Engineering Mathematics (4320002) - Summer 2024 Solution

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June 26, 2024

Question 1 [14 marks]

Fill in the blanks using appropriate choice from the given options.

Question 1.1 [1 marks]

Order of the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 4 \end{bmatrix}$ **is __________.**

Solution

Answer: (b) 3×2
Solution: Order of a matrix is given by (number of rows) $\times$ (number of columns) Matrix A has 3 rows and 2 columns Therefore, order $= 3 \times 2$

Question 1.2 [1 marks]

If $A = \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$ **then** $A^{-1} =$ __________

Solution

Answer: (d) A^T
Solution: For orthogonal matrices, $A^{-1} = A^T$ Since $AA^T = I$, we have $A^{-1} = A^T$

Question 1.3 [1 marks]

$\begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix} \times \begin{bmatrix} -1 & 6 \\ 2 & 1 \end{bmatrix} =$ __________

Solution

Answer: (a) $\begin{bmatrix} 3 & 8 \\ -5 & 30 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix} \times \begin{bmatrix} -1 & 6 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(-1)+2(2) & 1(6)+2(1) \\ 5(-1)+0(2) & 5(6)+0(1) \end{bmatrix}$$

$$= \begin{bmatrix} -1+4 & 6+2 \\ -5+0 & 30+0 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ -5 & 30 \end{bmatrix}$$

Question 1.4 [1 marks]

If $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ **then** $A^T =$ ____________

Solution

Answer: (b) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Solution: Transpose of a matrix is obtained by interchanging rows and columns

$$A^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Question 1.5 [1 marks]

$\frac{d}{dx}(4^x) =$ ____________

Solution

Answer: (a) $4^x \log_e 4$

Solution: $\frac{d}{dx}(a^x) = a^x \ln a$ Therefore, $\frac{d}{dx}(4^x) = 4^x \ln 4 = 4^x \log_e 4$

Question 1.6 [1 marks]

$\frac{d}{dx}(\sin^2 x + \cos^2 x) =$ ____________

Solution

Answer: (b) 0

Solution: $\sin^2 x + \cos^2 x = 1$ (trigonometric identity) $\frac{d}{dx}(1) = 0$

Question 1.7 [1 marks]

If $x = \sin\theta, y = \cos\theta$ **then** $\frac{dy}{dx} =$ ____________

Solution

Answer: (d) $-\cot\theta$
Solution: $\frac{dx}{d\theta} = \cos\theta$, $\frac{dy}{d\theta} = -\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta = -\cot\theta$$

Question 1.8 [1 marks]

$\int x^7 dx =$ __________

Solution

Answer: (c) $\frac{x^8}{8}$
Solution: $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ $\int x^7 dx = \frac{x^8}{8} + c$

Question 1.9 [1 marks]

$\int_{-2}^{2} x^5 dx =$ __________

Solution

Answer: (b) 0
Solution: x^5 is an odd function For odd functions, $\int_{-a}^{a} f(x)dx = 0$ Therefore, $\int_{-2}^{2} x^5 dx = 0$

Question 1.10 [1 marks]

$\int \frac{\cos x}{\sin x} dx =$ __________

Solution

Answer: (d) $\log|\sin x|$
Solution: Let $u = \sin x$, then $du = \cos x dx$

$$\int \frac{\cos x}{\sin x} dx = \int \frac{du}{u} = \log|u| + c = \log|\sin x| + c$$

Question 1.11 [1 marks]

The order of the differential equation $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^4 + y = 0$ **is** __________

Solution

Answer: (a) 3
Solution: Order of a differential equation is the highest order derivative present Highest derivative is $\frac{d^3y}{dx^3}$, so order $= 3$

Question 1.12 [1 marks]

An integrating factor of the differential equation $\frac{dy}{dx} + y = 3x$ is ________

Solution

Answer: (c) e^x
Solution: For linear differential equation $\frac{dy}{dx} + Py = Q$ Integrating factor $= e^{\int P dx} = e^{\int 1 dx} = e^x$

Question 1.13 [1 marks]

$i^7 =$ ________

Solution

Answer: (b) $-i$
Solution: $i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1$ $i^7 = i^4 \cdot i^3 = 1 \cdot (-i) = -i$

Question 1.14 [1 marks]

$\arg(1 + i) =$ ________

Solution

Answer: (c) $\frac{\pi}{4}$
Solution: $\arg(a + bi) = \tan^{-1}\left(\frac{b}{a}\right)$ $\arg(1 + i) = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

