Engineering Mathematics (4320002) - Summer 2022 Solution

Milav Dabgar

September 06, 2022

Question 1 [14 marks]

Fill in the blanks using appropriate choice from the given options

Question 1.1 [1 marks]

If $A_{2\times 3}$ and $B_{3\times 4}$ are two matrices then find order of AB =__________

Solution

Answer: b. 2×4
Solution: When multiplying matrices, if A is of order $m \times n$ and B is of order $n \times p$, then AB is of order $m \times p$. Given: $A_{2\times 3}$ and $B_{3\times 4}$ Therefore, AB will be of order 2×4.

Question 1.2 [1 marks]

If $A = [1\ 3\ 2]$ and $B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ then find AB =__________

Solution

Answer: b. 9
Solution:

$$AB = [1\ 3\ 2] \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 1(1) + 3(2) + 2(1) = 1 + 6 + 2 = 9$$

Question 1.3 [1 marks]

$A.I_2 = A$ **then** I_2 **=__________**

Solution

Answer: c. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Solution: I_2 is the identity matrix of order 2×2, which has 1's on the main diagonal and 0's elsewhere.

Question 1.4 [1 marks]

If $\frac{d}{dx}(\sin^2 x + \cos^2 x) =$ __________

Solution

Answer: b. 0
Solution: Since $\sin^2 x + \cos^2 x = 1$ (fundamental trigonometric identity)

$$\frac{d}{dx}(\sin^2 x + \cos^2 x) = \frac{d}{dx}(1) = 0$$

Question 1.5 [1 marks]

$\frac{d}{dx}(\cot x) =$ __________

Solution

Answer: d. $-\csc^2 x$
Solution:

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Question 1.6 [1 marks]

$\frac{d}{dx}\log(\sin x)$ **then find out** $\frac{d^2y}{dx^2} =$ __________

Solution

Answer: d. $-\cot^2 x$
Solution: Let $y = \log(\sin x)$

$$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x = \cot x$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\cot x) = -\csc^2 x$$

However, since $\csc^2 x = 1 + \cot^2 x$, the answer is $-\csc^2 x$.

Question 1.7 [1 marks]

$\frac{d}{dx}(\frac{1}{x}) =$ __________

Solution

Answer: c. $-\frac{1}{x^2}$
Solution:

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -1 \cdot x^{-2} = -\frac{1}{x^2}$$

Question 1.8 [1 marks]

If $\int x^5 dx =$ __________ **+ c**

Solution

Answer: a. $\frac{x^6}{6}$
Solution:

$$\int x^5 dx = \frac{x^{5+1}}{5+1} + c = \frac{x^6}{6} + c$$

Question 1.9 [1 marks]

$\int_0^{2\pi} (\sin^2\theta + \cos^2\theta) d\theta =$ __________ **+ c**

Solution

Answer: a. 2π
Solution:

$$\int_0^{2\pi} (\sin^2\theta + \cos^2\theta) d\theta = \int_0^{2\pi} 1\, d\theta = [\theta]_0^{2\pi} = 2\pi - 0 = 2\pi$$

Question 1.10 [1 marks]

$\int_{-1}^{1} x^3 dx =$ __________ **+ c**

Solution

Answer: c. 0
Solution:

$$\int_{-1}^{1} x^3 dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = \frac{1^4}{4} - \frac{(-1)^4}{4} = \frac{1}{4} - \frac{1}{4} = 0$$

Question 1.11 [1 marks]

The order and degree of the differential equation $x^2 \frac{d^2y}{dx^2} + 3y^2 = 0$ **is =**__________

Solution

Answer: c. 2 and 1
Solution: Order is the highest derivative present = 2 (from $\frac{d^2y}{dx^2}$) Degree is the power of the highest derivative = 1

Question 1.12 [1 marks]

An integrating factor of the differential equation $\frac{dy}{dx} + py = Q$ **is** __________

Solution

Answer: c. $e^{\int pdx}$
Solution: For a first-order linear differential equation $\frac{dy}{dx} + py = Q$, the integrating factor is $e^{\int pdx}$.

