

# Applied Mathematics (4320001) - Summer 2023 Solution

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## Question 1 [14 marks]

Fill in the blanks using appropriate choice from the given options

### Question 1.1 [1 marks]

If  $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 4 & 2 \end{bmatrix}$ , then  $A^T = \underline{\hspace{2cm}}$  Answer: b.  $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \end{bmatrix}$

#### Solution

For transpose of a matrix, rows become columns and columns become rows.  $A^T = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \end{bmatrix}$

### Question 1.2 [1 marks]

If  $\begin{bmatrix} x+y & 3 \\ -7 & x-y \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ -7 & 2 \end{bmatrix}$ , then  $(x, y) = \underline{\hspace{2cm}}$  Answer: c. (5, 3)

#### Solution

Comparing corresponding elements:  $x + y = 8 \dots (1)$   $x - y = 2 \dots (2)$

Adding equations (1) and (2):  $2x = 10 \implies x = 5$  Substituting in equation (1):  $5 + y = 8 \implies y = 3$

### Question 1.3 [1 marks]

If  $\begin{bmatrix} x & 3 \\ y & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 15 \\ 12 \end{bmatrix}$ , then  $y = \underline{\hspace{2cm}}$  Answer: c. 3

#### Solution

Matrix multiplication gives:  $2x + 9 = 15 \implies 2x = 6 \implies x = 3$   $2y + 6 = 12 \implies 2y = 6 \implies y = 3$

## Question 1.4 [1 marks]

Order of matrix  $\begin{bmatrix} 1 & -3 \\ -2 & 1 \\ 4 & 5 \end{bmatrix}$  is \_\_\_\_\_ Answer: b.  $3 \times 2$

### Solution

The matrix has 3 rows and 2 columns, so order is  $3 \times 2$ .

## Question 1.5 [1 marks]

$\frac{d}{dx}(x^2 + 2x + 3) =$  \_\_\_\_\_ Answer: b.  $2x + 2$

### Solution

Using power rule:  $\frac{d}{dx}(x^2 + 2x + 3) = 2x + 2 + 0 = 2x + 2$

## Question 1.6 [1 marks]

$\frac{d}{dx}(\sec x) =$  \_\_\_\_\_ Answer: a.  $\sec x \cdot \tan x$

### Solution

Standard derivative:  $\frac{d}{dx}(\sec x) = \sec x \tan x$

## Question 1.7 [1 marks]

If  $x^2 + y^2 = 1$ , then  $\frac{dy}{dx} =$  \_\_\_\_\_ Answer: b.  $-\frac{x}{y}$

### Solution

Differentiating implicitly:  $2x + 2y\frac{dy}{dx} = 0$  Therefore:  $\frac{dy}{dx} = -\frac{x}{y}$

## Question 1.8 [1 marks]

$\int \log x \, dx =$  \_\_\_\_\_ +c Answer: b.  $x \log x - x$

### Solution

Using integration by parts:  $\int \log x \, dx = x \log x - \int x \cdot \frac{1}{x} dx = x \log x - x + c$

## Question 1.9 [1 marks]

$\int \frac{1}{x^2} dx =$  \_\_\_\_\_ +c Answer: b.  $-\frac{1}{x}$

**Solution**

$$\int x^{-2} dx = \frac{x^{-1}}{-1} = -\frac{1}{x} + C$$

**Question 1.10 [1 marks]**

$$\int_{-1}^1 (x^2 + 1) dx = \text{_____} \quad \text{Answer: a. } \frac{8}{3}$$

**Solution**

$$\int_{-1}^1 (x^2 + 1) dx = \left[ \frac{x^3}{3} + x \right]_{-1}^1 = \left( \frac{1}{3} + 1 \right) - \left( -\frac{1}{3} - 1 \right) = \frac{4}{3} - \left( -\frac{4}{3} \right) = \frac{8}{3}$$

**Question 1.11 [1 marks]**

Order of the differential equation  $\left(\frac{d^2y}{dx^2}\right)^3 + 3\left(\frac{dy}{dx}\right)^2 - 6y = 0$  is \_\_\_\_\_ and degree is \_\_\_\_\_  
 Answer: a. 2, 3

**Solution**

Order = highest derivative = 2 Degree = power of highest derivative = 3

**Question 1.12 [1 marks]**

Integrating Factor of the differential equation  $\frac{dy}{dx} = y \tan x + e^x$  is \_\_\_\_\_ Answer: c.  $\sin x$

**Solution**

Rearranging:  $\frac{dy}{dx} - y \tan x = e^x$  This is a linear differential equation  $\frac{dy}{dx} + Py = Q$  with  $P = -\tan x$ . Integrating factor =  $e^{\int -\tan x dx} = e^{-\ln |\sec x|} = e^{\ln |\cos x|} = \cos x$ .

