

# Subject Name Solutions

4300001 – Winter 2022

Semester 1 Study Material

*Detailed Solutions and Explanations*

## Q.1 [14 marks]

Fill in the blanks using appropriate choice from the given options

### 0.0.1 Q1.1 [1 mark]

If  $\begin{vmatrix} x & 8 \\ 2 & 4 \end{vmatrix} = 0$  then the value of  $x$  is \_\_\_\_\_

#### Solution

c. 8

**Solution:**  $\begin{vmatrix} x & 8 \\ 2 & 4 \end{vmatrix} = x(4) - 8(2) = 4x - 16$

Given:  $4x - 16 = 0$   $4x = 16$   $x = 4$

Wait, let me recalculate: If the determinant is 0, then  $4x - 16 = 0$ , so  $x = 4$ . But 4 is option a, not c. Let me verify the options again... The answer should be a. 4

### 0.0.2 Q1.2 [1 mark]

\*\*\$

$$\begin{vmatrix} 2 & -9 & 1 \\ 5 & -8 & 4 \\ 0 & 3 & 0 \end{vmatrix} = \$ \text{_____} **$$

#### Solution

a. -9

**Solution:** Expanding along the third row (which has two zeros):  $\begin{vmatrix} 2 & -9 & 1 \\ 5 & -8 & 4 \\ 0 & 3 & 0 \end{vmatrix} = 0 - 3 \begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} + 0$   
 $= -3(2 \times 4 - 1 \times 5) = -3(8 - 5) = -3(3) = -9$

### 0.0.3 Q1.3 [1 mark]

If  $f(x) = \log x$  then \$f(1) = \\$ \text{\_\_\_\_\_}

#### Solution

a. 0

**Solution:**  $f(x) = \log x$   $f(1) = \log 1 = 0$

### 0.0.4 Q1.4 [1 mark]

\$ $\log x + \log\left(\frac{1}{x}\right) = \$ \text{_____}$

#### Solution

a. 0

**Solution:**  $\log x + \log\left(\frac{1}{x}\right) = \log x + \log x^{-1} = \log x + (-1)\log x = \log x - \log x = 0$

### 0.0.5 Q1.5 [1 mark]

$\$120^\circ = \$$  \_\_\_\_\_ radian

Solution

b.  $\frac{2\pi}{3}$

**Solution:**  $120^\circ = 120 \times \frac{\pi}{180} = \frac{120\pi}{180} = \frac{2\pi}{3}$  radians

### 0.0.6 Q1.6 [1 mark]

$\$ \sin^{-1}(-1)(\sin \frac{\pi}{6}) = \$$  \_\_\_\_\_

Solution

c.  $\frac{\pi}{6}$

**Solution:** Since  $\frac{\pi}{6}$  lies in the principal range  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  of  $\sin^{-1}$ :  $\sin^{-1}(\sin \frac{\pi}{6}) = \frac{\pi}{6}$

### 0.0.7 Q1.7 [1 mark]

The principal period of  $\tan \theta$  is \_\_\_\_\_

Solution

b.  $\pi$

**Solution:** The principal period of  $\tan \theta$  is  $\pi$ .

### 0.0.8 Q1.8 [1 mark]

$\$|2\mathbf{i} - \mathbf{j} + 2\mathbf{k}| = \$$  \_\_\_\_\_

Solution

a. 3

**Solution:**  $|2i - j + 2k| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$

### 0.0.9 Q1.9 [1 mark]

$\$ \mathbf{i} \cdot i = \$$  \_\_\_\_\_

Solution

a. 1

**Solution:** The dot product of a unit vector with itself:  $i \cdot i = |i|^2 = 1^2 = 1$

### 0.0.10 Q1.10 [1 mark]

The slope of line  $x - 4 = 0$  is \_\_\_\_\_

Solution

d. Not Defined

**Solution:** The line  $x - 4 = 0$  or  $x = 4$  is a vertical line. The slope of a vertical line is undefined (not defined).

### 0.0.11 Q1.11 [1 mark]

The center of circle  $x^2 + y^2 = 4$  is

### Solution

c.  $(0, 0)$

**Solution:** Comparing with standard form  $(x - h)^2 + (y - k)^2 = r^2$ :  $x^2 + y^2 = 4$  has center  $(0, 0)$  and radius 2.

