Tiling Formulas for $(2 \times n)$ Bracelets

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Abstract

The Fibonacci numbers, which are defined as $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$, are renowned for their unique pattern and have been studied for over a thousand years. Instead of using pure algebraic methods, in "Proofs that Really Count" [1], the authors represent the Fibonacci numbers and their derivations with tilings consisting of squares (of length 1) and dominos (of length 2). Though the algebraic calculation might be an easier technique in the proofs in most cases, the combinatorial methods can give us more insights into the patterns that the identities demonstrate.

As one of the derivations of the Fibonacci numbers, the Lucas numbers can be represented by the tilings of circular bracelets. For the Lucas numbers, which are denoted as l_n, we can easily find a formula and a recursion relation (which are $l_n = f_n + f_{n-2}$ and $l_n = l_{n-1} + l_{n-2}$ respectively) in a combinatorial way. However, it becomes trickier if we consider bracelets with an additional layer. In this paper, we find four tiling formulas for the number of tilings of 2-layer bracelets. The identities of two other related types of tilings would also be investigated. We make the proofs mainly by classifying tilings according to different conditions. The only induction proof, which is for Formula 3.7, is also based on the other results deducted with a combinatorial method.

1 Introduction

f_n is defined as the number of ways to tile a board of length n with squares and dominos. The first few numbers of f_n are listed in Table 1. And for simplicity, we call a board of length n an n-board. Therefore, $f_4 = 5$ because it enumerates the 5 tilings as shown in Figure 1. (Squares and dominos are represented by 1's and 2's respectively)

n	0	1	2	3	4	5	6	7	8	9	10	...
f_n	1	1	2	3	5	8	13	21	34	55	89	...

Table 1: The first few numbers in a f_n sequence.

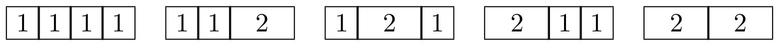

Figure 1: All five ways to tile a board of length 4 with squares and dominos.

As one of the derivations of the famous Fibonacci sequence, the Lucas numbers can be seen as the number of tilings of n-bracelets. They are tilings of circular n-boards. l_n is defined as the number of ways to tile a circular board composed of n cells with curved squares and dominos. The first few numbers of l_n are listed in Table 1. It is reasonable to infer l_0 to be 2 regarding its two "phases": in-phase and out-of-phase. There are clearly more ways to tile an n-bracelet than an n-board because a domino can occupy both Cell n and 1. Two phases are defined based on the connection between Cell n and 1. We call a bracelet in-phase if it breaks between Cell n and 1, and out-of-phase otherwise. For instance, we could see the seven tilings of a 4-bracelet in Figure 2. The first five are in-phase, and the last two are out-of-phase.

n	0	1	2	3	4	5	6	7	8	9	10	...
B_n	2	1	3	4	7	11	18	29	47	76	123	...

Table 2: The first few numbers in a l_n sequence.

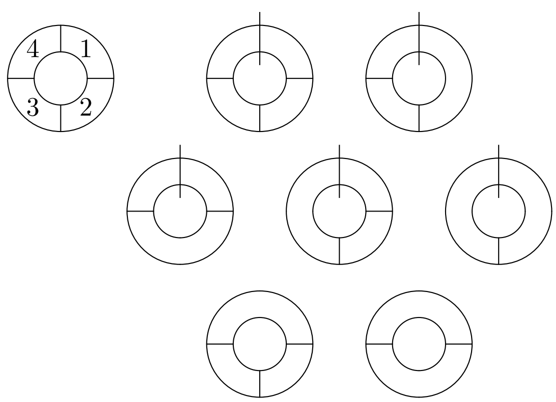

Figure 2: The 7 tilings of a 4-bracelet. The first five are in-phase and the last two are out-of-phase.