Question 1.13 [1 marks]

$i^4 =$ __________

Solution

Answer: a. 1
Solution:

$$i^4 = (i^2)^2 = (-1)^2 = 1$$

Question 1.14 [1 marks]

(3+4i)(4-5i) =__________

Solution

Answer: d. -32+ i
Solution:

$$\begin{aligned}(3+4i)(4-5i) &= 3(4) + 3(-5i) + 4i(4) + 4i(-5i) \\ &= 12 - 15i + 16i - 20i^2 \\ &= 12 + i - 20(-1) \\ &= 12 + i + 20 = 32 + i\end{aligned}$$

Wait, let me recalculate: $(3+4i)(4-5i) = 12 - 15i + 16i - 20i^2 = 12 + i + 20 = 32 + i$
The correct answer should be b. 32+ i, but option d shows -32+ i. There might be an error in the options.

Question 2(a) [6 marks]

Attempt any two

Question 2.1 [3 marks]

If $A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 1 & 7 \end{bmatrix}$ **then find out AB & BA.**

Solution

Solution:
AB calculation:

$$AB = \begin{bmatrix} 1 & -1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 1 & 7 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1(1) + (-1)(4) + 1(1) & 1(2) + (-1)(2) + 1(7) \\ 3(1) + 2(4) + 1(1) & 3(2) + 2(2) + 1(7) \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 - 4 + 1 & 2 - 2 + 7 \\ 3 + 8 + 1 & 6 + 4 + 7 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 12 & 17 \end{bmatrix}$$

BA calculation:

$$BA = \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1(1) + 2(3) & 1(-1) + 2(2) & 1(1) + 2(1) \\ 4(1) + 2(3) & 4(-1) + 2(2) & 4(1) + 2(1) \\ 1(1) + 7(3) & 1(-1) + 7(2) & 1(1) + 7(1) \end{bmatrix}$$

$$BA = \begin{bmatrix} 7 & 3 & 3 \\ 10 & 0 & 6 \\ 22 & 13 & 8 \end{bmatrix}$$

Question 2.2 [3 marks]

If $A = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$ **then prove that** $A^2 - 7I_2 = 0$

Solution

Solution:

$$A^2 = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} (-1)(-1) + (2)(3) & (-1)(2) + (2)(1) \\ (3)(-1) + (1)(3) & (3)(2) + (1)(1) \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 + 6 & -2 + 2 \\ -3 + 3 & 6 + 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$7I_2 = 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

Therefore,

$$A^2 - 7I_2 = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence proved.

Question 2.3 [3 marks]

Find the inverse complex number of $\frac{2+3i}{4-3i}$

Solution

Solution: First, let's find $\frac{2+3i}{4-3i}$:

$$\frac{2+3i}{4-3i} = \frac{(2+3i)(4+3i)}{(4-3i)(4+3i)} = \frac{8+6i+12i+9i^2}{16-9i^2}$$

$$= \frac{8+18i-9}{16+9} = \frac{-1+18i}{25} = -\frac{1}{25} + \frac{18}{25}i$$

The inverse of a complex number $z = a + bi$ is $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

Let $z = -\frac{1}{25} + \frac{18}{25}i$

$$|z|^2 = \left(-\frac{1}{25}\right)^2 + \left(\frac{18}{25}\right)^2 = \frac{1}{625} + \frac{324}{625} = \frac{325}{625} = \frac{13}{25}$$

$$\bar{z} = -\frac{1}{25} - \frac{18}{25}i$$

$$\frac{1}{z} = \frac{-\frac{1}{25} - \frac{18}{25}i}{\frac{13}{25}} = \frac{-1-18i}{13}$$

Question 2(b) [8 marks]

Attempt any two

Question 2.1 [4 marks]

2y+5x-4 =0 and 7x +3y = 5 solve the equations using matrix method.

Solution

Solution: The system can be written as:

$$5x + 2y = 4$$
$$7x + 3y = 5$$

In matrix form: $\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

Let $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$

$$|A| = 5(3) - 2(7) = 15 - 14 = 1$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3(4) + (-2)(5) \\ -7(4) + 5(5) \end{bmatrix} = \begin{bmatrix} 12 - 10 \\ -28 + 25 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Therefore, $x = 2$ and $y = -3$.

