Topological Optimization Using the SIMP Method

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Outline

- Heat Equation and Finite Volume Method
- Optimization and Method of Moving Asymptotes
- Solid Isotropic Material with Penalization (SIMP)
- Algorithm Results

Heat Equation

The temperature (T) at any point in such an object is a function of both space $(\mathbf{x})$ and time (t) coordinates: $T(\mathbf{x}, t)$.

Physical principles demand that such a temperature function must satisfy the equation

$$\frac{\partial T}{\partial t} = \nabla \cdot \left(k(\mathbf{x}) \nabla T \right), \tag{1}$$

where ∇ is the gradient operator and the function k represents the thermal diffusivity at a point in our object.

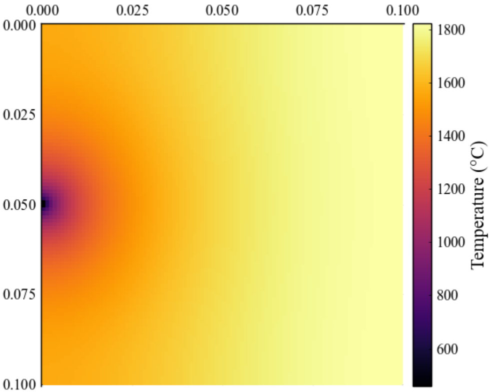

Figure 1: Heatmap for a 0.1m $\times$ 0.1m object with uniform heat generation and a heat sink at the center of its west boundary. This map was produced via the Finite Volume Method using 100×100 uniform control volumes.

Finite Volume Method (FVM)

Theorem (The Divergence Theorem)

Suppose that $\mathcal{V}$ is a compact subset of $\mathbb{R}^n$ that has a piecewise smooth boundary $\mathcal{S}$ (i.e. $\partial\mathcal{V} = \mathcal{S}$) with outward pointing normal vectors. If $\mathbf{F}$ is a continously differentiable vector field defined on a neighborhood of $\mathcal{V}$, then

$$\iiint_{\mathcal{V}} (\nabla \cdot \mathbf{F})\, \mathrm{d}\mathcal{V} = \oiint_{\mathcal{S}} (\mathbf{F} \cdot \hat{\mathbf{n}})\, \mathrm{d}\mathcal{S} \tag{2}$$

where $\hat{n}$ is the outwards pointing normal vector to the boundary.

Discretization of Heat Equation

$$\int_{V_i} \frac{\partial T}{\partial t}\,\mathrm{d}\mathbf{x} = \int_{V_i} \nabla \cdot \big(k(\mathbf{x})\nabla T\big)\,\mathrm{d}\mathbf{x} \underset{(2)}{=} \int_{\partial V_i} k(\mathbf{x})\nabla T \cdot \hat{\mathbf{n}}\,\mathrm{d}s \qquad (3)$$

In a square grid there are only four neighboring cells which we can label as North, South, East, West.

For a control volume V_i we'll label the North boundary as ∂V_N, the South boundary as ∂V_S, the East boundary as ∂V_E, and the West boundary as ∂V_W.

Additionally, let Δx be the length of the North and South boundaries, and Δy the length of the East and West boundaries.

Discretization of Heat Equation

$$\int_{V_i} \frac{\partial T}{\partial t}\,\mathrm{d}\mathbf{x} = \int_{V_i} \nabla \cdot \big(k(\mathbf{x})\nabla T\big)\,\mathrm{d}\mathbf{x} \underset{(2)}{=} \int_{\partial V_i} k(\mathbf{x})\nabla T \cdot \hat{\mathbf{n}}\,\mathrm{d}s$$

