University of Vienna
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Introductory Seminar Advanced Numerical Analysis

Exercise sheet 5, due date: 04.04.2022

Exercise 1 (SSOR preconditioning).
Let $A \in \mathbb{R}^{n\times n}$ be a SPD matrix with the additive decomposition $A = L + D + L^T$ where D consists of the diagonal entries of A and L is lower triangular. Consider for $\omega \in (0,2)$ the parameter dependent matrix

$$C_\omega = \frac{1}{2-\omega}\left(\frac{1}{\omega}D + L\right)\left(\frac{1}{\omega}D\right)^{-1}\left(\frac{1}{\omega}D + L^T\right).$$

(1) Write C_ω in the form KK^T with an invertible lower-triangular matrix K.
(2) Explain why C_ω^{-1} may be viewed as an approximation for A^{-1} and argue why C_ω serves as a candidate for a preconditioning matrix.

Exercise 2 (M-matrix).
Let $n, m \in \mathbb{N}$. Further, let $I \in \mathbb{R}^{m\times m}$ be the identity matrix in $\mathbb{R}^{m\times m}$ and Q the banded matrix

$$Q = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & & 4 \end{pmatrix} \in \mathbb{R}^{m\times m}.$$

Let $A \in \mathbb{R}^{nm\times nm}$ be the matrix which arises from I and Q via

$$A = \begin{pmatrix} Q & -I & & & \\ -I & Q & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & Q & -I \\ & & & & Q \end{pmatrix} \in \mathbb{R}^{nm\times nm}.$$

The matrix A originates from the 5-point discretization of the Poisson problem on the unit square.

(1) Show that Q is invertible.
(2) Show that A is a so-called ”(inverse) monotone” matrix, or M-matrix. That is, all entries of the inverse of A are non-negative.

Exercise 3 (Polynomial norm bound).
Let $A \in \mathbb{R}^{n\times n}$ be a SPD matrix and $b \in \mathbb{R}^n$ a right-hand side. Suppose we apply the CG-method for solving the system $Ax = b$ (as defined in the lecture notes p. 65). Recall that the k-th iterate x_k of the CG method satisfies the A-norm optimality condition

$$\|x_k - x\|_A = \min_{y \in x_0 + B_k} \|y - x\|_A$$

where

$$B_k = \operatorname{span}\{p_0, \ldots, p_{k-1}\} = \operatorname{span}\{r_0, Ar_0, \ldots, A^{k-1}r_0\}$$

and the search directions p_k form an A-orthogonal system. Use the above facts to prove the following statement: If the spectrum of A lies in the interval $[a, b] \subset (0, \infty)$ then for any polynomial $p \in \mathbb{P}_k$ with $p(0) = 1$ we have

$$\|x_k - x\|_A \leq \left(\sup_{t \in [a,b]} |p(t)| \right) \|x_0 - x\|_A.$$

Show that this, in particular, implies that

$$\|x_k - x\|_A \leq \inf_{p \in \mathbb{P}_k, p(0)=1} \left(\|p\|_{C[a,b]} \right) \|x_0 - x\|_A.$$

Exercise 4 (GMRES method).
Let $A \in \mathbb{R}^{n \times n}$ be a SPD matrix and $b \in \mathbb{R}^n$ a right-hand side. In a similar fashion as in Exercise 3 we can show that the iterates x_k of the CG method satisfy the A^{-1}-norm optimality

$$\|Ax_k - b\|_{A^{-1}} = \min_{y \in x_0 + C_k} \|Ay - b\|_{A^{-1}}$$

with $C_k = \operatorname{span}\{p_0, Ap_0, \ldots, A^{l-1}p_0\}$. The so-called "generalized minimal residual method" (GMRES), instead, formally constructs a sequence of iterates x_k^{G} by

$$\|Ax_k^{\mathrm{G}} - b\|_2 = \min_{y \in x_0 + C_k} \|Ay - b\|_2.$$

Prove that the GMRES method allows for an error inequality similar to the one that was derived for the CG-method, namely

$$\|Ax_k^{\mathrm{G}} - b\|_2 \leq \inf_{p \in \mathbb{P}_k, p(0)=1} \|p(A)\|_2 \|Ax_0 - b\|_2.$$

Exercise 5 (Richardson iteration).
Let $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$.

(1) Take A, b, n, ω as an input and implement the damped Richardson iteration with damping parameter ω.

(2) Now let

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}.$$

Find a suitable damping parameter ω and calculate a numerical approximation of the solution of the equation $Ax = b$ using the implementation from (1).