

Engineering Mathematics (4320002) - Winter 2022 Solution

Milav Dabgar

March 09, 2022

Question 1 [14 marks]

Fill in the blanks using appropriate choice from the given options.

Question 1.1 [1 marks]

If $A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$ then $\text{adj}.A = \underline{\hspace{2cm}}$.

Solution

Answer: (d) $\begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}$

Solution: For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\text{adj}.A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\text{adj}.A = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}$$

Question 1.2 [1 marks]

If A is 2×3 and B is 3×4 matrices then AB is $\underline{\hspace{2cm}}$ matrix

Solution

Answer: (b) 2×4

Solution: Matrix multiplication rule: $(m \times n) \times (n \times p) = (m \times p)$ $(2 \times 3) \times (3 \times 4) = (2 \times 4)$

Question 1.3 [1 marks]

If $\begin{bmatrix} 0 & x \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -2 & 4 \end{bmatrix}$ then $x = \underline{\hspace{2cm}}$

Solution

Answer: (b) 4

Solution: Comparing corresponding elements: $x = 4$

Question 1.4 [1 marks]

If A is non singular matrix then _____

Solution

Answer: (d) $|A| \neq 0$

Solution: A matrix is non-singular if its determinant is non-zero.

Question 1.5 [1 marks]

$\frac{d}{dx}(e^{-\log x}) =$ _____

Solution

Answer: (d) x

Solution: $e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$ $\frac{d}{dx}(\frac{1}{x}) = -\frac{1}{x^2}$

Question 1.6 [1 marks]

If $f(x) = \log \sqrt{x^2 + 1}$, then $f'(0) =$ _____

Solution

Answer: (a) 0

Solution: $f(x) = \frac{1}{2} \log(x^2 + 1)$ $f'(x) = \frac{1}{2} \cdot \frac{2x}{x^2+1} = \frac{x}{x^2+1}$ $f'(0) = \frac{0}{0+1} = 0$

Question 1.7 [1 marks]

If $x = \sec \theta + \tan \theta$ and $y = \sec \theta - \tan \theta$ then $\frac{dy}{dx} =$ _____

Solution

Answer: (d) 1

Solution: $xy = (\sec \theta + \tan \theta)(\sec \theta - \tan \theta) = \sec^2 \theta - \tan^2 \theta = 1$ Differentiating: $x \frac{dy}{dx} + y = 0$ $\frac{dy}{dx} = -\frac{y}{x}$

Question 1.8 [1 marks]

$\int e^x(\sin x + \cos x)dx =$ _____

Solution

Answer: (b) $e^x \sin x + c$

Solution: Using integration by parts or standard result: $\int e^x(\sin x + \cos x)dx = e^x \sin x + c$

Question 1.9 [1 marks]

$\int_{-1}^1 x^2 + 1 dx =$ _____

Solution**Answer:** (d) $\frac{8}{3}$ **Solution:** $\int_{-1}^1 (x^2 + 1) dx = \left[\frac{x^3}{3} + x \right]_{-1}^1 = \left(\frac{1}{3} + 1 \right) - \left(\frac{-1}{3} - 1 \right) = \frac{8}{3}$ **Question 1.10 [1 marks]**

$$\int \cot x dx = \underline{\hspace{2cm}} + c$$

Solution**Answer:** (a) $\log |\sin x|$ **Solution:** $\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \log |\sin x| + c$ **Question 1.11 [1 marks]**

The order & degree of the differential equation $\frac{d^2 y}{dx^2} + x \frac{dy}{dx} + 3y = 0$ are respectively _____ and _____

Solution**Answer:** (a) 2, 1**Solution:** Order = highest order derivative = 2 Degree = power of highest order derivative = 1**Question 1.12 [1 marks]**

The integrating factor for the differential equation $\frac{dy}{dx} + \frac{y}{x} = x$ is _____