Wait, the provided answer (c) is  $\sin x$ . Let's re-read the question carefully. "Integrating Factor of the differential equation  $\frac{dy}{dx} = y \tan x + e^x$  is..." If terms are rearranged as  $\frac{dy}{dx} - (\tan x)y = e^x$ , then  $P = -\tan x$ .  $\exp(\int -\tan x dx) = \exp(\ln(\cos x)) = \cos x$ .

However, strict fidelity requires me to follow the MDX content. MDX Solution says: "This is not in standard linear form. The given options suggest  $\sin x$  as integrating factor." Actually, looking at standard GTU papers, sometimes there are typos. I will follow the MDX text exactly.

**Question 1.13 [1 marks]**

Mean of the first five natural numbers is \_\_\_\_\_ Answer: c. 3

**Solution**

First five natural numbers: 1, 2, 3, 4, 5 Mean =  $\frac{1+2+3+4+5}{5} = \frac{15}{5} = 3$

## Question 1.14 [1 marks]

If the mean of observations 15, 7, 6, a, 3 is 7, then  $a = \underline{\hspace{2cm}}$  Answer: b. 4

### Solution

$$\frac{15+7+6+a+3}{5} = 7 \quad 31 + a = 35 \implies a = 4$$

## Question 2(a) [6 marks]

Attempt any two

### Question 2(a)(1) [3 marks]

If  $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} -2 & 1 & 2 \\ 2 & -1 & 3 \\ 0 & 2 & 4 \end{bmatrix}$  and  $C = \begin{bmatrix} 5 & 4 & 2 \\ -1 & 7 & 8 \\ 6 & 4 & 3 \end{bmatrix}$ , then Find  $2A - B + C$

### Solution

$$\begin{aligned} 2A &= 2 \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 2 & -2 & 0 \\ 6 & 4 & 2 \end{bmatrix} \\ 2A - B &= \begin{bmatrix} 2 & 4 & 2 \\ 2 & -2 & 0 \\ 6 & 4 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 1 & 2 \\ 2 & -1 & 3 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 0 \\ 0 & -1 & -3 \\ 6 & 2 & -2 \end{bmatrix} \\ 2A - B + C &= \begin{bmatrix} 4 & 3 & 0 \\ 0 & -1 & -3 \\ 6 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 4 & 2 \\ -1 & 7 & 8 \\ 6 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 7 & 2 \\ -1 & 6 & 5 \\ 12 & 6 & 1 \end{bmatrix} \end{aligned}$$

### Question 2(a)(2) [3 marks]

If  $A = \begin{bmatrix} 7 & 5 \\ -1 & 2 \end{bmatrix}$  and  $B = \begin{bmatrix} 6 & 0 \\ -2 & 3 \end{bmatrix}$ , then prove that  $(A + B)^T = A^T + B^T$

### Solution

$$\begin{aligned} A + B &= \begin{bmatrix} 7 & 5 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ -3 & 5 \end{bmatrix} \\ (A + B)^T &= \begin{bmatrix} 13 & -3 \\ 5 & 5 \end{bmatrix} \\ A^T &= \begin{bmatrix} 7 & -1 \\ 5 & 2 \end{bmatrix}, B^T = \begin{bmatrix} 6 & -2 \\ 0 & 3 \end{bmatrix} \\ A^T + B^T &= \begin{bmatrix} 7 & -1 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 6 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 13 & -3 \\ 5 & 5 \end{bmatrix} \end{aligned}$$

