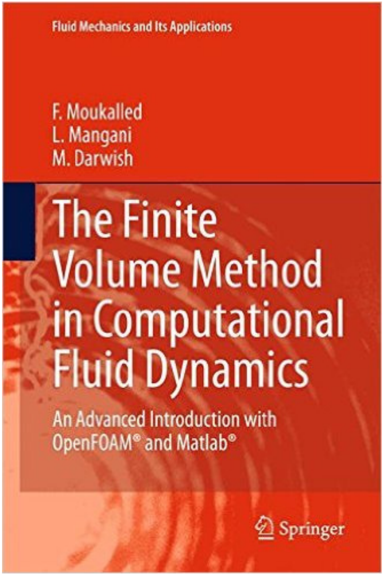
051176 - Computational Techniques for Thermochemical Propulsion
Master of Science in Aeronautical Engineering

Momentum Interpolation Methods for p-U Coupling In Colocated Grids

Prof. **Federico Piscaglia**
Dept. of Aerospace Science and Technology (DAER)
POLITECNICO DI MILANO, Italy
federico.piscaglia@polimi.it

Bibliography



Some figures in this slides are taken from the reference text book:

F. Moukalled, L. Mangani, M. Darwish. "**The Finite Volume Method in Computational Fluid Dynamics**", Springer International Publishing Switzerland 2016.

Prof. Marwan Darwish is greatly acknowledged for sharing the images from his book and for allowing to include them in this course's material.

Bibliography

OTHER REFERENCES:

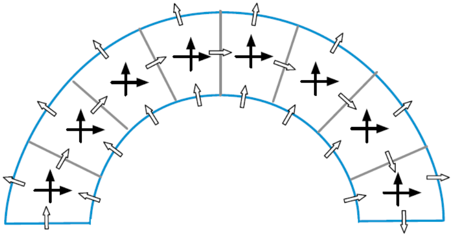
- C. Rhie, W.L. Chow, A numerical study of the turbulent flow past an isolated airfoil with trailing edge separation, AIAA J. 21 (1983) 1525–1532.
- S. Choi, H. Nam, M. Cho, Use of momentum interpolation method for numerical solution of incompressible flows in complex geometries: choosing cell face velocities, Numer. Heat Transfer B 23 (1) (1993).
- B. Yu, Y. Kawaguchi, W.-Q. Tao, H. Ozoe, Checkerboard pressure predictions due to the underrelaxation-factor and time step size for a nonstaggered grid with momentum interpolation method, Numer. Heat Transfer B 41 (1) (2002) 85–94.
- *J. Martínez, F. Piscaglia et al.*. "Influence of momentum interpolation methods on the accuracy and convergence of pressure–velocity coupling algorithms in OpenFOAM". J. of Computational and Applied Mathematics, Vol. 309, 2017.

Colocated grid arrangement

The use of a **cell-centered collocated grid system**, where all variables are stored at the same location (the cell centroid), is a more attractive solution for CFD.

- Unfortunately, complications arise when discretizing the momentum equations in curvilinear coordinates, due to the increased complexity in the treatment of the diffusion term, and because the equations gain non-conservative terms;
- It is worth nothing that while the velocity components are stored at the centroids of the elements as is the case for pressure or any other variable; **but the mass flux, a scalar value, in a collocated grid is stored at the element faces**.

Discretization of the Momentum Equation



- With Cartesian coordinates, the convection terms in the momentum equations are of the form $(\rho uu)_x + (\rho uu)_y$, which is fully conservative. However, when either the contravariant or the covariant velocity components are used as primary dependent variables, the fully conservative form can no longer be guaranteed: **linear momentum is conserved along a straight line and not over a curved line!**
- Furthermore, in the numerical implementation, the contravariant components ρU and ρv on each boundary of the mesh are defined as the mass flux between the two endpoints of the boundary and their values can artificially change with different grid systems.

