Applied Analysis

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Contents

Preface

These lecture notes only contain what I would actually write at the black board during the class. My intention is that you add comments and further details on your own while listening to my explanations and when you work through the notes after class.

Exercises are marked blue. Solutions are to be collected in a single pdf file to be submitted to monika.doerfler@univie.ac.at at the end of the semester (before the final exam).

Chapter 1

Introduction

1.1 Motivation: Sines and Cosines

A large part of this lecture will be concerned with Fourier Analysis in its different versions, that is, with Fourier series, and discrete and continuous Fourier transforms. At the base of all this are the trigonometric functions, which may be seen as the simplest periodic functions. We will now see, why.
"Sinusoids describe many natural, periodic processes"

1.1.1 Harmonic Oscillator and the Spring Equation

We will begin our study of wave phenomena by reviewing the harmonic oscillator. Consider a block with mass, m, free to slide on a frictionless air-track, but attached to a Hooke?s law spring with its other end attached to a fixed wall.

Light here means that the mass of the spring is small enough to be ignored in the analysis of the motion of the block. This system has only one relevant degree of freedom. In general, the number of degrees of freedom of a system is the number of coordinates that must be specified in order to determine the configuration completely. In this case, because the spring is light, we can assume that it is uniformly stretched from the fixed wall to the block. Then the only important coordinate is the position of the block. In this situation, gravity plays no role in the motion of the block.

Recall Newton's second law: The acceleration a of a body is parallel and directly proportional to the net force F and inversely proportional to the mass m, i.e., $F = ma = m \cdot x''$.
Now consider a tuning fork, that is struck and thus produces a sound. What happens as the tuning fork is struck? It will be deformed and a restoring force F strives to restore the equilibrium, the fork overshoots, etc. This motion produces air pressure waves that are picked up by our ears. How can the motion be modeled?

1. F is proportional to the displacement $x(t)$ from equilibrium:

$$F = -kx, \tag{1.1}$$

where k is an elasticity constant (whose unit would be $\frac{N}{m}$.)

2. Since F acts on the tine, it produces acceleration proportional to F:

$$F = ma, \tag{1.2}$$

where m is the mass of the tine.

3. Now recall that $a = d^2x/dt^2$, i.e. acceleration is the second derivative of displacement x with respect to time t. We thus obtain the ordinary differential equation for the harmonic oscillator:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x(t) \tag{1.3}$$

4. In order to understand the motion of the struck tine, we therefore have to find functions $x(t)$ that are proportional to their second derivative by a negative number $c = -\frac{k}{m}$:

$$\frac{d^2}{dt^2}\sin(wt) = -w^2\sin(wt) \tag{1.4}$$

and the $\cos(wt)$ fulfills (1.3) analogously. In both cases $w = \sqrt{\frac{k}{m}}$ and the period of the oscillation is then given by $T = 2\pi\sqrt{\frac{m}{k}} = \frac{2\pi}{w} = \frac{1}{f}$, where f is the usual frequency in Hertz (whereas w is measured in radians per seconds, thus $w = 2\pi f$).

Let us next look at some properties of the sinusoids, i.e. sinus and cosines, which we often don't discriminate since we have $\sin(wt + \pi/2) = \cos(wt)$.

Proposition 1.1.1. *1. Sinusoids are closed under time-shift.*

2. Sinusoids are closed under addition.

3. Adding sinusoids of close frequencies produces beats.

1.1.2 The Vibrating String and the Wave Equation

"Trigonometric series can represent arbitrary functions"

A vibration in a string is a wave. Usually a vibrating string produces a sound whose frequency in most cases is constant. Therefore, since frequency characterizes the pitch, the sound produced is a constant note. Vibrating strings are the basis of any string instrument like guitar, cello, or piano. Their behavior is described by the *wave equation*: Let the vertical position (displacement) of a point x on a given string at time t be described by the function $y(x,t)$. Then the vibrating string is guided by the following partial differential equation (PDE):

$$\frac{\partial^2 y}{\partial t^2} = \frac{k}{m}\frac{\partial^2 y}{\partial x^2} \tag{1.5}$$

This is the wave equation for $y(x,t)$ and, as above, k is a constant corresponding to tension (stiffness) and m is the mass of the string. We set $c = \sqrt{\frac{k}{m}}$. It is then easy to see (verify!) that a general solution of this PDE must satisfy $y(x,t) = f(t - x/c) + g(t + x/c)$, where f, g are smooth but arbitrary function of one variable. We now make use of the obvious initial conditions $y(0,t) = y(L,t) = 0$, for all (!) t, where L is the length of the string under investigation - these conditions simply state that the string is attached at both ends. We then deduce:

$$\begin{aligned} y(0,t) = 0 \Rightarrow & f(t) = -g(t) \\ y(L,t) = 0 \Rightarrow & 0 = f(t - L/c) + g(t + L/c) = f(t - L/c) - f(t + L/c) \text{ for all } t \\ \text{set } t' = t + L/c \Rightarrow & 0 = f(t') - f(t' + 2L/c), \text{ hence } f(t') = f(t' + 2L/c) \text{ for all } t' \end{aligned}$$

Therefore, the solutions of (1.5) must be periodic in t with period $\frac{2L}{c}$. We already know the general form of functions with this property, and therefore, we assume that y must be of the form $y(x,t) = e^{2\pi i \frac{c}{2L} t}\mathcal{Y}(x)$. We now plug this "Ansatz"' into our PDE (1.5) and obtain:

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} =& -4\pi^2 (\frac{c}{2L})^2 e^{2\pi i \frac{c}{2L} t}\mathcal{Y}(x) \\ \frac{\partial^2 y}{\partial x^2} =& e^{2\pi i \frac{c}{2L} t}\frac{\partial^2 \mathcal{Y}}{\partial x^2}(x) \\ \Rightarrow& -4\pi^2 (\frac{c}{2L})^2 \mathcal{Y}(x) = c^2 \frac{\partial^2 \mathcal{Y}}{\partial x^2}(x) \\ \Rightarrow& \frac{\partial^2 \mathcal{Y}}{\partial x^2}(x) = -\frac{\pi^2}{L^2}\mathcal{Y}(x) \end{aligned}$$

which is the differential equation we have solved in the previous section! We therefore know that the shape of our string is given by a general sinusoid: $\mathcal{Y}(x) = \sin(\frac{2\pi x}{L} + \Phi)$. However, in the assumption on the form of y, namely, that $y(x,t) = e^{2\pi i \frac{c}{2L} t}\mathcal{Y}(x)$, we have so far ignored, that not only the sinusoid $e^{2\pi i \frac{c}{2L} t}$ is periodic with period $\frac{2L}{c}$, but also $e^{2\pi i n \frac{c}{2L} t}$, for any integer $n \in \mathbb{Z}$, as we will investigate more precisely in the following.

Definition 1.1.2. *A function f on $\mathbb{R}$ is periodic with period p (p-periodic), for $p > 0$, if $f(x+p) = f(x)$ for all $x \in \mathbb{R}$.*

Example 1.1.3. *The functions $e^{2\pi i n \frac{c}{2L} t}$ are $\frac{2L}{c}$-periodic for any $n \in \mathbb{Z}$, since*

$$e^{2\pi i n \frac{c}{2L}(t + \frac{2L}{c})} = e^{2\pi i n \frac{c}{2L} t} \cdot e^{2\pi i n} = e^{2\pi i n \frac{c}{2L} t},$$

since $e^{2\pi i n} = 1$ for any $n \in \mathbb{Z}$.

Since we have shown that linear combinations of sinusoids are sinusoids, we come to the conclusion that the general solution for the shape of a string is in fact given by arbitrary combinations of sinusoids of the form $\sin(\frac{2\pi n x}{L} + \Phi)$.

Excursus: A primer on Lebesgue spaces

We refer to [1] for proofs of this section. Let $(X, \Sigma(X), \mu)$ be a measure space. For $1 \leq p < \infty$, define

$$\mathcal{L}^p(X,\mu) := \{f : X \to \mathbb{C} \mid f \text{ measurable}, \int_X |f(x)|^p \mathrm{d}\mu(x) < \infty\}.$$

For $f \in \mathcal{L}_p(X,\mu)$, let

$$\|f\|_{L^p} := \left(\int_X |f(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}.$$

The equivalence relation $f \sim g$ if and only if $\|f-g\|_{L^p} = 0$ leads to the Banach space

$$L^p(X,\mu) := \mathcal{L}^p(X,\mu)/\sim .$$

Exercise: Define $L^\infty(X,\mu)$.

If the choice of μ is clear from the context, we simply write $L^p(X)$.

Example 1.1.4. *The sequences of functions*

$$f_n = 1_{[n,n+1]}, \qquad g_n = \frac{1}{n}1_{[0,n]}, \qquad h_n = n1_{[\frac{1}{n},\frac{2}{n}]}$$

converge pointwise to 0 *but*

$$1 = \|f_n\|_{L^1} = \|g_n\|_{L^1} = \|h_n\|_{L^1},$$

so that they do not converge to 0 *in* $L^1(\mathbb{R})$. *Note that* $g_n \to 0$ *even uniformly.*

Theorem 1.1.5 (Dominated convergence theorem). *Let* $L^1(X,\mu) \ni f_k \to f : X \to \mathbb{C}$ *pointwise a.e.*

$$\exists g \in L^1(X,\mu) : |f_k| \leq g \text{ a.e.} \qquad \Rightarrow \qquad f \in L^1(X,\mu), \quad \int f_k \to \int f.$$

Theorem 1.1.6 (Young's convolutional inequality). *Let* $1 \leq p < \infty$. *For* $f \in L^1(\mathbb{R}^d)$ *and* $g \in L_p(\mathbb{R}^d)$,

$$(f * g)(\cdot) := \int f(y)g(\cdot - y)\mathrm{d}y \in L^p(\mathbb{R}^d)$$

is well-defined a.e. and it holds

$$\|f * g\|_{L^p} \leq \|f\|_{L^1}\|g\|_{L^p}. \tag{1.6}$$

Definition 1.1.7 (Approximate identity). *An approximate identity is a family* $(u_\epsilon)_{\epsilon>0} \subset L^1(\mathbb{R}^d)$ *such that it holds:*

(a) $\exists c > 0$ *such that* $\|u_\epsilon\|_{L^1} \leq c$, $\forall \epsilon > 0$,

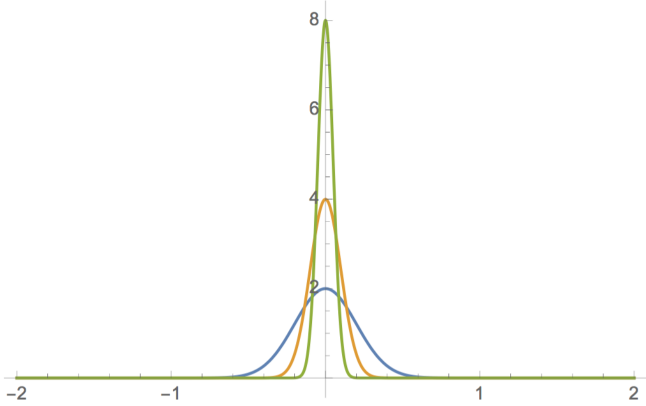

Figure 1.1: $u_\epsilon(x) = \frac{1}{\epsilon}\mathrm{e}^{-\pi\left(\frac{x}{\epsilon}\right)^2}$, for $|x| \leq 2$ and $\epsilon = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$.

(b) $\int u_\epsilon = 1$, $\forall \epsilon > 0$,

(c) for any neighborhood U *of* 0,

$$\int_{X \setminus U} |u_\epsilon| \xrightarrow{\epsilon \to 0} 0.$$

Example 1.1.8. *If* $u \in L^1(\mathbb{R}^d)$ *with* $\int_{\mathbb{R}^d} u(x)\mathrm{d}x = 1$, *then*

$$u_\epsilon(x) := \epsilon^{-d} u(\epsilon^{-1}x) \tag{1.7}$$

is an approximate identity, cf. Figure 1.1 for $d = 1$ *and* $u(x) = \mathrm{e}^{-\pi x^2}$. *Parts (a,b) are transformation identities and, Part (c) is an exercise.*

Theorem 1.1.9. *Let* $1 \leq p < \infty$ *and* $f \in L^p(\mathbb{R}^d)$. *If* $(u_\epsilon)_{\epsilon > 0}$ *is an approximate identity, then*

$$u_\epsilon * f \xrightarrow{\epsilon \to 0} f \quad \text{in } L^p(\mathbb{R}^d).$$

Proof. Let $p = 1$. We have

$$\begin{aligned} \|f - u_\epsilon * f\|_{L^1} &= \int |f(x) - u_\epsilon * f(x)| \mathrm{d}\lambda(x) \\ &= \int \left| f(x) \int u_\epsilon(y)\mathrm{d}\lambda(y) - \int u_\epsilon(y) f(y^{-1}x)\mathrm{d}\lambda(y) \right| \mathrm{d}\lambda(x) \\ &\leq \int \int \left| f(x) - f(y^{-1}x) \right| |u_\epsilon(y)| \, \mathrm{d}\lambda(x)\mathrm{d}\lambda(y). \end{aligned}$$

Splitting $X = U \cup (X \setminus U)$ with U as in Lemma **??** leads to

$$\|f - u_\epsilon * f\|_{L^1} \leq \int_U \int \left| f(x) - f(y^{-1}x) \right| \mathrm{d}\lambda(x) \, |u_\epsilon(y)| \, \mathrm{d}\lambda(y) + \int_{X \setminus U} \int \left| f(x) - f(y^{-1}x) \right| \mathrm{d}\lambda(x) \, |u_\epsilon(y)| \, \mathrm{d}\lambda$$

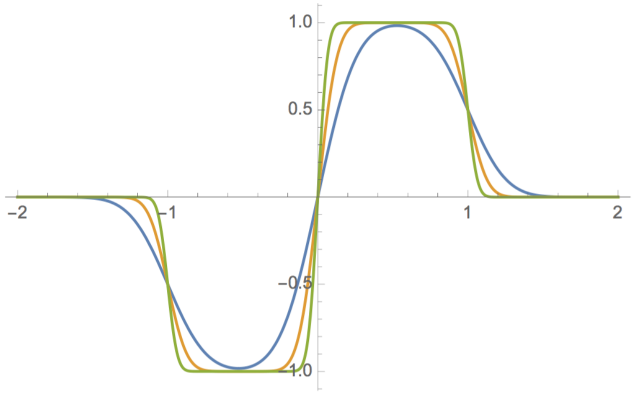

Figure 1.2: $f * u_\epsilon$ for $f = -1_{[-1,0)} + 1_{[0.1]}$ and $u_\epsilon(x) = \frac{1}{\epsilon} \mathrm{e}^{-\pi\left(\frac{x}{\epsilon}\right)^2}$, for $|x| \leq 2$ and $\epsilon = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$.

The case $1 < p < \infty$ is analogous. $\square$

We know that $\mathcal{C}_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, for $1 \leq p < \infty$.

Corollary 1.1.10. *The space $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, for $1 \leq p < \infty$.*

Chapter 2

Discrete Fourier analysis

2.1 Definition of Fourier series and examples

Definition 2.1.1 ($\mathbb{Z}^d$-periodic)**.** *A function* $f : \mathbb{R}^d \to \mathbb{C}$ *is called* $\mathbb{Z}^d$-periodic *if*

$$f(x+k) = f(x) \quad \forall k \in \mathbb{Z}^d .$$

Obviously, $x \mapsto \sin(2\pi x)$ is $\mathbb{Z}$-periodic, see Figure 2.1.

Definition 2.1.2 (d-torus)**.** *The* d*-torus is* $\mathbb{T}^d := \mathbb{R}^d \,/\, \mathbb{Z}^d$*.*

Any $\mathbb{Z}^d$-periodic function f can be considered as a function on the torus, i.e., $f : \mathbb{T}^d \to \mathbb{C}$ is well-defined.

Now we dive into the question of how periodic functions may be described. We start with the definition of the real Fourier series.

Definition 2.1.3. *For a* p*-periodic function* $f(x)$ *that is integrable on* $[-\frac{p}{2}, \frac{p}{2}]$*, the numbers*

$$a_n = \frac{2}{p} \int_{-p/2}^{p/2} f(x) \cos(\frac{2\pi n x}{p})\, dx, \quad n \geq 0 \tag{2.1}$$

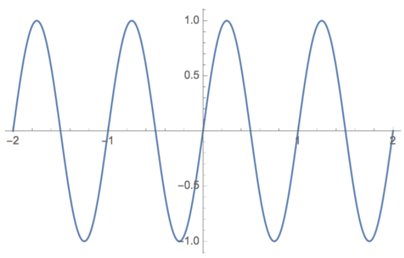

Figure 2.1: $\sin(2\pi x)$, for $|x| \leq 2$.

and

$$b_n = \frac{2}{p}\int_{-p/2}^{p/2} f(x)\sin(\frac{2\pi nx}{p})\,dx, \quad n \geq 1 \tag{2.2}$$

are called the Fourier coefficients of f. The expression

$$(S_N f)(x) = \frac{a_0}{2} + \sum_{n=1}^{N}[a_n\cos(\frac{2\pi nx}{p}) + b_n\sin(\frac{2\pi nx}{p})], \quad N \geq 0. \tag{2.3}$$

is called trigonometric polynomial of degree N. The infinite sum

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(\frac{2\pi nx}{p}) + b_n\sin(\frac{2\pi nx}{p})] \tag{2.4}$$

is called the Fourier series of f.

Example 2.1.4. *The Fourier series of a square wave Consider the $[0,1]$-periodic function*

$$f(x) := \begin{cases} 1 & \text{for } 0 \leq x < \frac{1}{2} \\ -1 & \text{for } \frac{1}{2} \leq x < 1 \end{cases}$$

Then its Fourier series is given by

$$f(x) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)}\sin(2\pi(2k-1)x)$$

Proposition 2.1.5. *For a bounded, piecewise continuous function f, the Fourier coefficients* (2.1) *and* (2.3) *yield the best approximation with a trigonometric polynomial of degree N. Furthermore, if f is piecewise smooth with finitely many discontinuities, its Fourier series converges pointwise.*

Remark 2.1.6. *Note that best approximation means, that the error which occurs, when approximating a given, p-periodic functions by a trigonometric polynomial of degree N, as in* (2.3)*, is minimal, if the coefficients a_n, b_n are the Fourier coefficients. This is an immediate consequence of the fact, that the sinusoids form an orthonormal basis for "all" periodic functions (which are sufficiently nice). We will consider this property in Proposition 2.1.8.*[1]

The complex version of Fourier series: We can use Euler's formula, $e^{2\pi i\frac{n}{p}x} = \cos(2\pi\frac{n}{p}x) + i\sin(2\pi\frac{n}{p}x)$ where i is the imaginary unit, to give a more concise formula of the Fourier series of a function f:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i\frac{n}{p}x}, \tag{2.5}$$

[1] In der Vorlesung haben wir mehrere Beweise für diese Tatsache der Minimalität des Fehlers besprochen. Diese Beweise werden ins Skriptum noch nachgeliefert, da sie strukturell auch durch ihre Verbindung zur Differentialrechnung (Minima finden...) interessant sind.

