Applied Analysis

Lecture notes by
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These are lecture notes for the 4 hours course "Applied Analysis" as designed for the Master of Mathematics at Univ. Wien, based on the **idea:**

"Applied Analysis" = 1/2 "Asymptotic Analysis" + 1/2 "Harmonic Analysis"

This way, master and PhD students in mathematics, physics, astronomy,..., computational and data science, ... get acquainted with 2 key aspects of the broad field of "Applied Analysis" as used in several groups at Univ. Wien who offer many interesting research themes and funding for masters and PhD theses.

The methods of this course have applications in acoustics, biology, climate research, fluid dynamics, magnetism, materials, medicine, (quantum) physics, semiconductors, sensors, signal processing, social sciences, etc.

In the first 2 hours "Asymptotic Analysis" part of the course we present a key tool to understand/derive model hierarchies in the sense that approximate models are "small" perturbations of an "exact" model, with possibly many levels of approximation between.

Of course, this procedure requires an important preparatory step (that is intertwined with the models and numerics) : find "scaled" equations after an "nondimensionalization". This "scaling" is not just "choosing the units" and it yields the dimensionless parameters of the model. A "small parameter" is usually called ε "epsilon", and "$\varepsilon \to 0$" is the essence of asymptotic analysis - which becomes extremely hard when you do more than "formal expansions" and ask: "which solution of the "exact" equation converges against which solution of the "reduced" equation in which sense ?"

Apart from the "reduced solution" for $\varepsilon = 0$, we can create a hierarchy of models by considering also $O(\varepsilon)$ corrections to it, and $O(\varepsilon^2)$ etc. Examples are the hierarchy from the relativistic Dirac equation to the nonrelativistic Schrödinger equation with "in between" the Pauli equation as a "semi-relativistic" equation at $O(\varepsilon)$ where $\varepsilon = 1/\mathrm{c}$ where c is the speed of light.

We shall develop the machinery of scaling and formal "asymptotic expansions", introduce "regular" and "singular" perturbations, "boundary layers" etc. and also sketch rigorous derivations.

The mathematical models are based on (ordinary or partial) differential equations.

As an important application we take continuum mechanics, as well as some elementary mathematical aerodynamics. The limit from the Navier-Stokes to the Euler equations when the "viscosity" vanishes is an example - where the Millenium ("Clay Inst.") problem of global classical solutions of hydrodynamics hits in.

The second half of the course is devoted to "Applied Harmonic Analysis".
We introduce the principles of Fourier analysis. That is, we discuss the Fourier transform

for L^1- and L^2-functions and then move on to discuss discretizations of the Fourier transform, thereby touching on algorithmic topics such as the fast Fourier transform (FFT). The question of discretization is intimately connected to problems in sampling. We are going to discuss the Poisson summation formula and prove the Shannon sampling theorem.

In the context of Fourier series, approximation questions will arise. We are going to discuss the quality of approximation by Fourier series in various spaces. Approximation in function spaces is a much more general topic and a myriad of representations and function spaces have been studied.
In this course we will get an idea of function approximation based on the wavelet transform and the short-time Fourier transform. We will also take a look at the discretization concepts of the latter, which are based on frames. Frames are systems of basic functions, often generated by operators based on some locally compact group, which relax the concept of basis by allowing for the replacement of Parseval's identity by the frame inequalities. The harmonic analysis part of this course will be illustrated by a lot of examples from image and audio processing.

1 Model Hierarchies by (formal) Asymptotic Expansions / Perturbation methods

1.1 Scaling ("Nondimensionalization") of equations - Dimensional analysis

Mathematical models/equations yield formal connections between quantities in physics, chemistry, biology,... that are represented by numbers, matrices etc. At first, these quantities bear "units", for example the unknown dependent variable $v(x,t)$ is a velocity measured in "kilometers per hour" (km/h), the independent variable x is a lenght measured in "meter", the independent variable t, the time, has the unit "second", etc.
In addition to the "variables", models usually contain "parameters" that are "arbitrary, but fixed".

The bracket notation $[x]$ denotes that we take the unit of a quantity, e.g. $[x] = m$ ("m like meter") or $[t] = s$ ("s like second").
For a dimensionless quantity we have $[q] = 1$; "q has unit 1".

Of course we have to make sure that all added terms in a model equation and it's right hand and left hand side consistently bear the same unit (a trivial test if a model equation can be correct). More than that, in order to make mathematical analysis or numerical computations we have to "nondimensionalize" (= "adimensionalize") the equation, i.e. to get rid of all ("physical") units by a procedure called "scaling".
Such a scaling simply means that we divide all variables by a reference quantity with the same unit, such that the new quantity "has unit 1".
Thus also all "parameters" automatically become dimensionless.

Note that such a scaling means much more than simply "chosing the appropriate units"; e.g. when "atomic" or "astronomic" units are chosen, like "Angström" /= 10^-10 m or "lightyears" as unit of lenght.
Further, this kind of scaling makes a "dimensional analysis" possible that yields insight into the connection and importance of the (physical) quantities in the equation. It is the basis for a "scaling analysis" which is a fundamental tool of applied mathematics for the motivation/derivation of "reduced", "simpler" models for complex problems, yielding "model hierarchies". In some situations it is possible to make this "formal asymptotics" rigorous by "asymptotic analysis", where the convergence of the solution of the exact problem to the solution of the reduced - approximate model(s) is proven in the sense of theorems.

We start with a seemingly simple example of the "scaling" :

Example 1. "The projectile problem with non-constant gravity"

We consider the movement of a body thrown vertically upwards from the surface of the earth, and we want to compute it's "trajectory", to answer questions like "how long it takes until it falls back to the ground ?". Let us consider a situation where we throw or shoot it really high, such that the "weight" of the body starts to differ from the weight at the surface of the earth.

The body will move along a ray through the center of the earth, and we denote its distance from the earth surface at time t^* by $x^*(t^*)$.

We assume that only gravity is acting on the body, and making many more model simplifications, the function x^* solves the following initial value problem of a 2nd order ODE (resulting from Newton's second law):

$$\frac{d^2x^*}{dt^{*2}} = -\frac{gR^2}{(x^*+R)^2}\,, \tag{1}$$

$$x^*(0) = 0\,, \qquad \frac{dx^*}{dt^*}(0) = V\,, \tag{2}$$

where g is the gravitational acceleration at the surface of the earth, R is the radius of the earth, and V is the initial speed. Of course some effects have been neglected, most importantly air resistance.

We want to compute the "trajectory" $x^*(t^*)$ and, e.g. T^* such that $x^*(T^*) = 0$.
Note that we use the notation $x^*(t^*)$ for the "unscaled" problem in order to free the notation $x(t)$ for the "scaled" problem where we do the mathematics/numerics. Physicists and other applied scientists tend to use the same symbols for the scaled and the unscaled equations.

For identifying "small parameters" we do a "dimensional analysis" of this problem.

Nondimensionalization ("Adimensionalization") of mathematical model problems consists of two steps:

preparation Step 1: Produce a list of all variables and parameters of the problem together with their physical dimensions.

Using the cgs-system (centimeter-gram-second) of units, we get for problem (1)

Quantity	Dimension	Unit
x^*	Length	cm
t^*	Time	s
g	Acceleration	cm s^{-2}
R	Length	cm
V	Velocity	cm s^{-1}

scaling Step 2 : "divide by reference quantity": For every independent variable, chose as a unit an *intrinsic reference quantity*, i.e. a combination of parameters with the same dimension.
Divide the independent variables by the respective reference quantity.
In general, there are several ways to find reference quantities, one can construct new ones from the parameters contained in the problem and one is allowed also to add reference quantities.
If possible, do it in a way that the intrinsic reference quantities match the “expected sizes” of the variables, such that you can expect to compute with moderate numbers.

After carrying out this procedure, the problem should contain only dimensionless quantities, in particular only *dimensionless parameters*, i.e. dimensionless combinations of the original quantities.

