Exercise Set 1.2

1. In each part, determine whether the matrix is in row echelon form, reduced row echelon form, both, or neither.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$

2. In each part, determine whether the matrix is in row echelon form, reduced row echelon form, both, or neither.

(a) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & 2 & 4 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

3. In each part, suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given row echelon form. Solve the system.

(a) $\begin{bmatrix} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 8 & -5 & 6 \\ 0 & 1 & 4 & -9 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 7 & -2 & 0 & -8 & -3 \\ 0 & 0 & 1 & 1 & 6 & 5 \\ 0 & 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4. In each part, suppose that the augmented matrix for a system of linear equations has been reduced by row operations to the given row echelon form. Solve the system.

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -4 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 & 2 & -2 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

► In Exercises **5–8**, solve the linear system by Gauss–Jordan elimination. ◄

5. $\begin{aligned} x_1 + 2x_2 - 3x_3 &= 6 \\ 2x_1 - x_2 + 4x_3 &= 1 \\ x_1 - x_2 + x_3 &= 3 \end{aligned}$

6. $\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{aligned}$

7. $\begin{aligned} 3x - y + z + 7w &= 13 \\ -2x + y - z - 3w &= -9 \\ -2x + y \qquad - 7w &= -8 \end{aligned}$

8. $\begin{aligned} -2b + 3c &= 1 \\ 3a + 6b - 3c &= -2 \\ 6a + 6b + 3c &= 5 \end{aligned}$

► In Exercises **9–12**, solve the linear system by Gaussian elimination. ◄

9. Exercise 5

10. Exercise 6

11. Exercise 7

12. Exercise 8

► In Exercises **13–16**, determine whether the homogeneous system has nontrivial solutions by inspection (without pencil and paper). ◄

13. $\begin{aligned} 2x_1 - 3x_2 + 4x_3 - x_4 &= 0 \\ 7x_1 + x_2 - 8x_3 + 9x_4 &= 0 \\ 2x_1 + 8x_2 + x_3 - x_4 &= 0 \end{aligned}$

14. $\begin{aligned} x_1 + 3x_2 - x_3 &= 0 \\ x_2 - 8x_3 &= 0 \\ 4x_3 &= 0 \end{aligned}$

15. $\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 0 \end{aligned}$

16. $\begin{aligned} 3x_1 - 2x_2 &= 0 \\ 6x_1 - 4x_2 &= 0 \end{aligned}$

► In Exercises **17–24**, solve the given linear system by any method. ◄

17. $\begin{aligned} 2x + y + 4z &= 0 \\ 3x + y + 6z &= 0 \\ 4x + y + 9z &= 0 \end{aligned}$

18. $\begin{aligned} 2x - y - 3z &= 0 \\ -x + 2y - 3z &= 0 \\ x + y + 4z &= 0 \end{aligned}$

19. $\begin{aligned} x_1 - x_2 + 7x_3 + x_4 &= 0 \\ x_1 + 2x_2 - 6x_3 - x_4 &= 0 \end{aligned}$

20. $\begin{aligned} v + 3w - 2x &= 0 \\ 2u + v - 4w + 3x &= 0 \\ 2u + 3v + 2w - x &= 0 \\ -4u - 3v + 5w - 4x &= 0 \end{aligned}$

21. $\begin{aligned} 2x + 2y + 4z &= 0 \\ w \qquad - y - 3z &= 0 \\ 2w + 3x + y + z &= 0 \\ -2w + x + 3y - 2z &= 0 \end{aligned}$

22. $\begin{aligned} x_1 + 3x_2 \qquad + x_4 &= 0 \\ x_1 + 4x_2 + 2x_3 \qquad &= 0 \\ - 2x_2 - 2x_3 - x_4 &= 0 \\ 2x_1 - 4x_2 + x_3 + x_4 &= 0 \\ x_1 - 2x_2 - x_3 + x_4 &= 0 \end{aligned}$

23. $\begin{aligned} 2I_1 - I_2 + 3I_3 + 4I_4 &= 9 \\ I_1 \qquad - 2I_3 + 7I_4 &= 11 \\ 3I_1 - 3I_2 + I_3 + 5I_4 &= 8 \\ 2I_1 + I_2 + 4I_3 + 4I_4 &= 10 \end{aligned}$

24. $\begin{aligned} Z_3 + Z_4 + Z_5 &= 0 \\ -Z_1 - Z_2 + 2Z_3 - 3Z_4 + Z_5 &= 0 \\ Z_1 + Z_2 - 2Z_3 \qquad - Z_5 &= 0 \\ 2Z_1 + 2Z_2 - Z_3 \qquad + Z_5 &= 0 \end{aligned}$

