DSNDE exam 4.6.2023

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1. Hyperbolic equilibria [5 points]:

   (a) State the definition of a hyperbolic equilibrium of an autonomuous ODE in $\mathbb{R}^n$.

   (b) Formulate the theorem of Grobman-Hartman.

2. Omega limit [5 points]:

   (a) Define the ω-limit $\omega(p)$.

   (b) List the most important properties of $\omega(p)$.

   (c) Prove one of these properties.

3. Stability [10 points]:
   Show that the origin of the system

   $$\dot{x} = \sigma(y - x)$$
   $$\dot{y} = \rho x - y - xz$$
   $$\dot{z} = xy - \beta z$$

   for $0 < \rho < 1$, $0 < \sigma$, $0 < \beta$ is globally asymptotically stable.
   (Hint: Use the function $V(x, y, z) = \rho x^2 + \sigma y^2 + \sigma z^2$.)

4. Newtonian system [5 points]:
   Let the Newtonian system $\ddot{x} = F(x)$ be given. Write it as a Hamiltonian system of order one and find H.

5. Limit cycle [10 points]:
   Prove that the system

   $$\dot{x} = x - y - x^3$$
   $$\dot{y} = x + y - y^3$$

   has a limit cycle.
   (Hint: Show that the square $|x|, |y| \leq 2$ is forward invariant.)

6. Rotation number [10 points]:
   Let $X := \mathbb{R}/\mathbb{Z} \cong (0, 1]$ and T be an orientation preserving homeomorphism.

   (a) What is a lift of T?

   (b) Define the rotation number $\rho(T)$.

   (c) List the most important properties of $\rho(T)$.

   (d) Let $T_\alpha, T_\beta : X \to X$ with $T_\alpha x = x + \alpha$ (mod 1) and $T_\beta x = x + \beta$ (mod 1). When are T_α and T_β topologically conjugate?

7. Orbits [10 points]:
   Let $X := \mathbb{R}/\mathbb{Z} \cong (0, 1]$ and $T : X \to X$ with $Tx = 2x$ (mod 1). Show that there are infinitely many periodic orbits and some dense orbits.