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Numerical Analysis Problems

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1 Sheet 1

1.1 Problem 1

Consider the following two matrices $A, L_1 \in \mathbb{R}^{4\times 4}$ defined in as

$$A := \begin{pmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{pmatrix}, \qquad L_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & 0 & 1 & 0 \\ z & 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

for $x, y, z\mathbb{R}$.

To show that A is invertible, we need to show it has maximal rank, that is $\text{rank}(A) = 4$. We can do this by doing Gaussian elimination steps until A is of the form of a upper triangular matrix

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} \begin{matrix} \\ -2\cdot I \\ -4\cdot I \\ -3\cdot I \end{matrix} \longrightarrow \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix} \begin{matrix} \\ \\ -3\cdot II \\ -4\cdot II \end{matrix} \longrightarrow \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{matrix} \\ \\ \\ -1\cdot III \end{matrix} \tag{2}$$

$$\longrightarrow \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \underset{\det}{\longrightarrow} 8. \tag{3}$$

Next we will determine x, y and z, s.t. $(L_1 A)_{\cdot,1} = \begin{pmatrix} 2 & 0 & 0 & 0 \end{pmatrix}$ by solving the linear system

$$L_1 A = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 2x+4 & x+3 & x+3 & 1 \\ 2y+8 & y+7 & y+9 & 5 \\ 2z+6 & z+7 & z+9 & 8 \end{pmatrix}, \tag{4}$$

we get $x = -2$, $y = -4$ and $z = -3$ and thereby

$$L_1 A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 3 & 5 & 8 \end{pmatrix}. \tag{5}$$

In an analogous structure we may define $L_2, L_3 \in \mathbb{R}^{4\times 4}$, s.t.

$$L_3 L_2 L_1 A = U, \tag{6}$$

where U is an upper triangular matrix. We may notice that this is an LU decompositions of a matrix and can be determined by the inversion of a single step of Gaussian elimination. By that the three steps needed to achieve the upper triangular by Gaussian elimination are introduced in 2 and 3, that is also why $-2, -4, -3$ aligns up with L_1. To summarize, by looking at 2 and 3 the matrices L_2, L_3 are the following

$$L_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & -4 & 0 & 1 \end{pmatrix}, \qquad L_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}. \tag{7}$$

And by no calculation we know that U needs to be the upper triangular found in 3, i.e.

$$L_3 L_2 L_1 A = U = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}. \tag{8}$$

We have indeed preformed an LU decomposition of A, which is indeed useful for solving a linear system of the form

$$Ax = b \qquad \text{and} \quad L_3 L_2 L_1 A = U, \tag{9}$$
$$(L_3 L_2 L_1 A)x = Ux = L_3 L_2 L_1 b = y \tag{10}$$
$$\Rightarrow Ux = y, \tag{11}$$

where the system is recursively solvable as U is the upper triangular and no additional transformation steps are required only "plug and play".

1.2 Problem 2

Next we consider $A_\varepsilon \in \mathbb{R}^{2\times 2}$ defined as

$$A_\varepsilon := \begin{pmatrix} \varepsilon & 1 \\ 1 & 1 \end{pmatrix}, \tag{12}$$

for $\varepsilon > 0$. The inverse of A_ε is

$$A_\varepsilon^{-1} = \frac{1}{\det(A_\varepsilon)} \operatorname{adj}(A_\varepsilon) = \frac{1}{\varepsilon - 1} \begin{pmatrix} 1 & -1 \\ -1 & \varepsilon \end{pmatrix} \tag{13}$$

Now let $\|x\|_\infty = \max\{|x_1|, |x_2|\}$ be the maximum norm of $x \in \mathbb{R}^2$, and $\|A_\varepsilon\|_\infty$ the induced matrix norm of A_ε. We can show that

$$\lim_{\varepsilon \to 0} K(A_\varepsilon) = 4, \tag{14}$$

where $K(A_\varepsilon) = \|A_\varepsilon\|_\infty \|A_\varepsilon^{-1}\|_\infty$ is the condition number of A_ε.

