

University of Vienna
Faculty of Physics

Theoretical Physics Lab-Course 2021S
Quantum Entanglement at High Energies

Milutin Popovic & Tim Vogel

Supervisor: Prof. Dr. Beatrix C. Hiesmayr

May the 9th, 2021

Abstract

Some text in the abstract

Contents

1	Bell Theorem and Bell Inequality	1
2	Quantum System of K-mesons	3
2.1	CP-Symmetry violation	4
2.2	Density Matrix Approach for decaying Quantum Systems	5

1 Bell Theorem and Bell Inequality

The Enstein-Podolsky-Rosen paradox argues that, because of hidden variables not considered, quantum mechanics is incomplete. J.S. Bell in his paper [1], published in 1964 discovered what is known as the Bell's Theorem. The theorem states in short that "In certain experiments all local realistic theories are incompatible with quantum mechanics" [1]. This is achieved by establishing a so called Bell inequality which satisfies the local realistic theories but violates quantum mechanics.

To demonstrate the results of Bell, we consider a Wiegner-type Bell inequality for spin- $\frac{1}{2}$ particles with a source that generates a Bell state

$$P(\uparrow \vec{a}, \uparrow \vec{b}) \leq P(\uparrow \vec{a}, \uparrow \vec{c}) + P(\uparrow \vec{c}, \uparrow \vec{b}) \quad (1)$$

here we denote the joint probability, of say Alice and Bob to find their particles in the spin-up state, with respect to their orientation \vec{a}, \vec{b} , $P(\uparrow \vec{a}, \uparrow \vec{b})$. For the source we consider an antisymmetric Bell state $|\psi^-\rangle$ in terms of \vec{a} .

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|\uparrow \vec{a}; \downarrow \vec{a}\rangle - |\downarrow \vec{a}; \uparrow \vec{a}\rangle). \quad (2)$$

Quantum Mechanics tells us how to calculate the probability

$$P^{QM}(\uparrow \vec{a}, \uparrow \vec{b}) = \left\| \langle \uparrow \vec{a}; \uparrow \vec{b} | \psi^- \rangle \right\|^2. \quad (3)$$

Furthermore quantum mechanics tells us how we can represent the ket $|\uparrow \vec{b}\rangle$ in terms $|\uparrow \vec{a}\rangle$ and the other way around.

$$|\uparrow \vec{b}\rangle = \cos\left(\frac{\phi_{ab}}{2}\right) |\uparrow \vec{a}\rangle + e^{i\theta_{ab}} \sin\left(\frac{\phi_{ab}}{2}\right) |\downarrow \vec{a}\rangle \quad (4)$$

$$|\uparrow \vec{a}\rangle = \sin\left(\frac{\phi_{ab}}{2}\right) |\uparrow \vec{b}\rangle - e^{i\theta_{ab}} \cos\left(\frac{\phi_{ab}}{2}\right) |\downarrow \vec{b}\rangle \quad (5)$$

where θ_{ab} is an unphysical phase in this case and is set to zero.

With this information we can derive the following results for the probability in Equation 3 (Task 1)

$$P^{QM}(\uparrow \vec{a}, \uparrow \vec{b}) = \left\| \langle \uparrow \vec{a}; \uparrow \vec{b} | \psi^- \rangle \right\|^2 \quad (6)$$

$$= \frac{1}{2} \left\| \langle \uparrow \vec{a}; \uparrow \vec{b} | \uparrow \vec{a}; \downarrow \vec{a} \rangle - \langle \uparrow \vec{a}; \uparrow \vec{b} | \downarrow \vec{a}; \uparrow \vec{a} \rangle \right\|^2 \quad (7)$$

$$= \frac{1}{2} \left\| (\langle \uparrow \vec{a} | \uparrow \vec{a} \rangle) (\langle \uparrow \vec{b} | \downarrow \vec{a} \rangle) - \underbrace{(\langle \uparrow \vec{a} | \downarrow \vec{a} \rangle)}_{\perp} (\langle \uparrow \vec{b} | \uparrow \vec{a} \rangle) \right\|^2 \quad (8)$$

$$= \frac{1}{2} \left\| \langle \uparrow \vec{b} | \downarrow \vec{a} \rangle \right\|^2 = \frac{1}{2} \sin^2\left(\frac{\phi_{ab}}{2}\right) \quad (9)$$

