DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **12**, Number **3**, October **2009**

doi:10.3934/dcdsb.2009.12.525

pp. **525–537**

ON THE PROPAGATION OF TSUNAMI WAVES, WITH EMPHASIS ON THE TSUNAMI OF 2004

Adrian Constantin

University of Vienna, Fakultät für Mathematik
Nordbergstraße 15, 1090 Vienna, Austria

Abstract. In this review paper we discuss the range of validity of nonlinear dispersive integrable equations for the modelling of the propagation of tsunami waves. For the 2004 tsunami the available measurements and the geophyiscal scales involved rule out a connection between integrable nonlinear wave equations and tsunami dynamics.

1. **Introduction.** The December 2004 tsunami, generated on Sunday, 26 December 2004 at $7:58\,am$ (local time, Indonesia) by the most powerful earthquake in decades, killed more than 275000 people [58] and made millions homeless, making it one of the most destructive natural disasters in history. The hypocenter of the earthquake was about $30\,km$ under the floor of the Indian Ocean, at $160\,km$ off the west coast of the Indonesian island of Sumatra, and the violent movement of the Earth's tectonic plates displaced an enormous amount of water, sending tsunami waves westwards across the Indian Ocean as well as eastwards across the Andaman Basin. Since the earthquake occured over about 10 minutes along a $1000\,km$ long approximately straight fault line, the generated tsunami waves were approximately two-dimensional [52, 53], that is, the motion was approximately identical in any direction parallel to the crest line. Within hours these waves crashed upon the coastline of 11 Indian Ocean countries, snatching people out to sea, drowning others in their homes or on beaches, and demolishing property from South Africa to Thailand. The catastrophic devastation wrought by the tsunami occured primarily on the shores of the Bay of Bengal and of the Andaman Basin but substantial damage was also documented in Somalia (some $5000\,km$ to the west of the epicenter) and large waves were noticed as far as Madagascar and South Africa. For modeling purposes, outside of the Bay of Bengal the two-dimensional character of the tsunami waves can not be taken any more more for granted since diffraction around islands and reflection from steep shores alter this feature considerably. The earthquake that generated the tsunami changed the shape of the ocean floor by raising it by a few m to the west of the epicenter and lowering it to the east (over $100\,km$ in the east-west direction and about $900\,km$ in the north-south direction) [53]. The initial shape of the wave pattern that developed into the tsunami wave featured therefore to the west of the epicenter a wave of elevation followed by a wave of depression (that is, with water levels higher, respectively lower than normal), while to the east of the epicenter the initial wave profile consisted of a depression followed by an elevation. The tsunami waves to the west of the epicenter propagated approximately $1600\,km$

2000 *Mathematics Subject Classification.* 35Q51, 35Q53, 76B15, 76B25.
Key words and phrases. Soliton, tsunami, shallow-water, dispersion.

across the Bay of Bengal in the Indian Ocean towards India and Sri Lanka, hitting in less than 3 hours the coastal regions of India and Sri Lanka, the first tsunami wave being a wave of elevation [20, 53]. The tsunami wave to the east of the epicenter crossed the 700 km distance across the Andaman Basin in less than 2 hours, with a leading wave of depression as it hit resorts in Thailand [28, 53]. Several news reports of seaside villages in Thailand confirmed that the first evidence of the tsunami was that the ocean receded rapidly and far. Many people were killed because they went to view the retreating ocean exposing the seafloor, unaware that the large wave of depression was followed by several large waves of elevation (photographs reproduced in [20] show that the shoreline receded before the arrival of the first wave front at Hat Ray Leh beach in southern Thailand and two fronts, one closely behind the other and the second considerably larger, occurred at this time with the maximum height of the tsunami, as it came ashore at this location, of about 10 m). The fact that as the tsunami waves reached the shore in either direction, the shape of the initial disturbance (first wave of elevation, then wave of depression, respectively *vice-versa*) was not altered is of utmost importance in validating a theory for the wave dynamics on this occasion. This observation suggests that perhaps the shape of the tsunami waves remained approximately constant as they propagated across the Bay of Bengal, respectively across the Andaman Basin. This hypothesis is further substantiated by the accurate measurements of the water's surface (the spatial scale of the coverage being about 800 km) performed by a radar altimeter on board a satellite about two hours after the earthquake took place, along a track traversing the Indian Ocean/Bay of Bengal [28]. These clearly show a leading wave of elevation, followed by a wave of depression, a feature common both to the initial wave profile west of the epicenter and to the tsunami as it entered the coastal regions of India and Sri Lanka. These measurements also confirm another essential feature of tsunami waves: even though these waves reach large amplitudes due to the diminishing depth effect as they approach the shore (waves as high as 30 m were observed near the city Banda Aceh on the west coast of the northern tip of Sumatra [41] about 160 km away from the epicenter of the earthquake), tsunami waves are barely noticeable at sea due to their small amplitude. Indeed, the satellite data shows that the maximum amplitude of the waves, whether positive or negative with respect to the usual sea level, was less than 0.8 m over distances of more than 100 km. To get a sense of how mild this disturbance is, we point out the delightful argument from [53]: sitting in a boat in the Bay of Bengal, midway between Sumatra and Sri Lanka, it would take a tsunami wave component (whether wave of elevation/depression) with a wavelength of 100 km about 10 minutes to move past the boat, time in which the boat would move up/down by 0.8 m, and then back down/up by 0.8m. For these estimates the assumption was made (see [53]) that the tsunami wave speed is at least 620 km/h, hypothesis that is confirmed by the considerations made in Section 4 below.

