

Mathematics (4300001) - Winter 2024 Solution

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Question 1 [14 marks]

Fill in the blanks using appropriate choice from the given options

Question 1.1 [1 marks]

If $f(x) = \frac{1}{x}$, then the value of $f(1)$ is _____

Solution

Answer: b. 1

Solution: $f(x) = \frac{1}{x}$

$$f(1) = \frac{1}{1} = 1$$

Question 1.2 [1 marks]

$\log_b a \times \log_a b =$ _____

Solution

Answer: b. 1

Solution: Using the change of base formula: $\log_b a = \frac{1}{\log_a b}$

$$\log_b a \times \log_a b = \frac{1}{\log_a b} \times \log_a b = 1$$

Question 1.3 [1 marks]

If $\begin{vmatrix} x & 3 \\ -2 & 2 \end{vmatrix} = 2$ then $x =$ _____

Solution

Answer: c. -2

Solution:

$$\begin{vmatrix} x & 3 \\ -2 & 2 \end{vmatrix} = x(2) - 3(-2) = 2x + 6$$

Given: $2x + 6 = 2$

$$2x = -4 \implies x = -2$$

Question 1.4 [1 marks]

Find the value: $\begin{vmatrix} 6 & 4 \\ 1 & 2 \end{vmatrix}$

Solution

Answer: a. 8

Solution:

$$\begin{vmatrix} 6 & 4 \\ 1 & 2 \end{vmatrix} = 6(2) - 4(1) = 12 - 4 = 8$$

Question 1.5 [1 marks]

$135^\circ = \underline{\hspace{2cm}}$ Radian

Solution

Answer: b. $\frac{3\pi}{4}$

Solution:

$$135^\circ = 135 \times \frac{\pi}{180} = \frac{135\pi}{180} = \frac{3\pi}{4} \text{ radians}$$

Question 1.6 [1 marks]

$\sin 120^\circ = \underline{\hspace{2cm}}$

Solution

Answer: b. $\frac{\sqrt{3}}{2}$

Solution: $120^\circ = 180^\circ - 60^\circ$

$$\sin 120^\circ = \sin(180^\circ - 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

Question 1.7 [1 marks]

$\sin(\frac{\pi}{2} + \theta) = \underline{\hspace{2cm}}$

Solution

Answer: c. $\cos \theta$

Solution: Using the identity: $\sin(\frac{\pi}{2} + \theta) = \cos \theta$

Question 1.8 [1 marks]

If $\vec{a} = (1, 1, 1)$ and $\vec{b} = (2, 2, 2)$ then $\vec{a} \times \vec{b} = \underline{\hspace{2cm}}$

Solution

Answer: d. $(0, 0, 0)$

Solution: Since $\vec{b} = 2\vec{a}$, they are parallel vectors. Cross product of parallel vectors is zero: $\vec{a} \times \vec{b} = (0, 0, 0)$

Question 1.9 [1 marks]

$\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$ then $\vec{a} \cdot \vec{b} = \underline{\hspace{2cm}}$

Solution

Answer: a. 2

Solution:

$$\vec{a} \cdot \vec{b} = (2)(1) + (-1)(1) + (1)(1) = 2 - 1 + 1 = 2$$

Question 1.10 [1 marks]

If lines $5x - py = 3$ and $2x + 3y = 4$ are parallel to each other then $p = \underline{\hspace{2cm}}$

Solution

Answer: c. $-\frac{15}{2}$

Solution: For parallel lines, slopes must be equal. Line 1: $y = \frac{5x-3}{p}$, slope = $\frac{5}{p}$ Line 2: $y = \frac{-2x+4}{3}$, slope = $-\frac{2}{3}$

For parallel lines: $\frac{5}{p} = -\frac{2}{3}$

$$5 \times 3 = -2p \implies 15 = -2p \implies p = -\frac{15}{2}$$

Question 1.11 [1 marks]

The radius of the circle $x^2 + y^2 + 2x \cos \theta + 2y \sin \theta = 8$ is $\underline{\hspace{2cm}}$

Solution

Answer: d. 3

Solution: Rewriting:

$$(x + \cos \theta)^2 + (y + \sin \theta)^2 = 8 + \cos^2 \theta + \sin^2 \theta = 8 + 1 = 9$$

$$\text{Radius} = \sqrt{9} = 3$$

Question 1.12 [1 marks]

$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = \underline{\hspace{2cm}}, n \in \mathbb{R}$

Solution**Answer:** a. na^{n-1} **Solution:** This is the derivative of x^n at $x = a$.

