Engineering Mathematics (4320002) - Winter 2023 Solution

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Question 1 [14 marks]

Fill in the blanks using appropriate choice from the given options.

Question 1.1 [1 marks]

Order of the matrix $\begin{bmatrix} 2 & 5 \\ 7 & 8 \end{bmatrix}$ **is** ________________

Solution

Answer: (d) 2×2
Solution: The matrix has 2 rows and 2 columns, so its order is 2×2.

Question 1.2 [1 marks]

$\begin{bmatrix} 4 & 3 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 5 \\ 5 & 8 \end{bmatrix} =$ ________________

Solution

Answer: (a) $\begin{bmatrix} 5 & 8 \\ 11 & 10 \end{bmatrix}$
Solution:

$$\begin{bmatrix} 4 & 3 \\ 6 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 5 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 4+1 & 3+5 \\ 6+5 & 2+8 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 11 & 10 \end{bmatrix}$$

Question 1.3 [1 marks]

Which of the following is a square matrix?

Solution

Answer: (c) $\begin{bmatrix} 1 & 3 \\ 5 & 4 \end{bmatrix}$
Solution: A square matrix has equal number of rows and columns. Only option (c) has 2×2 dimensions.

Question 1.4 [1 marks]

If $A = [3]$ **and** $B = [4]$ **then** $A \cdot B =$ ________________

Solution

Answer: (b) 12
Solution:

$$A \cdot B = [3] \times [4] = [3 \times 4] = [12] = 12$$

Question 1.5 [1 marks]

$\frac{d}{dx} \sin x =$ ________________

Solution

Answer: (d) $\cos x$
Solution: The derivative of $\sin x$ is $\cos x$.

Question 1.6 [1 marks]

If $f(x) = xe^x$ **then** $f'(0) =$ ________________

Solution

Answer: (b) 1
Solution: Using product rule: $f'(x) = \frac{d}{dx}(xe^x) = e^x + xe^x = e^x(1 + x)$

$$f'(0) = e^0(1 + 0) = 1 \times 1 = 1$$

Question 1.7 [1 marks]

If $y = x^2$ **then** $\frac{d^2y}{dx^2} =$ ________________

Solution

Answer: (b) 2
Solution:

$$y = x^2$$
$$\frac{dy}{dx} = 2x$$
$$\frac{d^2y}{dx^2} = 2$$

Question 1.8 [1 marks]

$\int \cos x dx =$ ________________ $+c$

Solution

Answer: (a) $\sin x$
Solution:

$$\int \cos x dx = \sin x + c$$

Question 1.9 [1 marks]

$\int_0^1 x dx =$ ______________

Solution

Answer: (c) $\frac{1}{2}$
Solution:

$$\int_0^1 x dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$$

Question 1.10 [1 marks]

$\int \frac{1}{1+x^2} dx =$ ______________ $+c$

Solution

Answer: (a) $\tan^{-1} x$
Solution:

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$$

Question 1.11 [1 marks]

Order of differential equation $x \sin y + xy = x$ **is** ______________

Solution

Answer: (b) 1
Solution: The equation can be written as $\frac{dy}{dx} = \frac{1-xy}{\sin y}$. The highest order derivative is first order.

Question 1.12 [1 marks]

Integration factor of $\frac{dy}{dx} + y = x$ **is** ______________

Solution

Answer: (d) e^x
Solution: For $\frac{dy}{dx} + Py = Q$, integration factor $= e^{\int P dx} = e^{\int 1 dx} = e^x$

Question 1.13 [1 marks]

$i^2 =$ ______________

Solution

Answer: (b) -1
Solution: By definition, $i^2 = -1$

Question 1.14 [1 marks]

$(2 + 3i)(2 - 3i) =$ ______________

Solution

Answer: (c) 13
Solution:

$$(2 + 3i)(2 - 3i) = 2^2 - (3i)^2 = 4 - 9i^2 = 4 - 9(-1) = 4 + 9 = 13$$

Question 2(A) [6 marks]

Attempt any two.

