Subject Name Solutions

4320002 – Summer 2024

Semester 1 Study Material

Detailed Solutions and Explanations

Engineering Mathematics (4320002) - Summer 2024 Solutions

Q.1 [14 marks]

Fill in the blanks using appropriate choice from the given options.

0.0.1 Q.1.1 [1 mark]

Order of the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 4 \end{bmatrix}$ **is _____.**

> **Solution**
>
> (b) 3×2
> **Solution**: Order of a matrix is given by (number of rows) $\times (number of columns) Matrix A has 3 rows and 2 columns Therefore,$
> 3×2

0.0.2 Q.1.2 [1 mark]

If $A = \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$ **then** $A^{-1} =$ **_____**

> **Solution**
>
> (d) A^T
> **Solution**: For orthogonal matrices, $A^{-1} = A^T$ Since $AA^T = I$, we have $A^{-1} = A^T$

0.0.3 Q.1.3 [1 mark]

\$
$\begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}$
$\times$
$\begin{bmatrix} -1 & 6 \\ 2 & 1 \end{bmatrix}$
$= \$$ **_____****

> **Solution**
>
> (a) $\begin{bmatrix} 3 & 8 \\ -5 & 30 \end{bmatrix}$
> **Solution**: $\begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix} \times \begin{bmatrix} -1 & 6 \\ 2 & 1 \end{bmatrix}$
> $= \begin{bmatrix} 1(-1) + 2(2) & 1(6) + 2(1) \\ 5(-1) + 0(2) & 5(6) + 0(1) \end{bmatrix}$
> $= \begin{bmatrix} -1+4 & 6+2 \\ -5+0 & 30+0 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ -5 & 30 \end{bmatrix}$

0.0.4 Q.1.4 [1 mark]

If $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ then $A^T = $ _____

> **Solution**
>
> (b) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$
>
> **Solution**: Transpose of a matrix is obtained by interchanging rows and columns $A^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

0.0.5 Q.1.5 [1 mark]

$\frac{d}{dx}(4^x) = $ _____

> **Solution**
>
> (a) $4^x \log_e 4$
> **Solution:** $\frac{d}{dx}(a^x) = a^x \ln a$ **Therefore,** $\frac{d}{dx}(4^x) = 4^x \ln 4 = 4^x \log_e 4$

0.0.6 Q.1.6 [1 mark]

$\frac{d}{dx}(\sin{^2}x + \cos{^2}x) = $ _____

> **Solution**
>
> (b) **0**
> **Solution:** $\sin^2 x + \cos^2 x = 1$ **(trigonometric identity)** $\frac{d}{dx}(1) = 0$

0.0.7 Q.1.7 [1 mark]

If $x = \sin\theta, y = \cos\theta$ then $\frac{dy}{dx} = $ _____

> **Solution**
>
> (d) $-\cot\theta$
> **Solution:** $\frac{dx}{d\theta} = \cos\theta$, $\frac{dy}{d\theta} = -\sin\theta$ $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta = -\cot\theta$

0.0.8 Q.1.8 [1 mark]

$\int x^7 dx = $ _____

> **Solution**
>
> (c) $\frac{x^8}{8}$
> **Solution:** $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ $\int x^7 dx = \frac{x^8}{8} + c$

0.0.9 Q.1.9 [1 mark]

$\int_{-2}^{2} x^5 \, dx = $ _____

> **Solution**
>
> (b) **0**
> **Solution:** x^5 **is an odd function For odd functions,** $\int_{-a}^{a} f(x)dx = 0$ **Therefore,** $\int_{-2}^{2} x^5 dx = 0$

0.0.10 Q.1.10 [1 mark]

$\int \frac{\cos x}{\sin x} dx = $ _____

> **Solution**
>
> **(d)** $\log |\sin x|$
> **Solution: Let** $u = \sin x$, **then** $du = \cos x dx$ $\int \frac{\cos x}{\sin x} dx = \int \frac{du}{u} = \log |u| + c = \log |\sin x| + c$

