

Advanced Mathematical Formatting Test

Matrix Operations

Problem: Matrix Multiplication

If $A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 1 & 7 \end{bmatrix}$ then find AB & BA .

Solution for AB :

$$AB = \begin{bmatrix} 1 & -1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 1 & 7 \end{bmatrix}$$

Calculating each element:

- $(AB)_{11} = (1)(1) + (-1)(4) + (1)(1) = 1 - 4 + 1 = -2$
- $(AB)_{12} = (1)(2) + (-1)(2) + (1)(7) = 2 - 2 + 7 = 7$
- $(AB)_{21} = (3)(1) + (2)(4) + (1)(1) = 3 + 8 + 1 = 12$
- $(AB)_{22} = (3)(2) + (2)(2) + (1)(7) = 6 + 4 + 7 = 17$

Therefore: $AB = \begin{bmatrix} -2 & 7 \\ 12 & 17 \end{bmatrix}$

Solution for BA :

$$BA = \begin{bmatrix} 1 & 2 \\ 4 & 2 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

Calculating each element:

- $(BA)_{11} = (1)(1) + (2)(3) = 1 + 6 = 7$
- $(BA)_{12} = (1)(-1) + (2)(2) = -1 + 4 = 3$
- $(BA)_{13} = (1)(1) + (2)(1) = 1 + 2 = 3$
- $(BA)_{21} = (4)(1) + (2)(3) = 4 + 6 = 10$
- $(BA)_{22} = (4)(-1) + (2)(2) = -4 + 4 = 0$
- $(BA)_{23} = (4)(1) + (2)(1) = 4 + 2 = 6$
- $(BA)_{31} = (1)(1) + (7)(3) = 1 + 21 = 22$
- $(BA)_{32} = (1)(-1) + (7)(2) = -1 + 14 = 13$
- $(BA)_{33} = (1)(1) + (7)(1) = 1 + 7 = 8$

Therefore: $BA = \begin{bmatrix} 7 & 3 & 3 \\ 10 & 0 & 6 \\ 22 & 13 & 8 \end{bmatrix}$

Calculus - Advanced Differentiation

Logarithmic Differentiation

If $y = \log\left(\frac{\cos x}{1+\sin x}\right)$ then find $\frac{dy}{dx}$

Solution:

$$y = \log\left(\frac{\cos x}{1+\sin x}\right)$$

Using logarithm properties:

$$y = \log(\cos x) - \log(1 + \sin x)$$

Differentiating both sides:

$$\frac{dy}{dx} = \frac{d}{dx}[\log(\cos x)] - \frac{d}{dx}[\log(1 + \sin x)]$$

$$\frac{dy}{dx} = \frac{1}{\cos x} \cdot (-\sin x) - \frac{1}{1+\sin x} \cdot \cos x$$

$$\frac{dy}{dx} = -\frac{\sin x}{\cos x} - \frac{\cos x}{1+\sin x}$$

$$\frac{dy}{dx} = -\tan x - \frac{\cos x}{1+\sin x}$$

Simplifying further:

$$\frac{dy}{dx} = -\frac{\sin x(1+\sin x)+\cos^2 x}{\cos x(1+\sin x)}$$

$$= -\frac{\sin x + \sin^2 x + \cos^2 x}{\cos x(1+\sin x)}$$

Since $\sin^2 x + \cos^2 x = 1$:

$$\frac{dy}{dx} = -\frac{\sin x + 1}{\cos x(1+\sin x)} = -\frac{1}{\cos x} = -\sec x$$

Integration Techniques

Integration by Parts

Evaluate: $\int x \log x \, dx$

Solution:

Using integration by parts: $\int u \, dv = uv - \int v \, du$

Let:

$$\bullet \quad u = \log x \Rightarrow du = \frac{1}{x} dx$$

$$\bullet \quad dv = x \, dx \Rightarrow v = \frac{x^2}{2}$$

$$\int x \log x \, dx = \log x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$$

$$= \frac{x^2 \log x}{2} - \int \frac{x}{2} dx$$

$$= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C$$

$$= \frac{x^2}{2} \left(\log x - \frac{1}{2} \right) + C$$

Complex Numbers - Polar Form

De Moivre's Theorem Application

Simplify: $\frac{(\cos 2\theta + i \sin 2\theta)^3 \cdot (\cos 3\theta - i \sin 3\theta)^2}{(\cos 2\theta + i \sin 2\theta)^7 \cdot (\cos 5\theta - i \sin 5\theta)^3}$

Solution:

Using Euler's formula: $\cos \theta + i \sin \theta = e^{i\theta}$ and $\cos \theta - i \sin \theta = e^{-i\theta}$

Numerator:

$$\begin{aligned} & (\cos 2\theta + i \sin 2\theta)^3 \cdot (\cos 3\theta - i \sin 3\theta)^2 = (e^{i2\theta})^3 \cdot (e^{-i3\theta})^2 \\ & = e^{i6\theta} \cdot e^{-i6\theta} = e^{i(6\theta-6\theta)} = e^{i0} = 1 \end{aligned}$$

Denominator:

$$\begin{aligned} & (\cos 2\theta + i \sin 2\theta)^7 \cdot (\cos 5\theta - i \sin 5\theta)^3 = (e^{i2\theta})^7 \cdot (e^{-i5\theta})^3 \\ & = e^{i14\theta} \cdot e^{-i15\theta} = e^{i(14\theta-15\theta)} = e^{-i\theta} \end{aligned}$$

Therefore:

$$\frac{1}{e^{-i\theta}} = e^{i\theta} = \cos \theta + i \sin \theta$$

Differential Equations

First Order Linear ODE

Solve: $\frac{dy}{dx} + \frac{y}{x} = e^x$, $y(0) = 2$

Solution:

This is a first-order linear ODE of the form $\frac{dy}{dx} + P(x)y = Q(x)$

Where $P(x) = \frac{1}{x}$ and $Q(x) = e^x$

Finding Integrating Factor:

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x| = x \text{ (for } x > 0\text{)}$$

Multiplying the equation by $\mu(x) = x$:

$$x \frac{dy}{dx} + y = xe^x$$

The left side is the derivative of xy :

$$\frac{d}{dx}(xy) = xe^x$$

Integrating both sides:

$$xy = \int xe^x dx$$

Using integration by parts with $u = x$, $dv = e^x dx$:

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C = e^x(x-1) + C$$

Therefore:

$$xy = e^x(x-1) + C$$

$$y = \frac{e^x(x-1)+C}{x}$$

Applying initial condition $y(0) = 2$:

This creates an issue since we have x in the denominator. Let's reconsider the problem setup or assume y approaches a limit as $x \rightarrow 0^+$.

For a proper solution, we typically need $x \neq 0$ in the domain.

Formulas Reference

Trigonometric Identities

- $\sin^2 \theta + \cos^2 \theta = 1$
- $\tan^2 \theta + 1 = \sec^2 \theta$
- $1 + \cot^2 \theta = \csc^2 \theta$
- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

Derivatives

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\log x) = \frac{1}{x}$
- $\frac{d}{dx}(e^x) = e^x$

Integrals

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ (where $n \neq -1$)
- $\int \frac{1}{x} dx = \log |x| + C$
- $\int e^x dx = e^x + C$
- $\int \sin x dx = -\cos x + C$
- $\int \cos x dx = \sin x + C$