

Applied Mathematics (4320001) - Summer 2023 Solution

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Question 1 [14 marks]

Fill in the blanks using appropriate choice from the given options

Question 1.1 [1 marks]

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 4 & 2 \end{bmatrix}$, then $A^T =$ _____ Answer: b. $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \end{bmatrix}$

Solution

For transpose of a matrix, rows become columns and columns become rows. $A^T = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \end{bmatrix}$

Question 1.2 [1 marks]

If $\begin{bmatrix} x+y & 3 \\ -7 & x-y \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ -7 & 2 \end{bmatrix}$, then $(x, y) =$ _____ Answer: c. (5, 3)

Solution

Comparing corresponding elements: $x + y = 8 \dots (1)$ $x - y = 2 \dots (2)$
Adding equations (1) and (2): $2x = 10 \implies x = 5$ Substituting in equation (1): $5 + y = 8 \implies y = 3$

Question 1.3 [1 marks]

If $\begin{bmatrix} x & 3 \\ y & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 15 \\ 12 \end{bmatrix}$, then $y =$ _____ Answer: c. 3

Solution

Matrix multiplication gives: $2x + 9 = 15 \implies 2x = 6 \implies x = 3$ $2y + 6 = 12 \implies 2y = 6 \implies y = 3$

Question 1.4 [1 marks]

Order of matrix $\begin{bmatrix} 1 & -3 \\ -2 & 1 \\ 4 & 5 \end{bmatrix}$ is _____ Answer: b. 3×2

Solution

The matrix has 3 rows and 2 columns, so order is 3×2 .

Question 1.5 [1 marks]

$\frac{d}{dx}(x^2 + 2x + 3) =$ _____ Answer: b. $2x + 2$

Solution

Using power rule: $\frac{d}{dx}(x^2 + 2x + 3) = 2x + 2 + 0 = 2x + 2$

Question 1.6 [1 marks]

$\frac{d}{dx}(\sec x) =$ _____ Answer: a. $\sec x \cdot \tan x$

Solution

Standard derivative: $\frac{d}{dx}(\sec x) = \sec x \tan x$

Question 1.7 [1 marks]

If $x^2 + y^2 = 1$, then $\frac{dy}{dx} =$ _____ Answer: b. $-\frac{x}{y}$

Solution

Differentiating implicitly: $2x + 2y \frac{dy}{dx} = 0$ Therefore: $\frac{dy}{dx} = -\frac{x}{y}$

Question 1.8 [1 marks]

$\int \log x \, dx =$ _____ +c Answer: b. $x \log x - x$

Solution

Using integration by parts: $\int \log x \, dx = x \log x - \int x \cdot \frac{1}{x} dx = x \log x - x + c$

Question 1.9 [1 marks]

$\int \frac{1}{x^2} dx =$ _____ +c Answer: b. $-\frac{1}{x}$

Solution

$$\int x^{-2} dx = \frac{x^{-1}}{-1} = -\frac{1}{x} + c$$

Question 1.10 [1 marks]

$$\int_{-1}^1 (x^2 + 1) dx = \underline{\hspace{2cm}} \text{ Answer: a. } \frac{8}{3}$$

Solution

$$\int_{-1}^1 (x^2 + 1) dx = \left[\frac{x^3}{3} + x \right]_{-1}^1 = \left(\frac{1}{3} + 1 \right) - \left(-\frac{1}{3} - 1 \right) = \frac{4}{3} - \left(-\frac{4}{3} \right) = \frac{8}{3}$$

Question 1.11 [1 marks]

Order of the differential equation $\left(\frac{d^2 y}{dx^2}\right)^3 + 3\left(\frac{dy}{dx}\right)^2 - 6y = 0$ is $\underline{\hspace{2cm}}$ and degree is $\underline{\hspace{2cm}}$ Answer: a. 2, 3

Solution

Order = highest derivative = 2 Degree = power of highest derivative = 3

Question 1.12 [1 marks]

Integrating Factor of the differential equation $\frac{dy}{dx} = y \tan x + e^x$ is $\underline{\hspace{2cm}}$ Answer: c. $\sin x$

Solution

Rearranging: $\frac{dy}{dx} - y \tan x = e^x$ This is a linear differential equation $\frac{dy}{dx} + Py = Q$ with $P = -\tan x$. Integrating factor $= e^{\int -\tan x dx} = e^{-\ln|\sec x|} = e^{\ln|\cos x|} = \cos x$.

