Systems Theory Volume I

Theory and Analysis of Linear Time Invariant Systems

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Milan Rother

Preface

This short script provides an introduction and overview of the use of the Laplace transformation for the analysis of linear time-invariant systems and ordinary differential equations. Elementary concepts of system theory, such as transfer functions, stability, impulse response, and step response are motivated and introduced. The presented methods enable the analysis of a linear system with respect to its stability, transient response behavior, and frequency response.

The concepts are primarily treated using single-input single-output systems, but without loss of generality. Some of the derivations should be understood more as motivations rather than strict proofs. In general, my goal in creating this short script was to clearly present some important concepts of linear system theory and the usefulness of the Laplace transformation. It was particularly important to me to support all conclusions and assertions with derivations, making them more accessible. Most intermediate steps are therefore deliberately presented in detail – as part of the path to understanding – and hopefully comprehensible. This document will continue to grow over time and is far from complete. Suggestions for better formulation and corrections are welcome.

Braunschweig, April 2021 Milan Rother

Structure of the Script

The first chapter deals with the description of linear time-invariant systems in the time domain and thus provides the fundamental structure and notation of the system, which is then assumed throughout the remainder. Then follows a general chapter on the Laplace transformation, which is quite detached from the other chapters and can be viewed more as a reference chapter. It provides important foundations and derivations related to the Laplace transformation. Some of the system analysis methods require the content of this chapter. The script concludes with two example chapters in which the presented methods are demonstrated. It is probably useful to switch back and forth between the examples and the corresponding general theory chapters.

Notation

The notation of mathematical symbols in this document is exemplified using the symbol a:

$a \in \mathbb{R}$	Real number
$\underline{a} \in \mathbb{C}$	Complex number, underline denotes complex-valued quantities
$a(t)$	Real-valued function with a real argument
$\underline{A}(\underline{s})$, $\underline{a}(\underline{s})$	Complex-valued function with complex argument
$\vec{a} \in \mathbb{R}^n$	Real-valued vector of dimension n
$\mathbf{A} \in \mathbb{R}^{n \times m}$	Real-valued matrix with n rows and m columns
$\mathcal{A}$	Transformation or operator, for example $\mathcal{H}$, $\mathcal{D}$, $\mathcal{L}$ and $\mathcal{F}$
$\square^T$	Transpose, applied to matrix or vector
$\square^{(n)}$	Abbreviation for the n-th derivative of a function

Contents

Chapter 1

Linear Time-Invariant Systems

In system theory, a linear system is a model for a sufficiently well-isolated part of nature in which all occurring functions are linear mappings. For many real systems, the description of behavior under approximations (*often the small-signal approximation or the well-known small-angle approximation for trigonometric functions*) is accurate enough with such a model. Linear time-invariant (*abbreviated LTI*) dynamic systems therefore play an extremely important role in physics and engineering.

In the development process, accurate but simple models are often sought to analyze, characterize, or optimize the product. Modeling real behavior with an LTI system is effective in many cases. These are continuous physical models whose describing functions and solutions can be given analytically (*in formula form, also symbolically*) and often provide much insight into the modeled system.

1.1 The System Concept

In general, a system can be described through the interaction between a real input function $y(t)$ and a real output function $x(t)$ (*later also generally* $u(t)$ *or* $\vec{u}(t)$ *for multiple outputs*), although of course there can be multiple inputs and outputs and correspondingly multiple input and output functions. The system itself is typically a mathematical model of reality with which the behavior in the form of an output function $x(t)$ is to be predicted based on a known input function $y(t)$ (*the generalization to systems with multiple inputs and outputs is also motivated later*).

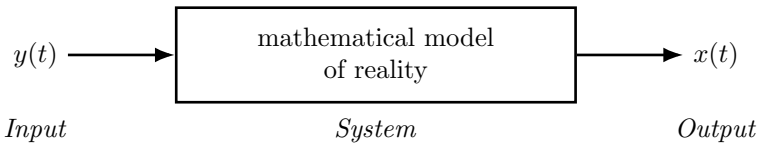


Figure 1.1: Visualization of a general system as a block diagram with system input $y(t)$ and system output $x(t)$

1.1.1 General System Description

Very generally, the connection between input and output can be expressed via a transformation $\mathcal{H}$ that characterizes the system and maps from the input $y(t)$ to the output $x(t)$:

$$\mathcal{H} : y(t) \mapsto x(t) \qquad or \qquad x(t) \;=\; \mathcal{H}\Big\{y(t)\Big\} \tag{1.1}$$

This general formulation for the relationship between input and output of the system appears somewhat abstract and not very meaningful at first. One goal of this script is, among other things, to explore the structure of this relationship and thus the effect or response of the system given an input. As we proceed, it will (*hopefully*) become clear what lies behind the transformation $\mathcal{H}$ in the case of linear time-invariant systems and how it can be systematically constructed from the system-describing parameters, the system parameters.

1.1.2 Linearity

Intuitively, the linearity of a system means that small changes in the input $y(t)$ lead to small changes in the output $x(t)$ (*what is meant here is rather a direct proportionality of the input amplitude to the output amplitude*). Particularly when there are multiple component signals at the input, the output can be described as a linear combination of the individual effects. Thus let:

$$y(t) \;=\; \sum_i \alpha_i \cdot y_i(t)$$

The transformation $\mathcal{H}$ and thus the system is called linear if the following equivalence is satisfied:

$$x(t) \;=\; \mathcal{H}\Big\{ \sum_i \alpha_i \cdot y_i(t) \Big\} \;=\; \sum_i \alpha_i \cdot \mathcal{H}\Big\{ y_i(t) \Big\} \tag{1.2}$$

In the case of system linearity, it does not matter in which order the transformation $\mathcal{H}$ is applied. The result remains unchanged. A consequence of this is the validity of the superposition principle, where parts of the solution can be calculated separately and then assembled/superposed to form a result.

1.1.3 Time Invariance

Time invariance of the system means that regardless of the shift by some T in time, the same excitation always results in the same system response, also shifted by T.

The transformation $\mathcal{H}$ and thus the system is called time-invariant if for any $T \in \mathbb{R}$ it holds that:

$$x(t) \;=\; \mathcal{H}\Big\{ y(t) \Big\} \quad \xrightarrow{(t) \to (t-T)} \quad x(t-T) \;=\; \mathcal{H}\Big\{ y(t-T) \Big\} \tag{1.3}$$

For temporally unchanging system parameters – i.e., constant coefficients – this is typically satisfied. The property of time invariance of the system makes it possible to calculate the system behavior generally for a period starting from $t = 0$ and then to perform the shift of the solution to the observation time. This usually simplifies the calculation considerably. In this script, therefore, the time $t = 0$ is always assumed to be the beginning of observation or the time of system creation. Special cases can be derived from the presented methods due to time invariance.

1.1.4 Causality

Causality (*Latin: causa* $\Leftrightarrow$ *cause*) is the relationship between cause and effect. It concerns the sequence of events and states that are related to one another. Thus A is the cause of effect B if B is brought about by A. An essential requirement for the realizability of a system is the demand that the values of the output signals at a given time depend only on values of the input signals at that time or an earlier time, but not on future values (*no time travel*).

Specifically, causality in linear system theory follows from an impulse response $h(t)$ that exists only for positive times t or is otherwise zero. What exactly the impulse response is will be discussed in detail later.

1.2 System Description in the Time Domain

How can linear time-invariant systems be described by mathematical formulas?

1.2.1 Systems and Differential Equations

The behavior of many physical systems (*e.g., mechanical oscillators, electrical networks, etc.*) can be described mathematically via linear ordinary differential equations with constant coefficients (*constant coefficients and linear ⇒ LTI*). The general form of a linear ordinary differential equation of n-th order with constant coefficients is:

$$\frac{d^n}{dt^n}x(t) + a_{n-1}\,\frac{d^{n-1}}{dt^{n-1}}x(t) + \cdots + a_2\,\frac{d^2}{dt^2}x(t) + a_1\,\frac{d}{dt}x(t) + a_0\,x(t) \;=\; y(t) \tag{1.4}$$

The coefficients $a_0, \ldots, a_{n-1}$ are constant (*but can also be zero*) and characterize the system. The order of the system is given by the highest derivative of the unknown function $x(t)$ contained. This is generally n here with the n-th derivative $\frac{d^n}{dt^n}x(t)$ as the highest derivative contained. System description via differential equations is so elegant because it is a local description. This means that at each time t, the equation shows how the sought function $x(t)$ will change with an infinitesimal change in time dt and under the influence of an externally acting quantity $y(t)$. With the additional specification of initial conditions $\vec{x}_0$, the system behavior is thus uniquely described by the differential equation at every time point.

1.2.2 Differential Equation as Operator Equation

The differential equation (1.4) can alternatively be interpreted as a linear operator equation. Let $\mathcal{D}$ be the differential operator that describes the system, given as:

$$\mathcal{D} \;:=\; \sum_{k=0}^{n} a_k\,\frac{d^k}{dt^k} \quad mit \quad a_n = 1 \tag{1.5}$$

Accordingly, the problem can now be reformulated as an operator equation. Then:

$$\mathcal{D}\,x(t) \;=\; y(t) \tag{1.6}$$

The operator $\mathcal{D}$ (*here a differential operator of n-th order*) describes the system, the function $y(t)$ is the known excitation or inhomogeneity, and $x(t)$ is the unknown solution function. In general, the solution function is always characteristic of the operator (*thus implicitly characteristic of the system*) and the respective excitation. A function $x(t)$ is sought as a solution that satisfies this equation.

1.2.3 State-Space Representation

A linear ordinary differential equation of n-th order with constant coefficients can always be converted into a system of n coupled first-order differential equations. For this step, the derivatives of the sought function $x(t)$ are abbreviated in the following as $\frac{d^k}{dt^k}x(t) = x^{(k)}(t)$ for $k = 0, \ldots, n-1$. In this notation, the differential equation (1.4) becomes:

$$\frac{d}{dt}x^{(n-1)}(t) + a_{n-1}\,x^{(n-1)}(t) + \cdots + a_2\,x^{(2)}(t) + a_1\,x^{(1)}(t) + a_0\,x(t) \;=\; y(t) \tag{1.7}$$

The new variables $x^{(k)}(t)$ of the system that resulted from this reformulation of the problem are linked to each other by derivatives and are also called state variables (*also basis functions of the state space*). The output variable $x(t)$ is, due to the linearity of the system, a linear combination of the state variables and lies in the so-called state space.

The coupling or superposition of the state variables can be interpreted as a linear system of equations and converted into a matrix equation system. This matrix equation system can be directly set up as a Frobenius companion matrix to the higher-order ODE. The differential equation (1.7) corresponds to the last row in the companion matrix, while the remaining rows link the derivatives to each other:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ x^{(1)}(t) \\ x^{(2)}(t) \\ \vdots \\ x^{(n-1)}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}}_{\mathbf{A}} \cdot \underbrace{\begin{bmatrix} x(t) \\ x^{(1)}(t) \\ x^{(2)}(t) \\ \vdots \\ x^{(n-1)}(t) \end{bmatrix}}_{\vec{x}(t)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{\vec{b}} y(t) \tag{1.8}$$

The system of differential equations (1.8) can now be characterized via the state matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ and the vector of state variables $\vec{x}(t) \in \mathbb{R}^n$. The system of equations is also called a state-space model. The dimension of the state-space model is the order of the system. The vector $\vec{b}$ describes the effect of the system's input variable on the state variables (*for MIMO systems with multiple inputs, the vector* $\vec{b}$ *becomes a matrix* $\mathbf{B}$). The system description can be abbreviated in this way:

$$\frac{d}{dt}\vec{x}(t) \;=\; \mathbf{A}\cdot\vec{x}(t) \;+\; \vec{b}\cdot y(t) \tag{1.9}$$

1.2.4 Properties of the State Matrix (WIP)

The eigenvalues of the system are exactly the eigenvalues of the state matrix $\mathbf{A}$. They are the zeros of the characteristic equation or the characteristic polynomial:

$$\underline{\Delta}(\underline{s}) \;=\; det\Big(\underline{s}\cdot\mathbf{I} - \mathbf{A}\Big) \tag{1.10}$$

Diagonalizability, etc ...

1.2.5 System Description and State-Space Representation

With the state-space representation (1.9), we now have a description of the system in which the states $\vec{x}(t)$ (*internal variables of the system*) are excited from outside by the excitation $y(t)$. In many cases, the state variables of the system cannot be measured directly. The measurable output variable $u(t)$ of the system is therefore generally a linear combination (*also superposition*) of the state variables and the input variables. In addition to the state-space representation, the system description therefore includes an additional equation that specifies the system output as a linear combination of the state variables and inputs:

$$\frac{d}{dt}\vec{x}(t) = \mathbf{A} \cdot \vec{x}(t) + \vec{b} \cdot y(t) \tag{1.11}$$

$$u(t) = \vec{c}^{\,T} \cdot \vec{x}(t) + d \cdot y(t) \tag{1.12}$$

The system description via the state-space model is thus characterized by the real state matrix $\mathbf{A}$ and the superposition vectors $\vec{b}$, $\vec{c}$ and the scalar d.

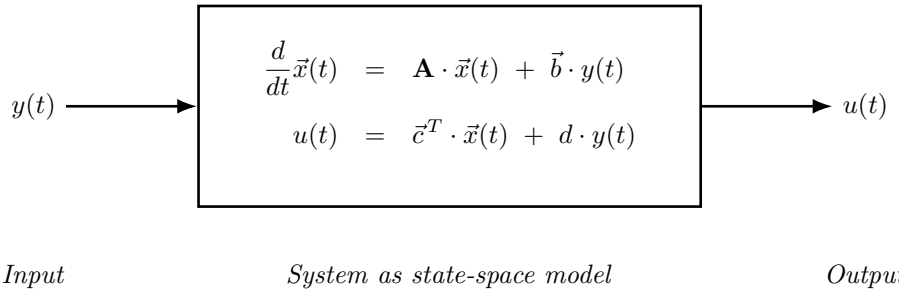


Figure 1.2: SISO system (*single input and single output*) as state-space model in a block diagram representation with system input $y(t)$ and system output $x(t)$

The input variable $y(t)$ acts via the vector $\vec{b}$ on the state variables $\vec{x}(t)$ of the system, which describe the dynamic behavior via a system of differential equations – the state-space model with the state matrix $\mathbf{A}$. Colloquially, we also speak of the states being excited. The output $u(t)$ of the system – i.e., the measurable quantity – is now composed of a superposition (*linear combination / superposition*) of the state variables with the vector $\vec{c}$ and the input variable via the scalar d. Referring to the general case of the system, the solution of the system of differential equations describing the system is thus part of the transformation $\mathcal{H}$, which maps the input $y(t)$ to the output $x(t)$.

1.2.6 Systems with Multiple Inputs and Outputs

The matrix formulation of the differential equation in the state-space representation suggests the possibility that a system with multiple inputs and outputs (*MIMO* $\Leftrightarrow$ *Multiple Input Multiple Output*) can also be described by a state-space representation. Then the vectors $\vec{b}$, $\vec{c}$ and the scalar d become matrices of corresponding dimension in the MIMO case:

$$\frac{d}{dt}\vec{x}(t) = \mathbf{A}\cdot\vec{x}(t) + \mathbf{B}\cdot\vec{y}(t) \tag{1.13}$$

$$\vec{u}(t) = \mathbf{C}^T\cdot\vec{x}(t) + \mathbf{D}\cdot\vec{y}(t) \tag{1.14}$$

Here $\vec{y}(t)$ are the input variables and $\vec{u}(t)$ are the output variables in vector notation. The variables $\vec{x}(t)$ are the state variables. The system description is accordingly via the square state matrix $\mathbf{A}$ and the rectangular coupling matrices $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$:

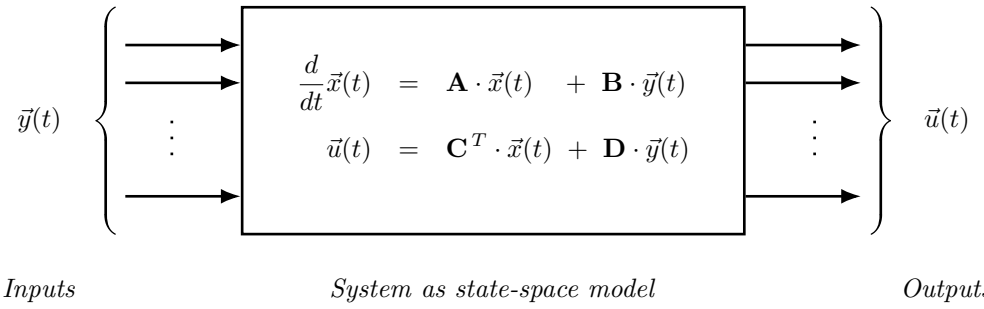


Figure 1.3: MIMO system as state-space model in a block diagram representation with system inputs $\vec{y}(t)$ and system outputs $\vec{u}(t)$

1.2.7 Differentiating Systems

In practice, derivatives of the excitations sometimes also act on the state variables and/or directly on the output variables. Then the formulation of the state-space representation must be extended by the corresponding derivatives $\vec{y}^{(k)}(t)$. Systems in which the derivative (*at least the first derivative*) of an excitation variable is visible as direct proportionality at the output are called differentiating[1].

$$\frac{d}{dt}\vec{x}(t) = \mathbf{A}\cdot\vec{x}(t) + \sum_{k=0}\mathbf{B}_k\cdot\vec{y}^{(k)}(t) \tag{1.15}$$

$$\vec{u}(t) = \mathbf{C}^T\cdot\vec{x}(t) + \underbrace{\sum_{k=0}\mathbf{D}_k\cdot\vec{y}^{(k)}(t)}_{\textit{differenzierende Terme}} \tag{1.16}$$

[1] More about differentiating systems can be found in the chapter on system analysis with the Laplace transformation

1.3 Linearization of Nonlinear Systems

In linearization, nonlinear functions or nonlinear differential equations are approximated by linear functions or linear differential equations. Since in reality many systems are at least approximately linear in a certain range, this step is often effective for analysis and can indeed yield good results. After linearization, the entire arsenal of linear system theory becomes available, which provides powerful analysis and solution methods (*see: the contents of this script*).

1.3.1 Nonlinear Model Description

For example, an ordinary nonlinear differential equation of first order, where f is a nonlinear function:

$$\frac{d}{dt}x(t) \;=\; f\big[x(t),t\big] \tag{1.17}$$

In the MIMO case and for higher-order systems, the model description is done through a nonlinear system of differential equations. Real systems can often be described by a nonlinear state-space model whose linear part with the state matrix $\mathbf{A}$ has been extended by a nonlinear part:

$$\frac{d}{dt}\vec{x}(t) \;=\; \underbrace{\mathbf{A}\cdot\vec{x}(t) \;+\; \mathbf{B}\cdot\vec{y}(t)}_{linearer\ Anteil} \;+\; \underbrace{\vec{F}\big[\vec{x}(t),\vec{y}(t)\big]}_{nichtlinearer\ Anteil} \tag{1.18}$$

1.3.2 Typical Nonlinear Effects

The small-signal approximation cannot always be maintained for real signals. Then higher-order terms (*e.g., quadratic or cubic*) of the Taylor expansion also become relevant for the behavior. When these nonlinear effects become significant compared to linear behavior, the real fun begins.

Harmonics then occur, i.e., additional frequencies beyond those introduced from outside, which can propagate in the system. This becomes particularly clear when considering one of the simplest nonlinear functions, the parabola:

$$f\big[x(t)\big] \;=\; x^2(t) \tag{1.19}$$

For the signal $x(t) = cos(\omega_0 t)$ it follows that:

$$f\big[x(t)\big] \;=\; cos^2(\omega_0 t) \;=\; \frac{1}{2}\Big(1 + cos(2\omega_0 t)\Big) \tag{1.20}$$

The frequencies $\omega = 0$ and $\omega = 2\omega_0$ are impressed on the system by the quadratic function. The simplified system description through phasors in the frequency domain in Chapter 5 is then no longer admissible.

An example from physics is the energy and power in the system, which goes quadratically with the signals. For example, $E_{kin}(t) = \frac{1}{2}mv^2(t)$ and in electrical resistance networks $P(t) = Ri^2(t)$. Energy functions are therefore nonlinear.

1.3.3 Approximation by Taylor Series Expansion

The Taylor series (*after Brook Taylor*) is used in analysis to represent a smooth function $f(x)$ (*smooth* $\Leftrightarrow$ *infinitely differentiable*) in the neighborhood of a point x_0 by a power series, which is the limit of the Taylor polynomials. This series expansion is called Taylor expansion. The expansion of the function $f(x)$ at the expansion point x_0 as a Taylor series can be generally given:

$$f(x) \;=\; \sum_{k=0}^{\infty} \frac{1}{k!} \cdot (x - x_0)^k \cdot \frac{d^k}{dx^k} f(x)\Big|_{x=x_0} \tag{1.21}$$

An approximation of the function around the expansion point arises when the series expansion is terminated prematurely – for example, up to some $k = n$. Then the Taylor approximation of n-th order of $f(x)$ for x near x_0 with the Taylor coefficients f_k is:

$$f(x) \;\approx\; \sum_{k=0}^{n} f_k \cdot (x - x_0)^k \qquad mit \qquad f_k = \frac{1}{k!} \cdot \frac{d^k}{dx^k} f(x)\Big|_{x=x_0} \tag{1.22}$$

The approximation of the function $f(x)$ by Taylor expansion yields a polynomial whose summands can be assigned to certain properties of the function at the expansion point x_0:

$$f(x) \;\approx\; \underbrace{\underbrace{f_0}_{konst.} + \underbrace{f_1 \cdot (x - x_0)}_{linearer\ Term}}_{Tangentengleichung} + \underbrace{f_2 \cdot (x - x_0)^2 \;+\; f_3 \cdot (x - x_0)^3 \;+\; \ldots}_{nichtlineare\ Anteile} \tag{1.23}$$

The coefficients f_0 and f_1 form a tangent equation that touches the function at the expansion point x_0 and represents the linear behavior. The coefficients f_2, f_3, ... represent the nonlinear component of the function at the expansion point.

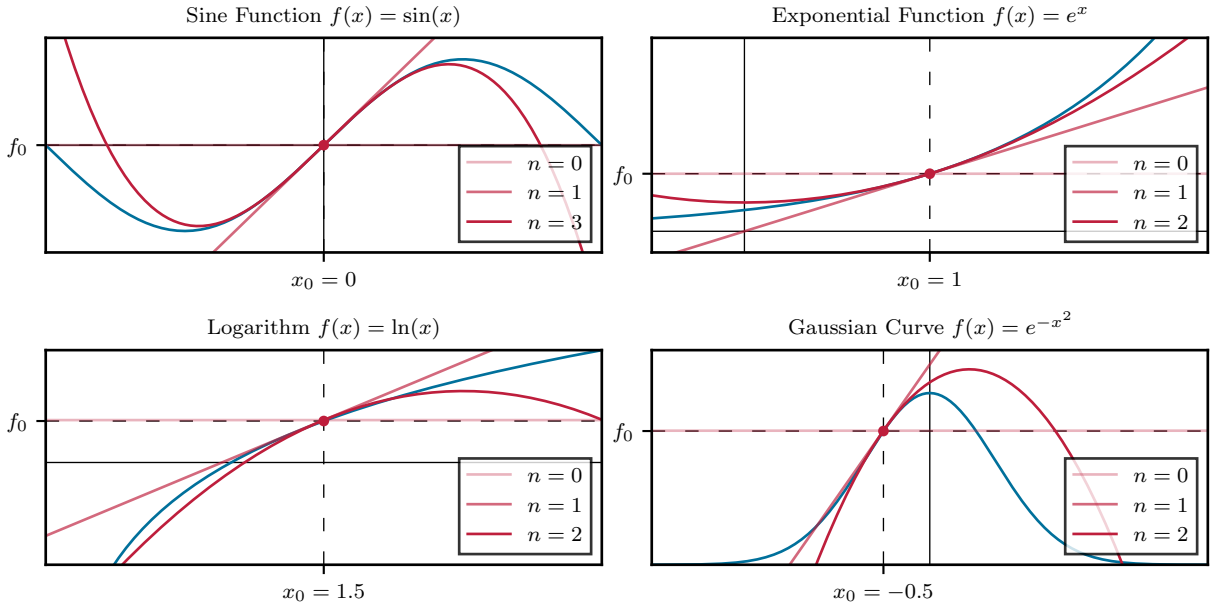


Figure 1.4: Exemplary Taylor series expansions of typical nonlinear functions at a generic expansion point x_0. Shown are sine, exponential function, logarithm, and Gaussian curve for various approximation levels n.

1.3.4 Small-Signal Approximation

Small-signal behavior describes the behavior of a system when driven with small signals, where the word *small* is to be understood not as a small distance from the origin, but from an operating point x_0. In a nonlinear relationship between input and output signal, signals are called small signals as long as an approximately linear transfer behavior results in a limited but task-relevant range. For this, the signal (*also called large signal*) can be represented as a constant DC component x_0 (*operating point*) and a time-dependent small-signal component $x_{ks}(t)$:

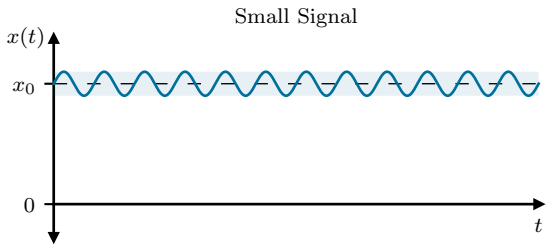


Figure 1.5: Visualization of small-signal approximation for input signals

$$x(t) \;=\; x_0 \;+\; x_{ks}(t) \tag{1.24}$$

1.3.5 How Small is Small Enough?

For the small-signal approximation and thus the linearization to remain valid, the maximum amplitude of the time-dependent small-signal component $x_{ks}(t)$ must be so small that the linear approximation of the function $f(x)$ by a tangent equation remains valid. This is the case when for the components of the Taylor series expansion (1.21) or (1.22) holds:

$$f_0 + f_1 \cdot \big(x_{ks}(t) - x_0\big) \;\gg\; \sum_{k=2}^{\infty} f_k \cdot \big(x_{ks}(t) - x_0\big)^k \tag{1.25}$$

That is, the linear component dominates, i.e., is significantly larger than the rest of the Taylor series. To derive an efficient rule of thumb for the maximum allowed amplitude of the small-signal component, the following equation can be established equivalent to (1.25):

$$f_0 + f_1 \cdot x_{ks}(t) \;\gg\; f\big[x_0 + x_{ks}(t)\big] - f_0 - f_1 \cdot x_{ks}(t) \tag{1.26}$$

After some transformations, it follows that:

$$x_{ks}(t) \;\gg\; \frac{1}{2f_1} \cdot f\big[x_0 + x_{ks}(t)\big] - \frac{f_0}{f_1} \tag{1.27}$$

At the given operating point, determined by x_0 and f_0 with the small-signal parameter f_1, the time dependence of the signal must therefore satisfy the inequality (1.27).

1.3.6 Linearization at the Operating Point

The assumption is now that the function near the expansion point or for small deviations $|x - x_0|$ around the expansion point is already well approximated by the tangent equation. This corresponds to the small-signal approximation introduced above. The expansion point or the constant component x_0 together with the function value $f_0 = f(x_0)$ is also called the operating point.

For small signals, the nonlinear function $f(x)$ can thus be linearized at an operating point $(f_0,\, x_0)$, since then the first two terms of the Taylor series expansion approximate the behavior well. Substituting the small-signal approach (*this is not yet an approximation*) into the argument of the nonlinear function yields:

$$f\big[x(t)\big] \;=\; f\big[x_0 + x_{ks}(t)\big] \tag{1.28}$$

For signals whose time-dependent component $x_{ks}(t)$ is small, $x(t) \approx x_0$ holds and the nonlinear function can be approximated according to (1.22) by a tangent equation, which is a linear function:

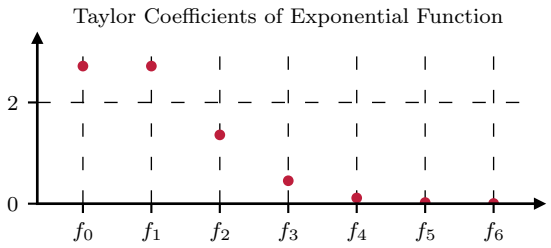


Figure 1.6: Taylor coefficients of the exponential function at expansion point $x_0 = 1$

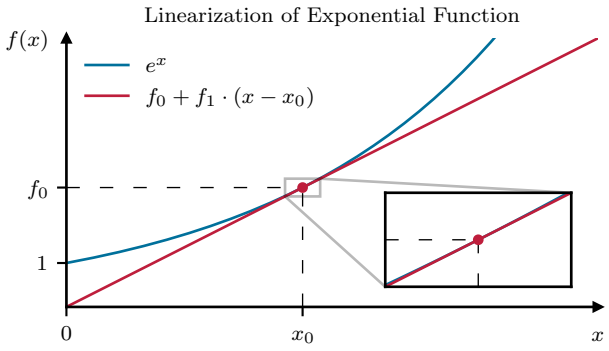


Figure 1.7: Linearization of the exponential function at point $x_0 = 1$ (*operating point* x_0, f_0)

$$f\big[x(t)\big] \;\approx\; f_0 + f_1 \cdot x_{ks}(t) \tag{1.29}$$

The temporal behavior is then determined only by the tangent slope f_1 and the time-dependent small-signal component $x_{ks}(t)$. Under these conditions, the behavior of nonlinear systems can be approximated by linear systems. Thus the entire arsenal of linear system theory methods becomes available again, which is the subject of this document.

1.4 General System Behavior

After discussing the description of the system by a mathematical model in the form of a linear system of differential equations (*or differential equation, operator equation*) with constant coefficients, we now discuss a solution. The following formulations are initially very general. It may therefore be useful to switch back and forth between this section and the examples at the end of the document (*especially the example of a first-order system*).

1.4.1 On the Dirac Delta Function

Now is probably the right time to introduce the Dirac delta function. This is a construct from distribution theory (*i.e., a distribution*) and not an ordinary function. Unlike ordinary functions, differentiation and integration are defined somewhat differently.

In particular, the integration property of the Dirac delta discussed below makes it a powerful tool, if not the workhorse of linear system theory.

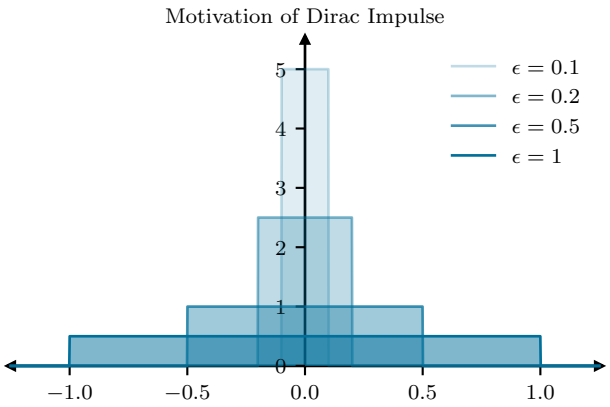


Figure 1.8: Motivation of the Dirac delta function through rectangles with area 1

Figure 1.8 shows an approach to derive the most important property of the Dirac delta. It has area one and becomes infinitely narrow in the limit. Thus let:

$$\delta_\epsilon(t) \;=\; \begin{cases} \frac{1}{2\epsilon} & \textit{for } -\epsilon \le t \le \epsilon \\ 0 & \textit{otherwise} \end{cases} \tag{1.30}$$

The area under the rectangular function $\delta_\epsilon(t)$ is:

$$\int_{-\infty}^{\infty} \delta_\epsilon(t)\,dt \;=\; \int_{-\epsilon}^{\epsilon} \frac{1}{2\epsilon}\,dt \;=\; \left[\frac{t}{2\epsilon}\right]_{t=-\epsilon}^{t=\epsilon} \;=\; 1 \tag{1.31}$$

Thus, the Dirac delta can be defined via a limit process of the above rectangular function:

$$\delta(t) \;=\; \lim_{\epsilon\to 0} \delta_\epsilon(t) \tag{1.32}$$

The Dirac delta assigns to each arbitrarily often differentiable function $g(t)$ (*often also called test function*) a real or complex number $\delta(g) = g(0)$. This corresponds to the evaluation of the function at the point $t = 0$. Formally, this evaluation property is expressed by an integral:

$$g(0) \;=\; \int_{-\infty}^{\infty} g(t)\cdot\delta(t)\,dt \tag{1.33}$$

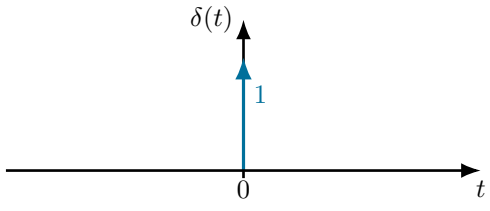


Figure 1.9: Visualization of the Dirac delta weighted by 1 in the time domain as an upward-pointing arrow.

1.4.2 General System Response and Convolution

Before we delve deeper into the solution/response of systems described by differential equations or even systems of differential equations, some statements can already be made from linearity and time invariance alone. Thus, according to (1.1), the transformation $\mathcal{H}$ describes the effect of the system. By exploiting the evaluation property of the Dirac delta, the input function $y(t)$ can be reformulated using the convolution integral (*more on this later*):

$$y(t) \;=\; \int_{-\infty}^{\infty} \delta(t-\tau)\cdot y(\tau)\,d\tau \tag{1.34}$$

Now substitute this notation of the excitation into equation (1.1):

$$\begin{aligned}
x(t) \;&=\; \mathcal{H}\Big\{y(t)\Big\} \\
&\quad\big|\quad (1.34)\ \textit{substitute} \\
&=\; \mathcal{H}\left\{\int_{-\infty}^{\infty} \delta(t-\tau)\cdot y(\tau)\,d\tau\right\} \\
&\quad\big|\quad \mathcal{H}\ \textit{acts only with respect to } t \textit{ and is linear} \\
&=\; \int_{-\infty}^{\infty} \mathcal{H}\big\{\delta(t-\tau)\big\}\cdot y(\tau)\,d\tau \\
&\quad\big|\quad h(t-\tau) := \mathcal{H}\big\{\delta(t-\tau)\big\}\ \textit{since transformation is time-invariant} \\
&=\; \int_{-\infty}^{\infty} h(t-\tau)\cdot y(\tau)\,d\tau
\end{aligned} \tag{1.35}$$

The notation (1.34) thus enables a representation of the system response without directly using the transformation $\mathcal{H}$. It is replaced by the function $h(t) = \mathcal{H}\big\{\delta(t)\big\}$, which is referred to hereafter as the impulse response of the system. This is a function characteristic of the system and elementary in the treatment of linear time-invariant systems.

The above formalism is particularly elegant because it completely replaces the cumbersome and partly abstract transformation $\mathcal{H}$ with a function that can in turn be analyzed and interpreted. The impulse response is virtually omnipresent in system theory, and its systematic determination and the discussion of its properties are an elementary component of this script.

1.4.3 Initial Conditions

The initial conditions describe the behavior of the system at the time from which a solution can be specified, i.e., the beginning of the solution. They contain the information already stored in the system at the beginning of the solution (*also beginning of observation*).

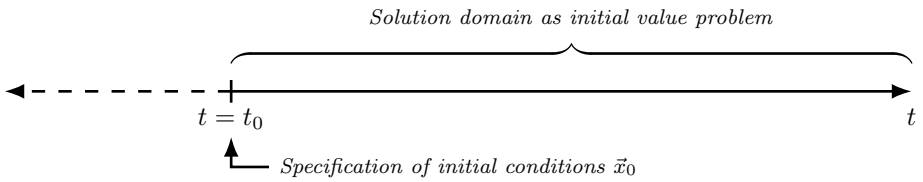


Figure 1.10: Visualization of the solution region for $t > t_0$ of initial value problems using a timeline

In physics and engineering, this is often the energy already stored in the system, for example the potential energy in a spring or the electric field energy in a capacitor. These energy storages are characterized by the state variables $\vec{x}(t)$. Due to energy conservation, it is clear that the energy in the system before the observation time $t < t_0$ is conserved and thus – unless it vanishes at time t_0 – must have an effect on the system behavior for the solution period $t > t_0$.

The time $t = t_0$ is also called the beginning of observation or the creation time of the system under consideration. The calculation of the effect of the initial conditions on the resulting system response is called an initial value problem. The initial conditions are specified in the initial state vector $\vec{x}_0 = \vec{x}(t_0)$. In the following, the beginning of observation is set to $t_0 = 0$.

Without loss of generality, other times $t_0 \neq 0$ can also be treated with the presented methods due to the time invariance of the system

1.4.4 Transient Processes at Observation Beginning

In practice, the transition to the beginning of observation t_0 is usually continuous and the initial conditions are consistent with $\vec{x}(t_0^-) = \vec{x}(t_0^+)$ since the limit values coincide. In general, however, this does not always have to hold. In particular, if the system has a different structure before and after the beginning of observation, which manifests itself in a change in the coupling of the state variables (*or even state variables disappear, for example by specifying a state variable through external excitations*), the limit values do not always coincide.

In these cases, transient processes can occur at time $t = t_0$ through jumps in the temporal course of the state variables. The jump height results as the difference of the limit values of the differentiable variables of the system:

$$\vec{x}_{Ausgleich} = \vec{x}(t_0^-) - \vec{x}(t_0^+) \tag{1.36}$$

Locally, the change in state variables at time $t = t_0$ can thus be represented by time-shifted step functions $\theta(t - t_0)$. Through the derivative of the state variables, weighted Dirac deltas are then impressed into the system at the beginning of observation, which in practice can lead to infinite powers (*infinitely fast change of finite energy*):

$$\frac{d}{dt}\vec{x}(t) \;=\; \delta(t - t_0) \cdot \vec{x}_{Ausgleich} \qquad for \qquad t = t_0 \tag{1.37}$$

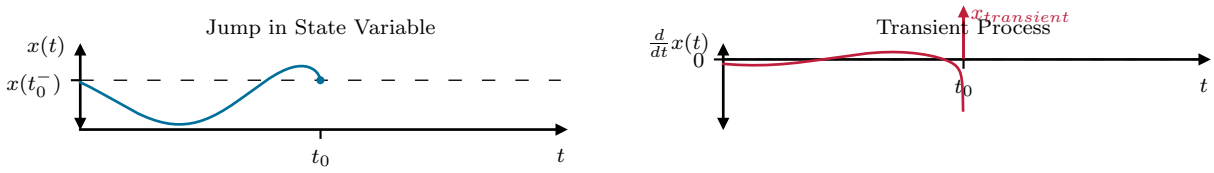


Figure 1.11: Visualization of a jump in a state variable at time $t = t_0$ and resulting impulse.

1.4.5 General Solution Approach

Apart from the general system response of linear time-invariant systems (1.35), we now look at the classical solution approach via solving the describing system of differential equations. In the later course it should become clear that both paths lead to the same goal and the general path via convolution is in many cases the faster, but at least the more elegant one.

A general approach to solving the system of differential equations (1.13) is the splitting of the solution function into a homogeneous $\vec{x}_h(t)$ and a particular $\vec{x}_p(t)$ part:

$$\vec{x}(t) \;=\; \underbrace{\vec{x}_h(t)}_{\substack{homogene\\ solution}} \;+\; \underbrace{\vec{x}_p(t)}_{\substack{particular\\ solution}} \tag{1.38}$$

This approach seems somewhat arbitrary at first glance, but makes sense when one considers the linearity of the system. Accordingly, the solution must be composed of a partial solution with respect to the initial values/initial conditions $\vec{x}_0$ (*homogeneous part*) and a partial solution with respect to the excitations $\vec{y}(t)$ (*particular part*). Linearity thus allows the application of the superposition principle here.

First, the general case for first-order systems that can be described by an ordinary linear first-order differential equation is treated here:

$$\frac{d}{dt}x(t) \;=\; \lambda \cdot x(t) + y(t) \tag{1.39}$$

The solution to this problem is then extended to higher-order systems.

