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Applied Analysis Problems

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Contents

1 Sheet 1

1.1 Fall from high

We consider a free fall ($\dot{x}(t=0)=0$) of an object with mass 20 kg from a height $x(0)=h=$ 20 km, such that the gravitational force depends on the hight $x(t)$ in the following way

$$\ddot{x} = -g\frac{R^2}{(x(t)+R)^2}, \tag{1}$$

where R is the radius of the earth $R \approx 6000$ km and $g \approx 9.91\ \frac{m}{s^2}$ is the gravitational acceleration on the surface of the earth. For this problem there are two possible non-dimensionalisations, but first let us rewrite the variables in terms of non-dimensional variables and some dimensional constants, a priori let

$$t = t_c\tau \quad \text{and} \tag{2}$$
$$x = x_c\xi. \tag{3}$$

With the above ansatz we get the following second derivative in time

$$\frac{d^2}{dt^2} = \frac{1}{t_c^2}\frac{d^2}{d\tau^2} \tag{4}$$
$$\Rightarrow \frac{d^2x}{dt^2} = \frac{x_c}{t_c^2}\frac{d^2\xi}{d\tau^2}, \tag{5}$$

and thus the initial conditions can be rewritten as

$$\xi(0) = \frac{h}{x_c}, \tag{6}$$
$$\dot{\xi} = 0. \tag{7}$$

Now we can rewrite the equation of the free fall in 1 in terms of $\xi(\tau)$ as

$$\frac{x_c}{gt_c^2}\ddot{\xi} = -\frac{1}{(\frac{x_c}{R}\xi+1)^2}. \tag{8}$$

Thereby we have three dimensional constants constants Π_1, Π_2, Π_3, as follows

$$\Pi_1 = \frac{x_c}{R}, \quad \Pi_2 = \frac{h}{x_c}, \quad \Pi_3 = \frac{x_c}{gt_c^2}. \tag{9}$$

The first scaling is done by reducing Π_1 and Π_3 to 1, by setting

$$x_c = R, \quad t_c = \sqrt{\frac{R}{g}}, \tag{10}$$

refolmulating the initial problem in equation 1 to

$$\begin{aligned} \ddot{\xi} &= -\frac{1}{(\xi+1)^2}, \quad \text{with} \\ \xi(0) &= \frac{h}{R}, \quad \dot{\xi}(0) = 0. \end{aligned} \tag{11}$$

Reducing the problem, meaning letting $R \to \infty$ makes the first initial condition $\xi(0) \to 0$. We can conclude that this scaling is bad since it changes the initial condition in the reduced problem.

The second scaling option reduces Π_2 and Π_3 to 1, by setting

$$x_c = h, \quad t_c = \sqrt{\frac{h}{g}}, \tag{12}$$

refolmulating the initial problem in equation 1 to

$$\begin{aligned} \ddot{\xi} &= -\frac{1}{(\frac{h}{R}\xi+1)^2}, \quad \text{with} \\ \xi(0) &= 1, \quad \dot{\xi}(0) = 0. \end{aligned} \tag{13}$$

By letting $R \to \infty$ we get the following reduced problem

$$\ddot{\xi} = -1. \tag{14}$$

Integrating and solving for $\xi(\tau = \frac{T}{t_c}) = 0$ for when the object hits the ground we get a familliar solution

$$T = \sqrt{\frac{2h}{g}} \tag{15}$$

Now in the reduced problem the time untill the object hits the ground is shorter since the acceleration is constant, but in the original one the acceleration increases as the object comes closer to earth. Additionally we can calculate the velocity at impact we need to integrate the reduced problem 14 once and put in the initial condition

$$\dot{\xi}(\tau = \frac{T}{t_c}) = -\tau = -\sqrt{2} \tag{16}$$

$$\text{and} \quad \dot{x} = \frac{x_c}{t_c}\dot{\xi} = \sqrt{gh}\,\dot{\xi} \tag{17}$$

$$\Rightarrow \dot{x}(T) = -\sqrt{2gh} \tag{18}$$

The vectical throw allows for different scaling because the initial conditions are different, and thus the solution too $x(0) = 0$ and $\dot{x}(0) = v$.