For the remaining probabilities in $|\uparrow \vec{a} \downarrow \vec{b}\rangle, |\downarrow \vec{a} \uparrow \vec{b}\rangle$ and $|\downarrow \vec{a} \downarrow \vec{b}\rangle$ we apply the same procedure to get

$$P^{QM}(\uparrow \vec{a}, \downarrow \vec{b}) = \left\| \langle \uparrow \vec{a}; \downarrow \vec{b} | \psi^- \rangle \right\|^2 \quad (10)$$

$$= \frac{1}{2} \left\| \langle \downarrow \vec{b} | \downarrow \vec{a} \rangle \right\|^2 = \frac{1}{2} \cos^2\left(\frac{\phi_{ab}}{2}\right) \quad (11)$$

$$(12)$$

$$P^{QM}(\downarrow \vec{a}, \downarrow \vec{b}) = \left\| \langle \downarrow \vec{a}; \downarrow \vec{b} | \psi^- \rangle \right\|^2 \quad (13)$$

$$= \frac{1}{2} \left\| \langle \downarrow \vec{b} | \uparrow \vec{a} \rangle \right\|^2 = \frac{1}{2} \sin^2\left(\frac{\phi_{ab}}{2}\right) \quad (14)$$

Thus for the Bell inequality we get

$$\frac{1}{2}\sin^2\left(\frac{\phi_{ab}}{2}\right) \leq \frac{1}{2}\sin^2\left(\frac{\phi_{ac}}{2}\right) + \frac{1}{2}\sin^2\left(\frac{\phi_{cb}}{2}\right) \quad (15)$$

We can take \vec{a}, \vec{b} to be in the same plane in \mathbb{R}^3 and \vec{c} ($\in \mathbb{R}^2$) to be the bisector of these two. This would imply $\phi_{ac} = \phi_{cb} = \phi$ and $\phi_{ab} = 2\phi$, which gives us.

$$\sin^2(\phi) \leq 2\sin^2\left(\frac{\phi}{2}\right) \quad (16)$$

$$\Leftrightarrow \cos\left(\frac{\phi}{2}\right) \leq \frac{1}{2} \quad (17)$$

This is not always the case and is thereby an obvious contradiction.

Now we will derive the original inequality of John Stuart Bell[1]. For this we consider the expectation values. The expectation value E is expressed in terms of probabilities of all possible outcomes.

$$E(\vec{a}, \vec{b}) = P(\uparrow \vec{a}, \uparrow \vec{b}) + P(\downarrow \vec{a}, \downarrow \vec{b}) - P(\uparrow \vec{a}, \downarrow \vec{b}) - P(\downarrow \vec{a}, \uparrow \vec{b}) \quad (18)$$

$$E(\vec{a}, \vec{c}) = P(\uparrow \vec{a}, \uparrow \vec{c}) + P(\downarrow \vec{a}, \downarrow \vec{c}) - P(\uparrow \vec{a}, \downarrow \vec{c}) - P(\downarrow \vec{a}, \uparrow \vec{c}) \quad (19)$$

$$E(\vec{c}, \vec{b}) = P(\uparrow \vec{c}, \uparrow \vec{b}) + P(\downarrow \vec{c}, \downarrow \vec{b}) - P(\uparrow \vec{c}, \downarrow \vec{b}) - P(\downarrow \vec{c}, \uparrow \vec{b}) \quad (20)$$

The sum of spin-up/spin-down and spin-down/spin-up probabilities needs to be one, with this we get.

$$P(\uparrow \vec{a}, \uparrow \vec{b}) = E(\vec{a}, \vec{b}) - P(\downarrow \vec{a}, \downarrow \vec{b}) + 1 \quad (21)$$

$$P(\uparrow \vec{a}, \uparrow \vec{c}) = E(\vec{a}, \vec{c}) - P(\downarrow \vec{a}, \downarrow \vec{c}) + 1 \quad (22)$$

$$P(\uparrow \vec{c}, \uparrow \vec{b}) = E(\vec{c}, \vec{b}) - P(\downarrow \vec{c}, \downarrow \vec{b}) + 1 \quad (23)$$

