# Applied Mathematics (4320001) - Summer 2023 Solution

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## Question 1 [14 marks]

**Fill in the blanks using appropriate choice from the given options**

## Question 1.1 [1 marks]

**If** $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 4 & 2 \end{bmatrix}$**, then** $A^T =$ ______________ **Answer**: b. $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \end{bmatrix}$

**Solution**

For transpose of a matrix, rows become columns and columns become rows. $A^T = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \end{bmatrix}$

## Question 1.2 [1 marks]

**If** $\begin{bmatrix} x+y & 3 \\ -7 & x-y \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ -7 & 2 \end{bmatrix}$**, then** $(x, y) =$ ______________ **Answer**: c. $(5, 3)$

**Solution**

Comparing corresponding elements: $x + y = 8$ ... (1) $x - y = 2$ ... (2)
Adding equations (1) and (2): $2x = 10 \implies x = 5$ Substituting in equation (1): $5 + y = 8 \implies y = 3$

## Question 1.3 [1 marks]

**If** $\begin{bmatrix} x & 3 \\ y & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 15 \\ 12 \end{bmatrix}$**, then** $y =$ ______________ **Answer**: c. 3

**Solution**

Matrix multiplication gives: $2x + 9 = 15 \implies 2x = 6 \implies x = 3$ $2y + 6 = 12 \implies 2y = 6 \implies y = 3$

## Question 1.4 [1 marks]

**Order of matrix** $\begin{bmatrix} 1 & -3 \\ -2 & 1 \\ 4 & 5 \end{bmatrix}$ **is** ______________ **Answer**: b. $3 \times 2$

**Solution**

The matrix has 3 rows and 2 columns, so order is $3 \times 2$.

## Question 1.5 [1 marks]

$\frac{d}{dx}(x^2 + 2x + 3) =$ ______________ **Answer**: b. $2x + 2$

**Solution**

Using power rule: $\frac{d}{dx}(x^2 + 2x + 3) = 2x + 2 + 0 = 2x + 2$

## Question 1.6 [1 marks]

$\frac{d}{dx}(\sec x) =$ ______________ **Answer**: a. $\sec x \cdot \tan x$

**Solution**

Standard derivative: $\frac{d}{dx}(\sec x) = \sec x \tan x$

## Question 1.7 [1 marks]

**If** $x^2 + y^2 = 1$**, then** $\frac{dy}{dx} =$ ______________ **Answer**: b. $-\frac{x}{y}$

**Solution**

Differentiating implicitly: $2x + 2y\frac{dy}{dx} = 0$ Therefore: $\frac{dy}{dx} = -\frac{x}{y}$

## Question 1.8 [1 marks]

$\int \log x \, dx =$ ______________ $+c$ **Answer**: b. $x \log x - x$

**Solution**

Using integration by parts: $\int \log x \, dx = x \log x - \int x \cdot \frac{1}{x} dx = x \log x - x + c$

## Question 1.9 [1 marks]

$\int \frac{1}{x^2} dx =$ ______________ $+c$ **Answer**: b. $-\frac{1}{x}$

**Solution**

$\int x^{-2}dx = \frac{x^{-1}}{-1} = -\frac{1}{x} + c$

## Question 1.10 [1 marks]

$\int_{-1}^{1}(x^2+1)dx =$ ______________ **Answer**: a. $\frac{8}{3}$

**Solution**

$\int_{-1}^{1}(x^2+1)dx = \left[\frac{x^3}{3}+x\right]_{-1}^{1} = \left(\frac{1}{3}+1\right) - \left(-\frac{1}{3}-1\right) = \frac{4}{3} - \left(-\frac{4}{3}\right) = \frac{8}{3}$

## Question 1.11 [1 marks]

**Order of the differential equation** $\left(\frac{d^2y}{dx^2}\right)^3 + 3\left(\frac{dy}{dx}\right)^2 - 6y = 0$ **is** ______________ **and degree is** ______________ **Answer**: a. 2, 3

**Solution**

Order = highest derivative = 2 Degree = power of highest derivative = 3

## Question 1.12 [1 marks]

**Integrating Factor of the differential equation** $\frac{dy}{dx} = y\tan x + e^x$ **is** ______________ **Answer**: c. $\sin x$

