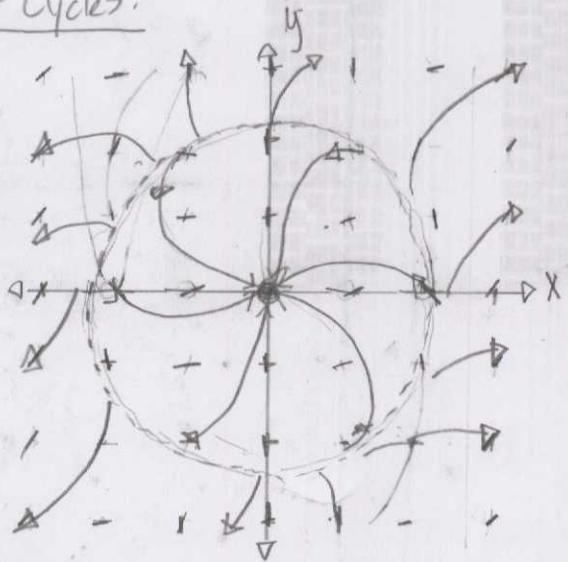


Chapter 7: Limit Cycles:

$$\dot{r} = r^3 - 4r \quad 7.1.1$$

$$\dot{\theta} = 1$$



$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r \cos \theta + \sin \theta \frac{dr}{d\theta}}{-r \sin \theta + \cos \theta \frac{dr}{d\theta}}$$

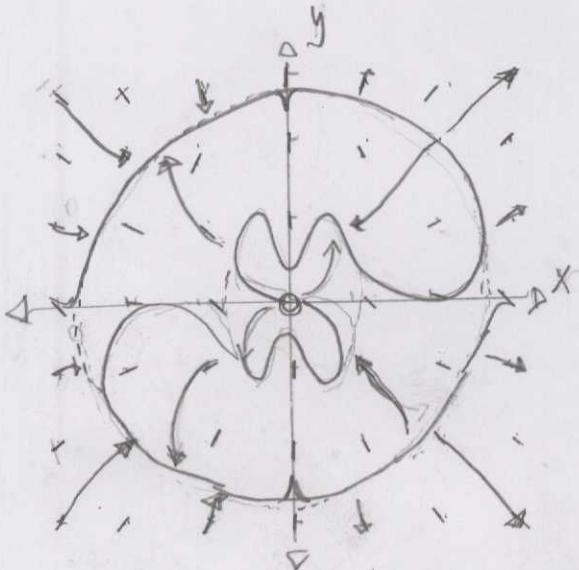
where $r = \sqrt{x^2 + y^2}$; $\theta = \arctan \frac{y}{x}$

$$\text{and } \frac{dr}{d\theta} = (\sqrt{x^2 + y^2})^3 - 4 \cdot \sqrt{x^2 + y^2}$$

$$\dot{r} = r(1 - r^2)(9 - r^2) \quad 7.1.2$$

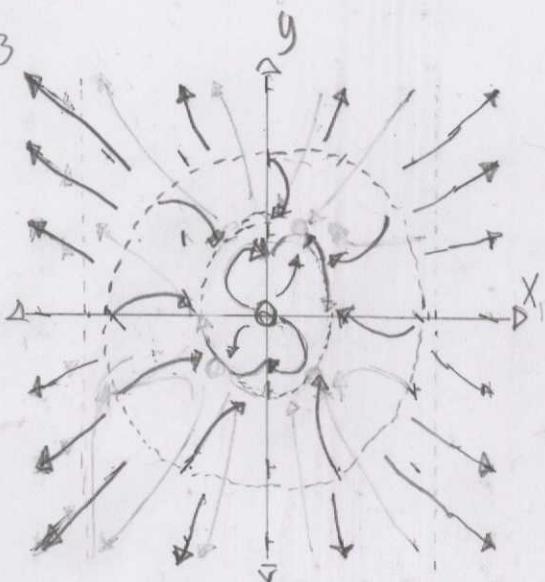
$$\dot{\theta} = 1$$

where $r = \sqrt{x^2 + y^2}$; $\theta = \arctan \left(\frac{y}{x} \right)$
and $\frac{dr}{d\theta} = r(1 - r^2)(9 - r^2)$



$$\dot{r} = r(1 - r^2)(4 - r^2) \quad 7.1.3$$

$$\dot{\theta} = 2 - r^2$$



$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r \cos \theta + \sin \theta \frac{dr}{d\theta}}{r \sin \theta + \cos \theta \frac{dr}{d\theta}}$$

where $r = \sqrt{x^2 + y^2}$; $\theta = \arctan \left(\frac{y}{x} \right)$

$$\text{and } \frac{dr}{d\theta} = \frac{r(1 - r^2)(4 - r^2)}{2 - r^2}$$

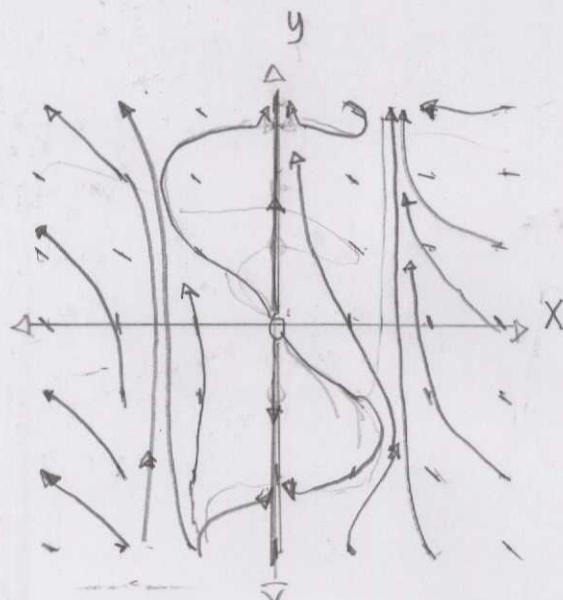
$$\dot{r} = r \sin \theta \quad 7.1.4$$

$$\dot{\theta} = 1$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r \cos \theta + \sin \theta \frac{dr}{d\theta}}{-r \cos \theta + \cos \theta \frac{dr}{d\theta}}$$

$$\text{where } r = \sqrt{x^2 + y^2}; \theta = \arctan \left(\frac{y}{x} \right)$$

$$\text{and } \frac{dr}{d\theta} = r \sin \theta$$



$$\dot{r} = r(1-r^2) \quad 7.1.5. \quad x = r\cos\theta; \quad y = r\sin\theta$$

$$\dot{\theta} = 1$$

$$\dot{x} = \frac{d}{dt} r\cos\theta = \dot{r}\cos\theta - r\sin\theta\dot{\theta}$$

$$= r(1-r^2)\cos\theta - r\sin\theta = x(1-x^2-y^2) - y$$

$$= x - x^3 - xy^2 - y = x - y - x(x^2+y^2)$$

$$\dot{y} = \frac{d}{dt} r\sin\theta = \dot{r}\sin\theta + r\cos\theta\dot{\theta}$$

$$= r(1-r^2)\sin\theta - x = y(1-x^2-y^2) + x$$

$$= x + y - y(x^2+y^2)$$

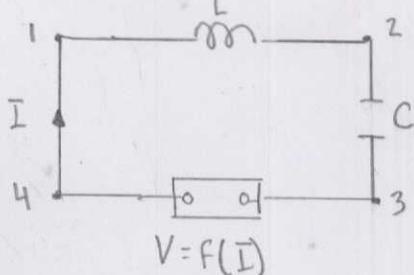
$$\dot{V} = -I/C$$

$$V = L\dot{I} + f(I)$$

7.1.6.

$$a) \quad V = V_{32} = -V_{23}$$

$$V_{41} - V_{12} - V_{23} - V_{34} = 0$$



~~$$V_I = V_L - V_C - V_F = 0$$~~

$$V_I = V_L + V_C + V_F$$

$$= L \frac{dI}{dt} + \frac{I}{C} + f(I)$$

$$= L \frac{dI}{dt} + f(I) - \frac{I}{C}$$

$$= V + \dot{V}$$

b. If $X = \sqrt{L} I$; $W = \sqrt{C} V$; $\tau = \frac{1}{\sqrt{LC}}$, and $F(X) = f(X/\sqrt{L})$ and $X = I$

then $\dot{V} = -I/C = \frac{dW}{dC} = -\frac{X}{\sqrt{L}} \left(\frac{1}{\tau}\right)$; $\frac{dX}{dC} = \frac{dW}{dC} = -\frac{X}{\sqrt{L}}$

and $V = L\dot{I} + f(I)$, $\frac{W}{\sqrt{C}} = L \frac{dX}{dC} \left(\frac{1}{\sqrt{L}}\right) + f(X/\sqrt{L})$

$$W = \sqrt{LC} \frac{dX}{dC} + \sqrt{C} f(X/\sqrt{L})$$

$$\frac{dX}{dC} = W - \mu F(X) ; \text{ where } \mu = \sqrt{C}$$

$$\dot{r} = r(4-r^2)$$

$$\dot{\theta} = 1$$

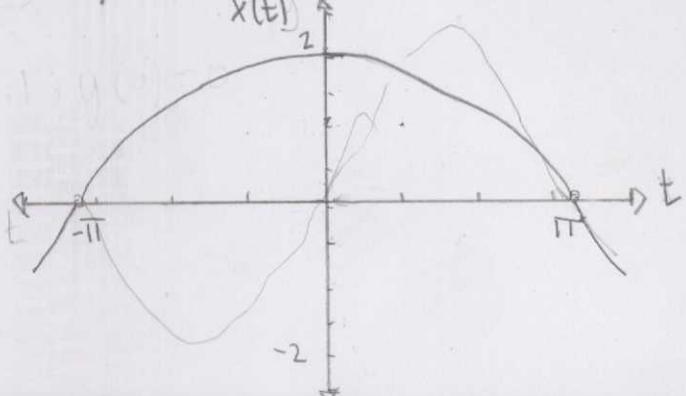
7.1.7.

$$x(t) = r(t)\cos\theta(t); \quad x(0) = 0.1$$

$$x(0) = 0.1; \quad y(0) = 0$$

$$r(t) = \int \frac{dr}{r(4-r^2)} = -2 \ln(4-r^2) + C$$

$$\theta(t) = t; \quad r(t) = \sqrt{4 - Ce^{-2t}}$$



$$\ddot{x} + \alpha \dot{x} (x^2 + \dot{x}^2 - 1) + x = 0$$

7.1.3:

$$a. \quad \overset{\circ}{u} = \overset{\circ}{x} = v$$

$$\overset{\circ}{v} = \ddot{\overset{\circ}{x}} = -\alpha \dot{x} (x^2 + \dot{x}^2 - 1) - x = -\alpha v (u^2 + v^2 - 1) - u$$

$$\text{Fixed Points: } \overset{\circ}{u} = \overset{\circ}{x} = 0$$

$$\overset{\circ}{v} = -\alpha v (u^2 + v^2 - 1) - u = 0$$

$$(u^*, v^*) = (0, 0)$$

$$\overset{\circ}{V} = A \cdot \overset{\circ}{U}; \quad A = \begin{pmatrix} 0 & 0 \\ -2vu-1 & -\alpha(u^2+3v^2-1) \end{pmatrix}$$

$$\lambda_1 = 0; \quad \lambda_2 = -\alpha(u^2 + 3v^2 - 1)$$

$$\Delta = 0; \quad \Gamma = -\alpha(u^2 + 3v^2 - 1)$$

$\Gamma^2 - 4\Delta > 0$ "Non-isolated
Fixed Point"

$$b. \quad \overset{\circ}{r} = \frac{V(u, v)}{u} = \frac{(r \cos \theta \circ r \sin \theta + r^2 \sin \theta \circ (-\alpha v (u^2 + v^2 - 1) - u))}{r}$$

$$V(u, v) = \frac{(r^2 \cos \theta \sin \theta - \alpha r \sin \theta \sin \theta (r^2 \cos^2 \theta + r^2 \sin^2 \theta - 1) - \alpha^2 \cos \theta \sin \theta)}{r}$$

$$= -\alpha r \sin^2 \theta (r^2 - 1)$$

$$\overset{\circ}{\theta} = \frac{(\overset{\circ}{v}u - \overset{\circ}{u}v)}{r^2} = \frac{(-\alpha r \sin \theta (r^2 - 1) - r \cos \theta) r \cos \theta - r^2 \sin \theta}{r^2}$$

$$= \frac{-\alpha r^2 \sin \theta \cos \theta (r^2 - 1) - r^2 \cos^2 \theta - r^2 \sin^2 \theta}{r^2}$$

$$\lambda_1 = \lambda_2 = -\alpha \sin \theta \cos \theta (r^2 - 1) - 1 - \frac{\alpha(u^2 + 3v^2 - 1)}{r}$$

$$\text{Amplitude: } \vec{x} = e^{i\theta} \vec{v}_1 + C_2 e^{i\theta} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_3 e^{i\theta} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

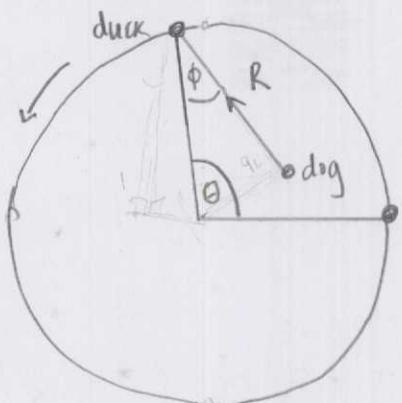
$$\text{Period (s)} \frac{2\pi}{|\overset{\circ}{\theta}|} = 2\pi + C_1$$

c. Stable limit cycle because larger α values generate a standard and periodic trajectory.

d. The limit cycle is unique because solutions containing a , b values and c initial conditions have many solutions, for similar initial conditions.

$$\frac{dR}{d\theta}, \frac{d\phi}{d\theta}$$

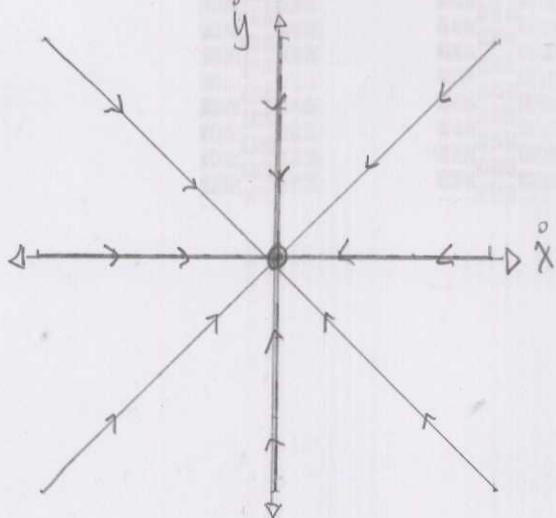
7.1.9. a.



$$\text{Duck: } \langle x, y \rangle = \langle r \cos \theta, r \sin \theta \rangle$$

$$\text{Dog: } \langle x, y \rangle =$$

$$V = x^2 + y^2 \quad 7.2.1: \ddot{\vec{x}} = -\nabla V; \quad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = - \begin{bmatrix} dx \\ dy \end{bmatrix} V; \quad \ddot{x} = - \int x^2 = -\frac{x^3}{3}$$



$$\ddot{x} = - \int x^2 = -\frac{x^3}{3}$$

$$V = x^2 - y^2 \quad 7.2.2: \ddot{\vec{x}} = -\nabla V; \quad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = - \begin{bmatrix} dx \\ dy \end{bmatrix} V; \quad \ddot{x} = - \int x^2 dx = -\frac{x^3}{3} + c$$

$$\dot{y} = + \int y^2 dx = +\frac{y^3}{3} + c$$

$$V = e^x \sin y \quad 7.2.3: \ddot{\vec{x}} = -\nabla V$$

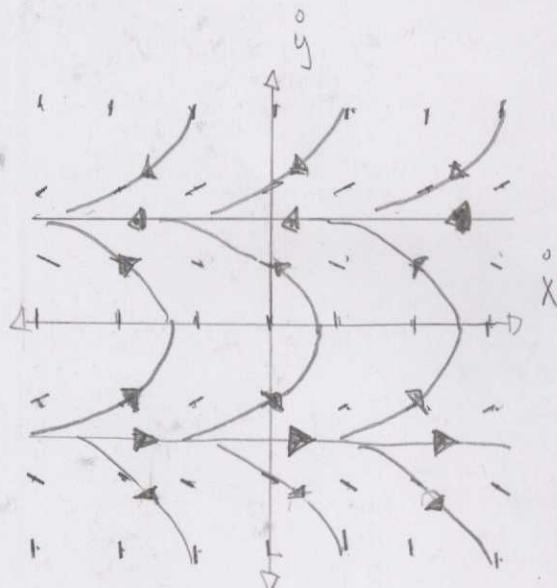
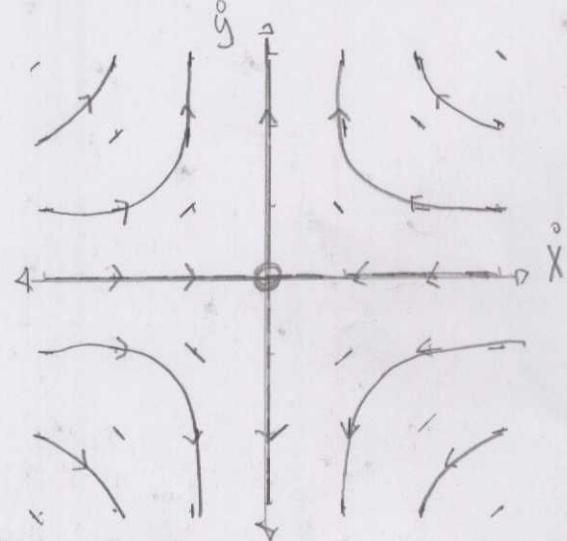
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = - \begin{bmatrix} dx \\ dy \end{bmatrix} V$$

$$\dot{x} = - \int e^x \sin y dx$$

$$= -e^x \sin y + c$$

$$\dot{y} = - \int e^x \sin y dy$$

$$= e^x \cos y + c$$



7.2.4. Gradient System: When a system can be written as $\dot{x} = -\nabla V$, for a continuously differentiable, single-valued scalar function

Line: A continuous function without curvature

Circle: A continuous and bounded function by an equilibrium center.

