Universität Wien
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Labcours Theoretical Physik 2021S
Detection of Quantum Entanglement with MUBs

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20. Juni, 2021

Abstract

In this lab course we go through the QM problem of detecting entangled and separable states. Even thought given a density matrix we cannot always know if the state is separable or entangled. Thus a new concept is introduced, a witness function build upon by the so called mutually unbiased bases short MUBs. With help of the witness we can experimentally test to prove entangled Bell states.

Contents

1 Background

1.1 Heisenberg's uncertainty relation-Robertson version

Given an observable $\mathscr{A}$ we can define a hermitian operator $\hat{A}$, given a state ψ, we can define the expectation value $\langle\hat{A}\rangle_\psi = \mathrm{Tr}(\hat{A}\psi)$ and thus a standard derivation $(\Delta\hat{A})^2_\psi = \langle\hat{A}^2\rangle_\psi - \langle\hat{A}\rangle^2_\psi$, where any such operator needs to satisfy.

$$\langle\hat{A}^\dagger\hat{A}\rangle_\psi = \langle\psi|\hat{A}^\dagger\hat{A}\rangle = \langle\hat{A}\psi|\hat{A}\psi\rangle \geq 0. \tag{1}$$

Furthermore two arbitrary hermitian operators $\hat{A}$ and $\hat{B}$ hold the following inequality

$$(\Delta\hat{A})_\psi \cdot (\Delta\hat{B})_\psi \geq \frac{1}{2}|\langle\hat{A},\hat{B}\rangle_\psi| \tag{2}$$

for any state ψ. This uncertainty is called the Heisenberg's uncertainty principle and forms a fundamental basis for quantum mechanics the unpredictability of quantum mechanics.

1.2 Entropic Uncertainty Relations-Quantum Information Theoretical Formulation

In quantum-information theory the entropic uncertainty is defined as the following

$$H(\hat{O}_n) + H(\hat{O}_m) \geq -\log_2\left(\max_{i,j}\{|\langle\chi^i_n|\chi^j_m\rangle|^2\}\right) = log_2(|\frac{1}{\sqrt{2}}|^2) \tag{3}$$

where $H(\bar{0})_n$ is the binary entropy for a pure state ψ

$$H(\bar{O}_n) = -p(n)\log_2(p(n)) - (1-p(n))\log_2(1-p(n)) \tag{4}$$

and $p(n) = |\langle\chi_n|\psi\rangle|^2$ is the probability for the outcome n of $\hat{O}_n$ for ψ.

The entropic uncertainty relation can be extended for an arbitrary number of outcomes, d, with the von-Neumann Entropy $S(\hat{O}_n)$

$$S(\hat{O}_n) + S(\hat{O}_m) \geq -\log_d\left(\max_{i,j}\{|\langle\chi^i_n|\chi^j_m\rangle|^2\}\right) \tag{5}$$

$$S(\hat{O}_n) = -\sum_{i=0}^{d-1} p_n(i)\ln(p_n(i)) \quad \text{or} \tag{6}$$

$$S(\hat{O}_n) = -\mathrm{Tr}(\hat{O}\ln(\hat{O})). \tag{7}$$

1.3 Mutually Unbiased Bases (MUBs)

A ONB of a d-dimensional Hilbert space is $B = \{|i\rangle\} = \{|0\rangle,\ldots,|(d-1)\rangle\}$. In quantum information theory a set of orthonormal bases $\{B_1,\ldots,B_m\}$ (each an ONB of the d-dimensional Hilbertspace H^d) is called mutually unbiased if

$$|\langle i_k|j_{k'}\rangle|^2 = \delta_{k,k'}\delta_{i,j}(1-\delta_{k,k'})\frac{1}{d} \tag{8}$$

Thus the maximum of the entropy uncertainty relation is

$$S(\hat{O}_n) + S(\hat{O}_m) \geq -\log_d(\frac{1}{d}) \tag{9}$$

1.4 Construction of MUBs

In this section we will show how to construct mutually unbiased bases (MUBs) using the Hadamard Matrix $\mathbb{H}$. In fact two orthonormal basies are connected by the Hadamard Matrix (unitary)

$$\mathbb{H} = \sum_{i,j} \frac{1}{\sqrt{d}} e^{i\phi_{ij}} |i\rangle\langle j|. \tag{10}$$

where $\phi_{i,j}$ is a phase chosen such that the $\mathbb{H}$ is unitary. A simple choice $e^{i\phi_{i,j}} = \omega^{-ij} = e^{\frac{2\pi i}{d}}$ always works. In this case the matrix is called the Fourier matrix

$$\mathbb{H} = \sum_{i,j} \frac{1}{\sqrt{d}} \omega^{-ij} |i\rangle\langle j|. \tag{11}$$

