N-BODY SIMULATION

Milena Yahya - Michael Elias

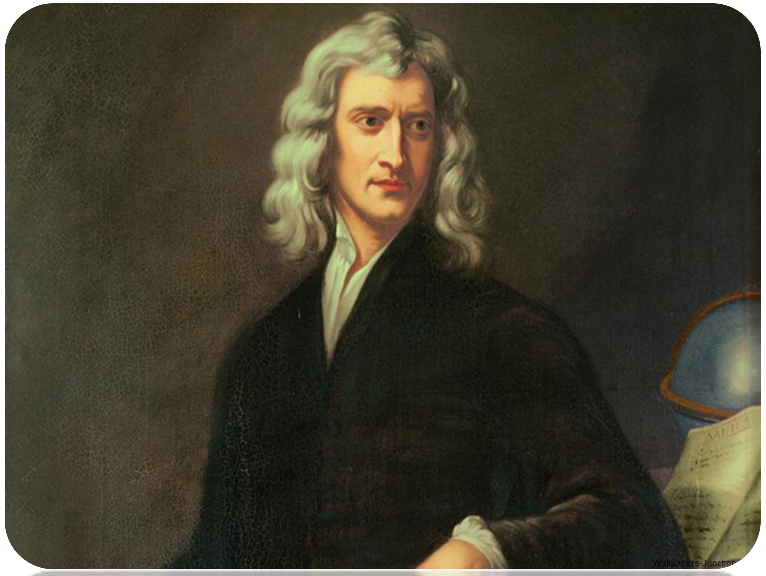
N-BODY PROBLEM

In astrophysics, the N-Body problem is the problem of predicting the individual motions of N celestial objects interacting with each other gravitationally. When given mass, initial position, and initial velocity of every body, it is possible to predict their orbits that are the result of the interaction of their gravitational fields.

NEWTON'S LAW OF UNIVERSAL GRAVITATION

The main force guarding the motion of every celestial body is:

$$\mathbf{F}_{21} = -G \frac{m_1 m_2}{|\mathbf{r}_{21}|^2} \hat{\mathbf{r}}_{21}$$



where

- F_{21} is the force applied on object 2 exerted by object 1,
- G is the gravitational constant,
- m_1 and m_2 are respectively the masses of objects 1 and 2,
- $|r_{21}| = |r_2 - r_1|$ is the distance between objects 1 and 2, and
- and $\hat{\mathbf{r}}_{21} \stackrel{\text{def}}{=} \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}$

COMPUTATIONAL DIFFICULTIES

When attempting to calculate the gravitational force of N-1 bodies acting on a single body, and repeating this calculation for N bodies at a single instant of time t, we obtain the position of the N bodies at the time t. However, to obtain their new positions every t+dt would require repeating this immense computation every dt, also known as the "Brute Force Calculation".

This elevates the complexity of the system to the order of N^2

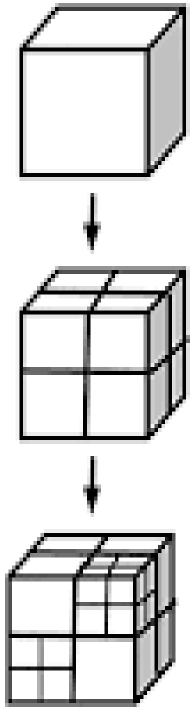
BARNES-HUT ALGORITHM

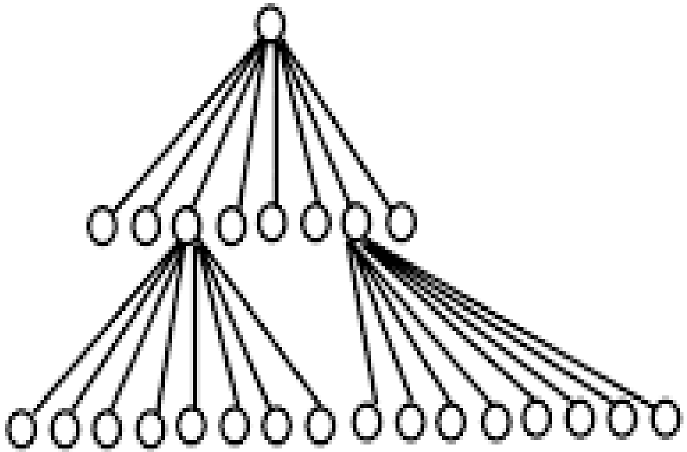
Instead of directly summing up all forces, the algorithm uses a tree-based approximation scheme which reduces the computational complexity of the problem from *$O(N^2)$* to *$O(N \log N)$*. It works by reducing the number of force calculations by grouping particles. The basic idea behind the algorithm is that the force which a particle group exerts on a single particle can be approximated by the force of a pseudo particle located at the group's center of mass.

OCT TREE ALGORITHM

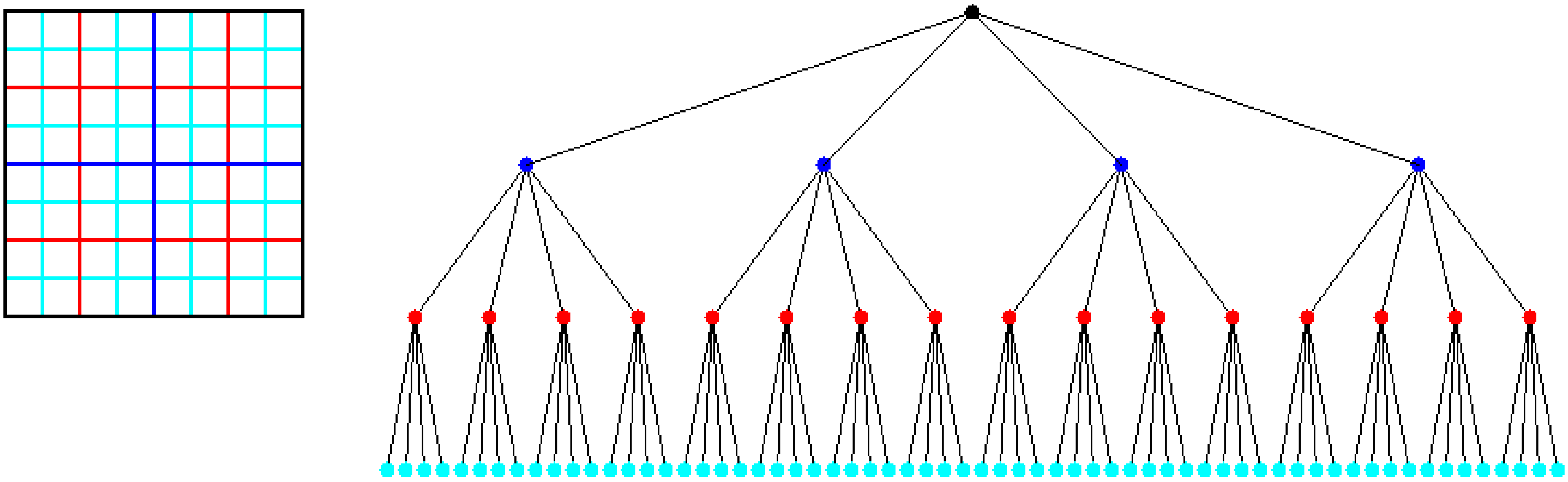
The Octree can be formed from 3D volume by doing the following steps:

- Divide the current 3D volume into eight boxes
- If any box has more than one point then divide it further into 8 boxes
- Do not divide the box which has one or zero points in it
- Do this process repeatedly until all the boxes contain one or zero points



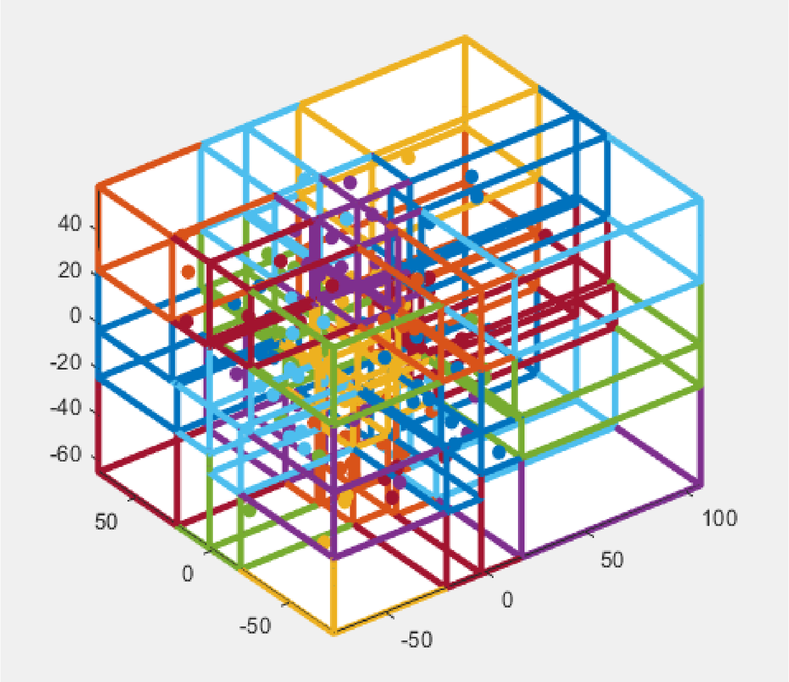


A Complete Quadtree with 4 Levels



APPLICATION ON OUR CODE

N=300



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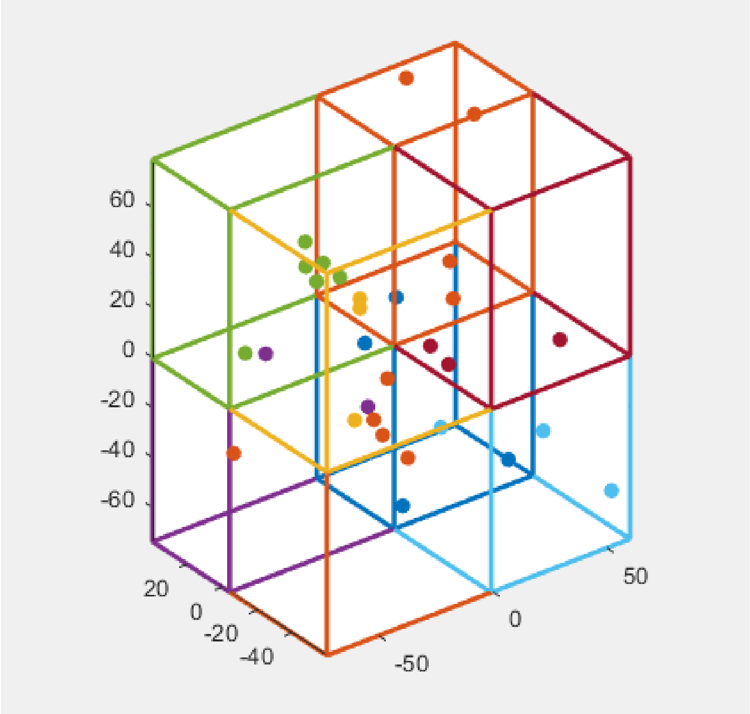
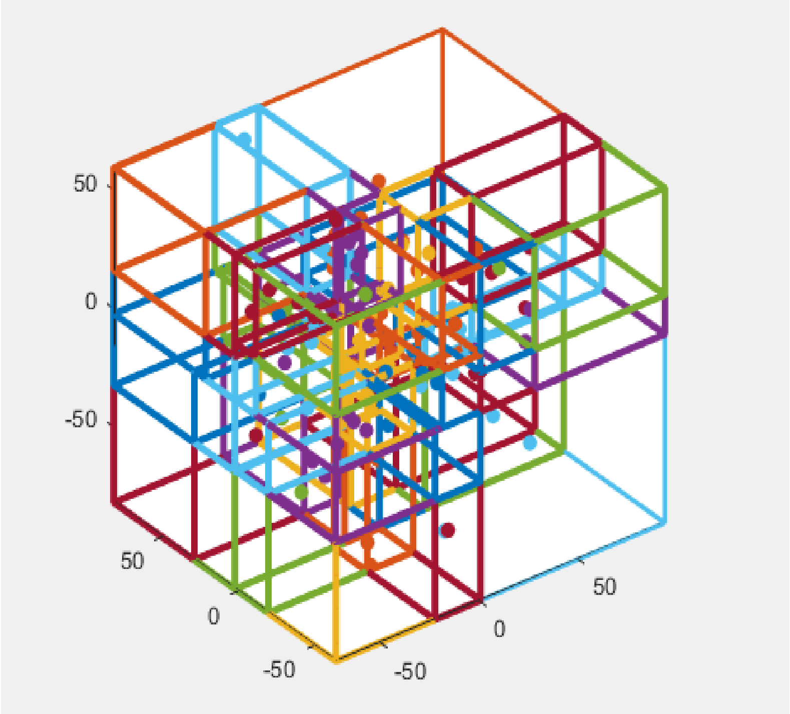