Line: $\dot{x} = -\nabla V = -1$; A gradient system.

Circle: $\sqrt{x^2 + y^2}$; $\dot{x} = -\nabla V = -\frac{x}{\sqrt{x^2 + y^2}}(x^2 + y^2)$ gradient system.

$$\begin{aligned}\dot{x} &= f(x, y) \\ \dot{y} &= g(x, y)\end{aligned}$$

$$7.2.5. \text{ a. } -\nabla V = \left(-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y} \right), \quad \frac{\partial V}{\partial x} = \dot{x} = f(x, y); \quad -\frac{\partial V}{\partial y} = \dot{y} = g(x, y)$$

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

b. A sufficient condition is $p \rightarrow q$ and $\neg p \rightarrow \neg q$,

so $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ is sufficient by $V = - \int f(x, y) dx = - \int g(x, y) dy$.

$$\begin{aligned}\dot{x} &= y^2 + y \cos x \\ \dot{y} &= 2xy + \sin x\end{aligned} \quad 7.2.6. \quad \text{a. } V = \int \dot{x} dx + \int \dot{y} dy = xy^2 + y \sin x + xy^2 + y \sin x \\ &= 2(xy^2 + y \sin x)$$

$$\begin{aligned}\dot{x} &= 3x^2 - 1 - e^{2y} \\ \dot{y} &= -2xe^{2y}\end{aligned} \quad \text{b. } V = \int \dot{x} dx + \int \dot{y} dy = x^3 - x - xe^{2y} + xe^{2y} \\ &= x(x^2 - 1 - 2e^{2y})$$

$$\begin{aligned}\dot{x} &= y + 2xy \\ \dot{y} &= x + x^2 - y^2\end{aligned} \quad 7.2.7. \quad \text{a. If } \dot{x} = f(x, y) \text{ and } \dot{y} = g(x, y), \text{ then}$$

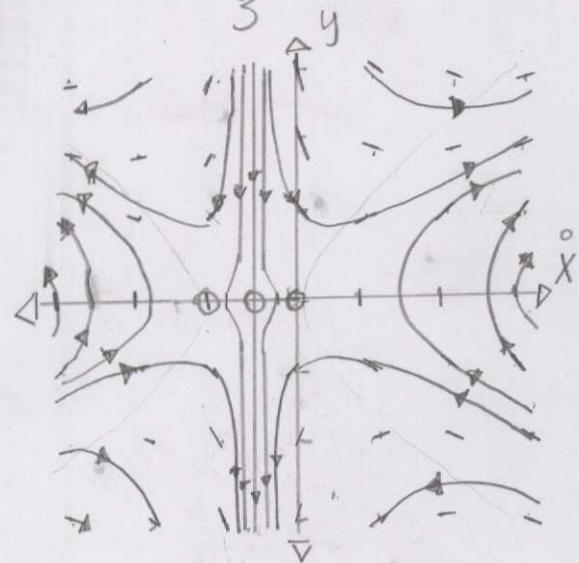
$$\text{then } \frac{\partial f}{\partial y} = 1 + 2x \text{ and } \frac{\partial g}{\partial x} = 1 + 2x$$

$$\text{and } \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

$$b. V = \int \dot{x} dx + \int \dot{y} dy = xy + xy^2 + xy + x^2y - \frac{y^3}{3}$$

$$= 2xy + xy^2 + x^2y - \frac{y^3}{3}$$

c.



7.2.8. If $\frac{df}{dy} = \frac{dy}{dx}$ at an equipotential, then $\frac{dy}{dx} = \frac{dy}{df}$.

The solution $\frac{dy}{dx} = \frac{dy}{df}$ is zero when $dy \neq df$

and one when $dy = df$, very similar to
orthogonal slopes (dy/dx).

$$\begin{aligned}\dot{x} &= y + x^2y \\ \dot{y} &= -x + 2xy\end{aligned}$$

7.2.9.

$$a. V = \int \dot{x} dx + \int \dot{y} dy = - \int y + x^2y dx + \int -x + 2xy dy$$

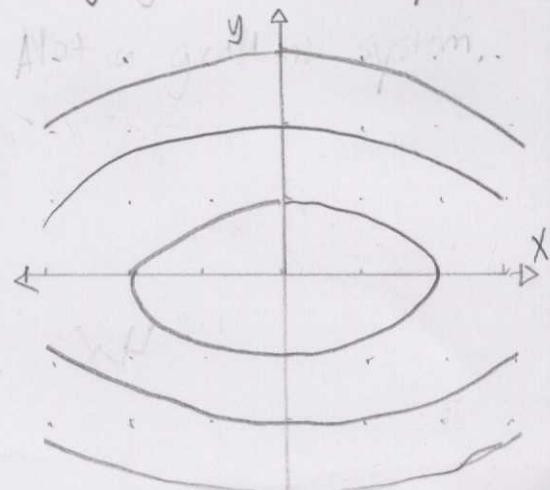
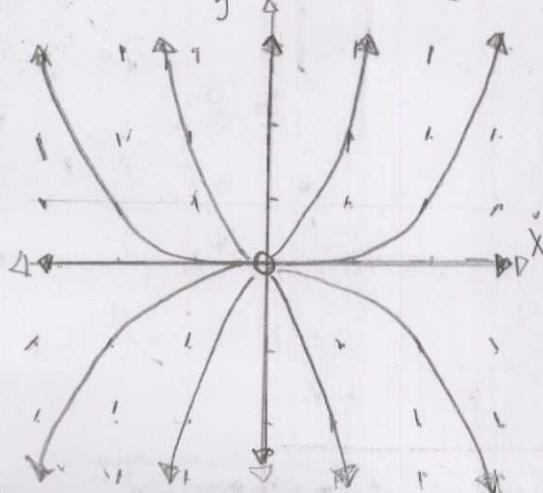
$$= -xy + \frac{x^3y}{3} + \frac{x^2}{2} = x^2y ; \frac{d\dot{x}}{dx} = \frac{dy}{dx} \quad (1)$$

Not a gradient system.

$$\begin{aligned}\dot{x} &= 2x \\ \dot{y} &= 3y\end{aligned}$$

$$b. V = - \int \dot{x} dx - \int \dot{y} dy = - \int 2x dx - \int 3y dy$$

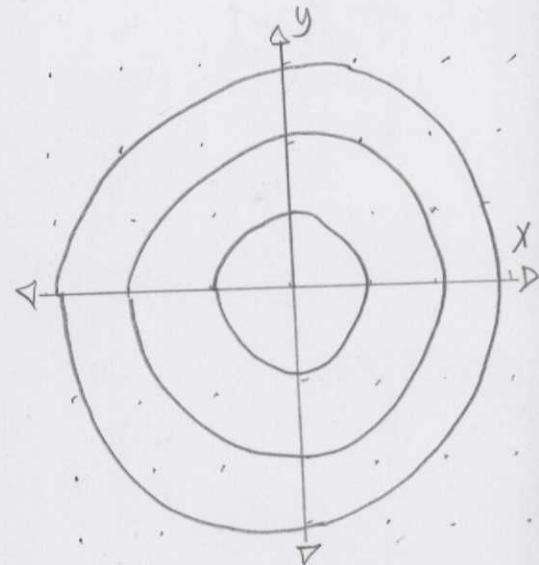
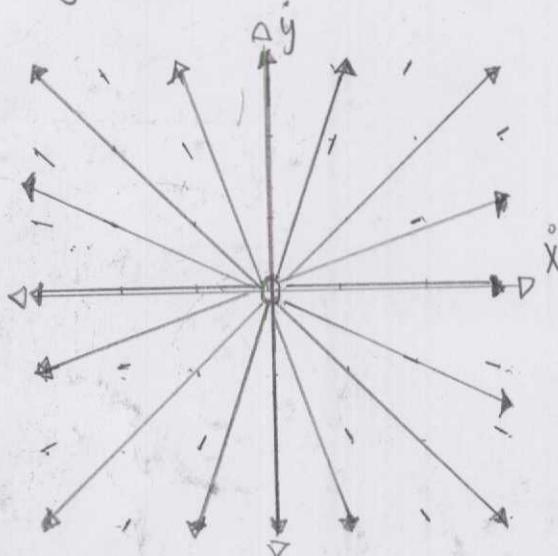
$$= -x^2 + 4y^2 ; x(1) \times \left(\frac{d\dot{x}}{dy} = \frac{dy}{dx} \right) \text{ Gradient System}$$



$$\begin{aligned}\dot{x} &= -2xe^{-x^2} \\ \dot{y} &= -2ye^{-y^2}\end{aligned}$$

$$\begin{aligned}\dot{x} &= -2xe^{x^2+y^2} \quad C. V(x,y) = -\int \dot{x} dx - \int \dot{y} dy = +2 \int xe^{x^2+y^2} dx + 2 \int ye^{x^2+y^2} dy \\ \dot{y} &= -2ye^{x^2+y^2} \quad = e^{x^2+y^2} + e^{x^2+y^2} = 2e^{x^2+y^2}\end{aligned}$$

$\frac{dx}{dy} (=) \frac{dy}{dx}$: Gradient system.



$$\begin{aligned}\dot{x} &= y - x^3 \quad 7.2.10. V = -\int \dot{x} dx - \int \dot{y} dy = -\int y - x^3 dx + \int x + y^3 dy \\ \dot{y} &= -x - y^3\end{aligned}$$

$$= -xy + \frac{x^4}{4} + xy + \frac{y^4}{4} = \frac{x^4}{4} + \frac{y^4}{4}$$

$$a = \frac{1}{4}, b = \frac{1}{4}$$

The potential function has a suitable a , and b , so this function is Liapunov stable with no closed orbits.

$$V = ax^2 + 2bxy + cy^2 \quad 7.2.11. \frac{\partial^2 V}{\partial x^2} \frac{\partial^2 V}{\partial y^2} - \left(\frac{\partial^2 V}{\partial x \partial y} \right)^2 = (2a)(2c) - (2b)^2 = 4(ac - b^2) @ (0,0)$$

Positive definite is a strictly positive, meaning strictly positive when $(ac - b^2) > 0$.

$$\begin{aligned}\dot{x} &= -x + 2y^3 - 2y^4 \quad 7.2.12 V = -\int \dot{x} dx - \int \dot{y} dy = \frac{x^2}{2} - 2y^3 x + 2y^4 x + xy + \frac{y^2}{2} - \frac{xy^2}{2} \\ \dot{y} &= -x - y + xy\end{aligned}$$

Fixed Points $(x^*, y^*) = (0,0), (-2,0), (-2,1), (-1,0)$

No periodic solutions

$$\dot{N}_1 = r_1 N_1 (1 - N_1/K_1) - b_1 N_1 N_2$$

$$\dot{N}_2 = r_2 N_2 (1 - N_2/K_2) - b_2 N_1 N_2$$

$$7.2.B. \quad g = (N_1 N_2)^{-1}$$

Dulac's Criterion: If $\dot{x} = f(x)$ is continuous, a real-valued function $g(x)$ such that $\nabla \cdot (g \dot{x})$ has one sign throughout \mathbb{R} in a closed orbit.

$$\begin{aligned}\nabla \cdot (g \dot{x}) &= \frac{\partial}{\partial N_1} (g \dot{N}_1) + \frac{\partial}{\partial N_2} (g \dot{N}_2) \\ &= \frac{\partial}{\partial N_1} \left[\frac{r_1}{N_2} (1 - N_1/K_1) - b_1 \right] - \frac{\partial}{\partial N_2} \left[\frac{r_2}{N_1} (1 - N_2/K_2) - b_2 \right] \\ &= \frac{r_2}{N_1 K_2} - \frac{r_1}{N_2 K_1}\end{aligned}$$

≤ 0

$$\begin{aligned}\dot{x} &= x^2 - y - 1 \\ \dot{y} &= y(x-2)\end{aligned}$$

$$7.2.14. \text{ a. Fixed Points: } \dot{x} = 0 = x^2 - y - 1 ;$$

$$\dot{y} = 0 = y(x-2)$$

$$(x^*, y^*) = (-1, 0); A_{(-1,0)} = \begin{pmatrix} -2 & -1 \\ 0 & -2 \end{pmatrix}; \text{ Stable Node}$$

$$(1, 0); A_{(1,0)} = \begin{pmatrix} 2 & -1 \\ 0 & -2 \end{pmatrix}; \text{ Saddle Point}$$

$$(2, 3); A_{(2,3)} = \begin{pmatrix} 4 & -1 \\ 3 & -2 \end{pmatrix}; \text{ Saddle Point}$$

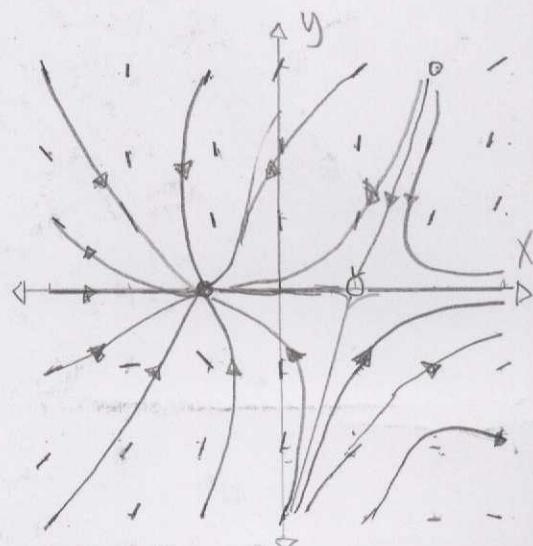
$$b. (-1, 0) \rightarrow (1, 0); \frac{dy}{dx} = 0$$

$$(1, 0) \rightarrow (2, 3); \frac{dy}{dx} = x^2 - 1$$

$$(2, 3) \rightarrow (-1, 0); \frac{dy}{dx} \cong -1$$

Closed orbits are nonexistent because of constant trajectory between fixed points.

c.



$$\begin{aligned} \dot{x} &= x(2-x-y) & 7.2.15. \text{ Fixed Points: } \dot{x} = 0 = x(2-x-y); A = \begin{pmatrix} 2-2x-y & -x \\ 4y-2yx & 4x-x^2-3 \end{pmatrix} \\ \dot{y} &= y(4x-x^2-3) & \dot{y} = 0 = y(4x-x^2-3) \\ (x^*, y^*) &: (0, 0); A_{(0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}; \text{Saddle Point} \end{aligned}$$

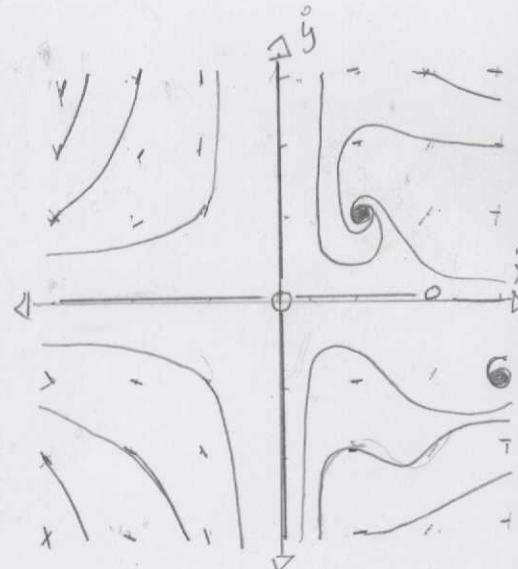
$$(1,1); A_{(1,1)} = \begin{pmatrix} -1 & 2-1 \\ 2 & 0 \end{pmatrix}; \begin{array}{l} \text{Stable spiral} \\ \text{Saddles} \end{array}$$

$$(2,0); A_{(2,0)} = \begin{pmatrix} -2 & -2 \\ 0 & 1 \end{pmatrix}; \text{Saddle Point}$$

$$(3,-1); A_{(3,-1)} = \begin{pmatrix} 9 & -3 \\ 2 & 0 \end{pmatrix}; \text{Stable spiral}$$

b. Phase Portrait

7.2.16. If R is a set of closed orbits, then $\iint_D \nabla \cdot (g\dot{x}) dA = \oint_C g\dot{x} dx$



7.2.17. If A is an annulus, then

Green's theorem holds true.