We could obtain a combinatorial identity of the Lucas numbers,

$$l_n = f_n + f_{n-2},$$

by conditioning on whether the tiling is in-phase or out-of-phase. The number of ways to tile an in-phase bracelet corresponds to f_n because the tiling can be straightened into an n-board. Likewise, since a domino covers Cell n and 1 when the bracelet is out-of-phase, the rest of the bracelet can be straightened into an $(n-2)$-board. Thus, the number of tilings corresponds to f_{n-2}.

Based on the identity, we can easily obtain the recursion relation of l_n:

$$l_n = l_{n-1} + l_{n-2}.$$

However, things become tricky when we add one more layer to either the board or the bracelet. Let us define the number of ways to tile a 2-layer board or bracelet as L_n and B_n respectively. The formula of L_n [2] has been proven to be

$$L_n = 3L_{n-1} + L_{n-2} - L_{n-3}.$$

However, we still do not know the formula of B_n.

In this paper we will find the general formulas of B_n, prove, and then explain them.

2 Definition

In this section we will define some terms and some basic identities of P_n, O_n and B_n.

2.1 Cell, Board, Bracelet, Gap, Breakability, Tile, Tiling

A *cell* is a notation of a certain position on a tiling. We can number a cell 1 to n depending on its position on a n-tiling.

A *board* is a straight strip. We call a board with n cells an *n-board*. Each cell is covered by either dominos or squares. A *$(2\times n)$-board* is a board with two layers.

A *bracelet* is a circular strip. We call a board with n cells an *n-bracelet*. Each cell is covered by either dominos or squares. A *$(2\times n)$-bracelet* is a bracelet with two layers.

A *gap* refers to the space between two cells regardless of whether there is an actual gap there. We can number a gap on a bracelet from 0 to $(n-1)$ depending on its position on the bracelet. We define the top gap as *Gap 0*.

We call a board or a bracelet *breakable* at gap n if there is no connection between the two adjacent cells.

To *tile* a board or a bracelet means to completely cover it by dominos and squares without any overlapping or leaving cells uncovered. A *tiling* is a correct solution to tile a board or a bracelet.

2.2 P_n Numbers and Tilings

P_n is the number of ways to tile a $(2 \times n)$-board with the upper-right cell removed. Let us call the board a P_n *board*. The first few numbers of P_n are listed in Table 3. Also, the first few numbers of L_n are listed in Table 4. Note that each number in P_n is a partial sum of L_n. This will be explained in Lemma 3.2.

n	0	1	2	3	4	5	6	7	8	9	...
P_n	1	3	10	32	103	331	1064	3420	10993	35335	...

Table 3: The first few numbers in a P_n sequence.

n	0	1	2	3	4	5	6	7	8	9	...
L_n	1	2	7	22	71	228	733	2356	7573	24342	...

Table 4: The first few numbers in a L_n sequence.

Take $n = 1$ for example, then $P_n = 3$. The three tilings are shown in Figure 3.

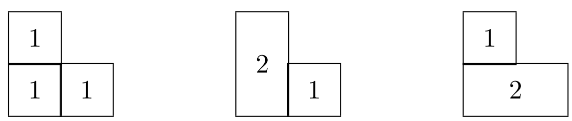

Figure 3: All three tilings of P_n.

2.3 O_n Numbers and Tilings

O_n is the number of ways to tile a $(2 \times n)$-board with the upper-left and upper-right cells removed. Let us call the board an O_n board. The first few numbers of O_n are listed in Table 5. Note that each number in O_n is a partial sum of P_n if n is odd, but it will be slightly larger by 1 if n is even, which will be explained in Lemma 3.3.

n	0	1	2	3	4	5	6	7	8	9	...
O_n	2	4	15	46	150	480	1545	4964	15958	51292	...

Table 5: The first few numbers in an O_n sequence.

Take $n = 1$ for example, then $O_1 = 4$. The four tilings are shown in Figure 4.