Wait, the provided answer (c) is $\sin x$. Let's re-read the question carefully. "Integrating Factor of the differential equation $\frac{dy}{dx} = y \tan x + e^x$ is..." If terms are rearranged as $\frac{dy}{dx} - (\tan x)y = e^x$, then $P = -\tan x$. $\exp(\int -\tan x dx) = \exp(\ln(\cos x)) = \cos x$.

However, strict fidelity requires me to follow the MDX content. MDX Solution says: "This is not in standard linear form. The given options suggest $\sin x$ as integrating factor." Actually, looking at standard GTU papers, sometimes there are typos. I will follow the MDX text exactly.

Question 1.13 [1 marks]

Mean of the first five natural numbers is $\underline{\hspace{2cm}}$ Answer: c. 3

Solution

First five natural numbers: 1, 2, 3, 4, 5 Mean $= \frac{1+2+3+4+5}{5} = \frac{15}{5} = 3$

Question 1.14 [1 marks]

If the mean of observations 15, 7, 6, a, 3 is 7, then $a =$ _____ Answer: b. 4

Solution

$$\frac{15+7+6+a+3}{5} = 7 \quad 31 + a = 35 \implies a = 4$$

Question 2(a) [6 marks]

Attempt any two

Question 2(a)(1) [3 marks]

If $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 & 2 \\ 2 & -1 & 3 \\ 0 & 2 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 5 & 4 & 2 \\ -1 & 7 & 8 \\ 6 & 4 & 3 \end{bmatrix}$, then Find $2A - B + C$

Solution

$$\begin{aligned} 2A &= 2 \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 2 & -2 & 0 \\ 6 & 4 & 2 \end{bmatrix} \\ 2A - B &= \begin{bmatrix} 2 & 4 & 2 \\ 2 & -2 & 0 \\ 6 & 4 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 1 & 2 \\ 2 & -1 & 3 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 0 \\ 0 & -1 & -3 \\ 6 & 2 & -2 \end{bmatrix} \\ 2A - B + C &= \begin{bmatrix} 4 & 3 & 0 \\ 0 & -1 & -3 \\ 6 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 4 & 2 \\ -1 & 7 & 8 \\ 6 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 7 & 2 \\ -1 & 6 & 5 \\ 12 & 6 & 1 \end{bmatrix} \end{aligned}$$

Question 2(a)(2) [3 marks]

If $A = \begin{bmatrix} 7 & 5 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 0 \\ -2 & 3 \end{bmatrix}$, then prove that $(A + B)^T = A^T + B^T$

Solution

$$\begin{aligned} A + B &= \begin{bmatrix} 7 & 5 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ -3 & 5 \end{bmatrix} \\ (A + B)^T &= \begin{bmatrix} 13 & -3 \\ 5 & 5 \end{bmatrix} \\ A^T &= \begin{bmatrix} 7 & -1 \\ 5 & 2 \end{bmatrix}, B^T = \begin{bmatrix} 6 & -2 \\ 0 & 3 \end{bmatrix} \\ A^T + B^T &= \begin{bmatrix} 7 & -1 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 6 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 13 & -3 \\ 5 & 5 \end{bmatrix} \end{aligned}$$

Therefore, $(A + B)^T = A^T + B^T$ ✓

Question 2(a)(3) [3 marks]

Solve: $(x + y)dy = dx$

Solution

$$(x + y)dy = dx \implies \frac{dx}{dy} = x + y \implies \frac{dx}{dy} - x = y$$

This is a linear differential equation in x . Integrating factor = $e^{\int -1 dy} = e^{-y}$

$$e^{-y} \cdot x = \int ye^{-y} dy$$

$$\text{Using integration by parts: } \int ye^{-y} dy = -ye^{-y} - \int -e^{-y} dy = -ye^{-y} - e^{-y} = -e^{-y}(y + 1)$$

$$\text{Therefore: } xe^{-y} = -e^{-y}(y + 1) + C \implies x = -(y + 1) + Ce^y$$

Question 2(b) [8 marks]

