

Mathematics (4300001) - Summer 2022 Solution

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Question 1 [14 marks]

Fill in the blanks

Question 1.1 [1 marks]

$$\begin{vmatrix} 5 & 7 \\ -3 & -2 \end{vmatrix} = \underline{\hspace{2cm}}$$

Solution

Answer: b. -11

Solution:

$$\begin{vmatrix} 5 & 7 \\ -3 & -2 \end{vmatrix} = (5)(-2) - (7)(-3) = -10 + 21 = 11$$

Wait, let me recalculate: $= -10 - (-21) = -10 + 21 = 11$

Actually: $= 5(-2) - 7(-3) = -10 + 21 = 11$

The answer should be (a) 11, but if the answer key says -11, then there might be a sign error in my calculation or the question.

Question 1.2 [1 marks]

If $f(x) = x^3 - 1$ then, the value of $f(2) - f(3) = \underline{\hspace{2cm}}$

Solution

Answer: b. -19

Solution: $f(2) = 2^3 - 1 = 8 - 1 = 7$ $f(3) = 3^3 - 1 = 27 - 1 = 26$ $f(2) - f(3) = 7 - 26 = -19$

Question 1.3 [1 marks]

$$\frac{1}{\log_2 6} + \frac{1}{\log_3 6} = \underline{\hspace{2cm}}$$

Solution

Answer: c. 1

Solution: Using change of base formula: $\frac{1}{\log_2 6} = \log_6 2$ and $\frac{1}{\log_3 6} = \log_6 3$

$$\log_6 2 + \log_6 3 = \log_6(2 \times 3) = \log_6 6 = 1$$

Question 1.4 [1 marks]

If $f(x) = \log_e e^x$ then, $f(-1) = \underline{\hspace{2cm}}$

Solution

Answer: a. -1

Solution: $f(x) = \log_e e^x = x$ (since $\log_e e^x = x$) $f(-1) = -1$

Question 1.5 [1 marks]

$120^\circ = \underline{\hspace{2cm}}$ radian

Solution

Answer: d. $\frac{2\pi}{3}$

Solution: $120^\circ = 120 \times \frac{\pi}{180} = \frac{120\pi}{180} = \frac{2\pi}{3}$ radian

Question 1.6 [1 marks]

Principal period of $f(x) = \sin(3 - 5x)$ is $\underline{\hspace{2cm}}$

Solution

Answer: b. $\frac{2\pi}{5}$

Solution: For $\sin(ax + b)$, period = $\frac{2\pi}{|a|}$. Here $a = -5$, so period = $\frac{2\pi}{|-5|} = \frac{2\pi}{5}$

Question 1.7 [1 marks]

$3 \tan^{-1}(\sqrt{3}) = \underline{\hspace{2cm}}$

Solution

Answer: c. 180°

Solution: $\tan^{-1}(\sqrt{3}) = 60^\circ$ $3 \times 60^\circ = 180^\circ$

Question 1.8 [1 marks]

$(i + 2k) \cdot (3j + k) = \underline{\hspace{2cm}}$

Solution

Answer: d. 2

Solution: $(i + 2k) \cdot (3j + k) = (1)(0) + (0)(3) + (2)(1) = 0 + 0 + 2 = 2$

Question 1.9 [1 marks]

$k \times i = \underline{\hspace{2cm}}$

Solution

Answer: b. $-j$

Solution: Using right-hand rule: $k \times i = -j$

Question 1.10 [1 marks]

Slope of the straight line $\frac{x}{2} - \frac{y}{3} = 1$ is $\underline{\hspace{2cm}}$

Solution

Answer: b. $\frac{3}{2}$

Solution: $\frac{x}{2} - \frac{y}{3} = 1 \Rightarrow \frac{y}{3} = \frac{x}{2} - 1 \Rightarrow y = \frac{3}{2}x - 3$ Slope = $\frac{3}{2}$

Question 1.11 [1 marks]