Plugging this into the inequality in equation 1 we get

$$E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{c}) \leq E(\vec{c}, \vec{b}) + 1 \quad (24)$$

$$+ P(\downarrow \vec{a}, \downarrow \vec{b}) - P(\downarrow \vec{a}, \downarrow \vec{c}) - P(\downarrow \vec{c}, \downarrow \vec{b}) \quad (25)$$

The term in equation 25 satisfies the bell inequality if the term on the left is strictly positive. Thus we get the original Bell inequality

$$|E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{c})| \leq E(\vec{c}, \vec{b}) + 1 \quad (26)$$

2 Quantum System of K-mesons

Kaons were discovered in 1947 by Rochester and Butler, where cosmic ray particles hit a lead plate and produced a neutral particle. The neutral particle is named kaon, it was noticed through its decay in two charged pions. K^0 , the kaon, is the first strange particle. The neutral kaons are states of quarks and anti-quarks, the ket $|K^0\rangle = |d\bar{s}\rangle$ denotes the neutral K-meson, and its antiparticle is $|\bar{K}^0\rangle = |\bar{d}s\rangle$. The state of the kaon is entangled in the sense that the particle K^0 can turn into an antiparticle \bar{K}^0 before decay through strangeness oscillation, a particle-antiparticle oscillation.

There are two states, the long lived kaon K_L and the short lived kaon K_S which diagonalize the Hamiltonian, the difference of their decay rates is about a factor of 600. The thing is K_L should decay into three pions, while

K_S should decay into two pions. But in 1964 Cronin and Fitch found out that the long lived kaon can also decay into two pions, which directly implies a CP symmetry violation. CP symmetry (charge conjugation parity symmetry) in particle physics states that physics is the same if instead of looking at an particle we looked at its antiparticle.

The short lived kaon K_S and the long lived kaon K_L can be expressed in terms of $|K^0\rangle$ and $|\bar{K}^0\rangle$ as follows,

$$|K_S\rangle = \frac{1}{N}(p|K^0\rangle - q|\bar{K}^0\rangle) \quad (27)$$

$$|K_L\rangle = \frac{1}{N}(p|K^0\rangle + q|\bar{K}^0\rangle) \quad (28)$$

where $p = 1 + \varepsilon$, $q = 1 - \varepsilon$ and $N^2 = |p|^2 + |q|^2$, ε is called the CP violating parameter and can be measured, with a magnitude of $|\varepsilon| \approx 10^{-3}$ [2]. Note that these two states are NOT orthogonal due to the CP violation.

Furthermore the connection to the CP basis is,

$$|K_1\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - e^{i\alpha}|\bar{K}^0\rangle) \quad (29)$$

$$|K_2\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + e^{i\alpha}|\bar{K}^0\rangle) \quad (30)$$

$$(31)$$

here α is an unphysical phase and is conventionally set to zero.

2.1 CP-Symmetry violation

The violation in symmetry can be seen considering the Bell inequality

$$P(K_S, \bar{K}^0) \leq P(K_S, K_1) + P(K_1, \bar{K}^0) \quad (32)$$

where $P(K_S, \bar{K}^0)$ is the probability of finding K_S on the right and \bar{K}^0 on the left at the time of measurement at $t = 0$, K_1 denotes a completely unphysical state and cannot be measured. Quantum mechanics gives us the tools to calculate these probabilities, the calculation gives

$$|p| \leq |q| \quad (33)$$

$$\Leftrightarrow \delta := \frac{|p|^2 - |q|^2}{|p|^2 + |q|^2} \leq 0 \quad (34)$$

