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Dispersion relations

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Abstract

This protocol will provide a first look at dispersion relations. We will show some trivial examples in the harmonic oscillator, where the mathematical basics are provided and applied. From this we will dive into more complex topics, derive a few equations within scattering processes in quantum dynamics and after that, show, that dispersion do not only work in the quantum physical world, but also find application in particle physics, where they can be used to obtain results in areas, where quantum chromodynamics fail. We explicitly calculate solutions to the so called Omnés-problem, and finally apply some numerical values to the found solutions.

Contents

1 Introduction

Within the tools of complex analysis, there exists the possibility, to form relations between observable quantities of physicals systems, for example dispersion in a dielectric medium. This can be taken even further, by using the same methods within particle-physical problems, where the now more popular methods of quantum chromodynamics do not apply, which would be low energy hadronic processes. We will firstly apply these concepts to the simple example of the harmonic oscillator and finally work out more complex problems, regarding the pion vector form factor. The reader is expected to be familiar with the subject of complex analysis, especially analyticity/holomorphicity of a function, integration of complex functions, the residue theorem and the Schwartz reflection principle.

2 Damped harmonic oscillator

Considering a free harmonic oscillator, the equation of motion accounts to:

$$\ddot{x}(t) + \gamma\dot{x}(t) + \omega_0^2 x(t) = 0 \tag{1}$$

where $\gamma > 0$ is the damping coefficient, and ω_0 the angular frequency of the oscillator. Using the exponential ansatz we can arrive at an general solution to this ordinary differential equation

$$x(t) = ae^{-i\omega_1 t} + be^{-i\omega_2 t} \tag{2}$$

with:

$$\omega_{1/2} = \pm\sqrt{\omega_0^2 - (\frac{\gamma}{2})^2} - i\frac{\gamma}{2} \tag{3}$$

where a and b are calculated based on the Couchy boundary conditions.

For the case $\omega_0 > \frac{\gamma}{2}$ we can rewrite the solution

$$x(t) = \left(ae^{-i\tilde{\omega}_0 t} + be^{-i\tilde{\omega}_0 t}\right) e^{-\frac{\gamma}{2}t} \tag{4}$$

with:

$$\tilde{\omega}_0 = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2} \tag{5}$$

2.1 External Force

Now consider a harmonic oscillator with an external force $F(t)$ driving it

$$\ddot{x}(t) + \gamma\dot{x}(t) + \omega_0^2 x(t) = \frac{F(t)}{m} =: f(t). \tag{6}$$

By Fourier transforming the equation we can arrive at an equation for the greens function in Fourier space. Note that the Fourier transform of $x(t)$ is

$$\hat{x}(t) = \frac{1}{2\pi}\int_{\infty}^{\infty} d\omega X(\omega) e^{-i\omega t} \tag{7}$$

so the Fourier transforms of $\dot{x}$ and $\ddot{x}$ are

$$\mathscr{F}(\dot{x}) = -i\omega X(\omega) \tag{8}$$
$$\mathscr{F}(\ddot{x}) = -\omega^2 X(\omega) \tag{9}$$
$$\tag{10}$$

and the equation 6 turns into

$$(-\omega^2 - i\gamma\omega + \omega_0^2)X(\omega) = F(\omega) \tag{11}$$

The Green's function can be represented in Fourier space like the following

$$G(\omega) = \frac{1}{-\omega^2 - i\gamma\omega + \omega_0^2} \tag{12}$$

The Maximum of the squared modulus $|G(\omega)|^2$ for $\gamma \ll \omega_0$ is roughly at ω_0, thus the width at half maximum can be calculated by looking for two ω's that satisfy

$$\frac{1}{2}|G(\omega_0)|^2 = |G(\omega)|^2 \tag{13}$$
$$\frac{1}{2}\frac{1}{\omega_0^2\gamma^2} = |G(\omega)|^2 \tag{14}$$

The exact solutions are

$$\tilde{\omega}_1 = \omega_0\sqrt{-0.5\left(\frac{\gamma}{\omega_0}\right)^2 - 1.0\frac{\gamma}{\omega_0}(0.25\left(\frac{\gamma}{\omega_0}\right)^2 + 1)^{\frac{1}{2}} + 1} \tag{15}$$
$$\tilde{\omega}_2 = \omega_0\sqrt{-0.5\left(\frac{\gamma}{\omega_0}\right)^2 + 1.0\frac{\gamma}{\omega_0}(0.25\left(\frac{\gamma}{\omega_0}\right)^2 + 1)^{\frac{1}{2}} + 1} \tag{16}$$

