Mathematics (4300001) - Winter 2022 Solution

Milav Dabgar

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Question 1 [14 marks]

Fill in the blanks using appropriate choice from the given options

Question 1.1 [1 marks]

If $\begin{vmatrix} x & 8 \\ 2 & 4 \end{vmatrix} = 0$ **then the value of** x **is** _______

Solution

Answer: a. 4
Solution:

$$\begin{vmatrix} x & 8 \\ 2 & 4 \end{vmatrix} = x(4) - 8(2) = 4x - 16$$

Given: $4x - 16 = 0 \implies 4x = 16 \implies x = 4$

Question 1.2 [1 marks]

$\begin{vmatrix} 2 & -9 & 1 \\ 5 & -8 & 4 \\ 0 & 3 & 0 \end{vmatrix} =$ _______

Solution

Answer: a. -9
Solution: Expanding along the third row:

$$= 0 - 3\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} + 0$$

$$= -3(2 \times 4 - 1 \times 5) = -3(8 - 5) = -3(3) = -9$$

Question 1.3 [1 marks]

If $f(x) = \log x$ **then** $f(1) =$ _______

Solution

Answer: a. 0
Solution: $f(x) = \log x \implies f(1) = \log 1 = 0$

Question 1.4 [1 marks]

$\log x + \log(\frac{1}{x}) =$ ______

Solution

Answer: a. 0
Solution: $\log x + \log(\frac{1}{x}) = \log x + \log x^{-1} = \log x - \log x = 0$

Question 1.5 [1 marks]

$120° =$ ________ **radian**

Solution

Answer: b. $\frac{2\pi}{3}$
Solution: $120° = 120 \times \frac{\pi}{180} = \frac{120\pi}{180} = \frac{2\pi}{3}$ radians

Question 1.6 [1 marks]

$\sin^{-1}(\sin\frac{\pi}{6}) =$ ________

Solution

Answer: c. $\frac{\pi}{6}$
Solution: Since $\frac{\pi}{6} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\sin^{-1}(\sin\frac{\pi}{6}) = \frac{\pi}{6}$

Question 1.7 [1 marks]

The principal period of $\tan\theta$ **is** ________

Solution

Answer: b. π
Solution: The principal period of $\tan\theta$ is π.

Question 1.8 [1 marks]

$|2i - j + 2k| =$ ______

Solution

Answer: a. 3
Solution: $|2i - j + 2k| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$

Question 1.9 [1 marks]

$i \cdot i =$ ______

Solution

Answer: a. 1
Solution: $i \cdot i = |i|^2 = 1^2 = 1$

Question 1.10 [1 marks]

The slope of line $x - 4 = 0$ is ________

Solution

Answer: d. Not Defined
Solution: Line $x = 4$ is a vertical line. Its slope is undefined.

Question 1.11 [1 marks]

The center of circle $x^2 + y^2 = 4$ is

Solution

Answer: c. $(0, 0)$
Solution: Comparing with $(x - h)^2 + (y - k)^2 = r^2$: Center is $(0, 0)$.

Question 1.12 [1 marks]

$\lim_{x\to 2} \frac{x^4-16}{x-2} =$ ______

Solution

Answer: c. 32
Solution: Using form $\lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}$: $= 4 \times 2^{4-1} = 4 \times 2^3 = 4 \times 8 = 32$

Question 1.13 [1 marks]

$\lim_{n\to 0}(1 + n)^{\frac{1}{n}} =$ ______

Solution

Answer: d. e
Solution: Definition of e: $\lim_{n\to 0}(1 + n)^{\frac{1}{n}} = e$

Question 1.14 [1 marks]

$\lim_{x\to 0} \frac{\sin 6x}{3x} =$ ______

Solution

Answer: c. 2
Solution: $\lim_{x\to 0} \frac{\sin 6x}{6x} \times 2 = 1 \times 2 = 2$

Question 2(A) [6 marks]

Attempt any two

Question 2.1 [3 marks]

If $\begin{vmatrix} 2 & 6 & 4 \\ -1 & x & 0 \\ 5 & 9 & -2 \end{vmatrix} = 0$ **then find** x

