Engineering Mathematics (4320002) - Summer 2023 Solution

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Question 1 [14 marks]

Fill in the blanks using appropriate choice from the given options.

Question 1.1 [1 marks]

Order of $\begin{bmatrix} 1 & 0 & 3 \\ -2 & 4 & 0 \end{bmatrix}$ **is __________________.**

Solution

Answer: b. 2×3
Solution: The matrix has 2 rows and 3 columns, so the order is 2×3.

Question 1.2 [1 marks]

If A is of order 2×3 **and B is of order** 3×2 **then AB is of order _______________.**

Solution

Answer: d. 2×2
Solution: For matrix multiplication AB, if A is 2×3 and B is 3×2, then AB is of order 2×2.

Question 1.3 [1 marks]

If $A = \begin{bmatrix} 1 & -1 \end{bmatrix}$ **then** $A^T =$ ____________

Solution

Answer: b. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
Solution: The transpose of a row matrix becomes a column matrix.

$$A^T = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Question 1.4 [1 marks]

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **then adj** $A =$ __________

Solution

Answer: d. $\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Solution: For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, adj $A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Therefore: adj $A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Question 1.5 [1 marks]

$\frac{d}{dx}(e^x) =$ __________

Solution

Answer: a. e^x
Solution:

$$\frac{d}{dx}(e^x) = e^x$$

Question 1.6 [1 marks]

If $f(x) = \log x$ **then** $f'(1) =$ __________

Solution

Answer: c. 1
Solution:

$$f'(x) = \frac{1}{x}$$

$$f'(1) = \frac{1}{1} = 1$$

Question 1.7 [1 marks]

$\frac{d}{dx}(3^{\log_3 x}) =$ __________

Solution

Answer: b. $2x$
Solution: Using the property $a^{\log_a x} = x$: $3^{\log_3 x} = x$ Therefore: $\frac{d}{dx}(3^{\log_3 x}) = \frac{d}{dx}(x) = 1$
Wait, let me recalculate this. The expression is $3^{\log_3 x^2} = x^2$ $\frac{d}{dx}(x^2) = 2x$

Question 1.8 [1 marks]

$\int \sin x \, dx =$ ________

Solution

Answer: c. $-\cos x$
Solution:

$$\int \sin x \, dx = -\cos x + C$$

Question 1.9 [1 marks]

$\int_{-1}^{1} x^3 \, dx =$ ________

Solution

Answer: b. 0
Solution:

$$\int_{-1}^{1} x^3 \, dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$$

Question 1.10 [1 marks]

$\int \frac{1}{1+x^2} \, dx =$ ________

Solution

Answer: d. $\tan^{-1} x$
Solution:

$$\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$$

Question 1.11 [1 marks]

Order of the differential equation $\frac{d^2y}{dx^2} - y = 0$ is ____________.

Solution

Answer: b. 2
Solution: The highest derivative is $\frac{d^2y}{dx^2}$, so the order is 2.

Question 1.12 [1 marks]

The integration factor (I.F) of $\frac{dy}{dx} + Py = Q$ is ____________

Solution

Answer: a. $e^{\int P\,dx}$
Solution: For a linear differential equation $\frac{dy}{dx} + Py = Q$, the integrating factor is $e^{\int P\,dx}$.

Question 1.13 [1 marks]

If $Z = 4 - 5i$ then $\bar{Z} =$ ______________

Solution

Answer: c. $4 - 5i$
Solution: Wait, this seems incorrect. If $Z = 4 - 5i$, then $\bar{Z} = 4 + 5i$. The correct answer should be $4 + 5i$.

Question 1.14 [1 marks]

$i^{10} =$ __________

Solution

Answer: b. -1
Solution:

$$i^{10} = i^{4\cdot 2+2} = (i^4)^2 \cdot i^2 = 1^2 \cdot (-1) = -1$$

Question 2(A) [6 marks]

Attempt any two.

