Subject Name Solutions

4300001 – Winter 2023

Semester 1 Study Material

Detailed Solutions and Explanations

Q.1 [14 marks]

Fill in the blanks using appropriate choice from the given options

0.0.1 Q1.1 [1 mark]

**$
$$\begin{vmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{vmatrix}$$
$= $ __________________**

Solution

c. 1

Solution: $\begin{vmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{vmatrix} = \sin\theta \cdot \sin\theta - (-\cos\theta) \cdot \cos\theta = \sin^2\theta + \cos^2\theta = 1$

0.0.2 Q1.2 [1 mark]

If $f(x) = x^3 - 1$ then $f(-1) = $ ______________

Solution

d. -2

Solution: $f(x) = x^3 - 1$ $f(-1) = (-1)^3 - 1 = -1 - 1 = -2$

0.0.3 Q1.3 [1 mark]

$\log 1 \times \log 2 \times \log 3 \times \log 4 = $______________________

Solution

a. 0

Solution: Since $\log 1 = 0$, we have: $\log 1 \times \log 2 \times \log 3 \times \log 4 = 0 \times \log 2 \times \log 3 \times \log 4 = 0$

0.0.4 Q1.4 [1 mark]

$\log x - \log y = $_____________________

Solution

b. $\log \frac{x}{y}$

Solution: Using logarithm property: $\log x - \log y = \log \frac{x}{y}$

0.0.5 Q1.5 [1 mark]

Principal Period of $\sin(2x + 7) = $______________

Solution

c. π

Solution: For $\sin(ax + b)$**, the period is** $\frac{2\pi}{|a|}$ **Here,** $a = 2$**, so period =** $\frac{2\pi}{2} = \pi$

0.0.6 Q1.6 [1 mark]

\$450° $= \$$_________________$radian$

Solution

c. $\frac{5\pi}{2}$
Solution: $450° = 450 \times \frac{\pi}{180} = \frac{450\pi}{180} = \frac{5\pi}{2}$ **radians**

0.0.7 Q1.7 [1 mark]

\$tan ˆ{−1}x + cot ˆ{−1}x = \$_______________

Solution

d. $\frac{\pi}{2}$
Solution: This is a standard identity: $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$ **for all** $x > 0$

0.0.8 Q1.8 [1 mark]

\$|2i - 3j + 4k| = \$ ____________

Solution

a. $\sqrt{29}$
Solution: $|2i - 3j + 4k| = \sqrt{2^2 + (-3)^2 + 4^2} = \sqrt{4 + 9 + 16} = \sqrt{29}$

0.0.9 Q1.9 [1 mark]

For vector \$ $\times \vec{a} = \$$_______________

Solution

d. 0
Solution: The cross product of any vector with itself is always zero: $\vec{a} \times \vec{a} = 0$

0.0.10 Q1.10 [1 mark]

If two lines having slopes m_1 **and** m_2 **are perpendicular to each other then** _______________

Solution

c. $m_1 \cdot m_2 = -1$
Solution: For perpendicular lines, the product of their slopes equals -1.

0.0.11 Q1.11 [1 mark]

If $x^2 + y^2 = 25$ **then its radius** __________

Solution

c. 5
Solution: Comparing with standard form $x^2 + y^2 = r^2$**:** $r^2 = 25$**, so** $r = 5$

0.0.12 Q1.12 [1 mark]

\$lim

0.0.12 Q1.13 [1 mark]

\$lim

0.0.12 Q1.14 [1 mark]

$lim

Q.2(A) [6 marks]

Attempt any two

0.0.13 Q2.1 [3 marks]

If $f(x) = \frac{1-x}{1+x}$ **then prove that (1)** $f(x) \cdot f(-x) = 1$ **(2)** $f(x) + f(\frac{1}{x}) = 0$

Solution

Solution:
Part (1): Prove $f(x) \cdot f(-x) = 1$
$f(x) = \frac{1-x}{1+x}$
$f(-x) = \frac{1-(-x)}{1+(-x)} = \frac{1+x}{1-x}$
$f(x) \cdot f(-x) = \frac{1-x}{1+x} \cdot \frac{1+x}{1-x} = \frac{(1-x)(1+x)}{(1+x)(1-x)} = 1$
Part (2): Prove $f(x) + f(\frac{1}{x}) = 0$
$f(\frac{1}{x}) = \frac{1-\frac{1}{x}}{1+\frac{1}{x}} = \frac{\frac{x-1}{x}}{\frac{x+1}{x}} = \frac{x-1}{x+1}$
$f(x) + f(\frac{1}{x}) = \frac{1-x}{1+x} + \frac{x-1}{x+1} = \frac{1-x}{1+x} - \frac{1-x}{1+x} = 0$

