ProblemSet3

February 13, 2024

1 Problem Set 3

2 Part I. Exercises for Chapter 5: Fixed and Period Points

2.1 Problem 1:

2.2 (a)

$$S(x) = \frac{\pi}{2}\sin x$$

Findng fixed points:

$$\begin{aligned} S(x) &= x && \text{Must be true for x to be a f.p.} \\ \frac{\pi}{2}\sin x &= x \\ x &= -\frac{\pi}{2}, 0, \frac{\pi}{2} \end{aligned}$$

Classifying $p = -\frac{\pi}{2}$:

$$\begin{aligned} |S'(-\frac{\pi}{2})| = |\frac{\pi}{2}\cos(-\frac{\pi}{2})| = 0 < 1 \\ \therefore p = -\frac{\pi}{2} \text{ is attracting.} \end{aligned}$$

Classifying $p = 0$:

$$\begin{aligned} |S'(0)| = |\frac{\pi}{2}\cos(0)| = \frac{\pi}{2} > 1 \\ \therefore p = 0 \text{ is repelling.} \end{aligned}$$

Classifying $p = \frac{\pi}{2}$:

$$\begin{aligned} |S'(\frac{\pi}{2})| = |\frac{\pi}{2}\cos(\frac{\pi}{2})| = 0 < 1 \\ \therefore p = \frac{\pi}{2} \text{ is attracting.} \end{aligned}$$

2.3 (b)

$$F(x) = x^4 - 4x^2 + 2$$

Finding fixed points:

$$\begin{aligned} F(x) &= x \\ x^4 - 4x^2 + 2 &= x \\ x^4 - 4x^2 - x + 2 &= 0 \\ (x-2)(x^3 + 2x^2 - 1) &= 0 \\ (x-2)(x+1)(x^2 + x - 1) &= 0 \\ x &= -1, 2, -\frac{1}{2} \pm \frac{\sqrt{5}}{2} \end{aligned}$$

Classifying $p = -1$:

$$|F'(-1)| = |4(-1)^3 - 8(-1)| = 4 > 1$$
$$\therefore p = -1 \text{ is repelling.}$$

Classifying $p = 2$:

$$|F'(2)| = |4(2)^3 - 8(2)| = 8 > 1$$
$$\therefore p = 2 \text{ is repelling.}$$

Classifying $p = -\frac{1}{2} - \frac{\sqrt{5}}{2}$:

$$|F'(-\frac{1}{2} - \frac{\sqrt{5}}{2})| = |4(-\frac{1}{2} - \frac{\sqrt{5}}{2})^3 - 8(-\frac{1}{2} - \frac{\sqrt{5}}{2})| = |-4| = 4 > 1$$
$$\therefore p = -\frac{1}{2} - \frac{\sqrt{5}}{2} \text{ is repelling.}$$

Classifying $p = -\frac{1}{2} + \frac{\sqrt{5}}{2}$:

$$|F'(-\frac{1}{2} + \frac{\sqrt{5}}{2})| = |4(-\frac{1}{2} + \frac{\sqrt{5}}{2})^3 - 8(-\frac{1}{2} + \frac{\sqrt{5}}{2})| = |-4| = 4 > 1$$
$$\therefore p = -\frac{1}{2} + \frac{\sqrt{5}}{2} \text{ is repelling.}$$

2.4 Problem 2

$$F(x) = \begin{cases} x + 1 & x \leq \frac{7}{2} \\ 2x - 8 & x > \frac{7}{2} \end{cases}$$

Finding cycle starting with $p = 0$:

$$F(0) = 1, F(1) = 2, F(2) = 3, F(3) = 4, F(4) = 0$$
$$\text{5-cycle: } (0, 1, 2, 3, 4)$$

Classifying 5-cycle:

$$F'(x) = \begin{cases} 1 & x < \frac{7}{2} \\ 2 & x > \frac{7}{2} \end{cases}$$

$$|(F^5)'(x)| = |F'(0)F'(1)F'(2)F'(3)F'(4)| = |(1)(1)(1)(1)(2)| = 2 > 0$$
$$\therefore \text{ the 5-cycle is repelling.}$$