Example 1 (continued)**.** For the equations (1), (2) there is more than one way to define reference lengths and times from the parameters.

If we assume a 'relatively small' (relative to what?) initial velocity, we expect the maximum height to be "small" in comparison to the radius of the earth. So the "easy/obvious" choice of R as reference lenght might not be "good" (in which sense?).

Example 1 - Scaling 1: With the above considerations, we make an effort to construct a reference length "more complicated" than R, namely V^2/g, and a reference time V/g. Dividing the unscaled variables by these "units" yields the dimensionless quantities :

$$x = \frac{x^*}{V^2/g}\,, \quad t = \frac{t^*}{V/g}\,, \qquad \text{and hence} \quad x(t) := \frac{x^*(Vt/g)}{V^2/g}\,. \tag{3}$$

The problem (1),(2) for $x(t)$ in scaled form thus reads

$$x'' = -\frac{1}{(1+\varepsilon x)^2}\,, \tag{4}$$

$$x(0) = 0\,, \qquad x'(0) = 1\,, \tag{5}$$

with the dimensionless parameter

$$\varepsilon = \frac{V^2}{gR}\,. \tag{6}$$

Now considering $\varepsilon \ll 1$ we see what 'relatively small initial velocity' means, namely that V is small compared to $\sqrt{gR}$, i.e. $V \ll \sqrt{gR}$.
Note that the number of parameters has been reduced from 3 to 1.

One can compute the (ε dependent) solution $x(t)$ of (4)-(5). To obtain the time T^*_{final} until the body falls back to the ground one has to solve $x(t) = 0$ to get T_{final} and rescale

$$T^*_{final}(\varepsilon) = \frac{V}{g} T_{final}\,.$$

Note that T^*_{final} depends on ε and $T^*_{final}(\varepsilon = 0)$ (hopefully) coincides with the $T^*_{final,0}$ obtained from the "good" reduced problem for $\varepsilon = 0$. a question that we shall examine in the sequel.

A general framework for the scaling procedure is the following general statement about dimensional variables and parameters, which can be seen as an "axiom of dimensional analysis", sometimes called **Buckingham-"Theorem"**:

Proposition 1. *Mathematical models of physical (chemical, biological, economic, ...) phenomena can be nondimensionalized. Dimensionless parameters and intrinsic reference quantities can be chosen as products of powers of the original parameters.*

Of course, this heuristic statement is not a "theorem" in the strict mathematical sense; under certain conditions it can be made more precise and proven.

Typically, there are certain relations between dimensions of variables and parameters, e.g. Velocity $= \frac{\text{Length}}{\text{Time}}$. These become clear when all dimensions are expressed in terms of a set of *fundamental dimensions*. An example is provided by the SI-system (the international system of units), consisting of a choice of units for 7 fundamental dimensions, i.e. meter (m) for *length,* kilogram (kg) for *mass,* second (s) for *time,* Ampere (A) for *electric current,* Kelvin (K) for *temperature,* mole (mol) for *amount of substance,* and candela (cd) for *luminous intensity.*

For models in mechanics, like in the example above, the 3 fundamental dimensions length, mass, and time are sufficient. We stay with this situation and abbreviate the dimensions length, mass, and time by L, M, and T, respectively. Suppose we have a

problem with N parameters $\alpha_1, \ldots, \alpha_N$, with dimensions $[\alpha_1], \ldots, [\alpha_N]$. They can be written as

$$[\alpha_k] = \mathrm{L}^{l_k}\ \mathrm{M}^{m_k}\ \mathrm{T}^{t_k}\,,$$

with $(l_k, m_k, t_k) \in \mathbb{R}^3$, $k = 1, \ldots, N$. Thus the dimension of a quantity is characterized by a 3-vector, e.g. velocity by $(1, 0, -1)$. Suppose we are looking for an intrinsic reference quantity for a variable α, whose dimension is characterized by (l, m, t). As a consequence of Proposition 1, there exists $(a_1, \ldots, a_N) \in \mathbb{R}^N$ such that

$$[\alpha] = \left[\prod_{k=1}^{N} \alpha_k^{a_k}\right]\,,$$

which is equivalent to

$$\mathrm{L}^l\mathrm{M}^m\mathrm{T}^t = \prod_{k=1}^{N} \mathrm{L}^{a_k l_k}\mathrm{M}^{a_k m_k}\mathrm{T}^{a_k t_k} = \mathrm{L}^{\sum a_k l_k}\mathrm{M}^{\sum a_k m_k}\mathrm{T}^{\sum a_k t_k}\,,$$

and, thus, to the linear system

$$\sum_{k=1}^{N} a_k l_k = l\,, \qquad \sum_{k=1}^{N} a_k m_k = m\,, \qquad \sum_{k=1}^{N} a_k t_k = t\,,$$

for the exponent vector $(a_1, \ldots, a_N)$. Solvability of this system is related to Proposition 1.

Dimensionless parameters correspond to elements of the "nullspace" of the coefficient matrix

$$\begin{pmatrix} l_1 & \ldots & l_N \\ m_1 & \ldots & m_N \\ t_1 & \ldots & t_N \end{pmatrix}\,. \tag{7}$$

Two linearly dependent elements of the nullspace correspond to two dimensionless parameters β and γ, where one can be expressed in terms of the other by the relation $\gamma = \beta^\lambda$. Therefore the number N^* of independent dimensionless parameters is equal to the dimension of the nullspace and, thus, $N - 3 \leq N^* \leq N$. The number of parameters is always reduced ($N^* < N$) by nondimensionalization as long as the matrix (7) contains nonzero entries, i.e. as long as not all the original parameters have been dimensionless already.

Example 1 (continued)**.** For the projectile problem, the matrix (7) is given by

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -2 & 0 & -1 \end{pmatrix}$$

which has rank 2.
The parameter ε corresponds to the element $(-1, -1, 2)$ of the nullspace. Of course the second row of the matrix does not contribute to the result, since obviously the fundamental dimension *mass* does not appear.

In this example, there are several ways to choose intrinsic reference units, cf. the nontrivial null-space of the above matrix.

Example 1 - Scaling 2: An alternative choice (#2) of reference lenght and time is

$$y = \frac{x^*}{R}\,, \quad \tau = \frac{t^*}{R/V} \tag{8}$$

and the problem for $y(\tau)$ takes the form

$$\varepsilon \frac{d^2y}{d\tau^2} = -\frac{1}{(y+1)^2}\,, \tag{9}$$

$$y(0) = 0\,, \qquad \frac{d}{d\tau}y(0) = 1\,, \tag{10}$$

with the same dimensionless parameter ε defined in (6).

Example 1 - Scaling 3: A third choice (#3) is

$$z = \frac{x^*}{R}\,, \quad \sigma = \frac{t^*}{\sqrt{R/g}} \tag{11}$$

Then the nondimensionalized problem for $z(\sigma)$ is

$$\frac{d^2z}{d\tau_1^2} = -\frac{1}{(z+1)^2}\,, \tag{12}$$

$$z(0) = 0\,, \quad \frac{d}{d\sigma}z(0) = \sqrt{\varepsilon}\,, \tag{13}$$

with the same dimensionless parameter ε defined in (6), of course.

The 3 initial value problems (4)-(5) and (9)-(10) and (12)-(13) are equivalent for finite $\varepsilon > 0$; however their reduced problems for $\varepsilon = 0$ are completely different.

Note that for doing the "scaling", i.e. the mere nondimensionalization procedure, we actually do not need the complete model problem/equation, but only the list of variables and parameters.