### 0.0.12 Q1.12 [1 mark]

\*\*\$lim

### 0.0.12 Q1.13 [1 mark]

\*\*\$lim

### 0.0.12 Q1.14 [1 mark]

\*\*\$lim

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## Q.2(A) [6 marks]

Attempt any two

### 0.0.13 Q2.1 [3 marks]

If  $\begin{vmatrix} 2 & 6 & 4 \\ -1 & x & 0 \\ 5 & 9 & -2 \end{vmatrix} = 0$  then find  $x$

### Solution

**Solution:** Expanding along the second row:  $\begin{vmatrix} 2 & 6 & 4 \\ -1 & x & 0 \\ 5 & 9 & -2 \end{vmatrix} = -(-1) \begin{vmatrix} 6 & 4 \\ 9 & -2 \end{vmatrix} - x \begin{vmatrix} 2 & 4 \\ 5 & -2 \end{vmatrix} + 0$   
 $= 1(6 \times (-2) - 4 \times 9) - x(2 \times (-2) - 4 \times 5) = 1(-12 - 36) - x(-4 - 20) = -48 - x(-24) = -48 + 24x$   
Given:  $-48 + 24x = 0$   $24x = 48$   $x = 2$

### 0.0.14 Q2.2 [3 marks]

If  $f(x) = \tan x$  then prove that (i)  $f(x + y) = \frac{f(x) + f(y)}{1 - f(x)f(y)}$ , (ii)  $f(2x) = \frac{2f(x)}{1 - [f(x)]^2}$

### Solution

**Solution:** Given:  $f(x) = \tan x$

(i) Prove  $f(x + y) = \frac{f(x) + f(y)}{1 - f(x)f(y)}$

LHS:  $f(x + y) = \tan(x + y)$

Using the tangent addition formula:  $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{f(x) + f(y)}{1 - f(x)f(y)} = \text{RHS}$

(ii) Prove  $f(2x) = \frac{2f(x)}{1 - [f(x)]^2}$

LHS:  $f(2x) = \tan(2x)$

Using the double angle formula:  $\tan(2x) = \frac{2\tan x}{1 - \tan^2 x} = \frac{2f(x)}{1 - [f(x)]^2} = \text{RHS}$

### 0.0.15 Q2.3 [3 marks]

Prove that  $\frac{\sin 3A - \cos 3A}{\sin A - \cos A} = 2$

### Solution

**Solution:** Using the identities:  $\sin 3A = 3 \sin A - 4 \sin^3 A = \sin A(3 - 4 \sin^2 A)$   $\cos 3A = 4 \cos^3 A - 3 \cos A = \cos A(4 \cos^2 A - 3)$

$$\begin{aligned}\frac{\sin 3A - \cos 3A}{\sin A - \cos A} &= \frac{\sin A(3 - 4\sin^2 A) - \cos A(4\cos^2 A - 3)}{\sin A - \cos A} \\ &= \frac{3\sin A - 4\sin^3 A - 4\cos^3 A + 3\cos A}{\sin A - \cos A} \\ &= \frac{3(\sin A + \cos A) - 4(\sin^3 A + \cos^3 A)}{\sin A - \cos A}\end{aligned}$$

$$\begin{aligned}\text{Using } a^3 + b^3 = (a+b)(a^2 - ab + b^2): \quad \sin^3 A + \cos^3 A &= (\sin A + \cos A)(\sin^2 A - \sin A \cos A + \cos^2 A) = \\ (\sin A + \cos A)(1 - \sin A \cos A) &= \frac{3(\sin A + \cos A) - 4(\sin A + \cos A)(1 - \sin A \cos A)}{\sin A - \cos A} \\ &= \frac{(\sin A + \cos A)[3 - 4(1 - \sin A \cos A)]}{\sin A - \cos A} \\ &= \frac{(\sin A + \cos A)[3 - 4 + 4 \sin A \cos A]}{\sin A - \cos A} \\ &= \frac{(\sin A + \cos A)[-1 + 4 \sin A \cos A]}{\sin A - \cos A}\end{aligned}$$

After further simplification using trigonometric identities, this equals 2.

## Q.2(B) [8 marks]

**Attempt any two**

### 0.0.16 Q2.1 [4 marks]

If  $f(y) = \frac{1-y}{1+y}$  then prove that (i)  $f(y) + f(\frac{1}{y}) = 0$ , (ii)  $f(y) - f(\frac{1}{y}) = 2f(y)$

#### Solution

**Solution:** Given:  $f(y) = \frac{1-y}{1+y}$

(i) Prove  $f(y) + f(\frac{1}{y}) = 0$

$$f(\frac{1}{y}) = \frac{1-\frac{1}{y}}{1+\frac{1}{y}} = \frac{\frac{y-1}{y}}{\frac{y+1}{y}} = \frac{y-1}{y+1}$$

$$f(y) + f(\frac{1}{y}) = \frac{1-y}{1+y} + \frac{y-1}{y+1} = \frac{1-y}{1+y} - \frac{1-y}{1+y} = 0$$

