Full-order Observers Design for Nonlinear Systems with Unknown Input

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Abstract— **In this note, a design of full-order observers is presented for a class of Lipschitz nonlinear systems with unknown input. Sufficient condition on the existence of the observers is formulated in terms of linear matrix inequality (LMI), and the design algorithm is also derived. A numerical example is included to illustrate the effectiveness of the proposed results.**

Index Terms— **Full-order observer design, Nonlinear system, Unknown input, LMI.**

I. INTRODUCTION

The observer design problem is a very important problem that has various applications, such as output feedback control, system monitoring, process identification and fault detection. The problem of observing the state vector of linear time-invariant multivariable systems, subjected to unknown inputs, has received considerable attention in the last decades. Many types of full-order and reduced-order unknown inputs observers are now available. Reduced-order unknown inputs observers design methods can be found in [1-6] and full-order observers have been designed in [7] and [8]. Sufficient and necessary conditions for the existence of unknown inputs observers have been established in [3], [4] and [6]. The approach presented in [3] remains to be one of the most systematic computational approaches for the design of reduced-order unknown inputs observers.

In the last decades, many fundamental results to extend the existing unknown inputs observers design from linear systems to nonlinear systems have been reported. The unknown inputs observers for bilinear systems were designed in [9-12]. And the unknown inputs observers for more general nonlinear systems were also proposed in [13-15]. The nonlinear unknown inputs observers in these papers require construction of a state transformation to change the original nonlinear systems into canonical forms. One problem for these construction is that the required state transformation only exists for a limited class of nonlinear systems. The other problem is that the construction of the state transformation, required solving partial differential equations and is quite difficult. As expected, the design of nonlinear unknown inputs observers is much more difficult than the design of linear unknown inputs observers. Because the unknown inputs observers design for general nonlinear systems is very difficult and no systematical design method is available, some authors have considered unknown inputs observers design for a class of Lipschitz nonlinear systems. [16] first extended linear unknown inputs observers design to a class of Lipschitz systems and gave the linear matrix inequalities (LMIs) and linear matrix equalities (LMEs) based sufficient conditions for the existence of the proposed observers. However, how to find a solution satisfying the LMIs and LMEs is not an easy task. [16] also proposed a unknown inputs observers design for fault diagnosis purpose. The difficulty here lies in solving a parametric Lyapunov equation, which is very hard because no systematic method could be used. In [20], a dynamic unknown inputs observer was designed for a class of Lipschitz nonlinear systems. Although the dynamic observers introduce extra design freedom, the total order of the unknown inputs observers is higher than the non-dynamic ones. Thus, to design their unknown inputs observers, though an iterative algorithm was proposed, appears to be very comp -licated.

In this paper, we propose an LMI-based approach to solve the nonlinear unknown inputs observers design problem for a class of Lipschitz nonlinear systems. The sufficient observer existence condition is derived in terms of LMI, and the construction of corresponding full-order observer also can be lead to by the feasible solutions of a set of LMIs. The sufficient condition developed in this paper, when applied to linear systems, is also necessary. The effectiveness of our algorithm is shown by a numerical example.

The organization of this paper is as follows. Following the introduction, section 2 introduces the nonlinear system under consideration and formulates the nonlinear unknown input observer design problem. In section 3, we present the sufficient existence condition of the observer in terms of LMI, and derive the observer design algorithm. In section 4, a numerical example is given to illustrate the validity of our results. Finally, section 5 concludes this paper.

Notation: Throughout this paper, the notation are very standard. We let R^n denote the n dimensional Euclidean space, while $R^{n\times m}$ refer to the set of all $n\times m$ real matrices. A^T represents the transpose of the matrix A, while A^{-1} denotes the inverse of A. The notation $M > 0$ is used to denote a symmetric positive-definite matrix. $||x||$ refers to the Euclidean norm of the vector x, that is $||x|| = \sqrt{x^T x}$. I and 0 denote the identity matrix and zero matrix with compatible dimensions.