0.0.14 Q2.2 [3 marks]

If $\begin{vmatrix} x & 2 & 3 \\ 5 & 0 & 7 \\ 3 & 1 & 2 \end{vmatrix} = 30$ **then find the value of** x

Solution

Solution: Expanding along the second row (which has a zero): $\begin{vmatrix} x & 2 & 3 \\ 5 & 0 & 7 \\ 3 & 1 & 2 \end{vmatrix} = -5\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + 0 - 7\begin{vmatrix} x & 2 \\ 3 & 1 \end{vmatrix}$
$= -5(2 \times 2 - 3 \times 1) - 7(x \times 1 - 2 \times 3) = -5(4-3) - 7(x-6) = -5(1) - 7x + 42 = -5 - 7x + 42 = 37 - 7x$
Given: $37 - 7x = 30$ $7x = 37 - 30 = 7$ $x = 1$

0.0.15 Q2.3 [3 marks]

Prove that $\tan 55° = \frac{\cos 10° + \sin 10°}{\cos 10° - \sin 10°}$

Solution

Solution: We know that $55° = 45° + 10°$
Using the tangent addition formula: $\tan(45° + 10°) = \frac{\tan 45° + \tan 10°}{1 - \tan 45° \tan 10°}$
Since $\tan 45° = 1$: $\tan 55° = \frac{1+\tan 10°}{1-\tan 10°}$
Now, $\tan 10° = \frac{\sin 10°}{\cos 10°}$
$\tan 55° = \frac{1+\frac{\sin 10°}{\cos 10°}}{1-\frac{\sin 10°}{\cos 10°}} = \frac{\frac{\cos 10° + \sin 10°}{\cos 10°}}{\frac{\cos 10° - \sin 10°}{\cos 10°}} = \frac{\cos 10° + \sin 10°}{\cos 10° - \sin 10°}$

Q.2(B) [8 marks]

Attempt any two

0.0.16 Q2.1 [4 marks]

Prove that $\frac{1}{\log_{xy} xyz} + \frac{1}{\log_{yz} xyz} + \frac{1}{\log_{zx} xyz} = 2$

Solution

Solution: Using the change of base formula: $\frac{1}{\log_a b} = \log_b a$
$\frac{1}{\log_{xy} xyz} = \log_{xyz}(xy)$ $\frac{1}{\log_{yz} xyz} = \log_{xyz}(yz)$ $\frac{1}{\log_{zx} xyz} = \log_{xyz}(zx)$
LHS $= \log_{xyz}(xy) + \log_{xyz}(yz) + \log_{xyz}(zx) = \log_{xyz}[(xy)(yz)(zx)] = \log_{xyz}(x^2y^2z^2) = \log_{xyz}(xyz)^2 = 2\log_{xyz}(xyz) = 2 \times 1 = 2 =$ RHS

0.0.17 Q2.2 [4 marks]

If $\log(\frac{a+b}{3}) = \frac{1}{2}(\log a + \log b)$ **then prove that** $a^2 + b^2 = 7ab$

Solution

Solution: Given: $\log(\frac{a+b}{3}) = \frac{1}{2}(\log a + \log b)$
RHS: $\frac{1}{2}(\log a + \log b) = \frac{1}{2}\log(ab) = \log(ab)^{1/2} = \log\sqrt{ab}$
So: $\log(\frac{a+b}{3}) = \log\sqrt{ab}$
Taking antilog: $\frac{a+b}{3} = \sqrt{ab}$
Squaring both sides: $(\frac{a+b}{3})^2 = ab$
$\frac{(a+b)^2}{9} = ab$
$(a+b)^2 = 9ab$
$a^2 + 2ab + b^2 = 9ab$
$a^2 + b^2 = 9ab - 2ab = 7ab$

0.0.18 Q2.3 [4 marks]

If $\log x \times \frac{\log 16}{\log 32} = \log 256$ **then find the value of** x