**Solution**

Rearranging: $\frac{dy}{dx} - y\tan x = e^x$ This is a linear differential equation $\frac{dy}{dx} + Py = Q$ with $P = -\tan x$. Integrating factor $= e^{\int -\tan x dx} = e^{-\ln|\sec x|} = e^{\ln|\cos x|} = \cos x$.
Wait, the provided answer (c) is $\sin x$. Let's re-read the question carefully. "Integrating Factor of the differential equation $\frac{dy}{dx} = y\tan x + e^x$ is..." If terms are rearranged as $\frac{dy}{dx} - (\tan x)y = e^x$, then $P = -\tan x$. $\exp(\int -\tan x dx) = \exp(\ln(\cos x)) = \cos x$.
However, strict fidelity requires me to follow the MDX content. MDX Solution says: "This is not in standard linear form. The given options suggest $\sin x$ as integrating factor." Actually, looking at standard GTU papers, sometimes there are typos. I will follow the MDX text exactly.

## Question 1.13 [1 marks]

**Mean of the first five natural numbers is** ______________ **Answer**: c. 3

**Solution**

First five natural numbers: 1, 2, 3, 4, 5 Mean $= \frac{1+2+3+4+5}{5} = \frac{15}{5} = 3$

## Question 1.14 [1 marks]

**If the mean of observations 15, 7, 6, a, 3 is 7, then** $a =$ ____________ **Answer**: b. 4

**Solution**

$\frac{15+7+6+a+3}{5} = 7$ $31 + a = 35 \implies a = 4$

## Question 2(a) [6 marks]

**Attempt any two**

## Question 2(a)(1) [3 marks]

**If** $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$**,** $B = \begin{bmatrix} -2 & 1 & 2 \\ 2 & -1 & 3 \\ 0 & 2 & 4 \end{bmatrix}$ **and** $C = \begin{bmatrix} 5 & 4 & 2 \\ -1 & 7 & 8 \\ 6 & 4 & 3 \end{bmatrix}$**, then Find** $2A - B + C$

**Solution**

$$2A = 2\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 2 & -2 & 0 \\ 6 & 4 & 2 \end{bmatrix}$$

$$2A - B = \begin{bmatrix} 2 & 4 & 2 \\ 2 & -2 & 0 \\ 6 & 4 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 1 & 2 \\ 2 & -1 & 3 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 0 \\ 0 & -1 & -3 \\ 6 & 2 & -2 \end{bmatrix}$$

$$2A - B + C = \begin{bmatrix} 4 & 3 & 0 \\ 0 & -1 & -3 \\ 6 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 4 & 2 \\ -1 & 7 & 8 \\ 6 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 7 & 2 \\ -1 & 6 & 5 \\ 12 & 6 & 1 \end{bmatrix}$$

## Question 2(a)(2) [3 marks]

**If** $A = \begin{bmatrix} 7 & 5 \\ -1 & 2 \end{bmatrix}$ **and** $B = \begin{bmatrix} 6 & 0 \\ -2 & 3 \end{bmatrix}$**, then prove that** $(A+B)^T = A^T + B^T$

**Solution**

$$A + B = \begin{bmatrix} 7 & 5 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ -3 & 5 \end{bmatrix}$$

$$(A+B)^T = \begin{bmatrix} 13 & -3 \\ 5 & 5 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 7 & -1 \\ 5 & 2 \end{bmatrix}, B^T = \begin{bmatrix} 6 & -2 \\ 0 & 3 \end{bmatrix}$$

$$A^T + B^T = \begin{bmatrix} 7 & -1 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 6 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 13 & -3 \\ 5 & 5 \end{bmatrix}$$

Therefore, $(A+B)^T = A^T + B^T$ ✓

## Question 2(a)(3) [3 marks]

**Solve:** $(x+y)dy = dx$

**Solution**

$(x+y)dy = dx \implies \frac{dx}{dy} = x + y$ $\frac{dx}{dy} - x = y$
This is a linear differential equation in $x$. Integrating factor $= e^{\int -1dy} = e^{-y}$
$e^{-y} \cdot x = \int ye^{-y}dy$
Using integration by parts: $\int ye^{-y}dy = -ye^{-y} - \int -e^{-y}dy = -ye^{-y} - e^{-y} = -e^{-y}(y+1)$
Therefore: $xe^{-y} = -e^{-y}(y+1) + C$ $x = -(y+1) + Ce^y$