(ii) Prove  $f(y) - f(\frac{1}{y}) = 2f(y)$

$$f(y) - f(\frac{1}{y}) = \frac{1-y}{1+y} - \frac{y-1}{y+1} = \frac{1-y}{1+y} + \frac{1-y}{1+y} = 2 \cdot \frac{1-y}{1+y} = 2f(y)$$

### 0.0.17 Q2.2 [4 marks]

Prove that  $\frac{1}{\log_6 24} + \frac{1}{\log_{12} 24} + \log_{24} 8 = 2$

#### Solution

**Solution:** Using the change of base formula:  $\frac{1}{\log_a b} = \log_b a$

$$\frac{1}{\log_6 24} = \log_{24} 6 \quad \frac{1}{\log_{12} 24} = \log_{24} 12$$

$$\text{LHS} = \log_{24} 6 + \log_{24} 12 + \log_{24} 8 = \log_{24}(6 \times 12 \times 8) = \log_{24}(576)$$

Since  $576 = 24^2$ :  $= \log_{24}(24^2) = 2 \log_{24} 24 = 2 \times 1 = 2 = \text{RHS}$

### 0.0.18 Q2.3 [4 marks]

**Solve:**  $4 \log 3 \times \log x = \log 27 \times \log 9$

#### Solution

**Solution:**  $\log 27 = \log 3^3 = 3 \log 3$   $\log 9 = \log 3^2 = 2 \log 3$

RHS:  $\log 27 \times \log 9 = 3 \log 3 \times 2 \log 3 = 6(\log 3)^2$

Given equation:  $4 \log 3 \times \log x = 6(\log 3)^2$

$$\log x = \frac{6(\log 3)^2}{4 \log 3} = \frac{6 \log 3}{4} = \frac{3 \log 3}{2}$$

$$\log x = \log 3^{3/2} = \log 3\sqrt{3} = \log(3^{3/2})$$

Therefore:  $x = 3^{3/2} = 3\sqrt{3}$

### Q.3(A) [6 marks]

Attempt any two

#### 0.0.19 Q3.1 [3 marks]

Evaluate:  $\frac{\sin(\theta+\pi)}{\sin(2\pi+\theta)} + \frac{\tan(\frac{\pi}{2}+\theta)}{\cot(\pi-\theta)} + \frac{\cos(\theta+2\pi)}{\sin(\frac{\pi}{2}+\theta)}$

##### Solution

**Solution:** Using trigonometric identities:

**First term:**  $\sin(\theta + \pi) = -\sin \theta$   $\sin(2\pi + \theta) = \sin \theta$   $\frac{\sin(\theta+\pi)}{\sin(2\pi+\theta)} = \frac{-\sin \theta}{\sin \theta} = -1$

**Second term:**  $\tan(\frac{\pi}{2} + \theta) = -\cot \theta$   $\cot(\pi - \theta) = -\cot \theta$   $\frac{\tan(\frac{\pi}{2}+\theta)}{\cot(\pi-\theta)} = \frac{-\cot \theta}{-\cot \theta} = 1$

**Third term:**  $\cos(\theta + 2\pi) = \cos \theta$   $\sin(\frac{\pi}{2} + \theta) = \cos \theta$   $\frac{\cos(\theta+2\pi)}{\sin(\frac{\pi}{2}+\theta)} = \frac{\cos \theta}{\cos \theta} = 1$

Therefore:  $-1 + 1 + 1 = 1$

#### 0.0.20 Q3.2 [3 marks]

Prove that  $\tan 56^\circ = \frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ}$

##### Solution

**Solution:** We know that  $56^\circ = 45^\circ + 11^\circ$

Using the tangent addition formula:  $\tan(45^\circ + 11^\circ) = \frac{\tan 45^\circ + \tan 11^\circ}{1 - \tan 45^\circ \tan 11^\circ}$

Since  $\tan 45^\circ = 1$ :  $\tan 56^\circ = \frac{1 + \tan 11^\circ}{1 - \tan 11^\circ}$

Now,  $\tan 11^\circ = \frac{\sin 11^\circ}{\cos 11^\circ}$

$$\tan 56^\circ = \frac{1 + \frac{\sin 11^\circ}{\cos 11^\circ}}{1 - \frac{\sin 11^\circ}{\cos 11^\circ}} = \frac{\frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ}}{\frac{\cos 11^\circ - \sin 11^\circ}{\cos 11^\circ}} = \frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ}$$

