Mathematics (4300001) - Winter 2023 Solution

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Question 1 [14 marks]

Fill in the blanks using appropriate choice from the given options

Question 1.1 [1 marks]

$\begin{vmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{vmatrix} =$ ______________________

Solution

Answer: c. 1
Solution:

$$\begin{vmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{vmatrix} = \sin\theta \cdot \sin\theta - (-\cos\theta) \cdot \cos\theta$$

$$= \sin^2\theta + \cos^2\theta = 1$$

Question 1.2 [1 marks]

If $f(x) = x^3 - 1$ **then** $f(-1) =$ _______________

Solution

Answer: d. -2
Solution: $f(x) = x^3 - 1$

$$f(-1) = (-1)^3 - 1 = -1 - 1 = -2$$

Question 1.3 [1 marks]

$\log 1 \times \log 2 \times \log 3 \times \log 4 =$ ________________________

Solution

Answer: a. 0
Solution: Since $\log 1 = 0$, the entire product is 0.

Question 1.4 [1 marks]

$\log x - \log y =$ ____________________

Solution

Answer: b. $\log \frac{x}{y}$
Solution: Property: $\log x - \log y = \log \frac{x}{y}$

Question 1.5 [1 marks]

Principal Period of $\sin(2x + 7) =$ ______________

Solution

Answer: c. π
Solution: For $\sin(ax + b)$, period is $\frac{2\pi}{|a|}$. Here $a = 2$. Period $= \frac{2\pi}{2} = \pi$.

Question 1.6 [1 marks]

$450° =$ _______________ **radian**

Solution

Answer: c. $\frac{5\pi}{2}$
Solution: $450° = 450 \times \frac{\pi}{180} = \frac{45\pi}{18} = \frac{5\pi}{2}$ radians.

Question 1.7 [1 marks]

$\tan^{-1} x + \cot^{-1} x =$ ______________

Solution

Answer: d. $\frac{\pi}{2}$
Solution: Standard identity: $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$.

Question 1.8 [1 marks]

$|2i - 3j + 4k| =$ ___________

Solution

Answer: a. $\sqrt{29}$
Solution: $|2i - 3j + 4k| = \sqrt{2^2 + (-3)^2 + 4^2} = \sqrt{4 + 9 + 16} = \sqrt{29}$.

Question 1.9 [1 marks]

For vector $\vec{a} \times \vec{a} =$ ______________

Solution

Answer: d. 0
Solution: Cross product of a vector with itself is always zero.

Question 1.10 [1 marks]

If two lines having slopes m_1 and m_2 are perpendicular to each other then ______________

Solution

Answer: c. $m_1 \cdot m_2 = -1$
Solution: Condition for perpendicular lines is $m_1 m_2 = -1$.

Question 1.11 [1 marks]

If $x^2 + y^2 = 25$ then its radius ______________

Solution

Answer: c. 5
Solution: $x^2 + y^2 = r^2$. Here $r^2 = 25 \implies r = 5$.

Question 1.12 [1 marks]

$\lim_{\theta \to 0} \frac{\sin 5\theta}{\tan 7\theta} =$ ______________

Solution

Answer: b. $\frac{5}{7}$
Solution:

$$\lim_{\theta \to 0} \frac{\sin 5\theta}{\tan 7\theta} = \lim_{\theta \to 0} \frac{\frac{\sin 5\theta}{5\theta} \cdot 5\theta}{\frac{\tan 7\theta}{7\theta} \cdot 7\theta} = \frac{1 \cdot 5}{1 \cdot 7} = \frac{5}{7}$$

Question 1.13 [1 marks]

$\lim_{x \to 0} \frac{e^x - 1}{x} =$ ______________

Solution

Answer: c. 1
Solution: Standard limit.

Question 1.14 [1 marks]

$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} =$ ______________

Solution

Answer: d. 2
Solution: $\lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1}(x+1) = 2$.

