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Applied Analysis Problems

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February 24, 2022

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1 Sheet 1

1.1 Fall from high

We consider a free fall ($\dot{x}(t=0)=0$) of an object with mass 20 kg from a height $x(0)=h=$ 20 km, such that the gravitational force depends on the height $x(t)$ in the following way

$$\ddot{x}(t) = -g\frac{R^2}{(x(t)+R)^2}, \tag{1}$$

where R is the radius of the earth $R \approx 6000$ km and $g \approx 9.81\ \frac{m}{s^2}$ is the gravitational acceleration on the surface of the earth. For this problem there are two possible non-dimensionalisations, but first let us rewrite the variables in terms of non-dimensional variables and some dimensional constants, a priori let

$$t = t_c\tau \quad \text{and} \tag{2}$$
$$x = x_c\xi. \tag{3}$$

With the above ansatz we get the following second derivative in time

$$\frac{d^2}{dt^2} = \frac{1}{t_c^2}\frac{d^2}{d\tau^2} \tag{4}$$
$$\Rightarrow \frac{d^2x}{dt^2} = \frac{x_c}{t_c^2}\frac{d^2\xi}{d\tau^2}, \tag{5}$$

and thus the initial conditions can be rewritten as

$$\xi(0) = \frac{h}{x_c}, \tag{6}$$
$$\dot{\xi} = 0. \tag{7}$$

Now we can rewrite the equation of the free fall in 1 in terms of $\xi(\tau)$ as

$$\frac{x_c}{gt_c^2}\ddot{\xi} = -\frac{1}{(\frac{x_c}{R}\xi+1)^2}. \tag{8}$$

Thereby we have three dimensional constants constants Π_1, Π_2, Π_3, as follows

$$\Pi_1 = \frac{x_c}{R}, \quad \Pi_2 = \frac{h}{x_c}, \quad \Pi_3 = \frac{x_c}{gt_c^2}. \tag{9}$$

The first scaling is done by reducing Π_1 and Π_3 to 1, by setting

$$x_c = R, \quad t_c = \sqrt{\frac{R}{g}}, \tag{10}$$

reformulating the initial problem in equation 1 to

$$\ddot{\xi} = -\frac{1}{(\xi+1)^2}, \quad \text{with}$$
$$\xi(0) = \frac{h}{R}, \quad \dot{\xi}(0) = 0. \tag{11}$$

Reducing the problem, meaning if $\frac{h}{R} \to 0$ makes the first initial condition $\xi(0) \to 0$. We can conclude that this scaling is bad since it changes the initial condition in the reduced problem.

The second scaling option reduces Π_2 and Π_3 to 1, by setting

$$x_c = h, \quad t_c = \sqrt{\frac{h}{g}}, \tag{12}$$

reformulating the initial problem in equation 1 to

$$\ddot{\xi} = -\frac{1}{(\frac{h}{R}\xi+1)^2}, \quad \text{with}$$
$$\xi(0) = 1, \quad \dot{\xi}(0) = 0. \tag{13}$$

By letting $R \to \infty$ we get the following reduced problem

$$\ddot{\xi} = -1. \tag{14}$$

Integrating and solving for $\xi(\tau = T\sqrt{\frac{g}{h}}) = 0$ for when the object hits the ground we get a familiar solution

$$T = \sqrt{\frac{2h}{g}} \tag{15}$$

Note that in the reduced problem the time until the object hits the ground is **(much) shorter** since the acceleration is at its maximum $\ddot{x}(t) = g$ for all t. Yet in the original problem the acceleration (gravitation force) **increases** as the object comes **closer** to earth . For instance, if we let an object fall down from the height $h = R$ then its gravitational force (acceleration) at that height would be $\ddot{x}(0) = g/2$ and upon landing on earth the gravitational force $\ddot{x}(T) = g$, while in the reduced solution its gravitational force would be $\ddot{x}(t) = g$ for all t.

Additionally we can calculate the velocity at impact we need to integrate the reduced problem 14 once and put in the initial condition

$$\dot{\xi}(\tau = \frac{T}{t_c}) = -\tau = -\sqrt{2} \tag{16}$$
$$\text{and} \quad \dot{x} = \frac{x_c}{t_c}\dot{\xi} = \sqrt{gh}\,\dot{\xi} \tag{17}$$
$$\Rightarrow \dot{x}(T) = -\sqrt{2gh}, \tag{18}$$

The result is exactly the same as we would get from energy conservation

$$\frac{m}{2}\dot{x}^2 = mgh \quad \Rightarrow \quad \dot{x} = \sqrt{2gh}. \tag{19}$$

The vertical throw allows for an additional scaling because the initial conditions are different, $x(0) = 0$ and $\dot{x}(0) = v$. Thus the solution too.

To summarize, the assumptions that used for modeling and simplifying the equation are

- no relativistic influence,
- closed system, no outside influence (gravitation of the sun, air resistance),
- spherical symmetry of the earth (thereby center of mass can be set in the middle of earth).