Dulac's Criterion fails for

multiple holes because the

closed orbit (line integral) has only one path.

$$\dot{x} = rx\left(1 - \frac{x}{2}\right) - \frac{2x}{1+x}y \quad 7.2.18 \text{ If } g(x,y) = \frac{1+x}{x}y^{x-1}; \text{ then } \dot{x} = rx\left(1 - \frac{x}{2}\right) - 2g(x,y)^{-1}$$

$$\dot{y} = -y + \frac{2x}{1+x}y$$

$$\dot{y} = -y + 2g(x,y)^{-1}$$

$$\text{where } x = 0$$

$$\nabla \cdot (g\dot{x}) = \frac{\partial}{\partial x}(g\dot{x}) + \frac{\partial}{\partial y}(g\dot{y})$$

$$= \frac{r(1-2x)}{2y} > 0; \text{ No closed orbits in the positive quadrants}$$

$$\begin{aligned}\dot{R} &= -R + A_s + kS e^{-s} \quad 7.2.19. \\ \dot{S} &= -S + Ar + kR e^{-R}\end{aligned}$$

Term:	Meaning:
$-R$	Rhett's decreasing love for Scarlett.
$+A_s$	Scarlett's love for Rhett.
$+kS e^{-s}$	Scarlett's decaying love for Rhett.
$-S$	Scarlett's decreasing love for Rhett.
$+Ar$	Rhett's love.
$+kR e^{-R}$	Rhett's decaying love for Scarlett.

b. $\dot{R} = 0 = -R + A_s + kS e^{-s}$; $(R^*, S^*) = (A_s + kS e^{-s}, Ar + kR e^{-R})$
 $\dot{S} = 0 = -S + Ar + kR e^{-R}$; which are greater than zero.

c. $\nabla \cdot (g \vec{x}) = \frac{\partial}{\partial R}(g \dot{R}) + \frac{\partial}{\partial S}(g \dot{S})$

$$= -1 - 1 = -2 < 0 ; \text{ Where } g = 1.$$

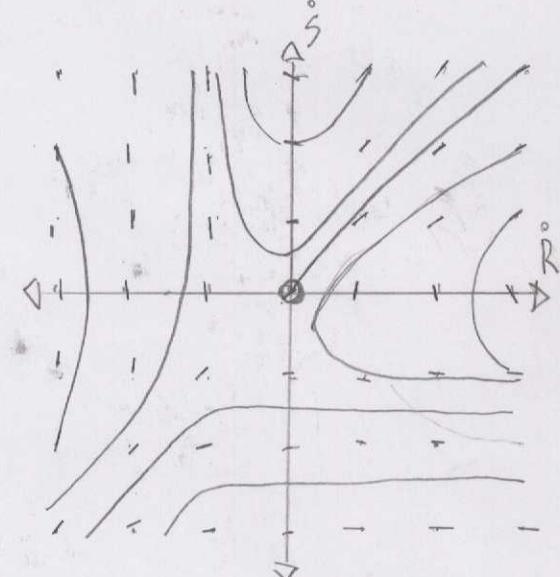
Since $x, y > 0$, there are no periodic solutions in the first quadrant.

d. Phase Portrait; $A_s = 1, 2$

$$Ar = 1$$

$$k = 15$$

$$R(0) = S(0) = 0$$



$$\begin{aligned}\dot{x} &= x - y - x(x^2 + 5y^2) \quad 7.3.1 \text{ a. } A = \begin{pmatrix} 1-x^2-5y^2-2x^2 & -1-10xy \\ 1-2xy & 1-x^2-y^2-2y^2 \end{pmatrix} \\ \dot{y} &= x + y - y(x^2 + y^2) \\ &= \begin{pmatrix} 1-3x^2-5y^2 & -1-10xy \\ 1-2xy & 1-x^2-3y^2 \end{pmatrix}\end{aligned}$$

$$A_{(0,0)} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}; \Delta = 2; \tau = 2; \tau^2 - 4\Delta < 0; \text{ Unstable Spiral}$$

$$\begin{aligned}
 b. \quad \dot{r} &= \frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{r} = \frac{r\cos\theta[x - y - x(x^2 + 5y^2)] + r\sin\theta[x + y - y(x^2 + y^2)]}{r} \\
 &= r\cos\theta[\cos\theta - \sin\theta - \cos\theta(r^2\cos^2\theta + 5r^2\sin^2\theta)] \\
 &\quad + r\sin\theta[\cos\theta + \sin\theta - \sin\theta(r^2)] \\
 &= r[\cos\theta(\cos\theta - \sin\theta - r^2\cos\theta(\cos^2\theta + 5\sin^2\theta)) \\
 &\quad + \sin\theta(\cos\theta + \sin\theta - r^2\sin^2\theta)] \\
 \dot{\theta} &= \frac{(x\dot{y} - y\dot{x})}{r^2} = \frac{\cos\theta(x + y - y(r^2)) - \sin\theta(x - y - x(x^2 + 5y^2))}{r} \\
 &= \cos\theta(\cos\theta + \sin\theta - r^2\sin\theta) - \sin\theta(\cos\theta - \sin\theta - r^2\cos\theta(\cos^2\theta + 5\sin^2\theta)) \\
 &= 4r^2\sin^3(\theta)\cos(\theta) + 1
 \end{aligned}$$

$$\begin{aligned}
 c. \quad r_{\min, \text{outward}} &: \dot{r} > 0 ; r[\cos\theta(\cos\theta - \sin\theta - r^2\cos\theta(\cos^2\theta + 5\sin^2\theta)) \\
 &\quad + \sin\theta(\cos\theta + \sin\theta - r^2\sin^2\theta)] > 0 \\
 r_{\min, \text{outward}} &\cong -\sqrt{\frac{\sin^2\theta - \sin\theta + \cos\theta}{\sin^3\theta + 3\cos\theta - 2\cos\theta\cos(2\theta)}}
 \end{aligned}$$

$$\begin{aligned}
 d. \quad r_{\max, \text{inward}} &: \dot{r} < 0 ; r[\cos\theta(\cos\theta - \sin\theta - r^2\cos\theta(\cos^2\theta + 5\sin^2\theta)) \\
 &\quad + \sin\theta(\cos\theta + \sin\theta - r^2\sin^2\theta)] < 0
 \end{aligned}$$

$$r_{\max, \text{inward}} \cong \sqrt{\frac{(\sin\theta - 1)\sin\theta + \cos\theta}{\sin^3\theta + \cos\theta(3 - 2\cos 2\theta)}}$$

$$e. \quad r_{\min, \text{outward}} < r < r_{\max, \text{inward}}$$

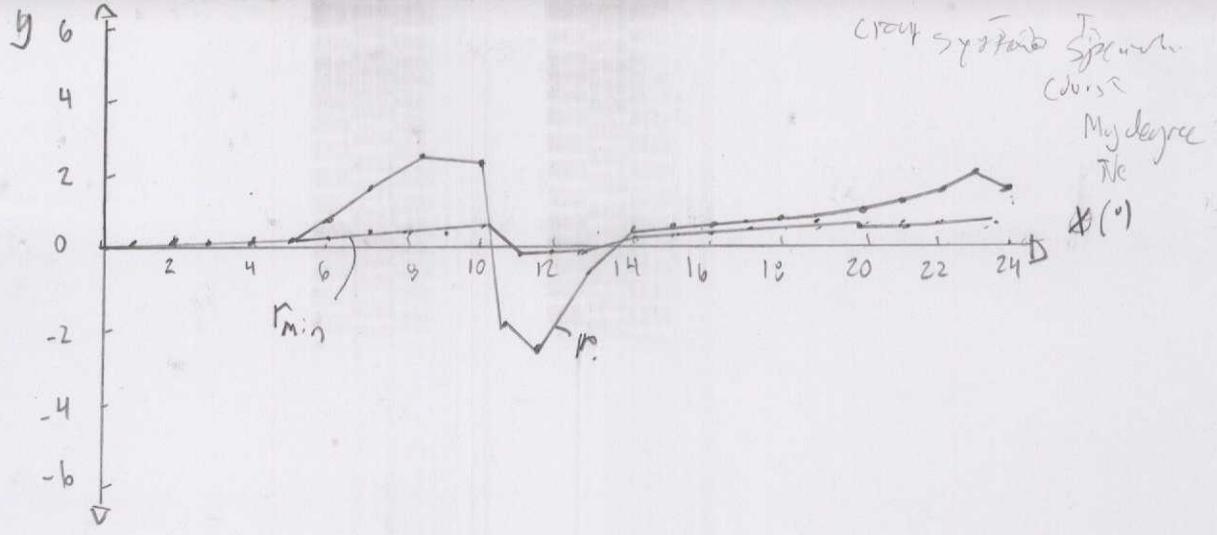
$$\theta \cong 2(n\pi - 4\pi/10); \text{ This solution is close to the books } \theta_2 = \frac{3\pi}{2}$$

7.3.2 Runge Kutta Method [4th Order]

$$\begin{array}{|c|c|c|c|} \hline
 r_0 & \theta_0 & k_1(r_0, \theta_0 + \theta\Delta h/2) & k_1(r_0 + r\Delta h/2, \theta_0) \\
 \hline
 0.1 & & k_1(r_0 + r\Delta h/2, \theta_0) & k_2(r_0 + r\Delta h/2, \theta_0) \\
 \hline
 \end{array} \dots$$

$$\begin{array}{|c|c|c|} \hline
 r_0 & \theta_0 + k_1\Delta h & k_2(r_0 + k_1\Delta h, \theta_0) \\
 \hline
 0.00 & k_2(r_0 + k_1\Delta h, \theta_0) & k_3(r_0 + k_2\Delta h, \theta_0) \\
 \hline
 \end{array}$$

$$r_n = r_{n-1} + \frac{\Delta h}{6}(r_0 + 2k_1 + 2k_2 + k_3) \Rightarrow \theta_n = \theta_{n-1} + \frac{\Delta h}{6}(\theta_0 + 2k_1 + 2k_2 + k_3)$$



$$\begin{aligned}\dot{x} &= x - y - x^3 \\ \dot{y} &= x + y - y^3\end{aligned}$$

7.3.3. $r = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{r \cos \theta (x - y - x^3) + r \sin \theta (x + y - y^3)}{r}$

$$= x \cos \theta (r \cos \theta - r \sin \theta - (r \cos \theta)^3) + y \sin \theta (r \cos \theta + r \sin \theta - (r \sin \theta)^3)$$

$$= r^2 (r (\cos^4 \theta + \sin^4 \theta) - r^3 \cos^3 \theta \sin \theta - r^3 \sin^3 \theta \cos \theta)$$

$r - r^3 < r < r - r^3/2$ Poincaré-Bendixson Theorem states
at least one periodic solution.

$$\dot{x} = x(1 - 4x^2 - y^2) - \frac{1}{2}y(1+x)$$

$$\dot{y} = y(1 - 4x^2 - y^2) + 2x(1+x)$$

7.3.4. a. $A = \begin{pmatrix} -12x^2 - y^2 - 1 & (-2x+1)y - 1/2 \\ x(4+y) + 2 & -4x^2 - 3y^2 + 1 \end{pmatrix}$

$$A_{(0,0)} = \begin{pmatrix} -1 & -1/2 \\ 2 & 1 \end{pmatrix}; \lambda_1 = 10\sqrt{2}; \lambda_2 = -10\sqrt{2}; \Delta = 0; \tau = 0; \tau^2 - 4\Delta = 0 \text{ "center"}$$

Although graph indicates an unstable spiral.

b. $V = (1 - 4x^2 - y^2)^2; \dot{V} = 2(1 - 4x^2 - y^2)(1 - 4x^2 - y^2)'$

$$\lim_{t \rightarrow \infty} \dot{V} = 0; 1 - 4x^2 - y^2 = 0; 4x^2 + y^2 = 1$$

$$\begin{aligned}\dot{x} &= -x - y + x(x^2 + 2y^2) \\ \dot{y} &= x - y + y(x^2 + 2y^2)\end{aligned}$$

7.3.5 $r = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{r \cos \theta (-x - y + x(x^2 + 2y^2)) + r \sin \theta (x - y + y(x^2 + 2y^2))}{r}$

$$= \frac{r \cos \theta (-r \cos \theta - r \sin \theta + r \cos \theta ((r \cos \theta)^2 + 2(r \sin \theta)^2))}{r} + \frac{r \sin \theta (r \cos \theta - r \sin \theta + r \sin \theta ((r \cos \theta)^2 + 2(r \sin \theta)^2))}{r}$$

$$= r^3 (\sin^2(x) + 1) \left(\frac{1}{2} \sin(2x) + \cos^2(x) \right) - r$$

$$\theta = \frac{38}{100} r^3 - r < r < \frac{152}{100} r^3 - r$$

A periodic solution exists by Poincare-Bendixson Theorem.

$$\ddot{x} + F(x, \dot{x})\dot{x} + x = 0$$

7.3.6. $F(x, \dot{x}) < 0$, $r \leq a$, else, $\dot{x}F(x, \dot{x}) > 0$ if $(r \geq b)$ where $r^2 = x^2 + \dot{x}^2$

a. $\dot{u} = \dot{x} = v$ A physical interpretation of
 $\dot{v} = \ddot{x} = -F(x, \dot{x})\dot{x} - x$ $F(x, \dot{x})$ is an additive force
 to acceleration, which increases
 or decreases.

$$b, r = \sqrt{x^2 + \dot{x}^2} = \sqrt{(r \cos \theta)^2 + ((\dot{r} \cos \theta - r \sin \theta \dot{\theta}))^2}$$

$$a < \sqrt{(r \cos \theta)^2 + (\dot{r} \cos \theta - r \sin \theta \dot{\theta})^2} < b$$

$$\dot{x} = y + ax(1-2b-r^2)$$

$$\dot{y} = -x + ay(1-r^2)$$

7.3.7. a. $(0 < a \leq 1, 0 \leq b < 1/2)$ and $r^2 = x^2 + y^2$

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \frac{r \cos \theta (y + ax(1-2b-r^2)) + r \sin \theta (-x + ay(1-r^2))}{r}$$

$$= r \cos \theta (r \sin \theta + a \cos \theta (1-2b-r^2)) + r \sin \theta (-r \cos \theta + a \sin \theta (1-r^2))$$

$$\dot{\theta} = \frac{\dot{x}y - \dot{y}x}{r^2} = \frac{r \cos \theta (-x + ay(1-r^2)) + r \sin \theta (y + ax(1-2b-r^2))}{r^2}$$

$$= \cos \theta (-\cos \theta + a \sin \theta (1-r^2)) - \sin \theta (\sin \theta + a \cos \theta (1-2b-r^2))$$

$$= ab \sin(2\theta) \rightarrow \sin(2\theta) = 1$$

b. A region of trapping $a \ar(1-r^2) \leq \dot{r} \leq a \ar(1-2b-r^2)$
 exists as an annular cycle, $T(a, b)$, i.e., $T(a, b)$.

c) If $b=0$, then $\alpha r(1-r^2) \leq \dot{r} \leq \alpha r(1-r^2)$, so \dot{r} must be $\alpha r(1-r^2)$.

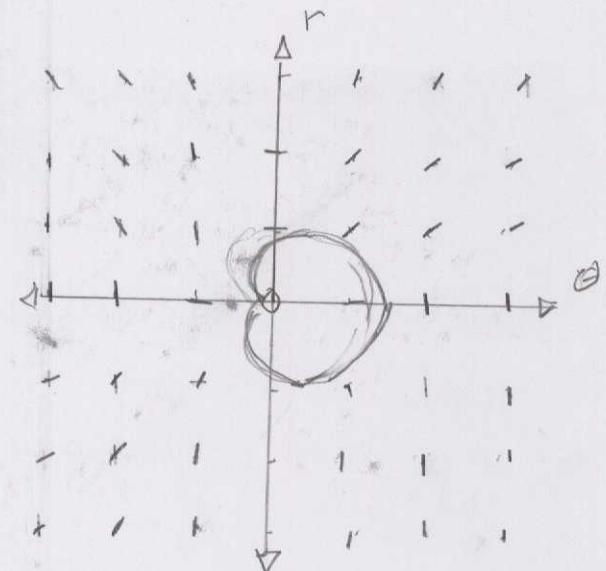
$$\dot{r} = r(1-r^2) + \mu r \cos \theta$$

$$\ddot{\theta} = 1$$

$$7.3.8. \quad \dot{r} = 0 = r(1-r^2) + \mu r \cos \theta$$

$$\mu = \frac{r(r^2-1)}{r \cos \theta}$$

If $r=0, 1, \text{ or } -1$, then the closed orbit known as the cardioid becomes absent, but a circular orbit remains.



$$7.3.9.$$

$$a. \quad r(\theta) = 1 + \mu r_1(\theta) + O(\mu^2)$$

$$\frac{dr}{d\theta} = r(1-r^2) + \mu r \cos \theta = \mu r'_1(\theta)$$

$$\mu r'_1(\theta) = (1 + \mu r_1(\theta))(1 - (1 + \mu r_1(\theta))^2) + \mu(1 + \mu r_1(\theta)) \cos \theta$$

$$r'_1(\theta) = -2r_1(\theta) + \cos(\theta) \quad \text{First-order linear differential equation.}$$

$$\cos(\theta) = r'_1(\theta) + 2r_1(\theta)$$

$$\cos(\theta) d\theta = r'_1(\theta) + 2r_1(\theta) d\theta$$

$$r(\theta) = \frac{e^{\theta} - 1}{\alpha}$$

$$b. \quad (\cos(\theta) - 2r'_1) d\theta - dr(\theta) = 0 \quad \text{Exact Differential Equation.}$$

$$N(r, \theta) d\theta + M(r, \theta) dr = 0$$

$N'(r, \theta) \neq M'(r, \theta)$, so a multiplication is necessary.

$$(2N'(r, \theta) = \frac{dN}{dr}) = 2; \quad M'(r, \theta) = \frac{dM}{d\theta} = 0$$

$$\text{The assumption; } \frac{dN}{dr} = \frac{dM}{d\theta}; \quad \frac{dN}{dr} - \frac{dM}{d\theta} = 0 = M \left(\frac{dN}{dr} - \frac{dM}{d\theta} \right)$$

$$\text{Also, } M(r, \theta) = \mu(r); \quad \frac{dM}{d\theta} = 0; \quad \text{so } \frac{1}{\mu} \frac{d\mu}{dr} = \frac{1}{M} \left(\frac{dN}{dr} - \frac{dM}{d\theta} \right)$$

$$\int \frac{d\mu}{\mu} = \int \frac{1}{M} \left(\frac{dN}{dr} - \frac{dM}{d\theta} \right) d\theta; \quad \ln(\mu) = 2\theta; \quad \mu = e^{2\theta}$$