Example 2. (The pendulum problem) We consider a rigid massless stick of length L with a point mass m at one end. It freely rotates around a horizontal axis through the other end under the action of a fixed downward gravitational acceleration. Friction effects are neglected. We bring the pendulum into a horizontal position, let it go, and are interested in the duration T^* of the first swing. The movement of the pendulum is described completely by the function $\theta^*(t^*)$, giving the angle between vertical lines and the pendulum. Parameters are the pendulum length L, the point mass m, and the gravitational acceleration g. This leads to the following list of variables and parameter:

Quantity	Dimension
θ^*	1
t^*	T
L	L
m	M
g	L T^{-2}

The invertibility of the matrix (7) ,

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

implies that there are no dimensionless parameters. The unique intrinsic reference time is given by $\sqrt{L/g}$. Thus the duration of the first swing is given by

$$T^* = \sqrt{\frac{L}{g}}\, T\,,$$

where T is determined from the nondimensionalized problem, which however does not contain any parameters. Therefore T is a number which does not depend on the physical realization of the pendulum besides it's lenght L. Instead of solving a mathematical problem, its value could also be determined from one experiment with known values of L and g. We also remark that the result is independent from the mass m.

1.2 Regular perturbations – asymptotic expansions

Example 1 (continued)**.** We recall the nondimensionalized projectile problem obtained by the Scaling (#1)

$$x'' = -\frac{1}{(1+\varepsilon x)^2}\,, \qquad x(0) = 0\,, \quad x'(0) = 1\,. \tag{14}$$

For relatively small initial speed, i.e. $\varepsilon \ll 1$, it is a natural idea to look at the problem with $\varepsilon = 0$, the so-called "reduced problem":

$$x_0'' = -1\,, \qquad x_0(0) = 0\,, \quad x_0'(0) = 1\,. \tag{15}$$

One can hope that the "reduced problem" should provide a good approximation for the "full problem" and that the reduced problem is simpler to solve.
For scaling (#1) this procedure works since problem (15) is much easier to solve: integrating the constant twice and using the 2 initial conditions we easily obtain the "reduced solution" $x_0(t) = t - t^2/2$.

Remark. Note that this procedure of the "reduced solution" does *not* work for the other two scalings (#2) and (#3). For equation (9), setting $\varepsilon = 0$ changes completely the nature of the equation and leads to a contradiction; the reduced problem has no solution. For equation (12), when setting $\varepsilon = 0$, the initial position and velocity are zero, with a negative acceleration, so the reduced solution would have negative values for all positive times and thus not describe an upwards throw.
In these cases, the reduced problem $\varepsilon = 0$ does not provide any approximation to the original problem, much less a good one.
So for the purpose of asymptotics, the different scalings are not at all equivalent.

In the limit $\varepsilon \to 0$, one is working under the assumption that the dimensionless variables, and the derivatives appearing in the problem are "of order 1" (an expression made more precise below, when we introduce the Landau symbols). The alternative nondimensionalization choices (#2) and (#3) do not satisfy this assumption, in contrast to scaling (#1).

Calling the full problem an ε-perturbation of the "reduced problem", it seems obviously appropriate to call it a "singular perturbation" for scaling (#2), since for $\varepsilon = 0$ we lose the highest derivative.
For scaling (#1) it seems appropriate to call it a "regular perturbation", since the reduced problem is a simplification of the full problem that keeps the main features. In such a situation it makes sense to look for "small corrections" of the reduced problem for getting better approximations.

In order to put the problem in an abstract mathematical setting, we define an operator

$$F(x, \varepsilon) = \Big(x'' + (1 + \varepsilon x)^{-2}, x(0), x'(0) - 1\Big) \tag{16}$$

mapping the real function x (belonging to some suitable function space) and the real number ε to a triple, whose first component is a real function, and the second and third components are real numbers.

We formulate the complete probem as $F(x, \varepsilon) = 0$, and now we determine x_0 by solving $F(x_0, 0) = 0$ as an approximation to the "exact" solution x.

Motivated by the example, we first consider the simple situation $F : \mathbb{R}^2 \to \mathbb{R}$ and the (algebraic) equation $F(x_\varepsilon, \varepsilon) = 0$ for the unknown number $x_\varepsilon \in \mathbb{R}$ and for the small parameter $\varepsilon \in \mathbb{R}$. This situation is covered by the *implicit function theorem:* If

1. F is smooth enough,
2. the approximate problem $F(x_0, 0) = 0$ has a solution $x_0 \in \mathbb{R}$, and
3. $\partial_x F(x_0, 0) \neq 0$,

then there exists a smooth real function $\varepsilon \mapsto x_\varepsilon$, defined for $|\varepsilon|$ small enough, such that $F(x_\varepsilon, \varepsilon) = 0$ and x_0 is the value of this function at $\varepsilon = 0$.

We can write an "asymptotic expansion" as

$$x_\varepsilon = \sum_{n=0}^{N} \varepsilon^n x_n + O(|\varepsilon|^{N+1}) \qquad \text{as } |\varepsilon| \to 0\,, \tag{17}$$

where the maximal order N depends on the smoothness of F.
Here we use the *'big O'* symbol, one of the **"Landau symbols"**, like *'little o'* or *'sharp O'*, defined as

Definition 2. *Let $(B, \|\cdot\|)$ denote a Banach space, let $\varepsilon_0 > 0$, and let $y_\varepsilon \in B$ and $g_\varepsilon > 0$ for $0 < \varepsilon \leq \varepsilon_0$. Then we write*
a) $y_\varepsilon = O(g_\varepsilon)$ as $\varepsilon \to 0$, iff there exists a constant $C > 0$ such that $\|y_\varepsilon\| \leq C g_\varepsilon$ for $0 < \varepsilon \leq \varepsilon_0$.
b) $y_\varepsilon = o(g_\varepsilon)$ as $\varepsilon \to 0$, iff $\lim_{\varepsilon \to 0} \|y_\varepsilon\|/g_\varepsilon = 0$.
c) $y_\varepsilon = O_s(g_\varepsilon)$ as $\varepsilon \to 0$, iff $y_\varepsilon = O(g_\varepsilon)$ and $g_\varepsilon = O(\|y_\varepsilon\|)$ as $\varepsilon \to 0$.

The following definition is a key to "perturbation theory".

Definition 3. *Let $(B, \|\cdot\|)$ denote a Banach space, let $\varepsilon_0 > 0$, $N \in \mathbb{N}$, let $x_0, \ldots, x_N \in B$ as well as $x_\varepsilon \in B$ for $0 < \varepsilon \leq \varepsilon_0$.*
The polynomial $x_0 + \varepsilon x_1 + \cdots + \varepsilon^N x_N$ is called an asymptotic expansion of order N for x_ε, iff

$$x_\varepsilon - \sum_{n=0}^{N} \varepsilon^n x_n = o(|\varepsilon|^N) \qquad \text{as } |\varepsilon| \to 0\,, \tag{18}$$

(which is slightly weaker than (17)).

Note that an "asymptotic expansion" is somewhat related to, but different than a "Taylor expansion": in a Taylor expansion, one considers the expansion order $N \to \infty$ with a fixed independent variable ε, whereas for the asymptotic expansion, the limit $\varepsilon \to 0$ is studied for a fixed order N.

There is a *formal* and a *rigorous* aspect (based e.g. on the implicit function theorem). The formal aspect is the mere computation of x_0 and, possibly, further coefficients $x_1, x_2, \ldots$ in the expansion. The rigorous aspect is the proof of the *validity* of the expansion.

In perturbation theory it is common to separate the two aspects, because they require ideas and methods of different nature, and also because the rigorous aspect is often too difficult and the formal result sufficient for application purposes. The situation is similar to numerical mathematics, where often the development of numerical methods can not be accompanied by rigorous convergence results.
We shall mainly discuss the formal procedure in the rest of this section and in the following two sections.
Section 1.5 will be concerned with rigorous aspects.

We start again with a *perturbation problem* of the form

$$F(x_\varepsilon, \varepsilon) = 0\,, \qquad 0 < \varepsilon \ll 1\,, \tag{19}$$

i.e. a problem containing a small dimensionless parameter, but now more generally with

$$F:\ B_1 \times [0, \varepsilon_0] \to B_2\,, \tag{20}$$

where $(B_1, \|\cdot\|_1)$ and $(B_2, \|\cdot\|_2)$ are Banach spaces.

