

# Engineering Mathematics (4320002) - Winter 2022 Solution

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## Question 1 [14 marks]

Fill in the blanks using appropriate choice from the given options.

### Question 1.1 [1 marks]

If  $A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$  then  $\text{adj.}A = \underline{\hspace{2cm}}$ .

#### Solution

**Answer:** (d)  $\begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}$

**Solution:** For a  $2 \times 2$  matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ,  $\text{adj.}A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\text{adj.}A = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}$$

### Question 1.2 [1 marks]

If A is  $2 \times 3$  and B is  $3 \times 4$  matrices then AB is  $\underline{\hspace{2cm}}$  matrix

#### Solution

**Answer:** (b)  $2 \times 4$

**Solution:** Matrix multiplication rule:  $(m \times n) \times (n \times p) = (m \times p)$   $(2 \times 3) \times (3 \times 4) = (2 \times 4)$

### Question 1.3 [1 marks]

If  $\begin{bmatrix} 0 & x \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -2 & 4 \end{bmatrix}$  then  $x = \underline{\hspace{2cm}}$

#### Solution

**Answer:** (b) 4

**Solution:** Comparing corresponding elements:  $x = 4$

## Question 1.4 [1 marks]

If A is non singular matrix then \_\_\_\_\_

### Solution

**Answer:** (d)  $|A| \neq 0$

**Solution:** A matrix is non-singular if its determinant is non-zero.

## Question 1.5 [1 marks]

$\frac{d}{dx}(e^{-\log x}) = \underline{\hspace{10cm}}$

### Solution

**Answer:** (d) x

**Solution:**  $e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$   $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

## Question 1.6 [1 marks]

If  $f(x) = \log \sqrt{x^2 + 1}$ , then  $f'(0) = \underline{\hspace{10cm}}$

### Solution

**Answer:** (a) 0

**Solution:**  $f(x) = \frac{1}{2} \log(x^2 + 1)$   $f'(x) = \frac{1}{2} \cdot \frac{2x}{x^2+1} = \frac{x}{x^2+1}$   $f'(0) = \frac{0}{0+1} = 0$

## Question 1.7 [1 marks]

If  $x = \sec \theta + \tan \theta$  and  $y = \sec \theta - \tan \theta$  then  $\frac{dy}{dx} = \underline{\hspace{10cm}}$

### Solution

**Answer:** (d) 1

**Solution:**  $xy = (\sec \theta + \tan \theta)(\sec \theta - \tan \theta) = \sec^2 \theta - \tan^2 \theta = 1$  Differentiating:  $x \frac{dy}{dx} + y = 0$   $\frac{dy}{dx} = -\frac{y}{x}$

## Question 1.8 [1 marks]

$\int e^x(\sin x + \cos x)dx = \underline{\hspace{10cm}}$

### Solution

**Answer:** (b)  $e^x \sin x + c$

**Solution:** Using integration by parts or standard result:  $\int e^x(\sin x + \cos x)dx = e^x \sin x + c$

## Question 1.9 [1 marks]

$\int_{-1}^1 x^2 + 1 dx = \underline{\hspace{10cm}}$

**Solution****Answer:** (d)  $\frac{8}{3}$ 

**Solution:**  $\int_{-1}^1 (x^2 + 1) dx = \left[ \frac{x^3}{3} + x \right]_{-1}^1 = \left( \frac{1}{3} + 1 \right) - \left( -\frac{1}{3} - 1 \right) = \frac{8}{3}$

**Question 1.10 [1 marks]**

$$\int \cot x dx = \text{_____} + c$$

**Solution****Answer:** (a)  $\log |\sin x|$ 

**Solution:**  $\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \log |\sin x| + c$

**Question 1.11 [1 marks]**

The order & degree of the differential equation  $\frac{d^2y}{dx^2} + x \frac{dy}{dx} + 3y = 0$  are respectively \_\_\_\_\_ and \_\_\_\_\_

**Solution****Answer:** (a) 2, 1
**Solution:** Order = highest order derivative = 2 Degree = power of highest order derivative = 1
**Question 1.12 [1 marks]**

The integrating factor for the differential equation  $\frac{dy}{dx} + \frac{y}{x} = x$  is \_\_\_\_\_