With help of Taylor expansion in the linear order in $\frac{\gamma}{\omega_0}$ gives us the approximation for the with at half maximum

$$\tilde{\omega}_2 - \tilde{\omega}_1 \simeq \gamma \tag{17}$$

In the figure below we plotted the squared modulus of $|G(\omega)|^2$

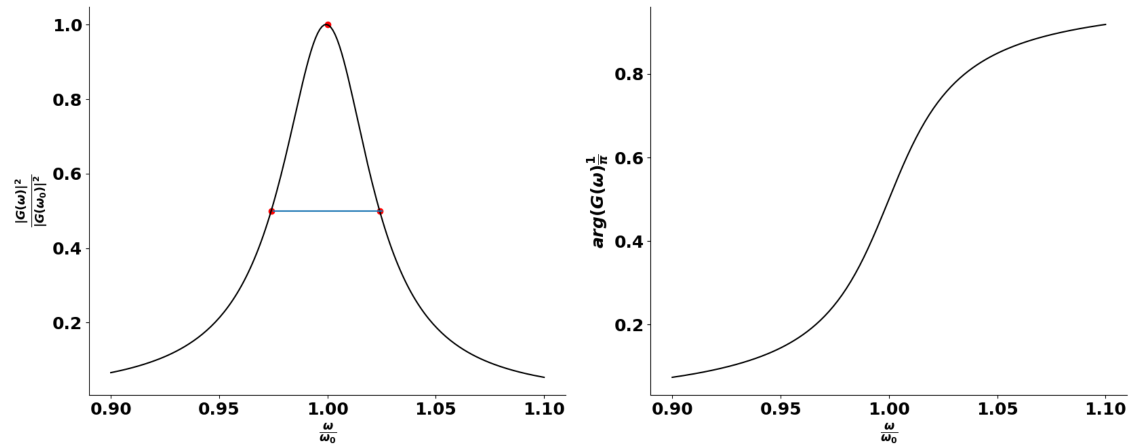
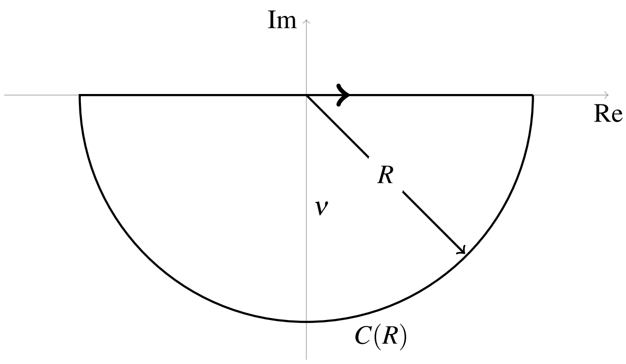


Figure 1: On the left the squared modulus $|G(\omega)|^2$ in $\omega \in [\omega_0 - 2\gamma, w_0 + w\gamma]$ for $\gamma \ll \omega_0$, precisely $\gamma = \omega_0/20$ for $\omega_0 = 1$ and on the right $arg(G(\omega))$

Next we want calculate the Green's function in terms of time

$$g(t) = \frac{1}{2\pi} \int_{\infty}^{\infty} d\omega G(\omega) e^{-i\omega t}. \tag{18}$$

Furthermore we can transform this to the complex integral where we have two singularities at We have two singularities at $z_{1/2} = -\frac{i\gamma}{2} \pm \tilde{\omega}_0$. We have the following integral path



The complex integral representation is

$$\oint_{\nu} dz G(z) e^{-izt} = \lim_{R\to\infty} \left(\int_{C(R)} + \int_{-R}^{R} \right) dz\, G(z) e^{-izt} \tag{19}$$

$$= 2\pi i \sum_j \mathrm{I}(C_j, z_j) \mathrm{Res}_j \tag{20}$$

Keep in mind that the integral from R to $-R$ is the integral we are trying to solve, that is pulling the limit we have one integral over the real axis. Because of Jordan's lemma, the integral over the complex curve vanishes

$$\left|\int_{C(R)} dz G(z) e^{-izt}\right| \leq \frac{\pi}{t} M_R \tag{21}$$

where $M_R := \max_{C(R)}\{G(Re^{i\varphi})\}$.It can easily be seen that M_R converges to 0 as R goes to infinity. Thus the only value the integral can take is 0 and we can calculate the real integral with the residues

$$\mathrm{Res}_1 = \frac{e^{izt}}{(z-z_1)(z-z_2)}(z-z_1)\bigg|_{z=z_1} \tag{22}$$

$$= -\frac{e^{-iz_1 t}}{z_1 - z_2} = \frac{e^{-\frac{\gamma}{2}t} e^{i\tilde{\omega}_0 t}}{2\tilde{\omega}_0} \tag{23}$$

$$\mathrm{Res}_2 = \frac{e^{izt}}{(z-z_1)(z-z_2)}(z-z_2)\bigg|_{z=z_2} \tag{24}$$

$$= -\frac{e^{-iz_1 t}}{z_2 - z_1} = -\frac{e^{-\frac{\gamma}{2}t} e^{-i\tilde{\omega}_0 t}}{2\tilde{\omega}_0} \tag{25}$$

$$\tag{26}$$

with the index $\mathrm{I}(C_R, z_i)$ being 1, because the curve goes around the singularities once.

