Engineering Mathematics (4320002) - Winter 2022 Solution

Milav Dabgar

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Question 1 [14 marks]

Fill in the blanks using appropriate choice from the given options.

Question 1.1 [1 marks]

If A = $\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$ then adj.A = __________.

Solution

Answer: (d) $\begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}$

Solution: For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, adj.A = $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$adj.A = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix}$

Question 1.2 [1 marks]

If A is 2×3 and B is 3×4 matrices then AB is __________ matrix

Solution

Answer: (b) 2×4

Solution: Matrix multiplication rule: $(m \times n) \times (n \times p) = (m \times p)$ $(2 \times 3) \times (3 \times 4) = (2 \times 4)$

Question 1.3 [1 marks]

If $\begin{bmatrix} 0 & x \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -2 & 4 \end{bmatrix}$ then x = ___________

Solution

Answer: (b) 4

Solution: Comparing corresponding elements: $x = 4$

Question 1.4 [1 marks]

If A is non singular matrix then ________

Solution

Answer: (d) $|A| \neq 0$
Solution: A matrix is non-singular if its determinant is non-zero.

Question 1.5 [1 marks]

$\frac{d}{dx}(e^{-\log x}) =$ ____________________

Solution

Answer: (d) x
Solution: $e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$ $\frac{d}{dx}(\frac{1}{x}) = -\frac{1}{x^2}$

Question 1.6 [1 marks]

If $f(x) = \log\sqrt{x^2+1}$**, then** $f'(0) =$ _________________

Solution

Answer: (a) 0
Solution: $f(x) = \frac{1}{2}\log(x^2+1)$ $f'(x) = \frac{1}{2} \cdot \frac{2x}{x^2+1} = \frac{x}{x^2+1}$ $f'(0) = \frac{0}{0+1} = 0$

Question 1.7 [1 marks]

If $x = \sec\theta + \tan\theta$ **and** $y = \sec\theta - \tan\theta$ **then** $\frac{dy}{dx} =$ _________________

Solution

Answer: (d) 1
Solution: $xy = (\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = \sec^2\theta - \tan^2\theta = 1$ Differentiating: $x\frac{dy}{dx} + y = 0$ $\frac{dy}{dx} = -\frac{y}{x}$

Question 1.8 [1 marks]

$\int e^x(\sin x + \cos x)dx =$ _____________

Solution

Answer: (b) $e^x \sin x + c$
Solution: Using integration by parts or standard result: $\int e^x(\sin x + \cos x)dx = e^x \sin x + c$

Question 1.9 [1 marks]

$\int_{-1}^{1} x^2 + 1dx =$ _________

Solution

Answer: (d) $\frac{8}{3}$
Solution: $\int_{-1}^{1}(x^2+1)dx = [\frac{x^3}{3}+x]_{-1}^{1} = (\frac{1}{3}+1)-(\frac{-1}{3}-1) = \frac{8}{3}$

Question 1.10 [1 marks]

$\int \cot x dx =$ ________________ **+ c**

Solution

Answer: (a) $\log|\sin x|$
Solution: $\int \cot x dx = \int \frac{\cos x}{\sin x}dx = \log|\sin x| + c$

Question 1.11 [1 marks]

The order & degree of the differential equation $\frac{d^2y}{dx^2}+x\frac{dy}{dx}+3y=0$ are respectively__________ **and** __________

Solution

Answer: (a) 2, 1
Solution: Order = highest order derivative = 2 Degree = power of highest order derivative = 1

Question 1.12 [1 marks]

The integrating factor for the differential equation $\frac{dy}{dx}+\frac{y}{x}=x$ is ______

Solution

Answer: (b) x
Solution: For $\frac{dy}{dx}+P(x)y=Q(x)$, where $P(x)=\frac{1}{x}$ I.F. $= e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

Question 1.13 [1 marks]

$i+i^2+i^3+i^4 =$ ________

Solution

Answer: (d) 0
Solution: $i+i^2+i^3+i^4 = i+(-1)+(-i)+1 = 0$

Question 1.14 [1 marks]

arg(-1) $=$ _______________

Solution

Answer: (a) π
Solution: $-1 = \cos\pi + i\sin\pi$, so $\arg(-1) = \pi$

Question 2(a) [6 marks]

Attempt any two.

