DATA130004: Homework 6

Due in class on December 6, 2021

1. Prove the following results about conjugate priors in Bayesian analysis.

   (a) Beta distribution is the conjugate prior for the success probability parameter p of a geometric distribution. That is, let the prior of p be $\text{Beta}(\alpha, \beta)$. Given n independent and identically distributed random samples $X_1, \ldots, X_n$ from the geometric distribution with parameter p, then the posterior distribution of p is still Beta. Recall that the probability density function of $\text{Beta}(\alpha, \beta)$ is

   $$f(y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}, \qquad 0 \le y \le 1 \text{ and } \alpha, \beta > 0.$$

   (b) Inverse Gamma (IG) distribution is the conjugate prior for variance parameter σ^2 of a normal distribution with known mean parameter μ_0. That is, let the prior of σ^2 be $\text{IG}(\alpha, \beta)$. Given n independent and identically distributed random samples $X_1, \ldots, X_n$ from $N(\mu_0, \sigma^2)$, then the posterior distribution of σ^2 is still IG. Recall that the probability density function of Inverse $\text{Gamma}(\alpha, \beta)$ is

   $$f(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} (1/y)^{\alpha+1} e^{-\beta/y}, \qquad y > 0 \text{ and } \alpha, \beta > 0.$$

2. Consider the Bayesian estimation of the success probability parameter for a rare event. Suppose n i.i.d. Bernoulli experiments with success probability $\theta \in [0, 1]$ are conducted. Then the number of successes y follows a binomial distribution $\text{Bin}(n, \theta)$. Our interest is in estimating θ. Take $\text{Beta}(a, b)$ as a prior for θ.

   (a) Derive the posterior distribution $\theta \mid y$.

   (b) Express the posterior mean of $\theta \mid y$ as a linear combination of the sample average $\bar{y} = y/n$ and the prior expectation of θ.

   (c) Comment on the effect of $\bar{y}$ on the shift of the posterior from the prior.