University of Vienna

Seminar:
Applied PDE Seminar

Mathematical Modeling of Some Water-Waves

Milutin Popovic

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Contents

1 Governing Equations of Fluid Dynamics

We first of start with a fluid with density/mass

$$\rho(\mathbf{x}, t), \tag{1}$$

with a position $\mathbf{x} = (x, y, z)$ in three dimensional space at time t. For water-wave applications, we should note that we take $\rho =$ constant. The fluid moves in time and space with a velocity field

$$\mathbf{u}(\mathbf{x}, t) = (u, v, w). \tag{2}$$

Additionally it is also described by the pressure of the fluid

$$P(\mathbf{x}, t), \tag{3}$$

generally depending on time and position. When thinking of e.g. water the pressure increases the deeper we go, that is with decreasing or increasing z coordinate (depending how we set up our system z pointing up or down respectively).

The general assumption in fluid dynamics is the **Continuum Hypothesis**, which assumes continuity of $\mathbf{u}, \rho$ and P in $\mathbf{x}$ and t. In other words, we premise that the velocity field, density and pressure are "nice enough" functions of position and time, such that we can do all the differential operations we desire in the framework of fluid dynamics.

1.1 Mass Conservation

Our aim is to derive a model of the fluid and its dynamics, with respect to time and position, in the most general way. This is generally done thinking of the density of a given fluid, which is a unit of mass per unit volume, intrinsically an integral representation to derive these equations is going to be used.

Let us now thing of an arbitrary fluid. Within this fluid we define a fixed volume V relative to a chosen inertial frame and bounded by a surface S within the fluid, such that the fluid motion $\mathbf{u}(\mathbf{x}, t)$ may cross the surface S. The fluid density is given by $\rho(\mathbf{x}, t)$, thereby the mass of the fluid in the defined Volume V is an integral expression

$$m = \int_V \rho(\mathbf{x}, t) dV. \tag{4}$$

The figure bellow 1, expresses the above described picture.

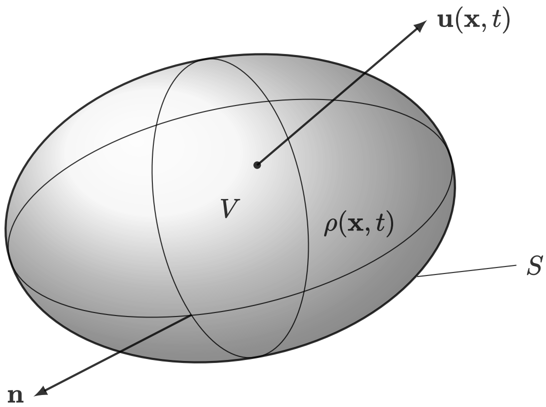

Figure 1: Volume bounded by a surface in a fluid with density and momentum, with a surface normal vector **n**

Since we want to figure out the fluid dynamics, we can consider now the rate of change of mass in of this completely arbitrary V, which needs to be disappear, i.e. is equal to zero since we cannot lose mass. Matter (mass) is neither created nor destroyed anywhere in the fluid

$$\frac{d}{dt}\left(\int_V \rho(\mathbf{x}, t)\ dV\right) = 0. \tag{5}$$

We may get more information with simply "differentiating under the integral sign", also known as the Leibniz Rule of Integration, see appendix A.1, the above integral equation reads

$$\frac{dm}{dt} = \int_V \frac{\partial \rho(\mathbf{x}, t)}{\partial t}\ dV + \int_{\partial V} \rho(\mathbf{x}, t)\mathbf{u} \cdot \mathbf{n}\ dS = 0. \tag{6}$$

The above equation 6 is an underlying equation, describing that the rate of change of mass in V is brought about, only by the rate of mass flowing into V across S, and thus the mass does not change.

