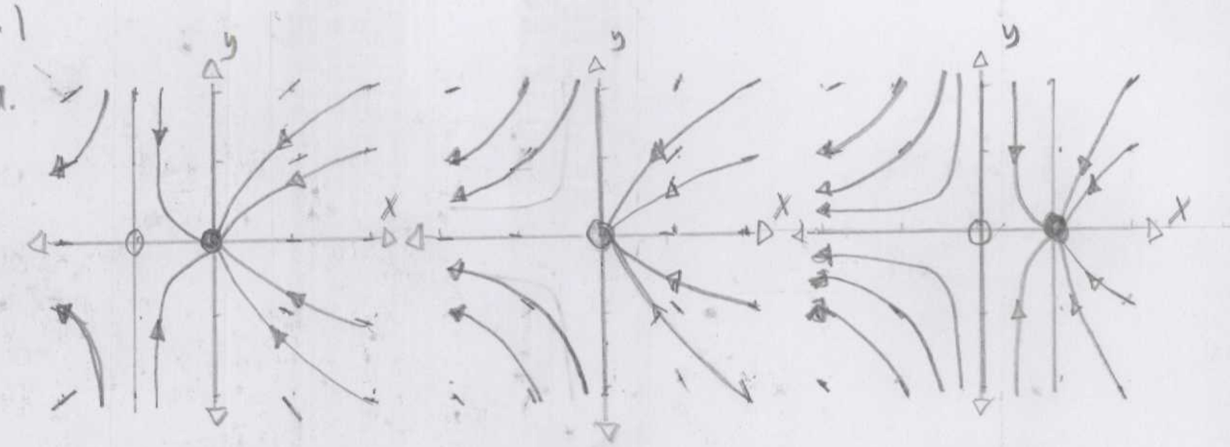
Chapter 8: Bifurcations Revisited!

$\dot{x} = \mu x - x^2$ 8.1.1
$\dot{y} = -y$

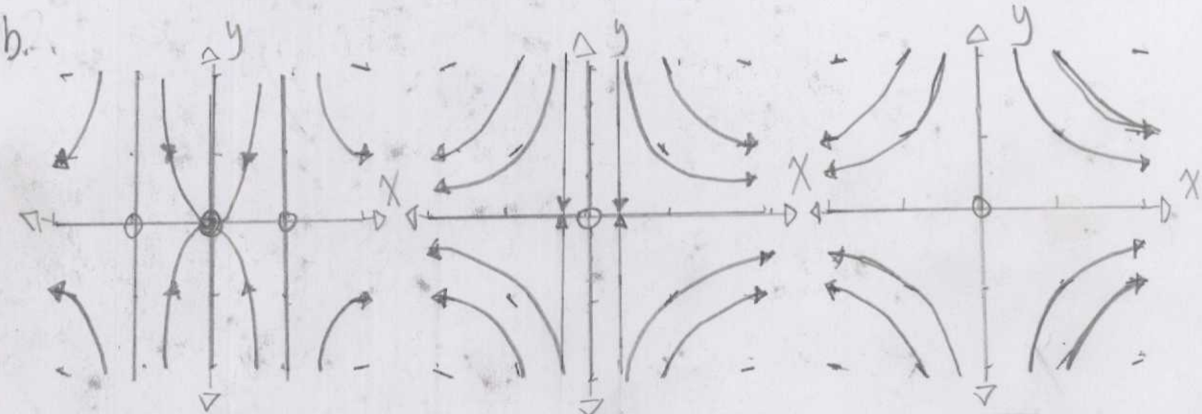
a.



$\boxed{\mu < 0}$ $\boxed{\mu = 0}$ $\boxed{\mu > 0}$

$\dot{x} = \mu x + x^3$
$\dot{y} = -y$

b.



$\boxed{\mu < 0}$ $\boxed{\mu = 0}$ $\boxed{\mu > 0}$

$\dot{x} = \mu - x^2$ 8.1.2. Eigenvalues: $\dot{\vec{x}} = A\vec{x} = 0$; $A\vec{x} = 0 = \lambda\vec{x}$; $(A - \lambda)\vec{x} = 0$
$\dot{y} = -y$

$$= \begin{pmatrix} -2x-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix} = (-2x-\lambda)(-1-\lambda) = 0$$

$$\lambda_1 = -2x \; ; \; \lambda_2 = -1$$

Fixed Points: $\dot{x} = 0 = \mu - x^2$
$\dot{y} = 0 = -y$; $(x^*, y^*) = (\sqrt{\mu}, 0)$

Eigenvalues + Fixed Points: $\lambda_1 = -2\sqrt{\mu}$; $\lim\limits_{\mu \to 0} \lambda_1 = 0$

$\dot{x} = \mu x - x^2$ 8.1.3. Eigenvalues: $\dot{\vec{x}} = A\vec{x} = 0$; $(A - \lambda)\vec{x} = \begin{pmatrix} \mu - 2x - \lambda_1 & 0 \\ 0 & -1-\lambda_2 \end{pmatrix}\vec{x} = 0$
$\dot{y} = -y$

$$\lambda_1 = \mu - 2x$$
$$\lambda_2 = -1$$

Fixed Points: $\dot{x} = 0 = \mu x - x^2$; $(x^*, y^*) = (\mu, 0)$

$\dot{y} = 0 = -y$

Eigenvalues + Fixed Points: $\lambda_1 = \mu - 2x$;

$$\lim_{\mu \to 0} \lambda_1 = \lim_{\mu \to 0} \mu - 2\mu = 0$$

$\dot{x} = \mu x + x^3$
$\dot{y} = -y$

8.1.4: Eigenvalues: $\dot{\vec{x}} = A\vec{x} = 0$; $(A - \lambda)\vec{x} = \begin{pmatrix} \mu + 3x^2 - \lambda_1 & 0 \\ 0 & -1-\lambda_2 \end{pmatrix} \vec{x}$

$$\lambda_1 = \mu + 3x^2$$
$$\lambda_2 = -1$$

Fixed Points: $\dot{x} = 0 = \mu x + x^3$; $(x^*, y^*) = (\sqrt{\mu}\, i, 0)$

$\dot{y} = 0 = -y$

Eigenvalues + Fixed Points: $\lambda_1 = \mu + 3(\sqrt{\mu}\, i)^2$

$$\lim_{\mu \to 0} \lambda_1 = \lim_{\mu \to 0} \mu + 3(\sqrt{\mu}\, i)^2 = \phi$$

8.1.5: True, since the zero-eigenvalue bifurcation is exampled by saddle-nodes, transcritical, and pitchfork bifurcations that each have tangential intersections about their nullclines.

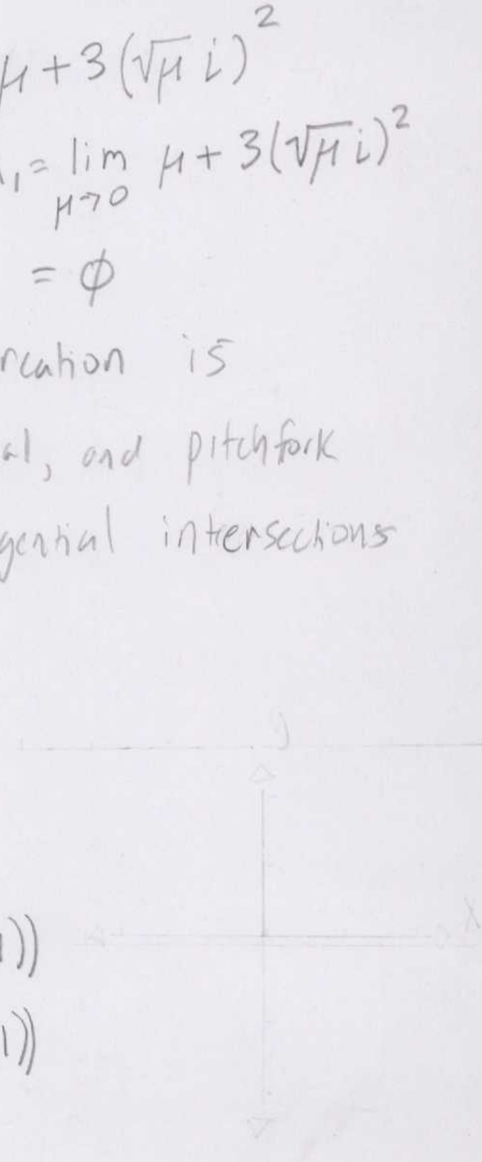
$\dot{x} = y - 2x$
$\dot{y} = \mu + x^2 - y$

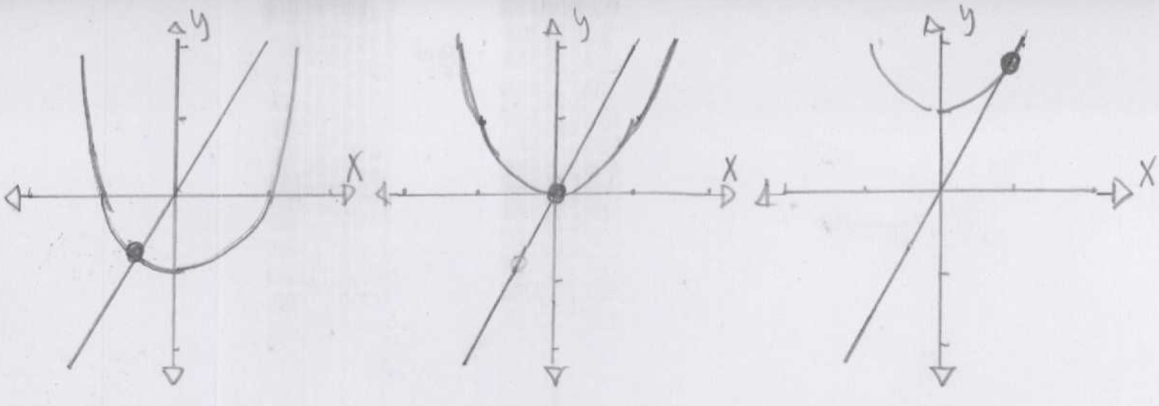
8.1.6: i

a. Nullclines: $\dot{x} = 0 = y - 2x$

$\dot{y} = 0 = \mu + x^2 - y$

$$(x^*, y^*) = (1 - \sqrt{1-\mu}, -2(\sqrt{1-\mu} - 1))$$
$$= (1 + \sqrt{1-\mu}, 2(\sqrt{1-\mu} + 1))$$

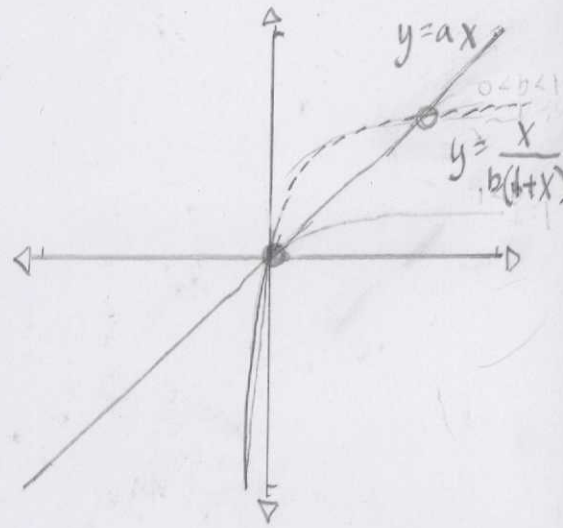




$\boxed{\mu<0}$ $\boxed{\mu=0}$ $\boxed{0<\mu\leq 1}$

b. Saddle-node

c. See part A.



$\dot{x}=y-ax$

$\dot{y}=-by+\frac{x}{1+x}$

8.1.7. $y=ax$; $y=\frac{x}{b(1+x)}$;

$$ax=\frac{x}{b(1+x)}$$

$$x=0,\ \frac{1}{ab}-1$$

The book shows a ... Jacobian method. This is an equation, table or equation graph. Maybe an equation, table, and graph.

Bifurcation Amount	Conditions
2	$ab<1$
1	$ab=1$
2	$ab>1$

Transcritical Bifurcation.

$$\varepsilon\frac{d^2\phi}{d^2\tau}=-\frac{d\phi}{d\tau}-\sin\phi+\gamma\sin\phi\cos\phi$$

8.1.8, $\varepsilon>0$; $\gamma>0$;

a.) $\dot{x}=\frac{d\phi}{d\tau}=y$

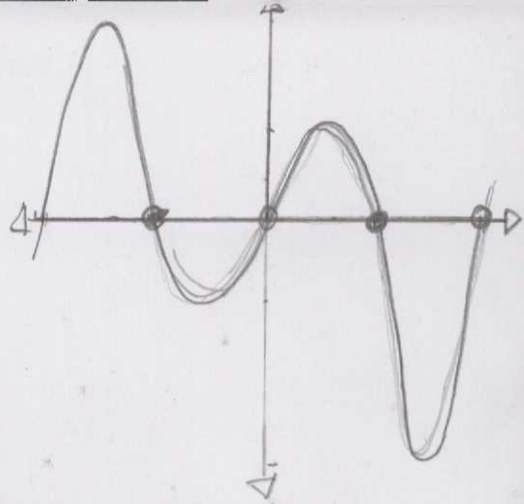
$$\dot{y}=\frac{d^2\phi}{d^2\tau}=\frac{-y+\sin x(\gamma\cos x-1)}{\varepsilon}$$

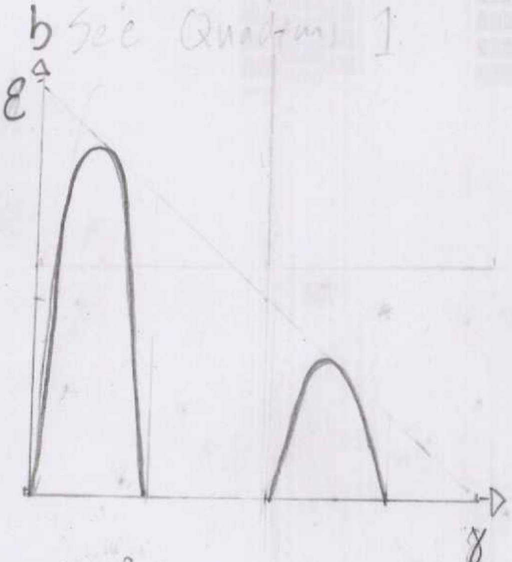
<u>Bifurcations</u>; $\dot{x}=0=y$

$$\dot{y}=0=\frac{-y+\sin x(\gamma\cos x-1)}{\varepsilon}$$

0 $\quad y = \sin x\,(\gamma \cos x - 1)$

$x = 0,\ \arccos\left(\frac{1}{\gamma}\right)$





Bifurcation Amount	Conditions
2	$0 \leq \gamma \leq 1$
4	$1 < \gamma$

"Pitch fork Bifurcation: Supercritical"

$\ddot{x} + b\dot{x} - kx + x^3 = 0$ 8.1.9. $\dot{x} = y$

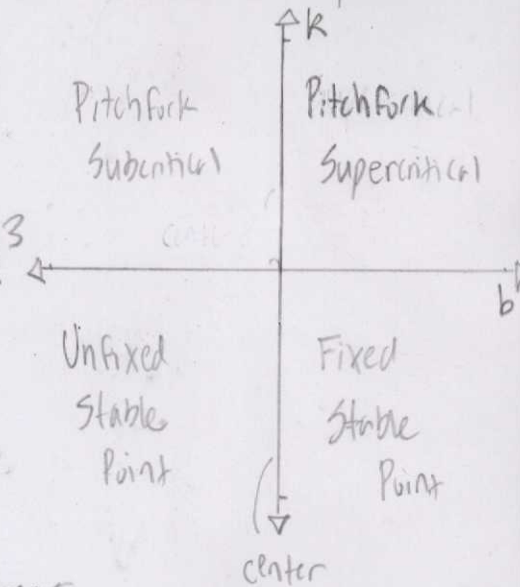
$\dot{y} = -b\dot{x} + kx - x^3 = -by + kx - x^3$

<u>Bifurcations:</u> $\dot{x} = 0 = y$

$\dot{y} = 0 = -b\dot{y} + kx - x^3$

$(x^*, y^*) = (0, 0)$

$(\pm\sqrt{k}, 0)$



$\dot{S} = r_S S\left(1 - \frac{S}{K_S}\frac{K_E}{E}\right)$

$\dot{E} = r_E E\left(1 - \frac{E}{K_S}\right) - P\frac{B}{S}$

8.1.10 $S(t)$ = Average Size of Trees

$E(t)$ = "Energy Reserve"

B = Budworm Population

$r_S, r_E, K_S, K_E, P > 0$

a. First term of $\dot{S}$ is rate of increase of average tree size
Seond term of $\dot{S}$ is rate of decrease of average tree size
First term of $\dot{E}$ is rate of increase of energy reserve
Seoond term of $\dot{E}$ is rate of decrease of energy reserve
Third term of $\dot{E}$ is rate of decrease of energy reserve from budworms

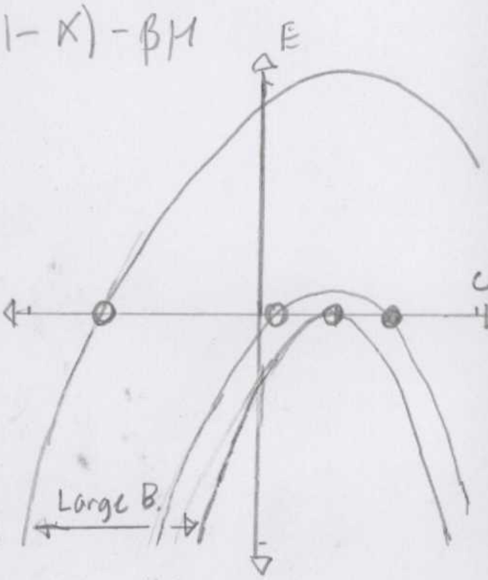
b. Scaled budworm density $\mu = \frac{P_B}{S}$; $\alpha = \frac{E}{K_E}$; $t = r_E \tau$

$$\dot{S} = r_E S' = r_S S\left(1 - \frac{S}{K_S}\frac{1}{\alpha}\right) ; \quad S' = \frac{r_S}{r_E} S\left(1 - \frac{S}{K_S \alpha}\right) ;$$

$$\dot{E} = r_E E' = r_E E(1-\alpha) - \beta\mu \quad E' = E(1-\alpha) - \beta\mu$$

If $R = \frac{r_S}{r_E}$; $C = \frac{S}{K_S}$; $C' = RC\left(1 - \frac{C}{\alpha}\right)$

$$E' = E(1-\alpha) - \beta\mu$$



c. <u>Nullclines:</u> $C' = 0 = RC\left(1 - \frac{C}{\alpha}\right)$

$$E' = 0 = E(1-\alpha) - \beta\mu$$

$$(C^*, E^*) = \left(0, \frac{\beta\mu}{(1-\alpha)}\right)$$

$$\left(\alpha, \frac{\beta\mu}{(1-\alpha)}\right)$$

d. See part c.

Bifurcation Amount	Conditions
0	$B > \frac{E(1-\alpha)}{\mu}$
1	$B = \frac{E(1-\alpha)}{\mu}$
2	$B < \frac{E(1-\alpha)}{\mu}$

$\dot{u} = a(1-u) - uv^2$
$\dot{v} = uv^2 - (a+k)v$

8.1.11. <u>Bifurcations:</u>

$$\dot{u} = 0 = a(1-u) - uv^2$$

$$\dot{v} = 0 = uv^2 - (a+k)v$$

$$(u^*, v^*) = \left(\frac{a \pm \sqrt{a^2 - 4(a+k)^2}}{2a}, \frac{a \mp \sqrt{a^2-4(a+k)^2}}{2(a+k)}\right)$$

$$0 = a^2 - 4(a+k)^2 ; \quad (a+k)^2 = \frac{a}{4} ; \quad k = -a \pm \frac{a^{1/2}}{v}$$

$\dot{\theta}_1 = K\sin(\theta_1 - \theta_2) - \sin\theta_1$
$\dot{\theta}_2 = K\sin(\theta_2 - \theta_1) - \sin\theta_2$

8.1.12 a. <u>Fixed Points:</u> $\dot{\theta}_1 = 0 = K\sin(\theta_1 - \theta_2) - \sin\theta_1$

$$\dot{\theta}_2 = 0 = K\sin(\theta_2 - \theta_1) - \sin\theta_2$$

$$(\theta_1^*, \theta_2^*) = (n_1\pi, n_2\pi) \quad n_1, n_2 \in \mathbb{Z}$$

b. Identity: $\sin(a-b) = \cos b \sin a - \sin b \cos a$

$$\dot{\theta}_1 = 0 = K[\sin\theta_1 \cos\theta_2 - \sin\theta_2 \cos\theta_1] - \sin\theta_1$$

$$\dot{\theta}_2 = 0 = K[\sin\theta_2 \cos\theta_1 - \sin\theta_1 \cos\theta_2] - \sin\theta_2$$

$$0 = 2K[\sin\theta_1 \cos\theta_2 - \sin\theta_2 \cos\theta_1] + \sin\theta_2 - \sin\theta_1$$

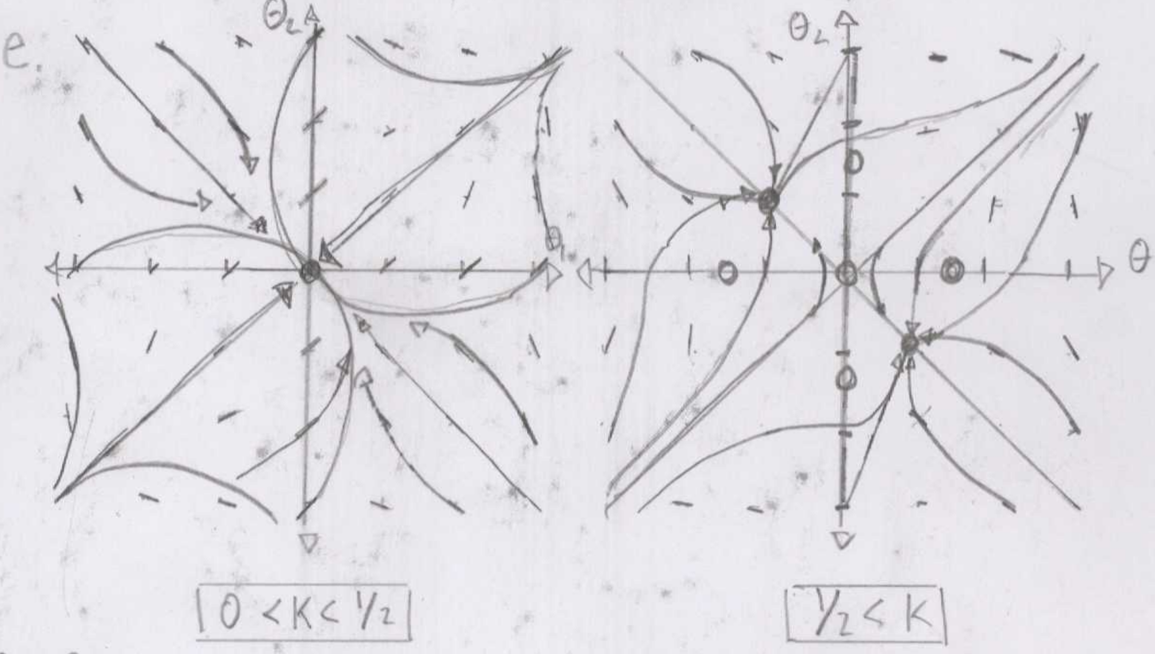
$$K = 1/2 \ @ \ (\theta_1, \theta_2) = (n_1\pi, n_2\pi)$$

c. $\dot{\theta} = -\frac{\partial V}{\partial \theta_1}$; $V(\theta_1, \theta_2) = K\cos(\theta_1 - \theta_2) - \cos\theta_1$

$= K\cos(\theta_2 - \theta_1) - \cos\theta_1$

d. $\dot{\theta}_1$ and $\dot{\theta}_2$ written as $V(\theta_1, \theta_2)$ imply a Lyapunov function. Gradient flows have no periodic orbits.

e.