Attempt any two

Question 2(b)(1) [4 marks]

If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, then prove that $A^2 - 4A - 5I_3 = 0$

Solution

$$\text{First, calculate } A^2: A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$4A = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$5I_3 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^2 - 4A - 5I_3 = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \checkmark$$

Question 2(b)(2) [4 marks]

If $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{bmatrix}$, then find A^{-1}

Solution

Using adjoint method: $A^{-1} = \frac{1}{|A|} \text{adj}(A)$

$$|A| = 1(0 - 3) - 2(0 - 3) + 1(2 - 1) = -3 + 6 + 1 = 4$$

Finding cofactors: $C_{11} = -3, C_{12} = 3, C_{13} = 1, C_{21} = 1, C_{22} = -1, C_{23} = 1, C_{31} = 5, C_{32} = -1, C_{33} = -3$

$$\text{adj}(A) = \begin{bmatrix} -3 & 1 & 5 \\ 3 & -1 & -1 \\ 1 & 1 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4} \begin{bmatrix} -3 & 1 & 5 \\ 3 & -1 & -1 \\ 1 & 1 & -3 \end{bmatrix}$$

Question 2(b)(3) [4 marks]

Solve the equations $2x + 3y = 7$ and $4x = 9 + y$ using matrix method

Solution

Rewriting: $2x + 3y = 7$ and $4x - y = 9$

In matrix form: $\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$

$$|A| = 2(-1) - 3(4) = -2 - 12 = -14$$

$$A^{-1} = \frac{1}{-14} \begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-14} \begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 9 \end{bmatrix} = \frac{1}{-14} \begin{bmatrix} -34 \\ -10 \end{bmatrix}$$

$$\text{Therefore: } x = \frac{34}{14} = \frac{17}{7}, y = \frac{10}{14} = \frac{5}{7}$$

Question 3(a) [6 marks]

Attempt any two

Question 3(a)(1) [3 marks]

If $y = x^x$, then find $\frac{dy}{dx}$

Solution

Taking natural logarithm: $\ln y = x \ln x$

Differentiating both sides: $\frac{1}{y} \frac{dy}{dx} = \ln x + x \cdot \frac{1}{x} = \ln x + 1$

$$\frac{dy}{dx} = y(\ln x + 1) = x^x(\ln x + 1)$$

Question 3(a)(2) [3 marks]

If $y = \log(x + \sqrt{x^2 + a^2})$, then find $\frac{dy}{dx}$

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \frac{d}{dx}(x + \sqrt{x^2 + a^2}) \\ \frac{d}{dx}(x + \sqrt{x^2 + a^2}) &= 1 + \frac{2x}{2\sqrt{x^2 + a^2}} = 1 + \frac{x}{\sqrt{x^2 + a^2}} = \frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}} \\ \frac{dy}{dx} &= \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}} = \frac{1}{\sqrt{x^2 + a^2}}\end{aligned}$$

Question 3(a)(3) [3 marks]

If $y = \operatorname{cosec}^{-1} x + \sec^{-1} x$, then find $\frac{dy}{dx}$

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\operatorname{cosec}^{-1} x) + \frac{d}{dx}(\sec^{-1} x) \\ &= -\frac{1}{|x|\sqrt{x^2 - 1}} + \frac{1}{|x|\sqrt{x^2 - 1}} = 0\end{aligned}$$

Question 3(b) [8 marks]

Attempt any two

Question 3(b)(1) [4 marks]

Differentiate $y = \cos x$ using the definition

Solution

$$\begin{aligned}\text{By definition: } \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ \frac{d}{dx}(\cos x) &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} \\ \text{Using the identity: } \cos(x+h) &= \cos x \cos h - \sin x \sin h \\ &= \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} = \lim_{h \rightarrow 0} \frac{\cos x (\cos h - 1) - \sin x \sin h}{h} = \cos x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \rightarrow 0} \frac{\sin h}{h} = \cos x \cdot 0 - \sin x \cdot 1 = -\sin x\end{aligned}$$

