waveEquation

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1 Project 2: 2D wave equation

Summary. The aim of this project is to develop a solver for the two-dimensional, standard, linear wave equation, with damping, and verify the solver.

1.1 Mathematical problem

The general wave equation in d space dimensions, with variable coefficients, can be written in the compact form

$$\varrho \frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = \nabla\cdot(q\nabla u) + f \text{ for } \boldsymbol{x}\in\Omega\subset\mathbb{R}^d,\ t\in(0,T],$$

which in 2D becomes

$$\varrho(x,y)\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + f(x,y,t)\,.$$

To save some writing and space we may use the index notation, where subscript t, x or y means differentiation with respect to that coordinate, i.e.

$$\begin{aligned}
u_t &= \frac{\partial u}{\partial t}, \quad u_{tt} = \frac{\partial^2 u}{\partial t^2}\\
u_x &= \frac{\partial u}{\partial x}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}\\
u_y &= \frac{\partial u}{\partial t}, \quad u_{yy} = \frac{\partial^2 u}{\partial y^2}\\
(qu_x)_x &= \frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right)\\
(qu_y)_y &= \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right)\,.
\end{aligned}$$

The 2D versions of the two model PDEs, with and without variable coefficients, can with now with the aid of the index notation for differentiation be stated as

$$\varrho u_{tt} + bu_t = (qu_x)_x + (qu_z)_z + (qu_z)_z + f$$

Since this PDE contains a second-order derivative in time, we need two initial conditions; (1) the initial shape of the string, I, and (2) the initial velocity of the string, V;

$$u(x, y, 0) = I(x, y),$$
$$u_t(x, y, 0) = V(x, y).$$

In addition, PDEs need boundary conditions, which are of three principal types:

- u is prescribed ($u = 0$ or a known time variation for an incoming wave)
- $\partial u/\partial n = \boldsymbol{n} \cdot \nabla u$ prescribed (zero for reflecting boundaries)
- An open boundary condition (also called radiation condition) is specified to let waves travel undisturbed out of the domain.

1.2 Discretization

In this section we will derive the discrete set of equations to be implemented in a program. We will for simplicity assume constant spacing between the mesh points. Our mesh points are

$$x_i = i\Delta x,\ i = 0, \ldots, N_x, y_j = j\Delta y,\ j = 0, \ldots, N_y, t_i = n\Delta t,\ n = 0, \ldots, N_t\,.$$

The solution $u(x, y, t)$ is sought at the mesh points. We introduce the mesh function $u^n_{i,j}$, which approximates the exact solution at the mesh point (x_i, y_j, t_n) for $i = 0, \ldots, N_x$, $j = 0, \ldots, N_y$ and $n = 0, \ldots, N_t$.

Discretizing the variable coefficient

The principal idea is to first discretize the outer derivative. We define

$$\phi^x = q(x, y)\frac{\partial u}{\partial x},$$

and use a centered derivative around $x = x_i$ for the derivative of ϕ:

$$\left[\frac{\partial \phi^x}{\partial x}\right]^n_i \approx \frac{\phi^x_{i+\frac{1}{2}} - \phi^x_{i-\frac{1}{2}}}{\Delta x} = [D_x\phi^x]^n_i\,.$$

Then discretize

$$\phi^x_{i+\frac{1}{2}} = q_{i+\frac{1}{2}}\left[\frac{\partial u}{\partial x}\right]^n_{i+\frac{1}{2}} \approx q_{i+\frac{1}{2}}\frac{u^n_{i+1} - u^n_i}{\Delta x} = [qD_xu]^n_{i+\frac{1}{2}}\,.$$

Similarly,

$$\phi^x_{i-\frac{1}{2}} = q_{i-\frac{1}{2}}\left[\frac{\partial u}{\partial x}\right]^n_{i-\frac{1}{2}} \approx q_{i-\frac{1}{2}}\frac{u^n_i - u^n_{i-1}}{\Delta x} = [qD_xu]^n_{i-\frac{1}{2}}\,.$$

These intermediate results are now combined to

$$\left[\frac{\partial}{\partial x}\left(q\frac{\partial u}{\partial x}\right)\right]^n_i \approx \frac{1}{\Delta x^2}\left(q_{i+\frac{1}{2}}\left(u^n_{i+1} - u^n_i\right) - q_{i-\frac{1}{2}}\left(u^n_i - u^n_{i-1}\right)\right)$$