Question 2.2 [4 marks]

If $A = \begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}$ **then Prove that** $(AB)^T = B^T.A^T$

Solution

Solution: First, let's find AB:

$$AB = \begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2(-1) + (-2)(4) & 2(5) + (-2)(-3) \\ 3(-1) + 1(4) & 3(5) + 1(-3) \end{bmatrix}$$

$$AB = \begin{bmatrix} -2-8 & 10+6 \\ -3+4 & 15-3 \end{bmatrix} = \begin{bmatrix} -10 & 16 \\ 1 & 12 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} -10 & 1 \\ 16 & 12 \end{bmatrix}$$

Now, let's find B^T and A^T: $A^T = \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}$, $B^T = \begin{bmatrix} -1 & 4 \\ 5 & -3 \end{bmatrix}$

$$B^T \cdot A^T = \begin{bmatrix} -1 & 4 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}$$

$$B^T \cdot A^T = \begin{bmatrix} -1(2) + 4(-2) & -1(3) + 4(1) \\ 5(2) + (-3)(-2) & 5(3) + (-3)(1) \end{bmatrix}$$

$$B^T \cdot A^T = \begin{bmatrix} -2-8 & -3+4 \\ 10+6 & 15-3 \end{bmatrix} = \begin{bmatrix} -10 & 1 \\ 16 & 12 \end{bmatrix}$$

Since $(AB)^T = B^T \cdot A^T$, the property is proved.

Question 2.3 [4 marks]

Simplify: $\frac{(\cos 2\theta + i\sin 2\theta)^{-3}.(\cos 3\theta - i\sin 3\theta)^2}{(\cos 2\theta + i\sin 2\theta)^{-7}.(\cos 5\theta - i\sin 5\theta)^3}$

Solution

Solution: Using De Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$
$(\cos 2\theta + i\sin 2\theta)^{-3} = \cos(-6\theta) + i\sin(-6\theta) = \cos(6\theta) - i\sin(6\theta)$
$(\cos 3\theta - i\sin 3\theta)^2 = (\cos(-3\theta) + i\sin(-3\theta))^2 = \cos(-6\theta) + i\sin(-6\theta) = \cos(6\theta) - i\sin(6\theta)$
$(\cos 2\theta + i\sin 2\theta)^{-7} = \cos(-14\theta) + i\sin(-14\theta) = \cos(14\theta) - i\sin(14\theta)$
$(\cos 5\theta - i\sin 5\theta)^3 = (\cos(-5\theta) + i\sin(-5\theta))^3 = \cos(-15\theta) + i\sin(-15\theta) = \cos(15\theta) - i\sin(15\theta)$
The expression becomes:

$$\frac{[\cos(6\theta) - i\sin(6\theta)][\cos(6\theta) - i\sin(6\theta)]}{[\cos(14\theta) - i\sin(14\theta)][\cos(15\theta) - i\sin(15\theta)]}$$

$$= \frac{[\cos(6\theta) - i\sin(6\theta)]^2}{[\cos(14\theta) - i\sin(14\theta)][\cos(15\theta) - i\sin(15\theta)]}$$

$$= \frac{\cos(12\theta) - i\sin(12\theta)}{\cos(29\theta) - i\sin(29\theta)}$$

$$= \cos(12\theta - 29\theta) + i\sin(12\theta - 29\theta) = \cos(-17\theta) + i\sin(-17\theta) = \cos(17\theta) - i\sin(17\theta)$$

Question 3(a) [6 marks]

Attempt any two

Question 3.1 [3 marks]

If $y = \frac{1+\tan x}{1-\tan x}$ then find $\frac{dy}{dx}$

Solution

Solution: Using quotient rule: $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
Let $u = 1 + \tan x$ and $v = 1 - \tan x$

$$\frac{du}{dx} = \sec^2 x \quad \text{and} \quad \frac{dv}{dx} = -\sec^2 x$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(1-\tan x)(\sec^2 x) - (1+\tan x)(-\sec^2 x)}{(1-\tan x)^2} \\
&= \frac{(1-\tan x)\sec^2 x + (1+\tan x)\sec^2 x}{(1-\tan x)^2} \\
&= \frac{\sec^2 x[(1-\tan x) + (1+\tan x)]}{(1-\tan x)^2} \\
&= \frac{2\sec^2 x}{(1-\tan x)^2}
\end{aligned}$$

Question 3.2 [3 marks]

Using Definition of differentiation differentiate x^3 with respect to x.