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II. PROBLEM STATEMENT AND PRELIMINARIES

Consider a class of nonlinear systems described by the following equations

$$\dot{x} = Ax + Bu + Dv + f(x), \qquad y = Cx \tag{1}$$

where $x \in R^n$ is the state vector, $u \in R^k$ is the known input vector, $v \in R^m$ is the unknown input vector, and $y \in R^p$ is the output vector of the systems. Matrices A, B, C and D are real constant and appropriate dimensions. Matrix D assumed, without loss of generality, to have a full column rank.

In this paper, it is assumed the nonlinear function $f(x)$ satisfies the following assumption.

Assumption: The nonlinear function $f(x)$ is assumed to be Lipschitz in its first argument with a Lipschitz constant γ, i.e. there exists a positive constant γ such that

$$||f(x) - f(\hat{x})|| \le \gamma ||x - \hat{x}|| \tag{2}$$

for all vectors $x \in R^n$ and $\hat{x} \in R^n$, and $|| \cdot ||$ denotes the the Euclidean norm.

Our objective is to design a full-order observer from the measured output signal $y(t)$ to estimate the states of the considered nonlinear systems asymptotically without any knowledge of the input vector v.

III. MAIN RESULTS

In this section, we derive the structure of the proposed full-order nonlinear unknown input observer, establish the sufficient existence condition of the corresponding observer in terms of LMI, and accomplish the observer design.

Following [5], the full-order nonlinear unknown input observer is described as

$$\begin{aligned} \dot{z} &= Nz + Gu + Ly + Mf(\hat{x}) \\ \hat{x} &= z - Ey \end{aligned} \tag{3}$$

where $z \in R^n$, $\hat{x} \in R^n$. N, L, G, and E are unknown matrices of appropriate dimensions, which must be determined such that $\hat{x}$ will asymptotically converge to x. Meanwhile, M satisfies $M = I_n + EC$.

Then the resulting dynamic estimation error vector is denoted by

$$e(t) = \hat{x} - x = z - x - Ey = z - Mx, \tag{4}$$

which satisfies

$$\begin{aligned} \dot{e}(t) \;\; = \;\; & Ne + (NM + LC - MA)x + (G - MB)u \\ & + M(f(\hat{x}) - f(x)) - MDv. \end{aligned} \tag{5}$$

When $MD = 0$, $G = MB$ and $NM + LC = MA$, equation (5) reduced to the homogeneous equation

$$\dot{e}(t) = Ne + M(f(\hat{x}) - f(x)). \tag{6}$$

Therefore, the conditions for $\hat{x}$ to be an asymptotic state observer of x are

$$\begin{aligned} NM + LC = MA,\; G = MB, \\ M = I_n + EC,\; MD = 0, \end{aligned} \tag{7}$$

and N must be a stability matrix, i.e. has all its eigenvalues in the left-hand side of the complex plane.

In order to use the well-known results obtained for the classical full-order observer without unknown inputs in [1], $NM + LC = MA$ can be written as

$$N = MA - KC \tag{8}$$

where

$$K = L + NE. \tag{9}$$

Substituting (7) into (8), we find

$$L = K(I_n + CE) - MAE. \tag{10}$$

Then the observer dynamical equation becomes

$$\dot{z} = (MA - KC)z + Gu + Ly + Mf(\hat{x}) \tag{11}$$

where matrices E, G, and L are obtained from (7) and (10), respectively.

Therefore, the problem of designing the full-order observer with unknown inputs is reduced to find a matrix E satisfying (7), and a matrix K such that $(MA - KC)$ is a stability matrix. This problem is equivalent to the standard problem of the observers design when all inputs are known.

Consequently, we present the following theorem, which insures the observer given by (3), (7) and (11) is indeed a nonlinear unknown input observer.

Theorem 1: If there exist two matrices E and K and a positive define symmetric matrix $P > 0$ such that

$$\begin{aligned} ECD = -D \\ N^T P + PN + \gamma PMM^T P + \gamma I < 0 \end{aligned} \tag{12}$$

then the observer given by (3), (7) and (4) can make $e(t)$ tend to zero asymptotically for any initial value $e(0)$.

