University of Vienna

Seminar:
Applied PDE Seminar

Mathematical Modeling of Some Water-Waves

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Contents

1 Governing Equations of Fluid Dynamics

We first start of with a fluid with a density

$$\rho(\mathbf{x}, t), \tag{1.1}$$

in three dimensional Cartesian coordinates $\mathbf{x} = (x, y, z)$ at time t. For water-wave applications, we should note that we take $\rho =$ constant, but we will go into this fact later. The fluid moves in time and space with a velocity field

$$\mathbf{u}(\mathbf{x}, t) = (u, v, w). \tag{1.2}$$

Additionally it is also described by its pressure

$$P(\mathbf{x}, t), \tag{1.3}$$

generally depending on time and position. When thinking of e.g. water the pressure increases the deeper we go, that is with decreasing or increasing z direction (depending how we set up our system z pointing up or down respectively).

The general assumption in fluid dynamics is the **Continuum Hypothesis**, which assumes continuity of $\mathbf{u}, \rho$ and P in $\mathbf{x}$ and t. In other words, we premise that the velocity field, density and pressure are "nice enough" functions of position and time, such that we can do all the differential operations we desire in the framework of differential analysis.

1.1 Mass Conservation

Our aim is to derive a model of the fluid and its dynamics, with respect to time and position, in the most general way. This is usually done thinking of the density of a given fluid, which is a unit mass per unit volume, intrinsically an integral representation to derive these equations suggests by itself.

Let us now thing of an arbitrary fluid. Within this fluid we define a fixed volume V relative to a chosen inertial frame and bound it by a surface S within the fluid, such that the fluid motion $\mathbf{u}(\mathbf{x}, t)$ may cross the surface S. The fluid density is given by $\rho(\mathbf{x}, t)$, thereby the mass of the fluid in the defined Volume V is an integral expression

$$m = \int_V \rho(\mathbf{x}, t) dV. \tag{1.4}$$

The figure bellow 1, expresses the above described picture.

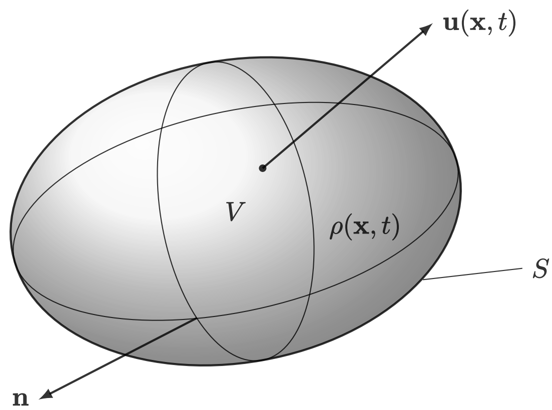

Figure 1: Volume bounded by a surface in a fluid with density and momentum, with a surface normal vector $\mathbf{n}$

Since we want to figure out the fluid's dynamics, we can consider the rate of change in the completely arbitrary V. The rate of change of mass needs to disappear, i.e. it is equal to zero since we cannot lose mass. Matter (mass) is neither created nor destroyed anywhere in the fluid, leading us to

$$\frac{d}{dt}\left(\int_V \rho(\mathbf{x},t)\ dV\right) = 0. \tag{1.5}$$

NOT SURE HERE YET!!!!!!!!!!!, CHECK LEIBINZ FORMULA To get more information we simply "differentiate under the integral sign", also known as the Leibniz Rule of Integration, see appendix A.1, the integral equation representing the rate of change of mass reads

$$\frac{dm}{dt} = \int_V \frac{\partial\rho(\mathbf{x},t)}{\partial t}\ dV + \int_{\partial V} \rho(\mathbf{x},t)\mathbf{u}\cdot\mathbf{n}\ dS = 0. \tag{1.6}$$

——————————- The above equation in 1.6 is an underlying equation, describing that the rate of change of mass in V is brought about, only by the rate of mass flowing into V across S, and thus the mass does not change.

For the second integral in 1.6 we utilize the Gaussian integration law to acquire an integral over the volume

$$\int_{\partial V} \rho(\mathbf{x},t)\mathbf{u}\cdot\mathbf{n}\ dS = \int_V \nabla(\rho\mathbf{u})\ dV. \tag{1.7}$$

Thereby we can put everything inside the volume integral

$$\frac{dm}{dt} = \int_V \left(\partial_t\rho + \nabla(\rho\mathbf{u})\right)\ dV = 0. \tag{1.8}$$

Everything under the integral sign needs to be zero, thus we obtain the **Equation of Mass Conservation** or in the general sense also called the **Continuity Equation**

$$\partial_t\rho + \nabla(\rho\mathbf{u}) = 0 \tag{1.9}$$

In light of the results of the equation of mass conservation in 1.9, an product rule gives

$$\partial_t\rho + (\nabla\rho)\mathbf{u} + \rho(\nabla\mathbf{u}), \tag{1.10}$$

for notational purposes, we define the **material/convective derivative** as follows

$$\frac{D}{Dt} = \partial_t + \mathbf{u}\nabla. \tag{1.11}$$

With the material derivative the equation of mass conservation reads

$$\frac{D\rho}{Dt} + \rho\nabla\mathbf{u} = 0 \tag{1.12}$$

We may undertake the first case separation, initiating $\rho =$ cosnt. called **incompressible flow** causes the material derivative of ρ to be zero, and thereby

$$\frac{D\rho}{Dt} = 0 \quad \Rightarrow \quad \nabla\mathbf{u} = 0, \tag{1.13}$$

following that the divergence of the velocity field is zero, in this case $\mathbf{u}$ is called **solenoidal**.

1.2 Euler's Equation of Motion

Additional consideration we undertake is the assumption of an **inviscid** fluid, that is we set viscosity to zero. Otherwise we would get a viscous contribution under the integral which results in the Navier-Stokes equation. In this regard we apply Newton's second law to our fluid in terms of infinitesimal pieces δV of the fluid. The acceleration divides into two terms, a **body force** given by gravity of earth in the z coordinate $\mathbf{F} = (0, 0, -g)$ and a **local/short-rage force** described by the stress tensor in the fluid. In the inviscid case we the local force retains the pressure P, producing a normal force, with respect to the surface, acting onto any infinitesimal element in the fluid. The integral formulation of the force would be

$$\int_V \rho \mathbf{F}\, dV - \int_S P\mathbf{n}\, dV. \tag{1.14}$$

Now applying the Gaussian rule of integration on the second integral over the surface, the resulting force in per unit volume is

$$\int_V (\rho\mathbf{F} - \nabla P)\ dV. \tag{1.15}$$

The acceleration of the fluid particles is given by $\frac{D\mathbf{u}}{Dt}$, and thus the total force per unit volume on the other hand is

$$\int_V \rho \frac{D\mathbf{u}}{Dt}\, dV = \int_V (\rho\mathbf{F} - \nabla P)\ dV. \tag{1.16}$$

Newton's Second Law for a fluid in an Volume is essentially saying that the rate of change of momentum of the fluid in the fixed volume V, which is the particle acceleration is the resulting force acting on V together with the rate of flow of momentum across the surface S into the volume V. Hence we arrive at the **Euler's Equation(s) of Motion**

$$\frac{D\mathbf{u}}{Dt} = \left(\frac{\partial \mathbf{u}}{\partial t}(\mathbf{u}\nabla)\mathbf{u}\right) = -\frac{1}{\rho}\nabla P + \mathbf{F}. \tag{1.17}$$

As a side note we have mentioned that there is another contribution if the fluid is viscid. Indeed there is a tangential force due to the velocity gradient, which into introduces the additional term

$$\mu\nabla^2\mathbf{u}, \qquad \mu = \text{viscosity of the Fluid}. \tag{1.18}$$

Thereby the equations become

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla P + \rho\mathbf{F} + \mu\nabla^2\mathbf{u}. \tag{1.19}$$

For now we have separated two simplifications, that define an **idealized/perfect fluid**

1. **incompressible** $\mu = 0$
2. **inviscid** $\rho = \text{const.},\ \nabla\mathbf{u} = 0$

1.3 Vorticity and irrotational Flow

The curl of the velocity field $\omega = \nabla \times \mathbf{u}$ of a fluid (i.e. the vorticity), describes a spinning motion of the fluid near a position $\mathbf{x}$ at time t. The vorticity is an important property of a fluid, flows or regions of flows where $\omega = 0$ are **irrotational**, and thus can be modeled and analyzed following well known routine methods. Even though real flows are rarely irrotational anywhere (!), in water wave theory wave problems, from the classical aspect of vorticity have a minor contribution. Hence we can assume irrotational flow modeling water waves. To arrive at the vorticity in the equations of motions derived in the last section we resort to a differential identity derived in appendix **??**, which gives for the material derivative

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t}\nabla(\frac{1}{2}\mathbf{u}\mathbf{u}) - (\mathbf{u} \times (\nabla \times \mathbf{u})\,. \tag{1.20}$$

Thus the equations of motion become

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{u}\mathbf{u} + \frac{P}{\rho} + \Omega\right) = \mathbf{u} \times \omega, \tag{1.21}$$

where Ω is the force potential per unite mass given by $\mathbf{F} = -\nabla\Omega$.

