Applied Mathematics (4320001) - Winter 2023 Solution

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Question 1 [14 marks]

Fill in the blanks using appropriate choice from the given options.

Question 1(1) [1 marks]

If $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ **then** $4A =$ **... Answer**: (b) $\begin{bmatrix} 4 & 8 \\ 12 & -4 \end{bmatrix}$

Solution

$$4A = 4 \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 12 & -4 \end{bmatrix}$$

Question 1(2) [1 marks]

Order of the matrix $\begin{bmatrix} 1 & 1 & 2 \\ -3 & 2 & 3 \end{bmatrix}$ **is ... Answer**: (a) 2×3

Solution

Matrix has 2 rows and 3 columns, so order is 2×3.

Question 1(3) [1 marks]

If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **then** $A^2 =$ **... Answer**: (d) $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

Solution

$$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Question 1(4) [1 marks]

If $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ **then adjoint of A = ... Answer**: (c) $\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$

Solution

For matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $adj(A) = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$

Question 1(5) [1 marks]

$\frac{d}{dx}(\tan x)$ **= ... Answer**: (d) $\sec^2 x$

Solution

$\frac{d}{dx}(\tan x) = \sec^2 x$

Question 1(6) [1 marks]

$\frac{d}{dx}(\sin 5x)$ **= ... Answer**: (b) $5 \cos 5x$

Solution

$\frac{d}{dx}(\sin 5x) = 5 \cos 5x$ (using chain rule)

Question 1(7) [1 marks]

If function $y = f(x)$ **is maximum at** $x = a$ **then** $f'(a)$ **= ... Answer**: (c) 0

Solution

At maximum point, first derivative equals zero: $f'(a) = 0$

Question 1(8) [1 marks]

$\int \sin x dx$ **= ... + C Answer**: (a) $-\cos x$

Solution

$\int \sin x \, dx = -\cos x + C$

Question 1(9) [1 marks]

$\int \frac{1}{x^2+4} dx$ **= ... + C Answer**: (d) $\frac{1}{2} \tan^{-1}(\frac{x}{2})$

Solution

$\int \frac{1}{x^2+4}dx = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

Question 1(10) [1 marks]

$\int_1^2 x^2dx =$ **... Answer**: (a) 7/3

Solution

$\int_1^2 x^2dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$

Question 1(11) [1 marks]

Order of differential equation $\left(\frac{d^3y}{dx^3}\right)^4 + \frac{dy}{dx} + 5y = 0$ **is ... Answer**: (c) 3

Solution

Order is the highest derivative present = 3

Question 1(12) [1 marks]

Integrating factor of $\frac{dy}{dx} + \frac{y}{x} = 1$ **is ... Answer**: (b) x

Solution

I.F. $= e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

Question 1(13) [1 marks]

Mean of 39,23,58,47,50,16,61 is ... Answer: (b) 42

Solution

Mean $= \frac{39+23+58+47+50+16+61}{7} = \frac{294}{7} = 42$

Question 1(14) [1 marks]

Mean of first five natural numbers is ... Answer: (a) 3

Solution

Mean $= \frac{1+2+3+4+5}{5} = \frac{15}{5} = 3$

Question 2 [14 marks]

Attempt any two

Question 2(a) [6 marks]

Question 2(a)(1) [3 marks]

If $A = \begin{bmatrix} 1 & 3 & 5 \\ -1 & 0 & 2 \\ 4 & 3 & 6 \end{bmatrix}$**,** $B = \begin{bmatrix} 3 & 4 & 5 \\ 5 & 4 & 3 \\ 3 & 5 & 4 \end{bmatrix}$**,** $C = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 3 \\ 4 & 5 & 6 \end{bmatrix}$**, find** $3A + 2B - 4C$