To predict accurately the appearance of a tsunami it is of paramount importance to model these powerful waves, explaining the propagation mechanism as well as the process by which they evolve from a small-amplitude disturbance of the sea level (albeit one of large wavelengths, in excess of 100 km) to become such devastating forces of nature as they approach the coast.

2. **Models for long waves in shallow water.**

Perhaps the most important scientific discovery of the last decades in the context of water waves was soliton theory

[32]. Solitons arise as special solutions of a widespread class of weakly nonlinear dispersive partial differential equations modeling water waves, such as the Korteweg-de Vries (KdV) [32] or the Camassa-Holm (CH) equation [9], representing to various degrees of accuracy approximations to the governing equations for water waves in the shallow water regime (see the discussion in [21]). Informally dispersion means that different harmonic components of a solution travel at different velocities determined by the frequency, so that within the framework of linear theory, even though energy is preserved due to the neglection of dissipative effects, the different components of a solution spread out and consequently the solution at later times tends to have much smaller amplitude than initially. At the weakly nonlinear level, however, in certain regimes, nonlinearity balances dispersion and permanent and localized wave forms traveling at constant speed ("solitary waves") arise as solutions. If such solitary waves present elastic interaction in the sense that as a result of the nonlinear interaction with other waves of this type, they emerge from the collision unchanged, except for a phase shift, we say that the solitary waves are solitons [32]. Examples of equations relevant for water waves with soliton solutions are KdV and CH. When first encountered, the situation we refer to might seem perplexing: in talking about the propagation of shallow water waves, in addition to KdV there exists the regularized long wave equation [49] (usually called BBM [5]), an equation which provides an approximation of the governing equations for water waves equations in the shallow water regime of the same accuracy as KdV (CH arises as a higher-order approximation [21]), whose solitary waves have even similar expressions but the solitary waves of KdV equation are solitons [32] while those of BBM are not [46]. This apparent paradox is however easily resolved. Given a physical situation, under certain simplifying assumptions established laws from physics can be applied to obtain a model of the physical process. Investigating the behaviour of the model by mathematical methods, our understanding of the physical phenomenon can be improved. The conclusions reached will reflect reality (that is, specific physical situations which may be observed experimentally) only insofar as the accuracy of the model permits: the value of a model depends on the number of physically useful deductions which can be made from it. The "truth" of the model is meaningless as all experiments contain inaccuracies of measurement and effects other than those accounted for cannot be totally excluded. Even with the aid of the most advanced computers it is not possible to find exact solitary wave solutions to the nonlinear governing equations for water waves. The progress towards understanding solitary waves based on the governing equations for water waves is noticeable: localized disturbances of a flat water surface propagating without change of form have to be two-dimensional [29], the existence of two-dimensional solitary wave solutions was established [2], these waves have to be waves of elevation with a profile symmetric about the crest [30], and a qualitative description of the particle motion beneath the solitary wave is available [15]. Since an in-depth study of solitary wave interactions using the governing equations is not within reach, to shed light on this important aspect one has to perform approximations leading to simplified model equations. The linear theory of waves of small amplitude does not provide any approximation to solitary waves (see [55]), so that nonlinear approximations to the governing equations for water waves have to be made: KdV and BBM arise as weakly nonlinear approximations, with CH capturing more nonlinear effects [21]. In assessing the relative importance of these model equations the benchmark is provided by the degree to which they provide a description/explanation of specific important water

wave phenomena encountered in nature. Soliton interactions of water waves occur in nature and can be reproduced in the laboratories, with the predictions made by KdV and BBM in close agreement with experimental measurements [52]. The solitary waves of KdV are orbitally stable (meaning that their shape is stable under small perturbations) [4, 44], a feature valid also for BBM [4] and CH [23, 33], which explains why these wave patterns are physically recognizable. The importance of KdV is further enhanced by the fact that an inverse scattering analysis which relies on structural properties of the equation (e.g. its Hamiltonian structure and associated integrals of motion) leads to the following dynamical picture: starting with arbitrary initial data that are smooth and sufficiently localized in space, the KdV solution that evolves from these data is developing into a finite number of localized solitary waves (solitons), plus an oscillatory tail (see [32]). Each solitary wave retains its localized identity and taller waves travel faster than smaller ones, while the oscillatory tail disperses and spreads out in space. Therefore the solution evolves in to an ordered set of solitons, with the tallest in front, followed by an oscillatory tail that gradually fades out. This shows that the solitons are the key to the understanding of the dynamics of water waves as modeled by KdV. BBM is not integrable [48] so that the mechanism of solitary wave interactions is not as plain as for KdV. It is thus no accident that KdV plays a more important role in water wave theory than BBM (which remains a valid model equation but of more limited interest). As for CH, while it is integrable [16, 22], the dynamics of its soliton interactions is more intricate than for KdV so that it is mostly in the context of breaking waves that CH gained importance [6, 7, 14]. We are not concerned with this aspect here: we concentrate on the propagation of tsunami waves across the sea, and in this regard KdV is the proper model equation among the variety of shallow water models, as pointed out above.

We are thus led to the fundamental question of whether tsunami waves enter the regime of validity of KdV as an approximation to the governing equations for water waves. A frequent view encountered throughout the research literature could be formulated as "... a tsunami is produced by a large enough soliton. There may exist tsunamis not directly related to solitons but experts agree that the majority of registered tsunamis were produced by solitons" [34]. However, it is not just because KdV is model arising in the shallow water regime for waves of small amplitude that one can regard tsunamis at sea as manifestations of solitons, even if this is implied by several classical as well as more recent research papers [28, 41, 51, 56]. The question is whether the geophysical scales involved lead to time- and space-scales that are compatible with those required for KdV theory. This question is answered in Section 4, after we specify the context of the validity of KdV as a model for water waves.