Therefore,  $(A + B)^T = A^T + B^T \checkmark$

## Question 2(a)(3) [3 marks]

**Solve:**  $(x + y)dy = dx$

**Solution**

$$(x + y)dy = dx \implies \frac{dx}{dy} = x + y \cdot \frac{dx}{dy} - x = y$$

This is a linear differential equation in  $x$ . Integrating factor  $= e^{\int -1 dy} = e^{-y}$

$$e^{-y} \cdot x = \int ye^{-y} dy$$

$$\text{Using integration by parts: } \int ye^{-y} dy = -ye^{-y} - \int -e^{-y} dy = -ye^{-y} - e^{-y} = -e^{-y}(y + 1)$$

$$\text{Therefore: } xe^{-y} = -e^{-y}(y + 1) + C \quad x = -(y + 1) + Ce^y$$

## Question 2(b) [8 marks]

**Attempt any two**

## Question 2(b)(1) [4 marks]

If  $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ , then prove that  $A^2 - 4A - 5I_3 = 0$

**Solution**

$$\text{First, calculate } A^2: A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$4A = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$5I_3 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^2 - 4A - 5I_3 = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \checkmark$$

## Question 2(b)(2) [4 marks]

If  $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{bmatrix}$ , then find  $A^{-1}$

### Solution

Using adjoint method:  $A^{-1} = \frac{1}{|A|} \text{adj}(A)$

$$|A| = 1(0 - 3) - 2(0 - 3) + 1(2 - 1) = -3 + 6 + 1 = 4$$

Finding cofactors:  $C_{11} = -3, C_{12} = 3, C_{13} = 1, C_{21} = 1, C_{22} = -1, C_{23} = 1, C_{31} = 5, C_{32} = -1, C_{33} = -3$

$$\text{adj}(A) = \begin{bmatrix} -3 & 1 & 5 \\ 3 & -1 & -1 \\ 1 & 1 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4} \begin{bmatrix} -3 & 1 & 5 \\ 3 & -1 & -1 \\ 1 & 1 & -3 \end{bmatrix}$$

## Question 2(b)(3) [4 marks]

Solve the equations  $2x + 3y = 7$  and  $4x - y = 9$  using matrix method

### Solution

Rewriting:  $2x + 3y = 7$  and  $4x - y = 9$

$$\text{In matrix form: } \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$

$$|A| = 2(-1) - 3(4) = -2 - 12 = -14$$

$$A^{-1} = \frac{1}{-14} \begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-14} \begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 9 \end{bmatrix} = \frac{1}{-14} \begin{bmatrix} -34 \\ -10 \end{bmatrix}$$

$$\text{Therefore: } x = \frac{34}{14} = \frac{17}{7}, y = \frac{10}{14} = \frac{5}{7}$$

## Question 3(a) [6 marks]

Attempt any two

## Question 3(a)(1) [3 marks]

If  $y = x^x$ , then find  $\frac{dy}{dx}$

### Solution

Taking natural logarithm:  $\ln y = x \ln x$

$$\text{Differentiating both sides: } \frac{1}{y} \frac{dy}{dx} = \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

$$\frac{dy}{dx} = y(\ln x + 1) = x^x(\ln x + 1)$$

### Question 3(a)(2) [3 marks]

If  $y = \log(x + \sqrt{x^2 + a^2})$ , then find  $\frac{dy}{dx}$

**Solution**

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{x+\sqrt{x^2+a^2}} \cdot \frac{d}{dx}(x + \sqrt{x^2 + a^2}) \\ \frac{d}{dx}(x + \sqrt{x^2 + a^2}) &= 1 + \frac{2x}{2\sqrt{x^2+a^2}} = 1 + \frac{x}{\sqrt{x^2+a^2}} = \frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}} \\ \frac{dy}{dx} &= \frac{1}{x+\sqrt{x^2+a^2}} \cdot \frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}} = \frac{1}{\sqrt{x^2+a^2}}\end{aligned}$$

### Question 3(a)(3) [3 marks]

If  $y = \operatorname{cosec}^{-1} x + \sec^{-1} x$ , then find  $\frac{dy}{dx}$

**Solution**

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\operatorname{cosec}^{-1} x) + \frac{d}{dx}(\sec^{-1} x) \\ &= -\frac{1}{|x|\sqrt{x^2-1}} + \frac{1}{|x|\sqrt{x^2-1}} = 0\end{aligned}$$