## Question 2(b) [8 marks]

**Attempt any two**

## Question 2(b)(1) [4 marks]

**If** $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$**, then prove that** $A^2 - 4A - 5I_3 = 0$

**Solution**

First, calculate $A^2$: $A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$

$4A = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$

$5I_3 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$A^2 - 4A - 5I_3 = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$ ✓

## Question 2(b)(2) [4 marks]

**If** $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{bmatrix}$**, then find** $A^{-1}$

**Solution**

Using adjoint method: $A^{-1} = \frac{1}{|A|}\text{adj}(A)$
$|A| = 1(0-3) - 2(0-3) + 1(2-1) = -3 + 6 + 1 = 4$
Finding cofactors: $C_{11} = -3$, $C_{12} = 3$, $C_{13} = 1$ $C_{21} = 1$, $C_{22} = -1$, $C_{23} = 1$ $C_{31} = 5$, $C_{32} = -1$, $C_{33} = -3$
$\text{adj}(A) = \begin{bmatrix} -3 & 1 & 5 \\ 3 & -1 & -1 \\ 1 & 1 & -3 \end{bmatrix}$
$A^{-1} = \frac{1}{4}\begin{bmatrix} -3 & 1 & 5 \\ 3 & -1 & -1 \\ 1 & 1 & -3 \end{bmatrix}$

## Question 2(b)(3) [4 marks]

**Solve the equations** $2x + 3y = 7$ **and** $4x = 9 + y$ **using matrix method**

**Solution**

Rewriting: $2x + 3y = 7$ and $4x - y = 9$
In matrix form: $\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$
$|A| = 2(-1) - 3(4) = -2 - 12 = -14$
$A^{-1} = \frac{1}{-14}\begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix}$
$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-14}\begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 9 \end{bmatrix} = \frac{1}{-14}\begin{bmatrix} -34 \\ -10 \end{bmatrix}$
Therefore: $x = \frac{34}{14} = \frac{17}{7}$, $y = \frac{10}{14} = \frac{5}{7}$

## Question 3(a) [6 marks]

**Attempt any two**

## Question 3(a)(1) [3 marks]

**If** $y = x^x$**, then find** $\frac{dy}{dx}$

**Solution**

Taking natural logarithm: $\ln y = x \ln x$
Differentiating both sides: $\frac{1}{y}\frac{dy}{dx} = \ln x + x \cdot \frac{1}{x} = \ln x + 1$
$\frac{dy}{dx} = y(\ln x + 1) = x^x(\ln x + 1)$

## Question 3(a)(2) [3 marks]

**If $y = \log(x + \sqrt{x^2 + a^2})$, then find $\frac{dy}{dx}$**

**Solution**

$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+a^2}} \cdot \frac{d}{dx}(x + \sqrt{x^2 + a^2})$
$\frac{d}{dx}(x + \sqrt{x^2 + a^2}) = 1 + \frac{2x}{2\sqrt{x^2+a^2}} = 1 + \frac{x}{\sqrt{x^2+a^2}} = \frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}$
$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+a^2}} \cdot \frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}} = \frac{1}{\sqrt{x^2+a^2}}$

## Question 3(a)(3) [3 marks]

**If $y = \operatorname{cosec}^{-1} x + \sec^{-1} x$, then find $\frac{dy}{dx}$**

**Solution**

$\frac{dy}{dx} = \frac{d}{dx}(\operatorname{cosec}^{-1} x) + \frac{d}{dx}(\sec^{-1} x)$
$= -\frac{1}{|x|\sqrt{x^2-1}} + \frac{1}{|x|\sqrt{x^2-1}} = 0$

## Question 3(b) [8 marks]

**Attempt any two**

## Question 3(b)(1) [4 marks]

**Differentiate $y = \cos x$ using the definition**

**Solution**

By definition: $\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
$\frac{d}{dx}(\cos x) = \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h}$
Using the identity: $\cos(x + h) = \cos x \cos h - \sin x \sin h$
$= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} = \lim_{h\to 0} \frac{\cos x(\cos h-1)-\sin x \sin h}{h} = \cos x \lim_{h\to 0} \frac{\cos h-1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h} = \cos x \cdot 0 - \sin x \cdot 1 = -\sin x$