The integral evolves to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega G(\omega) e^{-i\omega t} = \frac{\sin(\tilde{\omega}_0 t)}{\tilde{\omega}_0} e^{-\frac{\gamma}{2}t}. \tag{27}$$

Treating the cases $t < 0$ and $t > 0$ separately we can join them with the Heaviside-theta function $\theta(t)$, the Green's function for the damped harmonic oscillator is

$$g(t) = \frac{\sin(\tilde{\omega}_0 t)}{\tilde{\omega}_0} e^{-\frac{\gamma}{2}t}\theta(t) \tag{28}$$

With convolution we can arrive at a solution for the damped harmonic oscillator for an arbitrary driving force $f(t)$

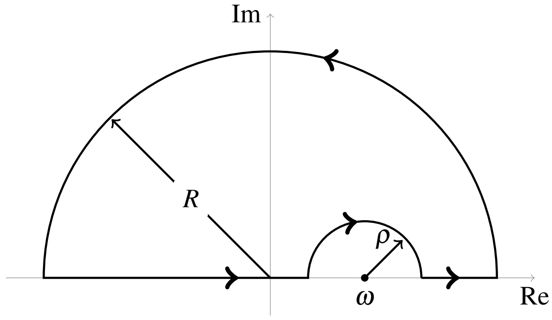
$$x(t) = \int_{-\infty}^{t} dt' \frac{\sin(\tilde{\omega}_0(t-t'))}{\tilde{\omega}_0} e^{-\frac{\gamma}{2}(t-t')} f(t'). \tag{29}$$

2.2 Green's Function and dispersion relations

Next we want to compute the following integral

$$0 = \oint_C d\omega' \frac{G(\omega')}{\omega - \omega'}, \quad \text{with} \quad \omega' = \omega_r + i\omega_i \tag{30}$$

along the following contour



so the integral representation is

$$\oint_C d\omega' \frac{G(\omega')}{\omega - \omega'} = \lim_{\substack{R\to\infty \\ \rho\to 0^+}} \left(\int_{C(R)} + \int_{(C(\rho)} + \int_{-R}^{\omega-\rho} + \int_{\omega+\rho}^{R} \right) d\omega' \, \frac{G(\omega')}{\omega - \omega'} \tag{31}$$

We need show that the integral over the big circle goes to 0. We know that for $\omega' \neq \omega$ we have

$$|\frac{G(\omega')}{\omega' - \omega}| = |\frac{1}{\omega'^3} \frac{1}{(1 - \frac{\omega_1}{\omega'})(1 - \frac{\omega_2}{\omega'})(1 - \frac{\omega}{\omega'})}| \leq \frac{1}{R^3} \tag{32}$$

thus

$$\left| \int_{C(R)} d\omega' \frac{G(\omega')}{\omega - \omega'} \right| \leq \frac{2\pi R}{R^3} = \frac{2\pi}{R^2} \xrightarrow[R\to\infty]{} 0. \tag{33}$$

The small circle can be calculated with the Residue theorem with the pole at ω

$$\int_{C(\rho)} d\omega' \frac{G(\omega')}{\omega - \omega'} = 2\pi i \mathrm{I}(C(\rho), \omega) \mathrm{Res}(\frac{G(\omega')}{\omega - \omega'}, \omega) = i\pi G(\omega). \tag{34}$$

Note that we go around ω only $1/2$ times. Reconstructing the integral equation we get

$$-i\pi G(\omega) = \lim_{\rho\to 0^+} \left(\int_{-R}^{\omega-\rho} + \int_{\omega+\rho}^{R} \right) d\omega' \, \frac{G(\omega')}{\omega' - \omega} \tag{35}$$

which is exactly the Cauchy Principal Value. Furthermore we can rewrite $G(\omega)$ into real and imaginary parts

$$\mathrm{Re}(G(\omega)) = \frac{1}{\pi} \fint d\omega' \frac{\mathrm{Im}(G(\omega'))}{\omega' - \omega} \tag{36}$$

$$\mathrm{Im}(G(\omega)) = \frac{1}{\pi} \fint d\omega' \frac{\mathrm{Re}(G(\omega'))}{\omega' - \omega} \tag{37}$$

$$\tag{38}$$

which are Hilbert transforms of each other, the equations are also known as “dispersion relations”. It should be noted that these equations also allow negative frequencies. Let us derive an representation for only positive frequencies. We start off by a simple statement

$$G(-\omega^*) = G(\omega)^*. \tag{39}$$

In our case this is obviously true

$$G(-\omega^*) = \frac{1}{-(\omega^*)^2 + i\gamma\omega^* + \omega_0^2} \tag{40}$$

$$G(\omega)^* = \frac{1}{-(\omega^2)^* + i\gamma\omega^* + \omega_0^2} = G(-\omega^*) \tag{41}$$

