Subject Name Solutions

4320002 – Winter 2023

Semester 1 Study Material

Detailed Solutions and Explanations

Q.1 Fill in the blanks [14 marks]

0.0.1 Q1.1 [1 mark]

Order of the matrix $\begin{bmatrix} 2 & 5 \\ 7 & 8 \end{bmatrix}$ is ______________

Solution

(d) 2×2
Solution: The matrix has 2 rows and 2 columns, so its order is 2×2.

0.0.2 Q1.2 [1 mark]

**$

$$\begin{bmatrix} 4 & 3 \\ 6 & 2 \end{bmatrix}$$

- $\begin{bmatrix} 1 & 5 \\ 5 & 8 \end{bmatrix}$

= $ ______________**

Solution

(a) $\begin{bmatrix} 5 & 8 \\ 11 & 10 \end{bmatrix}$
Solution: $\begin{bmatrix} 4 & 3 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 5 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 4+1 & 3+5 \\ 6+5 & 2+8 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 11 & 10 \end{bmatrix}$

0.0.3 Q1.3 [1 mark]

Which of the following is a square matrix?

Solution

(c) $\begin{bmatrix} 1 & 3 \\ 5 & 4 \end{bmatrix}$
Solution: A square matrix has equal number of rows and columns. Only option (c) has 2×2 dimensions.

0.0.4 Q1.4 [1 mark]

If $A = [3]$ and $B = [4]$ then $A $\cdot B = $______________

Solution

(b) 12
Solution: $A \cdot B = [3] \times [4] = [3 \times 4] = [12] = 12$

0.0.5 Q1.5 [1 mark]

$d $\frac{}{dx \sin x = \$\_\_\_\_\_\_\_\_\_\_\_\_}$

Solution

(d) $\cos x$
Solution: The derivative of $\sin x$ **is** $\cos x$**.**

0.0.6 Q1.6 [1 mark]

If $f(x) = xe^x$ **then $f'(0) = $** ______________

Solution

(b) 1
Solution: Using product rule: $f'(x) = \frac{d}{dx}(xe^x) = e^x + xe^x = e^x(1+x)$ $f'(0) = e^0(1+0) = 1 \times 1 = 1$

0.0.7 Q1.7 [1 mark]

If $y = x^2$ **then $d**$^2y_{\overline{dx^2=\$}}$______________

Solution

(b) 2
Solution: $y = x^2$ $\frac{dy}{dx} = 2x$ $\frac{d^2y}{dx^2} = 2$

0.0.8 Q1.8 [1 mark]

**$**$\int \cos x dx = \$$______________ $+c$

Solution

(a) $\sin x$
Solution: $\int \cos x dx = \sin x + c$

0.0.9 Q1.9 [1 mark]

$**$\int$ ***0^1 x dx = $* ______________

Solution

(c) $\frac{1}{2}$
Solution: $\int_0^1 x dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$

0.0.10 Q1.10 [1 mark]

**$**$\int \frac{1}{1+x^2} dx = \$$______________ $+c$

Solution

(a) $\tan^{-1} x$
Solution: $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

0.0.11 Q1.11 [1 mark]

Order of differential equation $x \sin y + xy = x$ **is** ______________

Solution

(b) 1
Solution: The equation can be written as $\frac{dy}{dx} = \frac{1-xy}{\sin y}$**. The highest order derivative is first order.**

0.0.12 Q1.12 [1 mark]

Integration factor of $\frac{dy}{dx} + y = x$ is ______________

Solution

(d) e^x
Solution: For $\frac{dy}{dx} + Py = Q$, **integration factor** $= e^{\int P dx} = e^{\int 1 dx} = e^x$

0.0.13 Q1.13 [1 mark]

$i^2 = $ ______________

Solution

(b) -1
Solution: By definition, $i^2 = -1$

0.0.14 Q1.14 [1 mark]

$(2+3i)(2-3i) = $ ______________

Solution

(c) 13
Solution: $(2+3i)(2-3i) = 2^2 - (3i)^2 = 4 - 9i^2 = 4 - 9(-1) = 4 + 9 = 13$

Q.2(A) Attempt any two [6 marks]

0.0.15 Q2.1(A)(1) [3 marks]

If $A = \begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 8 \\ 4 & 6 \end{bmatrix}$ **and** $C = \begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix}$ **then find** $2A + 3B - C$