## Question 3(b)(2) [4 marks]

**Find the maximum and minimum value of $f(x) = x^3 - 4x^2 + 5x + 7$**

**Solution**

$f'(x) = 3x^2 - 8x + 5$
Setting $f'(x) = 0$: $3x^2 - 8x + 5 = 0 \implies (3x - 5)(x - 1) = 0$ $x = \frac{5}{3}$ or $x = 1$
$f''(x) = 6x - 8$
At $x = 1$: $f''(1) = 6(1) - 8 = -2 < 0$ (Maximum) At $x = \frac{5}{3}$: $f''\left(\frac{5}{3}\right) = 6\left(\frac{5}{3}\right) - 8 = 2 > 0$ (Minimum)
Maximum value: $f(1) = 1 - 4 + 5 + 7 = 9$ Minimum value: $f\left(\frac{5}{3}\right) = \left(\frac{5}{3}\right)^3 - 4\left(\frac{5}{3}\right)^2 + 5\left(\frac{5}{3}\right) + 7 = \frac{158}{27}$

## Question 3(b)(3) [4 marks]

**If $y = (\tan^{-1} x)^2$, then prove that $(1 + x^2)y_2 + 2x(1 + x^2)y_1 = 2$**

**Solution**

$y = (\tan^{-1} x)^2 \implies y_1 = \frac{dy}{dx} = 2(\tan^{-1} x) \cdot \frac{1}{1+x^2}$
$y_2 = \frac{d^2y}{dx^2} = 2\left[\frac{1}{1+x^2} \cdot \frac{1}{1+x^2} + (\tan^{-1} x) \cdot \frac{-2x}{(1+x^2)^2}\right] = \frac{2}{(1+x^2)^2} - \frac{4x(\tan^{-1} x)}{(1+x^2)^2}$
Now substituting in LHS: $(1+x^2)y_2+2x(1+x^2)y_1 = (1+x^2)\cdot\frac{2-4x(\tan^{-1} x)}{(1+x^2)^2} + 2x(1+x^2)\cdot\frac{2(\tan^{-1} x)}{1+x^2} = \frac{2-4x(\tan^{-1} x)}{1+x^2} + 4x(\tan^{-1} x) = \frac{2-4x(\tan^{-1} x)+4x(\tan^{-1} x)(1+x^2)}{1+x^2} = \frac{2+4x^3(\tan^{-1} x)}{1+x^2}$
Wait, let me re-evaluate the substitution carefully. LHS $= (1+x^2)y_2+2x(1+x^2)y_1$. Note that the standard textbook problem is usually $(1+x^2)^2y_2+2x(1+x^2)y_1 = 2$. However, the question says $(1+x^2)y_2+2x(1+x^2)y_1$. Let's see if $(1+x^2)y_1 = 2\tan^{-1} x$. Diff again: $(1+x^2)y_2+2xy_1 = \frac{2}{1+x^2}$. Multiply by $(1+x^2)$: $(1+x^2)^2y_2+2x(1+x^2)y_1 = 2$. The question likely meant $(1+x^2)^2y_2$ OR it meant $(1+x^2)y_2+2xy_1$ equal to something else. BUT, looking at the MDX solution: The MDX solution ends with: " $= \frac{2}{1+x^2}\cdot(1+x^2) = 2$ ". This implies the term was indeed forming 2. Let's check the MDX step: "$= \frac{2-4x(\tan^{-1} x)}{1+x^2} + 4x(\tan^{-1} x)$" "$= \frac{2-4x(\tan^{-1} x)+4x(\tan^{-1} x)(1+x^2)}{1+x^2}$" This algebra seems weird in the MDX. $4x(\tan^{-1} x)(1+x^2)$ would not cancel $-4x(\tan^{-1} x)$ cleanly unless $x^2$ term is handled. Calculated value: $\frac{2-4x\tan^{-1} x+4x\tan^{-1} x+4x^3\tan^{-1} x}{1+x^2}$. This is not 2.
There is a discrepancy in the MDX algebra or the question statement. However, User requires strict fidelity to the MDX text. "Migrate the **EXACT** text content from MDX to LaTeX." I will copy the MDX solution steps exactly, even if they look mathematically dubious, as per instructions. MDX text: $= \frac{2-4x(\tan^{-1} x)+4x(\tan^{-1} x)(1+x^2)}{1+x^2} = \frac{2}{1+x^2}\cdot(1+x^2) = 2$ ✓

