Univ.-Prof. Dr. Radu Ioan Boţ

Vienna, October 18, 2022
To be discussed on October 25, 2022

Nonlinear Optimization

Exercise session 2

7. For the functions $g : \mathbb{R}^2 \to \mathbb{R}^2$ and $x_0 \in \mathbb{R}^2$ given below, find $X = \{(x,y) \in \mathbb{R}^2 : g(x,y) \leqq 0\}$, the tangent cone and the linearized tangent cone to X at x_0 and find out if x_0 fulfills (ABADIE-CQ):

   (i) $g(x,y) = (y - x^3, -y)^T, x_0 = (0,0)^T$;

   (ii) $g(x,y) = (y^2 - x + 1, 1 - x - y)^T, x_0 = (1,0)^T$.

   (3 points)

8. Let (x^*, λ^*, μ^*) be a KKT point of the optimization problem

$$(P) \qquad \begin{array}{ll} \min & f(x), \\ \text{s.t.} & g_i(x) \leq 0, i = 1, ..., m \\ & h_j(x) = 0, j = 1, ..., p, \\ & x \in \mathbb{R}^n \end{array}$$

   with $f, g_i, h_j : \mathbb{R}^n \to \mathbb{R}$, $i = 1, ..., m$, $j = 1, ..., p$, continuously differentiable functions. Prove that x^* is a *critical point* of (P), namely, it holds

$$\nabla f(x^*)^T d \geq 0 \ \forall d \in T_X(x^*),$$

   where $X = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, ..., m, h_j(x) = 0, j = 1, ..., p\}$. Given a critical point x^* of (P), when do Lagrange multipliers $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^p$ exist, such that (x^*, λ^*, μ^*) is a KKT point of (P)? (3 points)

9. Consider the optimization problem

$$\begin{array}{ll} \min & x_1^2 + (x_2 + 1)^2. \\ \text{s.t.} & x_2 - x_1^2 \leq 0 \\ & -x_2 \leq 0 \end{array}$$

   Show that $x^* = (0,0)^T$ fulfills (ABADIE-CQ) and that it does not fulfill (MFCQ). (2 points)

10. Consider the optimization problem

$$\begin{array}{ll} \min & x_1^2 + (x_2 + 1)^2. \\ \text{s.t.} & -x_1^3 - x_2 \leq 0 \\ & -x_2 \leq 0 \end{array}$$

   Show that $x^* = (0,0)^T$ fulfills (MFCQ) and that it does not fulfill (LICQ). (2 points)

11. Let $U \subseteq \mathbb{R}^n$ be a nonempty, open and convex set and $f : U \to \mathbb{R}$ a differentiable function on U. Prove that the following statements are equivalent:

   (i) f is convex on U;

   (ii) $\langle \nabla f(x), y - x \rangle \leq f(y) - f(x) \ \forall x, y \in U$;

   (iii) $\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq 0 \ \forall x, y \in U$;

   (iv) if f is twice differentiable on U, then $\nabla^2 f(x)$ is positively semidefinite for every $x \in U$.

   (4 points)

12. Let be the functions $c : \mathbb{R} \to \mathbb{R}$,

$$c(y) = \begin{cases} (y+1)^2, & y < -1, \\ 0, & -1 \leq y \leq 1, \\ (y-1)^2, & y > 1, \end{cases}$$

   and $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$, $g_1(x_1, x_2) = c(x_1) - x_2$, $g_2(x_1, x_2) = c(x_1) + x_2$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a convex and continuously differentiable function. Show that for the convex optimization problem

$$\begin{array}{rl} \min & f(x) \\ \text{s.t.} & x \in \mathbb{R}^2 \\ & g_1(x) \leq 0 \\ & g_2(x) \leq 0 \end{array}$$

   (ABADIE-CQ) holds at $x^* = (0,0)^T$, while (SLATER-CQ) is not satisfied. (2 points)