Solution: $2A = 2\begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ -2 & 6 \end{bmatrix}$

$3B = 3\begin{bmatrix} 5 & 8 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 15 & 24 \\ 12 & 18 \end{bmatrix}$

$2A + 3B = \begin{bmatrix} 4 & 10 \\ -2 & 6 \end{bmatrix} + \begin{bmatrix} 15 & 24 \\ 12 & 18 \end{bmatrix} = \begin{bmatrix} 19 & 34 \\ 10 & 24 \end{bmatrix}$

$2A + 3B - C = \begin{bmatrix} 19 & 34 \\ 10 & 24 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 15 & 32 \\ 9 & 19 \end{bmatrix}$

0.0.16 Q2.1(A)(2) [3 marks]

If $M = \begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix}$ **and** $N = \begin{bmatrix} 6 & 9 \\ 0 & 5 \end{bmatrix}$ **then prove that** $(M + N)^T = M^T + N^T$

Solution: $M + N = \begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix} + \begin{bmatrix} 6 & 9 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 13 \\ 3 & 12 \end{bmatrix}$

$(M + N)^T = \begin{bmatrix} 7 & 3 \\ 13 & 12 \end{bmatrix}$

$M^T = \begin{bmatrix} 1 & 3 \\ 4 & 7 \end{bmatrix}$, $N^T = \begin{bmatrix} 6 & 0 \\ 9 & 5 \end{bmatrix}$

$M^T + N^T = \begin{bmatrix} 1 & 3 \\ 4 & 7 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 9 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 13 & 12 \end{bmatrix}$

Hence, $(M + N)^T = M^T + N^T$ **is proved.**

0.0.17 Q2.1(A)(3) [3 marks]

Solve differential equation: $x\frac{dy}{dx} + y = xy$

Solution: $x\frac{dy}{dx} + y = xy$ $\frac{dy}{dx} + \frac{y}{x} = y$ $\frac{dy}{dx} = y - \frac{y}{x} = y\left(1 - \frac{1}{x}\right) = y\left(\frac{x-1}{x}\right)$

Separating variables: $\frac{dy}{y} = \frac{x-1}{x}dx$

Integrating: $\ln|y| = \int \frac{x-1}{x}dx = \int \left(1 - \frac{1}{x}\right)dx = x - \ln|x| + C$

$y = Ae^{x-\ln|x|} = A\frac{e^x}{x}$

Q.2(B) Attempt any two [8 marks]

0.0.18 Q2.1(B)(1) [4 marks]

Solve equations $2x + 3y = 8$**,** $3x + 4y = 11$ **using matrix method**

Solution: Writing in matrix form: $AX = B$ $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$

Finding A^{-1}**:** $|A| = 2(4) - 3(3) = 8 - 9 = -1$

$A^{-1} = \frac{1}{|A|}\begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$

$X = A^{-1}B = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 8 \\ 11 \end{bmatrix} = \begin{bmatrix} -32 + 33 \\ 24 - 22 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Therefore: $x = 1,$

y = 2

0.0.19 Q2.1(B)(2) [4 marks]

If $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ **then prove that** $(AB)^T = B^T A^T$

Solution: $AB = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 1 & 6 \end{bmatrix}$

$(AB)^T = \begin{bmatrix} 3 & 1 \\ 8 & 6 \end{bmatrix}$

$A^T = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$, $B^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$B^T A^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 6 \end{bmatrix}$

Hence, $(AB)^T = B^T A^T$ **is proved.**

0.0.20 Q2.1(B)(3) [4 marks]

If $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ **then prove that** $A^2 - 4A + 7I = O$

Solution: $A^2 = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix}$

$4A = 4\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ -4 & 8 \end{bmatrix}$

$7I = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$A^2 - 4A + 7I = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ -4 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

Hence proved.

