Vanilla Options Pricing Under GBM

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Abstract

This project develops a coherent and mathematically rigorous framework for pricing European and American vanilla options using the Black-Scholes model, dependent upon an underlying asset which follows a geometric Brownian motion in the risk-neutral measure. Starting from just an arbitrage-free, self-financing trading strategy, we derive the Black-Scholes partial differential equation for European options and formulate the American option price as a variational inequality tied to an optimal stopping problem. For European claims, we find the closed-form prices under the Black-Scholes-Merton model, analytic Greek sensitivities, and implied volatility from Newton's method with a bisection fallback, whilst also handling asymptotic behaviour. For American claims, we discretise the variational inequality to create a linear complementarity problem and analyse the Crank-Nicolson finite difference scheme with Rannacher time-stepping to ensure unconditional stability and thorough handling of oscillations at payoff kinks. The scheme is solved by the well-posed projected successive over-relaxation method. The accompanying Python code is an illustration of how the theoretical results in this paper can be translated into working numerical code.

Project Overview

After taking particular interest in my Mathematical Finance I and Numerical and Computational Methods with Python modules in my BSc Mathematics degree at King's College London, I sought to carry out a self-directed project which blended the topics contained in each of these modules and applied them within the realm of quantitative finance, specifically the options market. This paper forms the theoretical part of my project, which provides the mathematical framework necessary in order to implement robust and arbitrage-free derivatives pricing engines in Python.

My aim for this project was to introduce a level of mathematical analysis and rigour often not seen within an undergraduate degree, with a heavy focus on probability theory, stochastic calculus, and numerical schemes which can be applied to derivatives pricing.

The focus of this project is pricing vanilla contingent claims written on an underlying asset whose price follows a geometric Brownian motion under the risk-neutral measure. From an arbitrage-free self-financing trading strategy, we derive the Black-Scholes partial differential equation and pose ideas for the price of European options as a linear parabolic partial differential equation and the price of American options as a variational inequality tied to an optimal stopping problem.

Beyond the theory in this paper, an important goal of mine was to demonstrate how the aforementioned mathematical models can be translated into working code in Python. The accompanying implementation of these models includes the following:

- Closed-form European pricing engine with Greeks via the Black-Scholes-Merton model
- Put-call parity and lower and upper bounds for European options to assert no-arbitrage conditions
- Monte Carlo pricer with antithetic variates, finite difference Greeks, and confidence intervals as model validation for the Black-Scholes-Merton model
- Implied volatility solver based on Newton's method with a bisection fallback, with illustrations of volatility smiles and skews
- Crank-Nicholson with Rannacher startup finite difference scheme solved via projected successive over-relaxation to price American options

For the interested reader, the working code in Python for the above can be found in my GitHub repository here: `options-pricing`.

As of present day, I am working on the theory and implementation for pricing various exotic options under the same geometric Brownian motion dynamics, including discretely-monitored barriers corrected with a Brownian bridge and arithmetic Asians with geometric Asians as control variates. I am also looking at calibration for vanilla options under Heston's stochastic volatility model.

Notation

Below is a list of the main symbols and notation used throughout the project. It is not an exhaustive list of all notation used; however, it should cover the basics needed to ensure a coherent understanding of each section. Any notation or symbols which have not been described here will be introduced formally within the paper.

Symbol	Meaning
S_t	Underlying asset price at time t
K	Strike price of the option
T	Contract length of the option
τ	Time until maturity of the option
r	Constant risk-free interest rate
q	Continuous dividend yield of S_t
σ	Annualised volatility
W_t	Standard Brownian motion under a chosen measure
$\mathbb{P}$	Physical (real-world) probability measure
$\mathbb{Q}$	Risk-neutral (martingale) probability measure
$\mathcal{F}_t$	Filtration at time t
V_t	Arbitrage-free option price at time t

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1. Introduction

In this chapter, we will lay out the foundations needed to price vanilla and exotic options under a geometric Brownian motion. We will also explore any key ideas that will be reused or assumed throughout.

1.1. Market model & trading rules

We start by working in a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions and supporting a one-dimensional Brownian motion $W^{\mathbb{P}}$ [1]. We assume that time is continuous on the closed interval $[0, T]$.

We trade a money market account M with a progressively measurable and locally integrable short rate r_t [2]. Set $M_0 = 1$ and define the dynamics of our money market as

$$dM_t = r_t M_t dt \tag{1.1.1}$$

Hence, $M_t = e^{\int_0^t r_u du}$. We also trade a risky asset S with continuous dividend yield $q \geq 0$. The single-share cumulative gains process G_t is defined by

$$G_t := S_t + \int_0^t qS_u du \tag{1.1.2}$$

Let ζ_t be S-integrable and ξ_t be M-integrable, both predictable processes. Then, we can write any trading strategy as $\vartheta = (\zeta_t, \xi_t)_{t\in[0,T]}$ which holds ζ_t shares of S and ξ_t units of M to generate the wealth process $X_t = \zeta_t S_t + \xi_t M_t$. Our constraint is that our wealth process, X_t, is self-financing, so with dividends instantly swept into the money market, the dynamics of X_t become

$$dX_t = \zeta_t dG_t + \xi_t dM_t \tag{1.1.3}$$

We require our strategy ϑ to be admissible so that we can eliminate "double-or-nothing" strategies that almost surely guarantee gains with finite capital. We write $\tilde{X}_t := \frac{X_t}{M_t} \geq -A$ almost surely on $[0, T]$ for some finite $A < \infty$ [3]. Given the nature of this paper and the models we are looking to employ, it is necessary to adhere to the following market assumptions.

- Frictionless trading
- Continuous rebalancing
- No transaction costs or taxes
- Perfect divisibility of assets
- Constant risk free rate $r_t \equiv r$ and dividend yield q

A critical initial assumption which will determine all subsequent methods of pricing our contingent claim is that our ex-dividend price S_t follows a geometric Brownian motion in the physical (real-world) measure $\mathbb{P}$ [1]

$$dS_t = (\mu - q)S_t dt + \sigma S_t dW_t^{\mathbb{P}} \tag{1.1.4}$$

Suppose we set the money market M as the default numéraire. Then, for any process Y, we write $\tilde{Y_t} := \frac{Y_t}{M_t}$ [2]. In the next section, we will show that the discounted gains of any traded asset is a local martingale in an appropriate equivalent measure.

1.2. The $\mathbb{Q}$-martingale measure

For the market defined in Section 1.1, no-arbitrage forbids a self-financing strategy ϑ with wealth X if

$$X_0 = 0, \quad \mathbb{P}(X_T \geq 0) = 1, \quad \mathbb{P}(X_T > 0) > 0 \tag{1.2.1}$$

We enforce *no free lunch without vanishing risk (NFLVR)* as a strengthening for our no-arbitrage condition [3].

Let the continuous proportional dividend yield pay $D_t = \int_0^t qS_u du$ cash over $[0, t]$. Let B be our strictly positive numéraire, with $B_0 = 1$. An equivalent local martingale measure $\mathbb{Q}^B \sim \mathbb{P}$ is a probability measure such that the discounted gains of any traded asset

$$\tilde{G_t} := \frac{S_t}{B_t} + \int_0^t \frac{1}{B_u} dD_u \tag{1.2.2}$$

is a $\mathbb{Q}^B$-local martingale.

Theorem 1.2.1. *(Fundamental Theorem of Asset Pricing) A market satisfies NFLVR if and only if there exists at least one equivalent local martingale measure $\mathbb{Q} \sim \mathbb{P}$ for some numéraire. Furthermore, the market is complete if and only if this equivalent local martingale measure, $\mathbb{Q}$, is unique.*

Remark 1.2.2. If the Fundamental Theorem of Asset Pricing [4] is satisfied, then every contingent claim $V \in L^1(\mathbb{Q}^B)$ admits a unique replicating strategy.

Let $B = M$ be our numéraire. For simplicity of notation, we will now write $\mathbb{Q}^M := \mathbb{Q}$. For any contingent claim $V \in L^1(\mathbb{Q})$ with maturity T

$$\frac{\Pi_t(V)}{M_t} = \mathbb{E}^{\mathbb{Q}} \left[\frac{V}{M_T} \middle| \mathcal{F}_t \right], \quad t \in [0, T] \tag{1.2.3}$$

For a traded asset S with continuous dividends D, the ex-dividend price satisfies

$$\frac{S_t}{M_t} = \mathbb{E}^{\mathbb{Q}} \left[\frac{S_T}{M_T} + \int_t^T \frac{1}{M_u} dD_u \middle| \mathcal{F}_t \right] \tag{1.2.4}$$

These two pricing rules are universal throughout the project. From Chapter 2 onwards, we will specialise V and price accordingly under these identities.

Under GBM, we denote the market price of risk process by λ_t. The Doléans-Dade exponential

$$Z_t = e^{-\int_0^t \lambda_u dW_u^{\mathbb{P}} - \frac{1}{2}\int_0^t \lambda_u^2 du} \tag{1.2.5}$$

satisfies Novikov's condition

$$\mathbb{E}^{\mathbb{P}}\left[e^{\frac{1}{2}\int_0^T |\lambda_t|^2 dt}\right] < \infty \tag{1.2.6}$$

Hence, Z is a true $\mathbb{P}$-martingale with $\mathbb{E}^{\mathbb{P}}[Z_t] = 1$ [5].

N.B. In our case, we assume a constant market risk $\lambda_t := \frac{\mu - r}{\sigma}$.

In order to switch to the risk-neutral measure, we define $\mathbb{Q}$ on the filtration $\mathcal{F}_t$ by the Radon-Nikodym derivative

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = Z_t \tag{1.2.7}$$

By Girsanov's theorem, $W^{\mathbb{Q}} := W^{\mathbb{P}} + \int_0^t \lambda_u du$ is a $\mathbb{Q}$-Brownian motion. With numéraire M, every discounted gains process is thus a $\mathbb{Q}$-local martingale. Since we handle one Brownian motion driver for our risky asset with nondegenerate volatility, the market is complete. By the fundamental theorem of asset pricing (1.2.1), $\mathbb{Q}$ is then also unique [1].

1.3. Feynman-Kac formula

We can now model the dynamics of our ex-dividend price as a GBM in $\mathbb{Q}$ with numéraire M [1]

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t^{\mathbb{Q}} \tag{1.3.1}$$

Define the Black-Scholes spatial generator $\mathcal{L}$ [6] by

$$(\mathcal{L}g)(s) = (r - q)s\partial_s g(s) + \frac{1}{2}\sigma^2 s^2 \partial_{ss} g(s) \tag{1.3.2}$$

Theorem 1.3.1. *(Feynman-Kac Formula) Let Ψ be a function with at most polynomial growth and f be a continuous function. Let $u : [0, T] \times (0, \infty) \to \mathbb{R}$ with $u \in C^{1,2}$ be the function that satisfies*

$$\partial_t u + \mathcal{L}u - ru + f = 0, \quad u(T, s) = \Psi(s) \tag{1.3.3}$$

with well-behaved boundaries $\lim_{s\to 0^+} u(t, s)$ *and* $\lim_{s\to\infty} u(t, s)$*. Then*

$$u(t, s) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r du}\Psi(S_T) + \int_t^T e^{-\int_t^u r_v dv} f(u, S_u) du \middle| S_t = s\right] \tag{1.3.4}$$

Remark 1.3.2. For a terminal-only contingent claim V with discounted payoff $H(t, S_T) = e^{-r(T-t)}\Psi(S_T)$, no running cash flows ($f \equiv 0$), and constant r and q, our pricing identity under the GBM defined in (1.3.1) simplifies to

$$u(t, s) = \mathbb{E}^{\mathbb{Q}}\left[H(t, S_T) | S_t = s\right] \tag{1.3.5}$$

1.4. Monte Carlo simulation

There does not always exist a simple, closed-form solution for a contingent claim V with payoff Ψ. We will make use of Monte Carlo simulations in these instances to price V. Let $H := e^{-rT}\Psi$ denote the discounted payoff of the claim. We simulate N i.i.d. copies $H^{(1)}, \dots, H^{(N)}$ with finite mean and variance. That is, $\mathbb{E}[H^{(i)}] = \mu < \infty$ and $Var(H^{(i)}) = \sigma^2 < \infty$ respectively. Then

$$\hat{V}_N = \frac{1}{N}\sum_{k=1}^{N} H^{(k)} \xrightarrow{\text{a.s.}} \mu \tag{1.4.1}$$

[7] By the strong law of large numbers, $\hat{V}_N$ is a consistent estimator for μ. By the central limit theorem, $\hat{V}_N$ is asymptotically normal

$$\sqrt{N}(\hat{V}_N - \mu) \xrightarrow{d} \mathcal{N}(0, v^2) \tag{1.4.2}$$

v^2 is unknown, so we define the sample variance using the Bessel correction $s_N^2 = \frac{1}{N-1}\sum_{k=1}^{N}(H^{(k)} - \hat{V}_N)^2$, yielding a standard error $\mathrm{SE}(\hat{V}_N) = \frac{s_N}{\sqrt{N}}$. To create a confidence interval, we draw from the Student-t distribution with $N-1$ degrees of freedom

$$\hat{V}_N \pm t_{1-\frac{\alpha}{2};N-1}\frac{s_N}{\sqrt{N}} \tag{1.4.3}$$

1.4.1. Antithetic variates

To reduce variance, we generate M pairs $(H_+^{(k)}, H_-^{(k)})$ driven by standard Gaussian random variables Z and $-Z$. We treat $\{\bar{H}^{(k)}\}_{k=1}^{M} := \{\frac{1}{2}(H_+^{(k)} + H_-^{(k)})\}_{k=1}^{M}$ as i.i.d. and compute their expectation and variance

$$\hat{V}_M = \frac{1}{M}\sum_{k=1}^{M}\bar{H}^{(k)}, \quad s_{\text{anti}}^2 = \frac{1}{M-1}\sum_{k=1}^{M}(\bar{H}^{(k)} - \hat{V}_M)^2 \tag{1.4.4}$$

A naïve Monte Carlo simulation with N paths has variance of the mean $Var(\hat{V}_N) = \frac{\sigma^2}{N}$. With $\frac{N}{2}$ pairs and averaging within each pair, this becomes $Var(\hat{V}_N) = \frac{\sigma^2(1+\rho)}{N}$, where $\rho = Corr(X_+, X_-)$.