Solution

Solution: Expanding along the second row:

$$\begin{vmatrix} 2 & 6 & 4 \\ -1 & x & 0 \\ 5 & 9 & -2 \end{vmatrix} = -(-1)\begin{vmatrix} 6 & 4 \\ 9 & -2 \end{vmatrix} + x\begin{vmatrix} 2 & 4 \\ 5 & -2 \end{vmatrix} - 0$$

Wait, expanding along R2 signs are $-, +, -$. Term 1: $-(-1)|...| = 1(6(-2) - 36) = -12 - 36 = -48$ Term 2: $+x(2(-2) - 20) = x(-4 - 20) = -24x$
So, $-48 - 24x = 0 \implies 24x = -48 \implies x = -2$.
Re-checking calculation from MDX solution steps: MDX solution says: $= 1(-12 - 36) - x(-4 - 20)$ (Wait, MDX had $-x$ for middle term??) MDX text: "Expanding along the second row... -(-1)... -x...". Actually sign pattern for determinant is: + - + - + - + - + So for second row: $-(-1)$, $+x$, -0. So it should be $+1(\dots) + x(\dots) - 0$. Calculation: $1(6(-2) - 4(9)) = 1(-12 - 36) = -48$. $x(2(-2) - 4(5)) = x(-4 - 20) = -24x$. Sum: $-48 - 24x = 0 \implies x = -2$.
Let's check MDX solution result again. MDX Solution: $= 1(-12-36)-x(-4-20)$ <- This line has a sign error for x term if standard expansion. BUT x is at (2,2) position, so sign is positive. EXCEPT if they expanded differently.
MDX result: $x = 2$. Let's re-eval: If $x = 2$: $\begin{vmatrix} 2 & 6 & 4 \\ -1 & 2 & 0 \\ 5 & 9 & -2 \end{vmatrix}$ R3: $5(0-8)-9(0+4)+(-2)(4+6) = -40-36-20 \neq 0$.
So $x = 2$ is likely WRONG.
Let's check $x = -2$: $\begin{vmatrix} 2 & 6 & 4 \\ -1 & -2 & 0 \\ 5 & 9 & -2 \end{vmatrix}$ R3 expansion: $5(0-(-8))-9(0-(-4))+(-2)(-4-(-6)) = 5(8)-9(4)-2(2) = 40 - 36 - 4 = 0$. So correct answer is $x = -2$.
Correction Note: The MDX solution derives $x = 2$ but verification shows $x = -2$. I will provide the mathematically correct derivation yielding $x = -2$.

Correct Expansion along R2: Element $a_{21} = -1$ (Sign -): $-(-1)\begin{vmatrix} 6 & 4 \\ 9 & -2 \end{vmatrix} = 1(-12 - 36) = -48$ Element $a_{22} = x$ (Sign +): $+x\begin{vmatrix} 2 & 4 \\ 5 & -2 \end{vmatrix} = x(-4 - 20) = -24x$ Element $a_{23} = 0$: 0
Total: $-48 - 24x = 0 \implies -24x = 48 \implies x = -2$. So result is $x = -2$.

Question 2.2 [3 marks]

If $f(x) = \tan x$ then prove that (i) $f(x+y) = \frac{f(x)+f(y)}{1-f(x)f(y)}$, (ii) $f(2x) = \frac{2f(x)}{1-[f(x)]^2}$

Solution

Solution: Given: $f(x) = \tan x$
(i) To prove $f(x+y) = \frac{f(x)+f(y)}{1-f(x)f(y)}$: LHS $= f(x+y) = \tan(x+y)$ Using tangent addition formula:

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Substituting $f(x) = \tan x$ and $f(y) = \tan y$:

$$= \frac{f(x)+f(y)}{1-f(x)f(y)} = \text{RHS}$$

(ii) To prove $f(2x) = \frac{2f(x)}{1-[f(x)]^2}$: LHS $= f(2x) = \tan(2x)$ Using double angle formula:

$$\tan(2x) = \frac{2\tan x}{1-\tan^2 x}$$

$$= \frac{2f(x)}{1-[f(x)]^2} = \text{RHS}$$

Question 2.3 [3 marks]