Question 2(A) [6 marks]

Attempt any two

Question 2.1 [3 marks]

If $f(x) = \frac{1-x}{1+x}$ **then prove that (1)** $f(x) \cdot f(-x) = 1$ **(2)** $f(x) + f(\frac{1}{x}) = 0$

Solution

Solution: Given: $f(x) = \frac{1-x}{1+x}$
(1) Prove $f(x) \cdot f(-x) = 1$: $f(-x) = \frac{1-(-x)}{1+(-x)} = \frac{1+x}{1-x}$.

$$f(x) \cdot f(-x) = \frac{1-x}{1+x} \cdot \frac{1+x}{1-x} = 1$$

(2) Prove $f(x) + f(1/x) = 0$: $f(1/x) = \frac{1-1/x}{1+1/x} = \frac{(x-1)/x}{(x+1)/x} = \frac{x-1}{x+1} = -\frac{1-x}{1+x} = -f(x)$.

$$f(x) + f(1/x) = f(x) - f(x) = 0$$

Question 2.2 [3 marks]

If $\begin{vmatrix} x & 2 & 3 \\ 5 & 0 & 7 \\ 3 & 1 & 2 \end{vmatrix} = 30$ **then find the value of** x

Solution

Solution: Expand along R2 (since it has a zero): Signs for R2 are $-, +, -$.

$$-5\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + 0 - 7\begin{vmatrix} x & 2 \\ 3 & 1 \end{vmatrix} = 30$$

$$-5(4-3) - 7(x-6) = 30$$

$$-5(1) - 7x + 42 = 30$$

$$37 - 7x = 30$$

$$7x = 7 \implies x = 1$$

Question 2.3 [3 marks]

Prove that $\tan 55^\circ = \frac{\cos 10^\circ + \sin 10^\circ}{\cos 10^\circ - \sin 10^\circ}$

Solution

Solution: LHS $= \tan 55^\circ = \tan(45^\circ + 10^\circ)$. Using $\tan(A+B)$ formula:

$$= \frac{\tan 45^\circ + \tan 10^\circ}{1 - \tan 45^\circ \tan 10^\circ}$$

$$= \frac{1 + \tan 10^\circ}{1 - \tan 10^\circ}$$

Substitute $\tan 10^\circ = \frac{\sin 10^\circ}{\cos 10^\circ}$:

$$= \frac{1 + \frac{\sin 10^\circ}{\cos 10^\circ}}{1 - \frac{\sin 10^\circ}{\cos 10^\circ}}$$

Multiply num/den by $\cos 10^\circ$:

$$= \frac{\cos 10^\circ + \sin 10^\circ}{\cos 10^\circ - \sin 10^\circ} = \text{RHS}$$

Question 2(B) [8 marks]

Attempt any two

Question 2.1 [4 marks]

Prove that $\frac{1}{\log_{xy} xyz} + \frac{1}{\log_{yz} xyz} + \frac{1}{\log_{zx} xyz} = 2$

Solution

Solution: Using $\frac{1}{\log_a b} = \log_b a$: LHS $= \log_{xyz}(xy) + \log_{xyz}(yz) + \log_{xyz}(zx)$

$$= \log_{xyz}(xy \cdot yz \cdot zx)$$

$$= \log_{xyz}(x^2y^2z^2)$$

$$= \log_{xyz}((xyz)^2)$$

$$= 2\log_{xyz}(xyz) = 2(1) = 2 = \text{RHS}$$

Question 2.2 [4 marks]

If $\log(\frac{a+b}{3}) = \frac{1}{2}(\log a + \log b)$ **then prove that** $a^2 + b^2 = 7ab$

Solution

Solution:

$$\log\left(\frac{a+b}{3}\right) = \frac{1}{2}\log(ab) = \log(ab)^{1/2} = \log\sqrt{ab}$$

So, $\frac{a+b}{3} = \sqrt{ab}$. Squaring both sides:

$$\frac{(a+b)^2}{9} = ab$$

$$a^2 + 2ab + b^2 = 9ab$$

$$a^2 + b^2 = 9ab - 2ab$$

$$a^2 + b^2 = 7ab$$

Question 2.3 [4 marks]

If $\log x \times \frac{\log 16}{\log 32} = \log 256$ **then find the value of** x

Solution

Solution: Convert constants to base 2: $\log 16 = \log 2^4 = 4\log 2$, $\log 32 = \log 2^5 = 5\log 2$, $\log 256 = \log 2^8 = 8\log 2$.
Equation becomes:

$$\log x \times \frac{4\log 2}{5\log 2} = 8\log 2$$

$$\log x \times \frac{4}{5} = 8\log 2$$

$$\log x = \frac{5}{4} \cdot 8\log 2 = 10\log 2$$

$$\log x = \log(2^{10})$$

$x = 2^{10} = 1024$.