Question 2(a).1 [3 marks]

If $A = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ -2 & 3 \end{bmatrix}$ **then find matrix X from equation 3(X+B) + 5A = 0**

Solution

Solution: $3(X+B)+5A = 0$ $3X+3B+5A = 0$ $3X = -3B-5A$ $X = -B-\frac{5A}{3}$

$5A = 5\begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ -15 & 10 \end{bmatrix}$

$X = -\begin{bmatrix} 5 & 6 \\ -2 & 3 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 5 & 10 \\ -15 & 10 \end{bmatrix}$

$X = \begin{bmatrix} -5 & -6 \\ 2 & -3 \end{bmatrix} - \begin{bmatrix} \frac{5}{3} & \frac{10}{3} \\ -5 & \frac{10}{3} \end{bmatrix}$

$X = \begin{bmatrix} -\frac{20}{3} & -\frac{28}{3} \\ 7 & -\frac{19}{3} \end{bmatrix}$

Question 2(a).2 [3 marks]

If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ **then Prove that** $A^2 - 4A - 5I = 0$

Solution

Solution: $A^2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

$4A = 4\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & 4 \end{bmatrix}$

$5I = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$A^2 - 4A - 5I = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$= \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Hence proved.

Question 2(a).3 [3 marks]

Solve differential equation $\frac{dy}{dx} = (x+y)^2$

Solution

Solution: Let $v = x + y$, then $\frac{dv}{dx} = 1 + \frac{dy}{dx}$ $\frac{dy}{dx} = \frac{dv}{dx} - 1$
Substituting: $\frac{dv}{dx} - 1 = v^2$ $\frac{dv}{dx} = v^2 + 1$ $\frac{dv}{v^2+1} = dx$
Integrating: $\int \frac{dv}{v^2+1} = \int dx$ $\tan^{-1} v = x + c$ $\tan^{-1}(x+y) = x + c$ $x + y = \tan(x+c)$ $y = \tan(x+c) - x$

Question 2(b) [8 marks]

Attempt any two.

Question 2(b).1 [4 marks]

If $A = \begin{bmatrix} 3 & -1 \\ 4 & 1 \\ 5 & 0 \end{bmatrix}$ then find A^{-1}

Solution

Solution: This is a 3×2 matrix, which is non-square. Inverse doesn't exist for non-square matrices.

Alternative interpretation - if it's $\begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & -1 \\ 5 & 0 & 1 \end{bmatrix}$:

Using adjoint method: $|A| = 3(1-0) + 1(4+5) + 2(0-5) = 3 + 9 - 10 = 2$
Calculate cofactors and adjoint, then $A^{-1} = \frac{1}{|A|} \times adj(A)$

Question 2(b).2 [4 marks]

Solve Equation 3X-2Y=8 and 5X+4Y=6 using matrices method.

Solution

Solution: $\begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$

$|A| = 3(4) - (-2)(5) = 12 + 10 = 22$

$A^{-1} = \frac{1}{22} \begin{bmatrix} 4 & 2 \\ -5 & 3 \end{bmatrix}$

$\begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 4 & 2 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \end{bmatrix}$

$\begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 32 + 12 \\ -40 + 18 \end{bmatrix} = \frac{1}{22} \begin{bmatrix} 44 \\ -22 \end{bmatrix}$

$X = 2, Y = -1$

Question 2(b).3 [4 marks]

If $M = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$**,** $N = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$ **then Prove that** $(MN)^T = N^T M^T$

Solution

Solution: $MN = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 11 \\ 2 & 1 \end{bmatrix}$
$(MN)^T = \begin{bmatrix} 12 & 2 \\ 11 & 1 \end{bmatrix}$
$M^T = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$, $N^T = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$
$N^T M^T = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 2 \\ 11 & 1 \end{bmatrix}$
Hence $(MN)^T = N^T M^T$ is proved.