► In Exercises **25–28**, determine the values of a for which the system has no solutions, exactly one solution, or infinitely many solutions. ◄

25. $\begin{aligned} x + 2y + z &= 2 \\ 2x - 2y + 3z &= 1 \\ x + 2y - az &= a \end{aligned}$

26. $\begin{aligned} x + 2y + \qquad z &= 2 \\ 2x - 2y + \qquad 3z &= 1 \\ x + 2y - (a^2 - 3)z &= a \end{aligned}$

27. $\begin{aligned} x + 2y - \qquad 3z &= 4 \\ 3x - y + \qquad 5z &= 2 \\ 4x + y + (a^2 - 2)z &= a + 4 \end{aligned}$

28. $\begin{aligned} x + y + \qquad 7z &= -7 \\ 2x + 3y + \qquad 17z &= -16 \\ x + 2y + (a^2 + 1)z &= 3a \end{aligned}$

In Exercises **29–30**, solve the following systems, where a, b, and c are constants.

29. $$\begin{aligned} 2x + y &= a \\ 3x + 6y &= b \end{aligned}$$

30. $$\begin{aligned} x_1 + x_2 + x_3 &= a \\ 2x_1 + 2x_3 &= b \\ 3x_2 + 3x_3 &= c \end{aligned}$$

31. Find two different row echelon forms of

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$$

This exercise shows that a matrix can have multiple row echelon forms.

32. Reduce

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix}$$

to reduced row echelon form without introducing fractions at any intermediate stage.

33. Show that the following nonlinear system has 18 solutions if $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$, and $0 \le \gamma \le 2\pi$.

$$\begin{aligned} \sin\alpha + 2\cos\beta + 3\tan\gamma &= 0 \\ 2\sin\alpha + 5\cos\beta + 3\tan\gamma &= 0 \\ -\sin\alpha - 5\cos\beta + 5\tan\gamma &= 0 \end{aligned}$$

[*Hint:* Begin by making the substitutions $x = \sin\alpha$, $y = \cos\beta$, and $z = \tan\gamma$.]

34. Solve the following system of nonlinear equations for the unknown angles α, β, and γ, where $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$, and $0 \le \gamma < \pi$.

$$\begin{aligned} 2\sin\alpha - \cos\beta + 3\tan\gamma &= 3 \\ 4\sin\alpha + 2\cos\beta - 2\tan\gamma &= 2 \\ 6\sin\alpha - 3\cos\beta + \tan\gamma &= 9 \end{aligned}$$

35. Solve the following system of nonlinear equations for x, y, and z.

$$\begin{aligned} x^2 + y^2 + z^2 &= 6 \\ x^2 - y^2 + 2z^2 &= 2 \\ 2x^2 + y^2 - z^2 &= 3 \end{aligned}$$

[*Hint:* Begin by making the substitutions $X = x^2$, $Y = y^2$, $Z = z^2$.]

36. Solve the following system for x, y, and z.

$$\begin{aligned} \frac{1}{x} + \frac{2}{y} - \frac{4}{z} &= 1 \\ \frac{2}{x} + \frac{3}{y} + \frac{8}{z} &= 0 \\ -\frac{1}{x} + \frac{9}{y} + \frac{10}{z} &= 5 \end{aligned}$$

37. Find the coefficients a, b, c, and d so that the curve shown in the accompanying figure is the graph of the equation $y = ax^3 + bx^2 + cx + d$.

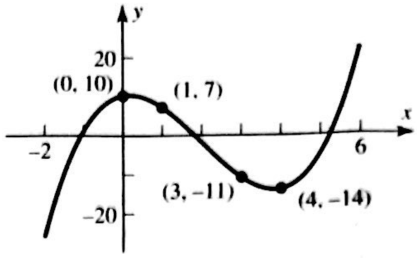

Figure Ex-37

38. Find the coefficients a, b, c, and d so that the curve shown in the accompanying figure is given by the equation $ax^2 + ay^2 + bx + cy + d = 0$.

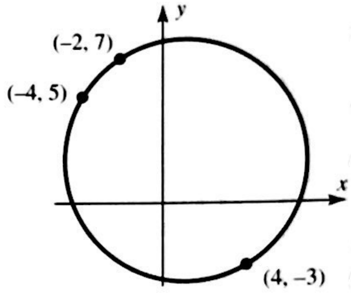

Figure Ex-38

39. If the linear system

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x - b_2y + c_2z &= 0 \\ a_3x + b_3y - c_3z &= 0 \end{aligned}$$

has only the trivial solution, what can be said about the solutions of the following system?

$$\begin{aligned} a_1x + b_1y + c_1z &= 3 \\ a_2x - b_2y + c_2z &= 7 \\ a_3x + b_3y - c_3z &= 11 \end{aligned}$$

40. (a) If A is a 3×5 matrix, then what is the maximum possible number of leading 1's in its reduced row echelon form?