N=100

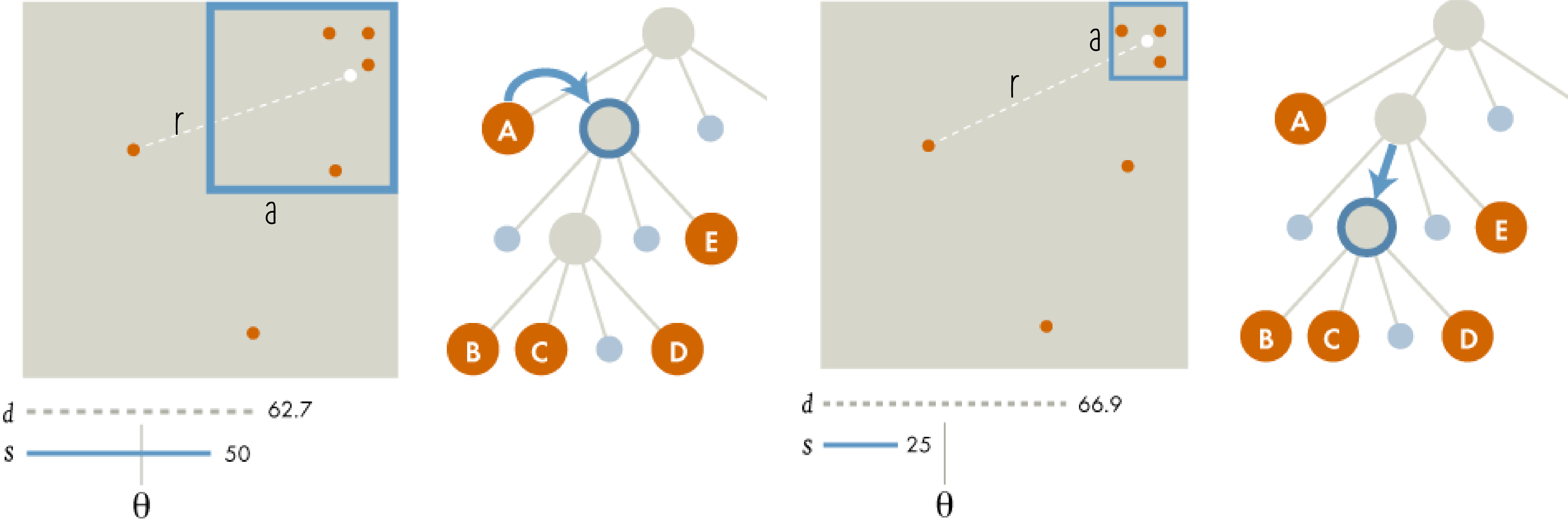


APPROXIMATION

The idea is to group bodies that are far-away enough from the considered body 'b' and to calculate their summed gravitational force on body 'b' by using their center of mass. The Barnes-Hut algorithm defines the term "far-away enough" as follows:

If the distance 'r' from a cluster of bodies to body 'b' is large , and the side 'a' of the cube of the octree containing the cluster is small, we calculate theta = a/r. Theta is knowns as the *Multipole-Acceptance-Criterion (MAC).*

The smaller theta, the better simulation results. If theta drops below a certain threshold the quality of the approximation starts to deteriorate, resulting in larger errors. Usually, theta = 0.5 is adopted in most applications.



CALCULATING THETA

Moving from the root of the tree downwards, we can see that the box in level 2 in the picture on the left is not far away enough from body b, and we have to move further downwards to a new level of the tree and test the approximation again.

FORCE CALCULATION

To calculate the net force acting on body *b at* time t, use the following recursive procedure, starting with the root of the quad-tree:

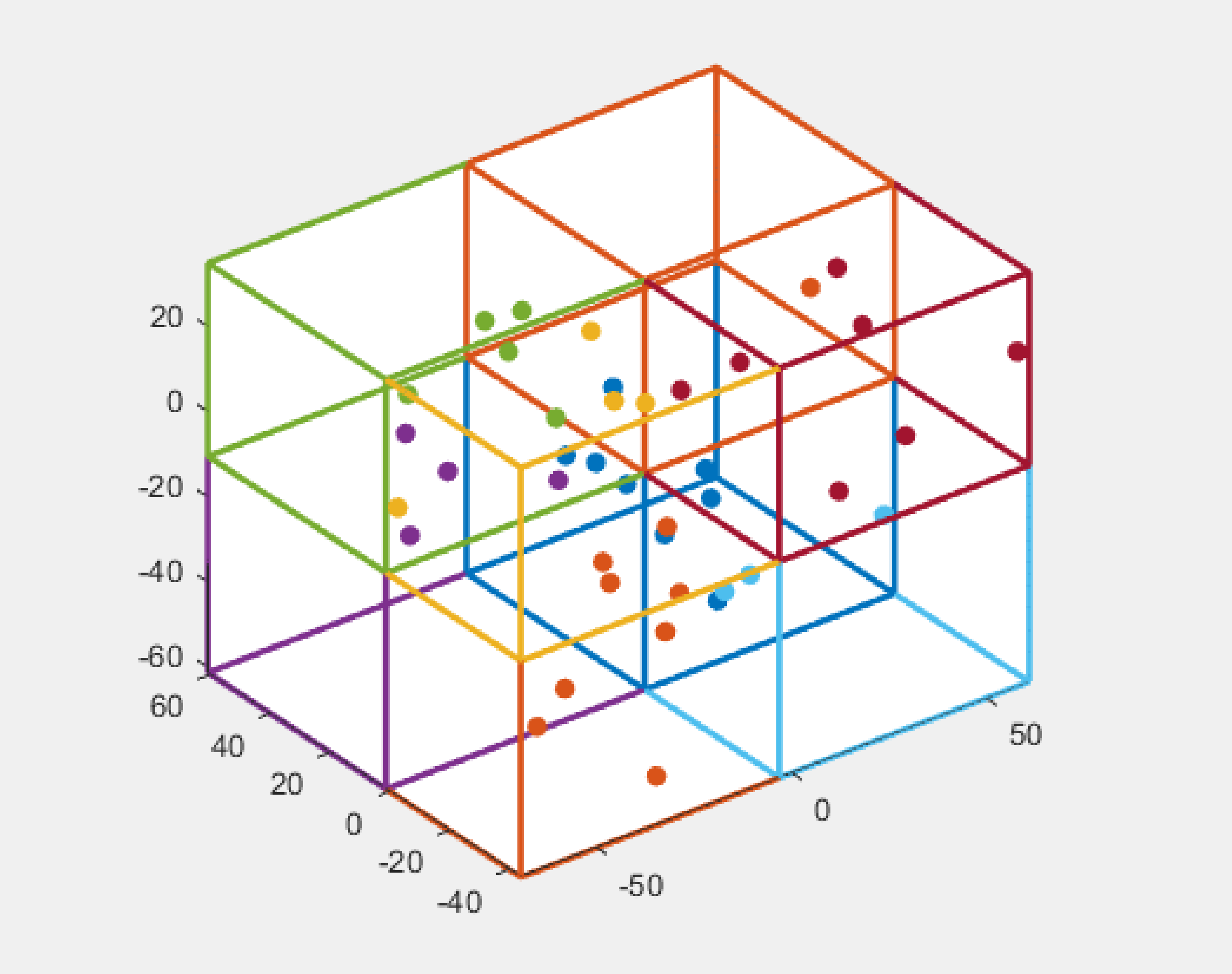
- *If the current node contains only 1 body and is not body b, calculate the force exerted by the current node on b, and add this amount to b's net force.*
- *Otherwise, calculate the ratio a/r. If a/r < $\boldsymbol{\theta}$, treat this internal node as a single body (its C-O-M) , and calculate the force it exerts on body b, and add this amount to b's net force.*
- *If a/r >$\boldsymbol{\theta}$, calculate the force every body on the cluster exerts on b individually.*

This calculation is to be done for each body at every instant of time t+dt.

In our project, the function Forcefun() calculates the total force acting on each body at a single instant of time according to the three cases mentioned above.

APPLICATION ON OUR CODE

For the mentioned calculation, we need to know the coordinates of the center of mass of every cube at every level of the octree, and the length of the side of this cube 'a'. This is achieved by our functions GetCubeSide() and GetCOM().