At this point we may differentiate between **stead and unsteady flow**. For **Steady Flow** we assume that $\mathbf{u}, P$ and Ω are time independent, thus we get

$$\nabla \left(\frac{1}{2}\mathbf{uu} + \frac{P}{\rho} + \Omega \right) = \mathbf{u} \times \omega. \tag{1.22}$$

It is general knowledge that the gradient of a function ∇f is perpendicular the level sets of $f(\mathbf{x})$, where $f(\mathbf{x}) = \text{const.}$. Thus $\mathbf{u} \times \omega$ is orthogonal to the surfaces where

$$\frac{1}{2}\mathbf{uu} + \frac{P}{\rho} + \Omega = \text{const.}, \tag{1.23}$$

The above equation is called **Bernoulli's Equation**.

Secondly **Unsteady Flow** but irrotational (+ incompressible), first of all gives us the condition for the existence of a velocity potential ϕ in the sense

$$\omega = \nabla \times \mathbf{u} = 0 \quad \Rightarrow \quad \mathbf{u} = \nabla \phi, \tag{1.24}$$

where ϕ needs to satisfy the Laplace equation

$$\Delta \phi = 0. \tag{1.25}$$

According to the Theorem of Schwartz we may exchange $\frac{\partial}{\partial t}$ and ∇, giving us an expression for the material derivative

$$\nabla \left(\frac{\partial \phi}{\partial t} + \frac{1}{2}\mathbf{uu} + \frac{P}{\rho} + \Omega \right) = 0 \tag{1.26}$$

Thus the expression differentiated by the ∇ operator is an arbitrary function $f(\mathbf{x}, t)$, writing

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\mathbf{uu} + \frac{P}{\rho} + \Omega = f(\mathbf{x}, t). \tag{1.27}$$

The function $f(\mathbf{x}, t)$ can be removed by gauge transformation of $\phi \to \phi + \int f(\mathbf{x}, t)\ dt$, never the less this is not further discussed and left to the reader in the reference.

1.4 Boundary Conditions for water waves

The boundary conditions for water-wave problems vary, generally on the simplification we undertake. At the surface, called the free surface as in free from the velocity conditions, we have the atmospheric stress on the fluid. The stress component would again have a viscid component, this however is only relevant when modeling surface wind, in this review we model the fluid as unaffectedly and within reason as inviscid. The atmosphere employs only a pressure on the surface, this pressure is taken to be the atmospheric pressure, dependent on time and point in space. Thereby any surface tension effects can also include a scenario at a curved surface (e.g. wave), giving rise to the pressure difference across the surface. A more precise description would use Thermodynamics to derive boundary conditions coupling water surface and the air above it, yet the density component of air compared to that of water makes our ansatz viable. The described conditions are called the **dynamic conditions**

An additional condition revolves around the fluid particles on the moving surface, called the **kinematic condition**. This condition bounds the vertical velocity component on the surface.

The logical step now is to define boundary conditions on the bod of the fluid, i.e. the bottom. If the viscid case bottom is impermeable, we a no slip condition to all fluid particles $\mathbf{u}_{\text{bottom}} = 0$. If we assume that the fluid is inviscid then the bottom becomes a surface of the fluid in the sense that the fluid particles in contact with the bed move in the surface, we more or less mirror the kinematic condition of the surface. For many problems the condition is going to vary, in most cases the bottom will be rigid and fixed not necessarily horizontal. This condition is simply called the **bottom condition**.

1.4.1 Kinematic Condition

Obtaining the free surface is the primary objective in the theory of modeling water waves, represented by

$$z = h(\mathbf{x}_\perp, t), \tag{1.28}$$

where $\mathbf{x}_\perp = (x, y)$ in Cartesian, or $\mathbf{x}_\perp = (r, \theta)$ in cylindrical coordinates. A surfaces that moves with the fluid, always contains the same fluid particles, described as

$$\frac{D}{Dt}\left(z - h(\mathbf{x}_\perp, t\right) = 0. \tag{1.29}$$

Upon expanding the derivative we get

$$\frac{Dz}{Dt} - \frac{Dh}{Dt} = \frac{\partial z}{\partial t} + (\mathbf{u}\nabla)z - \frac{\partial h}{\partial t} - (\mathbf{u}\nabla) \tag{1.30}$$

$$= w - (h_t - (\mathbf{u}_\perp \nabla_\perp)h) = 0, \tag{1.31}$$

where the subscript $\perp$ describes the components with regard to $\mathbf{x}_\perp$. The **kinematic condition** reads

$$w = h_t - (\mathbf{u}_\perp \nabla_\perp)h \qquad \text{on } \ z = h(\mathbf{u}_\perp, t). \tag{1.32}$$

1.4.2 Dynamic Condition

As described in the prescript of this section, the case of an inviscid fluid, requires that only the pressure P needs to be described on the free surface $z = h(\mathbf{x}_\perp, t)$. Assuming incompressible, irrotational, unsteady flow and setting $P = P_a$ for atmospheric pressure and $\Omega = g \cdot z$ for the force per unit mass potential the equations of motion are

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\mathbf{uu} + P_{\frac{a}{\rho}} + gh = f(t) \qquad \text{on } \ onz = h. \tag{1.33}$$

Somewhere $\|\mathbf{x}_\perp\| \to \infty$ the fluid reaches equilibrium and is thereby stationary, thereby has no motion and the pressure is $P = P_a$ and the surface is a constant $h = h_0$ $f(t)$ is

$$f(t) = \frac{P_a}{\rho} + gh_0. \tag{1.34}$$

The simplest description for the **dynamic condition** may be written as

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\mathbf{uu} + g(h - h_0) = 0 \qquad \text{on } \ z = h. \tag{1.35}$$

Regarding the pressure difference on a curved surface, we may expand the dynamic condition by introducing the pressure difference known as the **Young-Laplace Equation**

$$\Delta P = \frac{\Gamma}{R}, \tag{1.36}$$

where $\Gamma > 0$ is the coefficient of surface tension and $\frac{1}{R}$ is the curvature representing an implicit function, in our case the implicit function is $z - h(\mathbf{x}_\perp, t)$ for fixed time. The curvature in Cartesian coordinates takes the form

$$\frac{1}{R} = \frac{(1 + h_y^2)h_{xx} + (1 + h_y^2)h_{yy} - 2h_x h_y h_{xy}}{(h_x^2 + h_y^2 + 1)^{\frac{3}{2}}}, \tag{1.37}$$

the derivation is precisely described in **??**

1.4.3 The Bottom Condition

The representation for the bottom is

$$z = b(\mathbf{x}_\perp, t), \tag{1.38}$$

where the fluid surface needs to satisfy

$$\frac{D}{Dt}\,(z - b(\mathbf{x}_\perp)) = 0. \tag{1.39}$$

Hence we arrive at the bottom boundary conditions

$$w = b_t + (\mathbf{u}_\perp \nabla_\perp) b \qquad \text{on} \;\; z = b, \tag{1.40}$$

where $b(\mathbf{x}_\perp, t)$ is already known for most water wave problems. If we consider a stationary bottom then the time derivative vanishes, leaving us with the following condition

$$w = (\mathbf{u}_\perp \nabla_\perp) b \qquad \text{on} \;\; z = b \tag{1.41}$$

1.4.4 Integrated Mass Condition

In this section we want to combine the kinematics of both the free and the bottom surface with the mass conservation equation on the perpendicular components

$$\nabla \mathbf{u} = \nabla_\perp \mathbf{u}_\perp + w_z = 0. \tag{1.42}$$

Integrating the above expression from bottom to surface, i.e. from $z = b(\mathbf{x}_\perp, t)$ to $z = h(\mathbf{x}, t)$ gives

$$\int_b^h \nabla_\perp \mathbf{u}_\perp \; dz + w \Big|_{z=b}^{z=h} = 0, \tag{1.43}$$

where we insert the conditions on the free surface and on the bottom surface

$$w = h_t + (\mathbf{u}_{\perp \mathrm{s}} \nabla_\perp) h \quad \text{on} \;\; z = h \tag{1.44}$$
$$w = b_t + (\mathbf{u}_{\perp \mathrm{b}} \nabla_\perp) h \quad \text{on} \;\; z = b, \tag{1.45}$$

with the subscript s and b indicating the evaluation of a quantity on the free surface and the bottom surface respectively. Inserting the boundary conditions we get

$$\int_b^h \nabla_\perp \mathbf{u}_\perp + h_t + (\mathbf{u}_{\perp \mathrm{s}} \nabla_\perp) h - b_t - (\mathbf{u}_{\perp \mathrm{b}} \nabla_\perp) b = 0. \tag{1.46}$$