Solution

$$3A = \begin{bmatrix} 3 & 9 & 15 \\ -3 & 0 & 6 \\ 12 & 9 & 18 \end{bmatrix}$$
$$2B = \begin{bmatrix} 6 & 8 & 10 \\ 10 & 8 & 6 \\ 6 & 10 & 8 \end{bmatrix}$$
$$4C = \begin{bmatrix} 4 & 8 & 4 \\ 12 & 12 & 12 \\ 16 & 20 & 24 \end{bmatrix}$$
$$3A + 2B - 4C = \begin{bmatrix} 5 & 9 & 21 \\ -5 & -4 & 0 \\ 2 & -1 & 2 \end{bmatrix}$$

Question 2(a)(2) [3 marks]

If $A = \begin{bmatrix} 7 & 5 \\ -1 & 2 \end{bmatrix}$**,** $B = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$**, show that** $(A + B)^T = A^T + B^T$

Solution

$$A + B = \begin{bmatrix} 8 & 4 \\ 2 & 4 \end{bmatrix}$$
$$(A + B)^T = \begin{bmatrix} 8 & 2 \\ 4 & 4 \end{bmatrix}$$
$$A^T = \begin{bmatrix} 7 & -1 \\ 5 & 2 \end{bmatrix}, B^T = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$$
$$A^T + B^T = \begin{bmatrix} 8 & 2 \\ 4 & 4 \end{bmatrix}$$
Hence proved: $(A + B)^T = A^T + B^T$

Question 2(a)(3) [3 marks]

Solve the differential equation $xydy = (x+1)(y+1)dx$

Solution

Separating variables: $\frac{y}{y+1}dy = \frac{x+1}{x}dx$
$\left(1 - \frac{1}{y+1}\right)dy = \left(1 + \frac{1}{x}\right)dx$
Integrating: $y - \ln|y+1| = x + \ln|x| + C$
Final answer: $y - x = \ln|y+1| + \ln|x| + C$

Question 2(b) [8 marks]

Question 2(b)(1) [4 marks]

Find the inverse of matrix $\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

Solution

Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$
$|A| = 3(-3 - (-2)) - 1(2 - (-1)) + 2(4 - (-3)) = 3(-1) - 1(3) + 2(7) = -3 - 3 + 14 = 8$
Cofactors:

- $C_{11} = -1, C_{12} = -3, C_{13} = 7$
- $C_{21} = 3, C_{22} = 1, C_{23} = -5$
- $C_{31} = 5, C_{32} = 7, C_{33} = -11$

$adj(A) = \begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$
$A^{-1} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$

Question 2(b)(2) [4 marks]

Solve $3x - 2y = 8, 5x + 4y = 6$ **using matrix method**

Solution

$\begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$
$|A| = 3(4) - (-2)(5) = 12 + 10 = 22$

$A^{-1} = \frac{1}{22}\begin{bmatrix} 4 & 2 \\ -5 & 3 \end{bmatrix}$
$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 4 & 2 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ 6 \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 44 \\ -22 \end{bmatrix}$
Answer: $x = 2, y = -1$

Question 2(b)(3) [4 marks]

If $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$**, find** $A \cdot adj(A)$

Solution

$|A| = 1(6-2) - 2(4-1) + 1(4-3) = 4 - 6 + 1 = -1$
For any matrix A: $A \cdot adj(A) = |A| \cdot I$
$A \cdot adj(A) = (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Question 3 [14 marks]

Attempt any two

Question 3(a) [6 marks]

Question 3(a)(1) [3 marks]

If $y = \log(\frac{\sin x}{1+\cos x})$**, find** $\frac{dy}{dx}$

Solution

$y = \log(\sin x) - \log(1 + \cos x)$
$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x - \frac{1}{1+\cos x} \cdot (-\sin x)$
$= \frac{\cos x}{\sin x} + \frac{\sin x}{1+\cos x}$
$= \cot x + \frac{\sin x}{1+\cos x}$
Using identity: $\frac{\sin x}{1+\cos x} = \tan(\frac{x}{2})$
Answer: $\frac{dy}{dx} = \cot x + \tan(\frac{x}{2})$

Question 3(a)(2) [3 marks]