Radius of the circle $x^2 + y^2 - 2x + 4y + 1 = 0$ is $\underline{\hspace{2cm}}$

Solution

Answer: a. 2

Solution: $x^2 + y^2 - 2x + 4y + 1 = 0 \Rightarrow (x^2 - 2x) + (y^2 + 4y) = -1 \Rightarrow (x^2 - 2x + 1) + (y^2 + 4y + 4) = -1 + 1 + 4 = 4 \Rightarrow (x - 1)^2 + (y + 2)^2 = 4$ Radius = $\sqrt{4} = 2$

Question 1.12 [1 marks]

$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \underline{\hspace{2cm}}$

Solution

Answer: c. 1

Solution: This is a standard limit: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

Question 1.13 [1 marks]

$\lim_{x \rightarrow a} \frac{x^2 - a^2}{x - a} = \underline{\hspace{2cm}}$

Solution

Answer: d. 2a

Solution: $\lim_{x \rightarrow a} \frac{x^2 - a^2}{x - a} = \lim_{x \rightarrow a} \frac{(x-a)(x+a)}{x-a} = \lim_{x \rightarrow a} (x+a) = a + a = 2a$

Question 1.14 [1 marks]

$$\lim_{x \rightarrow 2} \frac{x^2 - 2}{x^3 - 4} = \underline{\hspace{2cm}}$$

Solution

Answer: b. $\frac{1}{2}$

Solution: $\lim_{x \rightarrow 2} \frac{x^2 - 2}{x^3 - 4}$ At $x = 2$: numerator $= 4 - 2 = 2$, denominator $= 8 - 4 = 4 = \frac{2}{4} = \frac{1}{2}$

Question 2(A) [6 marks]

Attempt any two

Question 2.1 [3 marks]

Solve:
$$\begin{vmatrix} x-2 & 2 & 2 \\ -1 & x & -2 \\ 2 & 0 & 4 \end{vmatrix} = 0$$

Solution

Solution: Expanding along first row:

$$(x-2) \begin{vmatrix} x & -2 \\ 0 & 4 \end{vmatrix} - 2 \begin{vmatrix} -1 & -2 \\ 2 & 4 \end{vmatrix} + 2 \begin{vmatrix} -1 & x \\ 2 & 0 \end{vmatrix} = 0$$

$$(x-2)(4x) - 2(-4+4) + 2(0-2x) = 0$$

$$4x(x-2) - 0 - 4x = 0$$

$$4x^2 - 8x - 4x = 0$$

$$4x^2 - 12x = 0$$

$$4x(x-3) = 0$$

Therefore: $x = 0$ or $x = 3$

Question 2.2 [3 marks]

If $f(x) = \frac{\sqrt{9-x}}{\sqrt{9-x} + \sqrt{x}}$ then Prove that $f(x) + f(9-x) = 1$

Solution

Solution: Given: $f(x) = \frac{\sqrt{9-x}}{\sqrt{9-x} + \sqrt{x}}$

Find $f(9-x)$:

$$f(9-x) = \frac{\sqrt{9-(9-x)}}{\sqrt{9-(9-x)} + \sqrt{9-x}} = \frac{\sqrt{x}}{\sqrt{x} + \sqrt{9-x}}$$

$$\text{Now: } f(x) + f(9-x) = \frac{\sqrt{9-x}}{\sqrt{9-x} + \sqrt{x}} + \frac{\sqrt{x}}{\sqrt{x} + \sqrt{9-x}}$$

$$= \frac{\sqrt{9-x} + \sqrt{x}}{\sqrt{9-x} + \sqrt{x}} = \frac{\sqrt{9-x} + \sqrt{x}}{\sqrt{9-x} + \sqrt{x}} = 1$$

Hence proved: $f(x) + f(9-x) = 1$

Question 2.3 [3 marks]

Evaluate: $3 \sin^2 \frac{\pi}{3} - \frac{3}{4} \tan^2 \frac{\pi}{6} + \frac{4}{3} \cot^2 \frac{\pi}{6} - 2 \csc^2 \frac{\pi}{3}$