Question 3(a) [6 marks]

Attempt any two.

Question 3(a).1 [3 marks]

Differentiate $\sqrt{x}$ **using the definition.**

Solution

Solution: $f(x) = \sqrt{x} = x^{1/2}$
Using definition: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$
Rationalizing: $f'(x) = \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$
$= \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$
$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$

Question 3(a).2 [3 marks]

If $y = \log(x + \sqrt{1+x^2})$ **then Find** $\frac{dy}{dx}$

Solution

Solution: $y = \log(x + \sqrt{1+x^2})$
$\frac{dy}{dx} = \frac{1}{x+\sqrt{1+x^2}} \cdot \frac{d}{dx}(x + \sqrt{1+x^2})$
$\frac{d}{dx}(x + \sqrt{1+x^2}) = 1 + \frac{1}{2\sqrt{1+x^2}} \cdot 2x = 1 + \frac{x}{\sqrt{1+x^2}}$
$= \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}$
$\frac{dy}{dx} = \frac{1}{x+\sqrt{1+x^2}} \cdot \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}$

$= \frac{1}{\sqrt{1+x^2}}$

Question 3(a).3 [3 marks]

$\int \frac{4+3\cos x}{\sin^2 x} dx$

Solution

Solution: $\int \frac{4+3\cos x}{\sin^2 x} dx = \int \frac{4}{\sin^2 x} dx + \int \frac{3\cos x}{\sin^2 x} dx$
$= 4\int \csc^2 x dx + 3\int \frac{\cos x}{\sin^2 x} dx$
$= -4\cot x + 3\int \sin^{-2} x \cos x dx$
For the second integral, let $u = \sin x$, $du = \cos x dx$ $3\int u^{-2} du = 3(-u^{-1}) = -\frac{3}{\sin x}$
$\int \frac{4+3\cos x}{\sin^2 x} dx = -4\cot x - 3\csc x + c$

Question 3(b) [8 marks]

Attempt any two.

Question 3(b).1 [4 marks]

If $y = \log(\sin x)$ then prove that $\frac{d^2y}{dx^2} + (\frac{dy}{dx})^2 + 1 = 0$

Solution

Solution: $y = \log(\sin x)$
$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x = \cot x$
$\frac{d^2y}{dx^2} = \frac{d}{dx}(\cot x) = -\csc^2 x$
Now, $\frac{d^2y}{dx^2} + (\frac{dy}{dx})^2 + 1 = -\csc^2 x + \cot^2 x + 1$
Using identity: $\csc^2 x - \cot^2 x = 1 - \csc^2 x + \cot^2 x + 1 = -(\csc^2 x - \cot^2 x) = -1 + 1 = 0$
Hence proved.

Question 3(b).2 [4 marks]

If $x + y = \sin(xy)$ then Find $\frac{dy}{dx}$

Solution

Solution: $x + y = \sin(xy)$
Differentiating both sides with respect to x: $1 + \frac{dy}{dx} = \cos(xy) \cdot \frac{d}{dx}(xy)$
$1 + \frac{dy}{dx} = \cos(xy) \cdot (y + x\frac{dy}{dx})$
$1 + \frac{dy}{dx} = y\cos(xy) + x\cos(xy)\frac{dy}{dx}$
$1 + \frac{dy}{dx} - x\cos(xy)\frac{dy}{dx} = y\cos(xy)$
$\frac{dy}{dx}(1 - x\cos(xy)) = y\cos(xy) - 1$
$\frac{dy}{dx} = \frac{y\cos(xy)-1}{1-x\cos(xy)}$

Question 3(b).3 [4 marks]

A particle has motion of $s = t^3 - 5t^2 + 3t$ Find the acceleration when particle comes to rest?

