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Applied Analysis Problems

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1 Sheet 4

1.1 Fourier Series

The Fourier series of a p periodic function f, integrable on $[-\frac{p}{2}, \frac{p}{2}]$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(\frac{2\pi n x}{p}) b_n sin(\frac{2\pi n x}{p}) \right). \tag{1}$$

The coefficients a_n and b_n are called the Fourier coefficients of f and are given by

$$a_n = \frac{2}{p} \int_{-\frac{p}{2}}^{\frac{p}{2}} f(x) \sin(\frac{2\pi n x}{p}) dx, \qquad n \geq 0 \tag{2}$$

$$b_n = \frac{2}{p} \int_{-\frac{p}{2}}^{\frac{p}{2}} f(x) \cos(\frac{2\pi n x}{p}) dx, \qquad n \geq 1 \tag{3}$$

Let us compute the Fourier series of $f(t) = t$ for $t \in [-\frac{1}{2}, \frac{1}{2}]$. The Fourier coefficients are

$$a_n = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} t \cos(2\pi n t)\; dt = 0 \quad \text{(odd: g(-t) = -g(t))}, \tag{4}$$

$$\begin{aligned} b_n &= 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} t \sin(2\pi n t)\; dt = && (5)\\ &= 2 \left(-\frac{1}{2\pi n} \cos(2\pi n t) \Big|_{-\frac{1}{2}}^{\frac{1}{2}} + \int_{\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2\pi n} \cos(2\pi n t)\; dt \right) = && (6)\\ &= -\frac{1}{\pi n} \left(-\cos(\pi n) + \frac{1}{\pi n} \sin(\pi n) \right) = \frac{\sin(\pi n) - \pi n \cos(\pi n)}{(\pi n)^2}. && (7) \end{aligned}$$

Thereby the Fourier series of $f(t) = t$ is

$$f(t) = \sum_{n=1}^{\infty} \left(\frac{\sin(\pi n) - \pi n \cos(\pi n)}{(\pi n)^2} \right) \sin(2\pi n t) = t \tag{8}$$

1.2 Truncation Error

The truncation error of the trigonometric polynomial (Sf_N) of degree N is

$$\sum_{|k|>N} |\hat{f}(k)|^2 = \|f - S_N\|_2^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |E_N(t)|^2 dt. \tag{9}$$

Computations for $N = 3$ and $N = 9$ were done in python with a integration error of around 10^{-15}, resulting in the overall truncation errors of

$$\sum_{|k|>3} |\hat{f}(k)|^2 = 0.0053, \tag{10}$$

$$\sum_{|k|>9} |\hat{f}(k)|^2 = 0.0143. \tag{11}$$

To achieve $\|E_N\|_2^2 < 0.1\|f\|_2^2$, the number of coefficients needed are about 61. This was done using a while loop and evaluating $\|E_N\|_2^2$ for N until the above condition is met.

1.3 Orthonormal Bases

Here we will go through the most important properties of orthonormal bases. So let $\{b_n\}_{n\in\mathbb{N}}$ be an ONB of a vector space $\mathcal{H}$, then for every $x \in \mathcal{H}$ we may write

$$x = \sum_{b_n} \langle b_n, x\rangle b_n, \tag{12}$$

and

$$\|x\|^2 = \sum_{b_n} |\langle b_n, x\rangle|^2. \tag{13}$$

For any $x, y \in \mathcal{H}$ we can write the scalar product as

$$\langle x, y\rangle = \sum_{b_n} \langle b_n, x\rangle \langle b_n, y\rangle, \tag{14}$$

Furthermore there exists a linear projection $\Phi\ : \mathcal{H} \to l^2(\{b_n\}_n)$ such that

$$\langle \Phi(x), \Phi(y)\rangle = \langle x, y\rangle \quad \forall x, y \in \mathcal{H}. \tag{15}$$

An example of an orthonormal basis, which spans $L^2([-\frac{p}{2}, \frac{p}{2}])$ is $\mathcal{T}_p = \{e_n := \frac{e^{2\pi i \frac{n}{p} x}}{\sqrt{p}}\}_{n\in\mathbb{Z}}$. The e_n's are orthonormal in L^2 which can be easily seen by using the scalar product of L^2, so for $n, m \in \mathbb{Z}$

$$\langle e_n, e_m\rangle_{L^2([-\frac{p}{2},\frac{p}{2}])} = \frac{1}{p} \int_{[-\frac{p}{2},\frac{p}{2}]} e_n \cdot e_m^*\ dx = \tag{16}$$

$$= \frac{1}{p} \int_{[-\frac{p}{2},\frac{p}{2}]} e^{2\pi i \frac{(n-m)}{p} x}\ dx = \tag{17}$$

$$= \frac{\sin(\pi(n-m))}{\pi(n-m)} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases} \tag{18}$$

1.4 Dirichlet Kernel

The function

$$D_t(x) := \sum_{\|k\|_\infty \le t} e_k(x), \qquad x \in \mathbb{R}^d \tag{19}$$

is called the Dirichlet Kernel. For $0 < t \in \mathbb{N}$ we have

$$(S_t f)(x) = \int_{I^d} f(y) D_t(x-y) dy, \tag{20}$$

where S_t represents the orthogonal projection onto the trigonometric polynomials Π_t of degree t, by

$$S_t : \ L^1(\mathbb{T}^d) \to \Pi_t \tag{21}$$

$$f \mapsto \sum_{\|k\| \le t} \langle f, e_k \rangle_{L^2(\mathbb{T}^d)} e_k \qquad k \in \mathbb{Z}^d \tag{22}$$

And furthermore the Dirichlet Kernel satisfies

$$D_t(x) = \prod_{i=1}^{d} \frac{e_{t+1}(x_i) - e_{-t}(x_i)}{e_1(x_i) - 1} \tag{23}$$

To show the convolution property, we start off by applying the orthogonal projection into the trigonometric polynomials S_t onto a function $f \in L(\mathbb{T}^d)$

$$(S_t f) = \sum_{|k\|_\infty \le t} \int_{I^d} f(y) e^{-2\pi i \langle k, y \rangle} \ dy \ e^{2\pi i \langle k, x \rangle} = \tag{24}$$

$$= \int_{I^d} f(y) \sum_{|k\|_\infty \le t} e^{2\pi i \langle k, (x-y) \rangle} \ dy = \tag{25}$$

$$= (f * D_t)(x) = \int_{I^d} f(y) D_t(x-y) \ dy. \tag{26}$$

To show the reformulation of the Dirichlet kernel, we need to simply calculate it directly

$$\sum_{\|k\|_\infty \le t} e^{2\pi i \langle k, x \rangle} = \prod_{j=1}^{d} \sum_{k_j = -t}^{t} e^{2\pi i k_j x_j} = \tag{27}$$

$$= \prod_{j=1}^{d} e^{-2\pi i t x_j} \sum_{k_j = 0}^{2t} e^{2\pi i k_j x_j} =; \quad \text{(trigonometric series)} \tag{28}$$

$$= \prod_{j=1}^{d} e^{-2\pi i t x_j} \frac{e^{2\pi i (2t+1) x_j} - 1}{e^{2\pi i x_j} - 1} = \tag{29}$$

$$= \prod_{j=1} \frac{e_{t+1}(x_j) - e_{-t}(x_j)}{e_1(x_j) - 1}. \tag{30}$$