Question 2(A).1 [3 marks]

If $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ then find the matrix X such that $2A + X = 3B$.

Solution

Solution: $2A + X = 3B \Rightarrow X = 3B - 2A$

$$2A = 2\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 8 & 6 \end{bmatrix}$$

$$3B = 3\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 3 & 12 \end{bmatrix}$$

$$X = \begin{bmatrix} 9 & 6 \\ 3 & 12 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ -5 & 6 \end{bmatrix}$$

Question 2(A).2 [3 marks]

If $A = \begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ **then find** $(AB)^T$**.**

Solution

Solution: First, find AB:

$$AB = \begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5(1) + 4(2) & 5(3) + 4(1) \\ 4(1) + 3(2) & 4(3) + 3(1) \end{bmatrix} = \begin{bmatrix} 13 & 19 \\ 10 & 15 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 13 & 10 \\ 19 & 15 \end{bmatrix}$$

Question 2(A).3 [3 marks]

Solve: $\frac{dy}{dx} = x^2 \cdot e^{-y}$**.**

Solution

Solution:

$$\frac{dy}{dx} = x^2 \cdot e^{-y}$$

Separating variables:

$$e^y \, dy = x^2 \, dx$$

Integrating both sides:

$$\int e^y \, dy = \int x^2 \, dx$$

$$e^y = \frac{x^3}{3} + C$$

$$y = \ln\left(\frac{x^3}{3} + C\right)$$

Question 2(B) [8 marks]

Attempt any two.

Question 2(B).1 [4 marks]

If $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 5 & 0 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \end{bmatrix}$ **then prove that** $(A + B)^T = A^T + B^T$**.**

Solution

Solution:

$$A + B = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 5 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 3 & 5 & 3 \\ 6 & 8 & 1 \end{bmatrix}$$

$$(A + B)^T = \begin{bmatrix} 3 & 6 \\ 5 & 8 \\ 3 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 4 \\ 3 & 5 \\ -1 & 0 \end{bmatrix}, \quad B^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$

$$A^T + B^T = \begin{bmatrix} 2 & 4 \\ 3 & 5 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 5 & 8 \\ 3 & 1 \end{bmatrix}$$

Therefore, $(A + B)^T = A^T + B^T$ is proved.

Question 2(B).2 [4 marks]

If $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 4 \\ 1 & -1 & 1 \end{bmatrix}$ **then find** A^{-1}**.**

Solution

Solution: To find A^{-1}, we use the formula $A^{-1} = \frac{1}{|A|} \cdot \text{adj}(A)$
First, find $|A|$: $|A| = 2(0 \cdot 1 - 4 \cdot (-1)) - (-1)(1 \cdot 1 - 4 \cdot 1) + 0(1 \cdot (-1) - 0 \cdot 1)$ $|A| = 2(4) + 1(-3) = 8 - 3 = 5$
Next, find cofactors: $C_{11} = (-1)^{1+1} \begin{vmatrix} 0 & 4 \\ -1 & 1 \end{vmatrix} = 4$ $C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} = -(-3) = 3$ $C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = -1$ $C_{21} = (-1)^{2+1} \begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} = -(-1) = 1$ $C_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2$ $C_{23} = (-1)^{2+3} \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = -(-1) = 1$ $C_{31} = (-1)^{3+1} \begin{vmatrix} -1 & 0 \\ 0 & 4 \end{vmatrix} = -4$ $C_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} = -(8) = -8$ $C_{33} = (-1)^{3+3} \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} = 1$

$$\text{adj}(A) = \begin{bmatrix} 4 & 1 & -4 \\ 3 & 2 & -8 \\ -1 & 1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 4 & 1 & -4 \\ 3 & 2 & -8 \\ -1 & 1 & 1 \end{bmatrix}$$

Question 2(B).3 [4 marks]

Solve the equations $3x - y = 1, x + 2y = 5$ **by matrix method.**

Solution

Solution: The system can be written as $AX = B$ where: $A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$$|A| = 3(2) - (-1)(1) = 6 + 1 = 7$$

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{7} \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$X = \frac{1}{7} \begin{bmatrix} 2+5 \\ -1+15 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 7 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore, $x = 1$ and $y = 2$.