Furthermore the Hadamard matrix is directly related to the generalized Pauli-matrices.

$$\sigma_{\mathbb{Z}} = \sum_i \omega^i |i\rangle\langle i| \tag{12}$$

$$\sigma_{\mathbb{X}} = \mathbb{H}\sigma_{\mathbb{Z}}\mathbb{H} = \sum_i |i+1\rangle\langle i| \tag{13}$$

$$\sigma_{\mathbb{X}}\sigma_{\mathbb{Z}} = i\sigma_{\mathbb{Y}}. \tag{14}$$

All this means, the problem of finding MUBs, essentially narrows down, to finding these Hadamad matrices.

A second way of constructing MUBs is the so called Heisenberg-Weyl construction. If d is prime, the eigenvectors of the operators, form a MUB, which looks like:

$$(\sigma_{\mathbb{Z}}, \sigma_{\mathbb{X}}, \sigma_{\mathbb{X}}.\sigma_{\mathbb{Z}}, \sigma_{\mathbb{X}}.\sigma_{\mathbb{Z}}^2, ..., \sigma_{\mathbb{X}}.\sigma_{\mathbb{Z}}^{d-1}) \tag{15}$$

Examples:

MUBs for qubits (d=2)

$$B_1 = \{|0_1\rangle, |1_1\rangle\} = \{|0\rangle, |1\rangle\} \tag{16}$$

$$B_2 = \{|0_2\rangle, |1_2\rangle\} = \frac{1}{\sqrt{2}}\{|0\rangle + |1\rangle, |0\rangle - |1\rangle\} \tag{17}$$

$$B_2 = \{|0_3\rangle, |1_3\rangle\} = \frac{1}{\sqrt{2}}\{|0\rangle + i|1\rangle, |0\rangle - i|1\rangle\} \tag{18}$$

$$\tag{19}$$

MUBs for qutrits (d=3)

$$B_1 = \{|0_1\rangle, |1_1\rangle, |2_1\rangle\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{20}$$

$$B_2 = \{|0_2\rangle, |1_2\rangle, |2_2\rangle\} = \frac{1}{\sqrt{3}} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}, \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix} \right\} \tag{21}$$

$$B_3 = \{|0_3\rangle, |1_3\rangle, |2_1\rangle\} = \frac{1}{\sqrt{3}} \left\{ \begin{pmatrix} 1 \\ \omega \\ \omega \end{pmatrix}, \begin{pmatrix} 1 \\ \omega^2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \omega^2 \end{pmatrix} \right\} \tag{22}$$

$$B_4 = \{|0_4\rangle, |1_1\rangle, |2_1\rangle\} = \frac{1}{\sqrt{3}} \left\{ \begin{pmatrix} 1 \\ \omega^2 \\ \omega^2 \end{pmatrix}, \begin{pmatrix} 1 \\ \omega \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \omega \end{pmatrix} \right\} \tag{23}$$

With these bases we can define an bell state seed $\Omega_{0,0}$ with $P_{0,0} = |\Omega_{0,0}\rangle\langle\Omega_{0,0}|$,

$$|\Omega_{0,0}\rangle = \frac{1}{\sqrt{d}} \sum_{s=0}^{d-1} |ss\rangle \tag{24}$$

extending this with the Wely operators W_{kl} we can arrive at an arbitrary bell state $P_{i,j}$

$$|\Omega_{k,l}\rangle = W_{kl} \otimes \mathbb{1} |\Omega_{0,0}\rangle \tag{25}$$

where:

$$W_{kl} = \sum_{j=0}^{d-1} \omega^{j \cdot k} |j\rangle\langle j+l| \tag{26}$$

where $\omega = e^{\frac{2\pi i}{d}}$ and $\sum_{j=0}^{d-1} \omega^j = 0$.