Now we choose $\omega \in \mathbb{R}^+$, our relation then becomes $G(-\omega) = G(\omega)^*$. Then we get

$$\mathrm{Re}(G(\omega)) = \frac{1}{\pi} \fint_0^\infty d\omega' \frac{2\omega' \mathrm{Im}(G(\omega'))}{\omega'^2 - \omega^2} \tag{42}$$

$$\mathrm{Im}(G(\omega)) = -\frac{1}{\pi} \fint_0^\infty d\omega' \frac{2\omega' \mathrm{Re}(G(\omega'))}{\omega'^2 - \omega^2} \tag{43}$$

To round this chapter up we would like to show one last identity in the sense of distributions

$$\lim_{\varepsilon\to 0^+} \frac{1}{\omega' - \omega \mp i\varepsilon} = \mathrm{P}(\frac{1}{\omega' - \omega}) \pm i\pi\delta(\omega' - \omega). \tag{44}$$

Let us extend the fraction with $\omega' - \omega \pm i\varepsilon$.

$$\frac{\omega' - \omega \pm i\varepsilon}{(\omega' - \omega \mp i\varepsilon)(\omega' - \omega \pm i\varepsilon)} = \frac{\omega' - \omega \pm i\varepsilon}{(\omega' - \omega)^2 + \varepsilon^2}. \tag{45}$$

That means for a test function $f(\omega')$ we have

$$\lim_{\varepsilon\to 0^+} \int_{-\infty}^{\infty} d\omega' \frac{f(\omega')}{\omega' - \omega \mp i\varepsilon} = \lim_{\varepsilon\to 0^+} \int_{-\infty}^{\infty} d\omega' \frac{(\omega' - \omega \pm i\varepsilon) f(\omega')}{(\omega' - \omega)^2 + \varepsilon^2} \tag{46}$$

$$= \lim_{\varepsilon\to 0^+} \left(\int_{-\infty}^{\infty} d\omega' \frac{f(\omega')(\omega' - \omega)}{(\omega' - \omega)^2 + \varepsilon^2} \pm i\varepsilon \int_{-\infty}^{\infty} d\omega' \frac{f(\omega')}{(\omega' - \omega)^2 + \varepsilon^2} \right). \tag{47}$$

Let us look into the first integral in equation 47, we can rewrite it

$$\lim_{\varepsilon\to 0^+} \int_{-\infty}^{\infty} d\omega' \frac{f(\omega')(\omega' - \omega)}{(\omega' - \omega)^2 + \varepsilon^2} = \lim_{\varepsilon,\rho\to 0^+} \left(\int_{-\infty}^{\omega-\rho} d\omega' \frac{f(\omega')(\omega' - \omega)}{(\omega' - \omega)^2 + \varepsilon^2} + \int_{\omega\rho}^{\infty} d\omega' \frac{f(\omega')(\omega' - \omega)}{(\omega' - \omega)^2 + \varepsilon^2} + \right. \tag{48}$$

$$\left. + \int_{\omega-\rho}^{\omega+\rho} d\omega' \frac{f(\omega')(\omega' - \omega)}{(\omega' - \omega)^2 + \varepsilon^2} \right) \tag{49}$$

$$= \fint_{-\infty}^{\infty} d\omega' \frac{f(\omega')}{(\omega' - \omega)} \tag{50}$$

The integral from $\omega - \rho$ to $\omega + \rho$ can be calculated by pulling out $f(\omega)$ out of the integral and directly computing it, which gives then vanishes. In second integral we approximate $f(\omega')$ to $f(\omega)$ in the region $\omega' \simeq \omega$

$$\varepsilon \int_{-\infty}^{\infty} d\omega' \frac{f(\omega')}{(\omega' - \omega)^2 + \varepsilon^2} \simeq \varepsilon f(\omega) \int_{-\infty}^{\infty} \frac{1}{(\omega' - \omega)^2 + \varepsilon^2} \tag{51}$$

$$= \pi f(\omega). \tag{52}$$

Which means the identity is

$$\lim_{\varepsilon\to 0^+} \int_{-\infty}^{\infty} d\omega' \frac{f(\omega')}{\omega' - \omega \mp i\varepsilon} = \fint_{-\infty}^{\infty} d\omega' \frac{f(\omega')}{\omega' - \omega} \pm i\pi f(\omega) \tag{53}$$

3 Potential scattering in quantum mechanics

If we consider elastic scattering of a spineless particle off a time-independent, spherically symmetric potential of finite range, we look for stationary solutions ψ of the Schrödinger equation

$$-\frac{\hbar^2}{2m}\vec{\nabla}^2\psi(\vec{x}) + V(r)\psi(\vec{x}) = E\psi(\vec{x}) \tag{54}$$