Question 3(b)(2) [4 marks]

Find the maximum and minimum value of $f(x) = x^3 - 4x^2 + 5x + 7$

Solution

$$\begin{aligned}f'(x) &= 3x^2 - 8x + 5 \\ \text{Setting } f'(x) = 0: 3x^2 - 8x + 5 &= 0 \implies (3x - 5)(x - 1) = 0 \implies x = \frac{5}{3} \text{ or } x = 1 \\ f''(x) &= 6x - 8 \\ \text{At } x = 1: f''(1) &= 6(1) - 8 = -2 < 0 \text{ (Maximum)} \quad \text{At } x = \frac{5}{3}: f''\left(\frac{5}{3}\right) = 6\left(\frac{5}{3}\right) - 8 = 2 > 0 \text{ (Minimum)} \\ \text{Maximum value: } f(1) &= 1 - 4 + 5 + 7 = 9 \quad \text{Minimum value: } f\left(\frac{5}{3}\right) = \left(\frac{5}{3}\right)^3 - 4\left(\frac{5}{3}\right)^2 + 5\left(\frac{5}{3}\right) + 7 = \frac{158}{27}\end{aligned}$$

Question 3(b)(3) [4 marks]

If $y = (\tan^{-1} x)^2$, then prove that $(1 + x^2)y_2 + 2x(1 + x^2)y_1 = 2$

Solution

$$y = (\tan^{-1} x)^2 \implies y_1 = \frac{dy}{dx} = 2(\tan^{-1} x) \cdot \frac{1}{1+x^2}$$

$$y_2 = \frac{d^2y}{dx^2} = 2 \left[\frac{1}{1+x^2} \cdot \frac{1}{1+x^2} + (\tan^{-1} x) \cdot \frac{-2x}{(1+x^2)^2} \right] = \frac{2}{(1+x^2)^2} - \frac{4x(\tan^{-1} x)}{(1+x^2)^2}$$

$$\text{Now substituting in LHS: } (1+x^2)y_2 + 2x(1+x^2)y_1 = (1+x^2) \cdot \frac{2-4x(\tan^{-1} x)}{(1+x^2)^2} + 2x(1+x^2) \cdot \frac{2(\tan^{-1} x)}{1+x^2} = \frac{2-4x(\tan^{-1} x)}{1+x^2} + 4x(\tan^{-1} x) = \frac{2-4x(\tan^{-1} x)+4x(\tan^{-1} x)(1+x^2)}{1+x^2} = \frac{2+4x^3(\tan^{-1} x)}{1+x^2}$$

Wait, let me re-evaluate the substitution carefully. LHS = $(1+x^2)y_2 + 2x(1+x^2)y_1$. Note that the standard textbook problem is usually $(1+x^2)^2 y_2 + 2x(1+x^2)y_1 = 2$. However, the question says $(1+x^2)y_2 + 2x(1+x^2)y_1$. Let's see if $(1+x^2)y_1 = 2 \tan^{-1} x$. Diff again: $(1+x^2)y_2 + 2xy_1 = \frac{2}{1+x^2}$. Multiply by $(1+x^2)$: $(1+x^2)^2 y_2 + 2x(1+x^2)y_1 = 2$. The question likely meant $(1+x^2)^2 y_2$ OR it meant $(1+x^2)y_2 + 2xy_1$ equal to something else. BUT, looking at the MDX solution: The MDX solution ends with: " $= \frac{2}{1+x^2} \cdot (1+x^2) = 2$ ". This implies the term was indeed forming 2. Let's check the MDX step: " $= \frac{2-4x(\tan^{-1} x)}{1+x^2} + 4x(\tan^{-1} x)$ " " $= \frac{2-4x(\tan^{-1} x)+4x(\tan^{-1} x)(1+x^2)}{1+x^2}$ ". This algebra seems weird in the MDX. $4x(\tan^{-1} x)(1+x^2)$ would not cancel $-4x(\tan^{-1} x)$ cleanly unless x^2 term is handled. Calculated value: $\frac{2-4x \tan^{-1} x + 4x \tan^{-1} x + 4x^3 \tan^{-1} x}{1+x^2}$. This is not 2.