Solution

Solution: First, let's simplify the logarithmic terms: $\log 16 = \log 2^4 = 4\log 2$ $\log 32 = \log 2^5 = 5\log 2$ $\log 256 = \log 2^8 = 8\log 2$
$\frac{\log 16}{\log 32} = \frac{4\log 2}{5\log 2} = \frac{4}{5}$
Given equation becomes: $\log x \times \frac{4}{5} = 8\log 2$
$\log x = \frac{5\times 8\log 2}{4} = 10\log 2$
$\log x = \log 2^{10} = \log 1024$
Therefore: $x = 1024$

Q.3(A) [6 marks]

Attempt any two

0.0.19 Q3.1 [3 marks]

Prove that $\frac{\sin(\frac{\pi}{2}+\theta)}{\cos(\pi-\theta)} + \frac{\cot(\frac{3\pi}{2}-\theta)}{\tan(\pi-\theta)} + \frac{(\frac{\pi}{2}-\theta)}{\sec(\pi+\theta)} = -3$

Solution

Solution: Using trigonometric identities:
First term: $\sin(\frac{\pi}{2}+\theta) = \cos\theta$ $\cos(\pi-\theta) = -\cos\theta$ $\frac{\sin(\frac{\pi}{2}+\theta)}{\cos(\pi-\theta)} = \frac{\cos\theta}{-\cos\theta} = -1$
Second term: $\cot(\frac{3\pi}{2}-\theta) = \cot(2\pi - \frac{\pi}{2} - \theta) = \cot(-(\frac{\pi}{2}+\theta)) = -\cot(\frac{\pi}{2}+\theta) = -(-\tan\theta) = \tan\theta$ $\tan(\pi-\theta) = -\tan\theta$ $\frac{\cot(\frac{3\pi}{2}-\theta)}{\tan(\pi-\theta)} = \frac{\tan\theta}{-\tan\theta} = -1$

Third term: $(\frac{\pi}{2}-\theta) = \frac{1}{\sin(\frac{\pi}{2}-\theta)} = \frac{1}{\cos\theta}$ $\sec(\pi+\theta) = \frac{1}{\cos(\pi+\theta)} = \frac{1}{-\cos\theta}$ $\frac{(\frac{\pi}{2}-\theta)}{\sec(\pi+\theta)} = \frac{\frac{1}{\cos\theta}}{\frac{1}{-\cos\theta}} = \frac{-\cos\theta}{\cos\theta} = -1$
Therefore: LHS $= (-1)+(-1)+(-1) = -3 =$ RHS

0.0.20 Q3.2 [3 marks]

Prove that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$

Solution

Solution: Using the formula: $\tan^{-1}a + \tan^{-1}b = \tan^{-1}(\frac{a+b}{1-ab})$ when $ab < 1$
Let $a = \frac{1}{2}$ and $b = \frac{1}{3}$
$ab = \frac{1}{2}\times\frac{1}{3} = \frac{1}{6} < 1$
$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}})$
$= \tan^{-1}(\frac{\frac{3+2}{6}}{1-\frac{1}{6}}) = \tan^{-1}(\frac{\frac{5}{6}}{\frac{5}{6}}) = \tan^{-1}(1) = \frac{\pi}{4}$

0.0.21 Q3.3 [3 marks]

Find the equation of the line passing through points $(1,6)$ **and** $(-2,5)$**. Also find the slope of the line.**

Solution

Solution: **Step 1: Find the slope** $m = \frac{y_2-y_1}{x_2-x_1} = \frac{5-6}{-2-1} = \frac{-1}{-3} = \frac{1}{3}$
Step 2: Find the equation using point-slope form Using point $(1,6)$: $y-6 = \frac{1}{3}(x-1)$ $3(y-6) = x-1$ $3y-18 = x-1$ $x-3y+17 = 0$

Table 1: Line Properties

Property	Value
Slope	$\frac{1}{3}$
Equation	$x-3y+17=0$

Q.3(B) [8 marks]

Attempt any two

0.0.22 Q3.1 [4 marks]

Draw the graph of $y = \sin x$**;** $0 \le x \le \pi$

Solution

Solution:
Table of Key Points:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$y = \sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

```
      y
      |
    1 +         *
      |       /   {}
  3/2+    /       {}
      |   /         {}
  2/2+  /           {}
      | /             {}
  1/2 +                 *
      |                   {}
    0 +{-{-}{-}*{-}{-}{-}+{-}{-}{-}+{-}{-}{-}+{-}{-}{-}*{-}{-}{-} x}
      0   /6  /4  /3  /2 2 /3 3 /4 5 /6
```