### Question 3(b) [8 marks]

Attempt any two

#### Question 3(b)(1) [4 marks]

Differentiate  $y = \cos x$  using the definition

**Solution**

$$\begin{aligned}\text{By definition: } \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \\ \frac{d}{dx}(\cos x) &= \lim_{h \rightarrow 0} \frac{\cos(x+h)-\cos x}{h} \\ \text{Using the identity: } \cos(x+h) &= \cos x \cos h - \sin x \sin h \\ &= \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} = \lim_{h \rightarrow 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} = \cos x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \rightarrow 0} \frac{\sin h}{h} = \cos x \cdot 0 - \sin x \cdot 1 = -\sin x\end{aligned}$$

#### Question 3(b)(2) [4 marks]

Find the maximum and minimum value of  $f(x) = x^3 - 4x^2 + 5x + 7$

**Solution**

$$\begin{aligned}f'(x) &= 3x^2 - 8x + 5 \\ \text{Setting } f'(x) = 0: 3x^2 - 8x + 5 &= 0 \implies (3x - 5)(x - 1) = 0 \quad x = \frac{5}{3} \text{ or } x = 1\end{aligned}$$

$$f''(x) = 6x - 8$$

At  $x = 1$ :  $f''(1) = 6(1) - 8 = -2 < 0$  (Maximum) At  $x = \frac{5}{3}$ :  $f''\left(\frac{5}{3}\right) = 6\left(\frac{5}{3}\right) - 8 = 2 > 0$  (Minimum)

$$\text{Maximum value: } f(1) = 1 - 4 + 5 + 7 = 9 \quad \text{Minimum value: } f\left(\frac{5}{3}\right) = \left(\frac{5}{3}\right)^3 - 4\left(\frac{5}{3}\right)^2 + 5\left(\frac{5}{3}\right) + 7 = \frac{158}{27}$$

## Question 3(b)(3) [4 marks]

If  $y = (\tan^{-1} x)^2$ , then prove that  $(1+x^2)y_2 + 2x(1+x^2)y_1 = 2$

### Solution

$$y = (\tan^{-1} x)^2 \implies y_1 = \frac{dy}{dx} = 2(\tan^{-1} x) \cdot \frac{1}{1+x^2}$$

$$y_2 = \frac{d^2y}{dx^2} = 2 \left[ \frac{1}{1+x^2} \cdot \frac{1}{1+x^2} + (\tan^{-1} x) \cdot \frac{-2x}{(1+x^2)^2} \right] = \frac{2}{(1+x^2)^2} - \frac{4x(\tan^{-1} x)}{(1+x^2)^2}$$

$$\text{Now substituting in LHS: } (1+x^2)y_2 + 2x(1+x^2)y_1 = (1+x^2) \cdot \frac{2-4x(\tan^{-1} x)}{(1+x^2)^2} + 2x(1+x^2) \cdot \frac{2(\tan^{-1} x)}{1+x^2} = \frac{2-4x(\tan^{-1} x)}{1+x^2} + 4x(\tan^{-1} x) = \frac{2-4x(\tan^{-1} x)+4x(\tan^{-1} x)(1+x^2)}{1+x^2} = \frac{2+4x^3(\tan^{-1} x)}{1+x^2}$$

Wait, let me re-evaluate the substitution carefully. LHS =  $(1+x^2)y_2 + 2x(1+x^2)y_1$ . Note that the standard textbook problem is usually  $(1+x^2)^2y_2 + 2x(1+x^2)y_1 = 2$ . However, the question says  $(1+x^2)y_2 + 2x(1+x^2)y_1$ . Let's see if  $(1+x^2)y_1 = 2 \tan^{-1} x$ . Diff again:  $(1+x^2)y_2 + 2xy_1 = \frac{2}{1+x^2}$ . Multiply by  $(1+x^2)$ :  $(1+x^2)^2y_2 + 2x(1+x^2)y_1 = 2$ . The question likely meant  $(1+x^2)^2y_2$  OR it meant  $(1+x^2)y_2 + 2xy_1$  equal to something else. BUT, looking at the MDX solution: The MDX solution ends with: " =  $\frac{2}{1+x^2} \cdot (1+x^2) = 2$ ". This implies the term was indeed forming 2. Let's check the MDX step: " =  $\frac{2-4x(\tan^{-1} x)}{1+x^2} + 4x(\tan^{-1} x)$ " " =  $\frac{2-4x(\tan^{-1} x)+4x(\tan^{-1} x)(1+x^2)}{1+x^2}$ ". This algebra seems weird in the MDX.  $4x(\tan^{-1} x)(1+x^2)$  would not cancel  $-4x(\tan^{-1} x)$  cleanly unless  $x^2$  term is handled. Calculated value:  $\frac{2-4x\tan^{-1} x+4x\tan^{-1} x+4x^3\tan^{-1} x}{1+x^2}$ . This is not 2.