Multiplying the equation by $e^{2\theta}$

$$(\cos(\theta) - 2r)e^{2\theta} d\theta - e^{2\theta} dr = 0$$

$$N(r, \theta) d\theta - M(r, \theta) dr = 0$$

$$(N'(r, \theta) = M'(r, \theta)) = -2e^{2\theta} \quad \text{Exact equation.}$$

$$dF(r, \theta) = N(r, \theta) d\theta + M(r, \theta) dr$$

$$\begin{aligned} F(r, \theta) &= \int N(r, \theta) dr = \int e^{2\theta} \cos(\theta) - 2e^{2\theta} \cdot r d\theta \\ &= \frac{e^{2\theta} \sin(\theta)}{5} + \frac{2e^{2\theta} \cos(\theta)}{5} - e^{2\theta} r + C \end{aligned}$$

$$r = \frac{\sin(\theta)}{5} + \frac{2\cos(\theta)}{5} + \frac{C}{e^{2\theta}}$$

$$r(\theta) = 1 + \mu \left(\frac{\sin(\theta)}{5} + \frac{2\cos(\theta)}{5} \right)$$

b. $\frac{dr}{d\theta} = 0 = \frac{\mu \cos \theta}{5} - \frac{2 \sin \theta}{5} \quad @ \theta = \arctan\left(\frac{1}{2}\right) + n\pi$

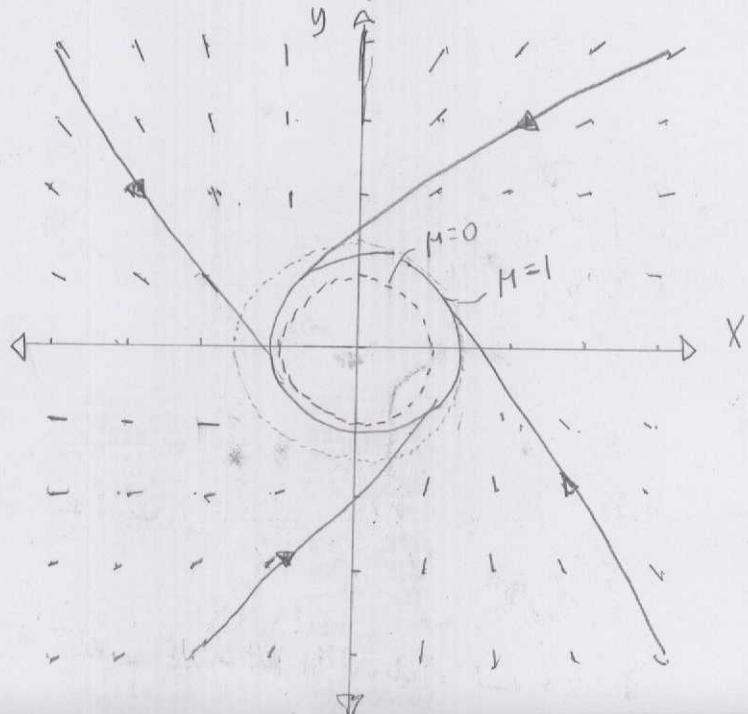
$$r(\arctan(\frac{1}{2})) = 1 + \mu \left(\frac{1}{5} \frac{1}{\sqrt{5}} + \frac{2}{5} \frac{2}{\sqrt{5}} \right) = 1 + \frac{\mu}{\sqrt{5}}$$

or

$$r(\arctan(\frac{1}{2}) + \pi) = 1 + \mu \left(\frac{1}{5} \frac{-1}{\sqrt{5}} + \frac{2}{5} \frac{-2}{\sqrt{5}} \right) = 1 - \frac{\mu}{\sqrt{5}}$$

$$\sqrt{1-\mu} < 1 - \frac{\mu}{\sqrt{5}} < r < 1 + \frac{\mu}{\sqrt{5}} < \sqrt{1+\mu}$$

c.



$$\dot{x} = Ax - r^2 x \quad 7.3.10. \quad r = \|x\| \geq A \in \mathbb{R}; \quad \lambda_{1,2} = \kappa \pm i\omega$$

$$\Delta = (\kappa^2 + \omega^2); \quad \Gamma = 2\kappa; \quad \Gamma^2 - 4\Delta < 0$$

"Unstable Spiral"

Fixed Points: $\dot{x} = 0 = Ax - r^2 x$; $x = 0$ if $x < 0$ "stable fixed point"

$$x = 0,$$

if $\kappa > 0$ "unstable spiral"

$$\dot{r} = r(1-r)[r^2 \sin^2 \theta + (r^2 \cos^2 \theta - 1)^2]$$

$$\dot{\theta} = r^2 \sin^2 \theta + (r^2 \cos^2 \theta - 1)^2$$

7.3.11. $\dot{x} = f(x)$ is a vector field on \mathbb{R}^2

Cycle graph: an invariant set containing a finite number of fixed points connected by a finite number of trajectories, all oriented clockwise or counter-clockwise.

a. Phase Portrait:

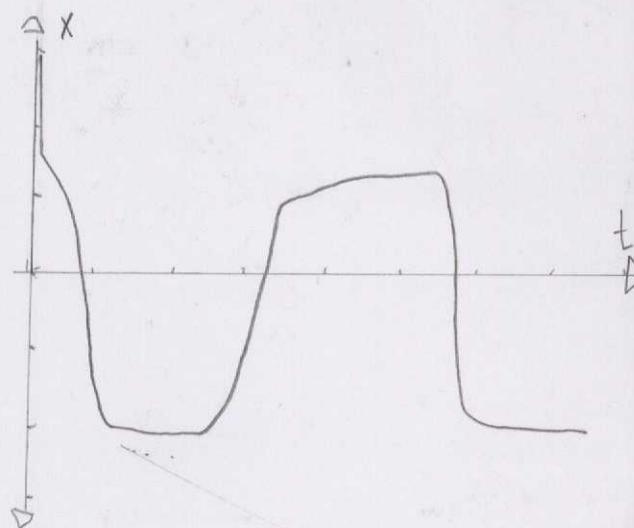
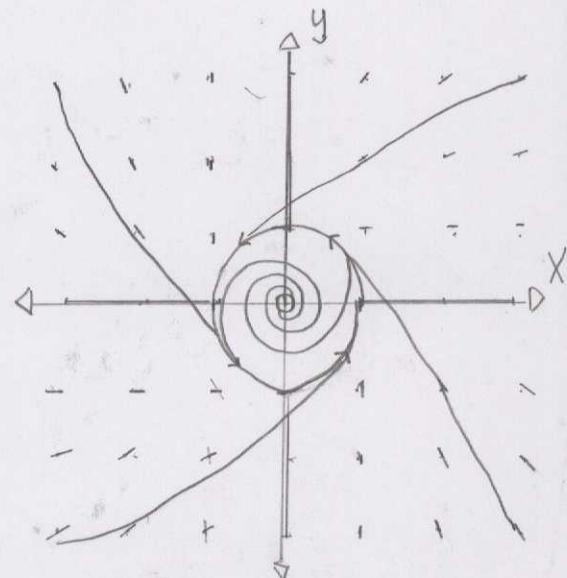
$$\begin{aligned} b. \dot{x} &= \frac{\dot{r}x}{r} - y \dot{\theta} \\ &= r(1-r)[r^2 \sin^2 \theta + (r^2 \cos^2 \theta - 1)^2] x \\ &\quad - y[r^2 \sin^2 \theta + (r^2 \cos^2 \theta - 1)^2] \end{aligned}$$

$$\text{where } r = \sqrt{x^2 + y^2} \quad \tan \theta = \frac{y}{x}$$

$$\text{The plot } \theta = \text{atan}(\frac{y}{x}) \text{ which}$$

b. Runge-Kutta performance

became absent because
of problem sets statement
"Sketch".



$$\dot{P} = P[(aR - S) - (a-1)(PR + RS + PS)] \quad E_1(P, RS) = P + R + S$$

$$\dot{R} = R[(aS - P) - (a-1)(PR + RS + PS)] \quad E_2(P, R, S) = PRS.$$

$$\dot{S} = S[(aP - R) - (a-1)(PR + RS + PS)]$$

7, 3, 12,

a. $\dot{E}_1 = (1 - E_1)(a-1)(PR + RS + PS)$; $P + R + S = 1$; $E_1 = 1$

$$\text{If } P = -1 \quad \dot{E}_1 = (1 - E_1)(a-1)(RS - PR - PS) = 0 \\ = (1 - E_1)(a-1)(PR + RS + PS) = 0$$

$$\text{If } R = -1 \quad \dot{E}_1 = (1 - E_1)(a-1)(PS - PR - RS) = 0 \\ = (1 - E_1)(a-1)(PR + RS + PS)$$

$$\text{If } S = -1 \quad \dot{E}_1 = (1 - E_1)(a-1)(PS - PR - RS) \quad \text{"Ellipsoid"} \\ = (1 - E_1)(a-1)(PR + RS + PS)$$

b. $P, R, S \geq 0 \quad \& \quad P + R + S = 1$
 $\text{If } (P = 1/2 \parallel R = 1/2 \parallel S = 1/2), \text{ then } \dot{E}_1 = 0 \quad \text{"sphere"}$

c. $\dot{E}_1 = 0 = (1 - E_1)(a-1)(PS + RS + PR)$

$$(P^*, R^*, S^*) = (P, -(P+S), -(P+R))$$

$$= (-S+R), R, -(P+R))$$

$$= (-S+P), -(S+R), S)$$

d. $\frac{dE_2}{dt} = \frac{d}{dt}(PRS) = R(S \cdot \dot{P} + P \cdot \dot{S}) + P \cdot S \cdot \dot{R}$
 $= R(S \cdot \frac{P^*}{2} + P \cdot \frac{S^*}{2}) + P \cdot S \cdot \frac{R^*}{2} = \frac{PRS(a-1)}{2} [(P-R)^2 + (R-S)^2 + (S-P)^2]$

e. $(P^*, R^*, S^*) = (1/3, 1/3, 1/3)$

$$\dot{E}_2 = \frac{(1/3)^3(a-1)}{2} [0^2 + 0^2 + 0^2] = 0$$

$$(P^*, R^*, S^*) = (3, 0, 0) ; \quad \dot{E}_2 = 0$$

f. If $a < 1$, then the model \dot{E}_2 trajectory is decreasing
else $a = 1$, then \dot{E}_2 model does not change.

g. If $a < 1$, then $\dot{E}_2 < 0$ is less than zero and trajectory to
fixed center.

$$\ddot{x} + \mu(x^2 - 1) \dot{x} + \tanh x = 0$$

$$7.4.1. \quad \dot{u} = \dot{x} = v$$

$$\text{Fixed Points: } \dot{u} = 0 = \dot{x} = v$$

$$\dot{v} = \mu(1-u^2)v - \tanh u$$

$$\begin{aligned} \dot{v} = 0 &= \mu(1-u^2)v - \tanh u \\ (u^*, v^*) &= (0, 0) \end{aligned}$$

① u and v are continuously differentiable

$$\text{② } \ddot{u} = 1; \quad \ddot{v} = \mu(1-2u\dot{u})v + \mu(1-u^2)\dot{v} - (1-\tanh^2 u) \quad \mu(1-u^2)$$

$$\text{③ } \dot{v}(-u, -v) = -\dot{v}(u, v) \text{ odd function.}$$

$$\dot{v}(-u, -v) = -\mu(1-u^2)v - \tanh(-u)$$

$$= -[\mu(1-u^2)v - \tanh(u)]$$

$$\text{④ } \dot{y}(u, v) > 0 \text{ for } u, v > 0$$

$$\dot{v}(u, v) = \underbrace{\mu(1-u^2)v}_{(+)} - \underbrace{\tanh u}_{(+)} \quad \text{unstable fixed point} \quad \Delta < 0$$

$$+ \text{ till } u^2 = 1 \quad (+)(+) ; \quad \mu(1-u^2)v > \tanh u \quad (+)$$

$$\text{⑤ } \dot{u}(-v) = \dot{u}(v) \text{ is an even function}$$

$$\dot{u}(-v) = -v = 0; \quad 0 = \dot{u}(v) \text{ at fixed point}$$

⑥ The odd function $F(v) = \int_0^v u(k) dk$ is positive
at $v=a$, and negative for $0 < v < a$, is
positive and nondecreasing for $v > 0$, and
 $F(v) \rightarrow \infty$ as $v \rightarrow \infty$

A ~~stable~~ limit cycle.

$$\ddot{x} + \mu(x^4 - 1)\dot{x} + x = 0 \quad 7.4.2. \quad f(x) = \ddot{x} + \mu(x^4 - 1)x \quad ; \quad g(x) = x$$

a. ① $f(x)$ and $g(x)$ are continuously differentiable

$$F(x) = \ddot{x} + \mu(4x^4) + \mu(x^4 - 1) \quad ; \quad \dot{g}(x) = 1$$

② $g(-x) = -g(x)$ is an odd function

$$g(-x) = -g(x) \text{ at } x=0$$

③ $f(-x) = f(x)$ is an even function

$$f(-x) = -\ddot{x} - \mu[(-x)^4 - 1]x = -f(x) \text{ at } x=0$$

④ $g(x) > 0$ for $x > 0$

$$⑤ F(x) = \int_0^x f(u) du = \int_0^x \ddot{u} + \mu(u^2 - 1) du = \frac{\ddot{x}}{2} + \mu\left(\frac{x^3}{3} - x\right)$$

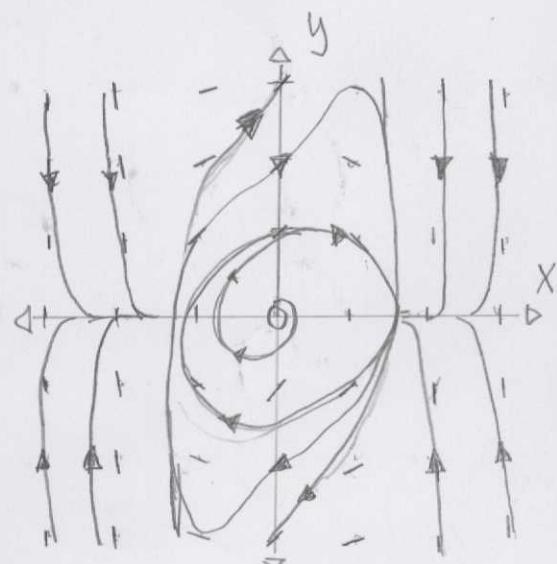
$F(x)$ has a positive root at $x = \sqrt{3}$

$F(x)$ is negative from $0 < x < \sqrt{3}$

$F(x)$ is positive and nondecreasing for $\sqrt{3} < x$
and $F(x) \rightarrow \infty$ as $x \rightarrow \infty$

b. Phase Portrait

c. If $\mu < 1$, then the function has an unstable periodic cycle in the opposite direction.



$$X_A = 2$$

$$7.5.1. \text{ Van der Pol Oscillator: } \ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

$$\mu > 1; \quad \ddot{x} + \mu(x^2 - 1)\dot{x} + x = \frac{d}{dt}\left(\ddot{x} + \mu\left(\frac{x^3}{3} - x\right) + \frac{x^2}{2}\right)$$

$$V = x = -H(x, \dot{x}) \quad ; \quad F(x) = \frac{x^3}{3} - x \quad ; \quad W = \dot{x} + \mu F(x) + \frac{x^2}{2}$$

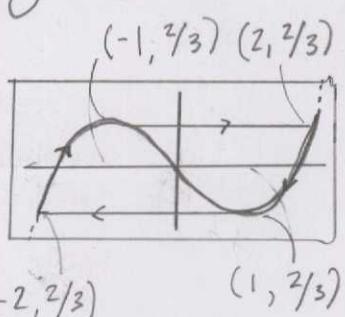
Fixed Point

$$\dot{x} = W - \mu F(x) - \frac{x^2}{2}; \quad \dot{W} = 0$$

If $y = \frac{\omega}{\mu}$, then $\dot{x} = \mu[y - F(x)]$ and $\dot{y} = 0$

Nullcline minimum: $F'(x) = x^2 - 1$; $x = \pm 1$

Nullcline intersection: $F(-1) = \frac{2}{3}$, $x = -1, 2$



7.5.2. Nullclines: $\dot{x} = 0 = y$

$$\dot{y} = 0 = -x - \mu(x^2 - 1)$$

$$(x^*, y^*) = \left(\frac{+1 \pm \sqrt{1 + 4\mu^2}}{-2\mu}, 0 \right)$$

A Liénard plane provides advantages, such as separation of timescales ($\propto \mu$) from the fixed point.