## Question 4(a) [6 marks]

**Attempt any two**

## Question 4(a)(1) [3 marks]

**Integrate:** $\int \frac{x^5}{1+x^{12}}dx$

**Solution**

Let $u = x^6$, then $du = 6x^5dx$, so $x^5dx = \frac{1}{6}du$
$\int \frac{x^5}{1+x^{12}}dx = \int \frac{1}{1+u^2}\cdot\frac{1}{6}du = \frac{1}{6}\tan^{-1} u + C = \frac{1}{6}\tan^{-1}(x^6) + C$

## Question 4(a)(2) [3 marks]

**Integrate:** $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$

**Solution**

Let $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$
Using property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$:
$I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2-x)}}{\sqrt{\sin(\pi/2-x)}+\sqrt{\cos(\pi/2-x)}}dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$
Adding both expressions: $2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx = \int_0^{\pi/2} 1\,dx = \frac{\pi}{2}$

Therefore: $I = \frac{\pi}{4}$

## Question 4(a)(3) [3 marks]

**If the mean of the following data is 19, then find missing frequency**

**Solution**

**Table 1.** Frequency Distribution

| $x_i$ | 6 | 10 | 14 | 18 | 24 | 28 | 30 |
|---|---|---|---|---|---|---|---|
| $f_i$ | 2 | 4 | 7 | f | 8 | 4 | 3 |

Mean $= \frac{\sum f_i x_i}{\sum f_i} = 19$
$\sum f_i = 2+4+7+f+8+4+3 = 28+f$ $\sum f_i x_i = 2(6)+4(10)+7(14)+f(18)+8(24)+4(28)+3(30)$
$= 12+40+98+18f+192+112+90 = 544+18f$
$\frac{544+18f}{28+f} = 19$ $544+18f = 19(28+f)$ $544+18f = 532+19f$ $12 = f$
Therefore, $f = 12$

## Question 4(b) [8 marks]

**Attempt any two**

## Question 4(b)(1) [4 marks]

**Integrate:** $\int \frac{x}{(x+1)(x+2)} dx$

**Solution**

Using partial fractions: $\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$
$x = A(x+2) + B(x+1)$
Setting $x = -1$: $-1 = A(1) \implies A = -1$ Setting $x = -2$: $-2 = B(-1) \implies B = 2$
$\int \frac{x}{(x+1)(x+2)} dx = \int \left(\frac{-1}{x+1} + \frac{2}{x+2}\right) dx = -\ln|x+1| + 2\ln|x+2| + C = \ln\left|\frac{(x+2)^2}{x+1}\right| + C$

## Question 4(b)(2) [4 marks]

**Integrate:** $\int \frac{x^2 \tan^{-1} x^3}{1+x^6} dx$

**Solution**

Let $u = x^3$, then $du = 3x^2 dx$, so $x^2 dx = \frac{1}{3} du$
$\int \frac{x^2 \tan^{-1} x^3}{1+x^6} dx = \int \frac{\tan^{-1} u}{1+u^2} \cdot \frac{1}{3} du$
Let $v = \tan^{-1} u$, then $dv = \frac{1}{1+u^2} du$
$= \frac{1}{3} \int v\, dv = \frac{1}{3} \cdot \frac{v^2}{2} + C = \frac{(\tan^{-1} u)^2}{6} + C$
$= \frac{(\tan^{-1} x^3)^2}{6} + C$

## Question 4(b)(3) [4 marks]

**Find the standard deviation for the following data: 10, 15, 7, 19, 9, 21, 23, 25, 26, 30**

**Solution**

First, find the mean: $\bar{x} = \frac{10+15+7+19+9+21+23+25+26+30}{10} = \frac{185}{10} = 18.5$
**Table for Standard Deviation:**

**Table 2.** Standard Deviation Calculation

| $x_i$ | $x_i - \bar{x}$ | $(x_i - \bar{x})^2$ |
|---|---|---|
| 10 | -8.5 | 72.25 |
| 15 | -3.5 | 12.25 |
| 7 | -11.5 | 132.25 |
| 19 | 0.5 | 0.25 |
| 9 | -9.5 | 90.25 |
| 21 | 2.5 | 6.25 |
| 23 | 4.5 | 20.25 |
| 25 | 6.5 | 42.25 |
| 26 | 7.5 | 56.25 |
| 30 | 11.5 | 132.25 |

$\sum(x_i - \bar{x})^2 = 564.5$
Standard deviation $= \sqrt{\frac{\sum(x_i-\bar{x})^2}{n}} = \sqrt{\frac{564.5}{10}} = \sqrt{56.45} = 7.51$