$\dot{n} = GnN - kn$

$\dot{N} = -GnN - fN + p$

8.1.13

a. $N(t)$ = # of excited atoms

$n(t)$ = # of photons in laser field

G = Gain coefficient for Stimulated Emission

k = Decay rate due to loss of photons by mirror transmission

f = Decay rate for spontaneous emission

p = pump strength

If $\dot{n} = GnN - kn$, then $\frac{G}{k^2}\dot{n} = \frac{G^2nN}{k^2} - \frac{Gn}{k}$

$\dot{N} = -GnN - fN + p$, then $\frac{G}{k^2}\dot{N} = -\frac{G^2nN}{k^2} - \frac{G}{k^2}(fN + p)$

If $\tau = k^2 t$, $x = \frac{Gn}{k}$, $y = \frac{GN}{k}$, $a = \frac{f}{k}$, $b = \frac{pG}{k^2}$

$\dot{x} = x(y-1)$, $\dot{y} = -xy - ay + b$

b. Fixed Points: $\dot{x} = 0 = x(y-1)$

$$\dot{y} = 0 = -xy - ay + b$$

$$(x^*, y^*) = (0, b/a)$$
$$= (b-a, 1)$$

Classification: $\dot{\vec{x}} = A\vec{x}$; $A = \begin{pmatrix} y-1 & x \\ -y & -(x+a) \end{pmatrix}$

$$(A_{(0,a/b)} - \lambda) = \begin{vmatrix} \frac{b}{a}-1-\lambda_1 & 0 \\ -b/a & -a-\lambda_2 \end{vmatrix} = 0$$

$$(A_{(b-a,1)}, -\lambda) = \begin{pmatrix} -\lambda_1 & b-a \\ -1 & -b-\lambda_2 \end{pmatrix}$$

$\lambda_1 = \frac{b}{a} - 1$; $\lambda_2 = -a$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4(b-a)}}{2}$$

$\Delta = a - (b-1)$

$$\Delta = \frac{(b^2 - (b-4)b - 4a^2)}{4}$$

$\tau = \frac{b}{a} - 1 - a$

$\tau^2 - 4\Delta < 0$ @ $a > b$

"Stable node"

$\tau = -b$

$\tau^2 - 4\Delta > 0$ @ $b < \frac{4}{3}a$

$\tau^2 - 4\Delta > 0$ @ $a < b$

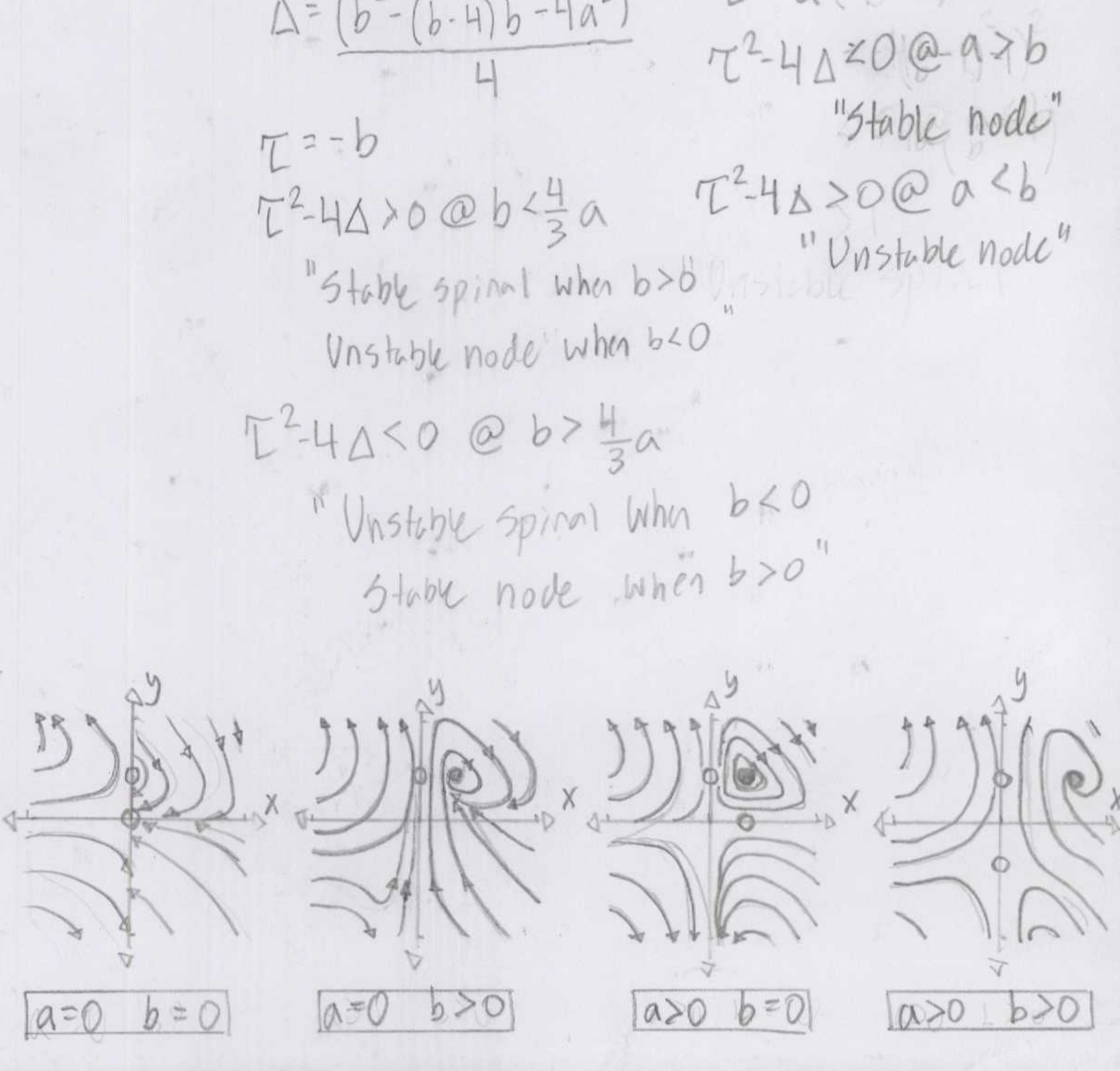
"Unstable node"

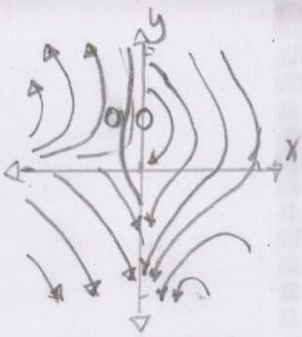
"Stable spiral when $b > 0$
Unstable node when $b < 0$"

$\tau^2 - 4\Delta < 0$ @ $b > \frac{4}{3}a$

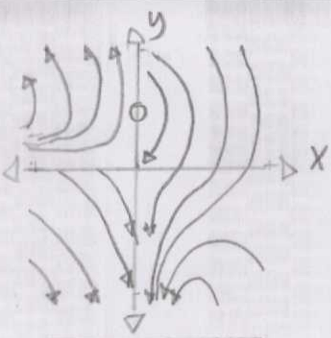
"Unstable spiral when $b < 0$
Stable node when $b > 0$"

c.

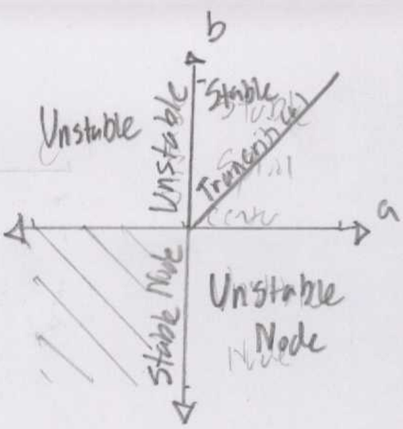




$\boxed{a=0 \quad b<0}$



$\boxed{a>0 \quad b<0}$



d. Upper - Right page.

$\dot{x}_1 = -x_1 + F(I - bx_2)$

$\dot{x}_2 = -x_2 + F(I - bx_1)$

8.1.14. If $F(x) = 1/(1+e^{-x})$: Gain Function

I : Strength of the Stimulus

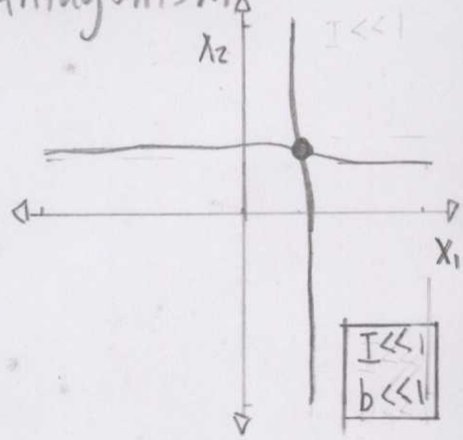
b : Strength of the Mutual Antagonism

a. Nullclines : $\dot{x}_1 = 0 = -x_1 + F(I - bx_2)$

$$= -x_1 + \frac{1}{1+e^{-I+bx_2}}$$

$$\dot{x}_2 = 0 = -x_2 + F(I - bx_1)$$

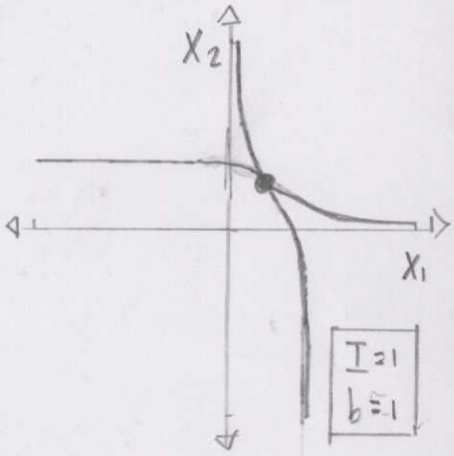
$$= -x_2 + \frac{1}{1+e^{-I+bx_1}}$$



$\boxed{I<<1 \; b<<1}$

$$x_2 = \frac{I + \log\left(\frac{1-x_1}{x_1}\right)}{b}$$

$$x_2 = \frac{1}{1+e^{-I+b\cdot x_1}}$$

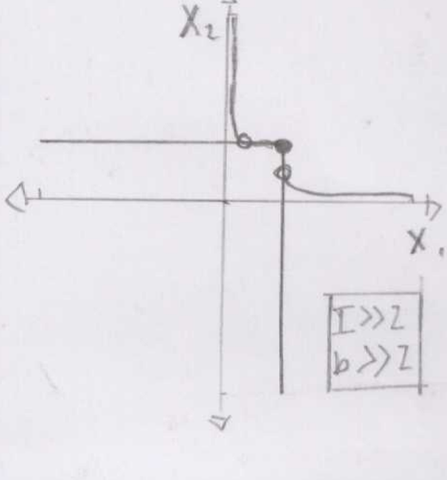


$\boxed{I=1 \; b=1}$

b. If $x_1^* = x_2^* = x^*$, then $\dot{x}_1 = -x_1 + \frac{1}{1+e^{-I+bx_1}}$

$$\dot{x}_2 = -x_1 + \frac{1}{1+e^{-I+bx_1}}$$

$$\dot{x}_1 = \dot{x}_2 = x^*$$



$\boxed{I>>2 \; b>>2}$

c. $\lim_{b\to\infty} \dot{x}_1 = \lim_{b\to\infty} -x_2 + \frac{1}{1+e^{-z+bx}} = -x_2$

d. See part a; Supercritical pitchfork because of the unstable center.

8.1.15.

$\dot{n}_A = (p+n_A)n_{AB} - n_A n_B$ where $n_{AB} = 1-(p+n_A)-n_B$

$\dot{n}_B = n_B n_{AB} - (p+n_A)n_{AB}$

a. The first term of $\dot{n}_A$ fits a constant (p) and changing (n_A) population of A-B.
The second term of $\dot{n}_A$ are the decrasing populations from A-B interaction.
The first term of $\dot{n}_B$ fits an increasing population from A-B interaction.
The second term of $\dot{n}_B$ fits a constant (p) and changing (n_A) population of A-B.

b. $n_B(0) = 1-p$; $n_A(0) = n_{AB}(0) = 0$

Numerical Integration: $\Delta h = 0.001$ $p = 0.15$

$k_{x1} = f(x_n)\Delta t$

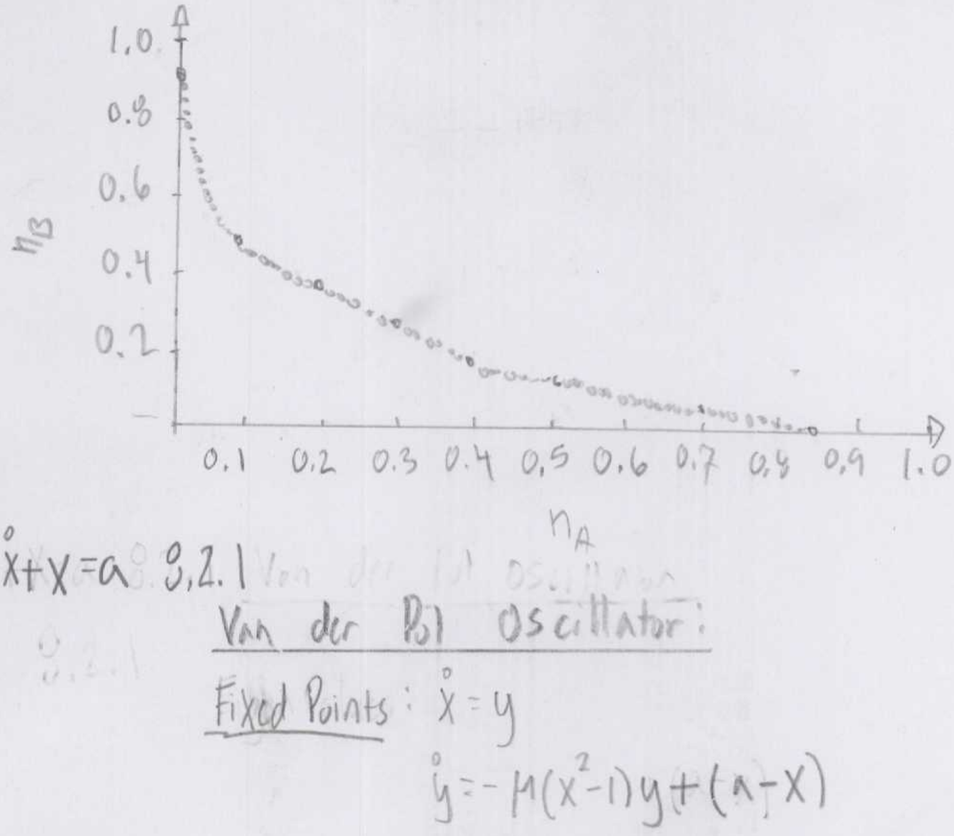
$k_{x2} = f(x_n + \frac{1}{2}k_{x1})\Delta t$

$k_{x3} = f(x_n + \frac{1}{2}k_{x2})\Delta t$

$k_{x4} = f(x_n + k_3)\Delta t$

$n_{x+1} = n_x + \frac{(k_1+2k_2+2k_3+k_4)}{6}$

n_A	n_B	n_{AB}	k_{A1}	k_{B1}	k_{A2}	k_{B2}	k_{A3}	k_{B3}	k_{A4}	k_{B4}
0	0.85	0	-	-	-	-	-	-	-	-
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.85	0.00	0.0001	0.00	-0.0001	0.00	-0.0001	0.00	-0.0001	0.00	0.00



$\ddot{x} + \mu(x^2-1)\dot{x} + x = a$ 8.2.1

Van der Pol oscillator:

Fixed Points: $\dot{x} = y$

$\dot{y} = -\mu(x^2-1)y + (a-x)$

$(x^*, y^*) = (a, 0)$

Eigenvalues: $(A-\lambda)\vec{x}=0$; $A-\lambda=\begin{pmatrix} -\lambda_1 & 1 \\ -2\mu xy-1 & -\mu(x^2-1)-\lambda_2 \end{pmatrix}$

$=(\lambda)(\mu(x^2-1)+\lambda)+2\mu xy-1$

$A_{(a,0)}=\lambda(\mu(a^2-1)+\lambda)=0.$

$\lambda_1=0$; $\lambda_2=-\mu(a^2-1)$

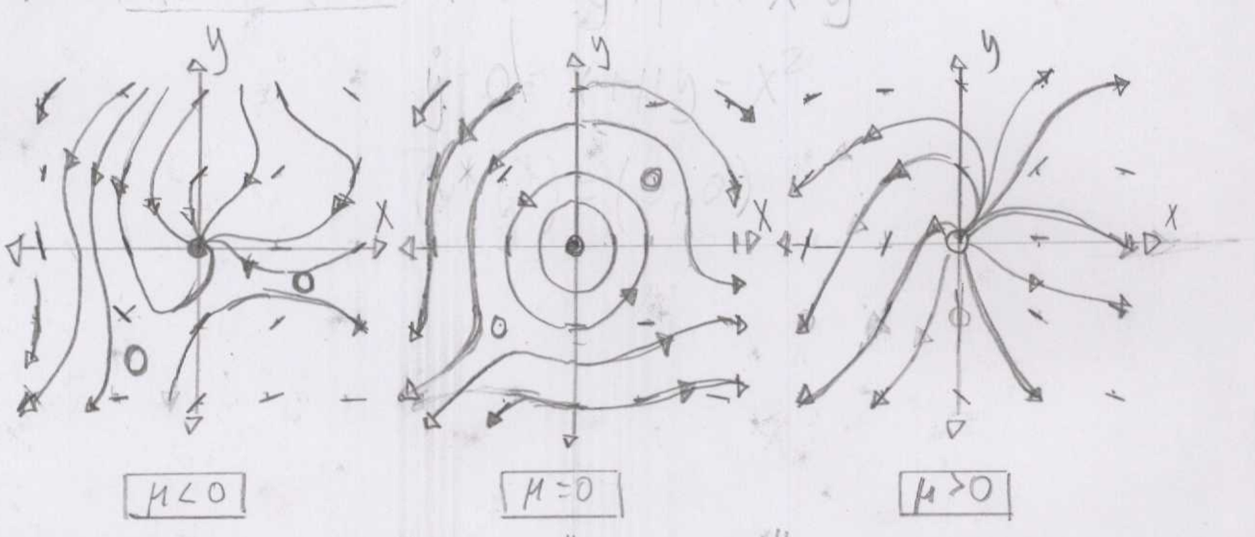
$\Delta=0$; $\tau=-\mu(a^2-1)$

$\tau^2-4\Delta>0$

Hopf Bifurcations: The phase plot changes when the sign of μ becomes positive, zero, or negative, in addition to, $a=\pm 1$.

$\dot{x}=-y+\mu x+xy^2$ 8.2.3. $\dot{x}=0=-y+\mu x+x^2y$

$\dot{y}=x+\mu y-x^2$



$\boxed{\mu<0}$ $\boxed{\mu=0}$ $\boxed{\mu>0}$

Pitchfork: "Super critical"

8.2.4.

a. $\dot{r}=\frac{d}{dt}\sqrt{x^2+y^2}=\frac{\dot{x}y+x\dot{y}}{\sqrt{x^2+y^2}}=\frac{(-y+\mu x+xy^2)y+x(x+\mu y-x^2)}{\sqrt{x^2+y^2}}$

$\dot{\theta}=\frac{d}{dt}\arctan\frac{y}{x}=\frac{x\dot{y}-y\dot{x}}{x^2+y^2}=\frac{x(x+\mu y-x^2)-y(-y+\mu x+xy^2)}{x^2+y^2}$

b. If $r\ll 1$, then $\dot{\theta}\approx\frac{x^2+y^2}{x^2+y^2}=1$

and $\dot{r} = \dfrac{x^2 - y^2 + 2\mu xy + xy^3 - x^3}{\sqrt{x^2+y^2}}$

c. $\dot{r} = 0 \approx \mu r + \frac{1}{8} r^3$; $r = \sqrt{-8\mu}$

If $\mu < 0$, then $r \in \mathbb{R}$, and if $\mu > 0$

$r \in \mathbb{I}$

$\dot{x} = y + \mu x$
$\dot{y} = -x + \mu y - x^2 y$

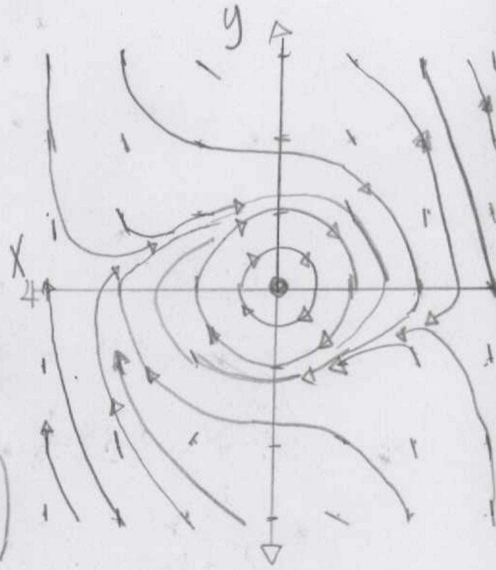
8.2.5. <u>Fixed Points:</u> $\dot{x} = 0 = y + \mu x$

$\dot{y} = 0 = -x + \mu y - x^2 y$

$(x^*, y^*) = (0, 0)$

$= \left(-\sqrt{\frac{\mu+1}{\mu}}, \sqrt{\mu(\mu^2+1)}\right)$

$= \left(\sqrt{\frac{\mu+1}{\mu}}, -\sqrt{\mu(\mu^2+1)}\right)$



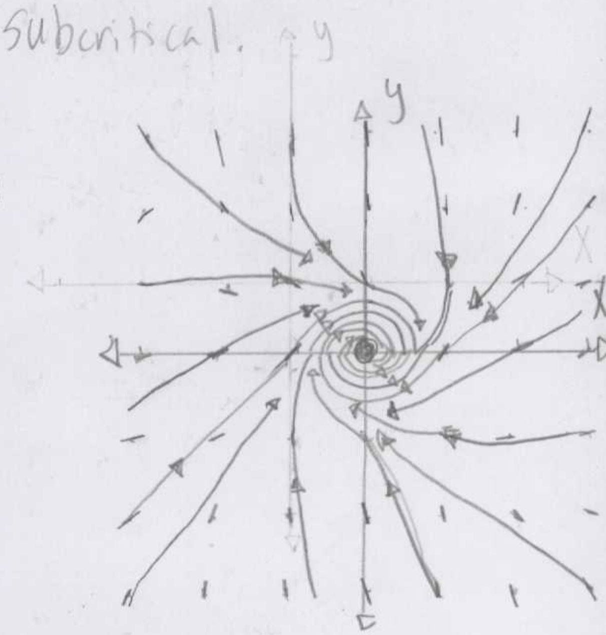
Pitchfork: Subcritical.