2.5 Problem 3:

2.6 (a)

$$E(x) = e^{x-1}$$

Finding fixed points:

$$\begin{aligned} E(x) &= x \\ e^{x-1} &= x \\ x &= 1 \end{aligned}$$

Classifying $p = 1$:

$$\begin{aligned} E'(1) = e^{1-1} = 1 \qquad & \text{neutral} \\ E''(1) = e^{1-1} = 1 \qquad & \text{weakly attracting to left, weakly repelling to right} \end{aligned}$$

$$\therefore p = 1 \text{ is neither weakly attracting nor weakly repelling.}$$

2.7 (b)

$$S(x) = \sin x$$

Finding fixed points:

$$\begin{aligned} S(x) &= x \\ \sin x &= x \\ x &= 0 \end{aligned}$$

Classifying $p = 0$:

$$\begin{aligned} S'(0) = \cos(0) = 1 \qquad & \text{neutral} \\ S''(0) = -\sin(0) = 0 \qquad & \\ S'''(0) = -\cos(0) = -1 \qquad & \text{weakly attracting} \end{aligned}$$

$$\therefore p = 0 \text{ is weakly attracting.}$$

2.8 (c)

$$L(x) = |x - 1|$$

Finding fixed points:

$$\begin{aligned} L(x) &= x \\ |x - 1| &= x \\ x &= \frac{1}{2} \end{aligned}$$

Classifying $p = \frac{1}{2}$:

$$L'(x) = \begin{cases} 1 & x > 1 \\ -1 & x < 1 \end{cases}$$

$$L'(\frac{1}{2}) = -1 \qquad \text{neutral}$$
$$L''(\frac{1}{2}) = 0$$
$$L'''(\frac{1}{2}) = 0$$
$$L^{(n)}(\frac{1}{2}) = 0$$

$$\therefore p = \frac{1}{2} \text{ is neither weakly attracting nor weakly repelling.}$$

3 Part II. Computer Exercise

Write code that will generate cobweb diagrams to graphically analyze discrete dynamical systems. Your code should accept as input the map, the seed value, and the number of iterations to plot and output a cobweb (staircase) plot.

3.1 Source Code:

[4]:
```
from math import *
import numpy as np
import matplotlib.pyplot as plt
```

[13]:
```
# Driver function to make cobweb diagram
def cobweb_diagram(f, x_0, iter=25, window=(-10,10,-10,10)):
    # Plot the function
    xs = np.linspace(window[0], window[1], 100)
    plt.plot(xs, f(xs), 'b')

    # Plot the line y = x
    plt.plot([window[0], window[1]], [window[0], window[1]], 'k')

    # Call plot_cobweb() on seed value(s)
    if isinstance(x_0, (list, tuple)):
        for seed in x_0:
            plot_cobweb(f, seed, iter, window)
    else:
        plot_cobweb(f, x_0, iter, window)

    # Display plot
    plt.axis('square')
    plt.axis(window)
    plt.grid()
    plt.show()

# Plots the "cobweb"
def plot_cobweb(f, x_0, iter, window):
    A = [] # stores x-coordinates of points on web
```

```
    B = [] # stores y-coordinates

    for i in range(iter):
        # If seed value is outside the window, stop looping
        if x_0 < window[0] or x_0 > window[1]:
            break

        # Add point on line y=x
        A.append(x_0)
        B.append(x_0)

        # Add point on line y=f(x)
        A.append(x_0)
        B.append(f(x_0))

        # Update seed value
        x_0 = f(x_0)

    # Plot cobweb
    plt.plot(A, B, 'r')
```

