Subject Name Solutions

4320002 – Summer 2022

Semester 1 Study Material

Detailed Solutions and Explanations

Q.1 [14 marks]

Fill in the blanks using appropriate choice from the given options

0.0.1 Q1.1 [1 mark]

If $A_{2\times 3}$ and $B_{3\times 4}$ are two matrices then find order of AB =__________

Solution

b. 2×4

Solution: When multiplying matrices, if A is of order m and B is of order n, then AB is of order m. Given: $A_{2\times 3}$ and $B_{3\times 4}$ Therefore, AB will be of order 2×4.

0.0.2 Q1.2 [1 mark]

If $A = [1\ 3\ 2]$ and $B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ then find AB =__________

Solution

b. 9

Solution: $AB = [1\ 3\ 2] \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 1(1) + 3(2) + 2(1) = 1 + 6 + 2 = 9$

0.0.3 Q1.3 [1 mark]

$A.I_2 = A$ **then** I_2 **=__________**

Solution

c. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Solution: I_2 is the identity matrix of order $2\times 2, which has 1's on the main diagonal and 0's elsewhere.$

0.0.4 Q1.4 [1 mark]

If $\frac{d}{dx}(\sin^2 x + \cos^2 x) =$ __________

Solution

b. 0

Solution: Since $\sin^2 x + \cos^2 x = 1$ (fundamental trigonometric identity) $\frac{d}{dx}(\sin^2 x + \cos^2 x) = \frac{d}{dx}(1) = 0$

0.0.5 Q1.5 [1 mark]

$\frac{d}{dx}(\cot x) =$ __________

Solution

d. $-\csc^2 x$
Solution: $\frac{d}{dx}(\cot x) = -\csc^2 x$

0.0.6 Q1.6 [1 mark]

$\frac{d}{dx}\log(\sin x)$ **then find out** $\frac{d^2y}{dx^2} =$ __________

Solution

d. $-\cot^2 x$
Solution: Let $y = \log(\sin x)$ $\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x = \cot x$ $\frac{d^2y}{dx^2} = \frac{d}{dx}(\cot x) = -\csc^2 x$
However, since $\csc^2 x = 1 + \cot^2 x$, the answer is $-\csc^2 x$.

0.0.7 Q1.7 [1 mark]

$\frac{d}{dx}(\frac{1}{x}) =$ __________

Solution

c. $-\frac{1}{x^2}$
Solution: $\frac{d}{dx}(\frac{1}{x}) = \frac{d}{dx}(x^{-1}) = -1 \cdot x^{-2} = -\frac{1}{x^2}$

0.0.8 Q1.8 [1 mark]

If $\int x^5 dx =$ __________**+ c**

Solution

a. $\frac{x^6}{6}$
Solution: $\int x^5 dx = \frac{x^{5+1}}{5+1} + c = \frac{x^6}{6} + c$

0.0.9 Q1.9 [1 mark]

$\int_0^{2\pi} (\sin^2 \ + \cos^2 \ ) d =$ __________**+ c**

Solution

a. 2
Solution: $\int_0^{2\pi} (\sin^2 \ + \cos^2 \ ) d = \int_0^{2\pi} 1 \, d = []_0^{2\pi} = 2\pi - 0 = 2\pi$

0.0.10 Q1.10 [1 mark]

$\int_{-1}^{1} x^3 dx =$ __________**+ c**

Solution

c. 0
Solution: $\int_{-1}^{1} x^3 dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = \frac{1^4}{4} - \frac{(-1)^4}{4} = \frac{1}{4} - \frac{1}{4} = 0$

0.0.11 Q1.11 [1 mark]

The order and degree of the differential equation $x^2\frac{d^2y}{dx^2} + 3y^2 = 0$ **is =**__________

Solution

c. 2 and 1
Solution: Order is the highest derivative present = 2 (from $\frac{d^2y}{dx^2}$) Degree is the power of the highest derivative = 1

0.0.12 Q1.12 [1 mark]

An integrating factor of the differential equation $\frac{dy}{dx} + py = Q$ is __________

Solution

c. $e^{\int pdx}$
Solution: For a first-order linear differential equation $\frac{dy}{dx} + py = Q$, the integrating factor is $e^{\int pdx}$.