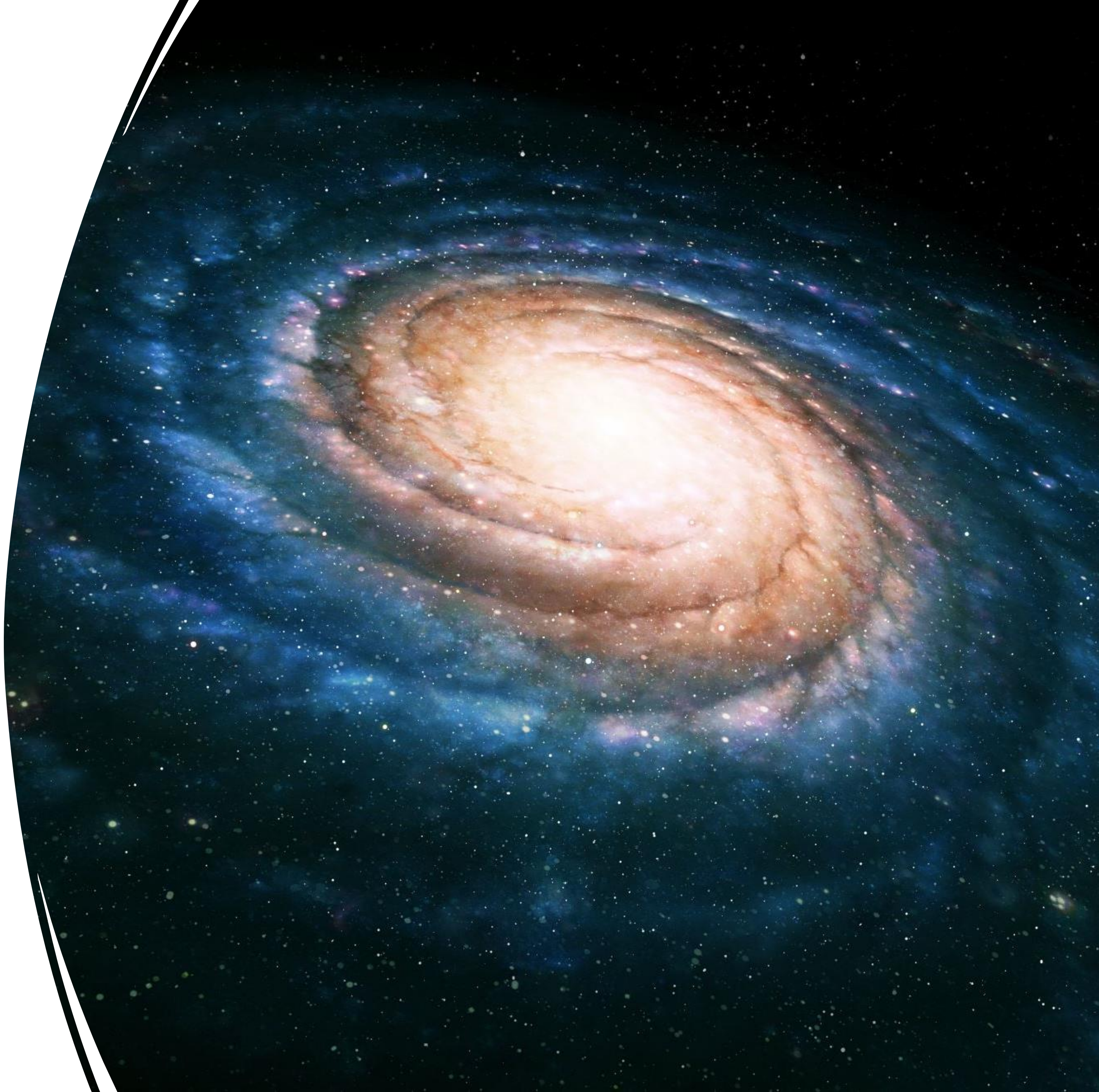


SO FAR, WE HAVE ONLY CALCULATED THE TOTAL GRAVITATIONAL FORCE ACTING ON A BODY B AT AN INSTANT OF TIME T. NEWTON'S 2ND LAW:

$$A_x = G \sum (m * \Delta x) / r^3$$

$$A_y = G \sum (m * \Delta y) / r^3$$

$$A_z = G \sum (m * \Delta z) / r^3$$



LEAPFROG FINITE DIFFERENCE APPROXIMATION SCHEME

We use the *leapfrog finite difference approximation scheme* to numerically integrate the above equations: this is the basis for most astrophysical simulations of gravitational systems. In the leapfrog scheme, we discretize time, and update the time variable *t* in increments of the *time quantum* Δt. We maintain the position and velocity of each particle, but they are half a time step out of phase.

$$\mathbf{v}_i = \mathbf{v}_i + \frac{\Delta t}{2} \times \mathbf{a}_i \qquad \mathbf{r}_i = \mathbf{r}_i + \Delta t \times \mathbf{v}_i$$

Notice the difference of $\Delta t/2$ between v and r.

CODE OVERVIEW

A random set of N points with random masses in the range [10^8 5.10^11] kg constitutes our initial conditions. An Oct-tree is constructed for these points, from which the total gravitational force acting on each body b is calculated using Forcefun() respecting the Barnes-Hut algorithm.

Using Newton's Second Law, we can equate the total force of the body and its acceleration. Applying the Leapfrog Scheme, we can integrate the obtained acceleration to find the exact position of each of the N bodies at the instant $t+\Delta t$.

From the new positions, we construct a new Oct-tree, and the algorithm is repeated recursively throughout the specified interval of time.

SIMULATION RESULTS

We will now exhibit simulation results for N : 7, 33, 100, 500.

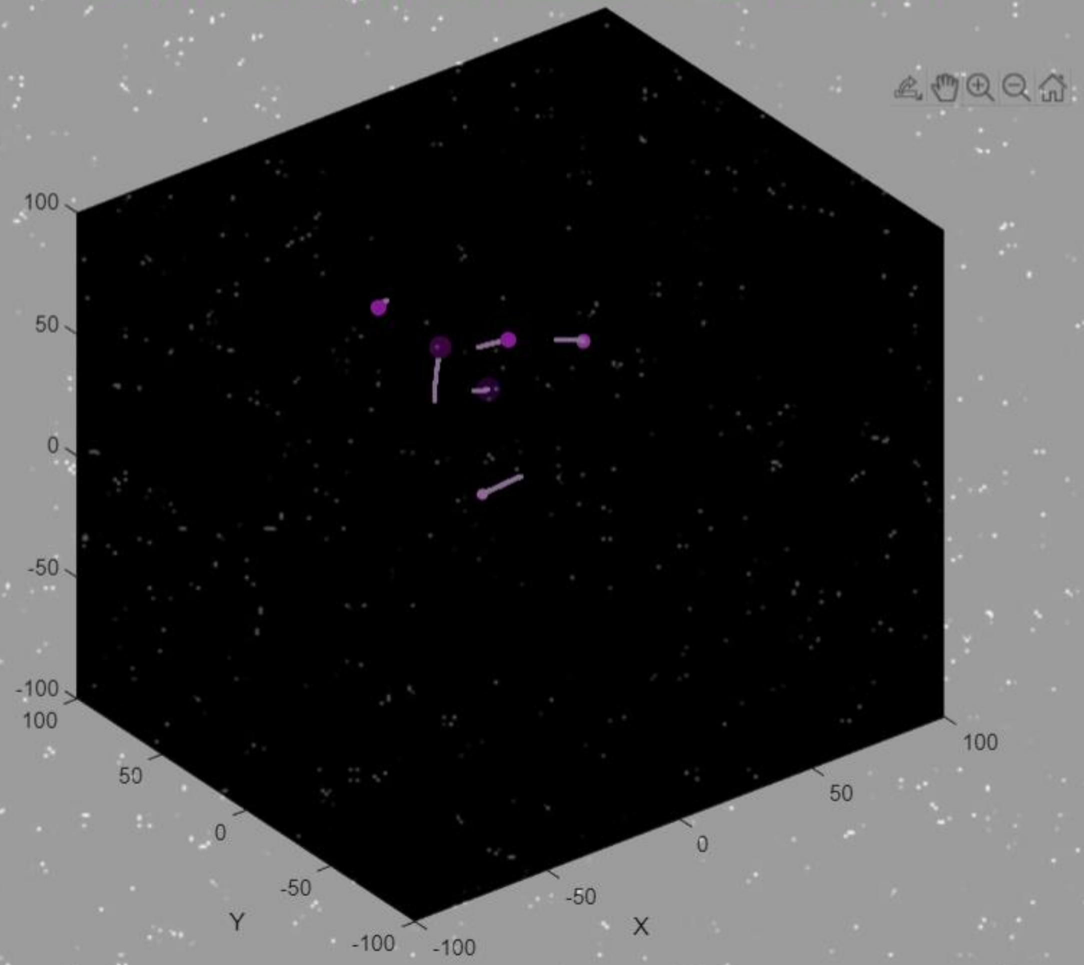
Change slides swiftly for animation effect.

N 7

θ 0.5

RUN

GRAVITATIONAL INTERACTION OF 7 CELESTIAL BODIES



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- 2.2555e+11 <mass<= 2.7066e+11
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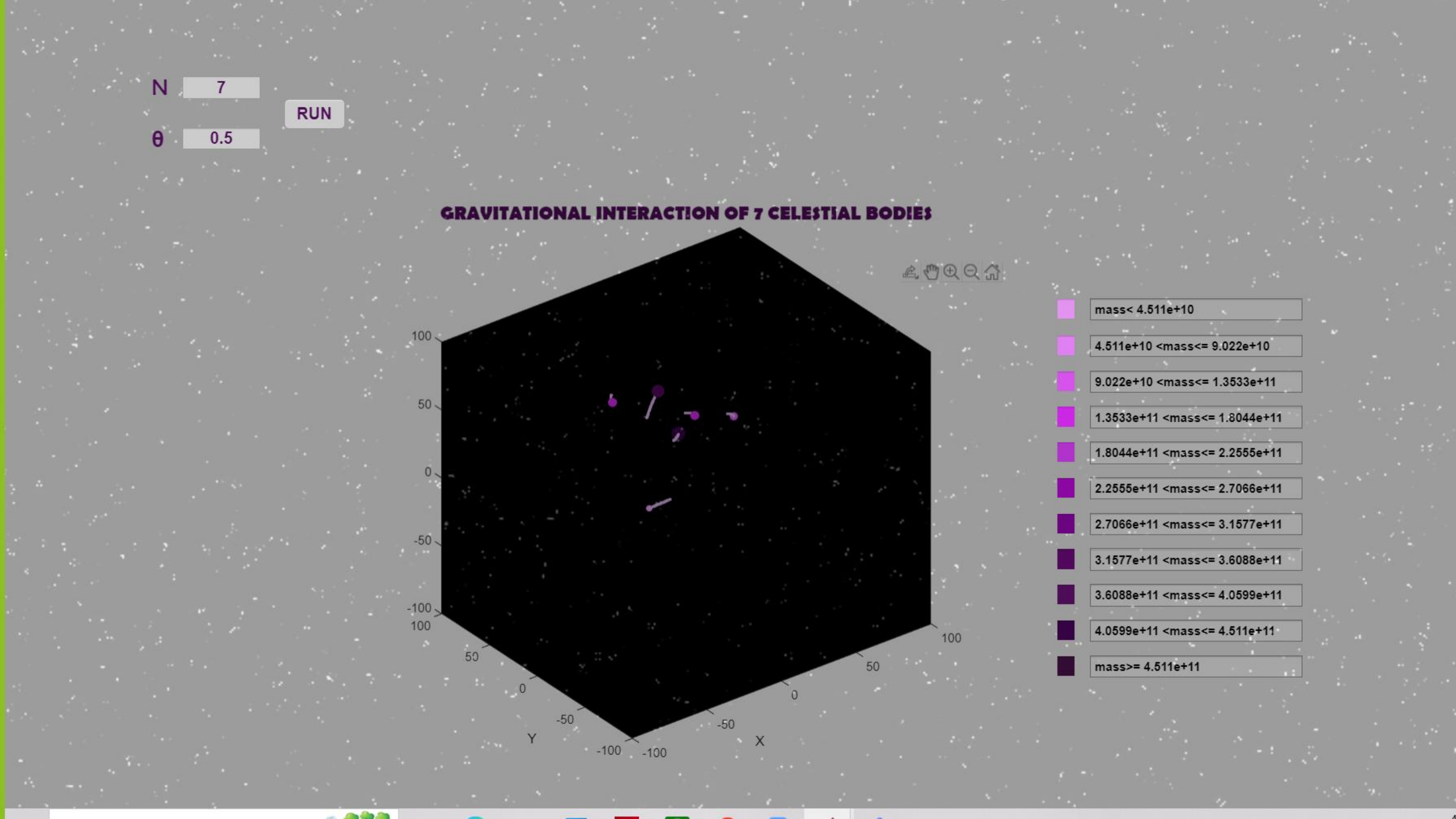
N 7
RUN
θ 0.5
GRAVITATIONAL INTERACTION OF 7 CELESTIAL BODIES
100
50
0
-50
-100
100
50
0
-50
Y
-100
-100
-50
0
50
100
X
mass< 4.511e+10
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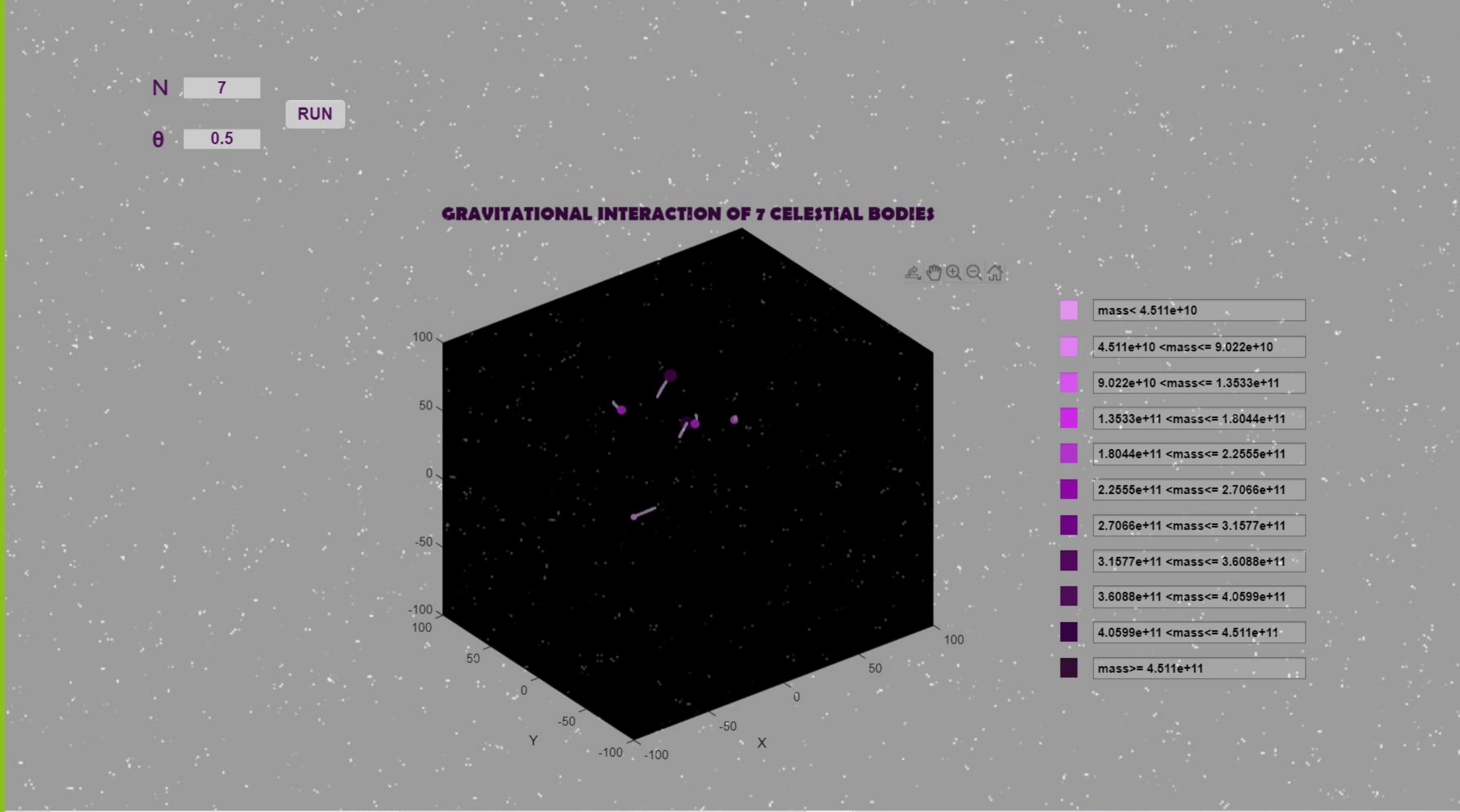