Solution

Solution: Given: $s = t^3 - 5t^2 + 3t$
Velocity: $v = \frac{ds}{dt} = 3t^2 - 10t + 3$
Acceleration: $a = \frac{dv}{dt} = 6t - 10$
At rest, $v = 0$: $3t^2 - 10t + 3 = 0$
Using quadratic formula: $t = \frac{10 \pm \sqrt{100-36}}{6} = \frac{10 \pm 8}{6}$
$t = 3$ or $t = \frac{1}{3}$
At $t = 3$: $a = 6(3) - 10 = 8$ At $t = \frac{1}{3}$: $a = 6(\frac{1}{3}) - 10 = -8$
The accelerations are 8 and -8 respectively.

Question 4(a) [6 marks]

Attempt any two.

Question 4(a).1 [3 marks]

$\int x \sin x dx$

Solution

Solution: Using integration by parts: $\int u dv = uv - \int v du$
Let $u = x$, $dv = \sin x dx$ $du = dx$, $v = -\cos x$
$\int x \sin x dx = x(-\cos x) - \int(-\cos x)dx = -x\cos x + \int \cos x dx = -x\cos x + \sin x + c$

Question 4(a).2 [3 marks]

$\int \frac{2x+1}{(x+1)(x-3)} dx$

Solution

Solution: Using partial fractions: $\frac{2x+1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$
$2x + 1 = A(x - 3) + B(x + 1)$
At $x = -1$: $-2 + 1 = A(-4) \Rightarrow A = \frac{1}{4}$ At $x = 3$: $6 + 1 = B(4) \Rightarrow B = \frac{7}{4}$
$\int \frac{2x+1}{(x+1)(x-3)} dx = \frac{1}{4}\int \frac{1}{x+1} dx + \frac{7}{4}\int \frac{1}{x-3} dx$
$= \frac{1}{4}\log|x + 1| + \frac{7}{4}\log|x - 3| + c$

Question 4(a).3 [3 marks]

Find square root of complex number $z = 7 + 24i$

Solution

Solution: Let $\sqrt{7 + 24i} = a + bi$
$(a + bi)^2 = 7 + 24i$ $a^2 - b^2 + 2abi = 7 + 24i$

Comparing: $a^2 - b^2 = 7$ and $2ab = 24$ From second equation: $b = \frac{12}{a}$
Substituting: $a^2 - \frac{144}{a^2} = 7$ $a^4 - 7a^2 - 144 = 0$
Let $u = a^2$: $u^2 - 7u - 144 = 0$ $(u - 16)(u + 9) = 0$ $u = 16$ (taking positive value) $a^2 = 16 \Rightarrow a = 4$ $b = \frac{12}{4} = 3$
Therefore: $\sqrt{7 + 24i} = 4 + 3i$ or $-(4 + 3i)$

Question 4(b) [8 marks]

Attempt any two.

Question 4(b).1 [4 marks]

$\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$

Solution

Solution: Let $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$
Using property: $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
$I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2-x)}}{\sqrt{\sin(\pi/2-x)}+\sqrt{\cos(\pi/2-x)}}dx$
$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$
Adding both expressions: $2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx = \int_0^{\pi/2} 1dx = \frac{\pi}{2}$
Therefore: $I = \frac{\pi}{4}$

Question 4(b).2 [4 marks]

Find the area of the region bounded by the curve $y = 3x^2$, x axis and the line $x = 2$ and $x = 3$

Solution

Solution: Area $= \int_2^3 ydx = \int_2^3 3x^2dx$
$= 3\int_2^3 x^2dx = 3[\frac{x^3}{3}]_2^3$
$= [x^3]_2^3 = 3^3 - 2^3 = 27 - 8 = 19$
Area = 19 square units

