Light Unbalanced Optimal Transport

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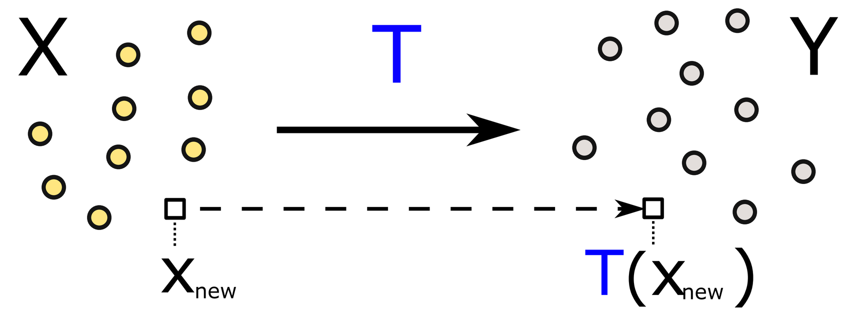
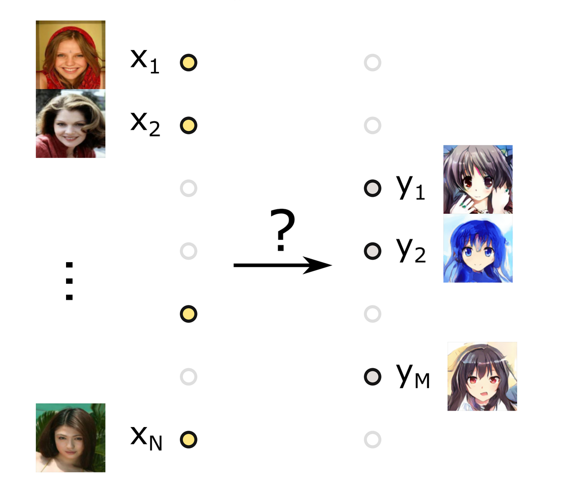
Domain Translation: The Problem which Motivated the Study



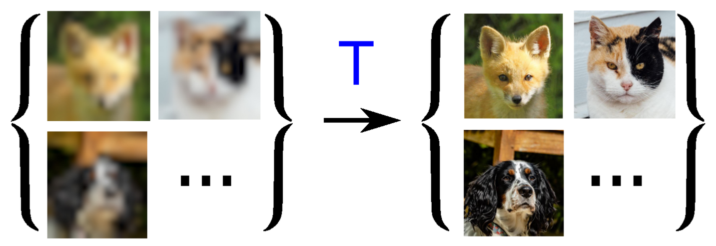
The (informal) task: given samples X, Y from two domains, construct a map T which can translate new samples from the input domain to the target domain.

Unpaired setup

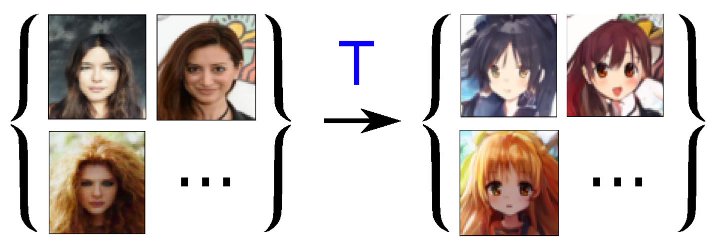
No paired training examples are available.



Example 1: Image Super-Resolution



Example 2: Style Translation



Our U-LightOT solver *: deriving an objective*

We propose a **novel *lightweight* and theoretically justified solver** to estimate continuous **unbalanced** EOT plans using Gaussian mixture parametrization.

The cornerstones of our solver:

1. We consider a **proper** parametrization of UEOT plans γ^*:

$$\gamma_{\theta,\omega}(x,y)=u_\omega(x)\gamma_\theta(y|x)=u_\omega(x)\frac{\exp\{\langle x,y\rangle/\varepsilon\}v_\theta(y)}{c_\theta(x)},$$

where u_w parametrizes marginal measure γ^*_x, and v_θ — variable v^* which entirely describes the *conditional measure* $\gamma^*(\cdot|x)$, c_θ is the normalization constant[2] depending only on v_θ.