Properties:

- **Domain**: $[0, \pi]$
- **Range**: $[0, 1]$
- **Maximum**: 1 at $x = \frac{\pi}{2}$
- **Zeros**: $x = 0$ and $x = \pi$

0.0.23 Q3.2 [4 marks]

Prove that $\frac{\sin\theta+\sin 2\theta+\sin 4\theta+\sin 5\theta}{\cos\theta+\cos 2\theta+\cos 4\theta+\cos 5\theta} = \tan 3\theta$

Solution

Solution: We can group the terms strategically:
Numerator: $(\sin\theta + \sin 5\theta) + (\sin 2\theta + \sin 4\theta)$
Using sum-to-product formula: $\sin A + \sin B = 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})$
$\sin\theta + \sin 5\theta = 2\sin(\frac{\theta+5\theta}{2})\cos(\frac{5\theta-\theta}{2}) = 2\sin(3\theta)\cos(2\theta)$
$\sin 2\theta + \sin 4\theta = 2\sin(\frac{2\theta+4\theta}{2})\cos(\frac{4\theta-2\theta}{2}) = 2\sin(3\theta)\cos(\theta)$
Numerator $= 2\sin(3\theta)\cos(2\theta) + 2\sin(3\theta)\cos(\theta) = 2\sin(3\theta)[\cos(2\theta) + \cos(\theta)]$
Denominator: $(\cos\theta + \cos 5\theta) + (\cos 2\theta + \cos 4\theta)$
$\cos\theta + \cos 5\theta = 2\cos(\frac{\theta+5\theta}{2})\cos(\frac{5\theta-\theta}{2}) = 2\cos(3\theta)\cos(2\theta)$
$\cos 2\theta + \cos 4\theta = 2\cos(\frac{2\theta+4\theta}{2})\cos(\frac{4\theta-2\theta}{2}) = 2\cos(3\theta)\cos(\theta)$
Denominator $= 2\cos(3\theta)\cos(2\theta) + 2\cos(3\theta)\cos(\theta) = 2\cos(3\theta)[\cos(2\theta) + \cos(\theta)]$
Therefore: $\frac{\text{Numerator}}{\text{Denominator}} = \frac{2\sin(3\theta)[\cos(2\theta)+\cos(\theta)]}{2\cos(3\theta)[\cos(2\theta)+\cos(\theta)]} = \frac{\sin(3\theta)}{\cos(3\theta)} = \tan(3\theta)$

0.0.24 Q3.3 [4 marks]

The constant forces $i - j + k$, $i + j - 3k$ and $4i + 5j - 6k$ act on a particle. Under the action of these forces, particle moves from point $3i - 2j + k$ to point $i + 3j - 4k$. Find the total work done by the forces.

Solution

Solution: **Step 1: Find resultant force** $\vec{F_{total}} = (i - j + k) + (i + j - 3k) + (4i + 5j - 6k) = (1 + 1 + 4)i + (-1 + 1 + 5)j + (1 - 3 - 6)k = 6i + 5j - 8k$
Step 2: Find displacement Initial position: $3i - 2j + k$ Final position: $i + 3j - 4k$ $\vec{d} = (i + 3j - 4k) - (3i - 2j + k) = -2i + 5j - 5k$
Step 3: Calculate work done $W = \vec{F_{total}} \cdot \vec{d} = (6i + 5j - 8k) \cdot (-2i + 5j - 5k)$ $W = 6(-2) + 5(5) + (-8)(-5) = -12 + 25 + 40 = 53$ units

Table 4: Work Calculation

Component	Force	Displacement	Work
x	6	-2	-12
y	5	5	25
z	-8	-5	40
Total			**53**

Q.4(A) [6 marks]

Attempt any two

0.0.25 Q4.1 [3 marks]

If $\vec{a} = 3i - j - 4k$, $\vec{b} = 4j - 2i - 3k$ and $\vec{c} = 2j - k - i$ then find $|3\vec{a} - 2\vec{b} + 4\vec{c}|$