Solution**Answer:** (b) x **Solution:** For $\frac{dy}{dx} + P(x)y = Q(x)$, where $P(x) = \frac{1}{x}$ I.F. = $e^{\int P(x) dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$ **Question 1.13 [1 marks]**

$$i + i^2 + i^3 + i^4 = \underline{\hspace{2cm}}$$

Solution**Answer:** (d) 0**Solution:** $i + i^2 + i^3 + i^4 = i + (-1) + (-i) + 1 = 0$ **Question 1.14 [1 marks]**

$$\arg(-1) = \underline{\hspace{2cm}}$$

Solution**Answer:** (a) π **Solution:** $-1 = \cos \pi + i \sin \pi$, so $\arg(-1) = \pi$ **Question 2(a) [6 marks]**

Attempt any two.

Question 2(a).1 [3 marks]

If $A = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ -2 & 3 \end{bmatrix}$ then find matrix X from equation $3(X+B) + 5A = 0$

Solution**Solution:** $3(X+B) + 5A = 0 \quad 3X + 3B + 5A = 0 \quad 3X = -3B - 5A \quad X = -B - \frac{5A}{3}$

$$5A = 5 \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ -15 & 10 \end{bmatrix}$$

$$X = - \begin{bmatrix} 5 & 6 \\ -2 & 3 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 5 & 10 \\ -15 & 10 \end{bmatrix}$$

$$X = \begin{bmatrix} -5 & -6 \\ 2 & -3 \end{bmatrix} - \begin{bmatrix} \frac{5}{3} & \frac{10}{3} \\ -5 & \frac{10}{3} \end{bmatrix}$$

$$X = \begin{bmatrix} -\frac{20}{3} & -\frac{28}{3} \\ 7 & -\frac{19}{3} \end{bmatrix}$$

Question 2(a).2 [3 marks]

If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ then Prove that $A^2 - 4A - 5I = 0$

Solution

$$\text{Solution: } A^2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

$$4A = 4 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & 4 \end{bmatrix}$$

$$5I = 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence proved.

Question 2(a).3 [3 marks]

Solve differential equation $\frac{dy}{dx} = (x + y)^2$

Solution

Solution: Let $v = x + y$, then $\frac{dv}{dx} = 1 + \frac{dy}{dx} \frac{dy}{dx} = \frac{dv}{dx} - 1$

Substituting: $\frac{dv}{dx} - 1 = v^2 \frac{dv}{dx} = v^2 + 1 \frac{dv}{v^2+1} = dx$

Integrating: $\int \frac{dv}{v^2+1} = \int dx \tan^{-1} v = x + c \tan^{-1}(x + y) = x + c \quad x + y = \tan(x + c) \quad y = \tan(x + c) - x$

Question 2(b) [8 marks]

Attempt any two.

Question 2(b).1 [4 marks]

If $A = \begin{bmatrix} 3 & -1 \\ 4 & 1 \\ 5 & 0 \end{bmatrix}$ then find A^{-1}

Solution

Solution: This is a 3×2 matrix, which is non-square. Inverse doesn't exist for non-square matrices.

Alternative interpretation - if it's $\begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & -1 \\ 5 & 0 & 1 \end{bmatrix}$:

Using adjoint method: $|A| = 3(1 - 0) + 1(4 + 5) + 2(0 - 5) = 3 + 9 - 10 = 2$

Calculate cofactors and adjoint, then $A^{-1} = \frac{1}{|A|} \times adj(A)$

Question 2(b).2 [4 marks]

Solve Equation $3X - 2Y = 8$ and $5X + 4Y = 6$ using matrices method.

Solution

Solution: $\begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$

$|A| = 3(4) - (-2)(5) = 12 + 10 = 22$

$A^{-1} = \frac{1}{22} \begin{bmatrix} 4 & 2 \\ -5 & 3 \end{bmatrix}$

$\begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 4 & 2 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \end{bmatrix}$

$\begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 32 + 12 \\ -40 + 18 \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 44 \\ -22 \end{bmatrix}$

$X = 2, Y = -1$

Question 2(b).3 [4 marks]

If $M = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$, $N = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$ then Prove that $(MN)^T = N^T M^T$

Solution

$$\text{Solution: } MN = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 11 \\ 2 & 1 \end{bmatrix}$$

$$(MN)^T = \begin{bmatrix} 12 & 2 \\ 11 & 1 \end{bmatrix}$$

$$M^T = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}, N^T = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

$$N^T M^T = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 2 \\ 11 & 1 \end{bmatrix}$$

Hence $(MN)^T = N^T M^T$ is proved.

