Applied Analysis, WS 2021 – Problem Sheet 3

P. 8 Define functions $f : \mathcal{D} \mapsto \mathbb{R}$ that show boundary layer behaviour at the following manifolds:

(a) $\mathcal{D} = \mathbb{R}^2$, $S = \{0\}$.

(b) $\mathcal{D} = \mathbb{R}^n$, $S = \{|x| = 1\}$.

(c) $\mathcal{D} = \mathbb{R}^3$, $S = \{x_1 = 1\}$, $x = (x_1, x_2, x_3)$.

P. 9 **(*)** (optional) We consider the linear boundary value problem

$$Lu := -\varepsilon u'' + b(x)u' + c(x)u = f(x), \qquad u(0) = u(1) = 0$$

with $0 < \varepsilon \ll 1$, b, c, $f \in C([0,1])$ and

$$c(x) \geq 0, \qquad b(x) \geq \beta > 0, \qquad x \in [0,1]$$

Let u_ε be the solution of this BVP, and u_0 the solution of the reduced problem

$$b(x)u' + c(x)u = f(x), \qquad u(0) = 0$$

Show that for all $x \in [0, x_0]$ with $x_0 < 1$

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = u_0(x)$$

holds, and that the convergence is uniform on $[0, x_0]$

Hint: Set $w_1(x) := \exp(\beta x)$ and $w_2(x) := \exp(-\beta(1-x)/\varepsilon)$. Then $Lw_1 \geq \gamma > 0$ for a suitable $\gamma > 0$ and $Lw_2 \geq 0$. For $v = \pm(u_\varepsilon - u_0)$ and $w = A\varepsilon w_1 + Bw_2$ with suitable constants A, B the following comparison principle is applicable: If

$$\begin{array}{lll} Lv(x) & \leq & Lw(x), \quad \forall x \in (0,1) \\ v(0) & \leq & w(0) \\ v(1) & \leq & w(1) \end{array}$$

then

$$\Longrightarrow v(x) \leq w(x) \quad \forall x \in (0,1)$$

which holds for $u, v \in C^2((0,1)) \cap C([0,1])$. So a boundary layer is possible only at $x = 1$. Analoguously, for $b(x) \leq \beta < 0$ it follows that a boundary layer is possible only at $x = 0$.

P. 10 Consider the following linear boundary value problem

$$-\varepsilon u'' - (1+x)u' + u = 2, \qquad u(0) = u(1) = 0, \qquad 0 < \varepsilon \ll 1$$

Where can this problem have a boundary layer? Hint: use the statement of problem 9, last sentence. Introduce the local variable $\xi = x\varepsilon^{-\alpha}$ and derive an asymptotic representation

$$u_\varepsilon(x) = \overline{u}(x) + \tilde{u}(\xi) + \mathrm{O}(\varepsilon)$$

by solving the reduced equation for $\overline{u}$, respectively the boundary layer equation for $\tilde{u}$.