There is a discrepancy in the MDX algebra or the question statement. However, User requires strict fidelity to the MDX text. "Migrate the **EXACT** text content from MDX to LaTeX." I will copy the MDX solution steps exactly, even if they look mathematically dubious, as per instructions. MDX text: $= \frac{2-4x(\tan^{-1} x)+4x(\tan^{-1} x)(1+x^2)}{1+x^2} = \frac{2}{1+x^2} \cdot (1+x^2) = 2$ ✓

Question 4(a) [6 marks]

Attempt any two

Question 4(a)(1) [3 marks]

Integrate: $\int \frac{x^5}{1+x^{12}} dx$

Solution

Let $u = x^6$, then $du = 6x^5 dx$, so $x^5 dx = \frac{1}{6} du$

$$\int \frac{x^5}{1+x^{12}} dx = \int \frac{1}{1+u^2} \cdot \frac{1}{6} du = \frac{1}{6} \tan^{-1} u + C = \frac{1}{6} \tan^{-1}(x^6) + C$$

Question 4(a)(2) [3 marks]

Integrate: $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

Solution

$$\text{Let } I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

Using property $\int_0^a f(x) dx = \int_0^a f(a-x) dx$:

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2-x)}}{\sqrt{\sin(\pi/2-x)} + \sqrt{\cos(\pi/2-x)}} dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$$

$$\text{Adding both expressions: } 2I = \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \int_0^{\pi/2} 1 dx = \frac{\pi}{2}$$

Therefore: $I = \frac{\pi}{4}$

Question 4(a)(3) [3 marks]

If the mean of the following data is 19, then find missing frequency

Solution

Table 1. Frequency Distribution

x_i	6	10	14	18	24	28	30
f_i	2	4	7	f	8	4	3

$$\text{Mean} = \frac{\sum f_i x_i}{\sum f_i} = 19$$

$$\sum f_i = 2 + 4 + 7 + f + 8 + 4 + 3 = 28 + f \quad \sum f_i x_i = 2(6) + 4(10) + 7(14) + f(18) + 8(24) + 4(28) + 3(30)$$

$$= 12 + 40 + 98 + 18f + 192 + 112 + 90 = 544 + 18f$$

$$\frac{544 + 18f}{28 + f} = 19 \quad 544 + 18f = 19(28 + f) \quad 544 + 18f = 532 + 19f \quad 12 = f$$

Therefore, $f = 12$

Question 4(b) [8 marks]

Attempt any two

Question 4(b)(1) [4 marks]

Integrate: $\int \frac{x}{(x+1)(x+2)} dx$

Solution

Using partial fractions: $\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$

$$x = A(x+2) + B(x+1)$$

Setting $x = -1$: $-1 = A(1) \implies A = -1$ Setting $x = -2$: $-2 = B(-1) \implies B = 2$

$$\int \frac{x}{(x+1)(x+2)} dx = \int \left(\frac{-1}{x+1} + \frac{2}{x+2} \right) dx = -\ln|x+1| + 2\ln|x+2| + C = \ln \left| \frac{(x+2)^2}{x+1} \right| + C$$

Question 4(b)(2) [4 marks]

Integrate: $\int \frac{x^2 \tan^{-1} x^3}{1+x^6} dx$

Solution

Let $u = x^3$, then $du = 3x^2 dx$, so $x^2 dx = \frac{1}{3} du$

$$\int \frac{x^2 \tan^{-1} x^3}{1+x^6} dx = \int \frac{\tan^{-1} u}{1+u^2} \cdot \frac{1}{3} du$$

Let $v = \tan^{-1} u$, then $dv = \frac{1}{1+u^2} du$

$$= \frac{1}{3} \int v dv = \frac{1}{3} \cdot \frac{v^2}{2} + C = \frac{(\tan^{-1} u)^2}{6} + C$$

$$= \frac{(\tan^{-1} x^3)^2}{6} + C$$

Question 4(b)(3) [4 marks]

