University of Vienna

Seminar:
Applied PDE Seminar

Mathematical Modeling of Some Water-Waves

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June 2, 2022

Contents

1 Governing Equations of Fluid Dynamics

We first start of with a fluid with a density

$$\rho(\mathbf{x}, t), \tag{1}$$

in three dimensional Cartesian coordinates $\mathbf{x} = (x, y, z)$ at time t. For water-wave applications, we should note that we take $\rho =$ constant, but we will go into this fact later. The fluid moves in time and space with a velocity field

$$\mathbf{u}(\mathbf{x}, t) = (u, v, w). \tag{2}$$

Additionally it is also described by its pressure

$$P(\mathbf{x}, t), \tag{3}$$

generally depending on time and position. When thinking of e.g. water the pressure increases the deeper we go, that is with decreasing or increasing z direction (depending how we set up our system z pointing up or down respectively).

The general assumption in fluid dynamics is the **Continuum Hypothesis**, which assumes continuity of $\mathbf{u}, \rho$ and P in $\mathbf{x}$ and t. In other words, we premise that the velocity field, density and pressure are "nice enough" functions of position and time, such that we can do all the differential operations we desire in the framework of differential analysis.

1.1 Mass Conservation

Our aim is to derive a model of the fluid and its dynamics, with respect to time and position, in the most general way. This is usually done thinking of the density of a given fluid, which is a unit mass per unit volume, intrinsically an integral representation to derive these equations suggests by itself.

Let us now thing of an arbitrary fluid. Within this fluid we define a fixed volume V relative to a chosen inertial frame and bound it by a surface S within the fluid, such that the fluid motion $\mathbf{u}(\mathbf{x}, t)$ may cross the surface S. The fluid density is given by $\rho(\mathbf{x}, t)$, thereby the mass of the fluid in the defined Volume V is an integral expression

$$m = \int_V \rho(\mathbf{x}, t) dV. \tag{4}$$

The figure bellow 1, expresses the above described picture.

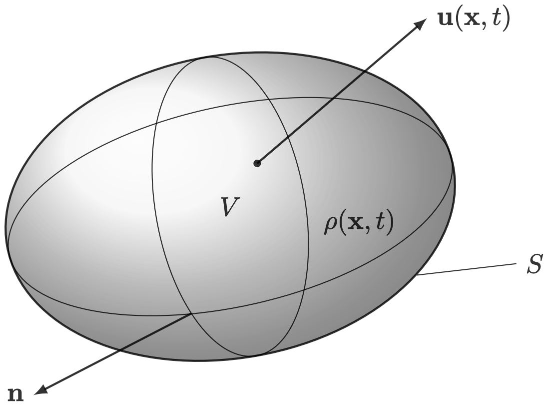

Figure 1: Volume bounded by a surface in a fluid with density and momentum, with a surface normal vector **n**

Since we want to figure out the fluid's dynamics, we can consider the rate of change in the completely arbitrary V. The rate of change of mass needs to disappear, i.e. it is equal to zero since we cannot lose mass. Matter (mass) is neither created nor destroyed anywhere in the fluid, leading us to

$$\frac{d}{dt}\left(\int_V \rho(\mathbf{x}, t)\ dV\right) = 0. \tag{5}$$

To get more information we simply "differentiate under the integral sign“, also known as the Leibniz Rule of Integration, see appendix A.1, the integral equation representing the rate of change of mass reads

$$\frac{dm}{dt} = \int_V \frac{\partial \rho(\mathbf{x}, t)}{\partial t}\ dV + \int_{\partial V} \rho(\mathbf{x}, t)\mathbf{u} \cdot \mathbf{n}\ dS = 0. \tag{6}$$

The above equation in 6 is an underlying equation, describing that the rate of change of mass in V is brought about, only by the rate of mass flowing into V across S, and thus the mass does not change.

For the second integral in 6 we utilize the Gaussian integration law to acquire an integral over the volume

$$\int_{\partial V} \rho(\mathbf{x}, t)\mathbf{u} \cdot \mathbf{n}\ dS = \int_V \nabla(\rho\mathbf{u})\ dV. \tag{7}$$

Thereby we can put everything inside the volume integral

$$\frac{dm}{dt} = \int_V \left(\partial_t \rho + \nabla(\rho\mathbf{u})\right)\ dV = 0. \tag{8}$$