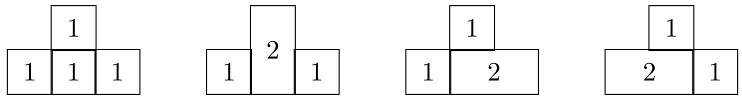

Figure 4: All three tilings of O_1.

2.4 B_n Numbers and Tilings

B_n is the number of ways to tile a $(2 \times n)$-bracelet. The first few numbers of B_n are listed in Table 6.

Since each $(2 \times n)$-bracelet has two layers, the phase conditions are more complicated than an n-bracelet's. A $(2 \times n)$-bracelet is *in-phase* if it breaks at Gap 0, or *out-of-phase* otherwise. We define a bracelet as *in-phase-top* if it only breaks at the upper layer at Gap 0, or *in-phase-bottom* otherwise. Note that we can obtain an in-phase-top bracelet by switching the two layers of an in-phase-bottom bracelet. Thus, there is a one-to-one correspondence relation between in-phase-top and in-phase-bottom. The four different phases are shown in Figure 5.

n	0	1	2	3	4	5	6	7	8	9	...
B_n	4	2	12	32	108	342	1104	3544	11396	36626	...

Table 6: The first few numbers in a B_n sequence.

Note that B_0 is equal to 4, which seems to be odd. But it is reasonable if we think about a 0-bracelet regarding its four phases.

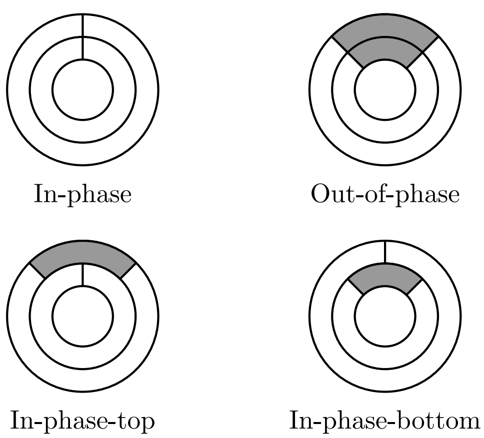

Figure 5: The four different phases of a bracelet.

3 Formulas for B_n

In this section, we will detail the proofs of four specific formulas for B_n.

3.1 Formula One

Theorem 3.1. *For $n \geq 2$, $B_n = 2O_{n-2} + L_n + L_{n-2}$.*

Proof. We can easily get the formula according to the four phases of a $2 \times n$-bracelet. See Figure 6. L_n and L_{n-2} can be obtained by unfolding the bracelet when it is in-phase or out-of-phase. (Remove the two dominos at the top when the bracelet is out-out-phase) And we can get the two O_{n-2} if we remove the domino and unfold the bracelet when the bracelet is either in-phase-top or in-phase-bottom.

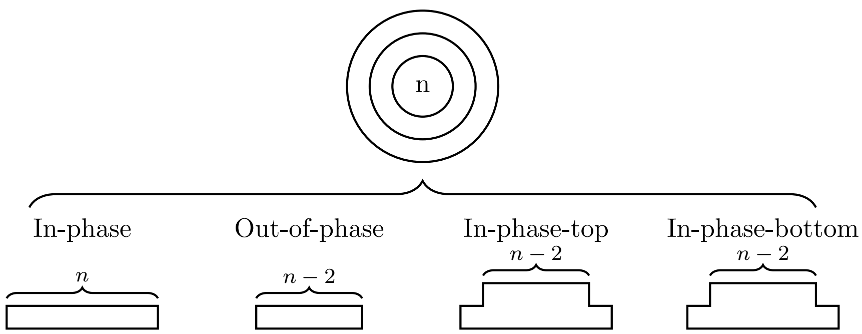

Figure 6: The four tilings that can be obtained according to the four different phases of a $(2 \times n)$-bracelet.

□

3.2 Formula Two

Though Theorem 3.1 is the fundamental combinatorial theorem of B_n, it requires 2 other sequences to calculate B_n. We can reduce the number of elements by deriving a formula for O_n in terms of L_n.

