1.1. Principles of Monte Carlo

Intro

* Mathematics of measure formalizes the intuitive no° of proba, associating an event w/ a set of outcomes & defining the proba of the event to be its volume or measure relative to that ~~measure~~ of a universe of possible outcomes. MC uses this identity in reverse, calculating the volume of a set by interpreting the volume as a proba !!

exp'. Consider a func° f s.t. $\alpha = \int_0^1 f(x)\,dx =: E[f(U)]$

where U is uniformly distributed btw 0 & 1.

let $U_1, U_2 \dots$ draws uniform independant $\in [0,1]$

MC estimate is $\hat{\alpha}_n = \frac{1}{n} \sum_{i=1}^{n} f(U_i)$

if f integrable over $[0,1]$, by law of large nbs we have

$$\hat{\alpha}_n \to \alpha \text{ w/ proba 1 as } n \to \infty$$

if f square integrable and we set

$$\sigma_f^2 = \int_0^1 \left(f(x) - \alpha\right)^2 dx$$

then error $\hat{\alpha}_n - \alpha$ is approx $\sim \mathcal{N}\left(0, \frac{\sigma_f}{\sqrt{n}}\right)$ when $n \uparrow$

We don't know σ_f or α but can estimate using sample std:

$$s_f = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} \left(f(U_i) - \hat{\alpha}_n\right)^2}$$

Comparing to the simple trapezoidal rule

$$\alpha \approx \frac{f(0) + f(1)}{2n} + \frac{1}{n}\sum_{i=1}^{n-1} f\left(\frac{i}{n}\right)$$

Trapezoidal rule is $O(n^{-2})$ for twice continuously differentiable f, MC is $O(n^{-1/2})$. MC not competitive for 1 dimension integral, but is for multidimensions: $\forall d$ over $[0,1]^d$ or $\mathbb{R}^d$.

Call option example

let $S(t)$ be spot price (stock in equity world) @ time t.
K strike price, r continuously ~~compounded~~ ~~rate~~ interest rate
T time to maturity $\quad t_0 = 0$.

$$\text{Payoff} = (S(T) - K)^+ = \max\{0, S(T) - K\}$$

We present value the payoff w/ discount factor e^{-rT} s.t.

$$PV = E[e^{-rT}(S(T) - K)^+]$$

Stock follows geometric brownian motion

$$\frac{dS(t)}{\cancel{\quad}} = S(t)\, r\, dt + S(t)\, \sigma\, dW(t) \qquad (1.1)$$

where $W(t)$ is a Wiener Process

mean rate of return $\qquad$ stock price vol

Sol^n of (1.1) is $S(T) = \underbrace{S(0)}_{\text{current price}} \exp\left\{\left[r - \frac{1}{2}\sigma^2\right]T + \sigma \underbrace{W(T)}_{\sim \mathcal{N}(0,T)}\right\}$

let $Z = \sqrt{T}^{-1} W(T)$ s.t. $Z \sim \mathcal{N}(0,1)$

Therefore,

$$S(T) = S(0) \exp\left\{\left[r - \frac{1}{2}\sigma^2\right]T + \sigma\sqrt{T}Z\right\}$$

Thus log of stock price is normally distributed
Stock price is lognormal

By representing $E[e^{-rT}(S(T)-K)^+]$ as an integral ~~the~~ w.r.t lognormal density of $S(T)$ using the cumulative dist. func., we obtain Black Scholes formula

$$BS(S, \sigma, T, r, K) = S\,\Phi\left(\frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - e^{-rT}K\,\Phi\left(\frac{\log(\frac{S}{K}) + [r - \frac{1}{2}\sigma^2]T}{\sigma\sqrt{T}}\right)$$

for call op^n w/ no divs.

Therefore, the monte Carlo algo ~~as~~ to compute $E[e^{-rT}(S(T)-K^+)]$, assuming a call op° w/ no dividends and assuming we can produce ~~the~~ a sequence $Z_1, Z_2, \dots,$ is:

for $i = 1, \dots, n$:
  generate Z_i
  set $S_i(T) = S(0)\exp\{(r - \frac{1}{2}\sigma^2)T + \sigma Z_i\sqrt{T}\}$
  set $C_i = e^{-rT}(S(T)-K)^+$
set $\hat{C}_n = (C_1 + \dots + C_n)/n$