If $y = \sin(x + y)$**, find** $\frac{dy}{dx}$

Solution

Differentiating both sides: $\frac{dy}{dx} = \cos(x+y) \cdot \left(1 + \frac{dy}{dx}\right)$
$\frac{dy}{dx} = \cos(x+y) + \cos(x+y) \cdot \frac{dy}{dx}$
$\frac{dy}{dx} - \cos(x+y) \cdot \frac{dy}{dx} = \cos(x+y)$
$\frac{dy}{dx}[1 - \cos(x+y)] = \cos(x+y)$
Answer: $\frac{dy}{dx} = \frac{\cos(x+y)}{1-\cos(x+y)}$

Question 3(a)(3) [3 marks]

Obtain $\int x^2 \log x dx$

Solution

Using integration by parts: $\int u dv = uv - \int v du$
Let $u = \log x$, $dv = x^2 dx$ Then $du = \frac{1}{x} dx$, $v = \frac{x^3}{3}$
$\int x^2 \log x \, dx = \log x \cdot \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x} dx$
$= \frac{x^3 \log x}{3} - \int \frac{x^2}{3} dx$
$= \frac{x^3 \log x}{3} - \frac{x^3}{9} + C$
Answer: $\frac{x^3}{3}(\log x - \frac{1}{3}) + C$

Question 3(b) [8 marks]

Question 3(b)(1) [4 marks]

Motion equation $s = 2t^3 - 3t^2 - 12t + 7$**. Find s and t when acceleration is zero**

Solution

$s = 2t^3 - 3t^2 - 12t + 7$
Velocity: $v = \frac{ds}{dt} = 6t^2 - 6t - 12$
Acceleration: $a = \frac{dv}{dt} = 12t - 6$
When acceleration $= 0$: $12t - 6 = 0$ $t = \frac{1}{2}$
At $t = 1/2$: $s = 2(\frac{1}{2})^3 - 3(\frac{1}{2})^2 - 12(\frac{1}{2}) + 7 = \frac{1}{4} - \frac{3}{4} - 6 + 7 = \frac{1}{2}$
Answer: $t = 1/2, s = 1/2$

Question 3(b)(2) [4 marks]

If $y = 2e^{3x} + 3e^{-2x}$**, prove** $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$

Solution

$y = 2e^{3x} + 3e^{-2x}$
$\frac{dy}{dx} = 6e^{3x} - 6e^{-2x}$
$\frac{d^2y}{dx^2} = 18e^{3x} + 12e^{-2x}$

Now: $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y$
$= (18e^{3x} + 12e^{-2x}) - (6e^{3x} - 6e^{-2x}) - 6(2e^{3x} + 3e^{-2x})$
$= 18e^{3x} + 12e^{-2x} - 6e^{3x} + 6e^{-2x} - 12e^{3x} - 18e^{-2x}$
$= (18 - 6 - 12)e^{3x} + (12 + 6 - 18)e^{-2x} = 0$
Hence proved

Question 3(b)(3) [4 marks]

Find maximum and minimum values of $f(x) = x^3 - 3x + 11$

Solution

$f(x) = x^3 - 3x + 11$
$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x - 1)(x + 1)$
Critical points: $x = 1, x = -1$
$f''(x) = 6x$
At $x = 1$: $f''(1) = 6 > 0 \rightarrow$ Local minimum At $x = -1$: $f''(-1) = -6 < 0 \rightarrow$ Local maximum
$f(1) = 1 - 3 + 11 = 9$ (minimum) $f(-1) = -1 + 3 + 11 = 13$ (maximum)
Answer: Maximum = 13 at $x = -1$, Minimum = 9 at $x = 1$

Question 4 [14 marks]

Attempt any two

Question 4(a) [6 marks]

Question 4(a)(1) [3 marks]