Question 2(A) [6 marks]

Attempt any two

Question 2(A).1 [3 marks]

If $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}$ then prove that $(A + B)^T = A^T + B^T$

Solution

Solution:

$$A + B = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 5 & 3 \end{bmatrix}$$

$$(A + B)^T = \begin{bmatrix} 6 & 5 \\ 0 & 3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}, \quad B^T = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}$$

$$A^T + B^T = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 0 & 3 \end{bmatrix}$$

Therefore, $(A+B)^T = A^T + B^T$ ✓**Proved**

Question 2(A).2 [3 marks]

If $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ then show that $A \cdot A^{-1} = I$

Solution

Solution: First, find A^{-1}: $|A| = 1(3) - 1(2) = 3 - 2 = 1$

$$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{1}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

Now verify $A \cdot A^{-1} = I$:

$$A \cdot A^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(3)+1(-2) & 1(-1)+1(1) \\ 2(3)+3(-2) & 2(-1)+3(1) \end{bmatrix}$$

$$= \begin{bmatrix} 3-2 & -1+1 \\ 6-6 & -2+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad \checkmark\textbf{Proved}$$

Question 2(A).3 [3 marks]

Solve the differential equation $xdy + ydx = 0$

Solution

Solution: $xdy + ydx = 0$ $xdy = -ydx$ $\frac{dy}{y} = -\frac{dx}{x}$
Integrating both sides:

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\ln|y| = -\ln|x| + c_1$$

$$\ln|y| + \ln|x| = c_1$$

$$\ln|xy| = c_1$$

$$|xy| = e^{c_1} = c \quad \text{(where } c = e^{c_1} \text{ is a constant)}$$

Therefore, $xy = \pm c$ or $xy = k$ where k is an arbitrary constant.

Question 2(B) [8 marks]

Attempt any two

Question 2(B).1 [4 marks]

If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ **then show that** $A^2 - 5A + 7I = 0$

Solution

Solution: First, calculate A^2:

$$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3(3) + 1(-1) & 3(1) + 1(2) \\ -1(3) + 2(-1) & -1(1) + 2(2) \end{bmatrix}$$

$$= \begin{bmatrix} 9 - 1 & 3 + 2 \\ -3 - 2 & -1 + 4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

Now calculate $5A$:

$$5A = 5 \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$$

And $7I$:

$$7I = 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

Now verify $A^2 - 5A + 7I = 0$:

$$A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8 - 15 + 7 & 5 - 5 + 0 \\ -5 + 5 + 0 & 3 - 10 + 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \quad \checkmark\textbf{Proved}$$

Question 2(B).2 [4 marks]

If $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$ **then prove that adj** $A = A$

Solution

Solution: To find adj A, we need to find the cofactor matrix and then transpose it.

Cofactors: $C_{11} = (-1)^{1+1} \begin{vmatrix} 0 & 1 \\ 4 & 3 \end{vmatrix} = 0(3) - 1(4) = -4$

$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = -(1(3) - 1(4)) = -(3 - 4) = 1$

$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 0 \\ 4 & 4 \end{vmatrix} = 1(4) - 0(4) = 4$

$C_{21} = (-1)^{2+1} \begin{vmatrix} -3 & -3 \\ 4 & 3 \end{vmatrix} = -((-3)(3) - (-3)(4)) = -(-9 + 12) = -3$

$$C_{22} = (-1)^{2+2}\begin{vmatrix}-4 & -3\\4 & 3\end{vmatrix} = (-4)(3) - (-3)(4) = -12 + 12 = 0$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix}-4 & -3\\4 & 4\end{vmatrix} = -((-4)(4) - (-3)(4)) = -(-16 + 12) = -(-4) = 4$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix}-3 & -3\\0 & 1\end{vmatrix} = (-3)(1) - (-3)(0) = -3$$

$$C_{32} = (-1)^{3+2}\begin{vmatrix}-4 & -3\\1 & 1\end{vmatrix} = -((-4)(1) - (-3)(1)) = -(-4 + 3) = -(-1) = 1$$

$$C_{33} = (-1)^{3+3}\begin{vmatrix}-4 & -3\\1 & 0\end{vmatrix} = (-4)(0) - (-3)(1) = 0 + 3 = 3$$

Cofactor matrix $= \begin{bmatrix}-4 & 1 & 4\\-3 & 0 & 4\\-3 & 1 & 3\end{bmatrix}$

$$\text{adj } A = (\text{Cofactor matrix})^T = \begin{bmatrix}-4 & -3 & -3\\1 & 0 & 1\\4 & 4 & 3\end{bmatrix} = A \quad \checkmark\textbf{Proved}$$