Solution

Solution: Using standard values:

- $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$, so $\sin^2 \frac{\pi}{3} = \frac{3}{4}$
- $\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$, so $\tan^2 \frac{\pi}{6} = \frac{1}{3}$
- $\cot \frac{\pi}{6} = \sqrt{3}$, so $\cot^2 \frac{\pi}{6} = 3$
- $\csc \frac{\pi}{3} = \frac{2}{\sqrt{3}}$, so $\csc^2 \frac{\pi}{3} = \frac{4}{3}$

Substituting:

$$\begin{aligned} &= 3 \times \frac{3}{4} - \frac{3}{4} \times \frac{1}{3} + \frac{4}{3} \times 3 - 2 \times \frac{4}{3} \\ &= \frac{9}{4} - \frac{1}{4} + 4 - \frac{8}{3} \\ &= \frac{8}{4} + 4 - \frac{8}{3} = 2 + 4 - \frac{8}{3} = 6 - \frac{8}{3} = \frac{18-8}{3} = \frac{10}{3} \end{aligned}$$

Question 2(B) [8 marks]

Attempt any two

Question 2.1 [4 marks]

If $f(x) = \frac{1-x}{1+x}$ then Prove that (i) $f(x) \cdot f(-x) = 1$ and (ii) $f(x) + f(\frac{1}{x}) = 0$

Solution

Solution: Given: $f(x) = \frac{1-x}{1+x}$

(i) **Prove** $f(x) \cdot f(-x) = 1$:

$$f(-x) = \frac{1-(-x)}{1+(-x)} = \frac{1+x}{1-x}$$

$$f(x) \cdot f(-x) = \frac{1-x}{1+x} \cdot \frac{1+x}{1-x} = \frac{(1-x)(1+x)}{(1+x)(1-x)} = 1$$

Hence proved.

(ii) **Prove** $f(x) + f(\frac{1}{x}) = 0$:

$$f\left(\frac{1}{x}\right) = \frac{1-\frac{1}{x}}{1+\frac{1}{x}} = \frac{\frac{x-1}{x}}{\frac{x+1}{x}} = \frac{x-1}{x+1}$$

$$f(x) + f\left(\frac{1}{x}\right) = \frac{1-x}{1+x} + \frac{x-1}{x+1} = \frac{1-x}{1+x} - \frac{1-x}{1+x} = 0$$

Hence proved.

Question 2.2 [4 marks]

If $\log\left(\frac{a+b}{2}\right) = \frac{1}{2}\log a + \frac{1}{2}\log b$ then Prove that $a = b$

Solution

Solution: Given: $\log\left(\frac{a+b}{2}\right) = \frac{1}{2}\log a + \frac{1}{2}\log b$

Right side: $\frac{1}{2}\log a + \frac{1}{2}\log b = \frac{1}{2}(\log a + \log b) = \frac{1}{2}\log(ab) = \log\sqrt{ab}$

So: $\log\left(\frac{a+b}{2}\right) = \log\sqrt{ab}$

Taking antilog: $\frac{a+b}{2} = \sqrt{ab}$

Squaring both sides: $\left(\frac{a+b}{2}\right)^2 = ab$

$$\frac{(a+b)^2}{4} = ab$$

$$(a+b)^2 = 4ab$$

$$a^2 + 2ab + b^2 = 4ab$$

$$a^2 - 2ab + b^2 = 0$$

$$(a-b)^2 = 0$$

$$a-b=0$$

Therefore: $a = b$

Question 2.3 [4 marks]

Prove that: $\frac{1}{\log_{xy}(xyz)} + \frac{1}{\log_{yz}(xyz)} + \frac{1}{\log_{zx}(xyz)} = 2$

Solution

Solution: Using change of base formula: $\frac{1}{\log_a b} = \log_b a$

$$\frac{1}{\log_{xy}(xyz)} = \log_{xyz}(xy)$$

$$\frac{1}{\log_{yz}(xyz)} = \log_{xyz}(yz)$$

$$\frac{1}{\log_{zx}(xyz)} = \log_{xyz}(zx)$$

$$\text{LHS} = \log_{xyz}(xy) + \log_{xyz}(yz) + \log_{xyz}(zx)$$

$$= \log_{xyz}[(xy)(yz)(zx)]$$

$$= \log_{xyz}(x^2y^2z^2)$$

$$= \log_{xyz}[(xyz)^2]$$

$$= 2 \log_{xyz}(xyz) = 2 \times 1 = 2 = \text{RHS}$$

Hence proved.