Each path under GBM (1.3.1) is built from symmetric Gaussian shocks. A property we revisit later is that the payoff Ψ is monotonically increasing in S_T for a call and monotonically decreasing in S_T for a put. Hence, X_- and X_+ are negatively correlated

$$Cov(X_+, X_-) \le 0 \implies \rho \le 0 \tag{1.4.5}$$

Thus, our variance reduction factor is $VRF(\rho) = \frac{1}{1+\rho}$ without additional computational cost or bias [7].

1.4.2. Control variates

Later, we will make use of a correlated asset Y with known with known mean $\mathbb{E}[Y] = \mu_Y$ under the same measure and model. This will serve as a control variate for the asset $X = e^{-rT}\Psi$ we are looking to price. For N i.i.d. paths, we form the adjusted samples

$$G_k(\beta) = X^{(k)} - \beta(Y^{(k)} - \mu_Y) \tag{1.4.6}$$

and estimate $V = \mathbb{E}[X]$ by $\hat{V}(\beta) = \frac{1}{N}\sum_{k=1}^{N} G_k(\beta)$. The optimal coefficient $\beta = \beta^\star \in \mathbb{R}$ is found by minimising $Var(\hat{V}(\beta))$

$$\beta^\star = \frac{Cov(X,Y)}{Var(Y)} \tag{1.4.7}$$

At $\beta^\star$, $Var(G(\beta^\star)) = Var(X)(1-\rho^2)$, so our variance reduction factor is $VRF(\rho) = \frac{1}{1-\rho^2}$.

1.4.3. Common random numbers (CRN)

When comparing two parameter sets, models, or payoffs, we drive both simulations from the same underlying shock, allowing their sampling errors to move in tangent with one another and largely cancel in the difference. Let $V(\theta) = \mathbb{E}[g(\theta, Z)]$, where Z is the source of all random inputs. To estimate the difference $D := V(\theta_\alpha) - V(\theta_\beta)$, we compute over N simulations

$$Y_\alpha^{(k)} = g(\theta_\alpha, Z^{(k)}), \quad Y_\beta^{(k)} = g(\theta_\beta, Z^{(k)}) \tag{1.4.8}$$

using the same common random number $Z^{(k)}$ for $Y_\alpha^{(k)}$ and $Y_\beta^{(k)}$. We compute the average

$$\hat{D} = \frac{1}{N}\sum_{k=1}^{N}(Y_\alpha^{(k)} - Y_\beta^{(k)}) \tag{1.4.9}$$

which has variance

$$Var(\hat{D}) = \frac{1}{N}(Var(Y_\alpha) + Var(Y_\beta) - 2Cov(Y_\alpha, Y_\beta)) \tag{1.4.10}$$

We see that the application of CRN is most effective when Y_α and Y_β are positively correlated assets.

1.5. Greeks

The Greeks measure the sensitivity of a contingent claim $V(\theta)$ to changes in one or more underlying parameters on which it is dependent upon. The payoff Ψ is not always continuous, so we seek alternative methods other than taking the analytic partial derivative $\partial_\theta V(\theta)$.

N.B. For pricing European and American vanilla options, we at most require a simple finite difference method with common random numbers. To understand further alternative methods which are useful for pricing claims with discontinuous payoffs, such as barrier options, see Appendix A.

1.5.1. Finite differences with CRN

The "bump-and-revalue" method is an alternative method to compute the Greeks when analytic formulae do not exist. In this paper, we will capitalise on this method to compute the Greeks for any American vanilla claim. For $V(\theta) = \mathbb{E}^{\mathbb{Q}}[H(\theta, Z)]$, let $h > 0$ be the bump in our parameter θ. We start with a central difference approximation of $\partial_\theta H(\theta, Z)$

$$\partial_\theta H(\theta, Z) \approx \frac{H(\theta+h, Z) - H(\theta-h, Z)}{2h} + \mathcal{O}(h^2) \tag{1.5.1}$$

For N Monte Carlo simulations, set $Z^{(i)}$ to be the common random number for both bumps and let $\Delta H(\theta, Z^{(i)}; h) = H(\theta + h, Z^{(i)}) - H(\theta - h, Z^{(i)})$. Then, our approximation of the Greek $\hat{G}_N(\theta)$ is

$$\hat{G}_N(\theta) \approx \frac{1}{N}\sum_{i=1}^{N} \frac{\Delta H(\theta, Z^{(i)}; h)}{2h} = \frac{1}{N}\sum_{i=1}^{N}(\partial_\theta H(\theta, Z^{(i)}) + \mathcal{O}(h^2)) \tag{1.5.2}$$

[7] Hence,

$$\mathbb{E}^{\mathbb{Q}}\left[\hat{G}_N(\theta)\right] = \partial_\theta V(\theta) + \mathcal{O}(h^2) \tag{1.5.3}$$

Because we used CRN, the variance is bounded as $h \to 0^+$

$$Var\left(\hat{G}_N(\theta)\right) = Var\left(\frac{1}{N}\sum_{i=1}^{N}(\partial_\theta H(\theta, Z^{(i)}) + \mathcal{O}(h^2))\right) = \frac{1}{N}Var(\partial_\theta H(\theta, Z)) + \mathcal{O}\left(\frac{h^2}{N}\right) \tag{1.5.4}$$

2. European Options

In 1973, Fischer Black, Myron Scholes and Robert C. Merton introduced a continuous-time, no-arbitrage framework to find a closed-form price for European options [8]. They assumed that the price of the underlying asset follows a GBM with constant drift and volatility, continuous and frictionless trading, and the ability to borrow and lend at a constant risk-free rate. The model provides a coherent valuation and hedging benchmark, analytic Greeks for sensitivity analysis, and consistency relations such as put-call parity.

However, the model is not without limitations and rather serves as a baseline; assumptions such as constant volatility, log-normal tails, and continuous trading ignore volatility smiles, jumps, and market friction, meaning that deviations from the Black-Scholes price often occur in practice [9][10]. These limitations are topics I wish to address, perhaps in another paper, by modelling the underlying asset dynamics with stochastic volatility.

2.1. Black-Scholes PDE

We work in the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$ under the market model and the trading rules defined in Section 1.1. The dynamics of our underlying asset price S_t follows the same GBM in $\mathbb{Q}$ with money-market numéraire M as in (1.3.1). For a fixed maturity T and current time $t \in [0, T]$, define $\tau := T - t$.

A European option V is one of the two vanilla contingent claims that we study. Let S_T be the terminal price of the underlying asset. For a pre-agreed strike price K and maturity T, the payoff function $\Psi : \mathbb{R} \to \mathbb{R}$ is defined as

$$\Psi(S_T) = \begin{cases} (S_T - K)^+, & V \text{ is a call option} \\ (K - S_T)^+, & V \text{ is a put option} \end{cases} \tag{2.1.1}$$

Let $V(t, S) = V$ be the price of the option at time t and construct a self-financing portfolio $\Pi(t, S) = V - \lambda S$. To compute the dynamics dV and $d\Pi$, we need to introduce Itô's lemma [6]

Lemma 2.1.1. *(Itô's Lemma) Let X_t be a process that satisfies $dX_t = \alpha_t dt + \beta_t dW_t$ with initial condition $X_0 = x_0$. Suppose that $f : [0, \infty] \times \mathbb{R} \to \mathbb{R}$ is sufficiently smooth such that its second derivative exists and define a new process $Y_t = f(t, X_t)$. Then*

$$dY_t = [\partial_t f(t, X_t) + \alpha_t \partial_x f(t, X_t) + \frac{1}{2}\beta_t^2 \partial_{xx} f(t, X_t)]dt + \beta_t \partial_x f(t, X_t) dW_t \tag{2.1.2}$$

For the continuous dividend yield q, an investor receives a cash flow of $qSdt$ over a closed interval $[t, t + dt]$. Using (2.1.1), we compute dV and $d\Pi$ respectively

$$dV = [\partial_t V + (r - q)S\partial_S V + \frac{1}{2}\sigma^2 S^2 \partial_{SS} V]dt + \sigma S \partial_S V dW^{\mathbb{Q}} \tag{2.1.3}$$

$$d\Pi = [\partial_t V + S(r-q)(\partial_S V - \zeta) + \frac{1}{2}\sigma^2 S^2 \partial_{SS} V - \zeta q S]dt + \sigma S(\partial_S V - \zeta)dW^{\mathbb{Q}} \tag{2.1.4}$$

For the self-financing strategy $\vartheta = (\zeta, \xi)$, choose $\zeta = \partial_S V$ [1][8]. This hedges away the stochastic element of the portfolio's dynamics, making Π locally riskless. This also means that the dynamics of Π must also follow the risk-free rate to ensure no-arbitrage

$$d\Pi = r(V - \partial_S V S)dt \tag{2.1.5}$$

We plug $\zeta = \partial_S V$ into (2.1.4) and set equal to (2.1.5), giving the Black-Scholes PDE [8][11] with terminal conditions

$$\partial_t V + (r-q)S\partial_S V + \frac{1}{2}\sigma^2 S^2 \partial_{SS} V - rV = 0, \quad V(T,S) = \Psi(S) \tag{2.1.6}$$

By the Feynman-Kac formula (1.3.1),

$$V(t, S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\Psi(S_T)|\mathcal{F}_t \equiv \{S_t = s\}] \tag{2.1.7}$$

is the solution to (2.1.6).

2.2. Black-Scholes formula

With S_t following the GBM defined in (1.3.1), define $Y_t = f(t, S_t) = \log(S_t) - (r - q - \frac{1}{2}\sigma^2)t$. Then, $dY_t = \sigma dW_t^{\mathbb{Q}}$ by Ito's lemma (2.1.2). Under the transformation, the initial condition for Y is $Y_0 = \log(S_0)$, and hence $Y_t = \log(S_0) + \sigma W_t^{\mathbb{Q}}$. Inverting the transformation yields the solution for S_t

$$S_t = S_0 e^{(r-q-\frac{1}{2}\sigma^2)t + \sigma W_t^{\mathbb{Q}}} = S_0 e^{(r-q-\frac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z} \tag{2.2.1}$$

where $Z : \Omega \to \mathbb{R}$ is a standard Gaussian random variable. This leads to the crucial idea that S_t follows a log-normal distribution [1]

$$\log(S_t) \sim \mathcal{N}(\log(S_0) + (r - q - \frac{1}{2}\sigma^2)t, \sigma^2 t) \tag{2.2.2}$$

which allows us to find an analytic solution for the price of the contingent claim. Let C_0 and P_0 denote the respective price of a European call and put option at time $t = 0$. Define $X : \Omega \to \mathbb{R}$ as the Gaussian random variable with mean $\mathbb{E}^{\mathbb{Q}}[X] = (r - q - \frac{1}{2}\sigma^2)T$ and variance $Var(X) = \sigma^2 T$. By (2.1.7),

$$C_0 = e^{-rT}\int_{\mathbb{R}} (S_0 e^x - K)^+ f_X(x)dx, \quad P_0 = e^{-rT}\int_{\mathbb{R}} (K - S_0 e^x)^+ f_X(x)dx \tag{2.2.3}$$

where $f_X : \mathbb{R} \to [0, 1]$ is the density function of X. Solving the integrals above gives us the famous Black-Scholes formula [8]

$$C_0 = S_0 e^{-qT}\Phi(d_1) - Ke^{-rT}\Phi(d_2), \quad P_0 = Ke^{-rT}\Phi(-d_2) - S_0 e^{-qT}\Phi(-d_1) \tag{2.2.4}$$

where $\Phi : \mathbb{R} \to [0, 1]$ is the distribution function of a standard Gaussian random variable $Z \sim \mathcal{N}(0, 1)$ and

$$d_1 = \frac{\log(S_0/K) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T} \tag{2.2.5}$$

2.3. Forward Form (Black-76 model)

To match market conventions, we introduce the forward form of the Black-Scholes model, also known as the Black-76 model [12]. To price any contingent claim using the forward price instead of the spot price, we need to switch to the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q}^T)$ with the T-forward measure $\mathbb{Q}^T$. Let $B(t,T) = e^{-r(T-t)}$ be the price at time t of a zero-coupon bond with maturity T. Currently, we have the money market numéraire M with risk-neutral measure $\mathbb{Q}$. Define the Radon-Nikodym density in $\mathcal{F}_t$

$$\left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{M_t B(0,T)}{M_0 B(t,T)} \tag{2.3.1}$$