Question 2(B).3 [4 marks]

Solve the following system of linear equations using matrix: $3x + 2y = 5$**,** $2x - y = 1$

Solution

Solution: The system can be written as $AX = B$ where: $A = \begin{bmatrix}3 & 2\\2 & -1\end{bmatrix}$, $X = \begin{bmatrix}x\\y\end{bmatrix}$, $B = \begin{bmatrix}5\\1\end{bmatrix}$

Find $|A| = 3(-1) - 2(2) = -3 - 4 = -7$

$$A^{-1} = \frac{1}{-7}\begin{bmatrix}-1 & -2\\-2 & 3\end{bmatrix} = \begin{bmatrix}\frac{1}{7} & \frac{2}{7}\\\frac{2}{7} & -\frac{3}{7}\end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix}\frac{1}{7} & \frac{2}{7}\\\frac{2}{7} & -\frac{3}{7}\end{bmatrix}\begin{bmatrix}5\\1\end{bmatrix}$$

$$= \begin{bmatrix}\frac{1}{7}(5) + \frac{2}{7}(1)\\\frac{2}{7}(5) - \frac{3}{7}(1)\end{bmatrix} = \begin{bmatrix}\frac{5+2}{7}\\\frac{10-3}{7}\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

Therefore, $x = 1, y = 1$

Question 3(A) [6 marks]

Attempt any two

Question 3(A).1 [3 marks]

Using definition of differentiation find the derivative of x^5 **with respect to** x

Solution

Solution: By definition: $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
For $f(x) = x^5$:

$$\frac{d}{dx}(x^5) = \lim_{h \to 0} \frac{(x+h)^5 - x^5}{h}$$

Using binomial theorem: $(x+h)^5 = x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5$

$$\begin{aligned}
\frac{d}{dx}(x^5) &= \lim_{h \to 0} \frac{x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5 - x^5}{h} \\
&= \lim_{h \to 0} \frac{5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5}{h} \\
&= \lim_{h \to 0} (5x^4 + 10x^3h + 10x^2h^2 + 5xh^3 + h^4) \\
&= 5x^4 + 0 + 0 + 0 + 0 = 5x^4
\end{aligned}$$

Therefore, $\frac{d}{dx}(x^5) = 5x^4$

Question 3(A).2 [3 marks]

Find $\frac{dy}{dx}$ if $y = \frac{x^2-1}{x^2+1}$

Solution

Solution: Using quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
Here, $u = x^2 - 1$, $v = x^2 + 1$ $\frac{du}{dx} = 2x$, $\frac{dv}{dx} = 2x$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2} \\
&= \frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2} \\
&= \frac{2x[(x^2+1) - (x^2-1)]}{(x^2+1)^2} \\
&= \frac{2x[x^2+1-x^2+1]}{(x^2+1)^2} \\
&= \frac{2x \cdot 2}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}
\end{aligned}$$

Therefore, $\frac{dy}{dx} = \frac{4x}{(x^2+1)^2}$

Question 3(A).3 [3 marks]

Evaluate the integral $\int \frac{x^2+5x+6}{x^2+2x} dx$

Solution

Solution: First, perform polynomial long division:

$$\frac{x^2+5x+6}{x^2+2x} = 1 + \frac{3x+6}{x^2+2x}$$

$$\int \frac{x^2+5x+6}{x^2+2x}dx = \int \left(1 + \frac{3x+6}{x^2+2x}\right) dx$$
$$= \int 1dx + \int \frac{3x+6}{x^2+2x}dx$$
$$= x + \int \frac{3x+6}{x(x+2)}dx$$

For the second integral: $\frac{3x+6}{x(x+2)} = \frac{3(x+2)}{x(x+2)} = \frac{3}{x}$

$$\int \frac{3x+6}{x(x+2)}dx = \int \frac{3}{x}dx = 3\ln|x| + c$$

Therefore: $\int \frac{x^2+5x+6}{x^2+2x}dx = x + 3\ln|x| + c$

Question 3(B) [8 marks]

Attempt any two

Question 3(B).1 [4 marks]

If $y = \log(\sec x + \tan x)$ then find $\frac{dy}{dx}$

Solution

Solution: $y = \log(\sec x + \tan x)$

$$\frac{dy}{dx} = \frac{1}{\sec x + \tan x} \cdot \frac{d}{dx}(\sec x + \tan x)$$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\frac{d}{dx}(\tan x) = \sec^2 x$

$$\frac{dy}{dx} = \frac{1}{\sec x + \tan x} \cdot (\sec x \tan x + \sec^2 x)$$
$$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$$
$$= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x$$

