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Vienna, January 10, 2023
To be discussed on January 17, 2023

Nonlinear Optimization

Exercise session 6

35. (a) Let $M \in \mathbb{R}^{n\times n}$ with $\|M\| < 1$. Show that $I - M$ is regular and

$$\|(I-M)^{-1}\| \leq \frac{1}{1-\|M\|}.$$

(b) Let $A, B \in \mathbb{R}^{n\times n}$ with $\|I - BA\| < 1$. Show that A and B are regular and that

$$\|B^{-1}\| \leq \frac{\|A\|}{1-\|I-BA\|} \text{ and } \|A^{-1}\| \leq \frac{\|B\|}{1-\|I-BA\|}.$$

(3 points)

36. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^4 + 2x^2y^2 + y^4$. Show that the local Newton algorithm (Algorithm 9.9) converges to the unique global minimum of f for every starting point $(x^0, y^0) \in \mathbb{R}^2 \setminus \{(0, 0)^T\}$. (2 points)

37. The local Newton algorithm (Algorithm 9.9) is invariant with respect to affine-linear transformation. In other words: let $A \in \mathbb{R}^{n\times n}$ be regular and $c \in \mathbb{R}^n$, $\{x^k\}_{k\geq 0}$ be the sequence by the local Newton algorithm for minimizing a function f with start vector x^0, and $\{y^k\}_{k\geq 0}$ be the sequence generated by the local Newton algorithm for minimizing the function $g(y) := f(Ay + c)$ with start vector y^0; then it holds

$$x^0 = Ay^0 + c \Rightarrow x^k = Ay^k + c \quad \forall k \geq 0$$

(3 points)

38. Let $M \in \mathbb{R}^{n\times n}$ be a regular matrix and $\{M_k\}_{k\geq 0} \in \mathbb{R}^{n\times n}$ a sequence of matrices which converges to M as $k \to +\infty$. Show that there exists $k_0 \geq 0$ such that M_k is regular for all $k \geq k_0$, and that the sequence $\{M_k^{-1}\}_{k\geq k_0}$ converges to M^{-1}. (3 points)

39. Let $H \in \mathbb{R}^{n\times n}$ be a regular matrix and $u, v \in \mathbb{R}^n$. Prove that the matrix $H + uv^T$ is regular if and only if $1 + v^T H^{-1} u \neq 0$. Show that under these assumptions the so-called *Sherman-Morrison formula* holds:

$$(H + uv^T)^{-1} = \left(I - \frac{1}{1 + v^T H^{-1} u} H^{-1} uv^T\right) H^{-1}.$$

(3 points)

40. Consider the quadratic optimization problem

$$(P) \qquad \begin{array}{ll} \min & f(x) := \gamma + c^T x + \frac{1}{2} x^T Q x, \\ \text{s.t.} & h(x) := b^T x = 0 \end{array}$$

with $Q \in \mathbb{R}^{n \times n}$ a symmetric and positive definite matrix, $b, c \in \mathbb{R}^n$, $b \neq 0$, and $\gamma \in \mathbb{R}$. For given $\alpha > 0$, find the minimum $x^*(\alpha)$ of the penalty function

$$P(x; \alpha) := f(x) + \frac{\alpha}{2} (h(x))^2,$$

determine $x^* := \lim\limits_{\alpha \to +\infty} x^*(\alpha)$, and prove that x^* is the unique optimal solution of the optimization problem (P).

Hint. Use the *Sherman-Morrison formula.* (3 points)

41. Implement the local Newton algorithm. Use as input data the starting vector x^0, the parameter for the stopping criterion ε and the parameter for the maximal number of allowed iterations $kmax$. The sequence $x^0, x^1, x^2, \ldots$ containing the iteration history and the number of performed iterations should be returned.

The implemented algorithm should be tested for $\varepsilon = 10^{-6}$ and $kmax = 200$, and the following functions and starting values:

(a) $f(x_1, x_2) = (1 - x_1)^2 + 100(x_2 - x_1^2)^2$ (Rosenbrock function) and $x^0 = (-1.2, 1)^T$.

(b) $f(x_1, x_2, x_3, x_4) = \sum_{i=1}^4 \left(4 - \sum_{j=1}^4 \cos x_j + i(1 - \cos x_i) - \sin x_i\right)^2$ (trigonometric function) and $x^0 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^T$.

(c) $f(x_1, x_2) = (x_1 - 10^6)^2 + (x_2 - 2 \cdot 10^6)^2 + (x_1 x_2 - 2)^2$ (Brown function) and $x^0 = (1, 1)^T$.

(d) $f(x_1, x_2, x_3, x_4) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 + 90(x_4 - x_3^2)^2 + (1 - x_3)^2 + 10(x_2 + x_4 - 2)^2 + \frac{1}{10}(x_2 - x_4)^2$ (Wood function) and $x^0 = (-3, -1, -3, -1)^T$.

(e) $f(x) = \sqrt{1 + x^2}$, $x^0 = 2$, $x^0 = 1$ and $x^0 = \frac{1}{2}$.

(4 points)

42. Implement the globalized Newton algorithm. Use as input data the starting vector x^0, the parameter for the stopping criterion ε, the parameter for the maximal number of allowed iterations $kmax$, the parameters for the determination of the Armijo step size σ and β, and the parameters ρ and p. The sequence $x^0, x^1, x^2, \ldots$ containing the iteration history and the number of performed iterations should be returned.

The implemented algorithm should be tested for ε=10^{-6}, $kmax$=200, ρ=10^{-8}, p=2.1, σ=10^{-4} and β=0.5, and the following functions and starting values:

(a) $f(x_1, x_2) = (1 - x_1)^2 + 100(x_2 - x_1^2)^2$ (Rosenbrock function) and $x^0 = (-1.2, 1)^T$.

(b) $f(x_1, x_2, x_3, x_4) = \sum_{i=1}^4 \left(4 - \sum_{j=1}^4 \cos x_j + i(1 - \cos x_i) - \sin x_i\right)^2$ (trigonometric function) and $x^0 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^T$.

(c) $f(x_1, x_2) = (x_1 - 10^6)^2 + (x_2 - 2 \cdot 10^6)^2 + (x_1 x_2 - 2)^2$ (Brown function) and $x^0 = (1, 1)^T$.

(d) $f(x_1, x_2, x_3, x_4) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 + 90(x_4 - x_3^2)^2 + (1 - x_3)^2 + 10(x_2 + x_4 - 2)^2 + \frac{1}{10}(x_2 - x_4)^2$ (Wood function) and $x^0 = (-3, -1, -3, -1)^T$.

(e) $f(x) = \sqrt{1 + x^2}$, $x^0 = 2$, $x^0 = 1$ and $x^0 = \frac{1}{2}$.

(5 points)