The mass flux can actually be viewed as a contra-variant component, except that in this case it is computed using a custom interpolation of the discrete momentum equation, known as the Rhie-Chow interpolation, which is the subject of the next section.

Discretization of the Momentum Equation

- Solution of the linearized momentum equations produces $\boldsymbol{u}^*$. For the discretized continuity equation, we need the cell face velocities which have to be calculated by interpolation; linear interpolation is the obvious choice.
- In SIMPLE-based methods, the interpolated cell face velocities needed in the continuity equation involve interpolated pressure gradients, so their correction is proportional to the interpolated pressure correction gradient:

$$u'_e = -\overline{\left(\frac{\Delta V}{A_P^u}\frac{\delta p'}{\delta x}\right)_e}$$

Discretization of the Momentum Equation

The Poisson equation reads:

$$\frac{\partial}{\partial x_i}\left(\frac{\partial p^n}{\partial x_i}\right) = \frac{\delta H_i^n}{\delta x_i}$$

By approximating the outer difference operator $\frac{\partial}{\partial x_i}$ in the pressure equation with the backward difference scheme:

$$\frac{\left(\frac{\delta p^n}{\delta x}\right)_P - \left(\frac{\delta p^n}{\delta x}\right)_W}{\Delta x} + \frac{\left(\frac{\delta p^n}{\delta y}\right)_P - \left(\frac{\delta p^n}{\delta y}\right)_S}{\Delta y} = \frac{H^n_{x,P} - H^n_{x,W}}{\Delta x} + \frac{H^n_{y,P} - H^n_{y,S}}{\Delta x}$$

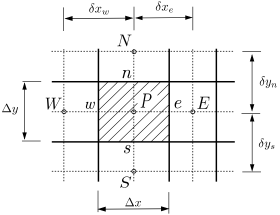
So:

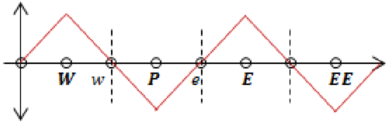
$$\frac{\frac{p_E^n - p_P^n}{\Delta x} - \frac{p_P^n - p_W^n}{\Delta x}}{\Delta x} + \frac{\frac{p_N^n - p_P^n}{\Delta y} - \frac{p_P^n - p_S^n}{\Delta y}}{\Delta x} = Q_P^H$$

Discretization of the Momentum Equation

$$\frac{\frac{p_E^n - p_P^n}{\Delta x} - \frac{p_P^n - p_W^n}{\Delta x}}{\Delta x} + \frac{\frac{p_N^n - p_P^n}{\Delta y} - \frac{p_P^n - p_S^n}{\Delta y}}{\Delta x} = Q_P^H$$

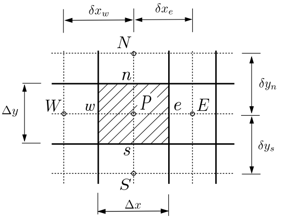
On non-uniform grids, the computational molecule of the pressure-correction equation involves the nodes P, E, W, N, S, EE, WW, NN and SS; this equation may have oscillatory solutions.

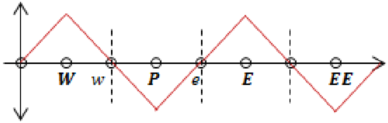




Although the oscillations can be filtered out (see Van der Wijngaart, 1990), the pressure-correction equation becomes complex on arbitrary grids and the convergence of the solution algorithm may be slow.

Discretization of the Momentum Equation





Derivation of equations for laminar steady state incompressible flows in the finite-volume approach with collocated grid arrangement:

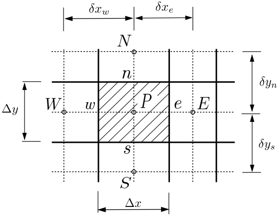
$$\int_V \nabla \cdot (\boldsymbol{u}\boldsymbol{u}) \, dV - \int_V \nabla \left(\nu \nabla \cdot \boldsymbol{u}\right) \, dV = -\int_V \nabla \left(\frac{p}{\rho}\right) \, dV$$

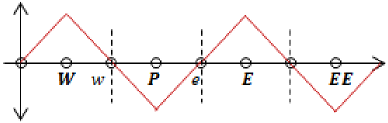
Assuming an uniform mesh, with $P = \left(\frac{p}{\rho}\right)$, it follows:

$$-\int_w^e \left(\frac{\partial P}{\partial x}\right) dx = -\frac{(P - P_W) + (P_E - P)}{\delta x} = -\frac{P_E - P_W}{\delta x}$$

The discretized pressure gradient is independent of P_P!