1.4.6 HOMOGENEOUS SOLUTION OF FIRST-ORDER SYSTEMS

The homogeneous solution describes the solution of the homogeneous differential equation. This corresponds to the unexcited system with input $y(t) = 0$. Thus, a solution (*a function that satisfies the operator equation*) is sought for the eigenvalue equation:

$$\frac{d}{dt}x_h(t) \;=\; \lambda \cdot x_h(t) \tag{1.40}$$

In the first step of the solution, a substitution of the function $x_h(t) = x$ is made, which allows a transformation (*often called separation of variables*) of the eigenvalue equation:

$$\frac{dx}{dt} \;=\; \lambda \cdot x \qquad \xrightarrow{\textit{separation of variables } x \textit{ and } t} \qquad \frac{1}{x}\,dx \;=\; \lambda\,dt$$

The right and left sides of the equation now depend on completely different variables. They can be interpreted as indefinite integrals:

$$\int \frac{1}{x}\,dx \;=\; \int \lambda\,dx \tag{1.41}$$

The eigenvalue λ does not depend on t. Now form the antiderivatives of the two indefinite integrals. The integration constants are – as the name says constant – and can therefore be combined into a common constant c:

$$ln\big(x\big) \;=\; \lambda \cdot t + c \tag{1.42}$$

The relationship of the natural logarithm on the left side to the exponential function can now be exploited to further transform the equation. With back-substitution it follows:

$$x \;=\; e^{\lambda \cdot t} \cdot e^{c} \qquad \xrightarrow{\textit{back-substitution } x = x_h(t)} \qquad x_h(t) \;=\; e^{\lambda \cdot t} \cdot e^{c}$$

The solution is not yet completely determined. The constant e^c remains. To find it, examine the homogeneous solution at point $t = 0$:

$$x_h(0) \;=\; e^{\lambda \cdot 0} \cdot e^{c} \;=\; \cdot e^{c} \tag{1.43}$$

Thus, the constant in the solution is the value of the sought function at point $t = t_0$. This value is hereafter referred to as the initial value and abbreviated as $x(0) = x_0$. It represents the starting value of the system and thus the initial situation at the beginning of system observation. The resulting solution depends via the initial value on the system behavior up to this point and can only be valid from this point on – i.e., for $t > 0$ – due to causality (*cause before effect*):

$$x_h(t) \;=\; x_0 \cdot e^{\lambda \cdot t} \qquad \textit{for} \qquad t > 0 \tag{1.44}$$

The homogeneous solution accordingly depends only on the initial value and the eigenvalue of the system, where the former depends on the system behavior for $t < 0$ and the latter is characteristic of the system itself. Due to the time invariance of the system (λ *is constant*), the specification of the initial value x_0 can also be made at any other time t_0. The solution then applies from this time of specification of the initial values

1.4.7 Particular Solution of First-Order Systems

To calculate the complete solution, the particular part of the solution is still missing. With the already known homogeneous solution, the following approach can be chosen for the entire solution:

$$x(t) \;=\; \underbrace{x_0 \cdot e^{\lambda \cdot t}}_{\textit{homogeneous solution}} \;+\; \underbrace{K(t) \cdot e^{\lambda \cdot t}}_{\textit{particular solution}} \tag{1.45}$$

The choice of the approach $K(t) \cdot e^{\lambda \cdot t}$ for the particular solution seems somewhat arbitrary at first glance. The usefulness of this notation becomes clear quickly, however, when the entire solution approach is differentiated:

$$\begin{aligned}
\frac{d}{dt}x(t) \;&=\; \frac{d}{dt}\Big(x_0 \cdot e^{\lambda \cdot t}\Big) \;+\; \frac{d}{dt}\Big(K(t) \cdot e^{\lambda \cdot t}\Big) && (1.46)\\
&\quad\;|\quad \textit{derivative of part. component via product rule}\\
&=\; \lambda \cdot \underbrace{x_0 \cdot e^{\lambda \cdot t}}_{x_h(t)} \;+\; \Big(\lambda \cdot \underbrace{K(t) \cdot e^{\lambda \cdot t}}_{x_p(t)} \;+\; e^{\lambda \cdot t} \cdot \frac{d}{dt}K(t)\Big) && (1.47)\\
&\quad\;|\quad \textit{with } x(t) = x_h(t) + x_p(t) \textit{ follows the substitution}\\
&=\; \lambda \cdot x(t) \;+\; \underbrace{e^{\lambda \cdot t} \cdot \frac{d}{dt}K(t)}_{y(t)} && (1.48)
\end{aligned}$$

Through this step, by comparison with the original differential equation, the excitation $y(t)$ can now be assigned to the right component. It follows:

$$e^{\lambda \cdot t} \cdot \frac{d}{dt}K(t) \;=\; y(t) \tag{1.49}$$

A brief transformation yields an expression for the derivative of the still unknown function $K(t)$:

$$\frac{d}{dt}K(t) \;=\; e^{-\lambda \cdot t} \cdot y(t) \tag{1.50}$$

By integration on both sides (*fundamental theorem of calculus*), the unknown function can be determined (*up to a constant*):

$$K(t) \;=\; \int_0^t e^{-\lambda \cdot \tau} \cdot y(\tau)\, d\tau \tag{1.51}$$

The lower integration limit $\tau = 0$ means that the integrand is only evaluated for $\tau > 0$ and thus only components of the excitation $y(\tau)$ for this time range can be contained in the solution. The solution can therefore again only be meaningful for $t > 0$ due to causality. For the particular solution, this yields for an arbitrary excitation:

$$x_p(t) \;=\; e^{\lambda \cdot t} \cdot K(t) \;=\; \int_0^t e^{\lambda (t-\tau)} \cdot y(\tau)\, d\tau \qquad \textit{for} \qquad t > 0 \tag{1.52}$$

Here too, the time invariance of the system again allows the choice of an arbitrary observation beginning t_0 for the solution.

1.4.8 NOTE ON THE MATRIX EXPONENTIAL FUNCTION

For extending the solution to higher-order linear systems of differential equations such as the state-space representation, the matrix exponential function is used, among other things. This is treated somewhat more precisely in this section.

Let the real-valued matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ be given. The matrix exponential function is defined via a power series:

$$e^{\mathbf{A} \cdot x} = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \mathbf{A}^k \cdot x^k \qquad mit \qquad \mathbf{A}^k = \underbrace{\mathbf{A} \cdots \mathbf{A}}_{k \ mal} \tag{1.53}$$

Completely equivalent to the scalar exponential function, the following holds for the derivative of the matrix exponential function via the representation as an infinite series:

$$\begin{aligned}
\frac{d}{dx} e^{\mathbf{A} \cdot x} &= \frac{d}{dx} \left(\mathbf{I} + \mathbf{A} \cdot x + \frac{1}{2} \cdot \mathbf{A}^2 \cdot x^2 + \frac{1}{6} \cdot \mathbf{A}^3 \cdot x^3 + \dots \right) \\
&\quad | \quad \textit{derivative of the matrix polynomial function} \\
&= \mathbf{0} + \mathbf{A} + \mathbf{A}^2 \cdot x + \frac{1}{2} \cdot \mathbf{A}^3 \cdot x^2 + \dots \\
&\quad | \quad \textit{factor out } \mathbf{A} \\
&= \mathbf{A} \cdot \left(\mathbf{I} + \mathbf{A} \cdot x + \frac{1}{2} \cdot \mathbf{A}^2 \cdot x^2 + \dots \right) \\
&\quad | \quad \textit{replace infinite series by sum} \\
&= \mathbf{A} \cdot \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \mathbf{A}^k \cdot x^k \\
&\quad | \quad \textit{replace the series by the exponential function} \\
&= \mathbf{A} \cdot e^{\mathbf{A} \cdot x} \qquad (1.54) \\
&\quad | \quad \textit{due to commutativity of } \mathbf{A} \textit{ with itself also holds} \\
&= e^{\mathbf{A} \cdot x} \cdot \mathbf{A}
\end{aligned}$$

By recursive application of the relation (1.54), a general expression for higher derivatives of the matrix exponential function follows directly:

$$\frac{d^n}{dx^n} e^{\mathbf{A} \cdot x} = \mathbf{A}^n \cdot e^{\mathbf{A} \cdot x} \tag{1.55}$$

The antiderivative of the matrix exponential function also follows directly and even relatively intuitively upon closer examination of (1.54). Analogous to the scalar exponential function, due to the matrix character, the inverse is formed instead of the reciprocal. The integration thus only works if $\mathbf{A}$ is invertible. But then:

$$\int_0^x e^{\mathbf{A} \cdot \sigma} \, d\sigma = \mathbf{A}^{-1} \cdot e^{\mathbf{A} \cdot x} \tag{1.56}$$

1.4.9 Homogeneous Solution of Higher-Order Systems

For the homogeneous solution, consider the unexcited linear time-invariant system with $\vec{y}(t) = \vec{0}$ in the state-space representation (1.13). The task is to solve the following linear system of differential equations with constant coefficients (*matrix entries*) for the state variables $\vec{x}(t)$ as an initial value problem:

$$\frac{d}{dt}\vec{x}(t) \;=\; \mathbf{A} \cdot \vec{x}(t) \tag{1.57}$$

The solution depends directly and exclusively on the initial conditions $\vec{x}(t_0)$ and the state matrix $\mathbf{A}$. It can be obtained directly by replacing the exponential function $e^{\lambda \cdot (t-t_0)} \rightarrow e^{\mathbf{A} \cdot (t-t_0)}$ with the matrix exponential function in the solution (1.44). The validity of this replacement follows directly from (1.54):

$$\vec{x}(t) \;=\; e^{\mathbf{A} \cdot (t-t_0)} \cdot \vec{x}_0 \quad \textit{for} \quad t > t_0 \tag{1.58}$$

The homogeneous solution describes the behavior of the system based on the values of the state variables $\vec{x}(t)$ stored in the system at the observation start time t_0. These values are also referred to as initial values or initial conditions and are abbreviated as $\vec{x}(t_0) = \vec{x}_0$. The system output or system outputs $\vec{u}(t)$ of the unexcited system then follow from the superposition equation (1.14):

$$\vec{u}(t) \;=\; \mathbf{C}^T \cdot e^{\mathbf{A} \cdot (t-t_0)} \cdot \vec{x}_0 \quad \textit{for} \quad t > t_0 \tag{1.59}$$

1.4.10 Particular Solution of Higher-Order Systems

The inhomogeneous solution refers to the part of the solution $x(t)$ that results solely from the effect of an external excitation $y(t)$ (*the inhomogeneity*). The initial conditions are completely neglected in this solution with $\vec{x}_0 = \vec{0}$. The inhomogeneous solution can be understood as the effect of the input on the output. Thus, we first seek the solution of the state variables $\vec{x}(t)$ of the inhomogeneous state-space model (1.13). Here initially without derivatives of source terms:

$$\frac{d}{dt}\vec{x}(t) \;=\; \mathbf{A} \cdot \vec{x}(t) \;+\; \mathbf{B} \cdot \vec{y}(t) \tag{1.60}$$

The general particular solution can be adapted from the solution for first-order systems (1.52). To do this, replace $e^{\lambda \cdot (t-\tau)} \rightarrow e^{\mathbf{A} \cdot (t-\tau)}$ and $y(\tau) \rightarrow \mathbf{B} \cdot \vec{y}(\tau)$ in the integral. Additionally, the multiplication order for the matrices must be observed. Then it follows:

$$\vec{x}(t) \;=\; \int_{t_0}^{t} e^{\mathbf{A} \cdot (t-\tau)} \cdot \mathbf{B} \cdot \vec{y}(\tau)\, d\tau \quad \textit{for} \quad t > t_0 \tag{1.61}$$

The system output then follows again via the superposition equation (1.14):

$$\vec{u}(t) \;=\; \mathbf{C}^T \cdot \int_{t_0}^{t} e^{\mathbf{A} \cdot (t-\tau)} \cdot \mathbf{B} \cdot \vec{y}(\tau)\, d\tau \;+\; \mathbf{D} \cdot \vec{y}(t) \quad \textit{for} \quad t > t_0 \tag{1.62}$$

1.4.11 General System Response of Linear Time-Invariant Systems

The complete inhomogeneous system solution results from the linearity of the system through the superposition of responses with respect to all excitations (*contained in the particular solution*) and the system response with respect to the initial values already stored in the system (*homogeneous solution*). This complete solution can also only be specified for a time range $t > t_0$:

$$\vec{x}(t) \;=\; \underbrace{e^{\mathbf{A}\cdot(t-t_0)}\cdot\vec{x}_0}_{\textit{homogeneous solution}} \;+\; \underbrace{\int_{t_0}^{t} e^{\mathbf{A}\cdot(t-\tau)}\cdot\mathbf{B}\cdot\vec{y}(\tau)\,d\tau}_{\textit{particular solution}} \quad \textit{for} \quad t > t_0 \tag{1.63}$$

Through the superposition matrix $\mathbf{C}^T$ and the excitations directly passed through with $\mathbf{D}$, the system outputs result as a superposition of the state variables and the excitations:

$$\vec{u}(t) \;=\; \mathbf{C}^T\cdot\underbrace{e^{\mathbf{A}\cdot(t-t_0)}\cdot\vec{x}_0}_{\textit{homogeneous solution}} + \mathbf{C}^T\cdot\underbrace{\int_{t_0}^{t} e^{\mathbf{A}\cdot(t-\tau)}\cdot\mathbf{B}\cdot\vec{y}(\tau)\,d\tau}_{\textit{particular solution}} + \mathbf{D}\cdot\vec{y}(t) \quad \textit{for} \quad t > t_0 \tag{1.64}$$

For systems with only one input $y(t)$ and one output $u(t)$ (*SISO* $\Leftrightarrow$ *Single Input Single Output*), the representation simplifies considerably. The matrices then become vectors representing the superposition of the state variables on the output. In the further course of this script, mainly systems with only one input and one output will be treated.

1.4.12 Representation of the Solution as an Integral Transformation

The general system response can now be further reformulated. The effect of the excitations $\vec{y}(t)$ can be written completely as an integration by introducing a Dirac delta function:

$$\vec{u}(t) \;=\; \underbrace{\mathbf{C}^T\cdot e^{\mathbf{A}\cdot(t-t_0)}\cdot\vec{x}_0}_{\textit{effect of initial cond.}} + \underbrace{\int_{t_0}^{t}\Big(\mathbf{C}^T\cdot e^{\mathbf{A}\cdot(t-\tau)}\cdot\mathbf{B}+\delta(t-\tau)\cdot\mathbf{D}\Big)\cdot\vec{y}(\tau)\,d\tau}_{\vec{u}_p(t)\;\textit{effect of excitations}} \quad \textit{for} \quad t > t_0$$

In this notation, a function (*here a matrix function*) can be identified that describes the effect of the inputs on the outputs and thus characterizes the system:

$$\mathbf{H}(t) \;=\; \theta(t)\cdot\mathbf{C}^T\cdot e^{\mathbf{A}\cdot t}\cdot\mathbf{B}+\delta(t)\cdot\mathbf{D} \tag{1.65}$$

This function is the impulse response of the system, which will be discussed much later. For the general MIMO system, this is a matrix function where the number of rows corresponds to the number of outputs and the number of columns corresponds to the number of inputs. The matrix entries thus each describe the effect of one input on one output. The effects are linearly superimposed through the matrix notation (*superposition*). The effect of the excitations $\vec{u}_p(t)$ (*particular component*) can be further abbreviated with the formulation of the impulse response (1.65). Furthermore, this allows us to identify the transformation $\mathcal{H}$ introduced at the beginning of this document, which is an integral transformation (*convolution integral*):

$$\vec{u}_p(t) \;=\; \mathcal{H}\Big\{\vec{y}(t)\Big\} \;=\; \int_0^t \mathbf{H}(t-\tau)\cdot\vec{y}(\tau)\,d\tau \quad \textit{for} \quad t > 0 \tag{1.66}$$

1.4.13 Solution for Differentiating Systems

For the case motivated in Section 1.2.7 of additional derivatives of the excitation quantities $\vec{y}(t)$ by the system, the general solution can also be specified. This is readily possible by substituting the excitation functions and superposition matrices with the corresponding derivatives:

$$\mathbf{B} \cdot \vec{y}(t) \quad \longrightarrow \quad \sum_{k=0} \mathbf{B}_k \cdot \vec{y}^{(k)}(t) \tag{1.67}$$

$$\mathbf{D} \cdot \vec{y}(t) \quad \longrightarrow \quad \sum_{k=0} \mathbf{D}_k \cdot \vec{y}^{(k)}(t) \tag{1.68}$$

It is easy to imagine that the solution formula for this general case becomes long and cumbersome.

1.4.14 Behavior in the Static Case

So far, linear time-invariant dynamic systems and their general solution have been treated. The dynamic (*time-dependent*) behavior was thereby attributed to the eigenvalues of the state-space model characteristic to the system. A special case occurs when the dimension of the state-space model and thus the order of the system is zero:

$$dim\{\mathbf{A}\} \; = \; 0 \qquad \Leftrightarrow \qquad dim\{\vec{x}(t)\} \; = \; 0 \tag{1.69}$$

Then it is a stateless system and thus a system without dynamics. Systems of this type are also referred to as static. For the complete system description, a purely algebraic system of equations is sufficient in this case, which directly links the inputs and outputs. No differential equation system is needed for the description (*or rather, there exists no differential algebraic description and the dimension of the state space is zero*).

From the system of equations (1.15, 1.16), all terms are omitted except for the direct feedthrough of the system inputs and their derivatives. Thus, for the system outputs:

$$\vec{u}(t) \; = \; \sum_{k=0} \mathbf{D}_k \cdot \vec{y}^{(k)}(t) \tag{1.70}$$

Chapter 2

The Laplace Transformation

The Laplace transformation is a one-sided integral transformation and maps from the continuous – mostly real – time domain to the complex Laplace domain. The properties of the Laplace transformation, such as the uniqueness of the forward and inverse transformation – under certain conditions – and the derivative theorem, make it possible to significantly simplify the calculations for solving linear ordinary differential equations. Furthermore, some properties of the system, such as stability and behavior when excited with harmonic oscillations of different frequencies – i.e., a frequency response – can be read directly from the structure of the representation in the Laplace domain.

2.1 Integral Transformations

Since the Laplace transformation is a (*one-sided*) integral transformation, general integral transformations are treated in somewhat more detail at this point. Additionally, some terminology in connection with integral transformations is introduced.

2.1.1 Definition of the Integral Transformation

A linear operator or a linear transformation $\mathcal{T}$ is called an integral transformation if the transformation rule can be written as follows:

$$\mathcal{T}\Big\{f(\tau)\Big\}(t) \;=\; \int K(\tau,t)\cdot f(\tau)\,d\tau \;=\; k(t) \tag{2.1}$$

Here, $K(\tau,t)$ is the integral kernel and the function $k(t)$ is the $\mathcal{T}$-transform of the function $f(t)$. The integral transformation is defined via the integral kernel. For the existence of the integral (*i.e., transformability*), requirements are typically placed on both the kernel and the function to be transformed. The transformation rule can be extended without loss of generality to complex functions and integral kernels. It can be recognized that the integral kernel $K(\tau,t)$ is the characterizing quantity of the transformation $\mathcal{T}$. The general solution of the operator equation via Green's functions in the previous chapter thus corresponds to an integral transformation, which corresponds to an inversion or reversal of the differential operator $\mathcal{D}^{-1}$.

2.1.2 The Convolution Integral as an Integral Transformation

The convolution integral is a very prominent example of an integral transformation; it is an essential component in many engineering disciplines. The convolution integral is defined as:

$$\Big(f(\tau)*g(\tau)\Big)(t) \;=\; \int_{-\infty}^{\infty} f(t-\tau)\cdot g(\tau)\,d\tau \tag{2.2}$$

The integral (2.7) is referred to as the convolution of the function $f(t)$ with the function $g(t)$. The integral kernel can be identified in this case as $K(\tau,t)=f(t-\tau)$. Intuitively, in the convolution, the two functions $f(\tau)$ and $g(\tau)$ are shifted/folded over each other in the integrand with the shift by t or $(t-\tau)$. This corresponds to a weighting of each function value of the function $f(t-\tau)$ shifted by t with each function value of $g(\tau)$ for all τ. The imaginary zero points $\tau=0$ of the two functions thereby move further and further apart with increasing t.

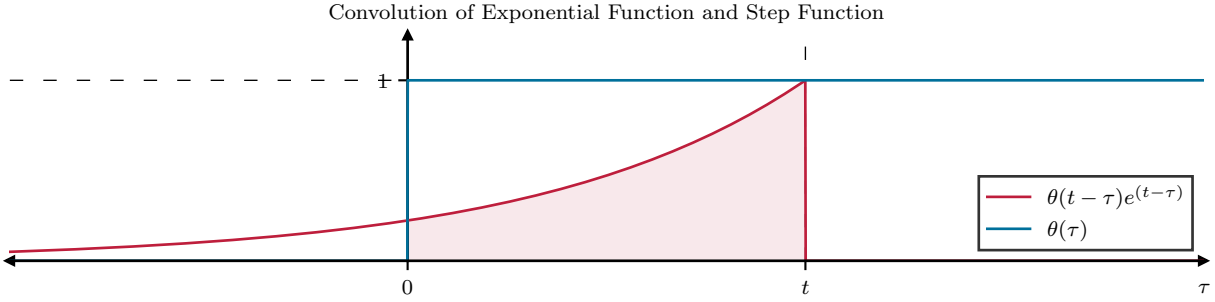


Figure 2.1: Visualization of the weighting of two functions (*here exponential function and step function*) in the integrand of the convolution integral.

2.2 Laplace Transformation and Laplace Domain

For a real function $f(t) : \mathbb{R} \mapsto \mathbb{R}$ in the time domain – a time-dependent function with the variable t – the application of the Laplace transformation $\mathcal{L}$ results in a transfer or transformation from the time domain to the Laplace domain. Here, $\underline{F}(\underline{s}) : \mathbb{C} \mapsto \mathbb{C}$ is the Laplace transform of $f(t)$ or the image function of $f(t)$ in the Laplace domain.

$$\mathcal{L} : f(t) \mapsto \underline{F}(\underline{s}) \qquad \Leftrightarrow \qquad f(t) \circ\!\!-\!\!\bullet\, \underline{F}(\underline{s}) \tag{2.3}$$

The complex variable $\underline{s} = \alpha + j\omega \in \mathbb{C}$ has a real part $\alpha \in \mathbb{R}$ and an imaginary part $\omega \in \mathbb{R}$. The argument has two degrees of freedom and a point in the Laplace domain can be interpreted as a point in the α-ω-plane. Therefore, the designation complex $\underline{s}$-plane or Laplace plane is often used for the domain of definition of the Laplace transformation.

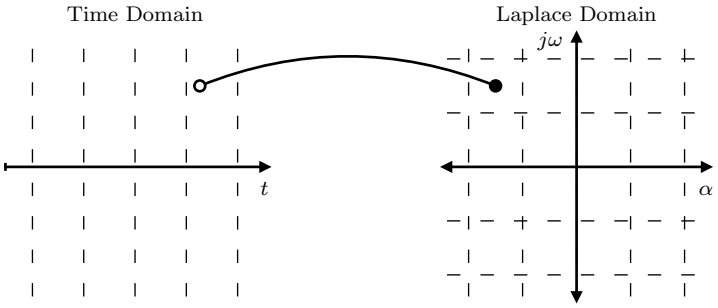


Figure 2.2: Visualization of the transformation from time domain to Laplace domain. Here, $\underline{s} = \alpha + j\omega$

2.2.1 Definition of the Laplace Transformation

The Laplace transformation links the two functions $f(t)$ and $\underline{F}(\underline{s})$:

$$\mathcal{L}\Big\{f(t)\Big\}(\underline{s}) \;=\; \int_0^\infty f(t) \cdot e^{-\underline{s}t} dt \;=\; \underline{F}(\underline{s}) \tag{2.4}$$

This is a one-sided integral transformation with the complex integration kernel $\underline{K}(t, \underline{s}) = e^{-\underline{s}t}$ and the integration interval $[0, \infty)$. Furthermore, the complex variable $\underline{s}$ in the Laplace domain has the unit of a frequency if t is time with the unit of seconds. This follows from the dimensionlessness of the exponent $\underline{s}t$ in the integration kernel. Therefore, $\underline{s}$ is also often referred to as the complex frequency.

2.2.2 Existence Conditions of the Laplace Transform

The Laplace transform does not exist for every arbitrary function $f(t)$. Fundamentally, the existence condition is based on the integral converging for the chosen limits. For this, the function to be transformed $f(t)$ must be of exponential order and satisfy:

$$\left|f(t)\right| \;\leq\; \left|C \cdot e^{\alpha\, t}\right| \qquad \textit{for all} \quad t > 0$$

Here, C and α are arbitrarily large selectable real constants. If this is satisfied, an $\underline{s} \in \mathbb{C}$ can always be chosen such that the Laplace integral converges. Thus, for the convergence of the Laplace integral, the function to be transformed $f(t)$ must always lie within a tube spanned by an arbitrary exponential function. Specifically, the improper Laplace integral (2.4) converges in a convergence plane starting from a real part $\Re\{\underline{s}\} > \alpha$ of the complex frequency.

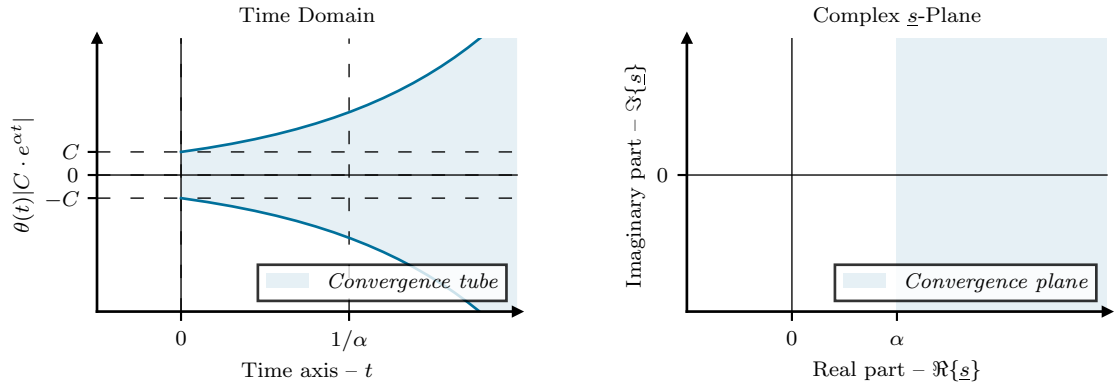


Figure 2.3: Visualization of the convergence criteria for the Laplace integral. Left: Convergence of functions of exponential order within a tube in the time domain, Right: Convergence from a real part α in the complex $\underline{s}$-plane

2.2.3 Motivation for Validity and Uniqueness

The special formulation of the Laplace integral (2.4) with the lower integration limit 0 means that the function $f(t)$ is only evaluated for positive values of t during the transformation. The Laplace transform $\underline{F}(\underline{s})$ is therefore merely a representation of the function for positive t, i.e., for $t > 0$. This means that no information about $f(t)$ for $t < 0$ is contained in the Laplace transform and accordingly statements about the behavior for positive t can only be made in the Laplace domain. The uniqueness of the Laplace transformation – in particular the inverse transformation $\mathcal{L}^{-1}$ – is thus not guaranteed for all t. For analysis with the Laplace transformation (*in the time domain* $t > 0$), the function is therefore assumed/presupposed as $f(t) = 0$ for all $t < 0$. Formally, this can be realized by cutting off the function for $t < 0$ using the step function $\theta(t)$.

2.3 Properties of the Laplace Transformation

From the definition of the Laplace transformation via the integral (2.4), several important properties follow directly (*assuming that the Laplace transform exists ...*). These properties are among the reasons why the Laplace transformation is so well suited for the analysis of linear differential equations.

2.3.1 Linearity of the Laplace Transformation

The importance and significance of linearity is often overlooked, so here it is presented once again in full detail. Let the function $\alpha \cdot f(t) + \beta \cdot g(t)$ be given, then this is a linear combination of the two functions $f(t)$ and $g(t)$ with the linear factors α and β (*here real, but can also be complex*). Then for the Laplace transform:

$$\begin{aligned}
\mathcal{L}\Big\{\alpha \cdot f(t) + \beta \cdot g(t)\Big\}(\underline{s}) &= \int_0^\infty (\alpha \cdot f(t) + \beta \cdot g(t)) \cdot e^{-\underline{s}t} dt \\
&\mid \textit{integral is linear} \\
&= \int_0^\infty \alpha \cdot f(t) \cdot e^{-\underline{s}t} + \beta \cdot g(t) \cdot e^{-\underline{s}t} dt \\
&\mid \textit{split and extract linear factors} \\
&= \alpha \cdot \int_0^\infty f(t) \cdot e^{-\underline{s}t} dt \; + \; \beta \cdot \int_0^\infty g(t) \cdot e^{-\underline{s}t} dt \\
&\mid \textit{from the definition follows} \\
&= \alpha \cdot \underline{F}(\underline{s}) \; + \; \beta \cdot \underline{G}(\underline{s})
\end{aligned}$$

The linearity of the Laplace transformation enables the superposition – i.e., the linear combination – of multiple functions into a total function. The partial functions – also basis functions – can therefore be individually analyzed / calculated / transformed and then reassembled. This form of superposition is also referred to as superposition and its application as the superposition principle.

2.3.2 The Derivative Theorem

The derivative theorem is probably the most important property for dealing with differential equations. The application of the Laplace transformation $\mathcal{L}$ to the derivative of a function $f(t)$, which is given for $t > 0$ and is at least once differentiable, yields:

$$\begin{aligned}
\mathcal{L}\Big\{\frac{d}{dt}f(t)\Big\}(\underline{s}) &= \int_0^\infty \frac{d}{dt}f(t)\cdot e^{-\underline{s}t}dt \\
&\quad | \quad \textit{integration by parts} \\
&= \Big[f(t)\cdot e^{-\underline{s}t}\Big]_{t=0}^{t=\infty} - \int_0^\infty f(t)\cdot(-\underline{s})e^{-\underline{s}t}dt \\
&\quad | \quad \textit{evaluation: } f(t) \textit{ of exponential order} \\
&= 0 - f(0^+) + \underline{s}\int_0^\infty f(t)\cdot e^{-\underline{s}t}dt \\
&\quad | \quad \textit{from the definition follows} \\
&= \underline{s}\cdot\underline{F}(\underline{s}) - f(0^+)
\end{aligned}$$

The derivative theorem is thus:

$$\mathcal{L}\Big\{\frac{d}{dt}f(t)\Big\}(\underline{s}) = \underline{s}\cdot\underline{F}(\underline{s}) - f(0^+) \tag{2.5}$$

Here, $f(0^+)$ is the right-sided limit value of the function $f(t)$ for $t \to 0$. This constant real value is the initial value of the function in the time domain. Often $f(0)$ is also specified as the initial value, but the equivalence $f(0) = f(0^+)$ only holds if the function $f(t)$ is continuous at the emergence time $t = 0$. However, this is not always guaranteed.

This relationship (2.5) now leads to the fact that when transitioning to the Laplace domain, the derivative becomes a multiplication with the variable $\underline{s}$. With regard to the analysis of differential equations, a differential-algebraic problem thus becomes a purely algebraic problem, which can be solved by purely algebraic operations – such as multiplication and addition. No integration (*except for the transformation step ... although one can also save this* → *correspondence tables*) is necessary and the initial values can be directly passed into the system of equations!

2.3.3 THE DERIVATIVE THEOREM FOR HIGHER DERIVATIVES

As a generalization for higher derivatives, a derivative theorem can also be formulated. Particularly for differential equations of higher order, this formulation can be used profitably. The notation for the derivative is abbreviated in the following for all $k \in \mathbb{N}_0$:

$$f^{(k)}(t) \;:=\; \frac{d^k}{dt^k} f(t) \qquad \text{and} \qquad f_0^{(k)} \;:=\; \frac{d^k}{dt^k} f(t)\Big|_{t\to 0}$$

For the n-th derivative of a function $f(t)$ given from $t > 0$ and at least n-times differentiable, it follows:

$$\begin{aligned}
\mathcal{L}\Big\{ f^{(n)}(t) \Big\}(\underline{s}) \;&=\; \int_0^\infty f^{(n)}(t) \cdot e^{-\underline{s}t} dt \\
&\mid \; \textit{integration by parts} \\
&=\; \Big[f^{(n-1)}(t) \cdot e^{-\underline{s}t} \Big]_{t=0}^{t=\infty} \;+\; \underline{s} \int_0^\infty f^{(n-1)}(t) \cdot e^{-\underline{s}t} dt \\
&\mid \; \textit{evaluation of the left term always has the same form} \\
&=\; -f_0^{(n-1)} \;+\; \underline{s} \int_0^\infty f^{(n-1)}(t) \cdot e^{-\underline{s}t} dt \\
&\mid \; \textit{integration by parts until } f^{(n-1)}(t) \textit{ is completely reduced} \\
&=\; -f_0^{(n-1)} + \underline{s}\Big(-f_0^{(n-2)} + \underline{s}\Big(\cdots \Big(-f_0^{(1)} + \underline{s}\Big(-f_0 + \underline{s} \cdot \underline{F}(\underline{s}) \Big) \Big) \cdots \Big) \Big) \\
&\mid \; \textit{expand each layer} \\
&=\; -f_0^{(n-1)} - \underline{s}\, f_0^{(n-2)} \cdots - \underline{s}^{n-2}\, f_0^{(1)} - \underline{s}^{n-1}\, f_0 + \underline{s}^n \cdot \underline{F}(\underline{s}) \\
&\mid \; \textit{representation as sum} \\
&=\; \underline{s}^n \cdot \underline{F}(\underline{s}) \;-\; \sum_{k=0}^{n-1} \underline{s}^{n-k-1}\, f_0^{(k)}
\end{aligned}$$

Thus, for the Laplace transform of the n-th derivative of a function:

$$\mathcal{L}\Big\{ \frac{d^n}{dt^n} f(t) \Big\}(\underline{s}) \;=\; \underline{s}^n \cdot \underline{F}(\underline{s}) \;-\; \sum_{k=0}^{n-1} \underline{s}^{n-k-1}\, f_0^{(k)} \tag{2.6}$$

Usually it is sufficient to remember the derivative theorem for the first derivative (2.5). Higher derivatives can then be constructed from it.

2.3.4 The Convolution Theorem in the Time Domain

The convolution product of two functions $f(t), g(t) : \mathbb{R} \mapsto \mathbb{R}$ is defined as an integral transformation:

$$\Big(f(\tau) * g(\tau)\Big)(t) \;:=\; \int_{-\infty}^{\infty} f(t-\tau) \cdot g(\tau) d\tau \tag{2.7}$$

The convolution integral (2.7) appears as part of the particular solution of ordinary differential equations and therefore has special significance. Under application of the Laplace transformation it follows:

$$\begin{aligned}
\mathcal{L}\Big\{\Big(f(\tau) * g(\tau)\Big)(t)\Big\}(\underline{s}) &= \int_0^{\infty} \int_{-\infty}^{\infty} f(t-\tau) \cdot g(\tau) d\tau \cdot e^{-\underline{s}t} dt \\
&\mid \textit{reformulate} \\
&= \int_0^{\infty} \int_{-\infty}^{\infty} f(t-\tau) \cdot g(\tau) \cdot e^{-\underline{s}t} d\tau \, dt \\
&\mid \textit{substitute: } t' = t - \tau \textit{ and } dt' = dt \\
&= \int_{-\tau}^{\infty} \int_{-\infty}^{\infty} f(t') \cdot g(\tau) \cdot e^{-\underline{s}(t'+\tau)} d\tau \, dt' \\
&\mid \textit{reformulate according to integration variables} \\
&= \int_{-\tau}^{\infty} f(t') \cdot e^{-\underline{s}t'} dt' \cdot \int_{-\infty}^{\infty} g(\tau) \cdot e^{-\underline{s}\tau} d\tau \\
&\left| \begin{array}{l} \textit{adjust integration limits, since } g(\tau) = 0 \textit{ for } \tau < 0 \\ \textit{and } f(t') = 0 \textit{ for } t' < 0 \end{array} \right. \\
&= \int_0^{\infty} f(t') \cdot e^{-\underline{s}t'} dt' \cdot \int_0^{\infty} g(\tau) \cdot e^{-\underline{s}\tau} d\tau \\
&\mid \textit{substituting the definition} \\
&= \underline{F}(\underline{s}) \cdot \underline{G}(\underline{s})
\end{aligned}$$

Thus follows the convolution theorem of the Laplace transformation for the convolution of functions in the time domain. The convolution of two functions corresponds to a multiplication in the Laplace domain:

$$\mathcal{L}\Big\{\Big(f(\tau) * g(\tau)\Big)(t)\Big\}(\underline{s}) \;=\; \underline{F}(\underline{s}) \cdot \underline{G}(\underline{s}) \tag{2.8}$$

2.3.5 The Initial Value Theorem

The initial value theorem links the behavior at infinity of the Laplace transform with the initial value of the function in the time domain. Here, the limit values for $t \to 0$ and $|\underline{s}| \to \infty$ are formed. Form the bilateral limit value $|\underline{s}| \to \infty$ with the derivative theorem (2.5):

$$\begin{aligned}
\lim_{|\underline{s}|\to\infty} \int_0^\infty \frac{d}{dt} f(t) \cdot e^{-\underline{s}t}\, dt &= \lim_{|\underline{s}|\to\infty} \underline{s} \cdot \underline{F}(\underline{s}) - f(0^+) \\
&\quad | \quad \textit{integrand is Riemann-integrable on } [0, \infty) \\
\int_0^\infty \frac{d}{dt} f(t) \cdot \lim_{|\underline{s}|\to\infty} e^{-\underline{s}t}\, dt &= \lim_{|\underline{s}|\to\infty} \underline{s} \cdot \underline{F}(\underline{s}) - f(0^+) \\
&\quad | \quad \textit{left term vanishes} \\
0 &= \lim_{|\underline{s}|\to\infty} \underline{s} \cdot \underline{F}(\underline{s}) - f(0^+)
\end{aligned}$$

From this follows the initial value theorem of the Laplace transformation:

$$\lim_{|\underline{s}|\to\infty} \underline{s} \cdot \underline{F}(\underline{s}) = f(0^+) \tag{2.9}$$

In particular, if the function $f(t)$ is not continuous at time $t = 0$, the initial conditions for $t > 0$ can be calculated from the Laplace transform via the initial value theorem.

2.3.6 The Final Value Theorem

The final value theorem links the behavior of the Laplace transform at the origin with the final value of the function in the time domain. For this, the limit values $t \to \infty$ and $\underline{s} \to 0$ are formed. Form the bilateral limit value $\underline{s} \to 0$ with the derivative theorem (2.5):

$$\begin{aligned}
\lim_{\underline{s}\to 0} \int_0^\infty \frac{d}{dt} f(t) \cdot e^{-\underline{s}t}\, dt &= \lim_{\underline{s}\to 0} \underline{s} \cdot \underline{F}(\underline{s}) - f(0^+) \\
&\quad | \quad \textit{integrand is Riemann-integrable on } [0, \infty) \\
\int_0^\infty \frac{d}{dt} f(t) \cdot \lim_{\underline{s}\to 0} e^{-\underline{s}t}\, dt &= \lim_{\underline{s}\to 0} \underline{s} \cdot \underline{F}(\underline{s}) - f(0^+) \\
&\quad | \quad \textit{reformulate left term} \\
\int_0^\infty \frac{d}{dt} f(t)\, dt &= \lim_{\underline{s}\to 0} \underline{s} \cdot \underline{F}(\underline{s}) - f(0^+) \\
&\quad | \quad \textit{solve integral via antiderivative} \\
\lim_{t\to\infty} f(t) - f(0^+) &= \lim_{\underline{s}\to 0} \underline{s} \cdot \underline{F}(\underline{s}) - f(0^+)
\end{aligned}$$

From this follows the final value theorem of the Laplace transformation:

$$\lim_{\underline{s}\to 0} \underline{s} \cdot \underline{F}(\underline{s}) = \lim_{t\to\infty} f(t) \tag{2.10}$$

2.3.7 The Shift Theorem

If a function $f(t)$ is shifted in the time domain by a constant $T > 0$, then this shift $f(t-T)$ can also be transferred to the Laplace domain:

$$\begin{aligned}
\mathcal{L}\Big\{f(t-T)\Big\}(\underline{s}) &= \int_0^\infty f(t-T)\cdot e^{-\underline{s}t}dt \\
&\mid \textit{substitute: } \tau = t-T \textit{ and } d\tau = dt \\
&= \int_{-T}^\infty f(\tau)\cdot e^{-\underline{s}(\tau+T)}d\tau \\
&\mid \textit{adjust integration limits, since } f(\tau)=0 \textit{ for } \tau<0 \\
&= e^{-\underline{s}T}\cdot\int_0^\infty f(\tau)\cdot e^{-\underline{s}\tau}d\tau \\
&\mid \textit{from the definition follows} \\
&= e^{-\underline{s}T}\cdot\underline{F}(\underline{s})
\end{aligned}$$

Thus follows the shift theorem:

$$\mathcal{L}\Big\{f(t-T)\Big\}(\underline{s}) \;=\; e^{-\underline{s}T}\cdot\underline{F}(\underline{s}) \tag{2.11}$$

Particularly in the decomposition of a function into multiple time ranges, such as for a rectangular or sawtooth function into individual shifted partial functions, the shift can be taken into account in the Laplace domain in this way (*this is where the time invariance of the system is contained*).

2.3.8 The Damping Theorem

Analogous to the shift in the time domain, a shift by a real constant a can also be performed in the Laplace domain. For $\underline{F}(\underline{s}-a)$ a representation in the time domain follows:

$$\begin{aligned}
\underline{F}(\underline{s}-a) &= \int_0^\infty f(t)\cdot e^{-(\underline{s}-a)t}dt \\
&\mid \textit{reformulate} \\
&= \int_0^\infty e^{at}\cdot f(t)\cdot e^{-\underline{s}t}dt \\
&\mid \textit{from the definition follows} \\
&= \mathcal{L}\Big\{e^{at}\cdot f(t)\Big\}(\underline{s})
\end{aligned}$$

A shift in the Laplace domain leads to an additional term e^{at} in the time domain. For $a<0$, this causes a damping of the function $f(t)$. Thus follows the damping theorem:

$$\underline{F}(\underline{s}-a) \;=\; \mathcal{L}\Big\{e^{at}\cdot f(t)\Big\}(\underline{s}) \tag{2.12}$$

2.3.9 The Similarity Theorem

Besides shifting the argument in the time domain, the argument can also be scaled. With the linear factor $\alpha \in \mathbb{R}^+$, the scaling of the argument $f(\alpha \cdot t)$ in the Laplace domain follows as:

$$\begin{aligned}
\mathcal{L}\Big\{f(\alpha \cdot t)\Big\}(\underline{s}) &= \int_0^\infty f(\alpha \cdot t) \cdot e^{-\underline{s}t} dt \\
&\Big| \quad \textit{substitute: } \tau = \alpha \cdot t \textit{ and } dt = \frac{d\tau}{\alpha} \\
&= \int_0^\infty f(\tau) \cdot e^{-\frac{\underline{s}}{\alpha}\tau} \frac{d\tau}{\alpha} \\
&| \quad \textit{reformulate} \\
&= \frac{1}{\alpha} \cdot \int_0^\infty f(\tau) \cdot e^{-\frac{\underline{s}}{\alpha}\tau} d\tau \\
&| \quad \textit{from the definition and linearity follows} \\
&= \frac{1}{\alpha} \cdot \underline{F}\left(\frac{\underline{s}}{\alpha}\right)
\end{aligned}$$

Thus, for a scaling of the argument in the time domain by a factor α, the scaling of the Laplace transform and the complex frequency $\underline{s}$ with the inverse factor $\frac{1}{\alpha}$ follows:

$$\mathcal{L}\Big\{f(\alpha \cdot t)\Big\}(\underline{s}) = \frac{1}{\alpha} \cdot \underline{F}\left(\frac{\underline{s}}{\alpha}\right) \tag{2.13}$$

Especially for sine and cosine terms, the similarity theorem can be useful, since then a scaling of the argument in the time domain corresponds to a scaling of the frequency.