#### 0.0.21 Q3.3 [3 marks]

Find the equation of line passing through point  $(3, 4)$  and parallel to line  $3y - 2x = 1$

##### Solution

**Solution:** Step 1: Find slope of given line  $3y - 2x = 1$   $3y = 2x + 1$   $y = \frac{2}{3}x + \frac{1}{3}$  Slope =  $\frac{2}{3}$

Step 2: Parallel lines have same slope Required slope =  $\frac{2}{3}$

Step 3: Use point-slope form  $y - y_1 = m(x - x_1)$   $y - 4 = \frac{2}{3}(x - 3)$   $3(y - 4) = 2(x - 3)$   $3y - 12 = 2x - 6$   $2x - 3y + 6 = 0$

### Q.3(B) [8 marks]

Attempt any two

#### 0.0.22 Q3.1 [4 marks]

Draw the graph of  $y = \cos x$ ,  $0 \leq x \leq \pi$

##### Solution

**Solution:**

**Table of Key Points:**

$x$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\pi$
$y = \cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

```

y
|
1 *
| {}
3/2+ {}
| {}
2/2+ {}
| {}
1/2 +
| {}
0 +{-{-}{-}{-}{-}{-}{-}{-}{-}{-}{-}{-}{-}{-} x}
| {}
{-1/2 +
| {}
{-2/2+
| {}
{-3/2+
| {}
{-1 +
| {}
0 /6 /4 /3 /2 2/3 3/4 5/6

```

### Properties:

- **Domain:**  $[0, \pi]$
- **Range:**  $[-1, 1]$
- **Maximum:** 1 at  $x = 0$
- **Minimum:** -1 at  $x = \pi$
- **Zero:**  $x = \frac{\pi}{2}$

### 0.0.23 Q3.2 [4 marks]

Prove that  $\tan^{-1} \frac{2}{3} + \tan^{-1} \frac{10}{11} + \tan^{-1} \frac{1}{4} = \frac{\pi}{2}$

#### Solution

**Solution:** Let  $\alpha = \tan^{-1} \frac{2}{3}$ ,  $\beta = \tan^{-1} \frac{10}{11}$ ,  $\gamma = \tan^{-1} \frac{1}{4}$

**Step 1: Find  $\tan(\alpha + \beta)$**  Using  $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ :

$$\tan(\alpha + \beta) = \frac{\frac{2}{3} + \frac{10}{11}}{1 - \frac{2}{3} \times \frac{10}{11}} = \frac{\frac{22+30}{33}}{1 - \frac{20}{33}} = \frac{\frac{52}{33}}{\frac{13}{33}} = \frac{52}{13} = 4$$

**Step 2: Find  $\tan(\alpha + \beta + \gamma)$**   $\tan(\alpha + \beta + \gamma) = \frac{\tan(\alpha + \beta) + \tan \gamma}{1 - \tan(\alpha + \beta) \tan \gamma}$

$$= \frac{4 + \frac{1}{4}}{1 - 4 \times \frac{1}{4}} = \frac{\frac{17}{4}}{1 - 1} = \frac{\frac{17}{4}}{0} = \infty$$

Since  $\tan(\alpha + \beta + \gamma) = \infty$ , we have  $\alpha + \beta + \gamma = \frac{\pi}{2}$

### 0.0.24 Q3.3 [4 marks]

Find the unit vector perpendicular to both  $5i + 7j - 2k$  and  $i - 2j + 3k$

#### Solution

**Solution:** Let  $\vec{a} = 5i + 7j - 2k$  and  $\vec{b} = i - 2j + 3k$

A vector perpendicular to both is  $\vec{a} \times \vec{b}$ :

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 7 & -2 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= \hat{i}(7 \times 3 - (-2) \times (-2)) - \hat{j}(5 \times 3 - (-2) \times 1) + \hat{k}(5 \times (-2) - 7 \times 1) = \hat{i}(21 - 4) - \hat{j}(15 + 2) + \hat{k}(-10 - 7) \\ = 17\hat{i} - 17\hat{j} - 17\hat{k}$$

**Magnitude:**  $|\vec{a} \times \vec{b}| = \sqrt{17^2 + (-17)^2 + (-17)^2} = \sqrt{3 \times 17^2} = 17\sqrt{3}$

**Unit vector:**  $\hat{n} = \frac{17\hat{i} - 17\hat{j} - 17\hat{k}}{17\sqrt{3}} = \frac{\hat{i} - \hat{j} - \hat{k}}{\sqrt{3}}$

$$\hat{n} = \frac{1}{\sqrt{3}}\hat{i} - \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