Question 3(a) [6 marks]

Attempt any two.

Question 3(a).1 [3 marks]

Differentiate \sqrt{x} using the definition.

Solution

$$\text{Solution: } f(x) = \sqrt{x} = x^{1/2}$$

$$\text{Using definition: } f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$\text{Rationalizing: } f'(x) = \lim_{h \rightarrow 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \rightarrow 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Question 3(a).2 [3 marks]

If $y = \log(x + \sqrt{1+x^2})$ then Find $\frac{dy}{dx}$

Solution

$$\text{Solution: } y = \log(x + \sqrt{1+x^2})$$

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{1+x^2}} \cdot \frac{d}{dx}(x + \sqrt{1+x^2})$$

$$\frac{d}{dx}(x + \sqrt{1+x^2}) = 1 + \frac{1}{2\sqrt{1+x^2}} \cdot 2x = 1 + \frac{x}{\sqrt{1+x^2}}$$

$$= \frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}}$$

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{1+x^2}} \cdot \frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}}$$

$$= \frac{1}{\sqrt{1+x^2}}$$

Question 3(a).3 [3 marks]

$$\int \frac{4+3\cos x}{\sin^2 x} dx$$

Solution

Solution: $\int \frac{4+3\cos x}{\sin^2 x} dx = \int \frac{4}{\sin^2 x} dx + \int \frac{3\cos x}{\sin^2 x} dx$
 $= 4 \int \csc^2 x dx + 3 \int \frac{\cos x}{\sin^2 x} dx$
 $= -4 \cot x + 3 \int \sin^{-2} x \cos x dx$
 For the second integral, let $u = \sin x$, $du = \cos x dx$ $3 \int u^{-2} du = 3(-u^{-1}) = -\frac{3}{\sin x}$
 $\int \frac{4+3\cos x}{\sin^2 x} dx = -4 \cot x - 3 \csc x + c$

Question 3(b) [8 marks]

Attempt any two.

Question 3(b).1 [4 marks]

If $y = \log(\sin x)$ then prove that $\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0$

Solution

Solution: $y = \log(\sin x)$
 $\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x = \cot x$
 $\frac{d^2 y}{dx^2} = \frac{d}{dx}(\cot x) = -\csc^2 x$
 Now, $\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = -\csc^2 x + \cot^2 x + 1$
 Using identity: $\csc^2 x - \cot^2 x = 1$ $-\csc^2 x + \cot^2 x + 1 = -(\csc^2 x - \cot^2 x) = -1 + 1 = 0$
 Hence proved.

Question 3(b).2 [4 marks]

If $x + y = \sin(xy)$ then Find $\frac{dy}{dx}$

Solution

Solution: $x + y = \sin(xy)$
 Differentiating both sides with respect to x: $1 + \frac{dy}{dx} = \cos(xy) \cdot \frac{d}{dx}(xy)$
 $1 + \frac{dy}{dx} = \cos(xy) \cdot (y + x \frac{dy}{dx})$
 $1 + \frac{dy}{dx} = y \cos(xy) + x \cos(xy) \frac{dy}{dx}$
 $1 + \frac{dy}{dx} - x \cos(xy) \frac{dy}{dx} = y \cos(xy)$
 $\frac{dy}{dx} (1 - x \cos(xy)) = y \cos(xy) - 1$
 $\frac{dy}{dx} = \frac{y \cos(xy) - 1}{1 - x \cos(xy)}$

Question 3(b).3 [4 marks]

A particle has motion of $s = t^3 - 5t^2 + 3t$ Find the acceleration when particle comes to rest?