0.0.13 Q1.13 [1 mark]

$i^4 =$ __________

Solution

a. 1
Solution: $i^4 = (i^2)^2 = (-1)^2 = 1$

0.0.14 Q1.14 [1 mark]

(3+4i)(4-5i) =__________

Solution

d. -32+ i
Solution: $(3 + 4i)(4 - 5i) = 3(4) + 3(-5i) + 4i(4) + 4i(-5i) = 12 - 15i + 16i - 20i^2 = 12 + i - 20(-1) = 12 + i + 20 = 32 + i$
Wait, let me recalculate: $(3 + 4i)(4 - 5i) = 12 - 15i + 16i - 20i^2 = 12 + i + 20 = 32 + i$
The correct answer should be b. 32+ i, but option d shows -32+ i. There might be an error in the options.

Q.2 [14 marks]

0.0.15 Q.2(a) [6 marks]

Attempt any two

Q2.1 [3 marks] If $A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 1 & 7 \end{bmatrix}$ then find out AB & BA.

Solution:

AB calculation: $AB = \begin{bmatrix} 1 & -1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 1 & 7 \end{bmatrix}$

$$AB = \begin{bmatrix} 1(1) + (-1)(4) + 1(1) & 1(2) + (-1)(2) + 1(7) \\ 3(1) + 2(4) + 1(1) & 3(2) + 2(2) + 1(7) \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 - 4 + 1 & 2 - 2 + 7 \\ 3 + 8 + 1 & 6 + 4 + 7 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 12 & 17 \end{bmatrix}$$

BA calculation: $BA = \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

$$BA = \begin{bmatrix} 1(1) + 2(3) & 1(-1) + 2(2) & 1(1) + 2(1) \\ 4(1) + 2(3) & 4(-1) + 2(2) & 4(1) + 2(1) \\ 1(1) + 7(3) & 1(-1) + 7(2) & 1(1) + 7(1) \end{bmatrix}$$

$$BA = \begin{bmatrix} 7 & 3 & 3 \\ 10 & 0 & 6 \\ 22 & 13 & 8 \end{bmatrix}$$

Q2.2 [3 marks] If $A = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$ then prove that $A^2 - 7I_2 = 0$

Solution: $A^2 = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$

$A^2 = \begin{bmatrix} (-1)(-1) + (2)(3) & (-1)(2) + (2)(1) \\ (3)(-1) + (1)(3) & (3)(2) + (1)(1) \end{bmatrix}$

$A^2 = \begin{bmatrix} 1+6 & -2+2 \\ -3+3 & 6+1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$7I_2 = 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

Therefore, $A^2 - 7I_2 = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Hence proved.

Q2.3 [3 marks] Find the inverse complex number of $\frac{2+3i}{4-3i}$

Solution: First, let's find $\frac{2+3i}{4-3i}$:

$\frac{2+3i}{4-3i} = \frac{(2+3i)(4+3i)}{(4-3i)(4+3i)} = \frac{8+6i+12i+9i^2}{16-9i^2}$

$= \frac{8+18i-9}{16+9} = \frac{-1+18i}{25} = -\frac{1}{25} + \frac{18}{25}i$

The inverse of a complex number $z = a + bi$ is $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

Let $z = -\frac{1}{25} + \frac{18}{25}i$

$|z|^2 = \left(-\frac{1}{25}\right)^2 + \left(\frac{18}{25}\right)^2 = \frac{1}{625} + \frac{324}{625} = \frac{325}{625} = \frac{13}{25}$

$\bar{z} = -\frac{1}{25} - \frac{18}{25}i$

$\frac{1}{z} = \frac{-\frac{1}{25} - \frac{18}{25}i}{\frac{13}{25}} = \frac{-1-18i}{13}$

0.0.16 Q.2(b) [8 marks]

Attempt any two

Q2.1 [4 marks] 2y+5x-4 =0 and 7x +3y = 5 solve the equations using matrix method.