Obtain $\int \sin 5x \sin 6x dx$

Solution

Using identity: $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$
$\sin 5x \sin 6x = \frac{1}{2}[\cos(5x - 6x) - \cos(5x + 6x)]$
$= \frac{1}{2}[\cos(-x) - \cos(11x)] = \frac{1}{2}[\cos x - \cos(11x)]$
$\int \sin 5x \sin 6x \, dx = \frac{1}{2} \int [\cos x - \cos(11x)]dx$
$= \frac{1}{2}[\sin x - \frac{\sin(11x)}{11}] + C$
Answer: $\frac{1}{2} \sin x - \frac{\sin(11x)}{22} + C$

Question 4(a)(2) [3 marks]

Obtain $\int \frac{(1+x)e^x}{\cos^2(xe^x)} dx$

Solution

Let $u = xe^x$, then $du = (1 + x)e^x dx$
The integral becomes: $\int \frac{du}{\cos^2 u} = \int \sec^2 u \, du = \tan u + C$
Substituting back: $= \tan(xe^x) + C$
Answer: $\tan(xe^x) + C$

Question 4(a)(3) [3 marks]

Find standard deviation for data: 6,7,10,12,13,4,8,12

Solution

Data: 6, 7, 10, 12, 13, 4, 8, 12 $n = 8$
Mean $= \frac{6+7+10+12+13+4+8+12}{8} = \frac{72}{8} = 9$

Table 1. Standard Deviation Calculation

x	x-9	(x-9)^2
6	-3	9
7	-2	4
10	1	1
12	3	9
13	4	16
4	-5	25
8	-1	1
12	3	9

$\sum(x - 9)^2 = 74$
Standard deviation $= \sqrt{\frac{\sum(x-\bar{x})^2}{n}} = \sqrt{\frac{74}{8}} = \sqrt{9.25} = 3.04$
Answer: $\sigma = 3.04$

Question 4(b) [8 marks]

Question 4(b)(1) [4 marks]

Obtain $\int \frac{2x+1}{(x+1)(x-3)} dx$

Solution

Using partial fractions: $\frac{2x+1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$
$2x + 1 = A(x - 3) + B(x + 1)$
When $x = -1$: $2(-1) + 1 = A(-4) \Rightarrow -1 = -4A \Rightarrow A = \frac{1}{4}$
When $x = 3$: $2(3) + 1 = B(4) \Rightarrow 7 = 4B \Rightarrow B = \frac{7}{4}$
$\int \frac{2x+1}{(x+1)(x-3)} dx = \frac{1}{4} \int \frac{1}{x+1} dx + \frac{7}{4} \int \frac{1}{x-3} dx$
$= \frac{1}{4} \ln |x + 1| + \frac{7}{4} \ln |x - 3| + C$
Answer: $\frac{1}{4} \ln |x + 1| + \frac{7}{4} \ln |x - 3| + C$

Question 4(b)(2) [4 marks]

Obtain $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}} dx$

Solution

Let $I = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}} dx$
Using property: $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
$I = \int_0^{\pi/2} \frac{\sqrt{\cot(\pi/2-x)}}{\sqrt{\cot(\pi/2-x)}+\sqrt{\tan(\pi/2-x)}} dx$
Since $\cot(\pi/2 - x) = \tan x$ and $\tan(\pi/2 - x) = \cot x$:
$I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}} dx$
Adding both expressions: $2I = \int_0^{\pi/2} \frac{\sqrt{\cot x}+\sqrt{\tan x}}{\sqrt{\cot x}+\sqrt{\tan x}} dx = \int_0^{\pi/2} 1\, dx = \frac{\pi}{2}$
Answer: $I = \frac{\pi}{4}$

Question 4(b)(3) [4 marks]

Find mean deviation for grouped data

Solution

Table 2. Grouped Data

x_i	4	8	11	17	20	24	32
f_i	3	5	9	5	4	3	1

$N = \sum f_i = 3 + 5 + 9 + 5 + 4 + 3 + 1 = 30$
Mean $= \frac{\sum f_i x_i}{N} = \frac{3(4)+5(8)+9(11)+5(17)+4(20)+3(24)+1(32)}{30}$
$= \frac{12+40+99+85+80+72+32}{30} = \frac{420}{30} = 14$