2. For this parametrization, we derive the following **bound**: $\varepsilon \mathrm{D}_{\mathrm{KL}}(\gamma^*\|\gamma_{\theta,w})\leq\mathcal{L}(\theta,w)-Const$, where

$$\mathcal{L}(\theta,w)\overset{def}{=}\int_{\mathbb{R}^d}\overline{f}_1\Big(-\varepsilon\log\frac{u_w(x)}{c_\theta(x)}-\frac{\|x\|^2}{2}\Big)p(x)dx+\int_{\mathbb{R}^d}\overline{f}_2\Big(-\varepsilon\log v_\theta(y)-\frac{\|y\|^2}{2}\Big)q(y)dy+\varepsilon\|u_w\|_1.$$

Here we use $\overline{f}$ to denote a Fenchel conjugate of a function f.

[2] Alexander Korotin, Nikita Gushchin, and Evgeny Burnaev (2024). "Light Schr\" odinger Bridge". In: *The Twelfth International Conference on Learning Representations*.

Unpaired Domain Translation: Class Imbalance

We perform comparison of our U-LightOT solver and other EOT/UEOT solvers in the unpaired domain translation problem between various classes of FFHQ images (*Man*↔*Woman*, *Adult*↔*Young*). The main challenge of the described translations is the imbalance of classes in the images from source and target subsets, see the Table below. The translation is performed in the latent space of the ALAE autoencoder.

Class	*Man*	*Woman*
Young	15K	23K
Adult	7K	3.5K

Table 1: Number of *train* FFHQ images for each subset.

Young → *Adult*

Our U-LightOT solver outperforms alternatives in dealing with class imbalance issues.

Continuous Optimal Transport Solvers: Main Challenges

Continuous Optimal Transport (OT) solvers constitute a promising novel framework for *unpaired* domain translation based on OT theory.

Idea of continuous OT solvers:

- estimate the OT plan between continuous measures given only empirical samples of data from them;
- use the learned conditional plans to perform the domain translation.

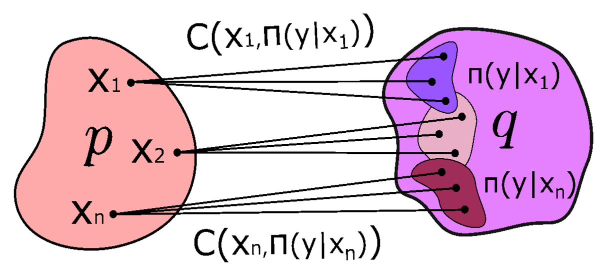
Note: Many papers consider an entropy-regularized OT (EOT) problem with stochastic plans.

Classic OT/EOT problems show high sensitivity

- to an imbalance of classes;
- to outliers in the source and target measures.

Classic EOT (quadratic cost)

$$\underbrace{\min_{\pi\in\Pi(p,q)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\|x-y\|^2}{2}\pi(x,y)dxdy-\varepsilon H(\pi)}_{\overset{def}{=}\mathrm{Cost}(p,q)}$$



The minimizer π^* is called the EOT plan.

Important: the problems ↑ can be resolved by considering **unbalanced** OT/EOT problem.

Our U-LightOT solver *: dealing with normalization constant*

3. **In practice**, we parametrize $u_\omega(x)$ and $v_\theta(y)$ as unnormalized Gaussian mixtures:

$$v_\theta(y)=\sum_{k=1}^{K}\alpha_k\mathcal{N}(y|r_k,\varepsilon S_k);\quad u_\omega(x)=\sum_{l=1}^{L}\beta_l\mathcal{N}(x|\mu_l,\varepsilon\Sigma_l).$$

For this parametrization, we get the closed-form formula for the normalization constant $c_\theta(x)$ and can easily estimate our objective.