1.2 Scaling The Van der Pol equation

The Van der Pol equation is a perturbation of the oscillation equation

$$LC\frac{d^2I}{dt^2} + (-g_1C + 3g_3CI^2)\frac{dI}{dt} = -I \tag{19}$$

with initial conditions

$$I(0) = I_0, \quad \dot{I}(0) = 0. \tag{20}$$

where $I(t)$ is the current at a time t, C is the capacity, L is the inductivity and g_1, g_3 are some parameters. The units of all the parameters are

$$[LC] = s^2 \tag{21}$$
$$[g_1 C] = s \tag{22}$$
$$[g_3 C] = sA^{-2} \tag{23}$$

The oscillation equation is

$$CL\ddot{I} + I = 0. \tag{24}$$

Solvable by the exponential ansatz of $I = Ae^{\lambda t}$, where $\lambda = \pm i\sqrt{\frac{1}{LC}}$, thereby

$$I(t) = A_1 e^{i\sqrt{\frac{1}{LC}}t} + A_2 e^{-i\sqrt{\frac{1}{LC}}t}. \tag{25}$$

With the initial conditions in equation 20 we get $A_1 = A_2$ and thus the solution to the oscillation equation is

$$I(t) = I_0 \cos(\frac{t}{\sqrt{LC}}) \tag{26}$$

Now that we know the reduced problem and the solution to it, we may work with the Van-Der-Pol equation 19, by determining all possible non-dimensionalisations. Let us begin by setting

$$I(t) = I_c \psi, \tag{27}$$
$$t = t_c \tau, \tag{28}$$

where I_c and t_c have the dimension of $I(t)$ and t accordingly and $\psi(\tau)$ and τ are dimensionless The first and second derivative in time is

$$\frac{d}{dt} = \frac{1}{t_c}\frac{d}{d\tau} \tag{29}$$
$$\frac{d^2}{dt^2} = \frac{1}{t_c^2}\frac{d^2}{d\tau^2}. \tag{30}$$

We can rewrite the Van-Der-Pol equation in terms of ψ and τ

$$\frac{LC}{t_c^2}\ddot{\psi} - \frac{g_1 C}{t_c}\dot{\psi}\frac{3g_3 C I_c}{t_c}\dot{\psi}\psi = -\psi \tag{31}$$
$$\psi(0) = \frac{I_0}{I_c} \quad \dot{\psi}(0) = 0 \tag{32}$$

There are a total of four constants that we can eliminate

$$\Pi_1 = \frac{I_0}{I_c}, \quad \Pi_2 = \frac{LC}{t_c^2},$$
$$\Pi_3 = \frac{-g_1 C}{t_c}, \quad \Pi_4 = \frac{3g_3 C I_C}{t_c}. \tag{33}$$

The first scaling is

$$I_c = I_0, \quad t_c = \frac{1}{\sqrt{LC}}. \tag{34}$$

Thereby we get the following problem

$$\ddot{\psi} - \sqrt{\frac{C}{L}} g_1 \dot{\psi} + \sqrt{\frac{C}{L}} 3g_3 I_0 \dot{\psi}\psi = -\psi \tag{35}$$
$$\psi(0) = 1 \quad \dot{\psi}(0) = 0 \tag{36}$$

The second scaling is

$$I_c = I_0, \quad t_c = g_1 C. \tag{37}$$

Thereby we get the following problem

$$\frac{L}{g_1^2 C}\ddot{\psi} - \dot{\psi} + \frac{3g_3}{g_1}\dot{\psi}\psi = -\psi \tag{38}$$

$$\psi(0) = 1 \quad \dot{\psi}(0) = 0 \tag{39}$$

The third scaling is

$$I_c = I_0, \quad t_c = g_3 C I_0. \tag{40}$$

Thereby we get the following problem

$$\frac{L}{g_3^2 C I_0^2}\ddot{\psi} - \frac{g_1}{g_3 I_0}\dot{\psi} + 3\dot{\psi}\psi = -\psi \tag{41}$$

$$\psi(0) = 1 \quad \dot{\psi}(0) = 0 \tag{42}$$

1.3 Scale the Schrödinger Equation

The well known Schrödinger equation that describes quantum physics of one particle can be written as

