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Applied Analysis Problems

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## 1 Sheet 3

### 1.1 Problem 8

Let us look at functions  $f : \mathcal{D} \mapsto \mathbb{R}$  that show boundary layer behavior at the following manifolds.

The **first** for  $\mathcal{D} = \mathbb{R}^2$  and  $S = \{0\}$  we have a function e.g.

$$f_\varepsilon(x, y) = e^{-\frac{x}{\varepsilon}} + y, \quad (1)$$

with the reduced equation

$$\lim_{\varepsilon \rightarrow 0} f_\varepsilon(x, y) = \begin{cases} y & x > 0 \\ 1 + y & x = 0 \end{cases} \quad (2)$$

The **second** example is  $\mathcal{D} = \mathbb{R}^n$  and  $S = \{|x| = 1\}$ .

$$f_\varepsilon(x_1, \dots, x_n) = \tanh\left(\frac{|x| - 1}{\varepsilon}\right), \quad (3)$$

with the reduced equation

$$\lim_{\varepsilon \rightarrow 0} f_\varepsilon(x_1, \dots, x_n) = \begin{cases} -1 & |x| < 0 \\ 1 & |x| > 0 \end{cases} \quad (4)$$

The **third** example is  $\mathcal{D} = \mathbb{R}^3$  and  $S = \{x_1 = 1\}$

$$f_\varepsilon(x_1, x_2, x_3) = \tanh\left(\frac{x_1 - 1}{\varepsilon}\right) + x_2 x_3 \quad (5)$$

with the reduced equation

$$\lim_{\varepsilon \rightarrow 0} f_\varepsilon(x_1, x_2, x_3) = \begin{cases} -1 + x_2 x_3 & x_1 < 0 \\ 1 + x_2 x_3 & x_1 > 0 \end{cases} \quad (6)$$

## 1.2 Problem 9

Consider a linear BVP

$$Lu := -\varepsilon u'' + b(x)u' + c(x)u = f(x), \quad (7)$$

$$u(0) = u(1) = 0, \quad (8)$$

for  $0 < \varepsilon \ll \varepsilon_0$  and  $b, c, f \in C([0, 1])$  with the conditions

$$c(x) \geq 0, \quad b(x) \geq \beta > 0 \quad x \in [0, 1] \quad (9)$$

We are to show that for all  $x \in [0, 1]$  the reduced solution  $u_0$  of the above BVP satisfies

$$\lim_{\varepsilon \rightarrow 0} u_\varepsilon(x) = u_0(x), \quad (10)$$

where the reduced solution  $u_0$  is the solution to the following differential equation

$$b(x)u' + c(x)u = f(x), \quad u(0) = 0. \quad (11)$$

The hint was given: Set

$$w_1(x) = e^{\beta x} \quad w_2(x) = e^{-\beta \frac{1-x}{\varepsilon}}, \quad (12)$$

such that  $Lw_1 \geq \gamma > 0$  for some suitable  $\gamma$  and  $Lw_2 \geq 0$ . Then for

$$v = \pm(u_\varepsilon - u_0), \quad w = A\varepsilon w_1 + B\varepsilon w_2, \quad (13)$$

for some suitable  $A, B$ . The following comparison principal is applicable: IF

$$Lv(x) \leq Lw(x) \quad \forall x \in (0, 1) \quad (14)$$

$$v(0) \leq w(0) \quad (15)$$

$$v(1) \leq w(1) \quad (16)$$

$$(17)$$

then

$$\implies v(x) \leq w(x) \quad \forall x \in (0, 1) \quad (18)$$

which holds for  $u, v \in C^2((0, 1)) \cap C([0, 1])$ . Thus a boundary layer is possible only at  $x = 1$ . On the other hand, for  $b(x) \leq \beta < 0$  it follows that the boundary layer is possible only at  $x = 0$ .

