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Applied Analysis Problems

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Contents

1 Sheet 6

1.1 Fourier Transform of the convolution

Consider the function $f(x)$, which has a Fourier Transform $\hat{f}(\xi)$, now let us compute the Fourier transform of

$$h(x) = f(3x-1)\sin(x). \tag{1}$$

We know that the Fourier transform of the convolution is (we use somewhat of the inverse convolution theorem).

$$\widehat{(f(3x-1) * g(x))} = \widehat{f(3x-1)} \cdot \hat{g}(\xi). \tag{2}$$

The Fourier transform of $f(3x-1)$ is simply done by substituting a new variable

$$\widehat{f(3x-1)} = \frac{1}{3}e^{2\pi i \frac{\xi}{3}}\ f\left(\frac{\xi}{3}\right). \tag{3}$$

The Fourier transform of $\sin(x)$ can be calculated when looking at the Fourier transform of the Dirac-delta function

$$\begin{aligned}
\widehat{\delta(ax-b)} &= \int_{\mathbb{R}} \delta(ax-b)e^{-2\pi i x\xi}\ dx \qquad (y = ax-b) && (4)\\
&= \int_{\mathbb{R}} \delta(y)e^{-2\pi i(y+b)\frac{\xi}{a}}\frac{dy}{a} && (5)\\
&= \frac{1}{a}e^{-2\pi i\xi\frac{b}{a}}. && (6)
\end{aligned}$$

We may plug in $\sin(x)$ in the definition of the Fourier transformation and observe where we can use the Dirac-delta to to the inverse Fourier transform

$$\widehat{\sin(x)} = \int_{\mathbb{R}} \sin(x) e^{-2\pi i x \xi} \, dx = \tag{7}$$

$$= \frac{1}{2i} \int_{\mathbb{R}} (e^{ix} - e^{-ix}) e^{-2\pi i \xi x} \, dx \tag{8}$$

$$= \frac{1}{2i} \left(\int_{\mathbb{R}} e^{ix} e^{-2\pi i \xi x} \, dx + \int_{\mathbb{R}} e^{-ix} e^{-2\pi i \xi x} \, dx \right). \tag{9}$$

Here we may use the above formula for the Fourier transform of the Dirac delta. We choose $a = 1$, $b = \pm\frac{1}{2\pi}$ and do some $y = -x$ substitutions and thereby get the following result

$$\widehat{\sin(x)} = \frac{1}{2i} \left(\delta(\xi - \frac{1}{2\pi}) - \delta(\xi + \frac{1}{2\pi}) \right) \tag{10}$$

The whole result is thereby

$$\widehat{f(3x-1)} * \widehat{sin(x)} = \frac{1}{6i} \left(e^{2\pi i (\frac{\xi}{3} - \frac{1}{6\pi})} \hat{f}(\frac{\xi}{3} - \frac{1}{6\pi}) - e^{2\pi i (\frac{\xi}{3} + \frac{1}{6\pi})} \hat{f}(\frac{\xi}{3} + \frac{1}{6\pi}) \right) \tag{11}$$

1.2 More Fourier Transforms

Consider the function

$$f(x) = e^{-|x|} \tag{12}$$

The Fourier transform of this function is

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-|x| e^{-2\pi i x \xi}} \, dx \tag{13}$$

$$= \int_{-\infty}^{0} e^{x} e^{-2\pi i x \xi} \, dx + \int_{0}^{\infty} e^{-x} e^{-2\pi i x \xi} \, dx = \tag{14}$$

$$= \frac{1}{1 - 2\pi i \xi} e^{(1-2\pi i \xi)x} \Big|_{-\infty}^{0} + \frac{-1}{1 + 2\pi i \xi} e^{-(1+2\pi i \xi)x} \Big|_{-\infty}^{0} = \tag{15}$$

$$= \frac{1}{1 - 2\pi i \xi} + \frac{1}{1 + 2\pi i \xi} = \tag{16}$$

$$= \frac{2}{1 + (2\pi\xi)^2}. \tag{17}$$

Let us use this result to solve the following integral

$$\int_{\mathbb{R}} \frac{\cos(a\xi)}{(2\pi\xi)^2 + 1} \, d\xi = \frac{1}{2} \int_{\mathbb{R}} \hat{f}(\xi) \mathrm{Re}(e^{ia\xi}) \, dx, \tag{18}$$

$$\tag{19}$$

where we used the fact that $\mathrm{Re}(e^{ia\xi}) = \cos(a\xi)$ and $\hat{f}(\xi) = \frac{2}{1+(2\pi\xi)^2}$, thereby

$$\frac{1}{2} \int_{\mathbb{R}} \hat{f}(\xi) \mathrm{Re}(e^{ia\xi}) \, dx = \frac{1}{2} \mathrm{Re} \left(\int_{\mathbb{R}} \hat{f}(\xi) e^{ia\xi} \, d\xi \right) = \tag{20}$$

$$= \frac{1}{2} \mathrm{Re} \left(\int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \frac{a}{2\pi} \xi} \, d\xi \right) = \tag{21}$$

$$= \frac{1}{2} \mathrm{Re} \left(f(\frac{a}{2\pi}) \right) = \tag{22}$$

$$= \frac{1}{2} e^{-\frac{|a|}{2\pi}}. \tag{23}$$

1.3 Finite discrete Fourier transform

Consider $s \in \mathbb{C}^N$ with entries

$$s[n] = \sin\left(2\pi\xi_0 \frac{n}{N}\right), \tag{24}$$

for same $0 < \xi_0 < N$. The finite discrete Fourier transform of s is

$$\hat{s}[k] = \frac{1}{N}\sum_{n=0}^{N-1} \sin\left(2\pi\xi_0 \frac{n}{N}\right) e^{-2\pi i \frac{k}{N} n} = \tag{25}$$

$$= \frac{1}{2iN}\left(\sum_{n=0}^{N-1} e^{2\pi i \frac{n}{N}(\xi_0 - k)} - e^{-2\pi i \frac{n}{N}(\xi_0 + k)}\right). \tag{26}$$

If we consider $\xi_0 \in \mathbb{Z}$, we have

$$\hat{s}[k] = \begin{cases} \frac{1}{2i} & \xi_0 = k \\ -\frac{1}{2i} & \xi_0 = -k \\ 0 & \text{else} \end{cases} \tag{27}$$

1.4 Discrete Matrix Notation

The convolution of two vectors $f, g \in \mathbb{C}^N$, can be expressed by a circulate matrix applied to f

$$(f * g)[n] = \sum_{k=0}^{N-1} f[k]g[n-k]. \tag{28}$$

Consider $g = s$, then the matrix takes the following values

$$s[n-k] = s_{nk} = \sin\left(2\pi\xi_0 \frac{n-k}{N}\right). \tag{29}$$

The convolution with an impulse input $f = \delta_{0k}$, a vector that is 1 for $k = 0$ and else 0 reads

$$\sum_k s_{nk} f_k = \sum_k s_{nk}\delta_{0k} = \tag{30}$$

$$= \sin\left(2\pi\xi_0 \frac{n}{N}\right). \tag{31}$$