University of Vienna
Lukas Liehr

Summer semester 2022

Introductory Seminar Advanced Numerical Analysis

Exercise sheet 2, due date: 14.03.2022

Exercise 1 (Convergence of the Jacobi method).
Let $\rho(A)$ denote spectral radius of $A \in \mathbb{R}^{n\times n}$. We say that a matrix norm $\|\cdot\|_1$ is consistent with the vector norm $\|\cdot\|_2$ if

$$\|Ax\|_2 \leq \|A\|_1 \|x\|_2$$

for every $x \in \mathbb{R}^n$ and every $A \in \mathbb{R}^{n\times n}$.

(1) Show that every matrix norm which is induced by a vector norm is consistent.

(2) Consider the splitting $A = D-(E+F)$ where D is the diagonal matrix of the diagonal entries of A, E is the lower triangual matrix of entries $e_{ij} = -a_{ij}$ if $i > j, e_{ij} = 0$ if $i \leq j$ and F is the upper triangular matrix of entries $f_{ij} = -a_{ij}$ if $j > i, f_{ij} = 0$ if $j \leq i$. Let $B_J = D^{-1}(E+F)$ be the iteration matrix of the Jacobi method. Show that if A is strictly diagonally dominant then

$$\rho(B_J) \leq \|B_J\|_\infty < 1.$$

(3) Conclude that the Jacobi method converges for every initial guess x^0 to the solution of the equation $Ax = b$ provided that A is strictly diagonally dominant. (Hint: show that the error $e^k = x - x^k$ converges to zero)

Exercise 2 (Jacobi and Gauss-Seidel method).
Consider a 2×2-matrix $A \in \mathbb{R}^{2\times 2}$,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Let $A = D - (E + F)$ be the splitting of A as considered in Exercise 1(2). Let $B_J = D^{-1}(E + F)$ and $B_G = (D - E)^{-1}F$ be the iteration matrix of the Jacobi method and the Gauss-Seidel method, respectively.

(1) Show that the spectral radius of B_J and B_G satisfies $\rho(B_J) = \sqrt{|\rho(B_G)|}$.

(2) Deduce that the Jacobi-method for A converges if and only if the Gauss-Seidel method for A converges.

(3) Let $r \in \mathbb{R}$ and

$$A_r = \begin{pmatrix} 1 & r & r \\ r & 1 & r \\ r & r & 1 \end{pmatrix}.$$

Show that the Gauss-Seidel method for A_r converges provided that $r \in (-\frac{1}{2}, 1)$. Show that the Jacobi-matrix for A_r does not converge if $r \in (\frac{1}{2}, 1)$.

Exercise 3 (Poisson matrix).
Let $Q \in \mathbb{R}^{n\times n}$ be the banded matrix

$$Q = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}.$$

(1) Show that all eigenvalues of Q lie in the interval $[0,4]$.
(2) Write a Python script which takes $n, m \in \mathbb{N}$ as an input and returns the matrix Q.
(3) Let $b = (1,1,\ldots,1)^T \in \mathbb{R}^n$ and $n = 20$. Implement the Gauss-Seidel method and apply 200 iteration for approximating the solution of $Ax = b$.

Exercise 4 (Eigenvalues).
Let $P_n \in \mathbb{R}^{n\times n}$ be the matrix

$$P_n = \begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{pmatrix}.$$

(1) Show that all eigenvalues of P_n lie in the interval $[0,4]$.
(2) Let $z := e^{\frac{2\pi i}{n}} \in \mathbb{C}$ ($i \in \mathbb{C}$ the imaginary unit) and define for $j, k \in \{1,\ldots,n\}$ the values

$$v_k^j := z^{jk} = e^{\frac{2\pi ijk}{n}}.$$

Let $v^j = (v_1^j, \ldots, v_n^j) \in \mathbb{C}^n$. Determine $\lambda(j)$ such that for every k-th component $(Av^j)_k$ of the matrix-vector product Av^j one has

$$(Av^j)_k = \lambda(j) v_k^j.$$

(3) Conclude that $\lambda(j)$ is an eigenvalue of A with eigenvector $\mathrm{Re}(v^j)$.
(4) Define the quantity $m(n) = \min\{|\lambda| : \lambda \text{ eigenvalue of } P_n\}$. Show that

$$\lim_{n\to\infty} m(n) = 0.$$

Exercise 5 (Neumann polynomial preconditioner).
Let $n \in \mathbb{N}$ and let the matrix $Q \in \mathbb{R}^{n\times n}$ be defined as in Exercise 3. Consider the splitting

$$Q = D - N$$

of Q where D is the matrix consisting of the diagonal entries of Q. Further, let $p \in \mathbb{N}_0$ and C_p be the matrix

$$C_p = D^{-1} \sum_{k=0}^{p} (ND^{-1})^k.$$

(1) Write a Python script which takes n, p as an input and returns the matrix C_p.
(2) Write a Python script which provides a table of two columns where in the first column we have the values $p = 1, \ldots, 10$ and in the second column the corresponding spectral condition number of the matrix C_pQ. What do you observe?