$\dot{x} = \mu x + y - x^3$
$\dot{y} = -x + \mu y - 2y^3$

8.2.6. <u>Fixed Points:</u> $\dot{x} = 0 = \mu x + y - x^3$

$\dot{y} = 0 = -x + \mu y - 2y^3$

$(x^*, y^*) = (0, 0)$



Pitchfork Supercritical

$\dot{x} = \mu x + y - x^2$
$\dot{y} = -x + \mu y - 2x^2$

8.2.7. Fixed Points: $\dot{x} = 0 = \mu x + y - x^2$

$\dot{y} = 0 = -x + \mu y - 2x^2$

$(x^*, y^*) = (0, 0)$

$= \left(\frac{\mu^2+1}{m-2}, \frac{(2m+1)(m^2+1)}{(m-2)^2}\right)$

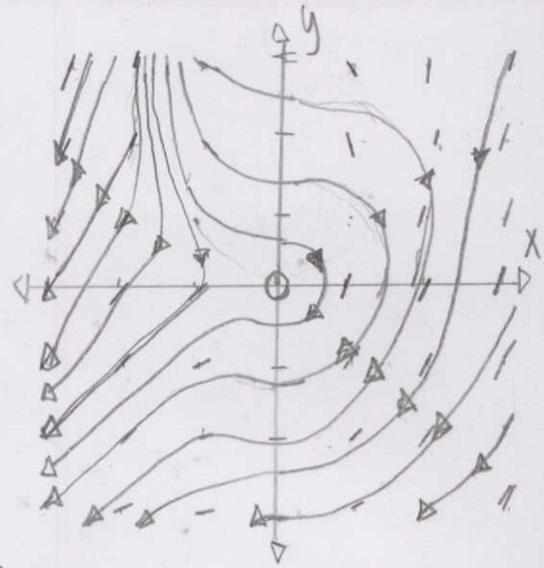
$\dot{x} = x[x(1-x)-y]$ "Pitchfork : Supercritical"

$\dot{y} = y(x-a)$ 8.2.8 a. <u>Nullclines</u>: $\dot{x} = 0 = x[x(1-x)-y]$

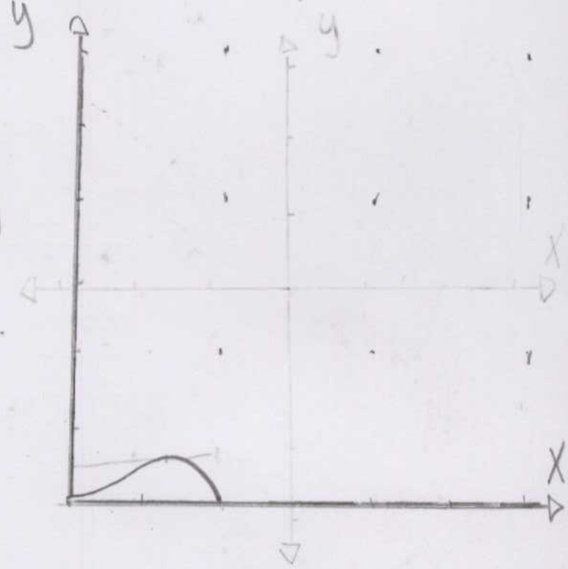
$\dot{y} = 0 = y(x-a)$

$x = 0 ; y = 0$

$x = a ; y = x \cdot (1-x)$



b. <u>Fixed Points</u>: $\dot{\vec{x}} = A\vec{x} = 0$



$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2x-3x^2 & -x \\ y & x-a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A_{(0,0)} - \lambda = \begin{bmatrix} -\lambda & 0 \\ 0 & -(\lambda+a) \end{bmatrix}$$

$\lambda_1 = 0 ; \lambda_2 = -a$

$\Delta = 0 ; \tau = -a ; \tau^2 - 4\Delta > 0$ "stable spiral"

$$A_{(1,0)} - \lambda = \begin{bmatrix} -(\lambda+1) & -1 \\ 0 & 1-(a+\lambda) \end{bmatrix}$$

$\lambda_1 = -1 , \lambda_2 = a-1$

$\Delta = 1-a ; \tau = a-2 ; \tau^2 - 4\Delta = a^2 - 6a$

If $a > 6$, line of unstable fixed points

If $a < 6$, saddle node.

If $a = 6$, unstable star / degenerate node.

$$A_{(a,a-a^2)} - \lambda = \begin{bmatrix} 2a-3a^2-\lambda & -a \\ a^2-a^2 & -\lambda \end{bmatrix}$$

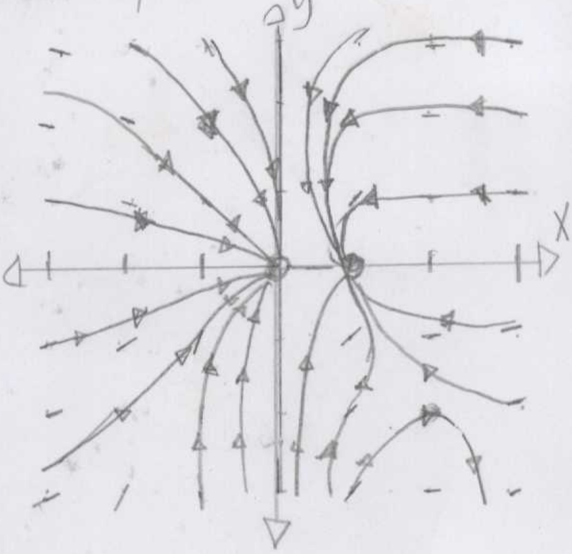
$$\lambda_1 = \frac{1}{2}\left(-\sqrt{9a-8}\,a^{3/2} - 3a^2 + 2a\right)$$

$$\lambda_2 = \frac{1}{2}\left(\sqrt{9a-8}\,a^{3/2} - 3a^2 + 2a\right)$$

$$\Delta = a^2 - a^3 ; \quad \tau = 2a - 3a^2$$

$\tau^2 - 4\Delta > 0$ "Stable Spiral"

c.) If $a > 1$, phase portrait



d)

a	Bifurcations
$<1/2$	2
$=1/2$	3
$>1/2$	3

Pitch fork: Supercritical Hopf

e) At Hopf Bifurcation $\left(\frac{1}{2}, \frac{1}{4}\right)$,

$$\lambda_1 = \frac{1}{2}\left(-\sqrt{9/2 - 8}\,(1/2)^{3/2} - 3(1/2)^2 + 2(1/2)\right)$$

$$= \frac{1}{2}\left(-\frac{35}{100}\left(\frac{7}{2}\right)i - (3/4) + 1\right)$$

$$= \frac{1}{8} + \frac{49}{80}i$$

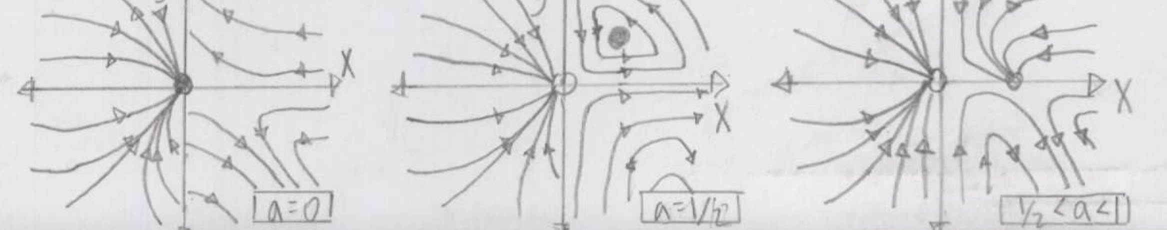
$$\lambda_2 = \frac{1}{2}\left(\sqrt{9/2 - 8}\,(1/2)^{3/2} - 3(1/2)^2 + 2(1/2)\right)$$

$$= \frac{1}{2}\left(-\frac{35}{100}\left(\frac{7}{2}\right)i - 3/4 + 1\right)$$

$$= \frac{1}{8} - \frac{49}{80}i$$

Frequency $= 2\pi\omega a = 2\pi\left(\frac{49}{80}\right)\left(\frac{1}{2}\right) = \frac{49}{80}\pi$

f)



$\dot{x} = x\left(b - x - \frac{y}{1+x}\right)$ 8.2.9. $x, y \geq 0$; $a, b > 0$

$\dot{y} = y\left(\frac{x}{1+x} - ay\right)$

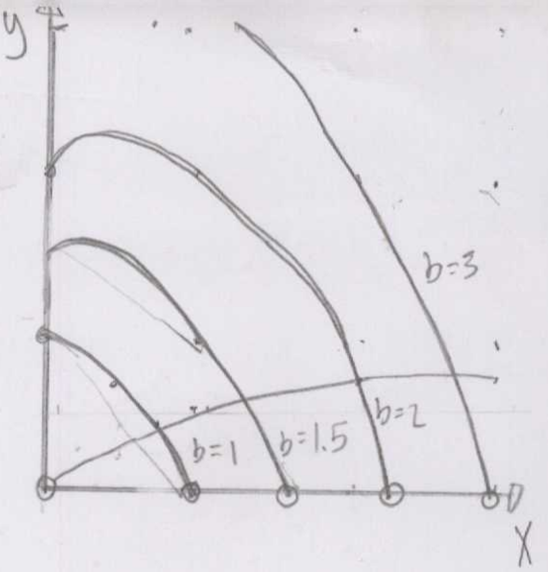
a. Nullclines: $\dot{x} = 0 = x\left(b - x - \frac{y}{1+x}\right)$

$\dot{y} = 0 = y\left(\frac{x}{1+x} - ay\right)$

$y = 0; x = 0$

$y = (1+x)\cdot(b-x)$

$y = \frac{x}{a(1+x)}$

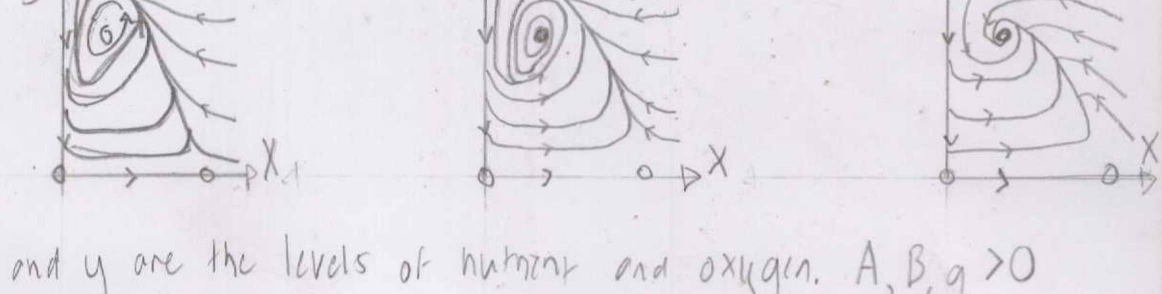


b. A graphical argument for the fixed point $x^* > 0$, $y^* > 0$ for all $a, b > 0$ displayed in part a, becomes an real-solution.

c. If $a_c = \frac{4(b-2)}{b^2(b+2)}$, then and $b<2$, $b=2$, or $b>2$, then

$x = a$	Number of Bifurcations
$< \frac{4(b-2)}{b^2(b+2)}$	3 "unstable"
$= \frac{4(b-2)}{b^2(b+2)}$	3 "stable"
$> \frac{4(b-2)}{b^2(b+2)}$	3 "stable"

d. Phase Portrait:



$\dot{x} = B - x - \frac{xy}{1+qx^2}$ 8.2.10. x and y are the levels of nutrient and oxygen. $A, B, q > 0$

$\dot{y} = A - \frac{xy}{1+qx^2}$

Fixed Points: $\dot{x} = 0 = B - x - \frac{x \cdot y}{1+q\cdot x^2}$

$\dot{y} = 0 = A - \frac{x\cdot y}{1+qx^2}$

$(x^*, y^*) = \left(A - B, \frac{A}{B-A}\left[1 + q(A-B)^2\right]\right)$

Nullclines: $x = A - B$

$$y = \frac{A}{B-A}\left(1 + q(A-B)^2\right)$$

Trapping Region: $\dot{\vec{x}} = A\vec{x}$; $0 = A\vec{x} = \lambda\vec{x}$; $0 = (A-\lambda)\vec{x}$

where $A = \begin{pmatrix} -1 - \frac{y(1-qx^2)}{(1+qx^2)^2} - \lambda & \frac{-x}{(1+qx^2)} \\ -\frac{y(1-qx^2)}{(1+qx^2)^2} & \frac{-x}{1+qx^2} - \lambda \end{pmatrix}$

$$0 = \left(-1 - \frac{y(1-qx^2)}{(1+qx^2)^2} - \lambda\right)\left(\frac{-x}{1+qx^2} - \lambda\right) - \left(\frac{x}{1+qx^2}\right)\left(\frac{y(1-qx^2)}{(1+qx^2)^2}\right)$$

$$\lambda_{1,2} = \frac{\left(q^3x^6(y-1) - q^2x^4(x+y-3) \pm \sqrt{(qx^2+1)^3\left(q^3x^6(y^2-2y+1) + q^2\left(x^5(2y-2) + x^4(3-y^2-2y+q)\right) - 4qx^3(y+1) + x^2q(y^2+2y+3) + x^2 + 2x(y-1) + y^2 + 2y + 1\right)} - 2qx^3 - qx^2y - 3qx^2 - x - y - 1\right)}{2(qx^2+1)^3}$$

A stable limit cycle has three parameters, μ the stability of a fixed point at the origin, ω the frequency, and b the dependence of frequency on amplitude.

The stability of μ depends on $q^3x^6(y-1) - q^2x^4(x+y-3)/2(qx^2+1)^3$ being positive or negative.

The frequency ω about the origin is the square root in the eigenvalue being real or imaginary.

$\ddot{x} + \mu\dot{x} + x - x^3 = 0$

8.2.11.

a. $\dot{x} = y$

$\dot{y} = -\mu y - x + x^3$

Fixed Points: $\dot{x} = 0 = y$

$\dot{y} = 0 = -\mu y - x + x^3$

$(x^*, y^*) = (0,0), (-1,0), (1,0)$

$\underline{\text{Bifurcations:}}$ $\dot{\vec{x}} = A\vec{x}$; $0 = A\vec{x} = \lambda\vec{x}$; $0 = (A-\lambda)\vec{x}$

$$A = \begin{pmatrix} -\lambda & 1 \\ -1+3x^2 & -\mu-\lambda \end{pmatrix} = \lambda(\mu+\lambda)+1-3x^2 = 0$$

$$\lambda_{1,2} = \frac{1}{2}\left(\pm\sqrt{\mu^2+12x^2-4} - m\right)$$

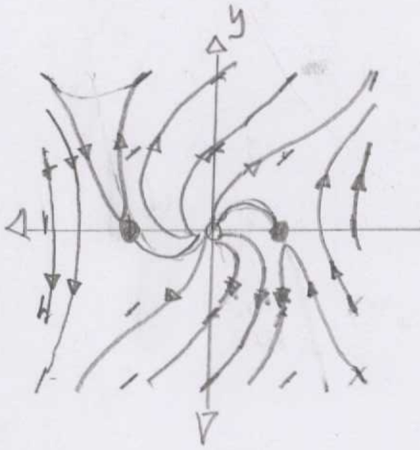
$\Delta = 1-3x^2$; $\tau = -\mu$; $\tau^2 - 4\Delta = \mu^2 - 4 - 12x^2$

If $\mu < 0$ and $x \geq \sqrt{\frac{1}{3}}$, unstable spiral

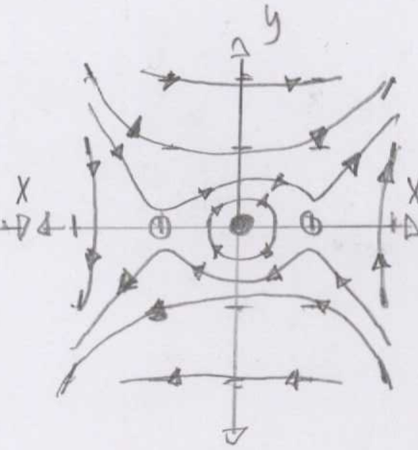
If $\mu = 0$ and $x > \sqrt{\frac{1}{3}}$, center

If $\mu > 0$ and $x > \sqrt{\frac{1}{3}}$, stable spiral.

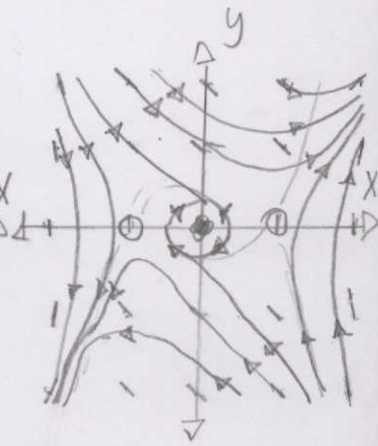
b. Phase Portraits:



$\boxed{\mu < 0}$



$\boxed{\mu = 0}$



$\boxed{\mu > 0}$

$\dot{x} = -\omega y + f(x,y)$ 8.2.12: $16a = f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}$

$\dot{y} = \omega x + g(x,y)$

$$+ \frac{1}{\omega}\left[f_{xy}(f_{xx}+f_{yy}) - g_{xy}(g_{xx}+g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\right]$$

If $a < 0$: supercritical, $a > 0$: subcritical.

a) $\dot{x} = -y + xy^2$ $\quad \dot{y} = \cancel{x} - x^2$

$f = xy^2$	$g = -x^2$	$f_{xy} = 2y$	$g_{xy} = 0$
$f_x = y^2$	$g_x = -2x$	$f_{xyy} = 2$	$g_{yy} = 0$
$f_{xx} = 0$	$g_{xx} = -2$	$f_{yy} = 2x$	$g_{xxy} = 0$
$f_{xxx} = 0$	$g_{xxx} = 0$		$g_{yyy} = 0$

$$16a = 0 + 2y + 0 + 0 + \frac{1}{\omega}\left[2y(0+2x) - 0(0+0) - 0\cdot 0 + 2x\cdot 0\right]$$
$$= 2y + \frac{4yx}{\omega}$$

An evaluation at the point $(0,0)$

$16a = \frac{2}{8}$; $a = \frac{1}{8} > 0$: subcritical

b. $\dot{x} = -y + \mu x + xy^2$; $\dot{y} = x + \mu y - x^2$

A subcritical Hopf Bifurcation occurs when $\mu = 0$.

$\dot{x} = y + \mu x$
$\dot{y} = -x + \mu y - x^2 y$

8.2.13. $f = 0$ $\qquad g = -x^2 y$

$f_x = 0 \quad f_{xy} = 0 \quad f_y = 0 \qquad g_x = -2xy \quad g_y = -x^2$

$f_{xx} = 0 \quad f_{xyy} = 0 \quad f_{yy} = 0 \qquad g_{xx} = -2y \quad g_{yy} = 0 \quad g_{xy} = -2x$

$f_{xxx} = 0 \qquad g_{xxy} = -2 \quad g_{yyy} = 0$

$$16a = 0 + 0 - 2 + 0 + \frac{1}{\omega}\left[0(0+0) + 2x(-2y+0) + 0\cdot 2y + 0\cdot 0\right]$$
$$= -2 - \frac{4xy}{\omega} = -2$$

An evaluation at the point $(0,0)$

$a = -\frac{1}{8}$; A supercritical Hopf Bifurcation

$\dot{x} = \mu x + y - x^3$
$\dot{y} = -x + \mu y + 2y^3$

8.2.14. $f = -x^3$ $\qquad g = 2y^3$

$f_x = -3x^2 \quad f_{xy} = 0 \quad f_y = 0 \qquad g_x = 0 \quad g_y = 6y^2$

$f_{xx} = -6x \quad f_{xyy} = 0 \quad f_{yy} = 0 \qquad g_{xx} = 0 \quad g_{yy} = 12y \quad g_{xy} = 0$

$f_{xxx} = -6 \qquad g_{xxy} = 0 \quad g_{yyy} = 12$

$$16a = -6 + 0 + 0 + 12 + \frac{1}{\omega}\left[0(-6x+0) - 0(0+12y) + 6x\cdot 0 + 0\cdot 12y\right]$$
$$= 6$$

An evaluation at the point $(0,0)$:

$a = \frac{3}{8}$; A subcritical Hopf Bifurcation.

