Applied Mathematics (4320001) - Summer 2024 Solution

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Question 1 [14 marks]

Fill in the blanks

Question 1.1 [1 marks]

Order of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \end{bmatrix}$ **is =** _____ **Answer**: (b) 2×3

> **Solution**
>
> A matrix with 2 rows and 3 columns has order 2×3.

Question 1.2 [1 marks]

If $\begin{bmatrix} x-3 & 2 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 4 & 0 \end{bmatrix}$ **then** $x =$ _____ **Answer**: (d) 8

> **Solution**
>
> For matrix equality, corresponding elements must be equal: $x - 3 = 5$ $x = 8$

Question 1.3 [1 marks]

The adjoint of $\begin{bmatrix} -3 & 2 \\ 0 & 1 \end{bmatrix} =$ _____ **Answer**: (b) $\begin{bmatrix} 1 & -2 \\ 0 & -3 \end{bmatrix}$

> **Solution**
>
> For matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $\text{adj} \begin{bmatrix} -3 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & -3 \end{bmatrix}$

Question 1.4 [1 marks]

For any square matrix A, $(A^{-1})^{-1} =$ _____ **Answer**: (b) A

> **Solution**
>
> By definition of inverse matrices: $(A^{-1})^{-1} = A$

Question 1.5 [1 marks]

$\frac{d}{dx}\log x =$ _____ **Answer:** (b) $\frac{1}{x}$

> **Solution**
>
> The derivative of natural logarithm: $\frac{d}{dx}\log x = \frac{1}{x}$

Question 1.6 [1 marks]

$\frac{d}{dx}(\tan^{-1} x + \cot^{-1} x) =$ _____ **Answer:** (d) 0

> **Solution**
>
> $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$ (constant) Therefore, $\frac{d}{dx}(\tan^{-1} x + \cot^{-1} x) = 0$

Question 1.7 [1 marks]

If $x = a\cos\theta$, $y = a\sin\theta$ **then** $\frac{dy}{dx} =$ _____ **Answer:** (a) $-\cot\theta$

> **Solution**
>
> $\frac{dx}{d\theta} = -a\sin\theta$, $\frac{dy}{d\theta} = a\cos\theta$ $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta$

Question 1.8 [1 marks]

$\int 5x^4 dx =$ _____ $+ c$ **Answer:** (d) x^5

> **Solution**
>
> $\int 5x^4 dx = 5 \cdot \frac{x^5}{5} = x^5 + c$

Question 1.9 [1 marks]

$\int_0^1 e^x dx =$ _____ **Answer:** (a) $e - 1$

> **Solution**
>
> $\int_0^1 e^x dx = [e^x]_0^1 = e^1 - e^0 = e - 1$

Question 1.10 [1 marks]

$\int_{-1}^{1} 3x^2 - 2x + 1 dx =$ _____ **Answer**: (c) 4

> **Solution**
>
> $\int_{-1}^{1}(3x^2 - 2x + 1)dx = [x^3 - x^2 + x]_{-1}^{1} = (1 - 1 + 1) - (-1 - 1 - 1) = 1 - (-3) = 4$

Question 1.11 [1 marks]

The order of differential equation $(\frac{dy}{dx})^2 + 4y = x$ is _____ **Answer**: (d) 1

> **Solution**
>
> Order is the highest derivative present. Here, only first derivative $\frac{dy}{dx}$ appears, so order $= 1$.

Question 1.12 [1 marks]

The integrating factor of $\frac{dy}{dx} + 3y = x$ is _____ **Answer**: (d) e^{3x}

> **Solution**
>
> For linear DE $\frac{dy}{dx} + Py = Q$, integrating factor $= e^{\int P dx}$ Here $P = 3$, so I.F. $= e^{\int 3dx} = e^{3x}$

Question 1.13 [1 marks]

The mean of first ten natural numbers is _____ **Answer**: (a) 5.5

> **Solution**
>
> Mean $= \frac{1+2+3+...+10}{10} = \frac{55}{10} = 5.5$

Question 1.14 [1 marks]

The range of the data 17, 15, 25, 34, 32 is _____ **Answer**: (d) 19

> **Solution**
>
> Range $=$ Maximum - Minimum $=$ 34 - 15 $=$ 19

Question 2(a) [6 marks]

Attempt any two

Question 2(a)(1) [3 marks]

If $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ **then find** $A + A^T + I$.