Rhie-Chow scheme

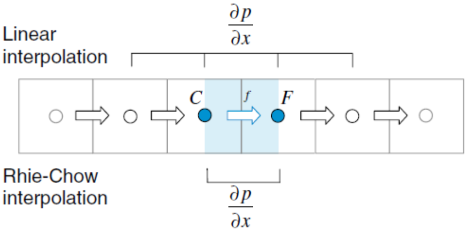




These aspects can cause difficulties in the way of preserving a high degree of numerical accuracy in satisfying the conservation laws.

- One of the main problems arising from the use of co-located grid arrangement is the checker-board pressure field. Due to the nature of the Navier-Stokes equations, pressure appears in the momentum equations inside a gradient term.
- Application of central-difference spatial discretization to this term in a colocated grid produces a decoupling of pressure and velocity cell values, leading to **saw-tooth pressure oscillations**.

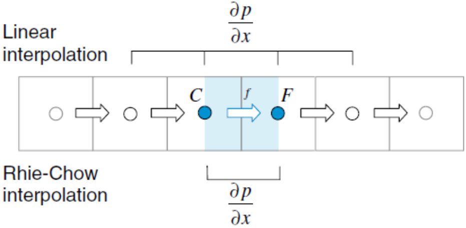
Discretization of the Momentum Equation



source: "The Finite Volume Method in Computational Fluid Dynamics", Springer.

A compact pressure- correction equation similar to the staggered grid equation can be obtained by replacing the interpolated pressure gradients by compact central-difference approximations at the cell faces. The interpolated cell face velocity is thus modified by the difference between the interpolated pressure gradient and the gradient calculated at the cell face.

Rhie-Chow scheme



In 1983, **Rhie and Chow proposed a technique for momentum-based interpolation of mass fluxes on cell faces, imitating the staggered-grid discretization**.

This this Momentum Interpolation Method (MIM):

- is based on formulating a discretized momentum equation for the face, so that the computation of the driving pressure force involves the pressure value at the nodes adjacent to the face in question and, therefore at the node itself;
- removes the checkerboarding problem for the most part, which is the reason of its wide acceptance and intensive use in unstructured grid solvers;
- undergoes extensive development for complex geometries through the years.

Theory of Momentum Interpolation Method

Let us now consider the equation for x-component of the velocity field, u. Before under-relaxation u_P would follow an equation in the form:

$$A_P\phi_P = A_E\phi_E + A_W\phi_W + A_N\phi_N + A_S\phi_S + b_P$$

If the pressure term is separated from the source we can write:

$$u_P = \frac{\sum_{nb} A_{nb}u_{nb} + b_P}{A_P} - \frac{\Delta y(p_e - p_w)}{A_P}$$

which can also be expressed in general form:

$$u_P = \frac{H_P}{A_p} - D_P(\nabla p)_P$$

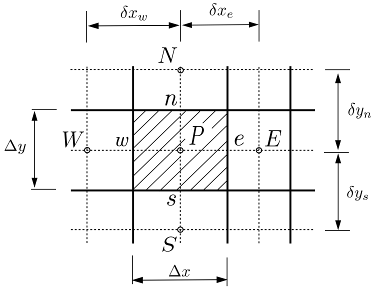
where

$$H_P = \frac{\sum_{nb} A_{nb}u_{nb} + b_P}{A_P}$$

$$D_P = \frac{\Delta y}{A_P}$$

and $(\nabla p)_P$ is the x component of the pressure gradient in cell P.