To simplify the equation we resort again to the Leibniz Rule of Integration

$$\int_b^h \nabla_\perp \mathbf{u}_\perp = \nabla_\perp \int_b^h \mathbf{u}_\perp \; dz - (\mathbf{u}_{\perp \mathrm{s}} \nabla_\perp) h - (\mathbf{u}_{\perp \mathrm{b}}) b. \tag{1.47}$$

As a consequence the **Integrated Mass Condition** is given by

$$\nabla_\perp \int_b^h \mathbf{u}_\perp \; dz + \underbrace{h_t - b_t}_{=d_t} = 0. \tag{1.48}$$

1.5 Energy Equation

To derive the energy equation we start off with Euler's Equation of Motion

$$\mathbf{u}_t + \nabla(\frac{1}{2}\mathbf{u}\mathbf{u} + \frac{P}{\rho} + \Omega) = \mathbf{u} \times \mathbf{w}, \tag{1.49}$$

multiplying the equation with $\mathbf{u}$ we get

$$\mathbf{u}\mathbf{u}_t \tag{1.50}$$

$$+(\mathbf{u}\nabla)(\frac{1}{2}\mathbf{u}\mathbf{u}+\frac{P}{\rho}+\Omega) \tag{1.51}$$

$$=\mathbf{u}(\mathbf{u}\times\mathbf{w}). \tag{1.52}$$

The first equation given in 1.50 can we rewritten using inverse product rule of differentiation

$$\mathbf{u}\frac{\partial \mathbf{u}}{\partial t}=\frac{\partial}{\partial t}(\mathbf{u}\mathbf{u})-\frac{\partial \mathbf{u}}{\partial t}\mathbf{u} \tag{1.53}$$

$$=\frac{\partial}{\partial t}(\mathbf{u}\mathbf{u})-\mathbf{u}\frac{\partial \mathbf{u}}{\partial t} \tag{1.54}$$

$$\Rightarrow \quad \mathbf{u}\frac{\partial \mathbf{u}}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}(\mathbf{u}\mathbf{u}). \tag{1.55}$$

Then we may add

$$\left(\frac{1}{2}\mathbf{u}\mathbf{u}+\frac{P}{\rho}+\Omega\right)\underbrace{(\nabla u)}_{=0}=0, \tag{1.56}$$

to above not changing anything. Thereby getting

$$\frac{\partial}{\partial t}(\frac{1}{2}\mathbf{u}\mathbf{u})+(\mathbf{u}\nabla\mathbf{u})\left(\frac{1}{2}\mathbf{u}\mathbf{u}+\frac{P}{\rho}\right)+\left(\frac{1}{2}\mathbf{u}\mathbf{u}+\frac{P}{\rho}+\Omega\right)(\nabla\mathbf{u})=0. \tag{1.57}$$

Applying the product rule we can simplify

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{u}\mathbf{u}\right)+\nabla\left(\mathbf{u}\left(\mathbf{u}(\frac{1}{2}\mathbf{u}\mathbf{u}+\frac{P}{\rho}\right)\right)=0, \tag{1.58}$$

additionally adding $\frac{\partial\Omega}{\partial t}=0$ leads us to

$$\underbrace{\frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{u}\mathbf{u}+\Omega\right)}_{\text{change of total energy density}}+\underbrace{\nabla\left(\mathbf{u}\left(\mathbf{u}(\frac{1}{2}\mathbf{u}\mathbf{u}+\frac{P}{\rho}\right)\right)}_{\text{energy flow of the velocity field}}=0. \tag{1.59}$$

This is called the **energy equation** and is a general result for a inviscid and incompressible fluids, which we can apply to study water waves. We start off with replacing $\nabla=\nabla_\perp+\frac{\partial}{\partial z}$ and $\Omega=gz$ and multiplying by ρ, then our energy equation in 1.59 becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}\mathbf{u}+\rho gz\right)+\nabla_\perp\left(\mathbf{u}_\perp\left(\frac{1}{2}\rho\mathbf{u}\mathbf{u}+P+\rho gz\right)\right)\frac{\partial}{\partial z}\left(w\left(\frac{1}{2}\rho\mathbf{u}\mathbf{u}+P+\rho gz\right)\right)=0. \tag{1.60}$$

Integrating from bottom to top, i.e. from bed to free surface gets us to

$$\int_b^h\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}\mathbf{u}+\rho gz\right)\ dz \tag{1.61}$$

$$+\int_b^h\nabla_\perp\left(\mathbf{u}_\perp\left(\frac{1}{2}\rho\mathbf{u}\mathbf{u}+P+\rho gz\right)\right)\ dz \tag{1.62}$$

$$+\left(\frac{\partial}{\partial z}\left(w\left(\frac{1}{2}\rho\mathbf{u}\mathbf{u}+P+\rho gz\right)\right)\right)\Bigg|_b^h=0. \tag{1.63}$$

For equation 1.61 we use Leibniz Rule of Integration, leaving us with

$$\int_b^h\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\mathbf{u}\mathbf{u}+\rho gz\right)\ dz=\frac{\partial}{\partial t}\int_b^h\frac{1}{2}\rho\mathbf{u}\mathbf{u}+\rho gz\ dz \tag{1.64}$$

$$+\left(\frac{1}{2}\rho\mathbf{u}_s\mathbf{u}_s+\rho gh\right)h_t \tag{1.65}$$

$$-\left(\frac{1}{2}\rho\mathbf{u}_b\mathbf{u}_b+\rho gb\right)b_t \tag{1.66}$$

For equation 1.62 we again take note of the Leibniz Rule of Integration, getting

$$\int_b^h \nabla_\perp \left(\mathbf{u}_\perp \left(\frac{1}{2}\rho\mathbf{u}\mathbf{u} + P + \rho g z \right) \right) \, dz = \nabla_\perp \int_b^h \mathbf{u}_\perp \left(\frac{1}{2}\rho\mathbf{u}\mathbf{u} + P + \rho g z \right) \, dz \tag{1.67}$$

$$- \left(\frac{1}{2}\rho\mathbf{u}_s\mathbf{u}_s + P + \rho g h \right) (\mathbf{u}_{\perp s}\nabla_\perp) \, h \tag{1.68}$$

$$+ \left(\frac{1}{2}\rho\mathbf{u}_b\mathbf{u}_b + P + \rho g b \right) (\mathbf{u}_{\perp b}\nabla_\perp) \, b \tag{1.69}$$

Thereby transforming our equation into

$$\underbrace{\frac{\partial}{\partial t} \int_b^h \frac{1}{2}\rho\mathbf{u}\mathbf{u} + \rho g z \; dz}_{=:\mathcal{E}} + \underbrace{\nabla_\perp \int_b^h \mathbf{u}_\perp \left(\frac{1}{2}\rho\mathbf{u}\mathbf{u} + \rho g z \right) \, dz}_{:=\mathcal{F}} + \underbrace{P_s h_t - P_b b_t}_{:=\mathcal{P}} = 0 \tag{1.70}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla_\perp \mathcal{F} + \mathcal{P} = 0, \tag{1.71}$$

where $\mathcal{E}$ represents the energy in the flow per unit horizontal area, since we are integrating from bed to free surface. Where $\mathcal{F}$ is the horizontal energy flux vector and lastly $\mathcal{P} = P_s h_t - P_b b_t$ is the net energy input due to the pressure forces doing work on the upper and lower boundaries, i.e. bottom and free surface of the fluid. Assuming stationary rigid bottom condition and constant surface pressure, we can set $P_s = 0$, such that $\mathcal{P} = 0$ leaving us with the equation

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla_\perp \mathcal{F} = 0. \tag{1.72}$$

We note that the assumption $P_s = 0$ is only possible if the coefficient of surface tension is set to 0, which usually is not the case.

