PS Dynamical Systems and Nonlinear DEs (2023S)
(Exercises for 27 June 2023)

30. Let (X, d) be a metric space. A map $T : X \to X$ is said to be *topologically mixing* if for any two nonempty open sets $U, V \subseteq X$ there is some $N \geq 1$ such that $U \cap T^{-n}V \neq \varnothing$ for all $n \geq N$. Assuming that X contains at least two points, show that no isometry on X has this property.

31. Let (X, d) be a metric space. A map $T : X \to X$ is said to have *sensitive dependence (on initial conditions)* if there is some $\delta > 0$ (a *sensitivity constant*) such that for every $x \in X$ and $\varepsilon > 0$ there is some $y \in X$ with $\mathrm{d}(x, y) < \varepsilon$ such that $\mathrm{d}(T^n x, T^n y) \geq \delta$ for some $n \geq 1$. Assuming that X contains at least two points, show that every topologically mixing map $T : X \to X$ has sensitive dependence.

32. Fix some integer $m \geq 2$ and consider the sequence space $\Omega_m := \{0, \ldots, m-1\}^{\mathbb{N}} = \{\omega = (\omega_j)_{j\geq 0} : \omega_j \in \{0, \ldots, m-1\}\}$. For $\omega, \widetilde{\omega} \in \Omega_m$ define the *separation time* $s(\omega, \widetilde{\omega}) := \inf\{j \geq 0 : \omega_j \neq \widetilde{\omega}_j\}$ (with $\inf \varnothing := \infty$). Take some constant $\lambda > 1$ and set
$$\mathrm{d}(\omega, \widetilde{\omega}) := \lambda^{-s(\omega,\widetilde{\omega})} \qquad \text{for } \omega, \widetilde{\omega} \in \Omega_m.$$
a) Show that d is a metric on Ω_m.
b) Show that each *cylinder set* $[\omega_0, \ldots, \omega_{r-1}] := \{\widetilde{\omega} \in \Omega_m : \omega_j = \widetilde{\omega}_j \text{ for } j < r\}$ in Ω_m is both open and closed.
c) Show that (Ω_m, d) is a compact space. (Hint: Either check that d induces the product topology, or show directly, by a diagonalization argument, that the space is sequentially compact.)

33. Consider the metric space (Ω_m, d) of the previous exercise.

    a) Show that the shift $\sigma : \Omega_m \to \Omega_m$ given by $\sigma((\omega_j)_{j\geq 0}) := (\omega_{j+1})_{j\geq 0}$ is continuous.
    b) Is the shift topologically mixing?
    c) Define $\eta : \Omega_m \to [0, 1]$ by letting $\eta(\omega) := \sum_{j\geq 0} \omega_j / m^{j+1}$. Prove that η is continuous.

34. A *subshift of finite type* (or *topological Markov chain*) is the dynamical system given by the restriction of σ to a subset $\Omega' \subseteq \Omega_m$ of the following type: Start with an $m \times m$ matrix $(\theta_{i,j})_{i,j\in\{0,\ldots,m-1\}}$ with all entries $\theta_{i,j} \in \{0, 1\}$. Let $\Omega' := \{\omega = (\omega_j)_{j\geq 0} \in \Omega_m : \theta_{\omega_j,\omega_{j+1}} = 1 \text{ for all } j \geq 0\}$ be the set of sequences which only contain pairs (ω_j, ω_{j+1}) of consecutive digits that are allowed by $(\theta_{i,j})$. Show that Ω' is a closed subset of Ω_m.