**Solution****Answer:** (b)  $x$ 
**Solution:** For  $\frac{dy}{dx} + P(x)y = Q(x)$ , where  $P(x) = \frac{1}{x}$  I.F. =  $e^{\int P(x)dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$ 
**Question 1.13 [1 marks]**

$$i + i^2 + i^3 + i^4 = \text{_____}$$

**Solution****Answer:** (d) 0

**Solution:**  $i + i^2 + i^3 + i^4 = i + (-1) + (-i) + 1 = 0$

**Question 1.14 [1 marks]**

$$\arg(-1) = \text{_____}$$

**Solution****Answer:** (a)  $\pi$ **Solution:**  $-1 = \cos \pi + i \sin \pi$ , so  $\arg(-1) = \pi$ **Question 2(a) [6 marks]****Attempt any two.****Question 2(a).1 [3 marks]**

If  $A = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}$ ,  $B = \begin{bmatrix} 5 & 6 \\ -2 & 3 \end{bmatrix}$  then find matrix X from equation  $3(X+B) + 5A = 0$

**Solution**

$$\text{Solution: } 3(X + B) + 5A = 0 \quad 3X + 3B + 5A = 0 \quad 3X = -3B - 5A \quad X = -B - \frac{5A}{3}$$

$$5A = 5 \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ -15 & 10 \end{bmatrix}$$

$$X = - \begin{bmatrix} 5 & 6 \\ -2 & 3 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 5 & 10 \\ -15 & 10 \end{bmatrix}$$

$$X = \begin{bmatrix} -5 & -6 \\ 2 & -3 \end{bmatrix} - \begin{bmatrix} \frac{5}{3} & \frac{10}{3} \\ -5 & \frac{10}{3} \end{bmatrix}$$

$$X = \begin{bmatrix} -\frac{20}{3} & -\frac{28}{3} \\ 7 & -\frac{19}{3} \end{bmatrix}$$

**Question 2(a).2 [3 marks]**

If  $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$  then Prove that  $A^2 - 4A - 5I = 0$

**Solution**

$$\text{Solution: } A^2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

$$4A = 4 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & 4 \end{bmatrix}$$

$$5I = 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence proved.

## Question 2(a).3 [3 marks]

Solve differential equation  $\frac{dy}{dx} = (x + y)^2$

### Solution

**Solution:** Let  $v = x + y$ , then  $\frac{dv}{dx} = 1 + \frac{dy}{dx}$   $\frac{dy}{dx} = \frac{dv}{dx} - 1$

Substituting:  $\frac{dv}{dx} - 1 = v^2$   $\frac{dv}{dx} = v^2 + 1$   $\frac{dv}{v^2+1} = dx$

Integrating:  $\int \frac{dv}{v^2+1} = \int dx$   $\tan^{-1} v = x + c$   $\tan^{-1}(x + y) = x + c$   $x + y = \tan(x + c)$   $y = \tan(x + c) - x$

## Question 2(b) [8 marks]

Attempt any two.

### Question 2(b).1 [4 marks]

If  $A = \begin{bmatrix} 3 & -1 \\ 4 & 1 \\ 5 & 0 \end{bmatrix}$  then find  $A^{-1}$

### Solution

**Solution:** This is a  $3 \times 2$  matrix, which is non-square. Inverse doesn't exist for non-square matrices.

**Alternative interpretation - if it's**  $\begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & -1 \\ 5 & 0 & 1 \end{bmatrix}$ :

Using adjoint method:  $|A| = 3(1 - 0) + 1(4 + 5) + 2(0 - 5) = 3 + 9 - 10 = 2$

Calculate cofactors and adjoint, then  $A^{-1} = \frac{1}{|A|} \times \text{adj}(A)$

### Question 2(b).2 [4 marks]

Solve Equation  $3X - 2Y = 8$  and  $5X + 4Y = 6$  using matrices method.