0.0.11 Q.1.11 [1 mark]

The order of the differential equation $\left(\frac{d^3 y}{dx^3}\right)^2 + \left(\frac{d^2 y}{dx^2}\right)^4 + y = 0$ is _____

> **Solution**
>
> **(a) 3**
> **Solution: Order of a differential equation is the highest order derivative present Highest deriva-**
> **tive is** $\frac{d^3 y}{dx^3}$, **so order** $= 3$

0.0.12 Q.1.12 [1 mark]

An integrating factor of the differential equation $\frac{dy}{dx} + y = 3x$ is _____

> **Solution**
>
> **(c)** e^x
> **Solution: For linear differential equation** $\frac{dy}{dx} + Py = Q$ **Integrating factor** $= e^{\int P dx} = e^{\int 1 dx} = e^x$

0.0.13 Q.1.13 [1 mark]

$i^7 = $ _____

> **Solution**
>
> **(b)** $-i$
> **Solution:** $i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1$ $i^7 = i^4 \cdot i^3 = 1 \cdot (-i) = -i$

0.0.14 Q.1.14 [1 mark]

$\arg(1 + i) = $ _____

> **Solution**
>
> **(c)** $\frac{\pi}{4}$
> **Solution:** $\arg(a + bi) = \tan^{-1}\left(\frac{b}{a}\right)$ $\arg(1 + i) = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

Q.2 (A) [6 marks]

Attempt any two

0.0.15 Q.2 (A).1 [3 marks]

If $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}$ then prove that $(A + B)^T = A^T + B^T$

     Solution: $A + B = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 5 & 3 \end{bmatrix}$

$(A + B)^T = \begin{bmatrix} 6 & 5 \\ 0 & 3 \end{bmatrix}$

$$A^T = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}, B^T = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}$$

$$A^T + B^T = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 0 & 3 \end{bmatrix}$$

Therefore, $(A + B)^T = A^T + B^T$ **Proved**

0.0.16 Q.2 (A).2 [3 marks]

If $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ then show that $A \cdot A^{-1} = I$

Solution: First, find A^{-1}: $|A| = 1(3) - 1(2) = 3 - 2 = 1$

$$A^{-1} = \frac{1}{|A|} \mathbf{adj}(A) = \frac{1}{1} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

Now verify $A \cdot A^{-1} = I$: $A \cdot A^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1(3) + 1(-2) & 1(-1) + 1(1) \\ 2(3) + 3(-2) & 2(-1) + 3(1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 - 2 & -1 + 1 \\ 6 - 6 & -2 + 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad \textbf{Proved}$$

0.0.17 Q.2 (A).3 [3 marks]

Solve the differential equation $xdy + ydx = 0$

Solution: $xdy + ydx = 0$ $xdy = -ydx$ $\frac{dy}{y} = -\frac{dx}{x}$

Integrating both sides: $\int \frac{dy}{y} = -\int \frac{dx}{x}$ $\ln|y| = -\ln|x| + c_1$ $\ln|y| + \ln|x| = c_1$ $\ln|xy| = c_1$ $|xy| = e^{c_1} = c$ (where $c = e^{c_1}$ is a constant)

Therefore, $xy = \pm c$ or $xy = k$ where k is an arbitrary constant.

Q.2 (B) [8 marks]

Attempt any two

0.0.18 Q.2 (B).1 [4 marks]

If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ then show that $A^2 - 5A + 7I = 0$

Solution: First, calculate A^2: $A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 3(3) + 1(-1) & 3(1) + 1(2) \\ -1(3) + 2(-1) & -1(1) + 2(2) \end{bmatrix}$$

$$= \begin{bmatrix} 9 - 1 & 3 + 2 \\ -3 - 2 & -1 + 4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

Now calculate $5A$: $5A = 5 \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$

And $7I$: $7I = 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

Now verify $A^2 - 5A + 7I = 0$: $A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$$= \begin{bmatrix} 8 - 15 + 7 & 5 - 5 + 0 \\ -5 + 5 + 0 & 3 - 10 + 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \quad \textbf{Proved}$$

0.0.19 Q.2 (B).2 [4 marks]

If $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ then prove that adj $A = A$

Solution: To find adj A, we need to find the cofactor matrix and then transpose it.