$\dot{x} = \mu x + y - x^2$ 8.2.15. $f = -x^2$ $g = 2x^2$

$\dot{y} = -x + \mu y + 2x^2$

$f_x = -2x$ $f_{xy} = 0$ $f_y = 0$ $g_x = 4x$ $g_y = 0$

$f_{xx} = -2$ $f_{xyy} = 0$ $f_{yy} = 0$ $g_{xx} = 4$ $g_{yy} = 0$ $g_{xy} = 0$

$f_{xxx} = 0$ $g_{xxy} = 0$ $g_{yyy} = 0$

$$16a = 0 + 0 + 0 + 0 + \frac{1}{\omega}\left[0(-2+0) - 0(4+0) + 2 \cdot 4 + 0 \cdot 0\right]$$

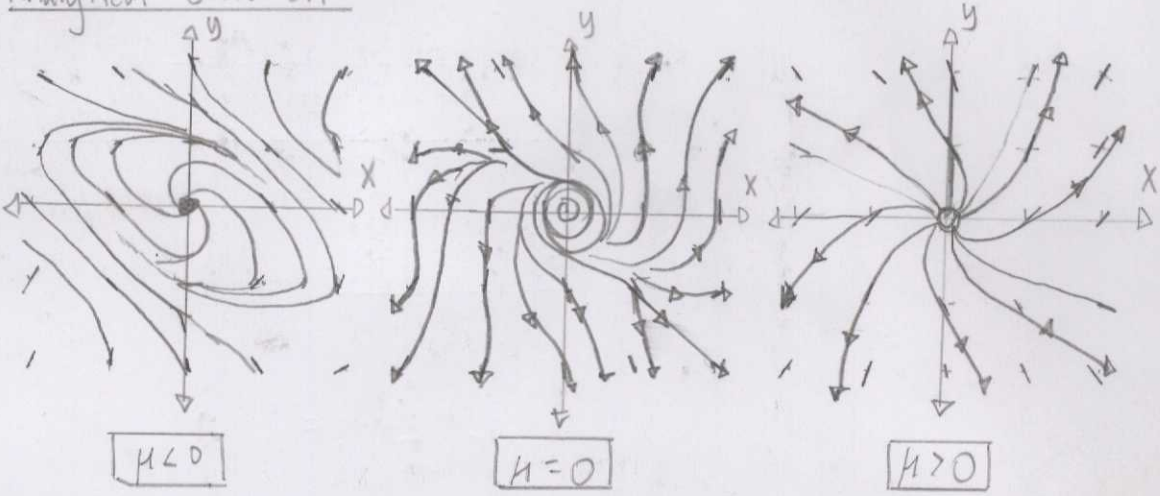
$$= \frac{8}{\omega}$$

An evaluation at point $(0,0)$:

$a = \frac{1}{2}\omega = \frac{1}{2}$; A subcritical Hopf Bifurcation.

$\dot{x} = \mu x - y + xy^2$ 8.2.16. Analytical Criterion:

$\dot{y} = x + \mu y + y^3$



Subcritical bifurcation when $\mu < 0$.

$\dot{x}_1 = -x_1 + F(I - bx_2 - gy_1)$ 8.2.17 y = adoption

$\dot{y}_1 = (-y_1 + x_1)/T$ T = Timescale

$\dot{x}_2 = -x_2 + F(I - bx_1 - gy_2)$ g = Associated neuronal population

$\dot{y}_2 = (-y_2 + x_2)/T$ $F(x) = 1/1+e^{-x}$ = Gain Function

b = mutual strength

a) $x_1^* = y_1^* = x_2^* = y_2^* = u = 0$; $u_1 = \dfrac{1}{1+e^{-(I - bx_2 - gy_1)}}$

b) $$A = \begin{bmatrix} -1 & -Fg & -Fb & 0 \\ 1/T & -1/T & 0 & 0 \\ -Fb & 0 & -1 & -Fg \\ 0 & 0 & 1/T & -1/T \end{bmatrix} = \begin{bmatrix} -c_1 & -c_2 & -c_3 & 0 \\ d_1 & -d_1 & 0 & 0 \\ -c_3 & 0 & -c_1 & -c_2 \\ 0 & 0 & d_1 & -d_1 \end{bmatrix}$$

Where $c_1 = 1$, $c_2 = Fg$, $c_3 = Fb$, $d_1 = 1/T$

If $A = \begin{bmatrix} -c_1 & -c_2 \\ d_1 & -d_1 \end{bmatrix}$ and $B = \begin{bmatrix} -c_3 & 0 \\ 0 & 0 \end{bmatrix}$

then $\begin{bmatrix} A & B \\ B & A \end{bmatrix} = A^2 - B^2 = (A+B)(A-B)$

Eigenvalues of a 4x4:

$$\begin{bmatrix} -c_1 - c_3 - \lambda & -c_2 \\ d_1 & -d_1 - \lambda \end{bmatrix} \begin{bmatrix} c_3 - c_1 - \lambda & -c_2 \\ d_1 & -d_1 - \lambda \end{bmatrix} = 0$$

$$\lambda_{1,2} = \frac{\pm\sqrt{4T(Fb - Fg - 1) + (FbT - T - 1)^2} + FbT - T - 1}{2T}$$

$$\lambda_{3,4} = \frac{\pm\sqrt{(FbT + T + 1)^2 - 4T(Fb + Fg + 1)} - FbT - T - 1}{2T}$$

c. $\Delta = \lambda_1 \cdot \lambda_2 = \dfrac{F(g+b)+1}{T} > 0$

$\tau = \lambda_1 + \lambda_2 = -Fb - \frac{1}{T} - 1 < 0$

So, λ_1 and λ_2 are each negative eigenvalues.

d. $\Delta = \lambda_3 \cdot \lambda_4 = \dfrac{-F(g-b)+1}{T}$; If $g > b$, then $\Delta > 0$; Hopf Bifurcation.
If $g < b$, then $\Delta < 0$; Pitchfork Bifurcation

$\tau = \lambda_3 + \lambda_4 = Fb - \frac{1}{T} - 1$; Positive or negative.

e.

$\dot{x} = 1-(b+1)x + ax^2y$ 8.3.1. $a, b > 0$ and $x, y \geq 0$
$\dot{y} = bx - ax^2y$

a) <u>Fixed Points:</u> $\dot{x} = 0 = 1-(b+1)x + ax^2y$

$$\dot{y} = 0 = bx - ax^2y$$

$$(x^*, y^*) = (0,0), (1, b/a)$$

$$(A-\lambda)\vec{x} = 0; \quad A = \begin{pmatrix} -(b+1)+2axy-\lambda & ax^2 \\ b-2axy & -ax^2-\lambda \end{pmatrix}$$

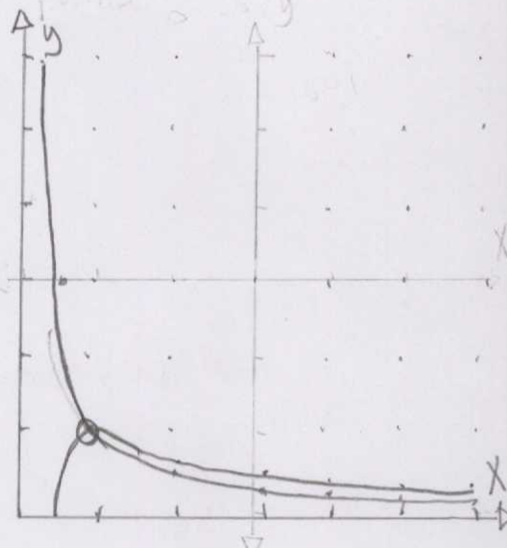
$$A_{(1,\frac{b}{a})} = \begin{pmatrix} -b-1 & a \\ -b & -a \end{pmatrix}$$

$$\Delta = a; \quad \tau = -b-(1+a); \quad \tau^2 - 4\Delta$$

b) <u>Nullclines:</u> $\dot{x} = 0 = 1-(b+1)x + ax^2y$

$$\dot{y} = 0 = bx - ax^2y$$

$$y = \frac{b}{a}\left(\frac{1}{x}\right); \quad y = \frac{-1+(b+1)x}{ax^2}$$



c) <u>Bifurcations:</u>

The bifurcation occurs at

$\tau = 0 = b-(1+a)$.

d) <u>Poincaré - Bendixson theorem:</u>

1) A single unstable node or spiral inside an invariant region

2) No critical points inside an invariant region.

If either case exists and non-periodic, then no limit cycle in the graph.

If the solution is periodic, then a limit cycle appears.

A critical point at $(1, b/a)$ fits type I and is a trapping region defined by the domain and range of the nullclines.

e) The period of the limit cycle appears from a transformation to polar coordinates or eigenvalues. An analysis about the Jacobian shows the eigenvalues, as$_{1,2}$ complex values proportional to $\sqrt{a}$, so a limit cycle has a period $\frac{2\pi}{\sqrt{a}}$.

$X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} A$

$B \xrightarrow{k_2} Y$

$2X + Y \xrightarrow{k_3} 3X$

$\dot{x} = a - x + x^2 y$

$\dot{y} = b - x^2 y$

8.3.2.

a) $\frac{\dot{y}}{\dot{x}} = \frac{b - x^2 y}{a - x + x^2 y}$; $\lim_{x \to \infty} \frac{\dot{y}}{\dot{x}} = -1$; $\lim_{x \to -\infty} \frac{\dot{y}}{\dot{x}} = -1$

$\lim_{y \to \infty} \frac{\dot{y}}{\dot{x}} = -1$; $\lim_{y \to -\infty} \frac{\dot{y}}{\dot{x}} = -1$

b) Fixed Points: $\dot{x} = 0 = a - x + x^2 y$

$\dot{y} = 0 = b - x^2 y$

$(x^*, y^*) = \left(a+b, \frac{b}{(a+b)^2}\right)$

$$A = \begin{pmatrix} -1 + 2xy & x^2 \\ -2xy & -x^2 \end{pmatrix}$$

$$A_{(x^*, y^*)} = \begin{pmatrix} -1 + \frac{2b}{(a+b)} & (a+b)^2 \\ \frac{-2b}{(a+b)} & -(a+b)^2 \end{pmatrix}$$

$\Delta = (a+b)^2 - 2b(a+b)$; $\tau = -1 + \frac{2b}{(a+b)} - (a+b)^2$

$= a^2 - b^2$

If $a>b$, "stable spiral"

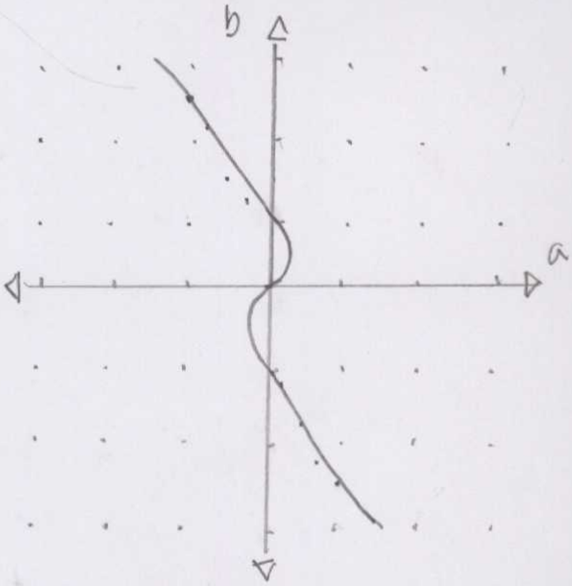
If $a<b$, "saddle point"

c) Bifurcations: $\tau = -1 + \dfrac{2b}{(a+b)} - (a+b)^2$

$$= \frac{-(a+b) + 2b - (a+b)^3}{(a+b)} = 0$$

$$b - a = (a+b)^3$$

d) The Hopf Bifurcation is subcritical because the center is stable.



e) Stability Diagram:

$$b = \frac{\sqrt[3]{\sqrt{3}\sqrt{27a^2-1} - 9a}}{3^{2/3}} + \frac{1}{\sqrt[3]{3}\sqrt[3]{\sqrt{3}\sqrt{27a^2-1} - 9a}} - a$$

If $x^* = (a+b)$, then

$$a = \frac{(a+b)}{2}\left(1 - (a+b)^2\right)$$

$$b = \frac{(a+b)}{2}\left(1 + (a+b)^2\right) \quad \text{and} \quad b - a = \frac{(a+b)}{2}\left[2(a+b)^2\right] = (a+b)^3$$

$$\dot{x} = a - x - \frac{4xy}{1+x^2}$$

$$\dot{y} = bx\left(1 - \frac{y}{1+x^2}\right)$$

8.3.3 Phase Plane:

$b \ll 1$; 0.5

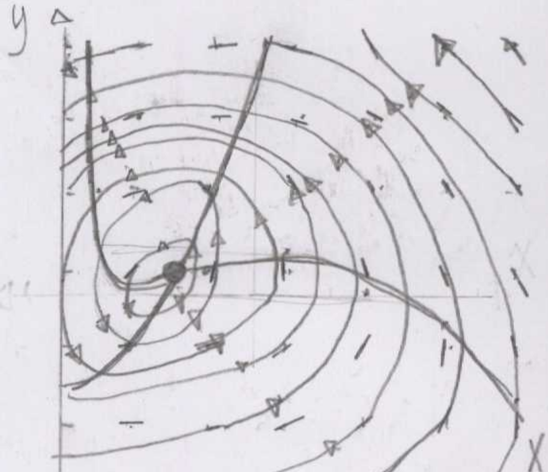
(8.3.7) $b < b_c = \dfrac{3a}{5} - \dfrac{25}{a}$

If $b = 0.5$, then $a = 6.8$

Nullclines:

$$y = 1 + x^2$$

$$y = \frac{(a-x)(1+x^2)}{4x}$$



Limit Cycle: $T = \int_{t_1}^{t_2} dt + \int_{t_2}^{t_3} dt + \int_{t_3}^{t_4} dt + \int_{t_4}^{t_1} dt$

where $\int_{t_2}^{t_3} dt = \int_{t_4}^{t_1} dt = 0$

$$T = 2\int_{t_1}^{t_2} dt + 2\int_{t_3}^{t_4} dt = \int_{t_1}^{t_2} \frac{dy}{dx}\frac{dt}{dy}dx + \int_{t_3}^{t_4} \frac{dy}{dx}\frac{dt}{dy}dx$$

$$= \int_{t_1}^{t_2} \frac{ax^2-a-2x^3}{4x^2} \frac{dx}{bx\left(1-\frac{a-x}{4x}\right)}$$

$$+ \int_{t_3}^{t_4} \frac{ax^2-a-2x^3}{4x^2} \cdot \frac{dx}{bx\left(1-\frac{a-x}{4x}\right)}$$

$$= \left.\frac{(3a^2-125)x\ln(5x-a)-5a(2x^2+5)+125x\ln(x)}{25abx}\right|_{t_1}^{t_2}$$

$$+ \left.\frac{(3a^2-125)x\ln(5x-a)-5a(2x^2+5)+125x\ln(x)}{25abx}\right|_{t_3}^{t_4}$$

$\dot{r} = r(1-r^2)$ 8.4.1 $\frac{dr}{dt} = r(1-r^2)$

$\dot{\theta} = \mu - \sin\theta$

$$t = \int \frac{dr}{r(1-r^2)} = \int \frac{A}{r}dr + \int \frac{B}{(1-r)}dr + \int \frac{C}{(1+r)}dr$$

$A(1-r)(1+r) + Br(1+r) + Cr(1-r) = 1$

$r = 1;\ B = 1/2$

$r = -1;\ C = -1/2$

$r = 0;\ A = 1$

$$= \int \frac{1}{r}dr + \int \frac{1}{(1-r)}dr - \int \frac{1}{(1+r)}dr = \ln r - \frac{\ln|1-r|}{2} - \frac{\ln|1+r|}{2}$$

$$= \ln \frac{r}{\sqrt{1-r^2}}$$

$$(1-r^2)e^{2t} = e^{2t} - r^2 e^{2t} = r^2;\quad -r^2 e^{2t} - r^2 + e^{2t} = -r^2(e^{2t}+1) + e^{2t}$$

$$r = \pm\frac{e^{t}}{\sqrt{e^{2t}+1}}$$

$$\frac{d\theta}{dt} = \mu - \sin\theta;\quad t = \int\frac{d\theta}{\mu-\sin\theta} = \int\frac{d\theta}{\mu-\sin\theta} = \int\frac{d\theta}{\mu - \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}}}$$

$$= \int\frac{d\theta}{\mu - \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}}$$

If $u = \tan\frac{\theta}{2}$; $\frac{du}{dx} = \frac{\sec^2\frac{\theta}{2}}{2}$; $dx = \frac{2\,du}{\sec^2\frac{\theta}{2}} = \frac{2}{u^2+1}du$

$$= \int\frac{d\theta}{\mu - \frac{2u}{1+u^2}}\cdot\frac{2}{u^2+1}du = 2\int\frac{du}{\mu u^2+\mu-2u}$$

$$= 2\int\frac{du}{\mu\left(u-\frac{1}{\mu}\right)^2 - \frac{1}{\mu^2}+\mu}$$

If $v = \frac{\mu u - 1}{\sqrt{\mu}\left(\sqrt{\mu - 1/\mu}\right)}$; $\frac{dv}{du} = \frac{\sqrt{\mu}}{\sqrt{\mu-1/\mu}}$; $du = \frac{\sqrt{\mu-1/\mu}}{\sqrt{\mu}}dv$

$$= 2\int\frac{\sqrt{\mu-1/\mu}}{\sqrt{\mu}\left(\left(\mu-\frac{1}{\mu}\right)v^2+\mu-\frac{1}{\mu}\right)}dv$$

$$= \frac{2}{\sqrt{\mu}\sqrt{\mu-\frac{1}{\mu}}}\int\frac{1}{v^2+1}dv = \frac{\arctan(v)}{\sqrt{\mu}\sqrt{\mu-1/\mu}}$$

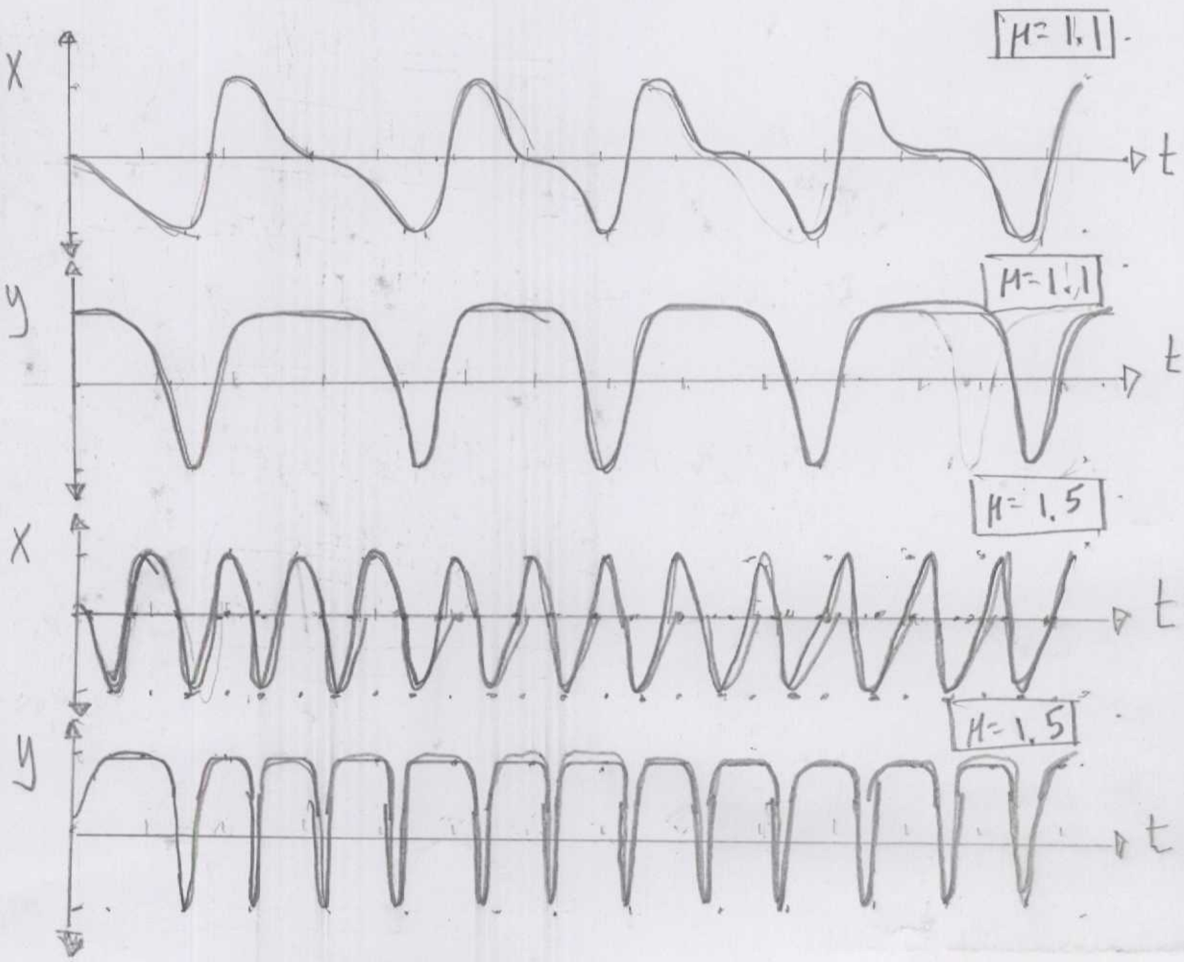
$$= \frac{2\arctan\left(\frac{\mu u-1}{\sqrt{\mu}\sqrt{\mu-1/\mu}}\right)}{\sqrt{\mu}\sqrt{\mu-1/\mu}}$$

$$= \frac{2\arctan\left(\frac{\mu\tan\left(\frac{\theta}{2}\right)-1}{\sqrt{\mu}\sqrt{\mu-1/\mu}}\right)}{\sqrt{\mu}\sqrt{\mu-1/\mu}}$$

$$\theta = 2\tan^{-1}\left(\frac{\sqrt{\mu}\tan\left(\frac{1}{2}\sqrt{\mu}\sqrt{\frac{\mu^2-1}{\mu}}\,t\right)}{\sqrt{\frac{\mu^2-1}{\mu}}} - \frac{\tan\left(\frac{1}{2}\sqrt{\mu}\sqrt{\frac{\mu^2-1}{\mu}}\,t\right)}{\mu^{3/2}\sqrt{\frac{\mu^2-1}{\mu}}} + \frac{1}{\mu} \right)$$

$$x(t) = r\cos\theta = \frac{e^{2t}}{\sqrt{e^{2t}+1}}\cos\left[2\tan^{-1}\left(\frac{\sqrt{\mu}\tan\left(\frac{1}{2}\sqrt{\mu}\sqrt{\frac{\mu^2-1}{\mu}}\,t\right)}{\sqrt{\frac{\mu^2-1}{\mu}}} - \frac{\tan\left(\frac{1}{2}\sqrt{\mu}\sqrt{\frac{\mu^2-1}{\mu}}\,t\right)}{\mu^{3/2}\sqrt{\frac{\mu^2-1}{\mu}}} + \frac{1}{\mu} \right)\right]$$

$$y(t) = r\sin\theta = \frac{e^{2t}}{\sqrt{e^{2t}+1}}\sin\left[2\tan^{-1}\left(\frac{\sqrt{\mu}\tan\left(\frac{1}{2}\sqrt{\mu}\sqrt{\frac{\mu^2-1}{\mu}}\,t\right)}{\sqrt{\frac{\mu^2-1}{\mu}}} - \frac{\tan\left(\frac{1}{2}\sqrt{\mu}\sqrt{\frac{\mu^2-1}{\mu}}\,t\right)}{\mu^{3/2}\sqrt{\frac{\mu^2-1}{\mu}}} + \frac{1}{\mu} \right)\right]$$



$\ddot{\theta} + (1-\mu\cos\theta)\dot{\theta} + \sin\theta = 0$

8.4.4. $\dot{\phi} = 4 =$

$\dot{4} = -(1-\mu\cos\phi)4 - \sin\phi$; $\dot{4} = 0 = -(1-\mu\cos\phi)4 - \sin 4$

$$\mu_c = \frac{\tan(\phi)}{4}$$

If $\mu < \mu_c$, Infinite period bifurcation.