Find the standard deviation for the following data: 10, 15, 7, 19, 9, 21, 23, 25, 26, 30

Solution

First, find the mean: $\bar{x} = \frac{10+15+7+19+9+21+23+25+26+30}{10} = \frac{185}{10} = 18.5$

Table for Standard Deviation:

Table 2. Standard Deviation Calculation

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
10	-8.5	72.25
15	-3.5	12.25
7	-11.5	132.25
19	0.5	0.25
9	-9.5	90.25
21	2.5	6.25
23	4.5	20.25
25	6.5	42.25
26	7.5	56.25
30	11.5	132.25

$$\sum (x_i - \bar{x})^2 = 564.5$$

$$\text{Standard deviation} = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n}} = \sqrt{\frac{564.5}{10}} = \sqrt{56.45} = 7.51$$

Question 5(a) [6 marks]

Attempt any two

Question 5(a)(1) [3 marks]

Find the standard deviation for the following data:

Solution

Table 3. Data

x_i	4	8	11	17	20	24	32
f_i	3	5	9	5	4	3	1

$$N = \sum f_i = 3 + 5 + 9 + 5 + 4 + 3 + 1 = 30$$

$$\text{Mean Calculation: } \bar{x} = \frac{\sum f_i x_i}{N} = \frac{3(4)+5(8)+9(11)+5(17)+4(20)+3(24)+1(32)}{30} = \frac{12+40+99+85+80+72+32}{30} = \frac{420}{30} = 14$$

Standard Deviation Table:

Table 4. Standard Deviation Calculation

x_i	f_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
4	3	-10	100	300
8	5	-6	36	180
11	9	-3	9	81
17	5	3	9	45
20	4	6	36	144
24	3	10	100	300
32	1	18	324	324

$$\sum f_i(x_i - \bar{x})^2 = 1374$$

$$\text{Standard deviation} = \sqrt{\frac{\sum f_i(x_i - \bar{x})^2}{N}} = \sqrt{\frac{1374}{30}} = \sqrt{45.8} = 6.77$$

Question 5(a)(2) [3 marks]

Find the standard deviation for the following data:

Solution

Table 5. Grouped Data

Class	0-10	10-20	20-30	30-40	40-50
Frequency	5	8	15	16	6

First, find class midpoints and calculate mean:

Table 6. Midpoint Calculation

Class	Midpoint (x_i)	f_i	$f_i x_i$
0-10	5	5	25
10-20	15	8	120
20-30	25	15	375
30-40	35	16	560
40-50	45	6	270

$$N = 50, \sum f_i x_i = 1350 \quad \bar{x} = \frac{1350}{50} = 27$$

Standard Deviation Table:

Table 7. Standard Deviation Calculation

x_i	f_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i(x_i - \bar{x})^2$
5	5	-22	484	2420
15	8	-12	144	1152
25	15	-2	4	60
35	16	8	64	1024
45	6	18	324	1944

$$\sum f_i(x_i - \bar{x})^2 = 6600$$

$$\text{Standard deviation} = \sqrt{\frac{6600}{50}} = \sqrt{132} = 11.49$$

Question 5(a)(3) [3 marks]

Find the mean for the following data:

Solution

Table 8. Grouped Frequency Distribution

Class	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	3	7	12	15	8	3	2

Using midpoint method:

Table 9. Mean Calculation

Class	Midpoint (x_i)	f_i	$f_i x_i$
30-40	35	3	105
40-50	45	7	315
50-60	55	12	660
60-70	65	15	975
70-80	75	8	600
80-90	85	3	255
90-100	95	2	190

$$N = \sum f_i = 50 \quad \sum f_i x_i = 3100$$

$$\text{Mean} = \frac{\sum f_i x_i}{N} = \frac{3100}{50} = 62$$

Question 5(b) [8 marks]

Attempt any two

Question 5(b)(1) [4 marks]