3. **The governing equations for water waves.** An important feature in the investigation of the 2004 tsunami is the fact that the depth of the Bay of Bengal between Sumatra and Sri Lanka is relatively uniform with average depth $h_0 = 4\,km$ [50], while the bathmetry of the Andaman Basin is also relatively flat (even if less uniform) with average depth $h_0 = 1\,km$ [53]. Moreover, as we specified in the Introduction, the tsunami waves of December 2004 in the Bay of Bengal and in the Andaman Basin were approximately two-dimensional. To describe such waves it suffices to consider a cross section of the flow that is perpendicular to the crest line. Choose Cartesian coordinates (X, Y) with the Y-axis pointing vertically

upwards, the X-axis being the direction of wave propagation, and with the origin located on the flat bed. Let $(U(X,Y,T),\,V(X,Y,T))$ be the velocity field of the two-dimensional flow propagating in the X-direction over the flat bed $Y=0$, and let $Y=h_0+H(X,T)$ be the water's free surface with mean water level $Y=h_0>0$.

The equation of mass conservation

$$U_X+V_Y=0. \tag{3.1}$$

is a consequence of assuming constant density, a physically reasonable assumption for gravity water waves [42]. The assumption of inviscid flow is realistic since experimental evidence confirms that the length scales associated with an adjustment of the velocity distribution due to laminar viscosity or turbulent mixing are long compared to typical wavelengths [31], so that the equation of motion is Euler's equation

$$\begin{cases} U_T+UU_X+VU_Y=-\dfrac{1}{\rho}\,P_X, \\[2ex] V_T+UV_X+VV_Y=-\dfrac{1}{\rho}\,P_Y-g, \end{cases} \tag{3.2}$$

where P is the pressure, g is the constant acceleration of gravity and ρ is the constant density of water. We also have the boundary conditions

$$P=P_{atm} \qquad \text{on} \quad Y=Y=h_0+H(X,T), \tag{3.3}$$

where P_{atm} is the (constant) atmospheric pressure at the water's free surface,

$$V=H_T+UH_X \qquad \text{on} \quad Y=h_0+H(X,T), \tag{3.4}$$

and

$$V=0 \qquad \text{on} \quad Y=0. \tag{3.5}$$

The conditions (3.4) and (3.5) express the fact that water particles can not cross the free surface, respectively the impermeable rigid bed, and (3.3) decouples the motion of the water from that of the air above it in the absence of surface tension (for tsunami waves the effects of surface tension are negligible [3]). We will consider irrotational flows with zero vorticity

$$U_Y-V_X=0, \tag{3.6}$$

meaning that we allow uniform currents but ignore the presence of non-uniform currents in the fluid [24, 25, 26, 27]. Starting at time $T=0$ with an initial wave profile, we analyze its subsequent motion governed by the balance between the restoring gravity force and the inertia of the system (1)-(6), the objective being to understand the behaviour in time of the free surface $Y=h_0+H(X,T)$.

3.1. **Non-dimensionalisation.** To overcome the difficult mathematical problems encountered in dealing with the governing equations for water waves (3.1)-(3.6) we seek linear or weakly nonlinear approximations. To this end we define a set of nondimensional variables which will enable us to derive approximate linear or weakly nonlinear versions of the governing equations: in non-dimensional form the terms involved can be compared and one can give a meaning to "small with respect to". Without restricting our attention to wave trains (that is, periodic waves traveling without change of shape at constant speed), we assume that the two-dimensional wave pattern under investigation changes very little during its propagation across the sea — a hypothesis that is realistic for the December 2004 tsunami waves in the Bay of Bengal and in the Andaman Basin, as we pointed out in the Introduction.

Since h_0 is the average depth of the water, the non-dimensionalisation Y_0 of Y should be

$$Y = h_0\, y. \tag{3.7}$$

We replace thus the dimensional, physical variable Y by $h_0\, y$, with y now a non-dimensional version of the original Y. The non-dimensionalisation of the horizontal spatial variable is also obvious: if λ is the typical wavelength of the wave, we set

$$X = \lambda\, x. \tag{3.8}$$

The non-dimensionalisation of the remaining variables T, U, V, P, is not that obvious but, once performed, will give at once the remaining non-dimensionalisation of the free surface H. To proceed, we fix the horizontal speed scale $c = \sqrt{gh_0}$; the reason for this reference value being that if one fixes c for the moment, goes through all the steps we are going to perform and requires that for an irrotational flow in non-dimensional variables the average linear wave propagation speed equals 1, the value $c = \sqrt{gh_0}$ emerges. To this speed scale corresponds the time scale

$$T = \frac{\lambda}{c}\, t, \tag{3.9}$$

originating in the observation that an average time of λ/c seconds (if λ is measured in m and c in m/s) is needed for the wave profile to advance the distance λ in the direction it is propagating — in the case of a wave train λ/c would represent the wave period. To obtain the non-dimensional velocity (U_0, V_0) we impose that in the resulting non-dimensionalisation the equation of mass conservation continues to hold true. This is best expressed by noticing that (3.1) can be understood as the necessary and sufficient condition for the existence of a stream function $\Psi(X, Y, T)$ defined uniquely up to an additive function of T by the requirements $\Psi_X = -V$ and $\Psi_Y = U$. We now set $\Psi(X, Y, T) = \alpha\, \Psi_0(x, y, t)$ with (x, y, t) defined by (3.7), (3.8), (3.9), and with $\alpha > 0$ to be determined later on, and set $U_0 = \partial_y \Psi_0$, $V_0 = -\,\partial_x \Psi_0$. This amounts to