Question 3(A) [6 marks]

Attempt any two.

Question 3(A).1 [3 marks]

If $y = \frac{e^x+1}{e^x-1}$ then find $\frac{dy}{dx}$.

Solution

Solution: Using quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
Let $u = e^x + 1$ and $v = e^x - 1$ $\frac{du}{dx} = e^x$ and $\frac{dv}{dx} = e^x$

$$\frac{dy}{dx} = \frac{(e^x - 1)(e^x) - (e^x + 1)(e^x)}{(e^x - 1)^2}$$

$$= \frac{e^{2x} - e^x - e^{2x} - e^x}{(e^x - 1)^2} = \frac{-2e^x}{(e^x - 1)^2}$$

Question 3(A).2 [3 marks]

If $x = a\cos\theta, y = b\sin\theta$ then find $\frac{dy}{dx}$.

Solution

Solution: $\frac{dx}{d\theta} = -a\sin\theta$ $\frac{dy}{d\theta} = b\cos\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b\cos\theta}{a\sin\theta} = -\frac{b}{a}\cot\theta$$

Question 3(A).3 [3 marks]

Evaluate: $\int \frac{\cos\sqrt{x}}{2\sqrt{x}}dx$.

Solution

Solution: Let $u = \sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}dx$

$$\int \frac{\cos\sqrt{x}}{2\sqrt{x}}dx = \int \cos u \, du = \sin u + C = \sin\sqrt{x} + C$$

Question 3(B) [8 marks]

Attempt any two.

Question 3(B).1 [4 marks]

Differentiate $y = x^{\cos x}$ with respect to x.

Solution

Solution: Taking natural logarithm on both sides:

$$\ln y = \cos x \ln x$$

Differentiating both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \cos x \cdot \frac{1}{x} + \ln x \cdot (-\sin x)$$

$$\frac{dy}{dx} = y\left(\frac{\cos x}{x} - \sin x \ln x\right)$$

$$\frac{dy}{dx} = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right)$$

Question 3(B).2 [4 marks]

If $y = A\cos pt + B\sin pt$, prove that $\frac{d^2y}{dt^2} + p^2y = 0$.

Solution

Solution: $y = A\cos pt + B\sin pt$

$$\frac{dy}{dt} = -Ap\sin pt + Bp\cos pt$$

$$\frac{d^2y}{dt^2} = -Ap^2\cos pt - Bp^2\sin pt = -p^2(A\cos pt + B\sin pt) = -p^2y$$

Therefore: $\frac{d^2y}{dt^2} + p^2y = -p^2y + p^2y = 0$

Question 3(B).3 [4 marks]

The equation of motion of a particle is $s = t^3 + 2t^2 - 3t + 5$. Find the velocity and acceleration of the particle at $t = 1$ and $t = 2$ seconds.

Solution

Solution: $s = t^3 + 2t^2 - 3t + 5$
Velocity: $v = \frac{ds}{dt} = 3t^2 + 4t - 3$
Acceleration: $a = \frac{dv}{dt} = 6t + 4$
At $t = 1$: $v(1) = 3(1)^2 + 4(1) - 3 = 3 + 4 - 3 = 4$ units/sec $a(1) = 6(1) + 4 = 10$ units/sec²
At $t = 2$: $v(2) = 3(2)^2 + 4(2) - 3 = 12 + 8 - 3 = 17$ units/sec $a(2) = 6(2) + 4 = 16$ units/sec²

Question 4(A) [6 marks]

Attempt any two.

Question 4(A).1 [3 marks]

Evaluate: $\int x \log x \, dx$.