Question 2(A).1 [3 marks]

If $A = \begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 8 \\ 4 & 6 \end{bmatrix}$ **and** $C = \begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix}$ **then find** $2A + 3B - C$

Solution

Solution:

$$2A = 2\begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ -2 & 6 \end{bmatrix}$$

$$3B = 3\begin{bmatrix} 5 & 8 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 15 & 24 \\ 12 & 18 \end{bmatrix}$$

$$2A + 3B = \begin{bmatrix} 4 & 10 \\ -2 & 6 \end{bmatrix} + \begin{bmatrix} 15 & 24 \\ 12 & 18 \end{bmatrix} = \begin{bmatrix} 19 & 34 \\ 10 & 24 \end{bmatrix}$$

$$2A + 3B - C = \begin{bmatrix} 19 & 34 \\ 10 & 24 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 15 & 32 \\ 9 & 19 \end{bmatrix}$$

Question 2(A).2 [3 marks]

If $M = \begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix}$ **and** $N = \begin{bmatrix} 6 & 9 \\ 0 & 5 \end{bmatrix}$ **then prove that** $(M + N)^T = M^T + N^T$

Solution

Solution:

$$M + N = \begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix} + \begin{bmatrix} 6 & 9 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 13 \\ 3 & 12 \end{bmatrix}$$

$$(M + N)^T = \begin{bmatrix} 7 & 3 \\ 13 & 12 \end{bmatrix}$$

$$M^T = \begin{bmatrix} 1 & 3 \\ 4 & 7 \end{bmatrix}, \quad N^T = \begin{bmatrix} 6 & 0 \\ 9 & 5 \end{bmatrix}$$

$$M^T + N^T = \begin{bmatrix} 1 & 3 \\ 4 & 7 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 9 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 13 & 12 \end{bmatrix}$$

Hence, $(M + N)^T = M^T + N^T$ is proved.

Question 2(A).3 [3 marks]

Solve differential equation: $x\frac{dy}{dx} + y = xy$

Solution

Solution:

$$x\frac{dy}{dx} + y = xy$$

$$\frac{dy}{dx} + \frac{y}{x} = y$$

$$\frac{dy}{dx} = y - \frac{y}{x} = y\left(1 - \frac{1}{x}\right) = y\left(\frac{x-1}{x}\right)$$

Separating variables:

$$\frac{dy}{y} = \frac{x-1}{x}dx$$

Integrating:

$$\ln|y| = \int \frac{x-1}{x}dx = \int \left(1 - \frac{1}{x}\right)dx = x - \ln|x| + C$$

$$y = Ae^{x - \ln|x|} = A\frac{e^x}{x}$$

Question 2(B) [8 marks]

Attempt any two.

Question 2(B).1 [4 marks]

Solve equations $2x + 3y = 8$**,** $3x + 4y = 11$ **using matrix method**

Solution

Solution: Writing in matrix form: $AX = B$

$$\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$$

Finding A^{-1}:

$$|A| = 2(4) - 3(3) = 8 - 9 = -1$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 8 \\ 11 \end{bmatrix} = \begin{bmatrix} -32 + 33 \\ 24 - 22 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore: $x = 1, y = 2$

Question 2(B).2 [4 marks]

If $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ **then prove that** $(AB)^T = B^T A^T$

Solution

Solution:

$$AB = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 1 & 6 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 3 & 1 \\ 8 & 6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}, \quad B^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 6 \end{bmatrix}$$

Hence, $(AB)^T = B^T A^T$ is proved.

Question 2(B).3 [4 marks]

If $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ **then prove that** $A^2 - 4A + 7I = O$

Solution

Solution:

$$A^2 = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix}$$

$$4A = 4\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ -4 & 8 \end{bmatrix}$$

$$7I = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$A^2 - 4A + 7I = \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ -4 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Hence proved.

Question 3(A) [6 marks]

Attempt any two.

