Mathematical Modeling of Water-Wave Problems
Applied PDE Seminar
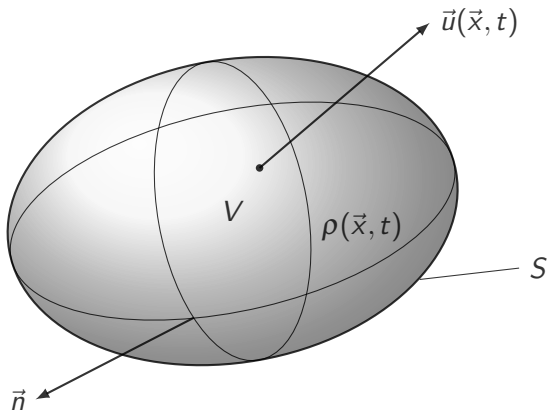
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Fluid Description

The fluid is described by

- Fluid density
  $\rho(\vec{x}, t)$
- Velocity Field
  $\vec{u}(\vec{x}, t) = (u, v, w)$
- Pressure
  $P(\vec{x}, t)$

Figure: Control volume of the fluid

○ Mass:

$$m(t) = \int_V \rho(\vec{x}, t) \; dV$$

○ Rate of change:

$$\int_V \frac{\partial \rho(\vec{x}, t)}{\partial t} \; dV = \frac{dm}{dt} = - \int_S \rho(\vec{x}, t) \vec{u} \cdot \vec{n} \; dS$$

○ Use Gauss's law to get the **Equation of Mass conservation**

$$\frac{\partial \rho}{\partial t} \nabla \cdot (\rho \vec{u}) = 0$$

$\rightarrow$ Apply Newton's second law to the Fluid

Body Force	**Local/Short-range Force**
$\vec{F} = (0, 0, -g)$	Stress tensor For inviscid fluid: $P(\vec{x}, t)$

$$\Rightarrow \int_V \rho \frac{D\vec{u}}{Dt} \, dV = \int_V \left(\rho \vec{F} - \nabla P \right) \, dV$$

$\rightarrow$ Leads us to **Euler's Equation of Motion**

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho} \nabla P + \vec{F}$$

Vorticity	**Irrotational Flow**
$\vec{\omega} = \nabla \times \vec{u}$	$\vec{\omega} = 0$

$\rightarrow$ Vorticity pops up in the acceleration of the fluid particles

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + \nabla \left(\frac{1}{2} \vec{u} \cdot \vec{u} \right) - (\vec{u} \times \vec{\omega})$$

$\rightarrow$ We can incorporate vorticity into Euler's Equation of Motion

$$\frac{\partial \vec{u}}{\partial t} + \nabla \left(\frac{1}{2} \vec{u} \cdot \vec{u} + \frac{P}{\rho} + \Omega \right) = \vec{u} \times \vec{\omega}$$

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- **Kinematic Condition**: Fluid particles at the surface
- **Dynamic Condition**: Atmospheric Pressure on the surface
- **Bottom Condition**: Rigid and fixed bottom
- (**Integrated Mass Condition**): Combination

- h_0 for the typical water depth
- λ for the typical wavelength
- $\frac{\lambda}{\sqrt{gh_0}}$ time scale of wave propagation
- $\sqrt{gh_0}$ velocity scale of waves in (x, y)
- $\frac{h_0\sqrt{gh_0}}{\lambda}$ velocity scale in z

$\rightarrow$ **Shallowness parameter** $\delta = \frac{h_0}{\lambda}$
$\rightarrow$ **Amplitude Parameter** $\varepsilon = \frac{a}{h_0}$

$\rightarrow$ Nondimensionalisation

$$x \rightarrow \lambda x, \quad u \rightarrow \sqrt{g h_0} u,$$
$$y \rightarrow \lambda y, \quad v \rightarrow \sqrt{g h_0} v, \qquad t \rightarrow \frac{\lambda}{\sqrt{g h_0}} t,$$
$$z \rightarrow h_0 z, \quad w \rightarrow \frac{h_0 \sqrt{g h_0}}{\lambda} w.$$

$\rightarrow$ Top and Bottom conditions

$$h = h_0 + a\eta(\vec{x}_\perp, t), \qquad b \rightarrow h_0 b(\vec{x}_\perp, t)$$

$\rightarrow$ Rewrite Pressure

$$P = P_a + \rho g(h_0 - z) + \rho g h_0 p(\vec{x})$$

$\rightarrow w$, p and the free surface z are $\propto \varepsilon$, leading to the scaling

$$p \rightarrow \varepsilon p, \quad w \rightarrow \varepsilon w, \quad \vec{u}_\perp \rightarrow \varepsilon \vec{u}_\perp$$