## Question 5(a) [6 marks]

**Attempt any two**

## Question 5(a)(1) [3 marks]

**Find the standard deviation for the following data:**

**Solution**

**Table 3.** Data

| $x_i$ | 4 | 8 | 11 | 17 | 20 | 24 | 32 |
|---|---|---|---|---|---|---|---|
| $f_i$ | 3 | 5 | 9 | 5 | 4 | 3 | 1 |

$N = \sum f_i = 3 + 5 + 9 + 5 + 4 + 3 + 1 = 30$
**Mean Calculation:** $\bar{x} = \frac{\sum f_i x_i}{N} = \frac{3(4)+5(8)+9(11)+5(17)+4(20)+3(24)+1(32)}{30} = \frac{12+40+99+85+80+72+32}{30} = \frac{420}{30} = 14$
**Standard Deviation Table:**

**Table 4.** Standard Deviation Calculation

| $x_i$ | $f_i$ | $x_i - \bar{x}$ | $(x_i - \bar{x})^2$ | $f_i(x_i - \bar{x})^2$ |
|---|---|---|---|---|
| 4 | 3 | -10 | 100 | 300 |
| 8 | 5 | -6 | 36 | 180 |
| 11 | 9 | -3 | 9 | 81 |
| 17 | 5 | 3 | 9 | 45 |
| 20 | 4 | 6 | 36 | 144 |
| 24 | 3 | 10 | 100 | 300 |
| 32 | 1 | 18 | 324 | 324 |

$\sum f_i(x_i - \bar{x})^2 = 1374$

Standard deviation $= \sqrt{\frac{\sum f_i(x_i-\bar{x})^2}{N}} = \sqrt{\frac{1374}{30}} = \sqrt{45.8} = 6.77$

## Question 5(a)(2) [3 marks]

**Find the standard deviation for the following data:**

**Solution**

**Table 5.** Grouped Data

| Class | 0-10 | 10-20 | 20-30 | 30-40 | 40-50 |
|---|---|---|---|---|---|
| Frequency | 5 | 8 | 15 | 16 | 6 |

First, find class midpoints and calculate mean:

**Table 6.** Midpoint Calculation

| Class | Midpoint ($x_i$) | $f_i$ | $f_i x_i$ |
|---|---|---|---|
| 0-10 | 5 | 5 | 25 |
| 10-20 | 15 | 8 | 120 |
| 20-30 | 25 | 15 | 375 |
| 30-40 | 35 | 16 | 560 |
| 40-50 | 45 | 6 | 270 |

$N = 50$, $\sum f_i x_i = 1350$ $\bar{x} = \frac{1350}{50} = 27$

**Standard Deviation Table:**

**Table 7.** Standard Deviation Calculation

| $x_i$ | $f_i$ | $x_i - \bar{x}$ | $(x_i - \bar{x})^2$ | $f_i(x_i - \bar{x})^2$ |
|---|---|---|---|---|
| 5 | 5 | -22 | 484 | 2420 |
| 15 | 8 | -12 | 144 | 1152 |
| 25 | 15 | -2 | 4 | 60 |
| 35 | 16 | 8 | 64 | 1024 |
| 45 | 6 | 18 | 324 | 1944 |

$\sum f_i(x_i - \bar{x})^2 = 6600$

Standard deviation $= \sqrt{\frac{6600}{50}} = \sqrt{132} = 11.49$

## Question 5(a)(3) [3 marks]

**Find the mean for the following data:**

**Solution**

**Table 8.** Grouped Frequency Distribution

| Class | 30-40 | 40-50 | 50-60 | 60-70 | 70-80 | 80-90 | 90-100 |
|---|---|---|---|---|---|---|---|
| Frequency | 3 | 7 | 12 | 15 | 8 | 3 | 2 |

Using midpoint method:

**Table 9.** Mean Calculation

| Class | Midpoint ($x_i$) | $f_i$ | $f_i x_i$ |
|---|---|---|---|
| 30-40 | 35 | 3 | 105 |
| 40-50 | 45 | 7 | 315 |
| 50-60 | 55 | 12 | 660 |
| 60-70 | 65 | 15 | 975 |
| 70-80 | 75 | 8 | 600 |
| 80-90 | 85 | 3 | 255 |
| 90-100 | 95 | 2 | 190 |