Question 3(A).1 [3 marks]

Find derivative of $f(x) = e^x$ using definition of differentiation

Solution

Solution: Using definition: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$f'(x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h}$$

$$= \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h}$$

Since $\lim_{h\to 0} \frac{e^h-1}{h} = 1$ Therefore: $f'(x) = e^x$

Question 3(A).2 [3 marks]

If $y = \log(\sin x)$ then find $\frac{dy}{dx}$

Solution

Solution:

$$y = \log(\sin x)$$

Using chain rule:

$$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x} = \cot x$$

Question 3(A).3 [3 marks]

Evaluate: $\int \left(4x^3 + 3x^2 + \frac{2}{x}\right) dx$

Solution

Solution:

$$\int \left(4x^3 + 3x^2 + \frac{2}{x}\right) dx$$
$$= \int 4x^3 dx + \int 3x^2 dx + \int \frac{2}{x} dx$$
$$= 4 \cdot \frac{x^4}{4} + 3 \cdot \frac{x^3}{3} + 2\ln|x| + C$$
$$= x^4 + x^3 + 2\ln|x| + C$$

Question 3(B) [8 marks]

Attempt any two.

Question 3(B).1 [4 marks]

If $y = e^{\tan x} + \log(\sin x)$ then find $\frac{dy}{dx}$

Solution

Solution:

$$y = e^{\tan x} + \log(\sin x)$$
$$\frac{dy}{dx} = \frac{d}{dx}[e^{\tan x}] + \frac{d}{dx}[\log(\sin x)]$$

For first term: $\frac{d}{dx}[e^{\tan x}] = e^{\tan x} \cdot \sec^2 x$ For second term: $\frac{d}{dx}[\log(\sin x)] = \frac{1}{\sin x} \cdot \cos x = \cot x$
Therefore: $\frac{dy}{dx} = e^{\tan x} \sec^2 x + \cot x$

Question 3(B).2 [4 marks]

The equation of motion of a particle is $s = t^4 + 3t$. Find its velocity and acceleration at $t = 2$ sec

Solution

Solution: Given: $s = t^4 + 3t$
Velocity: $v = \frac{ds}{dt} = 4t^3 + 3$ At $t = 2$: $v = 4(2)^3 + 3 = 4(8) + 3 = 32 + 3 = 35$ units/sec
Acceleration: $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 12t^2$ At $t = 2$: $a = 12(2)^2 = 12(4) = 48$ units/sec²

Question 3(B).3 [4 marks]

Find the maximum and minimum value of the function $f(x) = 2x^3 - 3x^2 - 12x + 5$

Solution

Solution:

$$f(x) = 2x^3 - 3x^2 - 12x + 5$$
$$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1)$$

For critical points: $f'(x) = 0 \Rightarrow 6(x-2)(x+1) = 0 \Rightarrow x = 2$ or $x = -1$

$$f''(x) = 12x - 6$$

At $x = -1$: $f''(-1) = 12(-1) - 6 = -18 < 0$ (Maximum) At $x = 2$: $f''(2) = 12(2) - 6 = 18 > 0$ (Minimum) $f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 5 = -2 - 3 + 12 + 5 = 12$ (Maximum) $f(2) = 2(8) - 3(4) - 12(2) + 5 = 16 - 12 - 24 + 5 = -15$ (Minimum)
Maximum value: 12 at $x = -1$ **Minimum value**: -15 at $x = 2$

Question 4(A) [6 marks]

Attempt any two.

Question 4(A).1 [3 marks]

Evaluate: $\int xe^x dx$

Solution

Solution: Using integration by parts: $\int udv = uv - \int vdu$
Let $u = x$, $dv = e^x dx$ Then $du = dx$, $v = e^x$

$$\int xe^x dx = x \cdot e^x - \int e^x dx = xe^x - e^x + C = e^x(x-1) + C$$

Question 4(A).2 [3 marks]

Evaluate: $\int \frac{dx}{\sqrt{9-4x^2}}$

Solution

Solution:

$$\int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{dx}{\sqrt{9(1-\frac{4x^2}{9})}} = \int \frac{dx}{3\sqrt{1-\left(\frac{2x}{3}\right)^2}}$$

Let $\frac{2x}{3} = \sin\theta$, then $x = \frac{3\sin\theta}{2}$, $dx = \frac{3\cos\theta}{2}d\theta$

$$= \int \frac{\frac{3\cos\theta}{2}d\theta}{3\sqrt{1-\sin^2\theta}} = \int \frac{\frac{3\cos\theta}{2}d\theta}{3\cos\theta} = \int \frac{1}{2}d\theta = \frac{\theta}{2} + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C$$

Question 4(A).3 [3 marks]

Find complex conjugate of $\frac{1-i}{1+i}$

Solution

Solution:

$$\frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{(1-i)^2}{1-i^2} = \frac{1-2i+i^2}{1-(-1)} = \frac{1-2i-1}{2} = \frac{-2i}{2} = -i$$

Complex conjugate of $-i$ is $\overline{-i} = i$

Question 4(B) [8 marks]

Attempt any two.

