Applied Mathematics (4320001) - Summer 2022 Solution

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Question 1 [14 marks]

Fill in the blanks using appropriate choice from the given options.

Question 1.1 [1 marks]

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **then** $A^2 =$ **.........** **Answer**: (c) $\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$

Solution

$$A^2 = A \times A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
$$A^2 = \begin{bmatrix} 1(1) + 2(3) & 1(2) + 2(4) \\ 3(1) + 4(3) & 3(2) + 4(4) \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

Question 1.2 [1 marks]

If $A = \begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix}$ **then** $2A - 2I =$ **.........** **Answer**: (a) $\begin{bmatrix} 0 & 6 \\ 8 & -6 \end{bmatrix}$

Solution

$$2A = 2\begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 8 & -4 \end{bmatrix}$$
$$2I = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$
$$2A - 2I = \begin{bmatrix} 2 & 6 \\ 8 & -4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ 8 & -6 \end{bmatrix}$$

Question 1.3 [1 marks]

If $A = \begin{bmatrix} -8 & -6 \\ 3 & 4 \end{bmatrix}$ **then Adj** $A =$ **.........** **Answer**: (a) $\begin{bmatrix} 4 & 6 \\ -3 & -8 \end{bmatrix}$

Solution

For a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, Adj $A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Adj $A = \begin{bmatrix} 4 & 6 \\ -3 & -8 \end{bmatrix}$

Question 1.4 [1 marks]

Order of the matrix $\begin{bmatrix} 5 & 2 & 20 & 41 & 0 \\ 15 & 4 & 30 & 40 & 1 \\ 25 & 6 & 40 & 39 & 2 \\ 35 & 8 & 50 & 38 & 3 \end{bmatrix}$ **is ....** **Answer**: (b) 4×5

Solution

The matrix has 4 rows and 5 columns, so the order is 4×5.

Question 1.5 [1 marks]

$\frac{d}{dx}(\cos^2 x + \sin^2 x) =$ **.........** **Answer**: (d) 0

Solution

Since $\cos^2 x + \sin^2 x = 1$ (trigonometric identity)
$\frac{d}{dx}(1) = 0$

Question 1.6 [1 marks]

If $f(x) = \log x$ **then** $f'(1) =$ **.........** **Answer**: (a) 1

Solution

$f(x) = \log x \implies f'(x) = \frac{1}{x}$
$f'(1) = \frac{1}{1} = 1$

Question 1.7 [1 marks]

If $x^2 + y^2 = a^2$ **then** $\frac{dy}{dx} =$ **.........** **Answer**: (b) $-\frac{x}{y}$

Solution

Differentiating both sides with respect to x: $2x + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}$

Question 1.8 [1 marks]

$\int x^2 dx =$ .... **Answer**: (b) $\frac{x^3}{3}$

Solution

$\int x^2 dx = \frac{x^{2+1}}{2+1} + c = \frac{x^3}{3} + c$

Question 1.9 [1 marks]

$\int e^{x \log a} dx =$ .... **Answer**: (d) $\frac{a^x}{\log a}$

Solution

$e^{x \log a} = a^x$
$\int a^x dx = \frac{a^x}{\log a} + c$

Question 1.10 [1 marks]

$\int \cot x dx =$ .... **Answer**: (a) $\log |\sin x|$

Solution

$\int \cot x dx = \int \frac{\cos x}{\sin x} dx$
Let $u = \sin x$, then $du = \cos x dx$. $\int \frac{du}{u} = \log |u| + c = \log |\sin x| + c$

Question 1.11 [1 marks]

Order of differential equation $\left(\frac{d^2y}{dx^2}\right)^4 + \left(\frac{d^2y}{dx^2}\right)^3 = 0$ **is ....** **Answer**: (b) 2

Solution

The highest derivative present is $\frac{d^2y}{dx^2}$, which is a second derivative. Therefore, the order is 2.