Solution: The system can be written as: $5x + 2y = 4$ $7x + 3y = 5$

In matrix form: $\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

Let $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$

$|A| = 5(3) - 2(7) = 15 - 14 = 1$

$A^{-1} = \frac{1}{|A|} \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3(4) + (-2)(5) \\ -7(4) + 5(5) \end{bmatrix} = \begin{bmatrix} 12 - 10 \\ -28 + 25 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

Therefore, $x = 2$ and $y = -3$.

Q2.2 [4 marks] If $A = \begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}$ then Prove that $(AB)^T = B^T.A^T$

Solution: First, let's find AB: $AB = \begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}$

$AB = \begin{bmatrix} 2(-1) + (-2)(4) & 2(5) + (-2)(-3) \\ 3(-1) + 1(4) & 3(5) + 1(-3) \end{bmatrix}$

$AB = \begin{bmatrix} -2-8 & 10+6 \\ -3+4 & 15-3 \end{bmatrix} = \begin{bmatrix} -10 & 16 \\ 1 & 12 \end{bmatrix}$

$(AB)^T = \begin{bmatrix} -10 & 1 \\ 16 & 12 \end{bmatrix}$

Now, let's find B^T and A^T: $A^T = \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}$, $B^T = \begin{bmatrix} -1 & 4 \\ 5 & -3 \end{bmatrix}$

$B^T \cdot A^T = \begin{bmatrix} -1 & 4 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}$

$B^T \cdot A^T = \begin{bmatrix} -1(2)+4(-2) & -1(3)+4(1) \\ 5(2)+(-3)(-2) & 5(3)+(-3)(1) \end{bmatrix}$

$B^T \cdot A^T = \begin{bmatrix} -2-8 & -3+4 \\ 10+6 & 15-3 \end{bmatrix} = \begin{bmatrix} -10 & 1 \\ 16 & 12 \end{bmatrix}$

Since $(AB)^T = B^T \cdot A^T$, the property is proved.

Q2.3 [4 marks] Simplify: $\frac{(cos2+isin2)^{-3}.(cos3-isin3)^2}{(cos2+isin2)^{-7}.(cos5-isin5)^3}$

Solution: Using De Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

$(\cos 2 + i\sin 2)^{-3} = \cos(-6) + i\sin(-6) = \cos(6) - i\sin(6)$

$(\cos 3 - i\sin 3)^2 = (\cos(-3) + i\sin(-3))^2 = \cos(-6) + i\sin(-6) = \cos(6) - i\sin(6)$

$(\cos 2 + i\sin 2)^{-7} = \cos(-14) + i\sin(-14) = \cos(14) - i\sin(14)$

$(\cos 5 - i\sin 5)^3 = (\cos(-5) + i\sin(-5))^3 = \cos(-15) + i\sin(-15) = \cos(15) - i\sin(15)$

The expression becomes: $\frac{[\cos(6)-i\sin(6)][\cos(6)-i\sin(6)]}{[\cos(14)-i\sin(14)][\cos(15)-i\sin(15)]}$

$= \frac{[\cos(6)-i\sin(6)]^2}{[\cos(14)-i\sin(14)][\cos(15)-i\sin(15)]}$

$= \frac{\cos(12)-i\sin(12)}{\cos(29)-i\sin(29)}$

$= \cos(12-29) + i\sin(12-29) = \cos(-17) + i\sin(-17) = \cos(17) - i\sin(17)$

Q.3 [14 marks]

0.0.17 Q.3(a) [6 marks]

Attempt any two

Q3.1 [3 marks] If $y = \frac{1+\tan x}{1-\tan x}$ **then find** $\frac{dy}{dx}$

Solution: Using quotient rule: $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Let $u = 1 + \tan x$ and $v = 1 - \tan x$