Cofactors: $C_{11} = (-1)^{1+1} \begin{vmatrix} 0 & 1 \\ 4 & 3 \end{vmatrix} = 0(3) - 1(4) = -4$

$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = -(1(3) - 1(4)) = -(3 - 4) = 1$

$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 0 \\ 4 & 4 \end{vmatrix} = 1(4) - 0(4) = 4$

$C_{21} = (-1)^{2+1} \begin{vmatrix} -3 & -3 \\ 4 & 3 \end{vmatrix} = -((-3)(3) - (-3)(4)) = -(-9 + 12) = -3$

$C_{22} = (-1)^{2+2} \begin{vmatrix} -4 & -3 \\ 4 & 3 \end{vmatrix} = (-4)(3) - (-3)(4) = -12 + 12 = 0$

$C_{23} = (-1)^{2+3} \begin{vmatrix} -4 & -3 \\ 4 & 4 \end{vmatrix} = -((-4)(4) - (-3)(4)) = -(-16 + 12) = -(-4) = 4$

$C_{31} = (-1)^{3+1} \begin{vmatrix} -3 & -3 \\ 0 & 1 \end{vmatrix} = (-3)(1) - (-3)(0) = -3$

$C_{32} = (-1)^{3+2} \begin{vmatrix} -4 & -3 \\ 1 & 1 \end{vmatrix} = -((-4)(1) - (-3)(1)) = -(-4 + 3) = -(-1) = 1$

$C_{33} = (-1)^{3+3} \begin{vmatrix} -4 & -3 \\ 1 & 0 \end{vmatrix} = (-4)(0) - (-3)(1) = 0 + 3 = 3$

Cofactor matrix $= \begin{bmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{bmatrix}$

adj $A = (\textbf{Cofactor matrix})^T = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix} = A$ **Proved**

0.0.20 Q.2 (B).3 [4 marks]

Solve the following system of linear equations using matrix: $3x + 2y = 5$, $2x - y = 1$

Solution: The system can be written as $AX = B$ **where:** $A = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

Find $|A| = 3(-1) - 2(2) = -3 - 4 = -7$

$A^{-1} = \frac{1}{-7} \begin{bmatrix} -1 & -2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{bmatrix}$

$X = A^{-1}B = \begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{2}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{7}(5) + \frac{2}{7}(1) \\ \frac{2}{7}(5) - \frac{3}{7}(1) \end{bmatrix} = \begin{bmatrix} \frac{5+2}{7} \\ \frac{10-3}{7} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Therefore,

y = 1 $x = 1$,

y = 1

Q.3 (A) [6 marks]

Attempt any two

0.0.21 Q.3 (A).1 [3 marks]

Using definition of differentiation find the derivative of x^5 **with respect to** x

Solution: By definition: $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

For $f(x) = x^5$: $\frac{d}{dx}(x^5) = \lim_{h \to 0} \frac{(x+h)^5 - x^5}{h}$

Using binomial theorem: $(x+h)^5 = x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5$

$\frac{d}{dx}(x^5) = \lim_{h \to 0} \frac{x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5 - x^5}{h}$

$= \lim_{h \to 0} \frac{5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5}{h}$