[2] Since the risk-free rate r is constant, $M_t B(0,T) = B(t,T)$, and $M_0 = 1$, we have $\mathbb{Q}^T = \mathbb{Q}$. Hence, for any contingent claim $V_T \in L^1(\mathbb{Q})$

$$\Pi(V_t) = M_t \mathbb{E}^{\mathbb{Q}}\left[\frac{V_T}{M_T}\middle|\mathcal{F}_t\right] = B(t,T)\mathbb{E}^{\mathbb{Q}^T}\left[\frac{V_T}{B(T,T)}\middle|\mathcal{F}_t\right] \tag{2.3.2}$$

Define the no-arbitrage forward price $F(t,T) = F_t^{(T)} := S_t e^{(r-q)\tau}$ and let $\tilde{G}_t$ be the discounted gains with continuous dividend yield q paying $D_t = \int_0^t qS_u du$ cash over $[0,t]$. Then

$$\tilde{G}_t = F_t^{(T)} + \int_0^t \frac{1}{B(u,T)} dD_u \tag{2.3.3}$$

is a $\mathbb{Q}^T$-local martingale, and so $F_t^{(T)}$ has 0 drift under $\mathbb{Q}^T$

$$dF_t^{(T)} = \sigma F_t^{(T)} dW_t^{\mathbb{Q}^T} \tag{2.3.4}$$

We now rewrite the Black-Scholes formula in terms of $F_t^{(T)}$. Let $\tau := T - t$ be as previously defined in Section 2.1. Then, the Black-76 model [12] is

$$C_t = e^{-r\tau}(F_t^{(T)}\Phi(\delta_1) - K\Phi(\delta_2)), \quad P_t = e^{-r\tau}(K\Phi(-\delta_2) - F_t^{(T)}\Phi(-\delta_1)) \tag{2.3.5}$$

where $\Phi : \mathbb{R} \to [0,1]$ is the same distribution function defined in (2.2.4) and

$$\delta_1 = \frac{\log(F_t^{(T)}/K) + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}, \quad \delta_2 = \delta_1 - \sigma\sqrt{\tau} \tag{2.3.6}$$

2.4. Parity and bounds

2.4.1. Put-call parity

Consider a portfolio Π_t at time $t \in [0,T]$ consisting of a long call option C_t and a short put option P_t, both with strike price K. Then Π_t has payoff $\Psi(S_T) = (S_T - K)^+ - (K - S_T)^+$. Let $\tau = T - t$. Discounting by time τ, we write put-call parity in forward price form or spot price form respectively [13]

$$\text{Forward form: } C_t - P_t = e^{-r\tau}(F_t^{(T)} - K) \tag{2.4.1}$$

$$\text{Spot form: } C_t - P_t = S_t e^{-q\tau} - K e^{-r\tau} \tag{2.4.2}$$

2.4.2. Upper and lower bounds

To prevent arbitrage opportunities from occurring, we enforce the following lower bounds from put-call parity (2.4.1) and non-negativity of the options $C_t, P_t \geq 0$

$$C_t \geq (S_t e^{-q\tau} - K e^{-r\tau})^+, \quad P_t \geq (K e^{-r\tau} - S_t e^{-q\tau})^+ \tag{2.4.3}$$

[13] For any $\alpha \geq 0$, it is true that $(\alpha - K)^+ \leq \alpha$ and $(K - \alpha)^+ \leq K$. Hence, C_t and P_t are bounded above by

$$C_t = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+ \middle| \mathcal{F}_t\right] \leq S_t e^{-q\tau}, \quad P_t = e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[(K - S_T)^+ \middle| \mathcal{F}_t\right] \leq K e^{-r\tau} \tag{2.4.4}$$

So we write

$$0 \leq C_t \leq S_t e^{-q\tau}, \quad 0 \leq P_t \leq K e^{-r\tau} \tag{2.4.5}$$

Subsequently, we define the asymptotic values for the European call and put price as the underlying asset price $S \to 0$ by

$$V(t, 0) = \begin{cases} 0, & V \text{ is a call option} \\ K e^{-r\tau}, & V \text{ is a put option} \end{cases} \tag{2.4.6}$$

[13] In Chapter 3, we will use Dirichlet bounds to formalise the behaviour of the American option price as the underlying asset price attains its minimum and maximum values on a discretised space-time grid. Thus, we will briefly discuss the Neumann boundary conditions here for completeness. As $S \to \infty$, the Neumann boundary values require that we use the derivative $\partial_S V(t, S)$ to price $V(t, S_{\max})$. For a sufficiently large $S_{\max}$ on a discretised grid, a European call option is almost surely in-the-money, with payoff $\Psi(S_T) = (S_T - K)^+ \to S_T - K$. Thus,

$$C(t, S_{\max}) \approx e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left([S_T|\mathcal{F}_t] - K\right) = S e^{-q\tau} - K e^{-r\tau} \tag{2.4.7}$$

And so, by put-call parity (2.4.1), the European put option price is almost surely out-the-money, with $P(t, S_{\max}) \approx 0$. Thus, the Neumann boundary conditions [13] can be computed using the following spatial derivatives

$$V(t, S_{\max}) \begin{cases} e^{-q\tau}, & V \text{ is a call option} \\ 0, & V \text{ is a put option} \end{cases} \tag{2.4.8}$$

N.B. The derivations for these upper and lower bounds can be found in Section 2.6.

2.4.3. Monotonicity and convexity

For each ω, the functions $K \mapsto (S_T(\omega) - K)^+$ and $K \mapsto (K - S_T(\omega))^+$ are piecewise-linear and convex [14]. Specifically, they are, respectively, non-increasing and non-decreasing. Thus, the rates of change of C or P with respect to strike price K always satisfies the following conditions

$$\partial_K C \leq 0, \ \partial_K^2 C \geq 0 \quad \text{and} \quad \partial_K P \geq 0, \ \partial_K^2 P \geq 0 \tag{2.4.9}$$

2.5. Greeks

The Greeks measure the option sensitivity to changes in a single parameter, $\vartheta \in \{\Delta, \Gamma, \mathcal{V}, \Theta, \rho\}$, while keeping the other parameters fixed. In practice, these sensitivities inform us about trading opportunities and serve as a guide for hedging strategies [13].

Let $\Phi, \phi : \mathbb{R} \to [0, 1]$ be the distribution and density functions of a standard Gaussian random variable, respectively, with $\delta_1, \delta_2 \in \mathbb{R}$ the same as defined in (2.3.6). With $\tau = T - t$; underlying asset price S_t; constant risk-free rate r, dividend yield q and annualised volatility σ, we define each Greek below.

Definition 2.5.1. Delta, denoted Δ, measures the sensitivity of a contingent claim to changes in the underlying asset S_t.

$$\Delta_{call}(t) = \partial_S C(t, S_t) = e^{-q\tau}\Phi(\delta_1), \tag{2.5.1}$$

$$\Delta_{put}(t) = \partial_S P(t, S_t) = e^{-q\tau}(\Phi(\delta_1) - 1) \tag{2.5.2}$$

Definition 2.5.2. Gamma, denoted Γ, measures the sensitivity of a contingent claim to changes in Δ.

$$\Gamma(t) = \partial_{SS} V(t, S_t) = \frac{e^{-q\tau}\phi(\delta_1)}{S_t \sigma \sqrt{\tau}} \tag{2.5.3}$$

Definition 2.5.3. Vega, denoted $\mathcal{V}$, measures the sensitivity of a contingent claim to changes in the annualised volatility σ.

$$\mathcal{V}(t) = \partial_\sigma V(t, S_t) = S_t e^{-q\tau}\phi(\delta_1)\sqrt{\tau} \tag{2.5.4}$$

Definition 2.5.4. Rho, denoted ρ, measures the sensitivity of a contingent claim to changes in the risk-free rate r.

$$\rho_{call}(t) = \partial_r C(t, S_t) = K\tau e^{-r\tau}\Phi(\delta_2), \tag{2.5.5}$$

$$\rho_{put}(t) = \partial_r P(t, S_t) = -K\tau e^{-r\tau}\Phi(-\delta_2) \tag{2.5.6}$$

Definition 2.5.5. Theta, denoted Θ, measures the sensitivity of a contingent claim to time decay t.

$$\Theta_{call}(t) = \partial_t C(t, S_t) = -\frac{S_t e^{-q\tau}\phi(\delta_1)\sigma}{2\sqrt{\tau}} - rKe^{-r\tau}\Phi(\delta_2) + qS_t e^{-q\tau}\Phi(\delta_1), \tag{2.5.7}$$

$$\Theta_{put}(t) = \partial_t P(t, S_t) = -\frac{S_t e^{-q\tau}\phi(\delta_1)\sigma}{2\sqrt{\tau}} + rKe^{-r\tau}\Phi(-\delta_2) - qS_t e^{-q\tau}\Phi(-\delta_1) \tag{2.5.8}$$

Remark 2.5.6. From the above definitions, notice that $\Gamma_{call}(t) = \Gamma_{put}(t)$ and $\mathcal{V}_{call}(t) = \mathcal{V}_{put}(t)$, so we simply write $\Gamma(t)$ and $\mathcal{V}(t)$ respectively for both call and put options.

2.6. Implied Volatility

Let $V^{\mathrm{BS}}(S_t, K, \tau, r, q, \sigma) \in L^1(\mathbb{Q})$ denote the theoretical price of a European contingent claim at time t under the Black-Scholes model. For a given set of fixed parameters $\mathcal{A} = \{S_t, K, \tau, r, q\}$, suppose that there exists a European contingent claim which is being traded at time t in the market at a price V^{mkt}.

Then, there exists a volatility $\sigma > 0$ that, when applied to the Black-Scholes model together with set $\mathcal{A}$, matches the price of V^{mkt}. We call this implied volatility, σ_{imp}

$$V^{\text{BS}}(S_t, \tau, K, r, q, \sigma_{\text{imp}}) = V^{\text{mkt}} \tag{2.6.1}$$

Since $\partial_\sigma V = S_t e^{-q\tau}\phi(d_1)\sqrt{\tau}$ exists and is strictly positive for all $\tau > 0$, the price V^{BS} is a continuous and strictly increasing function in σ. Hence, $\sigma_{\text{imp}} > 0$ is also unique [14].

To ensure well-posedness, we derive the bounds of $V^{\text{BS}}(\sigma)$ by analysing what happens when $\sigma \downarrow 0$ and $\sigma \uparrow \infty$. Recall the forward-form European call and put prices in Section 2.3. Let $\delta_1, \delta_2 \in \mathbb{R}$ be defined the same as in (2.3.6). For $\sigma \downarrow 0$, we divide into three cases: when the forward price $F := S_t e^{(r-q)\tau}$ is greater than, equal to, and less than the strike price K. Then

$$\lim_{\sigma \downarrow 0} \delta_1 = \begin{cases} \infty, & F > K \\ 0, & F = K \\ -\infty, & F < K \end{cases} \implies \lim_{\sigma \downarrow 0} \delta_2 = \begin{cases} \infty, & F > K \\ 0, & F = K \\ -\infty, & F < K \end{cases} \tag{2.6.2}$$

Plugging these limiting values of δ_1 and δ_2 into (2.3.5), we find the lower bounds for $C^{\text{BS}}(\sigma)$ and $P^{\text{BS}}(\sigma)$

$$C^{\text{BS}}(\sigma) \geq (e^{-r\tau}(F - K))^+, \quad P^{\text{BS}}(\sigma) \geq (e^{-r\tau}(K - F))^+ \tag{2.6.3}$$

For $\sigma \uparrow \infty$, it is clear that $\lim_{\sigma \uparrow \infty} \delta_1 = \infty$ and $\lim_{\sigma \uparrow \infty} \delta_2 = -\infty$. Hence,

$$C^{\text{BS}}(\sigma) \leq S_t e^{-q\tau}, \quad P^{\text{BS}}(\sigma) \leq K e^{-r\tau} \tag{2.6.4}$$

So, the unique solution σ_{imp} for (2.6.1) exists if and only if the observed market price V^{mkt} lies within the lower and upper bounds in (2.6.3) and (2.6.4).

2.6.1. Newton's method

We start by defining the continuous and strictly increasing function $f : [0, \infty) \to \mathbb{R}$ given by $f(\sigma) := V^{\text{BS}}(S_t, \tau, K, r, q, \sigma) - V^{\text{mkt}}$. We wish to find $\sigma > 0$ such that $f(\sigma) = 0$. It is clear that $f'(\sigma) = \partial_\sigma V^{\text{BS}}$. By a simple Taylor expansion of $f(\sigma)$ around an initial guess $\sigma_0 > 0$, the n^{th} iterate becomes

$$\sigma_n = \sigma_{n-1} - \frac{f(\sigma_{n-1})}{f'(\sigma_{n-1})} \tag{2.6.5}$$

We repeat until a predefined tolerance $|V^{\text{BS}}(\sigma) - V^{\text{mkt}}| \leq \varepsilon_V$ is met [15]. Newton's method has a quadratic convergence rate and thus is fast. However, for small Vega, the method becomes numerically unstable, and the steps between iterations are subsequently large. It is possible to dampen the step size relative to the size of $\partial_\sigma V^{\text{BS}}$, or define a volatility bracket so that Newton's method is applied if and only if $\sigma \in [0, \sigma_{max}]$ [14].