N 7
RUN
θ 0.5
GRAVITATIONAL INTERACTION OF 7 CELESTIAL BODIES
100
50
0
-50
-100
Y
X
mass< 4.511e+10
4.511e+10 <mass<= 9.022e+10
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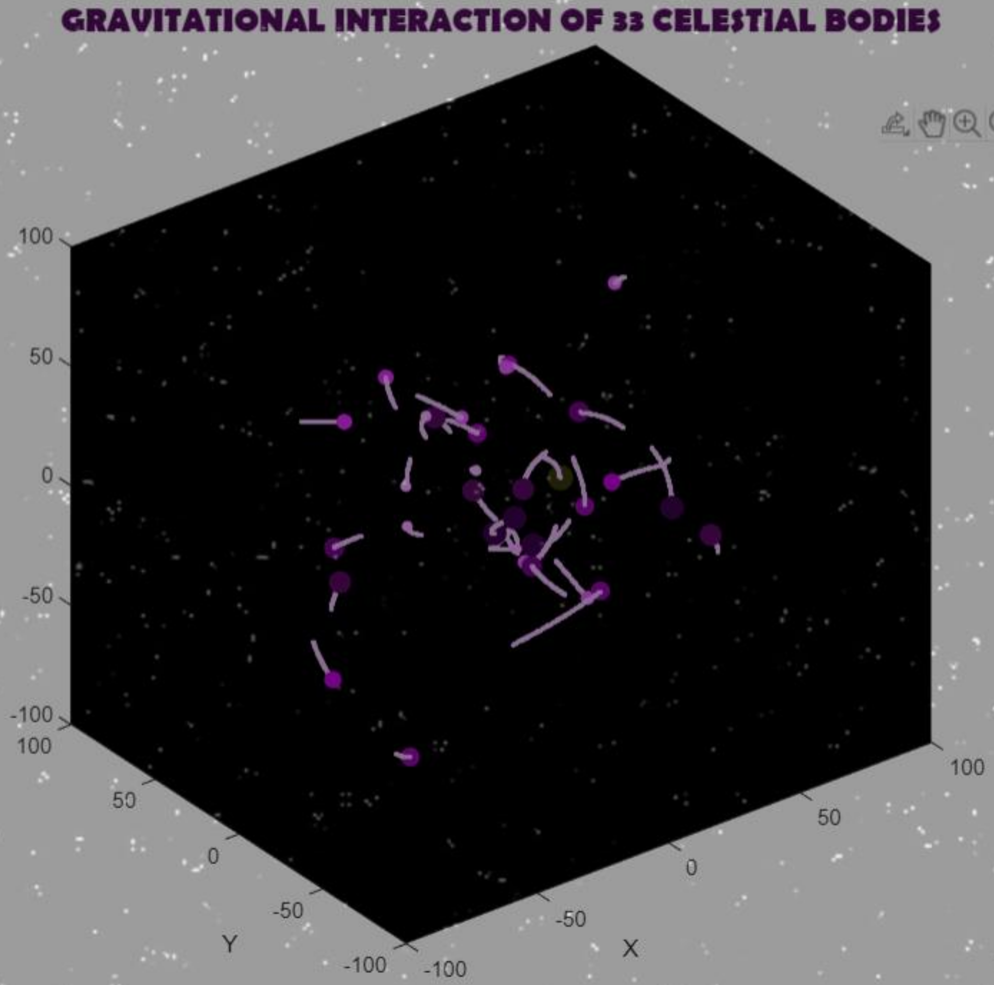
N 33