Lemma 3.2. *For $n \geq 0$, $P_n = \sum_{i=0}^{n} L_i$.*

Proof. Condition on whether Cell $(n+1)$ is occupied by a square or a domino. See Figure 7. If it is a square, we will get a $(2\times n)$-board by removing the square. If it is a domino, we will get a P_{n-1} board instead. Repeating the operation, we can derive the formula $P_n = \sum_{i=0}^{n} L_i$.

□

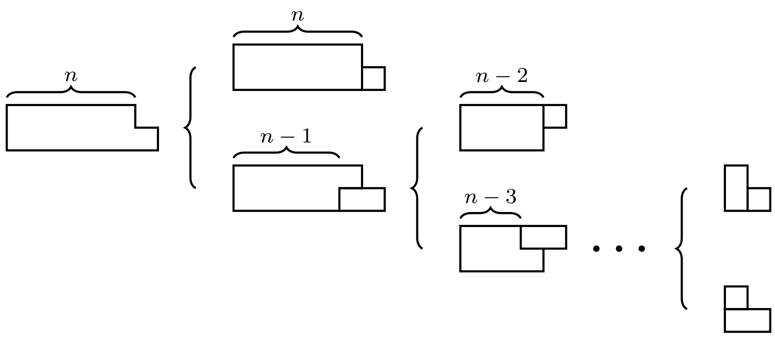

Figure 7: We break up the P_n board conditioning on whether a domino or a square occupies the last cell.

Lemma 3.3. *For* $n \geq 0$, $O_n = \sum_{i=0}^{n} P_i + \begin{cases} \textit{1 if n is even} \\ \textit{0 if n is odd} \end{cases}$.

Proof. Condition on whether Cell 1 is occupied by a square or a domino. See Figure 8. If it is occupied by a square, we will get a P_n board if we remove the square. If it is occupied by a domino, then condition on whether Cell $(n+1)$ is occupied by a square or a domino. If Cell $(n+1)$ is occupied by a square, we will get a P_{n-1} board by removing the square and the domino. If Cell $(n+1)$ is occupied by a domino, we will get a O_{n-2} board by removing the two dominos. Repeat the operation.

But when n is even, we should be careful about the last step of classification of the O_2 board, which generates a P_2 board, a P_1 board, and a 2-board (when Cell 1 and 4 are both occupied by dominos.) The ways to tile a 2-board is f_2, which is larger than P_0 by 1. See Figure 9. Then, we can derive the formula $O_n = \sum_{i=0}^{n} P_i + \begin{cases} \text{1 if n is even} \\ \text{0 if n is odd} \end{cases}$.

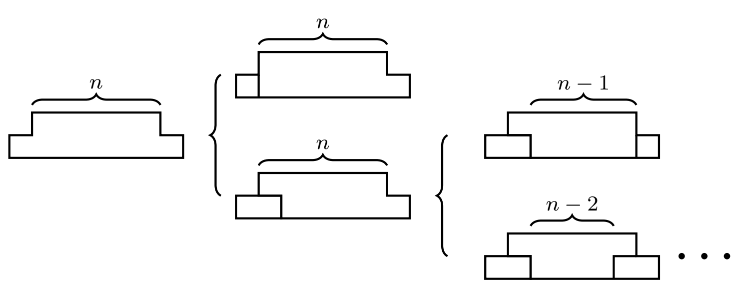

Figure 8: We break up the O_n board conditioning on whether a domino or a square occupies the first cell and the last cell respectively.

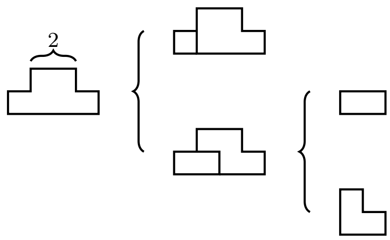

Figure 9: A O_2 board generates a P_2 board, a P_1 board, and a 2-board. And f_2 is larger than P_0 by 1.

