Mathematics-I Solutions

DI01000021 – Winter 2024

Semester 1 Study Material

Detailed Solutions and Explanations

Question 1 [14 marks]

Fill in the blanks/MCQs using appropriate choice from the given options.

Q1.1 [1 mark]

$\begin{vmatrix} 5 & 1 \\ 2 & 3 \end{vmatrix} =$ ___________

Solution

Answer: b. 13

Solution: For 2×2 determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$\begin{vmatrix} 5 & 1 \\ 2 & 3 \end{vmatrix} = (5 \times 3) - (1 \times 2) = 15 - 2 = 13$

Q1.2 [1 mark]

If $A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$**, then** $2A =$ ___________

Solution

Answer: c. $\begin{bmatrix} 2 & -4 \\ -6 & 8 \end{bmatrix}$

Solution: Scalar multiplication multiplies every element by the scalar. $2 \times \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(-2) \\ 2(-3) & 2(4) \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -6 & 8 \end{bmatrix}$

Q1.3 [1 mark]

$\log(xy) =$ ___________

Solution

Answer: a. $\log x + \log y$

Solution: Product rule of logarithms: $\log_b(mn) = \log_b(m) + \log_b(n)$

Q1.4 [1 mark]

The value of $\log_{10} 0.001$ **is** ___________

Solution

Answer: d. -3

Solution: $\log_{10}(0.001) = \log_{10}(10^{-3}) = -3\log_{10} 10 = -3(1) = -3$

Q1.5 [1 mark]

If $\sin\theta = \frac{3}{5}$**, then cosec** $\theta =$ ____________

Solution

Answer: b. $\frac{5}{3}$
Solution: cosec θ is the reciprocal of $\sin\theta$. cosec $\theta = \frac{1}{\sin\theta} = \frac{1}{3/5} = \frac{5}{3}$

Q1.6 [1 mark]

The period of $\sin(3x)$ **is** ____________

Solution

Answer: b. $2\pi/3$
Solution: The period of $\sin(bx)$ is $\frac{2\pi}{|b|}$. Here $b = 3$, so period $= 2\pi/3$.

Q1.7 [1 mark]

The value of $\sin^{-1}(\frac{1}{2})$ **is** ____________

Solution

Answer: a. $\pi/6$
Solution: $\sin(\pi/6) = \sin(30^\circ) = 1/2$. Therefore, $\sin^{-1}(1/2) = \pi/6$.

Q1.8 [1 mark]

The range of $\cos^{-1} x$ **is** ____________

Solution

Answer: b. $[0, \pi]$
Solution: By definition, the principal value range for arccos is $[0, \pi]$.

Q1.9 [1 mark]

The area of a triangle with vertices $(0, 0), (4, 0), (0, 3)$ **is** ____________

Solution

Answer: a. 6
Solution: Base $= 4$, Height $= 3$ (Right angled triangle). Area $= \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 4 \times 3 = 6$.

Q1.10 [1 mark]

The distance between points $(1, 2)$ **and** $(4, 6)$ **is** ____________

Solution

Answer: c. 5
Solution: Distance formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $d = \sqrt{(4 - 1)^2 + (6 - 2)^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

Q1.11 [1 mark]

$\lim_{x\to 2}(x^2 + 3) =$ ____________

Solution

Answer: d. 7
Solution: Direct substitution: $2^2 + 3 = 4 + 3 = 7$.

Q1.12 [1 mark]

$\lim_{x\to 0} \frac{\sin x}{x} =$ ____________

Solution

Answer: a. 1
Solution: Standard limit identity.

Q1.13 [1 mark]

If $f(x) = x^3$**, then** $f'(x) =$ ____________

Solution

Answer: b. $3x^2$
Solution: Power rule: $\frac{d}{dx}(x^n) = nx^{n-1}$. $n = 3$, so $3x^{3-1} = 3x^2$.

Q1.14 [1 mark]

$\int x^2 dx =$ ____________

Solution

Answer: c. $\frac{x^3}{3} + c$
Solution: Power rule for integration: $\int x^n dx = \frac{x^{n+1}}{n+1} + c$. $n = 2$, so $\frac{x^{2+1}}{2+1} = \frac{x^3}{3} + c$.