> **Solution**
>
> $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$
>
> $A^T = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$
>
> $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
>
> $A + A^T + I = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
>
> $= \begin{bmatrix} 3 & 1 \\ 1 & 7 \end{bmatrix}$

Question 2(a)(2) [3 marks]

If $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ **then prove that** $A^2 - 4A + 7I_2 = 0$ **Answer**: Proved

> **Solution**
>
> $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$
>
> $A^2 = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix}$
>
> $4A = 4\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ -4 & 8 \end{bmatrix}$
>
> $7I_2 = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$
>
> $A^2 - 4A + 7I_2 = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ -4 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$
>
> $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \checkmark$

Question 2(a)(3) [3 marks]

Solve differential equation $dy - 3x^2 e^{-y}dx = 0$ **Answer**: $e^y = x^3 + C$

> **Solution**
>
> $dy - 3x^2 e^{-y} dx = 0$ $dy = 3x^2 e^{-y} dx$ $e^y dy = 3x^2 dx$
> Integrating both sides: $\int e^y dy = \int 3x^2 dx$ $e^y = x^3 + C$

Question 2(b) [8 marks]

Attempt any two

Question 2(b)(1) [4 marks]

Find the inverse of matrix $\begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & -1 \\ 5 & 0 & 1 \end{bmatrix}$ **Answer:** $A^{-1} = \begin{bmatrix} 1/14 & 1/14 & -1/14 \\ -9/14 & -7/14 & 11/14 \\ -5/14 & -5/14 & 1/2 \end{bmatrix}$

> **Solution**
>
> Let $A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & -1 \\ 5 & 0 & 1 \end{bmatrix}$
>
> First, find $\det(A)$: $\det(A) = 3(1 \cdot 1 - (-1) \cdot 0) - (-1)(4 \cdot 1 - (-1) \cdot 5) + 2(4 \cdot 0 - 1 \cdot 5) = 3(1) + 1(9) + 2(-5) = 3 + 9 - 10 = 2$
> Since $\det(A) \neq 0$, inverse exists.
> Finding cofactors and adjoint matrix: $C_{11} = 1$, $C_{12} = -9$, $C_{13} = -5$ $C_{21} = 1$, $C_{22} = -7$, $C_{23} = -5$ $C_{31} = -1$, $C_{32} = 11$, $C_{33} = 7$
>
> $\text{adj}(A) = \begin{bmatrix} 1 & 1 & -1 \\ -9 & -7 & 11 \\ -5 & -5 & 7 \end{bmatrix}$
>
> $A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A) = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ -9 & -7 & 11 \\ -5 & -5 & 7 \end{bmatrix}$

Question 2(b)(2) [4 marks]

If $A + B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$ **and** $A - B = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$ **then find** AB. **Answer:** $AB = \begin{bmatrix} 0 & -1 \\ 4 & -2 \end{bmatrix}$

> **Solution**
>
> Adding the equations: $(A + B) + (A - B) = 2A$ $2A = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 4 & 4 \end{bmatrix}$ $A = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$
>
> Subtracting the equations: $(A + B) - (A - B) = 2B$ $2B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 2 & -4 \end{bmatrix}$ $B = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$
>
> $AB = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 0 & -6 \end{bmatrix}$

Question 2(b)(3) [4 marks]

Solve the system of linear equation $2x + 3y = 1$, $y - 4x = 2$ **using matrices.** **Answer**: $x = -\frac{1}{11}$, $y = \frac{13}{11}$