Solution

Solution: Using the definition: $\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
For $f(x) = x^3$:

$$\begin{aligned}
\frac{d}{dx}(x^3) &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2)}{h} \\
&= \lim_{h\to 0} (3x^2 + 3xh + h^2) \\
&= 3x^2 + 0 + 0 = 3x^2
\end{aligned}$$

Question 3.3 [3 marks]

Simplify: $\int \frac{4+3\cos x}{\sin^2 x}dx$

Solution

Solution:

$$\int \frac{4+3\cos x}{\sin^2 x}dx = \int \frac{4}{\sin^2 x}dx + \int \frac{3\cos x}{\sin^2 x}dx$$
$$= 4\int \csc^2 x\,dx + 3\int \frac{\cos x}{\sin^2 x}dx$$

For the first integral: $\int \csc^2 x\,dx = -\cot x$
For the second integral, let $u = \sin x$, then $du = \cos x\,dx$:

$$\int \frac{\cos x}{\sin^2 x}dx = \int \frac{1}{u^2}du = -\frac{1}{u} = -\frac{1}{\sin x} = -\csc x$$

Therefore:

$$\int \frac{4+3\cos x}{\sin^2 x}dx = 4(-\cot x) + 3(-\csc x) + C = -4\cot x - 3\csc x + C$$

Question 3(b) [8 marks]

Attempt any two

Question 3.1 [4 marks]

If $y = \log\left(\frac{\cos x}{1+\sin x}\right)$ **then find** $\frac{dy}{dx}$

Solution

Solution:

$$y = \log\left(\frac{\cos x}{1+\sin x}\right) = \log(\cos x) - \log(1+\sin x)$$
$$\frac{dy}{dx} = \frac{d}{dx}[\log(\cos x)] - \frac{d}{dx}[\log(1+\sin x)]$$
$$= \frac{1}{\cos x}\cdot(-\sin x) - \frac{1}{1+\sin x}\cdot\cos x$$
$$= -\frac{\sin x}{\cos x} - \frac{\cos x}{1+\sin x}$$
$$= -\tan x - \frac{\cos x}{1+\sin x}$$

To simplify further:

$$= -\frac{\sin x(1+\sin x)+\cos^2 x}{\cos x(1+\sin x)}$$
$$= -\frac{\sin x+\sin^2 x+\cos^2 x}{\cos x(1+\sin x)}$$
$$= -\frac{\sin x+1}{\cos x(1+\sin x)} = -\frac{1}{\cos x} = -\sec x$$

Question 3.2 [4 marks]

Find maximum and minimum value of function $f(x) = 2x^3 - 15x^2 + 36x + 10$.

Solution

Solution: To find extrema, we find where $f'(x) = 0$:

$$f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x-2)(x-3)$$

Setting $f'(x) = 0$: $x = 2$ or $x = 3$
To determine nature of critical points, we use the second derivative test: $f''(x) = 12x - 30$
At $x = 2$: $f''(2) = 24 - 30 = -6 < 0 \rightarrow$ Local maximum
At $x = 3$: $f''(3) = 36 - 30 = 6 > 0 \rightarrow$ Local minimum
Values:

$$f(2) = 2(8) - 15(4) + 36(2) + 10 = 16 - 60 + 72 + 10 = 38$$

$$f(3) = 2(27) - 15(9) + 36(3) + 10 = 54 - 135 + 108 + 10 = 37$$

Therefore:
- Local maximum value: 38 at $x = 2$
- Local minimum value: 37 at $x = 3$

Question 3.3 [4 marks]

If $y = 2e^{-3x} + 3e^{2x}$ then prove that $y_2 + y_1 - 6y = 0$.

Solution

Solution: Given: $y = 2e^{-3x} + 3e^{2x}$

$$y_1 = \frac{dy}{dx} = 2(-3)e^{-3x} + 3(2)e^{2x} = -6e^{-3x} + 6e^{2x}$$

$$y_2 = \frac{d^2y}{dx^2} = -6(-3)e^{-3x} + 6(2)e^{2x} = 18e^{-3x} + 12e^{2x}$$

Now let's verify $y_2 + y_1 - 6y = 0$:

$$\begin{aligned} y_2 + y_1 - 6y &= (18e^{-3x} + 12e^{2x}) + (-6e^{-3x} + 6e^{2x}) - 6(2e^{-3x} + 3e^{2x}) \\ &= 18e^{-3x} + 12e^{2x} - 6e^{-3x} + 6e^{2x} - 12e^{-3x} - 18e^{2x} \\ &= (18 - 6 - 12)e^{-3x} + (12 + 6 - 18)e^{2x} \\ &= 0 \cdot e^{-3x} + 0 \cdot e^{2x} = 0 \end{aligned}$$

Hence proved.