$= \lim_{h \to 0}(5x^4 + 10x^3h + 10x^2h^2 + 5xh^3 + h^4)$

$= 5x^4 + 0 + 0 + 0 + 0 = 5x^4$

Therefore, $\frac{d}{dx}(x^5) = 5x^4$

0.0.22   Q.3 (A).2 [3 marks]

Find $\frac{dy}{dx}$ **if** $y = \frac{x^2-1}{x^2+1}$

Solution: Using quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Here, $u = x^2 - 1$, $v = x^2 + 1$ $\frac{du}{dx} = 2x$, $\frac{dv}{dx} = 2x$

$\frac{dy}{dx} = \frac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2}$

$= \frac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2}$

$= \frac{2x[(x^2+1)-(x^2-1)]}{(x^2+1)^2}$

$= \frac{2x[x^2+1-x^2+1]}{(x^2+1)^2}$

$= \frac{2x\cdot2}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$

Therefore, $\frac{dy}{dx} = \frac{4x}{(x^2+1)^2}$

0.0.23   Q.3 (A).3 [3 marks]

Evaluate the integral $\int \frac{x^2+5x+6}{x^2+2x}dx$

Solution: First, perform polynomial long division: $\frac{x^2+5x+6}{x^2+2x} = 1 + \frac{3x+6}{x^2+2x}$

$\int \frac{x^2+5x+6}{x^2+2x}dx = \int \left(1 + \frac{3x+6}{x^2+2x}\right)dx$

$= \int 1dx + \int \frac{3x+6}{x^2+2x}dx$

$= x + \int \frac{3x+6}{x(x+2)}dx$

For the second integral, use partial fractions: $\frac{3x+6}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$

$3x + 6 = A(x + 2) + Bx$

When $x = 0$**:** $6 = 2A$**, so** $A = 3$ **When** $x = -2$**:** $-6 + 6 = -2B$**, so** $B = 0$

Wait, let me recalculate: When $x = -2$**:** $3(-2) + 6 = -6 + 6 = 0 = B(-2)$ **When** $x = 0$**:** $6 = 2A$**, so** $A = 3$

Actually: $3x + 6 = 3(x + 2)$ **So** $\frac{3x+6}{x(x+2)} = \frac{3(x+2)}{x(x+2)} = \frac{3}{x}$

$\int \frac{3x+6}{x(x+2)}dx = \int \frac{3}{x}dx = 3\ln|x| + c_1$

Therefore: $\int \frac{x^2+5x+6}{x^2+2x}dx = x + 3\ln|x| + c$

Q.3 (B) [8 marks]

Attempt any two

0.0.24   Q.3 (B).1 [4 marks]

If $y = \log(\sec x + \tan x)$ **then find** $\frac{dy}{dx}$

Solution: $y = \log(\sec x + \tan x)$

$\frac{dy}{dx} = \frac{1}{\sec x + \tan x} \cdot \frac{d}{dx}(\sec x + \tan x)$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{dy}{dx} = \frac{1}{\sec x + \tan x} \cdot (\sec x \tan x + \sec^2 x)$

$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$

$= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x$

Therefore, $\frac{dy}{dx} = \sec x$

0.0.25 Q.3 (B).2 [4 marks]

If $y = 2e^{3x} + 3e^{-2x}$ then prove that $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$

Solution: $y = 2e^{3x} + 3e^{-2x}$

First derivative: $\frac{dy}{dx} = 2(3e^{3x}) + 3(-2e^{-2x}) = 6e^{3x} - 6e^{-2x}$

Second derivative: $\frac{d^2y}{dx^2} = 6(3e^{3x}) - 6(-2e^{-2x}) = 18e^{3x} + 12e^{-2x}$

Now verify the equation: $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y$