2.4 Common Functions in the Laplace Domain

The Laplace transforms of some common functions follow. Particularly due to the uniqueness of the forward and inverse transformation – after cutting off by a step function $\theta(t)$ – and linearity, large, complex functions (*for example using the shift theorem, etc.*) in the time domain can often be decomposed into these simpler functions and transformed individually (*follows from linearity*). The inverse transformation can likewise be performed individually (*decomposition of the Laplace transform via partial fraction decomposition → correspondences of known functions*).

2.4.1 Step Function

According to the considerations on the uniqueness of the forward and inverse transformation, let us first look at the step function / unit step:

$$\theta(t) \;:=\; \begin{cases} 0 & \textit{for } t < 0 \\ 1 & \textit{for } t > 0 \end{cases}$$

Since the Laplace transformation – or the inverse transformation – is only defined for positive t, the step function can be used to cut off an arbitrary function for negative t. Thus, let the function to be transformed in the time domain be given as $f(t) = \theta(t)$, then with the definition it follows:

$$\begin{aligned}
\mathcal{L}\Big\{\theta(t)\Big\}(\underline{s}) &= \int_0^\infty \theta(t) \cdot e^{-\underline{s}t} dt \\
&\mid \quad \theta(t) = 1 \textit{ holds within the integration limits} \\
&= \int_0^\infty e^{-\underline{s}t} dt \\
&\mid \quad \textit{integration of the exponential function} \\
&= \left[-\frac{1}{\underline{s}} \cdot e^{-\underline{s}t} \right]_{t=0}^{t=\infty} \\
&\mid \quad \textit{evaluation of the antiderivative} \\
&= \frac{1}{\underline{s}}
\end{aligned}$$

The Laplace transform of the step function is thus:

$$\mathcal{L}\Big\{\theta(t)\Big\}(\underline{s}) \;=\; \frac{1}{\underline{s}} \tag{2.14}$$

2.4.2 Polynomial Function

Polynomial functions also appear repeatedly – for example in an approximation by a Taylor expansion – and are therefore treated in more detail here. Generally they have the form (*here cut off for negative t by a step function*):

$$f(t) \;=\; a_0\,\theta(t) + a_1\,\theta(t)\,t + a_2\,\theta(t)\,t^2 + a_3\,\theta(t)\,t^3 + \cdots \tag{2.15}$$

It is particularly noteworthy that due to the linearity of the Laplace transformation, the individual summands / polynomial terms (*sometimes also called monomials*) can be transformed individually and then summed again. Therefore, without loss of generality, a polynomial term $\theta(t)\,t^n$ for an arbitrary power $n \in \mathbb{N}$ is now considered:

$$\begin{aligned}
\mathcal{L}\Big\{\theta(t)\,t^n\Big\}(\underline{s}) \;&=\; \int_0^\infty \theta(t)\,t^n \cdot e^{-\underline{s}t}dt \\
&\mid\; \theta(t) = 1 \textit{ holds within the integration limits} \\
&=\; \int_0^\infty t^n \cdot e^{-\underline{s}t}dt \\
&\mid\; \textit{integration by parts until } t^n \textit{ is completely reduced} \\
&=\; \left[-\frac{1}{\underline{s}}t^n \cdot e^{-\underline{s}t}\right]_{t=0}^{t=\infty} - \int_0^\infty -\frac{n}{\underline{s}}t^{n-1} \cdot e^{-\underline{s}t}dt \\
&\mid\; \ldots \textit{ the left term always vanishes } \ldots \\
&=\; \int_0^\infty \frac{n!}{\underline{s}^n} \cdot e^{-\underline{s}t}dt \\
&\mid\; \textit{integrate exponential function} \\
&=\; \left[-\frac{n!}{\underline{s}^{n+1}} \cdot e^{-\underline{s}t}\right]_{t=0}^{t=\infty} \\
&\mid\; \textit{evaluate} \\
&=\; \frac{n!}{\underline{s}^{n+1}}
\end{aligned}$$

For the Laplace transform of a polynomial term of power n, thus:

$$\mathcal{L}\Big\{\theta(t)\,t^n\Big\}(\underline{s}) \;=\; \frac{n!}{\underline{s}^{n+1}} \tag{2.16}$$

Upon closer inspection, one can recognize the Laplace transform of the step function for the special case $n = 0$ in the general polynomial.

2.4.3 Exponential Function

Another common function (*perhaps even the most important, as will be seen later*) is the exponential function. Again, for reasons of uniqueness, this is cut off for negative t and with real a, the function to be transformed is then given as:

$$f(t) \;=\; \theta(t)\, e^{at} \tag{2.17}$$

Then for the Laplace transform follows:

$$\begin{aligned}
\mathcal{L}\Big\{\theta(t)\, e^{at}\Big\}(\underline{s}) \;&=\; \int_0^\infty \theta(t)\, e^{at} \cdot e^{-\underline{s}t} dt \\
&\quad\big|\quad \textit{combination of exponents} \\
&=\; \int_0^\infty e^{-(\underline{s}-a)t} dt \\
&\quad\big|\quad \textit{integration of the exponential function} \\
&=\; \left[-\frac{1}{\underline{s}-a} \cdot e^{-(\underline{s}-a)t}\right]_{t=0}^{t=\infty} \\
&\quad\big|\quad \textit{evaluation of the antiderivative} \\
&=\; \frac{1}{\underline{s}-a}
\end{aligned}$$

The Laplace transform of the exponential function is thus:

$$\mathcal{L}\Big\{\theta(t)\, e^{at}\Big\}(\underline{s}) \;=\; \frac{1}{\underline{s}-a} \tag{2.18}$$

2.4.4 Matrix Exponential Function

In mathematics, the matrix exponential function is defined analogously to the ordinary (*scalar*) exponential function. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a real $n \times n$ matrix and it holds:

$$e^{\mathbf{A}\cdot t} \;=\; \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \mathbf{A}^k \cdot t^k \tag{2.19}$$

Furthermore, the derivative and antiderivative also behave equivalently to the ordinary exponential function. Therefore, for the Laplace transformation, the result (2.18) can be used and extended to multidimensional quantities. The fraction then becomes the inverse:

$$\mathcal{L}\Big\{\theta(t)\, e^{\mathbf{A}\cdot t}\Big\}(\underline{s}) \;=\; \Big(\underline{s}\cdot\mathbf{I} - \mathbf{A}\Big)^{-1} \tag{2.20}$$

2.4.5 Complex Exponential Function – Sine and Cosine

The Laplace transformation following the same scheme can also be performed for an exponential function with imaginary exponent. To do this, replace the exponent $a \to j\omega$ in (2.18). Then:

$$\begin{aligned}
\mathcal{L}\Big\{\theta(t)\, e^{j\omega t}\Big\}(\underline{s}) &= \frac{1}{\underline{s} - j\omega} \\
&\mid \textit{extend with complex conjugate} \\
&= \frac{1}{\underline{s} - j\omega} \cdot \frac{\underline{s} + j\omega}{\underline{s} + j\omega} \\
&\mid \textit{multiply out} \\
&= \frac{\underline{s} + j\omega}{\underline{s}^2 + \omega^2} \\
&\mid \textit{split into real and imaginary parts} \\
&= \frac{\underline{s}}{\underline{s}^2 + \omega^2} + j \cdot \frac{\omega}{\underline{s}^2 + \omega^2}
\end{aligned}$$

Euler's formula provides a relationship of sine and cosine to the complex exponential function:

$$e^{j\omega t} = cos(\omega t) + j \cdot sin(\omega t) \tag{2.21}$$

Now replace the just transformed complex exponential function with Euler's formula:

$$\mathcal{L}\Big\{\theta(t)\, cos(\omega t) + j \cdot \theta(t)\, sin(\omega t)\Big\}(\underline{s}) = \frac{\underline{s}}{\underline{s}^2 + \omega^2} + j \cdot \frac{\omega}{\underline{s}^2 + \omega^2}$$

Through the linearity of the Laplace transformation, the real and imaginary parts can be transformed separately. This allows an assignment of the Laplace transforms of $\theta(t)\, cos(\omega t)$ and $\theta(t)\, sin(\omega t)$:

$$\mathcal{L}\Big\{\theta(t)\, cos(\omega t)\Big\}(\underline{s}) = \frac{\underline{s}}{\underline{s}^2 + \omega^2} \tag{2.22}$$

$$\mathcal{L}\Big\{\theta(t)\, sin(\omega t)\Big\}(\underline{s}) = \frac{\omega}{\underline{s}^2 + \omega^2} \tag{2.23}$$

2.4.6 DIRAC DELTA / DIRAC DISTRIBUTION

A somewhat more exotic construct than those already treated is the Dirac distribution, or the Dirac delta.

The Dirac delta assigns to each arbitrarily often differentiable function $g(t)$ (*often also called a test function*) a real or complex number $\delta(g) = g(0)$. This corresponds to the evaluation of the function at the point $t = 0$.

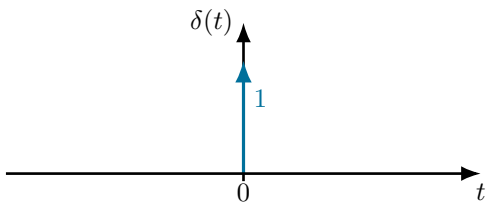


Figure 2.4: Visualization of the Dirac delta weighted with 1 in the time domain as an upward arrow.

If the function to be transformed is chosen as $f(t) = \delta(t)$, it follows with the definition of the Laplace transformation:

$$\begin{aligned}
\mathcal{L}\Big\{\delta(t)\Big\}(\underline{s}) &= \int_0^\infty \delta(t) \cdot e^{-\underline{s}t} dt \\
&\mid \textit{integral is evaluated for } t = 0 \\
&= e^{-\underline{s}t}\Big|_{t=0} \\
&\mid \textit{it follows} \\
&= 1
\end{aligned}$$

The Laplace transform of the Dirac delta is thus a constant in the Laplace domain:

$$\mathcal{L}\Big\{\delta(t)\Big\}(\underline{s}) = 1 \tag{2.24}$$

2.4.7 DERIVATIVE OF THE DIRAC DELTA

In some cases the Laplace transformation of the derivative of the Dirac delta is needed (*the derivation can be done oneself ... one should look at distributional differentiation for this*). However, for the n-th derivative of the Dirac delta:

$$\mathcal{L}\Big\{\frac{d^n}{dt^n}\delta(t)\Big\}(\underline{s}) = \underline{s}^n \tag{2.25}$$

Chapter 3

System Analysis in the Laplace Domain

3.1 General Procedure

The core of the following considerations is the derivative theorem (2.5). This transforms the differential-algebraic system of equations in the time domain into a purely algebraic system of equations in the Laplace domain. The basic assumption / claim is that the detour via the Laplace domain is significantly easier than the direct solution in the time domain and the alleged additional effort through the extra transformation steps is therefore justified.

The general procedure for solving with the Laplace transformation thus includes as the first step the transfer of the problem into the Laplace domain. More complicated functions can be decomposed into individual functions due to linearity. These can then be transformed individually with the already motivated correspondences and calculation rules (*here look up the notorious table book ...*). The problem in the Laplace domain can now be solved for the sought quantity through simple algebraic transformations (*plus, minus, times, divided*).

The solution in the time domain can then be determined via the inverse Laplace transformation of the solution in the Laplace domain. For this too, the correspondences and calculation rules can be used due to uniqueness. However, these often cannot be applied directly. A partial fraction decomposition then usually enables the assignment of the known functions and thus an inverse transformation.

If one knows the calculation rules and the correspondences of the most common functions, one can make the transition between the time domain and Laplace domain without great effort, without having to solve even a single integral. The additional effort for the transformations $\mathcal{L}$ and $\mathcal{L}^{-1}$ is assumed to be negligible, so that the main effort is the transformation of the algebraic system of equations in the Laplace domain into a suitable form for the inverse transformation.

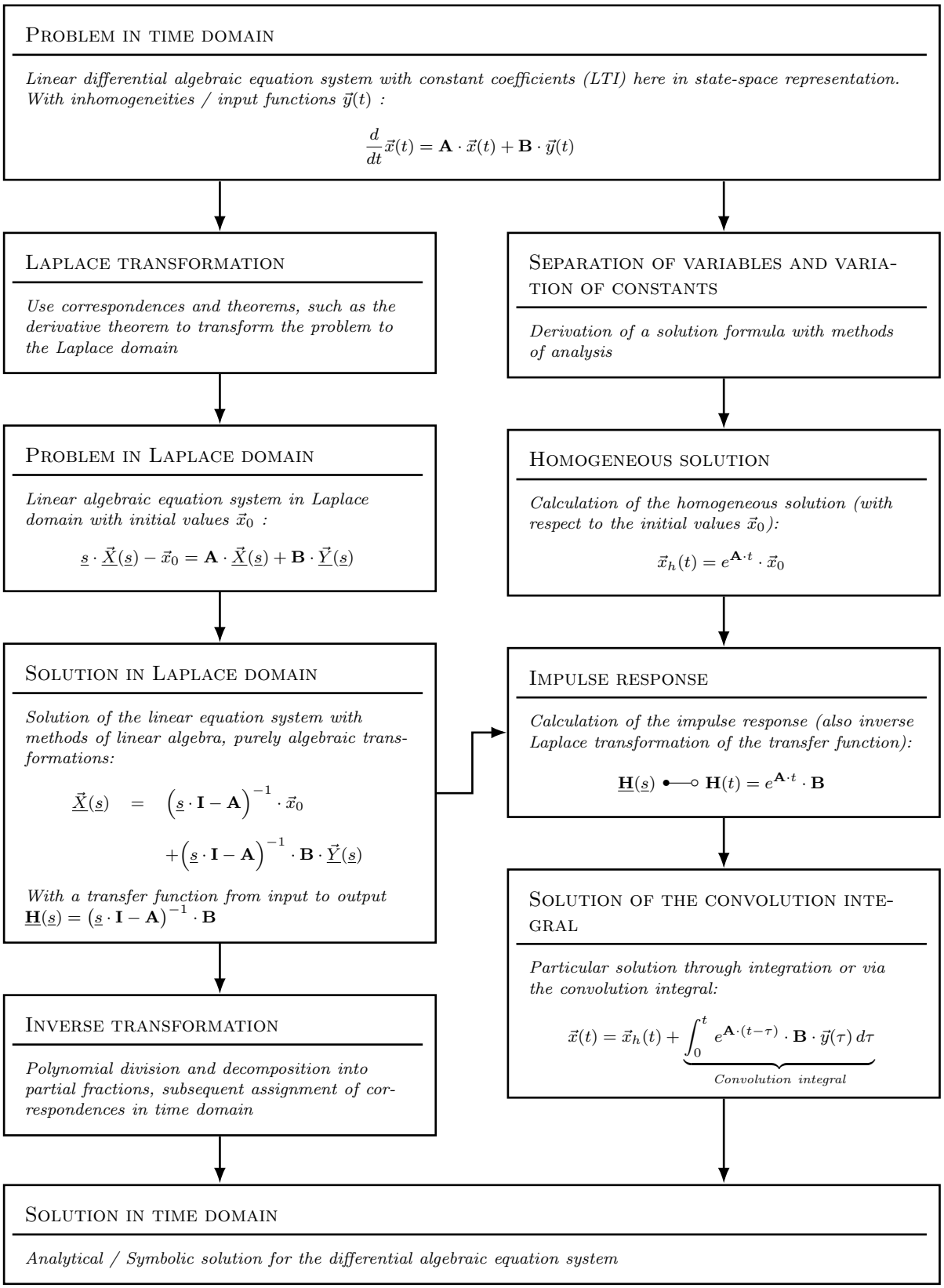


Figure 3.1: Representation of the general procedure for solving differential equations / initial value problems with the Laplace transformation and comparison with the solution in the time domain as a block diagram

3.2 The Transfer Function

The transfer function or also system function mathematically describes in engineering system theory the relationship between the input and output signal of a linear dynamic system in an image space (*here in the image domain of the Laplace transformation, also Laplace domain*). With the help of the transfer function, the output signal or the reaction of the system can be determined (*alternative to calculation in the time domain*) for any continuous input signal. This solution approach is usually simpler than solving differential equations in the time domain.

The properties of the transfer function, some fundamental relationships and analysis methods based on them are treated in this subsection.

3.2.1 Definition of the Transfer Function

The transfer function is a complex-valued function with a complex argument and is defined as the ratio of an input quantity $\underline{Y}(\underline{s})$ to an output quantity $\underline{X}(\underline{s})$ in the Laplace domain:

$$\underline{H}(\underline{s}) \;:=\; \frac{\underline{X}(\underline{s})}{\underline{Y}(\underline{s})} \tag{3.1}$$

The transfer function thus describes in the Laplace domain the effect of the system on the input quantity at the system output. For systems with multiple inputs and outputs, the transfer function becomes a transfer matrix and the superposition principle applies. The initial values $\vec{x}_0$ of the system are not contained in the transfer function. Thus it only describes the system behavior as a response from a rest state.

3.2.2 System Description via the Transfer Function

When neglecting the initial values $x_0 = 0$ (*... also called the response from the rest state*), the homogeneous solution is omitted and the system behavior is completely described by the particular solution. In the Laplace domain, the system description and characterization therefore takes place exclusively on the basis of the transfer function:

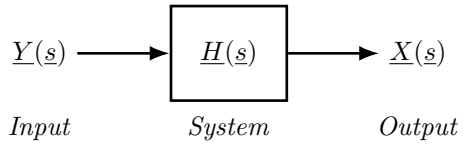


Figure 3.2: Visualization of the system as a block diagram in the Laplace domain, which is characterized by the transfer function $\underline{H}(\underline{s})$. Multiple subsystems can thus be combined in a signal flow graph.

The system response then results in the Laplace domain as a multiplication of the excitation (*the input signal*) with the transfer function:

$$\underline{X}(\underline{s}) \;=\; \underline{H}(\underline{s}) \cdot \underline{Y}(\underline{s}) \tag{3.2}$$

3.2.3 Representation Forms of the Transfer Function

The transfer function is characteristic of the system and can look different depending on the system. For linear time-invariant systems, it is a rational function with a numerator and denominator polynomial and therefore has various common representation forms:

$$\begin{aligned}
\underline{H}(\underline{s}) &= \frac{\underline{X}(\underline{s})}{\underline{Y}(\underline{s})} \quad \ldots \quad \textit{implicitly as ratio of input and output} \\
&= \frac{\underline{P}(\underline{s})}{\underline{Q}(\underline{s})} \quad \ldots \quad \textit{symbolically as numerator and denominator polynomial} \\
&= \frac{b_m\,\underline{s}^m + \cdots + b_2\,\underline{s}^2 + b_1\,\underline{s} + b_0}{a_n\,\underline{s}^n + \cdots + a_2\,\underline{s}^2 + a_1\,\underline{s} + a_0} \quad \ldots \quad \textit{explicitly in polynomial form} \\
&= \underline{c}\frac{(\underline{s} - \underline{z}_m)\cdots(\underline{s} - \underline{z}_1)}{(\underline{s} - \underline{p}_n)\cdots(\underline{s} - \underline{p}_1)} \quad \ldots \quad \textit{explicitly with poles / zeros}
\end{aligned}$$

3.2.4 Poles and Zeros of the Transfer Function

For polynomials, their degree determines the number of zeros. For the transfer function in polynomial form, the degree of the respective polynomial is the highest power of the argument, i.e., the variable $\underline{s}$:

$$grad\Big\{\underline{P}(\underline{s})\Big\} = m \qquad and \qquad grad\Big\{\underline{Q}(\underline{s})\Big\} = n \tag{3.3}$$

Here $\underline{P}(\underline{s})$ has exactly m zeros and $\underline{Q}(\underline{s})$ has exactly n zeros. With the fundamental theorem of algebra, the polynomials can thus also be expressed via their zeros. Let $\underline{z}_i$ with $i = 1, \ldots, m$ be the zeros of $\underline{P}(\underline{s})$ and $\underline{p}_i$ with $i = 1, \ldots, n$ be the zeros of $\underline{Q}(\underline{s})$:

$$\underline{H}(\underline{s}) \;=\; \frac{\underline{P}(\underline{s})}{\underline{Q}(\underline{s})} \;=\; \underline{c}\frac{\overbrace{(\underline{s} - \underline{z}_m)\cdots(\underline{s} - \underline{z}_2)(\underline{s} - \underline{z}_1)}^{zeros}}{\underbrace{(\underline{s} - \underline{p}_n)\cdots(\underline{s} - \underline{p}_2)(\underline{s} - \underline{p}_1)}_{poles}} \tag{3.4}$$

If $\underline{P}(\underline{s})$ and $\underline{Q}(\underline{s})$ now have common zeros, i.e., at least one combination i, j exists such that $\underline{z}_i = \underline{p}_j$, then the corresponding pair can be removed from the transfer function (*this is then polynomial division*). This step of removing equal zeros is absolutely necessary, since otherwise no statements about the system can be made based on the structure of the transfer function – i.e., the nature of the numerator and denominator polynomials! In the case of coprimality (*no common zeros*) of denominator and numerator, the zeros of the denominator $\underline{p}_i$ are the poles of the transfer function and the zeros of the numerator $\underline{z}_i$ are the zeros of the transfer function:

$$\Big|\underline{H}(\underline{s})\Big|\Big|_{\underline{s}=\underline{p}_i} \;\longrightarrow\; \infty \qquad and \qquad \underline{H}(\underline{s})\Big|_{\underline{s}=\underline{z}_i} \;=\; 0 \tag{3.5}$$

3.2.5 POLES AND ZEROS IN THE COMPLEX PLANE

The transfer function is also often represented graphically in the complex $\underline{s}$-plane. The poles $\underline{p}_i$ are represented according to real and imaginary parts as crosses and the zeros $\underline{z}_i$ as circles. This is a widespread convention. In the case of coprimality, a transfer function $\underline{H}(\underline{s})$ has the following general structure with respect to its poles and zeros:

$$\underline{H}(\underline{s}) = h_0 \cdot \frac{\overbrace{(\underline{s}-\alpha_z)}^{\text{real zero}} \cdots \overbrace{(\underline{s}-(\alpha_z+j\omega_z))(\underline{s}-(\alpha_z-j\omega_z))}^{\text{complex zero pair}} \cdots \overbrace{(\underline{s}-j\omega_z)(\underline{s}+j\omega_z)}^{\text{imaginary zero pair}}}{\underbrace{(\underline{s}-\alpha_p)}_{\text{real pole}} \cdots \underbrace{(\underline{s}-(\alpha_p+j\omega_p))(\underline{s}-(\alpha_p-j\omega_p))}_{\text{complex pole pair}} \cdots \underbrace{(\underline{s}-j\omega_p)(\underline{s}+j\omega_p)}_{\text{imaginary pole pair}}} \tag{3.6}$$

Most representations of the transfer function in the complex plane are limited to the position of the poles and zeros in the form of crosses and circles. They completely characterize the normalized transfer function (*without the constant* h_0) and are therefore sufficient for analysis.

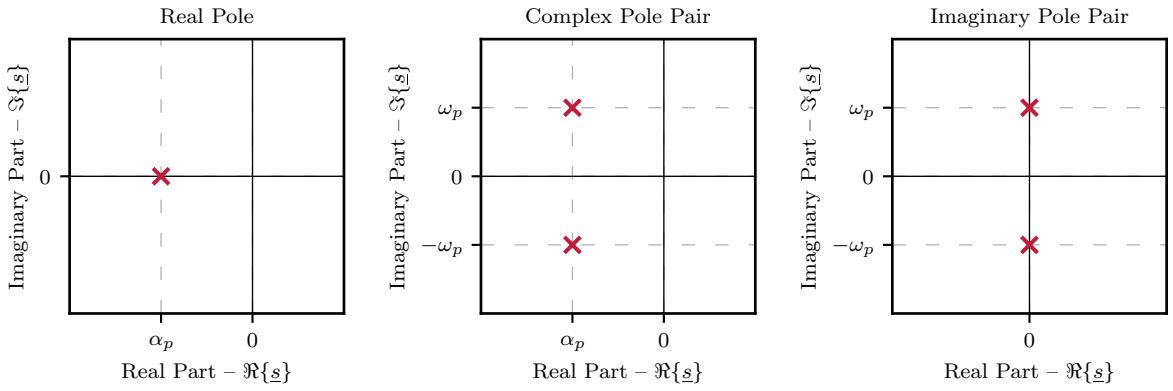


Figure 3.3: Visualization of the poles of the transfer function in the complex $\underline{s}$-plane by crosses

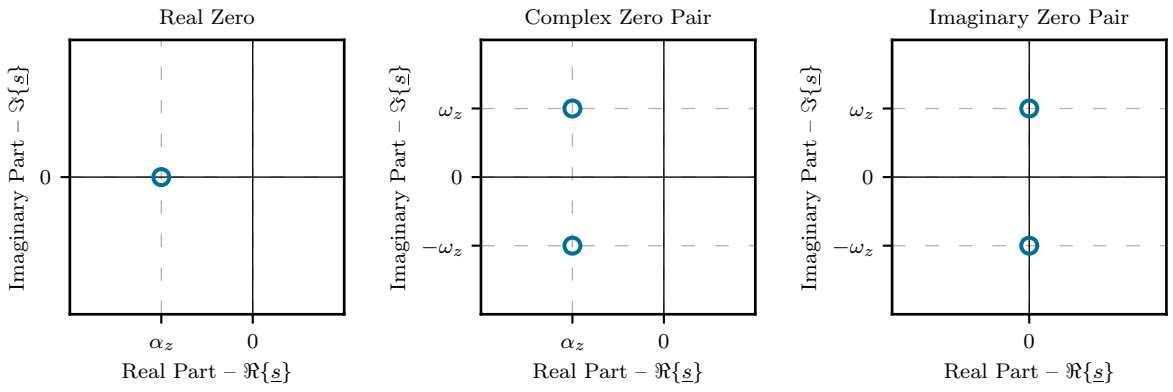


Figure 3.4: Visualization of the zeros of the transfer function in the complex $\underline{s}$-plane by circles (*sometimes also by points*)

Particularly the question of in which quadrant of the complex plane the poles and zeros lie and in what relationship they are to each other can be clearly recognized in this form of representation. Furthermore, one can relatively easily conclude an assessment of the ratio of real and imaginary parts of the poles and zeros, which represents a measure for damping and oscillation capability.

3.2.6 Relationship between Poles and Eigenvalues

The poles of the transfer function can be determined, when numerator and denominator polynomial are coprime, via the zeros of the denominator polynomial. These are eigenvalues of the system or eigenvalues of the state-space model:

$$\underline{Q}(\underline{s}) \;=\; 0 \qquad \Leftrightarrow \qquad \underline{s} \;=\; \underline{\lambda}_i \;=\; \underline{p}_i \quad \textit{for all} \quad i = 1, \ldots, n$$

The denominator polynomial is a sub-polynomial of the characteristic equation of the state-space model (1.10). In general, therefore, the degree of the denominator polynomial is a lower bound for the order of the system, since then:

$$dim\Big\{\, \mathbf{A} \,\Big\} \;=\; grad\Big\{\underline{\Delta}(\underline{s})\Big\} \;\geq\; grad\Big\{\underline{Q}(\underline{s})\Big\} \tag{3.7}$$

Intuitively, this means that not necessarily the entire system – that is, all state variables – must have an influence on the output. This is often the case particularly for systems with multiple inputs and outputs (*MIMO*) and for large systems that contain, for example, independent subcomponents (*although this does not apply to the examples in this script, one should still keep this in mind*).

3.2.7 Transfer Function in Pole-Residue Form

An important intermediate step in transferring the system from the Laplace domain back to the time domain is partial fraction decomposition. Since the transfer function is a rational function with numerator and denominator polynomials, the poles can also be specified as partial fractions with respective residues (*coefficients in the partial fraction decomposition*). For the general case, we now assume that the numerator and denominator polynomials are coprime and that multiple eigenvalues $\underline{\lambda}_i$ (*i.e., the poles*) are specified via their respective multiplicity μ_i. Then n_μ is the number of distinct eigenvalues and the transfer function has the representation:

$$\underline{H}(\underline{s}) \;=\; \frac{\underline{P}(\underline{s})}{\prod_{i=1}^{n_\mu} \left(\underline{s} - \underline{\lambda}_i\right)^{\mu_i}} \qquad mit \qquad n = \sum_{i=1}^{n_\mu} \mu_i \tag{3.8}$$

If the degree of the numerator is greater than or equal to the degree of the denominator, i.e., $m \geq n$ is satisfied, the degree of the numerator polynomial must be reduced by polynomial division before partial fraction decomposition. Taking into account the multiplicities in the partial fraction decomposition, the transfer function in pole-residue form is then obtained:

$$\underline{H}(\underline{s}) \;=\; \underbrace{\sum_{k=0}^{m-n} c_k \cdot \underline{s}^k}_{\textit{nur wenn } m \geq n} \;+\; \underbrace{\sum_{i=1}^{n_\mu} \Bigg(\underbrace{\sum_{k=1}^{\mu_i} \underline{r}_{i,k} \cdot \frac{1}{\left(\underline{s} - \underline{\lambda}_i\right)^k}}_{\textit{sum over multiplicities}} \Bigg)}_{\textit{sum over distinct poles}} \tag{3.9}$$

This general form of the transfer function is the most useful, since it clearly shows which contribution to the system output – represented by the residues $\underline{r}_{i,k}$ – the corresponding eigenvalues / state variables have. Furthermore, the transfer function in this form can be directly transferred to the time domain via the correspondences to the partial fractions.

3.2.8 NOTE ON PARTIAL FRACTION DECOMPOSITION

To bring the transfer function into pole-residue form, the following methods can be applied: polynomial division, residue theorem (cover-up method), and coefficient comparison.

3.2.9 SYMMETRY RELATION OF THE TRANSFER FUNCTION

The transfer function can be represented (*among others*) in the following general polynomial form:

$$\underline{H}(\underline{s}) \;=\; \frac{\underline{P}(\underline{s})}{\underline{Q}(\underline{s})} \;=\; \frac{b_m\,\underline{s}^m + \cdots + b_2\,\underline{s}^2 + b_1\,\underline{s} + b_0}{a_n\,\underline{s}^n + \cdots + a_2\,\underline{s}^2 + a_1\,\underline{s} + a_0} \tag{3.10}$$

Here, the coefficients b_i and a_i are constant and real. $\underline{P}(\underline{s})$ is the numerator polynomial and $\underline{Q}(\underline{s})$ is the denominator polynomial. For both polynomials (*here exemplified by the numerator polynomial*) holds:

$$\begin{aligned}
\underline{P}(\underline{s}^*) \;&=\; b_m\left(\underline{s}^*\right)^m + \cdots + b_2\left(\underline{s}^*\right)^2 + b_1\,\underline{s}^* + b_0 \\
&\quad\;|\quad \textit{Sum form of the polynomial} \\
&=\; \sum_{i=0}^{m} b_i\left(\underline{s}^*\right)^i \\
&\quad\;|\quad \textit{For complex numbers holds } (\underline{z}\cdot\underline{w})^* = \underline{z}^*\cdot\underline{w}^* \\
&=\; \sum_{i=0}^{m} b_i\left(\underline{s}^i\right)^* \\
&\quad\;|\quad \textit{Complex conjugation is } \mathbb{R}\textit{-linear} \\
&=\; \Big(\sum_{i=0}^{m} b_i\,\underline{s}^{\,i}\Big)^* \\
&\quad\;|\quad \textit{Substitute polynomial} \\
&=\; \underline{P}^*(\underline{s})
\end{aligned}$$

This symmetry relation for polynomials is sometimes useful and generally holds for linear systems with constant and real system parameters. It follows that this symmetry relation is also satisfied for the transfer function as a quotient of two polynomials with real coefficients:

$$\underline{H}(\underline{s}^*) \;=\; \frac{\underline{P}(\underline{s}^*)}{\underline{Q}(\underline{s}^*)} \;=\; \frac{\underline{P}^*(\underline{s})}{\underline{Q}^*(\underline{s})} \;=\; \left(\frac{\underline{P}(\underline{s})}{\underline{Q}(\underline{s})}\right)^* \;=\; \underline{H}^*(\underline{s}) \tag{3.11}$$

3.3 State-Space Representation (SISO) in the Laplace Domain

We have already motivated the description of the system with one input and one output (*SISO*) in the time domain via a state-space representation, in which the state variables $\vec{x}$ of the system are excited by the input $y(t)$ and the output $u(t)$ is a linear combination of these:

$$\begin{aligned} \frac{d}{dt}\vec{x}(t) &= \mathbf{A}\cdot\vec{x}(t) \;+\; \vec{b}\cdot y(t) \\ u(t) &= \vec{c}^{\,T}\cdot\vec{x}(t) \;+\; d\cdot y(t) \end{aligned}$$

3.3.1 Transition to the Laplace Domain

The transformation of the system of equations into the Laplace domain using the derivative theorem (2.5) together with the linearity of the Laplace transformation yields:

$$\underline{s}\cdot\vec{\underline{X}}(\underline{s}) - \vec{x}_0 = \mathbf{A}\cdot\vec{\underline{X}}(\underline{s}) \;+\; \vec{b}\cdot\underline{Y}(\underline{s}) \tag{3.12}$$

$$\underline{U}(\underline{s}) = \vec{c}^{\,T}\cdot\vec{\underline{X}}(\underline{s}) \;+\; d\cdot\underline{Y}(\underline{s}) \tag{3.13}$$

The system of differential equations of the state-space representation is thus a purely algebraic system of equations in the Laplace domain. A transformation of the first equation for the state variables yields:

$$\vec{\underline{X}}(\underline{s}) = \underbrace{\left(\underline{s}\cdot\mathbf{I}-\mathbf{A}\right)^{-1}\cdot\vec{x}_0}_{\substack{\textit{homogeneous solution}\\ \textit{in the Laplace domain}}} + \underbrace{\left(\underline{s}\cdot\mathbf{I}-\mathbf{A}\right)^{-1}\vec{b}\cdot\underline{Y}(\underline{s})}_{\substack{\textit{particular solution}\\ \textit{in the Laplace domain}}} \tag{3.14}$$

From this, a relationship between input and output can now be established by substituting into the second equation of the state-space model in the Laplace domain:

$$\underline{U}(\underline{s}) = \underbrace{\vec{c}^{\,T}\left(\underline{s}\cdot\mathbf{I}-\mathbf{A}\right)^{-1}\cdot\vec{x}_0}_{\textit{homogeneous component}} + \underbrace{\left(\vec{c}^{\,T}\left(\underline{s}\cdot\mathbf{I}-\mathbf{A}\right)^{-1}\vec{b}+d\right)}_{\underline{H}(\underline{s})\ \textit{transfer function}}\cdot\underline{Y}(\underline{s}) \tag{3.15}$$

3.3.2 Formulation of a Transfer Function

Based on this representation, when neglecting the initial values $\vec{x}_0 = 0$, a direct relationship is now obtained between input $\underline{Y}(\underline{s})$ and output $\underline{U}(\underline{s})$ in the Laplace domain. This connection is the transfer function of the system:

$$\underline{H}(\underline{s}) := \vec{c}^{\,T}\left(\underline{s}\cdot\mathbf{I}-\mathbf{A}\right)^{-1}\vec{b}+d \tag{3.16}$$

The dynamic behavior in this representation of the transfer function is determined solely by the matrix pencil $\left(\underline{s}\cdot\mathbf{I}-\mathbf{A}\right)$. The inverse of this matrix pencil exists (*is regular*) only if the determinant does not vanish. If the vector $\vec{c}$ has zero entries, not all state variables of the system are mapped to the output. Then not all eigenvalues of the system are contained in the poles.

3.3.3 Relationship between Eigenvalues and Poles

From the transfer function in equation (3.16) it becomes clear that the poles must be contained in the matrix pencil $\left(\underline{s} \cdot \mathbf{I} - \mathbf{A}\right)$, more precisely in its inverse. The inverse can be symbolically specified using Cramer's rule:

$$\left(\underline{s} \cdot \mathbf{I} - \mathbf{A}\right)^{-1} \;=\; \frac{1}{det\left(\underline{s} \cdot \mathbf{I} - \mathbf{A}\right)} \cdot adj\left(\underline{s} \cdot \mathbf{I} - \mathbf{A}\right) \tag{3.17}$$

This makes clear that the inverse of a matrix always contains the determinant of the matrix itself in the denominator. The matrix is therefore only invertible (*regular, non-singular*) if the determinant is not equal to zero, or singular if:

$$\underline{\Delta}(\underline{s}) \;=\; det\Big(\underline{s} \cdot \mathbf{I} - \mathbf{A}\Big) \;=\; 0 \tag{3.18}$$

The zeros of the characteristic equation $\underline{\Delta}(\underline{s})$ are simultaneously the eigenvalues of the state matrix $\mathbf{A}$ of the state-space model and the singularities of the transfer function. Since, intuitively speaking, the denominator becomes zero at the singularities and thus the transfer function goes to infinity in magnitude, the singularities are also the poles of the transfer function.

3.4 System Level and Block Diagrams

The representation of the system in the Laplace domain is particularly elegant, since especially large or complex systems composed of many interconnected subsystems can thereby be reduced in their behavior to their respective transfer functions. The interaction of the subsystems can then be represented by a connection of the transfer functions. At the system level, block diagrams are frequently used for the visualization of these connections in a signal flow diagram, which when interconnected represent the connections of the subsystems. The most important connections are presented below. From these cases, all possible connections of linear time-invariant subsystems with each other can be constructed.

3.4.1 Cascading

In a cascading (*also series connection or chain connection*) of two or more systems, the output of the first system forms the input of the second system.

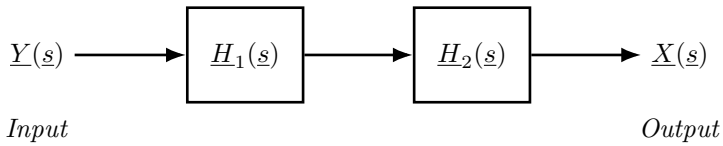


Figure 3.5: Block diagram of a cascading / series connection of two subsystems, characterized by the transfer functions $\underline{H}_1(\underline{s})$ and $\underline{H}_2(\underline{s})$.

Due to the multiplicative relationship between input and transfer function with the output in the Laplace domain, the following holds for the overall transfer function:

$$\underline{H}(\underline{s}) \;=\; \underline{H}_1(\underline{s}) \cdot \underline{H}_2(\underline{s}) \tag{3.19}$$

3.4.2 Parallel Connection

In a parallel connection of two or more subsystems, the input quantity acts equally on all inputs of the subsystems. Their outputs are superimposed to form a common output.

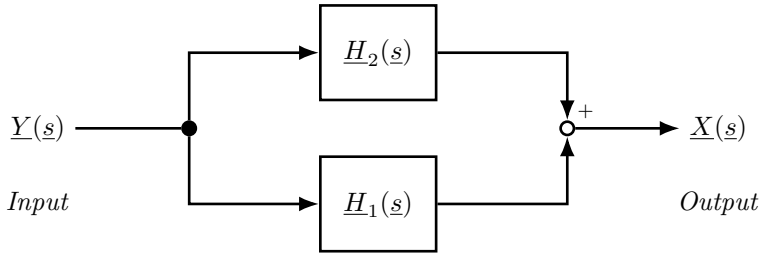


Figure 3.6: Block diagram of a parallel connection of two subsystems, characterized by the transfer functions $\underline{H}_1(\underline{s})$ and $\underline{H}_2(\underline{s})$.

The linear superposition of the partial outputs to a common output through summation corresponds to the application of the superposition principle, whose validity is a direct consequence of the linearity of the system. The overall transfer function is accordingly:

$$\underline{H}(\underline{s}) \;=\; \underline{H}_1(\underline{s}) \;+\; \underline{H}_2(\underline{s}) \tag{3.20}$$

3.4.3 Feedback

Feedback is of high relevance for many technical systems (*especially in control engineering*). Here, the signal from the system output is fed back to the input. This can lead to self-amplification and instability. Even if all subsystems are asymptotically stable, the overall system can still be unstable due to the feedback.

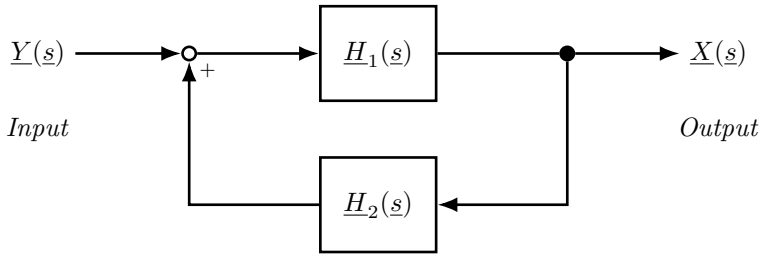


Figure 3.7: Block diagram of a feedback loop consisting of two subsystems, characterized by the transfer functions $\underline{H}_1(\underline{s})$ and $\underline{H}_2(\underline{s})$.

Due to the feedback, the overall transfer function is somewhat more difficult to establish. The following equation represents, as an intermediate step, the output quantity as a function of the input quantity and the output quantity itself (*through the feedback*):

$$\underline{X}(\underline{s}) \;=\; \underline{H}_1(\underline{s}) \cdot \underline{Y}(\underline{s}) \;+\; \underline{H}_1(\underline{s}) \cdot \underline{H}_2(\underline{s}) \cdot \underline{X}(\underline{s}) \quad \Rightarrow \quad \underline{X}(\underline{s}) \;=\; \underbrace{\frac{\underline{H}_1(\underline{s})}{1 - \underline{H}_1(\underline{s}) \cdot \underline{H}_2(\underline{s})}}_{\underline{H}(\underline{s})} \cdot \underline{Y}(\underline{s}) \tag{3.21}$$

3.4.4 Composite System Using the Example of Scattering Parameters

Often, complicated systems with multiple inputs and outputs are modeled by connecting multiple subsystems. A prominent example from electrical engineering is the port representation. Here, the interaction with the system (*for example, a transmission line or an amplifier*) occurs through ports, each with one input and one output. This representation enables the coupling and feedback of several interconnected systems through the two port quantities (*often current and voltage, or forward and backward traveling waves*).