If $\mu > \mu_c$, Infinite-period bifurcation.

If $\mu = \mu_c$, stable cycle.

$\ddot{x} + x + \epsilon(bx^3 + k\dot{x} - ax - F\cos t) = 0$ "Forced Duffing oscillator"

8.4.5. $r' = \langle h\sin\theta \rangle = \frac{1}{2\pi}\int_0^{2\pi} [bx^3 + k\dot{x} - ax - F\cos t]\sin\theta \, d\theta$

$$= \frac{1}{2\pi}\int_0^{2\pi} (br^3\cos^3\theta - kr\sin\theta - ar\cos\theta - F\cos(\theta-\phi))\sin\theta \, d\theta$$

$$= \frac{1}{2\pi}\left[br^3\int_0^{2\pi}\cos^3\theta\sin\theta \, d\theta - kr\int_0^{2\pi}\sin^2\theta \, d\theta - ar\int_0^{2\pi}\cos\theta\sin\theta \, d\theta - F\int_0^{2\pi}\cos(\theta-\phi)\sin\theta \, d\theta\right]$$

$u = \cos\theta$

$\frac{du}{d\theta}$

$$= \frac{1}{2\pi}\left[-br^3\int_0^{2\pi}u^3 du - kr\int_0^{2\pi}\frac{1-\cos 2\theta}{2}d\theta + ar\int_0^{2\pi}u \, du - \frac{F}{2}\int_0^{2\pi}\sin(2\theta-\phi) + \sin\phi \, d\theta\right]$$

$$= \frac{1}{2\pi}\left[-\frac{br^3\cos^4\theta}{4}\Big|_0^{2\pi} - \frac{kr(2\theta)}{4}\Big|_0^{2\pi} + \frac{kr\sin 2\theta}{4}\Big|_0^{2\pi} + ar\sin\theta\Big|_0^{2\pi} - \frac{F}{2}\left[\theta\sin\phi\Big|_0^{2\pi} - \frac{\cos(2\theta-\phi)}{2}\Big|_0^{2\pi}\right]\right]$$

$$= \frac{1}{2\pi}\left[-kr\pi - \frac{F}{2}\left[2\pi\sin\phi - \frac{\cos(4\pi-\phi)}{2} + \frac{\cos(\phi)}{2}\right]\right]$$

$$= \frac{-kr - F\sin\phi}{2}$$

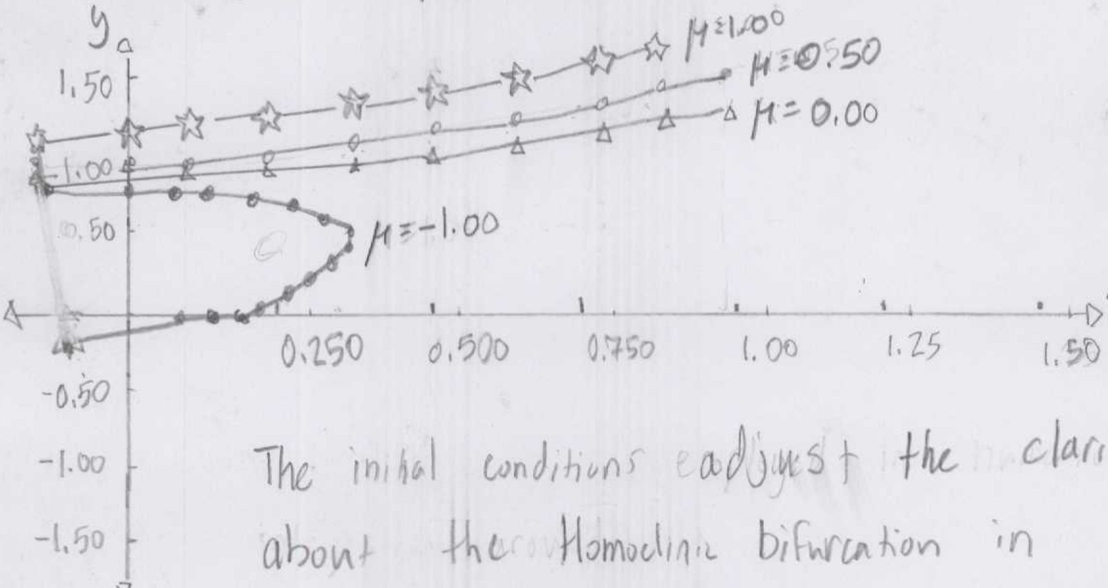
$\dot{r} = r(\mu - \sin r)$ 8.4.2. μ describes the frequency of radial nodes, about an
$\dot{\theta} = 1$ infinite-period bifurcation. When $|\mu| > 1$, the nodes appear in the graph.

$\boxed{\Delta h = 0.1}$

$\dot{x} = \mu x + y - x^2$
$\dot{y} = -x + \mu y + 2x^2$

8.4.3.

X_1 X_0	0
$f(X, y, t)$ y_0	0
KX_1	$f(X, y, t)$
Ky_1	$g(X, y, t)$
KX_2	$f(X_n + \Delta h \frac{KX_1}{2}, y_n + \Delta h \frac{Ky_1}{2}, t_n)$
Ky_2	$g(X_n + \Delta h \frac{KX_1}{2}, y_n + \Delta h \frac{Ky_1}{2}, t_n)$
KX_3	$f(X_n + \Delta h \frac{KX_2}{2}, y_n + \Delta h \frac{Ky_2}{2}, t_n)$
Ky_3	$g(X_n + \Delta h \frac{KX_2}{2}, y_n + \Delta h \frac{Ky_2}{2}, t_n)$
KX_4	$f(X_n + \Delta h KX_3, y_n + \Delta h Ky_3, t_n)$
Ky_4	$g(X_n + \Delta h KX_3, y_n + \Delta h Ky_3, t_n)$
X	$X_{n+1} = X_n + \frac{\Delta h}{6}(KX_1 + 2KX_2 + 2KX_3 + KX_4)$
y	$y_{n+1} = y_n + \frac{\Delta h}{6}(Ky_1 + 2Ky_2 + 2Ky_3 + Ky_4)$



$\boxed{\Delta h = 0.1}$

The initial conditions adjust the clarity about the Homoclinic bifurcation in Numerical Integration. Above, the graph begins at $(x = -0.1, y = -0.1)$, with an initial large step around the orbit.

$$r\phi' = \langle h\cos\theta\rangle = \frac{1}{2\pi}\int_0^{2\pi}\left(br^3\cos^3\theta - kr\sin\theta - ar\cos\theta - F\cos(\theta-\phi)\right)\cos\theta\, d\theta$$

$$= \frac{1}{2\pi}\left[br^3\int_0^{2\pi}\cos^4\theta\, d\theta - kr\int_0^{2\pi}\sin\theta\cos\theta\, d\theta - ar\int_0^{2\pi}\cos^2\theta\, d\theta - F\int_0^{2\pi}\cos(\theta-\phi)\cos\theta\, d\theta\right]$$

$$\int_0^{2\pi}\cos^4\theta\, d\theta = \int_0^{2\pi}\cos^2\theta - \cos^2\theta\sin^2\theta\, d\theta$$

$$= \int_0^{2\pi}\frac{1+\cos 2\theta}{2}\, d\theta - \int_0^{2\pi}\frac{\sin^2 2\theta}{4}\, d\theta$$

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4}\Big|_0^{2\pi} - \int_0^{2\pi}\frac{1-\cos 4\theta}{8}\, d\theta$$

$$= \pi - \frac{\theta}{8} + \frac{\sin 4\theta}{16}\Big|_0^{2\pi} = \pi - \frac{\pi}{4} = \frac{3}{4}\pi$$

$$\int_0^{2\pi}\cos(\theta-\phi)\cos\theta\, d\theta = \int_0^{2\pi}\frac{\cos(-\phi)+\cos(2\theta-\phi)}{2}\, d\theta$$

$$\frac{24}{32}$$

$$= \cos(-\phi)\cdot\pi$$

$$= \frac{1}{2\pi}\left[br^3\left(\frac{3}{4}\pi\right) - ar(\pi) - F\cos(\phi)\pi\right]$$

$$= \frac{3br^3}{8} - \frac{ar}{2} - \frac{F\cos(\phi)}{2} = \frac{3br^3 - 4ar - 4F\cos\phi}{8}$$

$$\phi' = \frac{3br^3 - 4ar - 4F\cos\phi}{8r}$$

8.4.6.

Averaged Equations: $r' = -\frac{1}{2}(kr + F\sin\phi)$

$$\phi' = -\frac{1}{8}\left(4a - 3br^2 + \frac{4F}{r}\cos\phi\right)$$

Fixed Points: $r' = 0 = -\frac{1}{2}(kr + F\sin\phi)$

$$\phi' = 0 = -\frac{1}{8}\left(4a - 3br^2 + \frac{4F}{r}\cos\phi\right)$$

$$(r^*, \phi^*) = \left(\frac{F}{\sqrt{k^2 + (\frac{3}{4}br^2 - a)^2}}, 2\pi n\right) \text{ where } n \in \mathbb{Z}$$

Phase-locked periodic solution correspondence:

The polar fixed points represent a closed orbit every 2π angles with radial ratios of $\frac{F}{\sqrt{k^2 + (\frac{3}{4}br^2 - a)^2}}$. The value is a solution to a forced duffing oscillator, because the derivation was from the oscillator equation

$$\nabla \cdot x' = \frac{1}{r}\frac{\partial}{\partial r}(rr') + \frac{1}{r}\frac{\partial}{\partial \phi}(r\phi')$$

8.4.7. Dulac's criterion $g(r,\phi) \equiv 1$; $x' = (r', r\phi')$

$$\nabla \cdot x' = \frac{1}{r}\frac{\partial}{\partial r}(rr') + \frac{1}{r}\frac{\partial}{\partial \phi}(r\phi')$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{-r}{2}(kr + F\sin\phi)\right] + \frac{1}{r}\frac{\partial}{\partial \phi}\left[-\frac{1}{2}ra - \frac{3br^2}{8} + 4F\cos\phi\right]$$

$$= -k - \frac{F\sin\phi}{2r} + \frac{F\sin\phi}{2r} = -k$$

So, $\nabla \cdot (gx') < 0$ because $k > 0$, and no closed orbits exist in the averaged system

8.4.8. A sink or saddle-node are the bifurcations. With the divergence from Dulac's criterion being negative, the slope points inward.

$r^2\left[k^2+\left(\frac{3}{4}br^2-a\right)^2\right]=F^2$

8.4.9. $r'=\frac{1}{2}(kr+F\sin\phi)$ $\qquad \phi'=-\frac{1}{8}\left(4a-3br^2+\frac{4F}{r}\cos\phi\right)$

$r'=0=\frac{1}{2}(kr+F\sin\phi)$; $\frac{1}{2}kr=-F\sin\phi$

$\cos\phi=\pm\sqrt{1-\frac{k^2r^2}{F^2}}$

$\phi'=0=-\frac{1}{8}\left(4a-3br^2+\frac{4F}{r}\cos\phi\right)$

$3br^3-4ar=4F\cos(\phi)$

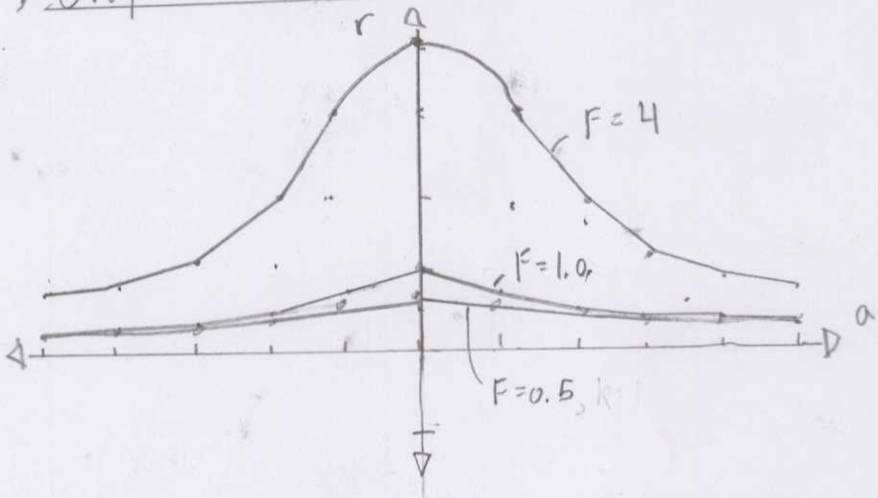
$(3br^3-4ar)^2=16F^2\cos^2(\phi)=16F^2\left(1-\frac{k^2r^3}{F^2}\right)$

$=16F^2-16k^2r^2$

$r^2\left[k^2+\left(\frac{3}{4}br^2-a\right)^2\right]=F^2$

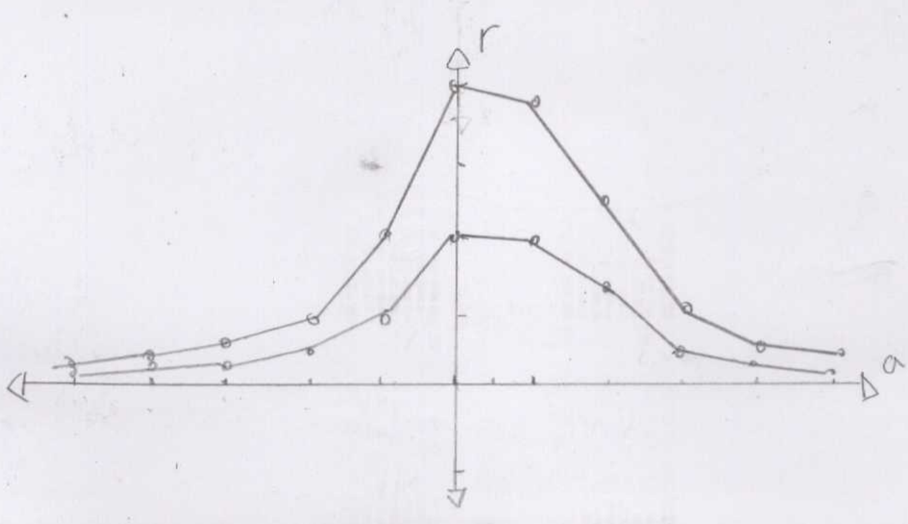
b) Graph of r vs. a at b = 0: $\quad r=\frac{F}{\sqrt{k^2+a^2}}$

$\boxed{k=1}$



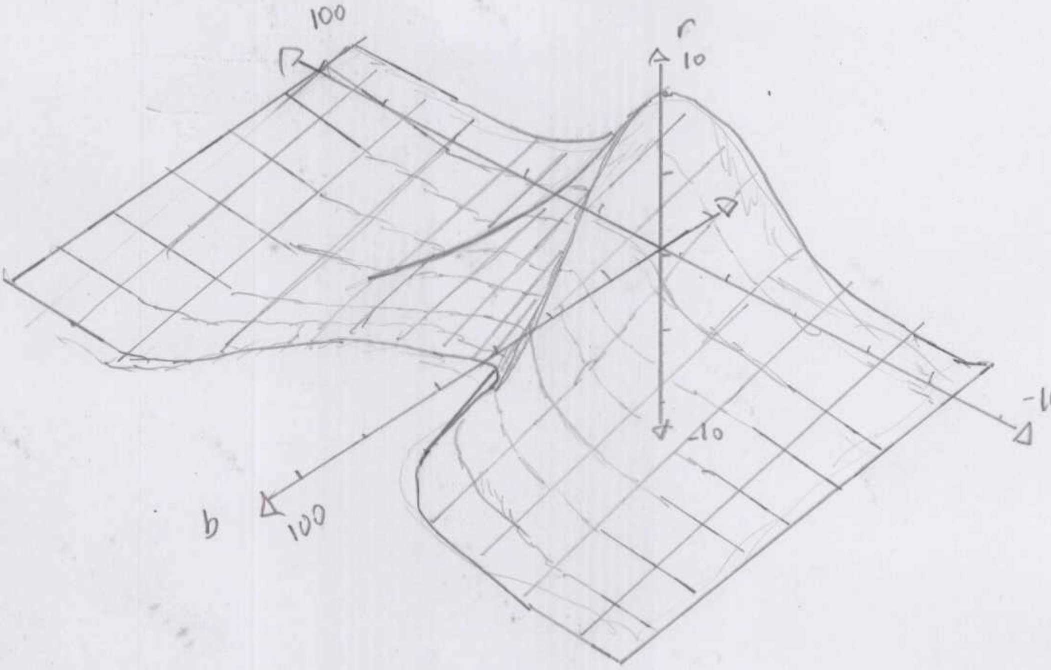
c. Graph of r vs. a at b ≠ 0: $\quad r=\frac{F}{\sqrt{k^2+\left(\frac{3}{4}br^2-a\right)^2}}$

$\boxed{\begin{matrix}k=1\\ b=1\end{matrix}}$



$$b_c = \frac{4\left(ar^4 \pm \sqrt{r^6(F-k^2r^2)}\right)}{3r^6}$$

d) Plot of (a,b) plane:



8.4.10.

a. $r' = 0 = -\frac{1}{2}(kr + F\sin\phi)$; $\phi = \arcsin\left[\frac{rK}{F}\right]$

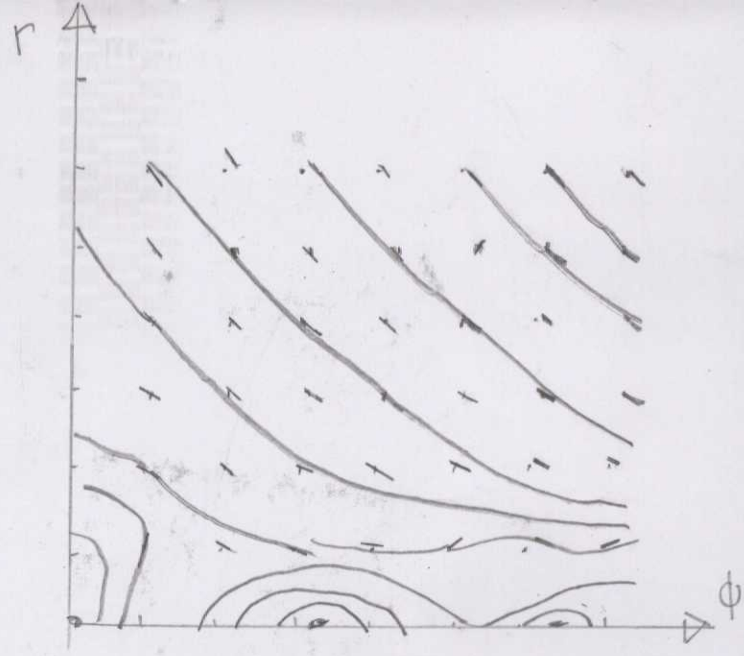
$\phi' = 0 = -\frac{1}{8}\left(4a - 3br^2 + \frac{4F}{r}\cos\phi\right)$; $\phi = \arccos\left[\frac{3br^3 - 4ar}{4F}\right]$

b.