Solve: $xy \, dx - (y^2 + x^2) \, dy = 0$

Solution

Rearranging: $xy \, dx = (y^2 + x^2) \, dy \quad \frac{dx}{dy} = \frac{y^2 + x^2}{xy} = \frac{y}{x} + \frac{x}{y}$

This is a homogeneous differential equation. Let $x = vy$, then $\frac{dx}{dy} = v + y \frac{dv}{dy}$

Substituting: $v + y \frac{dv}{dy} = \frac{y}{vy} + \frac{vy}{y} = \frac{1}{v} + v$

$$y \frac{dv}{dy} = \frac{1}{v} \implies v \, dv = \frac{dy}{y}$$

Integrating both sides: $\int v \, dv = \int \frac{dy}{y} \implies \frac{v^2}{2} = \ln |y| + C$

Substituting back $v = \frac{x}{y}$: $\frac{x^2}{2y^2} = \ln |y| + C \quad x^2 = 2y^2(\ln |y| + C)$

Question 5(b)(2) [4 marks]

Solve: $\frac{dy}{dx} + \frac{2y}{x} = \sin x$

Solution

This is a linear differential equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$ where $P(x) = \frac{2}{x}$ and $Q(x) = \sin x$

Integrating factor = $e^{\int P(x)dx} = e^{\int \frac{2}{x} dx} = e^{2 \ln |x|} = x^2$

Multiplying the equation by integrating factor: $x^2 \frac{dy}{dx} + 2xy = x^2 \sin x$

The left side is $\frac{d}{dx}(x^2 y)$: $\frac{d}{dx}(x^2 y) = x^2 \sin x$

Integrating both sides: $x^2 y = \int x^2 \sin x dx$

Using integration by parts twice: $\int x^2 \sin x dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$

Therefore: $x^2 y = -x^2 \cos x + 2x \sin x + 2 \cos x + C$ $y = -\cos x + \frac{2 \sin x}{x} + \frac{2 \cos x}{x^2} + \frac{C}{x^2}$

Question 5(b)(3) [4 marks]

Solve: $(1 + x^2) \frac{dy}{dx} + 2xy = \cos x$

Solution

Dividing by $(1 + x^2)$: $\frac{dy}{dx} + \frac{2x}{1+x^2} y = \frac{\cos x}{1+x^2}$

This is linear with $P(x) = \frac{2x}{1+x^2}$ and $Q(x) = \frac{\cos x}{1+x^2}$

Integrating factor = $e^{\int \frac{2x}{1+x^2} dx} = e^{\ln(1+x^2)} = 1 + x^2$

Multiplying by integrating factor: $(1 + x^2) \frac{dy}{dx} + 2xy = \cos x$

The left side is $\frac{d}{dx}[(1 + x^2)y]$: $\frac{d}{dx}[(1 + x^2)y] = \cos x$

Integrating: $(1 + x^2)y = \int \cos x dx = \sin x + C$

Therefore: $y = \frac{\sin x + C}{1+x^2}$

Complete Formula Sheet**Matrix Operations**

- **Transpose:** $(A^T)_{ij} = A_{ji}$
- **Inverse:** $A^{-1} = \frac{1}{|A|} \text{adj}(A)$
- **Properties:** $(A + B)^T = A^T + B^T$

Derivatives

- **Power Rule:** $\frac{d}{dx}(x^n) = nx^{n-1}$
- **Trigonometric:** $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$
- **Inverse Trig:** $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
- **Logarithmic:** $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Integration

- **By Parts:** $\int u dv = uv - \int v du$
- **Substitution:** If $u = g(x)$, then $\int f(g(x))g'(x)dx = \int f(u)du$
- **Definite Properties:** $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Differential Equations

- **Linear Form:** $\frac{dy}{dx} + P(x)y = Q(x)$
- **Integrating Factor:** $e^{\int P(x)dx}$
- **Variable Separable:** $\frac{dy}{dx} = f(x)g(y)$

Statistics

- **Mean:** $\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$

- **Standard Deviation:** $\sigma = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{N}}$
- **Variance:** $\sigma^2 = \frac{\sum f_i (x_i - \bar{x})^2}{N}$