Theory of Momentum Interpolation Method



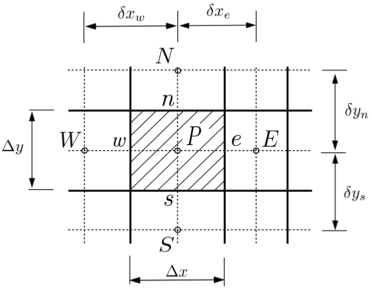
Therefore, for the two contiguous cells E and P we could write the following equations:

$$u_P = \frac{H_P}{A_P} - \Delta y \frac{(p_e - p_w)_P}{A_P} \tag{1}$$

$$u_E = \frac{H_E}{A_E} - \Delta y \frac{(p_e - p_w)_E}{A_E} \tag{2}$$

Rhie-Chow Momentum Interpolation

Theory of Momentum Interpolation Method



Mimicking the formulation followed for u_E and u_P, Rhie-Chow proposed a pseudo-equation for the face velocity u_e:

$$u_e = \frac{H_e}{A_e} - \Delta y \, \frac{(p_E - p_P)}{A_e} \tag{3}$$

which can also be expressed as:

$$u_e = \frac{H_e}{A_e} - D_e (\nabla p)_e \tag{4}$$

where H_e is the first term in the RHS of Eq. (4), $D_e = \frac{\Delta y}{(A_P)_e}$ and $(\nabla p)_e$ is the x component of pressure gradient in face e.

Theory of Momentum Interpolation Method

In the MIM of Rhie and Chow, unknown terms of RHS of Equation:

$$u_e = \frac{H_e}{A_e} - \frac{\Delta y}{A_e}(p_E - p_P) \tag{5}$$

are obtained by linear interpolation as follows:

$$H_e = f_e^+ \left(\frac{H_E}{A_E}\right) + (1 - f_e^+) \left(\frac{H_P}{A_P}\right)$$

which can be written as:

$$\frac{1}{A_e} = f_e^+ \frac{1}{A_E} + (1 - f_e^+) \frac{1}{A_P}$$

where f_e^+ is a **weighting factor depending on the discretization scheme**. For the case of uniform grid, $f_e^+ = \frac{\Delta x_P}{2\delta x_e}$.

Theory of Momentum Interpolation Method

Therefore, Eq. (4) becomes:

$$u_e = \overline{\left(\frac{H_e}{A_e}\right)} - \overline{\left(\frac{1}{A_e}\right)} \Delta y (p_E - p_P) \tag{6}$$

where the over-bar denotes linear interpolation. Combining Eq. (6) with Eqs. (1) and (2), and after re-ordering terms, it follows:

$$\begin{aligned} u_e = \; & \left[f_e^+ u_E + (1 - f_e^+)\, u_P \right] - \frac{\Delta y \, (p_E - p_P)}{A_e} \\ & + \left[f_e^+ \frac{\Delta y \, (p_e - p_w)_E}{A_E} + (1 - f_e^+) \frac{\Delta y \, (p_e - p_w)_P}{A_P} \right] \end{aligned} \tag{7}$$

Theory of Momentum Interpolation Method

$$u_e = \underbrace{\left[f_e^+ u_E + (1 - f_e^+) u_P\right] - \frac{\Delta y \, (p_E - p_P)}{A_e}}_{\text{linear interpolation of cell values}}$$

$$+ \underbrace{\left[f_e^+ \frac{\Delta y \, (p_e - p_w)_E}{(A_P)_E} + (1 - f_e^+) \frac{\Delta y \, (p_e - p_w)_P}{A_P}\right]}_{\text{correction terms}}$$

In the equation, the first term in the RHS corresponds to the linear interpolation of cell values, while the last two terms can be regarded as a correction term that smooths the pressure field, and removes the undesired checkerboard behavior.