Since the potential is spherically symmetric, it only depends on r and for large values of r, it can be shown, that the asymptotic form of ψ looks like:

$$\psi(r,\theta) \approx A[e^{ikz} + f(E,\theta)\frac{e^{ikr}}{r}] \tag{55}$$

Where $kr \gg 1$, and k given by

$$k = \frac{\sqrt{2mE}}{\hbar} \tag{56}$$

We also define the scattering angle θ by $z = r\cos\theta$, and since there is no dependence on the azimuthal angle ϕ. we can define the incoming and outgoing parts of the wave function as follows:

$$\psi_{in} = Ae^{ikz} \tag{57}$$

and

$$\psi_{out} = Af(E,\theta)\frac{e^{ikr}}{r} \tag{58}$$

Where the factor $\frac{1}{r}$ is carried, to conserve probability. The complex function $f(E,\theta)$ is the so called scattering amplitude. We are now interested in the differential cross section $\frac{d\sigma}{d\Omega}$, which is defined as the ratio of number of particles per unit time, that are scattered into the surface element $dS = r^2 d\Omega(\theta,\phi)$ and the number of incoming particles per unit time, per are orthogonal to the beam direction. Expressed via probability currents, we thus obtain:

$$\frac{d\sigma}{d\Omega} = \frac{\vec{j}_{out}\cdot\vec{e}_r r^2}{|\vec{j}_{in}|} \tag{59}$$

With $\vec{e}_r$ being a unit vector in direction of the radius, and the currents given as:

$$\vec{j} = \frac{\hbar}{2m}(\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi) \tag{60}$$

By applying these equations, we obtain the differential cross section as:

$$\frac{d\sigma}{d\Omega} = |f(E,\theta)|^2 \tag{61}$$

With the scattering amplitude, being given as:

$$f(E,\theta) = \sum_{l=0}^{\infty}(2l+1)f_l(E)P_l(cos\theta) \tag{62}$$

where l denotes the magnitude of orbital angular momentum, and $P_l(cos\theta)$ are the Legendre polynomials. We can now work out the total cross section σ via the integral:

$$\sigma = \int d\Omega\frac{d\sigma}{d\Omega} \tag{63}$$

We do this, by applying the orthogonality relation

$$\int d\Omega P_l(cos\theta)P_{l'}(cos\theta) = \frac{4\pi}{(2l+1)}\delta_{ll'} \tag{64}$$

Thus, we obtain:

$$\sigma = \sum_{l,l'}(2l+1)(2l'+1)f_l^*(E)f_{l'}(E)\int d\Omega P_l(cos\theta)P_{l'}(cos\theta) \tag{65}$$

Which, finally leads to:

$$\sigma = 4\pi\sum_l(2l+1)|f(E)|^2 \tag{66}$$

By now defining a phase shift of:

$$S_l(k) = 1 + 2ikf_l(E) \tag{67}$$

where, $S_l(k)$ is the l-th matrix element of the scattering operator, and can also be written as:

$$S_l = e^{2i\delta_l} \tag{68}$$

Thus, one can see pretty quickly, that

$$f_l(E) = \frac{e^{2i\delta_l} - 1}{2ik} = \frac{e^{i\delta_l}\sin\delta_l}{k} \tag{69}$$

With this result, we can now derive the optical theorem. We start, by plugging the result above into equation (66)

$$\sigma = 4\pi\sum_{l=0}^{\infty}(2l+1)|\frac{1}{k}e^{i\delta_l E}\sin\delta_l(E)|^2 = \frac{4\pi}{k^2}\sum_l(2l+1)\sin^2\delta_l \tag{70}$$

with

$$Im\, f(E,0) = \sum_{l=0}^{\infty}(2l+1)Im(\frac{e^{i\delta_l}}{k})\sin\delta_l = \frac{1}{k}\sum_l(2l+1)\sin^2\delta_l \tag{71}$$

Which, finally leads to:

$$\sigma_{el} = \frac{4\pi}{k}Im\, f(E,0) \tag{72}$$

We now consider a froward scattering amplitude, with the asymptotic behavior $[f(E,0) - f_\infty(0)] \to E - 1 - \varepsilon$ for $|E| \to \infty$, with $\varepsilon > 0$. We want to compute the integral, given by:

$$\oint_\Gamma dE'\frac{f(E',0) - f_\infty(0)}{E' - E} = 2\pi i\sum_i^N\frac{Res_{E'=E_i(f(E',0)-f_\infty(0)}}{E_i - E} \tag{73}$$

From the optical theorem we know, that the forward scattering amplitude is real on the negative real axis, and we also assume that $F_\infty(0)$ is real, as well. For further calculations the integrand in equation (73) will be named $F(E')$, to improve readability, also the arguments of the functions $f(E,0)$ and $f_\infty(0)$ will be left out, from this point forward. Firstly, we consider the parts of the curve, parallel to the real axis. We say $f(E,0)$ is analytical on the upper half plane, to the point E. By making use of the Schwarz reflection principle, we can write:

$$\int_\leftarrow dE'F(E'^*) + \int_\rightarrow dE'F(E') = \int_\leftarrow dE'F^*(E') + \int_\rightarrow dE'F(E') = 2i\int \to dE'\, ImF(E') \tag{74}$$

The imaginary part of $F(E')$ equates to:

$$Im(F) = Im(\frac{f - f_\infty}{E' - E}) = \frac{1}{|E' - E|^2}[Im(f)Re(E' - E) - Re(f)Im(E' - E) + f_\infty Im(E' - E)] \tag{75}$$