The scattering parameters presented here are particularly well suited for modeling systems in high-frequency applications, where electrical signals can be well described as propagating waves (*this also applies to optical systems, such as lenses*). The subsystems each characterize the interaction between the incident and outgoing waves. $\underline{S}_{11}(\underline{s})$ and $\underline{S}_{22}(\underline{s})$ can, for example, be interpreted as reflection transfer functions and $\underline{S}_{12}(\underline{s})$ and $\underline{S}_{21}(\underline{s})$ as transmission transfer functions.

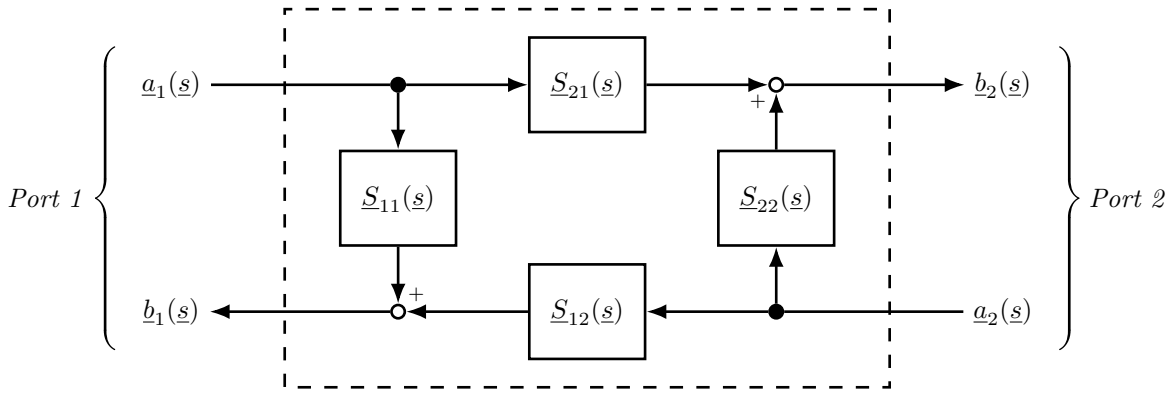


Figure 3.8: Block diagram of the subsystems of a two-port network in **S**-parameter representation. Connection of the subsystems to form an overall system with two inputs and two outputs.

The linear superposition of the outputs of the subnetworks to the outputs of the entire system yields a linear system of equations with two equations and two (*to be calculated given the input quantities*) output quantities. The system of equations can be given in matrix notation, where the matrix entries are the transfer functions of the subsystems. The overall system can thus be characterized by a transfer matrix, which allows a much more compact notation:

$$\begin{bmatrix} \underline{b}_1(\underline{s}) \\ \underline{b}_2(\underline{s}) \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{S}_{11}(\underline{s}) & \underline{S}_{12}(\underline{s}) \\ \underline{S}_{21}(\underline{s}) & \underline{S}_{22}(\underline{s}) \end{bmatrix}}_{\underline{\mathbf{S}}(\underline{s})} \cdot \begin{bmatrix} \underline{a}_1(\underline{s}) \\ \underline{a}_2(\underline{s}) \end{bmatrix} \tag{3.22}$$

Chapter 4

Transient Response Analysis

A transient is a settling process or a section in a signal that indicates a non-stationary process. The term transient comes from Latin and means transition. The transient behavior of the system is thus the settling behavior or the system behavior during the transition from one system state to another. Transient response analysis covers the system behavior close to the emergence time or the beginning of observation $t = 0$ of the system. For the analysis of transient behavior, the homogeneous solution – that is, the decay of initial values in stable systems – and the system behavior with changing excitation – such as the impulse and step response – are particularly interesting.

General system properties such as stability are also motivated in this chapter based on transient behavior (*here based on the impulse response*). Additionally, the settled state (*often also called stationary behavior or steady-state*) is introduced as a consequence of the settling process of stable systems, which allows a simplified solution representation after a settling time $t \gg \tau_{max}$.

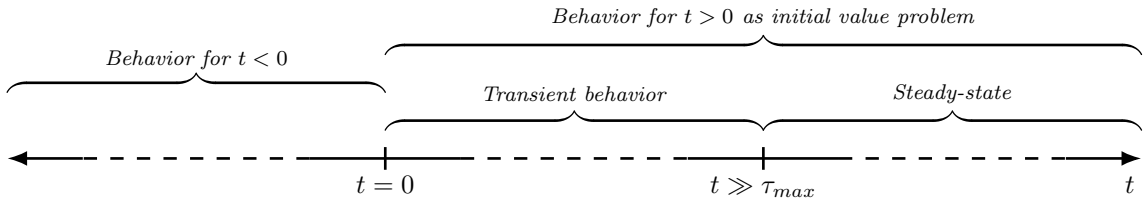


Figure 4.1: Visualization and division of the relevant time periods in transient response analysis using a timeline.

4.1 The Impulse Response

The impulse response plays a particularly important role in system theory. It depends exclusively on the system parameters – particularly the eigenvalues – (... *and not on their inputs*) and is therefore characteristic of the system. Moreover, it is a component of the homogeneous solution. In the time domain, the dynamic behavior of the system can therefore be characterized via the impulse response.

4.1.1 Motivation of the Impulse Response in the Laplace Domain

To further emphasize the significance of the impulse response, a general system is assumed in the following, which is described in the Laplace domain (*for this, the chapter on analysis in the Laplace domain should be read beforehand*) by the transfer function $\underline{H}(\underline{s})$. Now let the excitation of the system in the time domain be given as a Dirac delta and for the Laplace transform (2.24) holds:

$$y(t) \;=\; \delta(t) \qquad \circ\!\!-\!\!\bullet \qquad \underline{Y}(\underline{s}) \;=\; 1 \tag{4.1}$$

Correspondingly, the response / output in the Laplace domain with the transfer function is given by:

$$x(t) \;\; \circ\!\!-\!\!\bullet \;\; \underline{X}(\underline{s}) \;=\; \underline{H}(\underline{s}) \cdot \underline{Y}(\underline{s}) \;=\; \underline{H}(\underline{s}) \cdot 1 \;=\; \underline{H}(\underline{s}) \;\; \bullet\!\!-\!\!\circ \;\; h(t)$$

In the time domain, the following relationship follows due to the multiplication of two functions through the convolution theorem (2.8) and the evaluation of the convolution integral through the Dirac delta:

$$x(t) \;=\; \Big(h(\tau) * \delta(\tau)\Big)(t) \;=\; h(t) \tag{4.2}$$

When the system is excited by a Dirac delta, the output is thus dependent exclusively on the system parameters themselves via the transfer function. This allows a characterization of the system via the impulse response in the time domain and is part of the transformation $\mathcal{H}$ introduced at the beginning of this document.

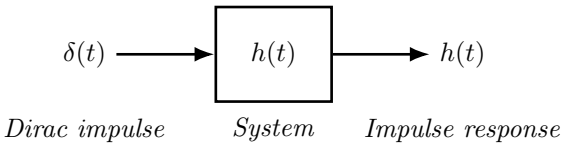


Figure 4.2: Representation of the system as a block diagram in the time domain. Excitation of the system by a Dirac delta at the input yields the impulse response at the output.

4.1.2 Definition of the Impulse Response

The excitation of the system by a Dirac delta $\delta(t)$ thus corresponds in the Laplace domain to a multiplication by 1. The response of the system in the time domain to a Dirac delta – that is, the impulse response – is therefore exactly the inverse Laplace transform of the transfer function:

$$\underline{H}(\underline{s}) \;\; \bullet\!\!-\!\!\circ \;\; h(t) \tag{4.3}$$

4.1.3 System Description via the Impulse Response

From the relationship between transfer function and impulse response (4.3), it becomes clear that if the transfer function completely describes the system behavior in the Laplace domain (*completely with respect to an excitation*), the same applies to the impulse response in the time domain. The system with the input function $y(t)$ and the output function $x(t)$ is thereby characterized by the impulse response $h(t)$. This representation in the time domain leads to a connection of input and output quantities via the convolution integral. The general system response in the time domain is therefore analogous to the observations in the chapter on system description in the time domain.

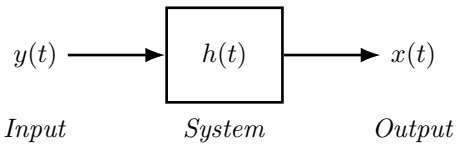


Figure 4.3: Characterization of the system in the time domain via the impulse response $h(t)$. Representation as a block diagram in the time domain.

4.1.4 General Form of the Impulse Response

With the general transfer function in pole-residue form in equation (3.9), a general form of the impulse response can be derived. The starting point is the completely partial-fraction-decomposed transfer function:

$$\underline{H}(\underline{s}) \;=\; \sum_{k=0}^{m-n} c_k \cdot \underline{s}^k \;+\; \sum_{i=1}^{n_\mu} \left(\sum_{k=1}^{\mu_i} \underline{r}_{i,k} \cdot \frac{1}{\left(\underline{s} - \underline{\lambda}_i\right)^k} \right)$$

For this, the individual partial fractions are transformed back to the time domain. In particular, the correspondence to the derivative of the Dirac delta (2.25) is used for the polynomial terms (*only in the case that the numerator degree is greater than the denominator degree*) and the correspondence to the polynomial term (2.16) in combination with the damping theorem (2.12):

$$c_k \cdot \underline{s}^k \quad \bullet\!\!-\!\!\circ \quad c_k \cdot \delta^{(k)}(t)$$

$$\underline{r}_{i,k} \cdot \frac{1}{\left(\underline{s} - \underline{\lambda}_i\right)^k} \quad \bullet\!\!-\!\!\circ \quad \underline{r}_{i,k} \cdot \theta(t) \cdot \frac{1}{(k-1)!} \cdot t^{k-1} \cdot e^{\underline{\lambda}_i t}$$

Thus it follows (*using the linearity of the Laplace transformation*) for the general form of the impulse response:

$$\underline{H}(\underline{s}) \quad \bullet\!\!-\!\!\circ \quad h(t) \;=\; \sum_{k=0}^{m-n} c_k \cdot \delta^{(k)}(t) \;+\; \theta(t) \sum_{i=1}^{n_\mu} \left(\sum_{k=1}^{\mu_i} \underline{r}_{i,k} \cdot \frac{1}{(k-1)!} \cdot t^{k-1} \cdot e^{\underline{\lambda}_i t} \right) \tag{4.4}$$

4.1.5 Prominent Special Case of the General Impulse Response

A common special case arises when the system is not differentiating and the excitation is not directly contained in the output of the system (*numerator degree less than denominator degree, i.e., $m < n$*) and furthermore all eigenvalues – that is, the poles – are simple with $\mu_i = 1$ for all i. Then the first term vanishes completely and with $n_\mu = n$ the second sum term simplifies to a single sum. Under these conditions, the general impulse response (4.4) simplifies to a sum of exponential terms with the eigenvalues $\underline{\lambda}_i$ and the associated residues $\underline{h}_i$:

$$h(t) \;=\; \theta(t) \sum_{i=1}^{n} \underline{h}_i \cdot e^{\underline{\lambda}_i t} \tag{4.5}$$

The residues $\underline{h}_i$ can also be interpreted here as weightings of the eigenvalues or their contribution to the dynamic behavior of the system. This simplified form of the impulse response can often be encountered (*for example, in the damped harmonic oscillator*). With the knowledge that the complex eigenvalues $\underline{\lambda}_i$ always occur in complex conjugate pairs, the expression for the impulse response can be broken down into purely real terms. Let the number of real eigenvalues λ_i now be denoted by n_r and the number of complex pairs $\underline{\lambda}_i = \alpha_i + j\omega_i$ and $\underline{\lambda}_i^* = \alpha_i - j\omega_i$ by n_k. The total number of eigenvalues is then $n = n_r + 2n_k$ and a corresponding splitting of the sum in (4.5) yields:

$$\begin{aligned}
h(t) \;&=\; \underbrace{\theta(t) \sum_{i=1}^{n_r} h_i \cdot e^{\lambda_i t}}_{\textit{real eigenvalues}} \;+\; \underbrace{\theta(t) \sum_{i=1}^{n_k} \left(\underline{h}_i \cdot e^{\underline{\lambda}_i t} + \underline{h}_i^* \cdot e^{\underline{\lambda}_i^* t} \right)}_{\textit{complex eigenvalue pairs}} \\
&\;\;\vert\;\; \textit{substitute complex eigenvalues } \underline{\lambda}_i = \alpha_i + j\omega_i \textit{ and rearrange} \\
&=\; \theta(t) \sum_{i=1}^{n_r} h_i \cdot e^{\lambda_i t} \;+\; \theta(t) \sum_{i=1}^{n_k} e^{\alpha_i t} \left(\underline{h}_i \cdot e^{j\omega_i t} + \underline{h}_i^* \cdot e^{-j\omega_i t} \right) \\
&\;\;\vert\;\; \textit{continue rearranging with } \underline{z} + \underline{z}^* = 2 \cdot \Re\{\underline{z}\} \\
&=\; \theta(t) \sum_{i=1}^{n_r} h_i \cdot e^{\lambda_i t} \;+\; \theta(t) \sum_{i=1}^{n_k} 2 \cdot e^{\alpha_i t} \cdot \Re\left\{ \underline{h}_i \cdot e^{j\omega_i t} \right\} \\
&\;\;\vert\;\; \textit{write out the right summand using Euler's formula} \\
&=\; \theta(t) \sum_{i=1}^{n_r} h_i \cdot e^{\lambda_i t} \;+\; \theta(t) \sum_{i=1}^{n_k} 2 \cdot e^{\alpha_i t} \underbrace{\left(\Re\{\underline{h}_i\} \cdot cos(\omega_i t) - \Im\{\underline{h}_i\} \cdot sin(\omega_i t) \right)}_{\textit{oscillation terms}}
\end{aligned} \tag{4.6}$$

It is clearly recognizable that the imaginary parts $\omega_i = \Im\{\underline{\lambda}_i\}$ of the complex eigenvalue pairs lead to sine and cosine oscillation terms with the corresponding angular frequency ω_i in the impulse response. The real parts $\alpha_i = \Re\{\underline{\lambda}_i\}$ and the real eigenvalues λ_i appear in the exponents of exponential functions and can be interpreted as damping depending on their sign.

4.2 The Step Response

The step response, in contrast to the impulse response, is the response of the system when excited by a unit step / step function. In many technical applications, real signals (*... or idealized real signals*), such as rectangular functions, signal edges, or (*instantaneous*) switch-on processes can be modeled by step functions. The step response is therefore often part of the solution when exciting the system with such functions and thus has high technical relevance.

4.2.1 The Step Function

The step function (*also Heaviside function, θ-function or σ-function*) finds numerous applications, for example in communications engineering, system theory, and control engineering.

In the time domain, the step function is defined by a case distinction as follows:

$$\theta(t) := \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases} \tag{4.7}$$

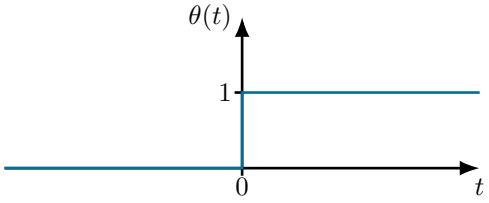


Figure 4.4: Step function in the time domain.

4.2.2 Truncating Arbitrary Functions with the Step Function

Especially as a mathematical filter (*often not realizable as an ideal filter in reality, yet practical*), the step function is frequently used. If one multiplies pointwise each value of an arbitrary continuous function $f(t)$ with the corresponding value of the step function, the result is a function that has the value zero to the left of $t = 0$, but agrees with the original function to the right of it. This property allows truncating arbitrary functions for negative times $t < 0$.

This filter property and some useful properties for the definition of the filter are exemplified in the following figures using an arbitrary continuous function $f(t)$:

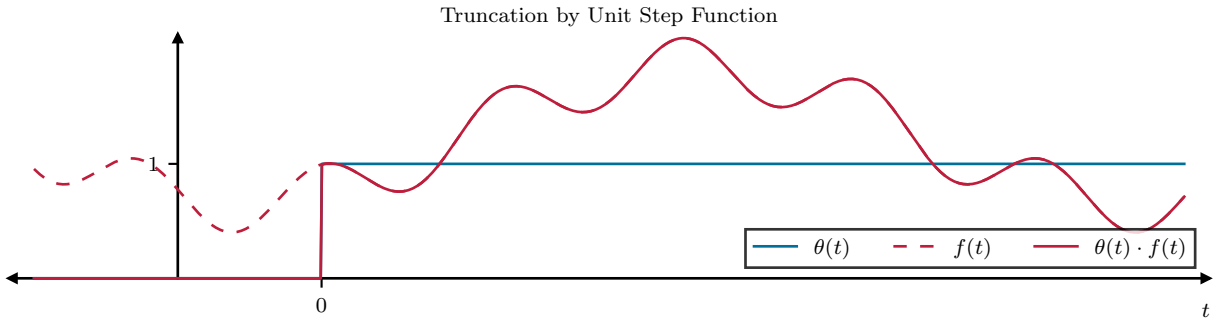


Figure 4.5: Visualization of the filter property of the step function. The arbitrary function $f(t)$ is thus truncated or set to zero for negative times $t < 0$.

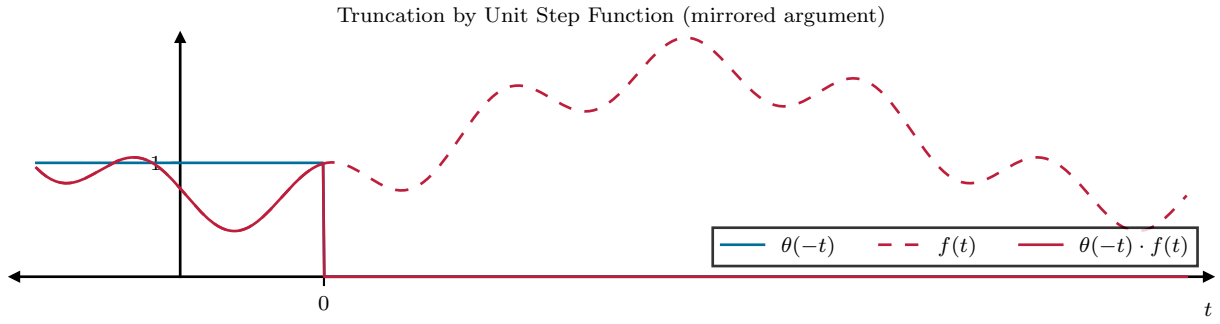


Figure 4.6: Visualization of the filter property of the step function with inverted argument. The arbitrary function $f(t)$ is thus truncated or set to zero for positive times $t > 0$.

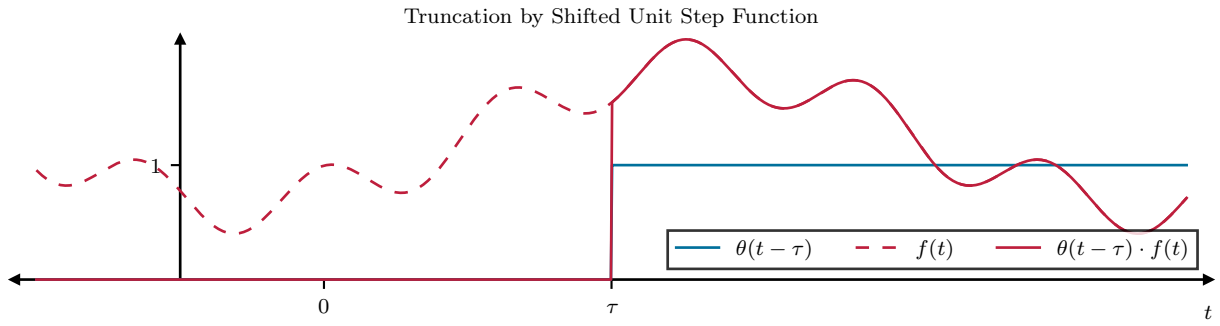


Figure 4.7: Visualization of the filter property of the step function with shift by τ. The arbitrary function $f(t)$ is thus truncated or set to zero for times $t < \tau$.

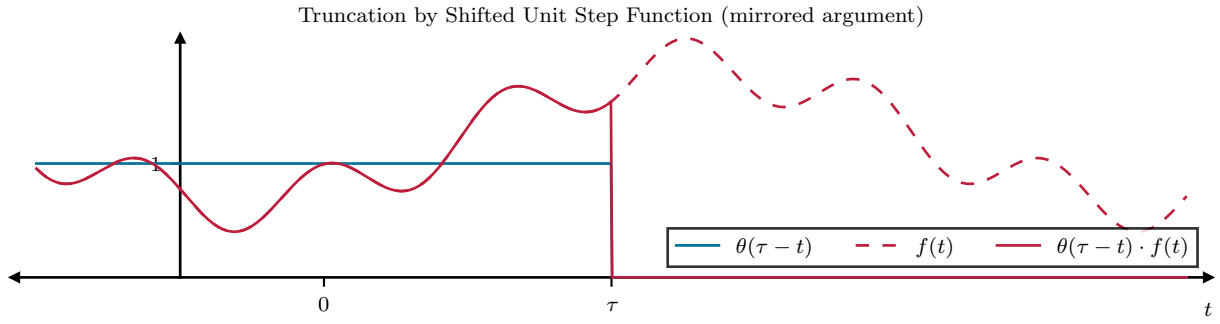


Figure 4.8: Visualization of the filter property of the step function with shift by τ and inverted argument. The arbitrary function $f(t)$ is thus truncated or set to zero for times $t > \tau$.

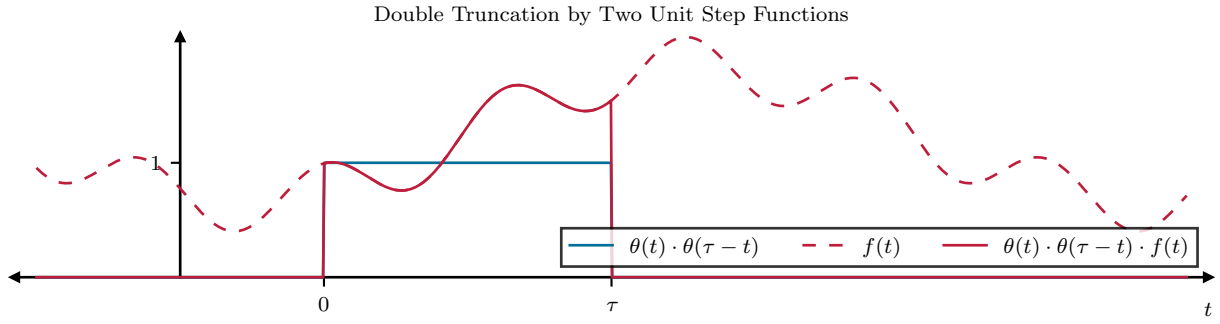


Figure 4.9: Visualization of the filter property of the step function. The arbitrary function $f(t)$ is set to zero by two step functions for times $t < 0$ and $t > \tau$.

4.2.3 Definition of the Step Response

The step response is the response of the system to a unit step. That is, the system output neglecting the initial values when the system is excited by a step function.

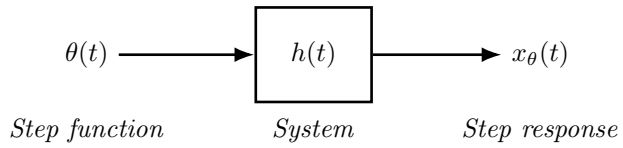


Figure 4.10: Schematic representation of the step response in the block diagram of the system. Excitation with a unit step gives the step response.

The step response is denoted in the following by $x_\theta(t)$ and can be obtained in several ways. In the time domain, the step response is obtained via the convolution integral:

$$x_\theta(t) \;=\; \Big(h(\tau) * \theta(\tau)\Big)(t) \;=\; \int_0^t h(t-\tau)\,d\tau \tag{4.8}$$

The step response thus corresponds in principle to the integration of the impulse response. In the Laplace domain, with the correspondence of the step function, the following holds for the Laplace transform of the step response:

$$\theta(t) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{\underline{s}} \qquad \Rightarrow \qquad x_\theta(t) \quad \circ\!\!-\!\!\bullet \quad \underline{H}(\underline{s}) \cdot \frac{1}{\underline{s}} \tag{4.9}$$

It should also be particularly emphasized that the particular solution, in which the step function is the inhomogeneity, is always part of the step response. The initial conditions $\vec{x}_0$ of the system are not contained therein (*and consequently neither is the homogeneous solution*), but exclusively the response of the system to a step function.

4.2.4 Derivative of the Step Response (WIP)

Another interesting property of the step response is that the impulse response can be directly recovered from it. Namely, the step function (4.7) has a discontinuity at the point $t = 0$ and its derivative at this point cannot be specified by an ordinary function. Instead, a Dirac delta as a distribution or generalized function provides the correct behavior:

$$\frac{d}{dt}\theta(t) \;=\; \delta(t) \tag{4.10}$$

The derivative of the step response thus yields the impulse response itself again.

4.3 STABILITY

The stability of linear dynamic systems is very important information in many application areas. It is a characteristic of the system and provides information about the dynamic behavior or the temporal evolution of the system quantities. If a system is unstable, the initial values do not decay and consequently there can be no steady state. Instability also leads to the system oscillating increasingly when excited, and thus can lead to irreversible damage through exceeding tolerances.

A statement about the stability of a system is possible through the analysis of the eigenvalues, which can be directly found in the impulse response (*by eigenvalues we mean here the eigenvalues of the state matrix* $\mathbf{A}$ *of the state-space model of the system, also known as the natural frequencies*).

4.3.1 MOTIVATION BASED ON THE IMPULSE RESPONSE AND THE EIGENVALUES

The relationship between stability and the eigenvalues of the system becomes clear when the general form of the impulse response is considered. This often has the following form (4.5) for non-differentiating systems with order n and only simple eigenvalues $\underline{\lambda}_i$:

$$h(t) \;=\; \theta(t) \sum_{i=1}^{n} h_i \cdot e^{\underline{\lambda}_i \, t} \tag{4.11}$$

The eigenvalues are generally complex and therefore have a real and an imaginary part with $\underline{\lambda}_i = \alpha_i + j\omega_i$. Purely real $\lambda_i = \alpha_i$ with $\omega_i = 0$ and purely imaginary $\underline{\lambda}_i = j\omega_i$ with $\alpha_i = 0$ eigenvalues are included. Complex eigenvalues are always contained in pairs as complex conjugate pairs. In general, however, holds:

$$h(t) \;=\; \theta(t) \sum_{i=1}^{n} h_i \cdot e^{(\alpha_i + j\omega_i) \, t} \tag{4.12}$$

If one now wants to know how large and in which range the impulse response moves, the magnitude can provide an estimate. With the triangle inequality and for $t > 0$ it follows:

$$|h(t)| \;\leq\; \sum_{i=1}^{n} \left| h_i \cdot e^{(\alpha_i + j\omega_i)} \right| \tag{4.13}$$

For the components of the imaginary part of the eigenvalues holds $\left|e^{j\omega_i \, t}\right| = 1$ and the temporal evolution of the magnitude can be completely attributed to the real part of the eigenvalues:

$$|h(t)| \;\leq\; \sum_{i=1}^{n} \left| h_i \cdot e^{\alpha_i \, t} \right| \tag{4.14}$$

In order to be able to assess how the individual eigenvalues have an effect, one of the eigenvalues – that is, one of the summands – with real part α_k is compared to the impulse response. It follows for all $t > 0$:

$$\left| h_k \cdot e^{\alpha_k \, t} \right| \;\leq\; |h(t)| \;\leq\; \sum_{i=1}^{n} \left| h_i \cdot e^{\alpha_i \, t} \right| \tag{4.15}$$

With the consideration in (4.15), the magnitude of the impulse response can be bounded through multiple limit formations. Each summand k in the impulse response can form a lower bound for the impulse response itself:

$$\lim_{t\to\infty} |h(t)| \;\geq\; \lim_{t\to\infty} \left|h_k \cdot e^{\alpha_k\, t}\right| \;=\; \begin{cases} 0 & \textit{for } \alpha_k < 0 \\ \infty & \textit{for } \alpha_k > 0 \end{cases} \tag{4.16}$$

The impulse response thus always diverges if at least one eigenvalue has a positive real part $\Re\{\underline{\lambda}_k\} = \alpha_k > 0$. The existence of such an eigenvalue is sufficient. Taking into account the insights from (4.16), the upper bound estimate of the magnitude of the impulse response yields:

$$\lim_{t\to\infty} |h(t)| \;\leq\; \lim_{t\to\infty} \sum_{i=1}^{n} \left|h_i \cdot e^{\alpha_i\, t}\right| \;=\; \begin{cases} 0 & \textit{for all } \alpha_i < 0 \\ \infty & \textit{for at least one } \alpha_i > 0 \end{cases} \tag{4.17}$$

The impulse response thus always converges when all eigenvalues have a strictly negative real part. That is, $\Re\{\underline{\lambda}_i\} = \alpha_i < 0$ is satisfied for $i = 1, \ldots, n$. Based on these considerations, it is clear that the real part of the eigenvalues decides whether the impulse response vanishes (*decays*), or diverges (*oscillates up*). This limit behavior is transferred through the convolution integral to the general system response when the excitation is bounded and different from zero. This results in two common stability definitions.

4.3.2 Asymptotic Stability

The system of order n is called asymptotically stable if for all eigenvalues $\underline{\lambda}_i$ holds:

$$\Re\{\underline{\lambda}_i\} \;<\; 0 \qquad \textit{for all} \qquad i = 1, \ldots, n \tag{4.18}$$

All eigenvalues must have a strictly negative real part. Equivalent to this is the requirement that the impulse response converges asymptotically to the time axis or vanishes:

$$\lim_{t\to\infty} h(t) \;=\; 0 \tag{4.19}$$

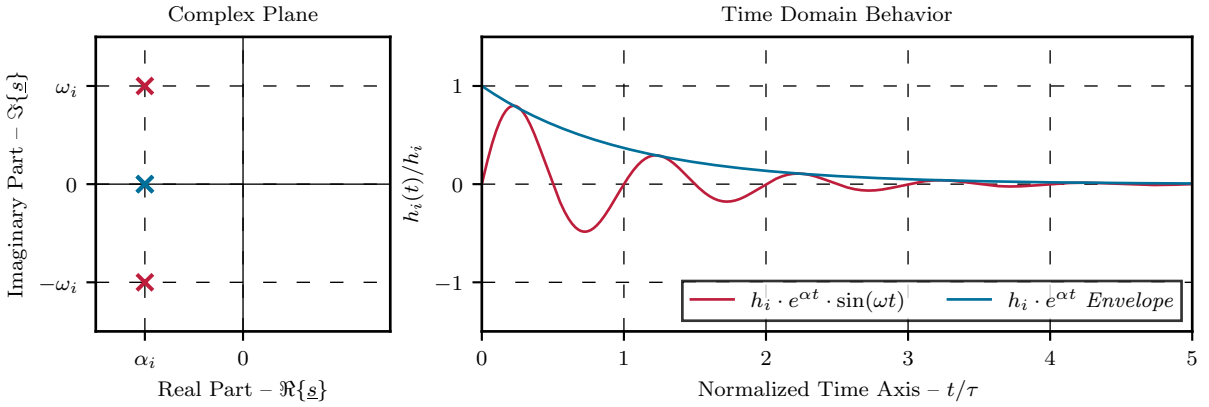


Figure 4.11: Visualization of asymptotically stable behavior. Shown is a summand of the impulse response with negative real part $\alpha_i < 0$. The function converges asymptotically to the time axis.

4.3.3 MARGINAL STABILITY

The system of order n is called marginally stable if for all eigenvalues $\underline{\lambda}_i$ holds:

$$\Re\{\underline{\lambda}_i\} \leq 0 \qquad \textit{for all} \qquad i = 1, \ldots, n \tag{4.20}$$

All eigenvalues must have a negative real part. Eigenvalues on the imaginary axis are permitted, but must be simple with multiplicity $\mu_i = 1$. Equivalent to this is the requirement that the limit of the impulse response is bounded and thus:

$$\lim_{t \to \infty} \left| h(t) \right| \leq h_\infty \tag{4.21}$$

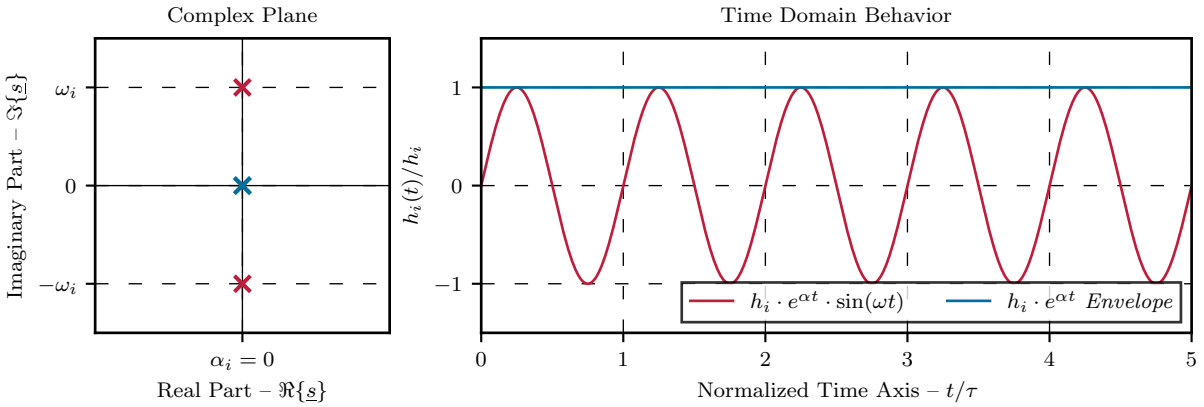


Figure 4.12: Visualization of marginally stable behavior. Shown is a summand of the impulse response with eigenvalue on the imaginary axis and thus $\alpha = 0$.

4.3.4 INSTABILITY

If the system is neither asymptotically stable nor marginally stable, it is unstable. For this it is sufficient that one eigenvalue has a positive real part. Furthermore, the impulse response then diverges. Unstable systems are generally undesirable. One of the goals in control engineering is, for example, to stabilize unstable systems through controllers.

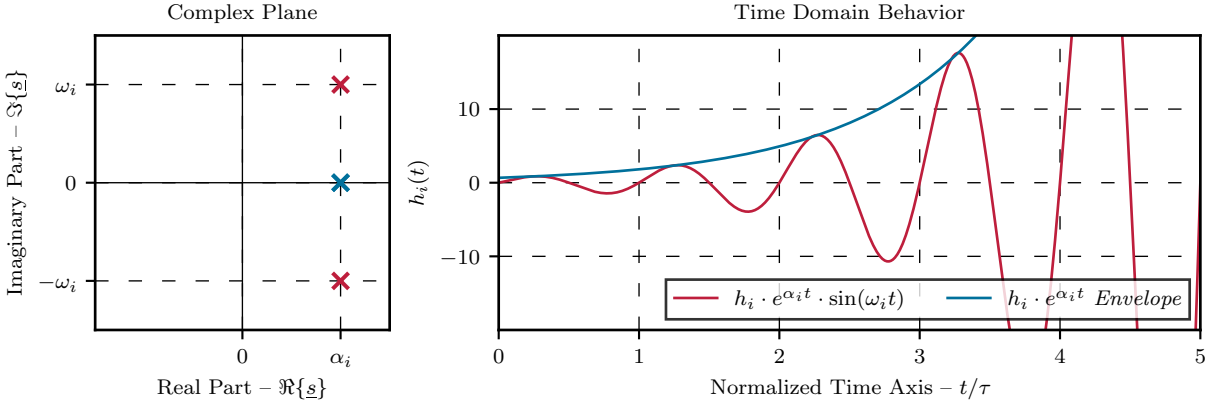


Figure 4.13: Visualization of unstable behavior. Shown is a summand of the impulse response with eigenvalue in the right half-plane and thus $\alpha > 0$. The divergence of the function is clearly recognizable.

4.4 Stability Criteria

To test linear time-invariant systems for their stability, it is sufficient to examine the eigenvalues of the state matrix $\mathbf{A}$, i.e., the natural frequencies of the system, since their real parts determine the convergence behavior of the system response. For systems of large order $n = \dim\{\mathbf{A}\}$, it is often not practicable to directly calculate and examine the eigenvalues. In order to still be able to make a statement about the stability of the system, some stability criteria are presented here, which primarily examine the denominator polynomial of the transfer function $\underline{H}(\underline{s})$ of the system. The analysis of the denominator polynomial is thereby sufficient, since its zeros are the poles of the transfer function and these are the eigenvalues of the system.

4.4.1 Hurwitz Polynomials

A Hurwitz polynomial (*after Adolf Hurwitz*) is a polynomial with real coefficients whose zeros all have a strictly negative real part. A polynomial:

$$\underline{Q}(\underline{s}) \;=\; a_n\,\underline{s}^n + a_{n-1}\,\underline{s}^{n-1} + \cdots + a_1\,\underline{s} + a_0 \tag{4.22}$$

Is thus called a Hurwitz polynomial if and only if for all $i = 1, \ldots, n$ holds:

$$\underline{Q}(\underline{\lambda}_i) \;=\; 0 \qquad \Rightarrow \qquad \Re\{\underline{\lambda}_i\} < 0 \tag{4.23}$$

Asymptotically stable systems thus have transfer functions whose denominator polynomials are Hurwitz polynomials. In the case that the polynomial is of degree 1 or 2 and has only coefficients of the same sign, it is automatically a Hurwitz polynomial and the investigation is complete. For polynomials of higher degree, this does not hold without further ado. However, the equality of the signs of the coefficients is always a necessary condition.

4.4.2 Hurwitz Determinant Criterion

Probably the best known method to test a polynomial of higher degree for its Hurwitz property is the Hurwitz matrix. This is constructed according to the following scheme from the coefficients of the polynomial:

$$\mathbf{H} \;=\; \begin{bmatrix} a_{n-1} & a_n & 0 & 0 & \cdots & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n & & \vdots \\ a_{n-5} & a_{n-4} & a_{n-3} & a_{n-2} & & \vdots \\ a_{n-7} & a_{n-6} & a_{n-5} & a_{n-4} & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & a_2 \\ 0 & 0 & 0 & 0 & \cdots & a_0 \end{bmatrix} \tag{4.24}$$

The underlying polynomial is then a Hurwitz polynomial if the Hurwitz matrix $\mathbf{H}$ is positive definite. For this, the determinants of all principal minors must be positive:

$$\left| \; a_{n-1} \; \right| > 0 \; , \quad \begin{vmatrix} a_{n-1} & a_n \\ a_{n-3} & a_{n-2} \end{vmatrix} > 0 \; , \quad \begin{vmatrix} a_{n-1} & a_n & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} \\ a_{n-5} & a_{n-4} & a_{n-3} \end{vmatrix} > 0 \; , \quad \ldots \tag{4.25}$$

4.4.3 Hurwitz Continued Fraction Criterion

Another method to examine a polynomial for its Hurwitz property is the continued fraction expansion. Here, the polynomial is first decomposed into an even and an odd part:

$$\underline{Q}(\underline{s}) \; = \; \underbrace{a_0 + a_2 \, \underline{s}^2 + a_4 \, \underline{s}^4 + \ldots}_{\underline{G}_Q(\underline{s}) \;\; even \;\; part} \; + \; \underbrace{a_1 \, \underline{s} + a_3 \, \underline{s}^3 + a_5 \, \underline{s}^5 + \ldots}_{\underline{U}_Q(\underline{s}) \;\; odd \;\; part} \tag{4.26}$$

The investigation is based on the development of the quotient of even and odd parts as a continued fraction. The polynomial with the larger degree must form the numerator. The development procedure consists of alternating splitting off a $\underline{s}$ in the numerator by polynomial division and subsequent flipping of the remainder term into the denominator:

$$\frac{\underline{G}_Q(\underline{s})}{\underline{U}_Q(\underline{s})} \; = \; b_1 \, \underline{s} + \cfrac{1}{b_2 \, \underline{s} + \cfrac{1}{b_3 \, \underline{s} + \cfrac{1}{b_4 \, \underline{s} + \cfrac{1}{\ldots}}}} \tag{4.27}$$

If the coefficients of the continued fraction are all positive $b_i > 0$ for all $i = 1, \ldots, n$, the underlying polynomial is a Hurwitz polynomial.

4.5 The Steady State

The analysis of the system in a steady state (*also steady-state analysis*) is a common method in engineering sciences. However, it is often omitted that the existence of a steady state is tied to strict prerequisites. Ultimately, the description of the system in the steady state is a simplified representation of the system solution, which only correctly describes the system behavior under these conditions.

4.5.1 Basic Prerequisites

The basic principle of steady-state analysis is the assumption that the system exhibits a proportionality to the excitation itself after a certain time when there is bounded excitation – that is, $|y(t)| < Y$. This then enables the description of the system behavior via magnitude and phase differences. For a steady state to even exist, the impulse response must converge asymptotically. This leads to the system output also being bounded when the excitation is bounded. The system must thus be asymptotically stable and the following must hold:

$$\Re\Big\{ \, \underline{\lambda}_i \, \Big\} \; < \; 0 \qquad \textit{for all} \quad i = 1, \ldots, n \tag{4.28}$$

When the system is asymptotically stable and the excitation $|y(t)| < Y$ is bounded, the system transitions into a steady state after the settling process and a certain settling time – that is, after the transient, and

when all initial values have decayed. This state allows a simplified representation of the solution. The validity of this claim follows in the following subsections.