$$\|A_\varepsilon\|_\infty = \| \begin{pmatrix} \varepsilon + 1 & 1 + 1 \end{pmatrix} \|_\infty = 2 \tag{15}$$
$$\|A_\varepsilon^{-1}\|_\infty = \| \begin{pmatrix} | -\frac{2}{\varepsilon - 1} | & 1 \end{pmatrix} \|_\infty = \frac{2}{1 - \varepsilon}, \tag{16}$$

and thereby

$$\lim_{\varepsilon\to 0} K(A_\varepsilon) = \lim_{\varepsilon\to 0} 2\cdot\frac{2}{1-\varepsilon} = 4 \tag{17}$$

If we preformed an LU decomposition of A_ε like in the first problem to get an upper diagonal the decompostion would be

$$LA_\varepsilon = \begin{pmatrix} 1 & 0 \\ -\frac{1}{\varepsilon} & 1 \end{pmatrix}\begin{pmatrix} \varepsilon & 1 \\ 1 & 1 \end{pmatrix} \tag{18}$$

$$= \begin{pmatrix} \varepsilon & 1 \\ 0 & 1-\frac{1}{\varepsilon} \end{pmatrix} = U_\varepsilon, \tag{19}$$

with the inverse

$$U_\varepsilon^{-1} = \frac{1}{\varepsilon-1}\begin{pmatrix} 1-\frac{1}{\varepsilon} & -1 \\ 0 & \varepsilon \end{pmatrix}. \tag{20}$$

The condition number of the resulting upper triangular matrix U_ε, $K(U_\varepsilon)$ as $\varepsilon\to 0$ is

$$\|U_\varepsilon\|_\infty = \|\left(\varepsilon+1 \quad |1-\tfrac{1}{\varepsilon}|\right)\|_\infty = \frac{1}{\varepsilon}-1 \tag{21}$$

$$\|U_\varepsilon^{-1}\|_\infty = \|\left(|\tfrac{1-\frac{1}{\varepsilon}}{\varepsilon-1}| \quad |\tfrac{\varepsilon}{\varepsilon-1}|\right)\|_\infty = \frac{1}{\varepsilon(\varepsilon-1)} \tag{22}$$

$$\implies \lim_{\varepsilon\to 0} K(U_\varepsilon) = \lim_{\varepsilon\to 0}\frac{1-\varepsilon}{\varepsilon}\frac{1}{\varepsilon(1-\varepsilon)} = \infty. \tag{23}$$

But if we on the other hand considered a pivoting step in which we exchange the rows of A_ε

$$PA_\varepsilon = A'_\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \varepsilon & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \varepsilon & 1 \end{pmatrix} \tag{24}$$

Then the P-LU decomposition is

$$L'A'_\varepsilon = \begin{pmatrix} 1 & 0 \\ -\varepsilon & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ \varepsilon & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1-\varepsilon \end{pmatrix} = U'_\varepsilon, \tag{25}$$

with the inverse

$$(U'_\varepsilon)^{-1} = \frac{1}{1-\varepsilon}\begin{pmatrix} 1-\varepsilon & -1 \\ 0 & 1 \end{pmatrix}. \tag{26}$$

Then the condition number as $\varepsilon\to 0$

$$\|U'_\varepsilon\|_\infty = \|\left(2 \quad 1-\varepsilon\right)\|_\infty = 2 \tag{27}$$

$$\|\left(U'_\varepsilon\right)^{-1}\|_\infty = \|\left(\tfrac{1-\varepsilon+1}{1-\varepsilon} \quad \tfrac{1}{1-\varepsilon}\right)\| = \frac{2-\varepsilon}{1-\varepsilon} \tag{28}$$

$$\implies \lim_{\varepsilon\to 0} K(U'_\varepsilon) = \lim_{\varepsilon\to 0} 2\cdot\frac{2-\varepsilon}{1-\varepsilon} = 2\cdot 2 = 4 \tag{29}$$