Question 4(B).1 [4 marks]

Evaluate: $\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx$

Solution

Solution: Let $I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx$
Using property: $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos(\pi/2-x)}}{\sqrt{\cos(\pi/2-x)}+\sqrt{\sin(\pi/2-x)}} dx = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$$

Adding both expressions:

$$2I = \int_0^{\pi/2} \frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx = \int_0^{\pi/2} 1 dx = \frac{\pi}{2}$$

Therefore: $I = \frac{\pi}{4}$

Question 4(B).2 [4 marks]

Find the area of circle $x^2 + y^2 = a^2$ using integration

Solution

Solution: For circle $x^2 + y^2 = a^2$, we have $y = \pm\sqrt{a^2 - x^2}$
Area of circle $= 4\times$ Area in first quadrant

$$= 4\int_0^a \sqrt{a^2 - x^2} dx$$

Let $x = a\sin\theta$, $dx = a\cos\theta d\theta$ When $x = 0$, $\theta = 0$; when $x = a$, $\theta = \pi/2$

$$= 4\int_0^{\pi/2} \sqrt{a^2 - a^2\sin^2\theta} \cdot a\cos\theta d\theta$$

$$= 4\int_0^{\pi/2} a\cos\theta \cdot a\cos\theta d\theta$$

$$= 4a^2 \int_0^{\pi/2} \cos^2\theta d\theta$$

$$= 4a^2 \cdot \frac{\pi}{4} = \pi a^2$$

Question 4(B).3 [4 marks]

Simplify: $\frac{(\cos 3\theta+i\sin 3\theta)^4\cdot(\cos\theta-i\sin\theta)^5}{(\cos 2\theta-i\sin 2\theta)^3\cdot(\cos 12\theta+i\sin 12\theta)}$

Solution

Solution: Using De Moivre's theorem: $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$
Numerator:

$$(\cos 3\theta+i\sin 3\theta)^4\cdot(\cos\theta-i\sin\theta)^5$$
$$=(\cos 12\theta+i\sin 12\theta)\cdot(\cos(-5\theta)+i\sin(-5\theta))$$
$$=\cos(12\theta-5\theta)+i\sin(12\theta-5\theta)$$
$$=\cos 7\theta+i\sin 7\theta$$

Denominator:

$$(\cos 2\theta-i\sin 2\theta)^3\cdot(\cos 12\theta+i\sin 12\theta)$$
$$=(\cos(-6\theta)+i\sin(-6\theta))\cdot(\cos 12\theta+i\sin 12\theta)$$
$$=\cos(-6\theta+12\theta)+i\sin(-6\theta+12\theta)$$
$$=\cos 6\theta+i\sin 6\theta$$

Result:

$$\frac{\cos 7\theta+i\sin 7\theta}{\cos 6\theta+i\sin 6\theta}=\cos(7\theta-6\theta)+i\sin(7\theta-6\theta)=\cos\theta+i\sin\theta$$

Question 5(A) [6 marks]

Attempt any two.