Question 4(a) [6 marks]

Attempt any two

Question 4.1 [3 marks]

Evaluate: $\int \frac{x^2}{1+x^6} dx$

Solution

Solution: Let $u = x^3$, then $du = 3x^2dx$, so $x^2dx = \frac{1}{3}du$

$$\int \frac{x^2}{1+x^6}dx = \int \frac{1}{1+(x^3)^2} \cdot x^2dx = \int \frac{1}{1+u^2} \cdot \frac{1}{3}du$$
$$= \frac{1}{3}\int \frac{1}{1+u^2}du = \frac{1}{3}\tan^{-1}(u) + C$$
$$= \frac{1}{3}\tan^{-1}(x^3) + C$$

Question 4.2 [3 marks]

Evaluate: $\int x \log x \, dx$

Solution

Solution: Using integration by parts: $\int u \, dv = uv - \int v \, du$
Let $u = \log x$ and $dv = x \, dx$ Then $du = \frac{1}{x}dx$ and $v = \frac{x^2}{2}$

$$\int x \log x \, dx = \log x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x}dx$$
$$= \frac{x^2 \log x}{2} - \int \frac{x}{2}dx$$
$$= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C$$
$$= \frac{x^2}{2}(\log x - \frac{1}{2}) + C$$

Question 4.3 [3 marks]

Solve the differential equation $xdy + ydx = 0$.

Solution

Solution: The given equation is: $xdy + ydx = 0$
This can be written as: $xdy = -ydx$
Separating variables: $\frac{dy}{y} = -\frac{dx}{x}$
Integrating both sides:

$$\int \frac{dy}{y} = \int -\frac{dx}{x}$$
$$\log|y| = -\log|x| + C_1$$
$$\log|y| + \log|x| = C_1$$
$$\log|xy| = C_1$$
$$|xy| = e^{C_1} = C \quad (\text{where } C = e^{C_1})$$

Therefore: $xy = \pm C$
The general solution is: $xy = k$ (where k is an arbitrary constant)

Question 4(b) [8 marks]

Attempt any two

Question 4.1 [4 marks]

Evaluate: $\int_1^e \frac{(\log x)^2}{x} dx$

Solution

Solution: Let $u = \log x$, then $du = \frac{1}{x} dx$
When $x = 1$: $u = \log 1 = 0$ When $x = e$: $u = \log e = 1$

$$\int_1^e \frac{(\log x)^2}{x} dx = \int_0^1 u^2 du$$

$$= \left[\frac{u^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

Question 4.2 [4 marks]

Evaluate: $\int_0^{\pi/2} \frac{\sec x}{\sec x + \cos x} dx$

Solution

Solution: Let $I = \int_0^{\pi/2} \frac{\sec x}{\sec x + \cos x} dx$
First, let's simplify the integrand:

$$\frac{\sec x}{\sec x + \cos x} = \frac{\frac{1}{\cos x}}{\frac{1}{\cos x} + \cos x} = \frac{\frac{1}{\cos x}}{\frac{1+\cos^2 x}{\cos x}} = \frac{1}{1 + \cos^2 x}$$

So $I = \int_0^{\pi/2} \frac{1}{1+\cos^2 x} dx$
Using the substitution $\tan(x/2) = t$: $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2dt}{1+t^2}$
When $x = 0$: $t = 0$ When $x = \pi/2$: $t = 1$

$$I = \int_0^1 \frac{1}{1 + \left(\frac{1-t^2}{1+t^2}\right)^2} \cdot \frac{2dt}{1+t^2}$$

After simplification (which involves significant algebra), this evaluates to:

$$I = \frac{\pi}{2\sqrt{2}}$$

Question 4.3 [4 marks]

Solve the differential equation $\frac{dy}{dx} + \frac{y}{x} = e^x$**,** $y(0) = 2$**.**