Example 1 (continued)**.** For the example (16), a possible choice of spaces would be $B_1 = C^2([0, t_0])$, $B_2 = C([0, t_0]) \times \mathbb{R}^2$ with some $t_0 > 0$.

Again, it seems reasonable to look for approximations x_0 for solutions x_ε by solving the *reduced problem* $F(x_0, 0) = 0$. More seriously, this idea relies on a convergence and a continuity assumption, i.e. we assume

$$\lim_{\varepsilon \to 0} x_\varepsilon = x_0 \qquad \text{and} \qquad \lim_{\varepsilon \to 0} F(x_\varepsilon, \varepsilon) = F(x_0, 0)\,,$$

where the first limit is in B_1 and the second in B_2.

The appropriate definitions are as follows:

Definition 4. *Let F be as in (20). Then $x_{as} \in B_1$, possibly depending on ε, is called a formal approximation for the solution of $F(x_\varepsilon, \varepsilon) = 0$, if the* **residual** *$r_\varepsilon := F(x_{as}, \varepsilon)$ tends to zero in B_2 as $\varepsilon \to 0$.*

Example 1 (continued)**.** For the projectile problem with the definition of F in (16) and the approximation x_0 defined in (15) we obtain the residual

$$F(x_0, \varepsilon) = (-1 + (\varepsilon x_0 + 1)^{-2}, 0, 0) = \varepsilon \left(-\frac{x_0(2 + \varepsilon x_0)}{(\varepsilon x_0 + 1)^2}, 0, 0 \right) ,$$

which converges to zero in $B_2 = C([0, t_0]) \times \mathbb{R}^2$ for every $t_0 > 0$.

As a next step we aim to improve the approximation by constructing a formal asymptotic expansion of x_ε for some suitable finite order N, so we use the ansatz:

$$x_{as} = \sum_{n=0}^{N} \varepsilon^n x_n \tag{21}$$

For the computation of the "order 1" asymptotic expansion $x_{as} = x_0 + \varepsilon x_1$ of a solution we assume sufficient regularity of F and compute formally

$$F(x_0 + \varepsilon x_1, \varepsilon) = F(x_0, 0) + \varepsilon \left(\partial_x F(x_0, 0) x_1 + \partial_\varepsilon F(x_0, 0) \right) + O(\varepsilon^2) ,$$

where $\partial_x F$ and $\partial_\varepsilon F$ are partial Fréchet derivatives of F. Note that $\partial_\varepsilon F(x_0, 0) \in B_2$, whereas $\partial_x F(x_0, 0)$ is a linear map from B_1 to B_2. A $O(\varepsilon^2)$-residual can be produced by solving the linear problem

$$\partial_x F(x_0, 0) x_1 = -\partial_\varepsilon F(x_0, 0)$$

for x_1. Unique solvability is guaranteed, if there exists the inverse $\partial_x F(x_0, 0)^{-1} : B_2 \to B_1$. For higher order expansions the general computations are somewhat involved and we only state the general result:

Lemma 5. *Let, for fixed $x \in B_1$, F possess a formal asymptotic expansion*

$$F(x, \varepsilon) = \sum_{k=0}^{N} F_k(x) \varepsilon^k + O\left(\varepsilon^{N+1}\right) \tag{22}$$

with $(N+1)$-times Fréchet differentiable coefficients F_k, and let $x_\varepsilon \in B_1$ have an asymptotic expansion

$$x_\varepsilon = \sum_{k=0}^{N} x_k \varepsilon^k + O\left(\varepsilon^{N+1}\right) . \tag{23}$$

Then $F(x_\varepsilon, \varepsilon)$ has a formal asymptotic expansion of the form

$$F(x_\varepsilon, \varepsilon) = F(x_0, 0) + \sum_{k=1}^{N} \left(\partial_x F_0(x_0) x_k + G_k(x_0, \ldots, x_{k-1}) \right) \varepsilon^k + O(\varepsilon^{N+1}) . \tag{24}$$

With an nth order asymptotic expansion of the solution, a $O(\varepsilon^{N+1})$-residual can be produced by consecutively solving the problems

$$\partial_x F_0(x_0) x_k = -G_k(x_0, \ldots, x_{k-1}) , \qquad k = 1, \ldots, N ,$$

which are all linearized versions of the leading order problem $F(x_0, 0) = 0$.

Definition 6. *Let $F : (B_1, \|\cdot\|_1) \times [0, \varepsilon_0] \to (B_2, \|\cdot\|_2)$ be continuously Fréchet differentiable. Then the problem $F(x_\varepsilon, \varepsilon) = 0$ is called regularly perturbed, if the reduced problem $F(x_0, 0) = 0$ has a solution $x_0 \in B_1$, such that the Fréchet derivative $\partial_x F(x_0, 0) : B_1 \to B_2$ has a bounded inverse. Otherwise the problem is called singularly perturbed.*

The bounded invertibility of the linearization is the generalization of the above mentioned condition $\partial_x F(x_0, 0) \neq 0$ for the application of the implicit function theorem in the case $B_1 = B_2 = \mathbb{R}$.

Example 1 (continued)**.** We return to the scaled projectile problem (4)- (5):

$$\begin{aligned} x'' &= -(\varepsilon x + 1)^{-2}\,, \\ x(0) = 0\,, &\qquad x'(0) = 1\,. \end{aligned} \tag{25}$$

Substituting the expansion - i.e. "plugging the ansatz

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O\left(\varepsilon^3\right) \tag{26}$$

into the equation" yields

$$\begin{aligned} x_0'' + \varepsilon x_1'' + \varepsilon^2 x_2'' + O\left(\varepsilon^3\right) &= -\left(1 + \varepsilon x_0 + \varepsilon^2 x_1 + \varepsilon^3 x_2 + O\left(\varepsilon^4\right)\right)^{-2}\,, \\ x_0\left(0\right) + \varepsilon x_1\left(0\right) + \varepsilon^2 x_2\left(0\right) + O\left(\varepsilon^3\right) &= 0\,, \\ x_0'\left(0\right) + \varepsilon x_1'\left(0\right) + \varepsilon^2 x_2'\left(0\right) + O\left(\varepsilon^3\right) &= 1\,. \end{aligned}$$

We use the Taylor expansion

$$-\left(1 + \varepsilon x_0 + \varepsilon^2 x_1 + \varepsilon^3 x_2 + O(\varepsilon^4)\right)^{-2} = -1 + \varepsilon 2x_0 + \varepsilon^2(2x_1 - 3x_0^2) + O(\varepsilon^3)$$

of the right hand side of the differential equation and compare coefficients of ε^0:

$$x_0'' = -1\,, \qquad x_0(0) = 0\,, \quad x_0'(0) = 1\,.$$

Coefficients of ε^1:

$$x_1'' = 2x_0\,, \qquad x_1(0) = x_1'(0) = 0\,.$$

Coefficients of ε^2:

$$x_2'' = 2x_1 - 3x_0^2\,, \qquad x_2(0) = x_2'(0) = 0\,.$$

The reduced problem has been solved already:

$$x_0(t) = t - \frac{t^2}{2}\,.$$

For the higher order coefficients we obtain

$$x_1(t) = \frac{t^3}{3}\left(1 - \frac{1}{4}t\right)\,, \qquad x_2(t) = -\frac{t^4}{4}\left(1 - \frac{11}{15}t + \frac{11}{90}t^2\right)\,,$$

and, thus, the formal expansion

$$x(t) = t - \frac{t^2}{2} + \varepsilon\frac{t^3}{3}\left(1 - \frac{1}{4}t\right) - \varepsilon^2\frac{t^4}{4}\left(1 - \frac{11}{15}t + \frac{11}{90}t^2\right) + O(\varepsilon^3)\,. \tag{27}$$

For finding an approximation of the time T, where the projectile hits the ground, we use the same procedure and make the ansatz

$$T = T_0 + \varepsilon T_1 + O(\varepsilon^2)\,.$$

Substitution into the expansion for x and re-expansion gives

$$x(T) = \frac{T_0}{2}(2 - T_0) + \varepsilon \left((1 - T_0)T_1 + \frac{T_0^3}{3} \left(1 - \frac{T_0}{4} \right) \right) + O(\varepsilon^2) \,,$$

and comparing coefficients of powers of ε in the equation $x(T) = 0$ leads to $T_0 = 2$, $T_1 = 4/3$, and therefore

$$T = 2 + \frac{4}{3}\varepsilon + O(\varepsilon^2) \,.$$

From the physical point of view, a positive correction to the leading order term seems plausible, since the full problem yields a weaker gravitation than the leading order approximation with the constant gravitation given at the surface of the earth, such that the projectile can be expected to fly higher and longer.

