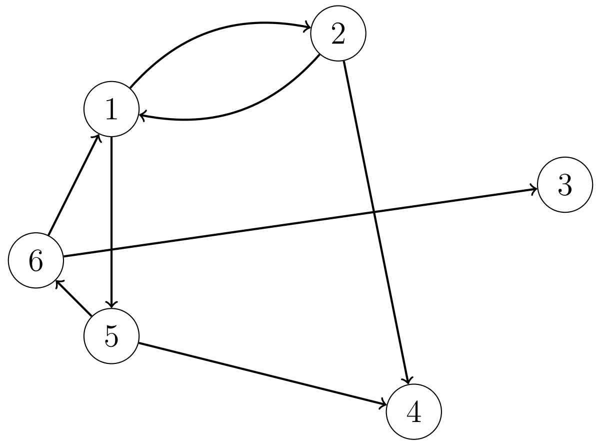
1 Graph

G_9 Formal Description

Below, provide a formal description of the directed graph G_9 below.



Solution:
Because G_9 is a directed graph, its vertex edges and node in-degree and out-degree must be described. Its formal description is then

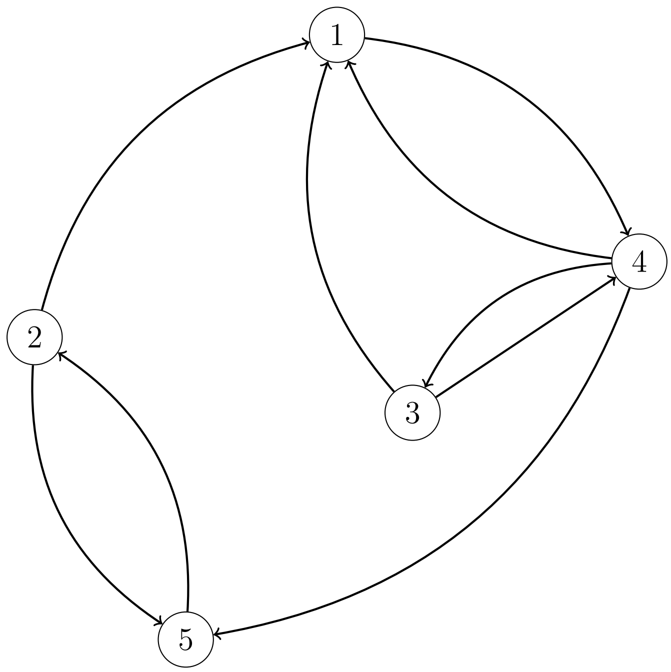
$$G_9 = (V_9, E_9)$$
$$V_9 = \{1, 2, 3, 4, 5, 6\}$$
$$E_9 = \{(1,2), (1,5), (2,1), (2,4), (5,4), (5,6), (6,1), (6,3)\}.$$

with the following in-degree and out-degree for each node $v \in V_9$.

v	In-degree	Out-degree
1	2	1
2	1	2
3	1	0
4	2	0
5	1	2
6	1	2

G_{10} Formal Description

Below, provide a formal description of the directed graph G_{10} below.



Solution:
Because G_{10} is a directed graph, its vertex edges and node in-degree and out-degree must be described. Its formal description is then

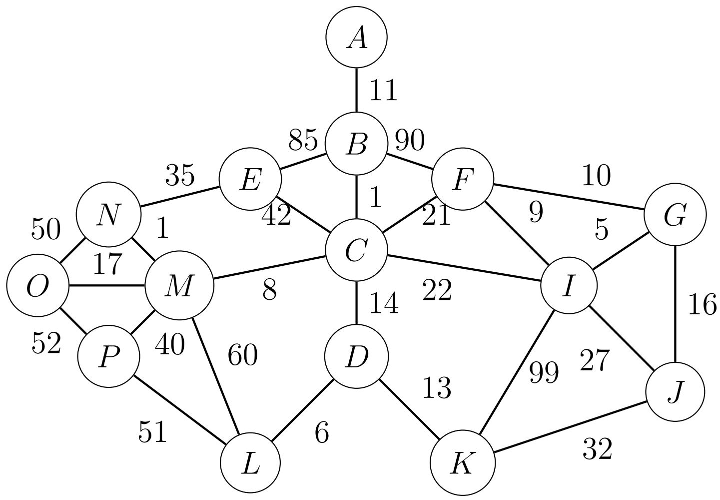
$$\begin{aligned} G_{10} &= (V_{10}, E_{10}) \\ V_{10} &= \{1, 2, 3, 4, 5\} \\ E_{10} &= \{(1,4), (2,1), (2,5), (3,1), (3,4), (4,1), (4,3), (4,5), (5,2)\}. \end{aligned}$$

with the following in-degree and out-degree for each node $v \in V_{10}$.

