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Applied Analysis Problems

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1 Sheet 6

1.1 Fourier Transform of the convolution

Consider the function  $f(x)$ , which has a Fourier Transform  $\hat{f}(\xi)$ , now let us compute the Fourier transform of

$$g(x) = f(3x - 1) \sin(x). \quad (1)$$

We know that the Fourier transform of the convolution is

$$\widehat{(f(3x - 1) * g(x))} = \widehat{f(3x - 1)} \cdot \hat{g}(\xi). \quad (2)$$

The Fourier transform of  $f(3x - 1)$  is simply done by substituting a new variable

$$\widehat{f(3x - 1)} = \frac{1}{3} e^{2\pi i \frac{\xi}{3}} \hat{f}\left(\frac{\xi}{3}\right). \quad (3)$$

The Fourier transform of  $\sin(x)$  can be calculated when looking at the Fourier transform of the Dirac-delta function

$$\widehat{\delta(ax - b)} = \int_{\mathbb{R}} \delta(ax - b) e^{-2\pi i x \xi} dx \quad (y = ax - b) \quad (4)$$

$$= \int_{\mathbb{R}} \delta(y) e^{-2\pi i (y+b) \frac{\xi}{a}} \frac{dy}{a} \quad (5)$$

$$= \frac{1}{a} e^{-2\pi i \xi \frac{b}{a}}. \quad (6)$$

We may plug in  $\sin(x)$  in the definition of the Fourier transformation and observe where we can use the Dirac-delta to to the inverse Fourier transform

$$\widehat{\sin(x)} = \int_{\mathbb{R}} \sin(x) e^{-2\pi i x \xi} dx = \quad (7)$$

$$= \frac{1}{2i} \int_{\mathbb{R}} (e^{ix} - e^{-ix}) e^{-2\pi i \xi x} dx \quad (8)$$

$$= \frac{1}{2i} \left( \int_{\mathbb{R}} e^{ix} e^{-2\pi i \xi x} dx + \int_{\mathbb{R}} e^{-ix} e^{-2\pi i \xi x} dx \right). \quad (9)$$

Here we may use the above formula for the Fourier transform of the Dirac delta. We choose  $a = 1$ ,  $b = \pm \frac{1}{2\pi}$  and do some  $y = -x$  substitutions and thereby get the following result

$$\widehat{\sin(x)} = \frac{1}{2i} \left( \delta\left(\xi - \frac{1}{2\pi}\right) - \delta\left(\xi + \frac{1}{2\pi}\right) \right) \quad (10)$$

The whole result is thereby

$$\widehat{f(3x-1) \cdot \sin(x)} = \frac{1}{6i} \left( e^{2\pi i(\frac{\xi}{3} - \frac{1}{2\pi})} \hat{f}\left(\frac{\xi}{3} - \frac{1}{2\pi}\right) - e^{2\pi i(\frac{\xi}{3} + \frac{1}{2\pi})} \hat{f}\left(\frac{\xi}{3} + \frac{1}{2\pi}\right) \right) \quad (11)$$

## 1.2 More Fourier Transforms

Consider the function

$$f(x) = e^{-|x|} \quad (12)$$

The Fourier transform of this function is

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-|x|} e^{-2\pi i x \xi} dx \quad (13)$$

$$= \int_{-\infty}^0 e^x e^{-2\pi i x \xi} dx + \int_0^{\infty} e^{-x} e^{-2\pi i x \xi} dx = \quad (14)$$

$$= \frac{1}{1 - 2\pi i \xi} e^{(1 - 2\pi i \xi)x} \Big|_{-\infty}^0 + \frac{-1}{1 + 2\pi i \xi} e^{-(1 + 2\pi i \xi)x} \Big|_0^{\infty} = \quad (15)$$

$$= \frac{1}{1 - 2\pi i \xi} + \frac{1}{1 + 2\pi i \xi} = \quad (16)$$

$$= \frac{2}{1 + (2\pi \xi)^2}. \quad (17)$$

Let us use this result to solve the following integral

$$\int_{\mathbb{R}} \frac{\cos(a\xi)}{(2\pi \xi)^2 + 1} d\xi = \frac{1}{2} \int_{\mathbb{R}} \hat{f}(\xi) \operatorname{Re}(e^{ia\xi}) dx = \quad (18)$$

$$= \frac{1}{2} \operatorname{Re} \left( \int_{\mathbb{R}} \hat{f}(\xi) e^{ia\xi} d\xi \right) = \quad (19)$$

$$= \frac{1}{2} \operatorname{Re} \left( \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \frac{a}{2\pi} \xi} d\xi \right) = \quad (20)$$

$$= \frac{1}{2} \operatorname{Re} \left( f\left(\frac{a}{2\pi}\right) \right) = \quad (21)$$

$$= \frac{1}{2} e^{-\frac{|a|}{2\pi}} \quad (22)$$

## 1.3 Finite discrete Fourier transform

Consider  $s \in \mathbb{C}^N$  with entries

$$s[n] = \sin\left(2\pi \xi_0 \frac{n}{N}\right), \quad (23)$$

for same  $0 < \xi_0 < N$ . The finite discrete Fourier transform of  $s$  is

$$\hat{s}[k] = \frac{1}{N} \sum_{n=0}^{N-1} \sin\left(2\pi \xi_0 \frac{n}{N}\right) e^{-2\pi i \frac{k}{N} n} = \quad (24)$$

$$= \frac{1}{2iN} \left( \sum_{n=0}^{N-1} e^{2\pi i \frac{\xi_0}{N} (n-k)} + e^{-2\pi i \frac{\xi_0}{N} (n+k)} \right). \quad (25)$$

If we consider  $\xi_0 \in \mathbb{Z}$ , we have

$$\hat{s}[k] = \begin{cases} \frac{1}{2i} & \xi_0 = k \\ -\frac{1}{2i} & \xi_0 = -k \\ 0 & \text{else} \end{cases} \quad (26)$$

## 1.4 Discrete Matrix Notation

The convolution of two vectors  $f, g \in \mathbb{C}^N$ , can be expressed by a circulate matrix applied to  $f$

$$(f * g)[n] = \sum_{k=0}^{N-1} f[k]g[n-k]. \quad (27)$$

Consider  $g = s$ , then the matrix takes the following values

$$s[n-k] = s_{nk} = \sin\left(2\pi\xi_0 \frac{n-k}{N}\right). \quad (28)$$

The convolution with an impulse input  $f = \delta_{0k}$ , a vector that is 1 for  $k = 0$  and else 0 reads

$$\sum_k s_{nk} f_k = \sum_k s_{nk} \delta_{0k} = \quad (29)$$

$$= \sin\left(2\pi\xi_0 \frac{n}{N}\right). \quad (30)$$