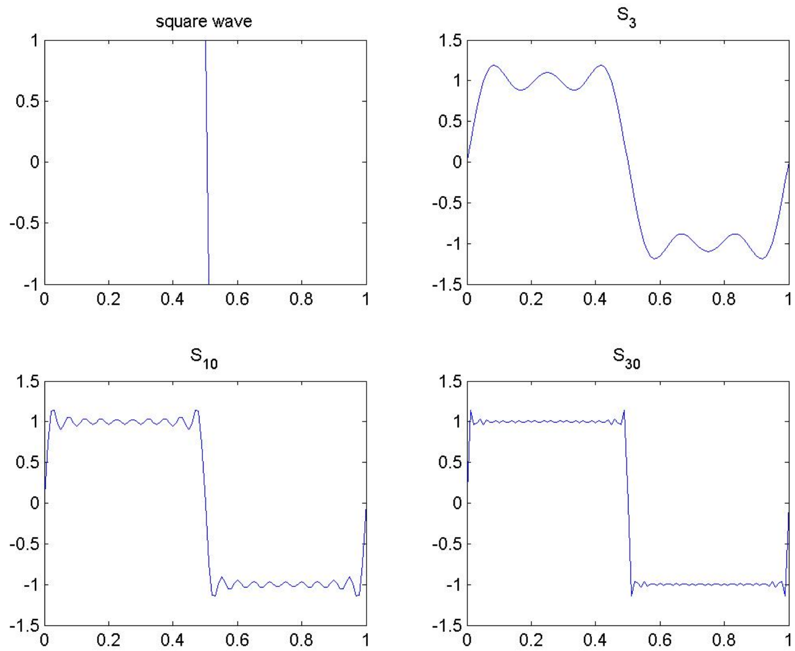

Figure 2.2: Fourier Series Square Wave

with the Fourier coefficients[2] given by:

$$\hat{f}[n] = c_n = \frac{1}{p}\int_{-1/p}^{1/p} f(x)e^{-2\pi i \frac{n}{p} x}\,dx. \tag{2.6}$$

If we assume here $p = 1$, the above formulas simplify further and we can use Euler's formula, $e^{2\pi inx} = \cos(2\pi nx) + i\sin(2\pi nx)$ where i is the imaginary unit, to give a more concise formula of the Fourier series of a function f:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi inx}, \tag{2.7}$$

and

$$\hat{f}[n] = c_n = \int_{-1/2}^{1/2} f(x)e^{-2\pi inx}\,dx. \tag{2.8}$$

Example 2.1.7. The Fourier series of a sum of sinusoids
We consider the functions $f_1(t) = \sin 2\pi\omega_0 t$ *and* $f_2(t) = \cos 2\pi 3\omega_0 t$*, for arbitrary* $\omega_0 \in \mathbb{N}$ *and* $h(t) = f_1(t) + \frac{1}{2}\cdot f_2(t)$*. We want to compute and interpret the Fourier series of*

[2] The Fourier coefficients c_n are often denoted by $\hat{f}[n]$, since $\hat{f}$ is the most common notation for the Fourier transform of f.

these three functions. Obviously, f_1 and f_2 are pure sinusoids with frequencies ω_0 and $3\omega_0$, respectively, hence, with periods $p_1 = \frac{1}{\omega_0}$ and $p_2 = \frac{1}{3\omega_0}$. It is clear, that f_2 is also periodic with the longer period $\frac{1}{\omega_0}$. (Check this by invoking the definition of periodic functions!)
We first consider the Fourier coefficients of f_1. Since its period is $\frac{1}{\omega_0}$, we are looking for the coefficients a_n, b_n in the expansion

$$f_1(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(2\pi n\omega_0 t) + b_n \sin(2\pi n\omega_0 t)], \quad n \geq 0. \tag{2.9}$$

We can now compute the coefficients (do it!!), according to the formulas given in Definition 2.1.3, to find, using the orthogonality relations, that $a_n = 0\,\forall n$ and $b_1 = 1$, $b_n = 0$ for $n \neq 1$. Alternatively we simply look at the given form of f_1 and argue that, since the expansion in an orthogonal system is unique, the coefficients have to be of the very same form! As a third version, compute the Fourier coefficients according to (2.6).

We immediately derive (do it!!) that $c_1 = \hat{f}[1] = \frac{1}{2i}$ and $c_{-1} = \hat{f}[-1] = -\frac{1}{2i}$, which leads us directly to the expression of the sine-function via Euler's formula:

$$f_1(t) = \sin 2\pi\omega_0 t = \frac{e^{2\pi i\omega_0 t} - e^{-2\pi i\omega_0 t}}{2i}!$$

Let us first interpret these findings: obviously, the coefficients a_n, b_n in the Fourier series express the "contribution" or energy of the cosine (or sine) function to the periodic signal we wish to express. If we use the complex form, we split the energy contained in one sinusoid into a positive and a negative part of equal absolute value (in the case of real functions). If we use the real part, the contributions to "one frequency component" may be split in cosine and sine parts. Since this is usually more complicated, the complex form is usually preferred.
We now turn to f_2, periodic with period p_2, hence, if we consider the orthonormal basis $\{\cos(2\pi n3\omega_0 t), \quad n \geq 0\} \cup \{\sin(2\pi n3\omega_0 t), \quad n \geq 1\}$ in complete analogy to before, we compute, or derive from the properties of our orthonormal basis, that $a_n = 0\,\forall n \neq 1$ and $a_1 = 1$, $b_n = 0 \forall n$. On the other hand, if we consider the basis $\{\cos(2\pi n\omega_0 t), \quad n \geq 0\} \cup \{\sin(2\pi n\omega_0 t), \quad n \geq 1\}$, which we will also have to use for $h(t)$, we find that

$$f_2(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(2\pi n\omega_0 t) + b_n \sin(2\pi n\omega_0 t)], \quad n \geq 0. \tag{2.10}$$

with $a_n = 0\,\forall n \neq 3$ and $a_3 = 1$, $b_n = 0 \forall n$! We can also derive the coefficients of the complex form:
Combining all the above considerations, we now derive the Fourier coefficients of $h(t)$ according to (2.6)*: $c_1 = \hat{f}[1] = \frac{1}{2i}$, $c_{-1} = \hat{f}[-1] = -\frac{1}{2i}, c_3 = \hat{f}[3] = \frac{1}{2}$, $c_{-3} = \hat{f}[-3] = \frac{1}{2}$. The absolute values of these Fourier coefficients as well as the functions $h(t)$ are shown in Figure 2.1.7 for $\omega_0 = 10$. Please also write out the real form of the Fourier series and verify that the two forms are identical. In the figure, note that the x-axis is labeled with the frequencies in Hertz. Of course, this is an interpretation of our observation that the*

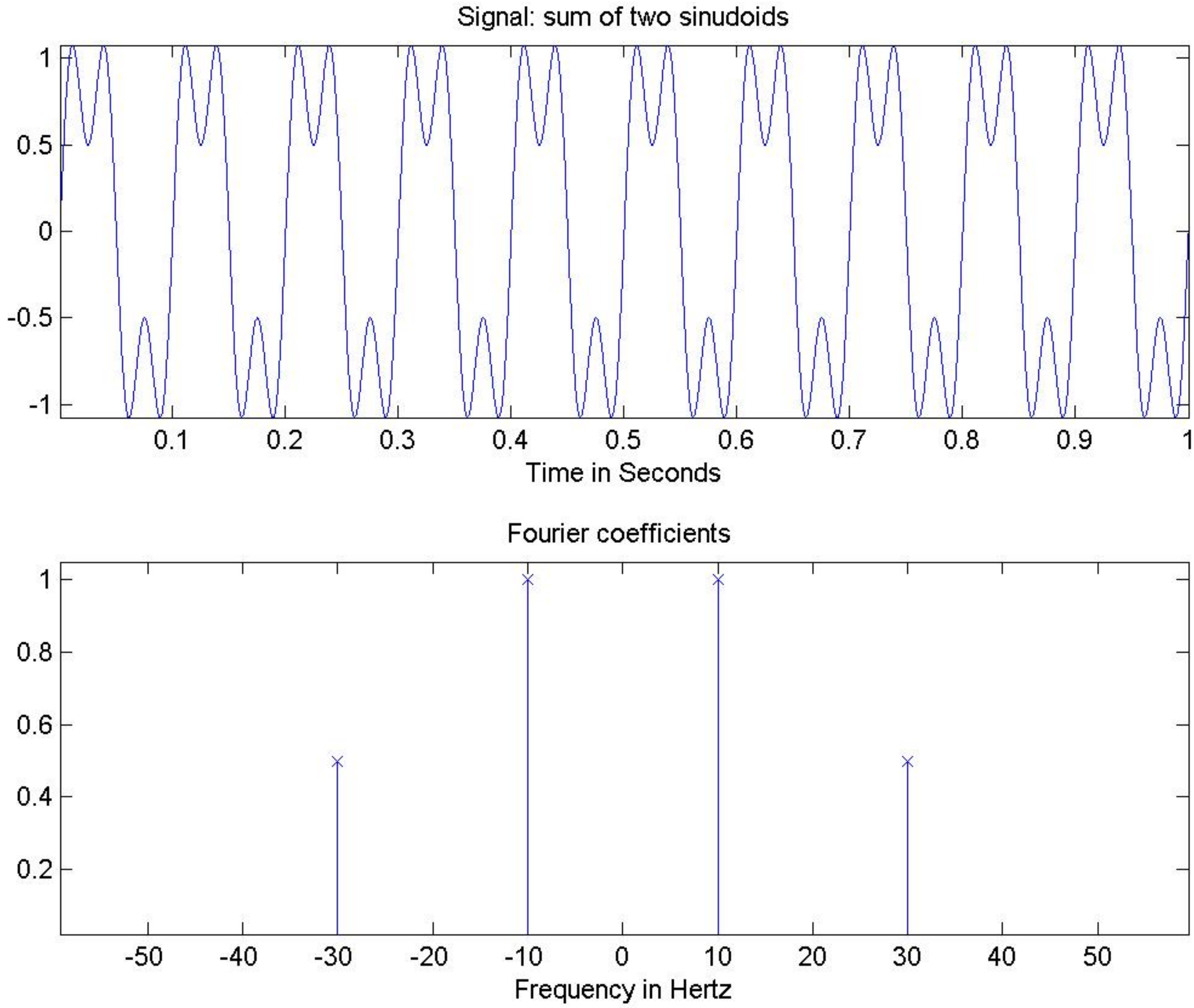

Figure 2.3: Fourier coefficients of the sum of two sinusoids

coefficients in the Fourier series correspond to the pure frequencies in the function (signal) of interest: c_0 corresponds to $0 \cdot \omega_0 Hz$, c_1 corresponds to $1 \cdot \omega_0 Hz$, etc.

We will often denote the Fourier coefficients of a function f by $F[n]$ or $\hat{f}(n)$. More precise explanations on scalar product, (norm and minimal error), orthogonality, and the correspondence to ONBs:

Proposition 2.1.8. *The family of functions $\{e^{2\pi i \frac{k}{p} x}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $L^2([-\frac{p}{2}, \frac{p}{2}])$.*

Alternatively, we state for the real sinusoids: The sines and cosines form an orthogonal set: (note that the constant function is $\cos(2\pi \frac{m}{p} x)$ for $m = 0$).

$$\int_{-\frac{p}{2}}^{\frac{p}{2}} \cos(2\pi \frac{m}{p} x)\, \cos(2\pi \frac{n}{p} x)\, dx = \delta_{mn}, \quad m \geq 0, n \geq 1 \tag{2.11}$$

$$\int_{-\frac{p}{2}}^{\frac{p}{2}} \sin(2\pi \frac{m}{p} x)\, \sin(2\pi \frac{n}{p} x)\, dx = \delta_{mn}, \quad m, n \geq 1 \tag{2.12}$$

(here δ_{mn} is the Kronecker delta), and

$$\int_{-\frac{p}{2}}^{\frac{p}{2}} \cos(2\pi \frac{m}{p} x)\, \sin(2\pi \frac{n}{p} x)\, dx = 0, \quad m \geq 0, n \geq 1 \tag{2.13}$$

The span of these sines is dense in $L^2([-\frac{p}{2}, \frac{p}{2}])$, hence they form an orthonormal basis of this vector space.

Exercise 2.1.9. *Please recall (or google) the most important properties of orthonormal bases, their relation to projections and minimal distance in Eucledian spaces. Formulate what you found concretely for the orthonormal basis of complex exponentials given in Proposition 2.1.8.*

Corollary 2.1.10. *Consider the subspace which is spanned by the first $2N + 1$ functions of the ONB $\{e^{2\pi ikt}\}_{k\in\mathbb{Z}}$, i.e. by $\{e^{2\pi ikt} : k = -N, \dots, N\}$. Then, the best approximation of $f \in C([-\frac{1}{2}, \frac{1}{2}], \mathbb{C})$ by an arbitrary linear combination in $M_N(t) = \sum_{k=-N}^{N} c_k e^{2\pi ikt}$ is given by $S_N(t) = \sum_{k=-N}^{N} \hat{f}[k] e^{2\pi ikt}$.*

Proof. Let us assume that, for some $c_k \neq \hat{f}[k] = \langle f, e^{2\pi ikt} \rangle$, we can achieve a better approximation than with S_N:

$$\|f - M_N\|_2^2 < \|f - S_N\|_2^2 \tag{2.14}$$

We now show that this leads to a contradiction. We compute:

$$\begin{aligned}
\|f-M_N\|_2^2 =&\langle f-M_N, f-M_N\rangle \\
=&\langle f,f\rangle+\langle M_N,M_N\rangle-2\mathcal{R}e\langle f,M_n\rangle \\
=&\|f\|_2^2+\int_{-\frac{1}{2}}^{\frac{1}{2}} M_N(t)\overline{M_N}(t)dt-2\mathcal{R}e[\langle f,\sum_{k=-N}^{N} c_k e^{2\pi ikt}\rangle] \\
=&\|f\|_2^2+\sum_k\sum_{k'} c_k\overline{c_{k'}}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi ikt}e^{-2\pi ik't}dt-2\mathcal{R}e[\sum_{k=-N}^{N} c_k\langle f,e^{2\pi ikt}\rangle] \\
=&\|f\|_2^2+\sum_k |c_k|^2-2\mathcal{R}e[\sum_{k=-N}^{N}\overline{c_k}\langle f,e^{2\pi ikt}\rangle],
\end{aligned}$$

where the last step follows from the orthogonality of the basis functions $\{e^{2\pi ikt}\}$. We carry out the same steps for S_n and obtain:

$$\begin{aligned}
\|f-S_N\|_2^2 =&\|f\|_2^2+\sum_k |\hat{f}[k]|^2-2\mathcal{R}e[\sum_{k=-N}^{N}\hat{f}[k]\langle f,e^{2\pi ikt}\rangle] \\
=&\|f\|_2^2+\sum_k |\hat{f}[k]|^2-2\sum_{k=-N}^{N}|\hat{f}[k]|^2=\|f\|_2^2-\sum_k|\hat{f}[k]|^2
\end{aligned}$$

Hence, our assumption (2.14) is equivalent to assuming

$$\sum_k |c_k|^2-2\mathcal{R}e[\sum_{k=-N}^{N}\overline{c_k}\langle f,e^{2\pi ikt}\rangle]<-\sum_k|\hat{f}[k]|^2$$

for some $c_k, k=-N,\dots,N$. We rewrite this as

$$\sum_k |c_k|^2-2\mathcal{R}e[\sum_{k=-N}^{N}\overline{c_k}\hat{f}[k]\rangle]+\sum_k|\hat{f}[k]|^2<0$$

hence

$$\sum_k\left[|c_k|^2-2\mathcal{R}e[\overline{c_k}\hat{f}[k]]+|\hat{f}[k]|^2\right]=\sum_k|c_k-\hat{f}[k]|^2<0$$

and obviously the sum of positive values can never be negative. This contradiction concludes the proof. $\square$

From general properties of ONBs we can now easily deduce the following properties of Fourier series:

Proposition 2.1.11 (Parseval Identity)**.**

$$\langle f, g\rangle_{L^2([-\frac{p}{2},\frac{p}{2}])} = p\sum_{k\in\mathbb{Z}} \hat{f}(k)\overline{\hat{g}(k)} =: p\langle \hat{f}, \hat{g}\rangle_{\ell^2} \tag{2.15}$$

In particular, setting $f = g$, it follows, that

$$\|f\|^2_{L^2([-\frac{p}{2},\frac{p}{2}])} = \langle f, f\rangle_{L^2([-\frac{p}{2},\frac{p}{2}])} = \int_{-p/2}^{p/2} |f(x)|^2 dx = p\sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2.$$

Proof. A direct proof, assuming that the interchange of sum and integral is justified:

$$\begin{aligned}
\langle f, g\rangle_{L^2([-\frac{p}{2},\frac{p}{2}])} &= \int_{-p/2}^{p/2} f(x)\overline{g(x)}dx \\
&= \int_{-p/2}^{p/2} f(x)\overline{\sum_{n\in\mathbb{Z}} G[n]e^{2\pi i\frac{n}{p}x}}dx \\
&= \sum_{n\in\mathbb{Z}} \overline{G[n]} \int_{-p/2}^{p/2} f(x)e^{-2\pi i\frac{n}{p}x}dx = p\sum_{k\in\mathbb{Z}} F[k]\overline{G[k]} \\
&= p\sum_{k\in\mathbb{Z}} \hat{f}(k)\overline{\hat{g}(k)} =: p\langle \hat{f}, \hat{g}\rangle_{\ell^2}
\end{aligned}$$

□

2.1.1 Computing the truncation error

$$f(t) = S_N(t) + \sum_{|k|>N} \hat{f}(k)e^{2\pi ikt}$$

How big is $E_N(t) = f(t) - S_N(t)$? And how do we measure?

$$\|f - S_N\|_2^2 = \int_{1/2}^{1/2} |E_N(t)|^2 dt = \sum_{|k|>N} |\hat{f}(k)|^2$$

Because of the isometry of the Fourier transform, the error term can be computed by

$$\sum_{|k|>N} |\hat{f}(k)|^2 = \|f\|_2^2 - \sum_{|k|\le N} |\hat{f}(k)|^2$$

2.2 Pointwise convergence of Fourier series in $\mathbb{R}^d$

Let $I := \left[-\frac{1}{2}, \frac{1}{2}\right]$, so that the inner product in $L^2(\mathbb{T}^d)$ is

$$\langle f, g\rangle_{L^2(\mathbb{T}^d)} = \int_{I^d} f(x)\overline{g(x)}\mathrm{d}x.$$

For $k \in \mathbb{Z}^d$, we consider

$$\mathrm{e}_k(x) := \mathrm{e}^{2\pi \mathrm{i}\langle k,x\rangle}.$$

It is well-known that $\{\mathrm{e}_k : k \in \mathbb{Z}^d\}$ is an orthonormal basis for $L^2(\mathbb{T}^d)$. Let

$$\Pi_t := \operatorname{span}\{\mathrm{e}_k : \|k\|_\infty \le t\}.$$

denote the trigonometric polynomials of degree $t \in \mathbb{N}$. The orthogonal projection onto Π_t is

$$S_t : L^2(\mathbb{T}^d) \to \Pi_t, \qquad f \mapsto \sum_{\substack{k\in\mathbb{Z}^d\\ \|k\|_\infty \le t}} \langle f, \mathrm{e}_k\rangle_{L^2(\mathbb{T}^d)} \mathrm{e}_k, \tag{2.16}$$

so that $S_t f \xrightarrow{t\to\infty} f$ in $L^2(\mathbb{T}^d)$.