Q.3(A) Attempt any two [6 marks]

0.0.21 Q3.1(A)(1) [3 marks]

Find derivative of $f(x) = e^x$ **using definition of differentiation**

Solution: Using definition: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
$f'(x) = \lim_{h\to 0} \frac{e^{x+h}-e^x}{h} = \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h}$
$= \lim_{h\to 0} \frac{e^x(e^h-1)}{h} = e^x \lim_{h\to 0} \frac{e^h-1}{h}$
Since $\lim_{h\to 0} \frac{e^h-1}{h} = 1$
Therefore: $f'(x) = e^x$

0.0.22 Q3.1(A)(2) [3 marks]

If $y = \log(\sin x)$ **then find** $\frac{dy}{dx}$
Solution: $y = \log(\sin x)$
Using chain rule: $\frac{dy}{dx} = \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x} = \cot x$

0.0.23 Q3.1(A)(3) [3 marks]

Evaluate: $\int \left(4x^3 + 3x^2 + \frac{2}{x}\right) dx$
Solution: $\int \left(4x^3 + 3x^2 + \frac{2}{x}\right) dx$
$= \int 4x^3 dx + \int 3x^2 dx + \int \frac{2}{x} dx$
$= 4 \cdot \frac{x^4}{4} + 3 \cdot \frac{x^3}{3} + 2\ln|x| + C$
$= x^4 + x^3 + 2\ln|x| + C$

Q.3(B) Attempt any two [8 marks]

0.0.24 Q3.1(B)(1) [4 marks]

If $y = e^{\tan x} + \log(\sin x)$ **then find** $\frac{dy}{dx}$
Solution: $y = e^{\tan x} + \log(\sin x)$
$\frac{dy}{dx} = \frac{d}{dx}[e^{\tan x}] + \frac{d}{dx}[\log(\sin x)]$
For first term: $\frac{d}{dx}[e^{\tan x}] = e^{\tan x} \cdot \sec^2 x$
For second term: $\frac{d}{dx}[\log(\sin x)] = \frac{1}{\sin x} \cdot \cos x = \cot x$
Therefore: $\frac{dy}{dx} = e^{\tan x} \sec^2 x + \cot x$

0.0.25 Q3.1(B)(2) [4 marks]

The equation of motion of a particle is $s = t^4 + 3t$**. Find its velocity and acceleration at** $t = 2$ **sec**
Solution: Given: $s = t^4 + 3t$
Velocity: $v = \frac{ds}{dt} = 4t^3 + 3$
At $t = 2$**:** $v = 4(2)^3 + 3 = 4(8) + 3 = 32 + 3 = 35$ **units/sec**
Acceleration: $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 12t^2$
At $t = 2$**:** $a = 12(2)^2 = 12(4) = 48$ **units/sec^2**

0.0.26 Q3.1(B)(3) [4 marks]

Find the maximum and minimum value of the function $f(x) = 2x^3 - 3x^2 - 12x + 5$
Solution: $f(x) = 2x^3 - 3x^2 - 12x + 5$
$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$
For critical points: $f'(x) = 0$ $6(x-2)(x+1) = 0$ $x = 2$ **or** $x = -1$
$f''(x) = 12x - 6$
At $x = -1$**:** $f''(-1) = 12(-1) - 6 = -18 < 0$ **(Maximum) At** $x = 2$**:** $f''(2) = 12(2) - 6 = 18 > 0$ **(Minimum)**
$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 5 = -2 - 3 + 12 + 5 = 12$ **(Maximum)** $f(2) = 2(8) - 3(4) - 12(2) + 5 = 16 - 12 - 24 + 5 = -15$ **(Minimum)**
Maximum value: 12 at $x = -1$ **Minimum value: -15 at** $x = 2$

Q.4(A) Attempt any two [6 marks]

0.0.27 Q4.1(A)(1) [3 marks]

Evaluate: $\int xe^x dx$

Solution: Using integration by parts: $\int udv = uv - \int vdu$

Let $u = x$, $dv = e^x dx$ **Then** $du = dx$, $v = e^x$

$\int xe^x dx = x \cdot e^x - \int e^x dx = xe^x - e^x + C = e^x(x-1) + C$

0.0.28 Q4.1(A)(2) [3 marks]

Evaluate: $\int \frac{dx}{\sqrt{9-4x^2}}$

Solution: $\int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{dx}{\sqrt{9(1-\frac{4x^2}{9})}} = \int \frac{dx}{3\sqrt{1-(\frac{2x}{3})^2}}$

Let $\frac{2x}{3} = \sin\theta$, **then** $x = \frac{3\sin\theta}{2}$, $dx = \frac{3\cos\theta}{2}d\theta$