1.3 Problem 3

Let $v\in\mathbb{R}^n$, $n\in\mathbb{N}$ and $v\neq 0$. We define the Housholder matrix

$$H = \mathrm{Id} - \frac{2}{\langle v,v\rangle}vv^T. \tag{30}$$

Indeed H is an orthogonal matrix, it satisfies $HH^T = H^TH = \mathrm{Id}$.

$$
\begin{aligned}
HH^T &= \left(\mathrm{Id} - \frac{2}{\langle v, v\rangle} vv^T\right)\left(\mathrm{Id} - \frac{2}{\langle v, v\rangle} vv^T\right)^T && (31)\\
&= \left(\mathrm{Id} - \frac{2}{\langle v, v\rangle} vv^T\right)\left(\mathrm{Id} - \frac{2}{\langle v, v\rangle} (vv^T)^T\right) && (32)\\
&= \left(\mathrm{Id} - \frac{2}{\langle v, v\rangle} vv^T\right)\left(\mathrm{Id} - \frac{2}{\langle v, v\rangle} vv^T\right) && (33)\\
&= \mathrm{Id} - \frac{4}{\langle v, v\rangle} vv^T + \frac{4}{\langle v, v\rangle^2}(vv^T)(vv^T) && (34)\\
&= \mathrm{Id} - \frac{4}{\langle v, v\rangle} vv^T + \frac{4}{\langle v, v\rangle}(vv^T) = \mathrm{Id} && (35)
\end{aligned}
$$

$$
\begin{aligned}
H^TH &= \left(\mathrm{Id} - \frac{2}{\langle v, v\rangle} vv^T\right)\left(\mathrm{Id} - \frac{2}{\langle v, v\rangle} vv^T\right) && (36)\\
&= \mathrm{Id} && (37)
\end{aligned}
$$

Let us look at the projection of some $x \in \mathbb{R}^n$ in the v direction is given by

$$\frac{\langle v, x\rangle}{\langle v, v\rangle} v, \tag{38}$$

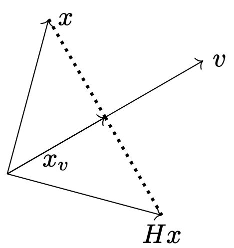
The projection of x in the orthogonal direction is

$$x - \frac{\langle v, x\rangle}{\langle v, v\rangle} v, \tag{39}$$

A reflection of x in v has -1 times the projection onto v, that x has onto v, so the orthogonal projection of the reflection onto v is the orthogonal projection of x onto v, therefor the reflection in v of x is

$$x - 2\frac{\langle v, x\rangle}{\langle v, v\rangle} v, \tag{40}$$

if we wanted to reflect x on the line spanned by v we would have to subtract the vector $\frac{\langle v, x\rangle}{\langle v, v\rangle} v$ twice, graphically it would look like this



The Household matrix acting on a vector x, Hx is exactly the above case since vector multiplication is associative we have

$$
\begin{aligned}
Hx &= x - \frac{2}{\langle v, v\rangle} vv^T x && (41)\\
&= x - 2\frac{\langle v, x\rangle}{\langle v, v\rangle} v && (42)
\end{aligned}
$$

The condition number of an orthogonal matrix A in the $\|\cdot\|_2$ induced norm is

$$K(A) = \|A\|_2\|A^{-1}\|_2 = 1, \tag{43}$$

because the orthogonal matrix preserves distance, i.e. $\|Ax\|_2 = \|x\|_2$ for all x. Also $A^{-1} = A^T$ is orthogonal as well

$$\|A\|_2 = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sup_{x\neq 0} \frac{\|x\|_2}{\|x\|_2} = 1 \tag{44}$$