Projectile Motion

Introduction to Projectile Motion

Projectile motion refers to the motion of an object projected into the air and subject only to acceleration as a result of gravity. The object is called a projectile, and its path is called its trajectory.

Equations of Motion in Projectile Motion

Recall that in two-dimensional space, we can treat the horizontal and vertical components of the motion separately.

In Projectile Motion, generally, we will assume that the object projected with an initial velocity of v_0 and an angle of θ. Therefore, the initial horizontal and vertical components of the motion are given by:

$$\begin{aligned} v_{x0} &= v_0 \cos(\theta) \\ v_{y0} &= v_0 \sin(\theta) \end{aligned} \tag{1}$$

Horizontal Part of Projectile Motion

The horizontal part of the projectile motion is not affected by gravity. Sometimes it might contain air resistance or drag, depending on the situation, but these are not considered here. Therefore, the acceleration can be neglected. Thus, we can write the equations of motion as:

$$v_x(t) = v_{x0} = v_0 \cos(\theta) \tag{2}$$

$$x(t) = x_0 + v_{x0}t = x_0 + v_0 \cos(\theta)t \tag{3}$$

Vertical Part of Projectile Motion

The vertical part of the projectile motion is affected by gravity, so the acceleration can be taken into account. Thus, we can write the equations of motion as:

$$v_y(t) = v_{y0} - gt = v_0 \sin(\theta) - gt \tag{4}$$

$$y(t) = y_0 + v_{y0}t - \frac{1}{2}gt^2 = y_0 + v_0 \sin(\theta)t - \frac{1}{2}gt^2 \tag{5}$$

where $g = 9.8 \ \mathrm{m/s}^2$ is the acceleration due to gravity.

We also have the following relations between height and vertical velocity:

$$v_y^2 = v_{y0}^2 - 2g(y - y_0) = v_0^2 \sin^2(\theta) - 2g(y - y_0) \tag{6}$$

Path of Projectile

With the equations of motion, we can also find the path of the projectile. We can use the the horizontal and vertical components of the position to find the path of the projectile:

If we consider the initial position of projectile as origin, we can use time to connect the horizontal and vertical components of the motion:

$$(3) \Rightarrow t = \frac{x}{v_0 \cos(\theta)} \tag{7}$$

Substitute equation (7) into equation (5)

$$(5) \Rightarrow y = v_0 \sin(\theta)\left(\frac{x}{v_0 \cos(\theta)}\right) - \frac{1}{2}g\left(\frac{x}{v_0 \cos(\theta)}\right)^2 = \tan(\theta)x - \left(\frac{gx^2}{2{v_0}^2 \cos^2(\theta)}\right) \tag{8}$$

As the formula shows, the path of the projectile is a parabola. (See Introduction to Parabola)

Sample Problem #1

A ball is launched with an initial velocity v_0 with magnitude $10\sqrt{2}\ \mathrm{m/s}$ and an angle of $\theta = 45^{\circ}$. What is the function of the position over time?

Sol.

What is the path of the ball (in the form of a parabola)?

Sol.

The Horizontal Range

If we consider a situation where a person is throwing a ball, it's not hard to find that both the angle and the initial velocity are important. If throwing a ball harder, we can find the horizontal range of the ball is longer. And if we throw a ball with different angle, it will also affect the horizontal range of the ball.

In Projectile Motion, we can find the horizontal range by following method.

Firstly, we can express the vertical range of the object as follows (assuming that the object lands at $R = x_L\hat{i} + 0\hat{j}$ when $t = t_L$):

$$(3) \Rightarrow x_L = v_0 \cos(\theta) t_L \tag{9}$$

Then we can find that the range is related to the time the object is in the air. If we use formula (5), we can precisely determine when the object will land (assuming it lands at $y = 0$).

$$(5) \Rightarrow y_L = v_0 \sin(\theta) t_L - \frac{1}{2} g t_L^2 = 0 \Rightarrow t_L = \frac{2v_0 \sin(\theta)}{g} \tag{10}$$

By substituting the time into formula (9), we can determine the horizontal range of the object:

$$(9) \Rightarrow x_L = \Big(v_0 \cos(\theta)\Big)\left(\frac{2v_0 \sin(\theta)}{g}\right) = \frac{v_0^2 \sin(2\theta)}{g} \tag{11}$$

What makes an object fly farther ?

From equation (11), we can see that if we want to make the object fly farther, we can either increase the initial velocity or change the launch angle to 45°.

Sample Problem #2

Assume that the object is thrown with same magnitude in the initial velocity, and the launch and landing heights is the same. (a) Show that the horizontal range at 30° and 60° is the same. (b) Show that the horizontal range at 45° is maximized.

Sol.