RUN

θ 0.5



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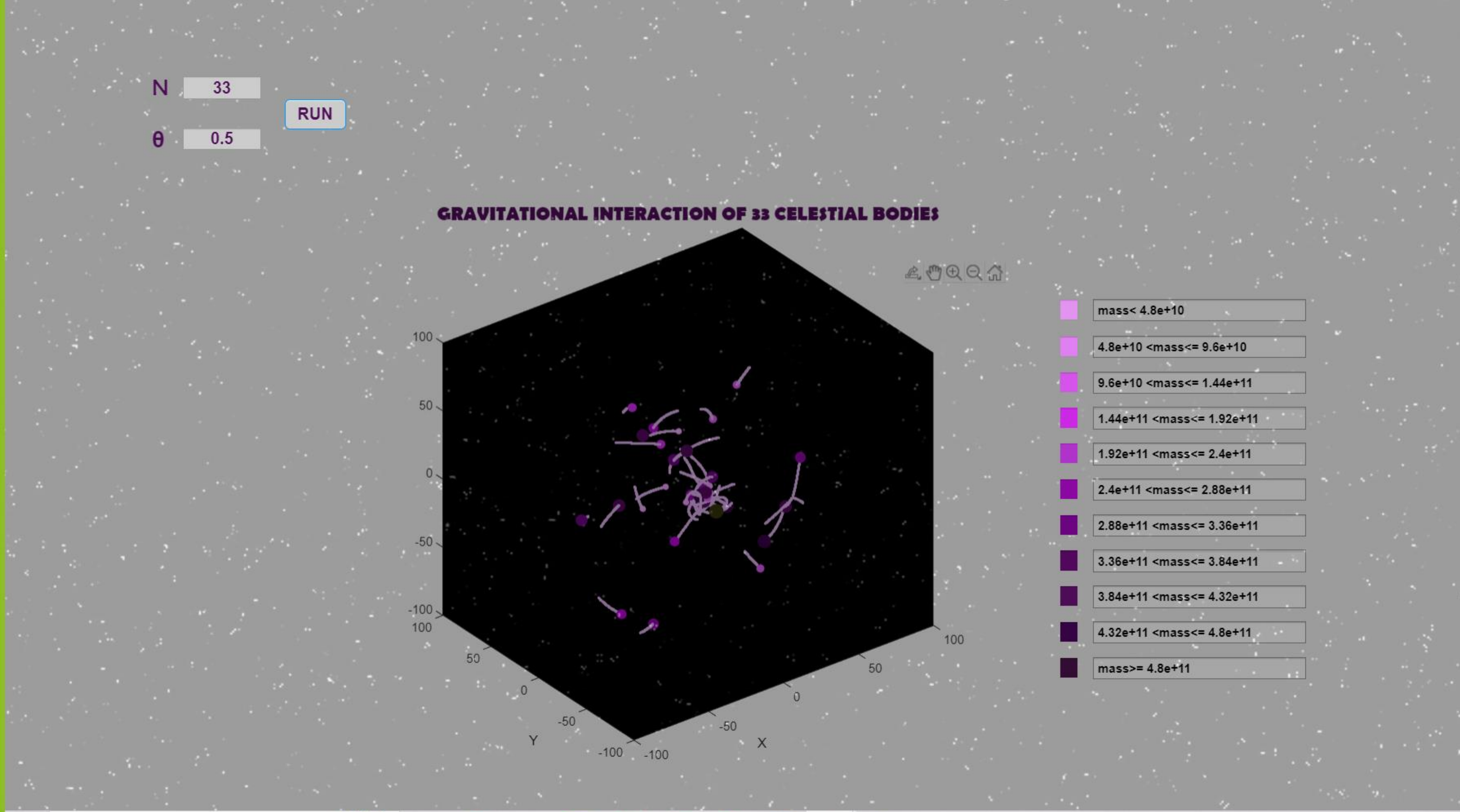
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RUN
θ
0.5
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0
-50
-100
Y
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-50
0
50
100
X
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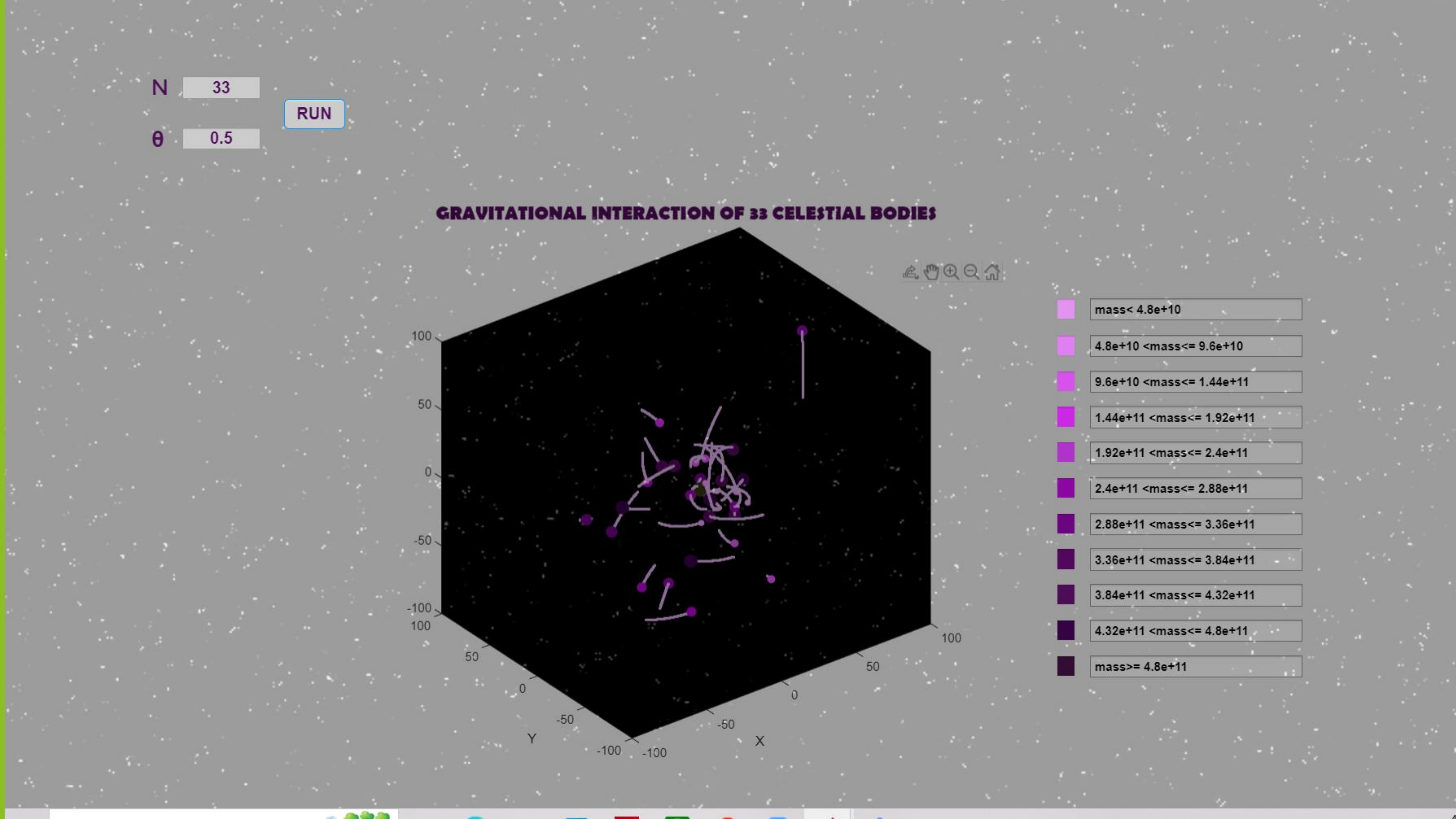
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RUN
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0.5
GRAVITATIONAL INTERACTION OF 33 CELESTIAL BODIES
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-50
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100
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0
-50
-100
Y
-100
-50
0
50
100
X
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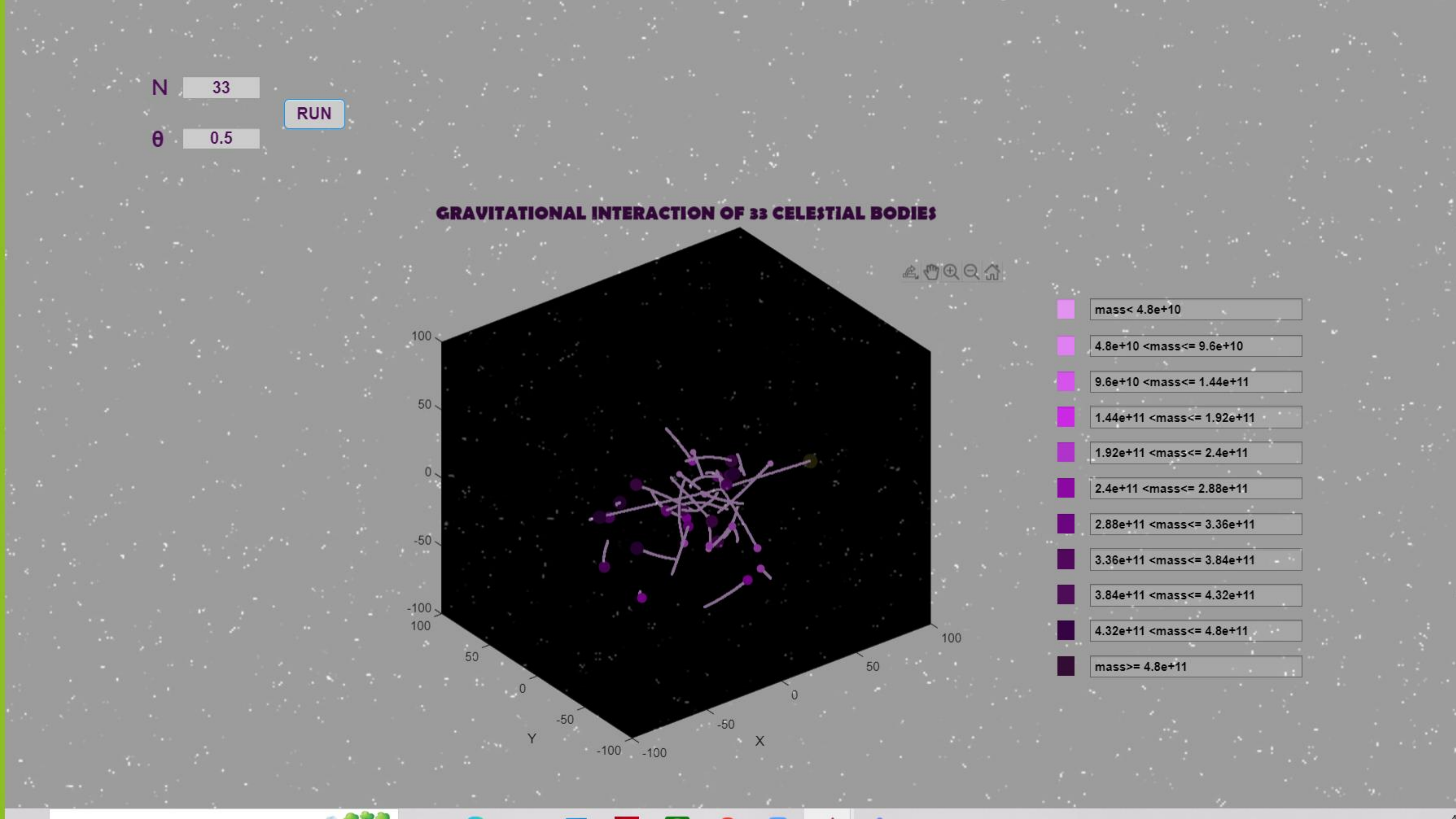
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RUN
θ
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GRAVITATIONAL INTERACTION OF 33 CELESTIAL BODIES
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-100
Y
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50
100
X
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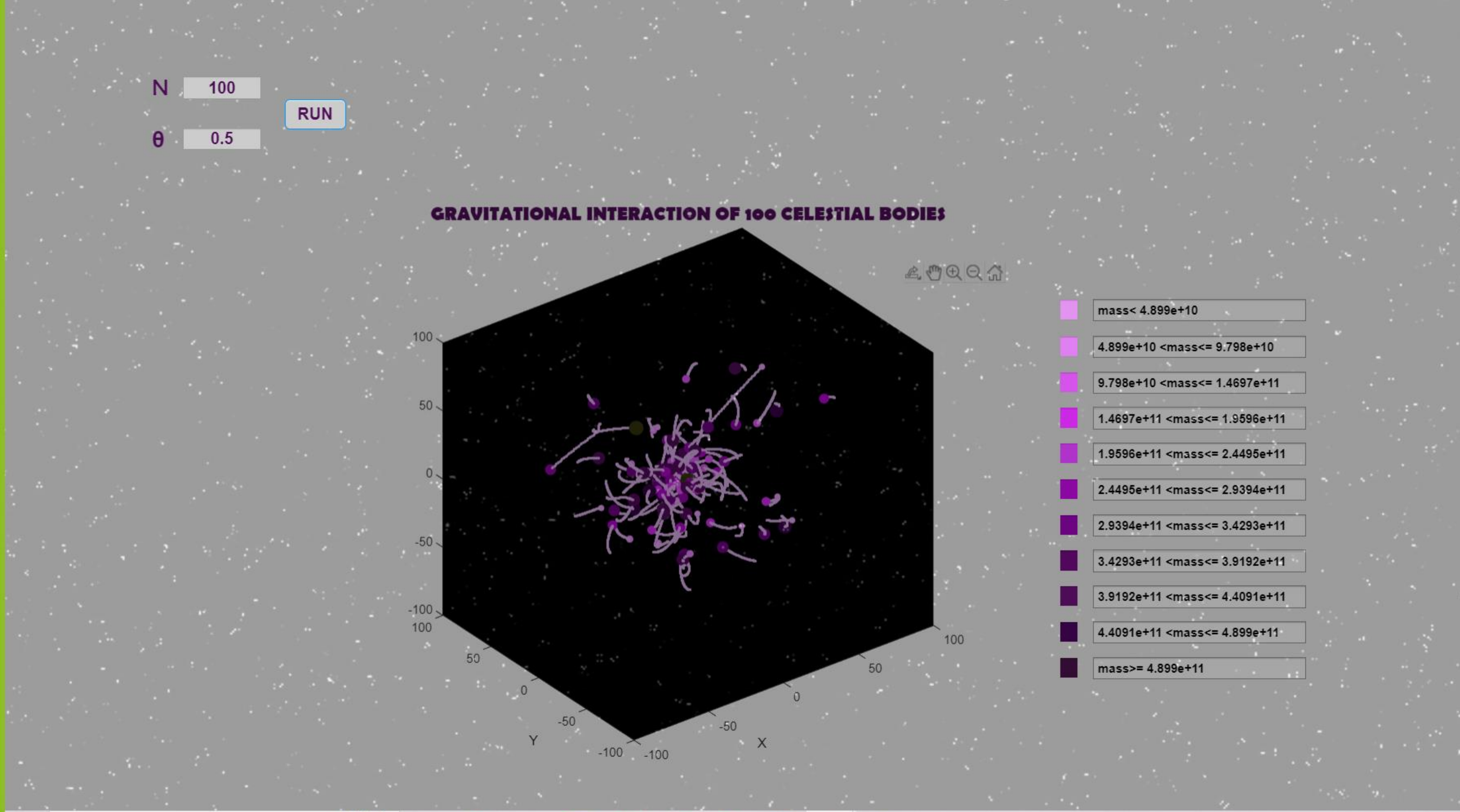