$= (18e^{3x} + 12e^{-2x}) - (6e^{3x} - 6e^{-2x}) - 6(2e^{3x} + 3e^{-2x})$

$= 18e^{3x} + 12e^{-2x} - 6e^{3x} + 6e^{-2x} - 12e^{3x} - 18e^{-2x}$

$= e^{3x}(18 - 6 - 12) + e^{-2x}(12 + 6 - 18)$

$= e^{3x}(0) + e^{-2x}(0) = 0$ **Proved**

0.0.26 Q.3 (B).3 [4 marks]

Find the maximum and minimum value of function $f(x) = x^3 - 3x + 11$

Solution: $f(x) = x^3 - 3x + 11$

First derivative: $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x - 1)(x + 1)$

For critical points, set $f'(x) = 0$: $3(x - 1)(x + 1) = 0$ $x = 1$ or $x = -1$

Second derivative: $f''(x) = 6x$

At $x = 1$: $f''(1) = 6 > 0 \to Local\, minimum\, At\, x = -1$: $f''(-1) = -6 < 0 \to Local\, maximum$

Function values: At $x = 1$: $f(1) = 1^3 - 3(1) + 11 = 1 - 3 + 11 = 9$ **At $x = -1$:** $f(-1) = (-1)^3 - 3(-1) + 11 = -1 + 3 + 11 = 13$

Therefore:

- **Local maximum value = 13 at $x = -1$**
- **Local minimum value = 9 at $x = 1$**

Q.4 (A) [6 marks]

Attempt any two

0.0.27 Q.4 (A).1 [3 marks]

Evaluate the integral $\int \frac{\cos(\log x)}{x} dx$

Solution: Let $u = \log x$, then $du = \frac{1}{x} dx$

$\int \frac{\cos(\log x)}{x} dx = \int \cos u \, du = \sin u + c$

Substituting back: $u = \log x$

Therefore, $\int \frac{\cos(\log x)}{x} dx = \sin(\log x) + c$

0.0.28 Q.4 (A).2 [3 marks]

Evaluate the integral $\int x \sin x \, dx$

Solution: Using integration by parts: $\int u \, dv = uv - \int v \, du$

Let $u = x$ and $dv = \sin x \, dx$ Then $du = dx$ and $v = -\cos x$

$\int x \sin x \, dx = x(-\cos x) - \int (-\cos x) dx$

$= -x \cos x + \int \cos x \, dx$

$= -x \cos x + \sin x + c$

Therefore, $\int x \sin x \, dx = \sin x - x \cos x + c$

0.0.29 Q.4 (A).3 [3 marks]

If $(2x - y) + 2yi = 6 + 4i$ then find x and y

Solution: $(2x - y) + 2yi = 6 + 4i$

Comparing real and imaginary parts: Real part: $2x - y = 6$ **... (1) Imaginary part:** $2y = 4$ **... (2)**

From equation (2): $y = 2$

Substituting in equation (1): $2x - 2 = 6$ $2x = 8$ $x = 4$

Therefore, $x = 4$ **and** $y = 2$

Q.4 (B) [8 marks]

Attempt any two

0.0.30 Q.4 (B).1 [4 marks]

Find the area of the region bounded by the curve $y = x^2$**, lines** $x = 1$**,** $x = 2$ **and X-axis**

   Solution: The required area is given by: $A = \int_1^2 x^2 \, dx$

$A = \left[\frac{x^3}{3} \right]_1^2$

$= \frac{2^3}{3} - \frac{1^3}{3}$

$= \frac{8}{3} - \frac{1}{3}$

$= \frac{7}{3}$ **square units**

   Therefore, Area $= \frac{7}{3}$ **square units**

0.0.31 Q.4 (B).2 [4 marks]

Evaluate the definite integral $\int_0^{\pi/2} \frac{\sec x}{\sec x + \csc x} dx$

   Solution: Let $I = \int_0^{\pi/2} \frac{\sec x}{\sec x + \csc x} dx$

   Using the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$**:**

   $I = \int_0^{\pi/2} \frac{\sec(\pi/2-x)}{\sec(\pi/2-x)+\csc(\pi/2-x)} dx$

   Since $\sec(\pi/2 - x) = \csc x$ **and** $\csc(\pi/2 - x) = \sec x$**:**

   $I = \int_0^{\pi/2} \frac{\csc x}{\csc x + \sec x} dx$

   Adding both expressions: $2I = \int_0^{\pi/2} \frac{\sec x}{\sec x + \csc x} dx + \int_0^{\pi/2} \frac{\csc x}{\sec x + \csc x} dx$