Max Height

The maximum height is also an important factor in projectile motion. We can find the maximum height by following methods.

Method 1: Use the equation of motion to find the point maximum height.

As we know, the maximum height is reached when the vertical velocity becomes zero. (Or in Math , the maximum height means the first derivative of the height function is zero, we also need to check the second derivative of the height function is negative , which means the acceleration is negative in physics.)

We can use the equation of motion to find the maximum height. In here, we use $t_{\max}$ and $y_{\max}$ to present the time and position reaching the max height.

$$\begin{cases} v_y = v_{y0} - gt \\ y = v_{y0}t - \frac{1}{2}gt^2 \end{cases} \Rightarrow t_{\max} = \frac{v_{y0}}{g} \Rightarrow y_{\max} = \frac{v_{y0}^2}{g} - \frac{v_{y0}^2}{2g} = \frac{v_{y0}^2}{2g} = \frac{v_0^2 \sin^2(\theta)}{2g}. \tag{12}$$

Method 2: Use the equation of path to find the maximum height.

As we know, the path of the projectile is a parabola, so the maximum height corresponds to the vertex of the parabola. We can simply derivate the equation of path to find where the maximum height is.

Sample Problem #3

Use Method 2 to find the maximum height of the projectile, and show that result are the same as Method 1.

Sol.

Sample Problem #4

Assume that Andrew is using an air gun to shoot a free-falling object. Show that Andrew can always hit the object if he aims at the object and shoots at the time it starts falling. (See picture 4-11 at page 72 in Halliday's book)

Sol.

Sample Problem #5

Assume that an archer located at $R = 0\hat{i} + 2\hat{j}$ is shooting an arrow at the target. The target is located at $R = 20\hat{i} + 2\hat{j}$. If the magnitude of the initial velocity is $v_0 = 20$, (a) what is the angle the archer needs to use to hit the target? (b) when does the arrow hit the target?

Sol.

Exercises

Exercise #1 [halliday 4.23]

Sol.

Exercise #2 [halliday 4.33]

Sol.

Exercise #3 [halliday 4.39]

Sol.

Exercise #4 [halliday 4.45]

Sol.

Exercise #5 [halliday 4.53]

Sol.

Exercise #5 [halliday 4.55]

Sol.

Solutions

Sample Problem #1

Initial Velocity Components

The initial velocity of the ball is given as $v_0 = 10\sqrt{2}\ \mathrm{m/s}$ and the launch angle is $\theta = 45^\circ$. We can decompose the initial velocity into its horizontal and vertical components using the launch angle. The horizontal component $v_{0x} = v_0 \cos(\theta)$ and the vertical component $v_{0y} = v_0 \sin(\theta)$. Since $\theta = 45^\circ$, we have $\cos(45^\circ) = \sin(45^\circ) = \frac{1}{\sqrt{2}}$. Therefore, $v_{0x} = v_{0y} = 10\ \mathrm{m/s}$.

Position Over Time

The position of the ball at any time t can be represented by the following equations:

$$x(t) = v_{0x}t = 10t, \quad y(t) = v_{0y}t - \frac{1}{2}gt^2 = 10t - \frac{1}{2}gt^2 \tag{1}$$

where g is the acceleration due to gravity.

Path of the Ball

The path of the ball can be determined by eliminating t from the above equations. Solving the equation $x(t) = 10t$ for t gives $t = \frac{x}{10}$. Substituting this into the equation for $y(t)$ gives the path of the ball:

$$y(x) = 10\left(\frac{x}{10}\right) - \frac{1}{2}g\left(\frac{x}{10}\right)^2 = x - \frac{1}{200}gx^2 \tag{2}$$

Sample Problem #2

(a) Horizontal Range at 30 and 60 Degrees

For $\theta = 30^\circ$ and $\theta = 60^\circ$, we have:

$$R_{30} = \frac{v_0^2}{g}\sin(60^\circ) = \frac{v_0^2}{g}\cdot\frac{\sqrt{3}}{2}, \quad R_{60} = \frac{v_0^2}{g}\sin(120^\circ) = \frac{v_0^2}{g}\cdot\frac{\sqrt{3}}{2} \tag{3}$$

So, $R_{30} = R_{60}$, which shows that the horizontal range at 30° and 60° is the same.

(b) Maximum Horizontal Range

To find the angle θ that maximizes the range R, we can take the derivative of R with respect to θ and set it equal to zero. This will give us the values of θ at which R is at a maximum or minimum.

$$\frac{dR}{d\theta} = \frac{v_0^2}{g}\cos(2\theta)\cdot 2 \tag{4}$$

Setting this equal to zero gives:

$$\cos(2\theta) = 0 \tag{5}$$

Solving for θ gives:

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4} \tag{6}$$

which correspond to 45° and 135°. However, since the launch angle θ is typically between 0° and 90°, we find that the range R is maximized at $\theta = 45^\circ$.