v	In-degree	Out-degree
1	1	3
2	2	1
3	2	1
4	3	2
5	1	2

G_{29} Minimum Spanning Tree

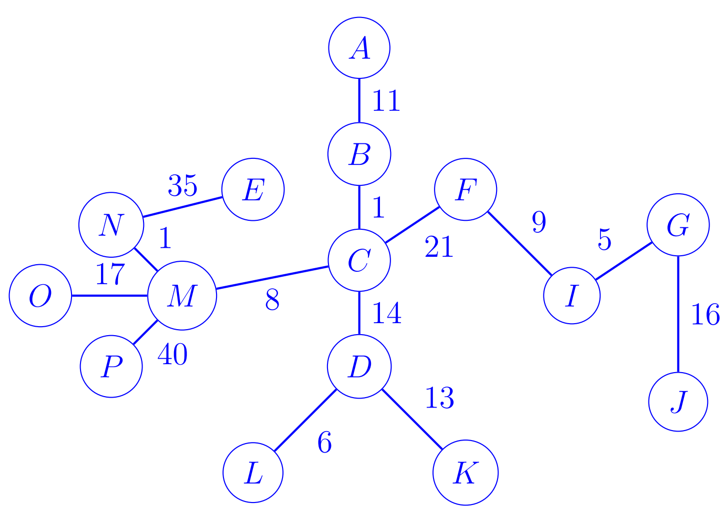
Find the minimum spanning tree of the graph G_{29} below using (a) Kruskal's Algorithm and (b) Prim's Algorithm.



Solution:

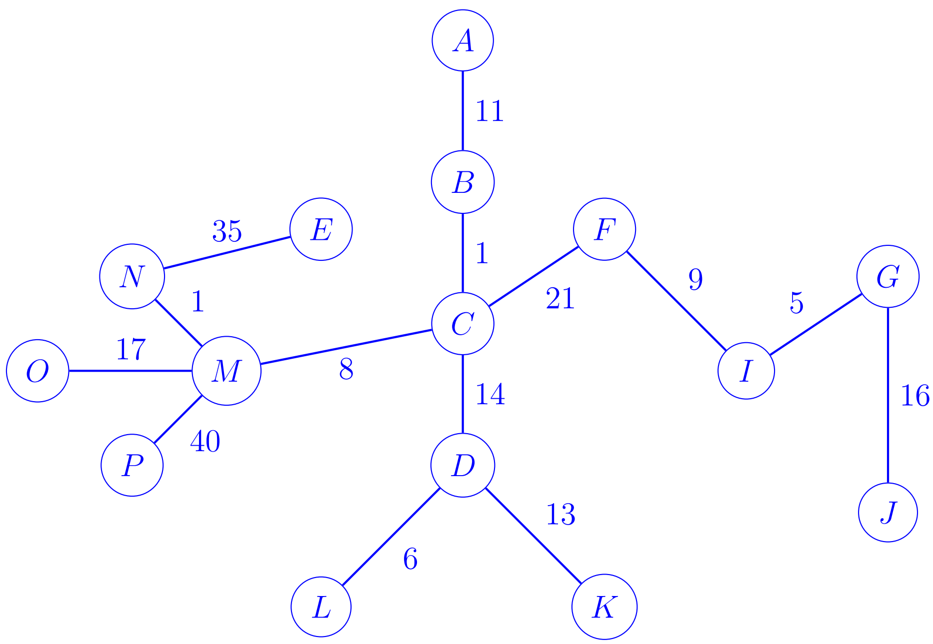
1. Kruskal's Algorithm $\left(\sum_{e\in E_{29}} w(e) = 197\right)$

$$\begin{array}{rrrr}
\underline{w(B,C)} = 1, & \underline{w(M,N)} = 1, & \underline{w(I,G)} = 5, & \underline{w(D,L)} = 6, \\
\underline{w(C,M)} = 8, & \underline{w(F,I)} = 9, & w(F,G) = 10, & \underline{w(A,B)} = 11, \\
\underline{w(D,K)} = 13, & \underline{w(C,D)} = 14, & \underline{w(G,J)} = 16, & \underline{w(M,O)} = 17, \\
\underline{w(C,F)} = 21, & w(C,I) = 22, & w(I,J) = 27, & w(J,K) = 32, \\
\underline{w(E,N)} = 35, & \underline{w(M,P)} = 40, & w(C,E) = 42, & w(N,O) = 50, \\
w(L,P) = 51, & w(O,P) = 52, & w(L,M) = 60, & w(B,E) = 85, \\
 & w(B,F) = 90, & w(I,K) = 99. &
\end{array}$$