Proof: From (7), we know $G = MB$ and $NM + LC = MA$. Since these facts, (5) can be rewritten as

$$\dot{e}(t) = Ne + M(f(\hat{x}) - f(x)) - MDv. \tag{13}$$

Because $ECD = -D$ implies that $MD = 0$, which leads to (6).

Define a Lyapunov function as $V(t) = e(t)^T Pe(t)$, then it follows from (6) that

$$\begin{aligned} \dot{V} \;\; &= \;\; e^T(N^T P + PN)e + 2e^T PM(f(\hat{x}) - f(x)) \\ &\le \;\; e^T(N^T P + PN)e + 2||e^T PM||||(f(\hat{x}) - f(x))|| \\ &\le \;\; e^T(N^T P + PN)e + 2||e^T PM||\gamma||e|| \\ &\le \;\; e^T(N^T P + PN)e + \gamma(||e^T PM||^2 + ||e||^2) \\ &= \;\; e^t(N^T P + PN + \gamma PMM^T P + \gamma I)e. \end{aligned} \tag{14}$$

Note that $N^T P + PN + \gamma PMM^T P + \gamma I < 0$ implies $e(t)$ tends to zero asymptotically for any initial value $e(0)$. This completes the proof. □

In order to design the nonlinear unknown input observer, from Theorem 1, it is clear that we only need to find E, K and P such that (12) is satisfied. One way to find them is try to solve (12) directly. However, this is very difficult because there is no systematic way to do it. This motivates us to reformulate (12) as LMIs.

For computational purposes, we first denote all possible solutions of E such that $ECD = -D$. Because D is of full column rank, the solution of $ECD = -D$ depends on the rank of matrix CD, and E exists if CD is of full column rank. If CD is of full column rank, then the general solution of $ECD = -D$ can be written as

$$E = -D(CD)^{+} + Y(I_n - (CD)(CD)^{+}) \tag{15}$$

where $(CD)^{+}$ is the generalized inverse matrix of CD, given by $(CD)^{+} = ((CD)^T(CD))^{-1}(CD)^T$, and Y is an arbitrary matrix of appropriate dimension. For notational convenience, defining $U = -D(CD)^{+}$ and $V = I_n - (CD)(CD)^{+}$, we can represented (15) as

$$E = U + YV. \tag{16}$$

Substituting E given by (16) into the matrix inequality in (12) yields

$$\begin{aligned} &((I+UC)A)^TP + P(I+UC)A \\ &+(VCA)^TY^TP + PY(VCA) - C^TK^TP - PKC \\ &+\gamma(P(I+UC) + PY(VC)) \cdot \\ &(P(I+UC) + PY(VC))^T + \gamma I < 0. \end{aligned} \tag{17}$$

Technically, the objective of finding matrices E, K and $P > 0$ is now equivalent to the problem of solving (17) for the feasible solutions of Y, K and $P > 0$. According to the following lemma, we are able to show that the matrix inequality presented in (17) can be reformulated as a LMI.

Lemma 1: The matrix inequality given by (17) has a solution for Y, K and $P > 0$ if and only if the following LMI has a solution for $\bar{Y}$, $\bar{K}$ as well as $P > 0$,

$$\begin{bmatrix} X & X_{12} \\ X_{12}^T & -I \end{bmatrix} < 0 \tag{18}$$

where X and X_{12} are defined as

$$\begin{aligned} X &= ((I+UC)A)^TP + P(I+UC)A \\ &\quad +(VCA)^T\bar{Y}^T + \bar{Y}(VCA) \\ &\quad -C^T\bar{K}^T - \bar{K}C + \gamma I, \\ X_{12} &= \sqrt{\gamma}[P(I+UC) + \bar{Y}(VC)], \end{aligned} \tag{19}$$

with $Y = P^{-1}\bar{Y}$ and $K = P^{-1}\bar{K}$.