$\frac{du}{dx} = \sec^2 x$ and $\frac{dv}{dx} = -\sec^2 x$

$\frac{dy}{dx} = \frac{(1-\tan x)(\sec^2 x)-(1+\tan x)(-\sec^2 x)}{(1-\tan x)^2}$

$= \frac{(1-\tan x)\sec^2 x+(1+\tan x)\sec^2 x}{(1-\tan x)^2}$

$= \frac{\sec^2 x[(1-\tan x)+(1+\tan x)]}{(1-\tan x)^2}$

$= \frac{2\sec^2 x}{(1-\tan x)^2}$

Q3.2 [3 marks] Using Definition of differentiation differentiate x^3 **with respect to** x**.**

Solution: Using the definition: $\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

For $f(x) = x^3$:

$\frac{d}{dx}(x^3) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$

$= \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$

$= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h}$

$= \lim_{h\to 0} \frac{h(3x^2+3xh+h^2)}{h}$

$= \lim_{h\to 0}(3x^2 + 3xh + h^2)$

$= 3x^2 + 0 + 0 = 3x^2$

Q3.3 [3 marks] Simplify: $\int \frac{4+3\cos x}{\sin^2 x} dx$

Solution: $\int \frac{4+3\cos x}{\sin^2 x} dx = \int \frac{4}{\sin^2 x} dx + \int \frac{3\cos x}{\sin^2 x} dx$

$= 4 \int \csc^2 x \, dx + 3 \int \frac{\cos x}{\sin^2 x} dx$

For the first integral: $\int \csc^2 x \, dx = -\cot x$

For the second integral, let $u = \sin x$, then $du = \cos x \, dx$: $\int \frac{\cos x}{\sin^2 x} dx = \int \frac{1}{u^2} du = -\frac{1}{u} = -\frac{1}{\sin x} = -\csc x$

Therefore: $\int \frac{4+3\cos x}{\sin^2 x} dx = 4(-\cot x) + 3(-\csc x) + C = -4\cot x - 3\csc x + C$

0.0.18 Q.3(b) [8 marks]

Attempt any two

Q3.1 [4 marks] If $y = \log\left(\frac{\cos x}{1+\sin x}\right)$ **then find** $\frac{dy}{dx}$

Solution: $y = \log\left(\frac{\cos x}{1+\sin x}\right) = \log(\cos x) - \log(1 + \sin x)$

$\frac{dy}{dx} = \frac{d}{dx}[\log(\cos x)] - \frac{d}{dx}[\log(1 + \sin x)]$

$= \frac{1}{\cos x} \cdot (-\sin x) - \frac{1}{1+\sin x} \cdot \cos x$

$= -\frac{\sin x}{\cos x} - \frac{\cos x}{1+\sin x}$

$= -\tan x - \frac{\cos x}{1+\sin x}$

To simplify further: $= -\frac{\sin x(1+\sin x)+\cos^2 x}{\cos x(1+\sin x)}$

$= -\frac{\sin x+\sin^2 x+\cos^2 x}{\cos x(1+\sin x)}$

$= -\frac{\sin x+1}{\cos x(1+\sin x)} = -\frac{1}{\cos x} = -\sec x$

Q3.2 [4 marks] Find maximum and minimum value of function $f(x) = 2x^3 - 15x^2 + 36x + 10$**.**

Solution: To find extrema, we find where $f'(x) = 0$:

$f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x - 2)(x - 3)$

Setting $f'(x) = 0$: $x = 2$ or $x = 3$

To determine nature of critical points, we use the second derivative test: $f''(x) = 12x - 30$

At $x = 2$: $f''(2) = 24 - 30 = -6 < 0 \to Local maximum At x = 3$: $f''(3) = 36 - 30 = 6 > 0 \to Local minimum$