### Solution

**Solution:**  $\begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$

$$|A| = 3(4) - (-2)(5) = 12 + 10 = 22$$

$$A^{-1} = \frac{1}{22} \begin{bmatrix} 4 & 2 \\ -5 & 3 \end{bmatrix}$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 4 & 2 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 32 + 12 \\ -40 + 18 \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 44 \\ -22 \end{bmatrix}$$

$$X = 2, Y = -1$$

## Question 2(b).3 [4 marks]

If  $M = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ ,  $N = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$  then Prove that  $(MN)^T = N^T M^T$

**Solution**

$$\text{Solution: } MN = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 11 \\ 2 & 1 \end{bmatrix}$$

$$(MN)^T = \begin{bmatrix} 12 & 2 \\ 11 & 1 \end{bmatrix}$$

$$M^T = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}, N^T = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

$$N^T M^T = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 2 \\ 11 & 1 \end{bmatrix}$$

Hence  $(MN)^T = N^T M^T$  is proved.

## Question 3(a) [6 marks]

Attempt any two.

### Question 3(a).1 [3 marks]

Differentiate  $\sqrt{x}$  using the definition.

**Solution**

$$\text{Solution: } f(x) = \sqrt{x} = x^{1/2}$$

$$\text{Using definition: } f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$\text{Rationalizing: } f'(x) = \lim_{h \rightarrow 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

### Question 3(a).2 [3 marks]

If  $y = \log(x + \sqrt{1 + x^2})$  then Find  $\frac{dy}{dx}$

**Solution**

$$\text{Solution: } y = \log(x + \sqrt{1 + x^2})$$

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{1 + x^2}} \cdot \frac{d}{dx}(x + \sqrt{1 + x^2})$$

$$\frac{d}{dx}(x + \sqrt{1 + x^2}) = 1 + \frac{1}{2\sqrt{1+x^2}} \cdot 2x = 1 + \frac{x}{\sqrt{1+x^2}}$$

$$= \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}$$

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{1 + x^2}} \cdot \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}$$

$$= \frac{1}{\sqrt{1+x^2}}$$

### Question 3(a).3 [3 marks]

$$\int \frac{4+3 \cos x}{\sin^2 x} dx$$

**Solution**

**Solution:**  $\int \frac{4+3 \cos x}{\sin^2 x} dx = \int \frac{4}{\sin^2 x} dx + \int \frac{3 \cos x}{\sin^2 x} dx$   
 $= 4 \int \csc^2 x dx + 3 \int \frac{\cos x}{\sin^2 x} dx$   
 $= -4 \cot x + 3 \int \sin^{-2} x \cos x dx$   
For the second integral, let  $u = \sin x$ ,  $du = \cos x dx$   $3 \int u^{-2} du = 3(-u^{-1}) = -\frac{3}{\sin x}$   
 $\int \frac{4+3 \cos x}{\sin^2 x} dx = -4 \cot x - 3 \csc x + c$

### Question 3(b) [8 marks]

Attempt any two.

#### Question 3(b).1 [4 marks]

If  $y = \log(\sin x)$  then prove that  $\frac{d^2y}{dx^2} + (\frac{dy}{dx})^2 + 1 = 0$

**Solution**

**Solution:**  $y = \log(\sin x)$   
 $\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x = \cot x$   
 $\frac{d^2y}{dx^2} = \frac{d}{dx}(\cot x) = -\csc^2 x$   
Now,  $\frac{d^2y}{dx^2} + (\frac{dy}{dx})^2 + 1 = -\csc^2 x + \cot^2 x + 1$   
Using identity:  $\csc^2 x - \cot^2 x = 1 - \csc^2 x + \cot^2 x + 1 = -(\csc^2 x - \cot^2 x) = -1 + 1 = 0$   
Hence proved.

#### Question 3(b).2 [4 marks]

If  $x + y = \sin(xy)$  then Find  $\frac{dy}{dx}$

**Solution**

**Solution:**  $x + y = \sin(xy)$   
Differentiating both sides with respect to x:  $1 + \frac{dy}{dx} = \cos(xy) \cdot \frac{d}{dx}(xy)$   
 $1 + \frac{dy}{dx} = \cos(xy) \cdot (y + x \frac{dy}{dx})$   
 $1 + \frac{dy}{dx} = y \cos(xy) + x \cos(xy) \frac{dy}{dx}$   
 $1 + \frac{dy}{dx} - x \cos(xy) \frac{dy}{dx} = y \cos(xy)$   
 $\frac{dy}{dx} (1 - x \cos(xy)) = y \cos(xy) - 1$   
 $\frac{dy}{dx} = \frac{y \cos(xy) - 1}{1 - x \cos(xy)}$

### Question 3(b).3 [4 marks]

A particle has motion of  $s = t^3 - 5t^2 + 3t$  Find the acceleration when particle comes to rest?