2.6.2. Bisection method

Let $f : [0, \infty) \to \mathbb{R}$ be the same as in Section 2.6.1. Choose $\sigma_\ell = 0$ so that $f(\sigma_\ell) \leq 0$ and increase σ_u geometrically until $f(\sigma_u) \geq 0$ to guarantee the existence of a root in the interval $[\sigma_\ell, \sigma_u]$. We compute

$\sigma_{mid} = \frac{1}{2}(\sigma_\ell + \sigma_u)$ and follow the algorithm below

$$\begin{aligned} &\textbf{1}.\quad \text{If } f(\sigma_{mid}) = 0, \text{ we have found the root and the iteration terminates} \\ &\textbf{2}.\quad \text{If } f(\sigma_{mid}) > 0, \text{ we set } \sigma_{mid} \to \sigma_u \\ &\textbf{3}.\quad \text{If } f(\sigma_{mid}) < 0, \text{ we set } \sigma_{mid} \to \sigma_\ell \end{aligned} \tag{2.6.6}$$

We repeat until we reach a predefined tolerance $|\sigma_\ell - \sigma_u| < \varepsilon_\sigma$. The bisection method has a linear convergence rate, which is slower than Newton's method. However, since the algorithm is derivative-independent, the process is extremely numerically stable and handles deep ITM/OTM quotes that Newton's method cannot [15].

2.7. Monte Carlo pricing

In Section 1.4, we introduced the notion of Monte Carlo methods to price a contingent claim $V = \mathbb{E}^{\mathbb{Q}}[H] \in L^1(\mathbb{Q})$ that does not have an analytic solution. In the case of European vanilla options, we obviously have an analytic solution via the Black-Scholes formula, which we presented earlier in the chapter. However, Monte Carlo simulations remain useful as a validation tool for the closed-form formula, especially for extreme strike values which may lead to deep in-the-money and out-the-money quotes [7].

Recall that at time t, the no-arbitrage price of the European option is $V(t, S_t) = \mathbb{E}^{\mathbb{Q}}[e^{-r\tau}\Psi(S_T)|\mathcal{F}_t]$, where $\tau := T - t$. We simulate M i.i.d. paths of the underlying asset at terminal time S_T under $\mathbb{Q}$-dynamics $dS_t = (r - q)S_t dt + \sigma S_t dW_t^{\mathbb{Q}}$. We define the corresponding discounted payoffs as

$$H^{(m)} := e^{-r\tau}\Psi(S_T^{(m)}), \quad m = 1, \dots, M \tag{2.7.1}$$

with the necessary assumption that we have finite mean $\mathbb{E}^{\mathbb{Q}}[H^{(m)}] = V(t, S_t) < \infty$ and finite variance $Var(H^{(m)}) < \infty$. The Monte Carlo estimator $\hat{V}(t, S_t)$ of the European option price is simply the sample mean

$$\hat{V}(t, S_t) := \frac{1}{M}\sum_{m=1}^{M} H^{(m)} \tag{2.7.2}$$

For notational simplicity, we write $\hat{V} := \hat{V}(t, S_t)$ and $V := V(t, S_t)$. By linearity of expectation, $\hat{V}$ is an unbiased estimator of V; by the strong law of large numbers, $\hat{V}(t, S_t) \to V(t, S_t)$ almost surely as $M \to \infty$. Thus, $\hat{V}$ is a consistent estimator. By the central limit theorem, it is also asymptotically normal

$$\sqrt{M}(\hat{V} - V) \xrightarrow{d} \mathcal{N}(0, \sigma_H^2) \tag{2.7.3}$$

where σ_H^2 is replaced by the sample variance using the Bessel correction $s_M^2 = \frac{1}{M-1}\sum_{m=1}^{M}(H^{(m)} - \hat{V})^2$. From here, we are able to construct standard errors and confidence intervals as described in Section 1.4.

To simulate M paths, we cast back to the closed form solution of our $\mathbb{Q}$-dynamics in (2.2.1), namely $S_T = S_t e^{(r-q-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}Z}$, where $Z \sim \mathcal{N}(0, 1)$. We can, for path-independent options such as the European vanilla, simulate S_T using this exact log-normal representation, so that no time or spatial discretisation bias is introduced

$$S_T^{(m)} = S_t e^{(r-q-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}Z^{(m)}}, \quad Z^{(m)} \overset{i.i.d.}{\sim} \mathcal{N}(0, 1) \tag{2.7.4}$$

If we wish to reduce the variance of the estimator without introducing any bias, we will make use of the antithetic variates defined in Section 1.4.1. For every draw $Z^{(m)}$, we also simulate a path driven by $-Z^{(m)}$ to obtain a pair of discounted payoffs $\left(H_+^{(m)}, H_-^{(m)}\right)$. Original samples $\{H^{(m)}\}_{m=1}^M$ are replaced by the pairwise average of the antithetic variates

$$\bar{H}^{(m)} := \frac{1}{2}\left(H_+^{(m)} + H_-^{(m)}\right) \tag{2.7.5}$$

The payoff $\Psi(S_T)$ is monotone for European call and put options, thus the discounted payoff generated by $Z^{(m)}$ and $-Z^{(m)}$ are negatively correlated. Hence, the variance of $\bar{H}^{(m)}$ must be strictly smaller than the original $H^{(m)}$. We re-write the new estimator $\hat{V}_{\text{anti}}(t, S_t)$ as

$$\hat{V}_{\text{anti}}(t, S_t) := \frac{1}{M}\sum_{m=1}^{M} \bar{H}^{(m)} \tag{2.7.6}$$

To compute the Greeks for our Monte Carlo estimates, we make use of the finite difference method set up in Section 1.5.1. Since we have simulated the path of the underlying asset, we use $\hat{V}(\theta) = \mathbb{E}^{\mathbb{Q}}[\hat{H}(\theta, Z)]$, applying the central difference approximation for delta, gamma, vega, rho and the backward difference approximation for theta. For N simulations, the approximation for the i^{th} simulation with small bump $h > 0$, is, respectively

$$\partial_\theta \hat{H}(\theta, Z^{(i)}) \approx \frac{\hat{H}(\theta + h, Z^{(i)}) - \hat{H}(\theta - h, Z^{(i)})}{2h}, \quad \partial_\theta \hat{H}(\theta, Z^{(i)}) \approx \frac{\hat{H}(\theta - h, Z^{(i)}) - \hat{H}(\theta, Z^{(i)})}{h} \tag{2.7.7}$$

[7] Then, the estimated Greek $\hat{G}_N(\theta)$ is

$$\hat{G}_N(\theta) \approx \frac{1}{N}\sum_{i=1}^{n} \partial_\theta \hat{H}(\theta, Z^{(i)}) \tag{2.7.8}$$

3. American Options

The American vanilla option is very similar in style to the European vanilla option priced in Chapter 2, except for one caveat: a holder of the American option $V(t, S_t)$ with maturity T has the right to exercise the contract at any time $t \in [0, T]$. This change in exercise conditions means that there no longer exists a closed-form solution to price the contingent claim. We use a finite difference scheme to approximate a solution, enforce no-arbitrage constraints at all points $(t, s) \in [0, T] \times [0, s_{\max}]$ in a space-time discretised grid, and produce a solution which is unconditionally stable in time [16].

3.1. Variational inequality

We remain in the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$ with risk-neutral measure $\mathbb{Q}$ and money-market numéraire M. We continue to assume that the underlying asset price S_t follows the same GBM defined in (1.3.1), that is

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t^{\mathbb{Q}} \tag{3.1.1}$$

For simplicity of notation, we will reuse the Black-Scholes spatial generator throughout, so it is worth defining again here

$$(\mathcal{L}g)(s) = (r - q)s\partial_s g(s) + \frac{1}{2}\sigma^2 s^2 \partial_{ss} g(s) \tag{3.1.2}$$

For an American option, the holder of V has the right but not obligation to exercise the contract at any time $t \in [0, T]$. For a pre-agreed strike price K and maturity T, we define the payoff function $\Psi : \mathbb{R} \to \mathbb{R}$ as

$$\Psi(s) = \begin{cases} (s - K)^+, & V \text{ is a call option} \\ (K - s)^+, & V \text{ is a put option} \end{cases} \tag{3.1.3}$$

Definition 3.1.1. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$ be a filtered probability space. Then, the random variable $\tau : \Omega \to [0, \infty)$ is a *stopping time* with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if

$$\{\tau \leq t\} \in \mathcal{F}_t \quad \forall t \geq 0 \tag{3.1.4}$$

Let $\mathcal{T}_{t,T}$ denote the set of stopping times in the closed interval $[t, T]$. The optimal stopping problem for an American option [1] can be written as

$$V(t, s) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{\mathbb{Q}}[e^{-r(\tau - t)}\Psi(S_\tau) \big| \mathcal{F}_t \equiv \{S_t = s\}] \tag{3.1.5}$$

Suppose that at some point (t, s), we have $V(t, s) < \Psi(s)$. Then, one could simply buy the American option at price $V(t, s)$, and exercise immediately to realise $\Psi(s)$, yielding a guaranteed profit of $\Psi(s) - V(t, s)$. To prevent this arbitrage from occurring, we must enforce the following condition

$$V(t, s) \geq \Psi(s) \quad \forall (t, s) \tag{3.1.6}$$

[1] In order to understand how early exercise affects the PDE for the price of the contingent claim, we must introduce the idea of a supermartingale.

Definition 3.1.2. Let X_t be an adapted process for the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ in the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$. X_t is a *supermartingale* if

$$\mathbb{E}^{\mathbb{Q}}[X_t|\mathcal{F}_s] \leq X_s \quad \forall t \geq s \tag{3.1.7}$$

At time t, the holder may defer exercise for a short interval $h > 0$ and then act optimally. Given that expiration does not occur within $[t, t+h]$, this is a feasible strategy. However, it cannot outperform the optimal value. Hence,

$$V(t,s) \geq \mathbb{E}^{\mathbb{Q}}[e^{-rh}V(t+h, S_{t+h})\big|\mathcal{F}_t] \tag{3.1.8}$$

Define the discounted value process $X_u := e^{-ru}V(u, S_u)$. It is clear that $X_t \geq \mathbb{E}^{\mathbb{Q}}[X_{t+h}|\mathcal{F}_t]$. Hence, $\{X_u\}_{u\geq t}$ is a supermartingale [5]. We compute dX_u using Itô's lemma (2.1.1)

$$dX_u = e^{-ru}(\partial_u V + \mathcal{L}V - rV)du + e^{-ru}\sigma S_u \partial_s V dW_u^{\mathbb{Q}} \tag{3.1.9}$$

An important property to note is that a process X_t is a supermartingale if and only if the drift is non-positive. Hence,

$$\partial_t V(t,s) + \mathcal{L}V(t,s) - rV(t,s) \leq 0 \tag{3.1.10}$$

A contingent claim holder acting optimally has two choices: remain in the continuation region or move into the exercise region. If they choose to remain in the continuation region, the option becomes European in nature

$$\partial_t V(t,s) + \mathcal{L}V(t,s) - rV(t,s) = 0, \quad V(t,s) > \Psi(s) \tag{3.1.11}$$

If the holder moves to the exercise region, the equations in (3.1.11) become

$$\partial_t V(t,s) + \mathcal{L}V(t,s) - rV(t,s) \leq 0, \quad V(t,s) = \Psi(s) \tag{3.1.12}$$

Hence, regardless of the holder's optimal choice to remain in the continuation region or move to the exercise region, we establish the following equality

$$(V(t,s) - \Psi(s))\left(\partial_t V(t,s) + \mathcal{L}V(t,s) - rV(t,s)\right) = 0 \tag{3.1.13}$$

At maturity T, the American option is identical to its European counterpart. Thus, its price is simply equal to the intrinsic value $V(T,s) = \Psi(s)$.

The four following conditions make up the variational inequality problem [16][17]

$$\boxed{\begin{array}{ll} \mathbf{1.} & V(t,s) \geq \Psi(s) \\ \mathbf{2.} & \partial_t V(t,s) + \mathcal{L}V(t,s) - rV(t,s) \leq 0 \\ \mathbf{3.} & (V(t,s) - \Psi(s))\left(\partial_t V(t,s) + \mathcal{L}V(t,s) - rV(t,s)\right) = 0 \\ \mathbf{4.} & V(T,s) = \Psi(s) \end{array}} \tag{3.1.14}$$

In the next section, we discretise this set of equations to turn it into a finite-dimensional linear complementarity problem and look to find an unconditionally stable solution at each time step.

3.2. Linear complementarity problem (LCP)

In order to discretise the variational inequality found in (3.1.14), we must construct a finite and discrete time-space grid to work in such that all nodes $(t, s) \in [0, T] \times [0, s_{\max}]$

$$0 = t_0 < t_1 < \cdots < t_N = T, \tag{3.2.1}$$

$$0 = s_0 < s_1 < \cdots < s_M = s_{\max} \tag{3.2.2}$$

Let $\Delta t_n := t_{n+1} - t_n$ for all $n = 0, \dots, N-1$ and $\Delta s_m := s_{m+1} - s_m$ for all $m = 0, \dots, M-1$. For a continuous function $g : [0, T] \times [0, \infty) \to \mathbb{R}$, we write $g(t_n, s_m) \equiv g_{n,m}$ for simplicity of notation.