There is a discrepancy in the MDX algebra or the question statement. However, User requires strict fidelity to the MDX text. "Migrate the \*\*EXACT\*\* text content from MDX to LaTeX." I will copy the MDX solution steps exactly, even if they look mathematically dubious, as per instructions. MDX text: " =  $\frac{2-4x(\tan^{-1} x)+4x(\tan^{-1} x)(1+x^2)}{1+x^2} = \frac{2}{1+x^2} \cdot (1+x^2) = 2$  ✓

## Question 4(a) [6 marks]

Attempt any two

### Question 4(a)(1) [3 marks]

Integrate:  $\int \frac{x^5}{1+x^{12}} dx$

### Solution

$$\text{Let } u = x^6, \text{ then } du = 6x^5 dx, \text{ so } x^5 dx = \frac{1}{6} du$$

$$\int \frac{x^5}{1+x^{12}} dx = \int \frac{1}{1+u^2} \cdot \frac{1}{6} du = \frac{1}{6} \tan^{-1} u + C = \frac{1}{6} \tan^{-1}(x^6) + C$$

### Question 4(a)(2) [3 marks]

Integrate:  $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

### Solution

$$\text{Let } I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

Using property  $\int_0^a f(x) dx = \int_0^a f(a-x) dx$ :

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2-x)}}{\sqrt{\sin(\pi/2-x)} + \sqrt{\cos(\pi/2-x)}} dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$$

$$\text{Adding both expressions: } 2I = \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \int_0^{\pi/2} 1 dx = \frac{\pi}{2}$$

Therefore:  $I = \frac{\pi}{4}$

## Question 4(a)(3) [3 marks]

If the mean of the following data is 19, then find missing frequency

**Solution**

**Table 1.** Frequency Distribution

$x_i$	6	10	14	18	24	28	30
$f_i$	2	4	7	f	8	4	3

$$\text{Mean} = \frac{\sum f_i x_i}{\sum f_i} = 19$$

$$\begin{aligned} \sum f_i &= 2 + 4 + 7 + f + 8 + 4 + 3 = 28 + f \quad \sum f_i x_i = 2(6) + 4(10) + 7(14) + f(18) + 8(24) + 4(28) + 3(30) \\ &= 12 + 40 + 98 + 18f + 192 + 112 + 90 = 544 + 18f \end{aligned}$$

$$\frac{544+18f}{28+f} = 19 \quad 544 + 18f = 19(28 + f) \quad 544 + 18f = 532 + 19f \quad 12 = f$$

Therefore,  $f = 12$

## Question 4(b) [8 marks]

Attempt any two

### Question 4(b)(1) [4 marks]

Integrate:  $\int \frac{x}{(x+1)(x+2)} dx$

**Solution**

Using partial fractions:  $\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$   
 $x = A(x+2) + B(x+1)$

Setting  $x = -1$ :  $-1 = A(1) \Rightarrow A = -1$  Setting  $x = -2$ :  $-2 = B(-1) \Rightarrow B = 2$

$$\int \frac{x}{(x+1)(x+2)} dx = \int \left( \frac{-1}{x+1} + \frac{2}{x+2} \right) dx = -\ln|x+1| + 2\ln|x+2| + C = \ln \left| \frac{(x+2)^2}{x+1} \right| + C$$

### Question 4(b)(2) [4 marks]

Integrate:  $\int \frac{x^2 \tan^{-1} x^3}{1+x^6} dx$

**Solution**

Let  $u = x^3$ , then  $du = 3x^2 dx$ , so  $x^2 dx = \frac{1}{3} du$

$$\int \frac{x^2 \tan^{-1} x^3}{1+x^6} dx = \int \frac{\tan^{-1} u}{1+u^2} \cdot \frac{1}{3} du$$

Let  $v = \tan^{-1} u$ , then  $dv = \frac{1}{1+u^2} du$

$$= \frac{1}{3} \int v dv = \frac{1}{3} \cdot \frac{v^2}{2} + C = \frac{(\tan^{-1} u)^2}{6} + C$$

$$= \frac{(\tan^{-1} x^3)^2}{6} + C$$

## Question 4(b)(3) [4 marks]

Find the standard deviation for the following data: 10, 15, 7, 19, 9, 21, 23, 25, 26, 30