Definition 2.2.1 (Fourier coefficients)**.** *The* Fourier coefficients *of $f \in L^1(\mathbb{T}^d)$ are*

$$\hat{f}_k := \int_{I^d} f(x)\mathrm{e}^{-2\pi\mathrm{i}\langle k,x\rangle}\mathrm{d}x, \quad k \in \mathbb{Z}^d.$$

Lemma 2.2.2 (Riemann-Lebesgue for $\mathbb{T}^d$)**.** *If $f \in L^1(\mathbb{T}^d)$, then $\hat{f}_k \xrightarrow{\|k\|\to\infty} 0$.*

Proof. Since $\mathbb{T}^d$ is compact, we have $\mathcal{C}(\mathbb{T}^d) \subset L^2(\mathbb{T}^d) \subset L^1(\mathbb{T}^d)$ are dense. For $f \in L^2(\mathbb{T}^d)$, we have $\left(\hat{f}_k\right)_{k\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$, so that $\hat{f}_k \xrightarrow{\|k\|\to\infty} 0$.

Exercise: complete the proof for $f \in L^1(\mathbb{T}^d)$. □

We have already used that $f \in L^2(\mathbb{T}^d)$ leads to $\hat{f}_k = \langle f, \mathrm{e}_k\rangle_{L^2(\mathbb{T}^d)}$. Note that S_t in (2.16) can be extended to

$$S_t : L^1(\mathbb{T}^d) \to \Pi_t, \qquad f \mapsto \sum_{\substack{k\in\mathbb{Z}^d\\ \|k\|_\infty \le t}} \hat{f}_k \mathrm{e}_k.$$

We shall verify convergence results.

Definition 2.2.3 (Dirichlet kernel)**.** *The function*

$$D_t(z) := \sum_{\|k\|_\infty \le t} \mathrm{e}_k(z), \qquad z \in \mathbb{R}^d,$$

is called the Dirichlet kernel.

Proposition 2.2.4. *For $0 < t \in \mathbb{N}$, we have*

$$(S_t f)(x) = \int_{I^d} f(y) D_t(x-y)\mathrm{d}y$$

and the Dirichlet kernel satisfies

$$D_t(z) = \prod_{i=1}^d \left(\frac{\mathrm{e}_{t+1}(z_i) - \mathrm{e}_t(z_i)}{\mathrm{e}_1(z_i) - 1}\right) \tag{2.17}$$

Proof. Exercise. (hint: prove (2.17) first for $d = 1$ via geometric progression and observe the tensor structure) □

Example 2.2.5. *For $d = 1$ and $k \in \mathbb{N}$, we have*

$$\operatorname{span}\{\mathrm{e}_k, \mathrm{e}_{-k}\} = \operatorname{span}\{\sin(2\pi k\cdot), \cos(2\pi k\cdot)\}.$$

Thus, the set

$$\{\sin(2\pi k\cdot) : k \in \mathbb{N}\} \cup \{\cos(2\pi k\cdot) : k \in \mathbb{N}\},$$

is also an orthonormal basis for $L^2(\mathbb{T})$.

Theorem 2.2.6. *If $f \in \mathcal{C}^1(\mathbb{T})$, then $S_t f \xrightarrow{t\to\infty} f$ pointwise.*

Proof. Let $x \in \mathbb{R}$ be fixed. Since $\mathrm{e}_1 = 1$, the orthogonality implies $\int_I \overline{D_t(z)}\mathrm{d}z = 1$, so that we derive with $\overline{D_t(z)} = D_t(-z)$,

$$S_t f(x) - f(x) = \int_I f(y) D_t(x-y)\mathrm{d}y - \int_I f(x) D_t(-z)\mathrm{d}z$$

Substitution and periodicity lead to

$$\begin{aligned}
S_t f(x) - f(x) &= -\int_{x+\frac{1}{2}}^{x-\frac{1}{2}} f(x-z) D_t(z)\mathrm{d}z - \int_I f(x) D_t(-z)\mathrm{d}z \\
&= \int_{-x-\frac{1}{2}}^{-x+\frac{1}{2}} f(x+z) D_t(-z)\mathrm{d}z - \int_I f(x) D_t(-z)\mathrm{d}z \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x+z) D_t(-z)\mathrm{d}z - \int_I f(x) D_t(-z)\mathrm{d}z \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(f(x+z) - f(x)\right) D_t(-z)\mathrm{d}z \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} g(z) \left(\mathrm{e}_{-(t+1)}(z) - \mathrm{e}_{-t}(z)\right) \mathrm{d}z,
\end{aligned}$$

where we have used (2.17) and

$$g(z) := \frac{f(x+z) - f(x)}{\mathrm{e}_1(z) - 1}.$$

We have $g \in L^1(\mathbb{T})$ because l'Hospital's rule yields

$$\lim_{0\neq z\to 0} g(z) = \lim_{0\neq z\to 0} \frac{f(x+z) - f(x)}{z} \cdot \underbrace{\frac{z}{\mathrm{e}_1(z) - 1}}_{\to\frac{1}{2\pi i}} = \frac{f'(x)}{2\pi i}.$$

The Riemann-Lebesgue Lemma 2.2.2 leads to

$$S_t f(x) - f(x) = \hat{g}_{t+1} - \hat{g}_t \to 0.$$

□

2.3 Approximation of Sobolev functions

Definition 2.3.1 (Sobolev space). *For $s > 0$, the Sobolev space $H^s(\mathbb{T}^d)$ is*

$$H^s(\mathbb{T}^d) := \{f \in L^2(\mathbb{T}^d) : \sum_{k\in\mathbb{Z}^d} \left(1 + \|k\|^2\right)^s \left|\hat{f}_k\right|^2 < \infty\}$$

with inner product

$$\langle f, g\rangle_{H^s} := \sum_{k\in\mathbb{Z}^d} \left(1 + \|k\|^2\right)^s \hat{f}_k\overline{\hat{g}_k}.$$

Theorem 2.3.2. *For $s, t > 0$ and $f \in H^s(\mathbb{T}^d)$, we have*

$$\|S_t f - f\|_{L^2} \le t^{-s}\|f\|_{H^s}.$$

Proof. Eventually, the Hölder inequality yields

$$\begin{aligned}
\|S_t f - f\|_{L^2}^2 &= \left\| \sum_{\|k\|_\infty > t} \hat{f}_k \mathrm{e}_k \right\|_{L^2}^2 = \sum_{\|k\|_\infty > t} \left|\hat{f}_k\right|^2 \\
&= \sum_{\|k\|_\infty > t} \left(1 + \|k\|^2\right)^{-s} \left(1 + \|k\|^2\right)^s \left|\hat{f}_k\right|^2 \\
&\le (1 + t^2)^{-s}\|f\|_{H^s}^2.
\end{aligned}$$ □

Lemma 2.3.3. *For $\alpha \in \mathbb{N}^d$ and $f \in \mathcal{C}^{|\alpha|}(\mathbb{T}^d)$, we have $\widehat{(\partial^\alpha f)}_k = (2\pi\mathrm{i}k)^\alpha \hat{f}_k$.*

Proof. For $d = 1 = \alpha$, integration by parts yields

$$\begin{aligned}
\widehat{\partial f}_k &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \partial f(x)\mathrm{e}^{-2\pi\mathrm{i}kx}\mathrm{d}x \\
&= \underbrace{\left[f(x)\mathrm{e}^{-2\pi\mathrm{i}kx}\right]_{-\frac{1}{2}}^{\frac{1}{2}}}_{0} - \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)(-2\pi\mathrm{i}k)\mathrm{e}^{-2\pi\mathrm{i}kx}\mathrm{d}x \\
&= 2\pi\mathrm{i}k\hat{f}_k.
\end{aligned}$$ □

Lemma 2.3.3 enables us to define $\partial^\alpha f$ via its Fourier coefficients. For $\alpha \in \mathbb{N}^d$ with

$s \geq |\alpha| \geq 1$ and $f \in H^s(\mathbb{T}^d)$, we observe

$$\begin{aligned}
\|\partial^\alpha f\|_{H^{s-|\alpha|}} &= \sum_{k\in\mathbb{Z}^d} (1+\|k\|^2)^{s-|\alpha|} |2\pi k|^{2\alpha} |\hat{f}_k|^2 \\
&= (2\pi)^{|\alpha|} \sum_{k\in\mathbb{Z}^d\setminus\{0\}} (1+\|k\|^2)^s |\hat{f}_k|^2 \frac{k^{2\alpha}}{(1+\|k\|^2)^{|\alpha|}} \\
&\leq (2\pi)^{|\alpha|} \sum_{k\in\mathbb{Z}^d\setminus\{0\}} (1+\|k\|^2)^s |\hat{f}_k|^2 \underbrace{\frac{\|k\|_\infty^{2|\alpha|}}{\|k\|^{2|\alpha|}}}_{\leq 1} \\
&\leq (2\pi)^{|\alpha|} \|f\|_{H^s}^2 .
\end{aligned}$$

Let us sumarize our computations:

Theorem 2.3.4. *For $\alpha \in \mathbb{N}^d$ with $s \geq |\alpha| \geq 1$, we have $\partial^\alpha : H^s(\mathbb{T}^d) \to H^{s-|\alpha|}(\mathbb{T}^d)$, where*

$$\widehat{(\partial^\alpha f)}_k = (2\pi \mathrm{i} k)^\alpha \, \hat{f}_k .$$

For $s \geq 2$ and $f \in H^{s-2}(\mathbb{T}^d)$, consider Poisson's equation

$$-\Delta u = f \tag{2.18}$$

in $L^2(\mathbb{T}^d)$. By applying $\Delta = \sum_{i=1}^d \partial_i^2$, the Fourier coefficients must satisfy

$$4\pi^2 \|k\|^2 \hat{u}_k = \hat{f}_k, \quad k \in \mathbb{Z}^d .$$

Hence, we may define u by

$$\hat{u}_k = \begin{cases} \frac{1}{4\pi^2\|k\|^2} \hat{f}_k, & k \in \mathbb{Z}^d \setminus \{0\}, \\ c, & k = 0, \end{cases} \tag{2.19}$$

where $c \in \mathbb{C}$ is an arbitrary constant. It yields $\int_{I^d} u(x) \mathrm{d}x = c$ and we indeed observe $u \in H^s(\mathbb{T}^d)$.

Theorem 2.3.5. *For $s \geq 2$ and $f \in H^{s-2}(\mathbb{T}^d)$, we may define $u^t \in \Pi_t$ by*

$$(\widehat{u^t})_k := \begin{cases} \frac{1}{4\pi^2\|k\|^2} \hat{f}_k, & 1 \leq \|k\|_\infty \leq t, \\ 0, & \textit{otherwise.} \end{cases}$$

If $u \in H^s(\mathbb{T}^d)$ solves Poisson's equation (2.18) with $c = 0$ in (2.19), then we have

$$\left\|u^t - u\right\|_{L^2} \leq t^{-s} \frac{\|f\|_{H^{s-2}}}{2\pi^2} .$$

Proof. Theorem 2.3.2 implies

$$\left\|u^t - u\right\|_{L^2} \le t^{-s}\|u\|_{H^s}.$$

We further estimate

$$\begin{aligned}
\|u\|_{H^s}^2 &= \sum_{k\in\mathbb{Z}^d\setminus\{0\}} (1+\|k\|^2)^s \left|\hat{u}_k\right|^2 \\
&= \sum_{k\in\mathbb{Z}^d\setminus\{0\}} (1+\|k\|^2)^{s-2} \left|\hat{f}_k\right|^2 \frac{(1+\|k\|^2)^2}{(4\pi^2\|k\|^2)^2} \\
&= \frac{1}{16\pi^4} \sum_{k\in\mathbb{Z}^d\setminus\{0\}} (1+\|k\|^2)^{s-2} \left|\hat{f}_k\right|^2 \left(\frac{1+\|k\|^2}{\|k\|^2}\right)^2 \\
&\le \frac{1}{4\pi^4} \sum_{k\in\mathbb{Z}^d\setminus\{0\}} (1+\|k\|^2)^{s-2} \left|\hat{f}_k\right|^2 \underbrace{\left(\frac{1+\|k\|^2}{2\|k\|^2}\right)^2}_{\le 1} \\
&\le \frac{1}{4\pi^4}\|f\|_{H^{s-2}}^2.
\end{aligned}$$

□

How do we compute $\hat{f}_k$?

Chapter 3

Fourier transform of functions on $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}^N$

3.1 The transition to other domains

We first introduced the Fourier series, since they are, in a certain sense, the most natural instance of Fourier transforms. The basic idea should be clear by now: a (periodic, so far) function can be represented by a sum of weighted sinusoids, and the sinusoids can be interpreted as the frequencies present in the function (signal). We will now push this concept a bit further. First, we will simply turn around the interpretation of the two variables involved in the definition of Fourier series, namely time t and frequency, which has so far been chosen to live on a discrete subset of $\mathbb{R}$ and labeled by the integers. The next step is of vital importance for understanding the world of digital signal processing, which is, in fact behind almost any modern technical tools we use. Indeed, if we assume, that the frequency information contained in a signal is contained in an interval of finite length, in other words, if the signal (function) is *band-limited*, we may - mutatis mutandis - expand its frequency-information in a "Fourier series", and the corresponding coefficients will then contain the time-information. However, as we have seen so far, this is discrete information, indexed by $k \in \mathbb{Z}$ - in Section 3.1.1 we thus arrive naturally at a concept of Fourier transform for discrete signals - as a dual concept of the Fourier series.
On the other hand, and complementary to the approach just described, we may think of a periodic time-signal for gradually growing period, which means that, for $p \to \infty$, the size of $\frac{k}{p}$ in the definition of the sinusoids $\{e^{2\pi i \frac{k}{p} x}\}_{k\in\mathbb{Z}}$ becomes infinitely small. This idea leads to Fourier transforms on $\mathbb{R}$, introduced in Section 3.1.2, with an integral replacing the sum in the representation.

3.1.1 The discrete Fourier transform

Let $f : \mathbb{Z} \mapsto \mathbb{C}$ be a function defined on the integers. We consider the complex exponentials $e^{2\pi isn}$, $n \in \mathbb{Z}$, $s \in \mathbb{R}$ and observe immediately, that

$$e^{2\pi i(s+m)n} = e^{2\pi isn} \text{ for all } n, m \in \mathbb{Z}.$$

In other words, the exponentials $e^{2\pi isn}, e^{2\pi i(s\pm1)n}, e^{2\pi i(s\pm2)n}, \ldots$ cannot be distinguished if n are integer values. Hence, in order to avoid ambiguity, we will synthesize f from $e^{2\pi isn/p}$, for $0 \leq s < p$ and $n \in \mathbb{Z}$.

Definition 3.1.1 (DFT)**.** *The discrete Fourier transform of a (suitably regular) function f on $\mathbb{Z}$ is defined as*

$$\hat{f}(s) = F(s) = \frac{1}{p} \sum_{n=-\infty}^{\infty} f[n] e^{-2\pi isn/p} \text{ for } 0 \leq s < p \tag{3.1}$$

f can then be written as

$$f[n] = \int_{s=0}^{p} F(s) e^{2\pi isn/p} ds. \tag{3.2}$$

Note that the Fourier transform $\hat{f} = F$ of a discrete-valued function is a function on the circle with diameter of length p.

Remark 3.1.2. *Note that normally p is set to 1, so that the frequencies that occur in a signal are normalized, on the unit circle. We will see later, when we discuss sampling (in fact, any signal on $\mathbb{Z}$ is a digital, hence sampled signal, unless it stems from a, inherently discrete process, e.g. a time-series of stock exchange values), that $\frac{1}{2}$ corresponds to the highest frequency that occurs in a real signal. The frequencies in the interval $]\frac{1}{2}, 1[$ are then the negative frequencies.*
We introduced p in the above definition to guarantee generality and to emphasize the parallelism to Fourier series.

Example 3.1.3 (Dirac Impulse)**.** *Consider the function δ defined on $\mathbb{Z}$, that is equal to 0 everywhere, except for $\delta[0] = 1$. It is then easy to see, that the DFT of δ is given by $\hat{\delta}(s) = 1/p$ for all $s \in [0, p]$.*

Example 3.1.4 (Sinusoid)**.**

3.1.2 The Fourier transform of functions on $\mathbb{R}$

We now consider integrable functions on $\mathbb{R}$.
Remark on the spaces L^1 and L^2 of integrable and square-integrable functions.

Definition 3.1.5. *The Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined by*

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i \omega t} dt. \tag{3.3}$$

Note that, if f is integrable, the integral in (3.3) converges and

$$|\hat{f}(\omega)| \leq \int_{-\infty}^{\infty} |f(t)| dt < \infty.$$

Proposition 3.1.6 (Inverse Fourier transform). *If $\hat{f} \in L^1(\mathbb{R})$, then f is given by the inverse Fourier transform of $\hat{f}$:*

$$f(t) = \int_{-\infty}^{\infty} \hat{f}\omega) e^{2\pi i \omega t} d\omega \tag{3.4}$$

Remark 3.1.7. *Note that the Fourier transform is usually extended to all functions in $L^2(\mathbb{R})$ by using a density argument, similar to our approach in the proof of Proposition 2.1.8. Then, as before, an inner product can be defined on $L^2(\mathbb{R})$, and most arguments work similar to the case of periodic functions.*

Example 3.1.8 (Fourier transform of the box function). *Consider the function*

$$\Pi(x) := \begin{cases} 1 & \textit{for } -\frac{1}{2} < x < \frac{1}{2} \\ 0 & \textit{else} \end{cases}$$

To compute the Fourier transform, first note that Π is even, so that we can omit the sine-part (generally we can observe, that the Fourier transform of even (symmetric) functions is always real! On the other hand, the Fourier transform of real functions is symmetric and we only have to consider the positive frequencies. This property is heavily exploited in the processing of speech and music signals, which are always real.) We therefore have

$$\begin{aligned} \hat{\Pi}(\omega) &= \int_{\mathbb{R}} \Pi(x) e^{-2\pi i \omega x} dx = \int_{\mathbb{R}} \Pi(x) \cos(2\pi\omega x) dx \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(2\pi\omega x) dx = \frac{\sin(2\pi\omega x)}{2\pi\omega} \Big|_{x=-\frac{1}{2}}^{\frac{1}{2}} \\ &= \frac{\sin(\pi\omega)}{2\pi\omega} - \frac{\sin(-\pi\omega)}{2\pi\omega} = \frac{\sin(\pi\omega)}{\pi\omega} =: sinc(\omega) \end{aligned}$$

Note that $sinc(x)$ can be defined as $sinc = \frac{\sin(\pi x)}{\pi x}$ only for $x \neq 0$. However according to L'Hôpital's rule, we have that $\lim_{x\to 0} sinc(x) = 1$, since, for any open interval I around 0, we have $h'(x) = 1 \neq 0$ for $h(x) = x$ and, since $\frac{d}{dx}\sin x = cosx$ and $\lim_{x\to 0}\cos(x) = 1$, we have $\lim_{x\to 0}\frac{\sin' x}{g'(x)} = 1$, and so $\lim_{x\to 0}\frac{\sin x}{x} = 1$. More directly, we may consider the Taylor series $\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$, such that $sinc(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n}$ for $x \neq 0$, and convergence to 1 is obvious.

