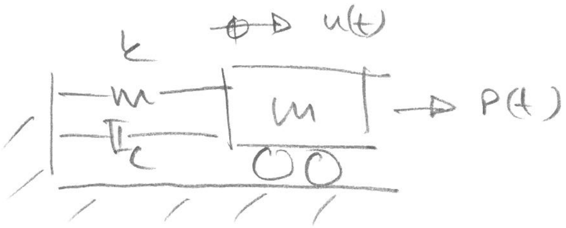
SINGLE-DEGREE OF FREEDOM SYSTEM

$$m\ddot{u} + c\dot{u} + ku = P \qquad (1)$$



(1) linear system $\Rightarrow$ general solution

$$u(t) = u_c(t) + u_p(t)$$

complementary — particular

$$u(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} + u_p(t)$$

<u>Case</u> : free vibrations $\Leftrightarrow$ $P(t) = 0$; $u_p(t) = 0$

$\Rightarrow$ Characteristic equation

$$s^2 + \frac{c}{m} s + \omega^2 = 0$$

$$s_{1,2} = \omega \left\{ -\frac{c}{2m\omega} \pm \sqrt{\left(\frac{c}{2m\omega}\right)^2 - 1} \right\}$$

ω - angular frequency (rad/s), $\omega > 0$

$\omega^2 = \frac{k}{m}$

critical damping, $c_{cr} = 2m\omega = 2\sqrt{km}$

relative damping $\xi = c/c_{cr}$

$\xi < 1$ - underdamped system

vibrations from a structural mechanics point of view $\Longleftrightarrow$ $\xi < 10\%$

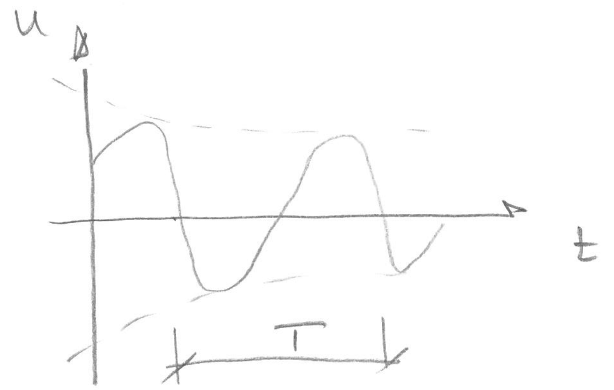
$$s_{1,2} = -\xi\omega \pm i\omega\sqrt{1-\xi^2}$$

damped angular frequency, $\omega_d = \omega\sqrt{1-\xi^2}$

$\Rightarrow$ general solution of free vibrations

$$u(t) = u_c(t) = e^{-\xi\omega t}\left(C_1 e^{i\omega_d t} + C_2 e^{-i\omega_d t}\right)$$

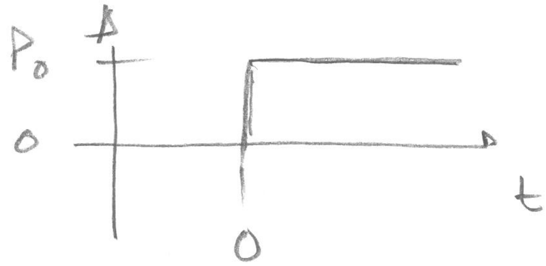
note: * C_1 and C_2 complex conjugate as $u(t)$ is a real quantity
* oscillary solution
* decaying exponential part



$$T = \frac{2\pi}{\omega_d} \Rightarrow f = \frac{1}{T} = \frac{\omega_d}{2\pi}$$

$$\xi \ll 1 \Rightarrow \omega_d \simeq \omega \Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Case: step load, $P(t) = P_0$, $\xi < 1$



the general solution

$$u(t) = u_c(t) + u_p(t)$$

$$u_c(t) = e^{-\xi\omega t}\left(C_1 e^{i\omega_d t} + C_2 e^{-i\omega_d t}\right) =$$

$$= // \; e^{ibt} = \cos bt + i\sin bt \; ; \; \sin(-\alpha) = -\sin(\alpha) \; ; \; \cos(-\alpha) = \cos\alpha \; //$$

$$= e^{-\xi\omega t}\left[(C_1 - C_2)\, i \sin\omega_d t + (C_1 + C_2)\cos\omega_d t\right] =$$

// C_1 and C_2 are complex conjugates // =

$$= e^{-\xi\omega t}\left(A_1 \sin\omega_d t + A_2 \cos\omega_d t\right)$$

The particular solution is a constant $\frac{P_0}{k}$.

assume initial conditions $u_0 = 0$ and $\dot{u}_0 = 0$ and solve for A_1 and A_2

$$\Rightarrow u(t) = \frac{P_0}{k}\left\{1 + e^{-\xi\omega t}\left[-\xi\frac{\omega}{\omega_d}\sin\omega_d t - \cos\omega_d t\right]\right\}$$

note: for $t \to \infty$ → $u(t \to \infty) = \frac{P_0}{k}$
the static solution!

for small damping values; $\xi \ll 1$

→ u(t) can be approximated as:

$$u(t) \simeq \frac{P_0}{k}\left(1 - e^{-\xi \omega t} \cos \omega_d t\right)$$

and if no damping is present then the exact solution is

$$u(t) = \frac{P_0}{k}\left(1 - \cos \omega t\right)$$

note: the stepload will generate a response twice the static response of P_0/k.

This is an important result and indicates that safety factors in structural mechanics should depend on how quickly the load is applied!

Case: harmonic excitation, $\xi < 1$

stationary (steady-state) solution where initial conditions are not considered

we have:

$$m\ddot{u} + c\dot{u} + ku = P(t)$$

$$P(t) = P_0 \sin(\omega_p t + \alpha)$$

ω_p — load angular frequency

α — arbitrary phase angle

define: $P^* = P_0 \cos\alpha + i P_0 \sin\alpha$

$$|P^*| = P_0$$

$$u = u^* e^{i\omega_p t} \quad ; \quad P = P^* e^{i\omega_p t}$$

$$\Rightarrow u^* = \frac{P^*}{k + i\omega_p c - \omega_p^2 m} =$$

$$= // \ \xi = \frac{c}{2m\omega} \ ; \ \omega^2 = \frac{k}{m} \ ; \ \beta = \frac{\omega}{\omega_p} \ // =$$

$$= \frac{P^*}{k(1 - \beta^2 + 2i\xi\beta)} = H(\beta)\, P^*$$

H – a frequency response function, FRF, and is complex.

multiply numerator and denominator with $(1-\beta^2-2i\xi\beta) \rightarrow$

$$\rightarrow H(\beta) = \frac{1}{K\sqrt{(1-\beta^2)+(2\xi\beta)^2}}$$

$$|u^*| = |H(\beta)|\,|P^*|$$

Define the dynamic amplification factor

$$D = \frac{1}{\sqrt{(1-\beta^2)^2+(2\xi\beta)^2}}$$

D indicates the response amplification with respect to the static response P_0/K $(\omega_p = 0)$

note: $\beta \simeq 1 \rightarrow D = \frac{1}{2\xi}$

$\xi = 0{,}04 \rightarrow D = 12.5$!!

It can be shown that exact max of D

$$D_{max} = \frac{1}{2\xi\sqrt{1-\xi^2}} \quad \text{for} \quad \beta = \sqrt{1-2\xi^2} \quad \text{and}$$

$$\xi \leq \frac{1}{\sqrt{2}}$$

Usually $\xi^2 << 1$