Values: $f(2) = 2(8) - 15(4) + 36(2) + 10 = 16 - 60 + 72 + 10 = 38$ $f(3) = 2(27) - 15(9) + 36(3) + 10 = 54 - 135 + 108 + 10 = 37$

Therefore:

- Local maximum value: 38 at $x = 2$
- Local minimum value: 37 at $x = 3$

Q3.3 [4 marks] If $y = 2e^{-3x} + 3e^{2x}$ **then prove that** $y_2 + y_1 - 6y = 0$**.**

Solution: Given: $y = 2e^{-3x} + 3e^{2x}$

$y_1 = \frac{dy}{dx} = 2(-3)e^{-3x} + 3(2)e^{2x} = -6e^{-3x} + 6e^{2x}$

$y_2 = \frac{d^2y}{dx^2} = -6(-3)e^{-3x} + 6(2)e^{2x} = 18e^{-3x} + 12e^{2x}$

Now let's verify $y_2 + y_1 - 6y = 0$:

$y_2 + y_1 - 6y = (18e^{-3x} + 12e^{2x}) + (-6e^{-3x} + 6e^{2x}) - 6(2e^{-3x} + 3e^{2x})$

$= 18e^{-3x} + 12e^{2x} - 6e^{-3x} + 6e^{2x} - 12e^{-3x} - 18e^{2x}$

$= (18 - 6 - 12)e^{-3x} + (12 + 6 - 18)e^{2x}$

$= 0 \cdot e^{-3x} + 0 \cdot e^{2x} = 0$

Hence proved.

Q.4 [14 marks]

0.0.19 Q.4(a) [6 marks]

Attempt any two

Q4.1 [3 marks] Evaluate: $\int \frac{x^2}{1+x^6} dx$

Solution: Let $u = x^3$, then $du = 3x^2 dx$, so $x^2 dx = \frac{1}{3} du$

$\int \frac{x^2}{1+x^6} dx = \int \frac{1}{1+(x^3)^2} \cdot x^2 dx = \int \frac{1}{1+u^2} \cdot \frac{1}{3} du$

$= \frac{1}{3} \int \frac{1}{1+u^2} du = \frac{1}{3} \tan^{-1}(u) + C$

$= \frac{1}{3} \tan^{-1}(x^3) + C$

Q4.2 [3 marks] Evaluate: $\int x \log x \, dx$

Solution: Using integration by parts: $\int u \, dv = uv - \int v \, du$

Let $u = \log x$ and $dv = x \, dx$ Then $du = \frac{1}{x} dx$ and $v = \frac{x^2}{2}$

$\int x \log x \, dx = \log x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$

$= \frac{x^2 \log x}{2} - \int \frac{x}{2} dx$

$= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C$

$= \frac{x^2}{2} (\log x - \frac{1}{2}) + C$

Q4.3 [3 marks] Solve the differential equation $xdy + ydx = 0$.

Solution: The given equation is: $xdy + ydx = 0$

This can be written as: $xdy = -ydx$

Separating variables: $\frac{dy}{y} = -\frac{dx}{x}$

Integrating both sides: $\int \frac{dy}{y} = \int -\frac{dx}{x}$

$\log |y| = -\log |x| + C_1$

$\log |y| + \log |x| = C_1$

$\log |xy| = C_1$

$|xy| = e^{C_1} = C$ (where $C = e^{C_1}$)

Therefore: $xy = \pm C$

The general solution is: $xy = k$ (where k is an arbitrary constant)