$$U = \frac{\alpha}{h_0}\, U_0, \qquad V = \frac{\alpha}{\lambda}\, V_0, \tag{3.10}$$

so that $\alpha = h_0\sqrt{gh_0}$ and $U = \sqrt{gh_0}\, U_0$, $V = \delta\sqrt{gh_0}\, V_0$. We now view the pressure P as a perturbation of the hydrostatic pressure $P_{atm} - \rho g\,(Y - h_0)$ which together with $U = V = 0$, and $H = 0$ provides a flat-surface solution to the governing equations (representing a still water surface). Thus we set

$$P(X, Y, T) = P_{atm} - \rho g h_0\,(y - 1) + \rho g h_0\, P_0(x, y, t), \tag{3.11}$$

where P_0 is the non-dimensional pressure and $\rho g h_0$ is its dimensional factor, representing the pressure difference between the value of the hydrostatic pressure on the flat bed and on the flat surface — the maximal oscillation of the hydrostatic pressure. Finally, we set

$$H(X, T) = a\, h(x, t) \tag{3.12}$$

with a being a typical amplitude and h being the non-dimensional surface profile. Performing now the changes of variables (3.7), (3.8), (3.9), 3.10), (3.11), (3.12), we transform the governing equations into the equivalent system of partial differential equations

$$\begin{cases} \dfrac{\partial U_0}{\partial x} + \dfrac{\partial V_0}{\partial y} = 0, \\ \dfrac{\partial U_0}{\partial t} + U_0\,\dfrac{\partial U_0}{\partial x} + V_0\,\dfrac{\partial U_0}{\partial y} = -\,\dfrac{\partial P_0}{\partial x}, \\ \delta^2\left(\dfrac{\partial V_0}{\partial t} + U_0\,\dfrac{\partial V_0}{\partial x} + V_0\,\dfrac{\partial V_0}{\partial y}\right) = -\,\dfrac{\partial P_0}{\partial y}, \\ \dfrac{\partial U_0}{\partial y} - \delta^2\,\dfrac{\partial V_0}{\partial x} = 0, \end{cases} \tag{3.13}$$

for $(x,y) \in \mathbb{R} \times [0, 1+\varepsilon\, h(x,t)]$, with the boundary conditions

$$\begin{cases} P_0 = \varepsilon\, h \quad \text{on} \quad y = 1 + \varepsilon\, h(x,t), \\ V_0 = \varepsilon\left(\dfrac{\partial h}{\partial t} + U_0\,\dfrac{\partial h}{\partial x}\right) \quad \text{on} \quad y = 1 + \varepsilon\, h(x,t), \\ V_0 = 0 \quad \text{on} \quad y = 0, \end{cases} \tag{3.14}$$

where we denoted by

$$\varepsilon = \frac{a}{h_0}, \qquad \delta = \frac{h_0}{\lambda}, \tag{3.15}$$

the amplitude parameter, respectively the shallowness parameter.

3.2. **Scaling.** For a Stokes wave[1], using maximum principles for elliptic equations (as in [10, 13]) one can easily show that the perturbation P_0 of the pressure with respect to the hydrostatic pressure distribution, given by (3.11), attains its maximum and its minimum throughout the fluid along the free surface, and a similar statement holds for the nondimensional vertical velocity component V_0 defined via (3.10). On the basis of this result and in view of the fact that the first two boundary conditions in (3.14) show that near the surface both V_0 and P_0 are proportional to ε, it is reasonable to use ε as a measure of the oscillation of V_0 and of the perturbation pressure P_0, and scale accordingly. Thus we set

$$P_0(x,y,t) = \varepsilon\, p(x,y,t), \qquad V_0(x,y,t) = \varepsilon\, v(x,y,t). \tag{3.16}$$

A Taylor expansion estimating the difference between the attained maximum and minimum of the function $(x,y) \mapsto U_0(x,y,t)$ as x and y remain bounded shows that the oscillation of this function is of order ε, the same order of magnitude as that of its first-order partial derivatives according to (3.16) and the first and last equation in (3.13). This leads us to the scaling

$$U_0(x,y,t) = \varepsilon\, u(x,y,t). \tag{3.17}$$

We change variables in (3.13)-(3.14) according to (3.16)-(3.17) and obtain the set of dimensionless, scaled equations

$$\begin{cases} u_x + v_y = 0, \\ u_t + \varepsilon\,(u u_x + v u_y) = -\,p_x, \\ \delta^2\Big(v_t + \varepsilon(u v_x + v v_y)\Big) = -\,p_y, \\ u_y - \delta^2\, v_x = 0, \end{cases} \tag{3.18}$$

[1]These are periodic traveling waves propagating in irrotational flow without change of shape and at constant speed at the surface of water with a flat bed [10].

for $(x,y) \in \mathbb{R} \times [0, 1+\varepsilon\, h(x,t)]$, with the boundary conditions

$$\begin{cases} p = h \quad \text{on} \quad y = 1 + \varepsilon\, h(x,t), \\ v = h_t + \varepsilon u\, h_x \quad \text{on} \quad y = 1 + \varepsilon\, h(x,t), \\ v = 0 \quad \text{on} \quad y = 0. \end{cases} \tag{3.19}$$