4.5.2 Homogeneous Solution in the Steady State

The settling process also includes the decay of the effect of the initial values or the homogeneous solution. The general homogeneous solution is given in (1.58). For eigenvalues with strictly negative real part and in the limit $t \to \infty$, it behaves as follows:

$$\lim_{t\to\infty} \vec{x}_h(t) \;=\; \lim_{t\to\infty} e^{\mathbf{A}t} \cdot \vec{x}_0 \tag{4.29}$$

Now the question arises about the convergence of the matrix exponential function. This depends on the eigenvalues of the matrix $\mathbf{A}$. This is initially assumed to be diagonalizable, then it has the representation:

$$\mathbf{A} \;=\; \mathbf{V} \cdot \underline{\mathbf{\Lambda}} \cdot \mathbf{V}^{-1} \qquad mit \qquad \underline{\mathbf{\Lambda}} \;=\; \begin{bmatrix} \underline{\lambda}_1 & & 0 \\ & \ddots & \\ 0 & & \underline{\lambda}_n \end{bmatrix} \tag{4.30}$$

For the matrix exponential function then holds (*this step follows directly from the series representation (1.53)*):

$$e^{\mathbf{A}t} \;=\; \mathbf{V} \cdot e^{\underline{\mathbf{\Lambda}}t} \cdot \mathbf{V}^{-1} \;=\; \mathbf{V} \cdot \begin{bmatrix} e^{\underline{\lambda}_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\underline{\lambda}_n t} \end{bmatrix} \cdot \mathbf{V}^{-1} \tag{4.31}$$

For each of the matrix entries i of the diagonal matrix in (4.31) now holds with $\underline{\lambda}_i = \alpha_i + j\omega_i$:

$$\lim_{t\to\infty} e^{\underline{\lambda}_i t} \;=\; \lim_{t\to\infty} e^{\alpha_i t} \cdot e^{j\omega_i t} \;=\; \begin{cases} 0 & for \quad \alpha_i < 0 \\ divergent & for \quad \alpha_i \geq 0 \end{cases} \tag{4.32}$$

The homogeneous solution thus converges or vanishes only in the case that the system is asymptotically stable. Then, however:

$$\lim_{t\to\infty} \vec{x}_h(t) \;=\; \vec{0} \tag{4.33}$$

4.5.3 MOTIVATION OF A PRACTICAL SETTLING TIME

In reality, an infinite waiting time is of course not practicable. Therefore, we now search for a criterion by which it can be determined when a steady state is assumed. Often a threshold is defined, from which the initial values – that is, the homogeneous solution – have decayed sufficiently.

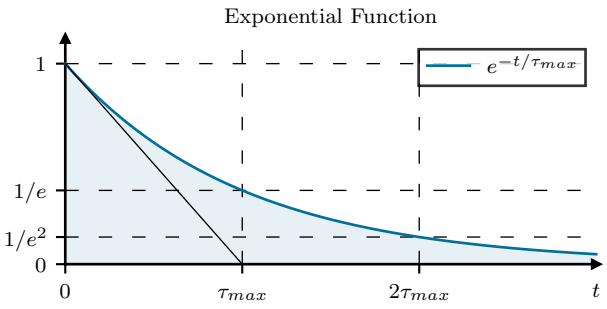


Figure 4.14: Envelope curve given by the largest time constant τ_{max} in the asymptotically stable system.

Since the decay behavior of the homogeneous solution depends on the real parts α_i of the eigenvalues $\underline{\lambda}_i = \alpha_i + j\omega_i$, the settling time is determined by the smallest real part – that is, the lowest natural frequency and thus the largest time constant. This is because for large times $t \gg 1$ the smallest eigenvalues dominate the homogeneous solution, since all other components have already decayed by then. With the time constants $\tau_i = \left|\frac{1}{\alpha_i}\right|$, the following thus holds for the largest time constant:

$$\tau_{max} \;=\; \max_{i=1,\ldots,n} \left\{ \; \tau_i \; \right\} \;=\; \max_{i=1,\ldots,n} \left\{ \; \frac{1}{\left|\Re\{\underline{\lambda}_i\}\right|} \; \right\} \tag{4.34}$$

For the homogeneous solution, an upper bound can thus be given. Specifically, with the assumption of diagonalizability of the state matrix $\mathbf{A}$, holds:

$$\vec{x}_h(t) \;=\; e^{\mathbf{A}t} \cdot \vec{x}_0 \;<\; e^{-t/\tau_{max}} \cdot \vec{x}_0 \tag{4.35}$$

With this consideration, one can conclude that the system assumes a steady state from $t \gg \tau_{max}$. In practice, this is often already guaranteed from about $t \approx 10\tau_{max}$, which is due to the exponential character of the system responses.

4.5.4 Boundedness of the Response in the Steady State

The simplified solution representation of systems in a steady state results from the assumption of bounded excitation. Thus let:

$$\big|y(t)\big| \;\leq\; Y \quad \textit{for all} \quad t>0 \qquad \Rightarrow \qquad \big|\underline{Y}(\underline{s})\big| \;\leq\; \Big|Y\cdot\frac{1}{\underline{s}}\Big| \tag{4.36}$$

If the excitation is thus bounded in magnitude from above by the constant Y, then for the system response $x(t)$ in a steady state follows:

$$\begin{aligned}
\lim_{t\to\infty}\big|x(t)\big| &= \lim_{t\to\infty}\Big|x_h(t)+\Big(h(\tau)*y(\tau)\Big)(t)\Big| \\
&\mid \textit{homogeneous component vanishes for asymptotically stable systems} \\
&= \lim_{t\to\infty}\Big|\Big(h(\tau)*y(\tau)\Big)(t)\Big| \\
&\mid \textit{substitute convolution integral and adjust limits} \\
&= \lim_{t\to\infty}\left|\int_0^t h(\tau)\cdot y(t-\tau)\,d\tau\right| \\
&\mid \textit{upper bound estimate, continuous integral kernel (impulse response)} \\
&\leq \lim_{t\to\infty}\int_0^t \big|h(\tau)\big|\cdot\big|y(t-\tau)\big|\,d\tau \\
&\mid \textit{upper bound estimate through bounded excitation} \\
&\leq Y\cdot\lim_{t\to\infty}\int_0^t \big|h(\tau)\big|\,d\tau \\
&\mid \textit{form limit, adjust upper bound} \\
&= Y\cdot\int_0^\infty \big|h(\tau)\big|\,d\tau \\
&\left|\begin{array}{l}\textit{integral converges for asymptotically stable systems}\\ \textit{and can be bounded by a constant } H\end{array}\right. \\
&\leq Y\cdot H \;<\; \infty
\end{aligned}$$

From this consideration it follows that when the excitation of an asymptotically stable system is bounded, this also holds for the system response in the steady state. This boundedness of the output can now be exploited to derive a simplified representation of the solution in the steady state. This then describes the system behavior after a certain settling time for $t \gg \tau_{max}$.

4.5.5 Note on Convergence and the Impulse Response

In the following sections, arguments are frequently made using the conditions of integrability of the impulse response $h(t)$. This property is fundamental for the discussion of the steady state.

But why exactly does the integral over the impulse response converge only for asymptotically stable systems? To investigate this, we consider the general form of the impulse response in equation (4.4), which has the following form:

$$h(t) \;=\; \sum_{k=0}^{m-n} c_k \cdot \delta^{(k)}(t) \;+\; \theta(t) \sum_{i=1}^{n_\mu} \left(\sum_{k=1}^{\mu_i} \frac{\underline{r}_{i,k}}{(k-1)!} \cdot t^{k-1} \cdot e^{\underline{\lambda}_i t} \right)$$

Integrated over the entire positive time axis for $t \in [0, \infty)$, due to the linearity of the integral yields:

$$\int_0^\infty h(t)\,dt \;=\; \sum_{k=0}^{m-n} c_k \underbrace{\int_0^\infty \delta^{(k)}(t)\,dt}_{\text{to solve}} \;+\; \sum_{i=1}^{n_\mu} \left(\sum_{k=1}^{\mu_i} \frac{\underline{r}_{i,k}}{(k-1)!} \underbrace{\int_0^\infty t^{k-1} \cdot e^{\underline{\lambda}_i t}\,dt}_{\text{to solve}} \right) \tag{4.37}$$

The summands of the impulse response can thus be integrated individually. The two functions to be integrated can be specified generally for an arbitrary index m up to scaling with complex constants:

$$\int_0^\infty \delta^{(m)}(t)\,dt \;=\; \begin{cases} \delta^{(m-1)}(t) & \textit{for} \quad m \geq 1 \\ 0 & \textit{otherwise} \end{cases} \tag{4.38}$$

$$\begin{aligned} \int_0^\infty t^m \cdot e^{\underline{\lambda}_i t}\,dt \;&=\; \left[\frac{1}{\underline{\lambda}_i} \cdot t^m \cdot e^{\underline{\lambda}_i t} \right]_0^\infty - \int_0^\infty m \cdot t^{m-1} \cdot e^{\underline{\lambda}_i t}\,dt \\ &\quad \mid \quad \textit{partial integration until } t^m \textit{ is completely reduced} \\ &=\; \sum_{j=0}^{m} \left[(-1)^j \cdot \frac{j!}{\underline{\lambda}_i^{j+1}} \cdot t^{m-j} \cdot e^{\underline{\lambda}_i t} \right]_0^\infty \\ &\quad \mid \quad \textit{evaluation of the antiderivative} \\ &=\; (-1)^{m+1} \cdot \frac{m!}{\underline{\lambda}_i^{m+1}} \;+\; \begin{cases} 0 & \textit{for} \quad \Re\{\underline{\lambda}_i\} < 0 \\ \textit{divergent for} & \Re\{\underline{\lambda}_i\} \geq 0 \end{cases} \end{aligned} \tag{4.39}$$

From the case distinction in the evaluation of the antiderivative it follows that the asymptotic stability of the system is a prerequisite for convergence at the upper integration limit. It should additionally be emphasized once more that even marginal stability, which is often used synonymously with stability, is not sufficient for the convergence of the integral.

4.5.6 The Time-Independent Steady State

A very special case for the steady state (*also stationary state, or DC behavior*) arises when the excitation for large t is a constant – that is, time-independent – function. Thus let:

$$y(t) \;=\; \theta(t) \cdot y_0 \;=\; \textit{const. for } t > 0$$

Under these conditions, the asymptotically stable system assumes a constant final value – also called stationary final value – for $t \gg \tau_{max}$ and thus transitions into a time-independent steady state. This final value is given by:

$$x_\infty \;:=\; \lim_{t\to\infty} x(t) \tag{4.40}$$

This final value can be calculated in the time domain by limit formation over the convolution integral:

$$\begin{aligned}
x_\infty \;&=\; \lim_{t\to\infty} x(t) \\
&\Big|\quad \textit{system response can be described via convolution integral,} \\
&\Big|\quad \textit{since initial values have already decayed} \\
&=\; \lim_{t\to\infty} \Big(h(\tau) * y(\tau)\Big)(t) \\
&\mid\quad \textit{substitute constant excitation and rearrange} \\
&=\; y_0 \cdot \lim_{t\to\infty} \Big(h(\tau) * \theta(\tau)\Big)(t) \\
&\mid\quad \textit{write out convolution integral and adjust lower bound} \\
&=\; y_0 \cdot \lim_{t\to\infty} \int_0^\infty h(\tau) \cdot \theta(t-\tau)\, d\tau \\
&\Big|\quad \textit{integrand is Riemann integrable on } [0,\infty) \textit{ if} \\
&\Big|\quad \textit{impulse response describes asymptotically stable system} \\
&=\; y_0 \cdot \int_0^\infty \lim_{t\to\infty} \Big(h(\tau) \cdot \theta(t-\tau)\Big)\, d\tau \\
&\mid\quad \textit{form limit and reformulate} \\
&=\; y_0 \cdot \int_0^\infty h(\tau)\, d\tau
\end{aligned}$$

For the convergence of the integral, the impulse response must describe an asymptotically stable system. Here it is thus again clearly recognizable that a time-independent steady state can only exist for asymptotically stable systems. The constant x_∞ arising from this limit formation now completely describes the system behavior under the prerequisite that the system is asymptotically stable and is in a steady state under constant excitation. This system behavior can often be observed in practice – for example, when excited by a step function and after the transients have decayed.

4.5.7 The Harmonic Steady State

Another technically extremely relevant solution formulation is the representation in the harmonic steady state (*also sinusoidal steady state*). It forms the basis for analysis in the frequency domain and the system description through a frequency response.

The prerequisites for the existence of a steady state apply again, that is, asymptotic stability and the disappearance of the transient component from a settling time $t \gg \tau_{max}$. Additionally, let the excitation be assumed as a harmonic oscillation, which is impressed into the system from the emergence time $t = 0$ (*generally* $t = t_0$) and can be described via the phasor $\underline{Y} \in \mathbb{C}$ (*characterizes amplitude and phase of the signal, corresponds to Fourier coefficients for the corresponding frequency of a periodic signal*) and an excitation frequency ω:

$$y(t) \;=\; \theta(t) \cdot \Re\{\underline{Y} \cdot e^{j\omega t}\} \;=\; \theta(t) \cdot |\underline{Y}| \cdot cos(\omega t + \varphi_y) \tag{4.41}$$

The response of the system based on the particular solution (*i.e., the part of the solution relevant for* $t \gg \tau_{max}$*, the homogeneous component is already neglected here since it decays for asymptotically stable systems*) results for $t > 0$ generally from the convolution integral and the impulse response $h(t)$ as:

$$\begin{aligned}
x(t) \;&=\; \Big(h(\tau) * y(\tau)\Big)(t) \\
&\quad\big|\; \textit{substitute harmonic excitation} \\
&=\; \Big(h(\tau) * \theta(\tau) \cdot \Re\{\underline{Y} \cdot e^{j\omega\tau}\}\Big)(t) \\
&\quad\Big|\; \begin{array}{l}\textit{reformulating the convolution is permitted} \\ \textit{due to the linearity of the system}\end{array} \\
&=\; \Re\Big\{\underline{Y} \cdot \Big(h(\tau) * \theta(\tau) \cdot e^{j\omega\tau}\Big)(t)\Big\} \\
&\quad\Big|\; \begin{array}{l}\textit{write out convolution and} \\ \textit{adjust integration limits}\end{array} \\
&=\; \Re\left\{\underline{Y} \cdot \int_0^t h(\tau) \cdot e^{j\omega(t-\tau)} d\tau\right\} \\
&\quad\big|\; \textit{rearrange according to integration variable} \\
&=\; \Re\left\{\underline{Y} \cdot \int_0^t h(\tau) \cdot e^{-j\omega\tau} d\tau \cdot e^{j\omega t}\right\}
\end{aligned}$$

Here it can already be seen that the fundamental oscillation of the system is at least preserved as an envelope curve / envelope oscillation in the system response through the term $e^{j\omega t}$. As will be seen shortly, the preservation of the excitation frequency ω in the system response is one of the reasons why the representation in the harmonic steady state can be so greatly simplified.

In order to now obtain the system response after the decay of the transients – that is, in the steady state – the integral is split into two intervals $[0, T]$ and $[T, t]$. This yields:

$$x(t) = \Re\left\{\underline{Y} \cdot \left(\int_0^T h(\tau) \cdot e^{-j\omega\tau} d\tau + \int_T^t h(\tau) \cdot e^{-j\omega\tau} d\tau\right) \cdot e^{j\omega t}\right\}$$

$$\mid \quad \textit{form the limit } T \to \infty \textit{ and exchange integration limits}$$

$$= \underbrace{\Re\left\{\underline{Y} \cdot \int_0^\infty h(\tau) \cdot e^{-j\omega\tau} d\tau \cdot e^{j\omega t}\right\}}_{x_{ez}(t) \textit{ steady state}} - \underbrace{\Re\left\{\underline{Y} \cdot \int_t^\infty h(\tau) \cdot e^{-j\omega\tau} d\tau \cdot e^{j\omega t}\right\}}_{\textit{transient} \to 0 \textit{ for } t \to \infty}$$

For the behavior for $t \to \infty$, the right integral vanishes. The representation in the steady state is thus exclusively dominated by the left integral:

$$x_{ez}(t) = \Re\left\{\underline{Y} \cdot \underbrace{\int_0^\infty h(\tau) \cdot e^{-j\omega\tau} d\tau}_{\textit{(one-sided) Fourier integral}} \cdot e^{j\omega t}\right\}$$

$$\left| \begin{array}{l} \textit{the Fourier integral converges, since the impulse response} \\ h(\tau) \textit{ describes an asymptotically stable system} \end{array} \right.$$

$$= \Re\left\{\underline{Y} \cdot \underline{H}(j\omega) \cdot e^{j\omega t}\right\}$$

The integral is also called the one-sided Fourier transformation. For the convergence of the Fourier integral, much stricter conditions apply than for the Laplace integral. The asymptotic convergence of the impulse response $h(t)$ is a necessary condition here and is only fulfilled for asymptotically stable systems (*see the section on stability*). The response of the system in the harmonic steady state then results as:

$$x(t) = x_{ez}(t) = \Re\left\{\underline{Y} \cdot \underline{H}(j\omega) \cdot e^{j\omega t}\right\} \quad \textit{for} \quad t \gg \tau_{max} \tag{4.42}$$

It should be emphasized once more that the amplitude of the output signal in the harmonic steady state is directly proportional to the magnitude of the function $\underline{H}(j\omega)$ and thus dependent on the exciting frequency ω.

The description of the system behavior in a harmonic steady state thus fulfills the basic assumptions in frequency domain analysis, in which all system quantities can be described as complex phasors and their magnitude and phase differences. This becomes particularly clear when the amplitude and phase of the response are also expressed via a corresponding phasor $\underline{X}$ (*more on phasors in section 5.1.4*):

$$x_{ez}(t) = \Re\left\{\underline{X} \cdot e^{j\omega t}\right\} \qquad \textit{with} \qquad \underline{X} = \underline{Y} \cdot \underline{H}(j\omega)$$

Under these conditions, $\underline{H}(j\omega)$ has the structure of the transfer function $\underline{H}(\underline{s})$ with the replacement $\underline{s} \to j\omega$ and is usually called the frequency response. The simplified description of the system behavior via frequency responses has its origin or justification in the assumption of a harmonic steady state. This thus forms the basis for the analysis of the system in the frequency domain in the following chapter.

Chapter 5

System Analysis in the Frequency Domain

Besides the analysis of the system in the Laplace domain, an analysis in the frequency domain is often expedient in practice. In this chapter, the fundamental aspects of system analysis in the frequency domain based on frequency responses are provided and the necessary theory (*as a subfield of Fourier analysis*) is conveyed.

5.1 Periodic Signals

Generally, in frequency domain analysis, all system quantities (*including input and output*) are assumed to be periodic.

5.1.1 Periodicity

A function $f(t)$ is called periodic with period T (*also T-periodic*), if for all t holds:

$$f(t) \;=\; f(t+T) \tag{5.1}$$

The function values of periodic functions repeat at regular intervals (*the period*). Also integer multiples of the period T are again a period of the function.

5.1.2 Fourier Series

A Fourier series (*after Joseph Fourier*) denotes the series expansion of a periodic, piecewise continuous function into a function series of sine and cosine functions with different frequencies, which are each ascending multiples of a fundamental frequency. Every continuously differentiable function $f(t)$ that is defined on the interval $t \in [0, T]$ can be developed into a Fourier series. This has in complex notation the general form:

$$f(t) \;=\; \frac{2}{T} \cdot \sum_{k=0}^{\infty} \Re\Big\{ \underline{F}_k \cdot e^{j\omega_k t} \Big\} \tag{5.2}$$

Here, $\underline{F}_k$ are the complex Fourier coefficients for the respective angular frequency ω_k, which is a multiple of the fundamental frequency $\frac{2\pi}{T}$ of the time interval.

$$\underline{F}_k \;=\; \int_0^T f(t) \cdot e^{-j\omega_k t}\, dt \qquad \textit{and} \qquad \omega_k \;=\; \frac{2\pi}{T} \cdot k \tag{5.3}$$

5.1.3 Fourier Synthesis and Approximation

Although the series expansion (5.2) represents the periodic signal exactly, the summation of infinitely many sine and cosine functions is of course not practicable in reality.

Instead, the series expansion is truncated at a certain $k = k_{max}$ previously determined by the requirements for the calculation, and the signal is thereby approximated.

The highest frequency components contained in the approximation are then given by the frequency $\omega_{max} = \frac{2\pi}{T} \cdot k_{max}$.

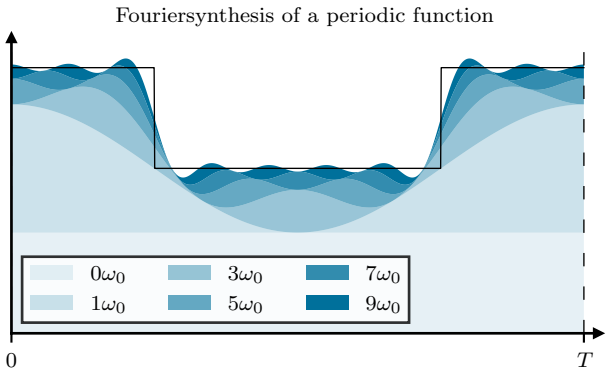


Figure 5.1: Superposition of harmonic oscillations to approximate a periodic function.

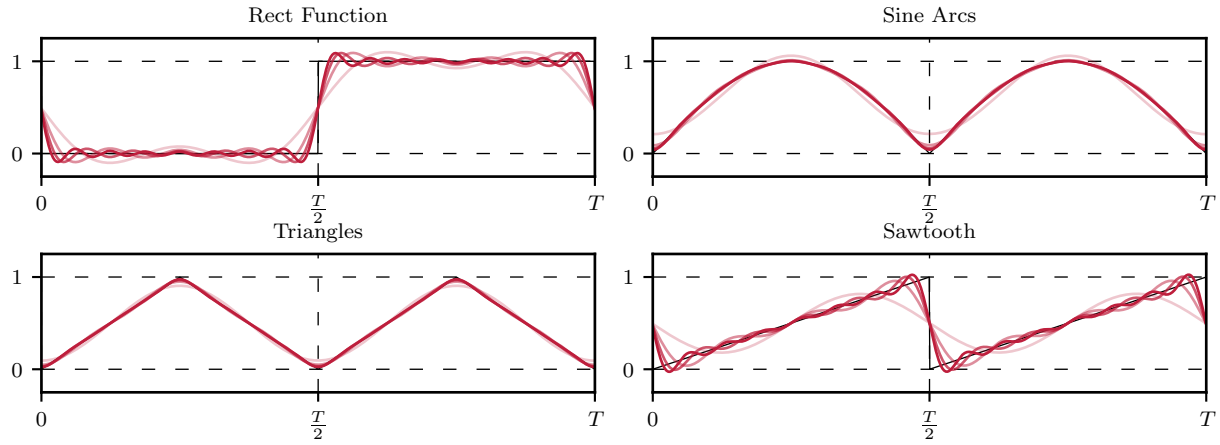


Figure 5.2: Exemplary Fourier approximation of T-periodic functions. Shown are square wave, sine arc, triangle, and sawtooth functions for various approximation levels k_{max}.

5.1.4 Complex Pointers and Phasors

The pointer model is a popular concept in physics. It represents periodic processes as the rotation of a pointer in the complex plane and finds application especially in oscillation theory, complex AC circuit analysis, wave optics, and quantum mechanics. In general, the pointer representation is particularly well-suited for describing harmonic processes, since amplitude, phase, and frequency of the signals can be interpreted geometrically in an intuitive way. Periodic signals are thereby described by a cosine function with a (*generally real frequency-dependent*) amplitude $Z(\omega) = |\underline{Z}(j\omega)|$ and phase $\varphi(\omega)$.

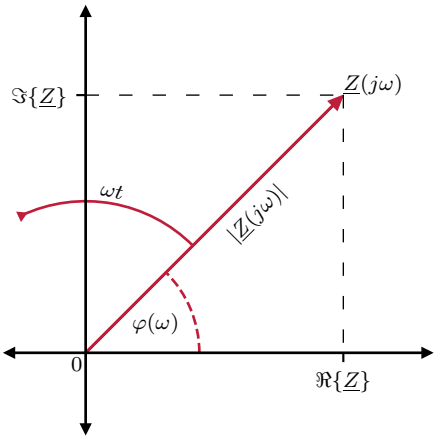


Figure 5.3: Geometric representation of a frequency-dependent phasor $\underline{Z}(j\omega)$ in the complex plane.

All magnitude and phase information of the oscillation is thus represented by the phasor $\underline{Z}(j\omega)$. The pointer representation of the oscillation is then:

$$z(t) = \underbrace{|\underline{Z}(j\omega)|}_{Amplitude} \cdot cos\big(\omega t + \underbrace{\varphi(\omega)}_{Phase}\big) = \Re\Big\{ \underbrace{\overbrace{\underline{Z}(j\omega)}^{Pointer}}_{Phasor} \cdot e^{j\omega t} \Big\} \tag{5.4}$$

This representation exploits the relationship between the trigonometric functions and the complex exponential function via Euler's formula. The phasor $\underline{Z}(j\omega)$ has the particular advantage that the (*sinusoidal*) time dependence does not appear in it. It depends only on the angular frequency ω and is thus a constant complex number for fixed frequencies $\omega = \omega_0$. The pointer, on the other hand, takes the time dependence into account and describes a circular path in the complex plane with frequency ω, whose radius and phase are determined by the phasor. In this notation, the relationships between oscillations and signals can be reduced to the relationships between the pointers (*or for fixed frequencies to the phasors*).

5.2 The Frequency Response

The frequency response of a system is often the centerpiece of analysis in the frequency domain and will therefore now be treated in somewhat more detail. But first some important fundamentals.

5.2.1 Basic Assumptions and Definition of the Frequency Response

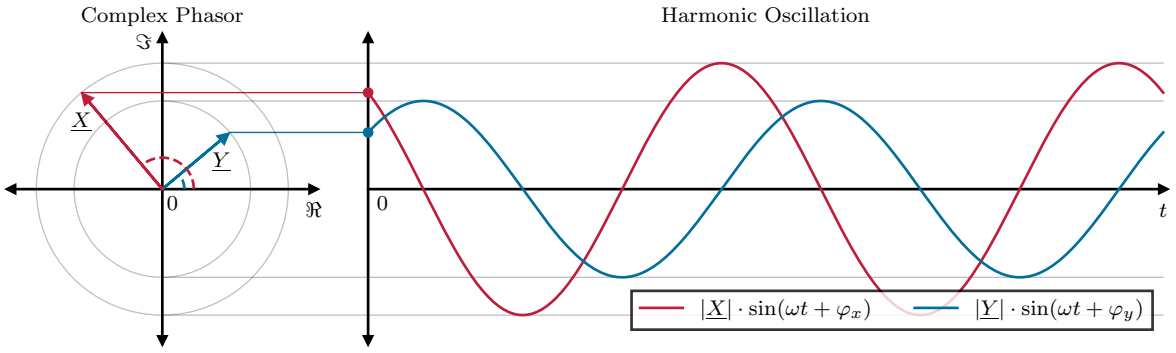


Figure 5.4: Visualization of the magnitude ratios and phase differences of the phasors $\underline{Y}$ and $\underline{X}$ in the phasor diagram. Projection of the imaginary part onto harmonic oscillations.

For linear time-invariant asymptotically stable systems in a steady state with harmonic excitation of frequency ω, it holds that all system quantities – including the output quantity – are based on the same oscillation with the same frequency ω (*see harmonic steady state*). All system quantities and thus all inputs:

$$y(t) \;=\; \Re\Big\{\underline{Y} \cdot e^{j\omega t}\Big\} \;=\; |\underline{Y}| \cdot cos\,(\omega t + \varphi_y) \tag{5.5}$$

And system outputs:

$$x(t) \;=\; \Re\Big\{\underline{X} \cdot e^{j\omega t}\Big\} \;=\; |\underline{X}| \cdot cos\,(\omega t + \varphi_x) \tag{5.6}$$

Can then be represented by complex pointers of the same frequency. The input and output signals can then be reduced to the phasors $\underline{Y}$ and $\underline{X}$. The superposition principle also applies here and each frequency can be described by a phasor (*Fourier coefficient*) and treated separately, entirely in the spirit of a Fourier series expansion. The phase and magnitude difference of the complex pointers can be traced directly back to the phasors and is generally dependent on the exciting frequency ω. The frequency-dependent ratio of the phasors $\underline{Y}$ and $\underline{X}$ in the pointer model is called the frequency response:

$$\underline{H}(j\omega) \;:=\; \frac{\underline{X}}{\underline{Y}} \qquad \textit{respectively} \qquad \underline{H}(j\omega) \cdot \underline{Y} \;=\; \underline{X} \tag{5.7}$$

In particular, there is an important relationship to the previously introduced impulse response $h(t)$. The frequency response can be calculated directly via the Fourier transformation $\mathcal{F}$ if the impulse response is known (*see: harmonic steady state in Section 4.5.7*):

$$\underline{H}(j\omega) \;=\; \mathcal{F}\Big\{h(t)\Big\}(j\omega) \;=\; \int_0^\infty h(t) \cdot e^{-j\omega t}\,dt \tag{5.8}$$

The integral transformation $\mathcal{F}$ here is the one-sided Fourier transformation. The evaluation of the inte-

grand and thus of the impulse response occurs only for positive $t > 0$, which is both a consequence of and a condition for the causality of the system. Moreover, the integral (*particularly the upper integration limit*) converges only if the impulse response describes an asymptotically stable system. Since the magnitude and phase difference is generally dependent on the exciting frequency ω in $y(t)$, both the magnitude and the phase of the frequency response are frequency-dependent. Thus it holds:

$$\underline{H}(j\omega) \;=\; \left|\underline{H}(j\omega)\right| \cdot e^{j\varphi(\omega)} \quad \textit{with} \quad \varphi(\omega) \;=\; \measuredangle\Big\{\underline{H}(j\omega)\Big\} \tag{5.9}$$

5.2.2 Note on the Existence of a Frequency Response

The impulse response of linear time-invariant systems often has the following form (*see the section on the general impulse response*), where $\underline{\lambda}_i$ are the natural frequencies or the eigenvalues of the state-space model:

$$h(t) \;=\; \theta(t) \sum_{i=1}^{n} \underline{h}_i \cdot e^{\underline{\lambda}_i t}$$

The calculation of the frequency response can, as motivated in the section on the harmonic steady state, be performed via the one-sided Fourier transformation of the impulse response. Therefore set:

$$\begin{aligned}
\mathcal{F}\Big\{h(t)\Big\}(j\omega) \;&=\; \int_0^\infty h(t) \cdot e^{-j\omega t}\, dt \\
&\quad\;\big|\; \textit{Substitute impulse response} \\
&=\; \int_0^\infty \sum_{i=1}^{n} \underline{h}_i \cdot e^{\underline{\lambda}_i t} \cdot e^{-j\omega t}\, dt \\
&\quad\;\big|\; \textit{Interchange integral and sum (linearity)} \\
&=\; \sum_{i=1}^{n} \left(\int_0^\infty \underline{h}_i \cdot e^{\underline{\lambda}_i t} \cdot e^{-j\omega t}\, dt \right)
\end{aligned}$$

For each summand of the impulse response, the following now holds, where the complex eigenvalue is split into real and imaginary parts with $\underline{\lambda}_i = \alpha_i + j\omega_i$ and the exponents are combined:

$$\int_0^\infty \underline{h}_i \cdot e^{\alpha_i t} \cdot e^{j(\omega_i - \omega)t}\, dt \;=\; \left[\frac{\underline{h}_i \cdot e^{\alpha_i t} \cdot e^{j(\omega_i-\omega)t}}{\alpha_i + j(\omega_i - \omega)} \right]_{t=0}^{t=\infty} \;=\; \begin{cases} \textit{divergent} & \textit{for } \alpha_i \geq 0 \\ \frac{\underline{h}_i}{\alpha_i + j(\omega_i - \omega)} & \textit{for } \alpha_i < 0 \end{cases} \tag{5.10}$$

It is therefore clear that the Fourier integral converges only for negative real parts $\Re\{\underline{\lambda}_i\} = \alpha_i$ of the eigenvalues and that a unique frequency response can exist only then. Decisive here is the necessary asymptotic convergence of the impulse response for $t \to \infty$ to the time axis. The asymptotic stability of the system can thus again be clearly recognized as a fundamental requirement for the existence of a frequency response. This holds generally, since due to $\left|e^{j\omega t}\right| = 1$ the convergence depends exclusively on the impulse response. This was shown in Section 4.5.5 for the general impulse response. Marginal stability is not sufficient, since then besides the exciting frequency, signal components of the natural frequencies are always contained in the system response, which do not decay, which in turn violates the basic assumption of analysis in the frequency domain.

5.2.3 System Description via the Frequency Response

The description of the system via the frequency response indicates how the amplitude and phase at the output behave when excited by a harmonic signal (*for example a real cosine*) at the input. Both are generally dependent on the frequency ω of the exciting signal. In the frequency domain, the following relationship holds for the phasors of input and output:

$$\underline{X} \;=\; \underline{H}(j\omega) \cdot \underline{Y} \tag{5.11}$$

The system response as a response in the harmonic steady state in the time domain can thereby be specified directly via the frequency response. Then:

$$x(t) \;=\; \Re\Big\{\underline{H}(j\omega) \cdot \underline{Y} \cdot e^{j\omega t}\Big\} \;=\; |\underline{H}(j\omega)| \cdot |\underline{Y}| \cdot cos\left(\omega t + \varphi(\omega) + \varphi_y\right) \tag{5.12}$$

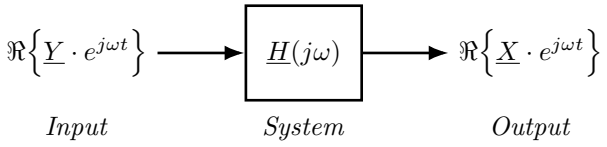


Figure 5.5: System in the frequency domain as a block diagram. Characterized by the frequency response $\underline{H}(j\omega)$.

5.2.4 Measurement of the Frequency Response

For excitation by a harmonic oscillation (*sine or cosine*) with phasor or amplitude $\underline{Y} = 1$, the response in the frequency domain follows, and then for the real physical quantity the response to the signal in the time domain holds:

$$\underline{X} \;=\; \underline{H}(j\omega) \cdot 1 \qquad \Rightarrow \qquad x(t) \;=\; \Re\Big\{\underline{H}(j\omega) \cdot e^{j\omega t}\Big\} \;=\; |\underline{H}(j\omega)| \cdot cos\left(\omega t + \varphi(\omega)\right) \tag{5.13}$$

The magnitude of the frequency response can be measured under these conditions as the amplitude directly at the output of the system. This is often done by measuring the amplitude for a desired frequency interval (*also sine sweep*). Thus the magnitude of the frequency response represents the frequency-dependent amplitude of the output. Therefore, an analysis of the magnitude can give a first estimate of the magnitude of the signal or oscillation. Vectorial measurement systems (*in electrical engineering Vector Network Analyzers VNA*) enable not only magnitude measurement but also measurement of the phase difference to the reference signal. Then the frequency response (*at least for certain discrete frequencies*) can even be completely determined.

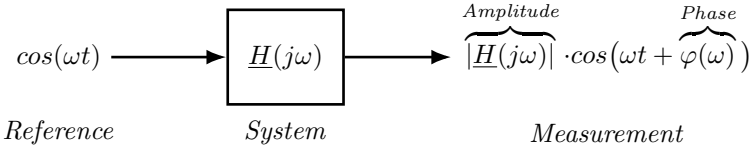


Figure 5.6: Excitation of the system with amplitude 1 and phase zero for direct measurement of the frequency response as amplitude at the system output.

5.2.5 Frequency Response and Transfer Function

The frequency response (5.7) describes, like the transfer function, the ratio of input and output of a system. For the substitution of the argument $\underline{s} \to j\omega$, the frequency response can be obtained from the transfer function. This corresponds to an evaluation of the transfer function along the imaginary axis:

$$\underline{H}(\underline{s}) \xrightarrow{substitute\ \underline{s} \to j\omega} \underline{H}(j\omega) \tag{5.14}$$

The transfer function can – as long as the system quantities are Laplace transformable – always be established and describes the complete behavior of the system with respect to an arbitrary excitation for all times $t > 0$ (*thus also the transient response / settling*). A frequency response, on the other hand, can deliver meaningful results only for stable systems that are in a steady state under harmonic excitation. Only then is the description exclusively via magnitude and phase differences permissible. The frequency response corresponds with the transition (5.14) essentially to an evaluation of the transfer function on the imaginary axis and therefore does not describe the complete behavior.

The similarity of the Laplace integral and the one-sided Fourier transformation leads to the fact that frequency response and transfer function generally have the same structure. Both are rational functions with a numerator and a denominator polynomial.

Table 5.1: Differences between Transfer Function and Frequency Response

Transfer Function	Frequency Response
Laplace transformation of the impulse response $h(t)$: $\underline{H}(\underline{s}) = \int_0^\infty h(t) \cdot e^{-\underline{s}t}\, dt$	One-sided Fourier transformation of the impulse response: $\underline{H}(j\omega) = \int_0^\infty h(t) \cdot e^{-j\omega t}\, dt$
Exists for arbitrary linear time-invariant systems, convergence of the integral can be ensured by the real part of $\underline{s}$.	Convergence of the integral depends exclusively on $h(t)$, since $\left\|e^{j\omega t}\right\| = 1$. Exists only for asymptotically stable systems.
The time derivative becomes in the transfer function (*which does not contain initial values!*) a multiplication with $\underline{s}$: $\frac{d}{dt} f(t) \xrightarrow{\mathcal{L}} \underline{s} \cdot \underline{F}(\underline{s})$	The time derivative becomes in the structure of the frequency response a multiplication with the purely imaginary variable $j\omega$: $\frac{d}{dt} f(t) \xrightarrow{\mathcal{F}} j\omega \underline{F}$
The system response in the Laplace domain results as multiplication of input and transfer function, where $\underline{X}(\underline{s})$ and $\underline{Y}(\underline{s})$ are the Laplace transforms of the time domain quantities: $\underline{X}(\underline{s}) = \underline{H}(\underline{s}) \cdot \underline{Y}(\underline{s})$	The system response in the frequency domain with respect to excitation with a frequency ω results as multiplication of the complex input amplitude (*phasor*) with the frequency response: $\underline{X} = \underline{H}(j\omega) \cdot \underline{Y}$

5.2.6 Frequency Response in Polar Representation

The frequency response as a complex-valued function can be represented in polar form like any complex quantity. This is done as a decomposition into magnitude and phase. Like the transfer function, the frequency response also has a numerator and a denominator polynomial:

$$\underline{H}(j\omega) \;=\; \frac{\underline{P}(j\omega)}{\underline{Q}(j\omega)}$$

Thus for the polar representation it follows:

$$\underline{H}(j\omega) \;=\; |\underline{H}(j\omega)| \cdot e^{j\varphi(\omega)} \;=\; \frac{|\underline{P}(j\omega)|}{|\underline{Q}(j\omega)|} \cdot e^{j\varphi_P(\omega)} \cdot e^{-j\varphi_Q(\omega)} \tag{5.15}$$

Both the magnitude $|\underline{H}(j\omega)|$ and the phase $\varphi(\omega) = \varphi_P(\omega) - \varphi_Q(\omega)$ are still frequency-dependent.

5.2.7 Magnitude of the Frequency Response

Since $\underline{P}(j\omega)$ and $\underline{Q}(j\omega)$ are generally polynomials, the real part and imaginary part can mostly be determined quite easily. The magnitude of the frequency response is calculated as follows:

$$|\underline{H}(j\omega)| \;=\; \sqrt{\Re\Big\{\underline{H}(j\omega)\Big\}^2 + \Im\Big\{\underline{H}(j\omega)\Big\}^2} \;=\; \sqrt{\frac{\Re\Big\{\underline{P}(j\omega)\Big\}^2 + \Im\Big\{\underline{P}(j\omega)\Big\}^2}{\Re\Big\{\underline{Q}(j\omega)\Big\}^2 + \Im\Big\{\underline{Q}(j\omega)\Big\}^2}} \tag{5.16}$$

5.2.8 Note on Complex Polynomial Functions

Specifically, the polynomials $\underline{P}(j\omega)$ and $\underline{Q}(j\omega)$ in the numerator and denominator of the frequency response have the following general form (*polynomial form*):

$$\underline{F}(j\omega) \;=\; b_0 + j\omega b_1 + (j\omega)^2 b_2 + (j\omega)^3 b_3 + (j\omega)^4 b_4 + (j\omega)^5 b_5 + \dots$$

In summation notation and after splitting into even and odd components, the polynomial can be represented as follows:

$$\underline{F}(j\omega) \;=\; \sum_{k=0} (j\omega)^k b_k \;=\; \underbrace{\sum_{k=0} (j\omega)^{2k}\, b_{2k}}_{even\ part} \;+\; \underbrace{\sum_{k=0} (j\omega)^{2k+1}\, b_{2k+1}}_{odd\ part}$$

The real part corresponds to the even terms with $(j\omega)^{2k} = (-1)^k \cdot \omega^{2k}$, since then the imaginary unit j disappears. The imaginary part corresponds to the odd terms with $(j\omega)^{2k+1} = j \cdot (-1)^k \cdot \omega^{2k+1}$, since then the imaginary unit is always contained. Thus for the real and imaginary parts of the polynomials it holds:

$$\Re\Big\{\underline{F}(j\omega)\Big\} \;=\; b_0 - \omega^2 b_2 + \omega^4 b_4 + \dots \qquad \textit{and} \qquad \Im\Big\{\underline{F}(j\omega)\Big\} \;=\; \omega b_1 - \omega^3 b_3 + \omega^5 b_5 + \dots$$

The treatment of the polynomials $\underline{P}(j\omega)$ and $\underline{Q}(j\omega)$ proceeds completely analogously to this general case.

5.2.9 Case Distinction for the Calculation of the Phase

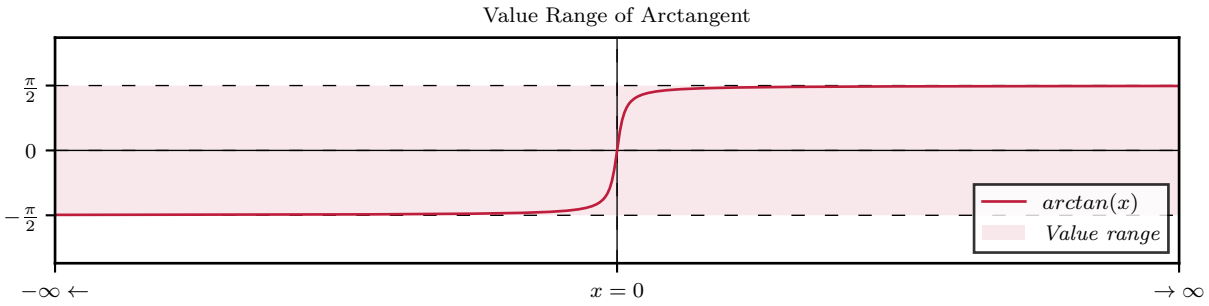


Figure 5.7: Visualization of the range of the arctangent

For each of the complex-valued polynomials – here $\underline{P}(j\omega)$ exemplarily – of the frequency response, the phase φ of the resulting complex number can be determined via the arctangent.