Question 1.12 [1 marks]

Integrating factor of differential equation $\frac{dy}{dx} + y = 3x$ **is ....** **Answer**: (c) e^x

Solution

For the linear differential equation $\frac{dy}{dx} + Py = Q$, where $P = 1$. Integrating factor $= e^{\int P dx} = e^{\int 1 dx} = e^x$

Question 1.13 [1 marks]

If given data is 6, 9, 7, 3, 8, 5, 4, 8, 7 and 8 then mean is .... **Answer**: (b) 6.5

Solution

Mean = $\frac{\text{Sum of all values}}{\text{Number of values}}$
Sum $= 6+9+7+3+8+5+4+8+7+8 = 65$
Number of values = 10. Mean = $\frac{65}{10} = 6.5$

Question 1.14 [1 marks]

The mean value of first eight natural numbers is .... Answer: (b) 4.5

Solution

First eight natural numbers: 1, 2, 3, 4, 5, 6, 7, 8. Sum $= 1+2+3+4+5+6+7+8 = 36$ Mean $= \frac{36}{8} = 4.5$

Question 2(a) [6 marks]

Attempt any two

Question 2(a)(1) [3 marks]

If $M = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$**,** $N = \begin{bmatrix} 4 & 1 \\ 2 & -3 \end{bmatrix}$ **then prove that** $(M+N)^T = M^T + N^T$

Solution

$M + N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 3 & -3 \end{bmatrix}$
$(M+N)^T = \begin{bmatrix} 6 & 3 \\ 4 & -3 \end{bmatrix}$
$M^T = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}, N^T = \begin{bmatrix} 4 & 2 \\ 1 & -3 \end{bmatrix}$
$M^T + N^T = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 4 & -3 \end{bmatrix}$
Therefore, $(M+N)^T = M^T + N^T$. **Proved.**

Question 2(a)(2) [3 marks]

If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ **then prove that** $A^2 - 5A + 7I = 0$

Solution

$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

$5A = 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$
$7I = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$
$A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$
$= \begin{bmatrix} 8 - 15 + 7 & 5 - 5 + 0 \\ -5 + 5 + 0 & 3 - 10 + 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
Therefore, $A^2 - 5A + 7I = 0$. **Proved.**

Question 2(a)(3) [3 marks]

Solve differential equation $\frac{dy}{dx} + x^2e^{-y} = 0$

Solution

$\frac{dy}{dx} = -x^2e^{-y} \implies e^y dy = -x^2dx$
Integrating both sides: $\int e^y dy = \int -x^2dx$
$e^y = -\frac{x^3}{3} + C$
$y = \log\left(-\frac{x^3}{3} + C\right)$

Question 2(b) [8 marks]

Attempt any two

Question 2(b)(1) [4 marks]

Solve $-5y + 3x = 1$, $x + 2y - 4 = 0$ using matrices

Solution

Rewriting the system: $3x - 5y = 1$ $x + 2y = 4$
In matrix form: $\begin{bmatrix} 3 & -5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$
Let $A = \begin{bmatrix} 3 & -5 \\ 1 & 2 \end{bmatrix}$
$|A| = 3(2) - (-5)(1) = 6 + 5 = 11$
$A^{-1} = \frac{1}{11}\begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix}$
$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$
$= \frac{1}{11}\begin{bmatrix} 2 + 20 \\ -1 + 12 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 22 \\ 11 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
Therefore, $x = 2$, $y = 1$

Question 2(b)(2) [4 marks]

If $A+B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$**,** $A-B = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$ **then find** $(AB)^{-1}$

Solution

Adding the equations: $2A = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 4 & 4 \end{bmatrix} \implies A = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$

Subtracting: $(A+B)-(A-B) = 2B$ $2B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 2 & -4 \end{bmatrix} \implies B = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$

$AB = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 0 & -6 \end{bmatrix}$

$|AB| = (-2)(-6) - (-2)(0) = 12$

$(AB)^{-1} = \frac{1}{12} \begin{bmatrix} -6 & 2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/6 \\ 0 & -1/6 \end{bmatrix}$

Question 2(b)(3) [4 marks]

If $B = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ **then prove that adj** $B = B$

Solution

For a 3×3 matrix, we need to find the cofactor matrix and then transpose it.