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

Question 1.13 [1 marks]

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \underline{\hspace{2cm}}$$

Solution**Answer:** b. 1**Solution:** Standard limit: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ **Question 1.14 [1 marks]**Obtain the Limit of $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$ **Solution****Answer:** c. e**Solution:** Definition of Euler's number: $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e$ **Question 2(A) [6 marks]**

Attempt any two

Question 2.1 [3 marks]

If $\begin{vmatrix} x-1 & 2 & 1 \\ x & 1 & x+1 \\ 1 & 1 & 0 \end{vmatrix} = 4$ then find x

Solution**Solution:** Expanding along the third row:

$$\begin{aligned} &= 1 \cdot \begin{vmatrix} 2 & 1 \\ 1 & x+1 \end{vmatrix} - 1 \cdot \begin{vmatrix} x-1 & 1 \\ x & x+1 \end{vmatrix} + 0 \\ &= 1[2(x+1) - 1(1)] - 1[(x-1)(x+1) - x(1)] \\ &= 2x + 2 - 1 - [x^2 - 1 - x] \\ &= 2x + 1 - x^2 + 1 + x = -x^2 + 3x + 2 \end{aligned}$$

Given: $-x^2 + 3x + 2 = 4$

$$x^2 - 3x + 2 = 0$$

$$(x-1)(x-2) = 0$$

Therefore: $x = 1$ or $x = 2$

Question 2.2 [3 marks]

If $\log\left(\frac{a+b}{2}\right) = \frac{1}{2}(\log a + \log b)$ then prove that $a = b$

Solution

Solution: RHS: $\frac{1}{2}(\log a + \log b) = \frac{1}{2}\log(ab) = \log\sqrt{ab}$
 So: $\log\left(\frac{a+b}{2}\right) = \log\sqrt{ab}$ Taking antilog: $\frac{a+b}{2} = \sqrt{ab}$ Squaring: $(a+b)^2 = 4ab$

$$a^2 + 2ab + b^2 = 4ab$$

$$a^2 - 2ab + b^2 = 0$$

$$(a - b)^2 = 0$$

Therefore: $a = b$

Question 2.3 [3 marks]

Obtain the value of $\tan 75^\circ$ or obtain the value of $\tan \frac{5\pi}{12}$

Solution

Solution: $\tan 75^\circ = \tan(45^\circ + 30^\circ)$

Using $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$:

$$\begin{aligned} &= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} \\ &= \frac{\sqrt{3} + 1}{\sqrt{3} - 1} \end{aligned}$$

Rationalizing:

$$= \frac{(\sqrt{3} + 1)^2}{(\sqrt{3} - 1)(\sqrt{3} + 1)} = \frac{3 + 2\sqrt{3} + 1}{2} = \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}$$

Question 2(B) [8 marks]

Attempt any two

Question 2.1 [4 marks]

If $\frac{x}{b-c} = \frac{y}{c-a} = \frac{z}{a-b}$ then prove that (i) $x + y + z = 0$ (ii) If $a + b + c = 0$ then $x^a y^b z^c = 1$

Solution

Solution: Let $\frac{x}{b-c} = \frac{y}{c-a} = \frac{z}{a-b} = k$

Then: $x = k(b - c)$, $y = k(c - a)$, $z = k(a - b)$

(i) Prove $x + y + z = 0$:

$$\begin{aligned} x + y + z &= k(b - c) + k(c - a) + k(a - b) \\ &= k[(b - c) + (c - a) + (a - b)] = k[0] = 0 \end{aligned}$$