With operator notation we can write the discretization as

$$\left[\frac{\partial}{\partial x}\left(q\frac{\partial u}{\partial x}\right)\right]_i^n \approx [D_x q D_x u]_i^n$$

Similarly we have

$$\left[\frac{\partial}{\partial y}\left(q\frac{\partial u}{\partial y}\right)\right]_i^n \approx [D_y q D_y u]_i^n$$

for

$$\phi^y = q(x,y)\frac{\partial u}{\partial y}.$$

In order to compute $[D_x q D_x u]_i^n$ and $[D_y q D_y u]_i^n$ we need to evaluate $q_{i\pm\frac{1}{2}}$. If q is a known function, we can easily evaluate $q_{i\pm\frac{1}{2}}$ simply as $q(x_{i\pm\frac{1}{2}})$ with $x_{i+\frac{1}{2}} = x_i + \frac{1}{2}\Delta x$. However, in many cases q, is only known as a discrete function, and evaluating must be done by averaging. The most commonly used averaging technique is the arithmetic mean:

$$q_{i+\frac{1}{2}} \approx \frac{1}{2}\left(q_{i+1} + q_i\right) = [\overline{q}^x]_{i+\frac{1}{2}}, q_{i-\frac{1}{2}} \approx \frac{1}{2}\left(q_i + q_{i-1}\right) = [\overline{q}^x]_{i-\frac{1}{2}},$$

which we are going to use in the following.

Replacing derivatives by finite differences

The second-order derivatives can be replaced by central differences. The most widely used difference approximation of the second-order derivative is

$$\frac{\partial^2}{\partial t^2}u(x_i, y_j, t_n) \approx \frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2}.$$

It is convenient to introduce the finite difference operator notation

$$[D_t D_t u]_{i,j}^n = \frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2}.$$

The first-order derivative in time in the damping term can be approximated by

$$[D_{2t} u]_{i,j}^n = \frac{u_{i,j}^{n+1} - u_{i,j}^{n-1}}{2\Delta t}.$$

Algebraic version of the initial conditions

The initial conditions are given as

$$u(x, y, t_0) = I(x, y),$$
$$u_t(x, y, t_0) = V(x, y).$$

The first condition can be computed by

$$u_i^0 = I(x_i), \quad i = 0, \ldots, N_x.$$

For the second one we use a centered difference of type

$$\frac{\partial}{\partial t}u(x_i, y_j, t_0) \approx \frac{u^1_{i,j} - u^{-1}_{i,j}}{2\Delta t} = [D_{2t}u]^0_{i,j} = V(x_i, y_j).$$

Algebraic version of the PDE

Interior spatial mesh points

The PDE with variable coefficients is discretized term by term:

$$[\varrho D_t D_t u + bD_{2t}u = (D_x \overline{q}^x D_x u + D_y \overline{q}^y D_y u) + f]^n_{i,j}\,.$$

When written out and solved for the unknown $u^{n+1}_{i,j}$ one gets the scheme

$$\begin{aligned}
u^{n+1}_{i,j} = \Bigg\{ & 2u^n_{i,j} + u^{n-1}_{i,j}\left[\frac{b\Delta t}{2\varrho_{i,j}} - 1\right] + \frac{\Delta t^2}{\varrho_{i,j}} f^n_{i,j} \\
& + \frac{1}{\varrho_{i,j}}\frac{\Delta t^2}{\Delta x^2}\left[\frac{1}{2}(q_{i,j} + q_{i+1,j})(u^n_{i+1,j} - u^n_{i,j}) - \frac{1}{2}(q_{i-1,j} + q_{i,j})(u^n_{i,j} - u^n_{i-1,j})\right] \\
& + \frac{1}{\varrho_{i,j}}\frac{\Delta t^2}{\Delta y^2}\left[\frac{1}{2}(q_{i,j} + q_{i,j+1})(u^n_{i,j+1} - u^n_{i,j}) - \frac{1}{2}(q_{i,j-1} + q_{i,j})(u^n_{i,j} - u^n_{i,j-1})\right]\Bigg\}\left(\frac{1}{1 + \frac{b\Delta t}{2\varrho_{i,j}}}\right).
\end{aligned}$$