### Solution

First, find the mean:  $\bar{x} = \frac{10+15+7+19+9+21+23+25+26+30}{10} = \frac{185}{10} = 18.5$

Table for Standard Deviation:

**Table 2.** Standard Deviation Calculation

$x_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
10	-8.5	72.25
15	-3.5	12.25
7	-11.5	132.25
19	0.5	0.25
9	-9.5	90.25
21	2.5	6.25
23	4.5	20.25
25	6.5	42.25
26	7.5	56.25
30	11.5	132.25

$$\sum(x_i - \bar{x})^2 = 564.5$$

$$\text{Standard deviation} = \sqrt{\frac{\sum(x_i - \bar{x})^2}{n}} = \sqrt{\frac{564.5}{10}} = \sqrt{56.45} = 7.51$$

## Question 5(a) [6 marks]

Attempt any two

## Question 5(a)(1) [3 marks]

Find the standard deviation for the following data:

### Solution

**Table 3.** Data

$x_i$	4	8	11	17	20	24	32
$f_i$	3	5	9	5	4	3	1

$$N = \sum f_i = 3 + 5 + 9 + 5 + 4 + 3 + 1 = 30$$

$$\text{Mean Calculation: } \bar{x} = \frac{\sum f_i x_i}{N} = \frac{3(4)+5(8)+9(11)+5(17)+4(20)+3(24)+1(32)}{30} = \frac{12+40+99+85+80+72+32}{30} = \frac{420}{30} = 14$$

Standard Deviation Table:

**Table 4.** Standard Deviation Calculation

$x_i$	$f_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
4	3	-10	100	300
8	5	-6	36	180
11	9	-3	9	81
17	5	3	9	45
20	4	6	36	144
24	3	10	100	300
32	1	18	324	324

$$\sum f_i(x_i - \bar{x})^2 = 1374$$

$$\text{Standard deviation} = \sqrt{\frac{\sum f_i(x_i - \bar{x})^2}{N}} = \sqrt{\frac{1374}{30}} = \sqrt{45.8} = 6.77$$

## Question 5(a)(2) [3 marks]

Find the standard deviation for the following data:

### Solution

**Table 5.** Grouped Data

Class	0-10	10-20	20-30	30-40	40-50
Frequency	5	8	15	16	6

First, find class midpoints and calculate mean:

**Table 6.** Midpoint Calculation

Class	Midpoint ( $x_i$ )	$f_i$	$f_i x_i$
0-10	5	5	25
10-20	15	8	120
20-30	25	15	375
30-40	35	16	560
40-50	45	6	270

$$N = 50, \sum f_i x_i = 1350 \bar{x} = \frac{1350}{50} = 27$$

### Standard Deviation Table:

**Table 7.** Standard Deviation Calculation

$x_i$	$f_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
5	5	-22	484	2420
15	8	-12	144	1152
25	15	-2	4	60
35	16	8	64	1024
45	6	18	324	1944

$$\sum f_i(x_i - \bar{x})^2 = 6600$$

$$\text{Standard deviation} = \sqrt{\frac{6600}{50}} = \sqrt{132} = 11.49$$

## Question 5(a)(3) [3 marks]

Find the mean for the following data:

### Solution

**Table 8.** Grouped Frequency Distribution

Class	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	3	7	12	15	8	3	2

Using midpoint method:

**Table 9.** Mean Calculation

Class	Midpoint ( $x_i$ )	$f_i$	$f_i x_i$
30-40	35	3	105
40-50	45	7	315
50-60	55	12	660
60-70	65	15	975
70-80	75	8	600
80-90	85	3	255
90-100	95	2	190

$$N = \sum f_i = 50$$

$$\text{Mean} = \frac{\sum f_i x_i}{N} = \frac{3100}{50} = 62$$

## Question 5(b) [8 marks]

Attempt any two

### Question 5(b)(1) [4 marks]

Solve:  $xy dx - (y^2 + x^2) dy = 0$

### Solution

$$\text{Rearranging: } xy dx = (y^2 + x^2) dy \quad \frac{dx}{dy} = \frac{y^2 + x^2}{xy} = \frac{y}{x} + \frac{x}{y}$$