0.0.20 Q.4(b) [8 marks]

Attempt any two

Q4.1 [4 marks] Evaluate: $\int_1^e \frac{(\log x)^2}{x} dx$

Solution: Let $u = \log x$, then $du = \frac{1}{x} dx$

When $x = 1$: $u = \log 1 = 0$ When $x = e$: $u = \log$

e = 1

$\int_1^e \frac{(\log x)^2}{x} dx = \int_0^1 u^2 du$

$= \left[\frac{u^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$

Q4.2 [4 marks] Evaluate: $\int_0^{\pi/2} \frac{\sec x}{\sec x + \cos x} dx$

Solution: Let $I = \int_0^{\pi/2} \frac{\sec x}{\sec x + \cos x} dx$

First, let's simplify the integrand: $\frac{\sec x}{\sec x + \cos x} = \frac{\frac{1}{\cos x}}{\frac{1}{\cos x} + \cos x} = \frac{\frac{1}{\cos x}}{\frac{1+\cos^2 x}{\cos x}} = \frac{1}{1+\cos^2 x}$

So $I = \int_0^{\pi/2} \frac{1}{1+\cos^2 x} dx$

Using the substitution $\tan(x/2) = t$: $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2dt}{1+t^2}$

When $x = 0$: $t = 0$ When $x = \pi/2$: $t = 1$

$I = \int_0^1 \frac{1}{1+\left(\frac{1-t^2}{1+t^2}\right)^2} \cdot \frac{2dt}{1+t^2}$

After simplification (which involves significant algebra), this evaluates to: $I = \frac{\pi}{2\sqrt{2}}$

Q4.3 [4 marks] Solve the differential equation $\frac{dy}{dx} + \frac{y}{x} = e^x$, $y(0) = 2$.

Solution: This is a first-order linear differential equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$

Here, $P(x) = \frac{1}{x}$ and $Q(x) = e^x$

The integrating factor is: $\mu(x) = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log|x|} = |x| = x$ (for $x > 0$)

Multiplying the equation by the integrating factor: $x\frac{dy}{dx} + y = xe^x$

The left side is $\frac{d}{dx}(xy)$, so: $\frac{d}{dx}(xy) = xe^x$

Integrating both sides: $xy = \int xe^x dx$

Using integration by parts for $\int xe^x dx$: Let $u = x$, $dv = e^x dx$ Then $du = dx$, $v = e^x$

$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C = e^x(x-1) + C$

Therefore: $xy = e^x(x-1) + C$

$y = \frac{e^x(x-1)+C}{x}$

Using the initial condition $y(0) = 2$: This presents a problem as the solution is undefined at $x = 0$. Let me reconsider the problem.

Actually, let's solve this more carefully. The equation should be valid for $x \neq 0$.

If we assume the initial condition is at $x = 1$ instead (as $x = 0$ makes the equation singular), and $y(1) = 2$:

$2 = \frac{e^1(1-1)+C}{1} = \frac{0+C}{1} = C$

So $C = 2$, and the solution is: $y = \frac{e^x(x-1)+2}{x}$

Q.5 [14 marks]

0.0.21 Q.5(a) [6 marks]

Attempt any two

Q5.1 [3 marks] Find the conjugate complex number and modulus of $\frac{3+7i}{1-i}$.

Solution: First, let's simplify $\frac{3+7i}{1-i}$:

$\frac{3+7i}{1-i} = \frac{(3+7i)(1+i)}{(1-i)(1+i)} = \frac{3+3i+7i+7i^2}{1-i^2}$

$= \frac{3+10i-7}{1+1} = \frac{-4+10i}{2} = -2 + 5i$

Conjugate: The conjugate of $-2 + 5i$ is $-2 - 5i$

Modulus: $|-2 + 5i| = \sqrt{(-2)^2 + (5)^2} = \sqrt{4 + 25} = \sqrt{29}$

Q5.2 [3 marks] Find the square root of complex number $3 - 4i$.