Therefore, $\frac{dy}{dx} = \sec x$

Question 3(B).2 [4 marks]

If $y = 2e^{3x} + 3e^{-2x}$ then prove that $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$

Solution

Solution: $y = 2e^{3x} + 3e^{-2x}$
First derivative: $\frac{dy}{dx} = 2(3e^{3x}) + 3(-2e^{-2x}) = 6e^{3x} - 6e^{-2x}$
Second derivative: $\frac{d^2y}{dx^2} = 6(3e^{3x}) - 6(-2e^{-2x}) = 18e^{3x} + 12e^{-2x}$
Now verify the equation: $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y$

$$= (18e^{3x} + 12e^{-2x}) - (6e^{3x} - 6e^{-2x}) - 6(2e^{3x} + 3e^{-2x})$$

$$= 18e^{3x} + 12e^{-2x} - 6e^{3x} + 6e^{-2x} - 12e^{3x} - 18e^{-2x}$$
$$= e^{3x}(18 - 6 - 12) + e^{-2x}(12 + 6 - 18)$$
$$= e^{3x}(0) + e^{-2x}(0) = 0 \quad \checkmark\textbf{Proved}$$

Question 3(B).3 [4 marks]

Find the maximum and minimum value of function $f(x) = x^3 - 3x + 11$

Solution

Solution: $f(x) = x^3 - 3x + 11$
First derivative: $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x - 1)(x + 1)$
For critical points, set $f'(x) = 0$: $3(x - 1)(x + 1) = 0$ $x = 1$ or $x = -1$
Second derivative: $f''(x) = 6x$
At $x = 1$: $f''(1) = 6 > 0 \to$ Local minimum At $x = -1$: $f''(-1) = -6 < 0 \to$ Local maximum
Function values: At $x = 1$: $f(1) = 1^3 - 3(1) + 11 = 1 - 3 + 11 = 9$ At $x = -1$: $f(-1) = (-1)^3 - 3(-1) + 11 = -1 + 3 + 11 = 13$
Therefore:

- **Local maximum value = 13 at** $x = -1$
- **Local minimum value = 9 at** $x = 1$

Question 4(A) [6 marks]

Attempt any two

Question 4(A).1 [3 marks]

Evaluate the integral $\int \frac{\cos(\log x)}{x} dx$

Solution

Solution: Let $u = \log x$, then $du = \frac{1}{x} dx$

$$\int \frac{\cos(\log x)}{x} dx = \int \cos u \, du = \sin u + c$$

Substituting back: $u = \log x$
Therefore, $\int \frac{\cos(\log x)}{x} dx = \sin(\log x) + c$

Question 4(A).2 [3 marks]

Evaluate the integral $\int x \sin x \, dx$

Solution

Solution: Using integration by parts: $\int u \, dv = uv - \int v \, du$
Let $u = x$ and $dv = \sin x \, dx$ Then $du = dx$ and $v = -\cos x$

$$\int x \sin x \, dx = x(-\cos x) - \int (-\cos x) dx$$

$$= -x\cos x + \int \cos x\,dx$$
$$= -x\cos x + \sin x + c$$

Therefore, $\int x\sin x\,dx = \sin x - x\cos x + c$

Question 4(A).3 [3 marks]

If $(2x - y) + 2yi = 6 + 4i$ then find x and y

Solution

Solution: $(2x - y) + 2yi = 6 + 4i$
Comparing real and imaginary parts: Real part: $2x - y = 6$ ... (1) Imaginary part: $2y = 4$ ... (2)
From equation (2): $y = 2$
Substituting in equation (1): $2x - 2 = 6$ $2x = 8$ $x = 4$
Therefore, $x = 4$ **and** $y = 2$

Question 4(B) [8 marks]

Attempt any two

Question 4(B).1 [4 marks]

Find the area of the region bounded by the curve $y = x^2$, lines $x = 1$, $x = 2$ and X-axis

Solution

Solution: The required area is given by:

$$A = \int_1^2 x^2\,dx$$
$$A = \left[\frac{x^3}{3}\right]_1^2$$
$$= \frac{2^3}{3} - \frac{1^3}{3}$$
$$= \frac{8}{3} - \frac{1}{3}$$
$$= \frac{7}{3} \text{ square units}$$