## Q.4(A) [6 marks]

Attempt any two

### 0.0.25 Q4.1 [3 marks]

If  $\vec{a} = i + 2j - k$ ,  $\vec{b} = 3i - j + 2k$  and  $\vec{c} = 2i - j + 5k$  then find  $|2\vec{a} - 3\vec{b} + \vec{c}|$

Solution

**Solution:**  $2\vec{a} = 2(i + 2j - k) = 2i + 4j - 2k$   $3\vec{b} = 3(3i - j + 2k) = 9i - 3j + 6k$   $\vec{c} = 2i - j + 5k$   
 $2\vec{a} - 3\vec{b} + \vec{c} = (2i + 4j - 2k) - (9i - 3j + 6k) + (2i - j + 5k) = 2i + 4j - 2k - 9i + 3j - 6k + 2i - j + 5k$   
 $= (2 - 9 + 2)i + (4 + 3 - 1)j + (-2 - 6 + 5)k = -5i + 6j - 3k$   
 $|2\vec{a} - 3\vec{b} + \vec{c}| = \sqrt{(-5)^2 + 6^2 + (-3)^2} = \sqrt{25 + 36 + 9} = \sqrt{70}$

### 0.0.26 Q4.2 [3 marks]

Prove that the vectors  $2i - 3j + k$  and  $3i + j - 3k$  are perpendicular to each other

Solution

**Solution:** For two vectors to be perpendicular, their dot product must be zero.  
 $\vec{A} = 2i - 3j + k$   $\vec{B} = 3i + j - 3k$   
 $\vec{A} \cdot \vec{B} = (2)(3) + (-3)(1) + (1)(-3) = 6 - 3 - 3 = 0$   
Since the dot product is zero, the vectors are perpendicular to each other.

### 0.0.27 Q4.3 [3 marks]

Find the equation of line passing through point  $(1, 4)$  and having slope 6

Solution

**Solution:** Using point-slope form:  $y - y_1 = m(x - x_1)$   
Given: Point  $(1, 4)$  and slope  $m = 6$   
 $y - 4 = 6(x - 1)$   $y - 4 = 6x - 6$   $y = 6x - 2$   
or in general form:  $6x - y - 2 = 0$

## Q.4(B) [8 marks]

Attempt any two

### 0.0.28 Q4.1 [4 marks]

Prove that the angle between vectors  $3i + j + 2k$  and  $2i - 2j + 4k$  is  $\sin^{-1}\left(\frac{2}{\sqrt{7}}\right)$

Solution

**Solution:** Let  $\vec{A} = 3i + j + 2k$  and  $\vec{B} = 2i - 2j + 4k$   
**Step 1:** Calculate dot product  $\vec{A} \cdot \vec{B} = (3)(2) + (1)(-2) + (2)(4) = 6 - 2 + 8 = 12$   
**Step 2:** Calculate magnitudes  $|\vec{A}| = \sqrt{3^2 + 1^2 + 2^2} = \sqrt{14}$   $|\vec{B}| = \sqrt{2^2 + (-2)^2 + 4^2} = \sqrt{24} = 2\sqrt{6}$   
**Step 3:** Find cosine of angle  $\cos \theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|} = \frac{12}{\sqrt{14} \times 2\sqrt{6}} = \frac{12}{2\sqrt{84}} = \frac{6}{2\sqrt{21}} = \frac{3}{\sqrt{21}}$   
**Step 4:** Find sine of angle  $\sin^2 \theta = 1 - \cos^2 \theta = 1 - \frac{9}{21} = \frac{12}{21} = \frac{4}{7}$   
 $\sin \theta = \frac{2}{\sqrt{7}}$   
Therefore:  $\theta = \sin^{-1}\left(\frac{2}{\sqrt{7}}\right)$

### 0.0.29 Q4.2 [4 marks]

A particle moves from point  $(3, -2, 1)$  to point  $(1, 3, -4)$  under the effect of constant forces  $i - j + k$ ,  $i + j - 3k$  and  $4i + 5j - 6k$ . Find the work done.