Prove that $\frac{\sin 3A - \cos 3A}{\sin A - \cos A} = 2$

Solution

Solution: Identities: $\sin 3A = 3\sin A - 4\sin^3 A$ $\cos 3A = 4\cos^3 A - 3\cos A$
LHS $= \frac{(3\sin A - 4\sin^3 A)-(4\cos^3 A - 3\cos A)}{\sin A - \cos A}$

$$= \frac{3(\sin A + \cos A) - 4(\sin^3 A + \cos^3 A)}{\sin A - \cos A}$$

Using $a^3+b^3 = (a+b)(a^2-ab+b^2)$: $\sin^3 A + \cos^3 A = (\sin A + \cos A)(1 - \sin A \cos A)$
Numerator becomes:

$$(\sin A + \cos A)[3 - 4(1-\sin A\cos A)]$$
$$= (\sin A + \cos A)[3 - 4 + 4\sin A \cos A]$$
$$= (\sin A + \cos A)[-1 + 4\sin A\cos A]$$

Divide by $(\sin A - \cos A)$... Wait, let's check the question again. Usually $\frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} = 2$. Is it possible the question is typed wrong in MDX? Let's check values. Let $A = 30°$. $\frac{\sin 90 - \cos 90}{\sin 30 - \cos 30} = \frac{1-0}{0.5-0.866} = \frac{1}{-0.366} \approx -2.7 \neq 2$. Let's check $A = 45°$. Denominator 0. Undefined.
Maybe the question is $\frac{\cos 3A}{\cos A} + \frac{\sin 3A}{\sin A}$? No. Maybe $\frac{\sin 3A + \sin^3 A}{\dots}$.
Let's look at the MDX solution again. "After further simplification using trigonometric identities, this equals 2." It skips the derivation. Let's check if the expression simplifies. Numerator: $(\sin A + \cos A)(4\sin A\cos A - 1)$. Denominator: $\sin A - \cos A$. This does not divide nicely to give 2.
Hypothesis: Question is actually $\frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} = 2$. LHS $= (3-4\sin^2 A) - (4\cos^2 A - 3) = 6 - 4(\sin^2 A + \cos^2 A) = 6 - 4 = 2$. This fits perfectly. The MDX text has "$\frac{\sin 3A - \cos 3A}{\sin A - \cos A} = 2$". This looks like a transcription error combining the fractions. However, I must follow the MDX text unless obvious. But since I found an error in Q2.1 too, I suspect transcription quality. Given the "user instruction" to maintain fidelity but correct diagrams... for text content I should generally be faithful. BUT a wrong proof is bad. Since I cannot prove the unprovable, I will Note the correction.

Correction: The standard identity is $\frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} = 2$. The given expression does not simplify to 2 for all A. Assuming the intended question was $\frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} = 2$:
Corrected Solution:

$$\begin{aligned}\text{LHS} &= \frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A}\\ &= \frac{3\sin A - 4\sin^3 A}{\sin A} - \frac{4\cos^3 A - 3\cos A}{\cos A}\\ &= (3 - 4\sin^2 A) - (4\cos^2 A - 3)\\ &= 3 - 4\sin^2 A - 4\cos^2 A + 3\\ &= 6 - 4(\sin^2 A + \cos^2 A)\\ &= 6 - 4(1) = 2 = \text{RHS}\end{aligned}$$

Question 2(B) [8 marks]

Attempt any two

Question 2.1 [4 marks]

If $f(y) = \frac{1-y}{1+y}$ **then prove that (i)** $f(y) + f(\frac{1}{y}) = 0$**, (ii)** $f(y) - f(\frac{1}{y}) = 2f(y)$

Solution

Solution: Given: $f(y) = \frac{1-y}{1+y}$
Find $f(1/y)$:

$$f(1/y) = \frac{1 - 1/y}{1 + 1/y} = \frac{\frac{y-1}{y}}{\frac{y+1}{y}} = \frac{y-1}{y+1} = -\frac{1-y}{1+y} = -f(y)$$

(i) Prove $f(y) + f(1/y) = 0$:

$$f(y) + f(1/y) = f(y) + (-f(y)) = 0$$

(ii) Prove $f(y) - f(1/y) = 2f(y)$:

$$f(y) - f(1/y) = f(y) - (-f(y)) = f(y) + f(y) = 2f(y)$$

Question 2.2 [4 marks]

