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Applied Analysis Problems

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1 Sheet 7

1.1 Dirac Comb

The Dirac train or Dirac comb on defined in the following way

$$\text{III}_m[n] = \begin{cases} 1 & n = 0, \pm m, \pm 2m, \dots \\ 0 & \text{else} \end{cases} \tag{1}$$

The discrete Fourier transform of the Dirac comb in $\mathbb{C}^N$ is

$$\begin{aligned}
\widehat{\text{III}_m[n]} &= \frac{1}{N}\sum_{n=0}^{N-1} \text{III}_m[n] e^{-2\pi i \frac{k}{N} n} = && (2)\\
&= \frac{1}{N}\sum_{n=0}^{N-1}\left(m \sum_{l\in\mathbb{Z}} \delta(n - lm)\right) e^{-2\pi i \frac{k}{N} n} = && (3)\\
&= \frac{m}{N}\sum_{l\in\mathbb{Z}} e^{-2\pi i \frac{k}{N} lm} = \qquad (m = \frac{N}{m'}) && (4)\\
&= \frac{1}{m'}\sum_{l\in\mathbb{Z}} e^{-2\pi i \frac{k}{m'} l} = && (5)\\
&= \frac{1}{m}\text{III}_{\frac{N}{m}}[k] && (6)
\end{aligned}$$

1.2 Schwartz Space

The Schwartz space $\mathcal{S}(\mathbb{R}^d)$, for $d \in \mathbb{N}$ is defined as

$$\mathcal{S} := \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) : \forall \alpha, \beta \in \mathbb{N}^d \;\; \|f\|_{\alpha,\beta} < \infty \right\}, \tag{7}$$

$$\|f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} \left| x^\alpha (D^\beta f)(x) \right|. \tag{8}$$

Our aim is to show that if $f \in \mathcal{S}(\mathbb{R})$ then $\hat{f} \in \mathcal{S}(\mathbb{R})$. The condition is obviously

$$\|\hat{f}\|_{\alpha,\beta} = \sup_{\xi \in \mathbb{R}} \left| \xi^\alpha (D^\beta \hat{f})(\xi) \right| < \infty, \tag{9}$$

for all $\alpha, \beta \in \mathbb{N}$. We can start with what we know about the Fourier transform

$$\xi^\alpha \hat{f}(\xi) = \mathcal{F}\left(\frac{1}{(2\pi i)^\alpha} (D^\alpha f)(x) \right) \tag{10}$$

$$D^\beta \hat{f}(\xi) = \mathcal{F}\left((-2\pi i x)^\beta f(x) \right). \tag{11}$$

Combining the two relations above we get

$$\xi^\alpha (D^\beta \hat{f})(\xi) = \mathcal{F}\left(\frac{(-2\pi i x)^\beta}{(2\pi i)^\alpha} x^\beta (D^\alpha f)(x) \right) =: \mathcal{F}(g(x)) \tag{12}$$

$$\tag{13}$$

If we call this function g, then $g \in \mathcal{S}(\mathbb{R})$ and $g \in L^1(\mathbb{R})$. Applying the Riemann-Lebesgue Lemma we get

$$\hat{g}(\xi) = \int_{\mathbb{R}} g(x) e^{-2\pi i x \xi} \, dx \longrightarrow 0 \quad \text{as } |\xi| \to \infty \tag{14}$$

Thereby $\hat{g} \in \mathcal{S}(\mathbb{R})$ and thus $\hat{f} \in \mathcal{S}(\mathbb{R})$.

1.3 Tempered Distributions

Tempered distributions are the elements of

$$\mathcal{S}'(\mathbb{R}^d) := \left\{ L : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C} | L \text{ is linear and continuous} \right\}. \tag{15}$$

Consider ξ as a tempered distribution, buy acting on $\varphi \in \mathcal{S}(\mathbb{R})$ we have

$$\xi(\phi) = \int_\xi \xi \varphi(\xi) \, d\xi. \tag{16}$$

The Fourier transform of ξ is

$$\begin{aligned}
\hat{\xi}(\varphi) = \xi(\hat{\varphi}) &= \int_{\mathbb{R}} \xi \hat{\varphi}(\xi) \, d\xi = && (17)\\
&= \int_{\mathbb{R}} \xi \int_{\mathbb{R}} \varphi(x) e^{-2\pi i \xi x} \, dx \, d\xi = && (18)\\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \xi e^{-2\pi i \xi x} \, d\xi \varphi(x) \, dx = && (19)\\
&= \int_{\mathbb{R}} \frac{i}{2\pi} \frac{d}{dx} \int_{\mathbb{R}} e^{-2\pi i \xi x} \, d\xi \varphi(x) \, dx = && (20)\\
&= \int_{\mathbb{R}} i \frac{1}{2\pi} \frac{d}{dx} \left(2\pi \delta(x) \right) \varphi(x) \, dx = && (21)\\
&= \int_{\mathbb{R}} i \delta'(x) \varphi(x) \, dx && (22)\\
&= i\delta'(\varphi). && (23)
\end{aligned}$$

1.4 Fourier transform of the Dirac Comb

The general case of the Dirac Comb as a distribution is

$$\text{III}_T = \sum_{n \in \mathbb{Z}} \delta_{nT}. \tag{24}$$

The Fourier transform of the III_T distribution for $\varphi \in \mathcal{S}(\mathbb{R})$ is

$$\widehat{\text{III}_T}(\varphi) = \sum_{n\in\mathbb{Z}} \hat{\delta}_{nT}(\varphi) \tag{25}$$
$$= \sum_{n\in\mathbb{Z}} \delta_{n\omega_0}(\varphi) \tag{26}$$
$$= \text{III}_{\omega_0}(\varphi). \tag{27}$$

The Fourier transform, transforms the period of the combs.

1.5 Shannon Sampling

The Fourier transform of $1_{[-\frac{a}{2},\frac{a}{2}]}(x)$ is

$$\mathcal{F}\left(1_{[-\frac{a}{2},\frac{a}{2}]}\right)(\xi) = \int_{\mathbb{R}} 1_{[-\frac{a}{2},\frac{a}{2}]} e^{-2\pi i x\xi}\, dx \tag{28}$$
$$= \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-2\pi i x\xi}\, dx \tag{29}$$
$$= \frac{-1}{2\pi i\xi} e^{-2\pi i x\xi}\Big|_{-\frac{a}{2}}^{\frac{a}{2}} \tag{30}$$
$$= \frac{1}{\pi\xi}\frac{1}{2i}\left(e^{piia\xi} - e^{-\pi i a\xi}\right) \tag{31}$$
$$= \frac{\sin(\pi\xi a)}{\pi\xi} \tag{32}$$