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Y
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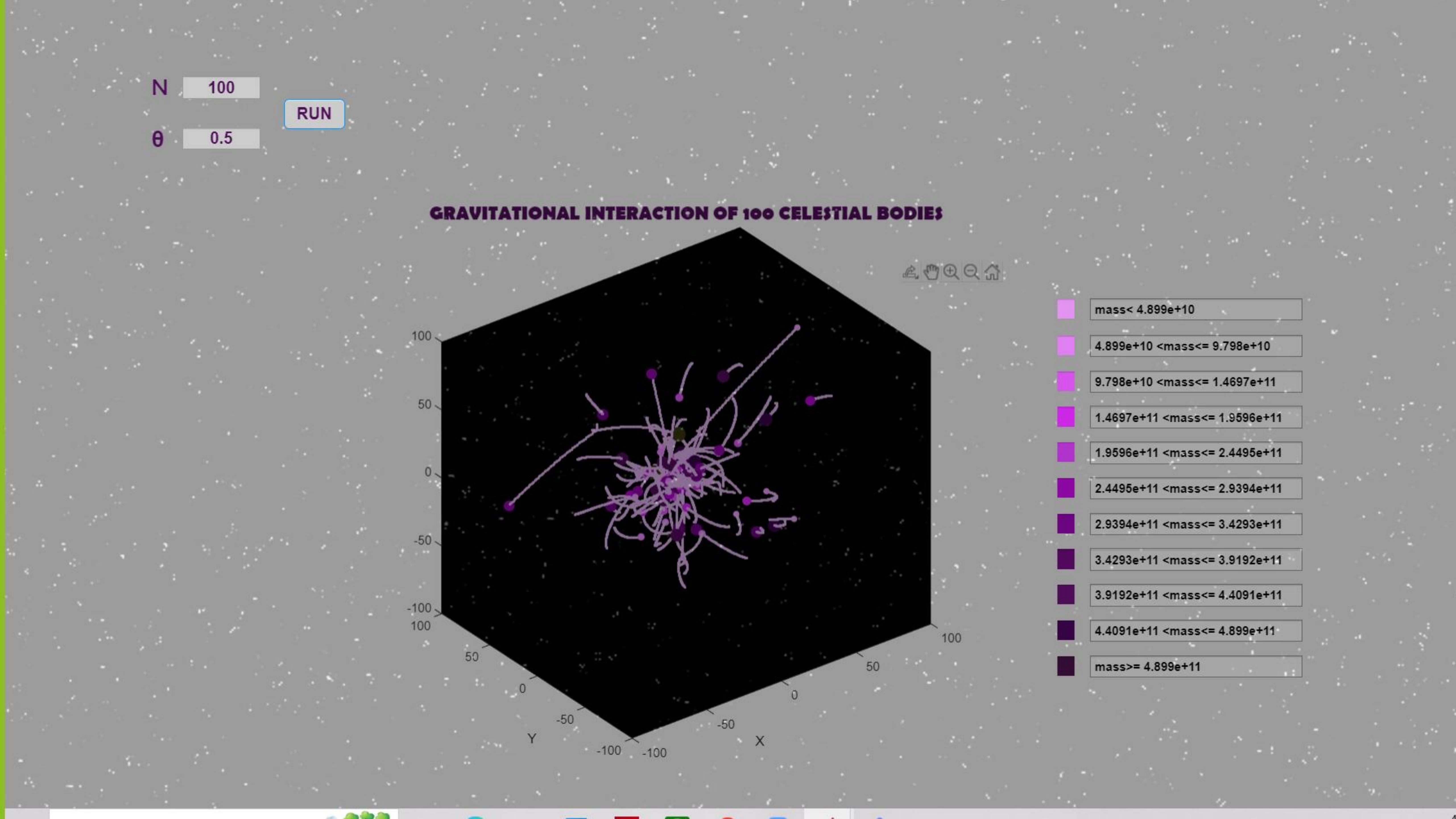
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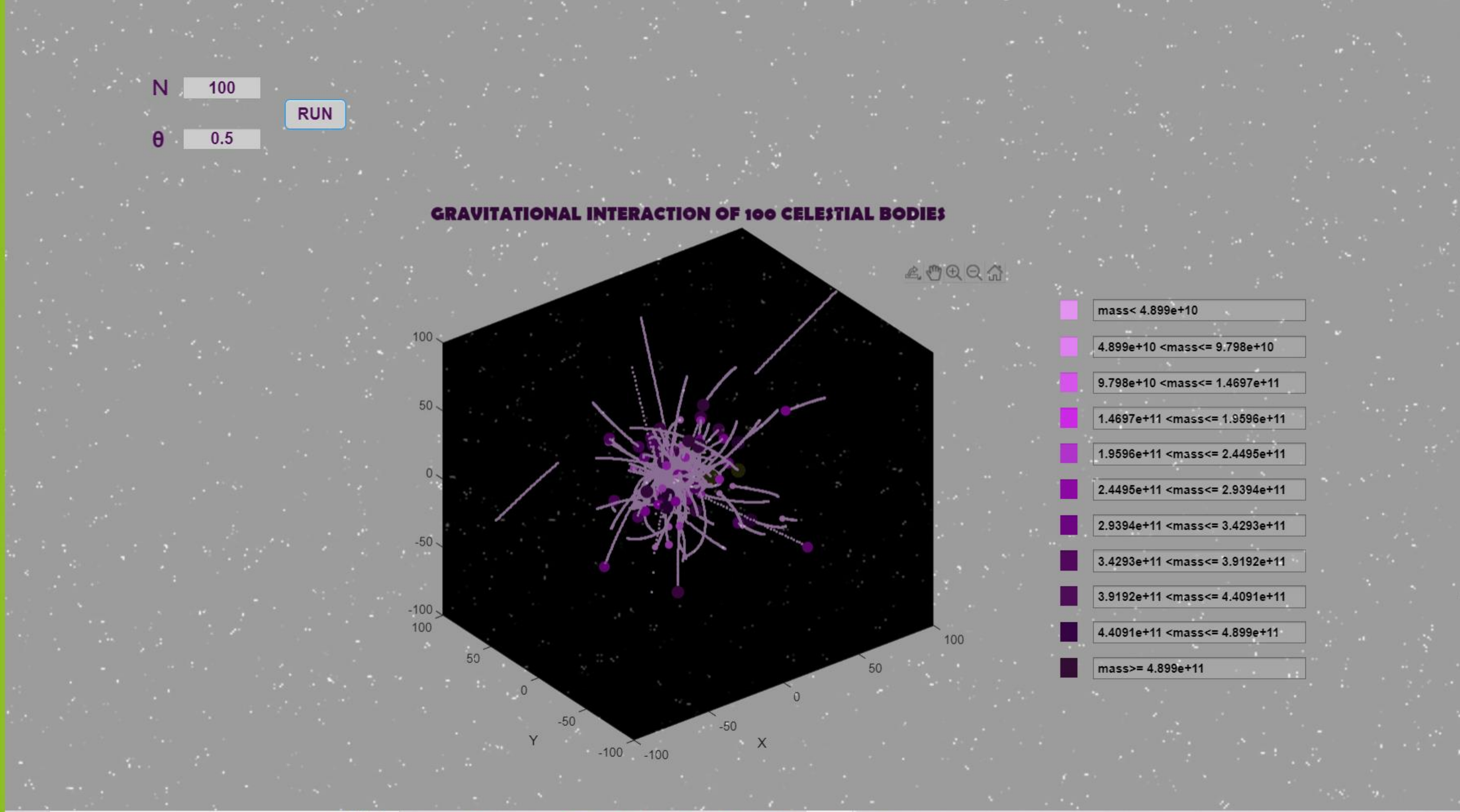
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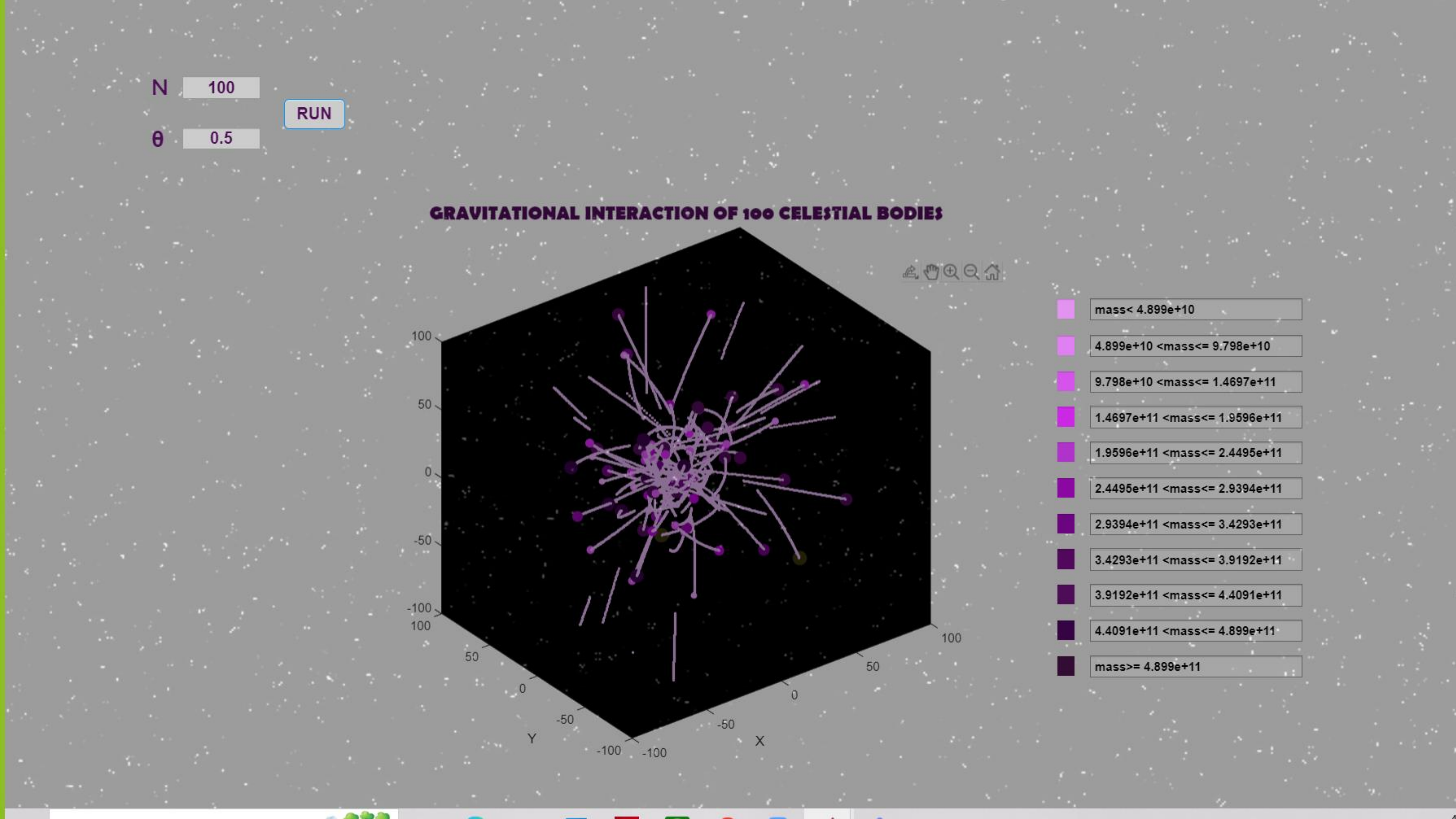
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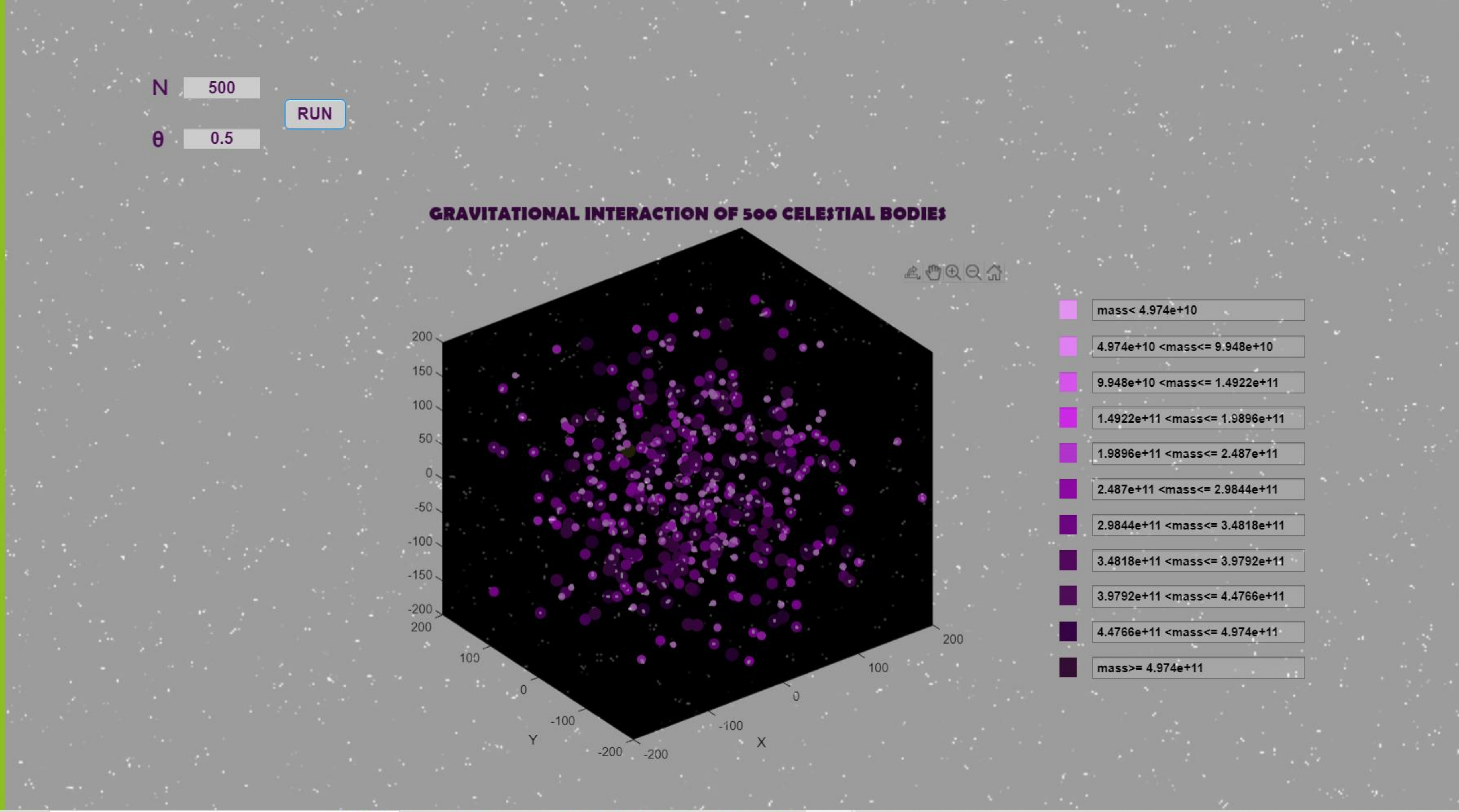