□

Theorem 3.4.

$$For\ n \geq 2,\ B_n = L_n + L_{n-2} + 2\sum_{j=0}^{n-2}\sum_{i=0}^{j} L_i + \begin{cases} \mathit{1\ if\ n\ is\ even} \\ \mathit{0\ if\ n\ is\ odd} \end{cases}. \tag{1}$$

Proof. Substitute the terms of Theorem 3.1 with Lemma 3.3:

$$B_n = L_n + L_{n-2} + 2\sum_{i=0}^{n-2} P_i + \begin{cases} \text{1 if n is even} \\ \text{0 if n is odd} \end{cases}. \tag{2}$$

Then substitute the P_i in Eq. 2 with Lemma 3.2, and we can derive:

$$B_n = L_n + L_{n-2} + 2\sum_{j=0}^{n-2}\sum_{i=0}^{j} L_i + \begin{cases} \text{1 if n is even} \\ \text{0 if n is odd} \end{cases}.$$

Corollary 3.4.1.

$$For\ n \geq 2,\ B_n = L_n + L_{n-2} + 2\sum_{i=1}^{n-2} i \cdot L_{n-i-1} + \begin{cases} \mathit{1\ if\ n\ is\ even} \\ \mathit{0\ if\ n\ is\ odd} \end{cases}. \tag{3}$$

Now we already have a nice formula for B_n in terms of L_n, but we can simplify the formula by noticing that

$$\sum_{j=0}^{n-2}\sum_{i=0}^{j} L_i = L_{n-2} + 2L_{n-3} + 3L_{n-4} + \ldots + (n-2)L_1 + (n-1)L_0. \tag{4}$$

By observing the right side of Eq. 4, we can find the pattern of the formula, which is

$$\sum_{j=0}^{n-2} \sum_{i=0}^{j} L_i = \sum_{i=1}^{n-2} i \cdot L_{n-i-1}.$$

Thus, we can derive Corollary 3.4.1:

$$B_n = L_n + L_{n-2} + 2 \sum_{i=1}^{n-2} i \cdot L_{n-i-1} + \begin{cases} 1 \text{ if n is even} \\ 0 \text{ if n is odd} \end{cases}.$$

□

3.3 Formula Three

In this section, we will derive a recursion relation for B_n, which can be proven by the recursive formula of L_n, P_n, and O_n step by step.

Lemma 3.5. *For $n \geq 4$, $P_n = 2P_{n-1} + 4P_{n-2} - P_{n-4}$.*

Proof. Let us prove this recursion relation by induction.

BASE CASE ($n = 4$):

$$P_4 = 103$$

$$2P_3 + 4P_2 - P_0 = 64 + 40 - 1 = 103.$$

Hence,

$$P_4 = 2P_3 + 4P_2 - P_0.$$

The statement is true when $n = 4$.

INDUCTIVE STEP: Assume it is true for $n = k$. By applying Lemma 3.2, then

$$\begin{aligned} P_{k+1} &= P_k + L_{k+1} \\ &= 2P_{k-1} + 4P_{k-2} - P_{k-4} + L_{k+1} \\ &= 2\sum_{i=0}^{k-1} L_i + 4\sum_{i=0}^{k-2} L_i - \sum_{i=0}^{k-4} L_i + L_{k+1}. \end{aligned} \tag{5}$$

Since

$$L_k = 3L_{k-1} + L_{k-2} - L_{k-3},$$

thus,

$$
\begin{aligned}
P_{k+1} &= 2\sum_{i=0}^{k-1} L_i + 4\sum_{i=0}^{k-2} L_i - \sum_{i=0}^{k-4} L_i + 3L_k + L_{k-1} - L_{k-2} \\
&= 2\sum_{i=0}^{k-1} L_i + 4\sum_{i=0}^{k-2} L_i - \sum_{i=0}^{k-4} L_i + 2L_k + 4L_{k-1} - L_{k-3} \\
&= 2\sum_{i=0}^{k} L_i + 4\sum_{i=0}^{k-1} L_i - \sum_{i=0}^{k-3} L_i \\
&= 2P_k + 4P_{k-1} - P_{k-3}.
\end{aligned} \tag{6}
$$

Thus, the statement is true when $n = k + 1$. □

Lemma 3.6. *For $n \geq 4$, $O_n = 2O_{n-1} + 4O_{n-2} - O_{n-4}$.*

Proof. Let us prove this recursion relation by induction.