Asymptotic Expansion vs. Taylor Expansion: $\varepsilon \to 0$ *vs.* $N \to \infty$

At first sight these 2 concepts might seem identical (and are sometimes mixed up by physicists), especially since in practice we often use Taylor expansions to construct asymptotic expansions. The fundamental difference is that with asymptotic series we are interested in the expansion in the limit $\varepsilon \to 0$ at fixed order N, whereas with Taylor series we are interested in the convergence of the infinite series as $N \to \infty$ for fixed ε.
There are functions for which asymptotic expansions up to arbitrary order exist, although the corresponding infinite series diverge for all $\varepsilon > 0$.

We give another definition in this context:

Definition 7. ***Asymptotic equivalence***
For $0 < \varepsilon \leq \varepsilon_0$ let x_ε and y_ε be ε-dependent elements of a normed space $(B, \|\cdot\|)$. Then ***we call x_ε and y_ε asymptotically equivalent for $\varepsilon \to 0$, if***

$$\|x_\varepsilon - y_\varepsilon\| = o(\|y_\varepsilon\|) \qquad \text{for} \quad \varepsilon \to 0 \tag{28}$$

i.e. if the relative error of the approximation of x_ε by y_ε (or the other way round) goes to zero when $\varepsilon \to 0$.

The name „equivalence“ is justified since obviously the above definition is symmetric w.r.t. x_ε and y_ε. Taking y_ε as an approximation for x_ε, it is common to call y_ε an *asymptotic approximation* for x_ε.

Example for asymptotic approximation/equivalence:
Let $B = \mathbb{R}$ and $x_\varepsilon = x(\varepsilon) = \sin(\varepsilon)$. Then the following 3 functions

$$y(\varepsilon) = \varepsilon, \qquad y(\varepsilon) = \varepsilon + 2\varepsilon^2, \qquad y(\varepsilon) = \varepsilon - \frac{1}{6}\varepsilon^3 \tag{29}$$

are each an asymptotic approximation of $x(\varepsilon)$, since we have $x(\varepsilon) = \varepsilon - \frac{1}{6}\varepsilon^3 + O(\varepsilon^5)$.

Example 3. "function with boundary layer at x=0"
Let $B = C([0, 1])$ and consider the real function $x_\varepsilon \in B$, $x_\varepsilon : [0, 1] \to \mathbb{R}$ given by

$$x_\varepsilon(t) = t + \exp\left(-\frac{t}{\varepsilon} \right) , \qquad 0 \leq t \leq 1 \,.$$

For positive $t > 0$ the sequence of functions $x_\varepsilon(t)$ converges with $\varepsilon \to 0$ towards $x_0(t) = t$. For $t = 0$, however, the value 1 is an asymptotic approximation - see fig 1.

a) If we consider x_ε as an element of the Banach-space $C([0,1])$ with norm

$$\|x\|_\infty = \max_{t\in[0,1]} |x(t)|\,, \tag{30}$$

then x_0 is not asymptotically equivalent to x_ε, since we have

$$\|x_\varepsilon - x_0\|_\infty = x_\varepsilon(0) - x_0(0) = 1\,, \qquad \|x_0\|_\infty = 1$$

Hence there is no ε independent asymptotic approximation for x_ε.

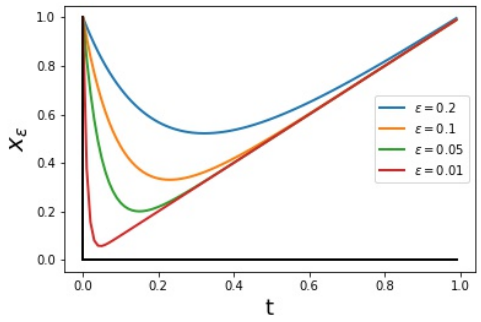

Figure 1: The function $x_\varepsilon(t)$ in $[0,1]$ for $\varepsilon = 0.2, 0.1, 0.05, 0.01$

b) If, however, we choose $B = L^p((0,1))$, with $1 \le p < \infty$, as the underlying Banach-space for this function $x_\varepsilon(t)$, the situation is completely different ! Now we use an integral-norm

$$\|f\|_p = \left(\int_0^1 |f(t)|^p dt\right)^{1/p}. \tag{31}$$

And for $p = 1$ we have

$$\|x_\varepsilon - x_0\|_1 = \int_0^1 e^{-\frac{t}{\varepsilon}} dt = -\varepsilon e^{-\frac{t}{\varepsilon}}\Big|_0^1 \le \varepsilon\,, \qquad \|x_0\|_1 = \frac{1}{2}\,.$$

Now x_0 is an asymptotic approximation of x_ε, in the sense of the L^1-Norm.
The point is that the quality of an approximation is "poorer" for integral norms than for pointwise norms. □

1.3 Validity of formal asymptotic approximations

Theorem 8. *Let the map $F : (B_1, \|\cdot\|_1) \to (B_2, \|\cdot\|_2)$ be Fréchet differentiable at $u_{as} \in B_1$. Let the Fréchet derivative $DF(u_{as})$ be boundedly invertible with*

$$\|DF(u_{as})^{-1} f\|_1 \le K\|f\|_2 \qquad \forall f \in B_2\,. \tag{32}$$

Let $P(v) := F(u_{as} + v) - F(u_{as}) - DF(u_{as})v$ satisfy

$$\|P(v_1) - P(v_2)\|_2 \le L\delta\|v_1 - v_2\|_1 \qquad \text{for } \|v_1\|_1\,, \|v_2\|_1 \le \delta\,, \tag{33}$$

and let the residual $\rho := F(u_{as})$ satisfy

$$\|\rho\|_2 \le \frac{1}{4K^2L}\,.$$

Then there exists a solution $u \in B_1$ of $F(u) = 0$, unique in the ball $K_{\overline{\delta}}(u_{as}) \subset B_1$ with center u_{as} and radius $\overline{\delta} = \frac{1}{2KL}$, satisfying

$$\|u - u_{as}\|_1 \leq 2K\|\rho\|_2 \,. \tag{34}$$

Proof: The proof is an application of the Banach fixed point theorem.