1.5 Detecting Entanglement via MUBs

One of the most important aspects of quantum theory, is the prediction of entanglement, and furthermore finding ways to construct experiments, that, with minimal effort allow the creation of so called entanglement witnesses for entanglement detection. Because, the bigger a system gets, the more measurements are needed, which for huge systems is often straight up impossible to realize. So, essentially, quantum theory tries to witness entanglement with as few measurements as possible, and without resorting to full state tomography.

	Lower Bounds		Upper Bounds	
m	$L_{m,2}^{MUB}$	$L_{m,3}^{MUB}$	$U_{m,2}^{MUB}$	$U_{m,3}^{MUB}$
2	1/2	0.211	3/2	4/3
3	1	1/2	2	5/3
4		1		2

Table 1: Lower L and upper U bounds for the MUB witness for $d = 2, 3$ and $m = 1, \ldots, d+1$

2 Exercises

Exercise 1

Compute the Heisenberg uncertainty relation for $\hat{A} = \hat{\sigma}_1$ and $\hat{B} = \hat{\sigma}_2$ (Pauli matrices) for an arbitrary pure state $|\psi\rangle = \cos\frac{\theta}{2}|\Uparrow\rangle + \sin\frac{\theta}{2}e^{i\phi}|\Downarrow\rangle$. Furthermore compute the quantum-information theoretical version of the inequality for $\hat{O}_{n,m} = \hat{\sigma}_{1,2}$.

To start of, the Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{27}$$

Now we have a straight forward calculation

$$\langle \sigma_1 \rangle_\psi^2 = \sin^2\theta \cos^2\phi \tag{28}$$
$$\langle (\sigma_1)^2 \rangle_\psi = 1 \tag{29}$$

$$\langle \sigma_2 \rangle_\psi^2 = \sin^2\theta \sin^2\phi \tag{30}$$
$$\langle (\sigma_2)^2 \rangle_\psi = 1 \tag{31}$$

$$\frac{1}{2}|\langle[\sigma_1, \sigma_2]\rangle = cos\theta \tag{32}$$

after some basic algebra with trigonometric functions we arrive at the following inequality

$$\sin^4\theta \sin^2(2\phi) \geq 0 \tag{33}$$

which holds true for all θ, ϕ.

For the quantum-theoretical version of the inequality we use Equation 7 to calculate the von Neumann entropy. The maximum of the right hand side is $\frac{1}{2}$

$$S(\sigma_1) = -\mathrm{Tr}(\sigma_1 \ln(\sigma_1)) = 0 \tag{34}$$
$$S(\sigma_2) = -\mathrm{Tr}(\sigma_2 \ln(\sigma_2)) = \pi \tag{35}$$

thus the inequality is

$$\pi \geq 1 \tag{36}$$

Since the Heisenberg's uncertainty principle is mathematically correct, because it holds true for all hermitian operators, a violation of the principle would put the basis of functional analysis and/or the axioms of quantum mechanics at question.

The quantum information theoretical approach to the uncertainty principal is convenient since the right hand side does not depend on any particular state.

Exercise 2

Compute

$$I_m^{MUB} = \sum_{k=1}^{m} \sum_{i=0}^{d-1} \mathbf{Tr}((|i_k\rangle\langle i_k| \otimes |i_k\rangle\langle i_k|)\rho) \quad \textbf{and} \tag{37}$$

$$I_m^{MUB} = \sum_{k=1}^{m} \sum_{i=0}^{d-1} \mathbf{Tr}((|i_k\rangle\langle i_k| \otimes (|i_k\rangle\langle i_k|)^*)\rho) \tag{38}$$

for two qubits ($d = 2$), for $m = 1,2,3$ and $|\psi\rangle = cos\alpha|00\rangle + sin\alpha|11>$. Here $|i_k\rangle$ is the eigenvector of the Pauli matrix σ_k.

The strategy to calculate the witness is to use the computer to loop over d and m for $m = 1,\dots,d+1$ then we compare the results with table 1. Note that $|i_k\rangle\langle i_k|$ is a d-dimensional matrix, the density matrix is a d^2-dimensional matrix and thus the matrix inside the trace is d^2.