   $2I = \int_0^{\pi/2} \frac{\sec x + \csc x}{\sec x + \csc x} dx = \int_0^{\pi/2} 1 \, dx = \frac{\pi}{2}$

   Therefore, $I = \frac{\pi}{4}$

> **Solution**
>
>   **Answer:** $\int_0^{\pi/2} \frac{\sec x}{\sec x + \csc x} dx = \frac{\pi}{4}$

0.0.32 Q.4 (B).3 [4 marks]

If $\alpha + i\beta = \frac{1}{a+ib}$ **then prove that** $(\alpha^2 + \beta^2)(a^2 + b^2) = 1$

   Solution: Given: $\alpha + i\beta = \frac{1}{a+ib}$

   Rationalizing the right side: $\alpha + i\beta = \frac{1}{a+ib} \cdot \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2}$

   $\alpha + i\beta = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$

   Comparing real and imaginary parts: $\alpha = \frac{a}{a^2+b^2}$ **and** $\beta = -\frac{b}{a^2+b^2}$

   Now calculating $\alpha^2 + \beta^2$**:** $\alpha^2 + \beta^2 = \left(\frac{a}{a^2+b^2}\right)^2 + \left(-\frac{b}{a^2+b^2}\right)^2$

   $= \frac{a^2}{(a^2+b^2)^2} + \frac{b^2}{(a^2+b^2)^2}$

   $= \frac{a^2+b^2}{(a^2+b^2)^2} = \frac{1}{a^2+b^2}$

   Therefore: $(\alpha^2 + \beta^2)(a^2 + b^2) = \frac{1}{a^2+b^2} \cdot (a^2 + b^2) = 1$ **Proved**

Q.5 (A) [6 marks]

Attempt any two

0.0.33 Q.5 (A).1 [3 marks]

Find conjugate and modulus of complex number $\frac{2+3i}{3+2i}$

Solution: First, simplify the complex number by rationalizing: $\frac{2+3i}{3+2i} = \frac{2+3i}{3+2i} \cdot \frac{3-2i}{3-2i}$

$= \frac{(2+3i)(3-2i)}{(3+2i)(3-2i)}$

$= \frac{6-4i+9i-6i^2}{9-4i^2}$

$= \frac{6+5i-6(-1)}{9-4(-1)}$

$= \frac{6+5i+6}{9+4} = \frac{12+5i}{13}$

So $\frac{2+3i}{3+2i} = \frac{12}{13} + \frac{5}{13}i$

Conjugate: $\overline{\frac{2+3i}{3+2i}} = \frac{12}{13} - \frac{5}{13}i$

Modulus: $\left|\frac{2+3i}{3+2i}\right| = \sqrt{\left(\frac{12}{13}\right)^2 + \left(\frac{5}{13}\right)^2}$

$= \sqrt{\frac{144}{169} + \frac{25}{169}} = \sqrt{\frac{169}{169}} = \sqrt{1} = 1$

0.0.34 Q.5 (A).2 [3 marks]

Simplify: $\frac{(\cos 3\theta + i \sin 3\theta)^{-4}(\cos \theta - i \sin \theta)^{-5}}{(\cos 2\theta - i \sin 2\theta)^7}$

Solution: Using De Moivre's theorem: $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$

Also, $\cos \theta - i \sin \theta = \cos(-\theta) + i \sin(-\theta)$

$(\cos 3\theta + i \sin 3\theta)^{-4} = \cos(-12\theta) + i \sin(-12\theta)$

$(\cos \theta - i \sin \theta)^{-5} = (\cos(-\theta) + i \sin(-\theta))^{-5} = \cos(5\theta) + i \sin(5\theta)$

