250085 VU
TENSOR METHODS FOR DATA SCIENCE AND SCIENTIFIC COMPUTING
WINTER SEMESTER 2021

HOMEWORK ASSIGNMENT 1

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Due by 13:00 on Monday, 25th October 2021. To be submitted via Moodle.

Throughout this assignment, we will work with two functions f and g defined pointwise on $[-1,1]^2$ as follows:

$$f(x,y) = \frac{1}{1+x^2+y^2} \quad \text{and} \quad g(x,y) = \sqrt{x^2+y^2} \cdot \Big(1 + \frac{1}{2}\cos(15x+22y)\Big)$$

for all $x, y \in [-1,1]$.

For $n = 901$, let us consider a grid of the points

$$t_i = 2\,\frac{i-1}{n-1} - 1 \quad \text{with} \quad i = 1, \ldots, n$$

and the matrices A and B of size $n \times n$ composed of the values of f and g at the corresponding grid points:

$$a_{ij} = f(t_i, t_j) \quad \text{and} \quad b_{ij} = g(t_i, t_j) \quad \text{for all} \quad i, j \in \{1, \ldots, n\}\,.$$

Throughout the assignment, $r \in \{1, \ldots, n\}$ will denote the ranks of approximate factorizations of A or B.

Any programming language and environment may be used. For each student submitting a solution, the submission should consist of (i) a single PDF file presenting the results in a self-contained fashion (so that running the student's code not be required for understanding the results) and (ii) the code, ready to run and reproduce the results presented in the PDF file, organized in any reasonable number of files.

1. **Matrix approximation by SVD.** Study the approximation of A and B in the Frobenius and max norms by the truncated SVD with ranks $r \in \{1, 2, \ldots, 80\}$. For each matrix, plot the errors (on logarithmic scale) in the two norms as functions of r.

What type of convergence do you observe, if any? Sketch how it can be explained (proven) for the given functions f and g?

2. **Matrix approximation by LU without pivoting.** Consider the same study for the cross approximation based on the leading principal submatrices or order r and constructed by the (standard, complete) LU factorization of the basis submatrix *without pivoting*.

First, choose $r = 300, 400, 500, 600, 900$. Present and explain your observations.

Second, consider $r \in 1, 2, \ldots, 80$ and present your results in the form of plots of the errors depending on r.

3. **Matrix approximation by LU with complete pivoting.** Use an implementation of the LU decomposition *with pivoting* to obtain the permutations of the rows and columns of A and B

produced by Gauelimination with *complete pivoting* (in which, at each step of elimination, an entry of the Schur complement with the largest absolute value is used as the pivot entry).

For $r \in 1, 2, \dots, 80$, perform the same study as above for the cross approximation based on the first r rows and the first r columns selected by the pivoted LU decomposition. The cross approximations should be constructed, for each r, only from the corresponding crosses and, (once the basis rows and columns have been identified by the pivoted LU algorithm), should not use any other entries of A and B. Use any suitable method for the inversion of the basis matrix.

What type of convergence do you observe, if any? Sketch how it can be explained (proven) for the given functions f and g?

4. Understanding the basis rows and columns.

Produce surface *and/or* contour plots for the values of f and g on $[-1, 1]^2$ (using the $n \times n$ grid defined above; for surface plots, use every third point in each dimension for efficiency). On these plots, for $r = 1, 2, 3, 4, 5, 10, 15, 20, 30, 40$, show the axis-parallel lines selected by the pivoted LU algorithm and used for the construction of the cross approximations (in surface plots, plot the graphs of the restrictions of f and g to these lines in a distinct color).

5. Conclusions.

For each of the two functions, Explain the choice of the basis rows and columns made by the pivoted LU algorithm.

For each of the two functions, compare the convergence of the SVD-based approximations and of the cross approximations based on the pivoted LU algorithm.