Since the arctangent itself can only assume values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, only phase angles in the right complex half-plane can be represented with it. The sign of the argument of the arctangent function is invariant under interchange of the signs of numerator and denominator, thus the determination of the phase via the arctangent is not unique.

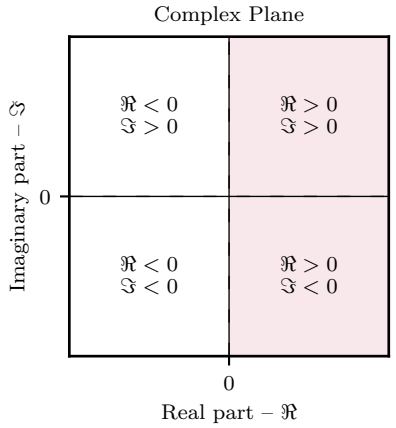


Figure 5.8: Complex plane with the four quadrants. In the right half-plane, the phase can be uniquely described by the arctangent.

Moreover, the phase angles $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ are only reached in the limit of the arctangent. Phase angles of values on the imaginary axis must therefore also be treated separately. For the unique determination of the phase angle, a case distinction must therefore be made that considers in which quadrant the complex number lies:

$$\varphi_P(\omega) \;=\; \measuredangle\Big\{\underline{P}(j\omega)\Big\} \;=\; \begin{cases} -\frac{\pi}{2} & \text{if } \Re = 0 \text{ and } \Im < 0 \\ \frac{\pi}{2} & \text{if } \Re = 0 \text{ and } \Im > 0 \\ arctan\left(\frac{\Im\left\{\underline{P}(j\omega)\right\}}{\Re\left\{\underline{P}(j\omega)\right\}}\right) \pm \pi & \text{if } \Re < 0 \\ arctan\left(\frac{\Im\left\{\underline{P}(j\omega)\right\}}{\Re\left\{\underline{P}(j\omega)\right\}}\right) & \text{otherwise} \end{cases} \tag{5.17}$$

Since the polynomials – and thus also the real and imaginary parts – are frequency-dependent, the case distinction must be carried out depending on the considered frequency ω.

5.2.10 Causality Relation of Linear Systems

Besides the symmetry relation – which applies not only to the transfer function but of course also to frequency responses – there is another, less well-known, but often used relation in optics and solid-state physics for the frequency response. The Kramers-Kronig relation is a causality relation and links the real and imaginary parts of rational functions with each other.

For the derivation, we recall that the impulse response $h(t)$ is defined only for positive times due to causality (*action before reaction*) and that it therefore makes no difference whether we additionally truncate it with a step function $\theta(t)$. Therefore, without restrictions it holds:

$$h(t) \;=\; h(t) \cdot \theta(t) \quad \textit{for} \quad t > 0$$

With the convolution theorem for a multiplication in the time domain[1], a convolution in the frequency domain results (*for this the impulse response must describe an asymptotically stable system*):

$$\underline{H}(j\omega) \;=\; \left(\underline{H}(j\omega_0) * \frac{1}{j\omega_0} \right)(j\omega)$$

This approach can now be exploited to derive a relationship between real and imaginary parts of the frequency response:

$$\begin{aligned}
\underline{H}(j\omega) \;&=\; \left(\underline{H}(j\omega_0) * \frac{1}{j\omega_0} \right)(j\omega) \\
&\mid \; \textit{Write out convolution integral and rearrange } j \\
&= \; -j \cdot \int_{-\infty}^{\infty} \frac{\underline{H}(j\omega_0)}{\omega - \omega_0} \, d\omega_0 \\
&\mid \; \textit{Cleverly extend denominator} \\
&= \; -j \cdot \int_{-\infty}^{\infty} \frac{\omega \underline{H}(j\omega_0) + \omega_0 \underline{H}(j\omega_0)}{\omega^2 - \omega_0^2} \, d\omega_0 \\
&\mid \; \textit{Split integral based on limits and interchange limits} \\
&= \; -j \cdot \int_{0}^{\infty} \frac{\omega \underline{H}(j\omega_0) + \omega_0 \underline{H}(j\omega_0)}{\omega^2 - \omega_0^2} \, d\omega_0 \;+\; j \cdot \int_{0}^{-\infty} \frac{\omega \underline{H}(j\omega_0) + \omega_0 \underline{H}(j\omega_0)}{\omega^2 - \omega_0^2} \, d\omega_0 \\
&\mid \; \textit{Reverse right upper integration limit (substitution)} \\
&= \; -j \cdot \int_{0}^{\infty} \frac{\omega \underline{H}(j\omega_0) + \omega_0 \underline{H}(j\omega_0)}{\omega^2 - \omega_0^2} \, d\omega_0 \;+\; j \cdot \int_{0}^{+\infty} \frac{\omega \underline{H}(-j\omega_0) - \omega_0 \underline{H}(-j\omega_0)}{\omega^2 - \omega_0^2} \, d\omega_0 \\
&\vdots
\end{aligned}$$

[1] yes I know, the convolution theorem for a multiplication in the time domain is not yet included in the script, but will certainly come at some point

$$\vdots$$

$$\begin{aligned}
&| \quad \textit{Combine integrals again} \\
&= \quad j \cdot \int_0^\infty -\frac{\omega \underline{H}(j\omega_0) + \omega_0 \underline{H}(j\omega_0)}{\omega^2 - \omega_0^2} \; + \; \frac{\omega \underline{H}(-j\omega_0) - \omega_0 \underline{H}(-j\omega_0)}{\omega^2 - \omega_0^2} \, d\omega_0 \\
&| \quad \textit{Apply symmetry relation } \underline{H}(-j\omega) = \underline{H}^*(j\omega) \\
&= \quad j \cdot \int_0^\infty -\frac{\omega \underline{H}(j\omega_0) + \omega_0 \underline{H}(j\omega_0)}{\omega^2 - \omega_0^2} \; + \; \frac{\omega \underline{H}^*(j\omega_0) - \omega_0 \underline{H}^*(j\omega_0)}{\omega^2 - \omega_0^2} \, d\omega_0 \\
&| \quad \textit{Combine fractions and sort by } \omega, \, \omega_0 \\
&= \quad j \cdot \int_0^\infty \frac{-\omega \big(\underline{H}(j\omega_0) + \underline{H}^*(j\omega_0)\big) + \omega_0 \big(\underline{H}(j\omega_0) - \underline{H}^*(j\omega_0)\big)}{\omega^2 - \omega_0^2} \, d\omega_0 \\
&| \quad \textit{use } \underline{z} + \underline{z}^* = 2\Re\{\underline{z}\} \textit{ and } \underline{z} - \underline{z}^* = j2\Im\{\underline{z}\} \textit{ for the frequency response} \\
&= \quad j \cdot \int_0^\infty \frac{-\omega 2\Re\big\{\underline{H}(j\omega_0)\big\} + j\omega_0 2\Im\big\{\underline{H}(j\omega_0)\big\}}{\omega^2 - \omega_0^2} \, d\omega_0 \\
&| \quad \textit{rearranging according to real and imaginary parts yields} \\
&= \quad \int_0^\infty -j \cdot \frac{\omega 2\Re\big\{\underline{H}(j\omega_0)\big\}}{\omega^2 - \omega_0^2} \; + \; \frac{\omega_0 2\Im\big\{\underline{H}(j\omega_0)\big\}}{\omega^2 - \omega_0^2} \, d\omega_0
\end{aligned}$$

In this form, the real and imaginary parts of the frequency response can now be assigned to the integrand and the two equations of the Kramers-Kronig relation result:

$$\Re\big\{\underline{H}(j\omega)\big\} \quad = \quad \int_0^\infty \Im\big\{\underline{H}(j\omega_0)\big\} \cdot \frac{2\omega_0}{\omega^2 - \omega_0^2} \, d\omega_0 \tag{5.18}$$

$$\Im\big\{\underline{H}(j\omega)\big\} \quad = \quad -\int_0^\infty \Re\big\{\underline{H}(j\omega_0)\big\} \cdot \frac{2\omega}{\omega^2 - \omega_0^2} \, d\omega_0 \tag{5.19}$$

This causality relation (*fulfillment of causality by prior truncation of the impulse response with a step function*) can alternatively also be obtained by applying the Hilbert transformation to an arbitrary frequency response.

5.3 The Bode Diagram

In the Bode diagram, magnitude and phase are plotted separately versus the exciting frequency ω. Customary here is a logarithmic plot of the phase and a double logarithmic plot for the magnitude versus frequency. The Bode diagram enables a quick visual analysis of the system behavior with respect to excitation by harmonic oscillations of different frequencies. In particular, the asymptotic behavior for low and high frequencies can be easily read off. Also, any resonance frequencies of the system can be identified as extrema in the Bode diagram.

5.3.1 Magnitude of the Frequency Response in the Bode Diagram

Particularly for manual analysis of the frequency response with a Bode diagram, the poles and zeros are needed. The magnitude is then:

$$|\underline{H}(j\omega)| \;=\; \left|\frac{(j\omega-\underline{z}_m)\cdots(j\omega-\underline{z}_1)}{(j\omega-\underline{p}_n)\cdots(j\omega-\underline{p}_1)}\right| \;=\; \frac{\prod_{k=1}^{m}|j\omega-\underline{z}_k|}{\prod_{k=1}^{n}|j\omega-\underline{p}_k|} \tag{5.20}$$

For a logarithmic plot of the magnitude in decibels, this yields:

$$\begin{aligned}
|\underline{H}(j\omega)| \;&=\; 20\cdot log_{10}\Big(\,|\underline{H}(j\omega)|\,\Big)\ \mathrm{dB}\\
&\quad\ |\quad \textit{Substitute pole/zero form}\\
&=\; 20\cdot log_{10}\left(\frac{\prod_{k=1}^{m}|j\omega-\underline{z}_k|}{\prod_{k=1}^{n}|j\omega-\underline{p}_k|}\right)\ \mathrm{dB}\\
&\quad\ |\quad \textit{by logarithm, product becomes sum}\\
&=\; \underbrace{20\cdot\sum_{k=1}^{m} log_{10}\Big(|j\omega-\underline{z}_k|\Big)\ \mathrm{dB}}_{zeros} \;-\; \underbrace{20\cdot\sum_{k=1}^{n} log_{10}\Big(|j\omega-\underline{p}_k|\Big)\ \mathrm{dB}}_{poles}
\end{aligned}$$

Thus zeros contribute positively to the magnitude and poles negatively. To get a better estimate of exactly how poles and zeros influence the magnitude, the asymptotic behavior of the individual elements with respect to frequency is now analyzed.

5.3.2 Phase of the Frequency Response in the Bode Diagram

The phase of the frequency response, on the other hand, results via the sum of the phase contributions:

$$\varphi(\omega) \;=\; \underbrace{\sum_{k=1}^{m}\measuredangle\Big\{j\omega-\underline{z}_k\Big\}}_{zeros} \;-\; \underbrace{\sum_{k=1}^{n}\measuredangle\Big\{j\omega-\underline{p}_k\Big\}}_{poles} \tag{5.21}$$

Also for the phase, the influence of the poles and zeros differs only by the sign. The phase is plotted in the Bode diagram only single-logarithmically versus frequency.

5.3.3 Contribution of Real Poles to the Magnitude

The partial frequency response – a factor of the frequency response – of a general real pole $\underline{p} = \alpha < 0$ is given by:

$$\underline{T}(j\omega) \;=\; \frac{1}{j\omega - \underline{p}} \;=\; \frac{1}{j\omega - \alpha} \tag{5.22}$$

The magnitude of the partial frequency response in decibels therefore results as:

$$\begin{aligned} |\underline{T}(j\omega)| \;&=\; 20 \cdot log_{10}\left(\left|\frac{1}{j\omega - \alpha}\right|\right) \text{ dB} \\ &\Big| \quad \textit{use the relation: } log\left(\tfrac{a}{b}\right) = log\,(a) - log\,(b) \\ &=\; -20 \cdot log_{10}\Big(|j\omega - \alpha|\Big) \text{ dB} \end{aligned} \tag{5.23}$$

For small frequencies and thus $\omega \ll |\alpha|$, the asymptotic behavior holds:

$$|\underline{T}(j\omega)|_{\omega \to 0} \;\approx\; -20 \cdot log_{10}\Big(|\alpha|\Big) \text{ dB} \;=\; T_0 \tag{5.24}$$

A particularly interesting result is obtained when the frequency is close to the magnitude of the pole with $\omega = |\alpha|$:

$$|\underline{T}(j\omega)|_{\omega = |\alpha|} \;=\; -20 \cdot log_{10}\Big(|\alpha|\Big) \text{ dB} - \underbrace{20 \cdot log_{10}\left(\sqrt{2}\right) \text{ dB}}_{\approx 3\,\text{dB}} \tag{5.25}$$

When the frequency $\omega = |\alpha|$ is reached, the magnitude has thus decayed by about 3dB compared to the zero value. For large frequencies $\omega \gg |\alpha|$, the asymptotic behavior holds:

$$|\underline{T}(j\omega)|_{\omega \to \infty} \;\approx\; -20 \cdot log_{10}\Big(\omega\Big) \text{ dB} \tag{5.26}$$

For large frequencies $\omega \gg |\alpha|$, the magnitude curve is thus only dependent on ω. This leads to a drop of $-20\,\text{dB}$ per decade in frequency. The two asymptotes resulting from these considerations can be identified in the Bode diagram of the magnitude. The famous 3dB cutoff frequency $\omega_{3\text{dB}} \approx |\alpha|$ is thus directly dependent on the eigenvalues of the system.

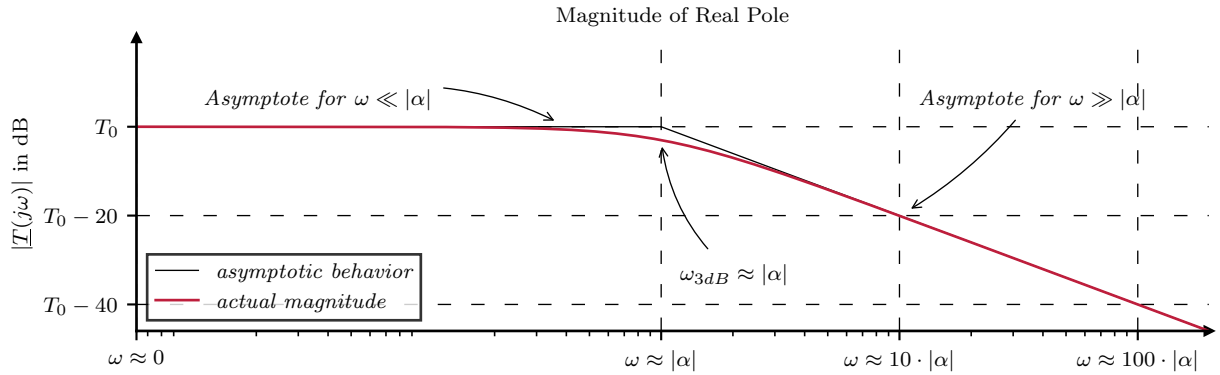


Figure 5.9: Double logarithmic plot of the contribution of a real pole to the magnitude of the frequency response. Also the asymptotic behavior for small and large frequencies.

5.3.4 Contribution of Real Poles to the Phase

The phase contribution of the partial frequency response of a general real pole $\underline{p} = \alpha < 0$ then results via:

$$\underline{T}(j\omega) \;=\; \frac{1}{j\omega - \alpha} \qquad \textit{with} \qquad \varphi(\omega) \;=\; -\measuredangle\Big\{j\omega - \alpha\Big\} \tag{5.27}$$

The calculation of the frequency-dependent phase angle is performed via the principal value of the arctangent and with the corresponding case distinction. To work out the asymptotic phase behavior, the phase for low frequencies $\omega \ll |\alpha|$ is first considered:

$$\varphi(\omega)\Big|_{\omega \ll |\alpha|} \;\approx\; -arctan\left(\frac{0}{-\alpha}\right) \;=\; 0 \tag{5.28}$$

For frequencies close to the magnitude of the pole $\omega = |\alpha|$ and $\alpha < 0$, correspondingly it holds:

$$\begin{aligned}
\varphi(\omega)\Big|_{\omega=|\alpha|} \;&=\; -\measuredangle\Big\{j|\alpha| - \alpha\Big\} \\
&\quad\;|\quad \textit{Phase calculation via arctangent} \\
&=\; -arctan\left(\frac{|\alpha|}{-\alpha}\right) \\
&\quad\;|\quad \textit{it follows} \\
&=\; -\frac{\pi}{4}
\end{aligned} \tag{5.29}$$

For high frequencies $\omega \gg |\alpha|$, then it follows:

$$\varphi(\omega)\Big|_{\omega \gg |\alpha|} \;\approx\; -\measuredangle\Big\{j\omega\Big\} \;=\; -\frac{\pi}{2} \tag{5.30}$$

This asymptotic behavior of the phase for low and high frequencies can also be found in the Bode diagram of the phase component. To draw the phase manually, the transition between the asymptotes is often assumed as a straight line through the inflection point at $\omega = |\alpha|$ with slope $-\frac{\pi}{4}$ per decade in frequency.

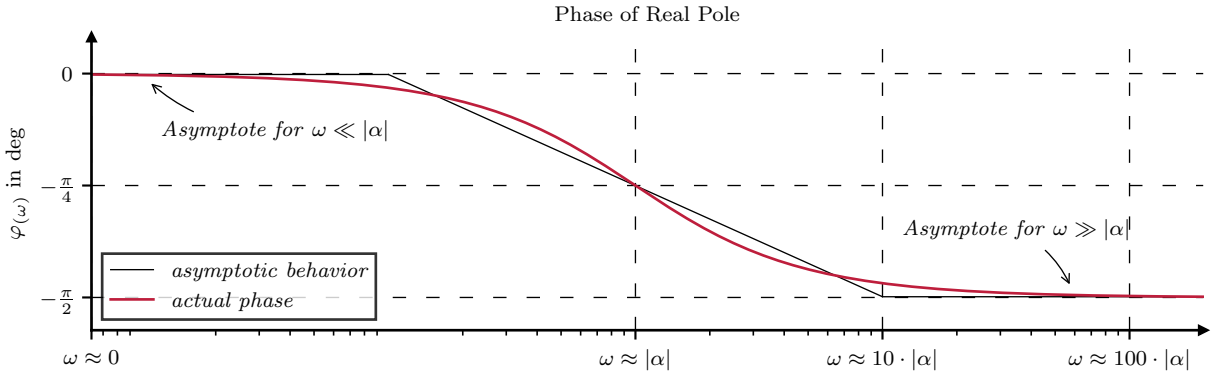


Figure 5.10: Logarithmic plot of the contribution of a real pole to the phase of the frequency response. Also shown is the asymptotic behavior for small and large frequencies.

5.3.5 Contribution of Complex Poles to the Magnitude

The partial frequency response – a factor of the frequency response – of a general complex conjugate pole pair $\underline{p} = \alpha_k + j\omega_k$ and $\underline{p}^* = \alpha_k - j\omega_k$ with $\alpha_k < 0$ is given by:

$$\underline{T}(j\omega) \;=\; \frac{1}{(j\omega - \underline{p})\,(j\omega - \underline{p}^*)} \;=\; \frac{1}{|\underline{p}|^2 - \omega^2 - j\omega\,2\,\alpha_k} \tag{5.31}$$

The magnitude of the partial frequency response in decibels therefore results as:

$$|\underline{T}(j\omega)| \;=\; -20 \cdot log_{10}\Big(\,\Big||\underline{p}|^2 - \omega^2 - j\omega\,2\,\alpha_k\Big|\,\Big)\;\text{dB}$$

For small frequencies and thus $\omega \ll |\underline{p}|$, the asymptotic behavior holds:

$$|\underline{T}(j\omega)|_{\omega \ll |\underline{p}|} \;\approx\; -40 \cdot log_{10}\Big(|\underline{p}|\Big)\;\text{dB} \;=\; T_0 \tag{5.32}$$

For frequencies close to the magnitude of the pole, i.e., $\omega = |\underline{p}|$, correspondingly it holds:

$$|\underline{T}(j\omega)|_{\omega = |\underline{p}|} \;=\; -20 \cdot log_{10}\Big(|\underline{p}|\Big)\;\text{dB}\;\; - 20 \cdot log_{10}\Big(|2\,\alpha_k|\Big)\;\text{dB} \tag{5.33}$$

For the special case $|\underline{p}| = 2\,|\alpha_k|$, this transitions into the asymptotic behavior for small frequencies. For $|\underline{p}| > 2\,|\alpha_k|$ and equivalently $\omega_k > \sqrt{3}\,\alpha_k$, an increase of the magnitude occurs compared to the zero value around the frequency $\omega = |\underline{p}|$ (*This corresponds to a peak or the well-known resonance peak*).

For large frequencies $\omega \gg |\underline{p}|$, the asymptotic behavior holds:

$$|\underline{T}(j\omega)|_{\omega \gg |\underline{p}|} \;\approx\; -40 \cdot log_{10}\Big(\omega\Big)\;\text{dB} \tag{5.34}$$

The magnitude of the frequency response thus drops for large frequencies (*after the peak*) with $-40\,\text{dB}$ per decade in frequency. For $\omega_k = 0$, this corresponds to two equal real poles. The asymptotic representation of the Bode diagram can be traced back to the behavior of the real pole in the previous section. When the imaginary part of the pole exceeds the real part proportionally with $\omega_k > \sqrt{3}\,\alpha_k$, an increase of the magnitude around $\omega = |\underline{p}|$ occurs.

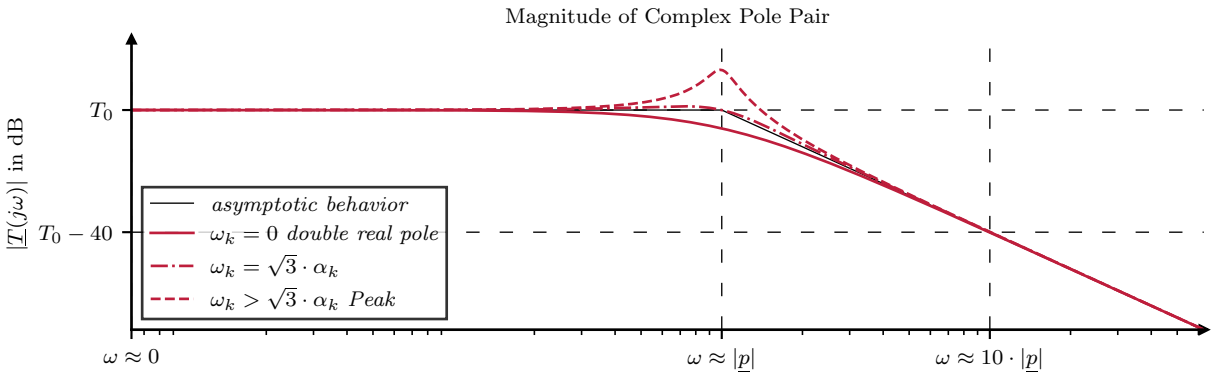


Figure 5.11: Double logarithmic plot of the contribution of a complex conjugate pole pair to the magnitude of the frequency response for different ratios of real and imaginary parts. Also shown is the asymptotic behavior for small and large frequencies.

5.3.6 CONTRIBUTION OF COMPLEX POLES TO THE PHASE

The phase component of the partial frequency response of a general complex conjugate pole pair $\underline{p} = \alpha_k + j\omega_k$ and $\underline{p}^* = \alpha_k - j\omega_k$ with $\alpha_k < 0$ then results via:

$$\underline{T}(j\omega) \;=\; \frac{1}{|\underline{p}|^2 - \omega^2 - j\omega\, 2\,\alpha_k} \qquad \textit{with} \qquad \varphi(\omega) \;=\; -\measuredangle\Big\{|\underline{p}|^2 - \omega^2 - j\omega\, 2\,\alpha_k\Big\} \tag{5.35}$$

The calculation of the frequency-dependent phase angle for the complex pole pair again proceeds via the principal value of the arctangent and with the corresponding case distinction. To work out the asymptotic phase behavior, the phase for low frequencies $\omega \ll |\underline{p}|$ is first considered:

$$\varphi(\omega)\Big|_{\omega \ll |\underline{p}|} \;\approx\; -arctan\left(\frac{-2\,\alpha_k \cdot 0}{|\underline{p}|^2}\right) \;=\; 0 \tag{5.36}$$

For frequencies close to the magnitude of the pole $\omega = |\underline{p}|$, with $\alpha_k < 0$ correspondingly it holds:

$$\varphi(\omega)\Big|_{\omega = |\underline{p}|} \;=\; -\measuredangle\Big\{-j|\underline{p}|\,2\,\alpha_k\Big\} \;=\; -\frac{\pi}{2} \tag{5.37}$$

For high frequencies $\omega \gg |\underline{p}|$, then with the case distinction for a negative real part $|\underline{p}|^2 - \omega^2 < 0$ and correspondingly in the limit it follows:

$$\varphi(\omega)\Big|_{\omega \gg |\underline{p}|} \;\approx\; \underbrace{\lim_{\omega\to\infty} -arctan\left(\frac{-\omega\, 2\,\alpha_k}{|\underline{p}|^2 - \omega^2}\right)}_{0} - \pi \;=\; -\pi \tag{5.38}$$

As with the real pole, the asymptotic behavior for low and high frequencies can also be clearly recognized in the Bode diagram for the complex pole pair. However, the ratio of real part α_k and imaginary part ω_k now influences how sharply the phase transition occurs around the inflection point $\omega = |\underline{p}|$.

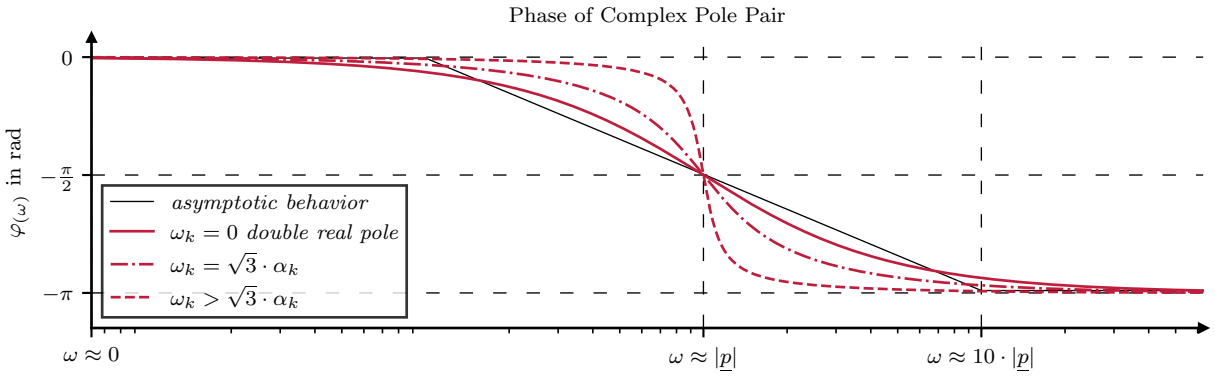


Figure 5.12: Logarithmic plot of the contribution of a complex conjugate pole pair to the phase of the frequency response for different ratios of real and imaginary parts. Also shown is the asymptotic behavior for small and large frequencies.

5.4 The Locus Curve

In systems theory, a locus curve refers to the graphical representation of a complex system quantity dependent on a real parameter (*usually the real angular frequency* ω), such as the frequency response or a corresponding output signal of the system in the frequency domain. Besides the Bode diagram, this is another method for graphical representation of the frequency response, which is used particularly in control engineering and there especially for feedback systems.

5.4.1 Definition

The path described by a parameter-dependent complex pointer $\underline{Z}(\omega)$ in the complex number plane is called a locus curve:

$$\underline{Z}(\omega) \;=\; \Re\Big\{\underline{Z}(\omega)\Big\} + j \cdot \Im\Big\{\underline{Z}(\omega)\Big\} \tag{5.39}$$

5.4.2 Nyquist Diagram

The graphical representation of the locus curve is often called a Nyquist diagram, Nyquist graph, or Nyquist plot. In it, the real part and imaginary part of the complex pointer are plotted against each other in the complex plane as a Cartesian coordinate system and evaluated over a wide parameter range.

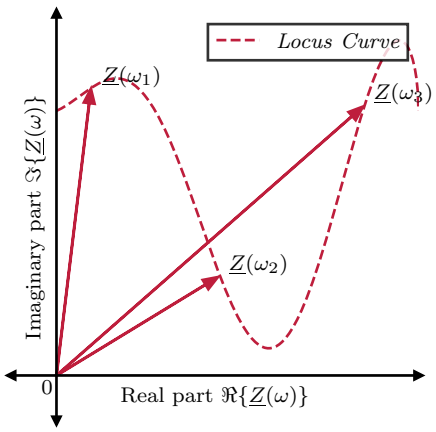


Figure 5.13: General locus curve of a complex function $\underline{Z}(\omega)$ in the complex plane. Phasors for ω_1, ω_2, and ω_3 shown as examples.

Usually the parameter is evaluated on the interval $\omega \in [0, \infty)$, although in practical terms the upper interval limit must of course be chosen finite and the evaluation can also only be performed for discrete parameter values from this interval.

Depending on the system, the interval must be chosen large enough to also capture well the asymptotic behavior for high frequencies. Connecting the evaluation points by a line yields the graphical representation of the locus curve and thus the Nyquist diagram. The locus curve thus corresponds to the line that the tip of the complex phasor describes in the pointer model as frequency varies.

5.5 RESONANCE

Resonance (*from Latin resonare ⇔ to resound*) is in physics and technology the amplified (*in general also the attenuated or extreme*) sympathetic oscillation of an oscillatory system when it is subject to a temporally variable influence. The system can oscillate many times more strongly than under constant influence with the excitation frequency $\omega = 0$. Resonances are often exploited in technology to filter out, amplify, or generally exploit the properties of a component at a certain frequency. Where amplification is not desired and would, for example, damage the component, unwanted resonances must be avoided.

5.5.1 RESONANCE AND HARMONIC STEADY STATE

An asymptotically stable system can be described by a frequency response $\underline{H}(j\omega)$ in a steady state under harmonic excitation. This characterizes the magnitude and phase of the output signal as a function of the respective excitation frequency ω. In the harmonic steady state (*see Section 4.5.7*), the output signal is directly proportional to the magnitude of the frequency response of the system:

$$x(t) \;\sim\; |\underline{H}(j\omega)| \tag{5.40}$$

Since the frequency response (*as the name suggests*) is frequency-dependent, in some cases extrema – both maxima and minima – can occur in the magnitude curve, which are directly reflected in the amplitude of the output signal. If a system is excited with the corresponding frequency, it is in resonance.

5.5.2 RESONANCE FREQUENCIES

The frequencies at which these extrema occur in the magnitude curve of the frequency response are generally referred to as resonance frequencies (*not to be confused with the eigenvalues and natural frequencies, which also have the dimension of a frequency but a different physical meaning*). Since in higher-order systems there can certainly also be multiple such maxima, there can of course also be multiple points at which resonance occurs. Consequently, in general there can also be multiple resonance frequencies.

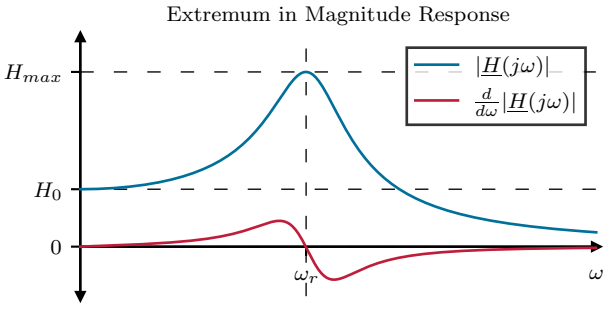


Figure 5.14: Extremum in the magnitude of a generic frequency response and resonance frequency ω_r

The resonance frequencies ω_r result directly from the extrema of the magnitude of the frequency response. That is, the points at which the first derivative vanishes:

$$\frac{d}{d\omega}\big|\underline{H}(j\omega)\big| \;=\; 0 \quad \textit{and} \quad \omega \;>\; 0 \qquad \Leftrightarrow \qquad \omega \;=\; \omega_r \tag{5.41}$$

For the extrema, a distinction is often made between resonance (*maxima, i.e., strong increase in amplitude*) and antiresonance (*minima, i.e., strong attenuation of amplitude*).

5.5.3 Note on Resonance Frequencies

Even in a system (*of only*) second order, there can be different resonance frequencies (*namely up to two*), since according to definition (5.41) they are the extrema of the frequency response and this can be established for each system quantity. Depending on which system quantity is regarded as the system output, a possibly different frequency response arises, which can have different extrema. The resonance frequency is thus always also dependent on the choice of system input and output.

As a prominent example, the electrical RLC oscillating circuit can be mentioned here.

5.5.4 General Calculation of Resonance Frequencies (WIP)

How are the resonance frequencies generally composed from the system parameters? Can an approach via the state-space representation or the general transfer function be derived?

5.5.5 Relationship of the Resonance Frequency to the Eigenvalues

The resonance frequencies are thus directly related to the extrema of the frequency response, quite similar to how the poles relate to the transfer function or eigenvalues of the system. Nevertheless, there are some crucial differences. The poles of the transfer function are defined via the zeros of the denominator polynomial and are generally complex-valued with $\underline{s} \in \mathbb{C}$ and $\underline{\lambda} \in \mathbb{C}$, whereas the frequency response can only take real frequencies $\omega \in \mathbb{R}$ as argument and can therefore only have real resonance frequencies. Particularly interesting is that the resonances occur approximately at the magnitude of the complex poles and zeros when the imaginary part is very large compared to the real part. Especially for oscillatory systems, this approximation is often applicable. The resonance frequencies can then be approximated as:

$$\omega_{r,i} \approx \left|\underline{p}_i\right| \textit{ maxima} \qquad \textit{respectively} \qquad \omega_{r,i} \approx \left|\underline{z}_i\right| \textit{ minima} \tag{5.42}$$

The resonance frequencies can also agree exactly with the poles/zeros of the transfer function if these lie on the imaginary axis (*have no real part*). In general, however, this is not the case, since they are distributed in the entire complex plane.

Appendix A

Electrical RC Network

Many physical effects, such as the discharge of a capacitor, growth, decay, and one-dimensional heat propagation can already be well described by a first-order system. In the following, the presented methods are demonstrated using a linear time-invariant dynamic first-order system, namely an electrical RC network model. The general structure of the results from this chapter can largely be directly transferred to higher-order systems.

A.1 Model Description in the Time Domain

The RC network model treated here consists of a linear time-invariant capacitance $C > 0$, a linear time-invariant resistance $R > 0$, and a time-dependent voltage source $v(t)$. The network elements are connected in a series circuit. Network models of this form model, for example, the charging and discharging process of a real capacitor. RC networks are often used as the simplest model for relaxation processes or saturation behavior in electrical engineering and physics.

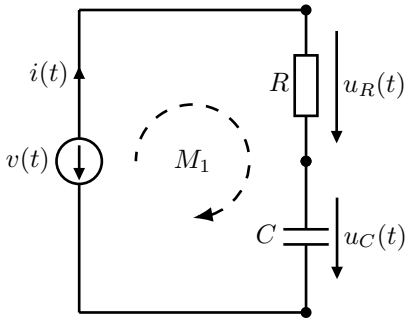


Figure A.1: RC network model in the time domain. Shown is the mesh M_1.

A.1.1 Network Equations of the RC Network

The branch equations link the branch current and branch voltage of a branch. The branch equations of the linear capacitance and the resistance are:

$$u_R(t) \;=\; R \cdot i_R(t) \qquad \textit{and} \qquad i_C(t) \;=\; C \cdot \frac{d}{dt} u_C(t) \tag{A.1}$$

The branch equations are linked via mesh and node equations (*also known as Kirchhoff's laws*). These have the character of equilibrium equations. In this example, all branch currents are equal as a consequence of continuity:

$$i(t) \;=\; i_R(t) \;=\; i_C(t) \tag{A.2}$$

The connection of the branch currents occurs via the mesh loop of mesh M_1. The mesh equation is then:

$$u_R(t) + u_C(t) - v(t) \;=\; 0 \tag{A.3}$$

A.1.2 Formulation of a Differential Equation

The network equations can now be used to obtain a description equation for the behavior of the model. To do this, start with the mesh equation and replace the voltage across the resistance $u_R(t)$ with the branch equation:

$$R \cdot i_R(t) + u_C(t) - v(t) \;=\; 0 \tag{A.4}$$

Using the equivalence of the currents $i_R(t) = i_C(t)$ (*node equation*), replace the current with the branch equation of the capacitance. A reformulation yields an inhomogeneous linear first-order differential equation with constant coefficients, where the capacitance voltage $u_C(t)$ is the state variable:

$$\frac{d}{dt} u_C(t) \;=\; -\frac{1}{RC} \cdot u_C(t) + \frac{1}{RC} \cdot v(t) \tag{A.5}$$

A.2 Transition to the General Case

To now transition to the analysis of a general first-order system, the general symbols from Chapter 1 are assigned to the components of the system description of the RC network.

$$\frac{d}{dt}\overbrace{u_C(t)}^{x(t)} = \underbrace{-\frac{1}{RC}}_{\lambda} \cdot \overbrace{u_C(t)}^{x(t)} + \underbrace{\frac{1}{RC} \cdot v(t)}_{y(t)} \tag{A.6}$$

A.2.1 Components of the Differential Equation

The components of the differential equation (A.6) are the differentiated variable $x(t)$, which is also called the state variable. The order of a system indicates the number of state variables – i.e., the number of internal variables characteristic of the system. In the system described by equation (A.6), there is exactly one state variable. Each of the state variables is assigned an eigenvalue that is also characteristic of the system – here $\lambda = -\frac{1}{RC}$. For LTI systems, this parameter is constant. The function $y(t) = \frac{1}{RC} \cdot v(t)$ is also called the inhomogeneity. This is a quantity acting on the system from outside (*here the voltage source*).

A.3 First-Order System in the Laplace Domain

As an example, the first-order system already discussed in the time domain is treated here in the Laplace domain. The starting point for this is the differential equation of the system (*... corresponds to a state-space representation with dimension* 1*, the state matrix is then essentially directly the eigenvalue*)

A.3.1 Laplace Transformation of the System

The linear ordinary first-order differential equation is:

$$\frac{d}{dt}x(t) \;=\; \lambda \cdot x(t) + y(t) \tag{A.7}$$

Apply the Laplace transformation to the left and right sides of the differential equation. Use the differentiation theorem (2.5) and linearity:

$$\frac{d}{dt}x(t) \quad \circ\!\!-\!\!\bullet \quad \underline{s} \cdot \underline{X}(\underline{s}) - x_0$$

$$\lambda \cdot x(t) + y(t) \quad \circ\!\!-\!\!\bullet \quad \lambda \cdot \underline{X}(\underline{s}) + \underline{Y}(\underline{s})$$

Thus for the Laplace transform of the differential equation it follows:

$$\underline{s} \cdot \underline{X}(\underline{s}) - x_0 \;=\; \lambda \cdot \underline{X}(\underline{s}) + \underline{Y}(\underline{s}) \tag{A.8}$$

A.3.2 Solution in the Laplace Domain

To solve the equation, one can now simply rearrange for the desired quantity (*purely algebraic manipulations*). This yields the solution $\underline{X}(\underline{s})$ in the Laplace domain:

$$\underline{X}(\underline{s}) \;=\; \frac{1}{\underline{s} - \lambda} \cdot x_0 \;+\; \frac{1}{\underline{s} - \lambda} \cdot \underline{Y}(\underline{s}) \tag{A.9}$$

A.3.3 Solution in the Time Domain

To obtain the solution in the time domain, the individual components of the solution in the Laplace domain can now be transferred to the time domain. Use the correspondence to the exponential function (2.18) and the convolution theorem (2.8):

$$\underline{X}(\underline{s}) \quad \bullet\!\!-\!\!\circ \quad x(t)$$

$$\frac{1}{\underline{s} - \lambda} \cdot x_0 \quad \bullet\!\!-\!\!\circ \quad \theta(t)\, e^{\lambda t} \cdot x_0$$

$$\frac{1}{\underline{s} - \lambda} \cdot \underline{Y}(\underline{s}) \quad \bullet\!\!-\!\!\circ \quad \Big(\theta(\tau)\, e^{\lambda \tau} * y(\tau)\Big)(t)$$

Thus for the solution in the time domain it follows:

$$x(t) \;=\; \underbrace{\theta(t) \cdot e^{\lambda t} \cdot x_0}_{\textit{homogeneous solution}} \;+\; \underbrace{\Big(\theta(\tau) \cdot e^{\lambda \tau} * y(\tau)\Big)(t)}_{\textit{particular solution}} \tag{A.10}$$

A.3.4 Transfer Function of the First-Order System

The first-order system, which was solved in the previous section with the Laplace transformation, can be described in the Laplace domain – neglecting the initial value $x_0 = 0$ – by the Laplace transform:

$$\underline{X}(\underline{s}) \;=\; \frac{1}{\underline{s} - \lambda} \cdot \underline{Y}(\underline{s}) \tag{A.11}$$

The term $\frac{1}{\underline{s}-\lambda}$ is obviously characteristic of the system and yields the system response $\underline{X}(\underline{s})$ in the Laplace domain through multiplication with the excitation $\underline{Y}(\underline{s})$.

The effect of the input quantity on the output quantity can be explicitly specified in the Laplace domain and has the form of a rational function ... the transfer function:

$$\underline{H}(\underline{s}) \;=\; \frac{\underline{X}(\underline{s})}{\underline{Y}(\underline{s})} \;=\; \frac{1}{\underline{s} - \lambda} \tag{A.12}$$

A.3.5 Structure of the Transfer Function

Some properties of the system can now be read directly from the transfer function. Since the numerator degree is 0, the transfer function has no zeros and the polynomials are in any case coprime. The denominator degree is 1 and the transfer function thus has exactly one pole. This can also be determined quite simply here:

$$\underline{Q}(\underline{p}) = 0 \quad \Leftrightarrow \quad \underline{p} = \lambda$$

The pole $\underline{p}$ is thus exactly the eigenvalue λ of the system and thus characterizes the dynamic behavior.