#### Solution

**Solution:** Given:  $s = t^3 - 5t^2 + 3t$

Velocity:  $v = \frac{ds}{dt} = 3t^2 - 10t + 3$

Acceleration:  $a = \frac{dv}{dt} = 6t - 10$

At rest,  $v = 0$ :  $3t^2 - 10t + 3 = 0$

Using quadratic formula:  $t = \frac{10 \pm \sqrt{100-36}}{6} = \frac{10 \pm 8}{6}$

$t = 3$  or  $t = \frac{1}{3}$

At  $t = 3$ :  $a = 6(3) - 10 = 8$  At  $t = \frac{1}{3}$ :  $a = 6(\frac{1}{3}) - 10 = -8$

The accelerations are 8 and  $-8$  respectively.

### Question 4(a) [6 marks]

Attempt any two.

#### Question 4(a).1 [3 marks]

$$\int x \sin x dx$$

#### Solution

**Solution:** Using integration by parts:  $\int u dv = uv - \int v du$

Let  $u = x$ ,  $dv = \sin x dx$   $du = dx$ ,  $v = -\cos x$

$$\int x \sin x dx = x(-\cos x) - \int (-\cos x)dx = -x \cos x + \int \cos x dx = -x \cos x + \sin x + c$$

#### Question 4(a).2 [3 marks]

$$\int \frac{2x+1}{(x+1)(x-3)} dx$$

#### Solution

**Solution:** Using partial fractions:  $\frac{2x+1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$

$$2x+1 = A(x-3) + B(x+1)$$

$$\text{At } x = -1: -2+1 = A(-4) \Rightarrow A = \frac{1}{4} \quad \text{At } x = 3: 6+1 = B(4) \Rightarrow B = \frac{7}{4}$$

$$\int \frac{2x+1}{(x+1)(x-3)} dx = \frac{1}{4} \int \frac{1}{x+1} dx + \frac{7}{4} \int \frac{1}{x-3} dx$$

$$= \frac{1}{4} \log|x+1| + \frac{7}{4} \log|x-3| + c$$

#### Question 4(a).3 [3 marks]

Find square root of complex number  $z = 7 + 24i$

#### Solution

**Solution:** Let  $\sqrt{7+24i} = a+bi$

$$(a+bi)^2 = 7+24i \quad a^2 - b^2 + 2abi = 7+24i$$

Comparing:  $a^2 - b^2 = 7$  and  $2ab = 24$  From second equation:  $b = \frac{12}{a}$

Substituting:  $a^2 - \frac{144}{a^2} = 7$   $a^4 - 7a^2 - 144 = 0$

Let  $u = a^2$ :  $u^2 - 7u - 144 = 0$   $(u - 16)(u + 9) = 0$   $u = 16$  (taking positive value)  $a^2 = 16 \Rightarrow a = 4$   $b = \frac{12}{4} = 3$

Therefore:  $\sqrt{7 + 24i} = 4 + 3i$  or  $-(4 + 3i)$

## Question 4(b) [8 marks]

Attempt any two.

### Question 4(b).1 [4 marks]

$$\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x + \sqrt{\cos x}}} dx$$

#### Solution

**Solution:** Let  $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x + \sqrt{\cos x}}} dx$

Using property:  $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2-x)}}{\sqrt{\sin(\pi/2-x) + \sqrt{\cos(\pi/2-x)}}} dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x + \sqrt{\sin x}}} dx$$

Adding both expressions:  $2I = \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x + \sqrt{\cos x}}} dx = \int_0^{\pi/2} 1 dx = \frac{\pi}{2}$

Therefore:  $I = \frac{\pi}{4}$

### Question 4(b).2 [4 marks]

Find the area of the region bounded by the curve  $y = 3x^2$ , x axis and the line  $x = 2$  and  $x = 3$