> **Solution**
>
> The system can be written as: $AX = B$ $\begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
>
> $\det(A) = 2(1) - 3(-4) = 2 + 12 = 14$
>
> $A^{-1} = \frac{1}{14} \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}$
>
> $X = A^{-1}B = \frac{1}{14} \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} -5 \\ 8 \end{bmatrix}$
>
> Therefore: $x = -\frac{5}{14}$, $y = \frac{8}{14} = \frac{4}{7}$
>
> Wait, MDX answer says $x = -1/11, y = 13/11$. Let's check the calculation in the MDX text. MDX Solution says: $X = \frac{1}{14} \begin{bmatrix} -5 \\ 8 \end{bmatrix}$ So $x = -5/14, y = 8/14 = 4/7$. But MDX Answer block says: $x = -1/11, y = 13/11$. There is a contradiction in the source MDX. Let's calculate $\det(A)$ for $2x + 3y = 1$ and $y - 4x = 2 \implies -4x + y = 2$.
>
> $A = \begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}$. $\det = 2 - (-12) = 14$. $A^{-1} = \frac{1}{14} \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}$. $X = \frac{1}{14} \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 1 - 6 \\ 4 + 4 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} -5 \\ 8 \end{bmatrix}$. So
>
> $x = -5/14, y = 4/7$. The MDX Answer block is wrong or describes a different problem. MDX says: **Answer**: $x = -\frac{1}{11}$, $y = \frac{13}{11}$ **Solution**: ... Therefore: $x = -\frac{5}{14}$, $y = \frac{8}{14} = \frac{4}{7}$
>
> I must follow the "Solution" part for the body, but usually checking what to do with the "Answer" block. The workflow says "Strict Fidelity". I will copy EXACTLY what is in MDX, including the contradiction. However, I will trust the computation in the Solution block for the steps. I'll just copy the text as is.

Question 3(a) [6 marks]

Attempt any two

Question 3(a)(1) [3 marks]

Find the derivative of $f(x) = e^x$ **using definition of derivative.** **Answer**: $f'(x) = e^x$

> **Solution**
>
> Using the definition: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
>
> $f'(x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x$

Question 3(a)(2) [3 marks]

If $\sqrt{x} + \sqrt{y} = \sqrt{a}$ **then prove that** $\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$ **Answer**: Proved

> **Solution**
>
> $\sqrt{x} + \sqrt{y} = \sqrt{a}$
>
> Differentiating both sides with respect to x: $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0$

$\frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$

$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} = -\sqrt{\frac{y}{x}}$ ✓

Question 3(a)(3) [3 marks]

Evaluate $\int \frac{\tan x}{\sec x + \tan x} dx$ **Answer:** $x - \ln|\sec x + \tan x| + C$

> **Solution**
>
> Let $I = \int \frac{\tan x}{\sec x + \tan x} dx$
>
> Multiply numerator and denominator by $(\sec x - \tan x)$: $I = \int \frac{\tan x(\sec x - \tan x)}{(\sec x + \tan x)(\sec x - \tan x)} dx = \int \frac{\tan x(\sec x - \tan x)}{\sec^2 x - \tan^2 x} dx$
>
> $= \int \frac{\tan x(\sec x - \tan x)}{1} dx = \int (\tan x \sec x - \tan^2 x) dx = \int \tan x \sec x dx - \int (\sec^2 x - 1) dx = \sec x - \tan x + x + C$

Question 3(b) [8 marks]

Attempt any two

Question 3(b)(1) [4 marks]

If $e^x + e^y = e^{x+y}$ **then find** $\frac{dy}{dx}$**. Answer:** $\frac{dy}{dx} = \frac{e^x(e^y - 1)}{e^y(e^x - 1)}$

> **Solution**
>
> $e^x + e^y = e^{x+y}$
>
> Differentiating both sides with respect to x: $e^x + e^y \frac{dy}{dx} = e^{x+y}(1 + \frac{dy}{dx})$ $e^x + e^y \frac{dy}{dx} = e^{x+y} + e^{x+y} \frac{dy}{dx}$
>
> Rearranging: $e^x - e^{x+y} = e^{x+y} \frac{dy}{dx} - e^y \frac{dy}{dx}$ $e^x - e^{x+y} = \frac{dy}{dx}(e^{x+y} - e^y)$
>
> $\frac{dy}{dx} = \frac{e^x - e^{x+y}}{e^{x+y} - e^y} = \frac{e^x(1 - e^y)}{e^y(e^x - 1)} = \frac{e^x(e^y - 1)}{e^y(e^x - 1)}$

Question 3(b)(2) [4 marks]

For $y = 2e^{3x} + 3e^{-2x}$**, prove that** $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$**. Answer:** Proved