Question 2 [14 marks]

Q2.a [3 marks]

Solve the system of linear equations using Matrix Inversion Method: $2x + y = 5$ $3x - 2y = 4$

Solution

Solution: System form $AX = B$: $A = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$
Step 1: Find $|A|$ (Determinant) $|A| = (2)(-2) - (1)(3) = -4 - 3 = -7$ Since $|A| \neq 0$, inverse exists.
Step 2: Find A^{-1} For 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, adj $A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ adj $A = \begin{bmatrix} -2 & -1 \\ -3 & 2 \end{bmatrix}$ $A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{-7}\begin{bmatrix} -2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2/7 & 1/7 \\ 3/7 & -2/7 \end{bmatrix}$
Step 3: Solve for $X = A^{-1}B$ $X = \frac{1}{-7}\begin{bmatrix} -2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ $X = \frac{1}{-7}\begin{bmatrix} (-2)(5) + (-1)(4) \\ (-3)(5) + (2)(4) \end{bmatrix}$ $X = \frac{1}{-7}\begin{bmatrix} -10 - 4 \\ -15 + 8 \end{bmatrix} = \frac{1}{-7}\begin{bmatrix} -14 \\ -7 \end{bmatrix}$ $X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
Therefore, $x = 2, y = 1$.

Q2.b [4 marks]

Prove that $\log(\frac{a^2}{bc}) + \log(\frac{b^2}{ac}) + \log(\frac{c^2}{ab}) = 0$

Solution

Proof: LHS $= \log(\frac{a^2}{bc}) + \log(\frac{b^2}{ac}) + \log(\frac{c^2}{ab})$
Using property $\log m + \log n + \log p = \log(mnp)$: $= \log\left(\frac{a^2}{bc} \times \frac{b^2}{ac} \times \frac{c^2}{ab}\right)$
Combine numerators and denominators: $= \log\left(\frac{a^2 \cdot b^2 \cdot c^2}{(bc)(ac)(ab)}\right) = \log\left(\frac{a^2b^2c^2}{a^2b^2c^2}\right)$
Simplify fraction: $= \log(1)$
We know $\log(1) = 0$. $= 0 =$ RHS
Hence Proved.

Q2.c [7 marks]

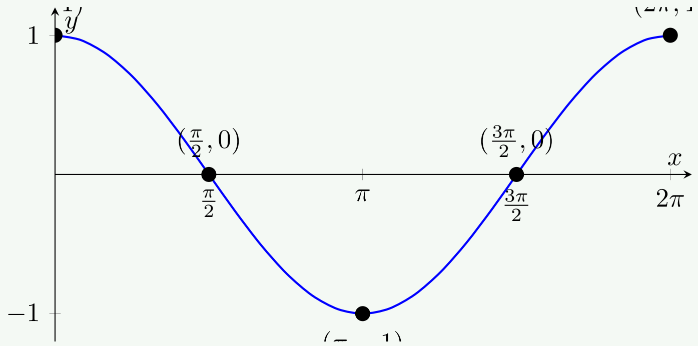
Draw the graph of $y = \cos x$ for $0 \le x \le 2\pi$.

Solution

Table of values:

x	0	$\pi/2$	π	$3\pi/2$	2π
$y = \cos x$	1	0	-1	0	1

Graph:



Question 3 [14 marks]

Q3.a [3 marks]

Find the value of $\sin(75°)$.

Solution

Solution: $\sin(75°) = \sin(45° + 30°)$
Using identity $\sin(A + B) = \sin A \cos B + \cos A \sin B$: $= \sin(45°)\cos(30°) + \cos(45°)\sin(30°)$
Substitute standard values: $= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

Q3.b [4 marks]

Prove that $\frac{1-\cos A}{\sin A} = \tan(\frac{A}{2})$.

Solution

Proof: LHS $= \frac{1-\cos A}{\sin A}$
Using half-angle identities: $1 - \cos A = 2\sin^2(A/2)$ $\sin A = 2\sin(A/2)\cos(A/2)$
Substitute into expression: $= \frac{2\sin^2(A/2)}{2\sin(A/2)\cos(A/2)}$
Cancel common terms (2 and $\sin(A/2)$): $= \frac{\sin(A/2)}{\cos(A/2)} = \tan(A/2) =$ RHS
Hence Proved.