Table 3. Mean Deviation Calculation

x_i	f_i	$\|x_i - 14\|$	$f_i\|x_i - 14\|$
4	3	10	30
8	5	6	30
11	9	3	27
17	5	3	15
20	4	6	24
24	3	10	30
32	1	18	18

$\sum f_i|x_i - 14| = 174$
Mean deviation $= \frac{\sum f_i|x_i-\bar{x}|}{N} = \frac{174}{30} = 5.8$
Answer: Mean deviation $= 5.8$

Question 5 [14 marks]

Attempt any two

Question 5(a) [6 marks]

Question 5(a)(1) [3 marks]

Find mean deviation for grouped data

Solution

Table 4. Grouped Data

Class	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Freq	3	7	12	15	8	3	2

$N = 50, \sum f_i x_i = 3100$
Mean $= 3100/50 = 62$

Table 5. Mean Deviation Calculation

Class	x_i	f_i	$\lvert x_i - 62\rvert$	$f_i\lvert x_i - 62\rvert$
30-40	35	3	27	81
40-50	45	7	17	119
50-60	55	12	7	84
60-70	65	15	3	45
70-80	75	8	13	104
80-90	85	3	23	69
90-100	95	2	33	66

Mean deviation $= 568/50 = 11.36$
Answer: Mean deviation $= 11.36$

Question 5(a)(2) [3 marks]

Find standard deviation for given data

Solution

Table 6. Grouped Data

Class	60	61	62	63	64	65	66	67	68
Freq	2	1	12	29	25	12	10	4	5

$N = 100$, Mean $= 63.8$

Table 7. Standard Deviation Calculation

x_i	f_i	$(x_i - 63.8)$	$(x_i - 63.8)^2$	$f_i(x_i - 63.8)^2$
60	2	-3.8	14.44	28.88
61	1	-2.8	7.84	7.84
62	12	-1.8	3.24	38.88
63	29	-0.8	0.64	18.56
64	25	0.2	0.04	1.00
65	12	1.2	1.44	17.28
66	10	2.2	4.84	48.40
67	4	3.2	10.24	40.96
68	5	4.2	17.64	88.20

$\sum f_i(x_i - \bar{x})^2 = 290$
Standard deviation $= \sqrt{290/100} = \sqrt{2.9} = 1.70$
Answer: $\sigma = 1.70$

Question 5(a)(3) [3 marks]

Find mean for grouped data

Solution

Table 8. Grouped Data

Class	0-20	20-40	40-60	60-80	80-100	100-120
Freq	26	31	35	42	82	71

Table 9. Mean Calculation

Class	Mid-value	f_i	$f_i x_i$
0-20	10	26	260
20-40	30	31	930
40-60	50	35	1750
60-80	70	42	2940
80-100	90	82	7380
100-120	110	71	7810

$N = 287, \sum f_i x_i = 21070$
Mean $= \frac{\sum f_i x_i}{N} = \frac{21070}{287} = 73.42$
Answer: Mean $= 73.42$

Question 5(b) [8 marks]

Question 5(b)(1) [4 marks]

Solve differential equation $(x + y + 1)^2 \frac{dy}{dx} = 1$

Solution

Let $z = x + y + 1$, then $\frac{dz}{dx} = 1 + \frac{dy}{dx}$ So $\frac{dy}{dx} = \frac{dz}{dx} - 1$
Substituting: $z^2(\frac{dz}{dx} - 1) = 1$ $z^2\frac{dz}{dx} - z^2 = 1$ $z^2\frac{dz}{dx} = 1 + z^2$ $\frac{z^2}{1+z^2}dz = dx$
Integrating: $\int \frac{z^2}{1+z^2}dz = \int dx$
$\int \left(1 - \frac{1}{1+z^2}\right) dz = x + C$
$z - \tan^{-1} z = x + C$
Substituting back $z = x + y + 1$: $(x + y + 1) - \tan^{-1}(x + y + 1) = x + C$
Answer: $y + 1 = \tan^{-1}(x + y + 1) + C$