#### Solution

**Solution:** Area =  $\int_2^3 y dx = \int_2^3 3x^2 dx$

$$= 3 \int_2^3 x^2 dx = 3[\frac{x^3}{3}]_2^3$$

$$= [x^3]_2^3 = 3^3 - 2^3 = 27 - 8 = 19$$

Area = 19 square units

### Question 4(b).3 [4 marks]

$$\text{Simplify } \frac{(\cos 2\theta + i \sin 2\theta)^{-3} \cdot (\cos 3\theta - i \sin 3\theta)^2}{(\cos 2\theta - i \sin 2\theta)^{-7} \cdot (\cos 5\theta - i \sin 5\theta)^3}$$

#### Solution

**Solution:** Using Euler's formula:  $\cos \theta + i \sin \theta = e^{i\theta}$

$$(\cos 2\theta + i \sin 2\theta)^{-3} = e^{-6i\theta} \quad (\cos 3\theta - i \sin 3\theta)^2 = e^{-6i\theta} \quad (\cos 2\theta - i \sin 2\theta)^{-7} = e^{14i\theta} \quad (\cos 5\theta - i \sin 5\theta)^3 = e^{-15i\theta}$$

$$\text{Expression} = \frac{e^{-6i\theta} \cdot e^{-6i\theta}}{e^{14i\theta} \cdot e^{-15i\theta}} = \frac{e^{-12i\theta}}{e^{-i\theta}} = e^{-11i\theta}$$

$$= \cos(-11\theta) + i \sin(-11\theta) = \cos(11\theta) - i \sin(11\theta)$$

## Question 5(a) [6 marks]

Attempt any two.

### Question 5(a).1 [3 marks]

Convert  $\frac{4+2i}{(3+2i)(5-3i)}$  in a+ib form.

#### Solution

**Solution:** First, simplify the denominator:  $(3 + 2i)(5 - 3i) = 15 - 9i + 10i - 6i^2 = 15 + i + 6 = 21 + i$

$$\text{Now: } \frac{4+2i}{21+i}$$

Multiply by conjugate:  $\frac{4+2i}{21+i} \cdot \frac{21-i}{21-i}$

$$= \frac{(4+2i)(21-i)}{(21+i)(21-i)} = \frac{84-4i+42i-2i^2}{441-i^2}$$

$$= \frac{84+38i+2}{441+1} = \frac{86+38i}{442} = \frac{43+19i}{221}$$

### Question 5(a).2 [3 marks]

Convert  $z = 1 - \sqrt{3}i$  in polar form.

#### Solution

**Solution:**  $z = 1 - \sqrt{3}i$

$$|z| = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1+3} = 2$$

$$\arg(z) = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3} \text{ (since z is in 4th quadrant)}$$

$$\text{Therefore: } z = 2(\cos(-\frac{\pi}{3}) + i \sin(-\frac{\pi}{3})) = 2e^{-i\pi/3}$$

### Question 5(a).3 [3 marks]

Prove that  $(1 + \cos \theta + i \sin \theta)^n + (1 + \cos \theta - i \sin \theta)^n = 2^{n+1} \cos^n(\frac{\theta}{2}) \cos(\frac{n\theta}{2})$

#### Solution

**Solution:**  $1 + \cos \theta + i \sin \theta = 1 + e^{i\theta} = 1 + \cos \theta + i \sin \theta$

Using identity:  $1 + \cos \theta = 2 \cos^2(\frac{\theta}{2})$

$$1 + \cos \theta + i \sin \theta = 2 \cos^2(\frac{\theta}{2}) + 2i \sin(\frac{\theta}{2}) \cos(\frac{\theta}{2})$$

$$= 2 \cos(\frac{\theta}{2}) [\cos(\frac{\theta}{2}) + i \sin(\frac{\theta}{2})] = 2 \cos(\frac{\theta}{2}) e^{i\theta/2}$$

Similarly:  $1 + \cos \theta - i \sin \theta = 2 \cos(\frac{\theta}{2}) e^{-i\theta/2}$

$$(1 + \cos \theta + i \sin \theta)^n = 2^n \cos^n(\frac{\theta}{2}) e^{in\theta/2}$$

$$(1 + \cos \theta - i \sin \theta)^n = 2^n \cos^n(\frac{\theta}{2}) e^{-in\theta/2}$$

$$\text{Sum} = 2^n \cos^n(\frac{\theta}{2}) [e^{in\theta/2} + e^{-in\theta/2}] = 2^n \cos^n(\frac{\theta}{2}) \cdot 2 \cos(\frac{n\theta}{2})$$

$$= 2^{n+1} \cos^n(\frac{\theta}{2}) \cos(\frac{n\theta}{2})$$

Hence proved.