Recall that $\partial_t V(t, s) + \mathcal{L}V(t, s) - rV(t, s) \leq 0$ must always hold. If we can discretise this, then all other conditions in the variational inequality should follow suit to yield the linear complementarity problem. Let us start by using a central difference method on the Black-Scholes spatial generator [16]

$$(\mathcal{L}g)(s_i) \approx (\mathcal{L}_h g)(s_i) = (r - q)s_i \frac{g_{i+1} - g_{i-1}}{2\Delta s_i} + \frac{1}{2}\sigma^2 s_i^2 \frac{g_{i+1} - 2g_i + g_{i-1}}{(\Delta s_i)^2} \tag{3.2.3}$$

Let $\alpha_i := \frac{\sigma^2 s_i^2}{2(\Delta s_i)^2}$ and $\beta_i := \frac{(r-q)s_i}{2\Delta s_i}$. Then

$$(\mathcal{L}_h g)_i = (\alpha_i - \beta_i)g_{i-1} - 2\alpha_i g_i + (\alpha_i + \beta_i)g_{i+1}, \quad i = 1, \dots, M-1 \tag{3.2.4}$$

$\mathcal{L}_h$ is the tridiagonal matrix, so $(\mathcal{L}_h g)_i$ approximates $(\mathcal{L}g)(s_i)$.

Returning to continuous time, we rewrite the PDE in terms of a single generator $\mathcal{A}$ as

$$\partial_t V(t, s) + \mathcal{A}V(t, s) \leq 0, \quad \mathcal{A} := \mathcal{L} - rI \tag{3.2.5}$$

By integrating both sides with respect to t over the closed interval $[t_n, t_{n+1}]$, we find

$$V(t_{n+1}, s) - V(t_n, s) = \int_{t_n}^{t_{n+1}} \partial_t V(t, s)dt \leq -\int_{t_n}^{t_{n+1}} \mathcal{A}V(t, s)dt \tag{3.2.6}$$

We take the convex approximation of $\int_{t_n}^{t_{n+1}} \mathcal{A}V(t, s)dt \approx \Delta t_n[\vartheta \mathcal{A}V(t_n, s) + (1 - \vartheta)\mathcal{A}V(t_{n+1}, s)]$, with adjustable parameter $\vartheta \in [0, 1]$ [18]. Hence,

$$V(t_n, s) - V(t_{n+1}, s) - \Delta t_n[\vartheta \mathcal{A}V(t_n, s) + (1 - \vartheta)\mathcal{A}V(t_{n+1}, s)] \geq 0 \tag{3.2.7}$$

N.B. For $\vartheta = 1$, this is the backward Euler method, and for $\vartheta = \frac{1}{2}$, this is the Crank-Nicolson method. We will see later that a mix of the two - specifically employing the Euler scheme in the first two steps and the rest Crank-Nicolson - yields an unconditionally stable, second-order accurate in time solution which handles oscillations at payoff kinks well.

To discretise (3.2.7), let $\boldsymbol{V}^n = (V(t_n, s_0), \dots, V(t_n, s_M))^T \in \mathbb{R}^{M+1}$ be the finite-dimensional vector of all prices in the spatial grid at time t_n, and define the spatial matrix $A_h := \mathcal{L}_h - rI$. Then

$$\Delta t_n[\vartheta A_h \boldsymbol{V}^n + (1 - \vartheta)A_h \boldsymbol{V}^{n+1}] \leq \boldsymbol{V}^n - \boldsymbol{V}^{n+1} \tag{3.2.8}$$

Rearranging, we see that

$$A_n \boldsymbol{V}^n \geq b_n, \quad A_n := I - \vartheta \Delta t_n A_h, \quad \boldsymbol{b}_n := (I + (1 - \vartheta)\Delta t_n A_h)\boldsymbol{V}^{n+1} \tag{3.2.9}$$

Define the payoff vector $\boldsymbol{\Psi} := (\Psi(s_0), \ldots, \Psi(s_M))^T \in \mathbb{R}^{M+1}$. To ensure no arbitrage on the space-time grid, we must impose the condition $\boldsymbol{V}^n \geq \boldsymbol{\Psi}$ for all times $t_n \in \{t_0, \ldots, t_N\}$.

At each node $(t_n, s_m) \in [0, T] \times [0, s_{\max}]$, a holder of the American option V_m^n acting optimally is either in the continuation region such that $V_m^n > \Psi_m$ and $(A_n \boldsymbol{V}^n - \boldsymbol{b}_n)_m = 0$, or in the exercise region such that $V_m^n = \Psi_m$ and $A_n \boldsymbol{V}^n - \boldsymbol{b}_n \geq 0$. We write this as $(A_n \boldsymbol{V}^n - \boldsymbol{b}_n)^T(\boldsymbol{V}^n - \boldsymbol{\Psi}) = 0$. Hence, our new discretised problem that we look to solve is the linear complementarity problem [19]

$$\boxed{\begin{array}{ll} \textbf{1.} & \boldsymbol{V}^n \geq \boldsymbol{\Psi} \\[6pt] \textbf{2.} & A_n \boldsymbol{V}^n \geq \boldsymbol{b}_n \\[6pt] \textbf{3.} & (A_n \boldsymbol{V}^n - \boldsymbol{b}_n)^T(\boldsymbol{V}^n - \boldsymbol{\Psi}) = 0 \end{array}} \tag{3.2.10}$$

3.3. Solving the LCP

3.3.1. Crank-Nicolson and backward Euler schemes

It is important to note that we assume that the discretised time set $\{0 = t_0, \ldots, t_N = T\}$ is uniformly spaced, for now, so that $\Delta t_{n+1} - \Delta t_n = \Delta t$ for all $n \in \{0, \ldots, N-1\}$.

The Crank-Nicolson scheme on its own takes $\vartheta = \frac{1}{2}$ and is simply the trapezoidal rule for the integral in (3.2.6). Hence, we recover a solution that has a global truncation error $\mathcal{O}((\Delta t)^2)$. However, since the terminal payoff $V(T, s) = \Psi(s)$ has a kink near the strike K, this scheme produces Gibbs-like oscillations that violate monotonicity. To fix this, we introduce the Rannacher startup: two starting half-steps using the backward Euler scheme ($\theta = 1$) [20]. We then use the Crank-Nicolson scheme for all remaining steps. This strongly damps the oscillations experienced and still provides the same global accuracy.

Since it is known that the price of the contingent claim at maturity is $V(T, s) = \Psi(s)$, we step backwards in time to price at each node until we reach initial time $t_0 = 0$. For the Rannacher startup, we have the following LCPs for each $n \in \{N - \frac{1}{2}, N - 1\}$

$$(I - \frac{1}{2}\Delta t A_h)\boldsymbol{V}^n \geq \boldsymbol{V}^{n+\frac{1}{2}}, \quad \boldsymbol{V}^n \geq \Psi \tag{3.3.1}$$

For each remaining $n \in \{N-2, \ldots, 0\}$ using the Crank-Nicholson scheme, we have the LCPs

$$(I - \frac{1}{2}\Delta t A_h)\boldsymbol{V}^n \geq (I + \frac{1}{2}\Delta t A_h)\boldsymbol{V}^{n+1}, \quad \boldsymbol{V}^n \geq \Psi \tag{3.3.2}$$

3.3.2. Projected successive over-relaxation

In order to solve the LCPs above, it is useful to introduce the general case of successive over-relaxation that we will employ [21].

Let $A\boldsymbol{x} = \boldsymbol{b}$ be a square system of n linear equations with entries $(A)_{ij} = a_{ij}$, $\boldsymbol{x}_i = x_i$, and $\boldsymbol{b}_i = b_i$. Decompose A into a diagonal matrix D, a strictly lower triangular matrix L, and a strictly upper triangular matrix U, so that $A = D + L + U$. Then, the system of linear equations may be written as

$$(D + \omega L)\boldsymbol{x} = \omega \boldsymbol{b} - [\omega U + (\omega - 1)D]\boldsymbol{x}, \quad \omega > 1 \tag{3.3.3}$$

where $\omega > 1$ is the *relaxation factor*. We iteratively solve the left side of the expression for $\boldsymbol{x}$, using the previous values of $\boldsymbol{x}$ on the right side. We write this analytically as

$$\boldsymbol{x}^{(k+1)} = (D+\omega L)^{-1}\left(\omega \boldsymbol{b} - [\omega U + (\omega-1)D]\boldsymbol{x}^{(k)}\right) = L_\omega \boldsymbol{x}^{(k)} + \boldsymbol{c} \tag{3.3.4}$$

Since $(D+\omega L)$ is a triangular matrix, it is possible to compute the elements of $\boldsymbol{x}^{(k+1)}$ sequentially using forward substitution

$$x_i^{(k+1)} = (1-\omega)x_i^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij}x_j^{(k+1)} - \sum_{j>i} a_{ij}x_j^{(k)}\right), \quad i = 1,\ldots,n \tag{3.3.5}$$

Returning to the tridiagonal matrix $\mathcal{L}_h$, let $\ell_i = \alpha_i - \beta_i$, $d_i = -2\alpha_i$, and $u_i = \alpha_i + \beta_i$ for $i = 1,\ldots,M-1$. To facilitate the boundary conditions, we extend $\mathcal{L}_h$ by adding a row of zeros at the top and bottom, so that $\mathcal{L}_h \in M_{M+1}(\mathbb{R})$, namely a square $(M+1)$ matrix. Using $A_n := I - \vartheta\Delta t(\mathcal{L}_h - rI)$, we see that A_n looks like

$$A_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\vartheta\Delta t\ell_1 & 1-\vartheta\Delta t(d_1 - r) & -\vartheta\Delta t u_1 & & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & -\vartheta\Delta t\ell_{M-1} & 1-\vartheta\Delta t(d_{M-1}-r) & -\vartheta\Delta t u_{M-1} \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \tag{3.3.6}$$

N.B. Any non-zero elements in the top and bottom rows are simply placeholders for the boundary conditions, so we replace these with a 1*.*

With $\boldsymbol{b}_n := (I + (1-\vartheta)\Delta t A_h)\boldsymbol{V}^{n+1}$, as defined in (3.2.9), we write it as

$$\boldsymbol{b}_n = \begin{bmatrix} V_0^{n+1} \\ V_1^{n+1} + \Delta t(1-\vartheta)\left(\ell_1 V_0^{n+1} + (d_1 - r)V_1^{n+1} + u_1 V_2^{n+1}\right) \\ \vdots \\ V_{M-1}^{n+1} + \Delta t(1-\vartheta)\left(\ell_{M-1}V_{M-2}^{n+1} + (d_{M-1}-r)V_{M-1}^{n+1} + u_{M-1}V_M^{n+1}\right) \\ V_M^{n+1} \end{bmatrix} \tag{3.3.7}$$

It indeed is possible to decompose A_n into a diagonal matrix $D = (A_n)_{i,i}$, a strictly lower triangular matrix $L = (A_n)_{i+1,i}$, and a strictly upper triangular matrix $U = (A_n)_{i,i+1}$. Thus, we are allowed to use the sequential SOR update defined in (3.3.5) to compute $\tilde{V}_i^{(k+1)}$. Let $b_i \equiv (\boldsymbol{b})_i$ and $a_{ij} \equiv (A)_{ij}$. Then

$$\tilde{V}_i^{(k+1)} = (1-\omega)V_i^{(k)} + \frac{\omega}{(A_n)_{ii}}\left(b_i - \sum_{j<i} a_{ij}\tilde{V}_j^{(k+1)} - \sum_{j>i} a_{ij}V_j^{(k)}\right), \quad i = 1,\ldots,n \tag{3.3.8}$$

[16] Once our iteration reaches a predefined tolerance or convergence, we label this value $\tilde{V}_i^n$.

We must also consider the projection $\tilde{V}_i^n \geq \Psi_i$ in every iteration to avoid immediate-exercise arbitrage. In practice, we set $V_i^n = \max\{\Psi_i, \tilde{V}_i^n\}$ for every node $(t_n, s_m) \in [0,T) \times (0, s_{\max})$. At any non-maturity time t_n, the nodes (t_n, s_0) and $(t_n, s_{\max})$ rely on a predefined set of boundary conditions, rather than PSOR. Thus, we set V_0^{n+1} and V_M^{n+1} equal to the Dirichlet boundary results obtained in Section 3.5.2.

3.4. Well-posedness

Before we discuss the stability, uniqueness, and convergence of PSOR, it is crucial to prove that A_n is a non-singular M-matrix [22]. This will facilitate everything that follows in the rest of this subsection. We require three conditions: non-positivity of both the lower diagonal and upper diagonal elements, positivity of the diagonal elements, and strict dominance of the diagonal elements.

Let us start with non-positivity of the lower diagonal elements. Any non-zero entry is $(A_n)_{i,i-1} = -\vartheta \Delta t(\alpha_i - \beta_i)$. In order for this to be non-positive, we require $\alpha_i > \beta_i$. By a simple rearranging, we see that

$$\frac{\sigma^2 s_i^2}{2(\Delta s)^2} > \frac{(r-q)s_i}{2\Delta s} \implies s_i > \frac{\Delta s(r-q)}{\sigma^2} \tag{3.4.1}$$

This is true, given that we have discretised our grid such that Δs is sufficiently small. Hence, $(A_n)_{i,i-1}$. Now, suppose that we observe the upper diagonal elements; any non-zero entry is $(A_n)_{i,i+1} = -\vartheta \Delta t(\alpha_i + \beta_i)$. Since $\alpha_i > 0$, this is trivially true. It is also easy to see that any diagonal element $(A_n)_{i,i} = 1 + \vartheta \Delta t(2\alpha_i + r) > 0$. In any row i, by subtracting the sum of any off-diagonal elements from the diagonal element is

$$(A_n)_{i,i} - (|(A_n)_{i,i-1}| + |(A_n)_{i,i+1}|) = 1 + \vartheta \Delta t r > 0 \tag{3.4.2}$$

Hence, A_n is a non-singular M-matrix and $A_n^{-1} \geq 0$ entry-wise.