> **Solution**
>
> $y = 2e^{3x} + 3e^{-2x}$
>
> $\frac{dy}{dx} = 6e^{3x} - 6e^{-2x}$
>
> $\frac{d^2y}{dx^2} = 18e^{3x} + 12e^{-2x}$
>
> Now checking the equation: $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = (18e^{3x} + 12e^{-2x}) - (6e^{3x} - 6e^{-2x}) - 6(2e^{3x} + 3e^{-2x}) = 18e^{3x} + 12e^{-2x} - 6e^{3x} + 6e^{-2x} - 12e^{3x} - 18e^{-2x} = (18 - 6 - 12)e^{3x} + (12 + 6 - 18)e^{-2x} = 0 \cdot e^{3x} + 0 \cdot e^{-2x} = 0$ ✓

Question 3(b)(3) [4 marks]

Equation of motion of a moving particle given by $s = t^3 + 3t$**,** $t > 0$**, when the velocity and acceleration will be equal? Answer:** At $t = 1$ second

Solution

Given: $s = t^3 + 3t$
Velocity: $v = \frac{ds}{dt} = 3t^2 + 3$ Acceleration: $a = \frac{dv}{dt} = 6t$
For velocity = acceleration: $3t^2 + 3 = 6t$ $3t^2 - 6t + 3 = 0$ $t^2 - 2t + 1 = 0$ $(t-1)^2 = 0$ $t = 1$
Therefore, velocity and acceleration are equal at $t = 1$ second.

Question 4(a) [6 marks]

Attempt any two

Question 4(a)(1) [3 marks]

Evaluate: $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$ **Answer:** $-2\cos\sqrt{x} + C$

Solution

Let $u = \sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}dx$, so $dx = 2\sqrt{x}du = 2udu$
$\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx = \int \frac{\sin u}{u} \cdot 2udu = 2\int \sin u\,du = -2\cos u + C = -2\cos\sqrt{x} + C$

Question 4(a)(2) [3 marks]

Evaluate: $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$ **Answer:** $\frac{\pi}{4}$

Solution

Let $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$

Using property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$: $I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2-x)}}{\sqrt{\cos(\pi/2-x)}+\sqrt{\sin(\pi/2-x)}}dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$

Adding both expressions: $2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx = \int_0^{\pi/2} 1dx = \frac{\pi}{2}$
Therefore: $I = \frac{\pi}{4}$

Question 4(a)(3) [3 marks]

Find the mean of the frequency distribution:

Solution

Table 1. Frequency Distribution

Age	20-24	25-29	30-34	35-39	40-44	45-49	50-54	55-59
Staff	5	7	9	11	10	8	6	4

Answer: Mean = 37.5 years

Table 2. Mean Calculation

Class	Midpoint (x)	Frequency (f)	fx
20-24	22	5	110
25-29	27	7	189
30-34	32	9	288
35-39	37	11	407
40-44	42	10	420
45-49	47	8	376
50-54	52	6	312
55-59	57	4	228
Total		**60**	**2330**

Mean $= \frac{\sum fx}{\sum f} = \frac{2330}{60} = 38.83$ years

Question 4(b) [8 marks]

Attempt any two

Question 4(b)(1) [4 marks]

Evaluate: $\int_0^1 \frac{x^2}{1+x^6} dx$ **Answer**: $\frac{\pi}{12}$

Solution

Let $u = x^3$, then $du = 3x^2 dx$, so $x^2 dx = \frac{1}{3} du$ When $x = 0$, $u = 0$; when $x = 1$, $u = 1$
$\int_0^1 \frac{x^2}{1+x^6} dx = \int_0^1 \frac{1}{1+u^2} \cdot \frac{1}{3} du = \frac{1}{3} \int_0^1 \frac{1}{1+u^2} du = \frac{1}{3} [\tan^{-1} u]_0^1 = \frac{1}{3} (\tan^{-1} 1 - \tan^{-1} 0) = \frac{1}{3} \cdot \frac{\pi}{4} = \frac{\pi}{12}$

Question 4(b)(2) [4 marks]

Find area enclosed by curve $y = x^2$, X-axis and $x = 2$ **Answer**: Area $= \frac{8}{3}$ square units

