}\|_2$ is the condition number of A.