$= \int \frac{\frac{3\cos\theta}{2}d\theta}{3\sqrt{1-\sin^2\theta}} = \int \frac{\frac{3\cos\theta}{2}d\theta}{3\cos\theta} = \int \frac{1}{2}d\theta = \frac{\theta}{2} + C$

$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C$

0.0.29 Q4.1(A)(3) [3 marks]

Find complex conjugate of $\frac{1-i}{1+i}$

Solution: $\frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{(1-i)^2}{1-i^2} = \frac{1-2i+i^2}{1-(-1)} = \frac{1-2i-1}{2} = \frac{-2i}{2} = -i$

Complex conjugate of $-i$ **is** $\overline{-i} = i$

Q.4(B) Attempt any two [8 marks]

0.0.30 Q4.1(B)(1) [4 marks]

Evaluate: $\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$

Solution: Let $I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$

Using property: $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$I = \int_0^{\pi/2} \frac{\sqrt{\cos(\pi/2-x)}}{\sqrt{\cos(\pi/2-x)}+\sqrt{\sin(\pi/2-x)}}dx = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$

Adding both expressions: $2I = \int_0^{\pi/2} \frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx = \int_0^{\pi/2} 1dx = \frac{\pi}{2}$

Therefore: $I = \frac{\pi}{4}$

0.0.31 Q4.1(B)(2) [4 marks]

Find the area of circle $x^2 + y^2 = a^2$ **using integration**

Solution: For circle $x^2 + y^2 = a^2$, **we have** $y = \pm\sqrt{a^2 - x^2}$

Area of circle $= 4\times$ **Area in first quadrant** $= 4\int_0^a \sqrt{a^2 - x^2}dx$

Let $x = a\sin\theta$, $dx = a\cos\theta d\theta$ **When** $x = 0$, $\theta = 0$; **when** $x = a$, $\theta = \pi/2$

$= 4\int_0^{\pi/2} \sqrt{a^2 - a^2\sin^2\theta} \cdot a\cos\theta d\theta = 4\int_0^{\pi/2} a\cos\theta \cdot a\cos\theta d\theta = 4a^2\int_0^{\pi/2} \cos^2\theta d\theta = 4a^2 \cdot \frac{\pi}{4} = \pi a^2$

0.0.32 Q4.1(B)(3) [4 marks]

Simplify: $\frac{(\cos 3\theta+i\sin 3\theta)^4 \cdot (\cos\theta - i\sin\theta)^5}{(\cos 2\theta - i\sin 2\theta)^3 \cdot (\cos 12\theta + i\sin 12\theta)}$

Solution: Using De Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Numerator: $(\cos 3\theta + i\sin 3\theta)^4 \cdot (\cos\theta - i\sin\theta)^5 = (\cos 12\theta + i\sin 12\theta) \cdot (\cos(-5\theta) + i\sin(-5\theta)) = \cos(12\theta - 5\theta) + i\sin(12\theta - 5\theta) = \cos 7\theta + i\sin 7\theta$

Denominator: $(\cos 2\theta - i\sin 2\theta)^3 \cdot (\cos 12\theta + i\sin 12\theta) = (\cos(-6\theta) + i\sin(-6\theta)) \cdot (\cos 12\theta + i\sin 12\theta) = \cos(-6\theta + 12\theta) + i\sin(-6\theta + 12\theta) = \cos 6\theta + i\sin 6\theta$

Result: $\frac{\cos 7\theta + i\sin 7\theta}{\cos 6\theta + i\sin 6\theta} = \cos(7\theta - 6\theta) + i\sin(7\theta - 6\theta) = \cos\theta + i\sin\theta$

Q.5(A) Attempt any two [6 marks]

0.0.33 Q5.1(A)(1) [3 marks]

If $(3x-7)+2iy=5y+(5+x)i$ **then find value of x and y**

Solution: $(3x-7)+2iy=5y+(5+x)i$

Comparing real and imaginary parts: Real parts: $3x-7=5y$ **... (1) Imaginary parts:** $2y=5+x$ **... (2)**

From equation (2): $x=2y-5$ **... (3)**

Substituting (3) in (1): $3(2y-5)-7=5y$ $6y-15-7=5y$ $6y-22=5y$ $y=22$

From (3): $x=2(22)-5=44-5=39$

Therefore: $x=39$,

y = 22

0.0.34 Q5.1(A)(2) [3 marks]

Convert $z=1+\sqrt{3}i$ **into polar form**

Solution: $z=1+\sqrt{3}i$

Modulus: $|z|=\sqrt{1^2+(\sqrt{3})^2}=\sqrt{1+3}=\sqrt{4}=2$

Argument: $\arg(z)=\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)=\tan^{-1}(\sqrt{3})=\frac{\pi}{3}$