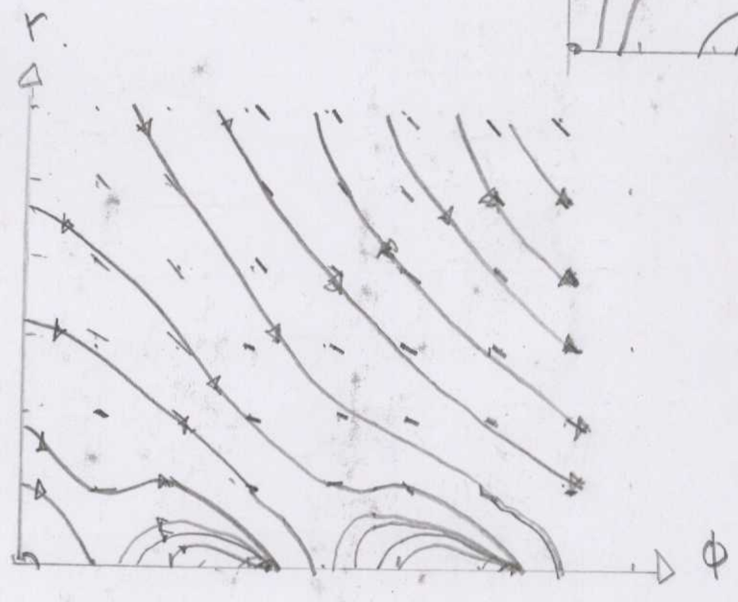
8.4.11 If $k=1$, $b=\frac{4}{3}$, $F=2$

a)

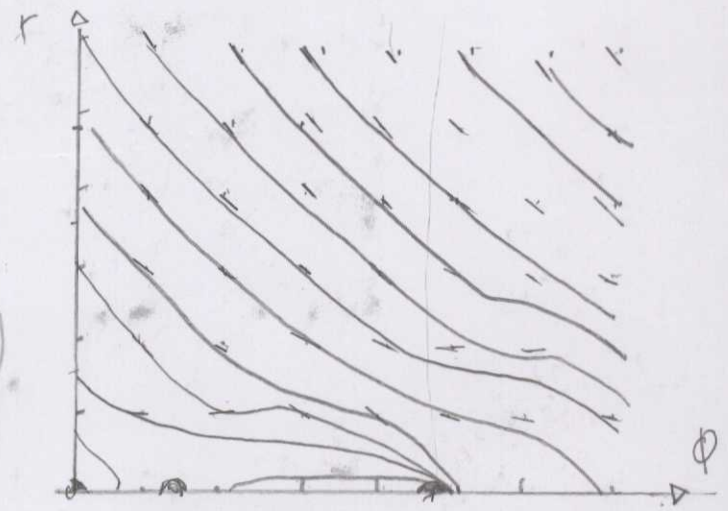
$\boxed{a=-1}$



$\boxed{a=0}$



$\boxed{a=1}$



$\boxed{a=2.8}$

b. Fixed Points:

$$r' = 0 = -\frac{1}{2}\left(kr + F\sin\phi\right)$$

$$\phi' = 0 = -\frac{1}{8}\left(4a - 3br^2 + \frac{4r}{r}\cos\phi\right)$$

$$\phi = 2\pi n \pm \cos^{-1}\left(\frac{1}{160} r\left(45r^2 - 224\right)\right)$$

C. Duffing Equation:

$$\ddot{x} + x + \varepsilon\left(bx^3 + k\dot{x} - ax - F\cos t\right) = 0$$

$$\ddot{x} = -x - \varepsilon\left(bx^3 + k\dot{x} - ax - F\cos t\right)$$

$$\dot{u} = \dot{x} = v$$

$$\dot{v} = \ddot{x} = -u - \varepsilon\left(bu^3 + k\cdot v - a\cdot u - F\cos t\right)$$

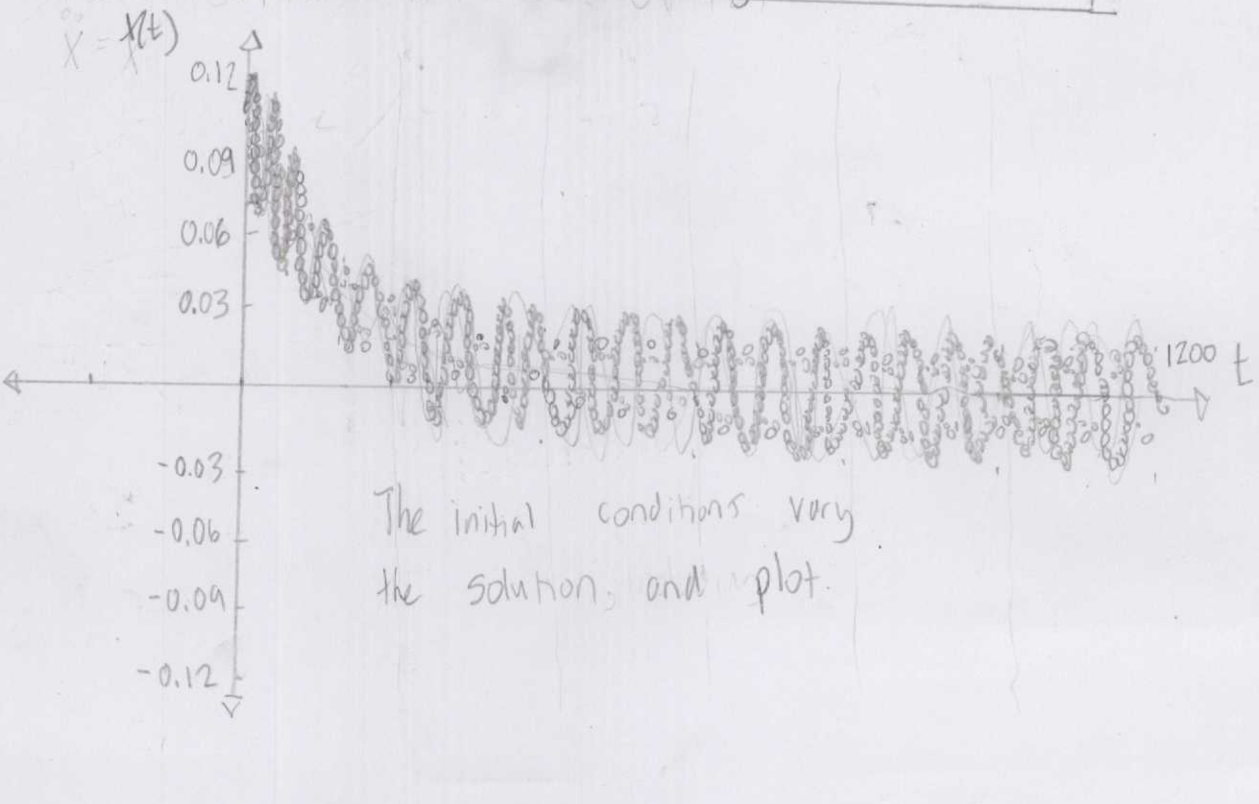
~or~

$\dot{u} = f(x,t)$

$$\dot{v} = g(u, \dot{u}, t) = -u - \varepsilon(bu^3 + k \circ f(x,t) - a \cdot u - F\cos t)$$
$$= -u(x) - \varepsilon(b u^3(x) + k \circ f(x,t) - a \circ u(x) - F\cos t)$$

$t=$

Term	Function
u	x
$\dot{u}$	$f(u,t) = f(x,t)$
k_{1f}	$f(u,t) = f(x,t)$
k_{2f}	$f(u+\Delta h\frac{k_{1f}}{2}, t+\frac{\Delta h}{2}) = f(x+\Delta h\frac{k_{1f}}{2}, t+\frac{\Delta h}{2})$
k_{3f}	$f(u+\Delta h\frac{k_{2f}}{2}, t+\frac{\Delta h}{2}) = f(x+\Delta h\frac{k_{2f}}{2}, t+\frac{\Delta h}{2})$
k_{4f}	$f(u+\Delta h k_{3f}, t+\Delta h) = f(x+\Delta h k_{3f}, t+\Delta h)$
$\dot{v}$	$g(u,\dot{u},t) = g(x, f(x,t), t)$
k_{1g}	$g(u,\dot{u},t) = g(x, f(x,t), t)$
k_{2g}	$g(u+\Delta h\frac{k_{1g}}{2}, f(u,t)+\frac{\Delta h k_{1g}}{2}, t+\Delta h/2)$ $= g(x+\Delta h\frac{k_{1g}}{2}, f(x,t)+\Delta h\frac{k_{1g}}{2}, t+\Delta h/2)$
k_{3g}	$g(u+\Delta h\frac{k_{2g}}{2}, f(u,t)+\Delta h\frac{k_{2g}}{2}, t+\Delta h/2)$ $= g(x+\Delta h\frac{k_{2g}}{2}, f(x,t)+\Delta h\frac{k_{2g}}{2}, t+\Delta h/2)$
k_{4g}	$g(u+\Delta h k_{3g}, f(u,t)+\Delta h \cdot k_{3g}, t+\Delta h/2$ $= g(x+\Delta h k_{3g}, f(x,t)+\Delta h k_{3g}, t+\Delta h/2$



The initial conditions vary the solution and plot.

8.4.12. $\dot{x} \approx \lambda_u x$; $\dot{y} \approx -\lambda_s y$; $(\mu, 1)$ where $\mu \ll 1$

$t\lambda = \ln x + C_1$; $t\lambda = \ln y + C_2$

$x(t) = C_1 e^{+\lambda t}$; $y(t) = C_2 e^{-\lambda t}$

$x(0) = \mu = C_1$; $y(0) = 1 = C_2$

$t = \frac{\ln \frac{x(t)}{\mu}}{\lambda}$; $t = \frac{\ln y(t)}{-\lambda}$

$t = \frac{-\ln \mu}{\lambda}$

$\ln(I - I_c)^{-1}$

8.5.1. If $f^{(n)}(I) = \left(\frac{d}{dI}\right)^n \ln(I - I_c)^{-1}$

$$f'(I) = \frac{d}{dI}\ln(I - I_c)^{-1} = \frac{-1}{(I-I_c)(\ln(I-I_c))^2}$$

$$f''(I) = \frac{d^2}{dI^2}\ln(I - I_c)^{-1} = \frac{2}{(I-I_c)^2(\ln(I-I_c))^3} + \frac{1}{(I-I_c)^2(\ln(I-I_c))^2}$$

$$f'''(I) = \frac{d^3}{dI^3}\ln(I - I_c)^{-1} = \frac{-6}{(I-I_c)^3\ln(I-I_c)^4} + \frac{-6}{(I-I_c)^3\ln(I-I_c)^3} + \frac{-2}{(I-I_c)^3\ln(I-I_c)^2}$$

$$f^{(1)} = \frac{-f}{I - I_c} \; ; \; f^{(2)} = \frac{2f}{(I-I_c)^2} + \frac{f^2}{(I-I_c)^2}$$

$$f^{(3)} = \frac{-6f^4 - 6f^3 - 2f}{(I-I_c)^3}$$

$$f^{(n)} = \frac{\sum_{k=2}^{n+1} (-1)^n n! \, f^k}{(I - I_c)^n}$$

$\phi'' + \alpha\phi' + \sin\phi = I$

8.5.2. $\dot{u} = \phi'$

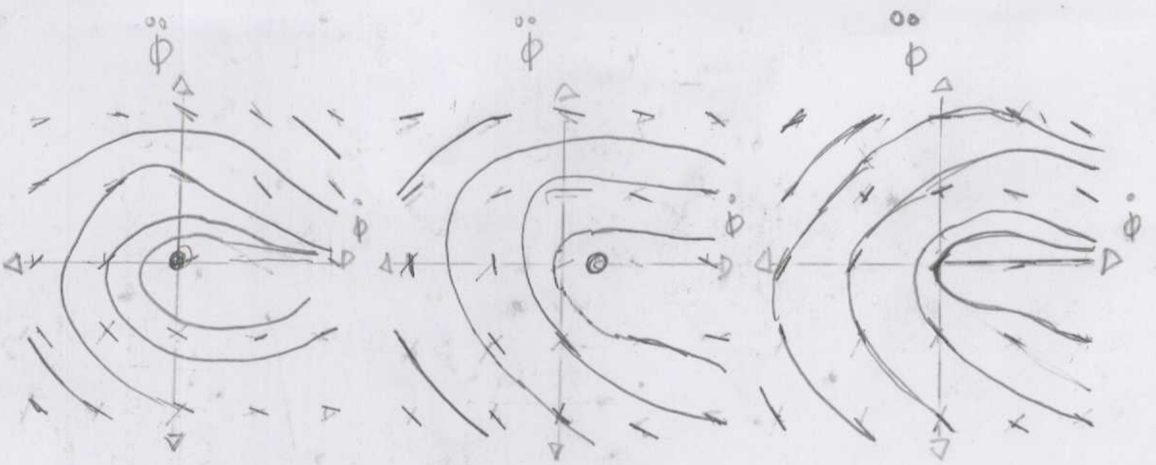
$\dot{v} = \phi'' = -\alpha v + \sin u + I$

<u>Fixed points:</u> $\dot{u} = 0$

$\dot{v} = 0 = -\alpha v + \sin u + I$

$(u^*, v^*) = (0, I/v)$

Phase Portrait:



$I < \alpha$ $\quad$ $I = \alpha$ $\quad$ $I > \alpha$

Large α and infinite-period bifurcation.

$\dot{N} = rN(1 - N/K(t))$

8.5.3.

a. Poincaré map: $-\dfrac{\dot{N}}{N(t)^2} + \dfrac{r}{N(t)^2} = \dfrac{r}{K(t)}$

$$\text{If } X = \frac{1}{N(t)} \; ; \; \dot{X} = \frac{-1}{N(t)^2}$$

$$\text{then, } \dot{X} + rX = \frac{r}{K(t)}$$

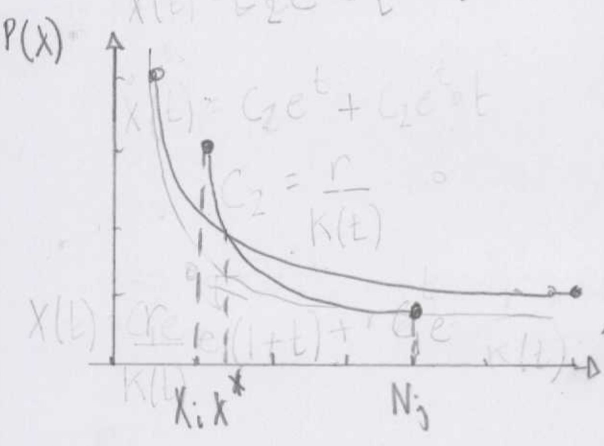
Integrating factor: e^{rt}

$$\dot{X}e^{rt} + r \cdot Xe^{rt} = \frac{re^{rt}}{K(t)}$$

$$\frac{d}{dt}\left(e^{rt} \cdot X\right) = \frac{re^{rt}}{K(t)}$$

$$X = \frac{1}{e^{rt}}\left(\int \frac{re^{rt}}{K(t)} dt + C\right)$$

Similar Solutions: $t = t + T$



b. The solution is unique because $\lim_{t \to \infty} X(t) = \text{constant}$.

$\dot{x} = rx(1-x) - h(1+\alpha \sin t)$; $r, h > 0$ and $0 < \alpha < 1$

8.5.4.

a. Solution to $\dot{x} = rx(1-x) - h(1+x\sin(t))$ is periodic if $x(t) = x(t+T)$ for all t.

If t ranges from zero to one, then

$$x(1) - x(0) = \int_0^1 rx(1-x) - h(1+\alpha \sin t)\, dt$$

$$\leq \frac{r}{4} - h$$

So, if $h > \frac{r}{4}$, then $x(n+1) - x(n) < 0$ and $x(t)$ is divergent.

b. If $h < \frac{r}{4(1+\alpha)}$, then $\dot{x} \geq r\left[x(1-x) - \frac{h}{r}(1+\alpha \sin t)\right]$

$$\geq r\left[x(1-x) - \frac{1}{4}(1+\alpha \sin t)\right]$$

$$\geq 0$$

$x > 1/2$ when $t = n\pi$

$x < 1$ when $t = n\pi$ "stable limit"

When $0 < x < 1/2$, then $\dot{x} < 0$ and divergent, such as an unstable limit cycle.

c. Biological systems with a stable limit cycle survive, while unstable diverge to zero populations.

c. If $\frac{r}{4(1+\alpha)} < h < \frac{r}{4}$, then zero, one, or two periodic solutions appear in the data.

$\ddot{\theta} + \alpha\dot{\theta}|\dot{\theta}| + \sin\theta = F$

8.5.5, $\alpha > 0$ and $F > 0$

a. $\ddot{\theta} + \alpha v|v| + \sin\theta = F$ where $v = \dot{\theta}$

$$\begin{bmatrix} \ddot{\theta} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} -\cos\theta & -2\alpha|v| \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ v \end{bmatrix}$$

Fixed Points: $\dot{\theta} = 0 = v$

$\dot{v} = 0 = F - \alpha v|v| - \sin\theta$

$(\ddot{\theta}, \dot{v}) = (\arcsin(F), 0); (\pi - \arcsin(F), 0)$

$$A_{(\arcsin(F),0)} = \begin{bmatrix} -\cos(\arcsin(F)) & \cancel{0} \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda = \begin{bmatrix} -\cos(\arcsin(F)) - \lambda & 0 \\ 0 & 1 - 1 - \lambda \end{bmatrix}$$

$$= [-\cos(\arcsin(F)) - \lambda][1 - \lambda] = 0$$

$$\lambda_1 = i(1-F^2)^{1/4} ; \lambda_2 = -i(1-F^2)^{1/4}$$

$$\Delta = (1-F^2)^{1/2} > 0 ; \tau = 0 ; \tau^2 - 4\Delta < 0$$

"Center"

$$A_{(\pi - \arcsin(F),0)} = \begin{bmatrix} -\cos(\pi - \arcsin(F)) & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda = \begin{bmatrix} -\cos(\pi - \arcsin(F)) - \lambda & 0 \\ 0 & 1 - \lambda \end{bmatrix}$$

$$= (-\cos(\pi - \arcsin(F)) \mp \lambda)(1 - \lambda) = 0$$

$$\lambda_1 = 1 ; \lambda_2 = +\cos(\pi - \arcsin(F)) = \sqrt{1-F^2}$$

$\Delta = \sqrt{1-F^2} > 0$; $\tau = 1 + \sqrt{1+F^2}$; $\tau^2 - 4\Delta > 0$ "unstable node"

$$V(\theta, v) = KE + PE$$

$$= \int \mathring{\theta}\, dv - \int \mathring{v}\, d\theta$$

$$= \frac{1}{2}v^2 - F\theta - \cos(\theta) + F \arcsin(F) + \sqrt{1-F^2}$$

$$\mathring{V}(\theta, v) = v\mathring{v} - F\mathring{\theta} + \sin(\theta)\mathring{\theta}$$

$$= v(F - \alpha v|v| - \sin(\theta)) - Fv + \sin(\theta) v$$

$$= -\alpha v^2 |v|$$

When $v = 0$, then $\mathring{V} = 0$ and the center is a Liapunov function.

b) A ~~f~~ stable limit cycle appears when $F > 1$.

Fixed Points: $\mathring{v} = F - \alpha v|v| - \sin(\theta) \geq 0$

$$F - \sin(\theta) \geq \alpha v|v|$$

where $\sin(\theta)$ oscillates between -1 to 1

Limit Cycle: $F - \sin(\theta) \geq F - 1 \geq \alpha v|v|$

and

$$F - \sin(\theta) \geq F + 1 \geq \alpha v|v|$$

$$\sqrt{\frac{F-1}{\alpha}} \leq v \leq \sqrt{\frac{F+1}{\alpha}}$$

Uniqueness: A second limit cycle justification is a contradiction to uniqueness. The minimum and maximum range are singular, positive, and unique.

c. When $u = \frac{1}{2}v^2$, $\frac{du}{d\theta} = \frac{1}{2}\frac{d}{d\theta}v^2(t(\theta))$

$$= v\dot{v}\cdot\frac{dt}{d\theta}$$

$$= \dot{v}$$

$\frac{du}{d\theta} + 2\alpha u + \sin\theta = F$

d. $\frac{du}{d\theta} = F - 2\alpha u - \sin\theta \leq 0$

$$u \geq \frac{F-\sin\theta}{2\alpha}$$

Limit Cycle:

$$\frac{F-\sin\theta}{2\alpha} \geq \frac{F-1}{2\alpha} \geq u$$

and

$$\frac{F-\sin\theta}{2\alpha} \geq \frac{F+1}{2\alpha} \geq u$$

$$\frac{F+1}{2\alpha} \geq u \geq \frac{F-1}{2\alpha}$$

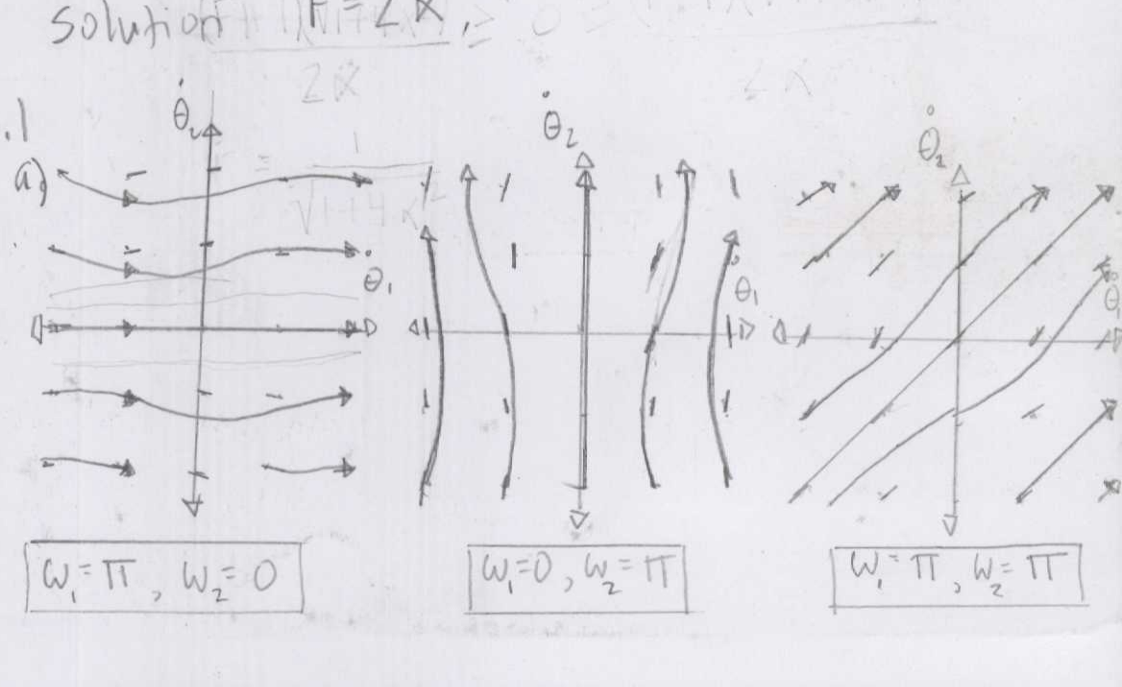
e. When $u = \frac{1}{2}v^2$, then the bifurcation occurs at $u = 0$, with the limit cycle $\frac{F+1}{2\alpha} \geq 0 \geq \frac{F-1}{2\alpha}$, and bifurcation solution $F = 2\alpha$,

$\dot{\theta}_1 = \omega_1 + \sin\theta_1\cos\theta_2$

$\dot{\theta}_2 = \omega_2 + \sin\theta_2\cos\theta_1$

8.6.1

a)



b) $\dot{\phi}_1 = \dot{\theta}_1 - \dot{\theta}_2 = \omega_1 - \omega_2 + \sin\theta_1 \cos\theta_2 - \sin\theta_2 \cos\theta_1$

$$= \omega_1 - \omega_2 + \sin(\theta_1 - \theta_2)$$

$$= \omega_1 - \omega_2 + \sin\phi_1$$

<u>Fixed Points:</u> $\dot{\phi}_1 = 0 = \omega_1 - \omega_2 + \sin\phi_1$

$$\phi_1^* = 2\pi - \arcsin(\omega_1 - \omega_2)$$

$$\dot{\phi}_2 = 0 = \omega_1 + \omega_2 + \sin\phi_2$$

$$\phi_2^* = 2\pi - \arcsin(\omega_1 + \omega_2)$$ "Sad

<u>Eigenvalues:</u>
$$\begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} = \begin{bmatrix} \cos\phi_1 & 0 \\ 0 & \cos\phi_2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