We will next address two very basic operators that can act on a function or signal, namely translation, or *time-shift* and modulation, or *frequency-shift*.

Example 3.1.9 (Translation and Modulation)**.** *For any real number x_0, if $g(x) = T_{x_0}f(x) := f(x-x_0)$, then $\hat{g}(\omega) = e^{-2\pi i x_0 \omega}\hat{f}(\omega)$.*
For any real number ω_0, if $g(x) = M_{\omega_0} := e^{2\pi i x \omega_0} f(x)$, then $\hat{g}(\omega) = \hat{f}(\omega - \omega_0)$.

Example 3.1.10 (Dilation)**.** *Let $a \neq 0 \in \mathbb{R}$. Set $g(x) = D_{\frac{1}{a}} f(x) := f(ax)$. Let $\hat{f}$ be the Fourier transform of f. Then, the Fourier transform of g is given by*

$$\hat{g}(\omega) = \frac{1}{a}\hat{f}(\frac{\omega}{a}).$$

3.2 Filters and convolution

We all know filters, since they are all around us. Every room is a filter, our own mouth is a filter, and of course filters are part of any modern audio equipment. Light is filtered by the air etc.
If we think about the characteristics of filters, then one of the most striking one is the fact that it shouldn't matter whether a signal is filter at an earlier time or later on. In other words, a filter is a time-invariant system. Let us denote our filter by L, and we assume that any input signal f is then mapped to an output Lf. We will hope to work with linear filters, so that we arrive at the class of linear, time-invariant systems.

Definition 3.2.1 (Linear, time-invariant (LTI) systems)**.** *A linear operator L that maps functions $f \in V$ to $Lf \in V$, where V is a vector space, is called time-invariant, if*

$$L(f(t-u)) = L(f)(t-u), \text{ equivalently: } L(T_u f) = T_u(Lf).$$

Let us now look at a very fundamental concept, the impulse response. Any LTI-system is completely characterized by its impulse response. That is, for any input function, the output function can be calculated in terms of the input and the impulse response. The impulse response of a linear transformation is the image of Dirac's delta function under the transformation. [1]
We now consider the mathematical derivation of impulse response of an LTI-system L . Note that

$$f(t) = \int_u f(u)\delta_u(t)du = \int_u f(u)\delta(t-u)du \tag{3.5}$$

hence, because L is linear

$$Lf(t) = \int_u f(u)L\delta_u(t)du \tag{3.6}$$

[1] In practical situations, it is not possible to produce a true impulse used for testing. Therefore, some other brief, explosive sound is sometimes used as an approximation of the impulse. In acoustic and audio applications, impulse responses enable the acoustic characteristics of a location, such as a concert hall, to be captured. These impulse responses can be used in applications to mimic the acoustic characteristics of a particular location.

Finally, we use the last property of L, namely time-invariance, to see that $L\delta_u(t) = L(\delta(t-u)) = (L\delta)(t-u)$, hence

$$Lf(t) = \int_u f(u)L\delta_u(t)du = \int_u f(u)(L\delta)(t-u)du \tag{3.7}$$

Setting $h(t) := (L\delta)(t)$, we achieve

$$Lf(t) = \int_u f(u)h(t-u)du =: h * f. \tag{3.8}$$

As we see from (3.8), an LTI-system is completely characterized by its impulse response.

Definition 3.2.2 (Impulse Response)**.** *Let L be an LIT-system. Its impulse response is defined as $h(t) = L\delta(t)$.*

Example 3.2.3 (Discrete Impulse response)**.**

Definition 3.2.4 (Convolution)**.** *The convolution of two functions $f, g \in L^1(\mathbb{R})$ is defined by*

$$(f * g)(t) = \int_u f(u)g(t-u)du = \int_u g(u)f(t-u)du = (g * f)(t) \tag{3.9}$$

For functions f, g on $\mathbb{Z}$, we define

$$(f * g)[n] = \sum_{m=-\infty}^{\infty} f[m]g[n-m] = \sum_{m=-\infty}^{\infty} g[m]f[n-m] = (g * f)[n] \tag{3.10}$$

For functions f, g on $\mathbb{Z}_N$, which are simply vectors in $\mathbb{C}^N$, periodically extended, i.e., we let $f[n] = f[n']$ and $g[n] = g[n']$ if $n \equiv n'(modN)$ and define

$$(f * g)[n] = \sum_{m=0}^{N-1} f[m]g[n-m]du = \sum_{m=0}^{N-1} g[m]f[n-m]du = (g * f)[n]. \tag{3.11}$$

Example 3.2.5. *Linear averaging over $[-T, T]$:*

$$Lf(t) = \frac{1}{2T}\int_{t-T}^{t+T} f(u)du = \frac{1}{2T}\int_t 1_{[-T,T]}(t-u)f(u)du = (1_{[-T,T]} * f)(t)$$

Excursus: eigenfunctions

Eigenfunctions of a linear operator L defined on some function space are non-zero functions h such that

$$Lh = \lambda h$$

for some scalar λ the corresponding eigenvalue. In the theory of of signals and systems, the eigenfunction of a system is the signal h which produces a scalar multiple (with a possibly

complex scalar $\lambda \in \mathbb{C}$ of itself as response to the system.
Now assume that there is an ONB $\{\varphi_k\}_{k\in\mathbb{Z}}$ of V of eigenfunctions of a mapping (system, operator) L, i.e.

$$f = \sum_k c_k \varphi_k, \quad \text{for all } f \in V.$$

Then:

$$Lf = \sum_k c_k L\varphi_k = \sum_k \lambda_k c_k \varphi_k$$

In other words, L can be evaluated as a multiplication on the coefficients of the function's expansion. Formel ..
End of excursus.
Since we have already seen, that complex exponentials provide nice signal expansion, with a natural interpretation, let us next consider, what happens to them, if we let an LTI system act on them. Intuitively, if we recall, that LTI systems "are filters", we should expect that complex exponentials corresponding to a particular frequency should merely be amplified, damped and maybe there phase can be shifted. Now, for an LTI system L with impulse response h we have in fact:

$$Le^{2\pi i\omega t} = \int_u e^{2\pi i\omega u} h(t-u)du \tag{3.12}$$

$$= \int_u e^{2\pi i\omega(t-u)} h(u)du \tag{3.13}$$

$$= e^{2\pi i\omega t} \int_u e^{-2\pi i\omega u} h(u)du = e^{2\pi i\omega t}\hat{h}(\omega) \tag{3.14}$$

More generally, we have the following

Proposition 3.2.6. *For $f, g \in L^1(\mathbb{R}^d)$, we have*

$$\widehat{f * g} = \hat{f}\hat{g}.$$

Proof. Theorem 1.1.6 ensures $f * g \in L^1(\mathbb{R}^d)$ and Fubini leads to

$$\begin{aligned}
\widehat{f * g}(\xi) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y)g(x-y)\mathrm{e}^{-2\pi\mathrm{i}\langle x,\xi\rangle}\mathrm{d}y\mathrm{d}x \\
&= \int_{\mathbb{R}^d} f(y)\int_{\mathbb{R}^d} g(x-y)\mathrm{e}^{-2\pi\mathrm{i}\langle x-y,\xi\rangle}\mathrm{d}x\, \mathrm{e}^{-2\pi\mathrm{i}\langle y,\xi\rangle}\mathrm{d}y \\
&= \int_{\mathbb{R}^d} f(y)\hat{g}(\xi)\mathrm{e}^{-2\pi\mathrm{i}\langle y,\xi\rangle}\mathrm{d}y = \hat{f}(\xi)\hat{g}(\xi).
\end{aligned}$$

□

Example 3.2.7. *Convolution of box functions. Define triangle function:*

$$\Lambda(x) = \begin{cases} 1-|x| & |x| \leq 1 \\ 0 & else \end{cases} \tag{3.15}$$

Then $(\Pi_I * \Pi_I)(x) = \Lambda(x)$*, if* $I = [-\frac{1}{2}, \frac{1}{2}]$*, since*

$$\begin{aligned}(\Pi_I * \Pi_I)(y) &= \int_{-\infty}^{\infty} \Pi_I(x)\Pi_I(y-x)dx \\ &= \begin{cases} \int_{-\frac{1}{2}}^{y+\frac{1}{2}} 1dx & -1 < y \leq 0 \\ \int_{y-\frac{1}{2}}^{\frac{1}{2}} 1dx & 0 < y < 1 \end{cases} \\ &= \begin{cases} y+1 & -1 < y \leq 0 \\ 1-y & 0 < y < 1 \\ 0 & else \end{cases},\end{aligned}$$

qed.
Cf. `https://en.wikipedia.org/wiki/File:Convolution_of_box_signal_with_itself2.gif`

Example 3.2.8. *Convolution with sinc*

3.2.1 The Fourier transform on L^2

Definition 3.2.9 (Fourier transform and inverse Fourier transform)**.** *Let* $\langle \xi, x\rangle$ *denote the standard inner product in* $\mathbb{R}^d$*. For* $f \in L^1(\mathbb{R}^d)$*, we call*

$$(\mathcal{F}f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)\mathrm{e}^{-2\pi\mathrm{i}\langle \xi, x\rangle}\mathrm{d}x \tag{3.16}$$

the Fourier transform *of* f*.*
For $f \in L^1(\mathbb{R}^d)$*, we define the inverse Fourier transform* $\check{f} : \mathbb{R}^d \to \mathbb{C}$ *by*

$$\check{f}(\xi) := \hat{f}(-\xi) = \int_{\mathbb{R}^d} f(x)\mathrm{e}^{2\pi i\langle x, \xi\rangle}\mathrm{d}x.$$

Lemma 3.2.10. *For* $f \in L^1(\mathbb{R}^d)$*, we have* $\hat{f} \in \mathcal{C}(\mathbb{R}^d)$ *and* $\|\hat{f}\|_\infty \leq \|f\|_{L^1}$*.*

Proof. We estimate

$$\left|\hat{f}(\xi)\right| = \left|\int_{\mathbb{R}^d} f(x)\mathrm{e}^{-2\pi\mathrm{i}\langle x, \xi\rangle}\mathrm{d}x\right| \leq \int_{\mathbb{R}^d} |f(x)|\mathrm{d}x.$$

If $\xi_n \to \xi$, then $f(x)\mathrm{e}^{-2\pi\mathrm{i}\langle \xi_n, x\rangle} \to f(x)\mathrm{e}^{-2\pi\mathrm{i}\langle \xi, x\rangle}$, for all $x \in \mathbb{R}^d$. Since $\left|f(x)\mathrm{e}^{-2\pi\mathrm{i}\langle \xi_n, x\rangle}\right| \leq |f(x)|$, the dominated convergence Theorem 1.1.5 implies $\hat{f}(\xi_n) \to \hat{f}(\xi)$. ☐

For $f \in L^1(\mathbb{R}^d)$, we would like to apply the "inverse" Fourier transform to $\hat{f}$ but Lemma 3.2.10 suggests that we could be faced with $\hat{f} \notin L^1(\mathbb{R}^d)$. Therefore, we replace $L^1(\mathbb{R}^d)$ with a "nicer" function space that better fits to the Fourier transform. For $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $\alpha, \beta \in \mathbb{N}^d$, define

$$p_{\alpha,\beta}(f) := \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta f(x)|$$

Definition 3.2.11 (Schwartz space)**.** *The* Schwartz space *is*

$$\mathcal{S}(\mathbb{R}^d) := \{f \in \mathcal{C}^\infty(\mathbb{R}^d) : \forall \alpha, \beta \in \mathbb{N}^d \; p_{\alpha,\beta}(f) < \infty\},$$

where $f_n \to f$ if and only if $p_{\alpha,\beta}(f_n - f) \to 0 \; \forall \alpha, \beta \in \mathbb{N}^d$.

Proposition 3.2.12. *For $f, g \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$\forall \alpha \in \mathbb{N}^d \qquad \partial^\alpha f, \quad f \cdot g, \quad f * g \quad \in \; \mathcal{S}(\mathbb{R}^d),$$

and $\partial^\alpha (f * g) = (\partial^\alpha f) * g$.

Proof. [1]. ☐

Although

$$\Delta : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d), \tag{3.17}$$

there are other issues...

Lemma 3.2.13. *For $f \in \mathcal{S}(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}^d$, we have*

$$\widehat{\partial^\alpha f}(\xi) = (2\pi \mathrm{i} \xi)^\alpha \hat{f}(\xi), \qquad \partial^\alpha \hat{f} = ((-2\pi \mathrm{i} \cdot)^\alpha f)\widehat{}\,.$$

Proof. Integration by parts with $f(\pm\infty) = 0$ yields

$$\begin{aligned} \widehat{\frac{\partial}{\partial x_k} f}(\xi) &= \int_{\mathbb{R}^d} \frac{\partial}{\partial x_k} f(x) \mathrm{e}^{-2\pi \mathrm{i}\langle x, \xi \rangle} \mathrm{d}x \\ &= -\int_{\mathbb{R}^d} f(x) \frac{\partial}{\partial x_k} \mathrm{e}^{-2\pi \mathrm{i}\langle x, \xi \rangle} \mathrm{d}x \\ &= -\int_{\mathbb{R}^d} f(x)(-2\pi \mathrm{i} \xi_k) \mathrm{e}^{-2\pi \mathrm{i}\langle x, \xi \rangle} \mathrm{d}x = 2\pi \mathrm{i} \xi_k \hat{f}(\xi). \end{aligned}$$

Interchange of differentiation and integration yields

$$\begin{aligned} (\frac{\partial}{\partial_k \xi} \hat{f})(\xi) &= \frac{\partial}{\partial_k \xi} \int_{\mathbb{R}^d} f(x) \mathrm{e}^{-2\pi \mathrm{i}\langle x, \xi \rangle} \mathrm{d}x \\ &= \int_{\mathbb{R}^d} \frac{\partial}{\partial_k \xi} \left(f(x) \mathrm{e}^{-2\pi \mathrm{i}\langle x, \xi \rangle} \right) \mathrm{d}x \\ &= -\int_{\mathbb{R}^d} 2\pi \mathrm{i} x_k f(x) \mathrm{e}^{-2\pi \mathrm{i}\langle x, \xi \rangle} \mathrm{d}x. \end{aligned}$$ ☐

Lemma 3.2.13 implies that (3.17) is equivalent to

$$4\pi^2 \|\xi\|^2 \hat{u}(\xi) = \hat{f}(\xi), \qquad \xi \in \mathbb{R}^d. \tag{3.18}$$

Lemma 3.2.14. *For $f : \mathbb{R}^d \to \mathbb{R}$, $x \mapsto \mathrm{e}^{-\pi\|x\|^2}$, we have $\hat{f} = f$.*

Proof. See Lecture. □

Proposition 3.2.15. *Let $\mathcal{U} = (u_\epsilon)_\varepsilon$ be an approximate identity. Then for $f \in L^1(\mathbb{R}^d)$, it holds that*

$$\lim_{\varepsilon \to 0} \|f - f * u_\epsilon\|_1 = 0$$

Furthermore, there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$, $\varepsilon_k > 0$ such that

$$\lim_{k\to\infty} f * u_\epsilon(t) = f(t) \qquad a.e.$$

The technical proof is omitted.

Example 3.2.16. *The family $\mathcal{U} = (g_\epsilon)_\varepsilon$, defined as*

$$g_\varepsilon(t) = e^{\frac{-\pi x^2}{\varepsilon}}$$

is an approximate identity.

Theorem 3.2.17. *Let $f \in L^1(\mathbb{R}^d)$ with $\hat{f} \in L^1(\mathbb{R}^d)$. Then*

$$f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega) \mathrm{e}^{2\pi i \langle \omega, x\rangle} \mathrm{d}\omega.$$

Proof. See lecture. □

Corollary 3.2.18. *Suppose that $f \in L^1(\mathbb{R})$ is continuous in 0 and that $\hat{f}(\omega) \geq 0$ for all $\omega \in \mathbb{R}$. Then*

$$f(0) = \int_{\mathbb{R}} \hat{f}(\omega) \mathrm{d}\omega.$$

Using the previous result we can show the following.

Theorem 3.2.19. *For $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, we have*

$$\|f\|_{L^2}^2 = \|\hat{f}\|_{L^2}^2. \tag{3.19}$$

Furthermore, we have, by polarization, for $f, g \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$,

$$\langle f, g\rangle_{L^2} = \langle \hat{f}, \hat{g}\rangle_{L^2}. \tag{3.20}$$

Proof. See lecture. □

We now extend $\mathcal{F}$ to $L^2(\mathbb{R}^d)$: for $f \in L^2(\mathbb{R}^d, \mathbb{C})$,

$$\int_{\mathbb{R}^d} \underbrace{f(x)\mathrm{e}^{-2\pi \mathrm{i}\langle x, \xi\rangle}}_{\notin L^1} \mathrm{d}x$$

is not well-defined.

Theorem 3.2.20. *$\mathcal{F}$ can be uniquely and continuously extended from $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ to an isometric isomorphism $\mathcal{F} : L^2(\mathbb{R}^d) \mapsto L^2(\mathbb{R}^d)$, that is, for all $f, g \in L^2(\mathbb{R}^d)$ we have*

$$\langle f, g\rangle_{L^2} = \langle \hat{f}, \hat{g}\rangle_{L^2}. \tag{3.21}$$

and

$$f = \mathcal{F}^*\mathcal{F}f,$$

where $\mathcal{F}^*$ is the adjoint of $\mathcal{F}$, as defined in Definition 3.2.9:

$$\mathcal{F}^*f(\xi) = \int_{\mathbb{R}^d} f(x)\mathrm{e}^{2\pi i\langle x,\xi\rangle}\mathrm{d}x.$$

Proof. See lecture. ☐

3.3 The finite discrete Fourier transform

So far, we have been dealing with functions/signals that are

1. continuous in time and have infinite duration; they have a Fourier transform that is continuous in time and has infinite bandwidth ("duration in frequency").
2. discrete in time and have infinite duration; they have a Fourier transform that is continuous in frequency and has finite bandwidth.
3. continuous in time and have finite duration (in other words; they are periodic); they have a Fourier transform that is discrete in frequency and has infinite duration.