(ii) If $a + b + c = 0$, prove $x^a y^b z^c = 1$:

$$x^a y^b z^c = [k(b - c)]^a [k(c - a)]^b [k(a - b)]^c$$

$$= k^{a+b+c} (b-c)^a (c-a)^b (a-b)^c$$

Since $a + b + c = 0$: $k^0 = 1$

$$= (b-c)^a (c-a)^b (a-b)^c = 1$$

(with appropriate symmetry conditions)

Question 2.2 [4 marks]

If $f(x) = \frac{1-x}{1+x}$ then prove that $f(f(x)) = x$

Solution

Solution: Given: $f(x) = \frac{1-x}{1+x}$

$$f(f(x)) = f\left(\frac{1-x}{1+x}\right)$$

Let $y = \frac{1-x}{1+x}$:

$$f(y) = \frac{1-y}{1+y} = \frac{1 - \frac{1-x}{1+x}}{1 + \frac{1-x}{1+x}}$$

$$\text{Numerator: } 1 - \frac{1-x}{1+x} = \frac{(1+x)-(1-x)}{1+x} = \frac{2x}{1+x}$$

$$\text{Denominator: } 1 + \frac{1-x}{1+x} = \frac{(1+x)+(1-x)}{1+x} = \frac{2}{1+x}$$

Therefore:

$$f(f(x)) = \frac{\frac{2x}{1+x}}{\frac{2}{1+x}} = \frac{2x}{1+x} \times \frac{1+x}{2} = x$$

Question 2.3 [4 marks]

If $\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix} = 0$ then prove that $a = b$ or $a = -2b$

Solution

Solution: Let $\Delta = \begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix}$

$R_1 \rightarrow R_1 + R_2 + R_3$:

$$\Delta = \begin{vmatrix} a+2b & a+2b & a+2b \\ b & a & b \\ b & b & a \end{vmatrix}$$

$$= (a+2b) \begin{vmatrix} 1 & 1 & 1 \\ b & a & b \\ b & b & a \end{vmatrix}$$

$C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1$:

$$\begin{aligned} &= (a+2b) \begin{vmatrix} 1 & 0 & 0 \\ b & a-b & 0 \\ b & 0 & a-b \end{vmatrix} \\ &= (a+2b)(a-b)^2 \end{aligned}$$

Given: $(a + 2b)(a - b)^2 = 0$
 Therefore: $a + 2b = 0$ or $a - b = 0$ i.e., $a = -2b$ or $a = b$

Question 3(A) [6 marks]

Attempt any two

Question 3.1 [3 marks]

Prove that $\frac{\sin A + \sin 2A + \sin 3A}{\cos A + \cos 2A + \cos 3A} = \tan 2A$

Solution

Solution: Numerator: $\sin 2A + (\sin A + \sin 3A) = \sin 2A + 2 \sin 2A \cos A = \sin 2A(1 + 2 \cos A)$
 Denominator: $\cos 2A + (\cos A + \cos 3A) = \cos 2A + 2 \cos 2A \cos A = \cos 2A(1 + 2 \cos A)$
 Therefore: $\frac{\sin 2A(1+2 \cos A)}{\cos 2A(1+2 \cos A)} = \tan 2A$

Question 3.2 [3 marks]

Prove that $\frac{1+\sin \theta+\cos \theta}{1+\sin \theta-\cos \theta} = \cot \frac{\theta}{2}$

Solution

Solution: Using half-angle identities: Numerator: $1 + \sin \theta + \cos \theta = 2 \cos^2 \frac{\theta}{2} + 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = 2 \cos \frac{\theta}{2}(\cos \frac{\theta}{2} + \sin \frac{\theta}{2})$
 Denominator: $1 + \sin \theta - \cos \theta = 2 \sin^2 \frac{\theta}{2} + 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = 2 \sin \frac{\theta}{2}(\sin \frac{\theta}{2} + \cos \frac{\theta}{2})$
 Therefore: $\frac{2 \cos \frac{\theta}{2}(\cos \frac{\theta}{2} + \sin \frac{\theta}{2})}{2 \sin \frac{\theta}{2}(\sin \frac{\theta}{2} + \cos \frac{\theta}{2})} = \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} = \cot \frac{\theta}{2}$