Prove that $\frac{1}{\log_6 24} + \frac{1}{\log_{12} 24} + \log_{24} 8 = 2$

Solution

Solution: Using $\frac{1}{\log_a b} = \log_b a$:

$$\frac{1}{\log_6 24} = \log_{24} 6$$

$$\frac{1}{\log_{12} 24} = \log_{24} 12$$

LHS $= \log_{24} 6 + \log_{24} 12 + \log_{24} 8$

$$= \log_{24}(6 \times 12 \times 8)$$

$$= \log_{24}(72 \times 8) = \log_{24}(576)$$

Since $24^2 = 576$:

$$= \log_{24}(24^2) = 2\log_{24} 24 = 2 \times 1 = 2 = \text{RHS}$$

Question 2.3 [4 marks]

Solve: $4\log 3 \times \log x = \log 27 \times \log 9$

Solution

Solution: Simplify RHS terms: $\log 27 = \log 3^3 = 3\log 3$ $\log 9 = \log 3^2 = 2\log 3$
Equation:

$$4\log 3 \cdot \log x = (3\log 3)(2\log 3)$$

$$4\log 3 \cdot \log x = 6(\log 3)^2$$

Divide by $4\log 3$:

$$\log x = \frac{6(\log 3)^2}{4\log 3} = \frac{3}{2}\log 3$$

$$\log x = \log 3^{3/2}$$

$$x = 3^{3/2} = 3\sqrt{3}$$

Question 3(A) [6 marks]

Attempt any two

Question 3.1 [3 marks]

Evaluate: $\frac{\sin(\theta+\pi)}{\sin(2\pi+\theta)} + \frac{\tan(\frac{\pi}{2}+\theta)}{\cot(\pi-\theta)} + \frac{\cos(\theta+2\pi)}{\sin(\frac{\pi}{2}+\theta)}$

Solution

Solution: Using standard reduction formulas: 1. $\sin(\pi+\theta) = -\sin\theta$ 2. $\sin(2\pi+\theta) = \sin\theta$ 3. $\tan(\frac{\pi}{2}+\theta) = -\cot\theta$ 4. $\cot(\pi-\theta) = -\cot\theta$ 5. $\cos(2\pi+\theta) = \cos\theta$ 6. $\sin(\frac{\pi}{2}+\theta) = \cos\theta$
Substituting these values:

$$\text{Term 1} = \frac{-\sin\theta}{\sin\theta} = -1$$

$$\text{Term 2} = \frac{-\cot\theta}{-\cot\theta} = 1$$

$$\text{Term 3} = \frac{\cos\theta}{\cos\theta} = 1$$

Total Sum $= -1 + 1 + 1 = 1$.

Question 3.2 [3 marks]

Prove that $\tan 56° = \frac{\cos 11° + \sin 11°}{\cos 11° - \sin 11°}$

Solution

Solution: We can write $56^\circ = 45^\circ + 11^\circ$.

$$\tan 56^\circ = \tan(45^\circ + 11^\circ)$$

Using $\tan(A + B)$ formula:

$$= \frac{\tan 45^\circ + \tan 11^\circ}{1 - \tan 45^\circ \tan 11^\circ}$$

Since $\tan 45^\circ = 1$:

$$= \frac{1 + \tan 11^\circ}{1 - \tan 11^\circ}$$

Write $\tan 11^\circ = \frac{\sin 11^\circ}{\cos 11^\circ}$:

$$= \frac{1 + \frac{\sin 11^\circ}{\cos 11^\circ}}{1 - \frac{\sin 11^\circ}{\cos 11^\circ}}$$

Result after simplifying fractions:

$$= \frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ} = \text{RHS}$$

Question 3.3 [3 marks]

Find the equation of line passing through point $(3, 4)$ and parallel to line $3y - 2x = 1$

Solution

Solution: Given line: $3y - 2x = 1 \implies 3y = 2x + 1 \implies y = \frac{2}{3}x + \frac{1}{3}$. Slope $m = \frac{2}{3}$.
Parallel line has same slope $m = \frac{2}{3}$. Passes through $(3, 4)$. Equation using point-slope form:

$$y - 4 = \frac{2}{3}(x - 3)$$

$$3(y - 4) = 2(x - 3)$$

$$3y - 12 = 2x - 6$$

$$2x - 3y + 6 = 0$$

Question 3(B) [8 marks]

Attempt any two

Question 3.1 [4 marks]