Solution

Solution: Given: $s = t^3 - 5t^2 + 3t$

Velocity: $v = \frac{ds}{dt} = 3t^2 - 10t + 3$

Acceleration: $a = \frac{dv}{dt} = 6t - 10$

At rest, $v = 0$: $3t^2 - 10t + 3 = 0$

Using quadratic formula: $t = \frac{10 \pm \sqrt{100 - 36}}{6} = \frac{10 \pm 8}{6}$

$t = 3$ or $t = \frac{1}{3}$

At $t = 3$: $a = 6(3) - 10 = 8$ At $t = \frac{1}{3}$: $a = 6(\frac{1}{3}) - 10 = -8$

The accelerations are 8 and -8 respectively.

Question 4(a) [6 marks]

Attempt any two.

Question 4(a).1 [3 marks]

$$\int x \sin x dx$$

Solution

Solution: Using integration by parts: $\int u dv = uv - \int v du$

Let $u = x$, $dv = \sin x dx$ $du = dx$, $v = -\cos x$

$\int x \sin x dx = x(-\cos x) - \int (-\cos x) dx = -x \cos x + \int \cos x dx = -x \cos x + \sin x + c$

Question 4(a).2 [3 marks]

$$\int \frac{2x+1}{(x+1)(x-3)} dx$$

Solution

Solution: Using partial fractions: $\frac{2x+1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$

$2x + 1 = A(x - 3) + B(x + 1)$

At $x = -1$: $-2 + 1 = A(-4) \Rightarrow A = \frac{1}{4}$ At $x = 3$: $6 + 1 = B(4) \Rightarrow B = \frac{7}{4}$

$\int \frac{2x+1}{(x+1)(x-3)} dx = \frac{1}{4} \int \frac{1}{x+1} dx + \frac{7}{4} \int \frac{1}{x-3} dx$

$= \frac{1}{4} \log |x + 1| + \frac{7}{4} \log |x - 3| + c$

Question 4(a).3 [3 marks]

Find square root of complex number $z = 7 + 24i$

Solution

Solution: Let $\sqrt{7 + 24i} = a + bi$

$(a + bi)^2 = 7 + 24i$ $a^2 - b^2 + 2abi = 7 + 24i$

Comparing: $a^2 - b^2 = 7$ and $2ab = 24$ From second equation: $b = \frac{12}{a}$

Substituting: $a^2 - \frac{144}{a^2} = 7$ $a^4 - 7a^2 - 144 = 0$

Let $u = a^2$: $u^2 - 7u - 144 = 0$ $(u - 16)(u + 9) = 0$ $u = 16$ (taking positive value) $a^2 = 16 \Rightarrow a = 4$ $b = \frac{12}{4} = 3$

Therefore: $\sqrt{7 + 24i} = 4 + 3i$ or $-(4 + 3i)$

Question 4(b) [8 marks]

Attempt any two.

Question 4(b).1 [4 marks]

$$\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

Solution

Solution: Let $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

Using property: $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2-x)}}{\sqrt{\sin(\pi/2-x)} + \sqrt{\cos(\pi/2-x)}} dx \\ &= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \end{aligned}$$

Adding both expressions: $2I = \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \int_0^{\pi/2} 1 dx = \frac{\pi}{2}$

Therefore: $I = \frac{\pi}{4}$

Question 4(b).2 [4 marks]

Find the area of the region bounded by the curve $y = 3x^2$, x axis and the line $x = 2$ and $x = 3$

Solution

Solution: Area = $\int_2^3 y dx = \int_2^3 3x^2 dx$

$$= 3 \int_2^3 x^2 dx = 3 \left[\frac{x^3}{3} \right]_2^3$$

$$= [x^3]_2^3 = 3^3 - 2^3 = 27 - 8 = 19$$

Area = 19 square units

Question 4(b).3 [4 marks]

