University of Vienna
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Numerical Analysis Problems

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1.1 Problem 1

For a matrix $A \in \mathbb{C}^{n\times n}$, define $B, C \in \mathbb{C}^{n\times n}$ in the following way

$$B = \frac{1}{2}(A + A^*), \quad C = \frac{1}{2i}(A - A^*). \tag{1}$$

The matrices B and C are Hermitian, which can be seen by directly calculating the adjoint.

$$\begin{aligned} B^* &= \frac{1}{2}(A + A^*)^* = \frac{1}{2}\left(A^* + (A^*)^*\right) &(2)\\ &= \frac{1}{2}(A^* + A) = \frac{1}{2}(A + A^*) = B, &(3) \end{aligned}$$

$$\begin{aligned} C^* &= -\frac{1}{2i}(A - A^*)^* = -\frac{1}{2i}\left(A^* - (A^*)^*\right) &(4)\\ &= -\frac{1}{2i}(A^* - A) = \frac{1}{2}(A - A^*) = C. &(5) \end{aligned}$$

Additionally we can bound the eigenvalues of A by the minimum and maximum eigenvalues of B and C by rewriting A as

$$A = (B + iC). \tag{6}$$

Now consider an arbitrary eigenpair of A, (λ, v), such that $\|v\| = 1$, the eigenvalue equation reads

$$\begin{aligned} Av &= (B + iC)v = \lambda v &(7)\\ &= Bv + iCv &(8)\\ \Leftrightarrow & v^*Bv + v^*(iC)v = \lambda. &(9) \end{aligned}$$

The real and the imaginary part of λ can be calculated by a simple identity

$$\begin{aligned}
\mathrm{Re}(\lambda) &= \frac{1}{2}(\lambda + \bar{\lambda}) && (10)\\
&= \frac{1}{2}(v^*Bv + v^*(iC)v + v^*B^*v - v^*(iC)v) && (11)\\
&= \frac{1}{2}(v^*Bv + v^*B^*v + v^*(iC)v - v^*(iC)v) && (12)\\
&= \frac{1}{2}(2v^*Bv) = v^*Bv && (13)
\end{aligned}$$

$$\begin{aligned}
\mathrm{Im}(\lambda) &= \frac{1}{2i}(\lambda - \bar{\lambda}) && (14)\\
&= v^*Cv && (15)
\end{aligned}$$

Putting the results from above with the Reighley-Ritz Theorem, which states that for all $D \in \mathbb{C}^{n\times n}$ Hermitian $\forall x \in \mathbb{C}^n$, where $x \neq 0$ we have a boundary from below and above by the minimum and maximum eigenvalue of D

$$\lambda_{\min}(D) \leq \frac{x^*Dx}{\|x\|^2} \leq \lambda_{\max}(D) \tag{16}$$

Then we have

$$\Rightarrow \begin{cases} \mathrm{Re}(\lambda) \in [\lambda_{\min}(B), \lambda_{\max}(B)] \\ \mathrm{Im}(\lambda) \in [\lambda_{\min}(C), \lambda_{\max}(C)] \end{cases} \tag{17}$$

1.2 Problem 2

Given two Hermitian matrices $A, B \in \mathbb{C}^{n\times n}$, denote $\{\lambda_j(A)\}_{j=1}^n$ and $\{\lambda_j(A+B)\}_{j=1}^n$ the eigenvalues of A and $A+B$ in increasing order. If B is positive semi-definite then we have a bound

$$\lambda_k(A) \leq \lambda_k(A+B) \qquad \forall k \in \{1, \ldots, n\}. \tag{18}$$

By the Courant Fischer Theorem, let $\mathcal{V}_k$ be the set of all k dimensional subsets of $\mathbb{C}^{n\times n}$ we have

$$\lambda_k(A) = \min_{v\in\mathcal{V}_k} \max_{v\in\mathbb{C}^{n\times n},\ \|v\|=1} \langle v, Av\rangle. \tag{19}$$

And if B is positive semi-definite we have

$$x^*Bx \geq 0 \qquad \forall x \in \mathbb{C}^n. \tag{20}$$

Since A and B are hermitian, then $A+B$ are hermitian too and we can write

$$\begin{aligned}
\lambda_k(A) &= \min_{v\in\mathcal{V}_k} \max_{v\in\mathbb{C}^{n\times n},\ \|v\|=1} \langle v, Av\rangle && (21)\\
&\geq \min_{v\in\mathcal{V}_k} \max_{v\in\mathbb{C}^{n\times n},\ \|v\|=1} \langle v, Av\rangle = \lambda_k(A) && (22)
\end{aligned}$$