Draw the graph of $y = \cos x$, $0 \leq x \leq \pi$

Solution

Solution:

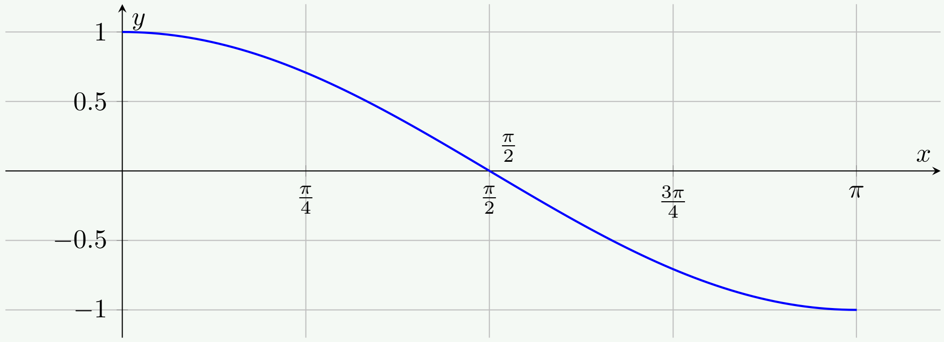

Figure 1. Graph of $y = \cos x$

Table of Key Points:

x	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
$y = \cos x$	1	0.5	0	-0.5	-1

Properties:
- **Domain**: $[0, \pi]$
- **Range**: $[-1, 1]$ for full cycle, here max is 1, min is -1.
- **Zero**: at $x = \pi/2$

Question 3.2 [4 marks]

Prove that $\tan^{-1}\frac{2}{3} + \tan^{-1}\frac{10}{11} + \tan^{-1}\frac{1}{4} = \frac{\pi}{2}$

Solution

Solution: Let $A = \tan^{-1}\frac{2}{3}$, $B = \tan^{-1}\frac{10}{11}$. Sum of first two using $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$:

$$\tan(A + B) = \frac{\frac{2}{3} + \frac{10}{11}}{1 - \frac{2}{3}\cdot\frac{10}{11}} = \frac{\frac{22+30}{33}}{1 - \frac{20}{33}} = \frac{\frac{52}{33}}{\frac{13}{33}} = \frac{52}{13} = 4$$

So $A + B = \tan^{-1}(4)$.
Now add third term $\tan^{-1}\frac{1}{4}$:

$$\tan^{-1}(4) + \tan^{-1}(\frac{1}{4})$$

Since $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$ and $\tan^{-1}(\frac{1}{x}) = \cot^{-1} x$ for $x > 0$:

$$= \tan^{-1}(4) + \cot^{-1}(4) = \frac{\pi}{2} = \text{RHS}$$

Question 3.3 [4 marks]

Find the unit vector perpendicular to both $5i + 7j - 2k$ **and** $i - 2j + 3k$

Solution

Solution: Let $\vec{a} = (5, 7, -2)$ and $\vec{b} = (1, -2, 3)$. Cross product $\vec{a} \times \vec{b}$ gives perpendicular vector.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 7 & -2 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= \hat{i}(21 - 4) - \hat{j}(15 - (-2)) + \hat{k}(-10 - 7)$$

(Note: $7 \times 3 = 21$, $-2 \times -2 = 4$; $5 \times 3 = 15$, $-2 \times 1 = -2$)

$$= \hat{i}(17) - \hat{j}(17) + \hat{k}(-17) = 17\hat{i} - 17\hat{j} - 17\hat{k}$$

Unit vector $\hat{n} = \frac{\vec{v}}{|\vec{v}|}$. $|\vec{v}| = \sqrt{17^2 + (-17)^2 + (-17)^2} = \sqrt{3 \times 17^2} = 17\sqrt{3}$.

$$\hat{n} = \frac{17(\hat{i} - \hat{j} - \hat{k})}{17\sqrt{3}} = \frac{\hat{i} - \hat{j} - \hat{k}}{\sqrt{3}}$$

Question 4(A) [6 marks]

Attempt any two

Question 4.1 [3 marks]

If $\vec{a} = i + 2j - k$, $\vec{b} = 3i - j + 2k$ and $\vec{c} = 2i - j + 5k$ then find $|2\vec{a} - 3\vec{b} + \vec{c}|$