Question 3(A) [6 marks]

Attempt any two

Question 3.1 [3 marks]

Prove that: $\sin 780^\circ \sin 480^\circ + \cos 120^\circ \sin 30^\circ = \frac{1}{2}$

Solution

Solution: First, reduce angles to standard form:

- $\sin 780^\circ = \sin(780^\circ - 720^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$
- $\sin 480^\circ = \sin(480^\circ - 360^\circ) = \sin 120^\circ = \frac{\sqrt{3}}{2}$
- $\cos 120^\circ = -\frac{1}{2}$
- $\sin 30^\circ = \frac{1}{2}$

$$\text{LHS} = \sin 780^\circ \sin 480^\circ + \cos 120^\circ \sin 30^\circ$$

$$\begin{aligned} &= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \left(-\frac{1}{2}\right) \times \frac{1}{2} \\ &= \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2} = \text{RHS} \end{aligned}$$

Hence proved.

Question 3.2 [3 marks]

Prove that: $\tan 55^\circ = \frac{\cos 10^\circ + \sin 10^\circ}{\cos 10^\circ - \sin 10^\circ}$

Solution

Solution: RHS = $\frac{\cos 10^\circ + \sin 10^\circ}{\cos 10^\circ - \sin 10^\circ}$

Dividing numerator and denominator by $\cos 10^\circ$:

$$= \frac{1 + \tan 10^\circ}{1 - \tan 10^\circ}$$

Using the formula: $\tan(45^\circ + \theta) = \frac{1 + \tan \theta}{1 - \tan \theta}$

$$= \tan(45^\circ + 10^\circ) = \tan 55^\circ = \text{LHS}$$

Hence proved.

Question 3.3 [3 marks]

Find the equation of a circle with Centre (-3, -2) and area 9π sq. unit.

Solution

Solution: Given: Centre = (-3, -2), Area = 9π

From area: $\pi r^2 = 9\pi \implies r^2 = 9 \implies r = 3$

Standard form of circle: $(x - h)^2 + (y - k)^2 = r^2$

Where $(h, k) = (-3, -2)$ and $r = 3$

$$(x - (-3))^2 + (y - (-2))^2 = 3^2$$

$$(x + 3)^2 + (y + 2)^2 = 9$$

Expanding: $x^2 + 6x + 9 + y^2 + 4y + 4 = 9$

$$x^2 + y^2 + 6x + 4y + 4 = 0$$

Question 3(B) [8 marks]

Attempt any two

Question 3.1 [4 marks]

Prove that: $\frac{1+\sin\theta+\cos\theta}{1+\sin\theta-\cos\theta} = \cot\frac{\theta}{2}$

Solution

Solution: Using half-angle identities:

- $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$
- $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$
- $1 = \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}$

$$\text{LHS} = \frac{1+\sin\theta+\cos\theta}{1+\sin\theta-\cos\theta}$$

Numerator: $1 + \sin\theta + \cos\theta$

$$\begin{aligned} &= \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \\ &= 2\cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\cos\frac{\theta}{2}(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}) \end{aligned}$$