We can now calculate the residues of each term of equation (75) individually. To achieve this, we consider the point $E = E_r + \alpha i E_i$. We will start by calculating the residues for the second and third term first, since the first term will be used differently. So, for the second term, we obtain the residue as follows:

$$Res_{E_2} = \lim_{\alpha\to 0} \frac{d}{dE'}(E' - E)^2 \frac{Re(f)Im(E' - E)}{(E' - E)(E' - E)^*}|_{E'=E} \tag{76}$$

$$= \lim_{\alpha\to 0} \frac{d}{dE'} \frac{Re(f)Im(E' - E)(E' - E)}{(E' - E)^*}|_{E'=E} \tag{77}$$

The product rule leaves four terms to calculate from this equation, but three of which can be disregarded, since they vanish, if the limit is taken before the derivation. The last term comes to:

$$\frac{d}{dE'}Im(E' - E) = -\frac{i}{2} \tag{78}$$

Which leaves the residue as:

$$Res_{E_2} = -\frac{i}{2}Ref(E) \tag{79}$$

By the same procedure, the other residue amounts to:

$$Res_{E_3} = -\frac{i}{2}f_\infty \tag{80}$$

For the first term, we calculate the integral

$$\lim_{\alpha\to 0}\int_0^\infty dE' \frac{Imf(E',0)}{|E' - E|^2} = P\int_0^\infty dE' \frac{Imf(E',0)}{E' - E} + \int_{circle} dE' \frac{Imf(E',0)}{|E' - E|^2} \tag{81}$$

Where $P\int$ denotes the principal value integral. With $E' = E_r - \rho e^{i\phi}$ and $E = E_r + i\alpha E_i$, the seconds integral in equation (81) can be written as

$$\lim_{\alpha\to 0}\lim_{\rho\to 0}\int_\pi^{2\pi} d\phi \frac{Imf(E_r - \rho e^{i\phi},0)(\rho\cos\phi)}{|(E_r - \rho e^{i\phi}) - (E_r + i\alpha E_i)|^2}(-i\rho e^{i\phi}) \to 0 \tag{82}$$

Thus, we finally obtain:

$$\oint_\Gamma dE' \frac{f(E',0) - f_\infty(0)}{E' - E} = 2i(-\pi Ref(E',0) + \pi f_\infty(0) + P\int_0^\infty dE' \frac{Imf(E',0)}{E' - E}) \tag{83}$$

Which is equal to

$$Ref(E',0) = f_\infty(0) + \frac{1}{\pi}P\int_0^\infty dE' \frac{Imf(E',0)}{E' - E} - \sum_i^N \frac{Res_{E'=E_i}(f(E',0) - f_\infty(0))}{E_i - E} \tag{84}$$

4 The pion vector from factor and the Omnès Problem

Not only do the principles of unitarity and analyticity apply in quantum physical scattering processes, they also work very well within relativistic ones in quantum field theory. For processes involving the strong nuclear force, quantum chromodynamics describe the involved theory. While for high energies, the very usual way, of perturbative expansion works very well, the smaller the energy the more unreliable the perturbative schemes get. The advantage of dispersion relations is now, that they hold non-perturbatively and thus enabling us, to obtain results by plugging in experimental input and deriving relations between observables. One such application is the pion vector form factor, which describes the non-perturbative effects of the strong interaction, affecting the transition of the virtual photon into the pion pair, in a collision between an electron and its anti-particle. The so called Omnés problem describes a way to find the pion VFF, via the $\pi\pi$ - phase shift.

4.1 Unitarity of the scattering matrix

As in quantum mechanics, we also define a scattering matrix in particle physics. This S-matrix describes the transition of incoming particles into outgoing particles in a scattering experiment. The S-operator must be unitary, in order to preserve probability and not to change the norm of the state:

$$SS^\dagger = S^\dagger S = 1 \tag{85}$$

This can be used to obtain Watson's final state theorem:

$$\mathrm{Im}(F_\pi^V(s) = F_\pi^V(s)e^{-i\delta_{\pi\pi}(s)}\sin\delta_{\pi\pi}(s) \tag{86}$$

4.2 The Omnès Problem

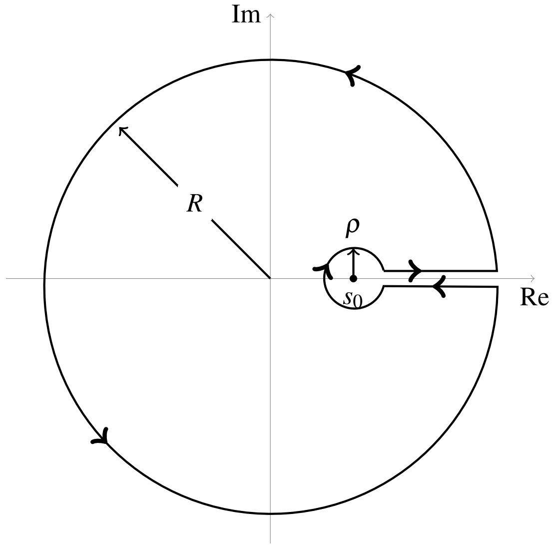
The equations 86 allow us to carefully reconstruct the pion Vector Form Factor, based on strictly formulated conditions. This is known as the Omnès Problem. First of all the equation tells us that $F_\pi^V(s)$ is a complex valued function, as s is an analytic variable in the complex plane, apart from a cut complex s-plane $\Gamma = [s_0, \infty) \subset \mathbb{R}$, where $s_0 = 4M_\pi^2 > 0$. To summarize e the conditions are