Solution

Solution: First, let's rewrite the vectors in standard form: $\vec{a} = 3i - j - 4k$ $\vec{b} = -2i + 4j - 3k$ $\vec{c} = -i + 2j - k$
$3\vec{a} = 3(3i - j - 4k) = 9i - 3j - 12k$ $2\vec{b} = 2(-2i + 4j - 3k) = -4i + 8j - 6k$ $4\vec{c} = 4(-i + 2j - k) = -4i + 8j - 4k$
$3\vec{a} - 2\vec{b} + 4\vec{c} = (9i - 3j - 12k) - (-4i + 8j - 6k) + (-4i + 8j - 4k) = 9i - 3j - 12k + 4i - 8j + 6k - 4i + 8j - 4k$
$= (9 + 4 - 4)i + (-3 - 8 + 8)j + (-12 + 6 - 4)k = 9i - 3j - 10k$
$|3\vec{a} - 2\vec{b} + 4\vec{c}| = \sqrt{9^2 + (-3)^2 + (-10)^2} = \sqrt{81 + 9 + 100} = \sqrt{190}$

0.0.26 Q4.2 [3 marks]

For what value of m, the vectors $2i - 3j + 5k$ and $mi - 6j - 8k$ are perpendicular to each other?

Solution

Solution: For two vectors to be perpendicular, their dot product must be zero.
$\vec{A} = 2i - 3j + 5k$ $\vec{B} = mi - 6j - 8k$
$\vec{A} \cdot \vec{B} = 0$ $(2)(m) + (-3)(-6) + (5)(-8) = 0$ $2m + 18 - 40 = 0$ $2m - 22 = 0$ $m = 11$

0.0.27 Q4.3 [3 marks]

Find the equation of the circle having center $(4, 3)$ and passing through point $(7, -2)$

Solution

Solution: **Step 1: Find radius** $r = \sqrt{(7-4)^2 + (-2-3)^2} = \sqrt{3^2 + (-5)^2} = \sqrt{9 + 25} = \sqrt{34}$
Step 2: Write equation Using standard form: $(x - h)^2 + (y - k)^2 = r^2$ $(x - 4)^2 + (y - 3)^2 = 34$
Step 3: Expand $x^2 - 8x + 16 + y^2 - 6y + 9 = 34$ $x^2 + y^2 - 8x - 6y + 25 - 34 = 0$ $x^2 + y^2 - 8x - 6y - 9 = 0$

Table 6: Circle Properties

Property	Value
Center	$(4, 3)$
Radius	$\sqrt{34}$
Standard Form	$(x - 4)^2 + (y - 3)^2 = 34$
General Form	$x^2 + y^2 - 8x - 6y - 9 = 0$

Q.4(B) [8 marks]

Attempt any two

0.0.28 Q4.1 [4 marks]

Prove that the angle between vectors $i + 2j$ and $i + j + 3k$ is $\sin^{-1}\sqrt{\frac{46}{55}}$

Solution

Solution: Let $\vec{A} = i + 2j$ and $\vec{B} = i + j + 3k$
Step 1: Calculate dot product $\vec{A} \cdot \vec{B} = (1)(1) + (2)(1) + (0)(3) = 1 + 2 + 0 = 3$
Step 2: Calculate magnitudes $|\vec{A}| = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$ $|\vec{B}| = \sqrt{1^2 + 1^2 + 3^2} = \sqrt{11}$
Step 3: Find cosine of angle $\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|} = \frac{3}{\sqrt{5} \times \sqrt{11}} = \frac{3}{\sqrt{55}}$

Step 4: Find sine of angle $\sin^2\theta = 1-\cos^2\theta = 1-\frac{9}{55} = \frac{55-9}{55} = \frac{46}{55}$
$\sin\theta = \sqrt{\frac{46}{55}}$
Therefore: $\theta = \sin^{-1}\sqrt{\frac{46}{55}}$

0.0.29 Q4.2 [4 marks]

If $\vec{x} = -2k + 3i$ **and** $\vec{y} = 5i + 2j - 4k$ **then find the value of** $|(\vec{x}+\vec{y}) \times (\vec{x}-\vec{y})|$