3.2 Use Cases:

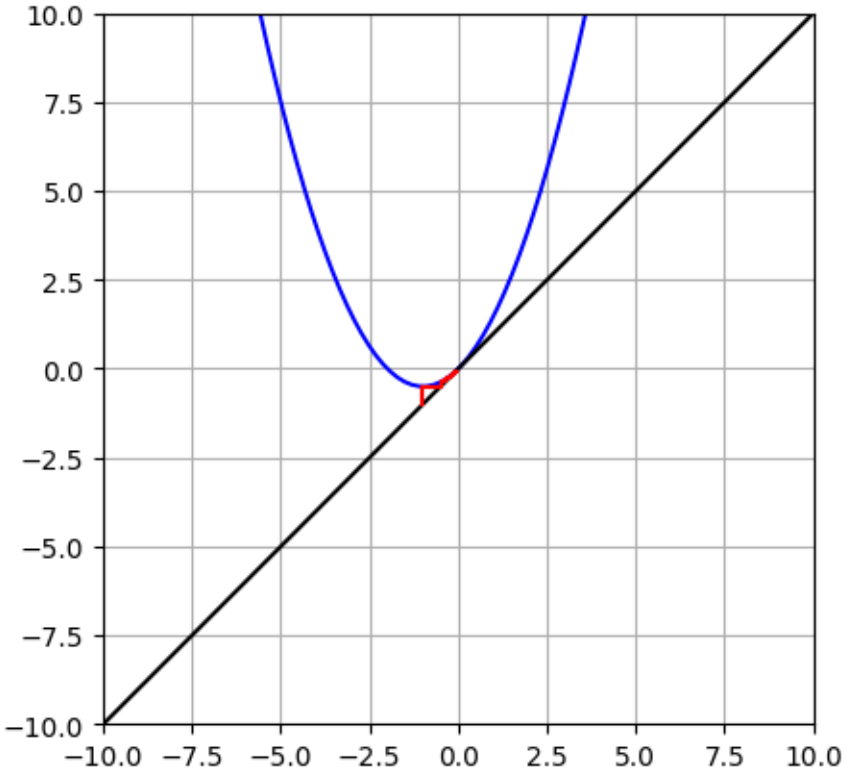
Use your code to perform the graphical analysis needed for exercises 5, 6, 7, and 8 on p. 60

3.3 Exercise 5:

Suppose that F has a neutral fixed point at x_0 with $F'(x_0) = 1$. Suppose also that $F''(x_0) > 0$. What can you say about x_0: is x_0 weakly attracting, weakly repelling, or neither?

We'll use the function $F(x) = 0.5x^2 + x$ with the fixed point $x_0 = 0$.

[18]:
```
F = lambda x: 0.5 * x**2 + x
# Cobweb diagram starting to left of fixed point
cobweb_diagram(F, -1)
# Cobweb diagram starting to right of fixed point
cobweb_diagram(F, 1)
```


10.0
7.5
5.0
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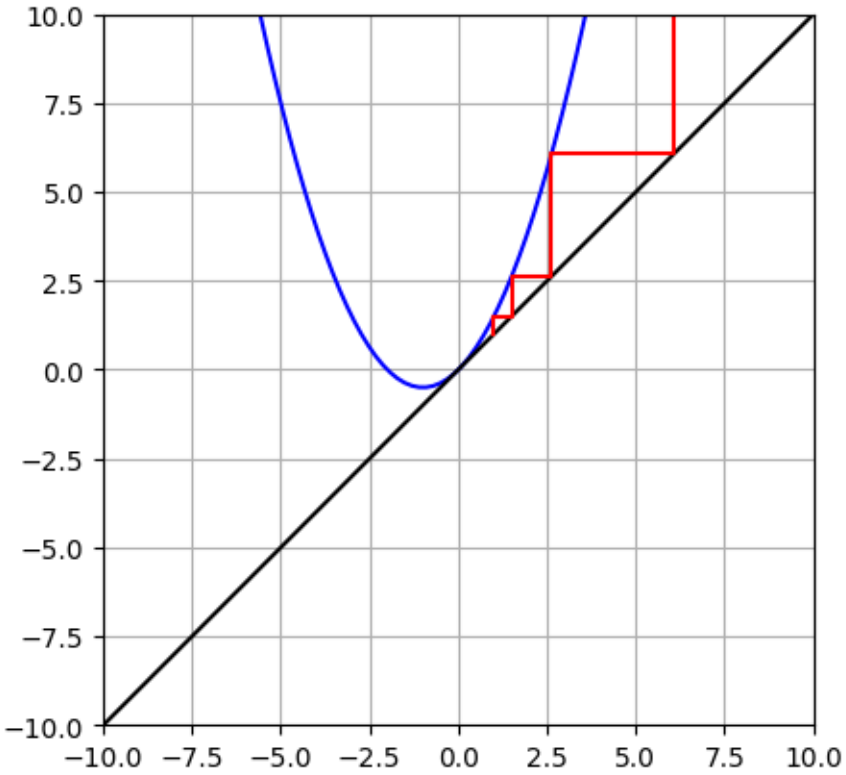
−10.0 −7.5 −5.0 −2.5 0.0 2.5 5.0 7.5 10.0