4. Then, in our Algorithm, we optimize the **empirical counterpart** of $\mathcal{L}(\theta,\omega)$:

$$\widehat{\mathcal{L}}(\theta,w)\overset{def}{=}\frac{1}{N}\sum_{i=1}^{N}\overline{f}_1\Big(-\varepsilon\log\frac{u_w(x_i)}{c_\theta(x_i)}-\frac{\|x_i\|^2}{2}\Big)+\frac{1}{M}\sum_{j=1}^{M}\overline{f}_2\Big(-\varepsilon\log v_\theta(y_j)-\frac{\|y_j\|^2}{2}\Big)+\varepsilon\|u_w\|_1 \quad (1)$$

using minibatch gradient descent procedure w.r.t. parameters (θ,w).

Typical choices of f_i $(i\in[1,2])$ are $f_i(t)=\tau_i f_{\mathrm{KL}}(t)$ or $f_i(t)=\tau_i f_{\chi^2}(t)$ $(\tau_i>0)$ yielding the scaled D_{KL} and D_{χ^2}, respectively. In this case, the bigger τ_1 (τ_2) is, the more γ_x (γ_y) is penalized for not matching the corresponding marginal distribution p (q).

Algorithm: Light Unbalanced Optimal Transport

Algorithm 0: Light Unbalanced Optimal Transport (U-LightOT).

repeat

Sample batches $X=\{x_1,...,x_N\}\sim p$, $Y=\{y_1,...,y_M\}\sim q$;

$\widehat{\mathcal{L}}(\theta,w)\leftarrow\frac{1}{N}\sum_{i=1}^{N}f_1(-\varepsilon\log\frac{u_w(x_i)}{c_\theta(x_i)}-\frac{\|x_i\|^2}{2})+\frac{1}{M}\sum_{j=1}^{M}f_2(-\varepsilon\log v_\theta(y_j)-\frac{\|y_j\|^2}{2})+\varepsilon\|u_w\|_1$;

Update $(\theta,\ \omega)$ by using $\frac{\partial\widehat{\mathcal{L}}_{\theta,\omega}}{\partial\theta}$ and $\frac{\partial\widehat{\mathcal{L}}_{\theta,\omega}}{\partial\omega}$;

until *not converged*;

Advantages:

1. Fast training - several minutes on 4 CPU cores, not hours of training on GPU like others;

2. Theoretical validity - the guarantees of the method's learning ability based on the statistical learning and approximation theories;

3. Practical importance - allows for mitigating bias issue and getting rid of outliers;

Unbalanced EOT Problem[1]

Formulation (quadratic cost):

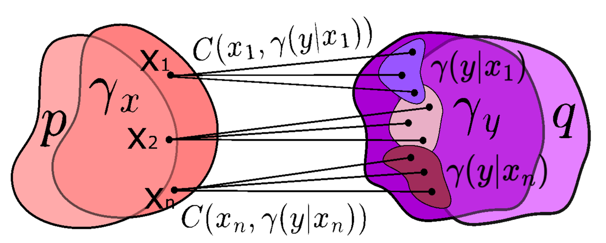
Let D_{f_1}, D_{f_2} be two f-divergences over $\mathbb{R}^d$. For two probability measures $p\in\mathcal{P}_{2,ac}(\mathbb{R}^d)$, $q\in\mathcal{P}_{2,ac}(\mathbb{R}^d)$ and $\varepsilon>0$, the unbalanced EOT (UEOT) problem between p and q consists of finding a minimizer of

$$\inf_{\gamma\in\mathcal{M}_{2,+}(\mathbb{R}^d\times\mathbb{R}^d)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\|x-y\|^2}{2}\gamma(x,y)dxdy-\varepsilon H(\gamma)+D_{f_1}(\gamma_x\|p)+D_{f_2}(\gamma_y\|q).$$