2 Dimensional Analysis

Our derived model of fluid dynamics yields formal connections between physical quantities. These quantities bear units, e.g. the velocity of fluid particles $\mathbf{u}$ has the "SI" unites of $\frac{m}{s}$, meters per second. The idea is the make use of these scales and formulate a model, where the quantities are nondimensionalized, i.e. to get rid of physical units by scaling each quantity appropriately. The appropriate length scales are that of the typical water depth h_0 and the typical wavelength λ of a surface wave.

2.1 Nondimensionalisation

In summary we use these adaptations

- h_0 for the typical water depth
- λ for the typical wavelength
- $\frac{\lambda}{\sqrt{gh_0}}$ time scale of wave propagation
- $\sqrt{gh_0}$ velocity scale of waves in (x, y)
- $\frac{h_0\sqrt{gh_0}}{\lambda}$ velocity scale in the z direction.

(x, z, t), then

$$u = \psi_z, \qquad w = -\psi_x; \tag{2.1}$$

and the scale of ψ must be $h_0\sqrt{gh_0}$. Additionally we write the boundary condition on the free surface as follows

$$h = h_0 + a\eta(\mathbf{x}_\perp, t) = z, \tag{2.2}$$

where a is the typical amplitude and η nondimensional function. All in all we have the following scaling for the physical quantities of our context

$$x \to \lambda x, \quad u \to \sqrt{gh_0}u, \tag{2.3}$$

$$y \to \lambda y, \quad v \to \sqrt{gh_0}v, \qquad t \to \frac{\lambda}{\sqrt{gh_0}}t, \tag{2.4}$$

$$z \to h_0 z, \quad w \to \frac{h_0\sqrt{gh_0}}{\lambda}w. \tag{2.5}$$

with

$$h = h_0 + a\eta, \qquad b \to h_0 b. \tag{2.6}$$

The pressure is also rewritten into

$$P = P_a + \rho g(h_0 - z) + \rho g h_0 p, \tag{2.7}$$

where P_a is the atmospheric pressure, the term $h_0 - z$ represent the hydrostatic pressure distribution, i.e. pressure at depth and the term with the pressure variable p measures the deviation from the hydrostatic pressure distribution. Indeed $p \neq 0$ for wave propagation. Now we can perform a rescaling of the Euler's Equation of Motion, we introduce the notation

$$t = \frac{\lambda}{\sqrt{gh_0}}\tau, \quad x = \lambda\xi, \quad u = \sqrt{gh_0}\tilde{u} \tag{2.8}$$

$$y = \lambda\chi, \quad v = \sqrt{gh_0}\tilde{v} \tag{2.9}$$

$$z = h_0\zeta, \quad w = \frac{h_0\sqrt{gh_0}}{\lambda}\tilde{w}. \tag{2.10}$$

We start off with the x coordinate, substitute and apply the chain rule leading us to

$$\begin{aligned}\frac{Du}{Dt} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} &(2.11)\\ &= \sqrt{gh_0}\frac{\partial \tilde{u}}{\partial \tau}\frac{\partial \tau}{\partial t} + gh_0\tilde{u}\frac{\partial \tilde{u}}{\partial \xi}\frac{\partial \xi}{\partial x} &(2.12)\\ &= \frac{gh_0}{\lambda}\left(\frac{\partial \tilde{u}}{\partial \tau}\tilde{u}\frac{\partial \tilde{u}}{\partial \xi}\right), &(2.13)\end{aligned}$$

on the other hand

$$\begin{aligned}\frac{gh_0}{\lambda}\left(\frac{\partial \tilde{u}}{\partial \tau}\tilde{u}\frac{\partial \tilde{u}}{\partial \xi}\right) &= -\frac{1}{\rho}\frac{1}{\lambda}\frac{\partial P}{\partial x} &(2.14)\\ &= -\frac{gh_0}{\lambda}\rho\frac{\partial p}{\partial \xi}. &(2.15)\end{aligned}$$

Thereby the rescaling evolves to

$$\frac{D\tilde{u}}{D\tau} = -\frac{\partial p}{\partial \xi}. \tag{2.16}$$

Because of the same scaling in y we get the same result as in x, that is

$$\frac{D\tilde{v}}{D\tau} = -\frac{\partial p}{\partial \chi}. \tag{2.17}$$

In the z coordinate we have

$$\begin{aligned}\frac{Dw}{Dt} &= \frac{\partial w}{\partial t} + w\frac{\partial w}{\partial \zeta} &(2.18)\\ &= \frac{h_0\sqrt{gh_0}}{\lambda}\frac{\sqrt{gh_0}}{\lambda}\frac{\partial \tilde{w}}{\partial \tau} + \frac{1}{h_0}\frac{h_0\sqrt{gh_0}}{\lambda}\frac{h_0\sqrt{gh_0}}{\lambda}\tilde{w}\frac{\partial \tilde{v}}{\partial \zeta} &(2.19)\\ &= \frac{h_0^2 g}{\lambda}\left(\frac{\partial \tilde{w}}{\partial \tau} + \tilde{w}\frac{\partial \tilde{w}}{\partial \zeta}\right). &(2.20)\end{aligned}$$

On the other side we have

$$\begin{aligned}\frac{h_0^2 g}{\lambda}\left(\frac{\partial \tilde{w}}{\partial \tau} + \tilde{w}\frac{\partial \tilde{w}}{\partial \zeta}\right) &= -\frac{1}{h_0\rho}\frac{\partial P}{\partial z} + g &(2.21)\\ &= -\frac{1}{h_0\rho}(-\rho g h_0\frac{\partial \zeta}{\partial \zeta}\rho g h_0\frac{\partial p}{\partial \zeta}) + g &(2.22)\\ &= -g\frac{\partial p}{\partial z}. &(2.23)\end{aligned}$$

In total for the z direction we get

$$\underbrace{\left(\frac{h_0}{\lambda}\right)^2}_{=:\delta^2}\frac{Dw}{Dt} = -\frac{\partial p}{\partial z}, \tag{2.24}$$

where δ is the **long wavelength** or **shallowness** parameter, a very important constant for developing model hierarchies. For clarity we resubstitute for x, y, z, t, u, v and w, and for completeness the we display the equations again, which are

$$\frac{Du}{Dt} = -\frac{\partial p}{\partial x}, \quad \frac{Dv}{Dt} = -\frac{\partial p}{\partial y}, \quad \delta^2\frac{Dw}{Dt} = -\frac{\partial p}{\partial z}, \tag{2.25}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2.26}$$

We can now turn our attention to the boundary conditions, on both free surface $z = h$ and the bottom $z = b$ we have $z \Rightarrow h_0 z$ and thereby

$$z = 1 + \underbrace{\frac{a}{h_0}}_{:=\varepsilon}\eta(\mathbf{x}_\perp, t) \quad \text{and} \quad z = b, \tag{2.27}$$

where we arrive at our second very important parameter ε called the **amplitude** parameter. As for the kinematic condition, we substitute the free surface $z = h = 1 + \varepsilon\eta$ and get

$$\frac{Dz}{Dt} = \varepsilon\left(\eta_t + (\mathbf{u}_\perp \nabla_\perp)\eta\right) \qquad \text{on } \; z = 1 + \varepsilon\eta. \tag{2.28}$$

Respectively the bottom condition is not changed

$$w = b_t + (\mathbf{u}_\perp \nabla_\perp)b \quad \text{on } \; z = b. \tag{2.29}$$

