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Applied Analysis Problems

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1 Sheet 7

1.1 Dirac Comb

The Dirac train or Dirac comb on defined in the following way

$$\text{III}_m[n] = \begin{cases} 1 & n = 0, \pm m, \pm 2m, \dots \\ 0 & \text{else} \end{cases} \tag{1}$$

The dirac comb can be represented in a series of discrete dirac delta's

$$\text{III}_m[n] = \sum_{l=-N}^{N} \delta[n - lm], \tag{2}$$

where $\delta[s] = 1$ if $s = 0$ else 0, for $s \in \mathbb{Z}$. The discrete Fourier transform of the Dirac comb in $\mathbb{C}^N$ is

$$\widehat{\text{III}_m[n]} = \frac{1}{N} \sum_{n=0}^{N-1} \text{III}_m[n] e^{-2\pi i \frac{k}{N} n} = \tag{3}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \left(\sum_{l=-N}^{N} \delta(n - lm) \right) e^{-2\pi i \frac{k}{N} n}, \tag{4}$$

where the summation happens exactly $\frac{N}{m}$ times, then

$$\frac{1}{m} \sum_{l=-N}^{N} e^{-2\pi i \frac{k}{N} lm} = \tag{5}$$

$$= \frac{1}{m} \sum_{l=-N}^{N} \delta[k - l \cdot \frac{N}{m}] \qquad \text{(Poisson's summation formula)} \tag{6}$$

$$= \frac{1}{m} \text{III}_{\frac{N}{m}}[k] \tag{7}$$

1.2 Schwartz Space

The Schwartz space $\mathcal{S}(\mathbb{R}^d)$, for $d \in \mathbb{N}$ is defined as

$$\mathcal{S} := \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) : \forall \alpha, \beta \in \mathbb{N}^d \ \ \|f\|_{\alpha,\beta} < \infty \right\}, \tag{8}$$

$$\|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^d} \left| x^\alpha (D^\beta f)(x) \right|. \tag{9}$$

Our aim is to show that if $f \in \mathcal{S}(\mathbb{R})$ then $\hat{f} \in \mathcal{S}(\mathbb{R})$. The condition is obviously

$$\|\hat{f}\|_{\alpha,\beta} = \sup_{\xi \in \mathbb{R}} \left| \xi^\alpha (D^\beta \hat{f})(\xi) \right| < \infty, \tag{10}$$

for all $\alpha, \beta \in \mathbb{N}$. We can start with what we know about the Fourier transform

$$\xi^\alpha \hat{f}(\xi) = \mathcal{F}\left(\frac{1}{(2\pi i)^\alpha} (D^\alpha f)(x) \right) \tag{11}$$

$$D^\beta \hat{f}(\xi) = \mathcal{F}\left((-2\pi i x)^\beta f(x) \right). \tag{12}$$

Combining the two relations above we get

$$\xi^\alpha (D^\beta \hat{f})(\xi) = \mathcal{F}\left(\frac{(-2\pi i x)^\beta}{(2\pi i)^\alpha} x^\beta (D^\alpha f)(x) \right) =: \mathcal{F}(g(x)) \tag{13}$$

$$\tag{14}$$

If we call this function g, then $g \in \mathcal{S}(\mathbb{R})$ and $g \in L^1(\mathbb{R})$. Applying the Riemann-Lebesgue Lemma we get

$$\hat{g}(\xi) = \int_{\mathbb{R}} g(x) e^{-2\pi i x \xi} \, dx \longrightarrow 0 \quad \text{as } |\xi| \to \infty \tag{15}$$

Thereby $\hat{g} \in \mathcal{S}(\mathbb{R})$ and thus $\hat{f} \in \mathcal{S}(\mathbb{R})$.

1.3 Tempered Distributions

Tempered distributions are the elements of

$$\mathcal{S}'(\mathbb{R}^d) := \left\{ L : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C} | L \text{ is linear and continuous} \right\}. \tag{16}$$

Consider ξ as a tempered distribution, buy acting on $\varphi \in \mathcal{S}(\mathbb{R})$ we have

$$\xi(\phi) = \int_\xi \xi \varphi(\xi) \, d\xi. \tag{17}$$

The Fourier transform of ξ is

$$\hat{\xi}(\varphi) = \xi(\hat{\varphi}) = \int_{\mathbb{R}} \xi \hat{\varphi}(\xi) \, d\xi \tag{18}$$

$$= \int_{\mathbb{R}^2} \xi \varphi(x) e^{2\pi i \xi x} \, dx d\xi \tag{19}$$

$$= \int_{\mathbb{R}^2} \varphi(x) \xi e^{2\pi i \xi x} \, dx d\xi \tag{20}$$

$$= \int_{\mathbb{R}^2} \varphi(x) \frac{i}{2\pi} \frac{\partial}{\partial x} e^{2\pi i \xi x} \, dx d\xi = \tag{21}$$

$$= \frac{i}{2\pi} \int_{\mathbb{R}^2} \varphi(x) \delta'(x) \, dx = \tag{22}$$

$$= \frac{i}{2\pi} \delta'(\varphi). \tag{23}$$

1.4 Fourier transform of the Dirac Comb

The general case of the Dirac Comb as a distribution is

$$\text{Ш}_T = \sum_{n\in\mathbb{Z}} \delta_{nT}. \tag{24}$$

The Fourier transform of the Ш_T distribution for $\varphi \in \mathcal{S}(\mathbb{R})$ is

$$\widehat{\text{Ш}_T}(\varphi) = \sum_{n\in\mathbb{Z}} \hat{\delta}_{nT}(\varphi) \tag{25}$$

$$= \sum_{n\in\mathbb{Z}} \delta_{n\omega_0}(\varphi) \tag{26}$$

$$= \text{Ш}_{\omega_0}(\varphi). \tag{27}$$

The Fourier transform, transforms the period of the combs.

1.5 Shannon Sampling

The Fourier transform of $1_{[-\frac{a}{2},\frac{a}{2}]}(x)$ is

$$\mathcal{F}\left(1_{[-\frac{a}{2},\frac{a}{2}]}\right)(\xi) = \int_{\mathbb{R}} 1_{[-\frac{a}{2},\frac{a}{2}]} e^{-2\pi i x\xi}\ dx \tag{28}$$

$$= \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-2\pi i x\xi}\ dx \tag{29}$$

$$= \frac{-1}{2\pi i\xi} e^{-2\pi i x\xi}\Big|_{-\frac{a}{2}}^{\frac{a}{2}} \tag{30}$$

$$= \frac{1}{\pi\xi}\frac{1}{2i}\left(e^{piia\xi} - e^{-\pi i a\xi}\right) \tag{31}$$

$$= \frac{\sin(\pi\xi a)}{\pi\xi} \tag{32}$$