2. Prim's Algorithm $\left(\sum_{e \in E_{29}} w(e) = 197\right)$

$$\begin{aligned} \underline{w(A,B)} &= 11, & \underline{w(M,O)} &= 17, \\ \underline{w(B,C)} &= 1, & \underline{w(C,F)} &= 21, \\ \underline{w(C,M)} &= 8, & \underline{w(F,I)} &= 9, \\ \underline{w(M,N)} &= 1, & \underline{w(I,G)} &= 5, \\ \underline{w(C,D)} &= 14, & \underline{w(G,J)} &= 16, \\ \underline{w(D,L)} &= 6, & \underline{w(E,N)} &= 35, \\ \underline{w(D,K)} &= 13, & \underline{w(M,P)} &= 40. \end{aligned}$$

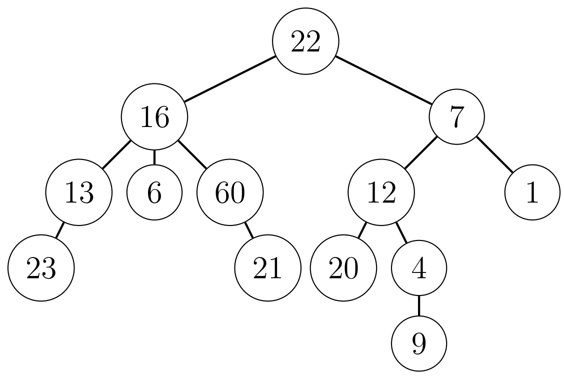


2 Trees

Short Quiz on Trees

1. Name the three properties of a tree.
   A tree have its weight, depth, and degree.
2. Is a tree a forest?
   Yes.
3. What do you call the special designated node in a tree?
   A root node.
4. What is the minimum number of nodes in a tree?
   0. Specifically, the null or empty tree.
5. Can a tree have no subtrees at all?
   Yes. An empty or null tree or a node tree doesn't have children and therefore, have no subtree.

Given the tree T_1 below, identify the following.



6. Children of node 16: 13, 6, 60
7. Parent of node 1: 7
8. Siblings of node 23: None
9. Ancestors of node 9: 4, 12, 7, 22
10. Descendants of node 16: 13,6,60, 23, 21
11. Leaves: 23, 6, 21, 20, 9, 1
12. Non-leaves: 22, 16, 7, 13, 60, 12, 4
13. Depth of node 4: $dep\{4\} = 3$
14. Degree of the tree: $deg\{T_1\} = 3$
15. Height of the tree: $h\{T_1\} = 4$
16. Weight of the tree: $w\{T_1\} = 6$
17. Is the tree a binary tree? No. Since $deg\{16\} = 3 > 2$.
18. Removing 6, is the tree a full binary tree?
    No, since $deg\{13\} = 1 \notin \{0, 2\}$ which is required for a node in a full binary tree.
19. Removing 6, is the tree a complete binary tree?
    No, since $dep\{9\} \neq dep\{20\}$ for leaf node 9 and 20. All leaf nodes in a complete tree must have the same depth.
20. Is a full binary tree complete?
    No. Since for two leaf nodes v_1 and v_2 of a full binary tree, it is possible for $dep\{v_1\} \neq dep\{v_2\}$, making it not complete.

21. Is a complete binary tree full?
    Yes, since a complete binary tree have all internal nodes have degree of 2, making a full tree.
22. How many leaves does a complete n-ary tree of height h have?
    Since the number of leaves for a complete n-ary tree with height 1 is n, n^2 for tree with height 2, and n^3 with height 3, it is safe to assume that the number of leaves for a complete n-ary tree is $\boxed{n^h}$.
23. What is the height of a complete n-ary tree with m leaves?
    It is known from the previous problem that a complete n-ary tree of height h have n^h. With m leaves for a complete n-ary tree, $n^h = m$ and $\boxed{h = \log_n m}$.
24. What is the number of internal nodes of a complete n-ary tree of height h?
    The number of internal nodes of a complete n-ary tree with height h is equal to the sum of the number of leaves of complete n-ary trees with height 0 to $h-1$. With n^h from question 22, $\displaystyle\sum_{i=0}^{h-1} n^i = \boxed{\frac{n^h - 1}{n - 1}}$.
25. What is the total number of nodes a complete n-ary tree of height h have?
    From question 22 and 24, the total number of nodes of a complete n-ary tree of height h is $\displaystyle\frac{n^h - 1}{n - 1} + n^h = \boxed{\frac{n^{h+1} - 1}{n - 1}}$.