Mathematical Modeling of Water-Wave Problems

Applied PDE Seminar

Popović Milutin

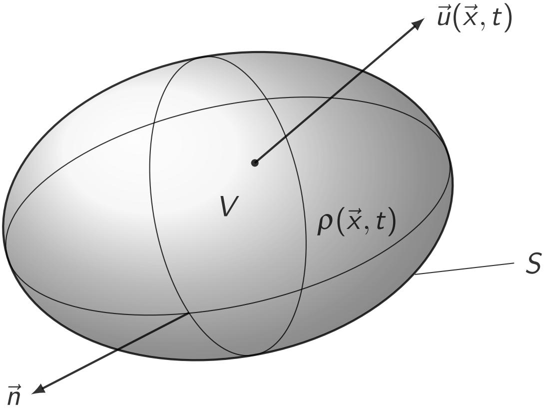
17. March 2021

Fluid Description

The fluid is described by

- Fluid density
  $\rho(\vec{x}, t)$
- Velocity Field
  $\vec{u}(\vec{x}, t) = (u, v, w)$
- Pressure
  $P(\vec{x}, t)$

Figure: Control volume of the fluid



Mass Conservation

- Mass:

$$m(t) = \int_V \rho(\vec{x}, t)\ dV$$

- Rate of change:

$$\int_V \frac{\partial \rho(\vec{x}, t)}{\partial t}\ dV = \frac{dm}{dt} = -\int_S \rho(\vec{x}, t)\vec{u} \cdot \vec{n}\ dS$$

- Use Gauss's law to get the **Equation of Mass conservation**

$$\frac{\partial \rho}{\partial t} \nabla \cdot (\rho \vec{u}) = 0$$

Euler's Equation of Motion

$\rightarrow$ Apply Newton's second law to the Fluid

Body Force

$\vec{F} = (0, 0, -g)$

Local/Short-range Force

Stress tensor

For inviscid fluid: $P(\vec{x}, t)$

$$\Rightarrow \int_V \rho \frac{D\vec{u}}{Dt} \, dV = \int_V \left(\rho \vec{F} - \nabla P \right) \, dV$$

$\rightarrow$ Leads us to **Euler's Equation of Motion**

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho} \nabla P + \vec{F}$$

Vorticity

$$\vec{\omega} = \nabla \times \vec{u}$$

Irrotational Flow

$$\vec{\omega} = 0$$

$\rightarrow$ Vorticity pops up in the acceleration of the fluid particles

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + \nabla\left(\frac{1}{2}\vec{u}\cdot\vec{u}\right) - (\vec{u}\times\vec{\omega})$$

$\rightarrow$ We can incorporate vorticity into Euler's Equation of Motion

$$\frac{\partial \vec{u}}{\partial t} + \nabla\left(\frac{1}{2}\vec{u}\cdot\vec{u} + \frac{P}{\rho} + \Omega\right) = \vec{u}\times\vec{\omega}$$

- **Kinematic Condition**: Fluid particles at the surface
- **Dynamic Condition**: Atmospheric Pressure on the surface
- **Bottom Condition**: Rigid and fixed bottom
- (**Integrated Mass Condition**): Combination

Nondimensionalisation

- h_0 for the typical water depth
- λ for the typical wavelength
- $\frac{\lambda}{\sqrt{gh_0}}$ time scale of wave propagation
- $\sqrt{gh_0}$ velocity scale of waves in (x, y)
- $\frac{h_0\sqrt{gh_0}}{\lambda}$ velocity scale in z

$\rightarrow$ **Shallowness parameter** $\delta = \frac{h_0}{\lambda}$ $\rightarrow$ **Amplitude Parameter** $\varepsilon = \frac{a}{h_0}$

Nondimensionalisation

$\rightarrow$ Nondimensionalisation

$$x \rightarrow \lambda x, \quad u \rightarrow \sqrt{gh_0}\,u,$$
$$y \rightarrow \lambda y, \quad v \rightarrow \sqrt{gh_0}\,v, \qquad t \rightarrow \frac{\lambda}{\sqrt{gh_0}}t,$$
$$z \rightarrow h_0 z, \quad w \rightarrow \frac{h_0\sqrt{gh_0}}{\lambda}w.$$