Proof: Now a simple calculation shows that

$$\begin{aligned} &\begin{bmatrix} I & W \\ 0 & I \end{bmatrix} \begin{bmatrix} X & W \\ W^T & -I \end{bmatrix} \begin{bmatrix} I & 0 \\ W^T & -I \end{bmatrix} \\ &= \begin{bmatrix} X + WW^T & 0 \\ 0 & -I \end{bmatrix}. \end{aligned} \tag{20}$$

From Schur complement, this implies that $\begin{bmatrix} X & W \\ W^T & -I \end{bmatrix} < 0$ is equivalent to $X + WW^T < 0$. Therefore, denoting $W = \sqrt{\gamma}[P(I+UC) + \bar{Y}(VC)]$, we can obtain (17). Thus the lemma is proved. □

Next, we can reformulate the sufficient condition given by Theorem 1 in terms of LMIs, and the feasible solutions to these LMIs guarantee the existence of a full-order nonlinear unknown input observer given by (3), (7) and (11).

Theorem 2: Assume that the matrix CD is of full column rank and the LMI defined by (18) and (19) has feasible solutions for $\bar{Y}$, then $\bar{K}$ can be designed such that the state estimation error $e(t)$ tends to zero asymptotically for any initial value of $e(0)$.

Proof: According to the conditions of this theorem and Lemma 1, we know the matrix inequality given by (17) has a solution of $Y = P^{-1}\bar{Y}$, $K = P^{-1}\bar{K}$ and $P > 0$. If we define $E = U + YV$ and $N = MA - KC$, (17) can be rewritten as $N^TP + PN + \gamma PMM^TP + \gamma I < 0$. Noticing that $E = U + YV$ implies $ECD = -D$, then the proof of this theorem follows from that of Theorem 1 immediately. The proof is complete. □

Based on Theorem 2, a design algorithm for the full-order observer is stated as follows.

Design Algorithm:

Step 1: Compute the matrices $U = -D(CD)^{+}$ and $V = I - (CD)(CD)^{+}$ respectively.

Step 2: Solve the LMIs defined by (18) and (19) for matrices $\bar{Y}$, $\bar{K}$, and a symmetric matrix $P > 0$.

Step 3: Let $Y = P^{-1}\bar{Y}$ and $K = P^{-1}\bar{K}$.

Step 4: Based on the Y and K obtained, the observer gains in (3) can yield as

$$\begin{aligned} &E = U + YV,\ M = I + EC,\ N = MA - KC, \\ &\qquad G = MB,\ L = K(I + CE) - MAE. \end{aligned}$$

Remark 1: Theorem 2 shows that the full-order nonlinear unknown input observer design can be accomplished systematically by solving LMIs defined by (18) and (19). And a reduced-order observer can also be designed for nonlinear Lipschitz system by combing the design technique in [4] and the technique in this paper.

Notice that (1) becomes a linear system when $f(x) = 0$. Hence the full-order unknown input observer design for linear systems can be considered as a special case of the observer design for Lipschitz nonlinear systems. In fact, for linear system case, we can derive a sufficient and necessary condition given in terms of LMI based on Theorem 2.

When $f(x) = 0$, the full-order nonlinear unknown input observer design method given by (3) becomes

$$\begin{aligned} \dot{z} &= Nz + Gu + Ly \\ \hat{x} &= z - Ey \end{aligned} \tag{21}$$

where N, G, L, M are defined as

$$\begin{aligned} N &= MA - KC,\ G = MB \\ L &= K(I + CE) - MAE,\ M = I + EC \end{aligned} \tag{22}$$

and E, K are chosen by the designers.

In order to obtain a sufficient and necessary condition presented in terms of LMIs, we introduce the following inequality as

$$\begin{aligned} &((I+UC)A)^TP + P(I+UC)A \\ &+(VCA)^T\bar{Y}^T + \bar{Y}(VCA) \\ &-C^T\bar{K}^T - \bar{K}C < 0 \end{aligned} \tag{23}$$

with $Y = P^{-1}\bar{Y}$ and $K = P^{-1}\bar{K}$.

Meanwhile, note that the LMI defined by (18) and (19) is reduced to (23) if we let $\gamma = 0$.

In the sequel, a sufficient and necessary condition given in terms of LMI for the existence of unknown input observer for linear systems will be proposed.