Question 3(A) [6 marks]

Attempt any two

Question 3.1 [3 marks]

Prove that $\frac{\sin(\frac{\pi}{2}+\theta)}{\cos(\pi-\theta)} + \frac{\cot(\frac{3\pi}{2}-\theta)}{\tan(\pi-\theta)} + \frac{(\frac{\pi}{2}-\theta)}{\sec(\pi+\theta)} = -3$

Solution

Solution: Term 1: $\sin(\frac{\pi}{2}+\theta) = \cos\theta$. $\cos(\pi-\theta) = -\cos\theta$. Ratio $= \frac{\cos\theta}{-\cos\theta} = -1$.
Term 2: $\cot(\frac{3\pi}{2}-\theta) = \tan\theta$ (Using 3rd quad reduction rule $\frac{3\pi}{2}-\theta$). $\tan(\pi-\theta) = -\tan\theta$. Ratio $= \frac{\tan\theta}{-\tan\theta} = -1$.
Term 3: $(\frac{\pi}{2}-\theta) = \sec\theta$. $\sec(\pi+\theta) = -\sec\theta$. Ratio $= \frac{\sec\theta}{-\sec\theta} = -1$.
Sum $= (-1)+(-1)+(-1) = -3 =$ RHS.

Question 3.2 [3 marks]

Prove that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$

Solution

Solution: Use $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$, with $xy < 1$. $xy = \frac{1}{6} < 1$.

$$\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}}\right) = \tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

Question 3.3 [3 marks]

Find the equation of the line passing through points $(1, 6)$ and $(-2, 5)$. Also find the slope of the line.

Solution

Solution: Slope $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-6}{-2-1} = \frac{-1}{-3} = \frac{1}{3}$.
Equation using point $(1, 6)$:

$$y - 6 = \frac{1}{3}(x - 1)$$

$$3(y - 6) = x - 1 \implies 3y - 18 = x - 1$$

$$x - 3y + 17 = 0$$

Question 3(B) [8 marks]

Attempt any two

Question 3.1 [4 marks]

Draw the graph of $y = \sin x$; $0 \leq x \leq \pi$

Solution

Solution:

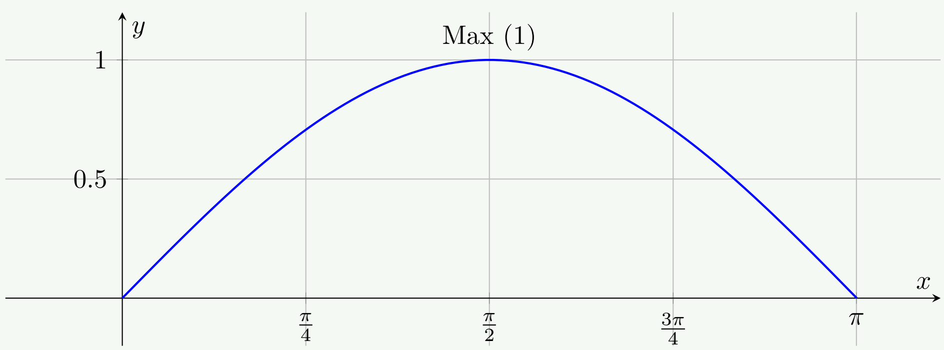

Figure 1. Graph of $y = \sin x$

Key Points:
- $x = 0, y = 0$
- $x = \pi/2, y = 1$
- $x = \pi, y = 0$

Question 3.2 [4 marks]

Prove that $\frac{\sin\theta + \sin 2\theta + \sin 4\theta + \sin 5\theta}{\cos\theta + \cos 2\theta + \cos 4\theta + \cos 5\theta} = \tan 3\theta$

Solution

Solution: Group terms: $(\sin 5\theta + \sin\theta) + (\sin 4\theta + \sin 2\theta)$ in numerator. Using $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$:
Num: $2\sin 3\theta\cos 2\theta + 2\sin 3\theta\cos\theta = 2\sin 3\theta(\cos 2\theta + \cos\theta)$.
Group denominator: $(\cos 5\theta + \cos\theta) + (\cos 4\theta + \cos 2\theta)$. Using $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$: Den: $2\cos 3\theta\cos 2\theta + 2\cos 3\theta\cos\theta = 2\cos 3\theta(\cos 2\theta + \cos\theta)$.
Ratio:

$$\frac{2\sin 3\theta(\cos 2\theta + \cos\theta)}{2\cos 3\theta(\cos 2\theta + \cos\theta)} = \frac{\sin 3\theta}{\cos 3\theta} = \tan 3\theta$$

Question 3.3 [4 marks]

The constant forces $i - j + k$, $i + j - 3k$ and $4i + 5j - 6k$ act on a particle. Under the action of these forces, particle moves from point $3i - 2j + k$ to point $i + 3j - 4k$. Find the total work done by the forces.