3.3. **Long waves of small amplitude.** Proving a well-posedness theorem for (3.18)-(3.19) which gives an estimate of the existence time of the solution which is uniform with respect to the two fundamental dimensionless numbers ε and δ, one obtains rigorously the validity ranges for the main physical regimes encountered in the modeling of two-dimensional water waves [1]:

1. shallow-water, large amplitude regime ($\delta \ll 1$, $\varepsilon \sim 1$) leading at first order to the shallow-water equations [40] and at second order to the Green-Naghdi model [35] which takes into account the dispersive effects neglected by the shallow water equations;
2. shallow-water, medium amplitude regime ($\delta \ll 1$, $\varepsilon \sim \delta$) leading to the Serre equations [54] and to the Camassa-Holm equation [9] (see [21]);
3. shallow-water, small amplitiude or long-wave regime ($\delta \ll 1$, $\varepsilon \sim \delta^2$) leading at first order to the linear wave equation
$$h_{tt} - h_{xx} = 0 \quad \text{on} \quad y = 1, \tag{3.20}$$
with the general solution
$$h(x,t) = h_+(x-t) + h_-(x+t), \tag{3.21}$$
where the sign $\pm$ refers to a wave of profile $h_\pm$ moving with unchanged shape to the right/left at constant unit speed by virtue of the non-dimensionalisation process, the corresponding dimensional speed being $\sqrt{gh_0}$. The small effects that were ignored at first order (small amplitude, long wave) build up on longer time/spatial scales to have a significant cumulative nonlinear effect so that on a longer time scale each of the waves that make up the solution (3.21) to (3.20) satisfies the KdV equation [53];
4. deep-water, small steepness regime ($\delta \gg 1$, $\varepsilon\delta \ll 1$) leading to the full-dispersion Matsuno equations [47].

Therefore KdV arises in the regime
$$\varepsilon = O(\delta^2) \tag{3.22}$$
which emerges naturally since by setting $\varepsilon = \beta\delta^2$ with $\beta = O(1)$ one transforms (3.18)-(3.19) into a problem involving only one small parameter, ε. The system (3.18)-(3.19) has actually a far-reaching property: the parameter δ can be scaled out in favor of ε as follows:
$$\begin{aligned} &x \mapsto x\,\delta/\sqrt{\varepsilon}, \quad y \to y, \quad t \mapsto t\,\delta/\sqrt{\varepsilon}, \\ &p \mapsto p, \quad \eta \mapsto \eta, \quad u \mapsto u, \quad v \mapsto v\,\sqrt{\varepsilon}/\delta, \end{aligned} \tag{3.23}$$
the additional scaling (3.23) producing therefore the system (3.18)-(3.19) with δ^2 replaced by ε, for arbitrary δ. This observation opens up the possibility to prove that, provided suitable length- and time-scales are available, for arbitrary wavelengths δ, and not just for those bound to ε via (3.22), KdV will arise as a valid approximation for the evolution of the free surface waves. However, no rigorous considerations are available to support this formal conclusion based on the scaling (3.23). Furthermore, on the basis of the satellite measurements available for the

December 2004 tsunami we know that for the propagation across the Bay of Bengal the choice

$$a = 1\, m, \qquad \lambda = 100\, km, \tag{3.24}$$

is appropriate [53]. Moreover, as argued at the beginning of Section 2, the Bay of Bengal between Sumatra and Sri Lanka has a relatively flat bed with average depth $h_0 = 4\, km$. Computing

$$\varepsilon = \frac{a}{h_0} = 25 \times 10^{-5}, \qquad \delta = \frac{h_0}{\lambda} \approx 4 \times 10^{-2},$$

we see that $\varepsilon \approx \delta^2$. Therefore the tsunami waves propagating westwards enter the regime (3.22) in which KdV arises as an approximation to the governing equations for water waves. Let us now specify how this approximation is to be understood. The sharpest rigorous result in this direction [1] ensures that, given $\varepsilon_0 > 0$, there exists $T_0 > 0$ such that in the region (3.22), if we define

$$\zeta^\varepsilon(x, t) = \zeta^+(x - t, \tau),$$

with $\tau = t\varepsilon$ and $\zeta^+(\xi, \tau)$ solving the KdV equation

$$\zeta^+_\tau + \frac{3}{2}\,\zeta^+_{\xi\xi\xi} + \frac{1}{6}\,\zeta^+\,\zeta^+_\xi = 0, \qquad \tau > 0, \quad \xi \in \mathbb{R},$$

with initial data

$$\zeta^+(\xi, 0) = h(\xi, 0), \qquad \xi \in \mathbb{R},$$

then for some $k > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ one has

$$|h(x, t) - \zeta^\varepsilon(x, t)| \le k\, \varepsilon^2 t, \qquad t \in [0, \frac{T_0}{\varepsilon}], \quad x \in \mathbb{R},$$

for the solution $h(x, t)$ entering in (3.18)-(3.19). Moreover, in the case of a non-flat bed with small variations of the order of the size of the surface waves, meaning that $\dfrac{b}{h_0} = O(\varepsilon)$ with b measuring the amplitude of the variations of the bottom topography, then the constant-coefficient KdV equation has to be replaced by a variable-coefficient KdV equation [36] and similar approximations of order $O(\varepsilon^2 t)$ over time intervals of length $O(\varepsilon^{-1})$ hold [39]. In these results an essential feature is the fact that the solutions to both KdV and (3.18)-(3.19) exist on the specified time intervals, provided that the initial profile $h(\cdot, 0)$ is sufficiently smooth and localized; the assumption of an initial profile in the Sobolev space $H^7(\mathbb{R})$ suffices [1].