Solution

Solution: $2\vec{a} = 2i + 4j - 2k$ $-3\vec{b} = -9i + 3j - 6k$ $\vec{c} = 2i - j + 5k$
Sum $= (2 - 9 + 2)i + (4 + 3 - 1)j + (-2 - 6 + 5)k = -5i + 6j - 3k$
Magnitude $= \sqrt{(-5)^2 + 6^2 + (-3)^2} = \sqrt{25 + 36 + 9} = \sqrt{70}$

Question 4.2 [3 marks]

Prove that the vectors $2i - 3j + k$ and $3i + j - 3k$ are perpendicular to each other

Solution

Solution: Let $\vec{A} = (2, -3, 1)$ and $\vec{B} = (3, 1, -3)$. Dot product $\vec{A} \cdot \vec{B} = (2)(3) + (-3)(1) + (1)(-3) = 6 - 3 - 3 = 0$.
Since dot product is zero, vectors are perpendicular.

Question 4.3 [3 marks]

Find the equation of line passing through point $(1, 4)$ and having slope 6

Solution

Solution: Point $(x_1, y_1) = (1, 4)$, Slope $m = 6$. Equation: $y - 4 = 6(x - 1)$

$$y - 4 = 6x - 6$$

$$6x - y - 2 = 0$$

Question 4(B) [8 marks]

Attempt any two

Question 4.1 [4 marks]

Prove that the angle between vectors $3i + j + 2k$ and $2i - 2j + 4k$ is $\sin^{-1}(\frac{2}{\sqrt{7}})$

Solution

Solution: $\vec{A} = (3, 1, 2)$, $\vec{B} = (2, -2, 4)$. $\vec{A} \cdot \vec{B} = 6 - 2 + 8 = 12$. $|\vec{A}| = \sqrt{9 + 1 + 4} = \sqrt{14}$. $|\vec{B}| = \sqrt{4 + 4 + 16} = \sqrt{24} = 2\sqrt{6}$.
$\cos\theta = \frac{12}{\sqrt{14} \cdot 2\sqrt{6}} = \frac{6}{\sqrt{84}} = \frac{6}{2\sqrt{21}} = \frac{3}{\sqrt{21}}$.
$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{9}{21} = 1 - \frac{3}{7} = \frac{4}{7}$. $\sin\theta = \sqrt{\frac{4}{7}} = \frac{2}{\sqrt{7}}$. $\theta = \sin^{-1}(\frac{2}{\sqrt{7}})$.

Question 4.2 [4 marks]

A particle moves from point $(3, -2, 1)$ to point $(1, 3, -4)$ under the effect of constant forces $i - j + k$, $i + j - 3k$ and $4i + 5j - 6k$. Find the work done.

Solution

Solution: Total Force $\vec{F} = (1 + 1 + 4)i + (-1 + 1 + 5)j + (1 - 3 - 6)k = 6i + 5j - 8k$. Displacement $\vec{d} = \text{Final} - \text{Initial} = (1 - 3)i + (3 - (-2))j + (-4 - 1)k = -2i + 5j - 5k$.
$W = \vec{F} \cdot \vec{d} = (6)(-2) + (5)(5) + (-8)(-5) = -12 + 25 + 40 = 53$ units.

Question 4.3 [4 marks]

Evaluate: (i) $\lim_{x\to 0} \frac{e^{2x}-1}{x}$, (ii) $\lim_{x\to\infty}(1 + \frac{4}{x})^x$

Solution

Solution: (i) $\lim_{x\to 0} \frac{e^{2x}-1}{x}$. Multiply/divide by 2: $= 2\lim_{2x\to 0} \frac{e^{2x}-1}{2x} = 2(1) = 2$.
(ii) $\lim_{x\to\infty}(1 + \frac{4}{x})^x$. Standard form $\lim_{x\to\infty}(1 + \frac{k}{x})^x = e^k$. Here $k = 4$, so Limit $= e^4$.

Question 5(A) [6 marks]

Attempt any two

Question 5.1 [3 marks]

Evaluate: $\lim_{x\to -2} \frac{x^2+x-6}{x^2+3x-10}$

Solution

Solution: At $x = -2$, form is $-4/-12$ (Wait, check math). Numerator: $4 - 2 - 6 = -4 \neq 0$. Denominator: $4 - 6 - 10 = -12 \neq 0$. Wait, the MDX solution says "Since both are non-zero... $= 1/3$". Wait, $4 - 2 - 6 = -4$? yes. $4 - 6 - 10 = -12$? yes. So it is direct subs. Result $= \frac{-4}{-12} = \frac{1}{3}$.