Therefore, **Area = $\frac{7}{3}$ square units**

Question 4(B).2 [4 marks]

Evaluate the definite integral $\int_0^{\pi/2} \frac{\sec x}{\sec x + \csc x} dx$

Solution

Solution: Let $I = \int_0^{\pi/2} \frac{\sec x}{\sec x + \csc x} dx$
Using the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$:

$$I = \int_0^{\pi/2} \frac{\sec(\pi/2 - x)}{\sec(\pi/2 - x) + \csc(\pi/2 - x)} dx$$

Since $\sec(\pi/2 - x) = \csc x$ and $\csc(\pi/2 - x) = \sec x$:

$$I = \int_0^{\pi/2} \frac{\csc x}{\csc x + \sec x} dx$$

Adding both expressions:

$$2I = \int_0^{\pi/2} \frac{\sec x}{\sec x + \csc x} dx + \int_0^{\pi/2} \frac{\csc x}{\sec x + \csc x} dx$$

$$2I = \int_0^{\pi/2} \frac{\sec x + \csc x}{\sec x + \csc x} dx = \int_0^{\pi/2} 1\, dx = \frac{\pi}{2}$$

Therefore, $I = \frac{\pi}{4}$
Answer: $\int_0^{\pi/2} \frac{\sec x}{\sec x + \csc x} dx = \frac{\pi}{4}$

Question 4(B).3 [4 marks]

If $\alpha + i\beta = \frac{1}{a+ib}$ then prove that $(\alpha^2 + \beta^2)(a^2 + b^2) = 1$

Solution

Solution: Given: $\alpha + i\beta = \frac{1}{a+ib}$
Rationalizing the right side:

$$\alpha + i\beta = \frac{1}{a+ib} \cdot \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2}$$

$$\alpha + i\beta = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$

Comparing real and imaginary parts: $\alpha = \frac{a}{a^2+b^2}$ and $\beta = -\frac{b}{a^2+b^2}$
Now calculating $\alpha^2 + \beta^2$:

$$\alpha^2 + \beta^2 = \left(\frac{a}{a^2+b^2}\right)^2 + \left(-\frac{b}{a^2+b^2}\right)^2$$

$$= \frac{a^2}{(a^2+b^2)^2} + \frac{b^2}{(a^2+b^2)^2}$$

$$= \frac{a^2+b^2}{(a^2+b^2)^2} = \frac{1}{a^2+b^2}$$

Therefore: $(\alpha^2 + \beta^2)(a^2 + b^2) = \frac{1}{a^2+b^2} \cdot (a^2 + b^2) = 1$ ✓**Proved**

Question 5(A) [6 marks]

Attempt any two

Question 5(A).1 [3 marks]

Find conjugate and modulus of complex number $\frac{2+3i}{3+2i}$

Solution

Solution: First, simplify the complex number by rationalizing:

$$\frac{2+3i}{3+2i} = \frac{2+3i}{3+2i} \cdot \frac{3-2i}{3-2i}$$

$$= \frac{(2+3i)(3-2i)}{(3+2i)(3-2i)}$$

$$= \frac{6-4i+9i-6i^2}{9-4i^2}$$

$$= \frac{6+5i-6(-1)}{9-4(-1)}$$

$$= \frac{6+5i+6}{9+4} = \frac{12+5i}{13}$$

So $\frac{2+3i}{3+2i} = \frac{12}{13} + \frac{5}{13}i$
Conjugate: $\overline{\frac{2+3i}{3+2i}} = \frac{12}{13} - \frac{5}{13}i$
Modulus: $\left|\frac{2+3i}{3+2i}\right| = \sqrt{\left(\frac{12}{13}\right)^2 + \left(\frac{5}{13}\right)^2}$

$$= \sqrt{\frac{144}{169} + \frac{25}{169}} = \sqrt{\frac{169}{169}} = \sqrt{1} = 1$$

Question 5(A).2 [3 marks]

Simplify: $\frac{(\cos 3\theta + i\sin 3\theta)^{-4}(\cos\theta - i\sin\theta)^{-5}}{(\cos 2\theta - i\sin 2\theta)^7}$