(b) If B is a 3×6 matrix whose last column has all zeros, then what is the maximum possible number of parameters in the general solution of the linear system with augmented matrix B?

(c) If C is a 5×3 matrix, then what is the minimum possible number of rows of zeros in any row echelon form of C?

41. (a) Prove that if $ad - bc \neq 0$, then the reduced row echelon form of

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{is} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) Use the result in part (a) to prove that if $ad - bc \neq 0$, then the linear system

$$\begin{aligned} ax + by &= k \\ cx + dy &= l \end{aligned}$$

has exactly one solution.

42. Consider the system of equations

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \\ ex + fy &= 0 \end{aligned}$$

Discuss the relative positions of the lines $ax + by = 0$, $cx + dy = 0$, and $ex + fy = 0$ when (a) the system has only the trivial solution, and (b) the system has nontrivial solutions.

43. Describe all possible reduced row echelon forms of

(a) $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ (b) $\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & p & q \end{bmatrix}$

True-False Exercises

In parts (a)–(i) determine whether the statement is true or false, and justify your answer.

(a) If a matrix is in reduced row echelon form, then it is also in row echelon form.

(b) If an elementary row operation is applied to a matrix that is in row echelon form, the resulting matrix will still be in row echelon form.

(c) Every matrix has a unique row echelon form.

(d) A homogeneous linear system in n unknowns whose corresponding augmented matrix has a reduced row echelon form with r leading 1's has $n - r$ free variables.

(e) All leading 1's in a matrix in row echelon form must occur in different columns.

(f) If every column of a matrix in row echelon form has a leading 1 then all entries that are not leading 1's are zero.

(g) If a homogeneous linear system of n equations in n unknowns has a corresponding augmented matrix with a reduced row echelon form containing n leading 1's, then the linear system has only the trivial solution.

(h) If the reduced row echelon form of the augmented matrix for a linear system has a row of zeros, then the system must have infinitely many solutions.

(i) If a linear system has more unknowns than equations, then it must have infinitely many solutions.

Exercise Set 1.2 (page 22)

1. **(a)** Both **(b)** Both **(c)** Both **(d)** Both **(e)** Both **(f)** Both **(g)** Row echelon

3. **(a)** $x_1 = -37,\ x_2 = -8,\ x_3 = 5$ **(b)** $x_1 = 13t - 10,\ x_2 = 13t - 5,\ x_3 = -t + 2,\ x_4 = t$
(c) $x_1 = -7s + 2t - 11,\ x_2 = s,\ x_3 = -3t - 4,\ x_4 = -3t + 9,\ x_5 = t$ **(d)** Inconsistent

5. $x_1 = \frac{17}{5}, x_2 = \frac{-7}{5}, x_3 = \frac{-9}{5}$. **7.** $x = 4 - 4t, y = -t, z = 1 + 4t, w = t$.

9. $x_1 = \frac{17}{5}, x_2 = \frac{-7}{5}, x_3 = \frac{-9}{5}$. **11.** $x = 4 - 4t, y = -t, z = 1 + 4t, w = t$.

13. Has nontrivial solutions **15.** Has nontrivial solutions **17.** $x_1 = 0, x_2 = 0, x_3 = 0$

19. $x_1 = -\frac{8}{3}s - \frac{7}{3}t, x_2 = \frac{13}{3}s + \frac{5}{3}t, x_3 = s, x_4 = t$.

21. $w = t, x = -t, y = t, z = 0$ **23.** $I_1 = -1,\ I_2 = 0,\ I_3 = 1,\ I_4 = 2$

25. No solutions if $a = -1$; unique solution if $a \neq -1$

27. inconsistent system if $a = -2$; infinitely many solutions if $a = 2$; unique solutions for any other value of a

29. $x = \frac{2a}{3} - \frac{b}{9},\ y = -\frac{a}{3} + \frac{2b}{9}$ **31.** $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are possible answers. **35.** $x = \pm 1,\ y = \pm\sqrt{3},\ z = \pm\sqrt{2}$

37. $a = 1, b = -6, c = 2, d = 10$ **39.** The nonhomogeneous system will have exactly one solution.

True/False 1.2

(a) True **(b)** False **(c)** False **(d)** True **(e)** True **(f)** False **(g)** True **(h)** False **(i)** False