Statistical Computing: Homework 6

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Question 1 Prove the following results about conjugate priors in Bayesian analysis.

(a) Beta distribution is the conjugate prior for the success probability parameter p of a geometric distribution. That is, let the prior of p be $Beta(\alpha,\ \beta)$. Given n independent and identically distributed random samples $X_1,\ \cdots,\ X_n$ from the geometric distribution with parameter p, then the posterior distribution of p is still Beta. Recall that the probability density function of $Beta(\alpha,\ \beta)$ is

$$f(y)\ =\ \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}y^{\alpha-1}(1-y)^{\beta-1},\ 0\le y\le 1\ and\ \alpha,\ \beta\ >0$$

Answer 1(a)

The poterior distribution of p is

$$\begin{aligned}
P(p\mid X_1,\ \cdots,X_n) &\propto \pi(p)f(X_1,\ \cdots,X_n|p)\\
&=\pi(p)\prod_{i=1}^{n}f(X_i|p)\\
&\propto p^{\alpha-1}(1-p)^{\beta-1}\prod_{i=1}^{n}(1-p)^{X_1}p\\
&=p^{\alpha+n-1}(1-p)^{\beta-1+\sum_i X_i}\\
&\rightsquigarrow Beta\left(\alpha+n,\ \beta+\sum_{i=1}^{n}X_i\right)
\end{aligned}$$

which is still Beta.

(b) Inverse Gamma (IG) distribution is the conjugate prior for variance parameter σ^2 of a normal distribution with known mean parameter μ_0. That is, let the prior of σ^2 be $IG(\alpha,\ \beta)$. Given n independent and identically distributed random samples $X_1,\ \cdots,X_n$ from $N\left(\mu_0,\ \sigma^2\right)$, then the posterior distribution of σ^2 is still IG. Recall that the probability density function of Inverse $Gamma(\alpha,\ \beta)$ is

$$f(y)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\left(\frac{1}{y}\right)^{\alpha+1}e^{-\frac{\beta}{y}},\ y>0\ and\ \alpha,\ \beta>0$$

Answer 1(b)

The posterior distribution of σ^2 is

$$\begin{aligned}
P\left(\sigma^2|X_1, \cdots, X_n\right) &\propto \pi\left(\sigma^2\right) f\left(X_1, \cdots, X_n|\sigma^2\right) \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)}\left(\frac{1}{\sigma^2}\right)^{\alpha+1} e^{-\frac{\beta}{\sigma^2}} \frac{1}{\left(\sqrt{2\pi\sigma^2}\right)^n} exp\left\{-\frac{\sum_i (X_i-\mu_0)^2}{2\sigma^2}\right\} \\
&\propto \left(\frac{1}{\sigma^2}\right)^{\alpha+\frac{n}{2}+1} e^{-\frac{2\beta+\sum_i (X_i-\mu_0)^2}{2\sigma^2}} \\
&\rightsquigarrow IG\left(\alpha+\frac{n}{2}, \beta+\frac{\sum_i (X_i-\mu_0)^2}{2}\right)
\end{aligned}$$

which is still IG.

Question 2 Consider the Bayesian estimation of the success probability parameter for a rare
event. Suppose n i.i.d. Bernoulli experiments with success probability $\theta \in [0, 1]$ are conducted. Then the number of successes y follows a binomial distribution $Bin(n, \theta)$. Our interest is in estimating θ. Take $Beta(a, b)$ as a prior for θ.

(a) Derive the posterior distribution $\theta|y$.

Answer 2(a)

The posterior distribution is

$$\begin{aligned}
P(\theta|y) &\propto \pi(\theta) f(y|\theta) \\
&\propto \theta^{\alpha-1}(1-\theta)^{\beta-1}\theta^y(1-\theta)^{n-y} \\
&= \theta^{\alpha+y-1}(1-\theta)^{n-y+\beta-1} \\
&\sim Beta(\alpha+y,\ n-y+\beta)
\end{aligned}$$

which is still Beta dstribution.

(b) Express the posterior mean of $\theta|y$ as a linear combination of the sample average $\overline{y} = \frac{y}{n}$ and the prior expectation of θ.

Answer 2(b)

the prior expectation of θ is $\mathbb{E}(\theta) = \frac{\alpha}{\alpha+\beta}$.

$$\begin{aligned}
\mathbb{E}(\theta|y) &= \frac{\alpha+y}{\alpha+\beta+n} \\
&= \frac{(\alpha+\beta)\mathbb{E}(\theta)+n\overline{y}}{\alpha+\beta+n} \\
&= \frac{\alpha+\beta}{\alpha+\beta+n}\mathbb{E}(\theta)+\frac{n}{\alpha+\beta+n}\overline{y}
\end{aligned}$$

(c) Comment on the effect of $\overline{y}$ on the shift of the posterior from the prior.

Answer 2(c)

if $\overline{y} > \mathbb{E}(\theta)$, i.e. the mean success ratio of the n i.i.d. Bernoulli experiments greater than the expected success probability of prior, which shows that the experiemnt result is more "positive" than prior guess, the expectation of posterior will increase;
if $\overline{y} < \mathbb{E}(\theta)$, the expectation of $\theta|y$ will decrease.