BASE CASE ($n = 4$):

$$O_4 = 150$$

$$2O_3 + 4O_2 - O_0 = 92 + 60 - 2 = 150.$$

Hence,

$$O_4 = 2O_3 + 4O_2 - O_0.$$

The statement is true when $n = 4$.

INDUCTIVE STEP: Assume it is true for $n = k$. By applying Lemma 3.3, if k is even, then

$$
\begin{aligned}
O_{k+1} &= O_k + P_{k+1} - 1 \\
&= 2O_{k-1} + 4O_{k-2} - O_{k-4} + P_{k+1} - 1 \\
&= 2\sum_{i=0}^{k-1} P_i + 4\sum_{i=0}^{k-2} P_i - \sum_{i=0}^{k-4} P_i + P_{k+1} + 2.
\end{aligned} \tag{7}
$$

Since

$$P_{k+1} = 2P_k + 4P_{k-1} - P_{k-3},$$

thus,

$$
\begin{aligned}
O_{k+1} &= 2\sum_{i=0}^{k-1} P_i + 4\sum_{i=0}^{k-2} P_i - \sum_{i=0}^{k-4} P_i + 2P_k + 4P_{k-1} - P_{k-3} + 2 \\
&= 2\sum_{i=0}^{k} P_i + 4\sum_{i=0}^{k-1} P_i - \sum_{i=0}^{k-3} P_i + 2 \\
&= 2O_k + 4O_{k-1} - O_{k-3}.
\end{aligned} \tag{8}
$$

Hence, the theorem is true when k is even.

If k is odd, then

$$\begin{aligned}
O_{k+1} &= O_k + P_{k+1} + 1 \\
&= 2O_{k-1} + 4O_{k-2} - O_{k-4} + P_{k+1} + 1 \\
&= 2\sum_{i=0}^{k-1} P_i + 4\sum_{i=0}^{k-2} P_i - \sum_{i=0}^{k-4} P_i + P_{k+1} + 3 \\
&= 2\sum_{i=0}^{k-1} P_i + 4\sum_{i=0}^{k-2} P_i - \sum_{i=0}^{k-4} P_i + 2P_k + 4P_{k-1} - P_{k-3} + 3 \\
&= 2\sum_{i=0}^{k} P_i + 4\sum_{i=0}^{k-1} P_i - \sum_{i=0}^{k-3} P_i + 3 \\
&= 2O_k + 4O_{k-1} - O_{k-3}.
\end{aligned} \tag{9}$$

Hence, the theorem is true when n is odd.

□

Theorem 3.7. *For $n \geq 4$, $B_n = 2B_{n-1} + 4B_{n-2} - B_{n-4}$.*

Proof. The formula can be proven by induction similar to the proof of Lemma 3.5 and Lemma 3.6.

BASE CASE ($n = 4$):

$$B_4 = 108$$

$$2B_3 + 4B_2 - B_0 = 64 + 48 - 4 = 108.$$

.

Hence,

$$B_4 = 2B_3 + 4B_2 - B_0.$$

The statement is true when $n = 4$.