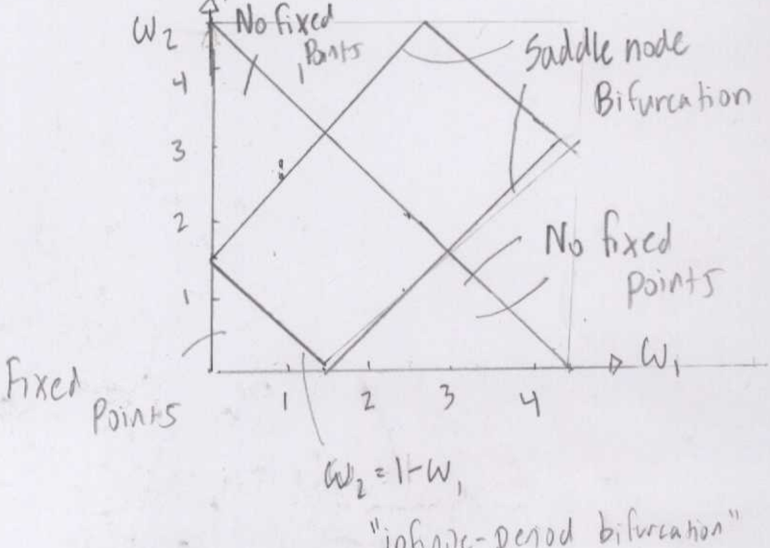
$$A - \lambda = (\cos\phi_1 - \lambda)(\cos\phi_2 - \lambda) = 0$$

$$\lambda_1 = \cos\phi_1 \; ; \; \lambda_2 = \cos\phi_2$$

$$\Delta = \cos\phi_1 \cos\phi_2 \; ; \; \tau = \cos\phi_1 + \cos\phi_2 \; ; \; \tau^2 - 4\Delta = (+)/(-)$$

If $\omega_1 + \omega_2 = 1$, then saddle node bifurcation or $\omega_1 - \omega_2 = 1$ an infinite-period bifurcation appears in the phase-plot.

c) <u>Parameter Space:</u>



$$\dot{\theta}_1 = \omega_1 + K_1 \sin(\theta_2 - \theta_1)$$
$$\dot{\theta}_2 = \omega_2 + K_2 \sin(\theta_1 - \theta_2)$$

8.6.2

a) <u>Fixed Points:</u> $\dot{\phi}_1 = 0 = \dot{\theta}_1 - \dot{\theta}_2$

$$= \omega_1 - \omega_2 - (K_1 + K_2)\sin(\phi)$$

$$\phi_1^* = \arcsin \frac{\omega_1 - \omega_2}{K_1 + K_2}$$

Although, the difference has a fixed point, the individual phases lack a positive frequency which a fixed point yields.

$$\dot{\phi}_1 = 0 = \dot{\theta}_1 - \dot{\theta}_2 \,;\; \dot{\theta}_1 = \theta_2 = \omega_2 + K_2 \sin\phi^\circ$$

$$= \omega_2 + K_2 \frac{(\omega_1 - \omega_2)}{K_1 + K_2}$$

$$= \frac{K_1 \omega_2 + K_2 \omega_2}{K_1 + K_2}$$

The comprimise frequency, ω^*, ~~is~~ ~~cannot be~~ a frequency at zero.

b. $\dot{\phi}_1 = \dot{\theta}_1 - \dot{\theta}_2 \,;\; \sin(\theta_1 - \theta_2) = \dfrac{\omega_1 - \omega_2}{K_1 + K_2}$

$\dot{\phi}_2 = \dot{\theta}_1 + \dot{\theta}_2 \,;\; \sin(\theta_1 + \theta_2) = \dfrac{\omega_2 + \omega_1}{K_1 + K_2}$

c) If $K_1 = K_2$, then

$\dot{\theta}_1 = \dfrac{d\theta}{dt} = \omega_1 + K \sin(\theta_2 - \theta_1)$ where $\tau = \omega_1 t$ and $a = \dfrac{K}{\omega_1}$

$$d\theta_1/d\tau = 1 + a \sin(\theta_2 - \theta_1)$$

and

$d\theta_2/d\tau = \dfrac{\omega_2}{\omega_1} + a\sin(\theta_1 - \theta_2)$ where $\omega = \dfrac{\omega_2}{\omega_1}$

$$= \omega + a \sin(\theta_1 - \theta_2)$$

d. Winding Number $\lim_{\tau \to \infty} \theta_1(\tau) / \theta_2(\tau)$

$$\langle d(\theta_1 + \theta_2)/d\tau \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T d(\theta_1 + \theta_2)/d\tau \, d\tau$$

$$= \lim_{T \to \infty} \frac{1}{T} \sum_0^T 1 + a\sin(\theta_2 - \theta_1) + w + a\sin(\theta_1 - \theta_2)$$

$$= 0$$

$$\langle d(\theta_1 - \theta_2)/d\tau \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T d(\theta_1 - \theta_2)/d\tau \, d\tau$$

$$= \lim_{T \to \infty} \frac{1}{T} \sum_0^T (1 - w) - 2a\sin(\theta_2 - \theta_1)$$

$$= 0$$

So the limit is also zero

$\mathring{\theta}_1 = \omega_1$

$\mathring{\theta}_2 = \omega_2$

8.6.3. $\frac{\omega_1}{\omega_2} \in \mathbb{R}$: $\frac{\omega_1}{\omega_2} = \frac{\mathring{\theta}_1}{\mathring{\theta}_2}$; Intersection $= \left| \frac{\mathring{\theta}_1}{\mathring{\theta}_2} - \varepsilon \right| = \frac{p}{q}$

$\mathring{\theta}_1 = E - \sin\theta_1 + K\sin(\theta_2 - \theta_1)$

$\mathring{\theta}_2 = E + \sin\theta_2 + K\sin(\theta_1 - \theta_2)$

8.6.4

a) Fixed Points: $\mathring{\phi}_1 = \mathring{\theta}_1 - \mathring{\theta}_2 = \sin\theta_2 - \sin\theta_1 - 2K\sin(\theta_1 - \theta_2)$

$= 0$

$\phi_2 = \mathring{\theta}_1 + \mathring{\theta}_2 = 2E + \sin\theta_1 + \sin\theta_2 = 0$

$(\theta_1, \theta_2) = (n\pi, m\pi) \quad n, m \in \mathbb{R}$

When $E = 0$.

Bifurcations: $\begin{bmatrix} \mathring{\theta}_1 \\ \mathring{\theta}_2 \end{bmatrix} = \begin{bmatrix} -\cos\theta_1 - K\cos(\theta_2 - \theta_1) & K\cos(\theta_2 - \theta_1) \\ -K\cos(\theta_1 - \theta_2) & \cos\theta_2 - K\cos(\theta_1 - \theta_2) \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_1 \end{bmatrix}$

$$A-\lambda = \begin{bmatrix} -\cos\theta_1 - K\cos(\theta_2-\theta_1) - \lambda & K\cos(\theta_2-\theta_1) \\ -K\cos(\theta_1-\theta_2) & \cos\theta_2 - K\cos(\theta_1-\theta_2) - \lambda \end{bmatrix}$$

$$= (-\cos\theta_1 - K\cos(\theta_2-\theta_1) - \lambda)(\cos\theta_2 - K\cos(\theta_1-\theta_2) - \lambda)$$
$$+ K^2\cos(\theta_2-\theta_1)\cos(\theta_1-\theta_2)$$
$$= 0$$

$$\lambda_1 = \frac{1}{2}\Big(-\sqrt{-4K^2\cos^2(\theta_1-\theta_2) + 2\cos(\theta_1)\cos(\theta_2) + \cos^2\theta_1 + \cos^2\theta_2}$$
$$- 2K\cos(\theta_1-\theta_2) - \cos\theta_1 + \cos\theta_2\Big)$$

$$\lambda_2 = \frac{1}{2}\Big(\sqrt{-4K^2\cos^2(\theta_1-\theta_2) + 2\cos\theta_1\cos\theta_2 + \cos^2\theta_1 + \cos^2\theta_2}$$
$$- 2K\cos(\theta_1-\theta_2) - \cos\theta_1 + \cos\theta_2\Big)$$

$$\Delta = 2K^2\cos^2(\theta_1-\theta_2) + K\cos(\theta_1)\cos(\theta_1-\theta_2) - K\cos(\theta_2)\cos(\theta_1-\theta_2) - \cos\theta_1\cos\theta_2$$

$$\tau = -2K\cos(\theta_1-\theta_2) - \cos(\theta_1) + \cos\theta_2$$

$$\tau^2 - 4\Delta = -4K^2\cos^2(\theta_1-\theta_2) + 2\cos\theta_1\cos\theta_2 + \cos^2\theta_1 + \cos^2\theta_2$$

$K=0$; $E=0$: Unstable saddle

$k>0$; $E=0$: Stable sinks and Unstable sources

$k=0$; $E>0$: Unstable saddles

$k>0$; $E>0$: Stable Fixed Points

A plot become the accurate method for classify fixed point analysis.

b) $\dot{\theta}_1 = 0 = E - \sin\theta_1 + K\sin(\theta_2-\theta_1)$

$E = \sin\theta_1 - K\sin(\theta_2-\theta_1)$

$\dot{\theta}_2 = 0 = E - \sin\theta_2 + K\sin(\theta_1-\theta_2)$

$E = \sin\theta_2 - K\sin(\theta_1-\theta_2)$

The type of periodic solution depends on K.

A $K=0$, unstable saddles become solutions,

While $K>0$, stable fixed points.

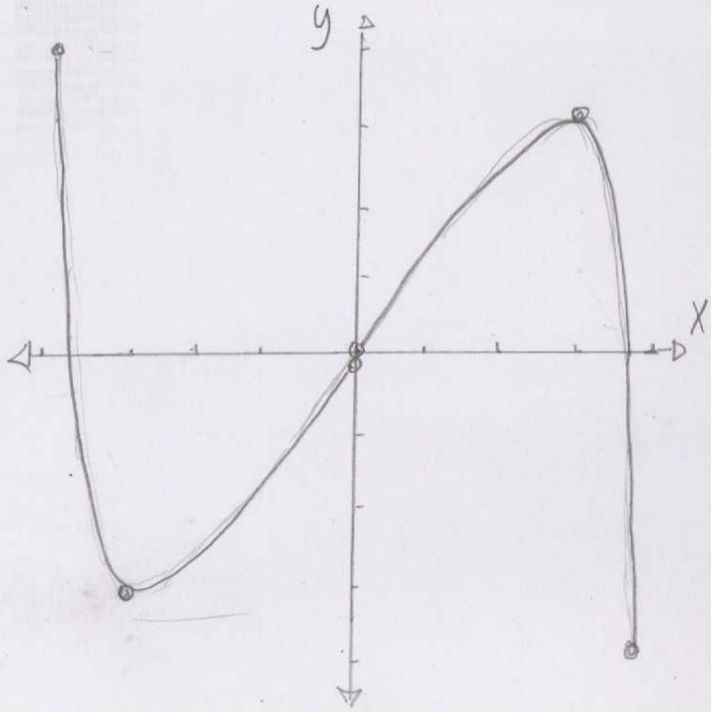
c) See part a.

$x(t) = \sin t$

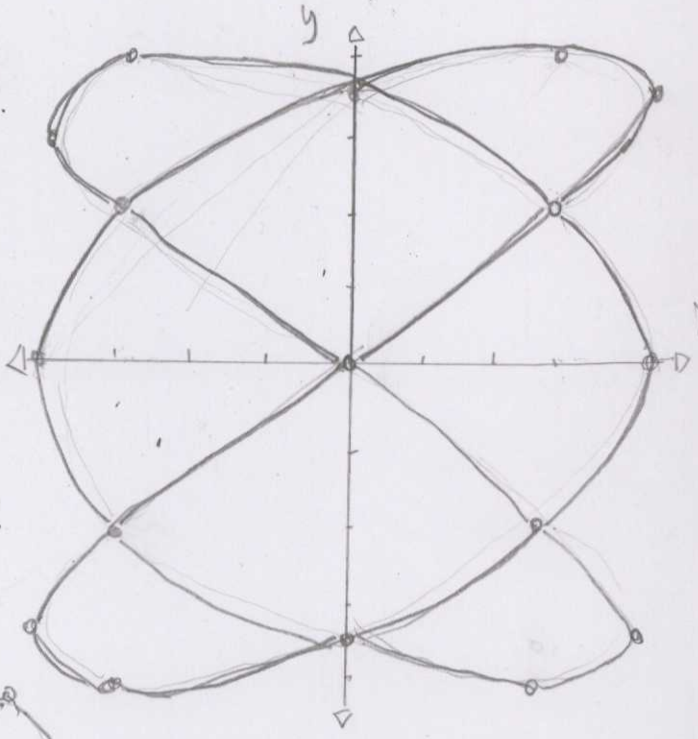
$y(t) = \sin \omega t$

8.6.5.

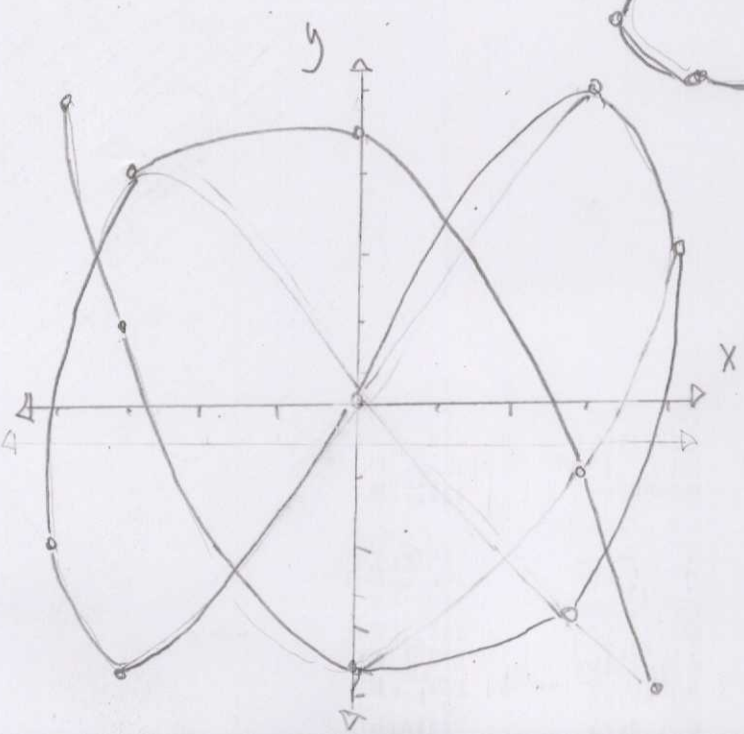
a) $\omega = 3$



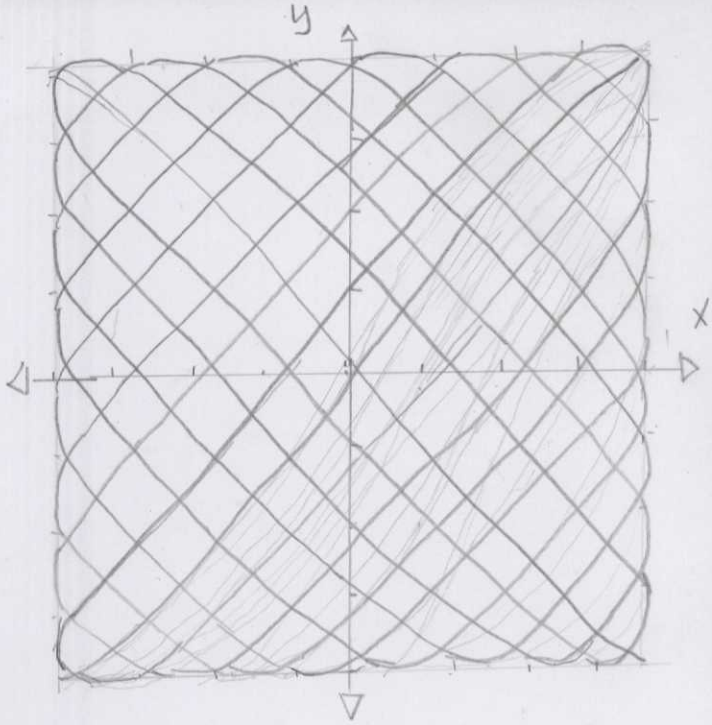
b) $\omega = \frac{2}{3}$



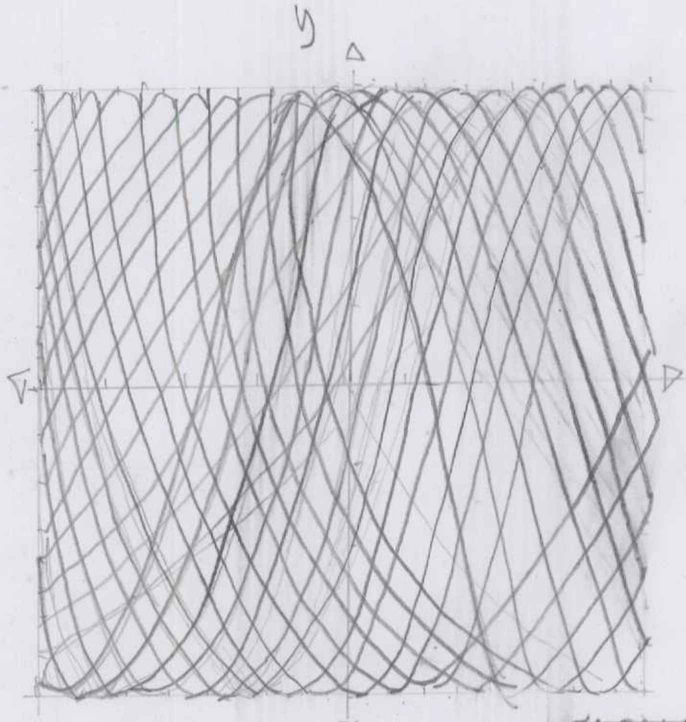
c) $\omega = 5/3$



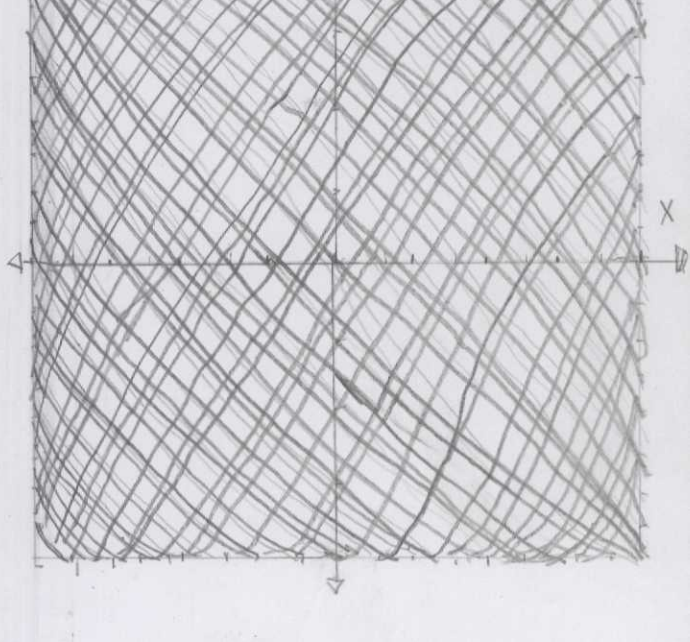
d.) $\omega = \sqrt{2}$



e) $\omega = \pi$



f) $\omega = \frac{1+\sqrt{5}}{2}$



$\ddot{x}+x=0$

$\ddot{y}+w^2y=0$

8.6:6

a) If $x=A(t)\sin\theta(t)$ and $y=B(t)\sin\phi(t)$, then

$$\dot{x}=A(t)\cos\theta(t)\cdot\dot{\theta}(t)+\dot{A}(t)\sin\theta(t)$$

$$\ddot{x}=\dot{A}(t)\cos\theta(t)\cdot\dot{\theta}(t)+A(t)\sin\theta(t)\,\dot{\theta}(t)^2+A(t)\cos\theta(t)\,\ddot{\theta}(t)$$
$$+\ddot{A}(t)\sin\theta(t)+\dot{A}(t)\cos\theta(t)\,\dot{\theta}(t)$$

$$\ddot{x}+x=\dot{A}(t)\cos\theta(t)\dot{\theta}(t)-A(t)\sin\theta(t)\,\dot{\theta}(t)^2+A(t)\cos\theta(t)\ddot{\theta}(t)$$
$$+\ddot{A}(t)\sin\theta(t)+\dot{A}(t)\cos\theta(t)\,\dot{\theta}(t)+A(t)\sin\theta(t)$$
$$=0\text{, where }\dot{\theta}=1\text{ and }\dot{A}(t)=0$$

$$y=B(t)\sin\phi(t)$$

$$\dot{y}=B(t)\cos\phi(t)\cdot\dot{\phi}(t)+\dot{B}(t)\sin\phi(t)$$

$$\ddot{y}=\dot{B}(t)\cos\phi(t)\cdot\dot{\phi}(t)-B(t)\sin\phi(t)\,\dot{\phi}(t)^2+B(t)\cos\phi(t)\,\ddot{\phi}(t)$$
$$+\ddot{B}(t)\sin\phi(t)+\dot{B}(t)\cos\phi(t)\,\dot{\phi}(t)$$

$$\ddot{y}+wy=\dot{B}(t)\cos\phi(t)\dot{\phi}(t)-B(t)\sin\phi(t)\,\dot{\phi}(t)^2+B(t)\cos\phi(t)\ddot{\phi}(t)$$
$$+\ddot{B}(t)\sin\phi(t)+\dot{B}(t)\cos\phi(t)\dot{\phi}(t)+wB(t)\sin\phi(t)$$
$$=0\text{, where }\dot{\phi}(t)=w\text{ and }\dot{B}(t)=0$$

b) A two-dimensional tori appears from the four-dimensional system because constraints ~~sim~~ the scaled equations.

c) Lissajous figures relate trajectories in the system through a constant period in the system

8.6.7

$m\ddot{r}=\dfrac{h^2}{mr^3}-K$

$\dot{\theta}=h/mr^2$

a) m = mass

K = central force of constant strength

h = constant (the angular momentum of the particle)

a) If $r = r_0$ and $\dot{\theta} = \omega_0$, then $m\ddot{r} = 0 = \dfrac{h^2}{mr_0^3} - k$

and $r_0 = \sqrt[3]{\dfrac{h^2}{mk}}$

Also, $\dot{\theta} = \dfrac{h}{mr_0^2} = \dfrac{h}{m}\left(\dfrac{mk}{h^2}\right)^{2/3} = \left(\dfrac{k^2}{mh}\right)^{1/3} = \omega_\theta$

b) $\omega_r = \sqrt{\dfrac{k}{m}}$; $\ddot{r} = \dfrac{h^2}{mr_0^3} - \dfrac{k}{m}$

$$\frac{d\ddot{r}}{dr_0} = \frac{-3h^2}{mr_0^4}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{3h^2}{mr_0^4} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$ confusing

$\omega_r^2 = \dfrac{3h^2}{mr_0^4}$; $\omega_r = \sqrt{\dfrac{3h^2}{mr_0^4}} = \sqrt{3}\,\omega_0$

c) Winding Number: $\dfrac{\omega_r}{\omega_\theta} = \dfrac{\sqrt{3}\,\omega_0}{\omega_\theta} = \sqrt{3}$ and irrational.

d) Eigenvalues: $(A - \lambda)I = \begin{bmatrix} -\lambda & 1 \\ -\dfrac{3h^2}{mr_0^4} & -\lambda \end{bmatrix} = \lambda^2 + \dfrac{3h^2}{mr_0^4} = 0$

$\lambda_{1,2} = \pm\sqrt{\dfrac{3h^2}{mr_0^4}}$; $\Delta = \dfrac{3h^2}{mr_0^4}$; $\tau = 0$; "center"

Also, the period: $\Delta\theta = \theta(t+T) - \theta(t)$

$= 2\pi$ ω_0... or a constant $\dot{\theta} = \omega_0$

Lastly, $\dot{\theta} = \omega_0$, which is a constant.

