

DATA130004: Homework 3

Due in class on October 25, 2021

1. Exercises 5.12, and 5.14.
2. Given two random variables X and Y , prove the law of total variance

$$\text{var}(Y) = \text{E}\{\text{var}(Y|X)\} + \text{var}\{\text{E}(Y|X)\}.$$

Be explicit at every step of your proof.

3. Define $\theta = \int_A g(x) dx$, where A is a bounded set and $g \in \mathcal{L}_2(A)$. Let f be an importance function which is a density function supported on the set A .
 - (a) Describe the steps to obtain the importance sampling estimator $\hat{\theta}_n$, where n is the number of random samples generated during the process.
 - (b) Show that the Monte Carlo variance of $\hat{\theta}_n$ is

$$\text{var}(\hat{\theta}_n) = \frac{1}{n} \left\{ \int_A \frac{g^2(x)}{f(x)} dx - \theta^2 \right\}$$

- (c) Show that the *optimal* importance function f^* , i.e., the minimizer of $\text{var}(\hat{\theta}_n)$, is

$$f^*(x) = \frac{|g(x)|}{\int_A |g(x)| dx},$$

and derive the theoretical lower bound of $\text{var}(\hat{\theta}_n)$.