Everything under the integral sign needs to be zero, thus we obtain the **Equation of Mass Conservation** or in the general sense also called the **Continuity Equation**

$$\partial_t \rho + \nabla(\rho \mathbf{u}) = 0 \tag{9}$$

In light of the results of the equation of mass conservation in 9, an product rule gives

$$\partial_t \rho + (\nabla \rho)\mathbf{u} + \rho(\nabla \mathbf{u}), \tag{10}$$

for notational purposes, we define the **material/convective derivative** as follows

$$\frac{D}{Dt} = \partial_t + \mathbf{u}\nabla. \tag{11}$$

With the material derivative the equation of mass conservation reads

$$\frac{D\rho}{Dt} + \rho \nabla \mathbf{u} = 0 \tag{12}$$

We may undertake the first case separation, initiating $\rho =$ cosnt. called **incompressible flow** causes the material derivative of ρ to be zero, and thereby

$$\frac{D\rho}{Dt} = 0 \quad \Rightarrow \quad \nabla \mathbf{u} = 0, \tag{13}$$

following that the divergence of the velocity field is zero, in this case $\mathbf{u}$ is called **solenoidal**.

1.2 Euler's Equation of Motion

Additional consideration we undertake is the assumption of an **inviscid** fluid, that is we set viscosity to zero. Otherwise we would get a viscous contribution under the integral which results in the Navier-Stokes equation. In this regard we apply Newton's second law to our fluid in terms of infinitesimal pieces δV of the fluid. The acceleration divides into two terms, a **body force** given by gravity of earth in the z coordinate $\mathbf{F} = (0, 0, -g)$ and a **local/short-rage force** described by the stress tensor in the fluid. In the inviscid case we the local force retains the pressure P, producing a normal force, with respect to the surface, acting onto any infinitesimal element in the fluid. The integral formulation of the force would be

$$\int_V \rho \mathbf{F} \, dV - \int_S P\mathbf{n} \, dV. \tag{14}$$

Now applying the Gaussian rule of integration on the second integral over the surface, the resulting force in per unit volume is

$$\int_V (\rho \mathbf{F} - \nabla P) \, dV. \tag{15}$$

The acceleration of the fluid particles is given by $\frac{D\mathbf{u}}{Dt}$, and thus the total force per unit volume on the other hand is

$$\int_V \rho \frac{D\mathbf{u}}{Dt} \, dV = \int_V (\rho \mathbf{F} - \nabla P) \, dV. \tag{16}$$

Newton's Second Law for a fluid in an Volume is essentially saying that the rate of change of momentum of the fluid in the fixed volume V, which is the particle acceleration is the resulting force acting on V together with the rate of flow of momentum across the surface S into the volume V. Hence we arrive at the **Euler's Equation(s) of Motion**

$$\frac{D\mathbf{u}}{Dt} = \left(\frac{\partial \mathbf{u}}{\partial t} (\mathbf{u}\nabla)\mathbf{u} \right) = -\frac{1}{\rho}\nabla P + \mathbf{F}. \tag{17}$$

As a side note we have mentioned that there is another contribution if the fluid is viscid. Indeed there is a tangential force due to the velocity gradient, which into introduces the additional term

$$\mu \nabla^2 \mathbf{u}, \qquad \mu = \text{viscosity of the Fluid.} \tag{18}$$

Thereby the equations become

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \rho \mathbf{F} + \mu \nabla^2 \mathbf{u}. \tag{19}$$

For now we have separated two simplifications, that define an **idealized/perfect fluid**

1. **incompressible** $\mu = 0$
2. **inviscid** $\rho = \text{const.},\ \nabla \mathbf{u} = 0$

1.3 Vorticity and irrotational Flow

The curl of the velocity field $\omega = \nabla \times \mathbf{u}$ of a fluid (i.e. the vorticity), describes a spinning motion of the fluid near a position $\mathbf{x}$ at time t. The vorticity is an important property of a fluid, flows or regions of flows where $\omega = 0$ are **irrotational**, and thus can be modeled and analyzed following well known routine methods. Even though real flows are rarely irrotational anywhere (!), in water wave theory wave problems, from the classical aspect of vorticity have a minor contribution. Hence we can assume irrotational flow modeling water waves. To arrive at the vorticity in the equations of motions derived in the last section we resort to a differential identity derived in appendix **??**, which gives for the material derivative

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} \nabla (\frac{1}{2}\mathbf{u}\mathbf{u}) - (\mathbf{u} \times (\nabla \times \mathbf{u})\,. \tag{20}$$

Thus the equations of motion become

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \left(\frac{1}{2}\mathbf{u}\mathbf{u} + \frac{P}{\rho} + \Omega \right) = \mathbf{u} \times \omega, \tag{21}$$

where Ω is the force potential per unite mass given by $\mathbf{F} = -\nabla \Omega$.