Solution: Let $\sqrt{3-4i} = a + bi$ where $a, b \in \mathbb{R}$

Then $(a + bi)^2 = 3 - 4i$

$a^2 + 2abi + (bi)^2 = 3 - 4i$

$a^2 - b^2 + 2abi = 3 - 4i$

Comparing real and imaginary parts: $a^2 - b^2 = 3$ ... (1) $2ab = -4$ ... (2)

From equation (2): $b = -\frac{2}{a}$

Substituting in equation (1): $a^2 - \left(-\frac{2}{a}\right)^2 = 3$

$a^2 - \frac{4}{a^2} = 3$

$a^4 - 3a^2 - 4 = 0$

Let $u = a^2$: $u^2 - 3u - 4 = 0$

$(u - 4)(u + 1) = 0$

So $u = 4$ or $u = -1$

Since $u = a^2 \geq 0$, we have $u = 4$, so $a^2 = 4$

Therefore $a = \pm 2$

If $a = 2$: $b = -\frac{2}{2} = -1$ If $a = -2$: $b = -\frac{2}{-2} = 1$

The two square roots are: $2 - i$ and $-2 + i$

Q5.3 [3 marks] Find $\frac{dy}{dx}$ for $y = (\sin x)^{\tan x}$

Solution: Taking logarithm of both sides: $\log y = \tan x \log(\sin x)$

Differentiating both sides with respect to x: $\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}[\tan x \log(\sin x)]$

Using product rule on the right side: $\frac{1}{y}\frac{dy}{dx} = \sec^2 x \log(\sin x) + \tan x \cdot \frac{\cos x}{\sin x}$

$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \log(\sin x) + \tan x \cdot \cot x$

$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \log(\sin x) + 1$

Therefore: $\frac{dy}{dx} = y[\sec^2 x \log(\sin x) + 1]$

$\frac{dy}{dx} = (\sin x)^{\tan x}[\sec^2 x \log(\sin x) + 1]$

0.0.22 Q.5(b) [8 marks]

Attempt any two

Q5.1 [4 marks] Find solution of the differential equation $\tan y\, dx + \tan x \sec^2 y\, dy = 0$.

Solution: The given equation is: $\tan y\, dx + \tan x \sec^2 y\, dy = 0$

Rearranging: $\tan y\, dx = -\tan x \sec^2 y\, dy$

$\frac{\tan y}{\sec^2 y} dy = -\tan x\, dx$

$\frac{\sin y/\cos y}{1/\cos^2 y} dy = -\tan x\, dx$

$\frac{\sin y}{\cos y} \cdot \cos^2 y\, dy = -\tan x\, dx$

$\sin y \cos y\, dy = -\tan x\, dx$

Integrating both sides: $\int \sin y \cos y\, dy = -\int \tan x\, dx$

For the left side, let $u = \sin y$, then $du = \cos y\, dy$: $\int \sin y \cos y\, dy = \int u\, du = \frac{u^2}{2} = \frac{\sin^2 y}{2}$

For the right side: $-\int \tan x\, dx = -\int \frac{\sin x}{\cos x} dx = \log|\cos x| + C_1$

Therefore: $\frac{\sin^2 y}{2} = \log|\cos x| + C$

$\sin^2 y = 2\log|\cos x| + K$ (where $K = 2C$)

Q5.2 [4 marks] If $A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & -1 \\ 5 & 0 & 1 \end{bmatrix}$ **then find** A^{-1}.

Solution: To find A^{-1}, we use the formula $A^{-1} = \frac{1}{|A|}\text{adj}(A)$

First, let's find $|A|$: $|A| = 3\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 4 & -1 \\ 5 & 1 \end{vmatrix} + 2\begin{vmatrix} 4 & 1 \\ 5 & 0 \end{vmatrix}$

$= 3(1 \cdot 1 - (-1) \cdot 0) + 1(4 \cdot 1 - (-1) \cdot 5) + 2(4 \cdot 0 - 1 \cdot 5)$

$= 3(1) + 1(4 + 5) + 2(0 - 5) = 3 + 9 - 10 = 2$

Now we find the cofactor matrix:

$C_{11} = +\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$

$C_{12} = -\begin{vmatrix} 4 & -1 \\ 5 & 1 \end{vmatrix} = -(4 - (-5)) = -9$

$C_{13} = +\begin{vmatrix} 4 & 1 \\ 5 & 0 \end{vmatrix} = 0 - 5 = -5$

$C_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -(-1 - 0) = 1$

$C_{22} = +\begin{vmatrix} 3 & 2 \\ 5 & 1 \end{vmatrix} = 3 - 10 = -7$

$C_{23} = -\begin{vmatrix} 3 & -1 \\ 5 & 0 \end{vmatrix} = -(0 - (-5)) = -5$

$C_{31} = +\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} = 1 - 2 = -1$

$C_{32} = -\begin{vmatrix} 3 & 2 \\ 4 & -1 \end{vmatrix} = -(-3 - 8) = 11$

$C_{33} = +\begin{vmatrix} 3 & -1 \\ 4 & 1 \end{vmatrix} = 3 - (-4) = 7$

The cofactor matrix is: $C = \begin{bmatrix} 1 & -9 & -5 \\ 1 & -7 & -5 \\ -1 & 11 & 7 \end{bmatrix}$

The adjugate is the transpose of the cofactor matrix: $\text{adj}(A) = \begin{bmatrix} 1 & 1 & -1 \\ -9 & -7 & 11 \\ -5 & -5 & 7 \end{bmatrix}$

Therefore: $A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ -9 & -7 & 11 \\ -5 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -9/2 & -7/2 & 11/2 \\ -5/2 & -5/2 & 7/2 \end{bmatrix}$

Q5.3 [4 marks] $x = a(\theta - \sin\theta)$**,** $y = a(1 - \cos\theta)$ **then find** $\frac{dy}{dx}$**.**

Solution: These are parametric equations. To find $\frac{dy}{dx}$, we use: $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

First, let's find $\frac{dx}{d\theta}$: $x = a(\theta - \sin\theta)$ $\frac{dx}{d\theta} = a(1 - \cos\theta)$

Next, let's find $\frac{dy}{d\theta}$: $y = a(1 - \cos\theta)$ $\frac{dy}{d\theta} = a\sin\theta$

Therefore: $\frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$

Using the identity $1 - \cos\theta = 2\sin^2(\theta/2)$ and $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$:

$\frac{dy}{dx} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = \frac{\cos(\theta/2)}{\sin(\theta/2)} = \cot(\theta/2)$

Formula Cheat Sheet

0.0.23 Differentiation Formulas

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\log x) = \frac{1}{x}$
- $\frac{d}{dx}(e^x) = e^x$

0.0.24 Integration Formulas

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ (for $n \neq -1$)
- $\int \frac{1}{x} dx = \log|x| + C$
- $\int e^x dx = e^x + C$
- $\int \sin x dx = -\cos x + C$
- $\int \cos x dx = \sin x + C$
- $\int \sec^2 x dx = \tan x + C$

0.0.25 Matrix Operations

- $(AB)^T = B^T A^T$
- $A^{-1} = \frac{1}{|A|}\text{adj}(A)$
- For $2 \times 2 matrix: \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

0.0.26 Complex Numbers

- $i^2 = -1$, $i^3 = -i$, $i^4 = 1$
- $|a + bi| = \sqrt{a^2 + b^2}$
- De Moivre's Theorem: $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

Problem-Solving Strategies

1. **For Matrix Problems**: Always check dimensions before multiplication
2. **For Differentiation**: Use appropriate rules (product, quotient, chain)
3. **For Integration**: Look for substitutions or integration by parts
4. **For Differential Equations**: Identify type (separable, linear, etc.)
5. **For Complex Numbers**: Convert to standard form before operations

Common Mistakes to Avoid

1. **Sign errors** in differentiation and integration
2. **Forgetting constant of integration**
3. **Matrix dimension mismatch**
4. **Not simplifying complex fractions**
5. **Missing absolute value signs** in logarithms

Exam Tips

1. **Show all steps** clearly
2. **Double-check calculations**
3. **Use proper mathematical notation**
4. **Manage time effectively**
5. **Attempt easier questions first**