10.0
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−10.0 −7.5 −5.0 −2.5 0.0 2.5 5.0 7.5 10.0

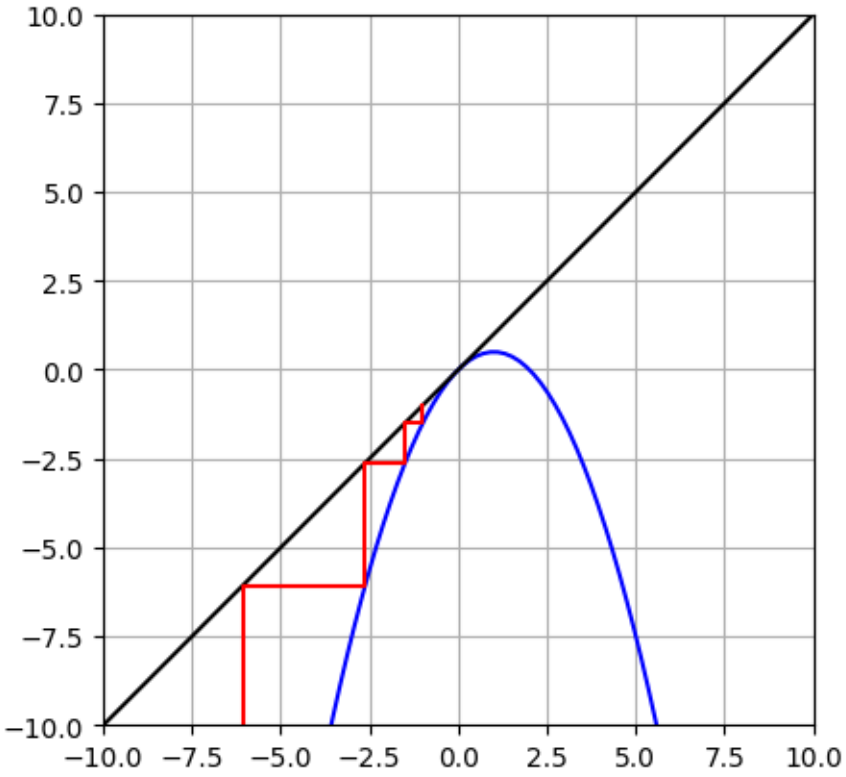
x_0 is weakly attracting to the left and weakly repelling to the right.