Solution

Solution: Resultant Force $\vec{F} = (1 + 1 + 4)i + (-1 + 1 + 5)j + (1 - 3 - 6)k = 6i + 5j - 8k$. Displacement $\vec{d} = \text{Final} - \text{Initial} = (1i + 3j - 4k) - (3i - 2j + k) = -2i + 5j - 5k$.
Work $W = \vec{F} \cdot \vec{d} = (6)(-2) + (5)(5) + (-8)(-5) = -12 + 25 + 40 = 53$ units.

Question 4(A) [6 marks]

Attempt any two

Question 4.1 [3 marks]

If $\vec{a} = 3i - j - 4k$, $\vec{b} = 4j - 2i - 3k$ and $\vec{c} = 2j - k - i$ then find $|3\vec{a} - 2\vec{b} + 4\vec{c}|$

Solution

Solution: Rewrite $\vec{b} = -2i + 4j - 3k$, $\vec{c} = -i + 2j - k$. Vector sum: $3\vec{a} = 9i - 3j - 12k$ $-2\vec{b} = 4i - 8j + 6k$ $4\vec{c} = -4i + 8j - 4k$ $3(3) - 2(-2) + 4(-1) = 9 + 4 - 4 = 9i$. $3(-1) - 2(4) + 4(2) = -3 - 8 + 8 = -3j$. $3(-4) - 2(-3) + 4(-1) = -12 + 6 - 4 = -10k$.
Resultant vector $\vec{R} = 9i - 3j - 10k$. Magnitude $|\vec{R}| = \sqrt{9^2 + (-3)^2 + (-10)^2} = \sqrt{81 + 9 + 100} = \sqrt{190}$.

Question 4.2 [3 marks]

For what value of m, the vectors $2i - 3j + 5k$ and $mi - 6j - 8k$ are perpendicular to each other?

Solution

Solution: Dot product must be zero. $(2)(m)+(-3)(-6)+(5)(-8) = 0$. $2m+18-40 = 0$. $2m-22 = 0 \implies m = 11$.

Question 4.3 [3 marks]

Find the equation of the circle having center $(4, 3)$ and passing through point $(7, -2)$

Solution

Solution: Radius r = distance between center and point. $r = \sqrt{(7-4)^2 + (-2-3)^2} = \sqrt{3^2 + (-5)^2} = \sqrt{9 + 25} = \sqrt{34}$. Equation: $(x - 4)^2 + (y - 3)^2 = 34$. $x^2 - 8x + 16 + y^2 - 6y + 9 = 34$. $x^2 + y^2 - 8x - 6y - 9 = 0$.

Question 4(B) [8 marks]

Attempt any two

Question 4.1 [4 marks]

Prove that the angle between vectors $i + 2j$ and $i + j + 3k$ is $\sin^{-1}\sqrt{\frac{46}{55}}$

Solution

Solution: $\vec{A} = (1, 2, 0)$, $\vec{B} = (1, 1, 3)$. $\vec{A} \cdot \vec{B} = 1 + 2 + 0 = 3$. $|\vec{A}| = \sqrt{1 + 4} = \sqrt{5}$. $|\vec{B}| = \sqrt{1 + 1 + 9} = \sqrt{11}$. $\cos\theta = \frac{3}{\sqrt{5}\sqrt{11}} = \frac{3}{\sqrt{55}}$. $\sin^2\theta = 1 - \frac{9}{55} = \frac{46}{55}$. $\sin\theta = \sqrt{\frac{46}{55}} \implies \theta = \sin^{-1}\sqrt{\frac{46}{55}}$.

Question 4.2 [4 marks]

If $\vec{x} = -2k + 3i$ and $\vec{y} = 5i + 2j - 4k$ then find the value of $|(\vec{x} + \vec{y}) \times (\vec{x} - \vec{y})|$

Solution

Solution: $\vec{x} = (3, 0, -2)$, $\vec{y} = (5, 2, -4)$. Using property $(\vec{x} + \vec{y}) \times (\vec{x} - \vec{y}) = -2(\vec{x} \times \vec{y})$. Let's calculate $\vec{x} \times \vec{y}$:

$$\begin{vmatrix} i & j & k \\ 3 & 0 & -2 \\ 5 & 2 & -4 \end{vmatrix} = i(4) - j(-12 + 10) + k(6) = 4i + 2j + 6k$$

So expression is $-2(4i + 2j + 6k) = -8i - 4j - 12k$. (Matches MDX calculation).
Magnitude $= \sqrt{(-8)^2 + (-4)^2 + (-12)^2} = \sqrt{64 + 16 + 144} = \sqrt{224}$. $\sqrt{16 \cdot 14} = 4\sqrt{14}$.

