University of Vienna
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Numerical Analysis Problems

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1 Sheet 4

1.1 Problem 1

Consider a linear system of equations $Ax = b$, where

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad b = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \tag{1}$$

we carry out iterations of the CG method by hand until we reach the solution with an initial guess $x_0 = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T$. For the sake of completeness the CG method has the following iteration at the k-th step

$$\alpha_k = \frac{r_k^T r_k}{p_k^T A p_k} \tag{2}$$

$$x_{k+1} = x_k + \alpha_k p_k \tag{3}$$

$$r_{k+1} = r_k - \alpha_k A p_k \tag{4}$$

$$\beta_k = \frac{r_{k+1}^T r_{k+1}}{r_k^T r_k} \tag{5}$$

$$p_{k+1} = r_{k+1} + \beta_k p_k \tag{6}$$

$$\tag{7}$$

For $k = 0$ we have

$$r_0 = b - Ax_0 = b, \tag{8}$$

$$p_0 = r_0 = b = \begin{pmatrix} 4 & 0 & 0 \end{pmatrix}^T. \tag{9}$$

For k=1 we have

$$\alpha_0 = \frac{1}{2}, \quad x_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \quad r_1 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \tag{10}$$

$$\beta_0 = \frac{1}{4}, \quad p_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}. \tag{11}$$

For k=2 we have

$$\alpha_1 = \frac{2}{3}, \quad x_2 = \frac{1}{3}\begin{pmatrix} 8 \\ 4 \\ 0 \end{pmatrix}, \quad r_2 = \frac{1}{3}\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}, \tag{12}$$

$$\beta_1 = \frac{4}{9}, \quad p_2 = \frac{1}{9}\begin{pmatrix} 4 \\ 8 \\ 12 \end{pmatrix}. \tag{13}$$

For k=3 we have

$$\alpha_2 = \frac{3}{4}, \quad x_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad r_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{14}$$

$$\beta_2 = 0, \quad p_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{15}$$

Since $r_3 = \mathbf{0}$ we can stop here, and $x_3 = x$ is the unique solution. The Krylov space of $\mathcal{K}_k(A, b)$ is defined for $k = 3$ as

$$\mathcal{K}_3(A, b) = \operatorname{span}\left\{b, Ab, A^2 b\right\} = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 8 \\ -4 \\ 0 \end{pmatrix}, \begin{pmatrix} 26 \\ -16 \\ 4 \end{pmatrix}\right\} \tag{16}$$

the rank of the span of $\mathcal{K}_k(A, b)$ is full thereby the $\dim(\mathcal{K}_k(A, b)) = 3$. Furthermore the residuals $r_0, \ldots, r_{k-1}$ form an orthogonal basis for $\mathcal{K}_k(A, b)$. This can be verified by checking that 'key' elements in $\mathcal{K}_k(A, b)$ can be expressed as a linear combination of r_0, r_1, r_2.

$$b = 3 \cdot r_2, \quad Ab = 2r_0 - 2r_1, \quad A^2 b = 6r_0 - 8r_1 + 3r_2. \tag{17}$$

1.2 Exercise 3, 4

Not important see notes is not easy