The general dynamic condition for $h = h(x, y, t)$ yields a rescaling of the curvature in terms of

$$\frac{1}{R} = \frac{(1 + h_y^2)h_{xx} + (1 + h_x^2)h_y y - 2h_x h_y h_{xy}}{\left(h_x^2 + h_y^2 + 1\right)^{\frac{3}{2}}} \tag{2.30}$$

$$= -\frac{\varepsilon h_0}{\lambda^2} \frac{(1 + \varepsilon^2\delta^2\eta_y^2)\eta_{xx} + (1 + \varepsilon^2\delta^2\eta_x^2)\eta_{yy} - 2\varepsilon^2\delta^2\eta_x\eta_y\eta_{xy}}{\left(1 + \varepsilon^2\delta^2\eta_x^2 + \varepsilon^2\delta^2\eta_y^2\right)^{\frac{3}{2}}}, \tag{2.31}$$

together with the pressure difference

$$\Delta P = \rho g h_0(p - \varepsilon\eta) = \frac{\Gamma}{R}, \tag{2.32}$$

leaving us ultimately with the dynamic condition

$$p - \varepsilon\eta = \varepsilon\left(\frac{\Gamma}{\rho g \lambda^2}\right)\left(\frac{\lambda^2}{\varepsilon h_0}\frac{1}{R}\right), \tag{2.33}$$

where $W_e = \frac{\Gamma}{\rho g h_0^2}$ is the **Weber number**. This dimensionless parameter can be considered as a measure of the fluid's inertia compered to its surface tension, which satisfies the relation

$$\delta^2 W_e = \frac{\Gamma}{\rho g \lambda^2}. \tag{2.34}$$

2.2 Scaling of Variables

Admits a simple observation of the governing equations in the last chapter we notice that w and p on the free surface $z = 1 + \varepsilon\eta$ are directly proportional to ε. Hence we want to "scale this way" by introducing the following transformation

$$p \to \varepsilon p, \quad w \to \varepsilon w, \quad \mathbf{u}_\perp \to \varepsilon \mathbf{u}_\perp. \tag{2.35}$$

Because of this scaling our material derivative changes slightly to

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \varepsilon\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right) \tag{2.36}$$

A simple recalculation yields the rescaled, nondimensionalized Euler's Equation of motion are the same as in equations 2.25 with the modified material derivative from 2.36, and the boundary conditions are

$$p = \eta - \frac{\delta^2\varepsilon h_0}{\lambda^2}\frac{W_e}{R} \tag{2.37}$$

$$w = \frac{1}{\varepsilon}\eta_t + (\mathbf{u}_\perp \nabla_\perp)\eta \quad \text{on } \; z = 1 + \varepsilon\eta \tag{2.38}$$

$$w = \frac{1}{\varepsilon}b_t + (\mathbf{u}_\perp \nabla_\perp)b \quad \text{on } \; z = b \tag{2.39}$$

2.3 Model Hierarchies

As we have derived a model of fluid dynamics, with small parameters ε and δ, we can conduct a series of classifications and perform asymptotic analysis on them. The main hierarchies important in this review are derived from the following problem classifications

- $\varepsilon \to 0$: linearized problem, small amplitude
- $\delta \to 0$: shallow Water, long-wave
- $\delta \to 0$; ε 1: shallow Water, large amplitude
- $\delta \ll 1$; ε δ: shallow water, medium amplitude
- $\delta \ll 1$; ε δ^2: shallow water, small amplitude
- $\delta \gg 1$; $\varepsilon\delta \ll 1$: deep water, small steepness.

3 The Solitary Wave and The KdV Equation

The solitary wave is a wave of translation, it is stable and can travel long distances additionally the speed depends on the size of the wave. An interesting feature is that two solitary waves do not merge together to form one solitary wave, rather the small wave is overtaken by a larger one. If a solitary wave is too big for the depth it splits into two, a big and a small one. Solitary waves arise in the region $\varepsilon = O(\delta^2)$.

3.1 Solitary Wave

To describe a solitary wave we begin with Euler's Equation of Motion, where we assume there is no surface tension we set $W_e = 0$ and additionally assume irrotational flow $\omega = \nabla \times \mathbf{u} = 0$. This means that there exists a velocity potential $\phi(\mathbf{x}, t)$ given by$\mathbf{u} = \nabla\phi$ satisfying the Laplace equation. In regard of a solitary wave being a plane wave, we rotate our coordinate system such that the propagation is in the x-direction and a stationary & fixed bottom $b = 0$. Ultimately leaving us with the following model

$$\left.\begin{array}{l} \phi_{zz} + \delta\phi_{xx} = 0, \\ \text{with the boundary conditions} \\ \quad \left.\begin{array}{l} \phi_z = \delta^2(\eta_t + \varepsilon\phi_x\eta_x) \\ \phi_t + \eta + \frac{1}{2}\varepsilon\left(\frac{1}{\delta^2}\phi_z^2 + \phi_x^2\right) = 0 \end{array}\right\} \quad \text{on } z = 1 + \varepsilon\eta, \\ \text{and} \\ \phi_z = 0 \quad \text{on } z = b = 0. \end{array}\right\} \tag{3.1}$$

Since the model arises $\varepsilon = O(\delta^2)$, for convince we set $\varepsilon = 1$. The fact of the matter is we are seeking a traveling wave solution, thereby we can go into the coordinate system of the traveling wave, one in the variable $\xi = x - ct$ for a from left to right traveling wave, where c is the nondimensional speed of the wave. Our goal is to find the solution for the velocity potential $\phi(\xi, z)$ and the wave profile $\eta(\xi)$. The chain rule gives us

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \xi}, \tag{3.2}$$

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} = -c\frac{\partial}{\partial \xi}. \tag{3.3}$$

Together with the equations in 3.1 we obtain

$$\left.\begin{array}{l} \phi_{zz} + \delta\phi_{\xi\xi} = 0, \\ \text{with the boundary conditions} \\ \quad \left.\begin{array}{l} \phi_z = \delta^2(\phi_\xi - c)\eta_\xi \\ -c\phi_\xi + \eta + \frac{1}{2}\varepsilon\left(\frac{1}{\delta^2}\phi_z^2 + \phi_\xi^2\right) = 0 \end{array}\right\} \quad \text{on } z = 1 + \eta, \\ \text{and} \\ \phi_z = 0 \quad \text{on } z = b = 0. \end{array}\right\} \tag{3.4}$$

3.1.1 Exponential Decay

We would like to analyze if the equation in 3.4 gives viable a solution that decays exponentially, we make the ansatz

$$\eta \simeq a e^{-\alpha|\psi|}, \quad \phi \simeq \psi(z) e^{-\alpha|\xi|}, \qquad |\, \xi \,| \to \infty, \tag{3.5}$$

where $\alpha > 0$ is the exponent. The equations in 3.4 transforms to

$$\psi'' + \alpha^2\delta^2\psi = 0. \tag{3.6}$$

The above equation is a standard well known ordinary differential equation reading

$$\psi = A\cos(\alpha\delta z), \tag{3.7}$$

where A is the integration constant. On the free surface $z \simeq 1$ gives

$$-cA\alpha\sin(\alpha\delta) = ca\alpha, \tag{3.8}$$

$$cA\alpha\cos(\alpha\delta) = -a. \tag{3.9}$$

Dividing equation 3.8 with equation 3.9 gives

$$c^2 = \frac{\tan(\alpha\delta)}{\alpha\delta}. \tag{3.10}$$

We conclude that the solution for such a wave exists provided that the dispersion relation on the wave propagation speed holds, thereby all solitary waves exhibit exponential decay in their tail and satisfy the dispersion relation in equation 3.10.

3.1.2 Asymptotic Analysis

The underlining equations in 3.1 extend from $-\infty$ to ∞, so the length scale is much greater than any finite depth of water. Therefore the classification $\delta \to 0$ is appropriate for a solitary wave, this however goes with the assumption $\varepsilon \to 0$ otherwise we cannot make an appropriate expansion. Let us look at the main equation

$$\phi_{zz} + \delta\phi_{xx} = 0. \tag{3.11}$$

For small δ we conduct the $\delta^2 = O(\varepsilon)$ standard ansatz in asymptotic analysis

$$\phi_\delta(x,t,z) \simeq \sum_{n=0}^{\infty} \delta^{2n}\phi_n(x,t,z). \tag{3.12}$$

Substituting ϕ_δ into equation 3.11 we get

$$\delta^{2\cdot 0}(\phi_{0zz}) + \delta^{2\cdot 1}(\phi_{1zz} + \phi_{0xx}) + \delta^{2\cdot 2}(\phi_{2zz} + \phi_{1xx}) + O(\delta^{2\cdot 3}) = 0. \tag{3.13}$$

We start off with $O(\delta^{2\cdot 0})$, which gives us an arbitrary function $\phi_0 = \theta(x,t)$. Next we may generalize the results for all $O(\delta^{2\cdot n})$ in the means of

$$\phi_{n+1zz} = -\phi_{nxx} \qquad \forall n \in \mathbb{N}. \tag{3.14}$$

Therefore leaving us for ϕ_1 and ϕ_2 with

$$\phi_1 = -\frac{1}{2}z^2\theta_0(x,t) + \theta(x,t), \tag{3.15}$$

$$\Rightarrow \phi_2 = \frac{1}{24}z^4\theta_0(x,t) - \frac{1}{2}z^2\theta_1(x,t) + \theta_2(x,t). \tag{3.16}$$