Question 5(A).1 [3 marks]

If $(3x-7)+2iy=5y+(5+x)i$ **then find value of x and y**

Solution

Solution:

$$(3x-7)+2iy=5y+(5+x)i$$

Comparing real and imaginary parts: Real parts: $3x-7=5y$ ... (1) Imaginary parts: $2y=5+x$ ... (2)
From equation (2): $x=2y-5$ ... (3)
Substituting (3) in (1):

$$3(2y-5)-7=5y$$
$$6y-15-7=5y$$
$$6y-22=5y\Rightarrow y=22$$

From (3): $x=2(22)-5=44-5=39$
Therefore: $x=39, y=22$

Question 5(A).2 [3 marks]

Convert $z=1+\sqrt{3}i$ **into polar form**

Solution

Solution:

$$z = 1 + \sqrt{3}i$$

Modulus: $|z| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$
Argument: $\arg(z) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$
Polar form: $z = |z|(\cos\theta + i\sin\theta) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

Question 5(A).3 [3 marks]

Express $\frac{4+2i}{(3+2i)(5-3i)}$ in $a + ib$ form

Solution

Solution: First, simplify denominator:

$$(3+2i)(5-3i) = 15 - 9i + 10i - 6i^2 = 15 + i - 6(-1) = 15 + i + 6 = 21 + i$$

$$\frac{4+2i}{21+i} = \frac{(4+2i)(21-i)}{(21+i)(21-i)} = \frac{84 - 4i + 42i - 2i^2}{21^2 - i^2} = \frac{84 + 38i + 2}{441 + 1} = \frac{86 + 38i}{442}$$

$$= \frac{86}{442} + \frac{38}{442}i = \frac{43}{221} + \frac{19}{221}i$$

Question 5(B) [8 marks]

Attempt any two.

Question 5(B).1 [4 marks]

Solve differential equation: $\frac{dy}{dx} + 2y = 3e^x$

Solution

Solution: This is a first-order linear differential equation of the form $\frac{dy}{dx} + Py = Q$
Here: $P = 2$, $Q = 3e^x$
Integration factor: $\mu = e^{\int P dx} = e^{\int 2 dx} = e^{2x}$
Multiplying equation by μ:

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = 3e^{2x} \cdot e^x = 3e^{3x}$$

This gives: $\frac{d}{dx}(ye^{2x}) = 3e^{3x}$
Integrating both sides:

$$ye^{2x} = \int 3e^{3x}dx = 3 \cdot \frac{e^{3x}}{3} + C = e^{3x} + C$$

Therefore: $y = \frac{e^{3x}+C}{e^{2x}} = e^x + Ce^{-2x}$

Question 5(B).2 [4 marks]

Solve differential equation: $\frac{dy}{dx} = (x+y)^2$

Solution

Solution: Let $v = x + y$, then $\frac{dv}{dx} = 1 + \frac{dy}{dx}$
So $\frac{dy}{dx} = \frac{dv}{dx} - 1$
Substituting in the original equation:

$$\frac{dv}{dx} - 1 = v^2$$

$$\frac{dv}{dx} = v^2 + 1$$

Separating variables:

$$\frac{dv}{v^2 + 1} = dx$$

Integrating both sides:

$$\int \frac{dv}{v^2 + 1} = \int dx$$

$$\tan^{-1}(v) = x + C$$

$$v = \tan(x + C)$$

Substituting back: $x + y = \tan(x + C)$ Therefore: $y = \tan(x + C) - x$

Question 5(B).3 [4 marks]

Solve differential equation: $\frac{dy}{dx} + \frac{y}{x} = e^x$, $y(0) = 2$

Solution

Solution: This is a first-order linear differential equation: $\frac{dy}{dx} + \frac{y}{x} = e^x$
Here: $P = \frac{1}{x}$, $Q = e^x$
Integration factor: $\mu = e^{\int \frac{1}{x} dx} = e^{\ln|x|} = |x| = x$ (for $x > 0$)
Multiplying equation by $\mu = x$:

$$x\frac{dy}{dx} + y = xe^x$$

This gives: $\frac{d}{dx}(xy) = xe^x$
Integrating both sides using integration by parts:

$$xy = \int xe^x dx$$

For $\int xe^x dx$: Let $u = x$, $dv = e^x dx$ Then $du = dx$, $v = e^x$

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x = e^x(x - 1)$$

So: $xy = e^x(x - 1) + C$

$$y = \frac{e^x(x - 1) + C}{x}$$

Using initial condition $y(0) = 2$: This presents a problem as we have division by zero. The equation needs to be solved more carefully near $x = 0$.
For the general solution: $y = e^x\left(1 - \frac{1}{x}\right) + \frac{C}{x}$