Solution

Solution: First, let's rewrite in standard form: $\vec{x} = 3i + 0j - 2k$ $\vec{y} = 5i + 2j - 4k$
$\vec{x}+\vec{y} = (3+5)i + (0+2)j + (-2-4)k = 8i + 2j - 6k$ $\vec{x}-\vec{y} = (3-5)i + (0-2)j + (-2+4)k = -2i - 2j + 2k$
$(\vec{x}+\vec{y}) \times (\vec{x}-\vec{y}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8 & 2 & -6 \\ -2 & -2 & 2 \end{vmatrix}$
$= \hat{i}(2\times 2-(-6)\times(-2)) - \hat{j}(8\times 2-(-6)\times(-2)) + \hat{k}(8\times(-2)-2\times(-2)) = \hat{i}(4-12) - \hat{j}(16-12) + \hat{k}(-16+4)$
$= -8\hat{i} - 4\hat{j} - 12\hat{k}$
$|(\vec{x}+\vec{y}) \times (\vec{x}-\vec{y})| = \sqrt{(-8)^2 + (-4)^2 + (-12)^2} = \sqrt{64+16+144} = \sqrt{224} = 4\sqrt{14}$

0.0.30 Q4.3 [4 marks]

Evaluate: $\lim_{n\to\infty}(\sqrt{n^2+n+1} - n)$

Solution

Solution: We have the indeterminate form $\infty - \infty$. Let's rationalize:
$\lim_{n\to\infty}(\sqrt{n^2+n+1} - n)$
Multiply and divide by the conjugate: $= \lim_{n\to\infty} \frac{(\sqrt{n^2+n+1}-n)(\sqrt{n^2+n+1}+n)}{\sqrt{n^2+n+1}+n}$
$= \lim_{n\to\infty} \frac{(n^2+n+1)-n^2}{\sqrt{n^2+n+1}+n}$
$= \lim_{n\to\infty} \frac{n+1}{\sqrt{n^2+n+1}+n}$
Divide numerator and denominator by n: $= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+1}$
$= \frac{1+0}{\sqrt{1+0+0}+1} = \frac{1}{1+1} = \frac{1}{2}$

Q.5(A) [6 marks]

Attempt any two

0.0.31 Q5.1 [3 marks]

Evaluate: $\lim_{x\to-2} \frac{x^3+2x^2+x+2}{x^2+x-2}$

Solution

Solution: Direct substitution at $x = -2$: Numerator: $(-2)^3 + 2(-2)^2 + (-2) + 2 = -8 + 8 - 2 + 2 = 0$
Denominator: $(-2)^2 + (-2) - 2 = 4 - 2 - 2 = 0$
We get $\frac{0}{0}$ form, so we need to factor.
Factoring numerator: $x^3 + 2x^2 + x + 2 = x^2(x+2) + 1(x+2) = (x+2)(x^2+1)$
Factoring denominator: $x^2 + x - 2 = (x+2)(x-1)$
$\lim_{x\to-2} \frac{x^3+2x^2+x+2}{x^2+x-2} = \lim_{x\to-2} \frac{(x+2)(x^2+1)}{(x+2)(x-1)}$
$= \lim_{x\to-2} \frac{x^2+1}{x-1} = \frac{(-2)^2+1}{-2-1} = \frac{4+1}{-3} = \frac{5}{-3} = -\frac{5}{3}$

0.0.32 Q5.2 [3 marks]

Evaluate: $\lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\cos^2 x}$

Solution

Solution: Direct substitution at $x = \frac{\pi}{2}$: Numerator: $1 - \sin\frac{\pi}{2} = 1 - 1 = 0$ Denominator: $\cos^2\frac{\pi}{2} = 0^2 = 0$
We get $\frac{0}{0}$ form.
Using the identity: $\cos^2 x = 1 - \sin^2 x$
$\lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\cos^2 x} = \lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{1-\sin^2 x}$
$= \lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{(1-\sin x)(1+\sin x)}$
$= \lim_{x\to\frac{\pi}{2}} \frac{1}{1+\sin x}$
Substituting $x = \frac{\pi}{2}$: $= \frac{1}{1+1} = \frac{1}{2}$

0.0.33 Q5.3 [3 marks]

Evaluate: $\lim_{x\to\infty}(1 + \frac{5}{x})^{2x}$

Solution

Solution: Let $y = (1 + \frac{5}{x})^{2x}$
Taking natural logarithm: $\ln y = 2x\ln(1 + \frac{5}{x})$
$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} 2x\ln(1 + \frac{5}{x})$
Let $t = \frac{5}{x}$, then as $x \to \infty$, $t \to 0$ and $x = \frac{5}{t}$
$= \lim_{t\to 0} 2 \cdot \frac{5}{t}\ln(1 + t) = \lim_{t\to 0} 10 \cdot \frac{\ln(1+t)}{t}$
Using the standard limit $\lim_{t\to 0} \frac{\ln(1+t)}{t} = 1$:
$= 10 \times 1 = 10$
Therefore: $\lim_{x\to\infty} y = e^{10}$