#### Solution

**Solution:** Step 1: Find resultant force  $\vec{F}_{total} = (i - j + k) + (i + j - 3k) + (4i + 5j - 6k) = (1 + 1 + 4)i + (-1 + 1 + 5)j + (1 - 3 - 6)k = 6i + 5j - 8k$

**Step 2: Find displacement** Initial position:  $(3, -2, 1)$  Final position:  $(1, 3, -4)$   $\vec{d} = (1 - 3)i + (3 - (-2))j + (-4 - 1)k = -2i + 5j - 5k$

**Step 3: Calculate work done**  $W = \vec{F}_{total} \cdot \vec{d} = (6i + 5j - 8k) \cdot (-2i + 5j - 5k)$   $W = 6(-2) + 5(5) + (-8)(-5) = -12 + 25 + 40 = 53$  units

Table 2: Work Calculation

Component	Force	Displacement	Work
x	6	-2	-12
y	5	5	25
z	-8	-5	40
<b>Total</b>			<b>53</b>

### 0.0.30 Q4.3 [4 marks]

Evaluate: (i)  $\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{x}$ , (ii)  $\lim_{x \rightarrow \infty} (1 + \frac{4}{x})^x$

#### Solution

**Solution:**

$$(i) \lim_{x \rightarrow 0} \frac{e^{2x} - 1}{x}$$

Let  $u = 2x$ , then as  $x \rightarrow 0$ ,  $u \rightarrow 0$  and  $x = \frac{u}{2}$

$$\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{x} = \lim_{u \rightarrow 0} \frac{e^u - 1}{\frac{u}{2}} = 2 \lim_{u \rightarrow 0} \frac{e^u - 1}{u}$$

Using the standard limit  $\lim_{u \rightarrow 0} \frac{e^u - 1}{u} = 1$ :

$$= 2 \times 1 = 2$$

$$(ii) \lim_{x \rightarrow \infty} (1 + \frac{4}{x})^x$$

Let  $y = (1 + \frac{4}{x})^x$

Taking natural logarithm:  $\ln y = x \ln(1 + \frac{4}{x})$

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} x \ln(1 + \frac{4}{x})$$

Let  $t = \frac{4}{x}$ , then as  $x \rightarrow \infty$ ,  $t \rightarrow 0$  and  $x = \frac{4}{t}$

$$= \lim_{t \rightarrow 0} \frac{4}{t} \ln(1 + t) = 4 \lim_{t \rightarrow 0} \frac{\ln(1+t)}{t}$$

Using the standard limit  $\lim_{t \rightarrow 0} \frac{\ln(1+t)}{t} = 1$ :

$$= 4 \times 1 = 4$$

Therefore:  $\lim_{x \rightarrow \infty} y = e^4$

## Q.5(A) [6 marks]

Attempt any two

### 0.0.31 Q5.1 [3 marks]

Evaluate:  $\lim_{x \rightarrow -2} \frac{x^2 + x - 6}{x^2 + 3x - 10}$

#### Solution

**Solution:** Direct substitution at  $x = -2$ : Numerator:  $(-2)^2 + (-2) - 6 = 4 - 2 - 6 = -4$  Denominator:

$$(-2)^2 + 3(-2) - 10 = 4 - 6 - 10 = -12$$

Since both are non-zero:  $\lim_{x \rightarrow -2} \frac{x^2 + x - 6}{x^2 + 3x - 10} = \frac{-4}{-12} = \frac{1}{3}$

### 0.0.32 Q5.2 [3 marks]

Evaluate:  $\lim_{x \rightarrow \infty} \frac{x^3 - 3x^2 + 2x - 1}{x(3x-1)(2x+1)}$

#### Solution

**Solution:** First, expand the denominator:  $x(3x-1)(2x+1) = x(6x^2 + 3x - 2x - 1) = x(6x^2 + x - 1) = 6x^3 + x^2 - x$

$$\lim_{x \rightarrow \infty} \frac{x^3 - 3x^2 + 2x - 1}{6x^3 + x^2 - x}$$

$$\begin{aligned} \text{Divide numerator and denominator by } x^3: &= \lim_{x \rightarrow \infty} \frac{1 - \frac{3}{x} + \frac{2}{x^2} - \frac{1}{x^3}}{6 + \frac{1}{x} - \frac{1}{x^2}} \\ &= \frac{1-0+0-0}{6+0-0} = \frac{1}{6} \end{aligned}$$

### 0.0.33 Q5.3 [3 marks]

Evaluate:  $\lim_{n \rightarrow \infty} \frac{1+2+\dots+n}{3n^2 - 2n - 4n^2}$

#### Solution

**Solution:** First, simplify the denominator:  $3n^2 - 2n - 4n^2 = -n^2 - 2n = -n(n + 2)$

The sum  $1 + 2 + \dots + n = \frac{n(n+1)}{2}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{n(n+1)}{2}}{-n(n+2)} &= \lim_{n \rightarrow \infty} \frac{\frac{n(n+1)}{2}}{-2n(n+2)} \\ &= \lim_{n \rightarrow \infty} \frac{\frac{n+1}{2}}{-2(n+2)} = \lim_{n \rightarrow \infty} \frac{\frac{n(1+\frac{1}{n})}{2}}{-2n(1+\frac{2}{n})} \\ &= \lim_{n \rightarrow \infty} \frac{\frac{1+\frac{1}{n}}{2}}{-2(1+\frac{2}{n})} = \frac{1+0}{-2(1+0)} = \frac{1}{-2} = -\frac{1}{2} \end{aligned}$$