By looking at our assumptions a question arises:**Is it a good approximation to replace the attractive force of the earth by the attraction of the whole mass concentrated at the center?**.

To answer this question more or less simply we look at the Poisson's equation for gravity,

$$\ddot{\vec{x}}(\vec{r}) = -\nabla\phi(\vec{r}) \tag{20}$$
$$\Delta\phi = 4\pi G\varrho(\vec{r}). \tag{21}$$

for a gravitational potential ϕ and the mass density of earth ϱ. We assume that **the earth can be approximated by a sphere** and then we integrate both sides along the sphere (and use the Gaussian law for integration)

$$\int_S \nabla\ddot{\vec{x}}\ dS = \int_{\partial S} \ddot{\vec{x}}\ d\vec{s} = -4\pi G \int_S \varrho(\vec{r})\ ds = -4\pi GM. \tag{22}$$

Obviously $\ddot{\vec{x}}$ and $d\vec{s}$ point in the same direction. We choose (rotate) the coordinate system such hat $\ddot{\vec{x}} = \ddot{x}\ \hat{\mathbf{n}}$ and $d\vec{s} = \hat{\mathbf{n}}\ ds$, thereby we get

$$\ddot{x}\int_{\partial S} ds = 4\pi r^2\ddot{x}, \tag{23}$$
$$\Rightarrow \ddot{x} = -\frac{GM}{r^2}. \tag{24}$$

The further derivation to get the exact equation of motion as in 1, we have to keep in mind that $r = x + R$, because by our assumptions we are not in the sphere only outside or on the border R. Lastly by reformulating the constants $gR^2 = GM$ gets us to our equation of motion

$$\ddot{x}(t) = -g\frac{R^2}{(x(t)+R)^2}. \tag{25}$$

1.2 Scaling The Van der Pol equation

The Van der Pol equation is a perturbation of the oscillation equation

$$LC\frac{d^2I}{dt^2} + (-g_1C + 3g_3CI^2)\frac{dI}{dt} = -I \tag{26}$$

with initial conditions

$$I(0) = I_0, \quad \dot{I}(0) = 0. \tag{27}$$

where $I(t)$ is the current at a time t, C is the capacity, L is the inductivity and g_1, g_3 are some parameters. The units of all the parameters are

$$[LC] = s^2 \tag{28}$$
$$[g_1C] = s \tag{29}$$
$$[g_3C] = sA^{-2} \tag{30}$$

The oscillation equation is

$$CL\ddot{I} + I = 0. \tag{31}$$

Solvable by the exponential ansatz of $I = Ae^{\lambda t}$, where $\lambda = \pm i\sqrt{\frac{1}{LC}}$, thereby

$$I(t) = A_1e^{i\sqrt{\frac{1}{LC}}t} + A_2e^{-i\sqrt{\frac{1}{LC}}t}. \tag{32}$$

With the initial conditions in equation 27 we get $A_1 = A_2$ and thus the solution to the oscillation equation is

$$I(t) = I_0 \cos(\frac{t}{\sqrt{LC}}) \tag{33}$$

Now that we know the reduced problem and the solution to it, we may work with the Van-Der-Pol equation 26, by determining all possible non-dimensionalisations. Let us begin by setting

$$I(t) = I_c \psi, \tag{34}$$
$$t = t_c \tau, \tag{35}$$

where I_c and t_c have the dimension of $I(t)$ and t accordingly and $\psi(\tau)$ and τ are dimensionless The **first** and second derivative in time is

$$\frac{d}{dt} = \frac{1}{t_c}\frac{d}{d\tau} \tag{36}$$
$$\frac{d^2}{dt^2} = \frac{1}{t_c^2}\frac{d^2}{d\tau^2}. \tag{37}$$

We can rewrite the Van-Der-Pol equation in terms of ψ and τ

$$\frac{LC}{t_c^2}\ddot{\psi} - \left(\frac{3g_3 I_c^2}{g_1}\psi^2 - 1\right)\frac{g_1 C}{t_c}\dot{\psi} = -\psi \tag{38}$$
$$\psi(0) = \frac{I_0}{I_c} \quad \dot{\psi}(0) = 0 \tag{39}$$

There are a total of four constants that we can eliminate

$$\Pi_1 = \frac{I_0}{I_c}, \qquad \Pi_2 = \frac{LC}{t_c^2},$$
$$\Pi_3 = \frac{3g_3 I_c^2}{g_1}, \qquad \Pi_4 = \frac{g_1 C}{t_c}. \tag{40}$$

The **first** scaling is

$$I_c = \sqrt{\frac{g_1}{3g_3}}, \quad t_c = \sqrt{LC}. \tag{41}$$

Thereby we get the following problem

$$\ddot{\psi} + (\psi^2 - 1)\varepsilon\dot{\psi} = -\psi, \qquad \psi(0) = \sqrt{\frac{3g_3}{g_1}} I_0, \tag{42}$$

where $\varepsilon = g_1\sqrt{\frac{C}{L}}$.

