Applied Analysis Problems

Milutin Popović

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1.1 Problem 4

We consider a quadratic equation with two ways to perturb it by ε:

$$x^2 + 2\varepsilon x - 1 = 0, \tag{1}$$

$$\varepsilon x^2 + 2x - 1 = 0. \tag{2}$$

Equation 2 is singular, because the reduced problem ($\varepsilon \to 0$) has only one solution at $x = \frac{1}{2}$. While the reduced problem in 1 has two solutions for $x = \pm 1$, which is the case for this non reduced equation. Let us thereby calculate the asymptotic expansion of the regular case up to $O(\varepsilon^2)$, we take the ansatz for the asymptotic expansion

$$x_\varepsilon = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3). \tag{3}$$

By substituting x_ε into 1 and factoring out the orders of ε we get

$$\varepsilon^0(x_0^2 - 1) + \varepsilon^1(2x_0 + 2x_0 x_1) + \varepsilon^2(x_1^2 + 2x_2 x_0 + 2x_1) + O(\varepsilon^3) = 0 \tag{4}$$

By solving the equations in order of ε, for the coefficients x_0, x_1 and x_2 we get

$$x_0 = \pm 1, \quad x_1 = -1, \quad x_2 = \pm\frac{1}{2}. \tag{5}$$

By substituting into the equation 3 we get

$$x_\varepsilon = \pm 1 - \varepsilon \pm \frac{1}{2}\varepsilon + O(\varepsilon^3). \tag{6}$$

For $\varepsilon = 0.001$ we get

$$x_\varepsilon = -1.0010005 + O(\varepsilon^3), \qquad x_\varepsilon = 0.9990005 + O(\varepsilon^3), \tag{7}$$
$$x_\varepsilon = -1.001 + O(\varepsilon^2), \qquad x_\varepsilon = 0.999 + O(\varepsilon^2). \tag{8}$$

1.2 Problem 5

Consider the following equations

$$\varepsilon y' + y = x \qquad y(0) = 1 \tag{9}$$
$$\varepsilon y' + y = x \qquad y(0) = 0 \tag{10}$$

$$\varepsilon y' + y = x \qquad y(0) = \varepsilon \tag{11}$$
$$\varepsilon^2 y' + y = x \qquad y(0) = \varepsilon \tag{12}$$

$$y' + \varepsilon y = x \qquad y(0) = 1 \tag{13}$$
$$y' + y = \varepsilon x \qquad y(0) = 1 \tag{14}$$

We will go through the equations and elaborate on if the perturbation is regular or singular, if regular we will compute the asymptotic expansion up to second order. Let us begin with equation 9. By the first look, the reduced problem does not agree with the boundary condition

$$y_0 = x \qquad y_0(0) = 1, \tag{15}$$

is a contradiction in $y_0(0) = 0 \neq 1$, thereby equation 9 is **singularly perturbed**.

The reduced problem of equation 10 on the other hand agrees with the boundary condition, since

$$y_0 = x \qquad y_0(0) = 0. \tag{16}$$

But by doing the ansatz for the asymptotic expansion

$$y_\varepsilon(x) = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + O(\varepsilon^3), \tag{17}$$

plugging in into 10 and separating coefficients in terms of ε, we get

$$\varepsilon^0(y_0 - x) + \varepsilon^1(y_0' + y_1) + \varepsilon^2(y_1' + y_2) + O(\varepsilon^3) = 0 \tag{18}$$

The solutions to these equations are

$$y_0 = x, \quad y_1 = 1, \quad y_2 = 0, \tag{19}$$

which is a contradiction to the boundary condition of $y_1(0) = 1 \neq 0$. Thereby we can conclude that equation 10 is **singularly perturbed**.

Next up is equation 11, where by the asymptotic expansion the first order coefficient of ε, y_2, has the boundary condition $y_2(0) = 0$. But by applying the ansatz of the asymptotic expansion and plugging into the equation we get

$$\varepsilon^2(y_0 - x) + \varepsilon^1(y_0' + y_1) + \varepsilon^2(y_1' + y_2) + O(\varepsilon^3) = 0. \tag{20}$$

Solving these equations we get

$$y_0 = 0, \quad y_1 = 1 \quad y_2 = 0, \tag{21}$$

which is a contradiction $y_1(0) = 1 \neq 0$, thus the equation 11 is **singularly perturbed**.

The next equation 12 is also singularly perturbed, we can see this by plugging the asymptotic expansion into the equation

$$\varepsilon^0(y_0 - x) + \varepsilon^1(y_1) + \varepsilon^2(y_0' + y_2) = O(\varepsilon^3), \tag{22}$$

solving for the coefficients we get

$$y_0 = x, \quad y_1 = 0, \quad y_2 = -1, \tag{23}$$

which is contradiction by the boundary condition $y_2(0) = -1 \neq 0$, thereby 12 is **singularly perturbed**.

Equation 13 on the first sight does not indicate for any contradictions, we may plug the ansatz of the asymptotic expansion into the equation and see what happens

$$\varepsilon^0(y_0 - x) + \varepsilon^1(y_1' + y_0) + \varepsilon^2(y_2' + y_1) + O(\varepsilon^2) = 0, \tag{24}$$

with the initial conditions $y_0(0) = 1$, $y_1(0) = y_2(0) = 0$.

$$y_0 = \frac{x^2}{2} + 1, \quad y_1 = -\frac{x^3}{6} + x, \quad y_2 = \frac{x^4}{24} + \frac{x^2}{2}. \tag{25}$$

Finally we get

$$y_\varepsilon(x) = (\frac{x^2}{2} + 1) + \varepsilon(-\frac{x^3}{6} - x) + \varepsilon^2(\frac{x^4}{24} + \frac{x^2}{2}) + O(\varepsilon^3). \tag{26}$$

Thereby we can conclude that 13 is **regularly perturbed**.