$\ddot{x} + k(x^2 - 4)\dot{x} + x = 1$ 7.5.2

$$7.5.3. \ddot{x} + k(x^2 - 4)\dot{x} + x - 1 = \frac{d}{dt} \left[\dot{x} + k \left(\frac{x^3}{3} - 4x \right) + \frac{x^2}{2} - x \right]$$

$$F(x) = \frac{x^3}{3} - 4x; \omega = \dot{x} + kF(x) + \frac{x^2}{2} - x$$

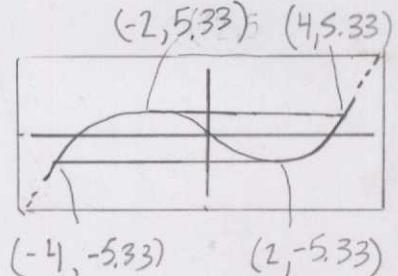
$$\dot{x} = \omega - kF(x) - \frac{x^2}{2} + x; \dot{\omega} = 0$$

$$\text{If } y = \frac{\omega}{k}, \text{ then } \dot{x} = k(y - F(x)) - \frac{x^2}{2} + x$$

$$\dot{y} = 0$$

Nullcline minimum: $F'(x) = x^2 - 4$; $x = \pm 2$

Nullcline Intersection: $F(-2) = \frac{16}{3}$; $x = 30$



$\ddot{x} + \mu f(x)\dot{x} + x = 0$ 7.5.4. $f(x) = -1$ for $|x| < 1$; $f(x) = 1$ for $|x| \geq 1$

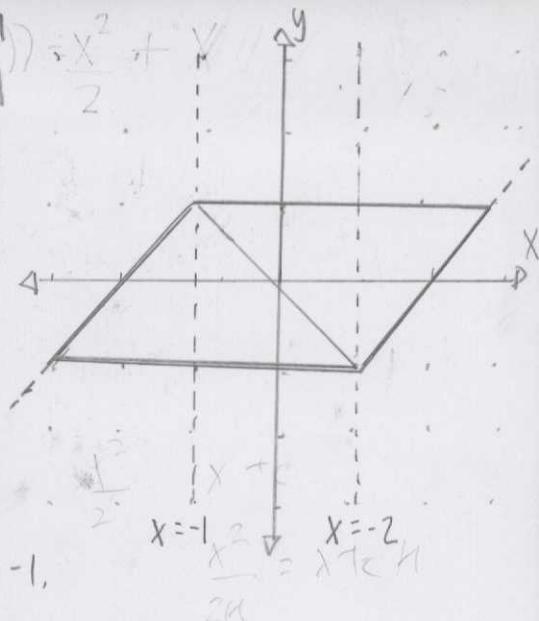
a. $\ddot{x} + \mu f(x)\dot{x} + x = \frac{d}{dt} \left[\dot{x} + \mu \int f(x) dx + \frac{x^2}{2} \right]$

$$= F(x) = \int f(x) dx; \omega = \dot{x} + \mu F(x) + \frac{x^2}{2}$$

$$\dot{x} = \omega - \mu F(x) - \frac{x^2}{2}; \dot{\omega} = 0$$

$$\text{If } y = -\frac{x^2}{2\mu}, \text{ then } \dot{x} = \mu(y - F(x)); \dot{y} = -\frac{x}{\mu}$$

$$F(x) = \int_{-\infty}^x f(x) dx = \begin{cases} x+2 & x \leq -1 \\ -x & -1 \leq x \leq 1 \\ x-2 & x \geq 1 \end{cases}$$



b. Nullclines in graph.

c. If $\mu \gg 1$, then nullcline minimum

is $y = -1$ and maximum ($y = 1$).

d. See graph about the limit

cycle about $F'(x) = -1$, $F(-1) = 1$ or -1 .

d. $|\dot{x}| \sim O(\mu) \gg 1$ and $|\dot{y}| \sim O(\mu^{-1}) \ll 1$

The period of the nullcline is $T \approx \mu \int_{-1}^2 \frac{y - F(x)}{-\mu} dx$

$$\approx \mu \int_{-1}^2 \frac{-x^3}{2\mu^2} + x dx$$

$$\approx \left[-\frac{x^4}{6\mu^2} + \frac{x^2}{2\mu^2} \right]_{-1}^2$$

$$\approx -\frac{(2^3 - 1)}{6} + \left[\frac{4}{2} + \frac{1}{2} \right] \mu$$

$$\approx -\frac{7}{6} + \frac{5}{2} \mu$$

$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$ 7.5.6.

a. Fixed Points: $\ddot{x} + \mu(x^2 - 1)\dot{x} + (x - a) = \frac{d}{dt} \left[\dot{x} + \mu F(x) + \frac{x^2}{2} - ax \right]$

$$F(x) = \frac{x^3}{3} - x ; \omega = \dot{x} + \mu F(x) + \frac{x^2}{2} - ax$$

$$\ddot{x} = \omega - \mu F(x) - \frac{x^2}{2} + ax ; \ddot{\omega} = 0$$

$$\text{If } y = -\frac{x^2}{2} + ax + \omega ; \dot{x} = \mu [y - F(x)]$$

$$\dot{y} = \frac{a - x}{\mu}$$

$$(x^*, y^*) = (a, F(x)) = (a, \frac{x^3}{3} - x)$$

b. Nullclines in the Liénard Plane.

The center seems stable not corresponding to the book.

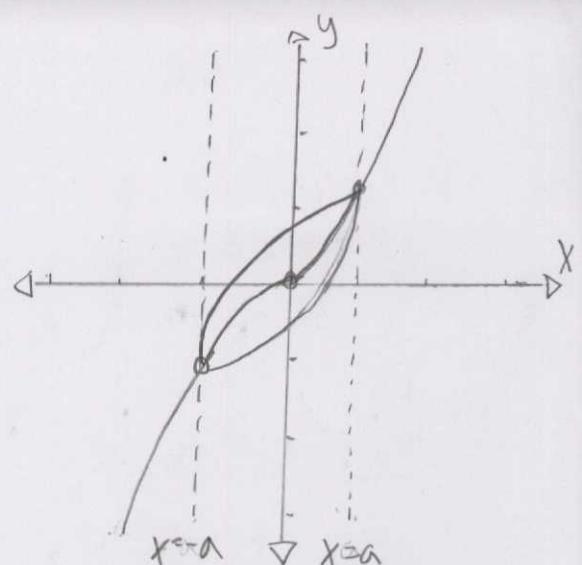
c. Nullcline Minimum: $F'(X) = X^2 - 1$

$$X = \pm 1$$

Nullcline Intersection: $F(-1) = \frac{2}{3}$

$|a| < 0$ because $\frac{dy}{dx} = 0$

d. See plot.



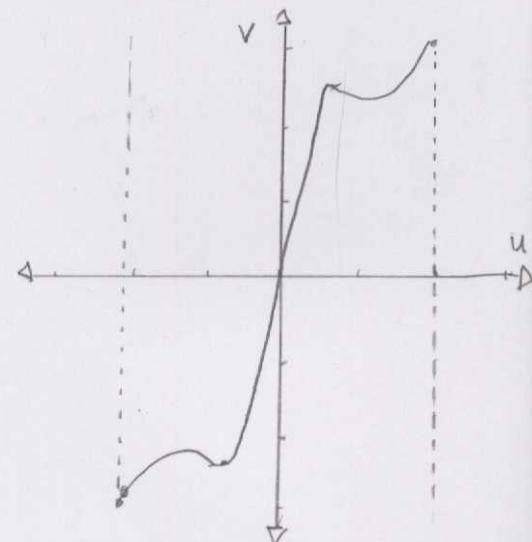
$$\dot{u} = b(v-u)(\alpha+u^2)-u$$

$$\dot{v} = c - u$$

7.5.7. a. Nullclines on graph.

$$b. |C_1| = |C_2| \neq C$$

c. A fixed point exists when c is beyond the inflection points.



$$X(t, \varepsilon) = (1 - \varepsilon^2)^{-1/2} e^{-\varepsilon t} \sin |(1 - \varepsilon^2)^{1/2} t|$$

$$X(t, \varepsilon) = \sin t - \varepsilon t \sin t + O(\varepsilon^2)$$

$$7.6.1 \quad X(t, \varepsilon) = (1 - \varepsilon^2)^{-1/2} \circ e^{-\varepsilon t} \circ \sin |(1 - \varepsilon^2)^{1/2} t|$$

$$\text{Identities: } (1+x)^a = 1 + ax + \frac{1}{2}(a-1)ax^2 + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$X(t, \varepsilon) = [1 + \frac{1}{2}\varepsilon^2 + \dots] [1 - \varepsilon t + \dots] \sin [(1 - \frac{1}{2}\varepsilon^2 + \dots)t]$$

$$= [1 - \varepsilon t + \frac{1}{2}\varepsilon^2 - \frac{\varepsilon^3 t}{2} + \dots] \sin [(1 - \frac{1}{2}\varepsilon^2 + \dots)t]$$

$$= \sin [t - O(\varepsilon^2)] - \varepsilon t \sin [t - O(\varepsilon^2)] + O(\varepsilon^2)$$

$$\cong \sin [t] - \varepsilon t \sin [t] + O(\varepsilon^2)$$

$$\overset{\circ}{X} + X + \varepsilon X = 0 \quad 7.6.2. \quad X(0) = 1; \quad \overset{\circ}{X}(0) = 0$$

$$\overset{\circ}{u} = \overset{\circ}{X} = V$$

$$\overset{\circ}{V} = -(1 + \varepsilon)X = -(1 + \varepsilon)u$$

$$\vec{x} = Ax = \begin{pmatrix} 0 & 1 \\ -(1+\varepsilon) & 0 \end{pmatrix} \begin{bmatrix} u \\ v \end{bmatrix} ; 0 = Ax = \lambda ; Ax - \lambda = 0$$

$$\begin{pmatrix} -\lambda & 1 \\ -(1+\varepsilon) & -\lambda \end{pmatrix} = 0 ; \lambda_1 = +\sqrt{1+\varepsilon} ; \lambda_2 = -\sqrt{1-\varepsilon}$$

$$\lambda_1 = +\sqrt{1+\varepsilon} ; \begin{pmatrix} -i(\sqrt{1+\varepsilon}) & 1 \\ -(1+\varepsilon) & -i(\sqrt{1+\varepsilon}) \end{pmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$-i(\sqrt{1+\varepsilon})v_{11} + v_{12} = 0 ; \vec{v}_1 = \begin{bmatrix} 1 \\ i\sqrt{1+\varepsilon} \end{bmatrix}$$

$$\lambda_2 = -i\sqrt{1+\varepsilon} ; \begin{pmatrix} i\sqrt{1+\varepsilon} & 1 \\ -(1+\varepsilon) & i\sqrt{1+\varepsilon} \end{pmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0$$

$$i\sqrt{1+\varepsilon}v_{21} + v_{22} = 0 ; \vec{v}_2 = \begin{bmatrix} 1 \\ -i\sqrt{1+\varepsilon} \end{bmatrix}$$

$$\vec{x} = \vec{v}_1 c_1 e^{-\lambda_1 t} + \vec{v}_2 c_2 e^{-\lambda_2 t}$$

$$= \begin{bmatrix} 1 \\ i\sqrt{1+\varepsilon} \end{bmatrix} c_1 e^{-\sqrt{1+\varepsilon}t} + \begin{bmatrix} 1 \\ -i\sqrt{1+\varepsilon} \end{bmatrix} c_2 e^{i\sqrt{1+\varepsilon}t}$$

$$u(t) = c_1 e^{-\sqrt{1+\varepsilon}t} + c_2 e^{i\sqrt{1+\varepsilon}t}$$

$$v(t) = \sqrt{1+\varepsilon} c_1 e^{-\sqrt{1+\varepsilon}t} - \sqrt{1+\varepsilon} c_2 e^{i\sqrt{1+\varepsilon}t}$$

$$x(0) = 0 = c_1 + c_2 ; v(x)(0) = 0 = -c_1 \sqrt{1+\varepsilon} + c_2 \sqrt{1+\varepsilon}$$

$$c_1 = c_2 ;$$

$$u(t) = e^{-\sqrt{1+\varepsilon}t} + e^{i\sqrt{1+\varepsilon}t}$$

$$v(t) = \sqrt{1+\varepsilon} e^{-\sqrt{1+\varepsilon}t} - \sqrt{1+\varepsilon} e^{i\sqrt{1+\varepsilon}t}$$

$$b. X(t, \varepsilon) = X_0 + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + O(\varepsilon^2)$$

$$0 = (X_0 + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + O(\varepsilon^2)) + \varepsilon \frac{d}{dt} [X_0 + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + O(\varepsilon^2)]$$

$$+ \varepsilon^2 \frac{d^2}{dt^2} [X_0 + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + O(\varepsilon^2)]$$

$$0 = (\overset{\circ}{X}_0 + \varepsilon X_1(t) + \varepsilon^2 X_2(t)) + \varepsilon [\overset{\circ}{\dot{X}}_0 + \varepsilon \overset{\circ}{\dot{X}}_1(t) + \varepsilon^2 \overset{\circ}{\dot{X}}_2(t)] \\ + \varepsilon^2 [\overset{\circ}{\ddot{X}}_0 + \varepsilon \overset{\circ}{\ddot{X}}_1(t) + \varepsilon^2 \overset{\circ}{\ddot{X}}_2(t)] \\ = X_0 + \varepsilon (X_1(t) + \overset{\circ}{X}_0) + \varepsilon^2 (X_2(t) + \overset{\circ}{X}_1(t) + \overset{\circ}{X}_0) + \dots$$

$$O(1) = X_0$$

$$O(\varepsilon) = X_1(t) + \overset{\circ}{X}_0$$

$$O(\varepsilon^2) = X_2(t) + \overset{\circ}{X}_1(t) + \overset{\circ}{X}_0$$

c. Secular terms consist of functions which go to infinity as t approaches infinity, because the function has many branches.

$$\overset{\circ}{X} + X = \varepsilon$$

$$7.6.3. X(0) = 1 ; \overset{\circ}{X}(0) = 0$$

$$\overset{\circ}{U} = \overset{\circ}{X} = V$$

$$\overset{\circ}{V} = \varepsilon - X = \varepsilon - U$$

$$\vec{X} \approx A'X + F(t) \Rightarrow AX + F(t) = \lambda X ; (A - \lambda)X + F(t) = 0$$

$$\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} ; \lambda_1 = +i ; \lambda_2 = -i$$

$$\lambda_1 = +i$$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ; -iV_{11} + V_{12} = 0$$

$$\vec{V}_1 = \begin{bmatrix} 1 \\ +i \end{bmatrix}$$

$$\lambda_2 = -i$$

$$\begin{bmatrix} +i & 1 \\ -1 & +i \end{bmatrix} \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ; iV_{21} + V_{22} = 0$$

$$\vec{V}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\vec{X} = \begin{bmatrix} U \\ V \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ +i \end{bmatrix} e^{-it} + C_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{it}$$

$$U(t) = C_1 e^{-it} + C_2 e^{it} = X(t)$$

$$V(t) = C_1 i e^{-it} + C_2 i e^{it}$$

$$u(0) = 1 = C_1 \sin(0) + C_2 \cos(0) + \mathcal{E} C_1 \cos(0) + i C_2 \sin(0)$$

$$\dot{u}(0) = 0 = C_1 \cos(0) + C_2 \sin(0) + i C_1 \cos(0) - i C_2 \sin(0)$$

$$1 = C_2 + \mathcal{E} ; \quad C_2 = 1 - \mathcal{E}$$

$$0 = C_1 + i C_2 \quad C_1 = -i C_2$$

$$u(t) = (1 - \mathcal{E}) e^{i t} + \mathcal{E}$$

$$v(t) = (1 - \mathcal{E}) i e^{i t}$$

$$b. \quad X(t, \mathcal{E}) = X_0(t) + \mathcal{E} X_1(t) + \mathcal{E}^2 X_2(t) + \mathcal{O}(\mathcal{E}^3)$$

$$0 = X_0 + \mathcal{E} X_1(t) + \mathcal{E}^2 X_2(t) + \frac{d}{dt} \mathcal{E} [X_0(t) + \mathcal{E} X_1(t) + \mathcal{E}^2 X_2(t)]$$

$$+ \frac{d^2}{dt^2} \mathcal{E}^2 [X_0(t) + \mathcal{E} X_1(t) + \mathcal{E}^2 X_2(t)]$$

$$= X_0 + \mathcal{E} [X_1(t) + \dot{X}_0(t)] + \mathcal{E}^2 [X_2(t) + \dot{X}_1(t) + \ddot{X}_0(t)] + \dots$$

$$0(1) = X_0$$

$$0(\mathcal{E}) = X_1(t) + \dot{X}_0(t)$$

$$0(\mathcal{E}^2) = X_2(t) + \dot{X}_1(t) + \ddot{X}_0(t)$$

c. Secular terms not present because this is a function of t , ~~which is~~ is periodic in the real-space.

$$\ddot{x} + x + \mathcal{E} h(x, \dot{x}) = 0$$

$$7.6.4. \quad h(x, \dot{x}) = x : \quad \ddot{x} + x + \mathcal{E} x = 0$$

$$\text{Averaged equation: } r' = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) \sin(\theta) d\theta \equiv \langle h \sin \theta \rangle$$

$$r\phi' = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) \cos(\theta) d\theta \equiv \langle h \cos \theta \rangle$$

$$r' = \langle h \sin \theta \rangle = \langle r \cos \theta \sin \theta \rangle = \frac{r}{2\pi} \int_0^{2\pi} \cos \theta \sin \theta d\theta = r \left[\frac{\sin^2 \theta}{2} \right]_0^{2\pi} = 0$$

$$r\phi' = \langle h \cos \theta \rangle = \langle r \cos^2 \theta \rangle = \frac{r}{2\pi} \int_0^{2\pi} \cos^2 \theta d\theta = \frac{r}{2\pi} \int_0^{2\pi} \frac{\cos(2\theta) + 1}{2} d\theta$$

$$= \frac{r}{2\pi} \left[\frac{\sin(2\theta)}{4} + \frac{1}{2} \theta \right]_0^{2\pi} = \frac{r}{2}$$