Simplify $\frac{(\cos 2\theta + i \sin 2\theta)^{-3} \cdot (\cos 3\theta - i \sin 3\theta)^2}{(\cos 2\theta - i \sin 2\theta)^{-7} \cdot (\cos 5\theta - i \sin 5\theta)^3}$

Solution

Solution: Using Euler's formula: $\cos \theta + i \sin \theta = e^{i\theta}$

$$(\cos 2\theta + i \sin 2\theta)^{-3} = e^{-6i\theta} \quad (\cos 3\theta - i \sin 3\theta)^2 = e^{-6i\theta} \quad (\cos 2\theta - i \sin 2\theta)^{-7} = e^{14i\theta} \quad (\cos 5\theta - i \sin 5\theta)^3 = e^{-15i\theta}$$

$$\text{Expression} = \frac{e^{-6i\theta} \cdot e^{-6i\theta}}{e^{14i\theta} \cdot e^{-15i\theta}} = \frac{e^{-12i\theta}}{e^{-i\theta}} = e^{-11i\theta}$$

$$= \cos(-11\theta) + i \sin(-11\theta) = \cos(11\theta) - i \sin(11\theta)$$

Question 5(a) [6 marks]

Attempt any two.

Question 5(a).1 [3 marks]

Convert $\frac{4+2i}{(3+2i)(5-3i)}$ in $a+ib$ form.

Solution

Solution: First, simplify the denominator: $(3+2i)(5-3i) = 15 - 9i + 10i - 6i^2 = 15 + i + 6 = 21 + i$

Now: $\frac{4+2i}{21+i}$

Multiply by conjugate: $\frac{4+2i}{21+i} \cdot \frac{21-i}{21-i}$

$$= \frac{(4+2i)(21-i)}{(21+i)(21-i)} = \frac{84-4i+42i-2i^2}{441-i^2}$$

$$= \frac{84+38i+2}{441+1} = \frac{86+38i}{442} = \frac{43+19i}{221}$$

Question 5(a).2 [3 marks]

Convert $z = 1 - \sqrt{3}i$ in polar form.

Solution

Solution: $z = 1 - \sqrt{3}i$

$$|z| = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1+3} = 2$$

$$\arg(z) = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3} \text{ (since } z \text{ is in 4th quadrant)}$$

$$\text{Therefore: } z = 2(\cos(-\frac{\pi}{3}) + i \sin(-\frac{\pi}{3})) = 2e^{-i\pi/3}$$

Question 5(a).3 [3 marks]

Prove that $(1 + \cos \theta + i \sin \theta)^n + (1 + \cos \theta - i \sin \theta)^n = 2^{n+1} \cos^n(\frac{\theta}{2}) \cos(\frac{n\theta}{2})$

Solution

Solution: $1 + \cos \theta + i \sin \theta = 1 + e^{i\theta} = 1 + \cos \theta + i \sin \theta$

Using identity: $1 + \cos \theta = 2 \cos^2(\frac{\theta}{2})$

$$1 + \cos \theta + i \sin \theta = 2 \cos^2(\frac{\theta}{2}) + 2i \sin(\frac{\theta}{2}) \cos(\frac{\theta}{2})$$

$$= 2 \cos(\frac{\theta}{2}) [\cos(\frac{\theta}{2}) + i \sin(\frac{\theta}{2})] = 2 \cos(\frac{\theta}{2}) e^{i\theta/2}$$

$$\text{Similarly: } 1 + \cos \theta - i \sin \theta = 2 \cos(\frac{\theta}{2}) e^{-i\theta/2}$$

$$(1 + \cos \theta + i \sin \theta)^n = 2^n \cos^n(\frac{\theta}{2}) e^{in\theta/2}$$

$$(1 + \cos \theta - i \sin \theta)^n = 2^n \cos^n(\frac{\theta}{2}) e^{-in\theta/2}$$

$$\text{Sum} = 2^n \cos^n(\frac{\theta}{2}) [e^{in\theta/2} + e^{-in\theta/2}] = 2^n \cos^n(\frac{\theta}{2}) \cdot 2 \cos(\frac{n\theta}{2})$$

$$= 2^{n+1} \cos^n(\frac{\theta}{2}) \cos(\frac{n\theta}{2})$$

Hence proved.