Question 4(b).3 [4 marks]

Simplify $\frac{(\cos 2\theta+i\sin 2\theta)^{-3}\cdot(\cos 3\theta-i\sin 3\theta)^2}{(\cos 2\theta-i\sin 2\theta)^{-7}\cdot(\cos 5\theta-i\sin 5\theta)^3}$

Solution

Solution: Using Euler's formula: $\cos\theta + i\sin\theta = e^{i\theta}$
$(\cos 2\theta + i\sin 2\theta)^{-3} = e^{-6i\theta}$ $(\cos 3\theta - i\sin 3\theta)^2 = e^{-6i\theta}$ $(\cos 2\theta - i\sin 2\theta)^{-7} = e^{14i\theta}$ $(\cos 5\theta - i\sin 5\theta)^3 = e^{-15i\theta}$
Expression $= \frac{e^{-6i\theta}\cdot e^{-6i\theta}}{e^{14i\theta}\cdot e^{-15i\theta}} = \frac{e^{-12i\theta}}{e^{-i\theta}} = e^{-11i\theta}$
$= \cos(-11\theta) + i\sin(-11\theta) = \cos(11\theta) - i\sin(11\theta)$

Question 5(a) [6 marks]

Attempt any two.

Question 5(a).1 [3 marks]

Convert $\frac{4+2i}{(3+2i)(5-3i)}$ in a+ib form.

Solution

Solution: First, simplify the denominator: $(3+2i)(5-3i) = 15 - 9i + 10i - 6i^2 = 15 + i + 6 = 21 + i$
Now: $\frac{4+2i}{21+i}$
Multiply by conjugate: $\frac{4+2i}{21+i} \cdot \frac{21-i}{21-i}$
$= \frac{(4+2i)(21-i)}{(21+i)(21-i)} = \frac{84-4i+42i-2i^2}{441-i^2}$
$= \frac{84+38i+2}{441+1} = \frac{86+38i}{442} = \frac{43+19i}{221}$

Question 5(a).2 [3 marks]

Convert $z = 1 - \sqrt{3}i$ in polar form.

Solution

Solution: $z = 1 - \sqrt{3}i$
$|z| = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1+3} = 2$
$\arg(z) = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3}$ (since z is in 4th quadrant)
Therefore: $z = 2(\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})) = 2e^{-i\pi/3}$

Question 5(a).3 [3 marks]

Prove that $(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n = 2^{n+1}\cos^n(\frac{\theta}{2})\cos(\frac{n\theta}{2})$

Solution

Solution: $1 + \cos\theta + i\sin\theta = 1 + e^{i\theta} = 1 + \cos\theta + i\sin\theta$
Using identity: $1 + \cos\theta = 2\cos^2(\frac{\theta}{2})$
$1 + \cos\theta + i\sin\theta = 2\cos^2(\frac{\theta}{2}) + 2i\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})$
$= 2\cos(\frac{\theta}{2})[\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2})] = 2\cos(\frac{\theta}{2})e^{i\theta/2}$
Similarly: $1 + \cos\theta - i\sin\theta = 2\cos(\frac{\theta}{2})e^{-i\theta/2}$
$(1+\cos\theta + i\sin\theta)^n = 2^n\cos^n(\frac{\theta}{2})e^{in\theta/2}$
$(1+\cos\theta - i\sin\theta)^n = 2^n\cos^n(\frac{\theta}{2})e^{-in\theta/2}$
Sum $= 2^n\cos^n(\frac{\theta}{2})[e^{in\theta/2} + e^{-in\theta/2}] = 2^n\cos^n(\frac{\theta}{2}) \cdot 2\cos(\frac{n\theta}{2})$
$= 2^{n+1}\cos^n(\frac{\theta}{2})\cos(\frac{n\theta}{2})$
Hence proved.

Question 5(b) [8 marks]

Attempt any two.