For the second integral in 6 we utilize the Gaussian integration law to acquire an integral over the volume

$$\int_{\partial V} \rho(\mathbf{x}, t)\mathbf{u} \cdot \mathbf{n}\ dS = \int_V \nabla(\rho\mathbf{u})\ dV. \tag{7}$$

Thereby we can put everything inside the volume integral

$$\frac{dm}{dt} = \int_V \left(\partial_t \rho + \nabla(\rho\mathbf{u})\right)\ dV = 0. \tag{8}$$

Everything under the integral sign needs to be zero, thus we obtain the **Equation of Mass Conservation** or in the general sense also called the **Continuity Equation**

$$\partial_t \rho + \nabla(\rho\mathbf{u}) = 0 \tag{9}$$

In the light of the results of the equation of mass conservation 9, an expansion, of the nabla gives

$$\partial_t \rho + (\nabla \rho)\mathbf{u} + \rho(\nabla \mathbf{u}), \tag{10}$$

for notational purposes, we define the **material/convective derivative** as follows

$$\frac{D}{Dt} = \partial_t + \mathbf{u}\nabla. \tag{11}$$

Thus the equation of mass conservation becomes

$$\frac{D\rho}{Dt} + \rho\nabla\mathbf{u} = 0 \tag{12}$$

We may undertake the first case separation, initiating $\rho =$ cosnt. called **incompressible flow** causes the material derivative of ρ to be zero, and thereby

$$\frac{D\rho}{Dt} = 0 \quad \Rightarrow \quad \nabla\mathbf{u} = 0, \tag{13}$$

following that the divergence of the velocity field is zero, in this case $\mathbf{u}$ is called **solenoidal**.

1.2 Euler's Equation of Motion

Additional consideration we undertake is the assumption of an **inviscid** fluid, that is we set viscosity to zero. Otherwise we would get a viscous contribution under the integral which results in the Navier-Stokes equation. In this regard we apply Newton's second law to our fluid in terms of infinitesimal pieces δV of the fluid. The acceleration divides into two terms, a **body force** given by gravity of earth in the z coordinate $\mathbf{F} = (0, 0, -g)$ and a **local/short-rage force** described by the stress tensor in the fluid. In the inviscid case we the local force retains the pressure P, producing a normal force, with respect to the surface, acting onto any infinitesimal element in the fluid. The integral formulation of the force would be

$$\int_V \rho\mathbf{F}\ dV - \int_S P\mathbf{n}\ dV. \tag{14}$$

Now applying the Gaussian rule of integration on the second integral over the surface, the resluting force in per unit volume is

$$\int_V (\rho\mathbf{F} - \nabla P)\ dV. \tag{15}$$

The acceleration of the fluid particles is given by $\frac{D\mathbf{u}}{Dt}$, and thus the total force per unit volume on the other hand is

$$\int_V \rho\frac{D\mathbf{u}}{Dt}\ dV = \int_V (\rho\mathbf{F} - \nabla P)\ dV \tag{16}$$

A Appendix: Mathematical Preliminaries

A.1 Leibniz Rule of Integration

The Leibniz integral rule for differentiation under the integral sign initiates with an integral

$$\mathcal{I}(t,x) = \int_{a(t)}^{b(t)} f(t,x)dx = \mathcal{I}(t, a(t, a(t), b(t))). \tag{17}$$

And upon differentiation w.r.t. t, utilizes the chain rule on $a(t)$ and $b(t)$ respectively, by

$$\frac{d\mathcal{I}}{dt} = \frac{\partial \mathcal{I}}{\partial t} + \frac{\partial \mathcal{I}}{\partial a}\frac{\partial a}{\partial t} + \frac{\partial \mathcal{I}}{\partial b}\frac{\partial b}{\partial t}. \tag{18}$$

Which in integral representation reads

$$\frac{d\mathcal{I}}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(t,x)}{\partial t}dx + f(t, b(t))\frac{\partial b(t)}{\partial t} - f(t, a(t))\frac{\partial a(t)}{\partial t} \tag{19}$$

A.2 Gaussian Integration Law

This should explain the Gaussian integration law

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