- $F_\pi^V(s)$ is analytic on the cut complex s-plane $\mathbb{C}\backslash\Gamma$
- $F_\pi^V(s) \in \mathbb{R} \quad \forall\, s \in \mathbb{R}\backslash\Gamma$
- $\lim_{\varepsilon\to 0}(F_\pi^V(s+i\varepsilon)e^{-i\delta_{\pi\pi}(s)}) \in \mathbb{R}$ on Γ for a real bounded function $\delta_{\pi\pi}(s)$
- We assume $F_\pi^V(0) = 1$ and $F_\pi^V(s)$ has no zeros.

We start off with the Couchy Integral

$$\ln(F(s)) = \frac{1}{2\pi i}\oint_C ds' \frac{\ln(F(s'))}{s'-s} \tag{87}$$

over the following contour



This means the integral can be separated into

$$\oint_C ds' \frac{\ln(F(s'))}{s'-s} = \lim_{\varepsilon\to 0}\left(\int_{C(R)} + \int_{C(\rho)} + \int_{s_0+i\varepsilon}^{\infty+i\varepsilon} + \int_{\infty-i\varepsilon}^{s_0-i\varepsilon}\right) ds' \frac{\ln(F(s'))}{s'-s} \tag{88}$$

The integrals over $C(R)$ and $C(\rho)$ disappear. For the last two integrals we can use the Schwarz reflection principle and then we get

$$\oint_C ds' \frac{\ln(F(s'))}{s'-s} = \frac{1}{\pi}\int_{s_0}^{\infty} ds' \mathrm{Im}\left(\frac{\ln(F(s'))}{s'-s}\right) \tag{89}$$

where we used $\mathrm{Im}(z) = \frac{z-z^*}{2i}$ to write the imaginary part here. We look now at equation 86 and refactor it

$$\frac{F_\pi^V(s) - F_\pi^V(s)^*}{2i} = F_\pi^V e^{i\delta_{\pi\pi}}(s)\sin(\delta_{\pi\pi}(s)) \tag{90}$$

$$F_\pi^V(s) = F_\pi^V(s)^* e^{2i\delta_{\pi\pi}(s)} \tag{91}$$

$$\ln(F_\pi^V(s)) = \ln((F_\pi^V e^{-i\delta_{\pi\pi}})^*) + i\delta_{\pi\pi}(s). \tag{92}$$

Now we use this equation to compute the integral with variation in $s \to s + i\varepsilon$ as ε goes to infinity.

$$\ln(F_\pi^V(s)) = \lim_{\varepsilon\to\infty} \ln(F_\pi^V(s+i\varepsilon)) = \tag{93}$$

$$= \lim_{\varepsilon\to\infty} \frac{1}{\pi}\int_{s_0}^{\infty} \mathrm{Im}\left(\frac{F_\pi^V(s')}{s'-s-i\varepsilon}\right) = \tag{94}$$

$$= \lim_{\varepsilon\to\infty} \frac{1}{\pi}\int_{s_0}^{\infty} \mathrm{Im}\left(\frac{\ln((F_\pi^V e^{-i\delta_{\pi\pi}})^*) + i\delta_{\pi\pi}}{s'-s-i\varepsilon}\right) \tag{95}$$

the part $F_\pi^V e^{-i\delta_{\pi\pi}}$ needs to be real that means

$$\ln(F_\pi^V(s)) = \lim_{\varepsilon\to\infty} \frac{1}{\pi}\int_{s_0}^{\infty} \frac{\delta_{\pi\pi}(s')}{s'-s-i\varepsilon} = \tag{96}$$

$$= \ln(F_\pi^V(0)) + \lim_{\varepsilon\to\infty} \frac{s}{\pi}\int_{s_0}^{\infty} \frac{\delta_{\pi\pi}(s')}{s'(s'-s-i\varepsilon)} \tag{97}$$

with the condition $F_\pi^V(0) = 1$ and the relation from 53 we get

$$F_\pi^V(s) = \exp\left(\frac{s}{\pi}\fint_{s_0}^{\infty} ds' \frac{\delta_{\pi\pi}(s')}{s'(s'-s)} + i\delta_{\pi\pi}(s)\right). \tag{98}$$