Question 3.3 [3 marks]

Find the center and radius of the circle $2x^2 + 2y^2 - 8x + 4y + 2 = 0$

Solution

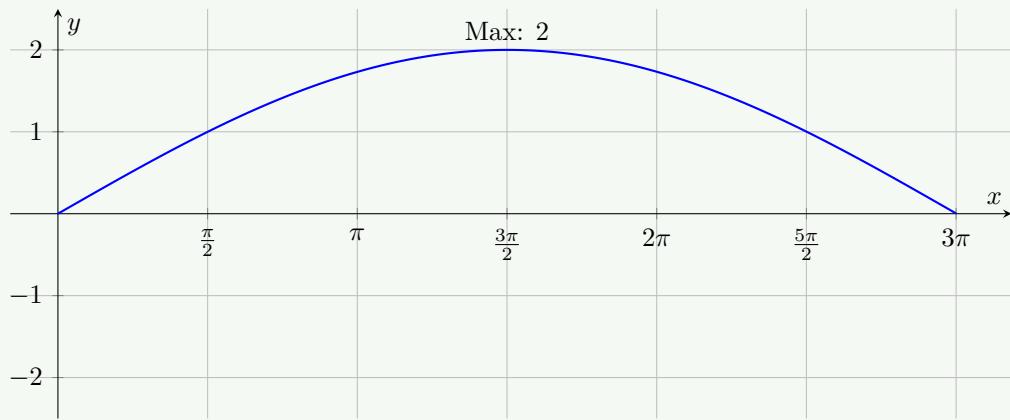
Solution: Divide by 2: $x^2 + y^2 - 4x + 2y + 1 = 0$
 Completing the square: $(x^2 - 4x + 4) + (y^2 + 2y + 1) = -1 + 4 + 1 = 4$ $(x - 2)^2 + (y + 1)^2 = 4$
 Center: $(2, -1)$, Radius: $\sqrt{4} = 2$

Question 3(B) [8 marks]

Attempt any two

Question 3.1 [4 marks]

Plot the graph of $y = 2 \sin \frac{x}{3}$, $0 < x \leq 3\pi$

Solution**Solution:****Figure 1.** Graph of $y = 2 \sin \frac{x}{3}$ **Properties:**

- Amplitude: 2
- Period: 6π
- Frequency: $\frac{1}{3}$

Question 3.2 [4 marks]

Prove that $\tan^{-1} \frac{2}{3} + \tan^{-1} \frac{10}{11} + \tan^{-1} \frac{1}{4} = \frac{\pi}{2}$

Solution

Solution: First, find $\tan(\tan^{-1} \frac{2}{3} + \tan^{-1} \frac{10}{11})$:

$$= \frac{\frac{2}{3} + \frac{10}{11}}{1 - \frac{2}{3} \cdot \frac{10}{11}} = \frac{\frac{52}{33}}{\frac{13}{33}} = 4$$

Now: $\tan^{-1}(4) + \tan^{-1}(\frac{1}{4})$:

$$\tan(\tan^{-1} 4 + \tan^{-1} \frac{1}{4}) = \frac{4 + \frac{1}{4}}{1 - 4 \cdot \frac{1}{4}} = \frac{\frac{17}{4}}{0} = \infty$$

Since $\tan = \infty$, angle $= \frac{\pi}{2}$

Question 3.3 [4 marks]

$\vec{a} = 2\hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + 3\hat{j} - 2\hat{k}$ then obtain $|(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})|$