We start of with I_m^{MUB} without conjugation

$$I_{m=1}^{MUB} = \cos^2\alpha \tag{39}$$

$$I_{m=2}^{MUB} = \frac{1}{4}(-\sin(2\alpha) + 2\cos(2\alpha) + 3) \tag{40}$$

$$I_{m=3}^{MUB} = \cos^2(\alpha) + \frac{1}{2} \tag{41}$$

For $m = 2$ entangled states for lower bound $\alpha = \frac{\pi}{4}$. For $m = 3$ entangled states for lower bound $\alpha = \frac{3\pi}{4}$.

with conjugation we get

$$I_{m=1}^{MUB} = \cos^2\alpha \tag{42}$$

$$I_{m=2}^{MUB} = \frac{1}{4}(\sin(2\alpha) + 2\cos(2\alpha) + 3) \tag{43}$$

$$I_{m=3}^{MUB} = \frac{1}{\sqrt{2}}\sin(2\alpha + \frac{\pi}{4}) + 1 \tag{44}$$

For $m = 2$ entangled states for lower bound $\alpha = \frac{\pi}{4}$. For $m = 3$ entangled states for lower bound $\alpha = -\frac{\pi}{8}$.

Exercise 3

Compute the same as in exercise 2 for the isotropic state

$$\rho_d^{iso}(p) = (1-p)\cdot\frac{1}{d^2}\mathbb{1}_{d^2} + pP_{i,j} \tag{45}$$

for a freely chosen bell state $P_{i,j}$, and for both $d = 2$ qubits and for $d = 3$ qutrits. For $p \in [-\frac{1}{d^2-1}, 1]$ we have the positivity condition and for $p \in [-\frac{1}{d^2-1}, \frac{1}{d+1}]$ we have a separable state else entangled.

We choose $P_{i,j} = P_{0,0} = |\Omega_{0,0}\rangle\langle\Omega_{0,0}|$. To calculate Ω we use the equation 24 and use the MUBs given in section 1.4.

For $d = 2$ we have the following for the standard I^{MUB}

$$I^{MUB}_{m=1} = \frac{1}{4}(3p+1) \tag{46}$$
$$I^{MUB}_{m=2} = \frac{1}{2}(3p+1) \tag{47}$$
$$I^{MUB}_{m=3} = \frac{1}{4}(5p+3) \tag{48}$$
$$\tag{49}$$

For $m = 2$ we have entanglement on the upper bound for $p = \frac{2}{3}$. For $m = 3$ we have entanglement on the upper bound for $p = 1$.

with conjugation we get

$$I^{MUB}_{m=1} = \frac{1}{4}(3p+1) \tag{50}$$
$$I^{MUB}_{m=2} = \frac{1}{2}(3p+1) \tag{51}$$
$$I^{MUB}_{m=3} = \frac{1}{4}(9p+3) \tag{52}$$
$$\tag{53}$$

For $m = 2$ we have entanglement on the upper bound for $p = \frac{2}{3}$. For $m = 3$ we have entanglement on the upper bound for $p = \frac{4}{9}$.

For $d = 3$ we have the following

$$I^{MUB}_{m=1} = \frac{1}{9}(16p+2) \tag{54}$$
$$I^{MUB}_{m=2} = \frac{1}{9}(23p+4) \tag{55}$$
$$I^{MUB}_{m=3} = 3p+\frac{1}{3} \tag{56}$$
$$I^{MUB}_{m=4} = \frac{1}{9}(31p+8) \tag{57}$$

For $m = 2$ we have entanglement on the upper bound for $p = \frac{8}{23}$. For $m = 3$ we have entanglement on the upper bound for $p = \frac{4}{9}$.

with conjugation we get

$$I_{m=1}^{MUB} = \frac{1}{9}(16p+2) \tag{58}$$

$$I_{m=2}^{MUB} = \frac{1}{9}(32p+4) \tag{59}$$

$$I_{m=3}^{MUB} = \frac{16}{3}p + \frac{2}{3} \tag{60}$$

$$I_{m=4}^{MUB} = \frac{1}{9}(64p+8) \tag{61}$$

For $m = 3$ we have entanglement on the upper bound for $p = \frac{5}{3}$.