Q.5(B) [8 marks]

Attempt any two

0.0.34 Q5.1 [4 marks]

Find the equation of the line passing through point $(2, 4)$ and perpendicular to line $5x - 7y + 11 = 0$

Solution

Solution: **Step 1: Find slope of given line** $5x - 7y + 11 = 0$ $7y = 5x + 11$ $y = \frac{5}{7}x + \frac{11}{7}$ Slope of given line $= \frac{5}{7}$
Step 2: Find slope of perpendicular line For perpendicular lines: $m_1 \times m_2 = -1$ $\frac{5}{7} \times m_2 = -1$ $m_2 = -\frac{7}{5}$
Step 3: Use point-slope form $y - y_1 = m(x - x_1)$ $y - 4 = -\frac{7}{5}(x - 2)$ $y - 4 = -\frac{7}{5}x + \frac{14}{5}$ $y = -\frac{7}{5}x + \frac{14}{5} + 4$ $y = -\frac{7}{5}x + \frac{14+20}{5}$ $y = -\frac{7}{5}x + \frac{34}{5}$
Multiplying by 5: $5y = -7x + 34$ $7x + 5y - 34 = 0$

0.0.35 Q5.2 [4 marks]

If the equation of circle is $2x^2 + 2y^2 + 4x - 8y - 6 = 0$ then find its center and radius

Solution

Solution: **Step 1: Simplify by dividing by 2** $x^2 + y^2 + 2x - 4y - 3 = 0$
Step 2: Complete the square $(x^2 + 2x) + (y^2 - 4y) = 3$ $(x^2 + 2x + 1) + (y^2 - 4y + 4) = 3 + 1 + 4$ $(x + 1)^2 + (y - 2)^2 = 8$

Table 8: Circle Properties

Property	Value
Center	$(-1, 2)$
Radius	$\sqrt{8} = 2\sqrt{2}$

0.0.36 Q5.3 [4 marks]

Find the equation of tangent and normal of circle $x^2 + y^2 - 2x + 4y - 20 = 0$ at point $(-2, 2)$

Solution

Solution: **Step 1: Find center of circle $x^2 + y^2 - 2x + 4y - 20 = 0$** Completing the square: $(x^2 - 2x + 1) + (y^2 + 4y + 4) = 20 + 1 + 4$ $(x-1)^2 + (y+2)^2 = 25$
Center: $(1, -2)$, Radius: 5
Step 2: Find slope of radius to point $(-2, 2)$ $m_{radius} = \frac{2-(-2)}{-2-1} = \frac{4}{-3} = -\frac{4}{3}$
Step 3: Find slope of tangent Tangent is perpendicular to radius: $m_{tangent} = -\frac{1}{m_{radius}} = -\frac{1}{-\frac{4}{3}} = \frac{3}{4}$
Step 4: Equation of tangent Using point-slope form at $(-2, 2)$: $y - 2 = \frac{3}{4}(x - (-2))$ $y - 2 = \frac{3}{4}(x + 2)$ $4(y - 2) = 3(x + 2)$ $4y - 8 = 3x + 6$ $3x - 4y + 14 = 0$
Step 5: Equation of normal Normal has slope $m_{radius} = -\frac{4}{3}$: $y - 2 = -\frac{4}{3}(x + 2)$ $3(y - 2) = -4(x + 2)$ $3y - 6 = -4x - 8$ $4x + 3y + 2 = 0$

Table 10: Line Equations

Line	Equation
Tangent	$3x - 4y + 14 = 0$
Normal	$4x + 3y + 2 = 0$

Mathematics Formula Cheat Sheet for Winter Exams

0.0.37 Determinants

- **2**$\times 2 Matrix : \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
- **3**$\times 3 Matrix : Expandalongrow/columnwithmostzeros$
- **Properties: $|A| = 0$ if any row/column is zero**

0.0.38 Functions

- **Composition**: $(f \circ g)(x) = f(g(x))$
- **Even function**: $f(-x) = f(x)$
- **Odd function**: $f(-x) = -f(x)$