Theorem 3: Assume that $f(x) = 0$, namely, (1) is a linear system and also that the matrix CD is full column rank, then the the full-order unknown input observer given by (21) and (22) exists if and only if the LMI (23) has feasible solutions of $\bar{Y}$, $\bar{K}$ and $P > 0$.

Proof: Obviously, the sufficient part is implied in the proof of Theorem 2. The necessity part can be proved easily based on the Theorem 2 in [8]. □

Remark 2: As mentioned in [8], we know that the design of the unknown input observer for linear systems is usually carried out in a trial and error manner in the literature. The new sufficient and necessary condition provided in this section presents a systematic approach since it is presented in terms of LMIs, which can be solved efficiently using Matlab LMI toolbox [20].

IV. NUMERICAL EXAMPLE

In order to demonstrate the applicability of the proposed method, we consider a simple example of Lipschitz nonlinear system described by

$$\begin{aligned} \dot{x} &= Ax + Bu + Dv + f(x) \\ y &= Cx \end{aligned}$$

where

$$A = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \ B = 0, \ D = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

and $f(x) = [0.6x_1 sin(2t), \ 0.6x_2 cos(2t), \ 0\ ]^T$, $v = 2sin(3t)$.

To construct the nonlinear unknown input observer, we choose and deduce that

$$U = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \ V = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$
$$||f(x) - f(\hat{x})|| \leq \ 0.6||x - \hat{x}||.$$

Noting that the Assumption in section 2, we may choose $\gamma = 0.65$.

Applying the Matlab LMI Control Toolbox [20], we solve the LMIs defined by (18) and (19) for matrices $P > 0$, $\bar{Y}$ and $\bar{K}$, and one feasible solution is found as follows

$$P = \begin{bmatrix} 50.25 & 0 & 0 \\ 0 & 0.8956 & 0 \\ 0 & 0 & 48.73 \end{bmatrix}, \ \bar{Y} = \begin{bmatrix} 0 & 0 \\ 0 & 1.3875 \\ 0 & -50.25 \end{bmatrix},$$
$$\bar{K} = \begin{bmatrix} 173.54.25 & 0.2854 \\ -0.8765 & -1.3874 \\ -0.2944 & -173.43 \end{bmatrix}.$$

Therefore, we have matrices Y and K respectively

$$Y = P^{-1}\bar{Y} = \begin{bmatrix} 0 & 0 \\ 0 & 1.538 \\ 0 & -1.071 \end{bmatrix},$$
$$K = P^{-1}\bar{K} = \begin{bmatrix} 3.4536 & 0.0061 \\ -1.248 & -1.543 \\ -0.071 & 3.4556 \end{bmatrix}.$$

Hence, with Y obtained, we have

$$E = U + YV = \begin{bmatrix} -1 & 0 \\ 0 & 1.329 \\ 0 & -1.642 \end{bmatrix}.$$

Thus, application of the design algorithm proposed in section 3 gives the following nonlinear unknown input observer gain matrices as

$$N = \begin{bmatrix} -3.453 & 0 & -0.0058 \\ 0 & -1.563 & 0 \\ -0.0058 & 0 & -3.453 \end{bmatrix}, \ G = 0,$$
$$M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1.358 & 1.569 \\ 0 & 0 & 0 \end{bmatrix}, \ L = \begin{bmatrix} 0 & 0 \\ -1 & 0.8379 \\ 0 & 0 \end{bmatrix}.$$

From the matrices N, M, L, E, the nonlinear unknown input observer given by (3) can be easily constructed as

$$\begin{aligned} &\dot{z} = Nz + Gu + Ly + Mf(\hat{x}) \\ &\hat{x} = z - Ey. \end{aligned}$$

V. CONCLUSIONS

In this paper, we have proposed the sufficient condition on the existence of full-order observer for a class of Lipschitz nonlinear systems with unknown input. The design method of observer has been presented in terms of linear matrix inequalities (LMIs), which can be easily determined using the existing tools in Matlab LMI toolbox [20]. The sufficient condition proposed in section 3, when applied to the linear systems, is also shown to be necessary. Finally, a numerical example has been given to illustrate the results established.

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