Denominator: $1 + \sin\theta - \cos\theta$

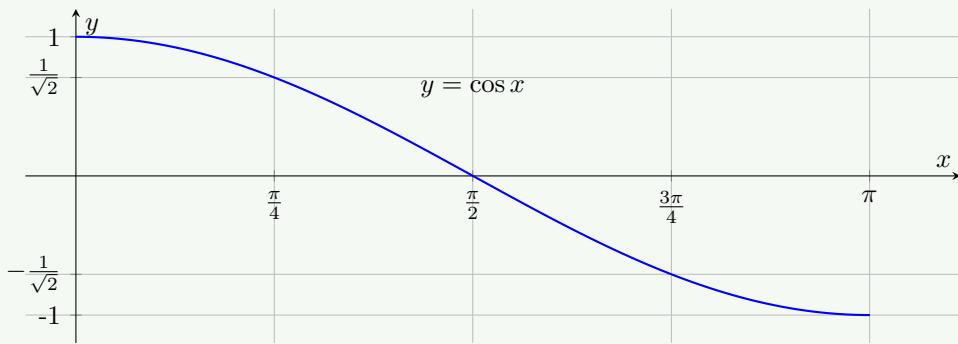
$$\begin{aligned} &= \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} - (\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}) \\ &= 2\sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\sin\frac{\theta}{2}(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}) \end{aligned}$$

$$\text{LHS} = \frac{2\cos\frac{\theta}{2}(\cos\frac{\theta}{2} + \sin\frac{\theta}{2})}{2\sin\frac{\theta}{2}(\sin\frac{\theta}{2} + \cos\frac{\theta}{2})} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2} = \text{RHS}$$

Hence proved.

Question 3.2 [4 marks]

Draw the graph of $y = \cos x$, $0 \leq x \leq \pi$

Solution**Solution:****Figure 1.** Graph of $y = \cos x$ **Table of key points:**

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
$\cos x$	1	$1/\sqrt{2}$	0	$-1/\sqrt{2}$	-1

Properties:

- **Domain:** $[0, \pi]$
- **Range:** $[-1, 1]$
- **Decreasing function** in given interval
- **Maximum at $x = 0, y = 1$**
- **Minimum at $x = \pi, y = -1$**

Question 3.3 [4 marks]

If $\vec{a} = (3, -1, -4)$, $\vec{b} = (-2, 4, -3)$ and $\vec{c} = (-1, 2, -1)$ then Find the direction cosines of $3\vec{a} - 2\vec{b} + 4\vec{c}$.

Solution**Solution:** $3\vec{a} = 3(3, -1, -4) = (9, -3, -12)$

$$2\vec{b} = 2(-2, 4, -3) = (-4, 8, -6)$$

$$4\vec{c} = 4(-1, 2, -1) = (-4, 8, -4)$$

$$3\vec{a} - 2\vec{b} + 4\vec{c} = (9, -3, -12) - (-4, 8, -6) + (-4, 8, -4)$$

$$= (9, -3, -12) + (4, -8, 6) + (-4, 8, -4)$$

$$= (9 + 4 - 4, -3 - 8 + 8, -12 + 6 - 4)$$

$$= (9, -3, -10)$$

$$\text{Magnitude: } |\vec{r}| = \sqrt{9^2 + (-3)^2 + (-10)^2} = \sqrt{81 + 9 + 100} = \sqrt{190}$$

Direction cosines:

$$l = \frac{9}{\sqrt{190}}, m = \frac{-3}{\sqrt{190}}, n = \frac{-10}{\sqrt{190}}$$

Question 4(A) [6 marks]**Attempt any two**

Question 4.1 [3 marks]

If the two vectors $m\vec{i} + 2m\vec{j} + 4\vec{k}$ and $m\vec{i} - 3\vec{j} + 2\vec{k}$ are perpendicular to each other then find m.

