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Applied Analysis Problems

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1 Sheet 4

1.1 Fourier Series

The Fourier series of a p periodic function f , integrable on $[-\frac{p}{2}, \frac{p}{2}]$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi nx}{p}\right) + b_n \sin\left(\frac{2\pi nx}{p}\right) \right). \quad (1)$$

The coefficients a_n and b_n are called the Fourier coefficients of f and are given by

$$a_n = \frac{2}{p} \int_{-\frac{p}{2}}^{\frac{p}{2}} f(x) \cos\left(\frac{2\pi nx}{p}\right) dx, \quad n \geq 0 \quad (2)$$

$$b_n = \frac{2}{p} \int_{-\frac{p}{2}}^{\frac{p}{2}} f(x) \sin\left(\frac{2\pi nx}{p}\right) dx, \quad n \geq 1 \quad (3)$$

Let us compute the Fourier series of $f(t) = t$ for $t \in [-\frac{1}{2}, \frac{1}{2}]$. The Fourier coefficients are

$$a_n = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) \cos(2\pi nt) dt = 0 \quad (\text{odd: } g(-t) = -g(t)), \quad (4)$$

$$b_n = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) \sin(2\pi nt) dt = \quad (5)$$

$$= 2 \left(-\frac{1}{2\pi n} \cos(2\pi nt) \right) \Big|_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2\pi n} \cos(2\pi nt) dt = \quad (6)$$

$$= -\frac{1}{\pi n} \left(-\cos(\pi n) + \frac{1}{\pi n} \sin(\pi n) \right) = \frac{\sin(\pi n) - \pi n \cos(\pi n)}{(\pi n)^2}. \quad (7)$$

Thereby the Fourier series of $f(t) = t$ is

$$f(t) = \sum_{n=1}^{\infty} \left(\frac{\sin(\pi n) - \pi n \cos(\pi n)}{(\pi n)^2} \right) \sin(2\pi n t) = t \quad (8)$$

1.2 Truncation Error

The truncation error of the trigonometric polynomial (Sf_N) of degree N is

$$\sum_{|k|>N} |\hat{f}(k)|^2 = \|f - S_N\|_2^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |E_N(t)|^2 dt. \quad (9)$$

Computations for $N = 3, 9$ were done in python with a integration error of around 10^{-15} , resulting in the overall truncation errors of

$$\sum_{|k|>3} |\hat{f}(k)|^2 = 0.0053, \quad (10)$$

$$\sum_{|k|>9} |\hat{f}(k)|^2 = 0.0143. \quad (11)$$

To achieve $\|E_N\|_2^2 < 0.1\|f\|_2^2$, the number of coefficients needed are about 61. This was done using a while loop and evaluating $\|E_N\|_2^2$ for N until the above condition is met.

1.3 Orthonormal Bases

Here we will go through the most important properties of orthonormal bases. So let $\{b_n\}_{n \in \mathbb{N}}$ be an ONB of a vector space \mathcal{H} , then for every $x \in \mathcal{H}$ we may write

$$x = \sum_{b_n} \langle b_n, x \rangle b_n, \quad (12)$$

and

$$\|x\|^2 = \sum_{b_n} |\langle b_n, x \rangle|^2. \quad (13)$$

For any $x, y \in \mathcal{H}$ we can write the scalar product as

$$\langle x, y \rangle = \sum_{b_n} \langle b_n, x \rangle \langle b_n, y \rangle, \quad (14)$$

Furthermore there exists a linear projection $\Phi : \mathcal{H} \rightarrow l^2(\{b_n\}_n)$ such that

$$\langle \Phi(x), \Phi(y) \rangle = \langle x, y \rangle \quad \forall x, y \in \mathcal{H}. \quad (15)$$

An example of an orthonormal basis, which spans $L^2([-\frac{p}{2}, \frac{p}{2}])$ is $\mathcal{T}_p = \{e_n := \frac{e^{2\pi i \frac{n}{p} x}}{\sqrt{p}}\}_{n \in \mathbb{Z}}$. The e_n are orthonormal in L^2 which can be easily seen by using the scalar product of L^2 , so for $n, m \in \mathbb{Z}$

$$\langle e_n, e_m \rangle_{L^2([-\frac{p}{2}, \frac{p}{2}])} = \frac{1}{p} \int_{[-\frac{p}{2}, \frac{p}{2}]} e_n \cdot \bar{e}_m \, dx = \quad (16)$$

$$= \frac{1}{p} \int_{[-\frac{p}{2}, \frac{p}{2}]} e^{2\pi i \frac{(n-m)}{p} x} \, dx = \quad (17)$$

$$= \frac{\sin(\pi(n-m))}{\pi(n-m)} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases} \quad (18)$$

1.4 Dirichlet Kernel

The function

$$D_t(x) := \sum_{\|k\|_\infty \leq t} e_k(x), \quad x \in \mathbb{R}^d \quad (19)$$

is called the Dirichlet Kernel. For $0 < t \in \mathbb{N}$ we have

$$(S_t f)(x) = \int_{I^d} f(y) D_t(x - y) dy, \quad (20)$$

where S_t represents the orthogonal projection onto the trigonometric polynomials Π_t of degree t , by

$$S_t : L^1(\mathbb{T}^d) \rightarrow \Pi_t \quad (21)$$

$$f \mapsto \sum_{\|k\| \leq t} \langle f, e_k \rangle_{L^2(\mathbb{T}^d)} e_k \quad k \in \mathbb{Z}^d \quad (22)$$

And furthermore the Dirichlet Kernel satisfies

$$D_t(x) = \prod_{i=1}^d \frac{e_{t+1}(x_i) - e_{-t}(x_i)}{e_1(x_i) - 1} \quad (23)$$

We start off by applying the orthogonal projection into the trigonometric polynomials S_t onto a function $f \in L(\mathbb{T}^d)$

$$(S_t f) = \sum_{\|k\|_\infty \leq t} \int_{I^d} f(y) e^{-2\pi i \langle k, y \rangle} dy e^{2\pi i \langle k, x \rangle} = \quad (24)$$

$$= \int_{I^d} f(y) \sum_{\|k\|_\infty \leq t} e^{2\pi i \langle k, (x-y) \rangle} dy = \quad (25)$$

$$= (f * D_t)(x) = \int_{I^d} f(y) D_t(x - y) dy. \quad (26)$$

The Dirichlet kernel can be simply rewritten

$$\sum_{\|k\|_\infty \leq t} e^{2\pi i \langle k, x \rangle} = \prod_{i=1}^d \sum_{k_i=-t}^t e^{2\pi i k_i x_i} = \quad (27)$$

$$= \prod_{i=1}^d e^{-2\pi i t x_i} \sum_{k_i=0}^{2t} e^{2\pi i k_i x_i} =; \quad (\text{trigonometric series}) \quad (28)$$

$$= \prod_{i=1}^d e^{-2\pi i t x_i} \frac{e^{2\pi i (2t+1)x_i} - 1}{e^{2\pi i x_i} - 1} = \quad (29)$$

$$= \prod_{i=1}^d \frac{e_{t+1}(x_i) - e_{-t}(x_i)}{e_1(x_i) - 1}. \quad (30)$$