INDUCTIVE STEP: Assume it is true for $n = k$. By applying Theorem 3.1, then

$$B_{k+1} = 2O_{k-1} + L_{k+1} + L_{k-1} \tag{10}$$

and

$$B_k = 2O_{k-2} + L_k + L_{k-2}. \tag{11}$$

Eq. 10 − Eq. 11

$$\begin{aligned}
\Rightarrow B_{k+1} - B_k &= 2(O_{k-1} - O_{k-2}) + L_{k+1} - L_k + L_{k-1} - L_{k-2} \\
&= 2(O_{k-2} + 4O_{k-3} - O_{k-5}) + L_{k+1} - L_k + L_{k-1} - L_{k-2}.
\end{aligned} \tag{12}$$

Since $B_{k+1} = B_k +$ Eq. 12, by applying Theorem 3.1, then

$$
\begin{aligned}
B_{k+1} &= 2B_{k-1} + 4B_{k-2} - B_{k-4} \\
&\quad + 2(O_{k-2} + 4O_{k-3} - O_{k-5}) + L_{k+1} - L_k + L_{k-1} - L_{k-2} \\
&= 2(2O_{k-3} + L_{k-1} + L_{k-3}) \\
&\quad + 4(2O_{k-4} + L_{k-2} + L_{k-4}) - (2O_{k-6} + L_{k-4} + L_{k-6}) \\
&\quad + 2(O_{k-2} + 4O_{k-3} - O_{k-5}) + L_{k+1} - L_k + L_{k-1} - L_{k-2}.
\end{aligned} \tag{13}
$$

Let us break down the terms by applying Lemma 3.6 and $L_k = 3L_{k-1} + L_{k-2} - L_{k-3}$, then combine the like terms:

$$
\begin{aligned}
B_{k+1} &= 2(2O_{k-2} + L_k + L_{k-2}) \\
&\quad + 4(2O_{k-3} + L_{k-1} + L_{k-3}) - (2O_{k-5} + L_{k-3} + L_{k-5}) \\
&= 2B_k + 4B_{k-1} - B_{k-3}.
\end{aligned} \tag{14}
$$

Thus, the statement is true for $n = k + 1$.

□

3.4 Formula Four

Theorem 3.8.

$$
For\ n \geq 2, B_n = 2n + 2\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1} L_{j-i} + 2L_{n-2} + \begin{cases} \textit{2 if n is even} \\ \textit{0 if n is odd} \end{cases}. \tag{15}
$$

Proof. The formula can be proven by conditioning on the breakability of the bracelet.

- If the bracelet is unbreakable through Cell 1 to n, the tilings should be entirely covered by dominos. Thus, when the bracelet is unbreakable, we can obtain two tilings if n is even, while we cannot get any tiling if n is odd. So n must be even if the bracelet is completely tiled by dominos. The example of the two unbreakable tilings of B_4 are shown in Figure 10. And we can conclude that $\begin{cases} \text{2 if n is even} \\ \text{0 if n is odd} \end{cases}$.

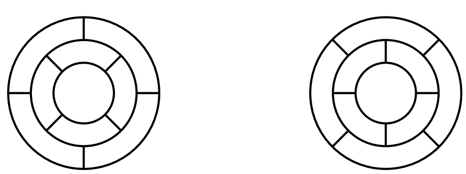

Figure 10: The two unbreakable tilings of B_4.

- If the bracelet is only breakable at one gap, then we can deduce that there are $2n$ tilings by conditioning on where the bracelet break. The example of four tilings of B_4, which break at Gap 0, 1, 2, and 3 respectively, are shown in Figure 11. We can get another group of four tilings by merely switching the two layers of the bracelets.

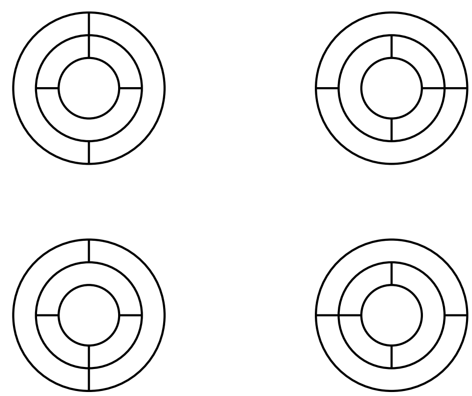

Figure 11: The four tilings of a B_4-board that is only breakable at one gap.