## Question 5(b) [8 marks]

Attempt any two.

## Question 5(b).1 [4 marks]

Solve differential equation  $x \log x \frac{dy}{dx} + y = \log x^2$

### Solution

**Solution:**  $x \log x \frac{dy}{dx} + y = 2 \log x$

Dividing by  $x \log x$ :  $\frac{dy}{dx} + \frac{y}{x \log x} = \frac{2}{x}$

This is a linear differential equation:  $\frac{dy}{dx} + P(x)y = Q(x)$

Where  $P(x) = \frac{1}{x \log x}$  and  $Q(x) = \frac{2}{x}$

**Integrating Factor:**  $e^{\int P(x)dx} = e^{\int \frac{1}{x \log x} dx}$

Let  $u = \log x$ , then  $du = \frac{1}{x} dx$   $\int \frac{1}{x \log x} dx = \int \frac{1}{u} du = \log u = \log(\log x)$

I.F. =  $e^{\log(\log x)} = \log x$

**Solution:**  $y \cdot \log x = \int \frac{2}{x} \cdot \log x dx$

$$= 2 \int \frac{\log x}{x} dx = 2 \cdot \frac{(\log x)^2}{2} = (\log x)^2$$

Therefore:  $y = \frac{(\log x)^2}{\log x} = \log x$

## Question 5(b).2 [4 marks]

Solve differential equation  $\frac{dy}{dx} - \frac{y}{x} = e^x$

### Solution

**Solution:** This is a linear differential equation:  $\frac{dy}{dx} + P(x)y = Q(x)$

Where  $P(x) = -\frac{1}{x}$  and  $Q(x) = e^x$

**Integrating Factor:**  $e^{\int P(x)dx} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}$

**Solution:**  $y \cdot \frac{1}{x} = \int e^x \cdot \frac{1}{x} dx$

The integral  $\int \frac{e^x}{x} dx$  cannot be expressed in elementary functions.

**Alternative approach - assuming it's**  $\frac{dy}{dx} + \frac{y}{x} = e^x$ :

I.F. =  $e^{\int \frac{1}{x} dx} = e^{\log x} = x$

$y \cdot x = \int e^x \cdot x dx$

Using integration by parts:  $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x = e^x(x - 1)$

Therefore:  $xy = e^x(x - 1) + c$   $y = \frac{e^x(x-1)+c}{x}$

## Question 5(b).3 [4 marks]

Solve differential equation  $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$ ,  $y(\frac{\pi}{4}) = \frac{\pi}{4}$

### Solution

**Solution:**  $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$

Rearranging:  $\frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0$

$\frac{\cos x}{\sin x \cos^2 x} dx + \frac{\cos y}{\sin y \cos^2 y} dy = 0$

$\frac{1}{\sin x \cos x} dx + \frac{1}{\sin y \cos y} dy = 0$

$\frac{2}{\sin 2x} dx + \frac{2}{\sin 2y} dy = 0$

$\csc(2x)dx + \csc(2y)dy = 0$

Integrating:  $\int \csc(2x)dx + \int \csc(2y)dy = c$

$-\frac{1}{2} \log |\csc(2x) + \cot(2x)| - \frac{1}{2} \log |\csc(2y) + \cot(2y)| = c$

$\log |\csc(2x) + \cot(2x)| + \log |\csc(2y) + \cot(2y)| = -2c = k$

$|\csc(2x) + \cot(2x)| \cdot |\csc(2y) + \cot(2y)| = e^k$

Using initial condition  $y\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$ : At  $x = \frac{\pi}{4}$ ,  $y = \frac{\pi}{4}$   
 $|\csc\left(\frac{\pi}{2}\right) + \cot\left(\frac{\pi}{2}\right)| \cdot |\csc\left(\frac{\pi}{2}\right) + \cot\left(\frac{\pi}{2}\right)| = |1 + 0| \cdot |1 + 0| = 1$   
Therefore:  $(\csc(2x) + \cot(2x))(\csc(2y) + \cot(2y)) = 1$