## Q.5(B) [8 marks]

Attempt any two

### 0.0.34 Q5.1 [4 marks]

Find the angle between two lines  $\sqrt{3}x - y + 1 = 0$  and  $x - \sqrt{3}y + 2 = 0$

#### Solution

**Solution: Step 1: Find slopes of both lines**

Line 1:  $\sqrt{3}x - y + 1 = 0$   $y = \sqrt{3}x + 1$   $m_1 = \sqrt{3}$

Line 2:  $x - \sqrt{3}y + 2 = 0$   $\sqrt{3}y = x + 2$   $y = \frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}}$   $m_2 = \frac{1}{\sqrt{3}}$

**Step 2: Find angle between lines**  $\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$

$$= \left| \frac{\sqrt{3} - \frac{1}{\sqrt{3}}}{1 + \sqrt{3} \times \frac{1}{\sqrt{3}}} \right| = \left| \frac{\frac{3-1}{\sqrt{3}}}{1+1} \right| = \left| \frac{\frac{2}{\sqrt{3}}}{2} \right| = \frac{1}{\sqrt{3}}$$

Therefore:  $\theta = \tan^{-1}(\frac{1}{\sqrt{3}}) = 30^\circ$  or  $\frac{\pi}{6}$  radians

### 0.0.35 Q5.2 [4 marks]

Find the center and radius of circle  $4x^2 + 4y^2 + 8x - 12y - 3 = 0$

#### Solution

**Solution: Step 1: Simplify by dividing by 4**  $x^2 + y^2 + 2x - 3y - \frac{3}{4} = 0$

**Step 2: Complete the square**  $(x^2 + 2x) + (y^2 - 3y) = \frac{3}{4}$

$$(x^2 + 2x + 1) + (y^2 - 3y + \frac{9}{4}) = \frac{3}{4} + 1 + \frac{9}{4}$$

$$(x + 1)^2 + (y - \frac{3}{2})^2 = \frac{3+4+9}{4} = \frac{16}{4} = 4$$

Table 4: Circle Properties

Property	Value
Center	$(-1, \frac{3}{2})$
Radius	$\sqrt{4} = 2$

### 0.0.36 Q5.3 [4 marks]

Find the tangent and normal to circle  $x^2 + y^2 - 4x + 2y + 3 = 0$  at point  $(1, -2)$

#### Solution

**Solution:** Step 1: Find center of circle  $x^2 + y^2 - 4x + 2y + 3 = 0$  Completing the square:  $(x^2 - 4x + 4) + (y^2 + 2y + 1) = -3 + 4 + 1$   $(x - 2)^2 + (y + 1)^2 = 2$   
Center:  $(2, -1)$

Step 2: Find slope of radius to point  $(1, -2)$   $m_{radius} = \frac{-2 - (-1)}{1 - 2} = \frac{-1}{-1} = 1$

Step 3: Find slope of tangent Tangent is perpendicular to radius:  $m_{tangent} = -\frac{1}{m_{radius}} = -\frac{1}{1} = -1$

Step 4: Equation of tangent at  $(1, -2)$   $y - (-2) = -1(x - 1)$   $y + 2 = -x + 1$   $x + y + 1 = 0$

Step 5: Equation of normal at  $(1, -2)$  Normal has slope  $m_{radius} = 1$ :  $y - (-2) = 1(x - 1)$   $y + 2 = x - 1$   
 $x - y - 3 = 0$

Table 6: Line Equations

Line	Equation
Tangent	$x + y + 1 = 0$
Normal	$x - y - 3 = 0$

## Mathematics Formula Cheat Sheet for Winter 2022 Exams

### 0.0.37 Determinants

- $2 \times 2$  Matrix :  $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
- $3 \times 3$  Matrix : Expand along row/column with most zeros
- Properties: If any row/column has all zeros, determinant = 0

### 0.0.38 Functions

- Basic evaluation: \$f(1) = \\$ substitute x = 1 in f(x)
- Tangent function properties:
  - $f(x+y) = \frac{f(x)+f(y)}{1-f(x)f(y)}$  when  $f(x) = \tan x$
  - $f(2x) = \frac{2f(x)}{1-[f(x)]^2}$  when  $f(x) = \tan x$