4. **KdV dynamics and the December 2004 tsunami.** We now examine the geophysical scales involved in the December 2004 tsunami to decide whether KdV models the propagation of these waves across the Bay of Bengal or across the Andaman Basin. Notice that dispersion is not relevant in the generation phase but it may modify the wave propagation when the motion takes place over a long time, become important [45] and the main issue is whether a balance between dispersion and nonlinearity can occur over the given geophysical scales so that KdV dynamics becomes possible. With the validity of the regime (3.22) already established, from the results in [1] we deduce that the region where a KdV-balance can occur is given by

$$x - t = O(1), \qquad \tau = O(1).$$

In view of the fact that $\tau = \varepsilon t$ and of the non-dimensionalisation (3.8), (3.9), in the original physical variables the region where KdV dynamics becomes relevant is given by

$$\frac{X - T\sqrt{gh_0}}{\lambda} = O(1), \quad \frac{\varepsilon T\sqrt{gh_0}}{\lambda} = O(1).$$

From the second relation above we get $\dfrac{T\sqrt{gh_0}}{\lambda} = O(\varepsilon^{-1})$, which in combination with the first relation above leads us to $\dfrac{X}{\lambda} = O(\varepsilon^{-1})$. Therefore the length scale on which KdV dynamics becomes relevant is

$$X = O(\varepsilon^{-1}\,\lambda). \tag{4.25}$$

This scale corrects the classical scale for the evolution of KdV, believed to be

$$X = O(\varepsilon^{-1}\,h_0) \tag{4.26}$$

cf. the discussion on pages 12-13 of the recent survey [53], where the classical theory put forward in the papers [37, 38] is presented. The correction (4.25) of (4.26) is significant in the case of tsunami waves due to the fact that throughout the propagation of these waves at sea we have $\dfrac{h_0}{\lambda} \ll 1$: tsunamis wavelengths are in excess of $100\,km$, while the deepest part of the world's seas is the Mariana Trench in the western North Pacific Ocean near Guar, with maximum depth of about $10911\,m$.

Let us now look in detail at the December 2004 tsunami. For the waves propagating across the Bay of Bengal towards India and Sri Lanka, the length scale (4.25) with $\lambda = 100\,km$ and $\varepsilon = 25 \times 10^{-5}$ shows that a propagation distance of about $4 \times 10^5\,km$ is needed for the occurence of a KdV balance. The distances of less than $1600\,km$ from the epicentre of the earthquake to the coasts of India and Sri Lanka therefore did not permit KdV dynamics to be of relevance on this occasion. For the waves propagating eastward across the Andaman Basin we start from the average depth $h_0 = 1\,km$ [28, 53], taking for the initial wave the same characteristics $a = 1\,m$ and $\lambda = 100\,km$ as in (3.24), data that was confirmed for the tsunami waves traveling westwards across the Bay of Bengal by the satellite measurements [53]. This gives $\varepsilon = 10^{-3}$ and $\delta = 10^{-2}$, showing that also these tsunami waves enter within the regime (3.22), with the corresponding length scale (4.25) required to see KdV dynamics of about $10^5\,km$. Our analysis invalidates the statement in [28] (see page 537) that dispersion played a key role in the westward propagation of the tsunami waves on December 2004, since we showed that a balance between dispersion and nonlinearity can not occur over the given geophysical scales (and thus dispersion has a negligible effect on the wave propagation). It is not enough that "the tsunami fits well into the regime (3.22) of dispersive nonlinear waves" [28] to ensure the validity of KdV as a model equation for the surface waves, only on length scales of order (4.25) can one see KdV dynamics. On the other hand, our analysis supports the conclusion made in [53] (see page 13) that "the propagation distances from the earthquake to India, Sri Lanka, or Thailand were much too short for KdV dynamics to develop". Notice, however, that this conclusion was reached in [53] by using the wrong scale (4.26), and it is only an artifact of the particular numbers involved that one can rule out the relevance of KdV: in [53] the distance required for the westward propagating wave to show KdV dynamics was estimated to be approximately $10^4\,km$, whereas the correct estimate is of the order of $4 \times 10^5\,km$, as computed above. We would also like to point out that our conclusion is in agreement with some other recent findings [17, 18, 19, 20] but in these papers the formal additional scaling (3.23) was used. In the present investigation we conclude from the satellite measurements, as in [28], that the tsunami propagation took place in the regime (3.22) and then the sharp results in [1] allow us to identify rigorously the exact length scales needed for KdV dynamics to play a role — this approach was also hinted out at in [12] but in the present paper we offer a more thorough analysis of the December 2004 tsunami.