3.4.1. Stability

To argue for the stability of our solution, we will construct a discrete supersolution to show that $||\boldsymbol{V}^n||_\infty$ is bounded by some function dependent on r, T, and Ψ for all $n \in \{0, \ldots, N\}$. First, it is useful to note that $(A_h \mathbf{1})_i = -r$ for all $i = 1, \ldots M+1$. Let $c := ||\boldsymbol{\Psi}||_\infty$ and propose that $\{\boldsymbol{W}^n\}$ given by $\boldsymbol{W}^n := c\mathbf{1} = (c, \ldots, c)^T$ is a discrete supersolution with $\boldsymbol{W}^n \geq \boldsymbol{\Psi}$ for all $n \in \{0, \ldots, N\}$. Plugging $\boldsymbol{W}^n$ into (3.2.8), we see that it indeed satisfies the inequality

$$\boldsymbol{W}^n - \boldsymbol{W}^{n+1} - \Delta t[\vartheta A_h \boldsymbol{W}^n + (1-\vartheta) A_h \boldsymbol{W}^{n+1}] = \Delta t c r \mathbf{1} \geq \mathbf{0} \tag{3.4.3}$$

Hence, $\{\boldsymbol{W}^n\}$ is a supersolution of the discrete variational inequality. Since the discrete operator A_n is monotone, the projection $\boldsymbol{V}^n = \max\{\boldsymbol{\Psi}, \tilde{\boldsymbol{V}}^n\}$ is also monotone. Suppose that now there exists $\{\boldsymbol{U}^n\}$, satisfying the same discrete inequality as $\{\boldsymbol{W}^n\}$ with $\boldsymbol{U}^{n+1} \leq \boldsymbol{W}^{n+1}$. Both satisfy $\boldsymbol{U}^n, \boldsymbol{W}^n \geq \boldsymbol{\Psi}$. Thus, $\boldsymbol{U}^n \leq \boldsymbol{W}^n$.

At maturity T, we have $\boldsymbol{V}^N = \boldsymbol{\Psi} \leq c\mathbf{1} = \boldsymbol{W}^N$. Because we have shown that $\{\boldsymbol{W^n}\}$ is a supersolution and the scheme is monotone, it is necessarily true by backward induction that $\boldsymbol{V}^n \leq \boldsymbol{W}^n$ for all $n \in \{0, \ldots, N\}$ and hence

$$||\boldsymbol{V}^n||_\infty \leq ||\boldsymbol{W}^n||_\infty = ||\boldsymbol{\Psi}||_\infty \quad \forall n \in \{0, \ldots, N\} \tag{3.4.4}$$

Hence, the Crank-Nicolson method with Rannacher startup yields a ℓ_∞-stable solution [16]. In particular, it is unconditionally stable in time and space, with the mesh condition $\alpha_i \geq |\beta_i|$.

3.4.2. Existence and uniqueness

Theorem 3.4.1. *Given a real matrix M and a vector $\boldsymbol{q}$, the linear complementarity problem is formulated as follows*

$$\boldsymbol{z} \geq 0, \quad \boldsymbol{w} := M\boldsymbol{z} + \boldsymbol{q}, \quad \boldsymbol{z}^T\boldsymbol{w} = 0 \tag{3.4.5}$$

If M is such that the LCP has a solution for every $\boldsymbol{q}$, then M is a Q-matrix. Furthermore, if M is such that the solution to the LCP is unique for every $\boldsymbol{q}$, then M is a P-matrix.

[19] To apply Theorem 3.4.1, we must show that the LCP in (3.2.10) can be written in this form. We will do so simply by a substitution of variables. Let $\boldsymbol{u}_n := \boldsymbol{V}^n - \boldsymbol{\Psi}$. Then, our LCP becomes

$$\boldsymbol{u}_n \geq 0, \quad \boldsymbol{w}_n := A_n\boldsymbol{u}_n + (A_n\boldsymbol{\Psi} - \boldsymbol{b}_n) \geq 0, \quad \boldsymbol{u}_n^T\boldsymbol{w}_n = 0 \tag{3.4.6}$$

It suffices to show that if A_n is a P-matrix, then there exists a unique solution for every $\boldsymbol{b}_n$. Let $\mathcal{P}$ denote the set of all P-matrices, and $\mathcal{M}$ denote the set of all non-singular M-matrices. However, it is true that $\mathcal{M} \subseteq \mathcal{P}$. We have already shown that $A_n \in \mathcal{M}$. Thus, $A_n \in \mathcal{P}$, and so there must exist a unique solution for every $\boldsymbol{b}_n$.

3.4.3. Convergence of PSOR

For simplicity of notation, we rewrite $A_n := A$, $\boldsymbol{V}^n := \boldsymbol{V}$, and $\boldsymbol{b}_n := \boldsymbol{b}_n$, so that the LCP becomes

$$\boldsymbol{V} \geq \boldsymbol{\Psi}, \quad A\boldsymbol{V} \geq \boldsymbol{b}, \quad (A\boldsymbol{V} - \boldsymbol{b})^T(\boldsymbol{V} - \boldsymbol{\Psi}) = 0 \tag{3.4.7}$$

To prove the convergence of PSOR, we will make use of the Banach fixed-point theorem [23]. First, we introduce the definition of a contraction mapping, as this will be needed in the theorem.

Definition 3.4.2. For a metric space (X, d), $T : X \to X$ is a contraction mapping on X if there exists $q \in [0, 1]$ such that

$$d(T(x), T(y)) \leq qd(x, y) \quad \forall x, y \in X \tag{3.4.8}$$

Theorem 3.4.3. *(Banach fixed-point theorem) Let (X, d) be a non-empty, complete metric space with a contraction mapping $T : X \to X$. Then T admits a unique fixed point $x^\star$ in X. Furthermore, $x^\star$ can be found starting with an arbitrary $x_0 \in X$ and defining a sequence $(x_n)_{n\in\mathbb{N}}$ given by $x_n = T(x_{n-1})$ for $n \geq 1$. Then, $\lim_{n\uparrow\infty} x_n = x^\star$.*

It would be great if we could characterise PSOR as a contraction mapping T, so that we can find the unique fixed point. First, let us introduce the SOR update and projection as separate transformations; PSOR is a composition of these two transformations. Define $S : \boldsymbol{V}^{(k)} \mapsto \tilde{\boldsymbol{V}}^{(k+1)}$ as the SOR update given by

$$S(\boldsymbol{V}^{(k)}) = (D + \omega L)^{-1}\left(\omega\boldsymbol{b} - [\omega U + (\omega - 1)D]\boldsymbol{V}^{(k)}\right) = L_\omega\boldsymbol{V}^{(k)} + \boldsymbol{c} \tag{3.4.9}$$

and define $P : \tilde{\boldsymbol{V}}^{(k+1)} \mapsto \boldsymbol{V}^{(k+1)}$ given by

$$P(\tilde{\boldsymbol{V}}_i^{(k+1)}) = \max\{\boldsymbol{\Psi}_i, \tilde{\boldsymbol{V}}_i^{(k+1)}\}, \quad i = 0, \ldots, M \tag{3.4.10}$$

Then, PSOR is the transformation $T \equiv P \circ S : \boldsymbol{V}^{(k)} \mapsto \boldsymbol{V}^{(k+1)}$. We claim the following:

$$\boldsymbol{V}^\star \text{ is a fixed point of } T \quad \iff \quad \boldsymbol{V}^\star \text{ solves the LCP} \tag{3.4.11}$$

If we can show this to be true, and that T is a contraction mapping, then the unique fixed point $\boldsymbol{V}^\star$ must also be the unique solution to the LCP.

For ($\Leftarrow$), we suppose that $\boldsymbol{V}^\star$ is a solution of the LCP, and our goal is to show that $\boldsymbol{V}^\star$ is necessarily a fixed point under T. We divide this into the continuation and exercise region. In the continuation region, $\boldsymbol{V}^\star > \boldsymbol{\Psi}$. By construction of SOR, iterating a known solution can only yield the solution again. Hence, $S(\boldsymbol{V}^\star) = \boldsymbol{V}^\star$. Since $\boldsymbol{V}^\star > \boldsymbol{\Psi}$, the projection P also returns the same known solution. Thus, $T(\boldsymbol{V}^\star) = \boldsymbol{V}^\star$ and $\boldsymbol{V}^\star$ is a fixed point here. In the exercise region, $\boldsymbol{V}^\star = \boldsymbol{\Psi}$. Define the residual $r(\boldsymbol{V}^\star) := A\boldsymbol{V}^\star - \boldsymbol{b} \geq 0$. Then, we write the SOR update as $S(\boldsymbol{V}^\star) = \boldsymbol{V} - (D + \omega L)^{-1} r(\boldsymbol{V}^\star)$. The change after this iteration is, therefore, $-(D + \omega L)^{-1} r(\boldsymbol{V}^\star) \leq 0$, since $(D + \omega L)$ is also an M-matrix, so that the entries of its inverse are non-negative. So, $S(\boldsymbol{V}^\star) \leq \boldsymbol{V}^\star$. If the SOR update decreased $V^\star$, then the projection will surely fix it back to $\boldsymbol{\Psi}$. Since $\boldsymbol{V}^\star = \boldsymbol{\Psi}$, we have $T(\boldsymbol{V}^\star) = \boldsymbol{V}^\star$. Thus, if $\boldsymbol{V}^\star$ solves the LCP, then $\boldsymbol{V}^\star$ is a fixed point of T.

For ($\Rightarrow$), suppose that $\boldsymbol{V}^\star$ is a fixed point of T, and we want to show that $\boldsymbol{V}^\star$ solves the LCP. It is evident $T(\boldsymbol{V}^\star) = \boldsymbol{V}^\star \implies \boldsymbol{V}^\star \geq \boldsymbol{\Psi}$. Again, we divide the problem into the continuation and exercise regions. When $\boldsymbol{V}^\star > \boldsymbol{\Psi}$, the projection P does nothing, and so $P(S(\boldsymbol{V}^\star)) = S(\boldsymbol{V}^\star)$. For the SOR update, the fixed point, $\boldsymbol{V}^\star$, of S is necessarily the solution of $A\boldsymbol{V} = \boldsymbol{b}$ and hence the LCP is satisfied here. If $\boldsymbol{V}^\star = \boldsymbol{\Psi}$, then the projection may not have been redundant, and so $S(\boldsymbol{V}^\star) \leq \boldsymbol{\Psi}$. These conditions imply that, for the fixed point $\boldsymbol{V}^\star$, $A\boldsymbol{V} \geq \boldsymbol{b}$ is satisfied. Between both the exercise and the continuation region, we have set up exactly the LCP defined in (3.4.7). Thus, if $\boldsymbol{V}^\star$ is a fixed point of T, then it necessarily solves the LCP.

We mentioned above that the SOR map can be written as $S(\boldsymbol{V}) = L_\omega \boldsymbol{V} + \boldsymbol{c}$, where L_ω is the SOR iteration matrix. It is known that for any $\omega \in (0, 2)$ with a non-singular M-matrix A, we have the spectral radius $\rho(L_\omega) < 1$ [21]. The proof of the following is beyond the scope of this paper; however, it can be shown that subsequently there exists a vector norm $||\cdot||_*$ in $\mathbb{R}^{M+1}$ such that the induced matrix norm satisfies $||L_\omega||_* < 1$. So, for any $\boldsymbol{V}, \boldsymbol{W} \in \mathbb{R}^{M+1}$

$$||S(\boldsymbol{V}) - S(\boldsymbol{W})||_* = ||L_\omega(\boldsymbol{V} - \boldsymbol{W})||_* \leq ||L_\omega||_* ||\boldsymbol{V} - \boldsymbol{W}||_* \leq q||\boldsymbol{V} - \boldsymbol{W}||_* \tag{3.4.12}$$

with $q \in (0, 1)$. Thus, S is a contraction mapping on $(\mathbb{R}^{M+1}, ||\cdot||_*)$.

Next, we aim to show that the projection P is non-expansive. Under the ℓ_∞-norm, for each component i, we have $|\max\{\boldsymbol{\Psi}_i, \boldsymbol{V}_i\} - \max\{\boldsymbol{\Psi}_i, \boldsymbol{W}_i\}| \leq |\boldsymbol{V}_i - \boldsymbol{W}_i|$, so it is true that P is non-expansive under $||\cdot||_\infty$

$$||P(\boldsymbol{V}) - P(\boldsymbol{W})||_\infty \leq ||\boldsymbol{V} - \boldsymbol{W}||_\infty \tag{3.4.13}$$

In a finite-dimensional space, all norms are equivalent [23], so there must exist constants $\delta, \varepsilon > 0$ such that for all $\boldsymbol{V}$, one has $\delta||\boldsymbol{V}||_* \leq ||\boldsymbol{V}||_\infty \leq \varepsilon||\boldsymbol{V}||_*$. Thus, for any $\boldsymbol{V}, \boldsymbol{W}$

$$||P(\boldsymbol{V}) - P(\boldsymbol{W})||_* \leq \gamma||\boldsymbol{V} - \boldsymbol{W}||_* \quad \gamma := \frac{\varepsilon}{\delta} \geq 1 \tag{3.4.14}$$

We are now able to use the idea that S is a contraction mapping and that P is 1-Lipschitz continuous in

$||\cdot||_*$ to show that T is also a contraction mapping

$$||T(\boldsymbol{V}) - T(\boldsymbol{W})||_* = ||P(S(\boldsymbol{V})) - P(S(\boldsymbol{W}))||_* \leq \gamma ||S(\boldsymbol{V}) - S(\boldsymbol{W})||_* \leq q\gamma ||\boldsymbol{V} - \boldsymbol{W}||_* \tag{3.4.15}$$

Since we can choose the norm $||\cdot||_*$, we pick one such that $q\gamma \leq 1$. Thus, T is a contraction mapping on $(\mathbb{R}^{M+1}, ||\cdot||_*)$. We now simply apply the Banach fixed-point theorem (3.4.3) to this complete metric space to claim that $\boldsymbol{V}^\star$ is the unique fixed point for T. From the if and only if statement in (3.4.11), $\boldsymbol{V}^\star$ must also be the unique solution to the LCP. Hence, for any arbitrary starting guess $\boldsymbol{V}^{(0)}$

$$\boldsymbol{V}^{(k)} \longrightarrow \boldsymbol{V}^\star \quad \text{as} \quad k \longrightarrow \infty \tag{3.4.16}$$

and so we can say that PSOR converges.