This is a homogeneous differential equation. Let  $x = vy$ , then  $\frac{dx}{dy} = v + y \frac{dv}{dy}$

$$\text{Substituting: } v + y \frac{dv}{dy} = \frac{y}{vy} + \frac{vy}{y} = \frac{1}{v} + v$$

$$y \frac{dv}{dy} = \frac{1}{v} \implies v dv = \frac{dy}{y}$$

$$\text{Integrating both sides: } \int v dv = \int \frac{dy}{y} \implies \frac{v^2}{2} = \ln |y| + C$$

$$\text{Substituting back } v = \frac{x}{y}: \frac{x^2}{2y^2} = \ln |y| + C \quad x^2 = 2y^2(\ln |y| + C)$$

### Question 5(b)(2) [4 marks]

Solve:  $\frac{dy}{dx} + \frac{2y}{x} = \sin x$

**Solution**

This is a linear differential equation of the form  $\frac{dy}{dx} + P(x)y = Q(x)$  where  $P(x) = \frac{2}{x}$  and  $Q(x) = \sin x$

Integrating factor =  $e^{\int P(x)dx} = e^{\int \frac{2}{x} dx} = e^{2\ln|x|} = x^2$

Multiplying the equation by integrating factor:  $x^2 \frac{dy}{dx} + 2xy = x^2 \sin x$

The left side is  $\frac{d}{dx}(x^2y)$ :  $\frac{d}{dx}(x^2y) = x^2 \sin x$

Integrating both sides:  $x^2y = \int x^2 \sin x dx$

Using integration by parts twice:  $\int x^2 \sin x dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$

Therefore:  $x^2y = -x^2 \cos x + 2x \sin x + 2 \cos x + C$   $y = -\cos x + \frac{2 \sin x}{x} + \frac{2 \cos x}{x^2} + \frac{C}{x^2}$

**Question 5(b)(3) [4 marks]**

**Solve:**  $(1+x^2)\frac{dy}{dx} + 2xy = \cos x$

**Solution**

Dividing by  $(1+x^2)$ :  $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{\cos x}{1+x^2}$

This is linear with  $P(x) = \frac{2x}{1+x^2}$  and  $Q(x) = \frac{\cos x}{1+x^2}$

Integrating factor =  $e^{\int \frac{2x}{1+x^2} dx} = e^{\ln(1+x^2)} = 1+x^2$

Multiplying by integrating factor:  $(1+x^2)\frac{dy}{dx} + 2xy = \cos x$

The left side is  $\frac{d}{dx}[(1+x^2)y]$ :  $\frac{d}{dx}[(1+x^2)y] = \cos x$

Integrating:  $(1+x^2)y = \int \cos x dx = \sin x + C$

Therefore:  $y = \frac{\sin x + C}{1+x^2}$

**Complete Formula Sheet****Matrix Operations**

- **Transpose:**  $(A^T)_{ij} = A_{ji}$
- **Inverse:**  $A^{-1} = \frac{1}{|A|} \text{adj}(A)$
- **Properties:**  $(A+B)^T = A^T + B^T$

**Derivatives**

- **Power Rule:**  $\frac{d}{dx}(x^n) = nx^{n-1}$
- **Trigonometric:**  $\frac{d}{dx}(\sin x) = \cos x$ ,  $\frac{d}{dx}(\cos x) = -\sin x$
- **Inverse Trig:**  $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
- **Logarithmic:**  $\frac{d}{dx}(\ln x) = \frac{1}{x}$

**Integration**

- **By Parts:**  $\int u dv = uv - \int v du$
- **Substitution:** If  $u = g(x)$ , then  $\int f(g(x))g'(x)dx = \int f(u)du$
- **Definite Properties:**  $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

**Differential Equations**

- **Linear Form:**  $\frac{dy}{dx} + P(x)y = Q(x)$
- **Integrating Factor:**  $e^{\int P(x)dx}$
- **Variable Separable:**  $\frac{dy}{dx} = f(x)g(y)$

**Statistics**

- **Mean:**  $\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$

- **Standard Deviation:**  $\sigma = \sqrt{\frac{\sum f_i(x_i - \bar{x})^2}{N}}$
- **Variance:**  $\sigma^2 = \frac{\sum f_i(x_i - \bar{x})^2}{N}$