Solution

The area is bounded by $y = x^2$, $y = 0$ (X-axis), $x = 0$ and $x = 2$
Area $= \int_0^2 x^2 dx = \left[\frac{x^3}{3} \right]_0^2 = \frac{8}{3} - 0 = \frac{8}{3}$ square units

Question 4(b)(3) [4 marks]

Calculate the standard deviation for the following continuous grouped data:

Solution

Table 3. Grouped Data

Class	0-10	10-20	20-30	30-40	40-50
Frequency	5	8	15	16	6

Answer: Standard deviation = 10.95

Table 4. Standard Deviation Calculation

Class	Midpoint (x)	f	fx	x^2	fx^2
0-10	5	5	25	25	125
10-20	15	8	120	225	1800
20-30	25	15	375	625	9375
30-40	35	16	560	1225	19600
40-50	45	6	270	2025	12150
Total		**50**	**1350**		**43050**

Mean $\bar{x} = \frac{1350}{50} = 27$

Variance $= \frac{\sum fx^2}{n} - (\bar{x})^2 = \frac{43050}{50} - (27)^2 = 861 - 729 = 132$

Standard deviation $= \sqrt{132} = 11.49$

Question 5(a) [6 marks]

Attempt any two

Question 5(a)(1) [3 marks]

If mean of 25 observation is 50 and mean of other 75 observation is 60. Considering all the observation then find the mean. **Answer**: Combined mean = 57.5

Solution

Combined mean $= \frac{n_1\bar{x_1}+n_2\bar{x_2}}{n_1+n_2} = \frac{25\times50+75\times60}{25+75} = \frac{1250+4500}{100} = \frac{5750}{100} = 57.5$

Question 5(a)(2) [3 marks]

Find the mean deviation for the following frequency distribution:

Solution

Table 5. Frequency Distribution

x_i	3	4	5	6	7	8
f_i	1	3	7	5	2	2

Answer: Mean deviation = 1.1

Table 6. Mean Deviation Calculation

x_i	f_i	$f_i x_i$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
3	1	3	2	2
4	3	12	1	3
5	7	35	0	0
6	5	30	1	5
7	2	14	2	4
8	2	16	3	6
Total	**20**	**110**		**20**

Mean $\bar{x} = \frac{110}{20} = 5.5$

Recalculating deviations from mean $= 5.5$: Mean deviation $= \frac{\sum f_i \lvert x_i - \bar{x} \rvert}{\sum f_i} = \frac{22}{20} = 1.1$

Question 5(a)(3) [3 marks]

Calculate the standard deviation for the following ungrouped data:
120, 132, 148, 136, 142, 140, 165, 153 Answer: Standard deviation $= 13.36$

Solution

Table 7. Data Table

x	$x - \bar{x}$	$(x - \bar{x})^2$
120	-19.5	380.25
132	-7.5	56.25
148	8.5	72.25
136	-3.5	12.25
142	2.5	6.25
140	0.5	0.25
165	25.5	650.25
153	13.5	182.25
Total	**0**	**1360**

$n = 8$, $\sum x = 1116$ Mean $\bar{x} = \frac{1116}{8} = 139.5$

Variance $= \frac{\sum (x - \bar{x})^2}{n} = \frac{1360}{8} = 170$

Standard deviation $= \sqrt{170} = 13.04$

Question 5(b) [8 marks]

Attempt any two

Question 5(b)(1) [4 marks]

Solve: $\frac{dy}{dx} + \tan x \cdot \tan y = 0$ **Answer**: $\ln \lvert \cos y \rvert = \ln \lvert \cos x \rvert + C$ or $\cos y = A \cos x$

> **Solution**
>
> $\frac{dy}{dx} + \tan x \cdot \tan y = 0$ $\frac{dy}{dx} = -\tan x \cdot \tan y$ $\frac{dy}{\tan y} = -\tan x \, dx$ $\cot y \, dy = -\tan x \, dx$
>
> Integrating both sides: $\int \cot y \, dy = -\int \tan x \, dx$ $\ln|\sin y| = \ln|\cos x| + C_1$ $\ln|\sin y| - \ln|\cos x| = C_1$ $\ln\left|\frac{\sin y}{\cos x}\right| = C_1$
>
> Taking exponential: $\frac{\sin y}{\cos x} = C$ (where $C = e^{C_1}$) $\sin y = C \cos x$
> Alternative form: $\cos y = A \cos x$ where A is a constant.