The boundary condition on the bottom comes around to be

$$\phi_{nz} = 0 \quad \text{on} \quad z = 0. \tag{3.17}$$

The free surface boundary condition $z = 1 + \varepsilon\eta$ n evolves more calculation, we consider only terms up the order of δ^2, initializing with

$$\phi_z = \delta^2(\eta_t + \varepsilon\phi_x\eta_x) \tag{3.18}$$

$$\Leftrightarrow \frac{1}{\delta}\phi_z = \eta_t + \varepsilon\phi_x\eta_x, \tag{3.19}$$

substituting ϕ_δ into the above proceeds to be

$$\frac{1}{\delta^2}\underbrace{\phi_{0z}}_{=0} + \phi_{1z} + \delta^2\phi_{zz}O(\delta^{2\cdot 2}) = -z\theta_{xx} + \delta^2\left(\frac{1}{6}z^3\theta_{0xxxx} - z\theta_{0xx}\right) + O(\delta^{2\cdot 2}) \tag{3.20}$$

$$= -(1+\varepsilon\eta)\theta_{0xx} + \delta^2\left(\frac{1}{6}(1+\varepsilon\eta)^3\theta_{0xx} - (1+\varepsilon\eta)\theta_{0xx}\right) \tag{3.21}$$

$$= \eta_t + \varepsilon\eta_x\left(\theta_{0x}\delta^2(\theta_{1x} - \frac{1}{2}(1+\varepsilon\eta)^2\theta_{0xxx}\right). \tag{3.22}$$

The second condition is

$$\phi_t + \eta + \frac{1}{2}\varepsilon\left(\frac{1}{\delta}\phi_z^2\phi_x^2\right) = 0, \tag{3.23}$$

becomes after substitution

$$\theta_{0t} + \delta^2\left(-\frac{1}{2}(1+\varepsilon\eta)^2\theta_{0xxt} + \theta_{1t}\right) + \eta + O(\delta^{2\cdot 2}) \tag{3.24}$$

$$= -\frac{1}{2}\delta^2\varepsilon(-(1+\varepsilon\eta)\theta_{0xx})^2 - \frac{1}{2}\left(\theta_{0x} + \delta^2\left(\theta_{1x} - \frac{1}{2}(1+\varepsilon\eta)^2\theta_{0xxxx}\right)\right)^2 \tag{3.25}$$

In the order of $O(\delta^0)$ as $\varepsilon \to 0$ gives us the conditions

$$-\theta_{0xx} = \eta_t \simeq \text{and} \quad \theta_{0t} \simeq -\eta \tag{3.26}$$

$$\Rightarrow \theta_{0xx} - \theta_{0tt} \simeq 0. \tag{3.27}$$

This gives us the wave equation, a simple solution in the frame of the right moving wave dependent on $\xi = x - t$ the chain rule gives us

$$\frac{\partial\theta_0(\xi(x,t))}{\partial t} = \frac{\partial\theta_0}{\partial\xi}\underbrace{\frac{\partial\xi}{\partial t}}_{=-1} + \frac{\partial\theta_0}{\partial t}\underbrace{\frac{\partial t}{\partial t}}_{=1} + \frac{\partial\theta_0}{\partial x}\underbrace{\frac{\partial x}{\partial t}}_{=0} \tag{3.28}$$

$$= -\theta_{0\xi} + \theta_{0t}. \tag{3.29}$$

substituting into we get

$$2\theta_{0t\xi} \simeq \theta_{0tt}, \tag{3.30}$$

$$\Rightarrow \eta = \theta_{0\xi} + O(\varepsilon). \tag{3.31}$$

As for the surface boundary condition we see that the derivatives in t are "small". So we can proceed by the scaling $\tau = \varepsilon t$ as $\varepsilon \to 0$, we proceed with equation given in 3.21 and 3.22 in the $O(\varepsilon), O(\delta^2)$

$$-(1+\varepsilon\eta)\theta_{0\xi\xi} + \delta^2\left(\frac{1}{6}\theta_{0\xi\xi\xi\xi} - \theta_{1\xi\xi}\right) \simeq \varepsilon\eta_\tau - \eta_\xi + \varepsilon\eta\theta_{0\xi} \tag{3.32}$$

and boundary equations in 3.24, 3.25 produce

$$\varepsilon\theta_{0\tau} - \theta_{0\xi} + \delta^2\left(\frac{1}{2}\theta_{0\xi\xi\xi} - \theta_{1\xi}\right) + \eta \simeq -\frac{1}{2}\varepsilon\theta_{0\xi}^2. \tag{3.33}$$

Doing the following operation to the above equations 3.32 $-$ $\frac{\partial}{\partial\xi}$3.33 turns out to be

$$-\theta_{0\xi\xi} - \varepsilon\eta\theta_{0\xi\xi} + \delta\left(\frac{1}{6}\theta_{0\xi\xi\xi\xi} - \theta_{1\xi\xi}\right) - \varepsilon\theta_{0\xi\tau} + \theta_{0\xi\xi} - \delta^2\left(\frac{1}{2}\theta_{0\xi\xi\xi\xi} - \theta_{1\xi\xi}\right) + \eta_\xi \tag{3.34}$$

$$\simeq \varepsilon\eta_t - \eta_\xi + \varepsilon\eta\theta_{0\xi} + \varepsilon\theta_{0\xi\xi}\theta_{0\xi}. \tag{3.35}$$

In the above equation we can simplify, i.e. short some terms out and substitute $\eta = \theta_{0\xi} + O(\varepsilon)$ and because of $\delta^2 = O(\varepsilon)$ we set $\delta^2 = K\varepsilon$ for constant K, leaving us with the equation for the surface profile, called the **Korteweg-de Vries**, KdV equation (1895)

$$2\eta_\tau + 3\eta\eta_\xi + \frac{K}{3}\eta_{\xi\xi\xi} = 0. \tag{3.36}$$

The KdV equation describes the balance between linearity and dispersion in the change of time of the wave profile. By rewriting $\eta = f(\xi - ct)$ we get

$$-2cf' + 3ff' + \frac{K}{3}f''' = 0 \tag{3.37}$$

$$\text{with} \quad f, f', f''' \to 0 \quad \text{as} \quad |\xi - ct| \Rightarrow \infty. \tag{3.38}$$

The solution is a sech function

$$f = 2c\,\mathrm{sech}^2\left(\sqrt{\frac{3}{2K}}(\xi - ct)\right) \tag{3.39}$$

3.2 KdV Equation

In this section we will go over the more general prerequisites and therefore a more convincing expedition for the Korteweg-de Vries equation. We still want to derive the wave profile of a wave in shallow water, small amplitude regime $\delta^2 = O(\varepsilon)$, where the bottom is horizontal & stationary. The propagating wave can be seen as a plane wave, therefore the coordinate system is rotated in such a way that the propagating direction is the x direction. For irrotational, inviscid flow without surface tension $W_e = 0$ that is for gravity waves, nondimensional and rescaled Euler's Equations of Motion for such a flow are

$$\left.\begin{array}{l} \dfrac{Du}{Dt} = -p_x, \qquad \delta^2 \dfrac{Dw}{Dt} = -p_z, \\ \text{where} \\ \dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + \varepsilon\left(u\dfrac{\partial u}{\partial x} + w\dfrac{\partial w}{\partial z}\right) \\ \text{with} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0 \end{array}\right\} \tag{3.40}$$

with free surface boundary conditions

$$\left.\begin{array}{l} p = \eta \\ w = \eta_t + \varepsilon u \eta_x \end{array}\right\} \text{ on } \; z = 1 + \varepsilon\eta, \tag{3.41}$$

and bottom boundary condition

$$w = 0 \quad \text{on} \;\; z = b = 0. \tag{3.42}$$

We note here thatthe soluition for such a wave is a solitary wave as in described in the previous section. In principel we expect to wind such waves reather rarely in nature, since $\delta^2 = O(\varepsilon)$ is a very special case. Never the less this is not the case. We demonstrate that $\forall\ \delta$ as ε goes to 0 there exists a region in the position space (x, t) where the KdV balance in terms of linearity and dispersion is observed. Indeed we can "generate" KdV solitary waves, provided a small enough amplitude in the sence of ε goes to 0. First of all we introduce a rescaling of the variables adjusted to our problem definition

$$x \to \frac{\delta}{\sqrt{\varepsilon}}\tilde{x}, \quad t \to \frac{\delta}{\sqrt{\varepsilon}}\tilde{t} \tag{3.43}$$

$$w \to \frac{\sqrt{\varepsilon}}{\delta}\tilde{w}. \tag{3.44}$$

Then the material derivative is transformed to be

$$\frac{D}{Dt} = \frac{\sqrt{\varepsilon}}{\delta}(\frac{\partial}{\partial \tilde{t}} + \varepsilon \tilde{\mathbf{u}} \nabla). \tag{3.45}$$