Let us conclude with a brief discussion of the wave dynamics as the tsunami propagated towards the coast. The previous considerations show that from initiation until reaching westwards the coastal regions of India and Sri Lanka, respectively eastwards those of Thailand, a good approximation in non-dimensional variables of the tsunami waves is provided by the solution $(x,t) \mapsto h_0(x \pm t)$ of the linear wave equation (3.20), where $h_0(x) = h(x,0)$ is the initial wave profile and the choice in $\pm$ determines the propagation direction (westward or eastward). In the original physical variables this means that up until near-shore the wave profile remained unaltered (explaining why in both directions the leading wave was a wave of elevation, respectively a wave of depression, in accordance with the shape of the initial profile), propagating at constant speed $\sqrt{gh_0}$. Setting $h_0 = 4\,km$ for the Bay of Bengal and $h_0 = 1\,km$ for the Andaman Basin, we conclude that the westward tsunami waves (which struck Sri Lanka) propagated at a speed of approximately $712\,km/h$, whereas the eastward propagating waves (which struck Thailand) had a wave speed of approximately $356\,km/h$. These values indicate that the tsunami waves propagating westwards crossed the $1550\,km$ in Bay of Bengal from the epicenter of the earthquake to Sri Lanka [28] in about $2h\,10\,min$, while those propagating eastwards towards Thailand crossed the $700\,km$ wide Andaman Basin in about $1h\,57\,min$. These estimates are quite accurate since the Thai resort at Hat Ray Leh was hit by the tsunami $2h$ after initiation [20], while the first tsunami waves hitting the southern tip of Sri Lanka were recorded $2h\,12\,min$ after initiation [43]. The linear model breaks down as the tsunami waves enter the shallower water of the coastal regions and for the understanding of the tsunamis close to the shore the appropriate equations are those modeling the propagation of long waves over variable depth (see the discussion in [20]). To quantify the dynamics of tsunami waves as they impact on coastal areas is a challenging mathematical and physical problem of outmost importance. It is in this regime that dispersion (that was insignificant during the tsunami propagation at sea) starts to play an important role: before the waves reach the breaking state, their front steepens and dispersion, no matter how weak, becomes relevant [20]. In this region faster wave fronts can catch up slower ones (but they can never overtake them) as a manifestation of the "confluence of shocks" (see [57]) and can result in large amplitude wave fronts building up behind smaller ones [20].

Acknowledgments. The author is grateful to the referee for constructive comments.

REFERENCES

[1] B. Alvarez-Samaniego and D. Lannes, *Large time existence for 3D water-waves and asymptotics*, Invent. Math., **171** (2008), 485–541.

[2] C. J. Amick and J. F. Toland, *On solitary water-waves of finite amplitude*, Arch. Ration. Mech. Anal., **76** (1981), 9–95.
[3] T. P. Barnett and K. E. Kenyon, *Recent advances in the study of wind waves*, Rep. Prog. Phys., **38** (1975), 667–729.
[4] T. B. Benjamin, *The stability of solitary waves*, Proc. Roy. Soc. London A, **328** (1972), 153–183.
[5] T. B. Benjamin, J. L. Bona and J. J. Mahony, *Model equations for long waves in nonlinear dispersive systems*, Philos. Trans. Roy. Soc. London Ser. A, **272** (1972), 47–78.
[6] A. Bressan and A. Constantin, *Global conservative solutions of the Camassa-Holm equation*, Arch. Ration. Mech. Anal., **183** (2007), 215–239.
[7] A. Bressan and A. Constantin, *Global dissipative solutions of the Camassa-Holm equation*, Anal. Appl., **5** (2007), 1–27.
[8] E. Bryant, "Tsunami: The Underrated Hazard," Springer, Heidelberg, 2008.
[9] R. Camassa and D. D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett., **71** (1993), 1661–1664.
[10] A. Constantin, *The trajectories of particles in Stokes waves*, Invent. Math., **166** (2006), 523–535.
[11] A. Constantin, *Solitons from the Lagrangian perspective*, Discrete Contin. Dyn. Syst., **19** (2007), 469–481.
[12] A. Constantin, *On the relevance of soliton theory to tsunami modelling*, Wave Motion, DOI: 10.1016/j.wavemoti.2009.05.002
[13] A. Constantin, M. Ehrnström and E. Wahlén, *Symmetry of steady periodic gravity water waves with vorticity*, Duke Math. J., **140** (2007), 591–603.
[14] A. Constantin and J. Escher, *Wave breaking for nonlinear nonlocal shallow water equations*, Acta Mathematica, **181** (1998), 229–243.
[15] A. Constantin and J. Escher, *Particle trajectories in solitary water waves*, Bull. Amer. Math. Soc., **44** (2007), 423–431.
[16] A. Constantin, V. Gerdjikov and R. Ivanov, *Inverse scattering transform for the Camassa-Holm equation*, Inverse Problems, **22** (2006), 2197–2207.
[17] A. Constantin and D. Henry, *Solitons and tsunamis*, Z. Naturforsch., **64a** (2009), 65–68.
[18] A. Constantin and R. S. Johnson, *Modelling tsunamis*, J. Phys. A, **39** (2006), L215–L217.
[19] A. Constantin and R. S. Johnson, *On the non-dimensionalisation, scaling and resulting interpretation of the classical governing equations for water waves*, J. Nonl. Math. Phys., **15** (2008), 58–73.
[20] A. Constantin and R. S. Johnson, *Propagation of very long water waves, with vorticity, over variable depth, with applications to tsunamis*, Fluid Dynam. Res., **40** (2008), 175–211.
[21] A. Constantin and D. Lannes, *The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations*, Arch. Ration. Mech. Anal., **192** (2009), 165–186.
[22] A. Constantin and H. P. McKean, *A shallow water equation on the circle*, Comm. Pure Appl. Math., **52** (1999), 949–982.
[23] A. Constantin and W. Strauss, *Stability of the Camassa-Holm solitons*, J. Nonlinear Sci., **12** (2002), 415–422.
[24] A. Constantin and W. Strauss, *Exact periodic water waves with vorticity*, C. R. Math. Acad. Sci. Paris, **335** (2002), 797–800.
[25] A. Constantin and W. Strauss, *Exact steady periodic water waves with vorticity*, Comm. Pure Appl. Math., **57** (2004), 481–527.
[26] A. Constantin and W. Strauss, *Stability properties of steady water waves with vorticity*, Comm. Pure Appl. Math., **60** (2007), 911–950.
[27] A. Constantin and W. Strauss, *Rotational steady water waves near stagnation*, Phil. Trans. Roy. Soc. London A, **365** (2007), 2227–2239.
[28] W. Craig, *Surface water waves and tsunamis*, J. Dynam. Differential Equations, **18** (2006), 525–549.
[29] W. Craig, *Non-existence of solitary water waves in three dimensions*, Phil. Trans. Roy. Soc. London Ser. A, **360** (2002), 2127–2135.
[30] W. Craig and P. Sternberg, *Symmetry of solitary waves*, Comm. Partial Differential Equations, **13** (1988), 603–633.
[31] A. F. Teles da Silva and D. H. Peregrine, *Steep, steady surface waves on water of finite depth with constant vorticity*, J. Fluid Mech., **195** (1988), 281–302.