Changing the choice of \bar{K}^0 and K^0 in equation 32 we calculate the probabilities again:

$$P^{QM}(K_S, K^0) = \|\langle K_S, K^0 | \psi^- \rangle\|^2 = \frac{|q|^2}{4N^2} \quad (35)$$

$$P^{QM}(K_S, K_1) = \|\langle K_S, K_1 | \psi^- \rangle\|^2 = \frac{1}{4N^2} |q - pe^{i\alpha}|^2 \quad (36)$$

$$P^{QM}(K_1, K^0) = \|\langle K_1, K^0 | \psi^- \rangle\|^2 = \frac{1}{4} |e^{i\alpha}|^2 \quad (37)$$

Inserting this into the inequality in equation 32 and applying some basic algebra we get

$$|q| \leq |p| \quad (38)$$

$$\Leftrightarrow \delta = \frac{|p|^2 - |q|^2}{|p|^2 + |q|^2} \geq 0 \quad (39)$$

implying a strict equality

$$\delta = 0. \quad (40)$$

This contradicts the experimental value of δ (citation from slides).

$$\delta_{exp} = (3.27 \pm 0.12) \cdot 10^{-1}. \quad (41)$$

We can say that the CP-violation is directly related to the violation of the Bell Inequality and entanglement. If CP-symmetry were true the long lived kaon would decay via two paths in equal amounts, which is not the case. Also according to the Big-Bang-Theory the amount of matter and antimatter initially created is equal, but according to experimental results CP-asymmetry means that there is an imbalance in matter and that physics differs for particles and antiparticles.

2.2 Density Matrix Approach for decaying Quantum Systems

In this section we describe an open quantum system with unstable particles (e.g. K-mesons) with the Lindbad-Gorini-Kossakowsky-Sudarhasanan master equation, an density matrix approach, by enlarging the Hilbertspace[3]. With this larger Hilbertspace $\mathbf{H}_{tot} = \mathbf{H}_s \oplus \mathbf{H}_f$ we take into consideration both the "surviving" (\mathbf{H}_s) and the "decaying" or "final" (\mathbf{H}_f) states and thus get a positive time evolution described by a non-hermitian Hamiltonian H_{eff} and a dissipator \mathcal{D} of the Lindbad operator L . The time evolution of the density matrix $\rho \in \mathbf{H}_{tot}$ is given by the master equation in the Lindbad form

$$\frac{d\rho}{dt} = -[H, \rho] - \mathcal{D}[\rho] \quad (42)$$

$$\text{with } \mathcal{D}[\rho] = \frac{1}{2} \sum_{j=0} (L_j^\dagger L_j \rho + \rho L_j^\dagger L_j - L_j \rho L_j^\dagger) \quad (43)$$

where the density matrix ρ is a 4x4 matrix with components ρ_{ij} ($i, j = s, f$) which are 2x2 matrices, with the property $\rho_{sf}^\dagger = \rho_{fs}$

$$\rho = \begin{pmatrix} \rho_{ss} & \rho_{sf} \\ \rho_{fs} & \rho_{ff} \end{pmatrix}. \quad (44)$$

The Hamiltonian H is an extension of the effective Hamiltonian H_{eff} on the total Hilbertspace \mathbf{H}_{tot}

$$H = \begin{pmatrix} H_{eff} & 0 \\ 0 & 0 \end{pmatrix}. \quad (45)$$

Furthermore the Lindbad generator L_0 is defined with $B : \mathbf{H}_s \rightarrow \mathbf{H}_f$, where $B^\dagger B = \Gamma$, decay matrix Γ from the effective Hamiltonian H_{eff} ,

$$L_0 = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \quad L_j = \begin{pmatrix} A_j & 0 \\ 0 & 0 \end{pmatrix} \quad (\text{with } j > 0). \quad (46)$$

Rewriting the master equation in 42 we get the following differential equations for the density matrix components

$$\dot{\rho}_{ss} = -i[H_{eff}, \rho_{ss}] - \frac{1}{2}\{B^\dagger B, \rho_{ss}\} - \tilde{D}[\rho_{ss}], \quad (47)$$

$$\dot{\rho}_{sf} = -iH_{eff}\rho_{sf} - \frac{1}{2}B^\dagger B\rho_{sf} - \frac{1}{2}\sum_j A_j^\dagger A_j \rho_{sf}, \quad (48)$$

$$\dot{\rho}_{ff} = B\rho_{ss}B^\dagger. \quad (49)$$

with $\tilde{D}[\rho_{ss}] = \frac{1}{2}\sum_{j=0}(A_j^\dagger A_j \rho_{ss} + \rho_{ss} A_j^\dagger A_j - A_j \rho_{ss} A_j^\dagger)$.