With $R := u - u_{as}$ the equation $F(u) = 0$ is equivalent to the fixed point problem

$$R = G(R) := -DF(u_{as})^{-1} \left(\rho + P(R)\right) \,.$$

We shall prove that $G(R)$ is a contraction in $K_{\overline{\delta}}(0)$. The estimate

$$\|G(R)\|_1 \leq K\|\rho + P(R)\|_2 \leq K\left(\frac{1}{4K^2L} + L\overline{\delta}^2\right) = \overline{\delta}$$

proves $G:\ K_{\overline{\delta}}(0) \to K_{\overline{\delta}}(0)$. The contraction property follows from

$$\|G(R_1) - G(R_2)\|_1 \leq K\|P(R_1) - P(R_2)\|_2 \leq KL\overline{\delta}\,\|R_1 - R_2\|_1 = \frac{1}{2}\|R_1 - R_2\|_1 \,.$$

This proves the existence of a fixed point R, unique in $K_{\overline{\delta}}(0)$, satisfying

$$\begin{aligned} \|R\|_1 &= \|G\,(R)\,\|_1 \leq K\,(\|\rho\|_2 + \|P(R)\|_2) \leq \\ &\leq K\left(\|\rho\|_2 + L\overline{\delta}\,\|R\|_1\right) = K\|\rho\|_2 + \frac{1}{2}\|R\|_1 \,, \end{aligned}$$

which implies (34). □

The assumption (33) is a smoothness assumption on F, equivalent to Lipschitz continuity of its Fréchet derivative.

The essential assumptions are smallness of the residual, a property that corresponds to *consistency* in numerical analysis, and the *stability* assumption (32). Together they imply smallness of the error, corresponding to *convergence* in numerical analysis ("Lax theorem" : Consistency + Stability => Convergence).

Applicability of Theorem 8 also implies *well posedness* in the sense of Hadamard: There exists a (locally in $K_{\overline{\delta}}(u_{as})$) unique solution, continuously depending on perturbations (ρ).

Example 1 (continued)**.**

By integrating twice, the projectile problem (14) of example 1 can be rewritten as the integral equation

$$x(t) = t - \int_0^t \int_0^\tau \frac{ds}{(1+\varepsilon x(s))^2} d\tau = t - \int_0^t \frac{(t-\tau)d\tau}{(1+\varepsilon x(\tau))^2} \,, \tag{35}$$

which is already in fixed point form. We attempt to show validity of the approximation $x_0(t) = t - t^2/2$. For $R(t) = x(t) - x_0(t)$ we have

$$R(t) = \int_0^t (t-\tau)\left(1 - \frac{1}{(1+\varepsilon(x_0(\tau) + R(\tau)))^2}\right) d\tau =: G(R)(t) \,,$$

where we consider G as a map from $(C([0,3]), \|\cdot\|_\infty)$ to itself. The right end 3 of the time interval is chosen such that the time, where the projectile hits the ground and which is

expected to be close to 2, is safely included. Under the assumption $\varepsilon|x_0 + R| \le \frac{1}{2}$ in $[0,3]$ we have

$$|G(R)(t)| \le 3\int_0^t (t-\tau)d\tau \le \frac{27}{2}\,.$$

The estimate $\varepsilon\|x_0+R\|_\infty \le \varepsilon(7/2+27/2) = 17\varepsilon$ for $\|R\|_\infty \le 27/2$ leads to the assumption $\varepsilon \le 1/34$, implying that G maps the ball $K_{27/2}(0) \subset C([0,3])$ into itself. By the mean value theorem we have

$$\begin{aligned} |G(R_1)(t) - G(R_2)(t)| &\le \int_0^t (t-\tau)\frac{2}{(1+\varepsilon(x_0(\tau)+\tilde{R}(\tau)))^3}\varepsilon|R_1(\tau)-R_2(\tau)|d\tau \\ &\le 16\varepsilon\|R_1-R_2\|_\infty \int_0^t (t-\tau)d\tau \le 72\varepsilon\|R_1-R_2\|_\infty\,. \end{aligned}$$

Thus for any $\varepsilon < 1/72$, G is a contraction on $K_{27/2}(0)$. This implies unique solvability of the projectile problem and the estimate (similar to above)

$$\|R\|_\infty = \|G(R)\|_\infty \le 72\varepsilon\|x_0+R\|_\infty \le 252\varepsilon + \frac{1}{2}\|R\|_\infty\,,$$

and therefore

$$\|x - x_0\|_\infty \le 504\varepsilon\,.$$

The constant 504 is unpleasantly large, but we have been rather generous in our estimates. Comparing with the formal asymptotic expansion (27), we would expect

$$\|x - x_0\|_\infty \le \varepsilon\|x_1\|_\infty + O(\varepsilon^2) = \frac{9}{4}\varepsilon + O(\varepsilon^2)\,.$$

A second approach starts with the full perturbed problem and uses compactness results.

Example 1 (continued)**.**

We start again with (35), the projectile problem written as a fixed point equation $x = H(x,\varepsilon)$ with

$$H(x,\varepsilon)(t) = t - \int_0^t \frac{(t-\tau)d\tau}{(1+\varepsilon x(\tau))^2}\,.$$

Similarly to above we note that for $\varepsilon\|x\|_\infty \le 1/2$ we have

$$|H(x,\varepsilon)(t)| \le t + 2t^2 \le 21\,.$$

Therefore under the assumption $\varepsilon \le 1/42$, the operator $x \mapsto H(x,\varepsilon)$ maps the closed ball $\overline{K_{21}(0)} \subset C([0,3])$ into itself. Combined with standard ODE results[1] this implies that the unique solution x_ε, $\varepsilon \le 1/42$, of the projectile problem exists at least in the time interval $[0,3]$. It satisfies

$$x_\varepsilon = H(x_\varepsilon,\varepsilon)\,, \qquad \|x_\varepsilon\|_\infty \le 21\,.$$

Furthermore x_ε is continuously differentiable and we have

$$\left|\frac{dx_\varepsilon}{dt}(t)\right| = \left|1 - \int_0^t \frac{d\tau}{(1+\varepsilon x_\varepsilon(\tau))^2}\right| \le 1 + 4t \le 13\,.$$

[1] a) the Picard theorem for existence and uniqueness in a small enough time interval and
b) existence at least up to a time T, where the solution is guaranteed to remain in a compact subset (here $[-21,21]$) of the domain of the right hand side.

By the Arzela-Ascoli theorem $\{x_\varepsilon : 0 < \varepsilon \leq 1/42\}$ is relatively compact in $C([0,3])$. As a consequence, there exists a sequence $\varepsilon_k \to 0$ as $k \to \infty$, such that $x_{\varepsilon_k} \to x_0$ in $C([0,3])$. Obviously, we can pass to the limit in $x_{\varepsilon_k} = H(x_{\varepsilon_k}, \varepsilon_k)$ and obtain

$$x_0(t) = H(x_0, 0)(t) = t - \frac{t^2}{2}\,.$$

The uniqueness of x_0 implies $\lim_{\varepsilon\to 0} x_\varepsilon = x_0$ in $C([0,3])$. The convergence has been proven, but without an error estimate.

1.4 Singular perturbations - boundary layers

Even in case that we perturb the highest derivative in the problem/equation and hence expect a singular perturbation, we can still use the idea of an asymptotic expansion with the reduced solution as the basis term. We can save the idea by adding extra "boundary layer terms" that remedy the problem that the reduced problem can not solve all boundary/initial conditions.

Example 4. *Consider the initial value problem*

$$\varepsilon u' = -u + x + \varepsilon\,, \qquad u(0) = 1\,. \tag{36}$$

Since the reduced problem

$$0 = -u_0 + x\,, \qquad u_0(0) = 1\,, \tag{37}$$

has no solution, the problem is singularly perturbed.

However, our previously studied function

$$u_\varepsilon(x) = e^{-x/\varepsilon} + x$$

is the solution of (36). And we see from Fig 1. that the solution $\overline{u}(x) = x$ of the reduced differential equation approximates the exact solution everywhere except close to $x = 0$.

In order to develop this idea into a well defined mathematical machinery where we introduce "(regularizing) local / layer variables" at (sets of) points like $x = 0$ in the above example, and then systematically find "boundary layer terms" that fix the problem at these (sets of) points, we need some clever but tedious definitions.