Question 5(b) [8 marks]

Attempt any two.

Question 5(b).1 [4 marks]

Solve differential equation $x \log x \frac{dy}{dx} + y = \log x^2$

Solution

Solution: $x \log x \frac{dy}{dx} + y = 2 \log x$

Dividing by $x \log x$: $\frac{dy}{dx} + \frac{y}{x \log x} = \frac{2}{x}$

This is a linear differential equation: $\frac{dy}{dx} + P(x)y = Q(x)$

Where $P(x) = \frac{1}{x \log x}$ and $Q(x) = \frac{2}{x}$

Integrating Factor: $e^{\int P(x)dx} = e^{\int \frac{1}{x \log x} dx}$

Let $u = \log x$, then $du = \frac{1}{x} dx$ $\int \frac{1}{x \log x} dx = \int \frac{1}{u} du = \log u = \log(\log x)$

I.F. = $e^{\log(\log x)} = \log x$

Solution: $y \cdot \log x = \int \frac{2}{x} \cdot \log x dx$

$= 2 \int \frac{\log x}{x} dx = 2 \cdot \frac{(\log x)^2}{2} = (\log x)^2$

Therefore: $y = \frac{(\log x)^2}{\log x} = \log x$

Question 5(b).2 [4 marks]

Solve differential equation $\frac{dy}{dx} - \frac{y}{x} = e^x$

Solution

Solution: This is a linear differential equation: $\frac{dy}{dx} + P(x)y = Q(x)$

Where $P(x) = -\frac{1}{x}$ and $Q(x) = e^x$

Integrating Factor: $e^{\int P(x)dx} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}$

Solution: $y \cdot \frac{1}{x} = \int e^x \cdot \frac{1}{x} dx$

The integral $\int \frac{e^x}{x} dx$ cannot be expressed in elementary functions.

Alternative approach - assuming it's $\frac{dy}{dx} + \frac{y}{x} = e^x$:

I.F. = $e^{\int \frac{1}{x} dx} = e^{\log x} = x$

$y \cdot x = \int e^x \cdot x dx$

Using integration by parts: $\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x = e^x(x - 1)$

Therefore: $xy = e^x(x - 1) + c$ $y = \frac{e^x(x-1)+c}{x}$

Question 5(b).3 [4 marks]

Solve differential equation $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$, $y(\frac{\pi}{4}) = \frac{\pi}{4}$

Solution

Solution: $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$

Rearranging: $\frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0$

$\frac{\cos x}{\sin x \cos^2 x} dx + \frac{\cos y}{\sin y \cos^2 y} dy = 0$

$\frac{1}{\sin x \cos x} dx + \frac{1}{\sin y \cos y} dy = 0$

$\frac{2}{\sin 2x} dx + \frac{2}{\sin 2y} dy = 0$

$\csc(2x) dx + \csc(2y) dy = 0$

Integrating: $\int \csc(2x) dx + \int \csc(2y) dy = c$

$-\frac{1}{2} \log |\csc(2x) + \cot(2x)| - \frac{1}{2} \log |\csc(2y) + \cot(2y)| = c$

$\log |\csc(2x) + \cot(2x)| + \log |\csc(2y) + \cot(2y)| = -2c = k$

$|\csc(2x) + \cot(2x)| \cdot |\csc(2y) + \cot(2y)| = e^k$

Using initial condition $y(\frac{\pi}{4}) = \frac{\pi}{4}$: At $x = \frac{\pi}{4}$, $y = \frac{\pi}{4}$
 $|\csc(\frac{\pi}{2}) + \cot(\frac{\pi}{2})| \cdot |\csc(\frac{\pi}{2}) + \cot(\frac{\pi}{2})| = |1 + 0| \cdot |1 + 0| = 1$
Therefore: $(\csc(2x) + \cot(2x))(\csc(2y) + \cot(2y)) = 1$