$C_{11} = +\begin{vmatrix} 0 & 1 \\ 4 & 3 \end{vmatrix} = -4$, $C_{12} = -\begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = 1$, $C_{13} = +\begin{vmatrix} 1 & 0 \\ 4 & 4 \end{vmatrix} = 4$

$C_{21} = -\begin{vmatrix} -3 & -3 \\ 4 & 3 \end{vmatrix} = -3$, $C_{22} = +\begin{vmatrix} -4 & -3 \\ 4 & 3 \end{vmatrix} = 0$, $C_{23} = -\begin{vmatrix} -4 & -3 \\ 4 & 4 \end{vmatrix} = 4$

$C_{31} = +\begin{vmatrix} -3 & -3 \\ 0 & 1 \end{vmatrix} = -3$, $C_{32} = -\begin{vmatrix} -4 & -3 \\ 1 & 1 \end{vmatrix} = 1$, $C_{33} = +\begin{vmatrix} -4 & -3 \\ 1 & 0 \end{vmatrix} = 3$

Cofactor matrix $= \begin{bmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{bmatrix}$

adj $B = (\text{Cofactor matrix})^T = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$

Since adj $B = B$. **Proved.**

Question 3(a) [6 marks]

Attempt any two

Question 3(a)(1) [3 marks]

If $y = \frac{1+\tan x}{1-\tan x}$ then find $\frac{dy}{dx}$

Solution

Using quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
Let $u = 1 + \tan x$, $v = 1 - \tan x$. $\frac{du}{dx} = \sec^2 x$, $\frac{dv}{dx} = -\sec^2 x$
$\frac{dy}{dx} = \frac{(1-\tan x)(\sec^2 x)-(1+\tan x)(-\sec^2 x)}{(1-\tan x)^2}$
$= \frac{\sec^2 x - \tan x \sec^2 x + \sec^2 x + \tan x \sec^2 x}{(1-\tan x)^2}$
$= \frac{2\sec^2 x}{(1-\tan x)^2}$

Question 3(a)(2) [3 marks]

If $x = a(t + \sin t)$, $y = a(1 - \cos t)$ then find $\frac{dy}{dx}$

Solution

$\frac{dx}{dt} = a(1 + \cos t)$, $\frac{dy}{dt} = a \sin t$
$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a(1+\cos t)} = \frac{\sin t}{1+\cos t}$
Using the identity $\sin t = 2\sin(t/2)\cos(t/2)$ and $1 + \cos t = 2\cos^2(t/2)$:
$\frac{dy}{dx} = \frac{2\sin(t/2)\cos(t/2)}{2\cos^2(t/2)} = \frac{\sin(t/2)}{\cos(t/2)} = \tan(t/2)$

Question 3(a)(3) [3 marks]

Evaluate $\int_0^{\pi/2} \sin x \cos x \, dx$

Solution

Method 1: Using substitution Let $u = \sin x$, then $du = \cos x \, dx$. When $x = 0$, $u = 0$; when $x = \pi/2$, $u = 1$.
$\int_0^{\pi/2} \sin x \cos x \, dx = \int_0^1 u \, du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}$
Method 2: Using double angle identity $\sin x \cos x = \frac{1}{2}\sin 2x$
$\int_0^{\pi/2} \sin x \cos x \, dx = \frac{1}{2}\int_0^{\pi/2} \sin 2x \, dx = \frac{1}{2}\left[-\frac{\cos 2x}{2}\right]_0^{\pi/2}$
$= -\frac{1}{4}[\cos \pi - \cos 0] = -\frac{1}{4}[-1 - 1] = \frac{1}{2}$

Question 3(b) [8 marks]

Attempt any two

Question 3(b)(1) [4 marks]

If $y = (\sin x)^{\tan x}$ then find $\frac{dy}{dx}$

Solution

Taking natural logarithm of both sides: $\ln y = \tan x \ln(\sin x)$
Differentiating both sides: $\frac{1}{y}\frac{dy}{dx} = \sec^2 x \ln(\sin x) + \tan x \cdot \frac{\cos x}{\sin x}$
$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \ln(\sin x) + \tan x \cot x$
$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \ln(\sin x) + 1$
$\frac{dy}{dx} = y[\sec^2 x \ln(\sin x) + 1]$
$\frac{dy}{dx} = (\sin x)^{\tan x}[\sec^2 x \ln(\sin x) + 1]$