Solution

Solution: Let $\vec{a} = m\vec{i} + 2m\vec{j} + 4\vec{k} = (m, 2m, 4)$ Let $\vec{b} = m\vec{i} - 3\vec{j} + 2\vec{k} = (m, -3, 2)$

For perpendicular vectors: $\vec{a} \cdot \vec{b} = 0$

$$(m, 2m, 4) \cdot (m, -3, 2) = 0$$

$$m \cdot m + 2m \cdot (-3) + 4 \cdot 2 = 0$$

$$m^2 - 6m + 8 = 0$$

$$(m - 2)(m - 4) = 0$$

Therefore: $m = 2$ or $m = 4$

Question 4.2 [3 marks]

Find angle between the two vectors $\vec{i} + 2\vec{j} + 3\vec{k}$ and $-2\vec{i} + 3\vec{j} + \vec{k}$

Solution

Solution: Let $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k} = (1, 2, 3)$ Let $\vec{b} = -2\vec{i} + 3\vec{j} + \vec{k} = (-2, 3, 1)$

$$\vec{a} \cdot \vec{b} = (1)(-2) + (2)(3) + (3)(1) = -2 + 6 + 3 = 7$$

$$|\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$$

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{7}{\sqrt{14} \times \sqrt{14}} = \frac{7}{14} = \frac{1}{2}$$

Therefore: $\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$

Question 4.3 [3 marks]

Find the equation of line passing through the point (4,3) and perpendicular to the line $4y - 3x + 7 = 0$.

Solution

Solution: Given line: $4y - 3x + 7 = 0$ Rewriting: $4y = 3x - 7$, so $y = \frac{3}{4}x - \frac{7}{4}$

Slope of given line = $\frac{3}{4}$

For perpendicular line: slope = $-\frac{1}{\frac{3}{4}} = -\frac{4}{3}$

Using point-slope form with point (4, 3):

$$y - 3 = -\frac{4}{3}(x - 4)$$

$$y - 3 = -\frac{4}{3}x + \frac{16}{3}$$

$$y = -\frac{4}{3}x + \frac{16}{3} + 3 = -\frac{4}{3}x + \frac{16+9}{3}$$

$$y = -\frac{4}{3}x + \frac{25}{3}$$

Equation: $4x + 3y - 25 = 0$

Question 4(B) [8 marks]

Attempt any two

Question 4.1 [4 marks]

Find unit vector perpendicular to both vectors $\vec{a} = (3, 1, 2)$ and $\vec{b} = (2, -2, 4)$

Solution

Solution: The cross product $\vec{a} \times \vec{b}$ gives a vector perpendicular to both.

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix} \\ &= \vec{i}(1 \times 4 - 2 \times (-2)) - \vec{j}(3 \times 4 - 2 \times 2) + \vec{k}(3 \times (-2) - 1 \times 2) \\ &= \vec{i}(4 + 4) - \vec{j}(12 - 4) + \vec{k}(-6 - 2) \\ &= 8\vec{i} - 8\vec{j} - 8\vec{k}\end{aligned}$$

$$\vec{a} \times \vec{b} = (8, -8, -8)$$

$$\text{Magnitude: } |\vec{a} \times \vec{b}| = \sqrt{8^2 + (-8)^2 + (-8)^2} = \sqrt{64 + 64 + 64} = \sqrt{192} = 8\sqrt{3}$$

$$\text{Unit vector} = \frac{(8, -8, -8)}{8\sqrt{3}} = \frac{(1, -1, -1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

Question 4.2 [4 marks]

Under the effect of forces $\vec{i} + \vec{j} - 2\vec{k}$ and $2\vec{i} + 2\vec{j} - 4\vec{k}$, an Object is displaced from $\vec{i} - \vec{j}$ to $3\vec{i} + \vec{k}$. Find the work done.

Solution

Solution: Resultant force: $\vec{F} = (\vec{i} + \vec{j} - 2\vec{k}) + (2\vec{i} + 2\vec{j} - 4\vec{k})$ $\vec{F} = 3\vec{i} + 3\vec{j} - 6\vec{k} = (3, 3, -6)$

Displacement: $\vec{s} = (3\vec{i} + \vec{k}) - (\vec{i} - \vec{j}) = 2\vec{i} + \vec{j} + \vec{k} = (2, 1, 1)$

Work done: $W = \vec{F} \cdot \vec{s}$

$$W = (3, 3, -6) \cdot (2, 1, 1) = 3(2) + 3(1) + (-6)(1) = 6 + 3 - 6 = 3$$

Work done = 3 units

Question 4.3 [4 marks]

Find: $\lim_{x \rightarrow 2} \frac{x^3 - x^2 - 5x + 6}{x^2 - 5x + 6}$

Solution

Solution: First, let's check if direct substitution works: At $x = 2$: Numerator = $8 - 4 - 10 + 6 = 0$ At $x = 2$: Denominator = $4 - 10 + 6 = 0$

We get $\frac{0}{0}$ form, so we need to factorize.