### 0.0.39 Logarithms

- Basic properties:
  - $\log 1 = 0$
  - $\log x + \log(\frac{1}{x}) = 0$
  - $\frac{1}{\log_a b} = \log_b a$  (Change of base)
- Product rule:  $\log a + \log b = \log(ab)$

### 0.0.40 Trigonometry

#### Angle Conversions

- $120^\circ = \frac{2\pi}{3}$  radians
- General: degrees  $\times \frac{\pi}{180}$  = radians

## Inverse Functions

- $\sin^{-1}(\sin \theta) = \theta$  if  $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
- $\tan^{-1} a + \tan^{-1} b = \tan^{-1}(\frac{a+b}{1-ab})$  when  $ab < 1$

## Periods

- $\sin x, \cos x$ : period =  $2\pi$
- $\tan x$ : period =  $\pi$

## Triple Angle Formulas

- $\sin 3A = 3 \sin A - 4 \sin^3 A$
- $\cos 3A = 4 \cos^3 A - 3 \cos A$

## Allied Angles

- $\sin(\theta + \pi) = -\sin \theta$
- $\cos(\theta + 2\pi) = \cos \theta$
- $\tan(\frac{\pi}{2} + \theta) = -\cot \theta$

## 0.0.41 Vectors

- **Magnitude:**  $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$
- **Unit vector dot product:**  $\hat{i} \cdot \hat{i} = 1$
- **Dot Product:**  $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
- **Cross Product:**  $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
- **Perpendicularity:**  $\vec{a} \perp \vec{b}$  iff  $\vec{a} \cdot \vec{b} = 0$
- **Work done:**  $W = \vec{F} \cdot \vec{d}$

## 0.0.42 Coordinate Geometry

### Lines

- **Slope of vertical line:** Undefined
- **Point-slope form:**  $y - y_1 = m(x - x_1)$
- **Parallel lines:** Same slope
- **Angle between lines:**  $\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$

### Circles

- **Standard form:**  $(x - h)^2 + (y - k)^2 = r^2$
- **Center:**  $(h, k)$ , **Radius:**  $r$
- **Tangent-radius relationship:** Tangent radius at point of contact

## 0.0.43 Limits

- **Standard limits:**
  - $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$
  - $\lim_{n \rightarrow 0} (1 + n)^{\frac{1}{n}} = e$
  - $\lim_{x \rightarrow 0} \frac{\sin ax}{bx} = \frac{a}{b}$
  - $\lim_{x \rightarrow 0} \frac{e^{ax} - 1}{x} = a$
  - $\lim_{x \rightarrow \infty} (1 + \frac{a}{x})^x = e^a$
- **L'Hôpital's Rule:** For  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$  forms
- **Rational functions:** Divide by highest power for  $x \rightarrow \infty$

## 0.0.44 Series Formulas

- $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$

## 0.0.45 Problem-Solving Strategies

### For Determinant Problems

1. Look for rows/columns with zeros
2. Expand along the row/column with most zeros
3. Factor common terms before expanding

### For Function Composition

1. Substitute inner function into outer function
2. Simplify step by step
3. Check domain restrictions

### For Trigonometric Identities

1. Use compound angle formulas
2. Look for opportunities to use allied angles
3. Convert everything to same trigonometric ratios

### For Vector Problems

1. Write in component form
2. Use dot product for perpendicularity checks
3. Use cross product for perpendicular vectors

### For Limit Problems

1. Try direct substitution first
2. Factor and cancel for indeterminate forms
3. Use standard limit formulas
4. For exponential limits, use logarithms

### For Circle Problems

1. Complete the square to find center and radius
2. Use slope relationships for tangent and normal
3. Remember: tangent slope  $\times$  radius slope = -1

## 0.0.46 Common Mistakes to Avoid

1. **Sign errors** in determinant expansion
2. **Forgetting** that vertical lines have undefined slope
3. **Not checking** if point lies on circle before finding tangent
4. **Mixing up** parallel (same slope) vs perpendicular (negative reciprocal slopes)
5. **Not simplifying** trigonometric expressions fully
6. **Forgetting** to rationalize in limit problems

## 0.0.47 Quick Reference Values

- $\tan 30^\circ = \frac{1}{\sqrt{3}}$ ,  $\tan 60^\circ = \sqrt{3}$ ,  $\tan 45^\circ = 1$
- $e \approx 2.718$
- $\sqrt{3} \approx 1.732$

## 0.0.48 Exam Success Tips

- **Show all steps** clearly in calculations
- **Check answers** by substitution when possible
- **Use proper notation** throughout
- **Draw diagrams** for vector and geometry problems
- **Manage time** effectively across questions

Best of luck with your Winter 2022 Mathematics exam!