Question 4.3 [4 marks]

Evaluate: $\lim_{n\to\infty}(\sqrt{n^2 + n + 1} - n)$

Solution

Solution: Rationalize:

$$\lim \frac{(\sqrt{n^2 + n + 1} - n)(\sqrt{n^2 + n + 1} + n)}{\sqrt{n^2 + n + 1} + n}$$

$$= \lim \frac{n^2 + n + 1 - n^2}{\sqrt{n^2 + n + 1} + n} = \lim \frac{n + 1}{n(\sqrt{1 + 1/n + 1/n^2} + 1)}$$

$$= \frac{1}{1 + 1} = \frac{1}{2}$$

Question 5(A) [6 marks]

Attempt any two

Question 5.1 [3 marks]

Evaluate: $\lim_{x\to-2} \frac{x^3 + 2x^2 + x + 2}{x^2 + x - 2}$

Solution

Solution: Factor numerator: $x^2(x+2)+1(x+2)=(x+2)(x^2+1)$. Factor denominator: $(x+2)(x-1)$. Limit $=\lim_{x\to-2}\frac{x^2+1}{x-1}=\frac{5}{-3}=-\frac{5}{3}$.

Question 5.2 [3 marks]

Evaluate: $\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\cos^2 x}$

Solution

Solution:

$$= \lim \frac{1-\sin x}{1-\sin^2 x} = \lim \frac{1}{1+\sin x}$$
$$= \frac{1}{1+1} = \frac{1}{2}$$

Question 5.3 [3 marks]

Evaluate: $\lim_{x\to\infty}(1+\frac{5}{x})^{2x}$

Solution

Solution: Form 1^∞. Let $L=e^{\lim 2x(\frac{5}{x})}=e^{10}$. Alternatively: limit is $((1+5/x)^{x/5\cdot 5})^2\to(e^5)^2=e^{10}$.

Question 5(B) [8 marks]

Attempt any two

Question 5.1 [4 marks]

Find the equation of the line passing through point $(2,4)$ and perpendicular to line $5x-7y+11=0$

Solution

Solution: Given line slope $m=5/7$. Perpendicular slope $m'=-7/5$. Equation: $y-4=-\frac{7}{5}(x-2)$. $5(y-4)=-7(x-2)$. $5y-20=-7x+14$. $7x+5y-34=0$.

Question 5.2 [4 marks]

If the equation of circle is $2x^2+2y^2+4x-8y-6=0$ then find its center and radius

Solution

Solution: Divide by 2: $x^2+y^2+2x-4y-3=0$. Compare with general eqn: $2g=2 \implies g=1$, $2f=-4 \implies f=-2$, $c=-3$. Center $(-g,-f)=(-1,2)$. Radius $\sqrt{g^2+f^2-c}=\sqrt{1+4-(-3)}=\sqrt{8}=2\sqrt{2}$.

Question 5.3 [4 marks]

Find the equation of tangent and normal of circle $x^2 + y^2 - 2x + 4y - 20 = 0$ at point $(-2, 2)$

Solution

Solution: Center $C(1, -2)$. Point $P(-2, 2)$. Slope of Normal (CP) $m_N = \frac{2-(-2)}{-2-1} = \frac{4}{-3}$. Slope of Tangent $m_T = 3/4$.
Normal Eq: $y - 2 = -\frac{4}{3}(x + 2) \implies 3y - 6 = -4x - 8 \implies 4x + 3y + 2 = 0$. Tangent Eq: $y - 2 = \frac{3}{4}(x + 2) \implies 4y - 8 = 3x + 6 \implies 3x - 4y + 14 = 0$.

Formula Cheat Sheet

Coordinate Geometry

- Slope $m = \frac{y_2 - y_1}{x_2 - x_1}$
- Circle: $(x - h)^2 + (y - k)^2 = r^2$
- Tangent/Normal slopes relationship

Trigonometry

- Sum-to-Product formulas
- Inverse trigonometric identities

Limits

- $\lim_{x \to \infty}(1 + k/x)^x = e^k$
- Rationalization technique for $\infty - \infty$

Vectors

- $(a + b) \times (a - b) = 2(b \times a)$
- Condition for perpendicular vectors: dot product is 0