$$\begin{aligned} i\hbar\partial_t\psi &= -\frac{\hbar}{2m}\Delta\psi + V\psi \\ \psi(t=0) &= \psi_0 \end{aligned} \tag{43}$$

where $\hbar$ is the Planks constant, $\psi = \psi(x,t)$ the wave function, m the mass and $V = V(x)$ the potential in which the wave function is. The dimensions are

$$[\hbar] = Js, \quad V = J, \quad [\psi] = m^{-d/2} \tag{44}$$

for the special dimension d. The standard scaling ansatz is

$$\psi = \psi_c\phi \tag{45}$$

$$t = t_c\tau \quad x = x_c\xi, \tag{46}$$

by that we get the following derivatives in time and in space

$$\partial_{x_i} = \frac{1}{x_{(i)c}}\partial_{\psi_i} \tag{47}$$

$$\partial^2_{x_i} = \frac{1}{x^2_{(i)c}}\partial^2_{\psi_i} \tag{48}$$

$$\partial_t = \frac{1}{t_c}\partial_{\psi_i} \tag{49}$$

$$\tag{50}$$

for $i = 1, 2, 3$, or depending on the dimension we are dealing with.

Let us consider $x \in \mathbb{R}^3$ and $V = 0$ to scale the equation. First we now have

$$i\partial_t\psi = -\frac{\hbar}{2m}\Delta\psi \tag{51}$$

with the initial condition $\phi(0) = \frac{\psi_0}{\psi_c}$. With our scaling the equation turns out to be

$$\frac{i\hbar t_c}{2m}\frac{1}{||\vec{x}_c||^2}\Delta_{\vec{\xi}}\,\phi = \partial_\tau\,\phi. \tag{52}$$

The constants we get are

$$\Pi_1 = \frac{t_c \hbar}{2m} \frac{1}{||\vec{x}_c||^2}, \quad \Pi_2 = \frac{\psi_0}{\psi_c}. \tag{53}$$

The simple choice of

$$\frac{1}{||\vec{x}_c||^2} = 1, \quad \psi_c = \psi_0, \quad t_c = \frac{2m}{\hbar}||\vec{x}_c||^2, \tag{54}$$

simplifies the Schrodinger equation without the potential to

$$i\Delta_{\vec{\xi}}\, \phi = \partial_\tau \phi, \tag{55}$$

with the initial condition $\phi(\tau = 0) = 1$.

Now consider $V = 0$, $x \in [0, L]$ and $t \in [0, T]$, the Schrodinger equation is the same only with one spacial dimension as above, we can set

$$\psi_c = \psi_0, \quad x_c = L, \quad t_c = \frac{2mL^2}{\hbar}. \tag{56}$$

Thus we get

$$i\partial_\xi^2 \phi = \partial_\tau \phi, \tag{57}$$

with the initial condition $\phi(\tau = 0) = 1$.

As a last example let us consider the quantum harmonic oscillator, that is $V(x) = m\omega^2 x^2$ for $x \in \mathbb{R}$, where ω is the frequency, the equation is the following

$$i\hbar\partial_t \psi = -\frac{\hbar^2}{2m}\partial_x^2 \psi m\omega^2 x^2 \psi. \tag{58}$$

By inserting the scaling

$$i\partial_\tau \phi = -\frac{t_c \hbar}{2m x_c^2}\partial_\xi^2 \phi \frac{t_c m\omega^2 x_c^2}{\hbar}\xi^2 \phi \tag{59}$$

The dimensional constants are

$$\Pi_1 = \frac{t_0 \hbar}{m x_c^2}, \quad \Pi_2 = \frac{m\omega^2 x_c^2 t_c}{\hbar}, \quad \Pi_3 = \frac{\psi_0}{\psi_c}. \tag{60}$$

The choice of scaling is

$$\psi_c = \psi_0, \quad t_c = \frac{1}{\omega}, \quad x_c = \sqrt{\frac{\hbar}{m\omega}}. \tag{61}$$

Thereby getting the following problem

$$i\partial_\tau \phi = -\frac{1}{2}\partial_\xi^2 \phi + \xi^2 \phi \tag{62}$$

with $\phi(\tau = 0) = 1$.