Q3.c [7 marks]

Inverse Trigonometry Problem: Prove $\tan^{-1}(\frac{1}{2}) + \tan^{-1}(\frac{1}{3}) = \frac{\pi}{4}$**.**

Solution

Proof: Using identity $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ if $xy < 1$.
Here $x = 1/2, y = 1/3$. $xy = (1/2)(1/3) = 1/6 < 1$. Condition satisfied.
LHS $= \tan^{-1}\left(\frac{1/2+1/3}{1-(1/2)(1/3)}\right)$ Numerator: $1/2 + 1/3 = \frac{3+2}{6} = 5/6$ Denominator: $1 - 1/6 = 5/6$
$= \tan^{-1}\left(\frac{5/6}{5/6}\right) = \tan^{-1}(1)$
Since $\tan(\pi/4) = 1$: $= \pi/4 =$ RHS
Hence Proved.

Question 4 [14 marks]

Q4.a [3 marks]

Find the midpoint of the line segment joining $A(2, 3)$ **and** $B(4, 7)$**.**

Solution

Solution: Midpoint formula $M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
$x_1 = 2, y_1 = 3$ $x_2 = 4, y_2 = 7$
$M = \left(\frac{2+4}{2}, \frac{3+7}{2}\right)$ $M = \left(\frac{6}{2}, \frac{10}{2}\right)$ $M = (3, 5)$

Q4.b [4 marks]

Find the equation of a line passing through $(2, -1)$ **with slope** 3**.**

Solution

Solution: Point-slope form: $y - y_1 = m(x - x_1)$ Given $m = 3, (x_1, y_1) = (2, -1)$.
$y - (-1) = 3(x - 2)$ $y + 1 = 3x - 6$ $y = 3x - 7$ or $3x - y - 7 = 0$

Q4.c [7 marks]

Evaluate $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$**.**

Solution

Solution: Direct substitution yields 0/0 (Indeterminate form). Rationalize the numerator: $= \lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} \times \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} = \lim_{x\to 0} \frac{(1+x)-1}{x(\sqrt{1+x}+1)} = \lim_{x\to 0} \frac{x}{x(\sqrt{1+x}+1)}$
Cancel x ($x \neq 0$): $= \lim_{x\to 0} \frac{1}{\sqrt{1+x}+1}$
Now substitute $x = 0$: $= \frac{1}{\sqrt{1+0}+1} = \frac{1}{1+1} = \frac{1}{2}$

Question 5 [14 marks]

Q5.a [3 marks]

Differentiate $y = e^x \sin x$ **with respect to** x**.**

Solution

Solution: Using Product Rule: $(uv)' = u'v + uv'$ Let $u = e^x \Rightarrow u' = e^x$ Let $v = \sin x \Rightarrow v' = \cos x$
$\frac{dy}{dx} = (e^x)(\sin x) + (e^x)(\cos x)$ $\frac{dy}{dx} = e^x(\sin x + \cos x)$

Q5.b [4 marks]

Evaluate $\int(3x^2 + 4x - 5)dx$.

Solution

Solution: Integrate term by term: $= \int 3x^2dx + \int 4xdx - \int 5dx = 3\frac{x^3}{3} + 4\frac{x^2}{2} - 5x + c = x^3 + 2x^2 - 5x + c$

Q5.c [7 marks]

Find the maximum and minimum values of $f(x) = x^3 - 3x^2 + 2$ on $[-1, 3]$.

Solution

Solution: Step 1: Find critical points ($f'(x) = 0$). $f'(x) = 3x^2 - 6x$ $3x(x-2) = 0 \Rightarrow x = 0, x = 2$ Both points are in $[-1, 3]$.
Step 2: Evaluate $f(x)$ at critical points and endpoints. Endpoints: $x = -1, x = 3$.
Calculate values: $f(-1) = (-1)^3 - 3(-1)^2 + 2 = -1 - 3 + 2 = -2$ $f(0) = 0^3 - 3(0)^2 + 2 = 2$ $f(2) = 2^3 - 3(2)^2 + 2 = 8 - 12 + 2 = -2$ $f(3) = 3^3 - 3(3)^2 + 2 = 27 - 27 + 2 = 2$
Maximum Value = 2 (at $x = 0$ and $x = 3$) Minimum Value = -2 (at $x = -1$ and $x = 2$)

Formula Cheat Sheet

Key Formula

Logarithms: $\log(xy) = \log x + \log y$ $\log(x/y) = \log x - \log y$ $\log(x^n) = n \log x$

Key Formula

Trigonometry: $\sin^2 x + \cos^2 x = 1$ $\sin(A + B) = \sin A \cos B + \cos A \sin B$

Key Formula

Differentiation: $\frac{d}{dx}(x^n) = nx^{n-1}$ $\frac{d}{dx}(\sin x) = \cos x$ Product Rule: $(uv)' = u'v + uv'$