Exercise 4

Compute I_m^{MUB} with conjugation and without for the Werner states for $d = 2,3$ and $m = 1,\dots,d+1$

$$\rho_W(q) = q\frac{P_{sym}}{d(d+1)} + (1-q)\frac{P_{asym}}{d(d-1)} \tag{62}$$

where $P_{sym} = (\mathbb{1}+\mathbb{P})$ and $P_{asym} = (\mathbb{1}-\mathbb{P})$ for $\mathbb{P} = \sum_{ij} |ji\rangle\langle ij|$. The state is separable for $q \in [0,\frac{1}{2}]$ and entangled for $q \in [\frac{1}{2},1]$

First we calculate for $d = 2$ we choose the basis B_1 to calculate the projection operator. And note that $|ij\rangle = |i\rangle \otimes |j\rangle$ we need the tensor product here.

Straightforward computation gives

$$I_{m=1}^{MUB} = \frac{q}{3} \tag{63}$$

$$I_{m=2}^{MUB} = \frac{2q}{3} \tag{64}$$

$$I_{m=3}^{MUB} = q \tag{65}$$

For $m = 3$ we have entanglement on the lower bound for $p = 1/2$.

with conjugation

$$I_{m=1}^{MUB} = \frac{q}{3} \tag{66}$$

$$I_{m=2}^{MUB} = \frac{2q}{3} \tag{67}$$

$$I_{m=3}^{MUB} = \frac{q}{3} + \frac{1}{2} \tag{68}$$

For $m = 3$ we have entanglement on the lower bound for $p = 0$.

For $d = 3$ we choose the basis B_1 to calculate the projection operator and straightforward computation gives

$$I_{m=1}^{MUB} = \frac{q}{3} \tag{69}$$

$$I_{m=2}^{MUB} = \frac{2q}{3} \tag{70}$$

$$I_{m=3}^{MUB} = q \tag{71}$$

$$I_{m=3}^{MUB} = \frac{4q}{3} \tag{72}$$

For $m = 2$ we have entanglement on the lower bound for $p = 0.3165$. For $m = 3$ we have entanglement on the lower bound for $p = \frac{1}{2}$.

with conjugation

$$I_{m=1}^{MUB} = \frac{q}{3} \tag{73}$$

$$I_{m=2}^{MUB} = \frac{1}{12}(5q+2) \tag{74}$$

$$I_{m=3}^{MUB} = \frac{1}{36}(15q+14) \tag{75}$$

$$I_{m=4}^{MUB} = \frac{1}{36}(15q+22) \tag{76}$$

(77)

For $m = 2$ we have entanglement on the lower bound for $p = 0.1064$. For $m = 3$ we have entanglement on the lower bound for $p = \frac{4}{15}$.

A simple comparison with exercise 3, we arrive at the conclusion that for the Werner states we detect entanglement only on the lower bound and for the isotropic states we detect entanglement only on the upper bound.

References

[1] David Deutsch. "Uncertainty in Quantum Measurements". In: *Phys. Rev. Lett.* 50 (9 Feb. 1983), pp. 631–633. DOI: 10.1103/PhysRevLett.50.631. URL: https://link.aps.org/doi/10.1103/PhysRevLett.50.631.

[2] Hans Maassen and J. B. M. Uffink. "Generalized entropic uncertainty relations". In: *Phys. Rev. Lett.* 60 (12 Mar. 1988), pp. 1103–1106. DOI: 10.1103/PhysRevLett.60.1103. URL: https://link.aps.org/doi/10.1103/PhysRevLett.60.1103.

[3] Beatrix C Hiesmayr and Wolfgang Löffler. "Complementarity reveals bound entanglement of two twisted photons". In: *New Journal of Physics* 15.8 (Aug. 2013), p. 083036. ISSN: 1367-2630. DOI: 10.1088/1367-2630/15/8/083036. URL: http://dx.doi.org/10.1088/1367-2630/15/8/083036.

[4] Beatrix Hiesmayr and Wolfgang Löffler. "Mutually Unbiased Bases and Bound Entanglement". In: *Physica Scripta* 2014 (Sept. 2013). DOI: 10.1088/0031-8949/2014/T160/014017.

[5] Christoph Spengler et al. "Entanglement detection via mutually unbiased bases". In: *Physical Review A* 86.2 (Aug. 2012). ISSN: 1094-1622. DOI: 10.1103/physreva.86.022311. URL: http://dx.doi.org/10.1103/PhysRevA.86.022311.