X
Y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RUN
θ 0.5
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150
100
50
0
-50
-100
-150
-200
Y
X
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9.948e+10 <mass<= 1.4922e+11
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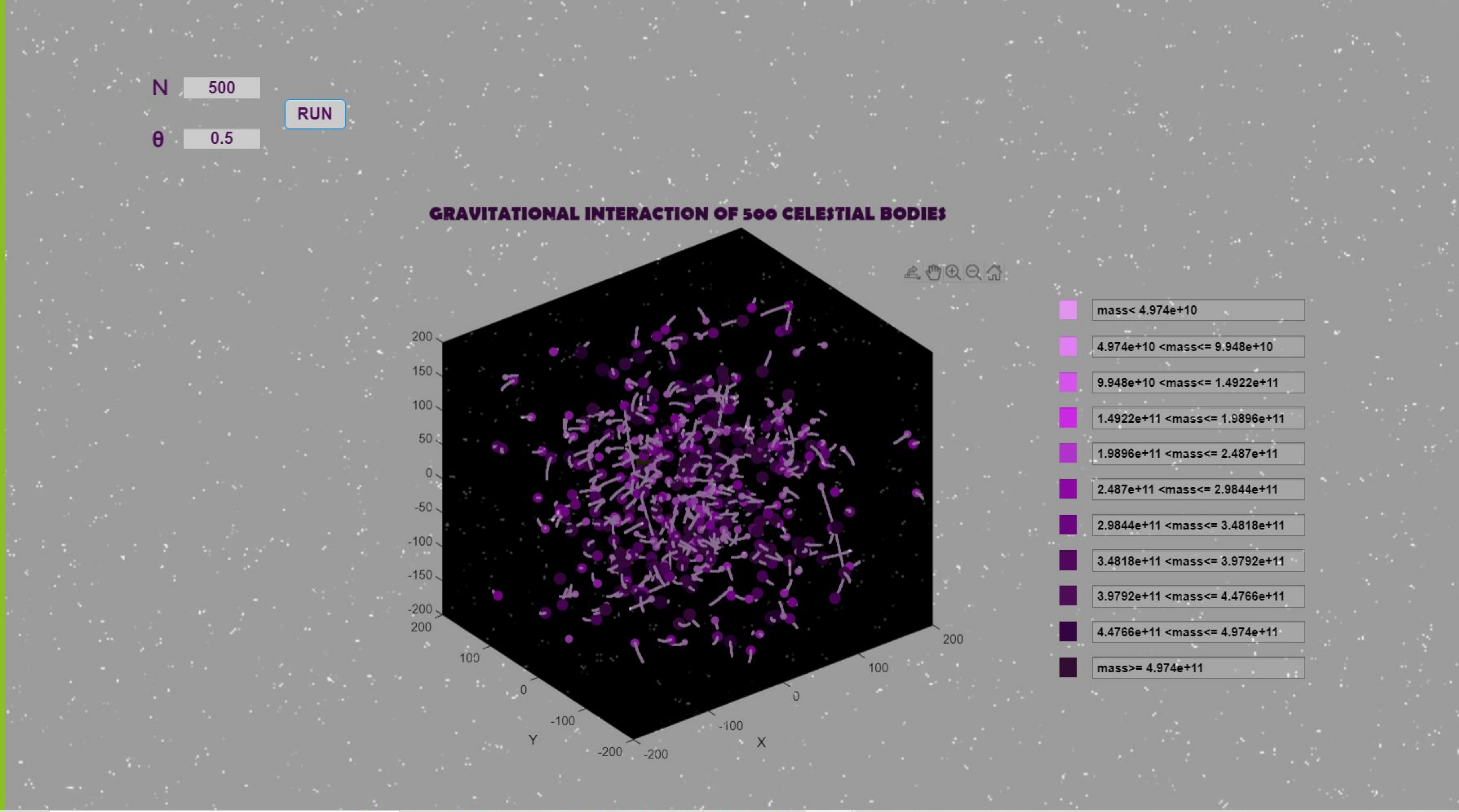
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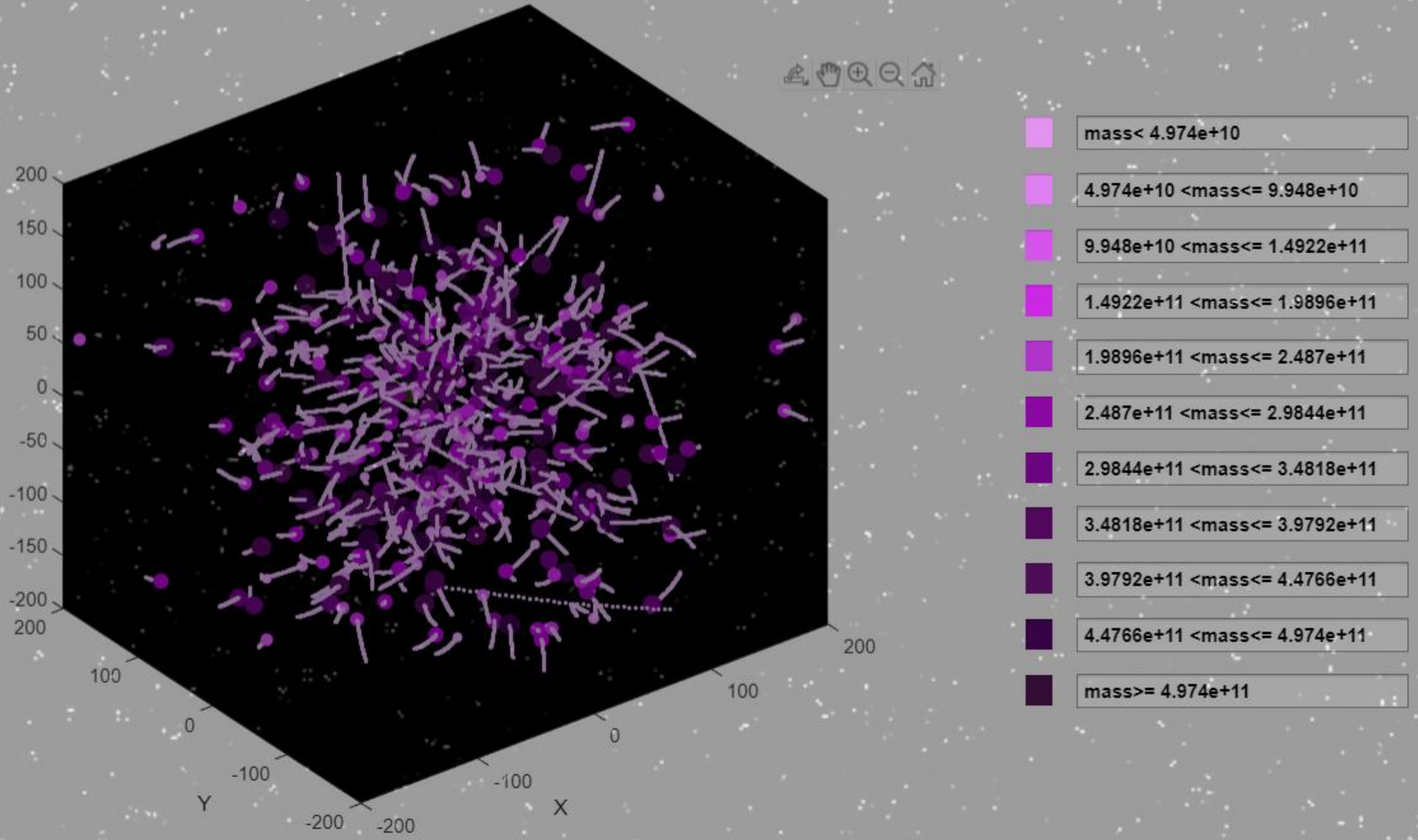
N 500