0.0.39 Logarithms

- **Basic properties**:
  - $\log_a a = 1$
  - $\log 1 = 0$
  - $\log x - \log y = \log \frac{x}{y}$
  - $\log x + \log y = \log(xy)$
- **Change of base**: $\frac{1}{\log_a b} = \log_b a$

0.0.40 Trigonometry

Periods

- $\sin(ax + b)$ has period $\frac{2\pi}{|a|}$
- $\cos(ax + b)$ has period $\frac{2\pi}{|a|}$
- $\tan(ax + b)$ has period $\frac{\pi}{|a|}$

Angle Conversions

- Degrees to radians: radians = degrees $\times \frac{\pi}{180}$

Inverse Trigonometric Identities

- $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$
- $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$
- $\tan^{-1} a + \tan^{-1} b = \tan^{-1}(\frac{a+b}{1-ab})$ when $ab < 1$

Allied Angles

- $\sin(\frac{\pi}{2} + \theta) = \cos\theta$
- $\cos(\pi - \theta) = -\cos\theta$
- $\tan(\pi - \theta) = -\tan\theta$
- $\cot(\frac{3\pi}{2} - \theta) = \tan\theta$

Sum-to-Product Formulas

- $\sin A + \sin B = 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})$
- $\cos A + \cos B = 2\cos(\frac{A+B}{2})\cos(\frac{A-B}{2})$

0.0.41 Vectors

- **Magnitude**: $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$
- **Dot Product**: $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
- **Cross Product**: $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
- **Properties**:
  - $\vec{a} \times \vec{a} = 0$
  - $\vec{a} \perp \vec{b}$ iff $\vec{a} \cdot \vec{b} = 0$
- **Work done**: $W = \vec{F} \cdot \vec{d}$

0.0.42 Coordinate Geometry

Lines

- **Slope**: $m = \frac{y_2 - y_1}{x_2 - x_1}$
- **Two-point form**: $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$
- **Perpendicular lines**: $m_1 \times m_2 = -1$
- **Point-slope form**: $y - y_1 = m(x - x_1)$

Circles

- **Standard form**: $(x-h)^2 + (y-k)^2 = r^2$
- **General form**: $x^2 + y^2 + 2gx + 2fy + c = 0$
- **Center**: $(-g, -f)$, **Radius**: $\sqrt{g^2 + f^2 - c}$
- **Tangent at point** (x_1, y_1): $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

0.0.43 Limits

- **Standard limits**:
  - $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$
  - $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$
  - $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$
  - $\lim_{x \to \infty} (1 + \frac{a}{x})^x = e^a$
- **Rationalization**: For expressions like $\sqrt{A} - \sqrt{B}$, multiply by $\frac{\sqrt{A}+\sqrt{B}}{\sqrt{A}+\sqrt{B}}$

0.0.44 Problem-Solving Strategies

For Function Problems

1. Check domain restrictions

2. Use algebraic manipulation for compositions
3. Verify results by substitution

For Logarithmic Proofs

1. Use change of base formula strategically
2. Convert complex expressions to simpler forms
3. Apply logarithm properties systematically

For Trigonometric Identities

1. Look for sum-to-product opportunities
2. Use allied angle formulas
3. Factor expressions when possible

For Vector Problems

1. Write vectors in component form
2. Use properties of dot and cross products
3. Check perpendicularity using dot product

For Limit Problems

1. Try direct substitution first
2. Factor and cancel for $\frac{0}{0}$ forms
3. Use rationalization for radical expressions
4. Apply standard limit formulas

For Circle Problems

1. Complete the square to find center and radius
2. Use slope relationships for tangent and normal
3. Remember tangent is perpendicular to radius

0.0.45 Common Mistakes to Avoid

1. **Sign errors** in determinant calculations
2. **Forgetting domain restrictions** in logarithmic functions
3. **Angle measure confusion** (degrees vs radians)
4. **Not simplifying** trigonometric expressions fully
5. **Calculation errors** in vector operations
6. **Incomplete factorization** in limit problems

0.0.46 Exam Success Tips

- **Show all working steps** clearly
- **Verify answers** when possible
- **Use proper mathematical notation**
- **Draw diagrams** for geometry problems
- **Manage time** effectively across all questions

Best of luck with your Winter 2023 Mathematics exam!