3.5. Boundary conditions and behaviour

3.5.1. Upper and lower bounds

To compute the lower bounds for the American call price $C^A(t,s)$ and the American put price $P^A(t,s)$, we begin by enforcing the inequality $V^A(t,s) \geq \Psi(s)$. Let $V^E(t,s)$ denote the price of a European call or put option at time t. Then, if the European price is feasible, that is, $\tau = T$, immediate exercise is optimal, and so $V^A(t,s) \geq V^E(t,s)$. Using the bounds defined in Section 2.4.2, we write the lower bounds as

$$C^A(t,s) \geq \max\{C^E(t,s), (Se^{-q\tau} - Ke^{-r\tau})^+\}, \quad P(t,s) \geq \max\{P^E(t,s), (Ke^{-r\tau} - se^{-q\tau})^+\} \tag{3.5.1}$$

[13] To compute the upper bounds, we can use the trivial bounds of the payoff function $(s-K)^+ \leq s$ and $(K-s)^+ \leq K$, together with $e^{-r(\tau - t)} \leq 1$ for any $r \geq 0$. Recall the optimal stopping problem defined in (3.1.5). Then

$$C^A(t,s) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}[e^{-r(\tau-t)}(S_\tau - K)^+ | S_t = s] \leq s, \quad P^A(t,s) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}[e^{-r(\tau-t)}(K - S_\tau)^+ | S_t = s] \leq K \tag{3.5.2}$$

3.5.2. Asymptotic behaviour

We saw that the vector $\boldsymbol{b}_n$ defined in (3.3.7) depends on $V_0^{n+1} = V(t_{n+1}, s_0)$ and $V_M^{n+1} = V(t_{n+1}, s_{\max})$. In order to solve the LCP, it is necessary to define the edge cases for any node that attains an underlying asset value of s_0 or $s_{\max}$. At $s_0 = 0$, we take $C^A(t,0) = 0$ and $P^A(t,0) = K$. An important note here is that if an American put reaches any s_0 node, immediate exercise is optimal.

Suppose that we discretise our grid appropriately, with $s_{\max}$ sufficiently large so that the following observation for $s \to \infty$ is reflective of our decision for the value of the option at these nodes. We have, for an American put option $P^A(t, s_{\max})$

$$P^A(t, s_{\max}) \approx \lim_{s \to \infty} \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}[e^{-r(\tau-t)}(K - S_\tau)^+ | S_t = s] = 0 \tag{3.5.3}$$

For the American call option $C^A(t, s_{\max})$, we divide into two cases: $q = 0$ and $q > 0$. If $q = 0$, early exercise is not optimal [13] and therefore our option is identical to its European counterpart. That is,

$C^A(t, s_{\max}) = C^E(t, s_{\max})$. If $q > 0$, we introduce a continuous carry benefit that is forfeited by holding the call. Hence, early exercise could be optimal. Subsequently, we define the price of C^A at any node that achieves $s_{\max}$ by

$$C^A(t, s_{\max}) = \begin{cases} s_{\max} - Ke^{-r\tau}, & q = 0 \\ s_{\max} - K, & q > 0 \end{cases} \tag{3.5.4}$$

3.6. Greeks

To compute the Greeks for an American option, we do not necessarily have a closed-form solution like we did for its European counterpart. Fortunately, our discretised time-space grid allows us to compute $\Delta(t, s) = \partial_S V(t, s)$, $\Gamma(t, s) = \partial_{SS} V(t, s)$ and $\Theta(t, s) = \partial_t V(t, s)$ using finite difference methods [16]. For delta and gamma, we use a central differences estimator. For theta, we use a backward differences estimator to match the natural passage of time. Then, for nodes $(t_n, s_m) \in [0, T] \times [0, s_{\max}]$

$$\Delta(t_n, s_m) \approx \frac{V_{m+1}^n - V_{m-1}^n}{2\Delta s}, \quad \Gamma(t_n, s_m) \approx \frac{V_{m+1}^n - 2V_m^n + V_{m-1}^n}{(\Delta s)^2}, \quad \Theta(t_n, s_m) \approx \frac{V_m^{n-1} - V_m^n}{\Delta t} \tag{3.6.1}$$

Since the grid we constructed is independent of the annualised volatility σ and risk-free interest rate r, we must employ a bump and revalue method [18] for $\mathcal{V}(t, s) = \partial_\sigma V(t, s)$ and $\rho(t, s) = \partial_r V(t, s)$. Let h_θ denote a small positive bump in the parameter $\theta \in \{\sigma, r\}$ and let $V_\theta^\pm(t_n, s_m)$ denote the value of the American option recomputed with $\theta \pm h_\theta$. Then we can approximate the vega and rho by

$$\mathcal{V}(t_n, s_m) \approx \frac{V_\sigma^+(t_n, s_m) - V_\sigma^-(t_n, s_m)}{2h_\sigma}, \quad \rho(t_n, s_m) \approx \frac{V_r^+(t_n, s_m) - V_r^-(t_n, s_m)}{2h_r} \tag{3.6.2}$$

A. Alternative Greek Estimators

In Section 1.5, we introduced the idea of computing the Greeks using finite difference methods with common random numbers. As briefly mentioned prior, it is not always possible to compute the analytic or finite difference Greeks, particularly for exotic claims with discontinuous payoffs, such as barrier or digital options. In these cases, using a finite difference method often becomes numerically unstable, unreasonably computationally expensive, or structurally unreliable. We introduce the pathwise and likelihood-ratio Greeks in a Monte Carlo setting to counteract the problems that otherwise would occur [7].

A.1. Pathwise (infinitesimal perturbation analysis)

Let $X_\theta = G(\theta, Z) \in \mathbb{R}^d$ be the random object simulated, with θ-independent base noise $Z \in \mathbb{R}^m$, which is almost surely differentiable in θ. Let $\Psi : \mathbb{R}^d \to \mathbb{R}$ denote the payoff function that is almost surely differentiable for all $x \in X_\theta$ and of at most polynomial growth. We define the discounted payoff as a function of the state

$$h(x) := e^{-rT}\Psi(x), \quad V(\theta) := \mathbb{E}^{\mathbb{Q}}[h(X_\theta)] \tag{A.1.1}$$

Assume that there exists $\varepsilon > 0$ and an integrable random variable $M(Z)$ with finite expectation such that $|\partial_\theta h(X_\vartheta)| \leq M(Z)$ for all $\vartheta \in (\theta - \varepsilon, \theta + \varepsilon)$. Then, the pathwise Greek is

$$\partial_\theta V(\theta) = \mathbb{E}^{\mathbb{Q}}[\nabla h(X_\theta) \cdot \partial_\theta G(\theta, Z] = \int_{\mathbb{R}^m} \nabla h(G(\theta, z)) \cdot \partial_\theta G(\theta, z)\phi(z)dz \tag{A.1.2}$$

A.2. Likelihood-ratio

For non-differentiable payoff functions $\Psi : \mathbb{R}^d \to \mathbb{R}$, the pathwise estimator may fail us. Instead, we propose that the simulated path $X_\theta \in \mathbb{R}^d$ is a random object with density function f_θ that is differentiable almost everywhere in θ and

$$\mathbb{E}^{\mathbb{Q}}[|h(X_\theta)\partial_\theta \log(f_\theta(X_\theta))|] < \infty \tag{A.2.1}$$

If h is not explicitly θ-dependent, then the claim $V(\theta) = \mathbb{E}^{\mathbb{Q}}[h(X_\theta)]$, has the likelihood-ratio Greek

$$\partial_\theta V(\theta) = \int_{\mathbb{R}^d} h(x)\partial_\theta f_\theta(x)dx = \int_{\mathbb{R}^d} h(x) f_\theta(x)\partial_\theta \log(f_\theta(x))dx = \mathbb{E}^{\mathbb{Q}}[h(X_\theta)\partial_\theta \log(f_\theta(X_\theta))] \tag{A.2.2}$$

The main takeaway here is that the payoff Ψ no longer needs to be almost surely differentiable. This is particularly useful for the aforementioned digital and barrier options.

B. Algorithms & Pseudocode

Beneath are the algorithms used for the iterative-based methods we used throughout the paper, namely Newton's method with the bisection fallback for finding the implied volatility and PSOR for solving the Crank-Nicolson finite difference scheme of the American option price [15][21][19][20].

B.1. Implied volatility: Newton with bisection fallback

Algorithm 1 Hybrid implied volatility solver

Input: Market price V^{mkt}, European option parameters (S_t, K, r, q, τ), price and volatility tolerances $(\varepsilon_V, \varepsilon_\sigma)$, maximum number of iterations $K_{\max}$

Output: σ_{imp} (or a failure flag if no-arbitrage bounds violated)

Compute no-arbitrage price bounds $(V_{\min}, V_{\max})$
if $V^{\text{mkt}} \notin [V_{\min}, V_{\max}]$ **then**
    return FAIL
end if
Bracket: set $\sigma_\ell \leftarrow 0$, choose $\sigma_u > 0$ and grow geometrically until $f(\sigma_u) \geq 0$
Initialise $\sigma \leftarrow \text{clip}(\sigma_0; [\sigma_\ell, \sigma_u])$
for $k = 1$ to $K_{\max}$ **do**
    $f(\sigma) \leftarrow V^{BS}(\sigma) - V^{\text{mkt}}, \quad \nu(\sigma) \leftarrow \partial_\sigma V^{BS}(\sigma)$
    if $|f(\sigma)| \leq \varepsilon_V$ **then**
        return σ
    end if
    if $\nu(\sigma)$ is too small **then**
        break
    end if
    $\sigma^{\text{new}} \leftarrow \sigma - f(\sigma)/\nu(\sigma)$
    if $\sigma^{\text{new}} \notin [\sigma_\ell, \sigma_u]$ **then**
        break
    end if
    Update bracket using monotonicity of f:
    if $f(\sigma^{\text{new}}) > 0$ **then**
        $\sigma_u \leftarrow \sigma^{\text{new}}$
    else
        $\sigma_\ell \leftarrow \sigma^{\text{new}}$
    end if
    $\sigma \leftarrow \sigma^{\text{new}}$
end for
return BISECTION$(f, \sigma_\ell, \sigma_u, \varepsilon_\sigma, K_{\max})$

Algorithm 2 Bisection for a monotone root (used as fallback)

Input: Monotone f, volatility bracket $[\sigma_\ell, \sigma_u]$ with $f(\sigma_\ell) \leq 0 \leq f(\sigma_u)$, volatility tolerance ε_σ, maximum number of iterations $K_{\max}$

Output: Approximate root $\hat{\sigma}$

for $k = 1$ to $K_{\max}$ **do**
    $\sigma_{\text{mid}} \leftarrow \frac{1}{2}(\sigma_\ell + \sigma_u)$
    if $|\sigma_u - \sigma_\ell| < \varepsilon_\sigma$ **then**
        return σ_{mid}
    end if
    if $f(\sigma_{\text{mid}}) > 0$ **then**
        $\sigma_u \leftarrow \sigma_{\text{mid}}$
    else
        $\sigma_\ell \leftarrow \sigma_{\text{mid}}$
    end if
end for
return $\frac{1}{2}(\sigma_\ell + \sigma_u)$

B.2. LCP: Projected successive over-relaxation (PSOR)

Algorithm 3 PSOR for the linear complementarity problem

Input: Matrix A, vector $\boldsymbol{b}$, payoff vector $\boldsymbol{\Psi}$, relaxation factor $\omega \in (1, 2)$, tolerance ε, maximum number of iterations $K_{\max}$

Output: Approximate solution $\boldsymbol{x}$ to LCP: $\boldsymbol{x} \geq \boldsymbol{\Psi}$, $A\boldsymbol{x} \geq \boldsymbol{b}$, $(A\boldsymbol{x} - \boldsymbol{b})^\top(\boldsymbol{x} - \boldsymbol{\Psi}) = 0$

Initialise $\boldsymbol{x}^{(0)} \leftarrow \boldsymbol{\Psi}$ (or previous time-step solution)
for $k = 0$ to $K_{\max} - 1$ **do**
    $x^{\text{old}} \leftarrow x^{(k)}$
    for $i = 1$ to n **do**
        Compute sequential SOR update:
        $\tilde{x}_i \leftarrow (1-\omega)x_i^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij}\tilde{x}_j^{(k+1)} - \sum_{j>i} a_{ij}x_j^{(k)}\right)$
        Project onto feasible set: $x_i^{(k+1)} \leftarrow \max\{\Psi_i, \tilde{x}_i\}$
    end for
    Compute stopping metric $\|x^{(k+1)} - x^{\text{old}}\|_\infty$
    if $\|x^{(k+1)} - x^{\text{old}}\|_\infty < \varepsilon$ **then**
        return $x^{(k+1)}$
    end if
end for
return $x^{(K_{\max})}$

B.3. American pricing: Rannacher + Crank-Nicolson + PSOR

Algorithm 4 American option pricing by finite differences with Rannacher startup

Input: Grid $\{t_n\}_{n=0}^{N}$, $\{s_m\}_{m=0}^{M}$, payoff $\boldsymbol{\Psi}$, matrix A_h, Dirichlet boundary conditions, PSOR parameters
Output: Discrete American price V^0

```
Set terminal condition: V^N ← Ψ
Rannacher startup (two half-steps):
for n = N − 1 down to N − 2 do
    Form LCP: (I − ½ΔtA_h)V^n ≥ V^{n+1/2}, V^n ≥ Ψ
    Assemble (A_n, b_n) including boundary contributions
    V^n ← PSOR(A_n, b_n, Ψ, ω, ε, K_max)
end for
Crank-Nicolson steps:
for n = N − 2 down to 0 do
    Form LCP: (I − ½ΔtA_h)V^n ≥ (I + ½ΔtA_h)V^{n+1}, V^n ≥ Ψ
    Assemble (A_n, b_n) including boundary contributions
    V^n ← PSOR(A_n, b_n, Ψ, ω, ε, K_max)
end for
return V^0
```

C. Supplementary Figures

Collected below are the representative diagnostic plots supporting the theoretical and numerical results in Chapters 2-3. The full set of experiments, parameter choices, and reproducible code can be found via my GitHub link here: `options-pricing`. The figures here are included to make this paper self-contained for interpretation and validation.