Rhie-Chow Momentum Interpolation

$$u_e = \underbrace{\left[f_e^+ u_E + (1 - f_e^+) u_P\right] - \frac{\Delta y (p_E - p_P)}{(A_P)_e}}_{\text{linear interpolation of cell values}}$$
$$+ \underbrace{\left[f_e^+ \frac{\Delta y (p_e - p_w)_E}{(A_P)_E} + (1 - f_e^+) \frac{\Delta y (p_e - p_w)_P}{(A_P)_P}\right]}_{\text{correction terms}}$$

Assuming

$$D_e \approx \overline{D_e}$$

and

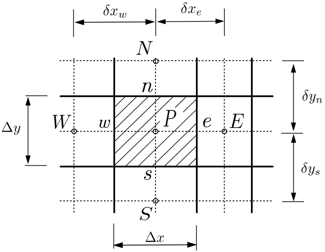
$$\overline{(D \nabla p)_e} \approx \overline{D_e}\, \overline{\nabla p_e}$$

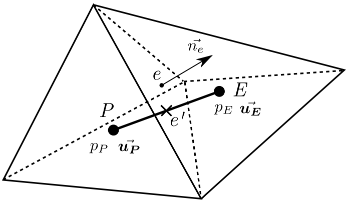
and considering Eqs. (13) and (4), the equation above can be re-written in the form of the so-called classical Rhie-Chow interpolation:

$$u_e = \overline{u_e} + \overline{D_e} \left(\overline{\nabla p_e} - (\nabla p)_e\right)$$

Rhie-Chow:
Under-Relaxation Factor Dependency

Under-relaxation in OpenFOAM





In segregated solvers, p-U coupling is often achieved by employing **under-relaxation of the primary variables**, which is applied both on velocity and pressure.

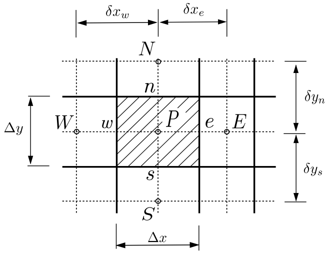
In the following, we will concentrate only on under-relaxation of **velocity**. For cell P:

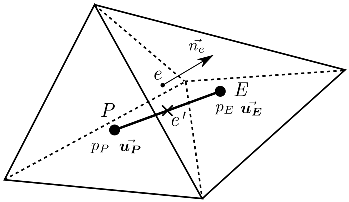
$$u_P = \underbrace{\alpha_u u_P + (1-\alpha_u)u_P^0}_{h_P} - \underbrace{\alpha_u \frac{\Delta y}{A_P}(p_e - p_w)_P}_{\texttt{momentumPredictor}}$$

$$= \alpha_u \left[u_P - \Delta y \frac{(p_e - p_w)_P}{A_P} \right] + (1-\alpha_u)u_P^0$$

$$= h_P - \alpha_u \frac{\Delta y}{A_P}(p_e - p_w)_P$$

Theory of Momentum Interpolation Method





If the Rhie-Chow momentum interpolation is applied, from Eq. (6) for cell E it follows:

$$
\begin{aligned}
u_e &= \overline{h_e} - \alpha_u \overline{\left(\frac{1}{A_e}\right)} \Delta y \; (p_E - p_P) \\
&= \left(f_e^+ h_E + (1 - f_e^+) h_P\right) - \alpha_u \overline{\left(\frac{1}{A_e}\right)} \Delta y (p_E - p_P) \qquad (8)
\end{aligned}
$$

Using definitions of h_E and h_P in terms of H_P and H_E, u_e can be written as:

$$
u_e = \alpha_u \underbrace{\left[\overline{\left(\frac{H_e}{A_e}\right)} - \overline{\left(\frac{1}{A_e}\right)} \Delta y (p_E - p_P)\right]}_{\text{Momentum interpolation}} + (1 - \alpha_u) \underbrace{\left[f_e^+ u_E^0 + (1 - f_e^+) u_P^0\right]}_{\text{Linear Interpolation}} \qquad (9)
$$