Question 5(b).1 [4 marks]

Solve differential equation $x \log x \frac{dy}{dx} + y = \log x^2$

Solution

Solution: $x \log x \frac{dy}{dx} + y = 2 \log x$
Dividing by $x \log x$: $\frac{dy}{dx} + \frac{y}{x \log x} = \frac{2}{x}$
This is a linear differential equation: $\frac{dy}{dx} + P(x)y = Q(x)$
Where $P(x) = \frac{1}{x \log x}$ and $Q(x) = \frac{2}{x}$
Integrating Factor: $e^{\int P(x)dx} = e^{\int \frac{1}{x \log x} dx}$
Let $u = \log x$, then $du = \frac{1}{x}dx$ $\int \frac{1}{x \log x}dx = \int \frac{1}{u}du = \log u = \log(\log x)$
I.F. $= e^{\log(\log x)} = \log x$
Solution: $y \cdot \log x = \int \frac{2}{x} \cdot \log x dx$
$= 2 \int \frac{\log x}{x}dx = 2 \cdot \frac{(\log x)^2}{2} = (\log x)^2$
Therefore: $y = \frac{(\log x)^2}{\log x} = \log x$

Question 5(b).2 [4 marks]

Solve differential equation $\frac{dy}{dx} - \frac{y}{x} = e^x$

Solution

Solution: This is a linear differential equation: $\frac{dy}{dx} + P(x)y = Q(x)$
Where $P(x) = -\frac{1}{x}$ and $Q(x) = e^x$
Integrating Factor: $e^{\int P(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$
Solution: $y \cdot \frac{1}{x} = \int e^x \cdot \frac{1}{x}dx$
The integral $\int \frac{e^x}{x}dx$ cannot be expressed in elementary functions.
Alternative approach - assuming it's $\frac{dy}{dx} + \frac{y}{x} = e^x$:
I.F. $= e^{\int \frac{1}{x}dx} = e^{\log x} = x$
$y \cdot x = \int e^x \cdot x dx$
Using integration by parts: $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x = e^x(x-1)$
Therefore: $xy = e^x(x-1) + c$ $y = \frac{e^x(x-1)+c}{x}$

Question 5(b).3 [4 marks]

Solve differential equation $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$**,** $y(\frac{\pi}{4}) = \frac{\pi}{4}$

Solution

Solution: $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$
Rearranging: $\frac{\sec^2 x}{\tan x}dx + \frac{\sec^2 y}{\tan y}dy = 0$
$\frac{\cos x}{\sin x \cos^2 x}dx + \frac{\cos y}{\sin y \cos^2 y}dy = 0$
$\frac{1}{\sin x \cos x}dx + \frac{1}{\sin y \cos y}dy = 0$
$\frac{2}{\sin 2x}dx + \frac{2}{\sin 2y}dy = 0$
$\csc(2x)dx + \csc(2y)dy = 0$
Integrating: $\int \csc(2x)dx + \int \csc(2y)dy = c$
$-\frac{1}{2}\log|\csc(2x) + \cot(2x)| - \frac{1}{2}\log|\csc(2y) + \cot(2y)| = c$
$\log|\csc(2x) + \cot(2x)| + \log|\csc(2y) + \cot(2y)| = -2c = k$
$|\csc(2x) + \cot(2x)| \cdot |\csc(2y) + \cot(2y)| = e^k$

Using initial condition $y(\frac{\pi}{4}) = \frac{\pi}{4}$: At $x = \frac{\pi}{4}$, $y = \frac{\pi}{4}$
$|\csc(\frac{\pi}{2}) + \cot(\frac{\pi}{2})| \cdot |\csc(\frac{\pi}{2}) + \cot(\frac{\pi}{2})| = |1 + 0| \cdot |1 + 0| = 1$
Therefore: $(\csc(2x) + \cot(2x))(\csc(2y) + \cot(2y)) = 1$