We shall go through the chronological order of the conditions [14](#), [15](#), [16](#) and check them. So for [14](#) we have that

$$Lw(x) = A\varepsilon Lw_1(x) + B\varepsilon Lw_2(x) \quad (19)$$

$$\geq A\varepsilon Lw_1(x) = A\varepsilon e^{\beta x} (-\varepsilon\beta^2 - b(x)\beta + c(x)) \quad (20)$$

$$\geq A\varepsilon\beta e^{\beta x} (1 - \varepsilon) \quad (21)$$

$$\geq \varepsilon A\beta^2 e^{\beta} (1 - \varepsilon) = \gamma > 0 \quad (22)$$

And obviously

$$Lv(x) \leq 0, \quad (23)$$

by that we have that

$$Lv(x) \leq \gamma \leq Lw(x). \quad (24)$$

For the condition [15](#) we have

$$w(0) = A\varepsilon w_1(0)Bw_2(0) = Be^{-\frac{\beta}{\varepsilon}}, \quad (25)$$

$$v(0) = \pm(u_\varepsilon(0) - u_0(0)) = 0. \quad (26)$$

By the simple choice  $B \geq 0$  we satisfy the condition

$$v(0) \leq w(0). \quad (27)$$

Now for the last condition 16 we have

$$w(1) = A\epsilon e^\beta + B \geq A\epsilon e^\beta, \quad (28)$$

$$v(1) = \mp u_0(1) = 0. \quad (29)$$

And choose  $A = \frac{\pm u(1)}{\epsilon} e^{-\beta}$ , which satisfies the last condition

$$v(1) \leq w(1). \quad (30)$$

Thereby we have

$$v(x) \leq w(x) \quad (31)$$

$$\Rightarrow \lim_{\epsilon \rightarrow 0} v(x) \leq \lim_{\epsilon \rightarrow 0} w(x) = 0 \quad (32)$$

$$\Rightarrow \lim_{\epsilon \rightarrow 0} v(x) = 0 \quad (33)$$

uniformly on  $(0, 1)$ .

### 1.3 Problem 10

Consider the following BVP

$$-\epsilon u'' + (1+x)u' + u = 2, \quad u(0) = u(1) = 0, \quad (34)$$

for  $0 < \epsilon \ll 1$ . **Where can this problem have a boundary layer?** To answer this question we need to look at the reduced problem

$$-(1+x)u' + u = 2. \quad (35)$$

The solution to the equation is

$$\bar{u}(x) = 2 + A(x+1). \quad (36)$$

According to the boundary conditions it is unclear what the value of the constant is, according to  $\bar{u}(0) = 0$  we get  $A = -2$  or according to  $\bar{u}(1) = 0$  we get  $A = -1$ . Ultimately this means that there exists a boundary layer near  $x = 1$  or  $x = 0$ . We choose  $x = 0$  and according to this the local variable  $\xi = x\epsilon^{-\alpha}$  ( $x = \xi\epsilon^{-\alpha}$ ). The derivatives of  $u$  are calculated using the chain rule

$$\frac{du}{dx} = \frac{du}{d\xi} \frac{d\xi}{dx} = \epsilon^{-\alpha} \dot{u} \quad (37)$$

$$\frac{d^2u}{dx^2} = \epsilon^{-\alpha} \frac{d^2u}{d\xi^2} \frac{d\xi}{dx} = \epsilon^{-2\alpha} \ddot{u}. \quad (38)$$

The BVP transforms as follows

$$-\epsilon^{1-\alpha} \ddot{u} - \dot{u} + \epsilon(u - \xi \dot{u} - 2) = \begin{cases} -\ddot{u} - \dot{u} = 0 & \alpha = 1 \\ -\dot{u} = 0 & 0 < \alpha < 1 \end{cases} \quad (39)$$

Choosing  $\alpha = 1$  for a reasonable solution

$$\hat{u}(\xi) = Be^{-\xi}, \quad (40)$$

which converges in the local limit (!). Thereby we have a asymptotic representation up to the degree of  $\epsilon$

$$u_\epsilon(x) = \bar{u}(x) + \hat{u}(\psi) + O(\epsilon) \quad (41)$$

$$= 2 + A(1+x) + Be^{-\frac{x}{\epsilon}} + O(\epsilon) \quad (42)$$

And by the boundary conditions

$$u_\varepsilon(0) = 2 + A + B = 0, \quad u_\varepsilon(1) = 2 + 2A + B = 0, \quad (43)$$

we get that the constants are

$$A = -4, \quad B = 2. \quad (44)$$

The asymptotic representation is thereby

$$u_\varepsilon(x) = 2 - 4(1 + x) + 2e^{-\frac{x}{\varepsilon}} + O(\varepsilon) \quad (45)$$