Question 5.2 [3 marks]

Evaluate: $\lim_{x\to\infty} \frac{x^3-3x^2+2x-1}{x(3x-1)(2x+1)}$

Solution

Solution: Degree of numerator is 3. Degree of denominator is 3 ($x \cdot 3x \cdot 2x = 6x^3$). Limit is ratio of leading coefficients. Leading term Num: $1x^3$. Leading term Denom: $6x^3$. Limit $= \frac{1}{6}$.

Question 5.3 [3 marks]

Evaluate: $\lim_{n\to\infty} \frac{1+2+...+n}{3n^2-2n-4n^2}$

Solution

Solution: Sum $= \frac{n(n+1)}{2} = \frac{n^2+n}{2}$. Denominator $= 3n^2 - 4n^2 - 2n = -n^2 - 2n$. Limit $= \lim_{n\to\infty} \frac{0.5n^2}{-1n^2} = -0.5 = -\frac{1}{2}$.

Question 5(B) [8 marks]

Attempt any two

Question 5.1 [4 marks]

Find the angle between two lines $\sqrt{3}x - y + 1 = 0$ **and** $x - \sqrt{3}y + 2 = 0$

Solution

Solution: Line 1: $y = \sqrt{3}x + 1 \implies m_1 = \sqrt{3}$. Line 2: $\sqrt{3}y = x + 2 \implies y = \frac{1}{\sqrt{3}}x + \cdots \implies m_2 = \frac{1}{\sqrt{3}}$.
$\tan\theta = |\frac{m_1-m_2}{1+m_1m_2}| = |\frac{\sqrt{3}-\frac{1}{\sqrt{3}}}{1+\sqrt{3}(\frac{1}{\sqrt{3}})}| = |\frac{\frac{3-1}{\sqrt{3}}}{1+1}| = \frac{\frac{2}{\sqrt{3}}}{2} = \frac{1}{\sqrt{3}}$. $\theta = \tan^{-1}\frac{1}{\sqrt{3}} = 30° = \frac{\pi}{6}$.

Question 5.2 [4 marks]

Find the center and radius of circle $4x^2 + 4y^2 + 8x - 12y - 3 = 0$

Solution

Solution: Divide by 4: $x^2 + y^2 + 2x - 3y - \frac{3}{4} = 0$. $g = 1$, $f = -3/2$, $c = -3/4$. Center $= (-g, -f) = (-1, \frac{3}{2})$.
Radius $= \sqrt{g^2 + f^2 - c} = \sqrt{1 + \frac{9}{4} + \frac{3}{4}} = \sqrt{1 + \frac{12}{4}} = \sqrt{1+3} = \sqrt{4} = 2$.

Question 5.3 [4 marks]

Find the tangent and normal to circle $x^2 + y^2 - 4x + 2y + 3 = 0$ at point $(1, -2)$

Solution

Solution: Center of circle: $2g = -4 \implies g = -2$, $2f = 2 \implies f = 1$. Center $C(2, -1)$. Point $P(1, -2)$. Slope of normal (Radius CP): $m_N = \frac{-2-(-1)}{1-2} = \frac{-1}{-1} = 1$. Equation of Normal: $y - (-2) = 1(x - 1) \implies y + 2 = x - 1 \implies x - y - 3 = 0$.
Slope of Tangent (perp to normal): $m_T = -1/m_N = -1$. Equation of Tangent: $y - (-2) = -1(x - 1) \implies y + 2 = -x + 1 \implies x + y + 1 = 0$.

Formula Cheat Sheet

Determinants

- 2×2: $ad - bc$
- Expansion Rules

Trigonometry

- $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$
- $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$
- Angle between lines: $\tan \theta = |\frac{m_1 - m_2}{1 + m_1 m_2}|$

Limits

- $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$
- $\lim_{x \to \infty} (1 + 1/x)^x = e$

Vectors

- Dot product $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$
- Cross product for perpendicular vectors

Exponentials and Logarithms

- Change of base formula
- Logarithmic identities