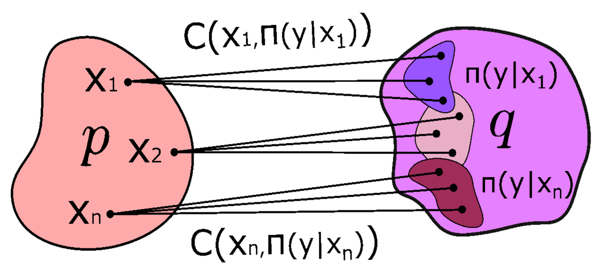
The minimizer γ^* is called the UEOT plan.

The UEOT problem resolves the issues of its classic counterpart by relaxing the marginal constraints.

Unbalanced EOT problem



Classic EOT problem



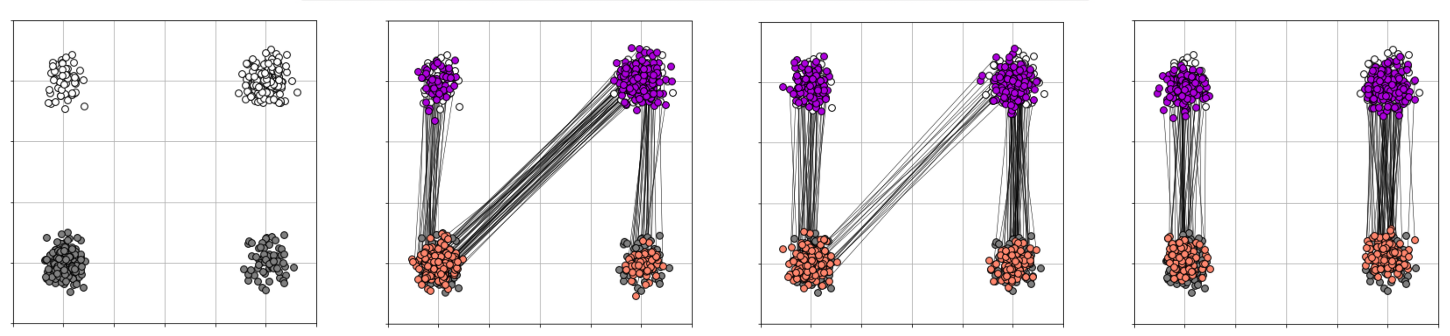
Existing UEOT solvers are either heuristical or require time-consuming optimization procedure.

[1] Lénaïc Chizat (2017). "Unbalanced optimal transport: Models, numerical methods, applications". PhD thesis. Université Paris sciences et lettres.

Toy Experiments: Class Imbalance Issue & Outliers

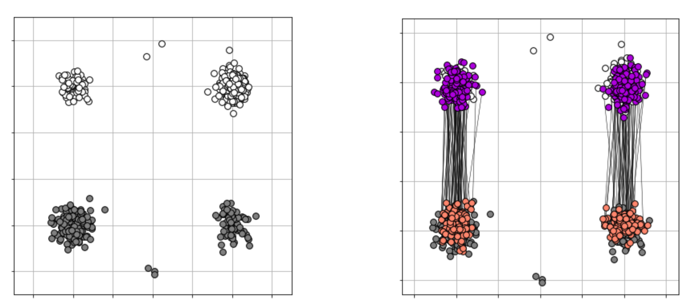
Gaussian Mixture with Class Imbalance

$x\sim p$ $y\sim q$ $x\sim p_\omega(\cdot)$ $y\sim\gamma_{\theta,\omega}(\cdot|x)$



Input,Target $\tau=100$ $\tau=10$ $\tau=1$

Gaussian Mixture with Outliers



U-LightOT allows for dealing with the class imbalance issue and getting rid of outliers. (Controlled by the unbalancedness parameter τ.)

Thank you

Light Unbalanced Optimal Transport (U-LightOT)

A novel light and fast algorithm to solve the Unbalanced EOT problem.

OpenReview.net

GitHub