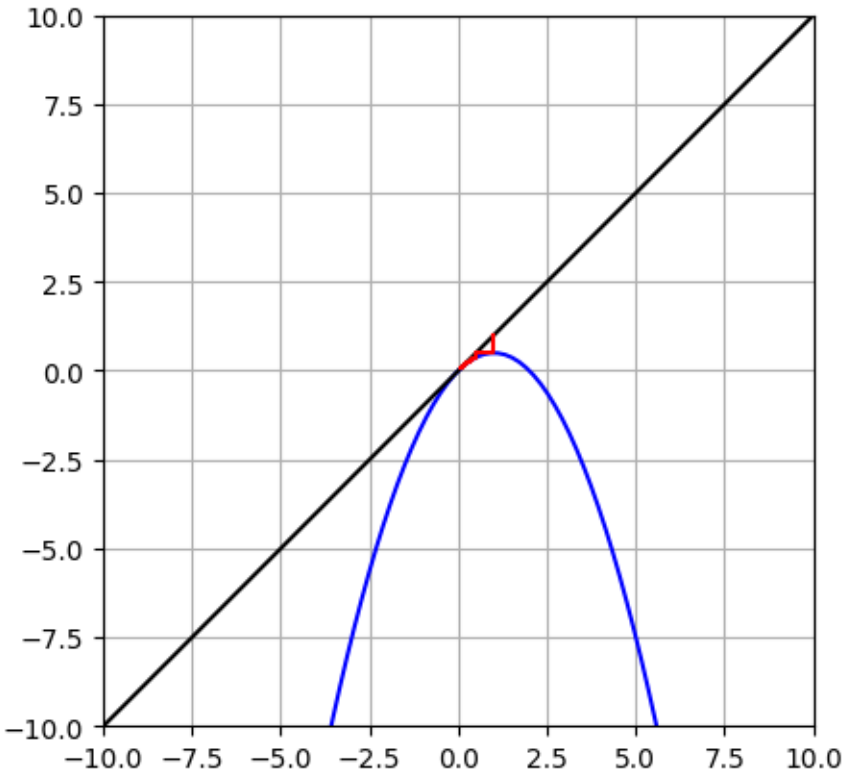
3.4 Exercise 6:

Repeat exercise 5, but this time assume $F''(x_0) < 0$.

We'll use the function $F(x) = -0.5x^2 + x$ with the fixed point $x_0 = 0$.

[19]:
```
F = lambda x: -0.5 * x**2 + x
# Cobweb diagram starting to left of fixed point
cobweb_diagram(F, -1)
# Cobweb diagram starting to right of fixed point
cobweb_diagram(F, 1)
```





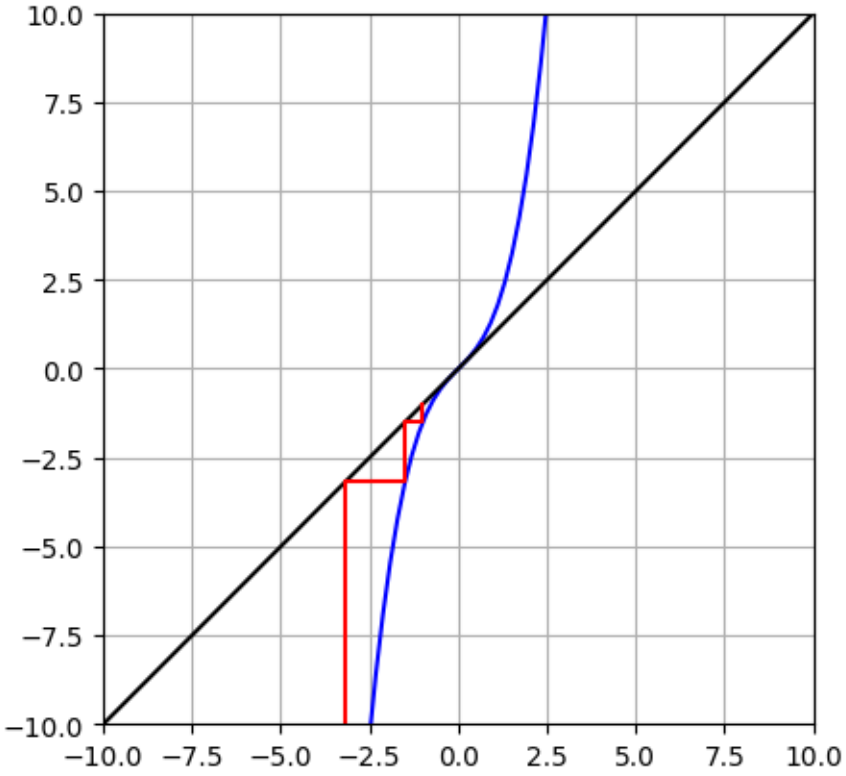
x_0 is weakly repelling to the left and weakly attracting to the right.