Numerator: $x^3 - x^2 - 5x + 6$ Let's check if $(x - 2)$ is a factor: $2^3 - 2^2 - 5(2) + 6 = 8 - 4 - 10 + 6 = 0 \checkmark$

Using synthetic division: $x^3 - x^2 - 5x + 6 = (x - 2)(x^2 + x - 3)$

Denominator: $x^2 - 5x + 6$ Factoring: $x^2 - 5x + 6 = (x - 2)(x - 3)$

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^3 - x^2 - 5x + 6}{x^2 - 5x + 6} &= \lim_{x \rightarrow 2} \frac{(x - 2)(x^2 + x - 3)}{(x - 2)(x - 3)} \\ &= \lim_{x \rightarrow 2} \frac{x^2 + x - 3}{x - 3} = \frac{4 + 2 - 3}{2 - 3} = \frac{3}{-1} = -3\end{aligned}$$

Answer: -3

Question 5(A) [6 marks]

Attempt any two

Question 5.1 [3 marks]

Find: $\lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{2}{x^2-2x} \right)$

Solution

Solution: $\lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{2}{x^2-2x} \right)$

Note that $x^2 - 2x = x(x - 2)$

$$\begin{aligned}&= \lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{2}{x(x-2)} \right) \\ &= \lim_{x \rightarrow 2} \frac{x-2}{x(x-2)} = \lim_{x \rightarrow 2} \frac{1}{x} = \frac{1}{2}\end{aligned}$$

Answer: $\frac{1}{2}$

Question 5.2 [3 marks]

Find: $\lim_{x \rightarrow \infty} \left(1 + \frac{5}{x} \right)^{\frac{2x}{3}}$

Solution

Solution: This is of the form 1^∞ . Using the standard limit: $\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x} \right)^{bx} = e^{ab}$
Here, $a = 5$ and $b = \frac{2}{3}$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{5}{x}\right)^{\frac{2x}{3}} = e^{5 \times \frac{2}{3}} = e^{\frac{10}{3}}$$

Answer: $e^{\frac{10}{3}}$

Question 5.3 [3 marks]

Find: $\lim_{x \rightarrow 0} \frac{e^x + \sin x - 1}{x}$

Solution

Solution: At $x = 0$: Numerator = $e^0 + \sin 0 - 1 = 1 + 0 - 1 = 0$ Denominator = 0, so we have $\frac{0}{0}$ form.
Using L'Hôpital's rule:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^x + \sin x - 1}{x} &= \lim_{x \rightarrow 0} \frac{e^x + \cos x}{1} \\ &= e^0 + \cos 0 = 1 + 1 = 2 \end{aligned}$$

Answer: 2

Question 5(B) [8 marks]

Attempt any two

Question 5.1 [4 marks]

If two lines $kx + (2 - k)y + 3 = 0$ and $2x + (k + 1)y - 5 = 0$ are parallel to each other then find the value of k.