Question 3(b)(2) [4 marks]

Find maximum and minimum value of $f(x) = 2x^3 - 3x^2 - 12x + 5$

Solution

$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$
For critical points: $f'(x) = 0 \implies x = 2$ or $x = -1$
$f''(x) = 12x - 6$
At $x = -1$: $f''(-1) = -12 - 6 = -18 < 0$ (Maximum) At $x = 2$: $f''(2) = 24 - 6 = 18 > 0$ (Minimum)
$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 5 = -2 - 3 + 12 + 5 = 12$
$f(2) = 2(8) - 3(4) - 12(2) + 5 = 16 - 12 - 24 + 5 = -15$
Maximum value = 12 at $x = -1$ Minimum value = -15 at $x = 2$

Question 3(b)(3) [4 marks]

The motion of a particle is given by $S = t^3 + 6t^2 + 3t + 5$. Find the velocity and acceleration at $t = 3$ sec.

Solution

Position: $S = t^3 + 6t^2 + 3t + 5$
Velocity: $v = \frac{dS}{dt} = 3t^2 + 12t + 3$
Acceleration: $a = \frac{dv}{dt} = 6t + 12$
At $t = 3$: Velocity: $v(3) = 3(9) + 12(3) + 3 = 27 + 36 + 3 = 66$ units/sec
Acceleration: $a(3) = 6(3) + 12 = 18 + 12 = 30$ units/sec^2

Question 4(a) [6 marks]

Attempt any two

Question 4(a)(1) [3 marks]

Evaluate $\int x^2 e^x dx$

Solution

Using integration by parts twice: Let $u = x^2$, $dv = e^x dx \implies du = 2xdx$, $v = e^x$
$\int x^2 e^x dx = x^2 e^x - \int 2xe^x dx$
For $\int 2xe^x dx$: Let $u_1 = 2x$, $dv_1 = e^x dx \implies du_1 = 2dx$, $v_1 = e^x$
$\int 2xe^x dx = 2xe^x - \int 2e^x dx = 2xe^x - 2e^x$

Therefore: $\int x^2 e^x dx = x^2 e^x - (2xe^x - 2e^x) + C = x^2 e^x - 2xe^x + 2e^x + C = e^x(x^2 - 2x + 2) + C$

Question 4(a)(2) [3 marks]

Evaluate $\int \frac{2x+3}{(x-1)(x+2)} dx$

Solution

Using partial fractions: $\frac{2x+3}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$
$2x + 3 = A(x + 2) + B(x - 1)$
Setting $x = 1$: $5 = 3A \implies A = \frac{5}{3}$ Setting $x = -2$: $-1 = -3B \implies B = \frac{1}{3}$
$\int \frac{2x+3}{(x-1)(x+2)} dx = \int \left(\frac{5/3}{x-1} + \frac{1/3}{x+2} \right) dx$
$= \frac{5}{3} \ln|x - 1| + \frac{1}{3} \ln|x + 2| + C$

Question 4(a)(3) [3 marks]

Find mean using the given information

Solution

Table 1. Frequency Distribution

x_i	52	55	58	62	79
f_i	5	3	2	3	6

Mean $= \frac{\sum f_i x_i}{\sum f_i}$
$\sum f_i x_i = 52(5) + 55(3) + 58(2) + 62(3) + 79(6) = 260 + 165 + 116 + 186 + 474 = 1201$
$\sum f_i = 5 + 3 + 2 + 3 + 6 = 19$
Mean $= \frac{1201}{19} = 63.21$

Question 4(b) [8 marks]

Attempt any two

Question 4(b)(1) [4 marks]