Initial Conditions: $r(t) = \sqrt{x(t)^2 + \dot{x}(0)^2}$; $\phi(t) \approx \arctan\left(\frac{\dot{x}(t)}{x(t)}\right)$

$$r(0) = \sqrt{a^2 + 0^2} = a; \phi(0) = \arctan(0) = 0$$

Amplitude/Frequency: Amplitude $\Rightarrow r(T) = \text{constant}$

$$\begin{aligned} \text{Frequency} : \omega &= \frac{d\theta}{dt} = 1 + \frac{d\phi}{dT} \frac{dT}{dt} = 1 + \varepsilon \dot{\phi} \\ &= 1 + \frac{\varepsilon r(T)}{2} \\ &= 1 + \frac{\varepsilon}{2} \end{aligned}$$

$$\begin{aligned} \text{Solution: } x_0 &= r(T) \cos(\bar{\omega} + \phi(T)) \\ &= a \cos(\bar{\omega} + \frac{1}{2}) \\ &\approx x(t, \varepsilon) \end{aligned}$$

$$h(x, \dot{x}) = \dot{x}^2 - \ddot{x} + x + \varepsilon x \dot{x}^2 = 0$$

$$\text{Averaged Equation: } r' = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) \sin(\theta) d\theta = \langle h \sin(\theta) \rangle$$

$$\begin{aligned} &= \frac{1}{2\pi} \int_0^{2\pi} r \cos(\theta) (-r \sin(\theta))^2 \sin(\theta) d\theta \\ &= -\frac{r^2}{2\pi} \int_0^{2\pi} \cos(\theta) \sin^3(\theta) d\theta = -\frac{r^2}{2\pi} \int_0^{2\pi} u^3 du \end{aligned}$$

$$= +\frac{r^3}{2\pi} \frac{\sin^2(\theta)}{4} \Big|_0^{2\pi} = 0$$

$$r\dot{\phi}' = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) \cos(\theta) d\theta = \langle h \cos(\theta) \rangle$$

$$= \frac{1}{2\pi} \int_0^{2\pi} r \cos(\theta) (-r \sin(\theta))^2 \cos(\theta) d\theta$$

$$= \frac{r^3}{2\pi} \int_0^{2\pi} \cos^2(\theta) \sin^2(\theta) d\theta = \frac{r^3}{2\pi} \int_0^{2\pi} (\cos(\theta) \sin(\theta))^2 d\theta$$

$$= \frac{r^3}{2\pi} \int_0^{2\pi} \frac{\sin^2(2\theta)}{4} d\theta = \frac{r^3}{2\pi} \int_0^{2\pi} \frac{1 - \cos(4\theta)}{8} d\theta$$

$$= \frac{r^3}{16\pi} \left[\theta - \frac{\sin(4\theta)}{4} \right]_0^{2\pi} = \frac{r^3}{16\pi} [2\pi] = \frac{r^3}{8}$$

$$\text{Initial conditions: } r(t) = \sqrt{x(t)^2 + \dot{x}(t)^2}; \phi(t) = \arctan(\dot{x}(t)/x(t)) \\ = a \quad = 0$$

Amplitude/Frequency: Amplitude: $r(T) = \text{constant}$

$$\text{Frequency: } \omega = 1 + \frac{d\phi}{dT} = 1 + \varepsilon \dot{\phi}' \\ = 1 + \frac{\varepsilon a^2}{8}; \phi(T) = \frac{\varepsilon a^2}{8} T$$

$$\text{Solution: } X_0 = r(T) \cos(\varphi + \phi(T)) \\ = a \cdot \cos(\varphi + \frac{\varepsilon a^2}{8} T)$$

$$X(t, \varepsilon) = a \cos((\frac{\varepsilon a^2}{8} + 1)t)$$

$$h(X, \ddot{X}) = \ddot{X} \ddot{X} - 7.6.6. \ddot{X} + X + X^2 \ddot{X} = 0$$

$$\text{Averaged Equations: } r' = \frac{1}{2\pi} \int_0^{2\pi} h \cos \theta d\theta \\ = -\frac{r^2}{2\pi} \int_0^{2\pi} \cos^2 \theta \sin \theta d\theta \\ = -\frac{r^2}{2\pi} \int_0^{2\pi} \frac{u^2}{2} du = \frac{-r^2}{2\pi} \frac{\cos^2(\theta)}{2} \Big|_0^{2\pi} = \frac{-r^2}{2\pi} \left[\frac{1}{2} - \frac{1}{2} \right] \\ = 0$$

$$r'\dot{\phi} = \frac{1}{2\pi} \int_0^{2\pi} h \sin \theta d\theta \\ = -\frac{r^2}{2\pi} \int_0^{2\pi} \cos \theta \sin^2 \theta d\theta = -\frac{r^2}{2\pi} \int_0^{2\pi} \frac{u^2}{2} du \\ = -\frac{r^2}{2\pi} \frac{\sin^3 \theta}{6} \Big|_0^{2\pi} = 0$$

$$\text{Initial conditions: } r(t) = \sqrt{x(t)^2 + \dot{x}(t)^2}; \phi(t) = \arctan(\dot{x}(t)/x(t)) \\ = a \quad = 0$$

Amplitude/Frequency: Amplitude: $r(T) = \text{constant}$

$$\text{Frequency: } \omega = 1 + \dot{\phi} \frac{dT}{dt} = 1 + \varepsilon \dot{\phi}' = 1; \phi(T) = 0$$

$$\phi(\tau) = \phi$$

$$(3)\phi + 1 = \phi 3 + 1 = M ; \text{ Frequency}$$

$$\frac{e^{\tau} + 1 + e^{-\tau}}{2}$$

$$T + C = (1 - e^{-\tau}) \ln \left(\frac{e^{\tau} + 1}{e^{-\tau} + 1} \right)$$

$$\int_{-1}^1 \frac{du}{1 - e^{-\tau u}} = \int_{-1}^1 \frac{du}{e^{\tau u} - 1}$$

$$\text{Amplitude/Frequency} : \text{Amplitude} : 16 \int \frac{du}{e^{2\tau u} - 1} = T + C$$

$$(T)X = (T)X : \phi = \phi : \underbrace{(T)X + (T)X}_{r(T)} = r(T) =$$

$$\phi = \left[-\theta \right] \frac{2\pi}{2\pi} = -\theta$$

Initial Conditions:

$$\int_{2\pi}^0 \left[\frac{2}{r^4} \int_{2\pi}^0 u^5 du + V dV \right] = \int_{2\pi}^0 \left[\frac{2}{r^4} \cos^6 \theta + \sin^2 \theta \right] =$$

$$= \int_{2\pi}^0 \left[\frac{2}{r^4} \int_{2\pi}^0 \cos^5 \theta \sin \theta d\theta + \sin \theta \cos \theta d\theta \right] =$$

$$= \frac{1}{2\pi} \int_{2\pi}^0 ((r^4 \cos^4 \theta - 1)(-\sin \theta) \cos \theta d\theta$$

$$\phi = \frac{1}{2\pi} \int_{2\pi}^0 h \cos \theta d\theta = \frac{1}{2\pi} \int_{2\pi}^0 (1 - X) \cos \theta d\theta$$

$$(r - r_4) = \frac{1}{r} - \frac{1}{r^5} = \frac{1}{r} (1 - \frac{1}{r^4})$$

$$= -\frac{1}{r} \left[r^4 \left(\frac{6\pi}{3} - \frac{5}{6} \left(\frac{6\pi}{3} \right) \right) - \pi \right] = \frac{2\pi}{r} \left(r^4 \left(\frac{3\pi}{3} \right) - \pi \right)$$

$$= -\frac{\theta}{2} - \frac{\cos \theta \sin \theta}{2}$$

$$= -\frac{5}{6} \left[\frac{3\theta}{8} + \frac{3 \cos \theta \sin \theta}{8} + \frac{\cos^3 \theta \sin \theta}{4} \right]$$

$$= -\frac{r}{2\pi} \left[r^4 \left(\frac{3\theta}{8} + \frac{3 \cos \theta \sin \theta}{8} + \frac{\cos^3 \theta \sin \theta}{4} \right) \right]$$

$$\text{Solution: } x_0 = r(\tau) \cos(\tau + \phi(\tau)) \\ = 0$$

$$x(t, \varepsilon) = 0$$

$$h(x, \dot{x}) = (x^4 - 1) \dot{x} \quad 7.6.7. \quad \ddot{x} + x + (x^4 - 1) \dot{x} = 0$$

$$\begin{aligned} \text{Averaged Equations: } r' &= \frac{1}{2\pi} \int_0^{2\pi} h \sin \theta d\theta = \langle h \sin \theta \rangle \\ &= \frac{1}{2\pi} \int_0^{2\pi} (x^4 - 1) \dot{x} \sin \theta d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} ([r \cos \theta]^4 - 1) (-r \sin \theta) \sin \theta d\theta \\ &= \frac{-r}{2\pi} \left[\int_0^{2\pi} r^4 \cos^4 \theta \sin^2 \theta d\theta - \int_0^{2\pi} \sin^2 \theta d\theta \right] \\ &= \frac{-r}{2\pi} \left[r^4 \int_0^{2\pi} \cos^4 \theta (1 - \cos^2 \theta) d\theta - \int_0^{2\pi} \sin^2 \theta d\theta \right] \end{aligned}$$

Reduction Formula:

$$\begin{aligned} \int \cos^n(x) dx &= \frac{n-1}{n} \int \cos^{n-2}(x) dx + \frac{\cos^{n-1}(x) \sin(x)}{n} \\ \int \sin^n(x) dx &= \frac{n-1}{n} \int \sin^{n-2}(x) dx + \frac{\cos(x) \sin^{n-1}(x)}{n} \\ &= \frac{-r}{2\pi} \left[r^4 \left(\int_0^{2\pi} \cos^4(\theta) d\theta - \int_0^{2\pi} \cos^6(\theta) d\theta \right) - \int_0^{2\pi} \sin^2(\theta) d\theta \right] \\ \int_0^{2\pi} \cos^4(\theta) d\theta &= \frac{3}{4} \int_0^{2\pi} \cos^2(\theta) d\theta + \frac{\cos^3(\theta) \sin(\theta)}{4} \\ &= \frac{-r}{2\pi} \left[\frac{3}{4} \left(\frac{\theta}{2} + \frac{\cos(\theta) \sin(\theta)}{2} \right) + \frac{\cos^3(\theta) \sin(\theta)}{4} \right] \\ \int_0^{2\pi} \cos^6(\theta) d\theta &= \frac{5}{6} \int_0^{2\pi} \cos^4(\theta) d\theta + \frac{\cos^5(\theta) \sin(\theta)}{6} \\ &= \frac{5}{6} \left[\frac{3\theta}{8} + \frac{3\cos(\theta) \sin(\theta)}{8} \right] + \frac{\cos^3(\theta) \sin(\theta)}{4} \end{aligned}$$

$$\int_0^{2\pi} \sin^2(\theta) d\theta = \left[\frac{\theta}{2} + \frac{\cos(\theta) \sin(\theta)}{2} \right]_{0}^{2\pi} = \pi$$

$$= 0.45 \text{ rad} \approx 0.39 \pi$$

$$\text{Solution: } x_0 = r(t) \cos(\tau + \phi(t)) \\ = \sqrt[4]{\frac{8}{e^{\tau/2} + 1 + C}} \cos(\tau + O(\epsilon^2))$$

$$x(t, \epsilon) = \sqrt[4]{\frac{8}{e^{\tau/2} - 1 + \frac{8}{\alpha^4}}} \cos(\tau + O(\epsilon^2))$$

$$h(x, \dot{x}) = (|x| - 1) \dot{x}$$

7.6.8 Averaged Equations:

$$r' = \frac{1}{2\pi} \int_0^{2\pi} h \sin \theta d\theta = \langle h \sin \theta \rangle$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (|x| - 1) \dot{x} \sin \theta d\theta = \frac{1}{2\pi} \int_0^{2\pi} (|r \cos \theta| - 1)(-r \sin \theta) \sin \theta d\theta$$

$$= \frac{-r}{2\pi} \left(\int_0^{2\pi} |r \cos \theta| \sin^2 \theta d\theta + \int_0^{2\pi} \sin^2 \theta d\theta \right)$$

$$= \frac{-r}{2\pi} \left(r \frac{\cos \theta}{|\cos \theta|} \int_0^{2\pi} \cos \theta \sin^2 \theta d\theta + \left(\frac{\theta}{2} - \frac{\cos(\theta)\sin(\theta)}{2} \right) \Big|_0^{2\pi} \right)$$

$$= \frac{-r}{2\pi} \left(r \frac{\cos \theta}{|\cos \theta|} \frac{\sin^3(\theta)}{3} + \frac{\theta}{2} - \frac{\cos(\theta)\sin(\theta)}{2} \right) \Big|_0^{2\pi}$$

$$= \frac{-r}{2\pi} [-\pi] = \frac{r}{2}$$

$$r'\phi' = \frac{1}{2\pi} \int_0^{2\pi} h \cos \theta d\theta = \langle h \cos \theta \rangle$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (|x| - 1) \dot{x} \cos \theta d\theta = \langle h \cos \theta \rangle$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (|r \cos \theta| - 1)(-r \sin \theta) \cos \theta d\theta$$

$$= \frac{-r}{2\pi} \left(r \int_0^{2\pi} |\cos \theta| \cos \theta \sin \theta d\theta - \int_0^{2\pi} \sin \theta \cos \theta d\theta \right)$$

$$= \frac{-r}{2\pi} \left(r \frac{\cos \theta}{|\cos \theta|} \int_0^{2\pi} \cos^2 \theta \sin \theta d\theta - \frac{\sin^2 \theta}{2} \Big|_0^{2\pi} \right)$$

$$= \frac{-r}{2\pi} \left(r \frac{\cos \theta}{|\cos \theta|} \frac{\cos^3 \theta}{3} \Big|_0^{2\pi} \right) = \frac{0}{2\pi} = 0$$

$$\text{Initial Conditions: } r(t) = \sqrt{x(t)^2 + \dot{x}(t)^2} = a$$

$$\phi(t) = \arctan(\dot{x}(t)/x(t)) = 0$$

Amplitude/Frequency: Amplitude $r(T) = a$
 Frequency $\omega(T) + \epsilon \dot{\phi} : \phi(T) = 0$

$$\text{Solution: } x_0 = r(T) \cos(\theta + T) = a \cos(t)$$

$$x(t, \epsilon) = a \cos(t)$$

$$h(x, \dot{x}) = (x^2 - 1) \dot{x}^3 \quad 7, 6, 9. \quad \ddot{x} + x + (x^2 - 1) \dot{x}^3 = 0$$

$$\text{Average Equations: } r' = \frac{1}{2\pi} \int_0^{2\pi} h(x, \dot{x}) \sin(\theta) d\theta = \langle h \sin(\theta) \rangle$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (x^2 - 1) \dot{x}^3 \sin(\theta) d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (r^2 \cos^2 \theta - 1) (-r^3) (\sin^3 \theta) \sin(\theta) d\theta$$

$$= -\frac{r^3}{2\pi} \left[\int_0^{2\pi} r^2 \cos^2 \theta \sin^4 \theta d\theta - \int_0^{2\pi} \sin^4 \theta d\theta \right]$$

Reduction Formula:

$$\int_0^{\pi} \cos^n(x) dx = \frac{n-1}{n} \int_0^{\pi} \cos^{n-2}(x) dx + \frac{\cos^{n-1}(x) \sin(x)}{n}$$

$$\int_0^{\pi} \sin^n(x) dx = \frac{n-1}{n} \int_0^{\pi} \sin^{n-2}(x) dx + \frac{\cos(x) \sin^{n-1}(x)}{n}$$

$$\int_0^{\pi} \sin^2(\theta) d\theta = \frac{\theta}{2} + \frac{\cos(\theta) \sin(\theta)}{2}$$

$$-\int_0^{\pi} \sin^6(\theta) d\theta = -\frac{5}{6} \int_0^{\pi} \sin^4(\theta) d\theta + \frac{\cos(\theta) \sin^5(\theta)}{6}$$

$$\int_0^{\pi} \sin^4(\theta) d\theta = \frac{5}{4} \int_0^{\pi} \sin^2(\theta) d\theta + \frac{\cos(\theta) \sin^3(\theta)}{4}$$

$$\int_0^{\pi} \sin^2(\theta) d\theta = \frac{\theta}{2} + \frac{\cos(\theta) \sin(\theta)}{2}$$

$$= \frac{-r^3}{2\pi} \left[r^2 \left[\frac{\theta}{2} + \frac{\cos(\theta)\sin(\theta)}{2} \right] - \frac{5}{6} \left[\frac{3}{4} \left[\frac{\theta}{2} + \frac{\cos(\theta)\sin(\theta)}{2} \right] + \frac{\cos(\theta)\sin^3(\theta)}{4} \right] \right. \\ \left. + \frac{\cos(\theta)\sin^5(\theta)}{6} \right] - \frac{3}{4} \left[\left[\frac{\theta}{2} + \frac{\cos(\theta)\sin(\theta)}{2} \right] + \frac{\cos(\theta)\sin^3(\theta)}{6} \right] \Big|_0^{2\pi}$$

$$= \frac{-r^3}{2\pi} \left[r^2 \left[\pi - \frac{15}{24}\pi \right] - \frac{3}{4}\pi \right]$$

$$= \frac{-r^3}{2\pi} \left[\frac{3}{8}\pi r^2 - \frac{3}{4}\pi \right] = \frac{r^3}{16} \left(\frac{3}{16} + r^2 \right) \pi$$

$$r\phi' = \frac{1}{2\pi} \int_0^{2\pi} h \cos(\theta) d\theta = \langle h \cos(\theta) \rangle$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (x^2 - 1) \dot{x}^3 \cos(\theta) d\theta = \frac{1}{2\pi} \int_0^{2\pi} (r^2 \cos^2 \theta - 1) (-r^3) \sin^3 \theta \cos(\theta) d\theta$$

$$= \frac{-r^3}{2\pi} \left[r^2 \int_0^{2\pi} \cos^3 \theta \sin^3 \theta d\theta - \int_0^{2\pi} \sin^3 \theta \cos \theta d\theta \right]$$

$$= \frac{-r^3}{2\pi} \left[r^2 \int_0^{2\pi} \cos(\theta) (1 - \sin^2 \theta) \sin^3 \theta d\theta - \frac{\sin^4 \theta}{4} \right]_0^{2\pi}$$

$$= \frac{-r^3}{2\pi} \left[r^2 \left[\int_0^{2\pi} \cos \theta \sin^3 \theta d\theta - \int_0^{2\pi} \cos \theta \sin^5 \theta d\theta \right] \right]$$

$$= \frac{-r^3}{2\pi} \left[r^2 \left[\frac{\sin^4 \theta}{4} - \frac{\sin^6 \theta}{6} \right] \right]_0^{2\pi} = 0$$

Initial Conditions: $r(t) = \sqrt{x(t)^2 + \dot{x}(t)^2} = a$

$$\phi(t) = \arctan(\dot{x}(t)/x(t)) = 0$$

Amplitude/Frequency: Amplitude: $\bar{r} = \sqrt{\frac{dr}{r^3(\frac{3}{16}r^2 - \frac{3}{32}\pi)}}$

$$= -16 \int \frac{dr}{r^3(r^2 - 6)}$$

$$= -16 \int \left(\frac{A}{r} + \frac{B}{r^2} + \frac{C}{r^3} + \frac{Dr^2 + E}{(r^2 - b)} \right) dr$$

$$A = -\frac{1}{3b}; B = 0; C = -\frac{1}{6}; D = \frac{1}{36}; E = 0$$

$$= -16 \int \frac{r}{36(r^2 - b)} = -\frac{1}{36} \int \frac{1}{r^2 - b} dr$$

$$= \frac{2}{9} \ln(r^2 - 6) + \frac{4}{9} \ln(|r|) - \frac{12}{9} \frac{1}{r^2} + T + C$$

Frequency: $\omega = 1 + \epsilon \phi'$; $\phi(T) = \phi_0$

Solution: $x_0 = r(T) \cos(\tau + \phi(T)) = 0$

$$\text{Solution: } x_0 = r \sqrt{6} \cos(\tau + \phi_0)$$

$$x(b, \epsilon) = \sqrt{6} \cos(\tau + \phi_0)$$

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

$$\sin \theta \cos^2 \theta = \frac{\sin \theta + \sin 3\theta}{4}$$

$$\begin{aligned} 7.6.10. \quad \sin \theta \cos^2 \theta &= \left[\frac{e^{i\theta} + e^{-i\theta}}{2} \right] \left[\frac{e^{i\theta} + e^{-i\theta}}{2} \right]^2 \\ &= \frac{1}{8} \left[e^{i\theta} - e^{-i\theta} + e^{3i\theta} - e^{-3i\theta} \right] \\ &= \frac{\sin \theta + \sin 3\theta}{4}. \end{aligned}$$

$$2\pi x_1 + x_1 = [-2r' + r - \frac{1}{4}r^3] \sin(\tau + \phi)$$

$$+ [-2r\phi'] \cos(\tau + \phi) - \frac{1}{4}r^3 \sin 3(\tau + \phi)$$

$$7.6.11 \quad x(0) = 2; \quad \dot{x}(0) = 0$$

$$r(\tau) = \sqrt{x(t)^2 + \dot{x}(t)^2} = 2; \quad \phi(\tau) = \arctan \left(\frac{\dot{x}(t)}{x(t)} \right) = 0$$

$$2\pi x_1 + x_1 = [2 - 2] \sin(\tau + \phi) + [0] \cos(\tau + \phi) - \frac{1}{2} \sin 3(\tau + \phi)$$

$$2\pi x_1 + x_1 = -\frac{1}{2} \sin 3(\tau + \phi)$$

$$\underbrace{2\pi x_1 + 2x_1}_{2} = -\sin 3(\tau + \phi)$$

Linear Equation with Constant Coefficients.