The last equation 14 is also regular, let us do the asymptotic expansion of the equation and order the equation in orders of ε.

$$\varepsilon^0(y_0' + y_0) + \varepsilon^1(y_1' + y_1 - x) + \varepsilon^2(y_2' + y_2) + O(\varepsilon^3) = 0. \tag{27}$$

by solving these differential equations with the boundary conditions $y_0(0) = 1$, $y_1(0) = y_2(0) = 0$ we get

$$y_0 = e^{-x} \quad y_1 = (x-1) + e^{-x} \quad y_2 = 0. \tag{28}$$

The equation we get

$$y_\varepsilon(x) = e^{-x} + \varepsilon(x - 1 + e^{-x}) + O(\varepsilon^3). \tag{29}$$

Thereby we can conclude that the last equation 14 is **regularly perturbed**.

1.3 Problem 6

In this section we will calculate the asymptotic expansion of a regularly perturbed equation in two ways, by doing the regular expansion ansatz and by substituting and expanding in terms of ε. The ordinary differential equation we are dealing with is

$$y' = -y + \varepsilon y^2 \quad y(0) = 1, \tag{30}$$

where $t > 0$ and $0 < \varepsilon \ll 1$. The standard expansion ansatz is

$$y_\varepsilon(x) = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + O(\varepsilon^3). \tag{31}$$

The ODE then expands to

$$\varepsilon^0(y_0' + y_0) + \varepsilon(y_1' + y_1 - y_0^2) + \varepsilon^2(y_2' + y_2 - 2y_0y_1) + O(\varepsilon^3) = 0. \tag{32}$$

Equations in order of ε and ε^2 are non-homogeneous ODE's. The solution to these three coefficients with the boundary conditions $y_0(0) = 1$, $y_1(0) = y_2(0) = 0$ we get

$$y_0 = e^{-x}, \quad y_1 = -e^{-2x} + e^{-x}, \quad y_2 = e^{-3x} - 2e^{-2x} + e^{-x}. \tag{33}$$

The expansion of y is then

$$y_\varepsilon(x) = e^{-x} + \varepsilon(-e^{-2x} + e^{-x}) + \varepsilon^2(e^{-3x} - 2e^{-2x} + e^{-x}) + O(\varepsilon^3). \tag{34}$$

The second ansatz, considers the substitution $z = \frac{1}{y}$, by calculating the first derivative and substituting the original problem we get

$$z' = \frac{-y'}{y^2} = \frac{y - \varepsilon y^2}{y^2} = \frac{1}{y} - \varepsilon = z - \varepsilon. \tag{35}$$

$$z(0) = \frac{1}{y(0)} = 1. \tag{36}$$

The solution is

$$z(x) = \varepsilon + (1-\varepsilon)e^{x}. \tag{37}$$

By substituting this into $y = \frac{1}{z}$ and expanding we get

$$\begin{aligned} y(x) &= \frac{1}{\varepsilon + (1-\varepsilon)e^{x}} = e^{-x}\frac{1}{1-(1-e^{-x})\varepsilon} && (38)\\ &= e^{-x}\sum_{n\geq 0}\varepsilon^{n}(1-e^{-x})^{n}. && (39) \end{aligned}$$

which is the geometric series.

1.4 Problem 7

The last problem consists of a perturbation of a partial differential equation (heat equation).

$$\begin{aligned} &\partial_t u(x,t) + \partial_x^2 u(x,t) - \varepsilon u(x,t)^2 = 0 && x \in (0,1),\ t>0, && (40)\\ &u(x,0) = \tilde{u}_0(x) && x \in (0,1), && (41)\\ &u(0,t) = u(1,t) = 0 && t > 0. && (42) \end{aligned}$$

The problem is regular because the reduced solution is the regular heat equation in the one special dimension on $x \in (0,1)$, we know this is solvable. By doing the expansion ansatz we can derive the first equations for the first three terms, the ansatz is always the same

$$u_\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^3). \tag{43}$$

Plugging this into the perturbed problem problem and factoring out the terms in the order of ε we get

$$\begin{aligned} &\varepsilon^0(\partial_t u_0 + \partial_x^2 u_0) + && (44)\\ &\varepsilon^1(\partial_t u_1 + \partial_x^2 u_1 - u_0^2) + && (45)\\ &\varepsilon^2(\partial_t u_2 + \partial_x^2 u_2 - 2u_1u_0) + O(\varepsilon^3) = 0. && (46) \end{aligned}$$

We can solve the reduced problem with the initial condition $\tilde{u}_0 = \sin(\pi x)$ by separation of variables. Setting $u(x,t) = \psi(x)\phi(t)$ and substituting into the equation we get two ordinary differential equation

$$\underbrace{\frac{\psi_{xx}}{\psi}}_{=k} + \underbrace{\frac{\phi_t}{\phi}}_{=-k} = 0, \tag{47}$$

for some k. Solving these two by the exponential ansatz.

$$\begin{aligned} \psi(x) &= A_1 e^{\sqrt{k}x} + A_2 e^{-\sqrt{k}x}, && (48)\\ \phi(t) &= A_3 e^{-kt}. && (49) \end{aligned}$$

With the initial condition we get the conditions that

$$\begin{aligned} A_1A_3 &= -A_2A_3, && (50)\\ k &= \pi^2 && (51) \end{aligned}$$

we choose $A_1 = A_3 = 1$, $A_2 = -1$. We get the following solution to the PDE

$$u(x,t) = \psi(x)\phi(t) = \sin(\pi x)e^{-\pi^2 t}. \tag{52}$$