A.4 Transient Analysis of the First-Order System

The first-order system is characterized by the differential equation:

$$\frac{d}{dt}x(t) \;=\; \lambda \cdot x(t) \;+\; y(t) \tag{A.13}$$

Several methods (*more precisely two: solution in the time domain and the Laplace transformation*) have now been presented for calculating the solution $x(t)$ for arbitrary excitations $y(t)$. In each case, a general solution formula in the time domain was formulated:

$$x(t) \;=\; \underbrace{\theta(t) \cdot x_0 \cdot e^{\lambda t}}_{\textit{homogeneous solution}} \;+\; \underbrace{\Big(\theta(\tau) \cdot e^{\lambda \tau} * y(\tau)\Big)(t)}_{\textit{particular solution}} \tag{A.14}$$

This solution formula will be analyzed in more detail in the following and the transient behavior of the system will be worked out.

A.4.1 HOMOGENEOUS SOLUTION OF THE FIRST-ORDER SYSTEM

The homogeneous solution depends exclusively on the initial value and thus represents the effects of the system behavior from $t < 0$ on the behavior from the observation time $t > 0$. The initial value $x_0 = x(0)$ is thus transferred from the system before the observation time to the system after the observation time (*sometimes also called the memory of the system*). The homogeneous solution of the first-order system is:

$$x_h(t) \;=\; \theta(t) \cdot x_0 \cdot e^{\lambda t} \tag{A.15}$$

In Figure A.2, the homogeneous solution of the system is shown for the case $\lambda > 0$ (*unstable*) and for the cases $\lambda = 0$ (*marginally stable*) and $\lambda < 0$ (*asymptotically stable*). The time constant dominating the system is given by the eigenvalue $\tau = \left|\frac{1}{\lambda}\right|$. For $t < 0$, the system behavior was assumed constant in this case $x(t) = x_0$. It is clearly visible that the homogeneous solution diverges in the unstable case. In the asymptotically stable case, however, it converges and the influence of the initial value decays with time.

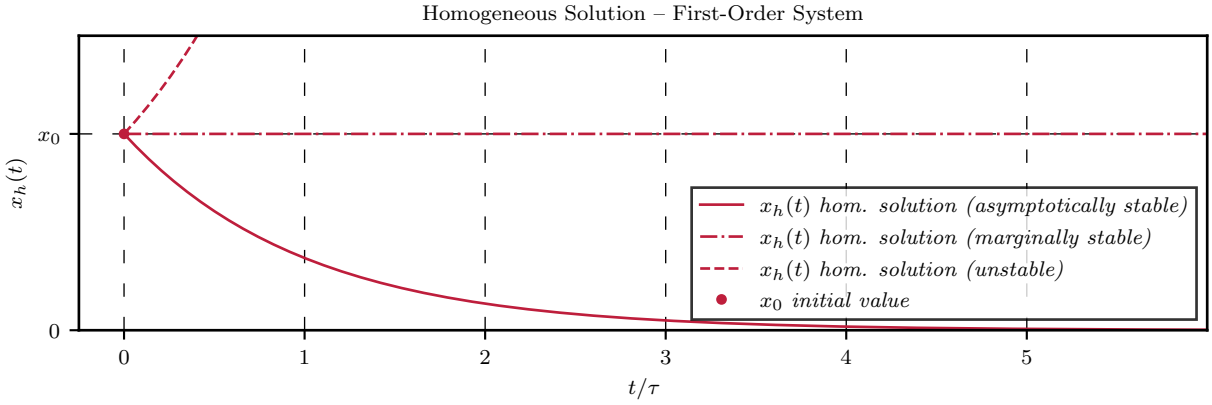


Figure A.2: Homogeneous solution of the dynamic first-order system for the specification of an arbitrary (*here positive*) initial value x_0. Shown are the three possible cases for unstable, marginally stable, and asymptotically stable behavior.

A.4.2 STABILITY OF THE FIRST-ORDER SYSTEM

The first-order system $n = 1$ has exactly one real eigenvalue. This can be read directly from the differential equation (*one-dimensional state-space representation*). There are three possible configurations:

$$\lambda \begin{cases} < 0 & \textit{asymptotically stable} \\ = 0 & \textit{marginally stable} \\ > 0 & \textit{unstable} \end{cases}$$

In the following, mainly the asymptotically stable case is treated, but hints about the behavior in the unstable case are also given.

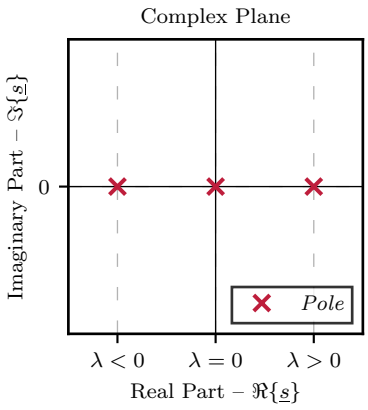


Figure A.3: Representation of the real pole λ in the complex plane for the three possible stability cases.

A.4.3 Impulse Response of the First-Order System

The impulse response of the first-order system results directly from the transfer function already established. The inverse Laplace transformation can be determined immediately with the correspondence of the exponential function and yields the impulse response:

$$\underline{H}(\underline{s}) \;=\; \frac{1}{\underline{s}-\lambda} \qquad \bullet\!\!-\!\!\circ \qquad h(t) \;=\; \theta(t)\cdot e^{\lambda t} \tag{A.16}$$

Shown is the impulse response $h(t)$ of the first-order system for the case $\lambda > 0$ (*unstable*) and for the cases $\lambda = 0$ (*marginally stable*) and $\lambda < 0$ (*asymptotically stable*). τ is the dominant time constant of the system and is given by $\tau = \left|\frac{1}{\lambda}\right|$ (*one also speaks of the dominant pole of the system, i.e., the eigenvalue with the smallest magnitude*). It can be clearly seen that the impulse response diverges for a positive eigenvalue and vanishes for a negative eigenvalue, according to the considerations on stability.

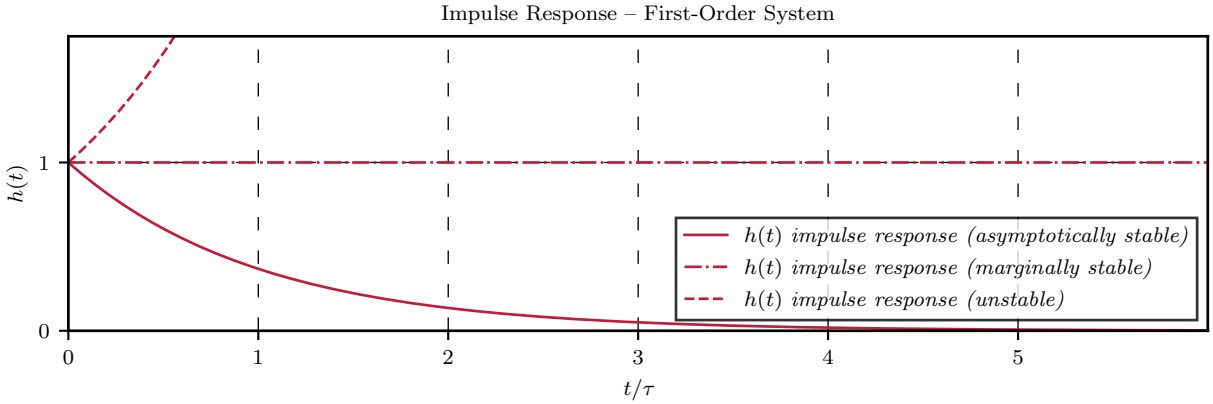


Figure A.4: Impulse response of the dynamic first-order system. Shown are the three possible cases for unstable, marginally stable, and asymptotically stable behavior.

A.4.4 STEP RESPONSE OF THE FIRST-ORDER SYSTEM

From the solution formula for the particular solution, the step response of the first-order system can now be calculated when excited by a unit step $y(t) = \theta(t)$. It thus follows via the convolution with the impulse response:

$$
\begin{aligned}
x_\theta(t) &= \Big(\theta(\tau) \cdot e^{\lambda\tau} * \theta(\tau)\Big)(t) \\
&\quad | \quad \textit{Substitute convolution integral} \\
&= \int_{-\infty}^{\infty} \theta(t-\tau) \cdot e^{\lambda(t-\tau)} \cdot \theta(\tau)\, d\tau \\
&\quad | \quad \textit{Adjust integration limits} \\
&= \theta(t) \cdot \int_{0}^{t} e^{\lambda(t-\tau)}\, d\tau \\
&\quad | \quad \textit{Form antiderivative} \\
&= \theta(t) \cdot \left[-\frac{1}{\lambda} \cdot e^{\lambda(t-\tau)}\right]_{\tau=0}^{\tau=t} \\
&\quad | \quad \textit{Evaluate antiderivative} \\
&= \theta(t) \cdot \frac{1}{\lambda} \cdot \Big(e^{\lambda t} - 1\Big)
\end{aligned}
\tag{A.17}
$$

Shown is the step response for the cases $\lambda > 0$ (*unstable*) and $\lambda < 0$ (*asymptotically stable*). Since the eigenvalue λ has the unit of a frequency, the amplitude (*or the final value*) of the step response was abbreviated by $x_\infty = \tau = \left|\frac{1}{\lambda}\right|$. The step response of the first-order system in the asymptotically stable case is thus an exponential function settling to the value $x_\infty = \tau$. It thus reaches the time-independent steady state with a stationary final value. In the unstable case, the step response diverges (*dashed red line*).

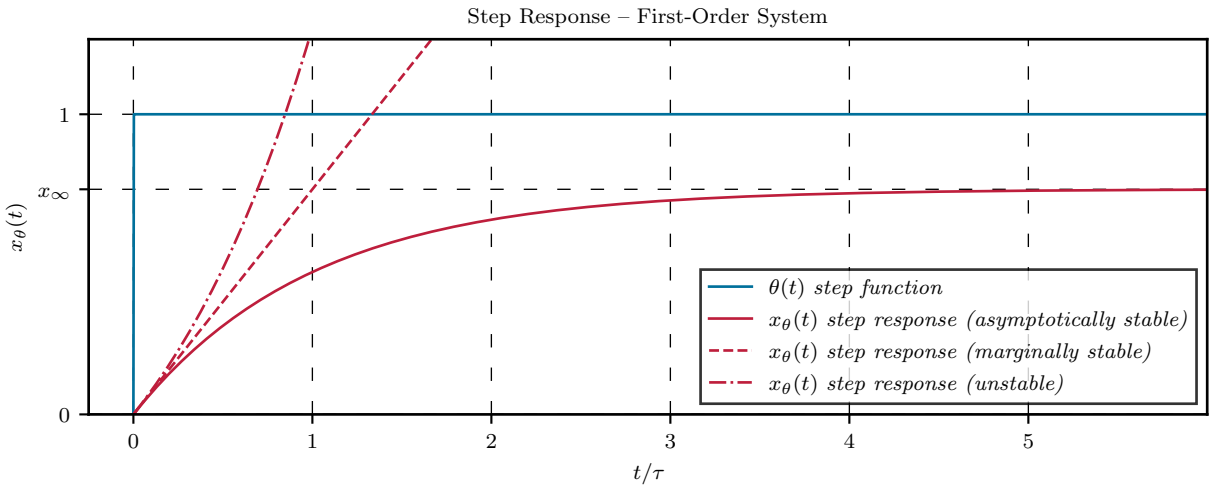


Figure A.5: Step response of the dynamic first-order system, shown here for the two cases of unstable and asymptotically stable behavior.

A.4.5 Transient Behavior of the First-Order System

The complete transient behavior of the first-order system when excited by a step now results from the superposition of the homogeneous solution with the step response:

$$x(t) \;=\; \underbrace{\theta(t)\cdot x_0\cdot e^{\lambda t}}_{\textit{homogeneous solution}} \;+\; \underbrace{\theta(t)\cdot\frac{1}{\lambda}\cdot\left(e^{\lambda t}-1\right)}_{\textit{step response}} \tag{A.18}$$

For an asymptotically stable system with $\lambda < 0$, the solution representation can be reformulated via the already introduced dominant time constant $\tau = \left|\frac{1}{\lambda}\right| = -\frac{1}{\lambda}$ and it follows:

$$x(t) \;=\; \theta(t)\cdot x_0\cdot e^{-\frac{t}{\tau}} \;+\; \theta(t)\cdot\tau\cdot\left(1-e^{-\frac{t}{\tau}}\right) \tag{A.19}$$

It can be clearly seen that the system near the observation start $t = 0$ transitions from behavior dominated by the initial value x_0 to behavior dominated by the excitation. Since the system is asymptotically stable, the response settles to the new final value, which is determined by the final value of the step response $x_\theta(t)$. It can also be seen that the system response assumes a constant value after a certain waiting time $t \gg \tau$ (*here already from about* $t > 5\tau$). The asymptotically stable first-order system thus assumes a stationary final value x_∞ after the settling process – i.e., the transient – and transitions to a time-independent steady state.

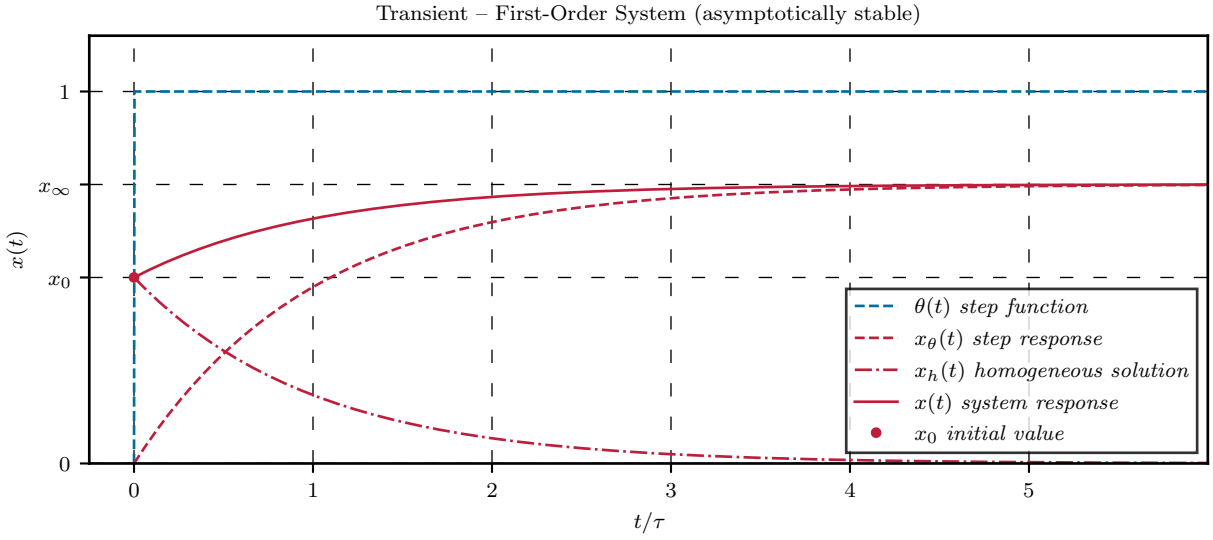


Figure A.6: Complete transient behavior of the dynamic first-order system when excited with a unit step. Also shown are the contributions that are superimposed to form the complete system response.

A.4.6 First-Order System in Time-Independent Steady State

In the case that the first-order system with $\lambda < 0$ is asymptotically stable and the excitation $y(t)$ is bounded, the system behavior transitions to a steady state for $t \to \infty$:

$$x_\infty = \lim_{t\to\infty} x(t)$$

In practice, an infinite waiting time is not realizable, therefore the beginning of the steady state is assumed when the homogeneous solution has decayed sufficiently. This can be assumed for $t \gg \tau_{max}$ – with τ_{max} the largest time constant of the system. Since the homogeneous solution has already decayed to below 1% from $t > 5\tau_{max}$, one can then reasonably assume that the system is in a steady state. Here in this example, when excited by a unit step, the asymptotically stable system assumes a time-independent steady state. The limit value (*final value of the step response*) can be calculated via the final value theorem of the Laplace transformation (2.10) and thus for $t \gg \tau_{max}$ it holds:

$$\begin{aligned}
x_\infty &= \lim_{\underline{s}\to 0} \left(\underline{s} \cdot \underline{X}(\underline{s}) \right) \\
&\mid \textit{Substitute Laplace transform} \\
&= \lim_{\underline{s}\to 0} \left(\underline{s} \cdot \underline{H}(\underline{s}) \cdot \frac{1}{\underline{s}} \right) \\
&\mid \textit{cancel} \\
&= \lim_{\underline{s}\to 0} \underline{H}(\underline{s}) \\
&\mid \textit{Substitute transfer function} \\
&= \lim_{\underline{s}\to 0} \left(\frac{1}{\underline{s} - \lambda} \right) \\
&\mid \textit{Evaluation of the transfer function yields} \\
&= -\frac{1}{\lambda} \\
&\mid \textit{with } \lambda < 0 \textit{ this corresponds to the dominant time constant} \\
&= \tau
\end{aligned}$$

The stationary final value that the asymptotically stable system assumes in the steady state when excited by a unit step is thus $x_\infty = \tau$.

A.5 First-Order System in the Frequency Domain

After motivating the frequency response and the Bode diagram for system analysis, we now examine both in more detail using the example of the first-order system. For this example, the system is assumed to be asymptotically stable with $\lambda < 0$, since only then is a system description via the frequency response valid.

A.5.1 Frequency Response of the First-Order System

Since the transfer function for the first-order system has already been established, the frequency response can be obtained through a simple substitution of the argument:

$$\underline{H}(\underline{s}) \;=\; \frac{1}{\underline{s}-\lambda} \quad \xrightarrow{\textit{Replace } \underline{s}\to j\omega} \quad \underline{H}(j\omega) \;=\; \frac{1}{j\omega-\lambda} \tag{A.20}$$

The magnitude of the frequency response is (*fortunately*) relatively simple in this example:

$$\left|\underline{H}(j\omega)\right| \;=\; \frac{1}{\sqrt{\omega^2+\lambda^2}} \tag{A.21}$$

Since the phase of the numerator is constantly 0, the phase of the frequency response is completely determined by the denominator. Due to $\lambda < 0$ (*asymptotically stable*), this has a positive real part. The phase of the frequency response can thus be uniquely determined via the arctangent (*no case distinction necessary*) and we have:

$$\varphi(\omega) \;=\; -arctan\left(\frac{\omega}{-\lambda}\right) \tag{A.22}$$

A.5.2 Response in Harmonic Steady State

The representation of the system via a frequency response is justified by the fact that the system is in a harmonic steady state. The system input is assumed to be a harmonic signal:

$$y(t) \;=\; Y\cdot cos\big(\omega t+\varphi_y\big) \quad for \quad t \gg \tau \tag{A.23}$$

The system output (*in steady state*) can then be directly specified using the frequency response, since this describes the magnitude and phase relationships between input and output signals:

$$x(t) \;=\; Y\cdot \underbrace{\frac{1}{\sqrt{\omega^2+\lambda^2}}}_{|\underline{H}(j\omega)|}\cdot cos\Big(\omega t+\varphi_y \underbrace{-arctan\Big(\frac{\omega}{-\lambda}\Big)}_{\varphi(\omega)}\Big) \tag{A.24}$$

A.5.3 BODE DIAGRAM OF THE FIRST-ORDER SYSTEM

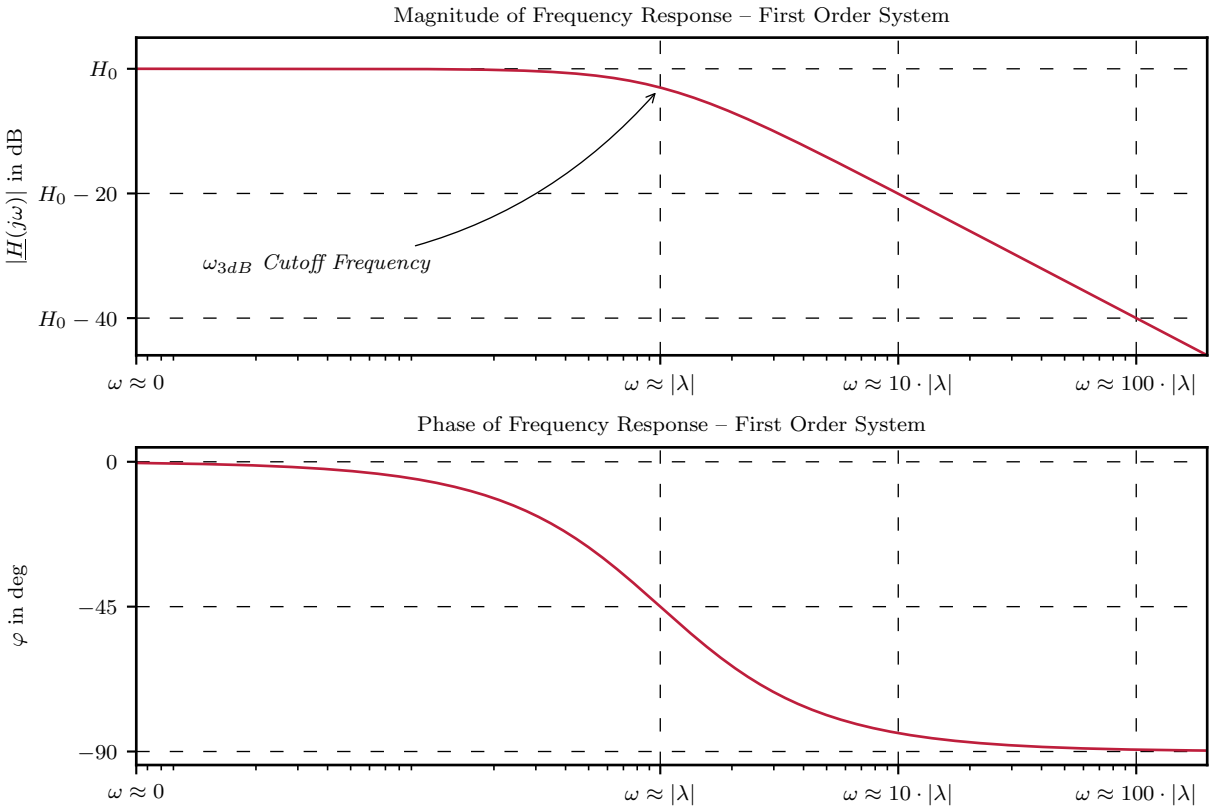


Figure A.7: Bode diagram of the frequency response of the dynamic (*asymptotically stable*) first-order system. Magnitude / amplitude and phase are plotted separately versus the angular frequency ω.

A.5.4 LOCUS CURVE OF THE FIRST-ORDER SYSTEM

Besides the Bode diagram, the frequency response can also be visualized via its locus curve in the complex plane (*Nyquist diagram*). Here, the frequency response is evaluated for $\omega \in [0, \infty)$ and the respective evaluation points are plotted in the complex plane. For the continuous interval, this results in a curve in the complex plane.

The connection to the Bode diagram can also be recognized. Each point can be directly assigned a magnitude and a phase in polar representation.

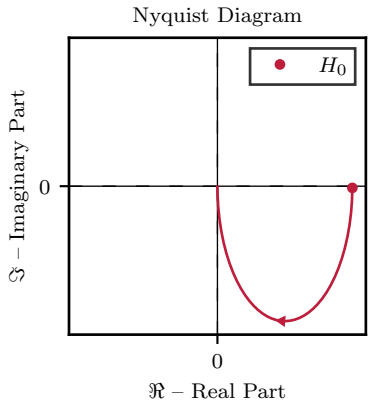


Figure A.8: Representation of the locus curve of the frequency response of the first-order system in the complex plane. The starting point H_0 for $\omega = 0$ is also marked.

Appendix B

The Damped Harmonic Oscillator

A harmonic oscillator is a system capable of oscillation that is characterized by a linear restoring quantity. For a mechanical system, this means that there is a force that counteracts an increasing deflection with proportionally increasing strength. After an external impulse, a harmonic oscillator oscillates sinusoidally (*or harmonically*) around its rest position, where the oscillation period is independent of the magnitude of the deflection. Examples of harmonic oscillators include spring pendulums, electrical oscillating circuits, and tuning forks.

The harmonic oscillator is an important model system in physics. Many more complex systems behave approximately like harmonic oscillators at small deflections (*linearization at the operating point*) *e.g., the simple pendulum.*

The term harmonic oscillator is also used for damped harmonic oscillators, even though these do not strictly perform a harmonic oscillation, but rather a damped oscillation.[1]

The methods described so far will now be demonstrated using the damped harmonic oscillator.

[1] Wikipedia provides a very nice introduction here

B.1 Model Description of the Mechanical System

In classical mechanics, the harmonic oscillator is represented by a one-dimensional spring-mass system with mass m and spring constant D. In idealized form, this system is lossless and is therefore often extended by a damper with damping constant k.

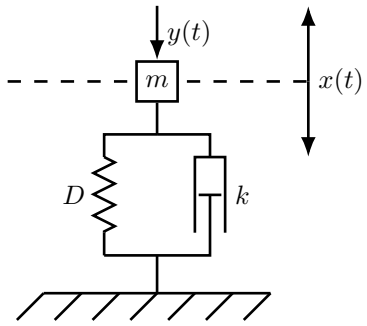


Figure B.1: Idealized representation of the one-dimensional damped harmonic oscillator.

The time-dependent input quantity $y(t)$ acts as a force on the mass m. The output quantity $x(t)$ is here the position or the deflection of the mass from the rest position. This refers to the operating point of the system / the spring, which already takes into account the influence of gravitational force, among other things. This corresponds to omitting the constant term of the Taylor expansion (*common in small-signal analysis*).

B.1.1 Force Equilibrium

For the one-dimensional mechanical system, all time-dependent forces $F_i(t)$ are scalar quantities and for force equilibrium, the following holds at every instant:

$$\sum_i F_i(t) = 0$$

The forces acting here are the restoring force of the spring, which according to Hooke's law behaves approximately linearly with the deflection $x(t)$ for small deflections (*here again the small-signal approximation ... this is now the linear term of the Taylor expansion*):

$$F_f(t) \;=\; -D \cdot x(t) \tag{B.1}$$

And the friction force, which acts through the damper. The friction force always acts opposite to the direction of motion and is – again linearly – dependent on the velocity (*for positive k, a positive force thus acts opposite to the direction of motion and brakes the mass ... what happens with negative k ... here one can already sense what happens to stability*):

$$F_d(t) \;=\; -k \cdot v(t) \;=\; -k \cdot \frac{d}{dt} x(t) \tag{B.2}$$

In addition, the system has an input $y(t)$, which also acts as a force on the mass. The sign here depends on how the direction of motion is oriented.

$$F_y(t) \;=\; y(t) \tag{B.3}$$

This results in a total force that ultimately acts on the mass and leads to an acceleration. This is oriented opposite to the sum of the other forces in the equilibrium equation:

$$F_{ges}(t) \;=\; -m \cdot a(t) \;=\; -m \cdot \frac{d^2}{dt^2} x(t) \tag{B.4}$$

The force equilibrium at the operating point of the spring results from the equilibrium equation as:

$$F_{ges}(t) \;+\; F_d(t) \;+\; F_f(t) \;+\; F_y(t) \;=\; 0$$

Now substitute the expressions for the time-dependent forces to obtain a differential equation:

$$-m \cdot \frac{d^2}{dt^2}x(t) \;-\; k \cdot \frac{d}{dt}x(t) \;-\; D \cdot x(t) \;+\; y(t) \;=\; 0 \tag{B.5}$$

B.1.2 Equation of Motion

A brief rearrangement yields the equation of motion for the damped harmonic oscillator, which is a linear ordinary differential equation of second order:

$$m \cdot \frac{d^2}{dt^2}x(t) \;+\; k \cdot \frac{d}{dt}x(t) \;+\; D \cdot x(t) \;=\; y(t) \tag{B.6}$$

We have now described the mechanical system of the damped harmonic oscillator as a linear system with the input quantity $y(t)$ and the output quantity $x(t)$. This model description now serves as the basis for the analysis of the system.

B.1.3 State-Space Representation

The equation of motion can be converted into a state-space model via the Frobenius companion matrix. The state variables are the deflection $x(t)$ and the velocity as the first derivative of the deflection $v(t) = \frac{d}{dt}x(t)$ combined in the vector $\vec{x}(t)$:

$$\frac{d}{dt}\vec{x}(t) \;=\; \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{D}{m} & -\frac{k}{m} \end{bmatrix}}_{\mathbf{A}} \cdot \underbrace{\begin{bmatrix} v(t) \\ x(t) \end{bmatrix}}_{\vec{x}(t)} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \cdot y(t) \tag{B.7}$$

The eigenvalues of the state matrix $\mathbf{A}$ are the eigenvalues of the system. They are the zeros of the characteristic equation:

$$\underline{\Delta}(\underline{s}) \;=\; det\Big(\underline{s} \cdot \mathbf{I} - \mathbf{A}\Big) \;=\; det\left(\begin{bmatrix} \underline{s} & -1 \\ \frac{D}{m} & \underline{s} + \frac{k}{m} \end{bmatrix}\right) \;=\; \underline{s}^2 + \underline{s} \cdot \frac{k}{m} + \frac{D}{m} \tag{B.8}$$

The form of this characteristic polynomial can now be kept in mind and later compared with the structure of the problem in the Laplace domain.

B.2 Model Description in the Laplace Domain

The – admittedly idealized – real mechanical system has now been mathematically described by a linear ordinary differential equation of second order with constant coefficients. For system analysis and ultimately the solution, the system is transferred to the Laplace domain.

B.2.1 Transformation of the Equation of Motion

The individual terms in (B.6) can be transformed separately due to linearity. Use the differentiation theorem for higher derivatives (2.6) for this:

$$\begin{aligned} m \cdot \frac{d^2}{dt^2} x(t) \quad &\circ\!\!-\!\!\bullet \quad m \cdot \left(\underline{s}^2 \cdot \underline{X}(\underline{s}) - \underline{s} \cdot x_0 - x_0^{(1)} \right) \\ k \cdot \frac{d}{dt} x(t) \quad &\circ\!\!-\!\!\bullet \quad k \cdot \left(\underline{s} \cdot \underline{X}(\underline{s}) - x_0 \right) \\ D \cdot x(t) \quad &\circ\!\!-\!\!\bullet \quad D \cdot \underline{X}(\underline{s}) \\ y(t) \quad &\circ\!\!-\!\!\bullet \quad \underline{Y}(\underline{s}) \end{aligned}$$

This results in the following Laplace transform of the equation of motion after a brief rearrangement of the initial values:

$$\underbrace{m \cdot \underline{s}^2 \cdot \underline{X}(\underline{s}) + k \cdot \underline{s} \cdot \underline{X}(\underline{s}) + D \cdot \underline{X}(\underline{s})}_{System} = \underbrace{\underline{Y}(\underline{s})}_{Input} + \underbrace{(m \cdot \underline{s} + k) \cdot x_0 + m \cdot x_0^{(1)}}_{Initial\ values,\ energy\ in\ the\ system} \tag{B.9}$$

B.2.2 Solution in the Laplace Domain

Equation (B.9) can now be rearranged for the output quantity $\underline{X}(\underline{s})$ (*only algebraic rearrangements*) and yields the solution in the Laplace domain:

$$\underline{X}(\underline{s}) = \underbrace{\underbrace{\frac{(m \cdot \underline{s} + k)}{m \cdot \underline{s}^2 + k \cdot \underline{s} + D} \cdot x_0}_{Component\ of\ initial\ deflection} + \underbrace{\frac{m}{m \cdot \underline{s}^2 + k \cdot \underline{s} + D} \cdot x_0^{(1)}}_{Component\ of\ initial\ velocity}}_{homogeneous\ component} + \underbrace{\frac{1}{m \cdot \underline{s}^2 + k \cdot \underline{s} + D} \cdot \underline{Y}(\underline{s})}_{Transfer\ function\ \ particular\ component} \tag{B.10}$$

In the solution representation (B.10), the homogeneous solution and the particular solution can now be identified. To transfer the solution from the Laplace domain to the time domain, the Laplace transform must first be brought into an appropriate form.

B.2.3 Transfer Function of the Harmonic Oscillator

The transfer function of the damped harmonic oscillator now results directly from the relationship (B.10) between input and output. It is advisable to normalize the denominator polynomial:

$$\underline{H}(\underline{s}) = \frac{\underline{X}(\underline{s})}{\underline{Y}(\underline{s})} = \frac{1}{m} \cdot \frac{1}{\underline{s}^2 + \underline{s} \cdot \frac{k}{m} + \frac{D}{m}} \tag{B.11}$$

B.2.4 Eigenvalues of the Second-Order System

The poles of the transfer function (B.11) are eigenvalues of the system. Since the numerator polynomial is of degree zero, the poles result directly from the zeros of the denominator polynomial (*... corresponds here to the characteristic equation of the state matrix*):

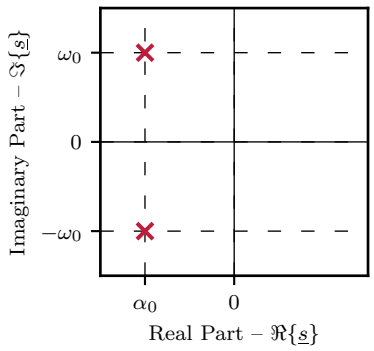
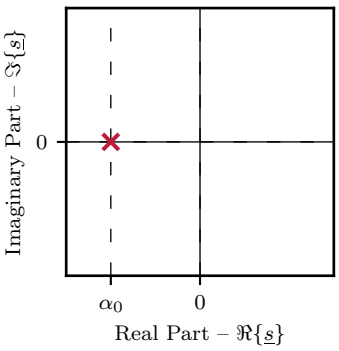
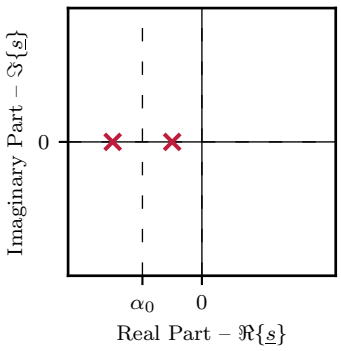
$$\underline{Q}(\underline{s}) \;=\; \underline{s}^2 + \underline{s} \cdot \frac{k}{m} + \frac{D}{m} \tag{B.12}$$

This is a polynomial of second degree, whose zeros can be determined with completing the square – or the well-known quadratic formula:

$$\underline{\lambda}_{1,2} \;=\; -\frac{k}{2m} \;\pm\; \sqrt{\frac{k^2}{4m^2} - \frac{D}{m}} \tag{B.13}$$

For different configurations of the system parameters m, k, D, several characteristic cases arise for the two eigenvalues $\underline{\lambda}_1$ and $\underline{\lambda}_2$:

Table B.1: Comparison of possible pole configurations for asymptotically stable second-order systems

complex conjugate pair	double real eigenvalue	two real eigenvalues
In the case of low damping $k^2 < 4 \cdot D \cdot m$ the root becomes negative and the eigenvalues are complex conjugate pairs: $\underline{\lambda}_{1,2} = \underbrace{-\frac{k}{2m}}_{\alpha_0} \pm j \cdot \underbrace{\sqrt{\frac{D}{m} - \frac{k^2}{4m^2}}}_{\omega_0}$	The term under the root vanishes for $k^2 = 4 \cdot D \cdot m$ and a double eigenvalue with multiplicity $\mu = 2$ results: $\underline{\lambda}_{1,2} = \underbrace{-\frac{k}{2m}}_{\alpha_0}$	For a strongly damped system with $k^2 > 4 \cdot D \cdot m$ two different real eigenvalues result with a positive root: $\underline{\lambda}_{1,2} = \underbrace{-\frac{k}{2m}}_{\alpha_0} \pm \underbrace{\sqrt{\frac{k^2}{4m^2} - \frac{D}{m}}}_{\beta_0}$
In the complex plane, the complex conjugate eigenvalue pair can be represented: 	The double real pole can be represented in the complex plane: 	In the complex plane, the real eigenvalues are distributed on the $\Re$-axis: 

B.2.5 Remark on Stability

The real parts of the eigenvalues are determined by this choice of system parameters:

$$\Re\{\underline{\lambda}_{1,2}\} \;=\; \alpha_0 \;=\; -\frac{k}{2m} \;<\; 0$$

All eigenvalues of the system therefore have a strictly negative real part. Thus, the system is asymptotically stable. The eigenvalues of the system determine the temporal evolution of the system response. In particular, for the homogeneous solution (*see equation* (1.44)), the proportionality holds:

$$x_h(t) \;\sim\; e^{\alpha_0 t} \tag{B.14}$$

The amplitude of the homogeneous solution thus converges asymptotically to the time axis according to this proportionality for $t \to \infty$. The following investigations are now based on the assumption of this characteristic parameter configuration. The plots are represented normalized in time to the dominant time constant (*i.e., the pole with the smallest magnitude*) $\tau = \left|\frac{1}{\alpha_0}\right| = \frac{2m}{k}$.

B.2.6 Remark on Oscillation Capability and Natural Frequency

For this example, the first and most common parameter configuration is considered here. The choice of positive system parameters under the condition $k^2 < 4 \cdot D \cdot m$ leads to a complex conjugate eigenvalue / pole pair:

$$\underline{\lambda}_{1,2} \;=\; \underbrace{-\frac{k}{2m}}_{\alpha_0} \pm j \cdot \underbrace{\sqrt{\frac{D}{m} - \frac{k^2}{4m^2}}}_{\omega_0} \qquad \textit{thus} \qquad \underline{\lambda}_1 \;=\; \underline{\lambda}_2^* \tag{B.15}$$

The eigenvalues of the system determine the temporal evolution of the system response. They appear in the exponents of exponential functions, and in the case of the second-order system, the proportionality holds for the homogeneous solution (*see equation* (1.44)):

$$x_h(t) \;\sim\; e^{\pm j\omega_0 t} \tag{B.16}$$

The imaginary part ω_0 of the eigenvalues thus determines the periodicity of the system response in the unforced case. This frequency is called the natural frequency (*the frequency of the natural oscillation*) of the system.

Such pairwise complex conjugate eigenvalues are particularly found in systems capable of oscillation. These are characterized by being damped just enough to be asymptotically stable, yet still containing an oscillatory term. This parameter configuration often applies in reality precisely to mechanical (*and also electrical*) oscillators.

B.3 TRANSIENT BEHAVIOR

The transient response behavior of the damped harmonic oscillator will now be analyzed in more detail. As an example, the system is again assumed to be asymptotically stable with the eigenvalue configuration already discussed.

B.3.1 HOMOGENEOUS SOLUTION

The homogeneous component represents the energy stored in the system at time $t = 0$. This is divided into the kinetic energy stored in the motion via the initial velocity $x_0^{(1)} = v_0$ and the potential energy stored in the spring through the initial deflection x_0. The homogeneous solution of the damped harmonic oscillator results from the inverse Laplace transformation for the case $\underline{Y}(\underline{s}) = 0$:

$$\underline{X}_h(\underline{s}) = \underbrace{\frac{(m \cdot \underline{s} + k)}{m \cdot \underline{s}^2 + k \cdot \underline{s} + D} \cdot x_0}_{\textit{Anteil der Anfangsauslenkung}} + \underbrace{\frac{m}{m \cdot \underline{s}^2 + k \cdot \underline{s} + D} \cdot x_0^{(1)}}_{\textit{Anteil der Anfangsgeschwindigkeit}} \tag{B.17}$$

A reformulation in pole-zero form with the calculated eigenvalues $\underline{\lambda}_1$ and $\underline{\lambda}_2$ yields:

$$\underline{X}_h(\underline{s}) = \frac{\left(\underline{s} + \frac{k}{m}\right)}{\left(\underline{s} - \underline{\lambda}_1\right)\left(\underline{s} - \underline{\lambda}_2\right)} \cdot x_0 + \frac{1}{\left(\underline{s} - \underline{\lambda}_1\right)\left(\underline{s} - \underline{\lambda}_2\right)} \cdot x_0^{(1)} \tag{B.18}$$

A partial fraction decomposition (*for example with the cover-up method, or by coefficient comparison ... only for $\underline{\lambda}_1 \neq \underline{\lambda}_2$!!!*) of both components yields:

$$\begin{aligned} \underline{X}_h(\underline{s}) = & \left(\frac{\left(\underline{\lambda}_1 + \frac{k}{m}\right)}{\left(\underline{\lambda}_1 - \underline{\lambda}_2\right)} \cdot \frac{1}{\underline{s} - \underline{\lambda}_1} + \frac{\left(\underline{\lambda}_2 + \frac{k}{m}\right)}{\left(\underline{\lambda}_2 - \underline{\lambda}_1\right)} \cdot \frac{1}{\underline{s} - \underline{\lambda}_2} \right) \cdot x_0 \\ & + \left(\frac{1}{\left(\underline{\lambda}_1 - \underline{\lambda}_2\right)} \cdot \frac{1}{\underline{s} - \underline{\lambda}_1} + \frac{1}{\left(\underline{\lambda}_2 - \underline{\lambda}_1\right)} \cdot \frac{1}{\underline{s} - \underline{\lambda}_2} \right) \cdot x_0^{(1)} \end{aligned} \tag{B.19}$$

To keep the further steps somewhat clearer, another rearrangement step is now performed after the partial fractions:

$$\underline{X}_h(\underline{s}) = \left(\frac{\left(\underline{\lambda}_1 + \frac{k}{m}\right) \cdot x_0 + x_0^{(1)}}{\left(\underline{\lambda}_1 - \underline{\lambda}_2\right)} \right) \cdot \frac{1}{\underline{s} - \underline{\lambda}_1} + \left(\frac{\left(\underline{\lambda}_2 + \frac{k}{m}\right) \cdot x_0 + x_0^{(1)}}{\left(\underline{\lambda}_2 - \underline{\lambda}_1\right)} \right) \cdot \frac{1}{\underline{s} - \underline{\lambda}_2} \tag{B.20}$$

The partial fractions can now be assigned functions in the time domain via the correspondence to the complex exponential function (2.18). The homogeneous solution of the damped harmonic oscillator is therefore:

$$\begin{aligned} x_h(t) &= \left(\frac{\left(\underline{\lambda}_1 + \frac{k}{m}\right) \cdot x_0 + x_0^{(1)}}{\left(\underline{\lambda}_1 - \underline{\lambda}_2\right)} \right) \cdot \theta(t) \cdot e^{\underline{\lambda}_1 \cdot t} + \left(\frac{\left(\underline{\lambda}_2 + \frac{k}{m}\right) \cdot x_0 + x_0^{(1)}}{\left(\underline{\lambda}_2 - \underline{\lambda}_1\right)} \right) \cdot \theta(t) \cdot e^{\underline{\lambda}_2 \cdot t} \\ &\quad | \quad \textit{weiter zusammenfassen mit } \underline{\lambda}_1 = \underline{\lambda}_2^* = \alpha_0 + j\omega_0 \\ &= \theta(t) \cdot e^{\alpha_0 t} \left(\frac{\left(\underline{\lambda}_1 + \frac{k}{m}\right) \cdot x_0 + x_0^{(1)}}{\left(\underline{\lambda}_1 - \underline{\lambda}_2\right)} \cdot e^{j\omega_0 \cdot t} + \frac{\left(\underline{\lambda}_2 + \frac{k}{m}\right) \cdot x_0 + x_0^{(1)}}{\left(\underline{\lambda}_2 - \underline{\lambda}_1\right)} \cdot e^{-j\omega_0 \cdot t} \right) \end{aligned} \tag{B.21}$$

The homogeneous solution with respect to the initial deflection x_0 corresponds to a deflection of the mass upward for $t < 0$ and release from $t > 0$. The homogeneous solution with respect to the negative initial velocity $x_0^{(1)}$ corresponds to a downward motion of the mass.