Sample Problem #3

Method 2: Use the equation of path to find the maximum height.

The maximum height can also be determined using the equation of the path of the projectile. The path of the projectile is a parabola, and the maximum height corresponds to the vertex of the parabola. The x-coordinate of the vertex of a parabola given by $y = ax^2 + bx + c$ is $-\frac{b}{2a}$. For the path of the projectile, $a = -\frac{g}{2v_0^2\cos^2(\theta)}$ and $b = \tan(\theta)$. Therefore, the maximum height is reached when $x = \frac{v_0^2\sin(\theta)\cos(\theta)}{g}$. Substituting this into the equation for y gives the maximum height:

$$\begin{aligned} y_{\max} &= x\tan(\theta) - \frac{g}{2v_0^2\cos^2(\theta)}x^2 \\ &= \left(\frac{v_0^2\sin(\theta)\cos(\theta)}{g}\right)\left(\frac{\sin\theta}{\cos\theta}\right) - \left(\frac{g}{2v_0^2\cos^2(\theta)}\right)\left(\frac{v_0^2\sin(\theta)\cos(\theta)}{g}\right)^2 = \frac{v_0^2\sin^2(\theta)}{2g} \end{aligned} \tag{7}$$

Sample Problem #4

Initial Conditions

At the moment Andrew fires, both the bullet and the object are at the same height, denoted as y_0. The bullet has an initial velocity v_0 and is fired at an angle α to the horizontal.

Equations of Motion

The horizontal and vertical positions of the bullet after time t are given by:

$$x_{\text{bullet}}(t) = v_0 \cos(\alpha)t, \quad y_{\text{bullet}}(t) = v_0 \sin(\alpha)t - \tfrac{1}{2}gt^2 \tag{8}$$

The vertical position of the object falling under gravity after time t is given by $y_{\text{object}}(t) = y_{0_{\text{object}}} - \frac{1}{2}gt^2$.

Collision Condition

For the bullet to hit the object, their vertical positions must be equal at some time t. Setting the vertical position equations equal to each other gives:

$$y_{\text{object}}(t) = y_{\text{bullet}}(t) \Rightarrow y_{0_{\text{object}}} - \tfrac{1}{2}gt^2 = v_0 \sin(\alpha)t - \tfrac{1}{2}gt^2 \tag{9}$$

Simplifying this equation gives the condition on the angle α for which the bullet will hit the object:

$$\sin(\alpha) = \frac{y_{0_{\text{object}}}}{v_0 t_{\text{collision}}} \tag{10}$$

If we aim at the target, $\sin(\alpha) = \frac{y_{0_{\text{object}}}}{x_{0_{\text{object}}}} = \frac{y_{0_{\text{object}}}}{v_0 t_{\text{collision}}}$, then it is clear that Andrew can always hit the object if he aims at the object and fires at the time it starts falling.

Sample Problem #5

Initial Conditions

The archer is located at $R = 0\hat{i} + 2\hat{j}$ and the target is located at $R = 20\hat{i} + 2\hat{j}$. The magnitude of the initial velocity is $v_0 = 20$.

Equations of Motion

The horizontal and vertical positions of the arrow after time t are given by:

$$x_{\text{arrow}}(t) = v_0 \cos(\theta)t, \quad y_{\text{arrow}}(t) = 2 + v_0 \sin(\theta)t - \tfrac{1}{2}gt^2 \tag{11}$$

The horizontal and vertical positions of the target are given by $x_{\text{target}} = 20, y_{\text{target}} = 2$.

Collision Condition

For the arrow to hit the target, their positions must be equal at some time t. Setting the position equations equal to each other gives:

$$x_{\text{target}} = x_{\text{arrow}}(t) \Rightarrow 20 = v_0 \cos(\theta)t \tag{12}$$

$$y_{\text{target}} = y_{\text{arrow}}(t) \Rightarrow 2 = 2 + v_0 \sin(\theta)t - \tfrac{1}{2}gt^2 \tag{13}$$

Solving these equations gives the conditions on the angle θ and the time t for which the arrow will hit the target:

$$\theta = \tfrac{1}{2}\arcsin(\tfrac{g}{v_0}) \tag{14}$$

$$t = \frac{x}{v_0 \cdot cos(\theta)} \tag{15}$$

The first possible angle (θ_1) the archer needs to use to hit the target is approximately 1.314 radians and the arrow will hit the target after approximately 3.94 seconds. The second possible angle (θ_2) the archer needs to use to hit the target is approximately 0.256 radians and the arrow will hit the target after approximately 1.03 seconds.

Exercises #1 to #6

You can find the answers to these exercises in Halliday's book.