Theory of Momentum Interpolation Method

The above discussion makes it clear that the direct application of Rhie-Chow technique to the under-relaxed momentum equations would yield a solution that will not converge to the desired momentum interpolation (first term), but will contain a portion of linear interpolation.

$$u_e = \alpha_u \underbrace{\left[\overline{\left(\frac{H_e}{A_e} \right)} - \overline{\left(\frac{1}{A_e} \right)} \Delta y (p_E - p_P) \right]}_{\text{Momentum interpolation}} + (1 - \alpha_u) \underbrace{\left[f_e^+ u_E^0 + (1 - f_e^+) u_P^0 \right]}_{\text{Linear Interpolation}}$$

Therefore, **when a small enough under-relaxation factor is used, the linear term will give the main contribution, and pressure oscillations may re-appear**, since no momentum interpolation is used. Hence, the final solution will depend on the value of the under-relaxation factor.

→ Nearly all of the recent SIMPLE-based (e.g. PIMPLE) unstructured grid methods employ the Rhie and Chow interpolation method to evaluate the cell face velocities.

Momentum Interpolation in OpenFOAM

In the OpenFOAM solvers, the Rhie and Chow momentum interpolation method may be found before the solution of the Poisson equation, usually in the file `pEqn.H`:

```
while (runTime.run())
{
    #include "readTimeControls.H"
    #include "CourantNo.H"
    #include "setDeltaT.H"

    runTime++;

    Info<< "Time = " << runTime.timeName() << nl << endl;

    // --- Pressure-velocity PIMPLE corrector loop
    while (pimple.loop())
    {
        #include "UEqn.H"

        // --- Pressure corrector loop
        while (pimple.correct())
        {
            #include "pEqn.H"
        }

        if (pimple.turbCorr())
        {
            laminarTransport.correct();
            turbulence->correct();
        }
    }
    ...
}
```

Rhie-Chow Momentum Interpolation

UEqn.H:

```
1 tmp<fvVectorMatrix> UEqn
2 (
3     ...
4 );
5
6 UEqn().relax();
7
8 solve(UEqn() == -fvc::grad(p));
```

- **Line 6**: `UEqn().relax()` → relax the velocity equation
- **Line 8**: `solve(UEqn() == -fvc::grad(p))` → known-pressure gradient treated explicitly in the momentum equation

Rhie-Chow Momentum Interpolation

pEqn.H:

```
1  volScalarField rAU(1.0/UEqn().A());
2  HbyA = rAU*UEqn().H();
3
4  surfaceScalarField phiHbyA("phiHbyA", fvc::interpolate(HbyA) &  mesh.Sf());
5
6  fvScalarMatrix pEqn
7  (
8      fvm::laplacian(rAU, p) == fvc::div(phiHbyA)
9  );
10
11 pEqn.solve();
12
13 if (simple.finalNonOrthogonalIter())
14 {
15     phi = phiHbyA - pEqn.flux();
16 }
17
18 p.relax();
19 U = HbyA - rAU*fvc::grad(p);
20 U.correctBoundaryConditions();
```

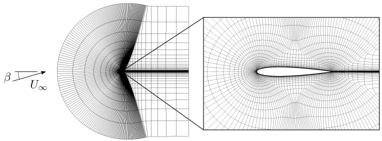
- **Line 2** $\rightarrow$ `UEqn().H()` = $\sum_i A_i \boldsymbol{u_i} + \boldsymbol{b}_p$
- **Line 4** $\rightarrow$ Rhie-Chow interpolation
- **Line 19** $\rightarrow$ velocity updated with the new pressure pressure field

Examples

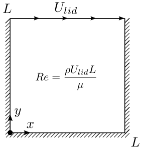
Rhie-Chow Momentum Interpolation Method

The Rhie-Chow momentum interpolation method presents some (well known) inherent limits:

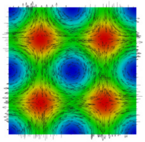
1. UNDER-RELAXATION FACTOR DEPENDENCY in the p-v coupling algorithm
2. TIME-STEP DEPENDENCY of the solution



a) NACA-012 airfoil



b) lid cavity flow

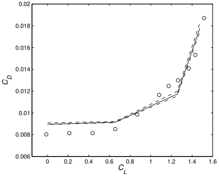


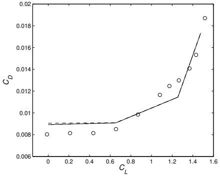
c) Taylor-Green vortex

The error is apparent in

1. **Flow field evolving <u>towards a STEADY solution</u>**:
   a) NACA-012 airfoil (`simpleFoam`);
   b) lid-cavity flow (`pimpleFoam` $\rightarrow$ steady solution)
2. **<u>UNSTEADY flow field</u>**:
   c) Taylor-Green vortex (`pimpleFoam`, transient mode)

Under-Relaxation Dependency





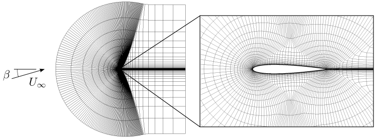
Coefficient		**Rhie-Chow (OMIM)**	**Majumdar**
C_L	$\alpha_u = 0.2$	1.2551915	1.2557273
	$\alpha_u = 0.8$	1.2552688	1.2554411
	Rel. Change	0.006 %	0.022%
C_D	$\alpha_u = 0.2$	0.011967419	0.011588629
	$\alpha_u = 0.8$	0.011548746	0.011450845
	Rel. Change	3.498 %	1.189%

Correction for under-relaxation factor dependency by *Majumdar* [1]:

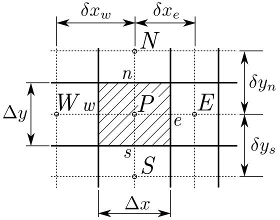
$$u_e = \alpha_u \left(\frac{\sum_i A_i u_i + B_P}{A_P} \right)_e - \Delta y \frac{\alpha_u (p_E - p_P)}{(A_P)_e} + \underbrace{(1-\alpha_u)\left[u_e^0 - (u_P^0)_e\right]}_{\text{correction}}$$

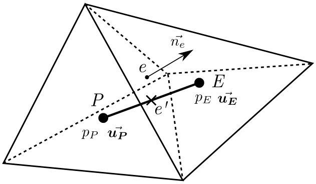
- u_e^0 is the face velocity of previous iteration
- $(u_P^0)_e$ is the linear interpolation of cell velocities in the face at `t-1`

Solver used: `simpleFoam`



Time-step size dependency





Velocity equation is formulated as follows (for a generic second order scheme)

$$A_P^t(c_0 u_P - c_1 u_P^{k-1} - c_2 u_P^{k-2}) + A'_P u_P = \sum_i A_i u_i + b_P + f_{x\,P}$$

When under-relaxation is applied:

$$u_P = \underbrace{\frac{\alpha_u}{1 + c_0 \beta_P^t}}_{(a)} \cdot \underbrace{\left(\frac{\sum_i A_i u_i + b_P}{A'_P}\right)}_{(b)} + \underbrace{\frac{\alpha_u}{1 + c_0 \beta_P^t}}_{(c)} + \underbrace{\left(\frac{f_{x\,P}}{A'_P}\right)}_{(d)} + \underbrace{\frac{\alpha_u c_1 \beta_P^t}{1 + c_0 \beta_P^t} u_P^{k-1} + \frac{\alpha_u c_2 \beta_P^t}{1 + c_0 \beta_P^t} u_P^{k-2} + (1 - \alpha_u) u_P^0}_{\text{out of the source term, interpolation by fluxes at } t-1}$$

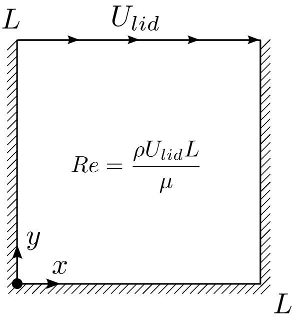
Mimicking the formulation face velocity is obtained:

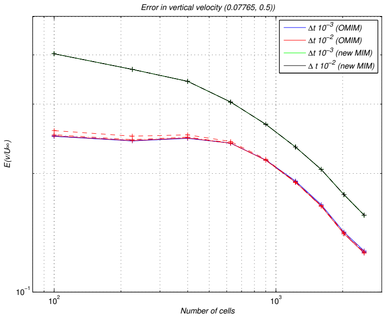
$$u_e = \frac{\alpha_u}{1 + c_0 \beta_e^t}\left(\frac{\sum_i A_i u_i + b_P}{A'_P}\right)_e - \frac{\alpha_u}{1 + c_0 \beta_e^t} \Delta y \frac{p_E - p_P}{(A'_P)_e} + \frac{\alpha_u c_1 \beta_e^t}{1 + c_0 \beta_e^t} u_e^{k-1} + \frac{\alpha_u c_2 \beta_e^t}{1 + c_0 \beta_e^t} u_e^{k-2} + (1 - \alpha_u) u_e^0$$

where "$_e$" coefficients are linearly interpolated in the faces, and u_e^0, u_e^{k-1}, u_e^{k-2} are face velocities at the previous iteration.

Time-step size dependency

Transient solution converging to steady state



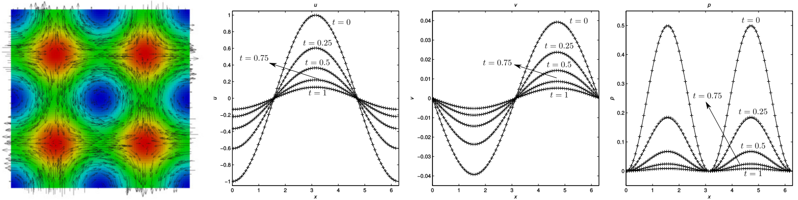


TEST CASE: lid cavity flow

- uniform grid (different resolutions)
- Re=5000
- solver: `pimple` (consistent)

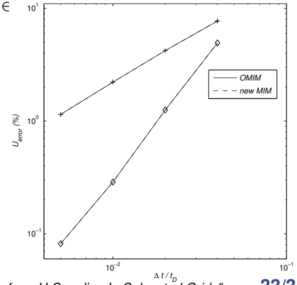
Time-step size dependency

Taylor-Green vortex: solution is initialized and vortices dissipate with time; $\alpha_u \in [0.2\ 0.8]$



Numerical solution at y$\simeq \frac{\pi}{2}$ vs analytical values at different time steps:

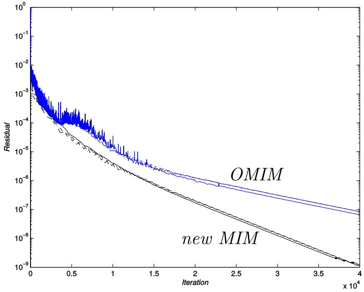
a) x-velocity b) y-velocity c) pre

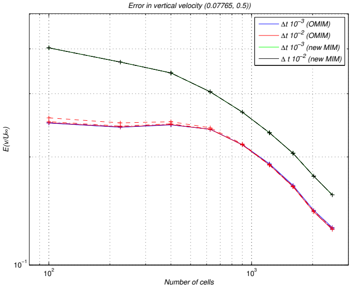


TEST CASE: Taylor-Green vortex

- existing analytical solution
- dimensionless study: L=2π; $u_{max} = v_{max} = 1$; ν=1
- solver: `pimple` (consistent), no turbulence
- under-relaxation on $\alpha_u \in [0.2\ 0.8]$

Solver performance





ADVANTAGES:

+ solution independence by the time-step size and under-relaxation factor

DRAWBACKS:

- slightly lower accuracy in the cases tested

Thank you for your attention!

contact: federico.piscaglia@polimi.it