Polar form: $z=|z|(\cos\theta+i\sin\theta)=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

0.0.35 Q5.1(A)(3) [3 marks]

Express $\frac{4+2i}{(3+2i)(5-3i)}$ **in** $a+ib$ **form**

Solution: First, simplify denominator: $(3+2i)(5-3i)=15-9i+10i-6i^2=15+i-6(-1)=15+i+6=21+i$

$\frac{4+2i}{21+i}=\frac{(4+2i)(21-i)}{(21+i)(21-i)}=\frac{84-4i+42i-2i^2}{21^2-i^2}=\frac{84+38i+2}{441+1}=\frac{86+38i}{442}$

$=\frac{86}{442}+\frac{38}{442}i=\frac{43}{221}+\frac{19}{221}i$

Q.5(B) Attempt any two [8 marks]

0.0.36 Q5.1(B)(1) [4 marks]

Solve differential equation: $\frac{dy}{dx}+2y=3e^x$

Solution: This is a first-order linear differential equation of the form $\frac{dy}{dx}+Py=Q$

Here: $P=2$, $Q=3e^x$

Integration factor: $\mu=e^{\int P\,dx}=e^{\int 2\,dx}=e^{2x}$

Multiplying equation by μ**:** $e^{2x}\frac{dy}{dx}+2e^{2x}y=3e^{2x}\cdot e^x=3e^{3x}$

This gives: $\frac{d}{dx}(ye^{2x})=3e^{3x}$

Integrating both sides: $ye^{2x}=\int 3e^{3x}dx=3\cdot\frac{e^{3x}}{3}+C=e^{3x}+C$

Therefore: $y=\frac{e^{3x}+C}{e^{2x}}=e^x+Ce^{-2x}$

0.0.37 Q5.1(B)(2) [4 marks]

Solve differential equation: $\frac{dy}{dx}=(x+y)^2$

Solution: Let $v=x+y$**, then** $\frac{dv}{dx}=1+\frac{dy}{dx}$

So $\frac{dy}{dx}=\frac{dv}{dx}-1$

Substituting in the original equation: $\frac{dv}{dx}-1=v^2$ $\frac{dv}{dx}=v^2+1$

Separating variables: $\frac{dv}{v^2+1}=dx$

Integrating both sides: $\int\frac{dv}{v^2+1}=\int dx$ $\tan^{-1}(v)=x+C$ $v=\tan(x+C)$

Substituting back: $x+y=\tan(x+C)$ **Therefore:** $y=\tan(x+C)-x$

0.0.38 Q5.1(B)(3) [4 marks]

Solve differential equation: $\frac{dy}{dx} + \frac{y}{x} = e^x$**,** $y(0) = 2$

Solution: This is a first-order linear differential equation: $\frac{dy}{dx} + \frac{y}{x} = e^x$

Here: $P = \frac{1}{x}$**,** $Q = e^x$

Integration factor: $\mu = e^{\int \frac{1}{x} dx} = e^{\ln |x|} = |x| = x$ **(for** $x > 0$**)**

Multiplying equation by $\mu = x$**:** $x\frac{dy}{dx} +$

y = xe^x

This gives: $\frac{d}{dx}(xy) = xe^x$

Integrating both sides using integration by parts: $xy = \int xe^x dx$

For $\int xe^x dx$**: Let** $u = x$**,** $dv = e^x dx$ **Then** $du = dx$**,** $v = e^x$ $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x = e^x(x - 1)$

So: $xy = e^x(x - 1) + C$ $y = \frac{e^x(x-1)+C}{x}$

Using initial condition $y(0) = 2$**: This presents a problem as we have division by zero. The equation needs to be solved more carefully near** $x = 0$**.**

For the general solution: $y = e^x \left(1 - \frac{1}{x}\right) + \frac{C}{x}$

Formula Cheat Sheet

0.0.39 Matrix Operations

- **Matrix addition:** $(A + B)_{ij} = A_{ij} + B_{ij}$
- **Matrix multiplication:** $(AB)_{ij} = \sum_k A_{ik}B_{kj}$
- **Transpose:** $(A^T)_{ij} = A_{ji}$
- **Inverse of 2**$\times 2 matrix : A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ **where** $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