$$\begin{aligned}\int_{\partial V_i} k(\mathbf{x})\nabla T \cdot \hat{\mathbf{n}}\,\mathrm{d}s \quad = \quad & \int_{\partial V_N} k(\mathbf{x})\nabla T \cdot \hat{\mathbf{n}}_N\,\mathrm{d}s + \int_{\partial V_S} k(\mathbf{x})\nabla T \cdot \hat{\mathbf{n}}_S\,\mathrm{d}s \\ & + \int_{\partial V_E} k(\mathbf{x})\nabla T \cdot \hat{\mathbf{n}}_E\,\mathrm{d}s + \int_{\partial V_W} k(\mathbf{x})\nabla T \cdot \hat{\mathbf{n}}_W\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
\int_{\partial V_i} k(\mathbf{x})\nabla T \cdot \hat{\mathbf{n}}\,\mathrm{d}s \quad = \quad & \int_{\partial V_N} k(\mathbf{x})\nabla T \cdot \hat{\mathbf{n}}_N\,\mathrm{d}s + \int_{\partial V_S} k(\mathbf{x})\nabla T \cdot \hat{\mathbf{n}}_S\,\mathrm{d}s \\
& + \int_{\partial V_E} k(\mathbf{x})\nabla T \cdot \hat{\mathbf{n}}_E\,\mathrm{d}s + \int_{\partial V_W} k(\mathbf{x})\nabla T \cdot \hat{\mathbf{n}}_W\,\mathrm{d}s
\end{aligned}
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\begin{aligned}
\int_{\partial V_i} k(\mathbf{x}) \nabla T \cdot \hat{\mathbf{n}}\, \mathrm{d}s \quad &= \quad \int_{\partial V_N} k(\mathbf{x}) \nabla T \cdot \hat{\mathbf{n}}_N\, \mathrm{d}s + \int_{\partial V_S} k(\mathbf{x}) \nabla T \cdot \hat{\mathbf{n}}_S\, \mathrm{d}s \\
&\quad + \int_{\partial V_E} k(\mathbf{x}) \nabla T \cdot \hat{\mathbf{n}}_E\, \mathrm{d}s + \int_{\partial V_W} k(\mathbf{x}) \nabla T \cdot \hat{\mathbf{n}}_W\, \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
\int_{V_i} \frac{\partial T}{\partial t}\, \mathrm{d}\mathbf{x} \quad &\approx \quad k_N \frac{T_N - T_i}{\|\mathbf{x}_N - \mathbf{x}_i\|} \Delta y + k_S \frac{T_S - T_i}{\|\mathbf{x}_S - \mathbf{x}_i\|} \Delta y \\
&\quad + k_E \frac{T_E - T_i}{\|\mathbf{x}_E - \mathbf{x}_i\|} \Delta x + k_W \frac{T_W - T_i}{\|\mathbf{x}_W - \mathbf{x}_i\|} \Delta x
\end{aligned}
$$

$$
\begin{aligned}
\int_{\partial V_i} k(\mathbf{x})\nabla T\cdot\hat{\mathbf{n}}\,\mathrm{d}s \quad &= \quad \int_{\partial V_N} k(\mathbf{x})\nabla T\cdot\hat{\mathbf{n}}_N\,\mathrm{d}s + \int_{\partial V_S} k(\mathbf{x})\nabla T\cdot\hat{\mathbf{n}}_S\,\mathrm{d}s \\
&\quad + \int_{\partial V_E} k(\mathbf{x})\nabla T\cdot\hat{\mathbf{n}}_E\,\mathrm{d}s + \int_{\partial V_W} k(\mathbf{x})\nabla T\cdot\hat{\mathbf{n}}_W\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
\int_{V_i} \frac{\partial T}{\partial t}\,\mathrm{d}\mathbf{x} \quad &\approx \quad k_N\frac{T_N-T_i}{\|\mathbf{x}_N-\mathbf{x}_i\|}\Delta y + k_S\frac{T_S-T_i}{\|\mathbf{x}_S-\mathbf{x}_i\|}\Delta y \\
&\quad + k_E\frac{T_E-T_i}{\|\mathbf{x}_E-\mathbf{x}_i\|}\Delta x + k_W\frac{T_W-T_i}{\|\mathbf{x}_W-\mathbf{x}_i\|}\Delta x
\end{aligned}
$$

$$
\begin{aligned}
\implies \Delta x\Delta y\frac{\mathrm{d}T_i}{\mathrm{d}t} \quad &= \quad \left(k_N\frac{T_N-T_i}{\|\mathbf{x}_N-\mathbf{x}_i\|} + k_S\frac{T_S-T_i}{\|\mathbf{x}_S-\mathbf{x}_i\|}\right)\Delta y \\
&\quad + \left(k_E\frac{T_E-T_i}{\|\mathbf{x}_E-\mathbf{x}_i\|} + k_W\frac{T_W-T_i}{\|\mathbf{x}_W-\mathbf{x}_i\|}\right)\Delta x
\end{aligned}
$$

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