Solution

Solution: Using De Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
Also, $\cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta)$
$(\cos 3\theta + i\sin 3\theta)^{-4} = \cos(-12\theta) + i\sin(-12\theta)$
$(\cos\theta - i\sin\theta)^{-5} = (\cos(-\theta) + i\sin(-\theta))^{-5} = \cos(5\theta) + i\sin(5\theta)$
$(\cos 2\theta - i\sin 2\theta)^7 = (\cos(-2\theta) + i\sin(-2\theta))^7 = \cos(-14\theta) + i\sin(-14\theta)$
Therefore:

$$\frac{(\cos 3\theta + i\sin 3\theta)^{-4}(\cos\theta - i\sin\theta)^{-5}}{(\cos 2\theta - i\sin 2\theta)^7}$$

$$= \frac{[\cos(-12\theta) + i\sin(-12\theta)][\cos(5\theta) + i\sin(5\theta)]}{\cos(-14\theta) + i\sin(-14\theta)}$$

$$= \frac{\cos(-12\theta + 5\theta) + i\sin(-12\theta + 5\theta)}{\cos(-14\theta) + i\sin(-14\theta)}$$

$$= \frac{\cos(-7\theta) + i\sin(-7\theta)}{\cos(-14\theta) + i\sin(-14\theta)}$$

$$= \cos(-7\theta + 14\theta) + i\sin(-7\theta + 14\theta)$$

$$= \cos(7\theta) + i\sin(7\theta)$$

Question 5(A).3 [3 marks]

Express Complex number $1+\sqrt{3}i$ into polar form

Solution

Solution: For complex number $z = a + bi$, polar form is $z = r(\cos\theta + i\sin\theta)$
Here, $a = 1$, $b = \sqrt{3}$
Modulus: $r = |z| = \sqrt{a^2+b^2} = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$
Argument: $\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$
Therefore, the polar form is: $1+\sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

Question 5(B) [8 marks]

Attempt any two

Question 5(B).1 [4 marks]

Solve: $\tan y\, dx + \tan x \sec^2 y\, dy = 0$

Solution

Solution: $\tan y\, dx + \tan x \sec^2 y\, dy = 0$
Rearranging: $\tan y\, dx = -\tan x \sec^2 y\, dy$
$\frac{dx}{\tan x} = -\frac{\sec^2 y\, dy}{\tan y}$
$\frac{\cos x}{\sin x} dx = -\frac{dy}{\sin y \cos y}$
$\cot x\, dx = -\frac{dy}{\sin y \cos y}$
Since $\frac{1}{\sin y \cos y} = \frac{2}{2\sin y \cos y} = \frac{2}{\sin 2y}$:
$\cot x\, dx = -\frac{2dy}{\sin 2y}$
Integrating both sides: $\int \cot x\, dx = -2\int \csc(2y)\, dy$
$\ln|\sin x| = -2 \cdot \left(-\frac{1}{2}\ln|\csc(2y)+\cot(2y)|\right) + c$
$\ln|\sin x| = \ln|\csc(2y)+\cot(2y)| + c$
Therefore: $\sin x \cdot [\csc(2y)+\cot(2y)] = k$ where k is a constant.

Question 5(B).2 [4 marks]

Solve: $x\frac{dy}{dx} - y = x^2$

Solution

Solution: $x\frac{dy}{dx} - y = x^2$
Dividing by x: $\frac{dy}{dx} - \frac{y}{x} = x$
This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$
Here, $P = -\frac{1}{x}$ and $Q = x$
Integrating factor: $I.F. = e^{\int P dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln|x|} = \frac{1}{x}$
Multiplying the equation by I.F.: $\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 1$
This can be written as: $\frac{d}{dx}\left(\frac{y}{x}\right) = 1$
Integrating: $\frac{y}{x} = x + c$

Therefore: $y = x^2 + cx$

Question 5(B).3 [4 marks]

Solve: $\frac{dy}{dx} + \frac{y}{x} = e^x$, $y(0) = 3$

Solution

Solution: This is a linear differential equation: $\frac{dy}{dx} + \frac{y}{x} = e^x$
Here, $P = \frac{1}{x}$ and $Q = e^x$
Integrating factor: $I.F. = e^{\int \frac{1}{x} dx} = e^{\ln|x|} = x$ (assuming $x > 0$)
Multiplying the equation by I.F.: $x\frac{dy}{dx} + y = xe^x$
This can be written as: $\frac{d}{dx}(xy) = xe^x$
Integrating both sides: $xy = \int xe^x dx$
Using integration by parts for $\int xe^x dx$: Let $u = x$, $dv = e^x dx$ Then $du = dx$, $v = e^x$
$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x = e^x(x-1)$
So: $xy = e^x(x-1) + c$
Therefore: $y = \frac{e^x(x-1)+c}{x}$
General solution: $y = \frac{e^x(x-1)+c}{x}$ **for** $x \neq 0$