Solution

Solution: This is a first-order linear differential equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$
Here, $P(x) = \frac{1}{x}$ and $Q(x) = e^x$
The integrating factor is: $\mu(x) = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log|x|} = |x| = x$ (for $x > 0$)
Multiplying the equation by the integrating factor:

$$x\frac{dy}{dx} + y = xe^x$$

The left side is $\frac{d}{dx}(xy)$, so:

$$\frac{d}{dx}(xy) = xe^x$$

Integrating both sides:

$$xy = \int xe^x dx$$

Using integration by parts for $\int xe^x dx$: Let $u = x$, $dv = e^x dx$ Then $du = dx$, $v = e^x$

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C = e^x(x-1) + C$$

Therefore: $xy = e^x(x-1) + C$

$$y = \frac{e^x(x-1) + C}{x}$$

Using the initial condition $y(0) = 2$: This presents a problem as the solution is undefined at $x = 0$. Let me reconsider the problem.
Actually, let's solve this more carefully. The equation should be valid for $x \neq 0$.
If we assume the initial condition is at $x = 1$ instead (as $x = 0$ makes the equation singular), and $y(1) = 2$:

$$2 = \frac{e^1(1-1) + C}{1} = \frac{0 + C}{1} = C$$

So $C = 2$, and the solution is:

$$y = \frac{e^x(x-1) + 2}{x}$$

Question 5(a) [6 marks]

Attempt any two

Question 5.1 [3 marks]

Find the conjugate complex number and modulus of $\frac{3+7i}{1-i}$.

Solution

Solution: First, let's simplify $\frac{3+7i}{1-i}$:

$$\frac{3+7i}{1-i} = \frac{(3+7i)(1+i)}{(1-i)(1+i)} = \frac{3+3i+7i+7i^2}{1-i^2}$$

$$= \frac{3+10i-7}{1+1} = \frac{-4+10i}{2} = -2+5i$$

Conjugate: The conjugate of $-2+5i$ is $-2-5i$
Modulus: $|-2+5i| = \sqrt{(-2)^2 + (5)^2} = \sqrt{4+25} = \sqrt{29}$

Question 5.2 [3 marks]

Find the square root of complex number $3 - 4i$.

Solution

Solution: Let $\sqrt{3-4i} = a + bi$ where $a, b \in \mathbb{R}$
Then $(a+bi)^2 = 3 - 4i$

$$a^2 + 2abi + (bi)^2 = 3 - 4i$$

$$a^2 - b^2 + 2abi = 3 - 4i$$

Comparing real and imaginary parts: $a^2 - b^2 = 3$ ... (1) $2ab = -4$ ... (2)
From equation (2): $b = -\frac{2}{a}$
Substituting in equation (1):

$$a^2 - \left(-\frac{2}{a}\right)^2 = 3$$

$$a^2 - \frac{4}{a^2} = 3$$

$$a^4 - 3a^2 - 4 = 0$$

Let $u = a^2$: $u^2 - 3u - 4 = 0$

$$(u-4)(u+1) = 0$$

So $u = 4$ or $u = -1$
Since $u = a^2 \geq 0$, we have $u = 4$, so $a^2 = 4$
Therefore $a = \pm 2$
If $a = 2$: $b = -\frac{2}{2} = -1$ If $a = -2$: $b = -\frac{2}{-2} = 1$
The two square roots are: $2 - i$ and $-2 + i$

Question 5.3 [3 marks]

Find $\frac{dy}{dx}$ for $y = (\sin x)^{\tan x}$

Solution

Solution: Taking logarithm of both sides:

$$\log y = \tan x \log(\sin x)$$

Differentiating both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}[\tan x \log(\sin x)]$$

Using product rule on the right side:

$$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \log(\sin x) + \tan x \cdot \frac{\cos x}{\sin x}$$

$$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \log(\sin x) + \tan x \cdot \cot x$$

$$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \log(\sin x) + 1$$

Therefore:

$$\frac{dy}{dx} = y[\sec^2 x \log(\sin x) + 1]$$

$$\frac{dy}{dx} = (\sin x)^{\tan x}[\sec^2 x \log(\sin x) + 1]$$

Question 5(b) [8 marks]

Attempt any two

Question 5.1 [4 marks]

Find solution of the differential equation $\tan y\, dx + \tan x \sec^2 y\, dy = 0$**.**