[32] P. G. Drazin and R. S. Johnson, "Solitons: An Introduction," Cambridge University Press, Cambridge, 1989.

[33] K. El Dika and L. Molinet, *Exponential decay of H^1-localized solutions and stability of the train of N solitary waves for the Camassa-Holm equation*, Philos. Trans. Roy. Soc. London A, **365** (2007), 2313–2331.

[34] A. T. Filippov, "The Versatile Soliton," Birkhäuser Boston, 2000.

[35] A. E. Green and P. M. Naghdi, *A derivation of equations for wave propagation in water of variable depth*, J. Fluid Mech., **78** (1976), 237–246.

[36] R. Grimshaw, *Solitary waves propagating over variable topography*, in "Tsunami and Nonlinear Waves" (ed. A. Kundu), Springer, (2007), 51–64.

[37] J. L. Hammack, *A note on tsunamis: their generation and propagation in an ocean of uniform depth*, J. Fluid Mech., **60** (1973), 769–799.

[38] J. L. Hammack and H. Segur, *The Korteweg-de Vries equation and water waves*, J. Fluid Mech., **84** (1978), 337–358.

[39] S. Israwi, *Variable depth KDV equations and generalizations to more nonlinear regimes*, preprint, arXiv:0901.3201v1.

[40] R. S. Johnson, "A Modern Introduction to the Mathematical Theory of Water Waves," Cambridge University Press, Cambridge, 1997.

[41] M. Lakshmanan, *Integrable nonlinear wave equations and possible connections to tsunami dynamics*, in "Tsunami and Nonlinear Waves," (ed. A. Kundu), Springer, (2007), 31–49.

[42] J. Lighthill, "Waves in Fluids," Cambridge University Press, Cambridge, 1978.

[43] L.-F. Liu Philip, P. Lynett et al., *Observations by the international Tsunami survey team in Sri Lanka*, Science, **308** (2005), 1595.

[44] J. H. Maddocks and R. L. Sachs, *On the stability of KdV multi-solitons*, Comm. Pure Appl. Math., **46** (1993), 867–901.

[45] P. A. Madsen, D. R. Fuhrman and H. A. Schaeffer, *On the solitary wave paradigm for tsunamis*, J. Geophys. Res. - Oceans, **113** (2008), Art. No. C12012.

[46] Y. Martel, F. Merle and T. Mizumachi, *Description of the inelastic collision of two solitary waves for the BBM equation*, preprint, arXiv:0803.4020v2.

[47] Y. Matsuno, *Nonlinear evolutions of surface gravity waves on fluid of finite depth*, Phys. Rev. Lett., **69** (1992), 609–611.

[48] P. J. Olver, *Euler operators and conservation laws of the BBM equation*, Math. Proc. Camb. Phil. Soc., **85** (1979), 143–160.

[49] D. H. Peregine, *Calculations of the development of an undular bore*, J. Fluid Mech., **25** (1966), 321–330.

[50] R. V. Sarma, M. V. Ramana et al., *Morphological features in the Bay of Bengal*, J. Ind. Geophys. Union, **4** (2000), 185–190.

[51] G. Schneider and C. E. Wayne, *On the validity of 2D-surface water wave models*, GAMM Mitt. Ges. Angew. Math. Mech., **25** (2002), 127–151.

[52] H. Segur, *Integrable models of waves in shallow water*, in "Probability, Geometry and Integrable Systems," (eds. M. Pinsky and B. Birnir), Math. Sci. Res. Inst. Publ., Cambridge University Press, 2008, 345–371.

[53] H. Segur, *Waves in shallow water, with emphasis on the tsunami of 2004*, in "Tsunami and Nonlinear Waves," (ed. A. Kundu), 2007, Springer, 3–29.

[54] F. Serre, *Contribution à l'étude des écoulements permanents et variables dans les canaux*, La Houille Blanche, **3** (1953), 374–388.

[55] J. J. Stoker, "Water Waves. The Mathematical Theory with Applications," Wiley-Interscience, New York, 1992.

[56] S. S. Voit, *Tsunamis*, Ann. Rev. Fluid Mech., **19** (1987), 217–236.

[57] G. B. Whitham, "Linear and Nonlinear Waves," John Wiley & Sons, Inc., New York, 1999.

[58] D. H. Zhang, T. L. Yip and C.-O. Ng, *Predicting tsunami arrivals: Estimates and policy implications*, Marine Policy, **33** (2009), 643–650.

Received April 2009; revised June 2009.

E-mail address: adrian.constantin@univie.ac.at