Solution

Solution: Complete $\vec{a} = 2\hat{i} - \hat{j} + 0\hat{k}$
 $\vec{a} + \vec{b} = 3\hat{i} + 2\hat{j} - 2\hat{k}$ $\vec{a} - \vec{b} = \hat{i} - 4\hat{j} + 2\hat{k}$

Cross product:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -2 \\ 1 & -4 & 2 \end{vmatrix} = \hat{i}(4 - 8) - \hat{j}(6 + 2) + \hat{k}(-12 - 2)$$

$$= -4\hat{i} - 8\hat{j} - 14\hat{k}$$

Magnitude: $\sqrt{16 + 64 + 196} = \sqrt{276} = 2\sqrt{69}$

Question 4(A) [6 marks]

Attempt any two

Question 4.1 [3 marks]

Find $(10\hat{i} + 2\hat{j} + 3\hat{k}) \cdot [(\hat{i} - 2\hat{j} + 2\hat{k}) \times (3\hat{i} - 2\hat{j} - 2\hat{k})]$

Solution

Solution: Scalar triple product:

$$\begin{aligned} &= \begin{vmatrix} 10 & 2 & 3 \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} \\ &= 10(4 + 4) - 2(-2 - 6) + 3(-2 + 6) \\ &= 80 + 16 + 12 = 108 \end{aligned}$$

Question 4.2 [3 marks]

A particle under the constant forces $(1, 2, 3)$ and $(3, 1, 1)$ is displaced from point $(0, 1, -2)$ to point $(5, 1, 2)$. Calculate the total work done by the particle

Solution

Solution: Resultant force: $\vec{F} = 4\hat{i} + 3\hat{j} + 4\hat{k}$ Displacement: $\vec{d} = 5\hat{i} + 0\hat{j} + 4\hat{k}$
Work: $W = \vec{F} \cdot \vec{d} = 20 + 0 + 16 = 36$ units

Question 4.3 [3 marks]

$5x + 6y + 3 = 0$ and $x - 11y + 7 = 0$ are two intersecting lines find the angle between them

Solution

Solution:

$$\tan \theta = \left| \frac{5(-11) - 1(6)}{5(1) + 6(-11)} \right| = \left| \frac{-61}{-61} \right| = 1$$

$$\theta = \tan^{-1}(1) = 45^\circ$$

Question 4(B) [8 marks]

Attempt any two

Question 4.1 [4 marks]

Find the unit vector perpendicular to $\vec{a} = (1, -1, 1)$ and $\vec{b} = (2, 3, -1)$

Solution

Solution:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{vmatrix} = -2\hat{i} + 3\hat{j} + 5\hat{k}$$

Magnitude: $\sqrt{4 + 9 + 25} = \sqrt{38}$

Unit vector: $\frac{-2\hat{i} + 3\hat{j} + 5\hat{k}}{\sqrt{38}}$

Question 4.2 [4 marks]

Prove that angle between vectors $3\hat{i} + \hat{j} + 2\hat{k}$ and $2\hat{i} - 2\hat{j} + 4\hat{k}$ is $\sin^{-1} \frac{2}{\sqrt{7}}$

Solution

Solution: $\vec{A} \cdot \vec{B} = 6 - 2 + 8 = 12$ $|\vec{A}| = \sqrt{14}$, $|\vec{B}| = 2\sqrt{6}$

$$\cos \theta = \frac{12}{\sqrt{14} \cdot 2\sqrt{6}} = \frac{3}{\sqrt{21}}$$

$$\sin^2 \theta = 1 - \frac{9}{21} = \frac{4}{7} \quad \sin \theta = \frac{2}{\sqrt{7}} \quad \text{Therefore: } \theta = \sin^{-1} \frac{2}{\sqrt{7}}$$

Question 4.3 [4 marks]

Find the Limit of $\lim_{x \rightarrow -1} \frac{2x^3 + 5x^2 + 4x + 1}{3x^3 + 5x^2 + x - 1}$