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = f(y)$$

$$a_0 \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n = 0$$

$$2\lambda^2 + 2 = 0$$

$$2(\lambda^2 + 1) = 0$$

$$\lambda_{1,2} = \pm i; R=1; \pi = C_1 \sin(y) + C_2 \cos(y)$$

Solution to a Homogeneous Differential Equation:

$$y = \sum P_{k-1}(y) e^{ky} \sin by + Q_{k-1}(y) e^{ky} \cos by$$

$$\text{Where } \lambda = k \pm bi$$

$$P_{k-1}(y) e^{ky} = C_1$$

$$Q_{k-1}(y) e^{ky} = C_2$$

Particular Solution: $2\partial_{tt}^3 X_1 + 2X_1 = -\sin 3(t+T)$

Assume $X_1 = A \sin 3t$ then $\partial_{tt}^3 X_1 = 0$

$$2A = -\sin 3(t+T)$$

$$A = -\frac{1}{2} \sin 3(t+T)$$

General Solution: $X = \text{Homogeneous} + \text{Particular}$

$$= C_1 \sin y + C_2 \cos y - \frac{\sin 3(t+T)}{2}$$

$$\langle f(\theta) \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta \quad 7.6.12$$

a. $\langle \cos k\theta \sin m\theta \rangle$ Method #1: even \times odd = odd; $\int_{-\infty}^{\infty} \text{odd} = \int_0^{\infty} \text{odd} = 0$

Method #2: Product to sum of two angles

$$\cos \phi \sin \theta = \frac{\sin(\theta+\phi) - \sin(\theta-\phi)}{2}$$

$$\int \cos k\theta \sin m\theta d\theta = \int \frac{\sin((k+m)\theta) - \sin((k-m)\theta)}{2} d\theta$$

b. $\langle \cos k\theta \cos m\theta \rangle$ Method #1: even \times even = even
odd \times odd = even

Method #2: Product to sum of Two Angles

$$\cos \theta \cos \phi = \frac{\cos(\theta-\phi) + \cos(\theta+\phi)}{2}$$

$$= \int \frac{\cos((k-m)\theta) + \cos((k+m)\theta)}{2} d\theta$$

$\neq 0$ for $k \neq m$

$$\langle \cos^2 k\theta \rangle = \langle \sin^2 k\theta \rangle = \frac{1}{2}; \text{ Method #1: even} \times \text{even} = \text{even}$$

$\int \text{even} = \text{constant}$,

Method #2: Product to Sum of Two Angles

$$\cos\theta \cos\phi = \frac{\cos(\theta-\phi) + \cos(\theta+\phi)}{2}$$

$$\int \cos^2 k\theta = \int \frac{1}{2} + \int \frac{\cos 2k\theta}{2} d\theta$$

$$= \frac{1}{2}$$

$$b, h(\theta) = \sum_{k=0}^{\infty} a_k \cos k\theta + \sum_{k=1}^{\infty} b_k \sin k\theta$$

$$\cos m\theta h(\theta) = \sum_{R=0}^{\infty} a_R \cos R\theta \cos m\theta + \sum_{R=1}^{\infty} b_R \sin R\theta \cos m\theta$$

$$\frac{\cos m\theta h(\theta)}{2\pi} = \frac{1}{2\pi} \sum_{k=0}^{2\pi} a_k \cos k\theta \cos m\theta + \frac{1}{2\pi} \sum_{R=0}^{2\pi} b_R \sin R\theta \cos m\theta$$

$$\langle h(\theta) \cos m\theta \rangle = \frac{1}{2} a_m$$

$$c, h(\theta) = \sum_{k=0}^{\infty} a_k \cos k\theta + \sum_{k=0}^{\infty} b_k \sin k\theta$$

$$\langle h(\theta) \sin k\theta \rangle = \sum_{R=0}^{2\pi} a_R \cos R\theta \sin + \sum_{R=0}^{2\pi} b_R \sin R\theta \sin k\theta$$

$$= \frac{1}{2} b_k$$

$$\langle h(\theta) \rangle = \sum_{k=0}^{2\pi} a_k \cos k\theta + \sum_{k=0}^{2\pi} b_k \sin k\theta$$

$$= a_0$$

$$\ddot{x} + x + \varepsilon x^3$$

$$7.6.13 \quad x(0) = a; \quad \dot{x}(0) = 0$$

a. $h = x^3; \quad r' = \frac{1}{2\pi} \int_0^{2\pi} h \sin \theta d\theta = \frac{1}{2\pi} \int_0^{2\pi} x^3 \sin \theta d\theta$

$$= \frac{1}{2\pi} \int_0^{2\pi} (r \cos \theta)^3 \sin \theta d\theta = \frac{r^3}{2\pi} \int_0^{2\pi} \cos^3 \theta \sin \theta d\theta$$

$$= \frac{r^3}{2\pi} \left[\frac{\cos^3 \theta}{3} \right]_0^{2\pi} = 0$$

$$T = r(T) = \text{constant.}$$

b. $r\dot{\phi} = \frac{1}{2\pi} \int_0^{2\pi} h \cos \theta d\theta - \int_0^{2\pi} (r^3 \cos \theta)^3 \cos \theta d\theta$

$$= \frac{r^3}{2\pi} \int_0^{2\pi} \cos^3 \theta \cos \theta d\theta = r^3 \int_0^{2\pi} \cos^4 \theta d\theta$$

Reduction Formula:

$$\int \cos^n \theta d\theta = \frac{n-1}{n} \int \cos^{n-2} \theta d\theta + \frac{\cos^{n-1} \theta \sin \theta}{n}$$

$$\int \cos^4 \theta d\theta = \frac{3}{4} \int \cos^2 \theta d\theta + \frac{\cos^3 \theta \sin \theta}{4}$$

$$\int \cos^2 \theta d\theta = \frac{\theta}{2} + \frac{\cos \theta \sin \theta}{2}$$

$$r\dot{\phi} = \frac{r^3}{2\pi} \left[\frac{3}{4} \left[\frac{\theta}{2} + \frac{\cos \theta \sin \theta}{2} \right] + \frac{\cos^3 \theta \sin \theta}{4} \right]_0^{2\pi}$$

$$= \frac{r^3}{2\pi} \left[\frac{3}{4} \pi \right] = \frac{3}{8} r^3$$

$$\dot{\phi} = \frac{3}{8} r^2; \quad \omega = 1 + \varepsilon \dot{\phi} = 1 + \frac{3}{8} r^2 \varepsilon = 1 + \frac{3}{8} \alpha^2 \varepsilon$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{1 + \frac{3}{8}\epsilon a^2} = 2\pi \left(1 - \frac{3}{8}\epsilon a^2 + \left(\frac{3}{8}\epsilon a^2\right)^2 + \dots\right)$$

b. Note: The book answer is a power series about kinetic energy, while the Fourier series arrived to the same answer.

$$\ddot{x} + \epsilon \dot{x} = 0$$

7.6.15.

$$a. \ddot{x} = -\sin(x); F = ma = m\ddot{x} = -kx = -\sin(x)$$

$$\approx -\left(x + \frac{1}{6}x^3\right)$$

$$= -\left(1 + \frac{1}{6}x^2\right)x$$

$$= -kx$$

$$r\phi' = \frac{1}{2\pi} \int_0^{2\pi} r \cos \theta d\theta = \frac{1}{2\pi} \int_0^{2\pi} r^3 \cos^3 \theta \cos \theta d\theta = \frac{3}{8} \left[r^3 \frac{\sin^2 \theta}{2} \right]_0^{2\pi} = 0$$

$$\omega = 1 + \epsilon \phi' = 1 + \epsilon \frac{3}{8} r^2 \approx 1 - \frac{1}{16} a^2$$

$$b. T = \frac{2\pi}{\omega} = \frac{2\pi}{1 - \frac{1}{16} a^2} = 2\pi \left(1 + \frac{1}{16} a^2 + \dots\right) \text{"Agreement!"}$$

$$\ddot{x} + \epsilon \dot{x}(x^2 - 1) + x = 0 \quad 7.6.16. \text{ Green's Theorem: } \oint_C \mathbf{v} \cdot \mathbf{n} dl = \iint_A \nabla \cdot \mathbf{v} dA$$

$$\mathbf{v} = \ddot{x} = (\ddot{x}, \dot{y})$$

$$\dot{v} = -x - \epsilon \dot{x}(x^2 - 1) = -x - \epsilon v(x^2 - 1)$$

$$\iint_A \nabla \cdot \mathbf{v} dA = \iint_A \frac{\partial v}{\partial r} (-x - \epsilon v(x^2 - 1)) dA = \iint_A -\epsilon(x^2 - 1) dA$$

$$= \int_0^{2\pi} \int_0^a -\epsilon(r^2 \cos^2 \theta - 1) r dr d\theta = \int_0^{2\pi} \left[-\frac{\epsilon}{4} r^4 \cos^2 \theta + \frac{\epsilon r^2}{2} \right]_0^a d\theta$$

$$= \int_0^{2\pi} \left[-\frac{\epsilon}{4} a^4 \cos^4 \theta + \frac{\epsilon a^2}{2} \right] d\theta = -\frac{\epsilon a^4 \pi}{4} + \epsilon a^2 \pi$$

$$\begin{aligned}
&= \frac{r}{2\pi} \int_0^{2\pi} [r(\gamma + \cos(2t)) \cos \theta \sin \theta] d\theta = \frac{\cos \theta}{3} \cos 2\phi \Big|_0^{2\pi} \\
&= \frac{r}{2\pi} \left[\int_0^{2\pi} 8 \sin \theta \cos \theta d\theta + \int_0^{2\pi} \cos(2\theta - 2\phi) \cos \theta \sin \theta d\theta \right] \\
&= \frac{r}{2\pi} \left[\int_0^{2\pi} \cos 2\theta \cos 2\phi \cos \theta \sin \theta d\theta + \int_0^{2\pi} \sin 2\theta \sin 2\phi \cos \theta \sin \theta d\theta \right] \\
&= \frac{r}{2\pi} \left[\int_0^{2\pi} (2\cos^2 \theta - 1) \cos 2\phi \cos \theta \sin \theta d\theta + \int_0^{2\pi} 2 \cos \theta \sin \theta \cos \theta \sin \theta d\theta \right] \\
&= \frac{r}{2\pi} \left[\frac{2\pi}{4} \sin 2\phi \right] = \frac{r}{4} \sin 2\phi
\end{aligned}$$

b. If $r=0$, $r\phi' = 0 = \frac{r}{2} (\gamma + \frac{\cos 2\phi}{2}) \Rightarrow \int \cos 2\phi = 28$

$$\begin{aligned}
r' &= 0 = \frac{r}{2} \sin(\arccos(-2\gamma)) \\
&= \frac{r}{2\pi} \left[\frac{2\pi}{4} \right] = \frac{r}{2} \sqrt{1 - (-2\gamma)^2} = \frac{r}{2} \sqrt{1 - 4\gamma^2}
\end{aligned}$$

When $\gamma < \frac{1}{2}$, then $r' > 0$

$\gamma \geq \frac{1}{2}$, then $r' = 0$

$\gamma > \frac{1}{2}$, then $r' \propto i$

Therefore $\gamma < \frac{1}{2}$ is a critical value.

c. $r' = \frac{dr}{dT} = \frac{r}{4} \sqrt{1 - 4\gamma^2}$ @ $r=0$, then $T = \int \frac{4 dr}{r \sqrt{1 - 4\gamma^2}}$

$$\begin{aligned}
T &= \frac{4}{\sqrt{1 - 4\gamma^2}} \ln r \\
&\quad + \frac{4}{\sqrt{1 - 4\gamma^2}} T/4
\end{aligned}$$

and $r(T) = e^T$

$$\text{where } R = \frac{\sqrt{1 - 4\gamma^2}}{4}$$

$$\int_V \mathbf{v} \cdot \mathbf{n} dL = \int_0^{2\pi} (\langle a \sin(t), -a \cos(t) - \epsilon a \sin(t)(a^2 \cos^2(t) - 1), \langle -a \sin(t), a \cos(t) \rangle dt$$

$$= \int_0^{2\pi} -\epsilon a^4 \sin(t) \cos^3(t) + \epsilon a^3 \sin(t) \cos(t)$$

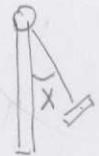
$$= \left. \frac{a^4}{4} \epsilon \cos^4(t) - \frac{a^2}{2} \epsilon \cos^2(t) \right|_0^{2\pi} = 0$$

$$\epsilon a^2 \pi \left(1 - \frac{1}{4} a^2\right) = 0 \Rightarrow a = 2$$

$$\ddot{x} + (1 + \epsilon \gamma + \epsilon \cos 2t) \sin x = 0$$

7.6.17. x = Angle Between swing and Downward Vertical

- $1 + \epsilon \gamma + \epsilon \cos 2t$ = Effect of gravity and periodic pumping



$$\dot{x} = 0; \ddot{x} = 0$$

$$a. \ddot{x} + (1 + \epsilon \gamma + \epsilon \cos 2t) x = 0$$

$$r \dot{\phi}^L = \frac{r}{2\pi} \int_0^{2\pi} ((\gamma + \cos(2t)) \cos^2 \theta d\theta$$

$$= \frac{r}{2\pi} \int_0^{2\pi} (\gamma \cos^2 \theta d\theta + \frac{r}{2\pi} \int_0^{2\pi} \cos(2\theta - 2\phi) \cos^2 \theta d\theta$$

$$= \frac{r}{2\pi} \left[\gamma \int_0^{2\pi} \cos^2 \theta d\theta + \int_0^{2\pi} (\cos 2\theta \cos 2\phi + \sin 2\theta \sin 2\phi) \cos^2 \theta d\theta \right]$$

$$= \frac{r}{2\pi} \left[\gamma \int_0^{2\pi} \cos^2 \theta d\theta + \int_0^{2\pi} \cos 2\theta \cos^2 \theta \cos 2\phi d\theta + \int_0^{2\pi} \sin 2\theta \sin 2\phi \cos^2 \theta d\theta \right]$$

$$= \frac{r}{2\pi} \left[\gamma \int_0^{2\pi} \cos^2 \theta d\theta + \int_0^{2\pi} (2 \cos^2 \theta - 1) \cos^2 \theta \cos 2\phi d\theta + \int_0^{2\pi} 2 \cos \theta \sin \theta \sin 2\phi \cos^2 \theta d\theta \right]$$

$$= \frac{r}{2\pi} \left[\gamma \int_0^{2\pi} \cos^2 \theta d\theta + \cos 2\phi \left[\frac{6\pi}{4} - \pi \right] \right] = \frac{r}{2} \left[\gamma + \frac{\cos 2\phi}{2} \right] \int_0^{2\pi} \cos^2 \theta d\theta$$

$$d. \frac{dr}{d\phi} = \frac{r'}{\phi'} = \frac{\left(\frac{r}{4}\sin(2\phi)\right)}{\left(\frac{1}{2}[8 + \frac{\cos 2\phi}{2}]\right)} = \frac{r \sin(2\phi)}{28 + \cos(2\phi)}$$

$$\int \frac{dr}{r} = \int \frac{\sin(2\phi)}{28 + \cos(2\phi)} d\phi$$

$$\ln(r) = -\frac{1}{2} \ln(28 + \cos(2\phi))$$

$$r(\phi) = \frac{c}{\sqrt{28 + \cos(2\phi)}} \text{ is a periodic function and closed orbit}$$