Evaluate $\int_{-1}^{1} \frac{x^5-6x}{x-4} dx$

Solution

First, let's perform polynomial long division: $\frac{x^5-6x}{x-4} = x^4 + 4x^3 + 16x^2 + 64x + 250 + \frac{1000}{x-4}$
$\int_{-1}^{1} \frac{x^5-6x}{x-4} dx = \int_{-1}^{1} \left(x^4 + 4x^3 + 16x^2 + 64x + 250 + \frac{1000}{x-4} \right) dx$
$= \left[\frac{x^5}{5} + x^4 + \frac{16x^3}{3} + 32x^2 + 250x + 1000 \ln|x - 4| \right]_{-1}^{1}$
At $x = 1$: $\frac{1}{5} + 1 + \frac{16}{3} + 32 + 250 + 1000 \ln 3$ At $x = -1$: $-\frac{1}{5} + 1 - \frac{16}{3} + 32 - 250 + 1000 \ln 5$
$= \left(\frac{2}{5} + \frac{32}{3} + 500 + 1000 \ln \frac{3}{5} \right)$
$= \frac{6+160+1500}{15} + 1000 \ln \frac{3}{5} = \frac{1666}{15} + 1000 \ln \frac{3}{5}$

Question 4(b)(2) [4 marks]

Evaluate $\int \sin 5x \sin 6x \, dx$

Solution

Using the product-to-sum formula: $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
$\sin 5x \sin 6x = \frac{1}{2}[\cos(5x-6x) - \cos(5x+6x)] = \frac{1}{2}[\cos(-x) - \cos(11x)] = \frac{1}{2}[\cos x - \cos(11x)]$
$\int \sin 5x \sin 6x \, dx = \frac{1}{2}\int[\cos x - \cos(11x)]dx$
$= \frac{1}{2}\left[\sin x - \frac{\sin(11x)}{11}\right] + C$
$= \frac{\sin x}{2} - \frac{\sin(11x)}{22} + C$

Question 4(b)(3) [4 marks]

Calculate the standard deviation for the following data: 6, 7, 9, 11, 13, 15, 8, 10

Solution

Data: 6, 7, 8, 9, 10, 11, 13, 15 (arranged in order) $n = 8$
Step 1: Calculate Mean $\bar{x} = \frac{6+7+8+9+10+11+13+15}{8} = \frac{79}{8} = 9.875$
Step 2: Calculate deviations and their squares

Table 2. Standard Deviation Calculation

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
6	-3.875	15.016
7	-2.875	8.266
8	-1.875	3.516
9	-0.875	0.766
10	0.125	0.016
11	1.125	1.266
13	3.125	9.766
15	5.125	26.266

$\sum(x_i - \bar{x})^2 = 64.878$
Step 3: Calculate Standard Deviation $\sigma = \sqrt{\frac{\sum(x_i-\bar{x})^2}{n}} = \sqrt{\frac{64.878}{8}} = \sqrt{8.11} = 2.85$
Standard Deviation = 2.85

Question 5(a) [6 marks]

Attempt any two

Question 5(a)(1) [3 marks]

Find the mean for the following data:

Solution

Table 3. Data

X_i	92	93	97	98	102	104
F_i	3	2	2	3	6	4

Mean $= \frac{\sum f_i x_i}{\sum f_i}$
$\sum f_i x_i = 92(3) + 93(2) + 97(2) + 98(3) + 102(6) + 104(4) = 276 + 186 + 194 + 294 + 612 + 416 = 1978$
$\sum f_i = 3 + 2 + 2 + 3 + 6 + 4 = 20$
Mean $= \frac{1978}{20} = 98.9$

Question 5(a)(2) [3 marks]

Calculate the standard deviation for the following data: 5, 9, 8, 12, 6, 10, 6, 8

Solution

Data: 5, 6, 6, 8, 8, 9, 10, 12 (arranged in order) $n = 8$
Step 1: Calculate Mean $\bar{x} = \frac{5+6+6+8+8+9+10+12}{8} = \frac{64}{8} = 8$
Step 2: Calculate Standard Deviation

Table 4. Deviations

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
5	-3	9
6	-2	4
6	-2	4
8	0	0
8	0	0
9	1	1
10	2	4
12	4	16

$\sum(x_i - \bar{x})^2 = 38$
$\sigma = \sqrt{\frac{38}{8}} = \sqrt{4.75} = 2.18$
Standard Deviation = 2.18

Question 5(a)(3) [3 marks]

Calculate the Mean for the following data: 5, 15, 25, 35, 45, 55, 65, 75, 85, 95, 75