C.1. European no-arbitrage checks

No-arbitrage shape constraints in strike

For fixed time until maturity t and underlying asset price S, European call prices should be non-increasing and convex in strike K. European put prices should be non-decreasing and convex in strike K. These reflect the standard no-arbitrage bounds and monotonicity/convexity properties.

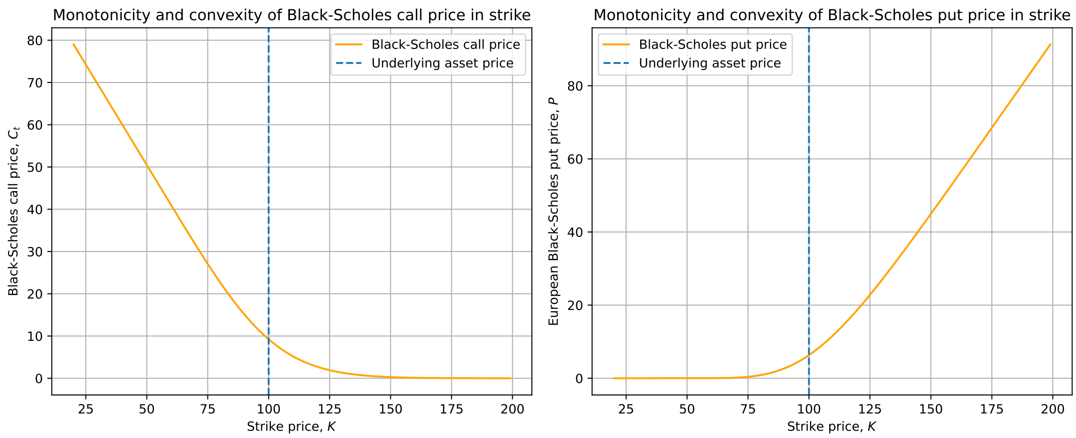

Figure C.1.: European option consistency checks in strike K: monotonicity and convexity. In particular, call prices decrease with K, put prices increase with K, and both exhibit convexity, consistent with no-arbitrage shape restrictions.

Asymptotic behaviour in underlying asset price

The European Black-Scholes prices of deep OTM and ITM quotes satisfy the expected limiting behaviour as $S \to 0$ and as $S \to \infty$, which is useful both as a sanity check and for motivating boundary conditions in PDE discretisations.

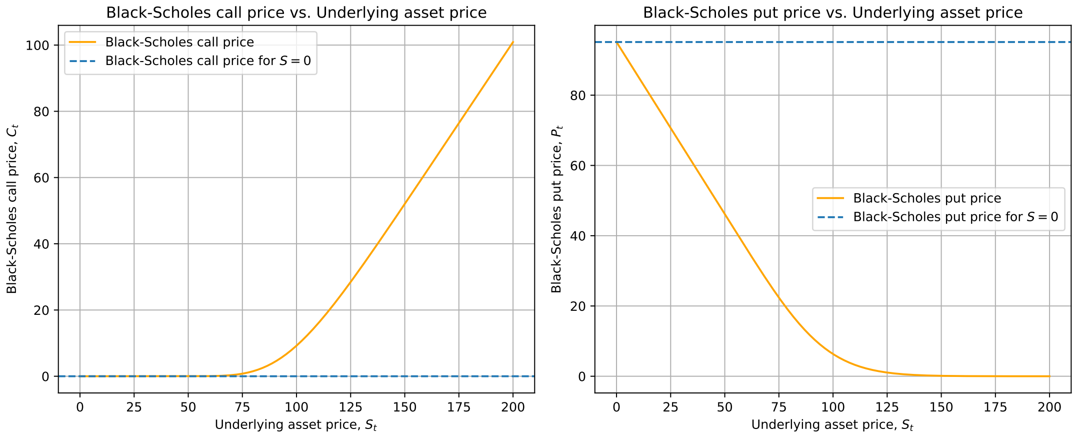

Figure C.2.: Asymptotic behaviour of European option prices as $S \to 0$ and for large S. The curves approach the standard deep OTM/ITM limits predicted by Black-Scholes.

C.2. Monte Carlo validation

Statistical convergence

Monte Carlo estimators converge at the canonical $\mathcal{O}(N^{-1/2})$ rate. The plot below shows convergence to the closed-form Black-Scholes benchmark as the number of simulated paths N increases, together with uncertainty quantification via confidence intervals as error bars.

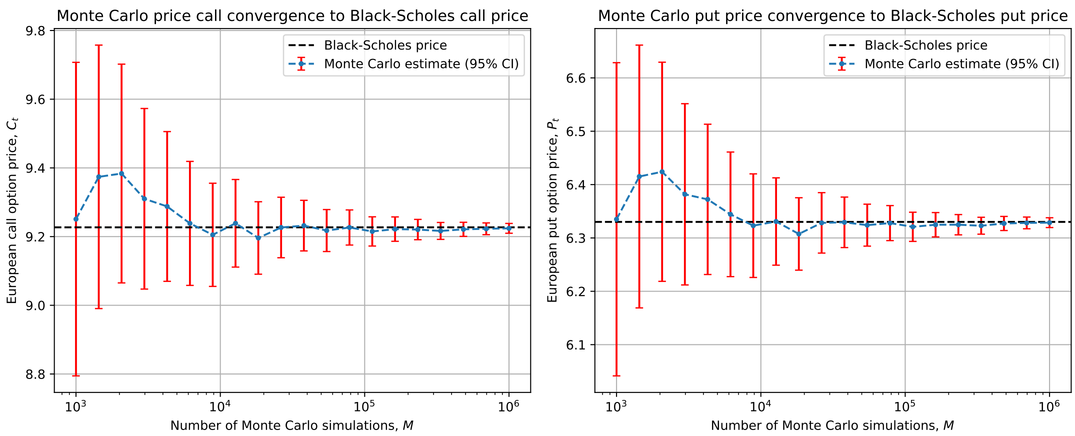

Figure C.3.: Monte Carlo convergence to the Black-Scholes benchmark as the number of simulations N increases. Error bars correspond to confidence intervals computed from the sample variance of discounted payoffs.

C.3. American free boundary curve

Exercise region and free boundary

For American options, the optimal exercise decision induces a continuation region and an exercise region, separated by a time-dependent free boundary. The figure below visualises the discretised free boundary recovered after solving for the American price via PSOR.

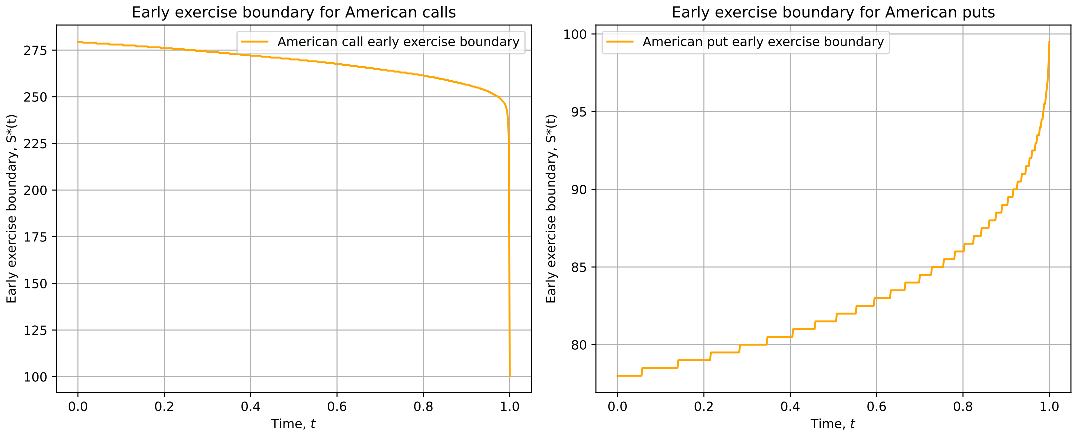

Figure C.4.: Discretised free boundary for an American vanilla option under the finite-difference LCP formulation. The boundary separates the exercise region, where $V_m^n = \Psi_m$, from the continuation region, where the discretised PDE holds.

References

[1] Steven E. Shreve. *Stochastic Calculus for Finance II: Continuous-Time Models*. Springer Finance. Springer, 2004.

[2] Tomas Björk. *Arbitrage Theory in Continuous Time*. 4th ed. Oxford University Press, 2020.

[3] Freddy Delbaen and Walter Schachermayer. *The Mathematics of Arbitrage*. Springer Finance. Springer, 2006.

[4] J. Michael Harrison and Stanley R. Pliska. “A Stochastic Calculus Model of Continuous Trading: Complete Markets”. In: *Stochastic Processes and their Applications* 15.3 (1983), pp. 313–316.

[5] Ioannis Karatzas and Steven E. Shreve. *Brownian Motion and Stochastic Calculus*. 2nd ed. Vol. 113. Graduate Texts in Mathematics. Springer, 1991.

[6] Bernt Øksendal. *Stochastic Differential Equations: An Introduction with Applications*. 6th ed. Universitext. Springer, 2003.

[7] Paul Glasserman. *Monte Carlo Methods in Financial Engineering*. Vol. 53. Stochastic Modelling and Applied Probability. Springer, 2004.

[8] Fischer Black and Myron Scholes. “The Pricing of Options and Corporate Liabilities”. In: *Journal of Political Economy* 81.3 (1973), pp. 637–654.

[9] Robert C. Merton. “Option Pricing When Underlying Stock Returns Are Discontinuous”. In: *Journal of Financial Economics* 3.1–2 (1976), pp. 125–144.

[10] Steven L. Heston. “A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options”. In: *Review of Financial Studies* 6.2 (1993), pp. 327–343.

[11] Robert C. Merton. “Theory of Rational Option Pricing”. In: *The Bell Journal of Economics and Management Science* 4.1 (1973), pp. 141–183.

[12] Fischer Black. “The Pricing of Commodity Contracts”. In: *Journal of Financial Economics* 3.1–2 (1976), pp. 167–179.

[13] John C. Hull. *Options, Futures, and Other Derivatives*. 10th ed. Pearson, 2018.

[14] Jim Gatheral. *The Volatility Surface: A Practitioner’s Guide*. Wiley Finance. Wiley, 2006.

[15] Richard L. Burden and J. Douglas Faires. *Numerical Analysis*. 9th ed. Brooks/Cole, Cengage Learning, 2010.

[16] Daniel J. Duffy. *Finite Difference Methods in Financial Engineering: A Partial Differential Equation Approach*. John Wiley & Sons, 2006.

[17] Paul Wilmott, Sam Howison, and Jeff Dewynne. *The Mathematics of Financial Derivatives: A Student Introduction*. Cambridge University Press, 1995.

[18] Domenico Tavella and Curt Randall. *Pricing Financial Instruments: The Finite Difference Method*. John Wiley & Sons, 2000.

[19] Richard W. Cottle, Jong-Shi Pang, and Richard E. Stone. *The Linear Complementarity Problem*. Society for Industrial and Applied Mathematics (SIAM), 2009.

[20] Rolf Rannacher. “Finite Element Solution of Diffusion Problems with Irregular Data”. In: *Numerische Mathematik* 43 (1984), pp. 309–327.

[21] Richard S. Varga. *Matrix Iterative Analysis*. 2nd ed. Springer, 2000.

[22] Abraham Berman and Robert J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. Society for Industrial and Applied Mathematics (SIAM), 1994.

[23] Erwin Kreyszig. *Introductory Functional Analysis with Applications*. John Wiley & Sons, 1978.