Solution

Solution: The given equation is: $\tan y\, dx + \tan x \sec^2 y\, dy = 0$
Rearranging: $\tan y\, dx = -\tan x \sec^2 y\, dy$

$$\frac{\tan y}{\sec^2 y} dy = -\tan x\, dx$$

$$\frac{\sin y / \cos y}{1/\cos^2 y} dy = -\tan x\, dx$$

$$\frac{\sin y}{\cos y} \cdot \cos^2 y\, dy = -\tan x\, dx$$

$$\sin y \cos y\, dy = -\tan x\, dx$$

Integrating both sides:

$$\int \sin y \cos y\, dy = -\int \tan x\, dx$$

For the left side, let $u = \sin y$, then $du = \cos y\, dy$: $\int \sin y \cos y\, dy = \int u\, du = \frac{u^2}{2} = \frac{\sin^2 y}{2}$
For the right side: $-\int \tan x\, dx = -\int \frac{\sin x}{\cos x} dx = \log|\cos x| + C_1$
Therefore:

$$\frac{\sin^2 y}{2} = \log|\cos x| + C$$

$$\sin^2 y = 2\log|\cos x| + K \quad \text{(where } K = 2C\text{)}$$

Question 5.2 [4 marks]

If $A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & -1 \\ 5 & 0 & 1 \end{bmatrix}$ **then find** A^{-1}**.**

Solution

Solution: To find A^{-1}, we use the formula $A^{-1} = \frac{1}{|A|}\text{adj}(A)$
First, let's find $|A|$:

$$|A| = 3\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 4 & -1 \\ 5 & 1 \end{vmatrix} + 2\begin{vmatrix} 4 & 1 \\ 5 & 0 \end{vmatrix}$$

$$= 3(1 \cdot 1 - (-1) \cdot 0) + 1(4 \cdot 1 - (-1) \cdot 5) + 2(4 \cdot 0 - 1 \cdot 5)$$

$$= 3(1) + 1(4 + 5) + 2(0 - 5) = 3 + 9 - 10 = 2$$

Now we find the cofactor matrix: $C_{11} = +\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$ $C_{12} = -\begin{vmatrix} 4 & -1 \\ 5 & 1 \end{vmatrix} = -(4 - (-5)) = -9$ $C_{13} = +\begin{vmatrix} 4 & 1 \\ 5 & 0 \end{vmatrix} = 0 - 5 = -5$ $C_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -(-1 - 0) = 1$ $C_{22} = +\begin{vmatrix} 3 & 2 \\ 5 & 1 \end{vmatrix} = 3 - 10 = -7$ $C_{23} = -\begin{vmatrix} 3 & -1 \\ 5 & 0 \end{vmatrix} = -(0 - (-5)) = -5$

$C_{31} = + \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} = 1 - 2 = -1$ $C_{32} = - \begin{vmatrix} 3 & 2 \\ 4 & -1 \end{vmatrix} = -(-3 - 8) = 11$ $C_{33} = + \begin{vmatrix} 3 & -1 \\ 4 & 1 \end{vmatrix} = 3 - (-4) = 7$

The cofactor matrix is: $C = \begin{bmatrix} 1 & -9 & -5 \\ 1 & -7 & -5 \\ -1 & 11 & 7 \end{bmatrix}$

The adjugate is the transpose of the cofactor matrix:

$$\text{adj}(A) = \begin{bmatrix} 1 & 1 & -1 \\ -9 & -7 & 11 \\ -5 & -5 & 7 \end{bmatrix}$$

Therefore:

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ -9 & -7 & 11 \\ -5 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -9/2 & -7/2 & 11/2 \\ -5/2 & -5/2 & 7/2 \end{bmatrix}$$

Question 5.3 [4 marks]

$x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ then find $\frac{dy}{dx}$.

Solution

Solution: These are parametric equations. To find $\frac{dy}{dx}$, we use:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

First, let's find $\frac{dx}{d\theta}$: $x = a(\theta - \sin\theta)$ $\frac{dx}{d\theta} = a(1 - \cos\theta)$
Next, let's find $\frac{dy}{d\theta}$: $y = a(1 - \cos\theta)$ $\frac{dy}{d\theta} = a\sin\theta$
Therefore:

$$\frac{dy}{dx} = \frac{a\sin\theta}{a(1 - \cos\theta)} = \frac{\sin\theta}{1 - \cos\theta}$$

Using the identity $1 - \cos\theta = 2\sin^2(\theta/2)$ and $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$:

$$\frac{dy}{dx} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = \frac{\cos(\theta/2)}{\sin(\theta/2)} = \cot(\theta/2)$$