Solution

$n = 11$
Sum $= 5 + 15 + 25 + 35 + 45 + 55 + 65 + 75 + 85 + 95 + 75 = 575$
Mean $= \frac{575}{11} = 52.27$

Question 5(b) [8 marks]

Attempt any two

Question 5(b)(1) [4 marks]

Solve differential equation $\frac{dy}{dx} + \frac{y}{x} = e^x$, $y(0) = 2$

Solution

This is a first-order linear differential equation of the form $\frac{dy}{dx} + Py = Q$
Here, $P = \frac{1}{x}$ and $Q = e^x$
Integrating Factor: $\mu = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$ (for $x > 0$)
Multiplying the equation by $\mu = x$: $x\frac{dy}{dx} + y = xe^x \implies \frac{d}{dx}(xy) = xe^x$
Integrating both sides: $xy = \int xe^x dx$
Using integration by parts for $\int xe^x dx$: Let $u = x$, $dv = e^x dx \implies du = dx$, $v = e^x$
$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x = e^x(x-1)$
Therefore: $xy = e^x(x-1) + C$
$y = \frac{e^x(x-1)+C}{x}$
Final Answer: $y = e^x + \frac{1}{x}$ (subject to domain restrictions)

Question 5(b)(2) [4 marks]

Solve differential equation $\frac{dy}{dx} + \frac{4x}{x^2+1}y = \frac{1}{(x^2+1)^2}$

Solution

This is a first-order linear differential equation. $P = \frac{4x}{x^2+1}$, $Q = \frac{1}{(x^2+1)^2}$
Integrating Factor: $\mu = e^{\int P dx} = e^{\int \frac{4x}{x^2+1} dx}$
Let $u = x^2 + 1$, then $du = 2xdx$. $\int \frac{4x}{x^2+1} dx = 2\int \frac{du}{u} = 2\ln u = 2\ln(x^2+1)$
$\mu = e^{2\ln(x^2+1)} = (x^2+1)^2$
Multiplying the equation by μ: $(x^2+1)^2\frac{dy}{dx} + 4x(x^2+1)y = 1$
This can be written as: $\frac{d}{dx}[y(x^2+1)^2] = 1$
Integrating: $y(x^2+1)^2 = x + C$
$y = \frac{x+C}{(x^2+1)^2}$

Question 5(b)(3) [4 marks]

Solve differential equation $\frac{dy}{dx} = \sin(x+y)$

Solution

Let $v = x + y$, then $\frac{dv}{dx} = 1 + \frac{dy}{dx} \implies \frac{dy}{dx} = \frac{dv}{dx} - 1$
Substituting into the original equation: $\frac{dv}{dx} - 1 = \sin v \implies \frac{dv}{dx} = 1 + \sin v$
Separating variables: $\frac{dv}{1+\sin v} = dx$
To integrate the left side, we use the identity: $\frac{1}{1+\sin v} = \frac{1-\sin v}{(1+\sin v)(1-\sin v)} = \frac{1-\sin v}{\cos^2 v}$
$\int \frac{dv}{1+\sin v} = \int \frac{1-\sin v}{\cos^2 v} dv = \int(\sec^2 v - \sec v\tan v)dv$
$= \tan v - \sec v + C_1$
Therefore: $\tan(x+y) - \sec(x+y) = x + C$

Formula Cheat Sheet

- **Matrix Operations**: $(A+B)^T = A^T + B^T$, $(AB)^T = B^T A^T$, $(A^{-1})^T = (A^T)^{-1}$
- **Differentiation**: $\frac{d}{dx}[x^n] = nx^{n-1}$, $\frac{d}{dx}[\ln x] = \frac{1}{x}$, $\frac{d}{dx}[e^x] = e^x$, $\frac{d}{dx}[\sin x] = \cos x$

- **Integration**: $\int x^n dx = \frac{x^{n+1}}{n+1}$, $\int e^x dx = e^x$, $\int \sin x dx = -\cos x$
- **Differential Equations**: Linear DE $\frac{dy}{dx} + Py = Q$, IF $\mu = e^{\int P dx}$
- **Statistics**: Mean $\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$, SD $\sigma = \sqrt{\frac{\sum(x_i - \bar{x})^2}{n}}$