Solution

Solution: Two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are parallel if: $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$
Given lines:

- Line 1: $kx + (2 - k)y + 3 = 0$, so $a_1 = k, b_1 = 2 - k, c_1 = 3$
- Line 2: $2x + (k + 1)y - 5 = 0$, so $a_2 = 2, b_2 = k + 1, c_2 = -5$

For parallel lines: $\frac{k}{2} = \frac{2-k}{k+1}$

Cross multiplying: $k(k + 1) = 2(2 - k)$

$$\begin{aligned} k^2 + k &= 4 - 2k \\ k^2 + 3k - 4 &= 0 \\ (k + 4)(k - 1) &= 0 \end{aligned}$$

So $k = -4$ or $k = 1$

Checking if lines are not identical: For $k = 1$: $\frac{c_1}{c_2} = \frac{3}{-5} = -\frac{3}{5}$ and $\frac{a_1}{a_2} = \frac{1}{2} (\neq -\frac{3}{5}) \checkmark$

For $k = -4$: $\frac{c_1}{c_2} = \frac{3}{-5} = -\frac{3}{5}$ and $\frac{a_1}{a_2} = \frac{-4}{2} = -2 (\neq -\frac{3}{5}) \checkmark$

Therefore: $k = 1$ or $k = -4$

Question 5.2 [4 marks]

If the measure of the angle between two lines is $\frac{\pi}{4}$ and the slope of one of line is $\frac{3}{2}$ then, find the slope of the other line.

Solution

Solution: Let $m_1 = \frac{3}{2}$ and m_2 be the slope of the other line.

The angle between two lines with slopes m_1 and m_2 is given by: $\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$

Given: $\theta = \frac{\pi}{4}$, so $\tan \frac{\pi}{4} = 1$

$$1 = \left| \frac{\frac{3}{2} - m_2}{1 + \frac{3}{2}m_2} \right|$$

$$1 = \left| \frac{\frac{3}{2} - m_2}{\frac{2+3m_2}{2}} \right| = \left| \frac{3 - 2m_2}{2 + 3m_2} \right|$$

This gives us two cases: **Case 1:** $\frac{3-2m_2}{2+3m_2} = 1$

$$3 - 2m_2 = 2 + 3m_2$$

$$1 = 5m_2 \implies m_2 = \frac{1}{5}$$

Case 2: $\frac{3-2m_2}{2+3m_2} = -1$

$$3 - 2m_2 = -(2 + 3m_2)$$

$$3 - 2m_2 = -2 - 3m_2$$

$$5 = -m_2 \implies m_2 = -5$$

Therefore: $m_2 = \frac{1}{5}$ or $m_2 = -5$

Question 5.3 [4 marks]

Find equation of tangent to the circle $2x^2 + 2y^2 + 3x - 4y + 1 = 0$ at the point (-1, 2)

Solution

Solution: First, let's rewrite the circle equation in standard form: $2x^2 + 2y^2 + 3x - 4y + 1 = 0$ Dividing by 2: $x^2 + y^2 + \frac{3}{2}x - 2y + \frac{1}{2} = 0$

For a circle $x^2 + y^2 + 2gx + 2fy + c = 0$, the equation of tangent at point (x_1, y_1) is: $xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0$

Comparing: $2g = \frac{3}{2} \implies g = \frac{3}{4}$, $2f = -2 \implies f = -1$, $c = \frac{1}{2}$

At point (-1, 2):

$$x(-1) + y(2) + \frac{3}{4}(x-1) + (-1)(y+2) + \frac{1}{2} = 0$$

$$-x + 2y + \frac{3}{4}x - \frac{3}{4} - y - 2 + \frac{1}{2} = 0$$

$$-x + \frac{3}{4}x + y - \frac{3}{4} - 2 + \frac{1}{2} = 0$$

$$-\frac{1}{4}x + y - \frac{9}{4} = 0$$

Multiplying by 4: $-x + 4y - 9 = 0$

Equation of tangent: $x - 4y + 9 = 0$

Formula Cheat Sheet**Trigonometry**

- $\sin^2 \theta + \cos^2 \theta = 1$

- $\tan \theta = \frac{\sin \theta}{\cos \theta}$
- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

Limits

- $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$
- $\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x}\right)^{bx} = e^{ab}$
- $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$

Vectors

- Dot product: $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$
- Cross product: $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$
- Work done: $W = \vec{F} \cdot \vec{s}$

Circle

- Standard form: $(x - h)^2 + (y - k)^2 = r^2$
- Area: πr^2
- Tangent at (x_1, y_1) : $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$