$\rightarrow$ Nondimensionalized Euler's Equation of motion

$$\frac{Du}{Dt} = -p_x \qquad \frac{Dv}{Dt} = -p_y \qquad \delta^2 \frac{Dw}{Dt} = -p_z$$

$$\nabla \cdot \vec{u} = 0$$

$\rightarrow$ With boundary conditions

$$\left.\begin{aligned} p &= \eta - \frac{\delta^2 \varepsilon h_0}{\lambda^2} \frac{W_e}{R} \\ w &= \frac{1}{\varepsilon}\eta_t + (u_\perp \nabla_\perp)\eta \end{aligned}\right\} \quad \text{on } z = 1 + \varepsilon\eta \tag{1}$$

$$w = \frac{1}{\varepsilon}b_t + (u_\perp \nabla_\perp)b \quad \text{on } z = b \tag{2}$$

History of the Soliton

- a **soliton** is a solitary wave that resists dispersion, maintaining its shape while it propagates at constant velocity

- John Scott Russell discovered the solitary wave in 1834, firstly calling it the **wave of translation**

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Korteweg-de Vries equation: nonlinear, dispersive PDE

$$2\eta_t + 3\eta\,\eta_\xi + \frac{K}{3}\eta_{\xi\xi\xi} = 0 \qquad (\xi = x - ct, \tau = \varepsilon t)$$

With Solution

$$\eta(\xi, \tau) = 2c^2 \mathrm{sech}^2\left(\sqrt{\frac{3}{2K}}(\xi - \tau)\right)$$

1) The KdV equation arises in the $\varepsilon = O(\delta^2)$
2) by rescaling δ in favor of ε in Euler's Equations of motion
3) going into the frame of the moving wave ($\xi = x - t, \tau = \varepsilon t$)
4) conducting an Asymptotic expansion of u, w, p and η.
5) KdV equation is present in the ε^1 term
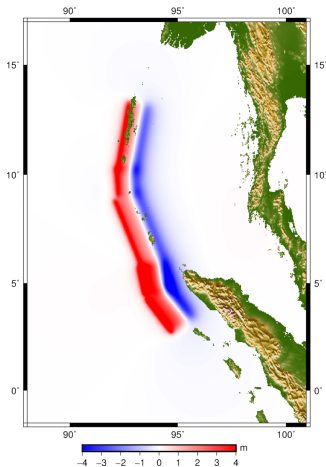
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Figure: Eearthquake generating a tsunami with $\lambda = 100$ km, $a = 1$ m

(3)

- $\varepsilon = \frac{a}{h_0}$ and $\delta = \frac{h_0}{\lambda}$ need to ender the regime $\varepsilon = O(\delta^2)$ for the KdV equation to become relevant
- But also the geophysical scales need to be $\xi = O(1)$ and $\tau = O(1)$ for the KdV-balance to become relevant, that is

$$x = O\left(\varepsilon^{-1}\lambda\right) \qquad (4)$$

- KdV balance is when the waves are ordered with the higest in front following an oscilatorry tale
- This happends because wave amplitude is proportional to wave speed

$$\lambda = 10 \text{ km} \qquad a = 1 \text{ m}$$

Waves propagating westwards to India/Sri Lanka

$$h_0 = 4 \text{ km} \Rightarrow \begin{cases} \varepsilon \simeq 25 \cdot 10^{-5} \\ \delta \simeq 4 \cdot 10^{-2} \end{cases}$$

Waves propagating eastwards to Thailand

$$h_0 = 1 \text{ km} \Rightarrow \begin{cases} \varepsilon \simeq 10^{-3} \\ \delta \simeq 10^{-2} \end{cases}$$

$$\Rightarrow \text{ Both enter the regime } \varepsilon = O(\delta^2)$$

Waves propagating westwards to
India/Sri Lanka ($\simeq 1600$ km)

Waves propagating eastwards to
Thailand ($\simeq 700$ km)

$$\left.\begin{array}{l} \varepsilon \simeq 25 \cdot 10^{-5} \\ \lambda = 100 \text{km} \end{array}\right\} \Rightarrow x \simeq 4 \cdot 10^{5} \text{ km}$$

$$\left.\begin{array}{l} \varepsilon \simeq 10^{-3} \\ \lambda = 100 \text{km} \end{array}\right\} \Rightarrow x \simeq 10^{5} \text{ km}$$

$\Rightarrow$ The propagation distance of the tsunami waves in both directions is not
enough for KdV dynamics to take place.

Bibliography

[1] R. S. Johnson. *A Modern Introduction to the Mathematical Theory of Water Waves*. Cambridge Texts in Applied Mathematics. Cambridge University Press, 1997. DOI: 10.1017/CBO9780511624056.

[2] Geoffrey K. Vallis. *Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-Scale Circulation*. 2nd ed. Cambridge University Press, 2017. DOI: 10.1017/9781107588417.

[3] Adrian Constantin. "On the propagation of tsunami waves, with emphasis on the tsunami of 2004". In: *Discrete and Continuous Dynamical Systems - B* 12.3 (2009), pp. 525–537.

[4] Rupert Klein. "Scale-Dependent Models for Atmospheric Flows". In: *Annual Review of Fluid Mechanics* 42 (Dec. 2009), pp. 249–274. DOI: 10.1146/annurev-fluid-121108-145537.

[5] Wolf Wahl Hans Kerner. *Mathemaik für Physiker*. 3rd ed. Springer-Verlag Berlin Heidelberg 2013, 2013. DOI: 10.1007/978-3-642-37654-2.

Thank you for listening!