PS Dynamical Systems and Nonlinear DEs (2023S)
(Exercises for 6 June 2023)

24. For each of the following maps $T_j : [0,1] \to [0,1]$ plot the (graphs of the) first four iterates T, T^2, T^3, T^4. Meditate over the images. (What changes/ does not change as you go to higher iterates?) Can you use the plots to find some periodic points of these maps?

 a) $T_1 x := x + 2/3 \pmod 1$,
 b) $T_2 x := x + \sqrt{5} \pmod 1$,
 c) $T_3 x := 2x \pmod 1$,
 d) $T_4 x := 4x(1-x)$,
 e) $T_5 x := \pi x(1-x)$.

25. Let (X, d) be a metric space. A map $T : X \to X$ is said to be *topologically transitive* if it has a dense (forward) orbit $(T^n x)_{n \geq 0}$. Consider also the following property which $T : X \to X$ may have:

$$\text{For any two nonempty open sets } U, V \subseteq X \text{ there is some } n \geq 1 \text{ such that } U \cap T^{-n} V \neq \varnothing. \qquad (\Diamond)$$

 a) Interpret this in terms of orbits. Show that, in general, topological transitivity does not imply $(\Diamond)$.
 b) Show that in the case of a continuous map on a *compact* metric space without isolated points topological transitivity is equivalent to $(\Diamond)$.
 c) Can you think of other metric spaces in which the two notions coincide?

26. Let (X, d) be a compact metric space without isolated points, and $T : X \to X$ a homeomorphism. Suppose there is some x with dense two-sided orbit $(T^n x)_{n \in \mathbb{Z}}$. Show that T is topologically transitive.

27. Let (X, d_X) and (Y, d_Y) be metric spaces, with maps $T : X \to X$ and $S : Y \to Y$ defining two discrete dynamical systems. A *(topological) semiconjugacy* (or *factor map*) between T and S is a continuous surjective map $\psi : X \to Y$ satisfying $\psi \circ T = S \circ \psi$. In this case S is said to be a *factor* of T.

 a) Show that a semiconjugacy preserves the following properties of orbits: Take $x \in X$ and let $y := \psi(x)$. If $x = T^p x$ is a periodic point for T, then $y = S^p y$ is a periodic point for S; if $T^n x \to x^*$, then $S^n y \to y^* := \psi(x^*)$; if $(T^n x)_{n \geq 0}$ is dense in X, then $(S^n y)_{n \geq 0}$ is dense in Y.
 b) Suppose that S is topologically transitive. Is the same necessarily true for T?
 c) If $x = T^p x$ is a periodic point for T, is p always the minimal period of $y := \psi(x)$?
 d) If $y := \psi(x)$ satisfies $y = S^p y$ for some $p \geq 1$, does this mean that x is a periodic point for T?

28. A semiconjugacy $\psi : X \to Y$ between $T : X \to X$ and $S : Y \to Y$ is a *topological conjugacy* (and T, S are said to be *topologically conjugate*) if it is a homeomorphism. (This is often best interpreted as a change of variables. The two systems are then essentially the same as far as topological properties are concerned.) Which of the following maps $T_j : X \to X$, with $X := (0,1] \simeq \mathbb{R}/\mathbb{Z}$ regarded as the circle, are topologically conjugate to each other? (One pair is quite difficult to decide. Don't worry if you cannot do that one.)

 a) $T_1 x := x + 2/3 \pmod 1$,
 b) $T_2 x := x + 1/3 \pmod 1$,
 c) $T_3 x := x + 1/4 \pmod 1$,
 d) $T_4 x := x + \pi/100 \pmod 1$,
 e) $T_5 x := x + \pi/99 \pmod 1$,
 f) $T_6 x := 2x \pmod 1$.

29. Consider an irrational rotation ($\alpha \notin \mathbb{Q}$), $Tx := x + \alpha \pmod 1$, $x \in X = (0,1] \simeq \mathbb{R}/\mathbb{Z}$. The following equidistribution result has been established during the lecture: If $A \subseteq X$ is an interval, and $\mathbf{S}_n(A) := \sum_{k=0}^{n-1} 1_A \circ T^k$, $n \geq 1$, counts the visits to A during $\{0, 1, \ldots, n-1\}$, then $\lim_{n\to\infty} \frac{1}{n} \mathbf{S}_n(A)(x) = \lambda(A)$ uniformly in $x \in X$.

 This can easily be generalized as follows: For $f : X \to \mathbb{R}$ write $\mathbf{S}_n(f) := \sum_{k=0}^{n-1} f \circ T^k$, $n \geq 1$, so that $\mathbf{S}_n(f)(x) = f(x) + f(Tx) + \ldots + f(T^{n-1}x)$ is the sum of the values of f which the orbit of x picks up during the time interval $\{0, 1, \ldots, n-1\}$. Show first that for any linear combination $f := \sum_{j=1}^{m} c_j 1_{A_j}$ with intervals $A_j \subseteq X$,

$$\lim_{n\to\infty} \frac{1}{n} \mathbf{S}_n(f)(x) = \int_0^1 f(y)\, dy \quad \text{uniformly in } x \in X. \qquad (\heartsuit)$$

 Then conclude that statement $(\heartsuit)$ is in fact correct for every Riemann-integrable function f.