The initial equations become

$$\frac{Du}{Dt} = \frac{\sqrt{\varepsilon}}{\delta} = -\frac{\sqrt{\varepsilon}}{\delta} p_{\tilde{x}} \;\Rightarrow\; u_{\tilde{t}} + \varepsilon(u u_{\tilde{x}} + w u_z) = -p_{\tilde{x}}. \tag{3.46}$$

$$\frac{Dw}{Dt} = \frac{\varepsilon}{\delta^2}\frac{D\tilde{w}}{D\tilde{t}} = -p_z \;\Rightarrow\; \varepsilon\left(\tilde{w}_{\tilde{t}} + \varepsilon\left(u\tilde{w}_{\tilde{x}} + \tilde{w}\tilde{w}_z\right)\right) = -p_z, \tag{3.47}$$

with

$$w = \frac{\varepsilon}{\delta}\tilde{w} = \frac{\sqrt{\varepsilon}}{\delta}\eta_{\tilde{t}} + \varepsilon u \frac{\sqrt{\varepsilon}}{\delta}\eta_{\tilde{x}} \tag{3.48}$$

$$\Rightarrow \left.\begin{array}{l} \tilde{w} = \eta_{\tilde{t}} + \varepsilon u \eta_{\tilde{x}} \\ p = \eta \end{array}\right\} \text{ on } \; z = 1 + \varepsilon\eta \tag{3.49}$$

and

$$w = 0 \quad \text{on} \;\; z = b = 0. \tag{3.50}$$

Now we replace the region δ^2 with $\varepsilon = \delta^2$, while we let ε go to 0. We conclude to the following equations

$$\left.\begin{aligned} &u_t = -p_x, \quad p_z = 0 \\ &u_x + w_z = 0, \\ &\text{with} \\ &w = \eta_t \quad p = \eta \quad \text{on} \;\; z = 1 \\ &w = 0 \quad \text{on} \;\; z = 0. \end{aligned}\right\} \tag{3.51}$$

Modification to these equations on the boundary condition, i.e. on $z = 1$ leaves us with

$$u = -p_x = -\eta_x \quad \Rightarrow \quad u_t + \eta_x = 0 \tag{3.52}$$

$$w = -zu_x\Big|_{z=1} = -u_x = \eta_t \quad \Rightarrow \quad u_x + \eta_t = 0. \tag{3.53}$$

By doing differentiation 3.52 with respect to x and subtracting the equation 3.53 differentiated with respect tot t we get the standard wave equation for the profile of the wave

$$\eta_{xx} - \eta_{tt} = 0. \tag{3.54}$$

We choose a solution for a right going wave and go into the frame of the moving wave by a coordinate transformation as in the last section to $\xi = x - t$. Additionally we want to introduce a long term variable, since we have a uniformity as t (or x) goes to infinity. This is usually done by rescaling $t = \varepsilon\tau$. In summary we have that $\xi = O(1)$ as well as $\tau = O(1)$. This is for **far field region** of the wave, and therefore the region, where we expect KdV type balance, between dispersion and linearity. The fact of this matter can be rigorously proven, it needs to be show that any sufficiently fast decaying smooth solution will eventually split into a finite superposition of two solitary waves traveling to the right and a decaying dispersive part traveling to the left. However will not go into this here. To transform the equations in 3.51, we look at the chain rule w.r.t ξ, τ evolving to

$$\frac{\partial}{\partial t} = -\frac{\partial}{\partial \xi} + \varepsilon \frac{\partial}{\partial \tau} \tag{3.55}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}. \tag{3.56}$$

Then we get

$$\left.\begin{aligned} &-u_\xi + \varepsilon\left(u_\tau + uu_\xi + wu_z\right) = -p_\xi \\ &\varepsilon\left(-w_\xi + \varepsilon\left(w_\tau + uw_\xi + ww_z\right)\right) = -p_z \\ &u_\xi + w_z = 0 \\ &\text{with} \\ &\left.\begin{aligned} &w = -\eta_\xi + \varepsilon(\eta_\tau + u\eta_\xi) \\ &p = \eta \end{aligned}\right\} \text{on} \;\; z = 1 + \varepsilon\eta \\ &\text{and} \\ &w = 0 \qquad z = b = 0. \end{aligned}\right\} \tag{3.57}$$

The crucial part now is to consider an asymptotic expansion of in ε for velocity of the fluid particles u, w and also the wave profile η and for the pressure variable p. The general asymptotic ansatz is of the form

$$q\left(\xi, \tau, z; \varepsilon\right) = \sum_{n=0}^{\infty} \varepsilon^n q_n\left(\xi, \tau, z\right). \tag{3.58}$$

The first equation in 3.57 up to the order of $O(\varepsilon^2)$ is of the form

$$\varepsilon^0\left(p_{0\xi} - u_{0\xi}\right) + \varepsilon^1\left(p_{1\xi} - u_{1\xi} + u_{0\tau} + u_0 u_{0\xi} + w_0 u_{0z}\right) + O(\varepsilon^2) = 0, \tag{3.59}$$

with the main condition $p_{0\xi} = u_{0\xi}$. For the second equation in 3.57 becomes

$$\varepsilon^0 \left(p_{0z}\right) + \varepsilon^1 \left(p_{1z} - w_{0\xi} + w_{0\tau} + u_0 w_{0\xi} + w_0 w_{0z}\right) + O(\varepsilon^2) = 0, \tag{3.60}$$

the main condition $p_{0z} = 0$. The third equation in 3.57 is the following

$$\varepsilon^0 (u_{0\xi} + w_{0z}) + \varepsilon^1 \left(u_{1\xi} + w_{1z}\right) O(\varepsilon^2) = 0, \tag{3.61}$$

where the main condition satisfies $u_{n\xi} = -w_{n\xi}$ for all $n \in \mathbb{N}$. Further the surface condition is expanded into

$$p_n = \eta_n \qquad \forall\, n \in \mathbb{N}, \tag{3.62}$$

and

$$\varepsilon^0 \left(w_0 + \eta_{0\xi}\right) + \varepsilon^1 \left(w_1 + \eta_{1\xi} + \eta_{0\tau} + \eta_0 \eta_{0\xi}\right) + O(\varepsilon^2) = 0, \tag{3.63}$$

Do note that the condition for for ε^0 is $z = 1$ and for ε^1 is $z = \varepsilon\eta$. The main conclusion from the above equation is however $w_0 = -\eta_{0\xi}$. And lastly the bottom condition remains unchanged for all n as

$$w_n = 0 \quad \text{on} \quad z = b = 0 \tag{3.64}$$

In essence $O(\varepsilon^0)$ give us the equations

$$u_{0\xi} = p_{0\xi}, \quad p_{0z} = 0, \quad u_{0\xi} + w_{0z} = 0, \tag{3.65}$$

with

$$p_0 = \eta_0, \quad w_0 = -\eta_{0\xi} \quad \text{on} \quad z = 1 \tag{3.66}$$

$$w_0 = 0 \quad \text{on} \quad z = b = 0. \tag{3.67}$$

They lead us tot he following solution which satisfies the boundary

$$p_0 = \eta_0, \quad u_0 = \eta_0, \quad w_0 = -z\eta_{0\xi} \quad \forall\, z \in [0, 1]. \tag{3.68}$$

Do notice who $w_0 = -z\eta_{0\xi}$ automatically satisfies the boundary conditions for both $z = 0$ and $z = 1$. Before we go on to consider $O(\varepsilon)$, we expand u, w and p around $z = 1$ via Taylor expansion. This makes only since in the case $\varepsilon \to 0$

$$\left.\begin{aligned} &p_0 + \varepsilon\eta_0 p_{0z} + \varepsilon p_1 = \eta_0 \varepsilon \eta_1 + O(\varepsilon^2) \\ &w_0 + \varepsilon\eta_0 w_{0z} + \varepsilon w_1 = -\eta_{0\xi} - \varepsilon\eta_{1\xi} + \varepsilon\left(\eta_0 + u_0\eta_{0\xi}\right) + O\left(\varepsilon^2\right). \end{aligned}\right\} \text{on} \quad z = 1 \tag{3.69}$$