$(\cos 2\theta - i \sin 2\theta)^7 = (\cos(-2\theta) + i \sin(-2\theta))^7 = \cos(-14\theta) + i \sin(-14\theta)$

Therefore: $\frac{(\cos 3\theta + i \sin 3\theta)^{-4}(\cos \theta - i \sin \theta)^{-5}}{(\cos 2\theta - i \sin 2\theta)^7}$

$= \frac{[\cos(-12\theta)+i\sin(-12\theta)][\cos(5\theta)+i\sin(5\theta)]}{\cos(-14\theta)+i\sin(-14\theta)}$

$= \frac{\cos(-12\theta+5\theta)+i\sin(-12\theta+5\theta)}{\cos(-14\theta)+i\sin(-14\theta)}$

$= \frac{\cos(-7\theta)+i\sin(-7\theta)}{\cos(-14\theta)+i\sin(-14\theta)}$

$= \cos(-7\theta + 14\theta) + i \sin(-7\theta + 14\theta)$

$= \cos(7\theta) + i \sin(7\theta)$

0.0.35 Q.5 (A).3 [3 marks]

Express Complex number $1 + \sqrt{3}i$ **into polar form**

Solution: For complex number $z = a + bi$, **polar form is** $z = r(\cos \theta + i \sin \theta)$

Here, $a = 1$, $b = \sqrt{3}$

Modulus: $r = |z| = \sqrt{a^2 + b^2} = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$

Argument: $\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

Therefore, the polar form is: $1 + \sqrt{3}i = 2\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$

Q.5 (B) [8 marks]

Attempt any two

0.0.36 Q.5 (B).1 [4 marks]

Solve: $\tan y \, dx + \tan x \sec^2 y \, dy = 0$

Solution: $\tan y \, dx + \tan x \sec^2 y \, dy = 0$

Rearranging: $\tan y \, dx = -\tan x \sec^2 y \, dy$

$\frac{dx}{\tan x} = -\frac{\sec^2 y \, dy}{\tan y}$

$\frac{\cos x}{\sin x} dx = -\frac{dy}{\sin y \cos y}$

$\cot x \, dx = -\frac{dy}{\sin y \cos y}$

Since $\frac{1}{\sin y \cos y} = \frac{2}{2 \sin y \cos y} = \frac{2}{\sin 2y}$**:**

$\cot x \, dx = -\frac{2dy}{\sin 2y}$

Integrating both sides: $\int \cot x \, dx = -2 \int \csc(2y) \, dy$

$\ln|\sin x| = -2 \cdot \left(-\frac{1}{2} \ln|\csc(2y) + \cot(2y)|\right) + c$

$\ln|\sin x| = \ln|\csc(2y) + \cot(2y)| + c$

Therefore: $\sin x \cdot [\csc(2y) + \cot(2y)] = k$ **where** k **is a constant.**

0.0.37   Q.5 (B).2 [4 marks]

Solve: $x\frac{dy}{dx} - y = x^2$

Solution: $x\frac{dy}{dx} - y = x^2$

Dividing by x**:** $\frac{dy}{dx} - \frac{y}{x} = x$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Here, $P = -\frac{1}{x}$ **and** $Q = x$

Integrating factor: $I.F. = e^{\int P dx} = e^{\int -\frac{1}{x} dx} = e^{-\ln|x|} = \frac{1}{x}$

Multiplying the equation by I.F.: $\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 1$

This can be written as: $\frac{d}{dx}\left(\frac{y}{x}\right) = 1$

Integrating: $\frac{y}{x} = x + c$

Therefore: $y = x^2 + cx$

0.0.38   Q.5 (B).3 [4 marks]

Solve: $\frac{dy}{dx} + \frac{y}{x} = e^x$, $y(0) = 3$

Solution: This is a linear differential equation: $\frac{dy}{dx} + \frac{y}{x} = e^x$