3.5 Exercise 7:

Suppose that F has a neutral fixed point at x_0 with $F'(x_0) = 1$ and $F''(x_0) = 0$. Suppose also that $F'''(x_0) > 0$. Show that x_0 is weakly repelling.

We'll use the function $F(x) = 0.5x^3 + x$ with the fixed point $x_0 = 0$.

[20]:
```
F = lambda x: 0.5 * x**3 + x
# Cobweb diagram starting to left of fixed point
cobweb_diagram(F, -1)
# Cobweb diagram starting to right of fixed point
cobweb_diagram(F, 1)
```


10.0
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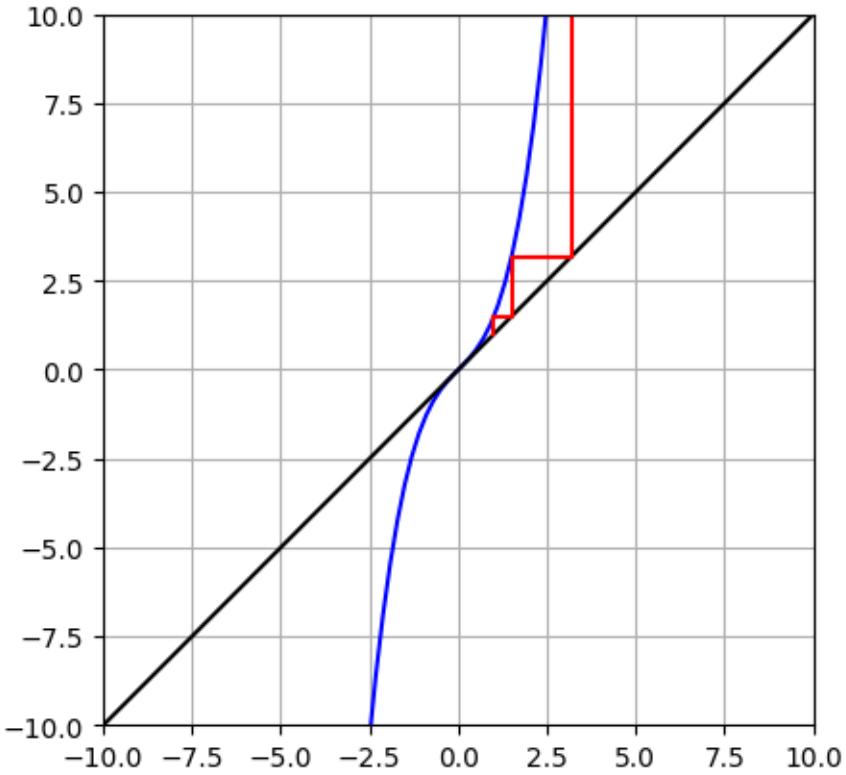
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−5.0
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−10.0
−10.0 −7.5 −5.0 −2.5 0.0 2.5 5.0 7.5 10.0