0.0.40 Differentiation Formulas

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- **Product rule:** $(uv)' = u'v + uv'$
- **Chain rule:** $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

0.0.41 Integration Formulas

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ **(for** $n \neq -1$**)**
- $\int \frac{1}{x} dx = \ln |x| + C$
- $\int e^x dx = e^x + C$
- $\int \sin x dx = -\cos x + C$
- $\int \cos x dx = \sin x + C$
- $\int \sec^2 x dx = \tan x + C$
- $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$
- $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

0.0.42 Differential Equations

- **First-order linear:** $\frac{dy}{dx} + Py = Q$
- **Integration factor:** $\mu = e^{\int P dx}$
- **Solution:** $y = \frac{1}{\mu} \left[\int \mu Q dx + C\right]$
- **Variable separable:** $\frac{dy}{dx} = f(x)g(y) \to \frac{dy}{g(y)} = f(x)dx$

0.0.43 Complex Numbers

- $i^2 = -1$, $i^3 = -i$, $i^4 = 1$
- **Modulus:** $|a + bi| = \sqrt{a^2 + b^2}$
- **Argument:** $\arg(a + bi) = \tan^{-1}\left(\frac{b}{a}\right)$
- **Polar form:** $z = r(\cos\theta + i\sin\theta)$
- **De Moivre's theorem:** $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Problem-Solving Strategies

0.0.44 Matrix Problems

1. Always check dimensions before performing operations
2. For matrix equations: Use inverse method $X = A^{-1}B$
3. For transpose properties: Use $(AB)^T = B^T A^T$
4. For matrix powers: Calculate step by step, look for patterns

0.0.45 Differentiation Problems

1. Identify the type: Product, quotient, chain rule, or implicit
2. For complex functions: Break down using appropriate rules
3. For applications: Remember $v = \frac{ds}{dt}$, $a = \frac{dv}{dt}$
4. For maxima/minima: Find critical points where $f'(x) = 0$

0.0.46 Integration Problems

1. Recognize standard forms first
2. For substitution: Look for $f'(x)$ when $f(x)$ appears
3. For integration by parts: Choose u as LIATE (Log, Inverse trig, Algebraic, Trig, Exponential)
4. For definite integrals: Use fundamental theorem or properties

0.0.47 Differential Equations

1. Identify the type: Linear, separable, or exact
2. For linear equations: Find integration factor systematically
3. For separable equations: Separate variables completely before integrating
4. Always check initial conditions if given

0.0.48 Complex Numbers

1. For operations: Convert to $a + bi$ form first
2. For polar form: Calculate modulus and argument carefully
3. For powers: Use De Moivre's theorem
4. For division: Multiply by conjugate of denominator

Common Mistakes to Avoid

0.0.49 Matrix Operations

- Don't assume $AB = BA$ (matrix multiplication is not commutative)
- Don't forget to check if matrices can be multiplied (inner dimensions must match)
- Don't confuse transpose with inverse

0.0.50 Differentiation

- Don't forget the chain rule for composite functions
- Don't mix up $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$
- Don't forget to use product rule when multiplying functions

0.0.51 Integration

- Don't forget the constant of integration $+C$
- Don't confuse indefinite and definite integrals
- Don't forget to substitute limits properly in definite integrals

0.0.52 Complex Numbers

- Don't forget $i^2 = -1$ when expanding
- Don't confuse modulus with real part
- Don't forget to rationalize denominators with complex numbers

---

Exam Tips

0.0.53 Time Management

- Spend 2-3 minutes reading the entire paper first
- Attempt easier questions first to build confidence
- Reserve 15 minutes at the end for review

0.0.54 Writing Strategy

- Show all steps clearly - partial marks are often awarded
- Draw diagrams where helpful - especially for geometry problems
- Write final answers clearly and box them if possible

0.0.55 Calculation Tips

- Double-check arithmetic - many marks are lost due to calculation errors
- Use calculator efficiently but don't become dependent on it
- Cross-verify answers using different methods when possible

0.0.56 Question Selection

- In OR questions, choose the one you're most confident about
- Don't spend too much time on any single question
- If stuck, move on and return later with fresh perspective

Good luck with your exam preparation!