Here, $P = \frac{1}{x}$ **and** $Q = e^x$

Integrating factor: $I.F. = e^{\int \frac{1}{x} dx} = e^{\ln|x|} = |x| = x$ **(assuming** $x > 0$**)**

Multiplying the equation by I.F.: $x\frac{dy}{dx} + y = xe^x$

This can be written as: $\frac{d}{dx}(xy) = xe^x$

Integrating both sides: $xy = \int xe^x dx$

Using integration by parts for $\int xe^x dx$**: Let** $u = x$**,** $dv = e^x dx$ **Then** $du = dx$**,** $v = e^x$

$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x = e^x(x - 1)$

So: $xy = e^x(x - 1) + c$

Therefore: $y = \frac{e^x(x-1)+c}{x}$

Using initial condition $y(0) = 3$**: This presents a problem as we have division by zero. Let me reconsider the approach.**

Actually, let's solve this more carefully. The equation $\frac{dy}{dx} + \frac{y}{x} = e^x$ **with** $y(0) = 3$ **has an issue because at** $x = 0$**, we have division by zero.**

For the general solution away from $x = 0$**:** $y = \frac{e^x(x-1)+c}{x}$

The initial condition suggests we need to examine the behavior near $x = 0$**.**

General solution: $y = \frac{e^x(x-1)+c}{x}$ **for** $x \neq 0$

Formula Cheat Sheet

0.0.39   Matrix Operations

- **Matrix multiplication:** $(AB)_{ij} = \sum_k A_{ik} B_{kj}$
- **Inverse of** $2 \times 2 matrix: A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ **for** $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- **Determinant:** $|A| = ad - bc$

0.0.40 Differentiation Rules

- **Power rule:** $\frac{d}{dx}(x^n) = nx^{n-1}$
- **Product rule:** $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
- **Quotient rule:** $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
- **Chain rule:** $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

0.0.41 Integration Rules

- **Power rule:** $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ **(for $n \neq -1$)**
- **Integration by parts:** $\int u\,dv = uv - \int v\,du$
- **Fundamental theorem:** $\int_a^b f(x)dx = F(b) - F(a)$

0.0.42 Differential Equations

- **Linear first order:** $\frac{dy}{dx} + Py = Q$, **Solution:** $y \cdot I.F. = \int Q \cdot I.F.\,dx$
- **Integrating factor:** $I.F. = e^{\int P dx}$
- **Variable separable:** $\frac{dy}{dx} = f(x)g(y) \rightarrow \frac{dy}{g(y)} = f(x)dx$

0.0.43 Complex Numbers

- **Polar form:** $z = r(\cos\theta + i\sin\theta)$
- **Modulus:** $|a + bi| = \sqrt{a^2 + b^2}$
- **Argument:** $\arg(a + bi) = \tan^{-1}(b/a)$
- **De Moivre's theorem:** $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

Problem-Solving Strategies

1. **Matrix Problems: Always check dimensions before multiplication**
2. **Differentiation: Identify which rule applies (product, quotient, chain)**
3. **Integration: Look for substitution opportunities first**
4. **Differential Equations: Identify type (separable vs linear) before solving**
5. **Complex Numbers: Convert to standard form before operations**

Common Mistakes to Avoid

1. **Matrix multiplication: Order matters - $AB \neq BA$ in general**
2. **Differentiation: Don't forget the chain rule for composite functions**
3. **Integration: Always add the constant of integration**
4. **Complex numbers: Be careful with signs when rationalizing**

Exam Tips

1. **Time management: Allocate time based on marks (1 mark = 2-3 minutes)**
2. **Show work: Partial marks are awarded for correct steps**
3. **Check units: Ensure final answers have appropriate units**
4. **Verify: When possible, substitute back to check answers**