To compute the principal value integral we use the following trick

$$\frac{s}{\pi}\fint_{s_0}^{\infty} ds' \frac{\delta_{\pi\pi}(s')}{s'(s'-s)} = \frac{s}{\pi}\int_{s_0}^{\infty} ds' \frac{\delta_{\pi\pi}(s') - \delta_{\pi\pi}(s)}{s'(s'-s)} + \delta_{\pi\pi}\frac{s}{\pi}\fint_{s_0}^{\infty} \frac{1}{s'(s'-s)}. \tag{99}$$

The first integral can be computed numerically, the second one has an analytic solution for $s > s_0$. We use the definition of the principal value and circle around s in a small half circle with the radius r.

$$\fint_{s_0}^{\infty} \frac{1}{s'(s'-s)} = \lim_{r\to 0}\left(\int_{s_0}^{s-r} ds' + \int_{s+r}^{\infty} ds'\right)\frac{1}{s'(s'-s)} \tag{100}$$

then we simply integrate and plug in

$$\fint_{s_0}^{\infty} \frac{1}{s'(s'-s)} = \lim_{r\to 0}\left(\frac{\ln(s'-s) - \ln(s')}{s}\Big|_{s'=s_0}^{s'=s-r} \frac{\ln(s'-s) - \ln(s')}{s}\Big|_{s'=s+r}^{s'=\infty}\right) = \tag{101}$$

$$= \frac{1}{s}\ln\left(\frac{s_0}{s_0 - s}\right) \tag{102}$$

that means the second integral is

$$\delta_{\pi\pi}\frac{s}{\pi}\fint_{s_0}^{\infty} \frac{1}{s'(s'-s)} = \delta_{\pi\pi}(s)\frac{1}{\pi}\ln\left(\frac{s_0}{s_0 - s}\right) \tag{103}$$

Lastly we would like to plot the modulus of the Omnès representation of the pion vector form factor. For the phase we would usually use experimental values, but in our case we will use the Breit-Wigner representation of the pion VVF to compute the phase $\delta_{\pi\pi}$. A reminder the Breit-Wigner representation is the following

$$F_\pi^V(s)_{BW} = \frac{M_\rho^2}{M_\rho^2 - s - iM_\rho\Gamma_\rho(s)} \tag{104}$$

where

$$\Gamma_\rho(s) := \Gamma_\rho\frac{s}{M_\rho^2}\left(\frac{\sigma_\pi(s)}{\sigma_\pi(M_\rho^2)}\right)^3\theta(s - 4M_\pi^2), \qquad \sigma_\pi(s) := \sqrt{1 - \frac{4M_\pi^2}{s}}. \tag{105}$$

Thus our phase shift will be

$$\delta_{\pi\pi}(s) := \arg\left(F_\pi^V(s)_{BW}\right) \tag{106}$$

where we will use numerical values for $M_\rho = 0.77$ GeV, $\Gamma_\rho = 0.15$ GeV, $M_\pi = 0.14$GeV.

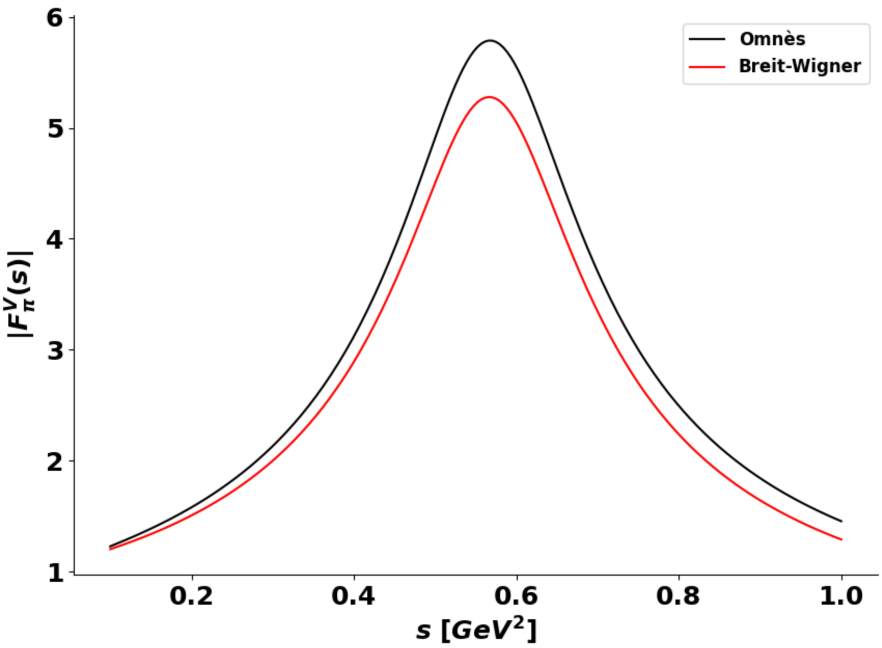


Figure 2: Plot of the modulus of the Breit-Wigner(red) and the Omnès representation(black) of the pion Vector From Factor for $s \in [0, 1]$ in GeV^2

The code for the plots and some minor calculations e.g. numerical integration can be found in [1].

References

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