For the asymptotically stable case, the unforced system (*i.e., only the homogeneous solution without externally acting forces*) assumes a stationary final value – namely the rest position $x(t) = 0$.

The plot of the two state variables position / deflection $x(t)$ and velocity $v(t)$ against each other over time for the harmonic oscillator is also called a phase diagram. Intuitively, the state vector describes paths (*so-called trajectories*) in the state space over time. In the asymptotically stable case, these are shrinking spirals (*decrease in amplitude, decay of initial conditions over time*); in the marginally stable case, they are closed paths since no energy is released by the system. In the unstable case, the spirals become larger, since the amplitudes then also grow or the system oscillates up.

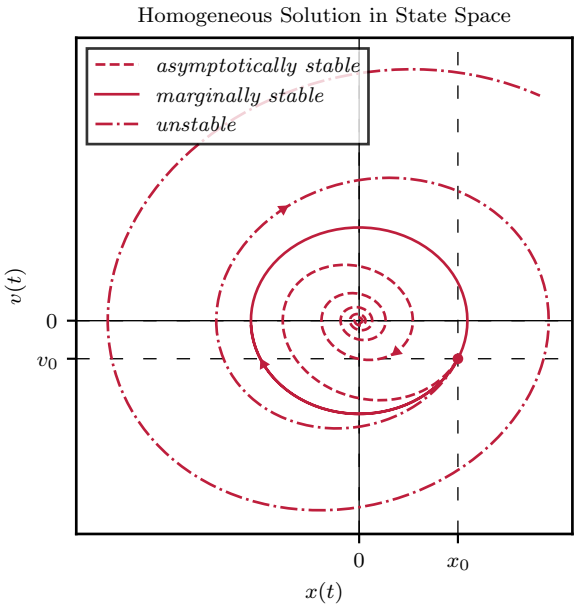


Figure B.2: Plot of the trajectory of the homogeneous solution of $\vec{x}$ of the state-space model (B.7) as a phase diagram. Showing $x(t)$ (*deflection*) and $v(t)$ (*velocity*) for the three possible stability cases.

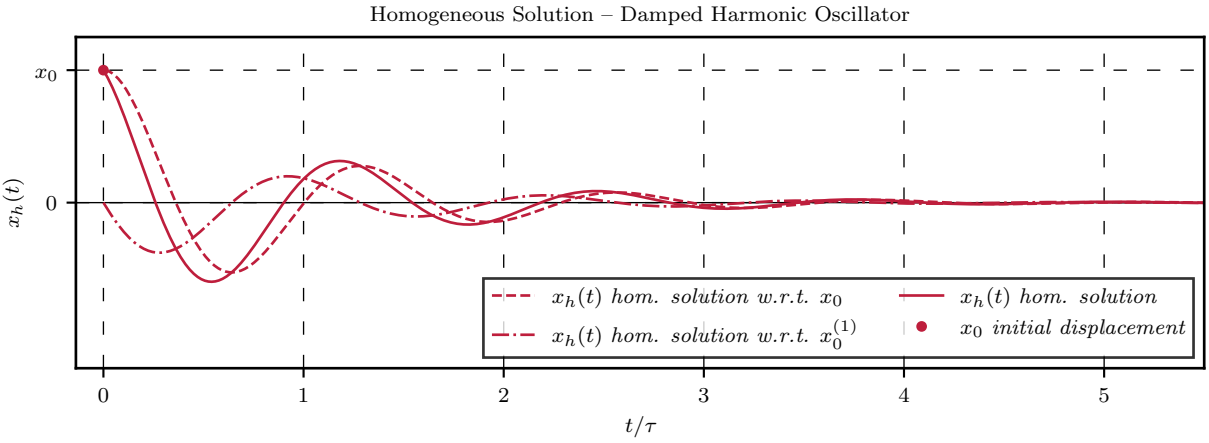


Figure B.3: Homogeneous solution (*asymptotically stable case*) of the damped harmonic oscillator for the specification of an arbitrary (*here positive*) initial deflection x_0 and initial velocity v_0. The individual contributions of the initial conditions are additionally plotted.

B.3.2 Homogeneous Solution and Energy

The initial deflection x_0 represents the potential energy $E_{pot} = \frac{1}{2} \cdot D \cdot x_0^2$, which is stored in the spring at the beginning of observation $t = 0$. The initial velocity $v_0 = x_0^{(1)}$ represents the kinetic energy $E_{kin} = \frac{1}{2} \cdot m \cdot v_0^2$, which is stored in the motion of the mass at the beginning of observation $t = 0$.

$$E_0 = \frac{1}{2} \cdot D \cdot x_0^2 + \frac{1}{2} \cdot m \cdot v_0^2 \tag{B.22}$$

The energy E_0 stored in the system at time $t = 0$ is consumed during the transient process due to asymptotic stability, or physically converted into heat.

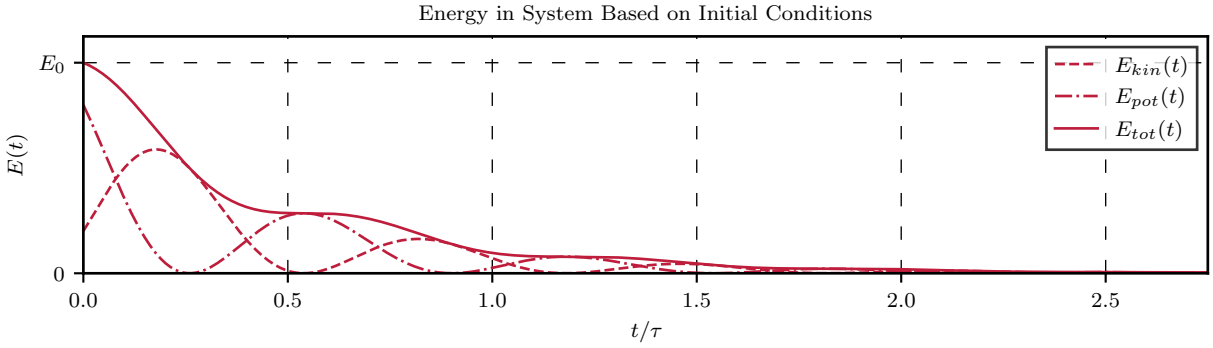


Figure B.4: Energy already stored in the system by the initial conditions. E_0 is supplied as the sum of potential and kinetic energy

B.3.3 Impulse Response

For the inverse transformation of the transfer function into the time domain via the correspondences, the transfer function must be converted into a pole-zero form (*for which the poles / eigenvalues are needed*). This enables the assignment of correspondences in the time domain.

$$\underline{H}(\underline{s}) \;=\; \frac{1}{m} \cdot \frac{1}{\left(\underline{s} - \underline{\lambda}_1\right)\left(\underline{s} - \underline{\lambda}_2\right)} \tag{B.23}$$

A partial fraction decomposition (*for example with the cover-up method, or by coefficient comparison*) yields:

$$\underline{H}(\underline{s}) \;=\; \frac{1}{m \cdot \left(\underline{\lambda}_1 - \underline{\lambda}_2\right)} \cdot \frac{1}{\underline{s} - \underline{\lambda}_1} \;+\; \frac{1}{m \cdot \left(\underline{\lambda}_1 - \underline{\lambda}_2\right)} \cdot \frac{1}{\underline{s} - \underline{\lambda}_2} \tag{B.24}$$

The partial fractions can now be assigned functions in the time domain via the correspondence to the complex exponential function (2.18). This yields the impulse response as the inverse Laplace transformation of the transfer function:

$$h(t) \;=\; \frac{1}{m \cdot \left(\underline{\lambda}_1 - \underline{\lambda}_2\right)} \cdot \theta(t) \cdot e^{\underline{\lambda}_1 \cdot t} \;+\; \frac{1}{m \cdot \left(\underline{\lambda}_2 - \underline{\lambda}_1\right)} \cdot \theta(t) \cdot e^{\underline{\lambda}_2 \cdot t} \tag{B.25}$$

The coefficients and exponents of the exponential functions in the impulse response (B.25) can be real or complex depending on the nature of the eigenvalues $\underline{\lambda}_1$ and $\underline{\lambda}_2$, thus suggesting different behavior of the system. Shown in Figure B.5 is the special case for the eigenvalue configuration $\underline{\lambda}_1 = \underline{\lambda}_2^*$ and with real part $\Re\{\underline{\lambda}_{1,2}\} < 0$.

The system is therefore asymptotically stable and has two complex-conjugate eigenvalues. It is particularly clear that the impulse response of the asymptotically stable system converges. The oscillation arises from the imaginary parts of the eigenvalues in the exponents of the exponential functions of the impulse response (B.25).

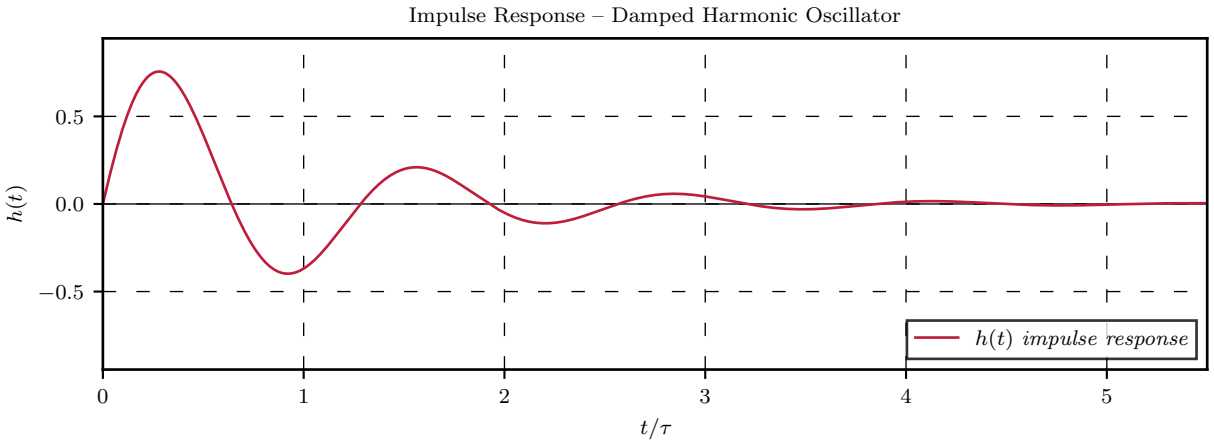


Figure B.5: Impulse response (*of the deflection to an external force*) of the damped harmonic oscillator for the asymptotically stable case.

B.3.4 General System Response and Duhamel Integral

The solution of the system for arbitrary excitations results from the homogeneous solution and the convolution of the impulse response with the excitation as the particular solution:

$$x(t) \;=\; x_h(t) \;+\; \underbrace{\Big(h(\tau) * y(\tau)\Big)(t)}_{x_p(t)} \qquad \textit{for} \qquad t > 0 \tag{B.26}$$

The homogeneous solution $x_h(t)$ and the impulse response $h(t)$ are already known. If an excitation function is now given, the convolution integral must still be solved to determine the system response. The solution formula for the particular solution for arbitrary input signals $y(t)$ and for $t > 0$ is then:

$$\begin{aligned}
x_p(t) \;&=\; \Big(h(\tau) * y(\tau)\Big)(t) \\
&\big|\; \textit{Faltungsintegral ausschreiben} \\
&=\; \int_{-\infty}^{\infty} h(t-\tau)\cdot y(\tau)\,d\tau \\
&\big|\; \textit{Impulse response (B.25) substitute} \\
&=\; \int_{-\infty}^{\infty} \left(\frac{1}{m\cdot\big(\underline{\lambda}_1-\underline{\lambda}_2\big)}\cdot\theta(t-\tau)\cdot e^{\underline{\lambda}_1\cdot(t-\tau)} \;+\; \frac{1}{m\cdot\big(\underline{\lambda}_2-\underline{\lambda}_1\big)}\cdot\theta(t-\tau)\cdot e^{\underline{\lambda}_2\cdot(t-\tau)} \right)\cdot y(\tau)\,d\tau \\
&\big|\; \textit{For the eigenvalues: } \underline{\lambda}_1=\underline{\lambda}_2^{*} \textit{ and therefore } \underline{\lambda}_1-\underline{\lambda}_2 = j2\Im\{\underline{\lambda}_1\} = j2\omega_0 \\
&=\; \int_{-\infty}^{\infty} \frac{1}{m\cdot j2\omega_0}\cdot\theta(t-\tau)\cdot\Big(e^{\underline{\lambda}_1\cdot(t-\tau)} \;-\; e^{\underline{\lambda}_1^{*}\cdot(t-\tau)}\Big)\cdot y(\tau)\,d\tau \\
&\big|\; \textit{Exponentialfunktionen zusammenfassen mit } \underline{\lambda}_1=\alpha_0+j\omega_0 \\
&=\; \int_{-\infty}^{\infty} \frac{1}{m\cdot j2\omega_0}\cdot\theta(t-\tau)\cdot e^{\alpha_0\cdot(t-\tau)}\Big(e^{j\omega_0\cdot(t-\tau)} \;-\; e^{-j\omega_0\cdot(t-\tau)}\Big)\cdot y(\tau)\,d\tau \\
&\big|\; \textit{adjust integration limits (causality)} \\
&=\; \int_{0}^{t} \frac{1}{m\cdot j2\omega_0}\cdot e^{\alpha_0\cdot(t-\tau)}\Big(e^{j\omega_0\cdot(t-\tau)} \;-\; e^{-j\omega_0\cdot(t-\tau)}\Big)\cdot y(\tau)\,d\tau \\
&\big|\; \textit{Substitute relationship to sine: } sin(x)=\tfrac{e^{jx}-e^{-jx}}{2j} \\
&=\; \int_{0}^{t} \frac{1}{m\cdot\omega_0}\cdot e^{\alpha_0\cdot(t-\tau)}\cdot sin\big(\omega_0\cdot(t-\tau)\big)\cdot y(\tau)\,d\tau
\end{aligned} \tag{B.27}$$

The general solution formula (B.27) for the damped, oscillatory harmonic oscillator is sometimes also referred to as the Duhamel integral or solution formula according to Duhamel.

Alternatively, the path via the Laplace transformation for the particular solution can also be taken. For this, the excitation must be Laplace-transformed, multiplied by the transfer function, then decomposed into partial fractions and transformed back into the time domain (*we will follow this path here, since the specific calculation of the impulse response via the convolution integral in the time domain was already demonstrated for the first-order example*).

B.3.5 Step Response

The step response of the damped harmonic oscillator can be calculated in the Laplace domain. With the correspondence of the unit step (2.14) and the transfer function (B.11), it follows:

$$x_\theta(t) \quad \circ\!\!-\!\!\bullet \quad \underline{X}_\theta(\underline{s}) \;=\; \underline{H}(\underline{s}) \cdot \frac{1}{\underline{s}} \tag{B.28}$$

In order to be able to transfer the Laplace transform of the step response into the time domain, a partial fraction decomposition must first be performed (*here again $\underline{\lambda}_1 \neq \underline{\lambda}_2$ is assumed, which allows the use of the cover-up method or the residue theorem*):

$$\begin{aligned}
\underline{X}_\theta(\underline{s}) &= \underline{H}(\underline{s}) \cdot \frac{1}{\underline{s}} \\
&\mid \textit{Substitute transfer function} \\
&= \frac{1}{m} \cdot \frac{1}{(\underline{s}-\underline{\lambda}_1)(\underline{s}-\underline{\lambda}_2)\,\underline{s}} \\
&\mid \textit{Determine PFD and residues via cover-up method} \\
&= \frac{1}{m\cdot\underline{\lambda}_1\cdot\underline{\lambda}_2}\cdot\frac{1}{\underline{s}} + \frac{1}{m\cdot(\underline{\lambda}_1-\underline{\lambda}_2)\cdot\underline{\lambda}_1}\cdot\frac{1}{\underline{s}-\underline{\lambda}_1} + \frac{1}{m\cdot(\underline{\lambda}_2-\underline{\lambda}_1)\cdot\underline{\lambda}_2}\cdot\frac{1}{\underline{s}-\underline{\lambda}_2}
\end{aligned}$$

With the correspondences of the step function (2.14) and the complex exponential function (2.18), the step response can be formulated in the time domain:

$$x_\theta(t) \;=\; \underbrace{\frac{1}{m\cdot\underline{\lambda}_1\cdot\underline{\lambda}_2}}_{x_\infty}\cdot\theta(t) + \frac{1}{m\cdot(\underline{\lambda}_1-\underline{\lambda}_2)\cdot\underline{\lambda}_1}\cdot\theta(t)\cdot e^{\underline{\lambda}_1\cdot t} + \frac{1}{m\cdot(\underline{\lambda}_2-\underline{\lambda}_1)\cdot\underline{\lambda}_2}\cdot\theta(t)\cdot e^{\underline{\lambda}_2\cdot t} \tag{B.29}$$

In Figure B.6, it can be clearly seen that the step response assumes a stationary final value x_∞ after a settling time of approximately 5τ. This behavior can also be directly read from the limit value of the step response (B.29). For $t \to \infty$, the exponential terms vanish and only the constant term remains. The system transitions to a time-independent steady state.

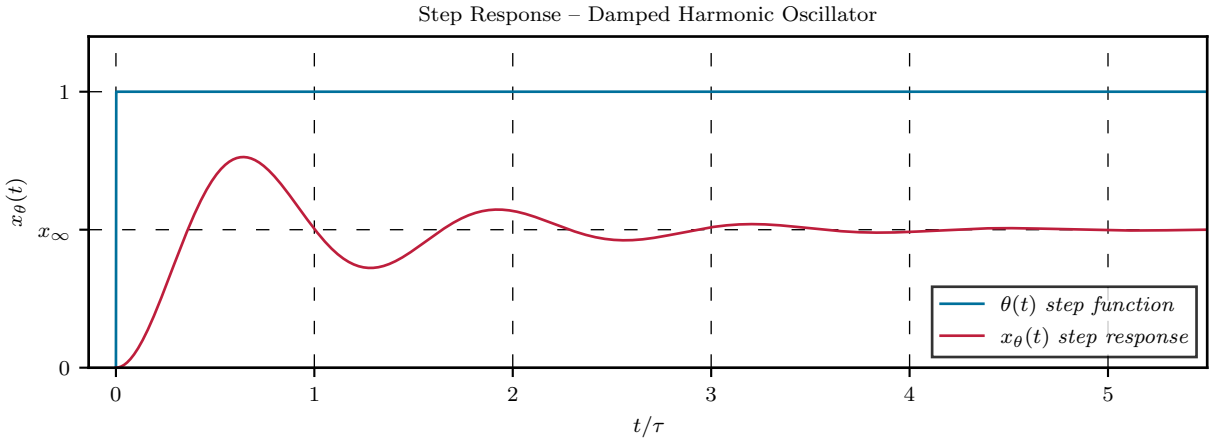


Figure B.6: Step response (*response of the deflection to a constant external force*) of the damped harmonic oscillator with two complex conjugate eigenvalues / poles for the asymptotically stable case.

B.3.6 Step Response and Energy

Through the exciting unit step, energy is fed into the system from time $t = 0$ by deflecting the mass against the spring with a constant force. The temporal evolution of this energy is plotted in Figure B.7.

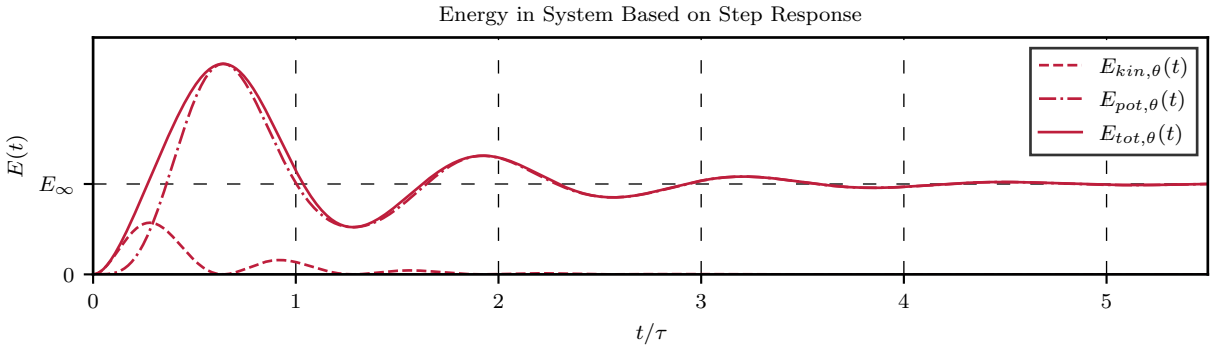


Figure B.7: Energy impressed into the system by excitation with a unit step. In the limit, a base energy $E_\infty = \frac{1}{2} \cdot D \cdot x_\infty^2$ is recognizable as potential energy due to a constant deflection against the spring.

B.3.7 Complete Transient Behavior

The complete transient behavior of the damped harmonic oscillator when excited by a unit step now results from the superposition of the homogeneous solution and the step response and corresponds to the red solid line in the following figure:

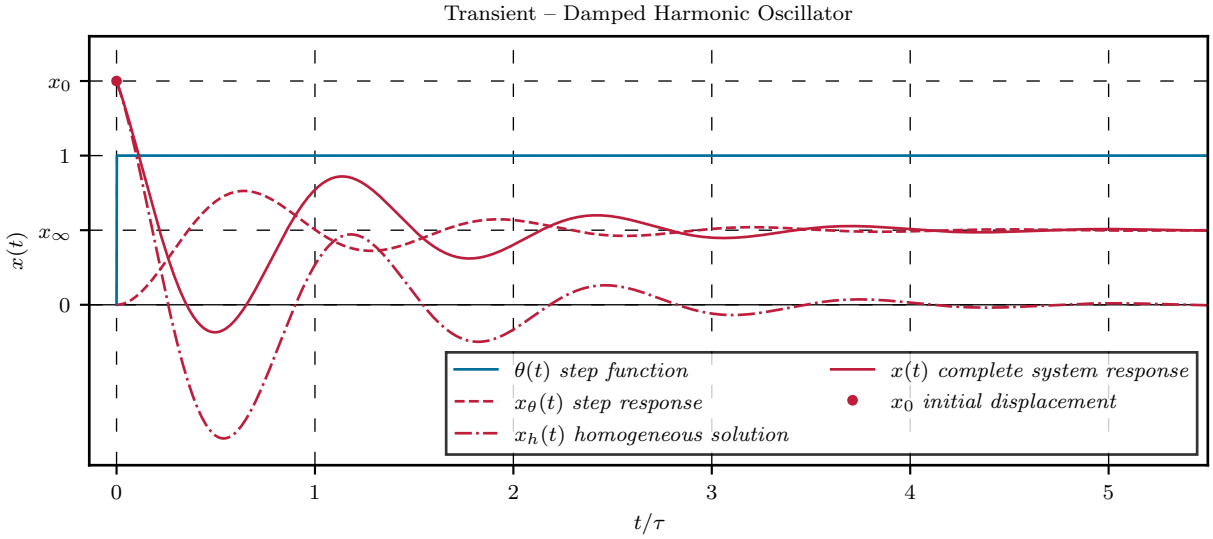


Figure B.8: Complete transient behavior (*of the deflection*) of the damped harmonic oscillator when excited with a unit step (*sudden constant normalized force action*). Also plotted are the individual contributions to the system behavior that are superimposed to form the complete system response.

B.3.8 TOTAL ENERGY IN THE SYSTEM

The total energy in the mechanical system is composed of potential and kinetic energy. The total energy at any time is also composed of the energy already stored in the system (*initial conditions*) and the energy impressed into the system (*excitation by unit step*). Just as for the deflection, there is also an initial value or initial energy E_0 and a final value E_∞ for the energy contained in the system. This situation is illustrated in Figure B.10.

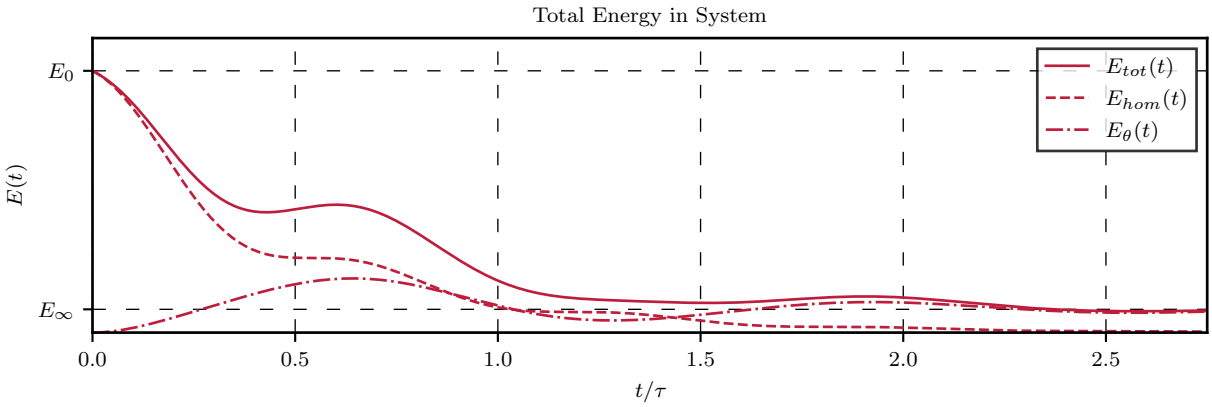


Figure B.9: Total energy in the system divided into the energy from the initial conditions (*homogeneous solution*) and the impressed energy (*step response*)

B.3.9 TRAJECTORIES IN STATE SPACE

The temporal evolution of the state variables $x(t)$ and $v(t) = x^{(1)}(t)$ completely determines the dynamics in the system at every time instant. The state vector describes paths in the state space over time (*trajectories*) that represent the system state at every instant. The superposition (*addition*) of the state vectors of the individual solutions yields the total solution of the system.

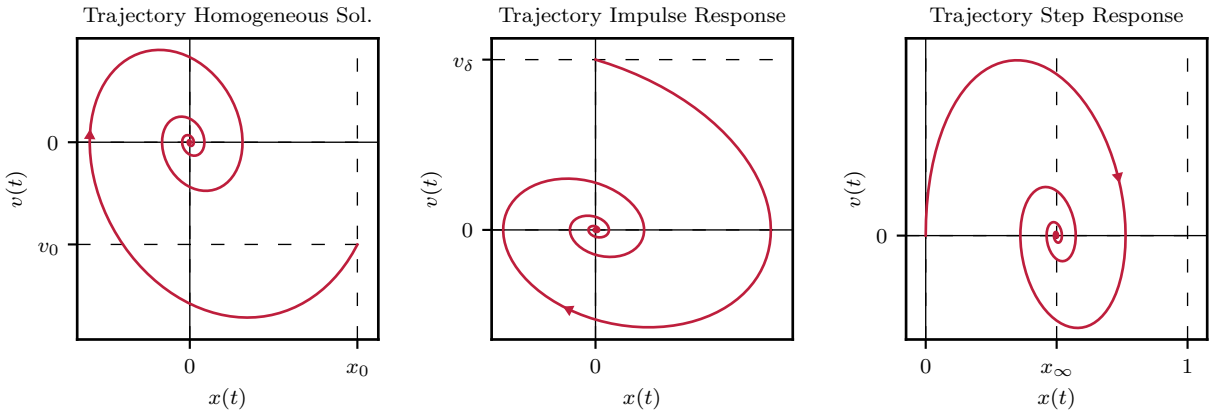


Figure B.10: Trajectories of the homogeneous solution (*left*), the impulse response (*center*), and the step response (*right*) in the state space, each plotted for the asymptotically stable case. Solutions as linear combinations of the state variables $x(t)$ (*position*) and $v(t)$ (*velocity*).

B.4 Behavior in the Frequency Domain

After the transient behavior of the damped harmonic oscillator was investigated using a stable parameter configuration, further analysis now follows in the frequency domain. With $\Re\{\underline{\lambda}_{1,2}\} = \alpha_0 < 0$, the system is asymptotically stable and transitions for $t \gg \tau = \frac{1}{|\alpha_0|}$ to a harmonic steady state under harmonic excitation. In this case, for an excitation of the form ...

$$y(t) \;=\; \Re\Big\{\underline{Y} \cdot e^{j\omega t}\Big\} \qquad \textit{with} \qquad \omega \geq 0 \tag{B.30}$$

... the system behavior can be completely described by a frequency response. The evaluation of the amplitude of the output over multiple frequencies ω (*often also called sine sweep or AC analysis*) yields the frequency response of the system for $\underline{Y} = 1$ and therefore has high relevance in practice.

B.4.1 Frequency Response

Since the transfer function (B.11) has already been established, the frequency response can be obtained by substituting the argument. This yields the transition:

$$\underline{H}(\underline{s}) \quad \xrightarrow{\textit{replace } \underline{s} \to j\omega} \quad \underline{H}(j\omega) \;=\; \frac{1}{m} \cdot \frac{1}{(j\omega)^2 + j\omega \cdot \frac{k}{m} + \frac{D}{m}} \tag{B.31}$$

Alternatively, the frequency response can of course be specified like the transfer function via the poles of the transfer function. However, one should keep in mind that these are then not necessarily poles of the frequency response, since this only allows purely imaginary arguments (*i.e., $j\omega$ as function argument*). Nevertheless, it holds:

$$\underline{H}(j\omega) \;=\; \frac{1}{m} \cdot \frac{1}{(j\omega)^2 + j\omega \cdot \frac{k}{m} + \frac{D}{m}} \;=\; \frac{1}{m} \cdot \frac{1}{\big(j\omega - \underline{\lambda}_1\big)\big(j\omega - \underline{\lambda}_2\big)} \tag{B.32}$$

B.4.2 Magnitude of the Frequency Response

The magnitude of the frequency response corresponds to the frequency-dependent proportionality in the amplitude of the output signal. The calculation is performed here directly via the polynomial form:

$$|\underline{H}(j\omega)| \;=\; \frac{1}{m} \cdot \frac{1}{\sqrt{\Big(\omega \cdot \frac{k}{m}\Big)^2 + \Big(\frac{D}{m} - \omega^2\Big)^2}} \tag{B.33}$$

The magnitude can alternatively also be specified via the poles. Let $\underline{\lambda}_1 = \alpha_0 + j\omega_0$ be written out:

$$|\underline{H}(j\omega)| \;=\; \frac{1}{m} \cdot \frac{1}{\sqrt{\omega^4 + 2\omega^2\left(\alpha_0^2 - \omega_0^2\right) + \omega_0^4 + 2\alpha_0^2\omega_0^2 + \alpha_0^4}} \tag{B.34}$$

B.4.3 Phase of the Frequency Response

The phase of the frequency response corresponds to the temporal delay of real physical systems between input and output (*also causality, action → reaction, also justified by the finite propagation speed of information in the universe*). Let $\omega^2 < \frac{D}{m}$ for this case to keep the case distinction simple, then:

$$\begin{aligned}
\varphi(\omega) &= \measuredangle\Big\{\underline{H}(j\omega)\Big\} \\
&\mid \textit{Substitute frequency response, only denominator relevant} \\
&= -\measuredangle\left\{(j\omega)^2 + j\omega\cdot\frac{k}{m} + \frac{D}{m}\right\} \\
&\mid \textit{Phase angle via arctangent} \\
&= -arctan\left(\frac{\omega\cdot\frac{k}{m}}{\frac{D}{m}-\omega^2}\right)
\end{aligned}$$

B.4.4 Response in Harmonic Steady State

The asymptotically stable system transitions after a settling time $t \gg \tau$ to a harmonic steady state under harmonic excitation. This enables the simplified representation of the system behavior:

$$x(t) \;=\; \Re\Big\{\underline{H}(j\omega)\cdot\underline{Y}\cdot e^{j\omega t}\Big\} \qquad for \qquad t \gg \tau \tag{B.35}$$

The representation can be further transformed and can be written as a real cosine function with the frequency-dependent magnitude $|\underline{H}(j\omega)|$ and the frequency-dependent phase $\varphi(\omega)$ of the frequency response (*already calculated*):

$$x(t) \;=\; \underbrace{|\underline{H}(j\omega)|\cdot|\underline{Y}|}_{Total\ amplitude}\cdot cos\big(\omega t + \underbrace{\varphi_y + \varphi(\omega)}_{Total\ phase}\big) \qquad for \qquad t \gg \tau \tag{B.36}$$

B.4.5 Response in Time-Independent Steady State

The asymptotic behavior for low frequencies near the origin $\omega \approx 0$ is often also technically relevant. For $\omega = 0$, the cosine in the excitation becomes the constant $cos(0) = 1$ and the system is then in a time-independent steady state with a stationary final value x_∞. The system's contribution to this final value can again be obtained from the frequency response: This DC component therefore results as:

$$H_0 \;:=\; |\underline{H}(0)| \;=\; \frac{1}{m}\cdot\frac{1}{\sqrt{\left(0\cdot\frac{k}{m}\right)^2 + \left(\frac{D}{m}-0^2\right)^2}} \;=\; \frac{1}{D}$$

Accordingly, the output in the time-independent steady state is determined by a likewise time-independent function (*i.e., a constant*):

$$x(t) \;=\; H_0\cdot|\underline{Y}|\cdot cos(\varphi_y) \;=\; x_\infty \tag{B.37}$$

B.4.6 Bode Diagram

The double-logarithmic plot of magnitude and phase occurs in Figure B.11 as a Bode diagram. The amplification of the output amplitude or the magnitude increase around the frequency $\omega \approx |\underline{\lambda}_{1,2}|$ is clearly recognizable. This maximum in the magnitude curve is often referred to as resonance and is typical for second-order systems that are damped enough to be asymptotically stable, yet still capable of oscillation.

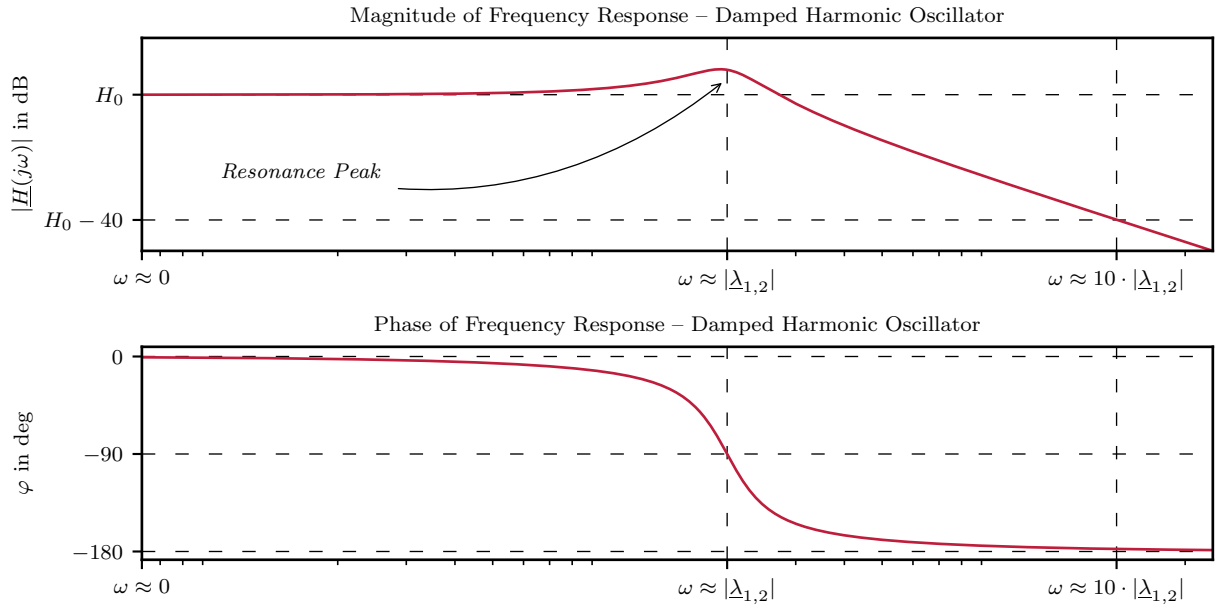


Figure B.11: Bode diagram of the frequency response of the harmonic oscillator. Magnitude and phase are plotted separately versus the angular frequency ω.

B.4.7 Nyquist Diagram

Besides the Bode diagram, the frequency response can also be visualized by its locus curve in the complex plane (*Nyquist diagram*).

Here, the frequency response is evaluated for $\omega \in [0, \infty)$ and the respective evaluation points are plotted in the complex plane. For the continuous interval, this results in a curve in the complex plane.

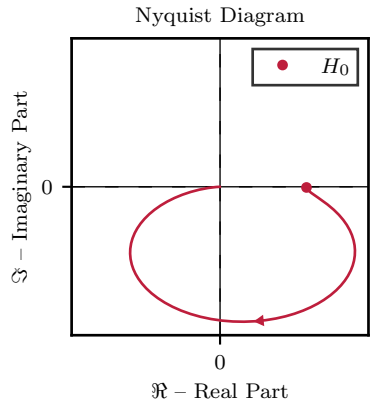


Figure B.12: Locus curve of the frequency response in the complex plane with starting point $H_0 = \underline{H}(0)$.

The connection to the Bode diagram can also be recognized. Each point can be directly assigned a magnitude and a phase in polar representation. In the case of the harmonic oscillator, the magnitude increase at the resonance frequency can also be recognized as a bulge in the Nyquist diagram. Furthermore, the phase rotation by π for high frequencies is evident.

B.4.8 Resonance Frequency

The frequency at which resonance occurs is the resonance frequency ω_{res} (*... in higher-order systems there can of course also be multiple extrema and thus multiple resonance frequencies*). This results from the extremal points of the magnitude of the frequency response (B.33). For this, form the derivative of the magnitude with respect to the frequency ω:

$$\begin{aligned}
\frac{d}{d\omega}|\underline{H}(j\omega)| &= \frac{d}{d\omega}\left(\frac{1}{m}\cdot\frac{1}{\left|(j\omega-\underline{\lambda}_1)(j\omega-\underline{\lambda}_2)\right|}\right) \\
&\quad | \quad \textit{Substitute real and imaginary parts of eigenvalues and transform} \\
&= \frac{d}{d\omega}\left(\frac{1}{m}\cdot\left(\left((\omega-\omega_0)^2+\alpha_0^2\right)\cdot\left((\omega+\omega_0)^2+\alpha_0^2\right)\right)^{-\frac{1}{2}}\right) \\
&\quad | \quad \textit{Multiply out} \\
&= \frac{d}{d\omega}\left(\frac{1}{m}\cdot\left(\omega^4+2\omega^2\left(\alpha_0^2-\omega_0^2\right)+\omega_0^4+2\alpha_0^2\omega_0^2+\alpha_0^4\right)^{-\frac{1}{2}}\right) \\
&\quad | \quad \textit{Apply chain rule} \\
&= -\frac{1}{m}\cdot\frac{2\omega\left(\omega^2+\left(\alpha_0^2-\omega_0^2\right)\right)}{\left(\omega^4+2\omega^2\left(\alpha_0^2-\omega_0^2\right)+\omega_0^4+2\alpha_0^2\omega_0^2+\alpha_0^4\right)^{\frac{3}{2}}}
\end{aligned} \tag{B.38}$$

The resonance frequency is the extremal point and thus a zero of the derivative. The calculation of the zeros is done via the numerator:

$$\frac{d}{d\omega}|\underline{H}(j\omega)| = 0 \qquad \Leftrightarrow \qquad \omega\left(\omega^2+\left(\alpha_0^2-\omega_0^2\right)\right) = 0 \tag{B.39}$$

From this, it can be seen that only one frequency can be a physically meaningful resonance frequency. The resonance frequency therefore results from the imaginary part ω_0 and real part α_0 of the eigenvalues. The resonance frequency is <u>not</u> the magnitude of the eigenvalues, but this can nevertheless provide a reasonable first estimate of the location of the resonance frequency:

$$\omega_{res} = \sqrt{\omega_0^2-\alpha_0^2} \qquad \approx \qquad \left|\underline{\lambda}_{1,2}\right| \textit{ when } \alpha_0 \ll \omega_0 \tag{B.40}$$

Depending on the ratio of real and imaginary parts of the eigenvalues, the resonance peak has different sharpness. The resonance peak becomes increasingly extreme the larger the imaginary part is compared to the real part. Furthermore, the peak disappears completely when the real part is correspondingly large, i.e., when the system is sufficiently damped.

The condition under which resonance can occur in the system can also be directly recognized here. Thus, $\omega_0^2 > \alpha_0^2$ must hold so that the root remains positive and yields a real resonance frequency. Otherwise, the derivative has no zero at this point and there is also no extremal point there, hence no resonance.