Solution

Solution: Using integration by parts: $\int u \, dv = uv - \int v \, du$
Let $u = \log x$ and $dv = x \, dx$ Then $du = \frac{1}{x} dx$ and $v = \frac{x^2}{2}$

$$\int x \log x \, dx = \log x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$$
$$= \frac{x^2 \log x}{2} - \int \frac{x}{2} dx$$
$$= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C$$
$$= \frac{x^2}{2}(\log x - \frac{1}{2}) + C$$

Question 4(A).2 [3 marks]

Evaluate: $\int_{-1}^{1} \frac{1}{1+x^2} dx$.

Solution

Solution:

$$\int_{-1}^{1} \frac{1}{1 + x^2} dx = [\tan^{-1} x]_{-1}^{1}$$
$$= \tan^{-1}(1) - \tan^{-1}(-1)$$
$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

Question 4(A).3 [3 marks]

Find inverse of $Z = 3 + 4i$.

Solution

Solution:

$$Z^{-1} = \frac{1}{Z} = \frac{1}{3+4i}$$

Multiply numerator and denominator by the conjugate:

$$Z^{-1} = \frac{1}{3+4i} \cdot \frac{3-4i}{3-4i} = \frac{3-4i}{(3)^2+(4)^2} = \frac{3-4i}{9+16} = \frac{3-4i}{25}$$

$$Z^{-1} = \frac{3}{25} - \frac{4}{25}i$$

Question 4(B) [8 marks]

Attempt any two.

Question 4(B).1 [4 marks]

Evaluate: $\int_0^{\pi/2} \frac{\tan x}{\tan x + \cot x} dx$.

Solution

Solution: Let $I = \int_0^{\pi/2} \frac{\tan x}{\tan x + \cot x} dx$
Using the property $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$:

$$I = \int_0^{\pi/2} \frac{\tan(\pi/2 - x)}{\tan(\pi/2-x) + \cot(\pi/2-x)} dx$$

$$= \int_0^{\pi/2} \frac{\cot x}{\cot x + \tan x} dx$$

Adding the two expressions:

$$2I = \int_0^{\pi/2} \frac{\tan x + \cot x}{\tan x + \cot x} dx = \int_0^{\pi/2} 1\, dx = \frac{\pi}{2}$$

Therefore: $I = \frac{\pi}{4}$

Question 4(B).2 [4 marks]

Find the area bounded by the line $y = x$, $x = 5$ and the X-axis.

Solution

Solution: The region is bounded by $y = x$, $x = 5$, and $y = 0$ (X-axis).
Area $= \int_0^5 x\, dx = \left[\frac{x^2}{2}\right]_0^5 = \frac{25}{2} - 0 = \frac{25}{2}$ square units

Question 4(B).3 [4 marks]

If $x + iy = \left(\frac{1+i}{2-i}\right)^2$, find the value of $x + y$.

Solution

Solution: First, simplify $\frac{1+i}{2-i}$:

$$\frac{1+i}{2-i} \cdot \frac{2+i}{2+i} = \frac{(1+i)(2+i)}{(2-i)(2+i)} = \frac{2+i+2i+i^2}{4-i^2} = \frac{2+3i-1}{4+1} = \frac{1+3i}{5}$$

Now:

$$\left(\frac{1+3i}{5}\right)^2 = \frac{(1+3i)^2}{25} = \frac{1+6i+9i^2}{25} = \frac{1+6i-9}{25} = \frac{-8+6i}{25}$$

Therefore: $x = -\frac{8}{25}$ and $y = \frac{6}{25}$

$$x + y = -\frac{8}{25} + \frac{6}{25} = -\frac{2}{25}$$

Question 5(A) [6 marks]

Attempt any two.

Question 5(A).1 [3 marks]

Find Square root of $Z = 5 + 12i$.