- If the bracelet is breakable at more than one gap, then condition on the first gap that the bracelet breaks on the two sides of Cell $(n-1)$. Let us define the first gap on the right side as Gap i and the first gap on the left side as Gap j, and $j > i$.

  The number of tilings of an unbreakable $(2 \times n)$-board is always 2 for $n \geq 1$ except for $n = 2$. There are 3 tilings for a (2×2)-board. The exceptional tiling consists of two horizontal dominos. Note that a $(2 \times n)$-board which is entirely tiled by dominos is unbreakable only when $n = 2$. Thus, we can obtain $2 \cdot L_{j-i}$ tilings from each pair of i and j except for $n - j = 2$.

  Hence, the total number of tilings of bracelets that are breakable at more than one gap is equal to $2\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1} L_{j-i} + 2L_{n-2}$.

  An example of the tilings of B_4 under this condition is shown in Figure 12.

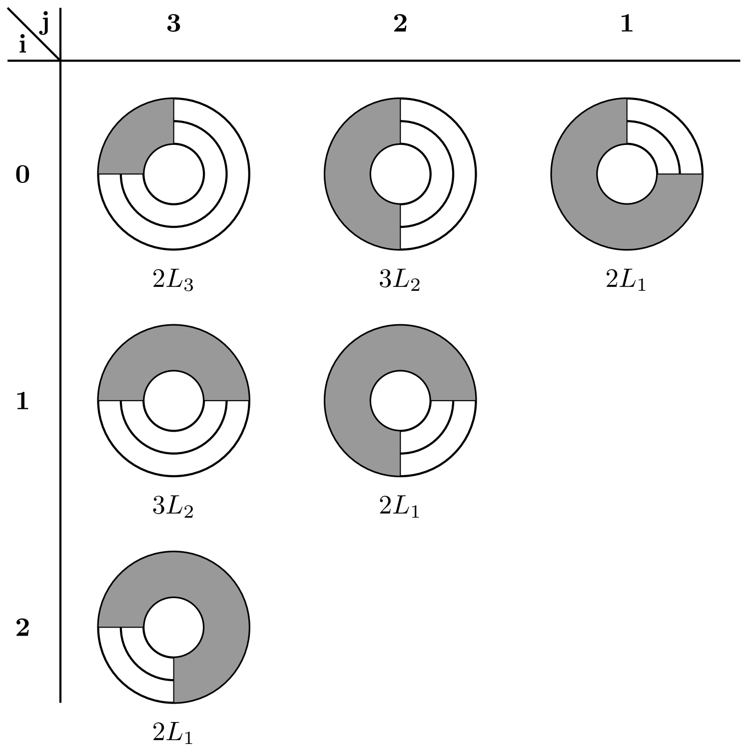

Figure 12: The shadow is the unbreakable part of the tiling. The total number of tilings can be represented by $2L_2 + \sum_{i=0}^{2} \sum_{j=i+1}^{3} L_{j-i}$.

We combine the counts of all three types of situations. Then we can derive the formula:

$$B_n = 2n + 2\sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} L_{j-i} + 2L_{n-2} + \begin{cases} 2 \text{ if n is even} \\ 0 \text{ if n is odd} \end{cases} . \tag{16}$$

□

4 Future Work

We have found four different theorems for the tiling formulas of B_n together with the formula expressions of P_n and O_n, which function as lemmas for the theorems. However, there is still more to be learned about $(2 \times n)$-bracelets.

As we have discovered numerous identities for the Fibonacci numbers, we might discover some ingenious identities by investigating the B_n sequence or making connection between the B_n sequence and the other related sequences. Moreover, we would also like to find new tiling formulas for B_n by making variations for the tilings. For example, we can change the length of the domino, then we could generate and study an entire new sequence as well as different tiling formulas. And the condition of phases would be more complicated in this case.

References

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