The motion is periodic for any amplitude.

d) A mechanical realization of this system is quasiperiodic vibrations or weather patterns.

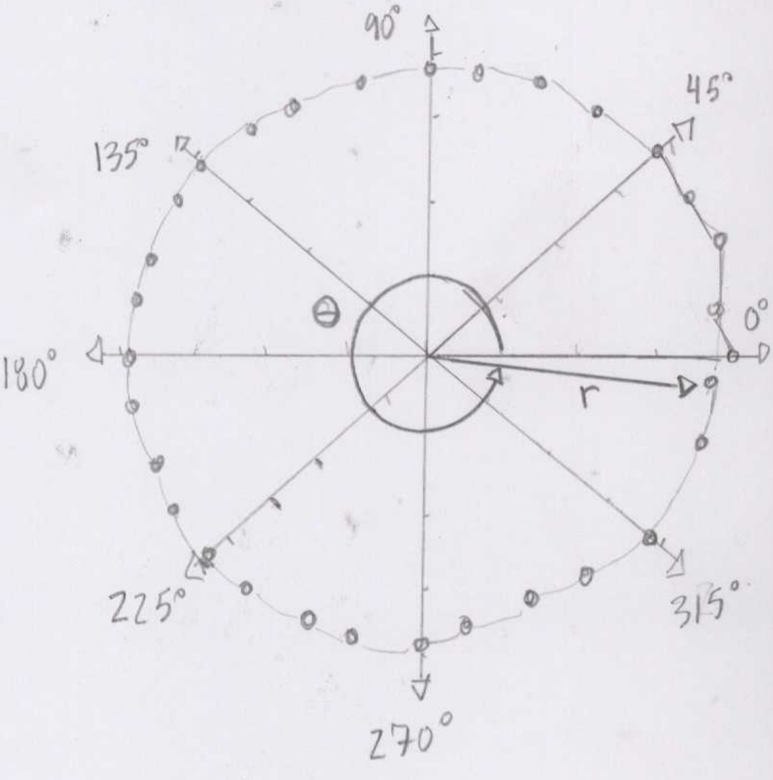
$m\ddot{r} = \frac{h^2}{mr^3} - k$

$\dot{\theta} = \frac{h}{mr^2}$

8.6.8. Runge-Kutta 4th-order:

$\dot{\Theta} = \omega_0$; $\Theta(t) = \omega_0 t$

Parameter	Function
r_0	1
k_1	$f(r_0, t)$
k_2	$f(r_0 + k_1/2, t)$
k_3	$f(r_0 + k_2/2, t)$
k_4	$f(r + k_3, t)$
r_{n+1}	$r_n + (k_1 + 2k_2 + 2k_3 + k_4)/6$



$\dot{\theta}_1 = \omega + H(\theta_2 - \theta_1)$

$\dot{\theta}_2 = \omega + H(\theta_1 - \theta_2)$

$\dot{\theta}_1 = \omega + H(\theta_2 - \theta_1) + H(\theta_3 - \theta_1)$

$\dot{\theta}_2 = \omega + H(\theta_1 - \theta_2) + H(\theta_3 - \theta_2)$

$\dot{\theta}_3 = \omega + H(\theta_1 - \theta_3) + H(\theta_2 - \theta_3)$

8.6.3. a) $\phi = \theta_1 - \theta_2$; $\dot{\phi} = \dot{\theta}_1 - \dot{\theta}_2 = H(\theta_2 - \theta_1) - H(\theta_1 - \theta_2)$

$\psi = \theta_2 - \theta_3$; $\dot{\psi} = \dot{\theta}_2 - \dot{\theta}_3 = H(\theta_1 - \theta_2) + H(\theta_3 - \theta_2) - H(\theta_1 - \theta_3) - H(\theta_2 - \theta_3)$

b) If $H(x) = a\sin x$, then $\dot{\phi} = H(\theta_1 - \theta_2) - H(\theta_2 - \theta_1)$

$$= a\sin(\theta_1 - \theta_2) - a\sin(\theta_2 - \theta_1)$$

$$= 2a\sin\phi$$

$$= 0, \text{ when } \phi = \pi.$$

$$\dot{\psi} = -2a\sin(\theta_2 - \theta_3) + a\sin(\theta_1 - \theta_2) - a\sin(\theta_1 - \theta_3)$$

$$= -2a\sin(\psi) + a\sin(\phi) - a\sin(\phi + \psi)$$

$$= 0, \text{ when } \phi = n\pi \text{ and } \psi = m\pi \quad n, m \in \mathbb{R}^3$$

$$\text{or } \phi = \frac{2n\pi}{3} \text{ and } \psi = \frac{2m\pi}{3}$$

c) $H(x) = a\sin x + b\sin 2x$

Fixed Points: $\dot{\phi} = 0 = H(-\phi) + H(-\phi - \psi) - H(\phi) - H(-\psi)$

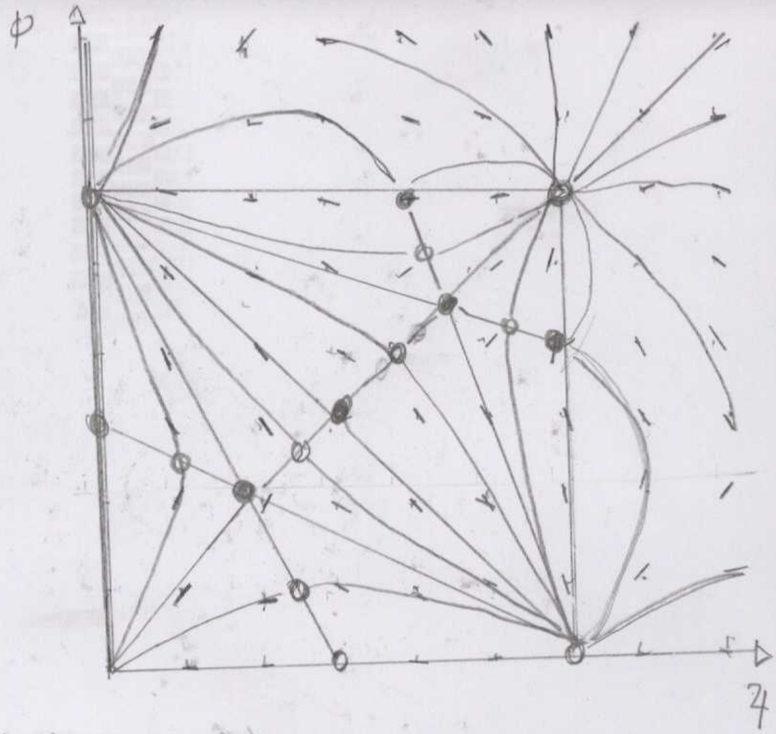
$$= a\sin(-\phi) + b\sin(-2\phi) + a\sin(-\phi - \psi) + b\sin(-2(\phi + \psi)) - a\sin(\phi) - b\sin(2\phi) - a\sin(-\psi) - b\sin(-2\psi)$$

$$= -2a\sin(\phi) - 2b\sin(2\phi) - a\sin(-(\phi + \psi)) - b\sin(-2(\phi - \psi)) + a\sin(\psi) + b\sin(2\psi)$$

$$\dot{\psi} = 0 = -2a\sin\psi + a\sin\phi - a\sin(\phi + \psi)$$

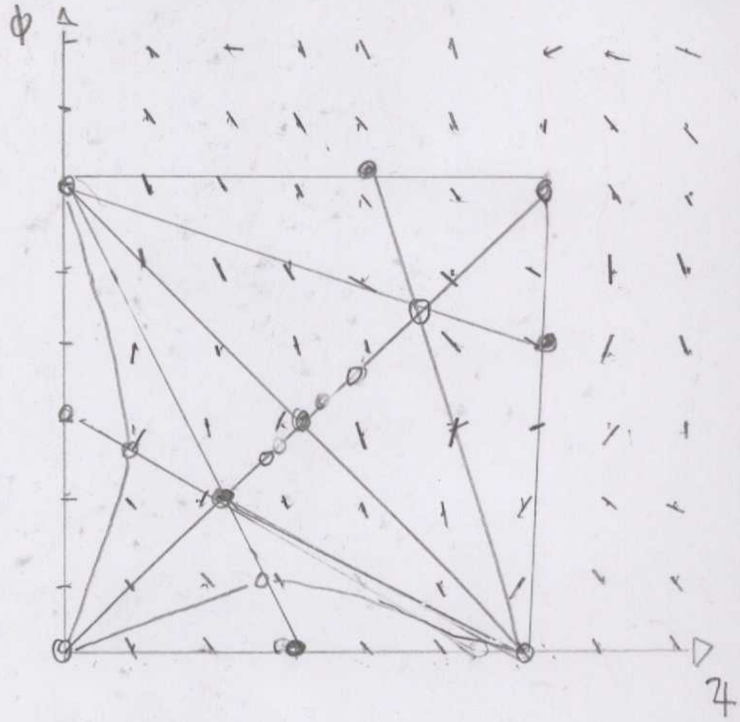
$$(a^*, b^*) = (0, 0)$$

Nullcline: $\dfrac{a}{b} = -\dfrac{2\sin(\phi)\cos(\phi - 2\psi)}{\sin(\phi + \psi) + 2\sin(\phi) - \sin(\psi)}$



$a=-5;b=1$

d) $H(X)=a\sin(X)+b\sin(2X)+c\cos(X)$



$a=-5;b=1;c=0.1$

The zero-points were hardly comprehensible because of a 30+ term function.

$r=[1+e^{-4\pi}(r_0^{-2}-1)]^{-1/2}$

8.7.1. $t=\int_{r_0}^{r_1}\frac{dr}{r(1-r^2)}=\int_{r_0}^{r_1}\frac{dr}{r(1+r)(1-r)}=\int_{r_0}^{r_1}\frac{A}{r}dr+\int_{r_0}^{r_1}\frac{B}{1+r}dr+\int_{r_0}^{r_1}\frac{C}{1-r}dr$

$=A(1+r)(1-r)+B\cdot(1-r)r+C\cdot(1+r)r=1$

If $r=0$, than $A=1$

If $r=1$, than $C=\frac{1}{2}$

If $r=-1$, then $B=\frac{-1}{2}$

$$= \int_{r_0}^{r_1} \frac{1}{r} dr - \frac{1}{2}\int_{r_0}^{r_1} \frac{dr}{1+r} - \frac{1}{2}\int_{r_0}^{r_1} \frac{dr}{r-1}$$

$$= \ln r_1/r_0 - \frac{1}{2} \ln \frac{1+r_1}{1+r_0} - \frac{1}{2} \ln \frac{r_1-1}{r_0-1} + C$$

$$= \ln \frac{r_1}{r_0} \frac{\sqrt{r_0^2 - 1}}{\sqrt{r_1^2 - 1}} = 2\pi$$

Solving for r_1:

$$\frac{r_1}{r_0} \frac{\sqrt{r_0^2 - 1}}{\sqrt{r_1^2 - 1}} = e^{2\pi}$$

$$r_1^2 = \frac{r_0^2 e^{4\pi}}{1 + r_0^2 e^{4\pi} - r_0^2} = \frac{1}{-e^{-4\pi} + r_0^{-2} e^{-4\pi} + 1}$$

$$= \frac{1}{1 + e^{-4\pi}(r_0^{-2} - 1)}$$

$$r_1 = \frac{1}{\sqrt{1 + e^{-4\pi}(r_0^{-2} - 1)}}$$

where $r_{n+1} = P(r_n) = \dfrac{1}{\sqrt{1 + e^{-4\pi}(r_n^{-2} - 1)}}$

and $\dfrac{dP(r)}{dr} = \dfrac{e^{-4\pi} r^{-3}}{\sqrt{1 + e^{-4\pi}(r^{-2} - 1)}}$

$$\frac{dP(1)}{dr} = e^{-4\pi}$$

$\dot{\theta} = 1$ 8.7.2. $\theta = t$; $y = C \cdot e^{at} = Ce^{a\theta}$ "Lyapunov stable = Periodic Orbit"

$\dot{y} = ay$

$\dot{x} + x = F(t)$ 8.7.3 $F(t) = \begin{cases} +A, & 0 < t < T/2 \\ -A, & T/2 < t < T \end{cases}$

a) $X(0) = X_0$: Bernoulli's Equation $\quad y' + P(x)y = Q(x)y^n$

$I(x) = \exp\left[\int [1-n] P(x) dx\right]$

$y^{1-n} = \frac{1}{I(x)} \left[\int [1-n] Q(x) I(x) dx\right]$

① $x' + x = F(t)$

② $I(t) = \exp\left[\int_0^T dt\right] = e^{T} - 1$

③ $x(t) = e^{-T} \circ \left[\int_0^T F(t) \circ e^{t}\, dt\right] = e^{-T}\left[\int_0^{T/2} A e^{t}\, dt + \int_{\frac{T}{2}}^{T} A e^{t}\, dt\right]$

$= e^{-T}\left[A(e^{T/2} - 1) - A(e^{T/2} - e^{T})\right] + C \circ e^{-T}$

④ Initial conditions: $x(0) = x_0$

$x(0) = 1 \cdot [A(0) - A(0)] + C(1) = x_0 \quad ; \; C = x_0$

$x(t) = x_0 e^{-T} - A(1 - e^{-T/2})^2$

b) Identity: $\tanh(x) = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

$x(t) = -x_0 e^{-T} - A(1 - e^{-T/2})^2$

$x_0 (1 + e^{-T}) = -A(1 - e^{-T/2})^2$

$x_0 = \dfrac{-A(1 - e^{-T/2})^2}{(1 - e^{-\frac{T}{2}})(1 + e^{-T/2})}$

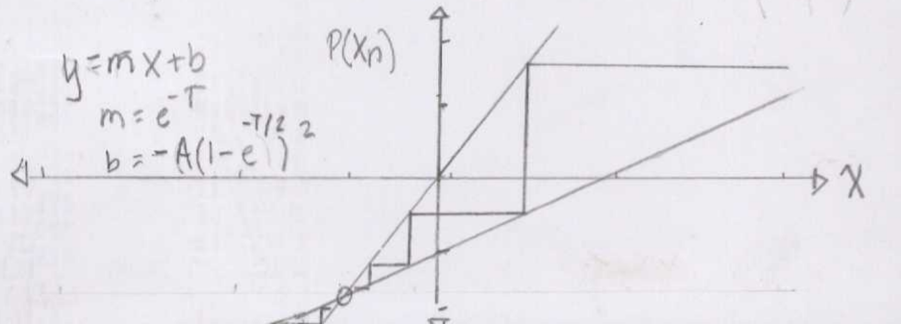
$= -A \tanh\left(\frac{T}{4}\right)$

c) $\lim_{T \to 0} x_0 = 0 \quad ; \quad \lim_{T \to \infty} x_0 = -A$

d) The results indicate a smaller period in an overdamped linear oscillator "strobe" little, while any always "a lot" for longer periods.

d) If $x_1 = x(T)$, then $x_1 = P(x_0)$ or $x_{n+1} = P(x_n) = x_n e^{-T} - A(1 - e^{-T/2})^2$

e)



$\dot{x}+x=A\sin\omega t$ 8.7.4. Solution: $P(x_0)=(x_0-c_3)e^{-2\pi/\omega}+c_3$

$$=x_0 e^{-2\pi/\omega}+c_4$$

The sign of $c_4=A$ is positive because the cobweb plot $(y=mx+b)$ has a slope $e^{-2\pi/\omega}$ and intercept $(b=c_4>0)$.

$\dot{\theta}+\sin\theta=\sin t$ 8.7.5. $\dot{t}=1$;

$\dot{\theta}=\sin t-\sin\theta$

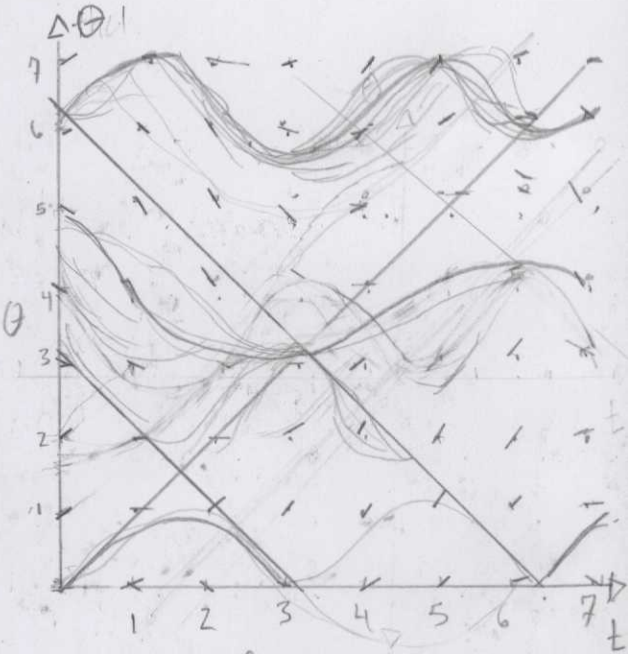
$\dot{\theta}=\sin t-\sin\theta$

or

Nullclines: $\dot{\theta}=0=\sin t-\sin\theta$

$\theta=t$

$=\pi-t$



8.7.6 A mechanical interpretation for $\dot{\theta}=\sin t-\sin\theta$ is a pendulum in a viscous medium.

8.7.7: See 8.75.

$\dot{x}+x=F(t)$ 8.7.8. Solution from 8.7.3: $x(T)=x_0e^{-T}-A(1-e^{-T/2})^2$

A T-periodic system has similar solutions with new T-values. At the limit of T to infinity, $x(T)$ equals negative one, and moreover, zero at some period (T). Since multiple solutions exist for $x(T)=0$ during different parameters and initial conditions, then yes, the system is T-periodic.

$\mathring{r} = r - r^2$ 8.7.9.

$\mathring{\theta} = 1$

a) $t = \int \frac{dr}{r(1-r)} = \int \frac{A}{r} dr + \int \frac{B}{1-r} dr = \ln r + \ln(1-r) + C$

$t = \ln \frac{r}{1-r} + r_0$

$r = \frac{e^{t-r_0}}{1+e^{t-r_0}}$

$P(r_{n+1}) = \frac{e^{t-r_n}}{1+e^{t-r_n}}$ or $P(r_0) = \frac{e^{t-r_0}}{1+e^{t-r_0}}$

b) $P(r^* + v_0) = P(r^*) + DP(r^*) v_0 + O(\|v_0\|^2)$

$(n-1)\times(n-1)$ Matrix: Linearized Poincare Map.

$v_1 = [DP(r^*)] v_0$

$= [DP(r^*)] \sum_{j=1}^{n-1} v_j e_j = \sum_{j=1}^{n-1} v_j \lambda_j e_j$

$v_k = \sum_{j=1}^{n-1} v_j (\lambda_j)^k e_j$; Goal: Characteristic multipliers during a small perturbation.

Fixed Points: $\mathring{r} = 0 = r - r^2$

$r^* = -1, 0, 1$

If $r = 1 + \eta$, where η is infinitesimal.

$\mathring{r} = \mathring{\eta} = (1+\eta) - (1+\eta)^2 = 1 + \eta - 1 - 2\eta + \eta^2$

$= -\eta^2 - \eta$

$\eta(t) = \frac{-e^c}{e^c - e^t} = \frac{-1}{1 - e^{t-c}} = \frac{-1}{1 - Ce^t}$

The characteristic multiplier is $e^{2\pi}$, and $P(1) > 1$, unstable.

c. The characteristic multiplier is $e^{2\pi}$.

8.7.10. Floquet multipliers:

① Find the fixed points about a differential

② Perturb the system by a small η

③ Solve the differential shifted by η

④ Determine the multipliers as coefficients about η_0.

⑤ Evaluate the multipliers at 2π or $2\pi i$ intervals

$$\int_{r_0}^{r_1} \frac{dr}{r(1-r^2)}$$

8.7.11. $\dot{r} = r(1-r^2)$; Fixed Points: $\dot{r} = 0 = r(1-r^2)$

$$r^* = 0, 1$$

Perturbations: $\dot{r} = \dot{\eta} = (1+\eta)(1-(1+\eta)^2)$

$$\eta(t) = \eta_0 e^{-2t}$$

Poincare Map: $P'(r^*) = e^{-4\pi} \times 1$, unstable

Note: A shift of -2π, rather than 2π changes the nodes stability.

$$\dot{\theta}_i = f(\theta_i) + \frac{K}{N}\sum_{j=1}^{N} f(\theta_j)$$

8.7.12. If $\theta(t) = \theta^*(t) + \eta(t)$, then the oscillator becomes:

$$\dot{\eta} = f(\theta_i^*)\eta_i + f(\theta_j^*)\frac{K}{N}\sum_{j=1}^{N}\eta_j$$

A substitution $\mu = \frac{K}{N}\sum_{j=1}^{N}\eta_j$ and $\varepsilon = \eta_{i+1} - \eta_i$

then, $\dfrac{d\varepsilon}{\varepsilon} = f(\theta_i^*)dt = \dfrac{f(\theta_i^*)\,d\theta^*}{f(\theta_i) + \frac{K}{N}\sum_{j=1}^{N} f(\theta_j)}$

$$\oint \frac{d\epsilon}{\epsilon} = \int_0^{2\pi} \frac{f(\theta^*)\,d\theta^*}{f(\theta_i)^* + \frac{K}{N}\sum_{j=1}^{N} f(\theta_j)^*}$$

$$\ln \frac{\epsilon(T)}{\epsilon(0)} = \frac{2\pi}{\frac{K}{N} + 1}$$

If $\epsilon(T) = \epsilon(0)$ for a periodic system, then

$0 = \frac{2\pi}{\frac{K}{N} + 1}$ and a characteristic multiplier is $\lambda = +1$

for K approaching infinity cycles.