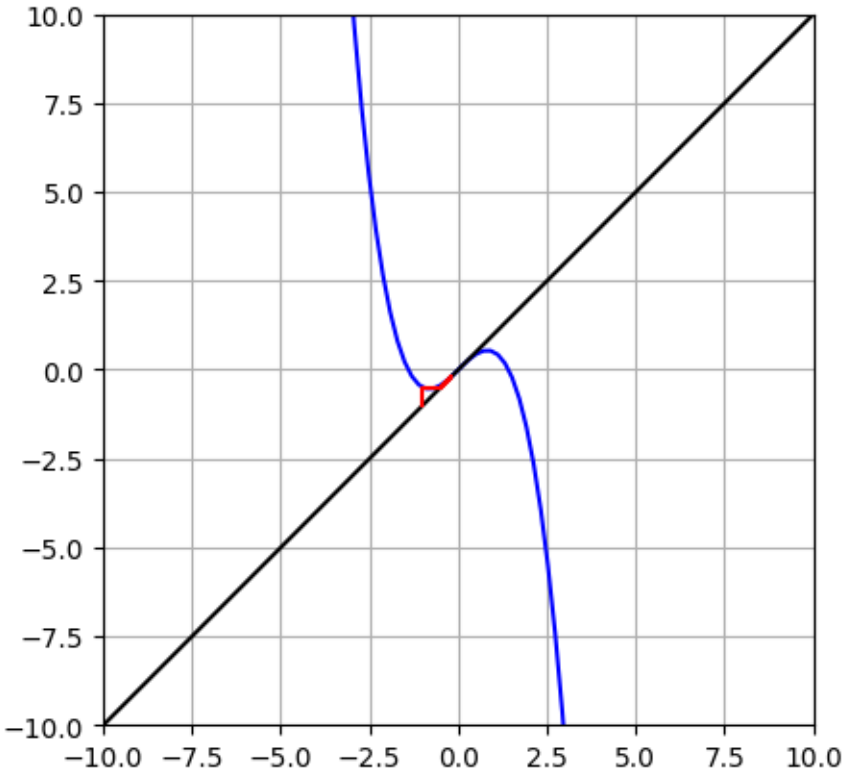
x_0 is weakly repelling.

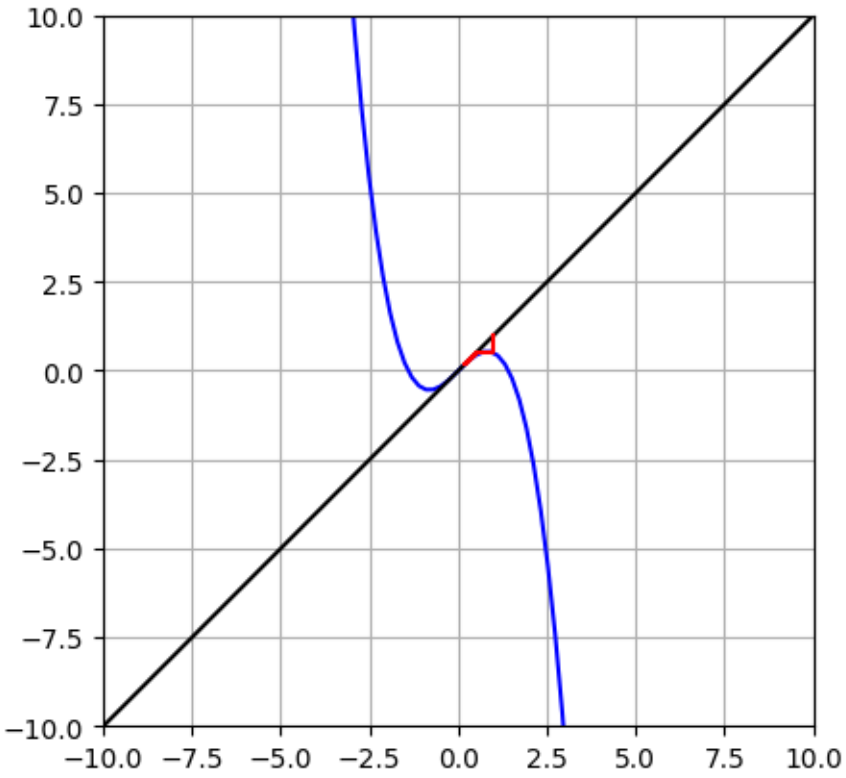
3.6 Exercise 8

Repeat exercise 7, but this time assume that $F'''(x_0) < 0$. Show that x_0 is weakly attracting.

We'll use the function $F(x) = -0.5x^3 + x$ with the fixed point $x_0 = 0$.

[21]:
```
F = lambda x: -0.5 * x**3 + x
# Cobweb diagram starting to left of fixed point
cobweb_diagram(F, -1)
# Cobweb diagram starting to right of fixed point
cobweb_diagram(F, 1)
```





x_0 is weakly attracting.