We notice that finite duration (or, equivalently, periodicity) in one domain (i.e. in time or frequency) leads to discreteness in the other domain. We now address the fourth case, which is the case of signals, that are both finite in duration *and* discrete, or, discrete and periodic.

Remark 3.3.1. *We want to point out that finite duration is equivalent to periodicity only in the sense, that the entire information that is contained in the signal is actually contained in an interval of finite length. in this sense, a periodic function can be identified with a function supported on an interval of finite length or on the torus $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. As we will see later, in the context of sampling, we may have to distinguish meticulously between periodic signals and signals that are supported in an interval of finite duration and are zero elsewhere.*

Definition 3.3.2 (Finite discrete Fourier transform)**.** *The finite discrete Fourier transform of $f \in \mathbb{C}^N$, i.e. of a vector of N complex numbers is given by*

$$\mathcal{F}'f[k] = \hat{f}[k] = F[k] = \frac{1}{N}\sum_{n=0}^{N-1} f[n] \cdot e^{-i2\pi\frac{k}{N}n}. \tag{3.22}$$

The inverse transform yields the expansion of f as

$$\mathcal{F}\mathcal{F}' f[n] = \mathcal{F}\hat{f}[n] = f[n] = \sum_{k=0}^{N-1} F[k] \cdot e^{i2\pi \frac{n}{N} k}. \tag{3.23}$$

Remark 3.3.3. *The Fourier transforms we discussed so far, gave us information about the amount of any pure frequency, i.e. complex exponentials, present in a given signal. Obviously, the sinusoid must have the same basic properties as the underlying signal under consideration: for periodic signals with a certain period p, we only considered sinusoids with the same period, for discrete-time signals we only considered discrete-time sinusoids, whereas, for continuous time signals of infinite duration, any complex exponential is a candidate in the expansion* (3.3).
For the finite, discrete signals, which are in fact vectors in $\mathbb{C}^N$, we may ask, how many complex exponentials are eligible for the definition of a corresponding Fourier transform. We have two criteria:
(a) they should be periodic with length N, that is, we require that

$$e^{2\pi i s(n+N)} = e^{2\pi i s n}, \text{ for all } n,$$

which means that $s = \frac{k}{N}$.
(b) we observe that for all $m \in \mathbb{Z}$:

$$e^{2\pi i k \frac{n}{N}} = e^{2\pi i (k+mN) \frac{n}{N}}, \text{ for all } n,$$

which is a similar phenomenon as observed for discrete-time sinusoids before. This means, that, since $s = \frac{k}{N}, s = \frac{k \pm N}{N}, s = \frac{k \pm 2N}{N}, \ldots$ all give the same signals, we have only N distinct sinusoid adequate for analyzing our N-finite, discrete signals, namely $e^{2\pi i k \frac{n}{N}}$, for $k = 0, \ldots, N-1$.
Of course, this is exactly what you should have expected: since the complex exponentials have provided ONBs so far, they should provide an orthonormal basis for $\mathbb{C}^N$ as well. Obviously, this means, that there should be N of them.

Example 3.3.4. *Note that for $k = 0$, $e^{2\pi i k \frac{n}{N}}$ is constantly equal to 1, then, the rotation of the vector $e^{2\pi i k \frac{n}{N}}$, that rotates, as n goes from 0 to $N-1$, accelerates with growing k: $k = 1$ corresponds to a single rotation, $k = 2$ to 2 rotations, etc., up to $N/2$, from where the frequencies decrease, since they become negative.*

We next show, that the vectors $e^{2\pi i k \frac{n}{N}}$ in fact form an orthogonal basis.

Proposition 3.3.5. *The vectors s_k, $k = 0, \ldots, N-1$, with entries $s_k[n] = e^{2\pi i k \frac{n}{N}}$ are orthogonal in $\mathbb{C}^N$. The set $\{\frac{1}{\sqrt{N}} s_k, k = 0, \ldots, N-1\}$ is an ONB.*

Proof.

$$\begin{aligned}\langle s_k, s_l\rangle &= \sum_{n=0}^{N-1} s_k[n]\overline{s_l[n]} \\ &= \sum_{n=0}^{N-1} e^{2\pi i k \frac{n}{N}} e^{-2\pi i l \frac{n}{N}} \\ &= \sum_{n=0}^{N-1} e^{2\pi i (k-l)\frac{n}{N}} = \frac{1-e^{2\pi i(k-l)}}{1-e^{2\pi i(k-l)/N}} \end{aligned} \tag{3.24}$$

where the last step follows from the well-known formula for geometric series: $\sum_{n=0}^{N-1} z^n = \frac{1-z^N}{1-z}$. Now, (3.24) is zero, if $k \neq l$, and for $k = l$, we evaluate the sum as $\sum_{n=0}^{N-1} e^{2\pi i(k-l)\frac{n}{N}} = \sum_{n=0}^{N-1} 1 = N$, therefore, the normalization $\frac{1}{\sqrt{N}} s_k$ leads to $\langle \frac{1}{\sqrt{N}} s_k, \frac{1}{\sqrt{N}} s_k\rangle = \frac{1}{N}\|s_k\|_2^2 = 1$. □

Example 3.3.6. • *The Delta Function*

- *The Constant Function*

- *The Delta train or Dirac comb on $\mathbb{C}^N$*
  When $m = 1, 2, \dots$ divides N , we define the Dirac comb as

$$Ш_m[n] = \begin{cases} 1 & \text{if } n = 0, \pm m, \pm 2m, \dots \\ 0 & \text{otherwise} \end{cases} \tag{3.25}$$

  Here, m specifies the spacing between the "teeth" hence $m' := N/m$ is the number of teeth. We can easily verify that $Ш_m$ has the Fourier transform

$$\widehat{Ш}_m[k] = \frac{1}{m} Ш_{N/m}[k]. \tag{3.26}$$

- *Periodicity on $\mathbb{C}^N$:*
  Let $N = m \cdot m'$, $m, m' \in \mathbb{N}^+$ and assume that f is m-periodic on $\mathbb{C}^N$. We show that $\hat{f}[k] = 0$ if k is not a multiple of m':
  Since f is m-periodic, we have $f[n+m] - f[n] = 0$. Now, since, for $n_0 \in \mathbb{Z}$

$$g[n] = f[n-n_0] \text{ has the Fourier transform } \hat{g}[k] = e^{-2\pi i k n_0/N}\hat{f}[k],$$

  we may write

$$(e^{2\pi i k m/N} - 1)\hat{f}[k] = (e^{2\pi i k/m'} - 1)\hat{f}[k] = 0$$

  hence $\hat{f}[k] = 0$ if $m' \neq k$.

3.3.1 Fast (discrete) Fourier Transform

Among the top 10 algorithms of the 20th century by the IEEE magazine Computing in Science & Engineering.

...the most important numerical algorithm of our lifetime.
(Gilbert Strang)

We restrict us to $d = 1$. For $2 \le N \in 2\mathbb{N}$, we consider

$$\hat{f}_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\mathrm{e}^{-2\pi\mathrm{i}kx}\mathrm{d}x \approx \frac{1}{N} \sum_{j=-\frac{N}{2}}^{\frac{N}{2}-1} f(\tfrac{j}{N})\mathrm{e}^{-2\pi\mathrm{i}k\frac{j}{N}}.$$

To bound the quadrature error, we define

$$\hat{\mathbf{f}}_k := \frac{1}{N} \sum_{j=-\frac{N}{2}}^{\frac{N}{2}-1} f(\tfrac{j}{N})\mathrm{e}^{-2\pi\mathrm{i}k\frac{j}{N}}, \qquad k \in \mathbb{Z}\,.$$

Lemma 3.3.7 (Character sums). *For $2 \le N \in 2\mathbb{N}$ and $m \in \mathbb{Z}$, we have*

$$\frac{1}{N} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \mathrm{e}^{-2\pi\mathrm{i}m\frac{n}{N}} = \begin{cases} 1, & m \in N\,\mathbb{Z}, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. We compute

$$\begin{aligned} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \mathrm{e}^{-2\pi\mathrm{i}m\frac{n}{N}} &= \sum_{n=0}^{\frac{N}{2}-1} \mathrm{e}^{-2\pi\mathrm{i}m\frac{n}{N}} + \sum_{n=-\frac{N}{2}}^{-1} \mathrm{e}^{-2\pi\mathrm{i}m\frac{n}{N}} \\ &= \sum_{n=0}^{\frac{N}{2}-1} \mathrm{e}^{-2\pi\mathrm{i}m\frac{n}{N}} + \sum_{n=\frac{N}{2}}^{N-1} \mathrm{e}^{-2\pi\mathrm{i}m\frac{n-N}{N}} \\ &= \sum_{n=0}^{N-1} \mathrm{e}^{-2\pi\mathrm{i}m\frac{n}{N}} \end{aligned}$$

The case $m \in N\,\mathbb{Z}$ is now obvious. For $m \notin N\,\mathbb{Z}$, we have $\mathrm{e}^{-2\pi\mathrm{i}\frac{m}{N}} \neq 1$ and the geometric series leads to

$$\sum_{n=0}^{N-1} \mathrm{e}^{-2\pi\mathrm{i}m\frac{n}{N}} = \frac{1 - \mathrm{e}^{-2\pi\mathrm{i}\frac{N}{N}}}{1 - \mathrm{e}^{-2\pi\mathrm{i}\frac{m}{N}}} = 0.$$

□

Theorem 3.3.8 (Aliasing). *For $f \in \mathcal{C}^1(\mathbb{T})$, we have*

$$\hat{\mathbf{f}}_k = \sum_{l\in\mathbb{Z}} \hat{f}_{k+lN}, \qquad k \in \mathbb{Z}\,.$$

Proof. According to Theorem 2.2.6, we observe

$$\begin{aligned}
\hat{\mathbf{f}}_k &= \frac{1}{N}\sum_{j=-\frac{N}{2}}^{\frac{N}{2}-1} f(\tfrac{j}{N})\mathrm{e}^{-2\pi\mathrm{i}k\frac{j}{N}} \\
&= \frac{1}{N}\sum_{j=-\frac{N}{2}}^{\frac{N}{2}-1}\sum_{m\in\mathbb{Z}} \hat{f}_m \mathrm{e}^{2\pi\mathrm{i}m\frac{j}{N}}\mathrm{e}^{-2\pi\mathrm{i}k\frac{j}{N}} \\
&= \sum_{m\in\mathbb{Z}} \hat{f}_m \frac{1}{N}\sum_{j=-\frac{N}{2}}^{\frac{N}{2}-1} \mathrm{e}^{2\pi\mathrm{i}(m-k)\frac{j}{N}} \\
&= \sum_{m\in\mathbb{Z}} \hat{f}_m \cdot \begin{cases} 1, & m-k\in N\,\mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases}
\end{aligned}$$

where we have applied Lemma 3.3.7. The requirement $m \in k+N\,\mathbb{Z}$ concludes the proof. □

Corollary 3.3.9. *For $f \in \Pi_t$ with $t < \frac{N}{2}$, we have*

$$\hat{\mathbf{f}}_k = \hat{f}_k, \qquad k = -\frac{N}{2},\dots,\frac{N}{2}-1.$$

Corollary 3.3.10. *For $f \in \mathcal{C}^1(\mathbb{T})$, we have*

$$\left|\hat{\mathbf{f}}_k - \hat{f}_k\right| \le \sum_{0\neq l\in\mathbb{Z}} \left|\hat{f}_{k+lN}\right|, \qquad k = -\frac{N}{2},\dots,\frac{N}{2}-1.$$

Remark 3.3.11. *One can show that $f \in \mathcal{C}(\mathbb{T})$ with $(\hat{f}_k)_{k\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ leads to $S_t f \to f$ uniformly. Moreover, $f \in \mathcal{C}^1(\mathbb{T})$ implies $(\hat{f}_k)_{k\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ (not trivial either), so that uniform convergence holds in Theorem 2.2.6.*

We define

$$\begin{aligned}
\mathbf{f} &:= \left(f(\tfrac{j}{N})\right)_{j=-\frac{N}{2},\dots,\frac{N}{2}-1} \in \mathbb{C}^N, \\
\hat{\mathbf{f}} &:= \left(\hat{\mathbf{f}}_k\right)_{k=-\frac{N}{2},\dots,\frac{N}{2}-1} \in \mathbb{C}^N, \\
\mathbf{D}_N &:= \left(\mathrm{e}^{-2\pi\mathrm{i}\frac{kj}{N}}\right)_{k,j=-\frac{N}{2},\dots,\frac{N}{2}-1} \in \mathbb{C}^{N\times N},
\end{aligned}$$

so that $\hat{\mathbf{f}} = \frac{1}{N}\mathbf{D}_N\,\mathbf{f}$. We observe

$$\mathbf{D}_N\mathbf{D}_N^* = \left(\sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \mathrm{e}^{-2\pi\mathrm{i}(k-j)\frac{n}{N}}\right)_{k,j=-\frac{N}{2},\dots,\frac{N}{2}-1} = N\cdot\mathbf{I}_N$$

by Lemma 3.3.7. This implies $\mathbf{f} = \mathbf{D}_N^*\,\hat{\mathbf{f}}$.

Remark 3.3.12. *To compute* $\hat{\mathbf{f}}$ *from* $\mathbf{f}$*, naive implementations require* $\mathcal{O}(N^2)$ *flops. By using the structure of* $\mathbf{D}_N$*, we can do better.*

For $4 \leq N \in 4\mathbb{N}$, we derive, for $k = -\frac{N}{2}, \dots, \frac{N}{2} - 1$,

$$\begin{aligned} N\hat{\mathbf{f}}_k &= \sum_{j=-\frac{N}{2}}^{\frac{N}{2}-1} \mathbf{f}_j \mathrm{e}^{-2\pi \mathrm{i} k \frac{j}{N}} \\ &= \sum_{j=-\frac{N}{4}}^{\frac{N}{4}-1} \mathbf{f}_{2j} \mathrm{e}^{-2\pi \mathrm{i} k \frac{2j}{N}} + \sum_{j=-\frac{N}{4}}^{\frac{N}{4}-1} \mathbf{f}_{2j+1} \mathrm{e}^{-2\pi \mathrm{i} k \frac{2j+1}{N}} \\ &= \sum_{j=-\frac{N}{4}}^{\frac{N}{4}-1} \mathbf{f}_{2j} \mathrm{e}^{-2\pi \mathrm{i} k \frac{j}{N/2}} + \mathrm{e}^{-2\pi \mathrm{i} k/N} \sum_{j=-\frac{N}{4}}^{\frac{N}{4}-1} \mathbf{f}_{2j+1} \mathrm{e}^{-2\pi \mathrm{i} k \frac{j}{N/2}} \\ &= \left(\mathbf{D}_{N/2} \mathbf{f}^{even}\right)_{(k \mod N/2)} + \mathrm{e}^{-2\pi \mathrm{i} k/N} \left(\mathbf{D}_{N/2} \mathbf{f}^{odd}\right)_{(k \mod N/2)}, \end{aligned}$$

where

$$\mathbf{f}^{even} = (\mathbf{f}_{2j})_{j=-\frac{N}{4},\dots,\frac{N}{4}-1} \in \mathbb{C}^{N/2}, \qquad \mathbf{f}^{odd} = (\mathbf{f}_{2j+1})_{j=-\frac{N}{4},\dots,\frac{N}{4}-1} \in \mathbb{C}^{N/2}.$$

Here, (k mod $N/2$) means we map $k = -\frac{N}{2}, \dots, \frac{N}{2} - 1$ onto $-\frac{N}{4}, \dots, \frac{N}{4} - 1$ via

$$\begin{aligned} -\frac{N}{2}, \dots, -\frac{N}{4} - 1 &\quad \mapsto \quad 0, \dots, \frac{N}{4} - 1 \\ -\frac{N}{4}, \dots, \frac{N}{4} - 1 &\quad \mapsto \quad -\frac{N}{4}, \dots, \frac{N}{4} - 1 \\ \frac{N}{4}, \dots, \frac{N}{2} - 1 &\quad \mapsto \quad -\frac{N}{4}, \dots, -1. \end{aligned}$$

For $N = 2^m$, we may iterate this idea, which yields the Fast Fourier Transform (FFT) algorithm.

Theorem 3.3.13. *To compute* $\hat{\mathbf{f}}$ *from* $\mathbf{f}$*, for* $N = 2^m$*, the FFT requires* $\mathcal{O}(N \log_2(N))$ *flops.*

We ignore (k mod $N/2$).

Proof. If T_N denotes the flops required by the FFT to compute $\hat{\mathbf{f}}$, then we observe

$$T_N \leq 2T_{N/2} + 2N, \tag{3.27}$$

because even and odd part require $T_{N/2}$, each, and sum and multiplications N each. If we put $T_1 := 1$, then (3.27) also holds for $N = 2$ and iteration yields

$$\begin{aligned} T_N \leq 2(2T_{N/4} + N) + 2N &= 2^2 T_{N/4} + 2(2N) \\ &\leq 2^2(2T_{N/2^3} + N/2) + 2(2N) = 2^3 T_{N/2^3} + 3(2N) \\ &\vdots \\ &\leq 2^m T_1 + m2N = N + \log_2(N)2N. \end{aligned}$$

□

3.4 Tempered distributions

Recall the definition of Schwartz space $\mathcal{S}$ (Definition 3.2.11). Since neither $\mathcal{S}(\mathbb{R}^d)$ nor $L^2(\mathbb{R}^d)$ are suitable to deal with certain derivatives [2], we need to introduce a new space.

Definition 3.4.1. *The* tempered distributions *are the elements of*

$$\mathcal{S}'(\mathbb{R}^d) = \{L : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C} \mid U \text{ is linear and continuous}\}.$$

We endow $\mathcal{S}'(\mathbb{R}^d)$ with the weak*-topology, so that, for $L_n, L \in \mathcal{S}'(\mathbb{R}^d)$,

$$L_n \to L \qquad \Leftrightarrow \qquad \forall \eta \in \mathcal{S}(\mathbb{R}^d): \quad L_n(\eta) \to L(\eta).$$

For $f : \mathbb{R}^d \to \mathbb{C}$ measurable, we write

$$L_f(\eta) := \int_{\mathbb{R}^d} f(x)\eta(x)\mathrm{d}x, \qquad \eta \in \mathcal{S}(\mathbb{R}^d).$$

If $L_f \in \mathcal{S}'(\mathbb{R}^d)$, we simply write $f \in \mathcal{S}'(\mathbb{R}^d)$. We also define

$$L^1_{loc}(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{C} \mid f \text{ measurable, } 1_K \cdot f \in L^1(\mathbb{R}^d)\ \forall K \subset \mathbb{R}^d \text{ compact}\}$$

Theorem 3.4.2. *If $f \in L^1_{loc}(\mathbb{R}^d)$ and $\exists N \in \mathbb{N}$: $\lim_{\|x\|\to\infty} \frac{|f(x)|}{\|x\|^N} \to 0$, then $f \in \mathcal{S}'(\mathbb{R}^d)$.*

Proof. [1]. □

Example 3.4.3. - $1 \in \mathcal{S}'(\mathbb{R}^d)$, $1_{[0,\infty)} \in \mathcal{S}'(\mathbb{R})$.