At this point we may differentiate between **stead and unsteady flow**. For **Steady Flow** we assume that $\mathbf{u}, P$ and Ω are time independent, thus we get

$$\nabla \left(\frac{1}{2}\mathbf{u}\mathbf{u} + \frac{P}{\rho} + \Omega \right) = \mathbf{u} \times \omega. \tag{22}$$

It is general knowledge that the gradient of a function ∇f is perpendicular the level sets of $f(\mathbf{x})$, where $f(\mathbf{x}) = \text{const.}$. Thus $\mathbf{u} \times \omega$ is orthogonal to the surfaces where

$$\frac{1}{2}\mathbf{u}\mathbf{u} + \frac{P}{\rho} + \Omega = \text{const.}, \tag{23}$$

The above equation is called **Bernoulli's Equation**.

Secondly **Unsteady Flow** but irrotational (+ incompressible), first of all gives us the condition for the existence of a velocity potential ϕ in the sense

$$\omega = \nabla \times \mathbf{u} = 0 \quad \Rightarrow \quad \mathbf{u} = \nabla \phi, \tag{24}$$

where ϕ needs to satisfy the Laplace equation

$$\Delta \phi = 0. \tag{25}$$

According to the Theorem of Schwartz we may exchange $\frac{\partial}{\partial t}$ and ∇, giving us an expression for the material derivative

$$\nabla \left(\frac{\partial \phi}{\partial t} + \frac{1}{2}\mathbf{u}\mathbf{u} + \frac{P}{\rho} + \Omega \right) = 0 \tag{26}$$

Thus the expression differentiated by the ∇ operator is an arbitrary function $f(\mathbf{x}, t)$, writing

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\mathbf{u}\mathbf{u} + \frac{P}{\rho} + \Omega = f(\mathbf{x}, t). \tag{27}$$

The function $f(\mathbf{x}, t)$ can be removed by gauge transformation of $\phi \to \phi + \int f(\mathbf{x}, t)\ dt$, never the less this is not further discussed and left to the reader in the reference.

1.4 Boundary Conditions for water waves

Surface, Bottom, Pressure

A Appendix: Mathematical Preliminaries

A.1 Leibniz Rule of Integration

The Leibniz integral rule for differentiation under the integral sign initiates with an integral

$$\mathcal{I}(t,x) = \int_{a(t)}^{b(t)} f(t,x)dx = \mathcal{I}(t, a(t, a(t), b(t))). \tag{28}$$

And upon differentiation w.r.t. t, utilizes the chain rule on $a(t)$ and $b(t)$ respectively, by

$$\frac{d\mathcal{I}}{dt} = \frac{\partial \mathcal{I}}{\partial t} + \frac{\partial \mathcal{I}}{\partial a}\frac{\partial a}{\partial t} + \frac{\partial \mathcal{I}}{\partial b}\frac{\partial b}{\partial t}. \tag{29}$$

Which in integral representation reads

$$\frac{d\mathcal{I}}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(t,x)}{\partial t}dx + f(t,b(t))\frac{\partial b(t)}{\partial t} - f(t,a(t))\frac{\partial a(t)}{\partial t} \tag{30}$$

A.2 Gaussian Integration Law

This should explain the Gaussian integration law

A.3 Identity for Vorticity

We start off with the standard material derivative

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t}(\mathbf{u}\nabla)\mathbf{u}. \tag{31}$$

We will use Einstein's Summation Convention, where we sum over indices that both appear at as the bottom as the top index, to rewrite the second part of the material derivative $(\mathbf{u}\nabla)\mathbf{u}$ into

$$\begin{aligned}
(\mathbf{u} \times (\nabla \times \mathbf{u}))_k &= \varepsilon^{ijk} u_j (\nabla \times \mathbf{u})_k && (32)\\
&= \varepsilon^{ijk} u_j \varepsilon_{klm} \partial^l u^m && (33)\\
&= (\delta^i_l \delta^j_m - \delta^i_m \delta^j_l) u_j \partial^l u^m && (34)\\
&= u_m \partial^i u^m - u_l \partial^l u^i. && (35)
\end{aligned}$$

Now the first part in equation 35 can be rewritten into

$$u_m \partial^i u^m = \partial^i (\frac{1}{2} u_m u^m). \tag{36}$$

Thus we get

$$(\mathbf{u} \times (\nabla \times \mathbf{u}))_k = \frac{1}{2}\partial^i (u_m u^m) + u_l \partial^l u^i, \tag{37}$$

which is

$$(\mathbf{u}\nabla)\mathbf{u} = \nabla(\frac{1}{2}\mathbf{u}\mathbf{u}) - (\mathbf{u} \times (\nabla \times \mathbf{u})) \tag{38}$$

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