Solution

Solution: Let $\sqrt{5+12i} = a + bi$ where $a, b \in \mathbb{R}$

$$(a+bi)^2 = 5 + 12i$$

$$a^2 + 2abi + b^2i^2 = 5 + 12i$$

$$(a^2 - b^2) + 2abi = 5 + 12i$$

Comparing real and imaginary parts: $a^2 - b^2 = 5$ ... (1) $2ab = 12$ ... (2)
From (2): $b = \frac{6}{a}$
Substituting in (1): $a^2 - \frac{36}{a^2} = 5$

$$a^4 - 5a^2 - 36 = 0$$

Let $u = a^2$: $u^2 - 5u - 36 = 0$

$$(u-9)(u+4) = 0$$

Since $u = a^2 \geq 0$, we have $u = 9$, so $a = \pm 3$
If $a = 3$, then $b = 2$ If $a = -3$, then $b = -2$
Therefore: $\sqrt{5+12i} = \pm(3+2i)$

Question 5(A).2 [3 marks]

Find $x, y \in \mathbb{R}$ from the equation $(2x - y) + yi = 6 + 4i$.

Solution

Solution: Comparing real and imaginary parts: Real part: $2x - y = 6$ ... (1) Imaginary part: $y = 4$ ... (2)
Substituting (2) into (1): $2x - 4 = 6$ $2x = 10$ $x = 5$
Therefore: $x = 5$ and $y = 4$

Question 5(A).3 [3 marks]

Find the modulus and principal argument of $Z = 1 + i$, and express Z into the polar form.

Solution

Solution: $Z = 1 + i$
Modulus: $|Z| = \sqrt{1^2 + 1^2} = \sqrt{2}$
Principal argument: $\arg(Z) = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$
Polar form: $Z = |Z|(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Question 5(B) [8 marks]

Attempt any two.

Question 5(B).1 [4 marks]

Solve: $\frac{dy}{dx} = 1 + x + y + xy$.

Solution

Solution:

$$\frac{dy}{dx} = 1 + x + y + xy = (1 + x) + y(1 + x) = (1 + x)(1 + y)$$

Separating variables:

$$\frac{dy}{1 + y} = (1 + x)dx$$

Integrating both sides:

$$\int \frac{dy}{1 + y} = \int (1 + x)dx$$

$$\ln|1 + y| = x + \frac{x^2}{2} + C$$

$$1 + y = Ae^{x + x^2/2} \quad \text{where } A = e^C$$

$$y = Ae^{x + x^2/2} - 1$$

Question 5(B).2 [4 marks]

Solve the differential equation: $\frac{dy}{dx} + y = e^x$.

Solution

Solution: This is a first-order linear differential equation of the form $\frac{dy}{dx} + Py = Q$ where $P = 1$ and $Q = e^x$.
Integrating factor: $I.F. = e^{\int P\,dx} = e^{\int 1\,dx} = e^x$
Multiplying the equation by e^x:

$$e^x\frac{dy}{dx} + e^x y = e^{2x}$$

$$\frac{d}{dx}(ye^x) = e^{2x}$$

Integrating both sides:

$$ye^x = \int e^{2x} dx = \frac{e^{2x}}{2} + C$$

$$y = \frac{e^x}{2} + Ce^{-x}$$

Question 5(B).3 [4 marks]

Solve the differential equation: $\frac{dy}{dx} - y \tan x = 1$.

Solution

Solution: This is a first-order linear differential equation where $P = -\tan x$ and $Q = 1$.
Integrating factor: $I.F. = e^{\int(-\tan x)dx} = e^{\ln|\cos x|} = \cos x$
Multiplying the equation by $\cos x$:

$$\cos x \frac{dy}{dx} - y \cos x \tan x = \cos x$$

$$\cos x \frac{dy}{dx} - y \sin x = \cos x$$

$$\frac{d}{dx}(y \cos x) = \cos x$$

Integrating both sides:

$$y \cos x = \int \cos x \, dx = \sin x + C$$

$$y = \tan x + \frac{C}{\cos x} = \tan x + C \sec x$$