1.3 Problem 3

Let $A \in \mathbb{C}^{n\times n}$ be diagonalizable by $X = (x_1, \ldots, x_n) \in \mathbb{C}^{n\times n}$ the matrix of right-eigenvectors $x_j \in \mathbb{C}^n$ of A. For all $\varepsilon > 0$, let ν be the eigenvalues of $A + \varepsilon A$, then there exists and eigenvalue λ of A with

$$\frac{|\lambda - \nu|}{|\lambda|} \leq K_p(X)\varepsilon \tag{23}$$

Let us rewrite

$$A + \varepsilon A = (1+\varepsilon)A, \tag{24}$$

then the eigenvalue $\nu \in \lambda(A + \varepsilon A)$ can be written as an eigenvalue of A with

$$\frac{\nu}{1+\varepsilon} \in \lambda(A). \tag{25}$$

Then the bound reads

$$\begin{aligned}
\frac{|\lambda - \nu|}{|\lambda|} &= \frac{|\frac{\nu}{1+\varepsilon} - \nu|}{|\frac{\nu}{1+\varepsilon}|} && (26)\\
&= \frac{|\nu - (1+\varepsilon)\nu|}{|\nu|} && (27)\\
&= \varepsilon \leq \varepsilon K_p(X), && (28)
\end{aligned}$$

since $K_p(X) \geq 1$ for all X that diagonalize A, if A is invertible !.

1.4 Exercise 4

Given some $\mu \in \mathbb{R}$ the shifted QR-algorithm is defined as: Let Q_0 be orthogonal, such that $T_0 = Q_0^T A Q_0$ is upper Hessenberg form. For $k \in \mathbb{N}$ determine a sequence of the matrices T_k by

- Determine Q_k and R_k, s.t. $Q_k R_k = T_{k-1} - \mu I$, as a QR-decomposition of $T_{k-1} - \mu I$
- Let $T_k = R_k Q_k + \mu I$

The sequence of these matrices T_k is infact similar to A, in the following way

$$\begin{aligned}
T_{k+1} &= R_k Q_k + \mu I && (29)\\
&= Q_k^T (T_k - \mu I) Q_k + \mu I && (30)\\
&= Q_k^T T_k Q_k - \mu I + \mu I && (31)\\
&= Q_k^T T_k Q_k && (32)\\
&= Q_k^T \cdots Q_1^T T_0 Q_1 \cdots Q_k && (33)\\
&= \underbrace{Q_k^T \cdots Q_0^T}_{=Q^T} A \underbrace{Q_0 \cdots Q_k}_{=Q} && (34)
\end{aligned}$$

Furthermore if A is an unreduced Hessenberg matrix and μ an eigenvalue of A. Then let $QR = A - \mu I$ be the QR-decomposition of $A - \mu I$, define

$$\overline{A} = RQ + \mu I, \tag{35}$$

then

$$\overline{A}_{n,n} = \mu \quad \& \quad \overline{A}_{n-1,n} = 0 \tag{36}$$

To start, if A is an irreducible Hessenber then

$$A_{i+1,i} \neq 0 \qquad \forall i \in \{1, \ldots, n-1\}. \tag{37}$$

Then $A - \mu I$ is singular since μ is Eigenvalue of A, $\det(A - \mu I) = 0$ is an eigenvalue equation. And additionally 0 is an eigenvalue of $A - \mu I$, then

$$\Rightarrow \overline{A} = RQ + \mu I. \tag{38}$$

Where $A - \mu I$ is singular and the first $n-1$ columns are linearly independent, since $R = Q^T(A - \mu I)$. Then the first $n-1$ columns of R are linearly independent and because R is also singular perserved by rotation of Q^T the last row needs to be 0, i.e. $R_{n,\cdot} = 0^T$, then

$$\begin{aligned}
R_{n,n-1} &= 0, \qquad (RQ)_{n,n-1} = 0, && (39)\\
R_{n,n} &= 0, \qquad (RQ)_{n,n} = 0. && (40)\\
& && (41)
\end{aligned}$$

By this we conlude

$$\begin{aligned}
\overline{A}_{n,n} &= (RQ)_{n,n} + \mu = \mu && (42)\\
\overline{A}_{n,n-1} &= (RQ)_{n,n-1} = 0 && (43)
\end{aligned}$$