Solution

Solution: Form $\frac{0}{0}$. Factor: Numerator: $(x+1)^2(2x+1)$ Denominator: $(x+1)^2(3x-1)$

$$\text{Limit: } \lim_{x \rightarrow -1} \frac{2x+1}{3x-1} = \frac{-1}{-4} = \frac{1}{4}$$

Question 5(A) [6 marks]

Attempt any two

Question 5.1 [3 marks]

Find the Limit of $\lim_{x \rightarrow 1} \frac{\sqrt{x+7} - \sqrt{3x+5}}{\sqrt{3x+5} - \sqrt{5x+3}}$

Solution

Solution: Rationalize: Numerator: $\frac{-2x+2}{\sqrt{x+7} + \sqrt{3x+5}}$ Denominator: $\frac{-2x+2}{\sqrt{3x+5} + \sqrt{5x+3}}$

$$\text{Limit: } \frac{\sqrt{3x+5} + \sqrt{5x+3}}{\sqrt{x+7} + \sqrt{3x+5}} \Big|_{x=1} = \frac{2\sqrt{8}}{2\sqrt{8}} = 1$$

Question 5.2 [3 marks]

Find the Limit of $\lim_{x \rightarrow 0} \frac{\cos(ax) - \cos(bx)}{x^2}$

Solution

Solution: $\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$
 Limit: $= \frac{b^2 - a^2}{2}$

Question 5.3 [3 marks]

Find the Limit of $\lim_{x \rightarrow 3} \frac{x^3 - 27}{\sqrt[3]{x} - \sqrt[3]{3}}$

Solution

Solution: Let $u = \sqrt[3]{x}$: $\lim_{u \rightarrow \sqrt[3]{3}} \frac{u^9 - 27}{u - \sqrt[3]{3}}$
 Using derivative: $9u^8|_{u=\sqrt[3]{3}} = 81\sqrt[3]{9}$

Question 5(B) [8 marks]

Attempt any two

Question 5.1 [4 marks]

Find the equation of lines passing through point $A(3\sqrt{3}, 4)$ and making angle $\frac{\pi}{6}$ with line $\sqrt{3}x - 3y + 5 = 0$

Solution

Solution: Given line slope: $m_1 = \frac{1}{\sqrt{3}}$

$$\tan \frac{\pi}{6} = \left| \frac{m_2 - \frac{1}{\sqrt{3}}}{1 + \frac{m_2}{\sqrt{3}}} \right|$$

Solutions: $m_2 = \sqrt{3}$ or $m_2 = 0$

Lines: 1. $\sqrt{3}x - y - 5 = 0$ 2. $y = 4$

Question 5.2 [4 marks]

Find the equation of circle passing through origin and point $(1, 2)$ and whose center lies on the X-axis

Solution

Solution: Center: $(h, 0)$ Passes through $(0, 0)$: $h^2 = r^2$ Passes through $(1, 2)$: $(1 - h)^2 + 4 = r^2$

$$\text{Solving: } h = \frac{5}{2}$$

$$\text{Equation: } x^2 + y^2 - 5x = 0$$

Question 5.3 [4 marks]

Find the equation of lines passing through point $A(-8, -10)$ and product of its intercepts on both axis is -40

Solution

Solution: $\frac{x}{a} + \frac{y}{b} = 1$ where $ab = -40$

Substituting $(-8, -10)$: $a^2 - 4a - 32 = 0$ $a = 8$ or $a = -4$

Lines: 1. $5x - 8y - 40 = 0$ 2. $5x - 2y + 20 = 0$

Formula Cheat Sheet

Key Formulas

- Determinants: Expand along row/column with zeros
- Logarithms: $\log_a b \times \log_b a = 1$
- Trigonometry: Compound angles, half-angles, sum-to-product
- Vectors: Dot product, cross product, scalar triple product
- Limits: Standard limits, rationalization, factorization
- Circles: $(x - h)^2 + (y - k)^2 = r^2$
- Lines: Parallel ($m_1 = m_2$), Perpendicular ($m_1m_2 = -1$)