- *$\mathcal{S}(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ by Theorem 3.4.2.*
- *For $x_0 \in \mathbb{R}^d$, we have $\delta_{x_0} \in \mathcal{S}'(\mathbb{R}^d)$, where $\delta_{x_0}(\eta) := \eta(x_0)$, for $\eta \in \mathcal{S}'(\mathbb{R}^d)$.*

Definition 3.4.4. *For $L \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$, we define $L \cdot g \in \mathcal{S}'(\mathbb{R}^d)$ by*

$$L \cdot g(\eta) := L(g \cdot \eta), \qquad \eta \in \mathcal{S}(\mathbb{R}^d).$$

For $f, g \in \mathcal{S}(\mathbb{R}^d)$, we observe $L_f \cdot g = L_{f\cdot g}$.

Definition 3.4.5. *For $L \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$,*

$$L * g(\eta) := L(g^- * \eta), \qquad \eta \in \mathcal{S}(\mathbb{R}^d),$$

yields $L * g \in \mathcal{S}'(\mathbb{R}^d)$, where $g^-(x) := g(-x)$, for $x \in \mathbb{R}^d$.

This extends the convolution of functions:

[2]Note that, according to Lemma 3.2.13, given $f \in \mathcal{S}(\mathbb{R}^d)$, it could be that $\frac{1}{4\pi^2\|\cdot\|^2}\hat{f}(\cdot) \notin \mathcal{S}(\mathbb{R}^d)$ (and $\notin L^2(\mathbb{R}^d)$).

Lemma 3.4.6. *For $f, g \in \mathcal{S}(\mathbb{R}^d)$, we have $L_f * g = L_{f*g}$.*

Proof. Exercise (Fubini, substitution) □

Lemma 3.4.7. *For $g \in \mathcal{S}(\mathbb{R}^d)$, we have $\delta_0 * g = L_g$.*

Hence, $\delta_0 * g = g$ in $\mathcal{S}'(\mathbb{R}^d)$.

Proof. For $\eta \in \mathcal{S}(\mathbb{R}^d)$, we compute

$$\delta_0 * g(\eta) = \delta_0(g^- * \eta) = \int_{\mathbb{R}^d} g(-y)\eta(0-y)\mathrm{d}y = \int_{\mathbb{R}^d} g(y)\eta(y)\mathrm{d}y = L_g(\eta). \qquad \square$$

Definition 3.4.8. *For $L \in \mathcal{S}'(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}^d$, we define $\partial^\alpha L \in \mathcal{S}'(\mathbb{R}^d)$ by*

$$\partial^\alpha L(\eta) := (-1)^{|\alpha|} L(\partial^\alpha \eta), \qquad \eta \in \mathcal{S}(\mathbb{R}^d).$$

Lemma 3.4.9. *For $f \in \mathcal{S}(\mathbb{R}^d)$, we have $L_{\partial^\alpha f} = \partial^\alpha L_f$.*

Proof. For $d = \alpha = 1$, integration by parts yields

$$L_{f'}(\eta) = \int_{-\infty}^{\infty} f'(x)\eta(x)\mathrm{d}x = \underbrace{[f(x)\eta(x)]_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} f(x)g'(x)\mathrm{d}x.$$

By iteration, we obtain the general formula. □

Example 3.4.10. *We have $\partial^1 1_{[0,\infty)} = \delta_0$ since*

$$\partial^1 1_{[0,\infty)}(\eta) = -\int_{\mathbb{R}} 1_{[0,\infty)}(x)\partial^1\eta(x)\mathrm{d}x = -\int_0^\infty \partial^1\eta(x)\mathrm{d}x = \eta(0) = \delta_0(\eta).$$

Definition 3.4.11. *For $L \in \mathcal{S}'(\mathbb{R}^d)$, we define $\hat{L}, \check{L} \in \mathcal{S}'(\mathbb{R}^d)$ by*

$$\hat{L}(\eta) := L(\hat{\eta}), \qquad \check{L}(\eta) := L(\check{\eta}) \qquad \eta \in \mathcal{S}(\mathbb{R}^d).$$

Lemma 3.4.12. *For $f \in \mathcal{S}(\mathbb{R}^d)$, we have $\widehat{L_f} = L_{\hat{f}}$ and $\check{L_f} = L_{\check{f}}$.*

Proof. Fubini leads to

$$\begin{aligned}
\widehat{L_f}(\eta) = L_f(\hat{\eta}) &= \int_{\mathbb{R}^d} f(x)\hat{\eta}(x)\mathrm{d}x \\
&= \int_{\mathbb{R}^d} f(x) \int_{\mathbb{R}^d} \eta(y)\mathrm{e}^{-2\pi i\langle y,x\rangle}\mathrm{d}y\mathrm{d}x \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)\mathrm{e}^{-2\pi i\langle y,x\rangle}\mathrm{d}x\eta(y)\mathrm{d}y \\
&= \int_{\mathbb{R}^d} \hat{f}(y)\eta(y)\mathrm{d}y = L_{\hat{f}}.
\end{aligned}$$

The proof for ˇ is analogous. □

Example 3.4.13. *We have $\hat{1} = \delta_0$ since, for $\eta \in \mathcal{S}(\mathbb{R}^d)$,*

$$\hat{1}(\eta) = \int_{\mathbb{R}^d} 1 \cdot \hat{\eta}(x)\mathrm{d}x = \int_{\mathbb{R}^d} \hat{\eta}(x)\mathrm{e}^{2\pi\mathrm{i}\langle 0,x\rangle}\mathrm{d}x = \check{\hat{\eta}}(0) = \eta(0) = \delta_0(\eta).$$

Proposition 3.4.14. *For $L \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$\widehat{L * g} = \hat{L} \cdot \hat{g}.$$

Proof. For $\eta \in \mathcal{S}(\mathbb{R}^d)$, we compute

$$\begin{aligned}\widehat{L * g}(\eta) &= L * g(\hat{\eta}) = L(g^- * \hat{\eta}) \\ &= \hat{L}(\mathcal{F}^{-1}(g^- * \hat{\eta})) \\ &= \hat{L}(\hat{g} \cdot \eta) = (\hat{L} \cdot \hat{g})(\eta).\end{aligned}$$

□

Theorem 3.4.15. *The Fourier transform $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, $L \mapsto \hat{L}$ is a topological isomorphism with inverse $\mathcal{F}^{-1} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, $L \mapsto \check{L}$.*

Proof. [1]. □

3.4.1 An Application: Distributional Poisson's equation

For $s > 2$ and $f \in \mathcal{S}(\mathbb{R}^d)$, consider

$$-\Delta u = f \qquad \text{in } \mathcal{S}'(\mathbb{R}^d). \tag{3.28}$$

Lemma 3.4.16. *For $d \geq 3$, $f : \mathbb{R}^d \to \mathbb{C}$, $x \mapsto \frac{1}{\|x\|^2}$ satisfies $f \in \mathcal{S}'(\mathbb{R}^d)$.*

Proof. Let $B \subset \mathbb{R}^d$ denote the unit ball. By spherical coordinates, we observe

$$\int_B \frac{1}{\|x\|^2}\mathrm{d}x = \mathrm{vol}(\mathbb{S}^{d-1}) \int_0^1 \frac{1}{r^2} r^{d-1}\mathrm{d}r = \int_0^1 r^{d-3}\mathrm{d}r < \infty,$$

so that $f \in L^1_{loc}(\mathbb{R}^d)$. Theorem 3.4.2 concludes the proof. □

According to Lemma 3.4.16, there is $E \in \mathcal{S}'(\mathbb{R}^d)$ such that $\hat{E} = \frac{1}{4\pi^2\|\cdot\|^2}$. For $f \in \mathcal{S}(\mathbb{R}^d)$, we solve (3.28) by putting

$$u := E * f$$

since then $\hat{u} = \hat{E}\hat{f} = \frac{1}{4\pi^2\|\xi\|^2}\hat{f}$, so that (3.18) is satisfied.

Remark 3.4.17. *In a course on PDE one observes that $E(x) = \frac{c_d}{\|x\|^{d-2}}$, for some suitable constant $c_d \in \mathbb{R}$.*

Definition 3.4.18. *For $s \in \mathbb{R}$, the Sobolev space is*

$$H^s(\mathbb{R}^d) := \{f \in \mathcal{S}'(\mathbb{R}^d) : (1 + \|\cdot\|^2)^{s/2}\hat{f} \in L^2(\mathbb{R}^d)\}$$

with inner product

$$\langle f, g\rangle_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1 + \|\xi\|^2)^s \hat{f}(\xi)\overline{\hat{g}(\xi)}\mathrm{d}\xi.$$

Lemma 3.4.19. *The Sobolev space $H^s(\mathbb{R}^d)$ is a Hilbert space.*

Proof. Due to Cauchy-Schwartz, the inner product is well-defined. To verify completeness, let $(f_n)_{n\in\mathbb{N}} \subset H^s(\mathbb{R}^d)$ be Cauchy. We observe that $(1 + \|\xi\|^2)^{s/2}\hat{f}_n(\xi)$ is Cauchy in $L^2(\mathbb{R}^d)$, hence, converges towards $g \in L^2(\mathbb{R}^d)$. Define

$$f := \mathcal{F}^{-1}(g(\xi)(1 + \|\xi\|^2)^{-s/2}),$$

so that $\hat{f}(\xi)(1 + \|\xi\|^2)^{s/2} \in L^2(\mathbb{R}^d)$, hence, $f \in H^s(\mathbb{R}^d)$. We estimate

$$\begin{aligned}\|f_n - f\|^2_{H^s(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} (1 + \|\xi\|^2)^s |\hat{f}_n(\xi) - \hat{f}(\xi)|^2 \mathrm{d}(\xi) \\ &= \int_{\mathbb{R}^d} |(1 + \|\xi\|^2)^{s/2}\hat{f}_n(\xi) - g(\xi)|^2 \mathrm{d}(\xi) \to 0. \qquad \square\end{aligned}$$

Theorem 3.4.20. *For $s > 2$, we have $\Delta : H^s(\mathbb{R}^d) \to H^{s-2}(\mathbb{R}^d)$ continuously.*

Proof. For $u \in H^s(\mathbb{R}^d)$, we observe

$$\begin{aligned}\|\Delta u\|^2_{H^{s-2}(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} (1 + \|\xi\|^2)^{s-2} \left|\widehat{\Delta u}(\xi)\right|^2 \mathrm{d}\xi \\ &= 16\pi^4 \int_{\mathbb{R}^d} (1 + \|\xi\|^2)^{s-2} \underbrace{\|\xi\|^4}_{\leq (1+\|\xi\|^2)^2} |\hat{u}(\xi)|^2 \mathrm{d}\xi \\ &\leq 16\pi^4 \int_{\mathbb{R}^d} (1 + \|\xi\|^2)^s |\hat{u}(\xi)|^2 \mathrm{d}\xi \\ &= 16\pi^4 \|u\|^2_{H^s(\mathbb{R}^d)}. \qquad \square\end{aligned}$$

For $f \in H^s(\mathbb{R}^d)$, consider

$$(I - \Delta)u = f.$$

A solution is given by

$$u := \mathcal{F}^{-1}\left(\frac{1}{1 + 4\pi^2\|\cdot\|^2}\hat{f}\right)$$

and the obvious inequality $\frac{1+\|\xi\|^2}{1+4\pi^2\|\xi\|^2} \leq 1$ leads to $u \in H^{s+2}(\mathbb{R}^d)$.

Chapter 4

Sampling

4.1 How does the Music end up on a CD? Sampling and Filtering

In the previous chapter we introduced continuous and discrete-time signals in a somewhat unrelated manner. This chapter deals with the core of modern digital signal processing: the idea and the basic theory of sampled signals. Th principal idea is the following: which conditions of a continuous signal guarantee perfect reconstructions from discrete signal samples?

In order to understand the principal idea, let us first look at what happens to the Fourier transform, if we sample a signal as to obtain f_d from f. In Figure 4.1, you see the plot of the excerpt of a (pseudo-)continuous signal (a piano sound), with its Fourier transform. In the lower plots, a rather coarsely sampled signal (Sampling rate 11025 samples per second) and its Fourier transform are shown. It should be immediately obvious, what happens to the Fourier transform, if we sample f: the Fourier transform $\hat{f}$ of f is periodized!

So, the answer to the next question, namely, how to obtain the original signal from the sampled version, should be really easy: since the sampling process leads to repeated copies of the (hopefully bandlimited) spectrum, all we need to do is multiply with a lowpass filter in order to get rid of the unwanted copies: $\hat{f} = \hat{f}_d \cdot \Pi$, hence $f = f_d * \Pi$. Here we are intentionally sloppy and don't specify any of the involved parameters, since we only want to get across the basic idea - and this seems almost perfect!

However, if we look a bit closer at the spectrum of f, namely, if we apply a logarithmic scale (which actually corresponds to our perception of audio), we can see, that the spectrum of f has not actually dropped to anything close to 0, see Figure 4.2 so, what will happen to the frequencies above the cut-off? In fact, if we don't suppress them by highpass-filtering before the sampling process, those samples will show up as - usually unwanted - aliases in the lower frequency bands.

Aliasing is an effect which is one of the limitations of discrete-time sampling. An example of aliasing can be seen in old movies, e.g. when watching wagon but also car wheels: the wheels appear to go in reverse. This phenomenon can be observed if the

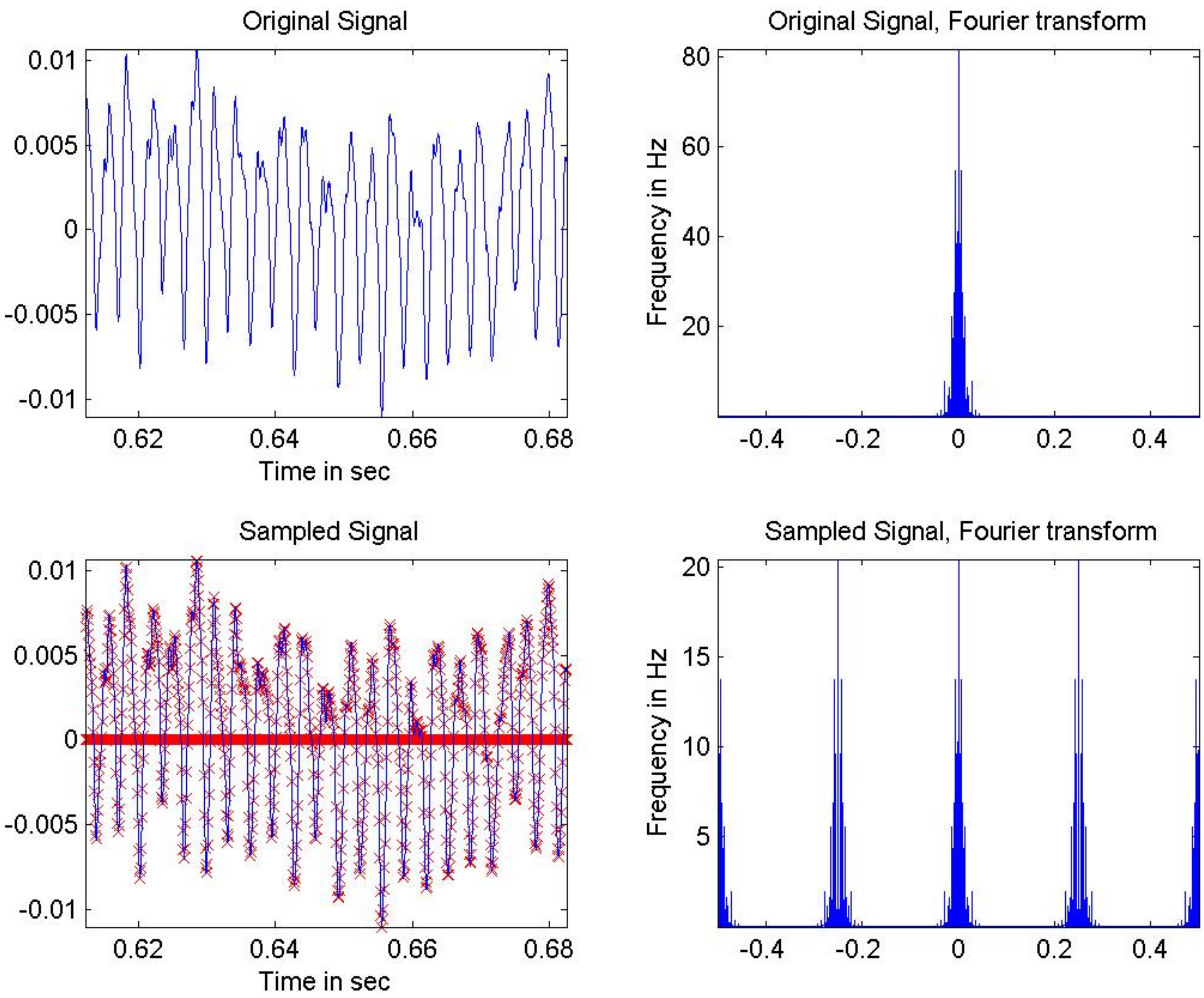

Figure 4.1: Subsampling and resulting Fourier transform

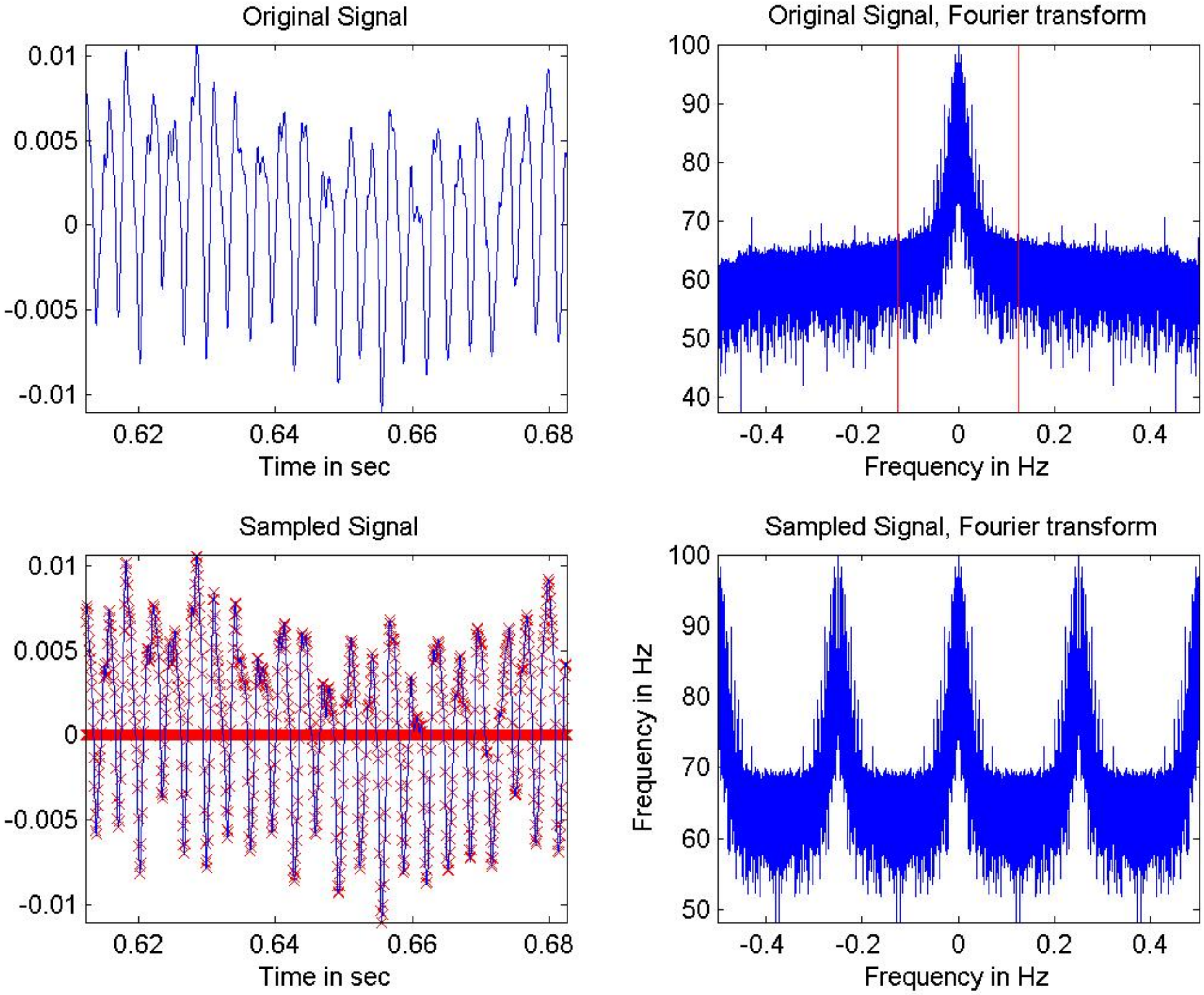

Figure 4.2: Subsampling and resulting Fourier transform

rate of the wagon wheel's spokes spinning approaches the rate of the sampler (the camera operating at about 30 frames per second) [1].