RUN

θ 0.5

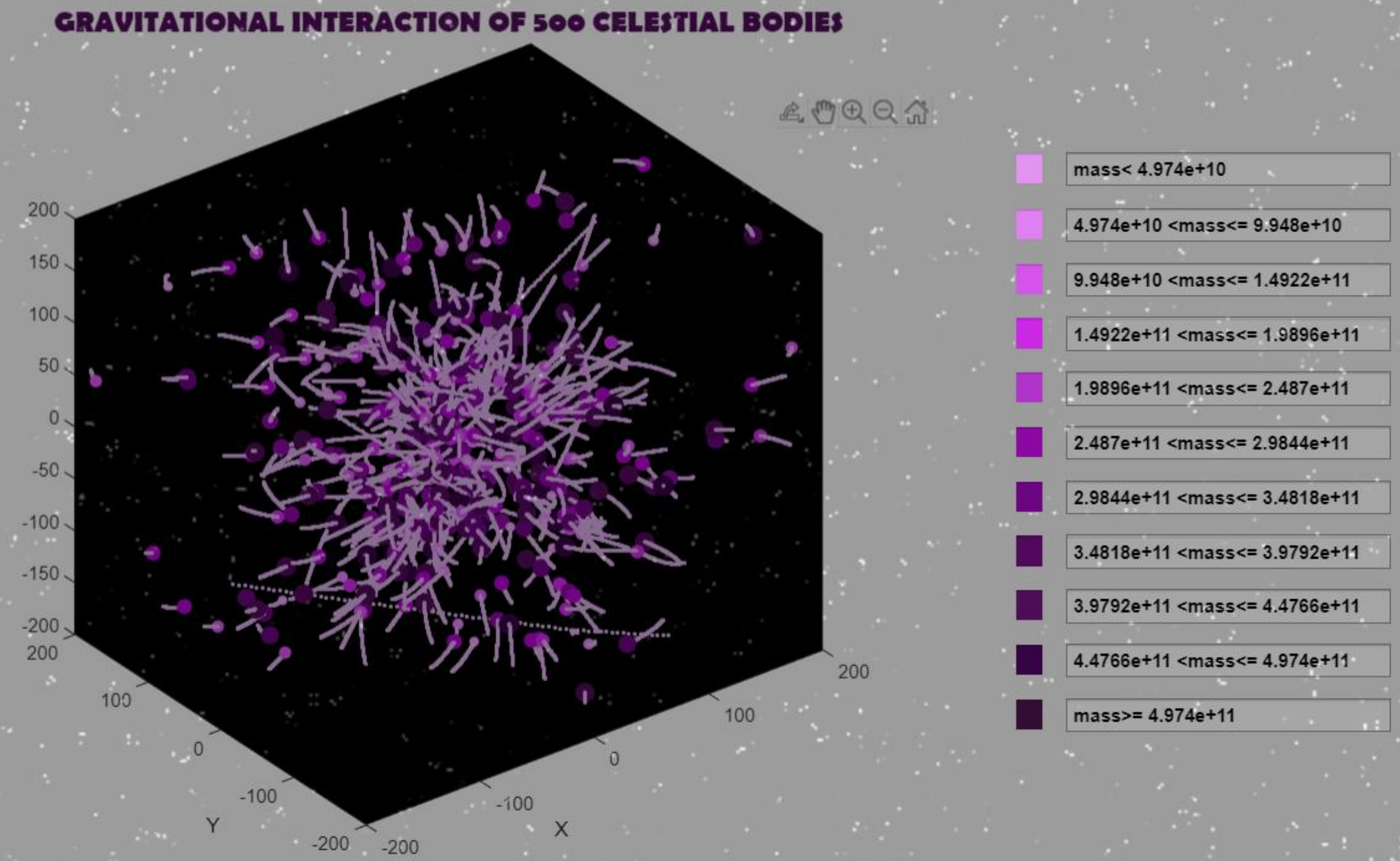
GRAVITATIONAL INTERACTION OF 500 CELESTIAL BODIES



N 500

RUN

θ 0.5


GRAVITATIONAL INTERACTION OF 500 CELESTIAL BODIES
200
150
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-150
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Y
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100
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X
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4.974e+10 <mass<= 9.948e+10
9.948e+10 <mass<= 1.4922e+11
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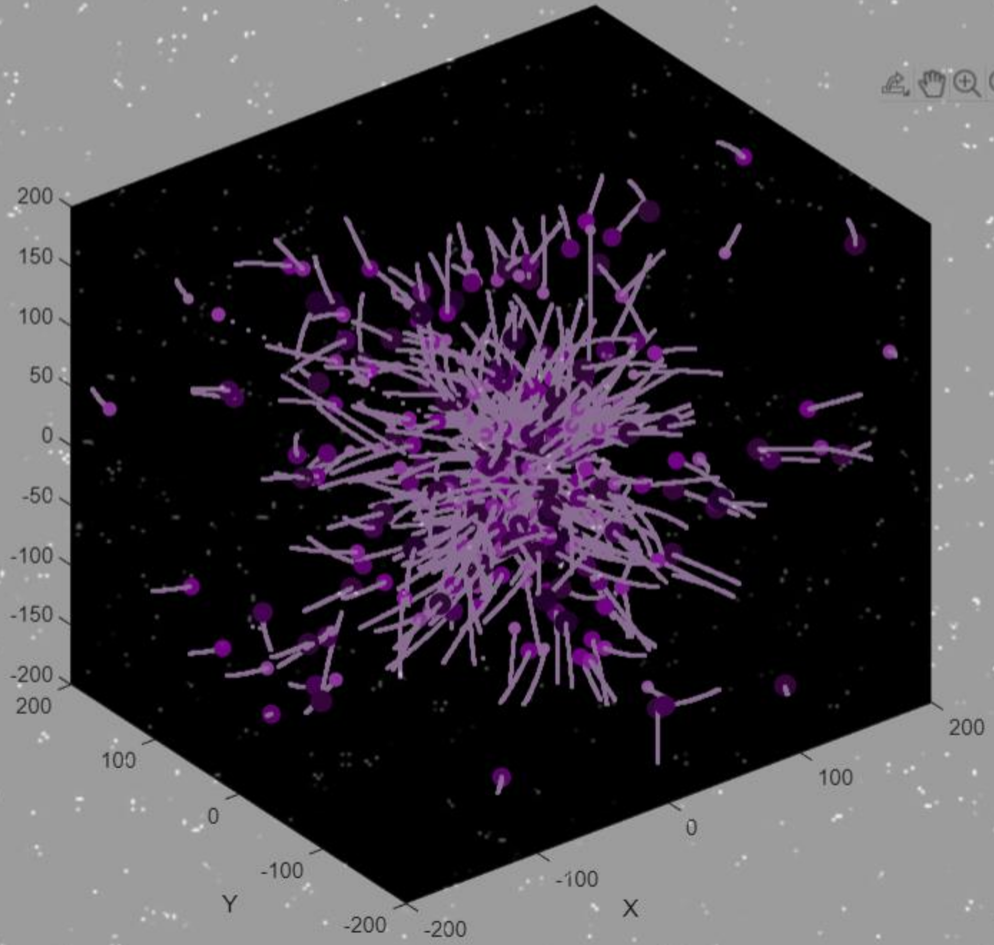
N 500

RUN

θ 0.5

GRAVITATIONAL INTERACTION OF 500 CELESTIAL BODIES



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2.487e+11 <mass<= 2.9844e+11

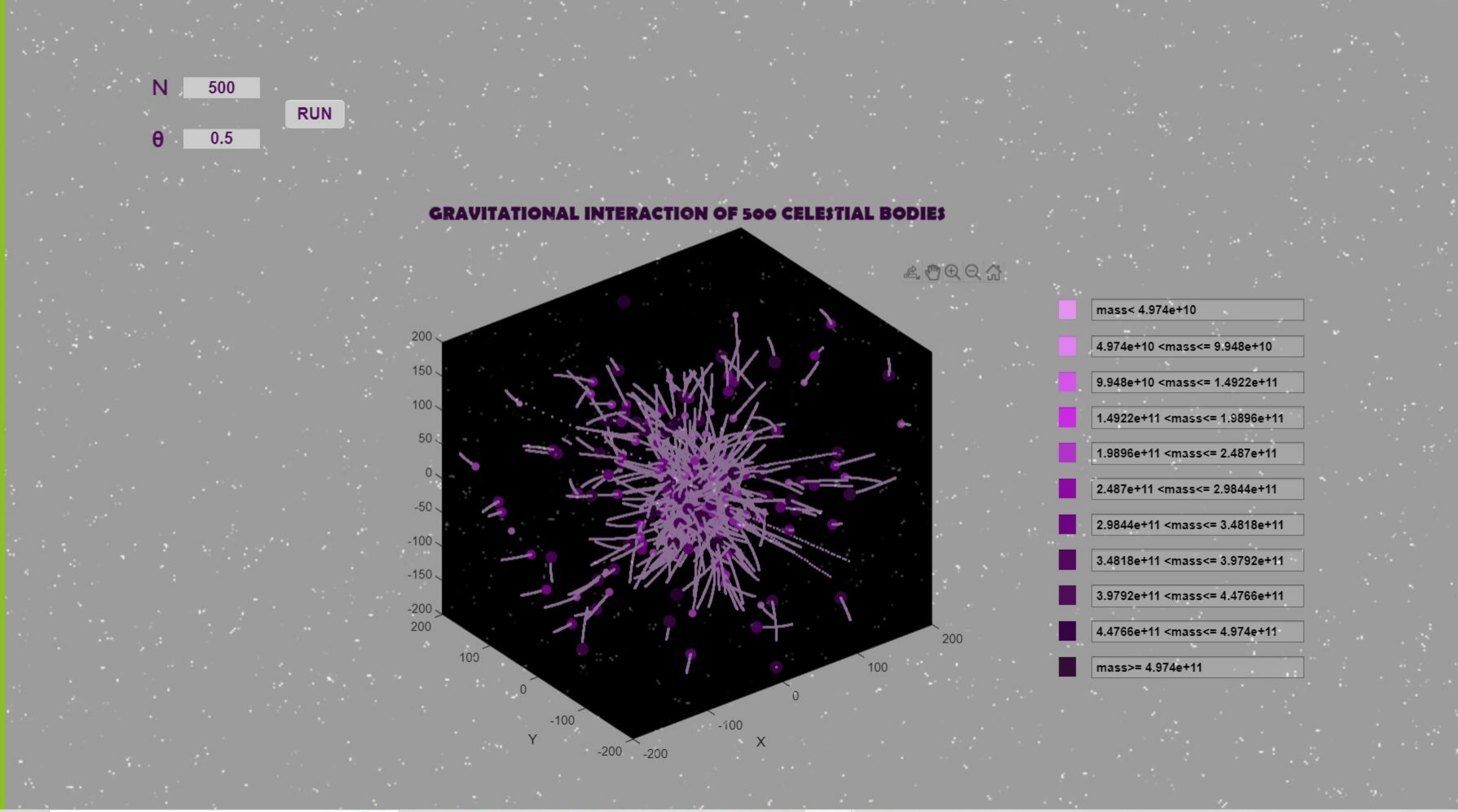
2.9844e+11 <mass<= 3.4818e+11

3.4818e+11 <mass<= 3.9792e+11

3.9792e+11 <mass<= 4.4766e+11

4.4766e+11 <mass<= 4.974e+11

mass>= 4.974e+11


N
500
RUN
θ
0.5
GRAVITATIONAL INTERACTION OF 500 CELESTIAL BODIES
200
150
100
50
0
-50
-100
-150
-200
Y
X
mass< 4.974e+10
4.974e+10 <mass<= 9.948e+10
9.948e+10 <mass<= 1.4922e+11
1.4922e+11 <mass<= 1.9896e+11
1.9896e+11 <mass<= 2.487e+11
2.487e+11 <mass<= 2.9844e+11
2.9844e+11 <mass<= 3.4818e+11
3.4818e+11 <mass<= 3.9792e+11
3.9792e+11 <mass<= 4.4766e+11
4.4766e+11 <mass<= 4.974e+11
mass>= 4.974e+11

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