Exercises for Applied Analysis; Part 5

Assignment 5; for 26th of January

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FDFT denotes the finite discrete Fourier transform defined in (3.22) in the lecture notes.

1. Let (a, b, c, d) be a vector in $\mathbb{C}^4$ with FDFT (A, B, C, D), where $a, b, c, d, A, B, C, D \in \mathbb{C}$. Show that the vector
$$(a, 0, b, 0, c, 0, d, 0)$$
has the FDFT
$$\frac{1}{2}(A, B, C, D, A, B, C, D)$$
Generalize by inserting more then one 0, i.e. $(a, 0, 0, 0, b, 0 \ldots)$ etc.

2. Compute the FDFT of the complex exponential of frequency $1Hz$:
$$exp = [1, (1 + i)/\sqrt{2}, i, (-1 + i)/\sqrt{2}, -1, (-1 - i)/\sqrt{2}, -i, (1 - i)/\sqrt{2}] \in \mathbb{C}^8,$$
that is, $exp[n] = e^{2\pi in/8}$, $n = 0, \ldots, 7$. Compute the FDFT for an arbitrary frequency $k_0 \in \{0, 1, \ldots, 7\}$ and generalize to arbitrary $N \in \mathbb{N}$ (here $k_0 \in \{0, 1, \ldots, N - 1\}$). Check in Octave/Python/Matlab!

3. Consider the continuous signal
$$f(t) = \sin(20\pi t) + \sin(40\pi t)$$
and sketch its Fourier transform. What is the minimum samplingrate to supress aliases. Which aliases occur if this signal is sampled at $50Hz$? Verify in Octave/Python/Matlab!

4. Rely on Definition 5.2.1 of the STFT in the lecture notes and show that
   (a) $\mathcal{S}_\varphi(T_u M_\eta f)(x, \omega) = e^{-2\pi iu\cdot\omega}\mathcal{S}_\varphi f(x - u, \omega - \eta)$.
   (b) $\mathcal{S}_\varphi(f)(x, \omega) = e^{-2\pi ix\cdot\omega}\mathcal{S}_{\hat{\varphi}}\hat{f}(\omega, -x)$.

   Give an interpretation of these equalities.

5. Optional: Practical/Playful Sampling and Aliasing Exercises:
   (a) Choose tone 1 at 800 Hz and tone 2 at 300 Hz. Set the sampling frequency at 2000 Hz. Play both the original signal and the recovered signal after sampling. Do you hear any differences? Did aliasing occur in this case?
   (b) For the same tones, set now the sampling frequency at 1000 Hz. Listen to both sounds. Do you hear any differences now? Which tone suffered aliasing? What is the cutoff frequency of the low-pass filter? What are the frequencies of the tones that are present in the signal after low-pass filtering?
   (c) Now set the sampling frequency at 500 Hz and listen again to both sounds. Which tones are now aliased? What are the frequencies that are present in the signal after low-pass filtering?
   (d) Choose two tones and a sampling frequency so that only one of the tones is aliased and the aliased tone has the same frequency as the non-aliased tone!