Modified scheme for the first step

A problem with algebraic version of the PDE equation arises when $n = 0$ since the formula for $u^1_{i,j}$ involves $u^{-1}_{i,j}$, which is an undefined quantity outside the time mesh (and the time domain). However, we can use the initial condition to arrive at a special formula for $u^1_{i,j}$. From initial condition we have

$$u^{-1}_{i,j} = u^1_{i,j} - 2\Delta t V.$$

Inserting this into the algebraic version of the PDE equation for $n = 0$ gives:

$$\begin{aligned}
u^1_{i,j} = \Bigg\{ & 2u^0_{i,j} + \left(u^1_{i,j} - 2\Delta t V\right)\left[\frac{b\Delta t}{2\varrho_{i,j}} - 1\right] + \frac{\Delta t^2}{\varrho_{i,j}} f^0_{i,j} \\
& + \frac{1}{\varrho_{i,j}}\frac{\Delta t^2}{\Delta x^2}\left[\frac{1}{2}(q_{i,j} + q_{i+1,j})(u^0_{i+1,j} - u^0_{i,j}) - \frac{1}{2}(q_{i-1,j} + q_{i,j})(u^0_{i,j} - u^0_{i-1,j})\right] \\
& + \frac{1}{\varrho_{i,j}}\frac{\Delta t^2}{\Delta y^2}\left[\frac{1}{2}(q_{i,j} + q_{i,j+1})(u^0_{i,j+1} - u^0_{i,j}) - \frac{1}{2}(q_{i,j-1} + q_{i,j})(u^0_{i,j} - u^0_{i,j-1})\right]\Bigg\}\left(\frac{1}{1 + \frac{b\Delta t}{2\varrho_{i,j}}}\right).
\end{aligned}$$

Thus, the special formula for $u^1_{i,j}$ is

$$
\begin{aligned}
u^1_{i,j} = &\Bigg\{ 2u^0_{i,j} - 2\Delta t V \left[\frac{b\Delta t}{2\varrho_{i,j}} - 1\right] + \frac{\Delta t^2}{\varrho_{i,j}} f^0_{i,j} \\
&+ \frac{1}{\varrho_{i,j}} \frac{\Delta t^2}{\Delta x^2} \left[\frac{1}{2}(q_{i,j} + q_{i+1,j})(u^0_{i+1,j} - u^0_{i,j}) - \frac{1}{2}(q_{i-1,j} + q_{i,j})(u^0_{i,j} - u^0_{i-1,j})\right] \\
&+ \frac{1}{\varrho_{i,j}} \frac{\Delta t^2}{\Delta y^2} \left[\frac{1}{2}(q_{i,j} + q_{i,j+1})(u^0_{i,j+1} - u^0_{i,j}) - \frac{1}{2}(q_{i,j-1} + q_{i,j})(u^0_{i,j} - u^0_{i,j-1})\right] \Bigg\} \left(\frac{1}{1 + \frac{b\Delta t}{2\varrho_{i,j}}}\right) \left(\frac{1}{2 - \frac{b\Delta t}{2\varrho_{i,j}}}\right).
\end{aligned}
$$

1.3 Boundary condition

In a rectangular spatial domain $\Omega = [0, L_x] \times [0, L_y]$, the homogeneous Dirichlet condition, is given by

$$u^1_{i,j} = u^{n+1}_{i,j} = 0$$

for $n = 1, 2, \ldots, N_t - 1$, when $i = 0,\ i = N_x,\ j \in [0, L_y]$ and when $j = 0,\ j = N_y,\ i \in [0, L_x]$. Note that $u^0_{i,j}$ is given by I, i.e.

$$u^0_{i,j} = I(x_i, y_j) \quad \text{for } i = 0, \ldots, N_x \quad j = 0, \ldots, Ny$$

The Neumann boundary condition is given by

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0, \tag{1}$$

at $x = 0, N_x$ and $y = 0, N_y$. Since we have used central differences in all the other approximations to derivatives in the scheme, it is tempting to implement this condition by the difference

$$
\begin{aligned}
[-D_{2x} u = 0]^n_{0,j}, \quad [D_{2x} u = 0]^n_{L_x,j} \Rightarrow &\quad u^n_{-1,j} = u^n_{1,j}, \quad u^n_{N_x+1,j} = u^n_{N_x-1,j} \\
[-D_{2y} u = 0]^n_{i,0}, \quad [D_{2y} u = 0]^n_{i,L_y} \Rightarrow &\quad u^n_{i,-1} = u^n_{i,1}, \quad u^n_{i,N_y+1} = u^n_{i,N_y-1}\,.
\end{aligned}
$$

The problem is that $u^n_{-1,j}$, $u^n_{i,-1}$, $u^n_{N_x+1,j}$ and is not a u value that is being computed since the point is outside the mesh.

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