The same thing happens in data acquisition between the sampler and the signal we are sampling. For an example, have a look at Figure 4.3. Here, the effect of undersampling is immediately obvious: the sinusoid of 330Hz appears as a sinudoid with much lower frequency, namely 30Hz. In the lower plot, the wrong sampling rate of 320Hz maps the frequency 330Hz to $10Hz$. We will now study a simple case of this phenomenon mathematically.

Example 4.1.1. *Consider a complex exponential (a phasor) with frequency ω_0, i.e. $\varphi(t) = e^{2\pi i\omega_0 t}$. Now, assume that we sub-sample this phasor to obtain*

$$\varphi_d(n) = e^{2\pi i\omega_0(nT)},$$

i.e., T is the sampling interval. We have seen many times by now, that adding $2\pi ink$ to exponent doesn't change this (discrete) function:

$$\varphi_d(n) = e^{2\pi i\omega_0(nT)+2\pi ink} = e^{2\pi iTn(\omega_0+k/T)}, \text{ for all } k \in \mathbb{Z}.$$

This equation tells us that, after sampling, a sinusoid with frequency ω_0 cannot be distinguished from a sinusoid with frequency $\omega_0 + k/T$, $k \in \mathbb{Z}$. Note that $F_s = 1/T$ is the sampling rate.
If we sample real-valued signals, however, we always have to consider positive and negative frequencies, so, to a real sinusoid (sine or cosine) with frequency ω_0, we have in fact aliases at $\pm + \omega_0 k/T$.

Example 4.1.2. *Recall now the Fourier series of a square wave as defined in Example 2.1.4:*

$$f(x) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)}\sin(2\pi(2k-1)x)$$

Obviously, this periodic function does NOT have a finite number of frequencies in it: its spectrum, i.e. the frequencies contained in the square wave decay like $1/n$ - and this really slow! We note that, due to this infinite bandwidth, the square-wave cannot be sampled properly: sampling, no matter how densely must always lead to aliasing, as we shall see next.
We assume a sampling rate of $F_s = 44100Hz$ and consider a square wave with fundamental frequency $F = 700Hz$. Then, the Fourier series of this function is simply

$$f(x) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)}\sin(2\pi * 700 * (2k-1)x)$$

[1] This effect is even called "wagon-wheel effect".

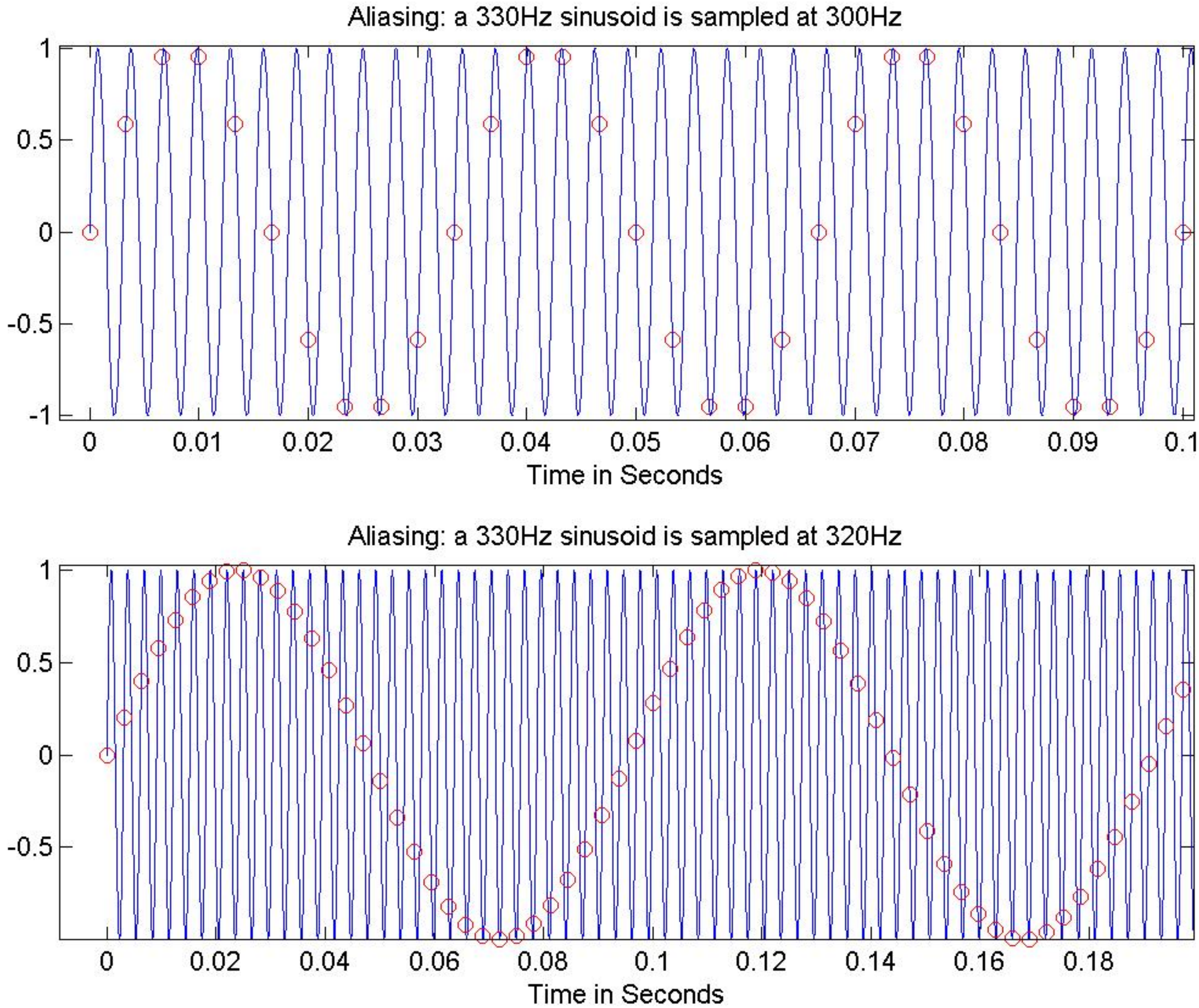

Figure 4.3: Aliasing by subsampling

since now we have 700 *oscillations per second. In this case, the highest frequency that is still below the Nyquist frequency of* $22050Hz$ *is the* $31st$ *harmonic which belongs to the frequency* $31 \cdot 700 = 21700$*. The next frequency contained in the signal, with index* 33 *is* $23100Hz$ *and is above Nyquist. It will therefore show up as an alias at* $(23100 - 44100)Hz = -21000Hz$*. This effect continues for all higher frequencies, and of course, all the negative frequencies turn into positive aliases accordingly. The phenomenon is shown on Figure 4.4.*

Note that in the current case, the fundamental frequency divides the Sampling rate, and the aliases become quasi-harmonics. In contrast, if we choose $F = 800Hz$*, the aliases will occur in frequencies that are not related not the fundamental frequencies, see Figure 4.5. While aliases should be avoided in usual sampling procedure, intentional aliasing can lead to interesting sound effects.*

We now turn to a more technical approach to sampling.

4.2 Formal Sampling

4.2.1 Poisson summation formula

We start by connecting the Fourier transform with Fourier coefficients of a periodic function-

Lemma 4.2.1. *If* $f \in L^1(\mathbb{T}^d)$ *and* $\left(\hat{f}_k\right)_{k\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d)$*, then* $f \in \mathcal{C}(\mathbb{T}^d)$ *with*

$$f = \sum_{k\in\mathbb{Z}^d} \hat{f}_k e^{2\pi i\langle k,\cdot\rangle} \tag{4.1}$$

converging uniformly and in $L^1(\mathbb{T}^d)$ *and* $L^2(\mathbb{T}^d)$*.*

Proof. The condition $\left(\hat{f}_k\right)_{k\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d)$ implies uniform convergence, so that

$$g := \sum_{k\in\mathbb{Z}^d} \hat{f}_k e^{2\pi i\langle k,\cdot\rangle} \in \mathcal{C}(\mathbb{T}^d) \subset L^2(\mathbb{T}^d) \subset L^1(\mathbb{T}^d).$$

Since $\ell^1(\mathbb{Z}^d) \subset \ell^2(\mathbb{Z}^d)$, we have $f \in L^2(\mathbb{T}^d)$. Obviously, $\hat{g}_k = \hat{f}_k$, so that we deduce $g = f$ in $L^2(\mathbb{T}^d)$, hence, $f = g$ almost everywhere. □

Proposition 4.2.2. *If* $f \in L^1(\mathbb{R}^d)$*, then*

$$\overline{\omega} f := \sum_{k\in\mathbb{Z}^d} f(\cdot + k)$$

converges pointwise almost everywhere and $\overline{\omega} f \in L^1(\mathbb{T}^d)$ *with*

$$\|\overline{\omega} f\|_{L^1(\mathbb{T}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)}.$$

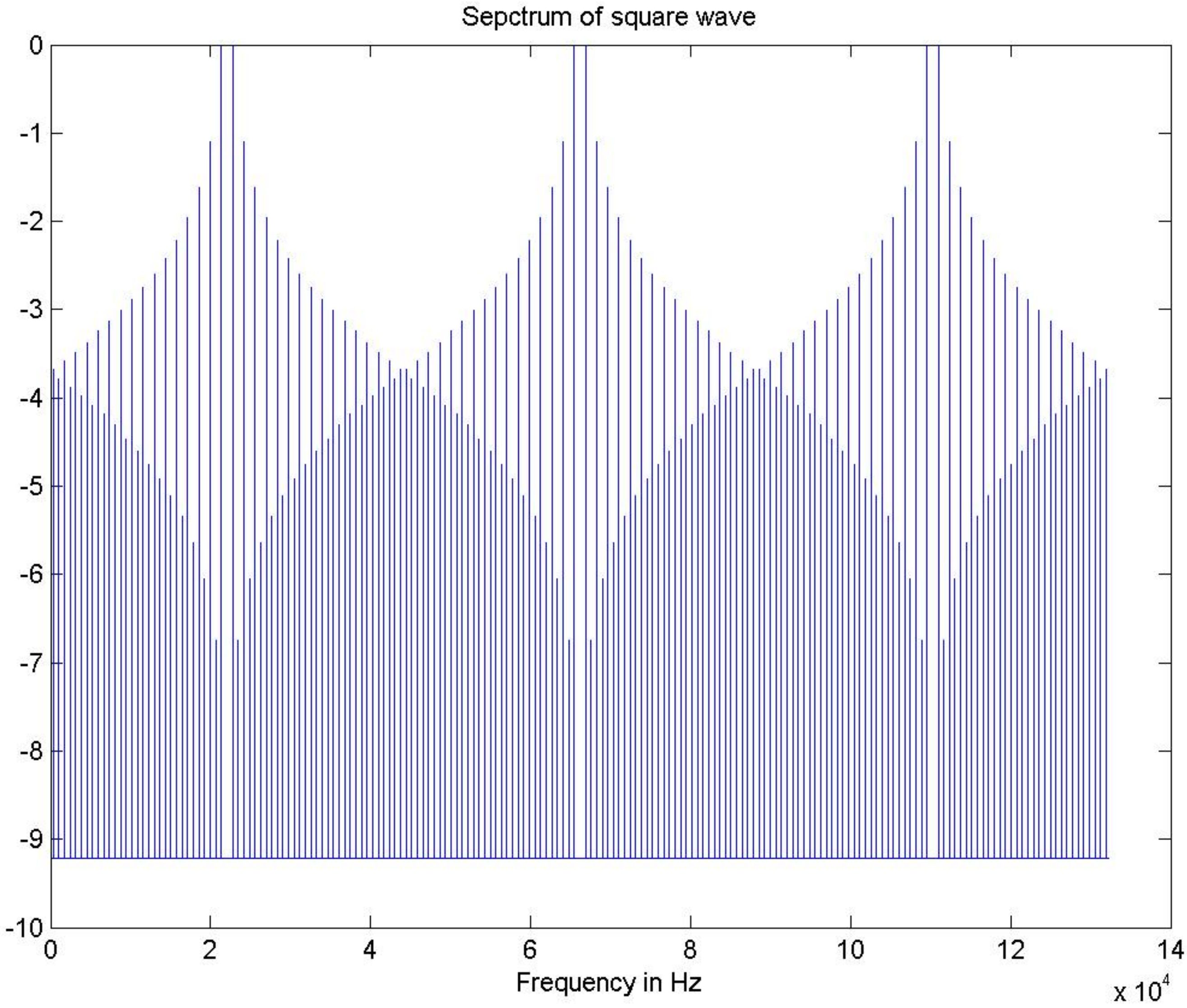

Figure 4.4: Aliasing by discretization of Square Wave

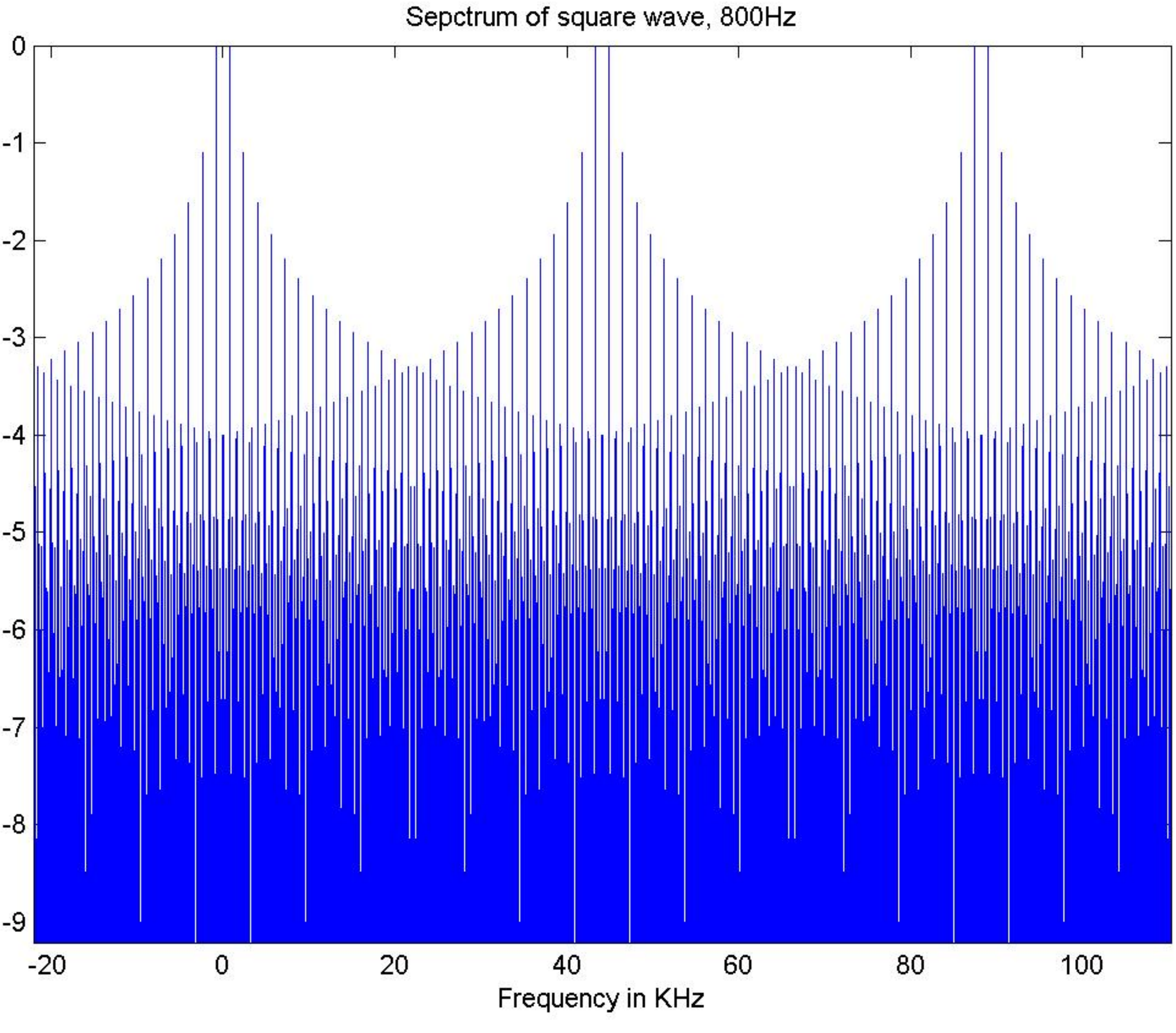

Figure 4.5: Aliasing by discretization of Square Wave

Proof. According to Theorem 1.1.5 and the monoton convergence theorem,

$$\begin{aligned}\|f\|_{L^1(\mathbb{R}^d)} &= \sum_{k\in\mathbb{Z}^d}\int_{I^d+k}|f| \\ &= \sum_{k\in\mathbb{Z}^d}\int_{I^d}|f(\cdot+k)| \\ &= \int_{I^d}\sum_{k\in\mathbb{Z}^d}|f(\cdot+k)|,\end{aligned}$$

where $\sum_{k\in\mathbb{Z}^d}|f(\cdot+k)|$ converges almost everywhere (and in $L^1(\mathbb{R}^d)$. Hence, $\sum_{k\in\mathbb{Z}^d} f(\cdot+k)$ converges almost everywhere with upper bound $\overline{\omega}|f|$ and $\overline{\omega}f$ is $\mathbb{Z}^d$-periodic, and we have

$$\|\overline{\omega}f\|_{L^1(\mathbb{T}^d)} \le \|\overline{\omega}|f|\|_{L^1(\mathbb{T}^d)} = \|f\|_{L^1(\mathbb{R}^d)}. \qquad \square$$

For $f\in L^1(\mathbb{R}^d)$, how do $\hat{f}(k)$ and $(\widehat{\overline{\omega}f})_k$ relate to each other?