Definition 9. *Let $D \subset \mathbb{R}^n$ be a domain and let $u_\varepsilon \in C(\overline{D})$ for $0 < \varepsilon \leq \varepsilon_0$.*
a) The family/sequence of functions $\{u_\varepsilon : 0 < \varepsilon \leq \varepsilon_0\}$ is called regular in D, if u_ε converges uniformly, i.e. with respect to the supremum norm $\|u\|_D := \sup_{x\in D} |u(x)|$, in D as $\varepsilon \to 0$.
b) Let $S \subset \overline{D}$ be a C^1-manifold with dimension smaller than n. The family $\{u_\varepsilon\}$ has layer behavior at S if it is not regular in D, but regular in each D_1 with $\overline{D}_1 \subset D \setminus S$.

Lemma 10. *Let $\{u_\varepsilon\}$ have layer behavior at S, then there exists an outer limit $\overline{u} \in C(\overline{D} \setminus S)$ such that*

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - \overline{u}\|_{D_1} = 0\,, \qquad \text{for } \overline{D}_1 \subset \overline{D} \setminus S\,. \tag{38}$$

Proof: Since $\overline{D} \setminus S$ is open relative to $\overline{D}$, every point in $\overline{D} \setminus S$ lies in a suitable D_1. This implies the existence of a pointwise limit $\overline{u}$, which is continuous as a consequence of the uniform convergence in D_1. $\square$

Example 4 (continued)**.** In example 4,

$$u_\varepsilon(x) = e^{-x/\varepsilon} + x\,, \qquad D = (0,1)\,, \quad S = \{0\}\,, \quad \overline{u}(x) = x\,. \tag{39}$$

Example 5.

$$u_\varepsilon(x,y) = \tanh\frac{x-y^2}{\varepsilon}, \qquad D = \mathbb{R}^2\,, \quad S = \{(x,y)\,\big|\, x-y^2 = 0\}\,, \tag{40}$$

$$\overline{u}(x,y) = \begin{cases} -1 & \text{für} \quad x-y^2 < 0\,, \\ 1 & \text{für} \quad x-y^2 > 0\,. \end{cases} \tag{41}$$

The function $\overline{u}$ describes the behavior of u_ε away from layers. For studying u_ε inside a layer, i.e. close to S, we use a mathematical magnifying glass in the form of a coordinate transformation:
For a C^1-manifold S with $\dim(S) = n-k$ we introduce "local coordinates" $z = z(x)$, defined in a neighborhood of S, such that

$$S = \{x : z_1(x) = \cdots = z_k(x) = 0\}\,.$$

Local variables $\xi(x)$ close to S are then given by a "rescaling with powers of ε" :

$$\begin{aligned} \xi_i &= z_i\varepsilon^{-\alpha_i}\,, \qquad \text{with } \alpha_i > 0\,, \text{ for } i = 1,\ldots,k\,, \\ \xi_i &= z_i\,, \qquad \text{for } i = k+1,\ldots,n\,. \end{aligned}$$

Local variables serve their purpose if they are *regularizing*, i.e. if the transformed family

$$U_\varepsilon(\xi) := u_\varepsilon\left(x(\xi,\varepsilon)\right)$$

is regular in an appropriate region.

Example 4 (continued)**.** For $u_\varepsilon(x) = e^{-x/\varepsilon} + x$, a local variable close to $x = 0$ is given by $\xi = x\varepsilon^{-\alpha}$. The transformed function

$$U_\varepsilon(\xi) = \exp(-\xi\varepsilon^{\alpha-1}) + \varepsilon^\alpha\xi$$

is for $\alpha \geq 1$ regular on ξ-intervals of the form $(0,X)$ with

$$\lim_{\varepsilon\to 0} U_\varepsilon(\xi) = \begin{cases} e^{-\xi} & \text{für} \quad \alpha = 1\,, \\ 1 & \text{für} \quad \alpha > 1\,, \end{cases}$$

and for $0 < \alpha < 1$ regular on intervals of the form (X_1, X_2) with $X_1 > 0$, and $\lim_{\varepsilon\to 0} U_\varepsilon(\xi) = 0$.

Example 6. Not always a reasonable regularization can be achieved by local variables. In the previous section we have been interested in the behavior of functions like

$$u_\varepsilon(t) = e^{-\varepsilon t}\sin t$$

as $t \to \infty$. This is equivalent to studying the local behavior of $s = 1/t$ near $s = 0$ of

$$v_\varepsilon(s) = e^{-\varepsilon/s} \sin \frac{1}{s} \,.$$

Obviously $\{v_\varepsilon\}$ has layer behavior with the outer limit $\overline{v}(s) = \sin(1/s)$. The local variable $\sigma = s\varepsilon^{-\alpha}$ leads to

$$V_\varepsilon(\sigma) = \exp(-\varepsilon^{1-\alpha}/\sigma) \sin\left(\frac{1}{\varepsilon^\alpha \sigma}\right) .$$

For $0 < \alpha \leq 1$ there is no σ-interval, where V_ε is regular. For $\alpha > 1$ we have $\lim_{\varepsilon\to 0} V_\varepsilon(\sigma) = 0 = v_\varepsilon(0)$, which is not very informative.

In the example $u_\varepsilon(x) = e^{-x/\varepsilon} + x$ the local variable $\xi = x/\varepsilon$ seems most appropriate, since the corresponding approximation $e^{-\xi}$ is the most informative. The other approximations 0 and 1 can be recovered from it with $\xi = 0$ and $\xi \to \infty$. These ideas are formalized in the following.

Definition 11.
a) Let ξ be a local variable. Then we define the local limit of u_ε with respect to ξ by

$$\left(\lim_{\xi} u_\varepsilon\right)(\xi) := \lim_{\varepsilon\to 0} u_\varepsilon(x(\xi,\varepsilon)) \,. \tag{42}$$

The domain of $\lim_\xi u_\varepsilon$ is the union of all compact sets, where the above limit is uniform.

b) Let ξ_1 and ξ_2 be local variables, and let D_2 and D_{12} be the domains of the local limits of u_ε and, respectively, of $\lim_{\xi_1} u_\varepsilon$ with respect to ξ_2. Then the local limit of u_ε with respect to ξ_2 is contained in the local limit with respect to ξ_1, if $D_2 \subset D_{12}$ and

$$\lim_{\xi_2}\left(\lim_{\xi_1} u_\varepsilon\right) = \lim_{\xi_2} u_\varepsilon \,, \tag{43}$$

uniformly in compact subsets of D_2.

c) A local limit is called significant, if it is not contained in any other local limit, i.e. if it is a maximal element with respect to the order relation defined in b). Local variables corresponding to significant limits are called layer variables.

Example 7. There can be more than one significant limit. Consider

$$u_\varepsilon(x) = x\left(\frac{e^{-x/\varepsilon}}{x+\varepsilon^2} + 1\right) \qquad 0 \leq x \leq 1 \,. \tag{44}$$

Obviously $\overline{u}(x) = x$, and $\{u_\varepsilon\}$ has layer behavior since $\lim_{\varepsilon\to 0} u_\varepsilon(\varepsilon^2) = 1/2$. It is easily seen that there are two layer variables: $\xi_1 = x/\varepsilon$ and $\xi_2 = x/\varepsilon^2$. The corresponding limits are $e^{-\xi_1}$ and, respectively, $\frac{\xi_2}{1+\xi_2}$.