Question 5(b)(2) [4 marks]

Solve $\frac{dy}{dx} + \frac{y}{x} = e^x$**,** $y(0) = 2$

Solution

This is a linear differential equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$
Here $P(x) = \frac{1}{x}, Q(x) = e^x$
Integrating factor: $I.F. = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x| = x$ (for $x > 0$)
Multiplying the equation by x: $x\frac{dy}{dx} + y = xe^x$
$\frac{d}{dx}(xy) = xe^x$
Integrating both sides: $xy = \int xe^x dx$
Using integration by parts for $\int xe^x dx$: Let $u = x, dv = e^x dx$ Then $du = dx, v = e^x$
$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x = e^x(x - 1)$
So: $xy = e^x(x - 1) + C$ $y = \frac{e^x(x-1)+C}{x}$
Using initial condition $y(0) = 2$: As $x \to 0$, we need to use L'Hôpital's rule or series expansion.
From the original equation at $x = 0$: $\frac{dy}{dx} = e^x - \frac{y}{x}$ This suggests we need to be more careful with the initial condition.
Alternative approach: Since the equation has a singularity at $x = 0$, we solve in the neighborhood where $x \neq 0$.
Answer: $y = \frac{e^x(x-1)+C}{x}$ where C is determined by boundary conditions.

Question 5(b)(3) [4 marks]

Solve $y\frac{dy}{dx} = \sqrt{1 + x^2 + y^2 + x^2y^2}$

Solution

$y\frac{dy}{dx} = \sqrt{1 + x^2 + y^2 + x^2y^2}$
$y\frac{dy}{dx} = \sqrt{(1 + x^2)(1 + y^2)}$
$\frac{ydy}{\sqrt{1+y^2}} = \sqrt{1 + x^2}dx$
Integrating both sides: $\int \frac{ydy}{\sqrt{1+y^2}} = \int \sqrt{1 + x^2}dx$
For the left side, let $u = 1 + y^2$, then $du = 2ydy$: $\int \frac{ydy}{\sqrt{1+y^2}} = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u} = \sqrt{1 + y^2}$
For the right side: $\int \sqrt{1 + x^2}dx = \frac{x\sqrt{1+x^2}}{2} + \frac{1}{2}\ln|x + \sqrt{1 + x^2}| + C$
Therefore: **Answer**: $\sqrt{1 + y^2} = \frac{x\sqrt{1+x^2}}{2} + \frac{1}{2}\ln|x + \sqrt{1 + x^2}| + C$

Formula Cheat Sheet

Matrix Operations

- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $A \cdot adj(A) = |A| \cdot I$
- For 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $adj = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Differentiation Formulas

- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\log x) = \frac{1}{x}$
- $\frac{d}{dx}(e^x) = e^x$
- Chain rule: $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$

Integration Formulas

- $\int \sin x \, dx = -\cos x + C$
- $\int \cos x \, dx = \sin x + C$
- $\int \sec^2 x \, dx = \tan x + C$
- $\int \frac{1}{x} dx = \ln |x| + C$
- $\int e^x dx = e^x + C$
- $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}(\frac{x}{a}) + C$

Differential Equations

- **Linear DE**: $\frac{dy}{dx} + P(x)y = Q(x)$
- **Integrating Factor**: $I.F. = e^{\int P(x)dx}$
- **Variable Separable**: $\frac{dy}{dx} = f(x)g(y) \Rightarrow \frac{dy}{g(y)} = f(x)dx$

Statistics

- **Mean**: $\bar{x} = \frac{\sum x_i}{n}$ (ungrouped), $\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$ (grouped)
- **Mean Deviation**: $M.D. = \frac{\sum |x_i - \bar{x}|}{n}$
- **Standard Deviation**: $\sigma = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n}}$