$\rightarrow$ Top and Bottom conditions

$$h = h_0 + a\eta(\vec{x}_\perp, t), \qquad b \rightarrow h_0 b(\vec{x}_\perp, t)$$

$\rightarrow$ Rewrite Pressure

$$P = P_a + \rho g(h_0 - z) + \rho g h_0 p(\vec{x})$$

Scaling

$\rightarrow$ w, p and the free surface z are $\propto \varepsilon$, leading to the scaling

$$p \rightarrow \varepsilon p, \quad w \rightarrow \varepsilon w, \quad \vec{u}_{\perp} \rightarrow \varepsilon \vec{u}_{\perp}$$

Results

→ Nondimensionalized Euler's Equation of motion

$$\frac{Du}{Dt} = -p_x \quad \frac{Dv}{Dt} = -p_y \quad \delta^2 \frac{Dw}{Dt} = -p_z$$

$$\nabla \cdot \vec{u} = 0$$

→ With boundary conditions

$$\left.\begin{aligned} p &= \eta - \frac{\delta^2 \varepsilon h_0}{\lambda^2} \frac{W_e}{R} \\ w &= \frac{1}{\varepsilon}\eta_t + (\mathrm{u}_\perp \nabla_\perp)\eta \end{aligned}\right\} \quad \text{on } z = 1 + \varepsilon\eta \tag{1}$$

$$w = \frac{1}{\varepsilon} b_t + (\mathrm{u}_\perp \nabla_\perp) b \quad \text{on } z = b \tag{2}$$

History of the Soliton

- a **soliton** is a solitary wave that resists dispersion, maintaining its shape while it propagates at constant velocity
- John Scott Russell discovered the solitary wave in 1834, firstly calling it the **wave of translation**

Korteweg-de Vries equation (KdV)

Korteweg-de Vries equation: nonlinear, dispersive PDE

$$2\eta_t + 3\eta\eta_\xi + \frac{K}{3}\eta_{\xi\xi\xi} = 0 \qquad (\xi = x - ct, \tau = \varepsilon t)$$

With Solution

$$\eta(\xi,\tau) = 2c^2 \mathrm{sech}^2\left(\sqrt{\frac{3}{2K}}(\xi - \tau)\right)$$

KdV Regime

1) The KdV equation arises in the $\varepsilon = O(\delta^2)$
2) by rescaling δ in favor of ε in Euler's Equations of motion
3) going into the frame of the moving wave $(\xi = x - t, \tau = \varepsilon t)$
4) conducting an Asymptotic expansion of u, w, p and η.
5) KdV arises in the $O(\varepsilon)$ term

2004 Tsunami: Description

2004 Tsunami: $\varepsilon = O\left(\delta^2\right)$

2004 Tsunami: KdV Equation

2004 Tsunami: Regime of Validity

Bibliography

[1] R. S. Johnson. *A Modern Introduction to the Mathematical Theory of Water Waves*. Cambridge Texts in Applied Mathematics. Cambridge University Press, 1997. DOI: 10.1017/CBO9780511624056.

[2] Geoffrey K. Vallis. *Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-Scale Circulation*. 2nd ed. Cambridge University Press, 2017. DOI: 10.1017/9781107588417.

[3] Adrian Constantin. "On the propagation of tsunami waves, with emphasis on the tsunami of 2004". In: *Discrete and Continuous Dynamical Systems - B* 12.3 (2009), pp. 525–537.

[4] Rupert Klein. "Scale-Dependent Models for Atmospheric Flows". In: *Annual Review of Fluid Mechanics* 42 (Dec. 2009), pp. 249–274. DOI: 10.1146/annurev-fluid-121108-145537.

[5] Wolf Wahl Hans Kerner. *Mathemaik für Physiker*. 3rd ed. Springer-Verlag Berlin Heidelberg 2013, 2013. DOI: 10.1007/978-3-642-37654-2.

To be continued. . .

Thank You!