Theorem 4.2.3 (Poisson formula)**.** *Let $f\in L^1(\mathbb{R}^d)$.*

a) For $k\in\mathbb{Z}^d$,

$$(\widehat{\overline{\omega}f})_k = \hat{f}(k), \qquad k\in\mathbb{Z}^d\,.$$

b) If $\left((\widehat{\overline{\omega}f})_k\right)_{k\in\mathbb{Z}^d}\in\ell^1(\mathbb{Z}^d)$ and $\overline{\omega}f$ is continuous, then

$$(\overline{\omega}f)(x) = \sum_{k\in\mathbb{Z}^d}\hat{f}(k)\mathrm{e}^{2\pi\mathrm{i}\langle k,x\rangle}, \qquad x\in\mathbb{T}^d.$$

Proof. By Proposition 4.2.2 and Theorem 1.1.5, we have

$$\begin{aligned}(\widehat{\overline{\omega}f})_k &= \int_{I^d}(\overline{\omega}f)(x)\,\mathrm{e}^{-2\pi\mathrm{i}\langle k,x\rangle}\mathrm{d}x \\ &= \sum_{l\in\mathbb{Z}^d}\int_{I^d}f(x+l)\mathrm{e}^{-2\pi\mathrm{i}\langle k,x\rangle}\mathrm{d}x \\ &= \sum_{l\in\mathbb{Z}^d}\int_{I^d+l}f(x)\,\mathrm{e}^{-2\pi\mathrm{i}\langle k,x\rangle}\mathrm{d}x \\ &= \int_{\mathbb{R}^d}f(x)\mathrm{e}^{-2\pi\mathrm{i}\langle k,x\rangle}\mathrm{d}x = \hat{f}(k).\end{aligned}$$

Part b) follows from (4.1) since both sides are continuous. $\square$

Corollary 4.2.4. *For $f\in\mathcal{S}(\mathbb{R}^d)$, we have*

$$(\overline{\omega}f)(x) = \sum_{k\in\mathbb{Z}^d}\hat{f}(k)\mathrm{e}^{2\pi\mathrm{i}\langle k,x\rangle}, \qquad x\in\mathbb{T}^d.$$

Proof. Since $\overline{\omega}f$ converges uniformly, Theorem 4.2.3 implies the claim. $\square$

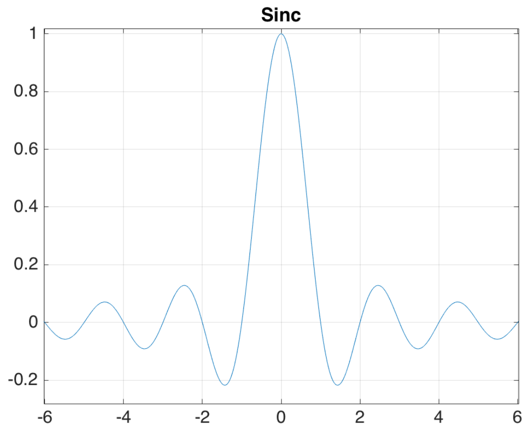

Figure 4.6: $\operatorname{sinc}(\xi) = \frac{\sin(\pi\xi)}{\pi\xi}$, for $|x| \leq 10$.

4.2.2 The Shannon Sampling Theorem

We define

$$\operatorname{sinc}(\xi) := \frac{\sin(\pi\xi)}{\pi\xi}, \quad \xi \in \mathbb{R},$$

see Figure 4.6.

Lemma 4.2.5. *For $d = 1, 2, \ldots$, we have*

$$\mathcal{F}^{\pm 1}\left(1_{[-\frac{1}{2},\frac{1}{2}]^d}\right)(\xi) = \prod_{i=1}^{d} \operatorname{sinc}(\xi_i), \qquad \xi \in \mathbb{R}^d. \tag{4.2}$$

Proof. For $d = 1$, we compute

$$\begin{aligned}
\mathcal{F}\left(1_{[-\frac{1}{2},\frac{1}{2}]}\right)(\xi) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{-2\pi \mathrm{i} x\xi} \mathrm{d}x \\
&= \frac{1}{-2\pi \mathrm{i}\xi} \left[\mathrm{e}^{-2\pi \mathrm{i} x\xi}\right]_{-\frac{1}{2}}^{\frac{1}{2}} \\
&= \frac{\mathrm{e}^{-\pi \mathrm{i}\xi} - \mathrm{e}^{\pi \mathrm{i}\xi}}{-2\pi \mathrm{i}\xi} \\
&= \frac{\cos(-\pi\xi) + \mathrm{i}\sin(-\pi\xi) - \cos(\pi\xi) - \mathrm{i}\sin(\pi\xi)}{-2\pi\xi} \\
&= \frac{\sin(\pi\xi)}{\pi\xi}.
\end{aligned}$$

For $d \geq 2$, we observe

$$\begin{aligned}
1_{[-\frac{1}{2},\frac{1}{2}]^d}(x) &= 1_{[-\frac{1}{2},\frac{1}{2}]}(x_1) \cdots 1_{[-\frac{1}{2},\frac{1}{2}]}(x_d), \\
\mathrm{e}^{-2\pi \mathrm{i}\langle x,\xi\rangle} &= \mathrm{e}^{-2\pi \mathrm{i} x_1\xi_1} \cdots \mathrm{e}^{-2\pi \mathrm{i} x_d\xi_d},
\end{aligned}$$

so that we derive

$$\mathcal{F}\left(1_{[-\frac{1}{2},\frac{1}{2}]^d}\right)(\xi) = \int_{\mathbb{R}} 1_{[-\frac{1}{2},\frac{1}{2}]}(x_1)\mathrm{e}^{-2\pi \mathrm{i} x_1\xi_1}\mathrm{d}x_1 \cdots \int_{\mathbb{R}} 1_{[-\frac{1}{2},\frac{1}{2}]}(x_d)\mathrm{e}^{-2\pi \mathrm{i} x_d\xi_d}\mathrm{d}x_d.$$

Since $\operatorname{sinc}(-\xi_i) = \operatorname{sinc}(\xi_i)$ and $\mathcal{F}^{-1}f(\xi) = \mathcal{F}f(-\xi)$, we conclude the proof. $\square$

Definition 4.2.6. *For $t > 0$, the Paley-Wiener space* $\mathrm{PW}(t)$ *is*

$$\mathrm{PW}(t) = \left\{ f \in L^2(\mathbb{R}^d) : \operatorname{supp}(\hat{f}) \subset [-t,t]^d \right\}.$$

Theorem 4.2.7 (Shannon's sampling theorem)**.** *Let* $\varphi(\xi) := \prod_{i=1}^d \operatorname{sinc}(\xi_i)$. *If* $f \in \mathrm{PW}\left(\frac{1}{2}\right)$, *then*

$$f = \sum_{k\in\mathbb{Z}^d} f(k)\varphi(\cdot - k) \tag{4.3}$$

holds in $L^2(\mathbb{R}^d)$ and uniformly in $\mathbb{R}^d$.

Proof. Since $f \in \mathrm{PW}\left(\frac{1}{2}\right)$ implies $\hat{f} \in L_1(\mathbb{R}^d)$, Poisson's formula yields

$$f(-k) = (\mathcal{F}\hat{f})(k) = (\widehat{\overline{\omega}\hat{f}})_k. \tag{4.4}$$

Due to $\operatorname{supp}(\hat{f}) \subset I^d$, we have

$$1_{[-\frac{1}{2},\frac{1}{2}]^d}\,\overline{\omega}\hat{f} = \hat{f},$$

so that $\overline{\omega}\hat{f} \in L^2(\mathbb{T}^d)$ because

$$\|\overline{\omega}\hat{f}\|^2_{L^2(\mathbb{T}^d)} = \int_{I^d} \left|(\overline{\omega}\hat{f})(x)\right|^2 \mathrm{d}x = \int_{I^d} \left|\hat{f}(x)\right|^2 \mathrm{d}x = \|f\|^2_{L^2(\mathbb{R}^d)} < \infty.$$

Thus, (4.4) implies $(f(k))_{k\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$. We further derive in $L^2(\mathbb{R}^d)$

$$\begin{aligned}
\hat{f} = 1_{[-\frac{1}{2},\frac{1}{2}]^d}\,\overline{\omega}\hat{f} &= 1_{[-\frac{1}{2},\frac{1}{2}]^d} \sum_{k\in\mathbb{Z}^d} f(-k)\mathrm{e}_k \\
&= \sum_{k\in\mathbb{Z}^d} f(k) 1_{[-\frac{1}{2},\frac{1}{2}]^d}\,\mathrm{e}_{-k} \\
&= \sum_{k\in\mathbb{Z}^d} f(k)\mathcal{F}\left(\varphi(\cdot - k)\right),
\end{aligned}$$

where we have used Lemma 4.2.5 for $\mathcal{F}^{-1}$. Applying $\mathcal{F}^{-1}$ to both sides implies (4.3) in $L^2(\mathbb{R}^d)$.

To verify uniform convergence, recall $|\varphi|^2 \in L^1(\mathbb{R}^d)$ with

$$\left(\widehat{\overline{\omega}|\varphi|^2}\right)_k = \mathcal{F}\left(|\varphi|^2\right)(k), \qquad k \in \mathbb{Z}^d. \tag{4.5}$$

Since $\mathcal{F}(|\varphi|^2) = \mathcal{F}(\varphi \cdot \overline{\varphi}) = \widehat{\varphi} * \widehat{\overline{\varphi}} = \widehat{\varphi} * \overline{\widehat{\varphi}(-\cdot)}$, we obtain

$$\operatorname{supp} \mathcal{F}\left(|\varphi|^2\right) \subset I^d + I^d = [-1,1]^d.$$

The function $\mathcal{F}(|\varphi|^2)$ is continuous, so that (4.5) yields $\left(\widehat{\overline{\omega}|\varphi|^2}\right)_k = 0$ for all $0 \neq k \in \mathbb{Z}^d$. We deduce $\overline{\omega}|\varphi|^2 = c \in \mathbb{C}$ is constant. Cauchy-Schwartz leads to

$$\left| \sum_{\|k\| \geq m} f(k)\varphi(\cdot - k) \right| \leq \left(\sum_{\|k\| \geq m} |f(k)|^2 \right)^{1/2} c \quad \xrightarrow{m\to\infty} \quad 0$$

uniformly since $(f(k))_{k\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$. $\square$

We now allow for $f \in \mathrm{PW}(t)$.

Corollary 4.2.8. *Suppose that* $0 < a \leq \frac{1}{2t}$ *and* $\varphi \in \mathrm{PW}\left(\frac{1}{2}\right)$ *satisfies*

$$\widehat{\varphi}(\xi) = 1, \quad \xi \in [-at, at]^d. \tag{4.6}$$

If $f \in \mathrm{PW}(t)$, *then*

$$f = \sum_{k\in\mathbb{Z}^d} f(ka)\varphi\left(\frac{\cdot}{a} - k\right)$$

holds in $L^2(\mathbb{R}^d)$ *and uniformly in* $\mathbb{R}^d$.

Note that φ can be chosen as in (4.2), but it is not necessary if $0 < a < \frac{1}{2t}$. The number $\frac{1}{2t}$ is called *Shannon's sampling rate.*

Proof. Put $g = f(\cdot a)$, so that $g \in \mathrm{PW}(at) \subset \mathrm{PW}\left(\frac{1}{2}\right)$. EXERCISE: Go through the proof of Theorem 4.2.7 and observe that the condition (4.6) with $0 < a \leq \frac{1}{2t}$ is sufficient. $\square$

4.2.3 Aliasing

If we do not sample sufficiently dense, then we cannot expect to reconstruct the correct function, see Figure 4.7.

If $a > \frac{1}{2t}$, then *aliasing* occurs, i.e., higher frequency components are falsely taken as lower frequency contributions. More specifically, write $f \in \mathrm{PW}(t)$ as the finite sum

$$f = \sum_m f_m, \qquad \text{with} \qquad \operatorname{supp}(\hat{f}_m) \subset \left[-\frac{1}{2}, \frac{1}{2}\right]^d + m, \qquad m \in \mathbb{Z}^d,$$

which is derived from $f_m := \mathcal{F}^{-1}\left(\hat{f} \cdot 1_{[-\frac{1}{2},\frac{1}{2}]^d + m}\right)$. For the sampling rate $a = 1 > \frac{1}{2t}$, we consider

$$F(x) := \sum_{k\in\mathbb{Z}^d} f(k)\varphi(\cdot - k).$$

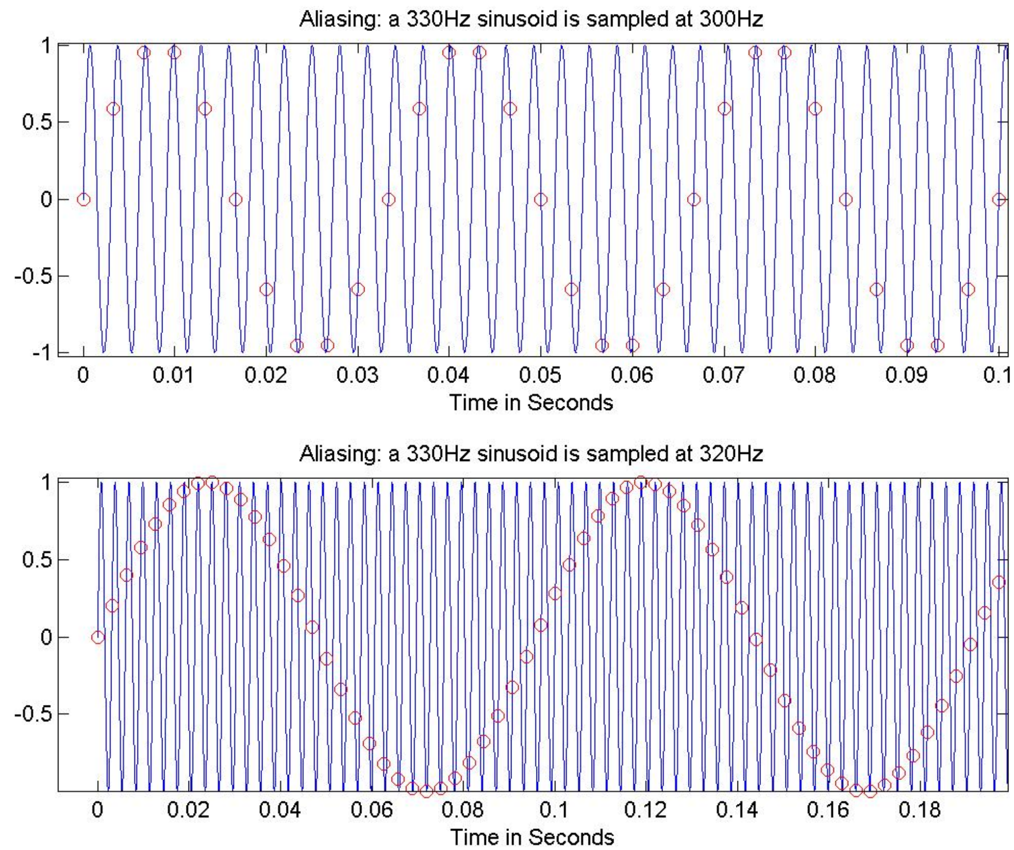

Figure 4.7: Sampling too sparsely.

The periodization of $\hat{f}$ satisfies

$$\overline{\omega}\hat{f} = \sum_{l\in\mathbb{Z}^d}\sum_m \hat{f}_m(\cdot - l) = \sum_m \hat{f}_m(\cdot + m).$$

Therefore, we derive

$$\hat{F}(\xi) = 1_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\,\overline{\omega}\hat{f} = 1_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\sum_m \hat{f}_m(\cdot + m),$$

so that F does not contain any frequencies beyond $\left[-\frac{1}{2},\frac{1}{2}\right]^d$. The Fourier reconstruction leads to

$$F(x) = \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d}\sum_m \hat{f}_m(\xi + m)\mathrm{e}^{2\pi\mathrm{i}\langle x,\xi\rangle}\mathrm{d}\xi.$$

The lower frequency components with $\xi \in \left[-\frac{1}{2},\frac{1}{2}\right]^d$ should be $\hat{f}_0(\xi)$. However, there are the additional contributions $\sum_{m\neq 0}\hat{f}_m(\xi+m)$ that are induced from higher frequencies. Further, we obtain

$$\begin{aligned}
F(x) &= \sum_m \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} \hat{f}_m(\xi + m)\mathrm{e}^{2\pi\mathrm{i}\langle x,\xi\rangle}\mathrm{d}\xi \\
&= \sum_m \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d+m} \hat{f}_m(\xi)\mathrm{e}^{2\pi\mathrm{i}\langle x,\xi\rangle}\mathrm{e}^{-2\pi\mathrm{i}\langle x,m\rangle}\mathrm{d}\xi \\
&= \sum_m f_m(x)\mathrm{e}^{-2\pi\mathrm{i}\langle x,m\rangle}.
\end{aligned}$$

Thus, instead of $f = f_0 + \sum_{m\neq 0} f_m$, we obtain $F = f_0 + \sum_{m\neq 0} f_m(x)\mathrm{e}^{-2\pi\mathrm{i}\langle x,m\rangle}$.

Bibliography

[1] L. Grafakos, *Classical fourier analysis*, Graduate Texts in Mathematics, 2014.