The **second** scaling is

$$I_c = \sqrt{\frac{g_1}{3g_3}}, \quad t_c = g_1 C. \tag{43}$$

Thereby we get the following

$$\varepsilon\psi'' + (\psi^2 + 1)\psi' = -\psi, \qquad \psi(0) = \sqrt{\frac{3g_3}{g_1}} I_0, \tag{44}$$

where $\varepsilon = \frac{L}{g_1^2 C}$. We could also consider scaling $I_c = I_0$ with $t_c = \sqrt{LC}$ or $t_c = g_1 C$ but they wouldn't develop significant model hierarchies like the above two scaling.

1.3 Scale the Schrödinger Equation

The well known Schrödinger equation that describes quantum physics of the one particle system is

$$\begin{aligned} i\hbar\partial_t\psi &= -\frac{\hbar^2}{2m}\Delta\psi + V\psi \\ \psi(t=0) &= \psi_0 \end{aligned} \tag{45}$$

where $\hbar$ is the reduced Plank constant, $\psi = \psi(x,t)$ the wave function, m the mass and $V = V(x)$ the potential in which the wave function is. The dimensions are

$$[\hbar] = js, \quad [V] = j, \quad [\psi] = m^{-d/2} \tag{46}$$

where d is the spacial dimension. The standard scaling ansatz is

$$\psi = \psi_c\phi \tag{47}$$

$$t = t_c\tau \quad x = x_c\xi, \tag{48}$$

by that we get the following derivatives in time and in space

$$\partial_{x_i} = \frac{1}{x_{(i)c}}\partial_{\psi_i} \tag{49}$$

$$\partial^2_{x_i} = \frac{1}{x^2_{(i)c}}\partial^2_{\psi_i} \tag{50}$$

$$\partial_t = \frac{1}{t_c}\partial_{\psi_i} \tag{51}$$

$$\tag{52}$$

for $i = 1,2,3$, or depending on the dimension we are dealing with.

Let us consider $x \in \mathbb{R}^3$ and $V = 0$ to scale the equation. First we now have

$$i\partial_t\psi = -\frac{\hbar}{2m}\Delta\psi \tag{53}$$

with the initial condition $\phi(0) = \frac{\psi_0}{\psi_c}$. With our scaling the equation turns out to be

$$\frac{i\hbar t_c}{2m}\frac{1}{||\vec{x}_c||^2}\Delta_{\vec{\xi}}\ \phi = \partial_\tau\ \phi. \tag{54}$$

The constants we get are

$$\Pi_1 = \frac{t_c\hbar}{2m}\frac{1}{||\vec{x}_c||^2}, \quad \Pi_2 = \frac{\psi_0}{\psi_c}. \tag{55}$$

The simple choice of

$$\frac{1}{||\vec{x}_c||^2} = 1, \quad \psi_c = \psi_0, \quad t_c = \frac{2m}{\hbar}||\vec{x}_c||^2, \tag{56}$$

simplifies the Schrodinger equation without the potential to

$$i\Delta_{\vec{\xi}}\ \phi = \partial_\tau\phi, \tag{57}$$

with the initial condition $\phi(\tau = 0) = 1.$.

Now consider $V = 0$, $x \in [0,L]$ and $t \in [0,T]$, the Schrodinger equation is the same only with one spacial dimension as above, we can set

$$\psi_c = \psi_0, \quad x_c = L, \quad t_c = \frac{2mL^2}{\hbar}. \tag{58}$$

Thus we get

$$i\partial_\xi^2\phi = \partial_\tau\phi, \tag{59}$$

with the initial condition $\phi(\tau = 0) = 1$, where $\xi \in [0,1]$ and $\tau \in [0, \frac{\hbar T}{2mL^2}]$. .

In the last example let us consider the quantum harmonic oscillator represented by the potential $V(x) = m\omega^2x^2$ for $x \in \mathbb{R}$, where ω is the frequency. The equation is the following

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\partial_x^2\psi + m\omega^2x^2\psi. \tag{60}$$

By inserting the standard scaling ansatz we get

$$i\partial_\tau\phi = -\frac{t_c\hbar}{2mx_c^2}\partial_\xi^2\phi + \frac{t_cm\omega^2x_c^2}{\hbar}\xi^2\phi, \tag{61}$$

The dimensional constants are

$$\Pi_1 = \frac{t_0\hbar}{mx_c^2}, \quad \Pi_2 = \frac{m\omega^2x_c^2t_c}{\hbar}, \quad \Pi_3 = \frac{\psi_0}{\psi_c}. \tag{62}$$

The choice of scaling is

$$\psi_c = \psi_0, \quad t_c = \frac{1}{\omega}, \quad x_c = \sqrt{\frac{\hbar}{m\omega}}. \tag{63}$$

Thereby getting the following problem

$$i\partial_\tau\phi = -\frac{1}{2}\partial_\xi^2\phi + \xi^2\phi \tag{64}$$

with $\phi(\tau = 0) = 1$.