We propose a procedure for finding approximations, which are uniformly valid within and away from layer regions. We consider the situation with exactly one layer variable ξ. The function

$$\widetilde{u}_\varepsilon = u_\varepsilon - \overline{u}$$

is small away from S. We shall use the *heuristic principle* that $\widetilde{u}_\varepsilon$ can be approximated uniformly by the significant local limit. This means that

$$\overline{u} + \lim_{\xi}(u_\varepsilon - \overline{u})$$

is expected to be a uniform asymptotic approximation for u_ε. The second term, which is small away from S, is called a *layer correction.*

Example 8 (continued)**.** The above procedure can be iterated in the case of more than one layer variables. We return to example (44) and approximate u_ε by

$$\overline{u} + \lim_{\xi_1}(u_\varepsilon - \overline{u}) + \lim_{\xi_2}\left(u_\varepsilon - \overline{u} - \lim_{\xi_1}(u_\varepsilon - \overline{u})\right) .$$

The three contribution depend on the variables x, ξ_1, and, respectively, ξ_2. Straightforward computations show

$$\left\| u_\varepsilon - x - e^{-x/\varepsilon} + \frac{1}{1 + x/\varepsilon^2} \right\|_{(0,1)} = O(\varepsilon) . \tag{45}$$

The approximation can be improved by dividing the error by ε and repeating the procedure. Iteration leads to an asymptotic expansion for u_ε in the form

$$u_\varepsilon(x) = \sum_{k=0}^{N} \left(\overline{u}_k(x) + \widetilde{u}_k\left(\frac{x}{\varepsilon}\right) + \widehat{u}_k\left(\frac{x}{\varepsilon^2}\right)\right) \varepsilon^k + O(\varepsilon^{N+1}) . \tag{46}$$

The coefficients of ε^0 have already appeared above:

$$\overline{u}_0(x) = \overline{u}(x) = x , \qquad \widetilde{u}_0(\xi_1) = e^{-\xi_1} , \qquad \widehat{u}_0(\xi_2) = -\frac{1}{1+\xi_2} . \tag{47}$$

For the coefficients of ε^1 we obtain

$$\overline{u}_1(x) = 0 , \qquad \widetilde{u}_1(\xi_1) = \frac{1 - e^{-\xi_1}}{\xi_1} , \qquad \widehat{u}_1(\xi_2) = -\frac{1}{1+\xi_2} . \tag{48}$$

The fact that the layer corrections do not contribute outside of the layer can be written as

$$\lim_{\xi_1 \to \infty} \widetilde{u}_k(\xi_1) = \lim_{\xi_2 \to \infty} \widehat{u}_k(\xi_2) = 0 .$$

We now turn our attention to problems whose solutions have layer behavior.

Example 8. Consider the boundary value problem

$$\begin{aligned} -\varepsilon u'' + u' + u = 0 , \qquad 0 < x < 1 , \\ u(0) = 1 , \qquad u(1) = 0 . \end{aligned} \tag{49}$$

The problem is singularly perturbed since solutions of the reduced differential equation

$$u' + u = 0 , \tag{50}$$

cannot satisfy both boundary conditions.

We shall call a differential equation containing a small parameter ε *singularly perturbed,* if setting $\varepsilon = 0$ changes the *type* of the equation. By 'type' we mean the class of auxiliary conditions which, together with the differential equation, produce a well posed problem. The type of an ODE is determined by its order. Thus, an ODE is singularly perturbed, if the reduced equation is of lower order. For PDEs a definition of type is a more subtle matter.

Problems consisting of singularly perturbed differential equations and auxiliary (i.e. initial or boundary) conditions are typically singularly perturbed problems. Solutions of the reduced differential equation typically

1. cannot satisfy all auxiliary conditions and/or
2. do not have the smoothness properties expected from the exact solution of the perturbed problem.

Heuristic rule: *Solutions of problems involving singularly perturbed differential equations have layer behavior at manifolds, where auxiliary conditions are posed or where a loss of smoothness occurs.*

As a consequence, the outer limit of the solution can be chosen as a solution of the reduced differential equation.

Example 9 (continued)**.** The general solution of the reduced ODE (50) is given by

$$\overline{u}(x) = Ae^{-x}\,. \tag{51}$$

It is not clear yet, how the constant A has to be chosen. According to the heuristic assumption boundary layers may occur at $x = 0$ and/or $x = 1$. In terms of the local variable $\xi = x\varepsilon^{-\alpha}$ near $x = 0$, the differential equation takes the form

$$-\varepsilon^{1-2\alpha}\ddot{u} + \varepsilon^{-\alpha}\dot{u} + u = 0\,, \qquad \left(\dot{u} = \frac{du}{d\xi}\right)\,. \tag{52}$$

For passing to the limit $\varepsilon \to 0$ we need to multiply by a power of ε in dependence of α. In the limit we obtain

$$\begin{aligned} \dot{u} &= 0 \qquad \text{for } 0 < \alpha < 1\,,\\ -\ddot{u} + \dot{u} &= 0 \qquad \text{for } \alpha = 1\,,\\ -\ddot{u} &= 0 \qquad \text{for } \alpha > 1\,. \end{aligned}$$

Definition 12.
a) Let L_ε be a linear differential operator, which becomes $\mathcal{L}_\varepsilon$ when written in terms of the local variable ξ. Let $\gamma \in \mathbb{R}$ be chosen such that $\varepsilon^\gamma \mathcal{L}_\varepsilon(u) = O_s(1)$. Then the local degeneration of L_ε with respect to ξ is defined by

$$\lim_{\xi} L_\varepsilon(u) := \lim_{\varepsilon\to 0} \varepsilon^\gamma \mathcal{L}_\varepsilon(u)\,.$$

b) The local degeneration of L_ε with respect to ξ_2 is contained in the local degeneration with respect to ξ_1, iff

$$\lim_{\xi_2}\left(\lim_{\xi_1} L_\varepsilon(u)\right) = \lim_{\xi_2} L_\varepsilon(u)\,.$$

A significant degeneration is a local degeneration, which is not contained in any other.

Correspondence principle: *When u_ε solves a linear differential equation $L_\varepsilon(u_\varepsilon) = 0$, then we expect that every significant limit of u_ε solves the corresponding significant degeneration of the differential equation.*

Example 9 (continued)**.** In the example (52) the local variable $\xi = x/\varepsilon$ leads to a significant degeneration near $x = 0$. For a boundary layer near $x = 1$ we find similarly $\eta = (1 - x)/\varepsilon$. The above heuristic principles lead to the ansatz

$$u(x, \varepsilon) = \overline{u}(x) + \widetilde{u}(\xi) + \widehat{u}(\eta) + O(\varepsilon)\,,$$

with

$$\lim_{\xi\to\infty} \widetilde{u}(\xi) = \lim_{\eta\to\infty} \widehat{u}(\eta) = 0\,.$$

Actually we shall need the stronger assumption that also the derivatives of $\widetilde{u}$ and of $\widehat{u}$ tend to zero as the arguments tend to infinity. We obtain the equations for $\overline{u}$, $\widetilde{u}$ and $\widehat{u}$ by inserting the ansatz into (49) and then writing the resulting expression solely in terms of x, ξ and η. Passing to the limit $\varepsilon \to 0$, we get for $\overline{u}$:

$$\overline{u}_x + \overline{u} = 0$$

so we get back the reduced equation (50). Now we write the whole equation in terms of the independent variable ξ, i.e we set $x = \varepsilon\xi$ and $\eta = \frac{1}{\varepsilon} - \xi$. Passing to the limit $\varepsilon \to 0$ and using the decay assumption on $\widehat{u}$ implies

$$-\frac{d\widetilde{u}^2}{d\xi^2} + \frac{d\widetilde{u}}{d\xi} = 0\,,$$

which is the significant degeneration of the problem corresponding to the boundary layer $\{x = 0\}$. Therefore $\widetilde{u} = 0$, since this is the only decaying solution. The analogous procedure with the independent variable η leads to

$$-\frac{d\widehat{u}^2}{d\eta^2} - \frac{d\widehat{u}}{d\eta} = 0\,,$$

and, thus, $\widehat{u}(\eta) = Be^{-\eta}$ with an arbitrary constant $B \in \mathbb{R}$. Combining these results with the general solution (51) of the reduced equation gives

$$u(x) = Ae^{-x} + Be^{(x-1)/\varepsilon} + O(\varepsilon)\,,$$

and, with the boundary conditions in (49),

$$u(x) = e^{-x} - e^{-1+(x-1)/\varepsilon} + O(\varepsilon)\,.$$

This result can be easily verified by comparison with the exact solution of (49).