Right off the equations of order $O(\varepsilon^1)$ become

$$-u_{1\xi} + u_{0\tau} + u_0 u_{0\xi} + w_0 u_{0z} = -p_{1\xi}, \tag{3.70}$$

$$p_{1z} = w_{0\xi} \quad \text{and} \quad u_{1\xi} + w_{1z} = 0, \tag{3.71}$$

with the boundary conditions

$$\left.\begin{aligned} &p_1 + \eta_0 p_{0z} = \eta_1 \\ &w_1 + \eta_0 w_{0z} = -\eta_{1\xi} + \eta_{0t} + u \end{aligned}\right\} \text{on} \quad z = 1 \tag{3.72}$$

$$w_1 = 0 \quad \text{on} \quad z = b = 0. \tag{3.73}$$

Thus

$$p_{1z} = w_{0\xi} = -z\eta_{0\xi} \tag{3.74}$$

$$\Rightarrow p_1 = -\frac{1}{2} z^2 \eta_{0\xi\xi} + c, \tag{3.75}$$

where c is the integration constant, together with the boundary condition on $z = 1$ we get that

$$c = \eta_1 + \frac{1}{2}\eta_{0\xi\xi}, \tag{3.76}$$

for p_1 leaving is with

$$p_1 = \frac{1}{2}\left(1 - z^2\right)\eta_{0\xi\xi} + \eta_1 \tag{3.77}$$

As for the condition $w_{1z} = -u_{1\xi}$ we get

$$w_{1z} = -u_{1\xi} = -p_{1\xi} - u_{0\tau} - u_0 u_{0\xi} - u_0 u_{0z} \tag{3.78}$$

$$= \frac{1}{2}(1 - z^2)\eta_{0\xi\xi\xi} - \eta_{1\xi} - \eta_{0\tau} - \eta_{0\xi}. \tag{3.79}$$

By integration and evaluation on $z = 1$ of the above equation we get

$$w_1\Big|_{z=1} = -\frac{1}{3}\eta_{0\xi\xi\xi} - \eta_{1\xi} - \eta_{0\tau} - \eta_0\eta_{0\xi}, \tag{3.80}$$

on the other hand we have the original boundary condition

$$w_1\Big|_{z=1} = -\eta_{1\xi} + \eta_{0\tau} + 2\eta_0\eta_{0\xi}. \tag{3.81}$$

Subtracting equation 3.80 from 3.81 we get the KdV equation

$$\frac{1}{3}\eta_{0\xi\xi\xi} - 2\eta_{0\tau} - 3\eta_0\eta_{0\xi} = 0. \tag{3.82}$$

A Appendix: Mathematical Preliminaries

A.1 Leibniz Rule of Integration

The Leibniz integral rule for differentiation under the integral sign initiates with an integral

$$\mathcal{I}(t,x) = \int_{a(t)}^{b(t)} f(t,x)dx = \mathcal{I}(t, a(t, a(t), b(t))). \tag{A.1}$$

And upon differentiation w.r.t. t, utilizes the chain rule on $a(t)$ and $b(t)$ respectively, by

$$\frac{d\mathcal{I}}{dt} = \frac{\partial \mathcal{I}}{\partial t} + \frac{\partial \mathcal{I}}{\partial a}\frac{\partial a}{\partial t} + \frac{\partial \mathcal{I}}{\partial b}\frac{\partial b}{\partial t}. \tag{A.2}$$

Which in integral representation reads

$$\frac{d\mathcal{I}}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(t,x)}{\partial t} dx + f(t, b(t))\frac{\partial b(t)}{\partial t} - f(t, a(t))\frac{\partial a(t)}{\partial t} \tag{A.3}$$

A.2 Gaussian Integration Law

This should explain the Gaussian integration law

A.3 Identity for Vorticity

We start off with the standard material derivative

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t}(\mathbf{u}\nabla)\mathbf{u}. \tag{A.4}$$

We will use Einstein's Summation Convention, where we sum over indices that both appear at as the bottom as the top index, to rewrite the second part of the material derivative $(\mathbf{u}\nabla)\mathbf{u}$ into

$$\begin{aligned}(\mathbf{u} \times (\nabla \times \mathbf{u}))_k &= \varepsilon^{ijk} u_j (\nabla \times \mathbf{u})_k & \text{(A.5)}\\ &= \varepsilon^{ijk} u_j \varepsilon_{klm} \partial^l u^m & \text{(A.6)}\\ &= (\delta^i_l \delta^j_m - \delta^i_m \delta^j_l) u_j \partial^l u^m & \text{(A.7)}\\ &= u_m \partial^i u^m - u_l \partial^l u^i. & \text{(A.8)}\end{aligned}$$

Now the first part in equation A.8 can be rewritten into

$$u_m \partial^i u^m = \partial^i (\frac{1}{2} u_m u^m). \tag{A.9}$$

Thus we get

$$(\mathbf{u} \times (\nabla \times \mathbf{u}))_k = \frac{1}{2}\partial^i (u_m u^m) + u_l \partial^l u^i, \tag{A.10}$$

which is

$$(\mathbf{u}\nabla)\mathbf{u} = \nabla(\frac{1}{2}\mathbf{u}\mathbf{u}) - (\mathbf{u} \times (\nabla \times \mathbf{u})) \tag{A.11}$$

A.4 Middle Curvature of an Implicit Function

In our case the implicit function for fixed time reads

$$z - h(x_1, x_2) = 0. \tag{A.12}$$

The parametric representation is

$$\sigma = \begin{pmatrix} x_1 \\ x_2 \\ h \end{pmatrix}. \tag{A.13}$$

The middle curvature of the surface parametrized by σ is

$$\frac{1}{R} = \mathrm{Tr}(G^{-1}B), \tag{A.14}$$

where G and B are given by

$$G_{ij} = \frac{\partial \sigma}{\partial x_i}\frac{\partial \sigma}{\partial x_j}, \tag{A.15}$$

$$B_{ij} = -\mathbf{N}\frac{\partial^2 \sigma}{\partial x_i \partial x_j}, \tag{A.16}$$

where $i, j = 1, 2$ and $\mathbf{N}$ is the normal, normalized surface vector given by

$$\mathbf{N} = \frac{\frac{\partial \sigma}{\partial x_1} \times \frac{\partial \sigma}{\partial x_2}}{\| \frac{\partial \sigma}{\partial x_1} \times \frac{\partial \sigma}{\partial x_2} \|} \tag{A.17}$$

$$= \frac{1}{\sqrt{h_x^2 + h_y^2 + 1}} \begin{pmatrix} -h_x \\ -h_y \\ 1 \end{pmatrix}. \tag{A.18}$$

Thereby the matrices B and G are calculated to be

$$G = \begin{pmatrix} 1 + h_x^2 & h_x h_y \\ h_x h_y & 1 + h_y^2 \end{pmatrix} \qquad B = \frac{1}{\sqrt{h_x^2 + h_y^2 + 1}} \begin{pmatrix} h_{xx} & h_{yx} \\ h_{xy} & h_{yy} \end{pmatrix}. \tag{A.19}$$

The inverse of G is

$$G^{-1} = \frac{1}{\det(G)} \mathrm{adj}(G) \tag{A.20}$$

$$= \frac{1}{h_x^2 + h_y^2 + 1} \begin{pmatrix} 1 + h_y^2 & -h_x h_y \\ -h_x h_y & 1 + h_x^2 \end{pmatrix}. \tag{A.21}$$

Hence the middle curvature is given by the follwing

$$\frac{1}{R} = \mathrm{Tr}(G^{-1}B) \tag{A.22}$$

$$= \frac{1}{(h_x^2 + h_y^2 + 1)^{\frac{3}{2}}} \mathrm{Tr} \begin{pmatrix} (1 + h_y)^2 h_{xx} - h_x h_y h_{xy} & * \\ * & (1 + h_x^2) h_{yy} - h_x h_y h_{xy} \end{pmatrix} \tag{A.23}$$

$$= \frac{(1 + h_y^2) h_{xx} + (1 + h_y^2) h_{yy} - 2 h_x h_y h_{xy}}{\left(h_x^2 + h_y^2 + 1\right)^{\frac{3}{2}}}. \tag{A.24}$$

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