# Heat equation demo

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This project demonstrates the solving of partial differential equations in Python using the finite element method. The FEniCS-package for Python performs the discretization required in the finite element method and computes the solution.

Let us consider thermal diffusion in a uniform cylinder. The initial temperature in the cylinder is $T_i = 300$ K. The temperatures at the top and bottom surfaces of the sphere as set to be constant as Dirichlet boundaries, with values $T_{bot} = 280$ K and $T_{top} = 320K$. The heat transfer in the cylinder is described by the heat equation without a heating source:

$$\frac{\partial T(r,t)}{\partial t} = \alpha \nabla^2 T(r,t), \tag{1}$$

where $T$ is temperature, $\alpha$ is the thermal diffusivity, $t$ is time and $r$ is location. For simplicity, let us set $\alpha = 1$.

The weak formulation of the problem required for the finite element method is derived as follows. The time-derivative of temperature can be approximated to be:

$$\frac{\partial T}{\partial t} = \frac{T - T_i}{\Delta t}, \tag{2}$$

where $T_i$ is the temperature at the previous value of time. Inserting Eq. (2) to Eq. (1) gives:

$$\frac{T - T_i}{\Delta t} = \nabla^2 T \tag{3}$$

Multiplying with a test function $v$, integrating over the domain $\Omega$ and applying Green's theorem gives:

$$\int_\Omega \frac{T - T_i}{\Delta t} v \, d\Omega = \int_\Gamma v \nabla T \cdot \hat{n} \, d\Gamma - \int_\Omega \nabla T \cdot \nabla v \, d\Omega \tag{4}$$

With only Dirichlet conditions the boundary term vanishes. By reorganizing we get the weak form:

$$\int_\Omega T v \, d\Omega + \int_\Omega \Delta t \nabla T \cdot \nabla v \, d\Omega = \int_\Omega T_i v \, d\Omega \tag{5}$$