Now we solve these equations for the case without decoherence, meaning the Lindbad Operators operators A_j disappear and we can rewrite the equations for ρ_{ss} above in

$$\dot{\rho}_{ss} = -[H_{eff}, \rho_{ss}] - \frac{1}{2}\{\Gamma, \rho_{ss}\} = \quad (50)$$

$$= -i((M - \frac{i}{2}\Gamma)\rho_{ss} - \rho_{ss}(M - \frac{i}{2}\Gamma)) - \frac{1}{2}(\Gamma\rho_{ss} + \rho_{ss}\Gamma) \quad (51)$$

$$= -i\underbrace{[M, \rho_{ss}]}_{=0} - \rho_{ss}\Gamma \quad (52)$$

$$= -\rho_{ss}\Gamma \quad (53)$$

$$\Rightarrow \rho_{ss} = \rho_{ss}(0)e^{-\Gamma t}. \quad (54)$$

For ρ_{sf} we get

$$\dot{\rho}_{sf} = -iH_{eff}\rho_{sf} - \frac{1}{2}\Gamma\rho_{sf} = \quad (55)$$

$$= -iM\rho_{sf} \quad (56)$$

$$\Rightarrow \rho_{sf} = \rho_{sf}(0)e^{-iMt}. \quad (57)$$

And for ρ_{ff}

$$\dot{\rho}_{ff} = B\rho_{ss}B^\dagger \quad (58)$$

$$\Rightarrow \rho_{ff} = B \int \rho_{ss} dt B^\dagger \quad (59)$$

$$= -B\rho_{ss}(0)\Gamma^{-1}e^{-\Gamma t}B^\dagger + \rho_{ff}(0) \quad (60)$$

In reality the decay rates of particles differ e.g. K_S and K_L , the density matrix allows such things to be taken care of by mathematically extending the Hilbertspace and including the Lindbad operator. We could also consider a particle with three different decay rates, though the Hamiltonian would be a nine dimensional. In this regard we might say that the master equation 42 is a more general Schrödinger equation, because it not only describes pure quantum states but also mixed states.

References

- [1] J. S. Bell. “On the Einstein Podolsky Rosen paradox”. In: *Physics Physique Fizika* 1 (3 Nov. 1964), pp. 195–200. DOI: [10.1103/PhysicsPhysiqueFizika.1.195](https://doi.org/10.1103/PhysicsPhysiqueFizika.1.195). URL: <https://link.aps.org/doi/10.1103/PhysicsPhysiqueFizika.1.195>.

- [2] R.A. Bertlmann. “Entanglement, Bell Inequalities and Decoherence in Particle Physics”. In: *Lecture Notes in Physics* (), pp. 1–45. DOI: [10.1007/11398448_1](https://doi.org/10.1007/11398448_1). URL: http://dx.doi.org/10.1007/11398448_1.
- [3] Reinhold A. Bertlmann, Walter Grimus, and Beatrix C. Hiesmayr. “Open-quantum-system formulation of particle decay”. In: *Phys. Rev. A* 73 (5 May 2006), p. 054101. DOI: [10.1103/PhysRevA.73.054101](https://doi.org/10.1103/PhysRevA.73.054101). URL: <https://link.aps.org/doi/10.1103/PhysRevA.73.054101>.
- [4] Carla Schuler. “Entanglement, Bell Inequalities and Decoherence in Neutral K-Meson Systems”. In: (2014). URL: https://homepage.univie.ac.at/reinhold.bertlmann/pdfs/dipl_diss/CarlaSchuler_BA_v2.pdf.
- [5] Miguel Socolovsky. “On Bell’s theorem”. In: *Revista Mexicana De Fisica - REV MEX FIS* 48 (Aug. 2002), pp. 384–390.