Question 5(b)(2) [4 marks]

Solve: $\frac{dy}{dx} + 2y = 3e^x$ **Answer**: $y = e^x + Ce^{-2x}$

> **Solution**
>
> This is a first-order linear differential equation of the form $\frac{dy}{dx} + Py = Q$ where $P = 2$ and $Q = 3e^x$
> Integrating factor: $I.F. = e^{\int P \, dx} = e^{\int 2 \, dx} = e^{2x}$
> Multiplying the equation by e^{2x}: $e^{2x}\frac{dy}{dx} + 2e^{2x}y = 3e^{3x}$
> The left side is the derivative of ye^{2x}: $\frac{d}{dx}(ye^{2x}) = 3e^{3x}$
> Integrating both sides: $ye^{2x} = \int 3e^{3x} \, dx = e^{3x} + C$
> Therefore: $y = e^x + Ce^{-2x}$

Question 5(b)(3) [4 marks]

Solve: $dy + 4xy^2 dx = 0$; $y(0) = 1$ **Answer**: $y = \frac{1}{1+2x^2}$

> **Solution**
>
> $dy + 4xy^2 dx = 0$ $dy = -4xy^2 dx$ $\frac{dy}{y^2} = -4x \, dx$
> Integrating both sides: $\int y^{-2} \, dy = \int -4x \, dx$ $-\frac{1}{y} = -2x^2 + C$ $\frac{1}{y} = 2x^2 - C$
> Using initial condition $y(0) = 1$: $\frac{1}{1} = 2(0)^2 - C$ $1 = -C$ $C = -1$
> Therefore: $\frac{1}{y} = 2x^2 + 1$ $y = \frac{1}{2x^2+1}$

Formula Cheat Sheet

Matrix Operations

- **Matrix Addition/Subtraction**: Element-wise operation
- **Matrix Multiplication**: $(AB)_{ij} = \sum_k a_{ik}b_{kj}$
- **Transpose**: $(A^T)_{ij} = A_{ji}$
- **Determinant (2×2)**: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$
- **Inverse (2×2)**: $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- **Adjoint (2×2)**: $\text{adj} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Differentiation Formulas

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(\ln x) = \frac{1}{x}$

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
- **Chain Rule**: $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$
- **Product Rule**: $(uv)' = u'v + uv'$
- **Quotient Rule**: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

Integration Formulas

- $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$ (for $n \neq -1$)
- $\int \frac{1}{x} \, dx = \ln|x| + C$
- $\int e^x \, dx = e^x + C$
- $\int \sin x \, dx = -\cos x + C$
- $\int \cos x \, dx = \sin x + C$
- $\int \sec^2 x \, dx = \tan x + C$
- $\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$
- **Integration by Parts**: $\int u \, dv = uv - \int v \, du$

Differential Equations

- **Variable Separable**: $\frac{dy}{dx} = f(x)g(y) \Rightarrow \frac{dy}{g(y)} = f(x)dx$
- **Linear DE**: $\frac{dy}{dx} + Py = Q$, Solution: $y \cdot I.F. = \int Q \cdot I.F. \, dx$
- **Integrating Factor**: $I.F. = e^{\int P \, dx}$

Statistics Formulas

- **Mean**: $\bar{x} = \frac{\sum x_i}{n}$ (ungrouped), $\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$ (grouped)
- **Mean Deviation**: $M.D. = \frac{\sum |x_i - \bar{x}|}{n}$ (ungrouped), $M.D. = \frac{\sum f_i |x_i - \bar{x}|}{\sum f_i}$ (grouped)
- **Standard Deviation**: $\sigma = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n}}$ (ungrouped)
- **Variance**: $\sigma^2 = \frac{\sum (x_i - \bar{x})^2}{n}$
- **Range**: Maximum value - Minimum value
- **Combined Mean**: $\bar{x} = \frac{n_1 \bar{x_1} + n_2 \bar{x_2}}{n_1 + n_2}$