$N = \sum f_i = 50$ $\sum f_i x_i = 3100$
Mean $= \frac{\sum f_i x_i}{N} = \frac{3100}{50} = 62$

## Question 5(b) [8 marks]

**Attempt any two**

## Question 5(b)(1) [4 marks]

**Solve:** $xy\,dx - (y^2 + x^2)\,dy = 0$

**Solution**

Rearranging: $xy\,dx = (y^2 + x^2)\,dy$ $\frac{dx}{dy} = \frac{y^2+x^2}{xy} = \frac{y}{x} + \frac{x}{y}$
This is a homogeneous differential equation. Let $x = vy$, then $\frac{dx}{dy} = v + y\frac{dv}{dy}$
Substituting: $v + y\frac{dv}{dy} = \frac{y}{vy} + \frac{vy}{y} = \frac{1}{v} + v$
$y\frac{dv}{dy} = \frac{1}{v} \implies v\,dv = \frac{dy}{y}$
Integrating both sides: $\int v\,dv = \int \frac{dy}{y} \implies \frac{v^2}{2} = \ln|y| + C$
Substituting back $v = \frac{x}{y}$: $\frac{x^2}{2y^2} = \ln|y| + C$ $x^2 = 2y^2(\ln|y| + C)$

## Question 5(b)(2) [4 marks]

**Solve:** $\frac{dy}{dx} + \frac{2y}{x} = \sin x$

**Solution**

This is a linear differential equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$ where $P(x) = \frac{2}{x}$ and $Q(x) = \sin x$
Integrating factor $= e^{\int P(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = x^2$
Multiplying the equation by integrating factor: $x^2\frac{dy}{dx} + 2xy = x^2 \sin x$
The left side is $\frac{d}{dx}(x^2 y)$: $\frac{d}{dx}(x^2 y) = x^2 \sin x$
Integrating both sides: $x^2 y = \int x^2 \sin x\, dx$
Using integration by parts twice: $\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2\cos x + C$
Therefore: $x^2 y = -x^2 \cos x + 2x \sin x + 2\cos x + C$ $y = -\cos x + \frac{2\sin x}{x} + \frac{2\cos x}{x^2} + \frac{C}{x^2}$

## Question 5(b)(3) [4 marks]

**Solve:** $(1 + x^2)\frac{dy}{dx} + 2xy = \cos x$

**Solution**

Dividing by $(1 + x^2)$: $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{\cos x}{1+x^2}$
This is linear with $P(x) = \frac{2x}{1+x^2}$ and $Q(x) = \frac{\cos x}{1+x^2}$
Integrating factor $= e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)} = 1 + x^2$
Multiplying by integrating factor: $(1 + x^2)\frac{dy}{dx} + 2xy = \cos x$
The left side is $\frac{d}{dx}[(1 + x^2)y]$: $\frac{d}{dx}[(1 + x^2)y] = \cos x$
Integrating: $(1 + x^2)y = \int \cos x\, dx = \sin x + C$
Therefore: $y = \frac{\sin x + C}{1+x^2}$

## Complete Formula Sheet

### Matrix Operations

- **Transpose**: $(A^T)_{ij} = A_{ji}$
- **Inverse**: $A^{-1} = \frac{1}{|A|}\text{adj}(A)$
- **Properties**: $(A + B)^T = A^T + B^T$

### Derivatives

- **Power Rule**: $\frac{d}{dx}(x^n) = nx^{n-1}$
- **Trigonometric**: $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$
- **Inverse Trig**: $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
- **Logarithmic**: $\frac{d}{dx}(\ln x) = \frac{1}{x}$

### Integration

- **By Parts**: $\int u\, dv = uv - \int v\, du$
- **Substitution**: If $u = g(x)$, then $\int f(g(x))g'(x)dx = \int f(u)du$
- **Definite Properties**: $\int_0^a f(x)dx = \int_0^a f(a - x)dx$

### Differential Equations

- **Linear Form**: $\frac{dy}{dx} + P(x)y = Q(x)$
- **Integrating Factor**: $e^{\int P(x)dx}$
- **Variable Separable**: $\frac{dy}{dx} = f(x)g(y)$

### Statistics

- **Mean**: $\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$

- **Standard Deviation**: $\sigma = \sqrt{\frac{\sum f_i(x_i - \bar{x})^2}{N}}$
- **Variance**: $\sigma^2 = \frac{\sum f_i(x_i - \bar{x})^2}{N}$