e. A physical interpretation of the form

$\ddot{x} + kx = 0$ is synonymous with Hooke's law, a pendulum, or swing. In this problem,

$$k = \frac{\sqrt{1-48^2}}{4} = r(8 + \cos 2\theta) \cos \theta$$

$$\ddot{x} + (a + \epsilon \cos t)x = 0$$

7.6.13 Mathieu Equation; $a \approx 1$; $T = \epsilon^2 t$

$$r \ddot{x} + [a + \epsilon \cos t]x = [x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \dots]$$

$$= \frac{r}{2\pi} \left[a \cos(2t) + \epsilon \left(x_0''(t) + \epsilon x_1''(t) + \epsilon^2 x_2''(t) \right) \right]$$

$$= [x_0 + \epsilon [x_1 + x_0]] + \epsilon [x_0'' + [x_1'' + x_0'']] + \dots$$

$$+ \epsilon \left[x_0''(t) + \alpha x_1(t) + \cos t x_0(t) \right]$$

$$O(\epsilon) = x_0'' + \cos t x_0 + \epsilon^2 [x_1''(t) + x_2(t) + \cos t x_1(t)]$$

$$O(\epsilon^2) = x_2 + x_1 + x_0 + \epsilon^3$$

$$O(1); \ddot{x}_0(t) + a x_0(t) = 0$$

$$O(\epsilon); \ddot{x}_1(t) + a x_1(t) + \cos(t) x_0(t) = 0$$

$$O(\epsilon^2); \ddot{x}_2(t) + a x_2(t) + \cos(t) x_1(t) = 0$$

$$\ddot{x} + x + \varepsilon x^3 = 0 \quad 7.6.19. \quad x(0) = a; \quad \dot{x}(0) = 0; \quad \boxed{\text{Poincare-Lindstedt Method}}$$

$$\begin{aligned} a. \quad \tau &= \omega t; \quad \frac{d^2x}{dt^2} + x + \varepsilon x^3 = \omega \frac{d^2x}{d\tau^2} + x + \varepsilon x^3 \\ &= \omega x'' + x + \varepsilon x^3 \\ &= 0 \end{aligned}$$

$$b. \quad x(\tau, \varepsilon) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + O(\varepsilon^3)$$

$$\omega = 1 + \varepsilon w_1 + \varepsilon^2 w_2 + O(\varepsilon^3)$$

$$(1 + \varepsilon w_1 + \varepsilon^2 w_2)^2 (x_0''(\tau) + \varepsilon x_1''(\tau) + \varepsilon^2 x_2''(\tau))$$

$$+ (x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau)) + (x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau))^3 = 0$$

$$(1 + 2\varepsilon w_1 + 2\varepsilon^2 w_2 + \dots) (x_0''(\tau) + \varepsilon x_1''(\tau) + \varepsilon^2 x_2''(\tau) + \dots)$$

$$+ (x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \dots) + \varepsilon (x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon x_2(\tau))^3 = 0$$

$$O(1) = x_0''(\tau) + x_0(\tau) = 0$$

$$O(\varepsilon) = x_1''(\tau) + 2w_1 x_0''(\tau) + x_1(\tau) + \varepsilon x_0^3(\tau) = 0$$

$$c. \quad x_0(0) = a; \quad \dot{x}_0(0) = 0; \quad x_K(0) = \dot{x}_K(0) = 0$$

From the blurb, $x(0) = a; \quad \dot{x}(0) = 0; \quad \ddot{x}_0 = (a); \quad \dot{x}_0(0) = 0$

$$d. \quad x_0''(\tau) + x_0(\tau) = 0; \quad x_0 = a \cos(\tau)$$

$$e. \quad x_1''(\tau) + x_1(\tau) = -2w_1 x_0''(\tau) - x_0^3(\tau) = -\frac{2a^3 \cos^3(\tau)}{2} + C$$

$$= +2aw_1 \cos(\tau) - a^3 \cos^3(\tau) = 2aw_1 \cos(\tau)$$

$$= 2aw_1 \cos(\tau) - a^3 \left[\frac{1}{4} (3 \cos(\tau) + \cos(3\tau)) \right]$$

$$= (2aw_1 - \frac{3}{4}a^3)\cos(\tau) - \frac{a^3}{4}\cos(3\tau) ; w_1 = a^2$$

$$F. X_1''(\tau) + X_1(\tau) = (2aw_1 - \frac{3}{4}a^3)\cos(\tau) - \frac{a^3}{4}\cos(3\tau)$$

$$X_1''(\tau) + X_1(\tau) = -\frac{1}{4}a^3\cos(3\tau) \quad 3aw_1\cos(\tau) - 3a^3\cos(\tau)$$

$$4X_1''(\tau) + 4X_1(\tau) = -a^3\cos(3\tau)$$

Linear Equation of Constant Coefficients. "Homogeneous"

$$a_0\lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n = 0$$

$$4(\lambda^2 + 1) = 0 ; \lambda^2 + 1 = 0 ; \lambda_{1,2} = \pm i$$

Solution of Homogeneous Equations

$$t = \sum P_{k-1}(t) e^{\lambda t} \sin \beta t + Q_{k-1}(t) e^{\lambda t} \cos \beta t$$

$$\lambda = \kappa \pm \beta i ; X = C_1 \sin(t) + C_2 \cos(t)$$

Method of Undetermined Coefficients.

$$X_1 = t^5 e^{\kappa t} (R_m(t) \cos \beta t + T_m(t) \sin \beta t) = -a^3 \cos(3t)$$

Lagrange $s=0 ; \kappa=0 ; \beta=3$ and then -o parameters

$$X = B \sin(3t) + A \cos(3t)$$

$$X'' = -9B \sin(3t) - 9A \cos(3t)$$

$$-32B \sin(3t) - 32A \cos(3t) = -a^3 \cos(3t)$$

$$\text{From } A = \frac{a^3}{32} ; B = 0 ; X = \frac{a^3}{32} \cos(3t)$$

General Solution

$$X = \frac{a^3}{32} \cos(3t) + C_1 \sin(t) + C_2 \cos(t)$$

$$X(0) = \frac{a^3}{32} + C_2 = 0 ;$$

$$= \frac{a^3}{32} \cos(3t) \sin(y) - (a^3 - 3a^3) \cos(t) \sin(y)$$

$$X = \frac{a^3}{32} \cos(3t) - \frac{a^3}{32} \cos(t)$$

$$X(t, \varepsilon) = a \cos t + \varepsilon a^3 \left[-\frac{3}{8} t \sin t + \frac{1}{32} (\cos 3t - \cos t) \right] + O(\varepsilon^2)$$

$$\text{7.6.20, } \ddot{X} + X + \varepsilon X^3 = (\ddot{X}_0 + \varepsilon \ddot{X}_1 + \varepsilon \ddot{X}_2) + (\dot{X}_0 + \varepsilon \dot{X}_1 + \varepsilon \dot{X}_2) \\ + \varepsilon (X_0 + \varepsilon X_1 + \varepsilon X_2)$$

$$C = \left(\ddot{X}_0 + X_0 \right) + \varepsilon \left(\ddot{X}_1 + X_1 + X_0 \right)$$

$$+ \varepsilon^2 \left(\ddot{X}_2 + X_2 + 3X_0^2 X_1 \right) + O(\varepsilon^3)$$

$$O(1); \ddot{X}_0 + X_0 = -\frac{a^3 \cos(3t) \sin(y)}{4} + 2aw \cos(t) \cos(y)$$

$$O(\varepsilon); \ddot{X}_1 + X_1 + X_0^3 = 0 \quad -\frac{3a^3 \cos(t) \cos(y)}{4}$$

$$O(\varepsilon^2); \ddot{X}_2 + X_2 + 3X_0^2 X_1 = 0$$

$$x_2(y) = C(y) dy = -\frac{a \cos(3t) \cos(y)}{4} - \frac{(3a^3 - 3aw)(t - \pi/3)}{4} + C_2$$

Solving for X_0 ; $\ddot{X}_0 + X_0 = 0 \Rightarrow X_0 = a \cos(t)$

Solving for X_1 ; $\ddot{X}_1 + X_1 = -a^3 \cos^3(t)$

Linear Equation of constant coefficients

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \dots + a_{n-1} t' + a_n t = f(t)$$

Solving the Homogeneous Equation

$$a_0 \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda' + a_n \lambda = 0$$

$$(\lambda^2 + 1) = 0 \Rightarrow \lambda_{1,2} = \pm i$$

Solution of a Homogeneous Equation; Summand

$$t = \sum P_{k-1}(t) e^{\lambda t} \sin \beta t + Q_{k-1}(t) e^{\lambda t} \cos \beta t$$

$$\text{where } \lambda = \alpha + \beta i \Rightarrow \alpha = 0 \Rightarrow \beta = 1 \Rightarrow k = 1$$

$$\text{so } X = A \sin(t) + B \cos(t)$$

Method for Undetermined Coefficients: Particular Solution

$$X_i = t^s e^{\lambda t} (R_m(t) \cos \beta t + T_m(t) \sin \beta t)$$

If $S=R=1$, $X=t(B\sin t + A\cos t)$

$$\ddot{X} = (-Bt - 2A)\sin t + (2B - At)\cos t$$

$$\ddot{X}_1 + X_1 = 2B\cos(t) - 2A\sin(t) = -a^3 \cos t$$

$$2B = -a^3 ; A = 0$$

$$-2A = 0 ; B = -\frac{a^3}{2}$$

$$X_1 = -\frac{a^3 t \sin t}{2}$$

[General Solution = Particular + Homogeneous Equation]

$$X(t) = -\frac{a^3 t \sin t}{2} + C_1 \sin t + C_2 \cos t$$

With initial conditions: $X(0) = a^3 ; \dot{X}(0) = 0$

$$X_1(t) = a^3 \cos t + \frac{a^3 t}{2} - 3a^2 \cos t$$

$$\text{Solving for } X_2: \ddot{X}_2 + X_2 = -3X_1^2$$

[Linear Equation with constant coefficients]

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \dots + a_{n-1} t^1 + a_n t^0 = f(t)$$

[Solving the Homogeneous Equation]

$$a_0 \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda^1 + a_n \lambda^0 = 0$$

$$(\lambda^2 + 1) = 0 ; \lambda_{1,2} = \pm i ; K = 1$$

[Solution of a Homogeneous Equation: Summand]

$$t = \sum P_{K-1}(t) e^{kt} \sin \beta t + Q_{K-1}(t) e^{kt} \cos \beta t$$

Where $\lambda = k + \beta i ; K = 0 ; \beta = 1 ; K = 1$

$$\text{So } X = A \sin(t) + B \cos(t)$$

[Method for Undetermined Coefficients: Particular Solution]

$$X_p = t^s e^{xt} (R_m(t) \cos \beta t + T_m(t) \sin \beta t)$$

$$= t (B \sin t + A \cos t)$$

$$0 = X + (1 - \frac{1}{2}X)M^2 + \frac{1}{2}MX$$

① Define a new time $\tau = \omega t$

7.6.21. [Duality - Linds' Method]

frequency depends on amplitude.

so why the Duffing oscillator has a

This solution is not an exact answer

$$= a \cos t + \epsilon a^3 \left[\frac{3}{2} t \cos(t) + \frac{3}{2} t \cos 3t - 3 \cos t \right]$$

$$(7) \cos\left(\frac{\omega}{\sqrt{a^2 + 3\epsilon}} t\right) + \left[(a + 3\epsilon a^2 t) \cos(t) + \frac{1}{4} [3 \cos 3t + \cos 3t] \right] =$$

$$X_1(t) = -3 \epsilon a^2 t \cos(t) + (a + 3\epsilon a^2 t) \cos(t)$$

$$A = 0$$

$$\dot{X}_1(t) = +9 \epsilon a^3 t \cos^2(t) \sin(t) + A \cos(t) - B \sin(t)$$

$$\text{With initial conditions; } B = a + 3\epsilon a^2 t$$

$$X_1(t) = -3 \epsilon a^2 t \cos^3(t) + A \sin(t) + B \cos(t)$$

General solution = Particular + Homogeneous

$$X_2 = -3 \epsilon a^2 t \cos^3(t)$$

$$B = -3 \epsilon a^2 \cos^2(t)$$

$$2B = -3 \epsilon a^3 \cos^2(t); A = 0$$

$$X_1^2 + X_2^2 = 2B \cos(t) - 2A \sin(t) = -3 \epsilon a^3 \cos^3(t)$$

$$X = (-Bt - A) \sin t + (2B - At) \cos t$$

$$X_1(0) = 0 ; \dot{X}_1(0) = 0 ; \frac{3}{4} + \frac{\alpha^2}{4} \text{Eqn}(3t)$$

$$\ddot{X}_1 + X_1 = 0 = 2\omega t, \alpha = 3 \Rightarrow \omega_1 = 0, \lambda = 0$$

$$\ddot{X}_1 + X_1 = \frac{\pm \alpha^3}{4} [3\sin(t) - \sin(3t)] \quad s=1$$

Solving the Linear Equation with constant coefficients: Summand

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \dots + a_{n-1} t^1 + a_n t^0 = f(t)$$

Solving the Homogeneous Equation: \ddot{X}_1

$$a_0 \lambda^{(n)} + a_1 \lambda^{(n-1)} + \dots + a_{n-1} \lambda^1 + a_n \lambda^0 = 0$$

$$(\lambda^2 + 1) = 0 ; \lambda = \pm i \rightarrow K \text{ Biost}$$

Solution to a Homogeneous Equation: Summand

$$t = \sum P_{k=1}^{\infty} e^{\lambda k t} \sin \beta t + Q_{k=1}^{\infty} e^{\lambda k t} \cos \beta t$$

$$\text{where } \lambda = \kappa + \beta i \Rightarrow \kappa = 0; \beta = 1; K = 1$$

$$\text{so } X = A \sin(t) + B \cos(t)$$

Solving the Particular Equation: Method for Undetermined Coefficients.

$$X_p = t e^{\lambda k t} (R_m(t) \cos \beta t + T_m(t) \sin \beta t)$$

$$X_p = t [A_2 \cos t + B_2 \sin t]$$

$$\ddot{X}_p = (-B_2 - 2A_2) \sin(t) + (2B_2 - A_2) \cos(t)$$

$$\ddot{X}_p + X_p = 2B_2 \cos(t) - 2A_2 \sin(t) = -\frac{\alpha^3}{4} [3\sin(t) - \sin(3t)]$$

$$2B_2 = 0 ; -2A_2 = \frac{\alpha^3}{8} \left[1 - \frac{2\sin(t)\sin(3t)}{\cos(2t)-1} \right]$$

General Solution = Particular + Homogeneous

$$X_p(t) = A_2 \sin(t) + B_2 \cos(t) - \frac{\alpha^3}{4} \sin(3t)$$

With initial conditions $\ddot{X}_p(0) = 0 ; \dot{X}_p(0) = 0$

$$A_2 = -\frac{\alpha^3}{4} ; B_2 = 0$$

$$X_p(t) = -\frac{\alpha^3}{4} [3\sin(t) + \sin(3t)]$$

$$[(7w_1 \cos(t) - \frac{3}{4} \sin(t)) + i(7w_1 \sin(t) - \frac{3}{4} \cos(t))] = a[2w_1 \cos(t)] + b[3\sin(t)]$$

$$\text{Linear System} = 2w_1 \alpha \cos(t) - (1 - \alpha^2) \sin(t) \quad \alpha \sin(t)$$

$$x_1'' + x_1' = -2w_1 x_0'' + (1 - \alpha^2)x_0'$$

Solving for x_1' :

$$x_0''(t) = \alpha \cos(t)$$

$$0 = (0)x_1' + x_1'' \quad ; \quad 0 = (0)x_1' + x_0'' \quad ; \quad 0 = x_1' + x_0''$$

$$(x_0'' + w_1 x_1') (x_0'' - 1) +$$

$$x_0'' - 2x_0'' x_1' - 2x_0'' x_1'' - 2x_1' w_1^2 - 2x_1'' w_1^2 - x_1'' (2w_1^2 + 2w_1^2) = x_1' + x_1''$$

$$0 = x_1' + x_1'' + x_1' - x_1'' - x_0'' x_1'' + x_0'' x_1' +$$

$$0 = 2x_0'' w_1^2 + 2x_1' w_1^2 + 2x_1'' w_1^2 + x_0'' x_1'' + x_0'' x_1' + 0$$

$$0 = 2w_1 x_0'' + x_0'' + x_1' + x_1'' = 0$$

$$0 = x_1' + x_1'' \quad (1) \quad 0$$

$$(2x_1' + x_1'' + x_0'') + (2x_1' + x_1'' + x_0'') +$$

$$+ (1 - \frac{1}{2}[x_1' + x_1'' + x_0'']) (2w_1^2 + M_1^2 + M_2^2 + \dots + 3 +$$

$$+ \dots + 2x_1' + x_1'' + x_0'') = (2w_1^2 + M_1^2 + M_2^2 + \dots + 3 +$$

$$w_1^2 x_1'' + w_1 x_1' (1 - x_0'')$$

③ Solve the perturbation equation

$$\dots + 2w_1^2 + M_1^2 + M_2^2 + \dots = M$$

② Assign a new $x = (3w_1 x_0'') \alpha$

Solving for X_2 :

$$\ddot{X}_2 + X_2 = -(w_1^2 + 2w_2) \ddot{X}_0 - 2\dot{X}_1 w_1^2 - 2X_0 \dot{X}_0 \dot{X}_1 + (1-X_0^2)(\dot{X}_1 + w_1 \ddot{X}_0)$$

$$= +2w_2 a \cos(t) + 2a^2 \cos(t) \sin(t) \left[-\frac{1}{4} a^3 [3 \sin(t) + \sin(3t)] \right]$$

$$+ \frac{3}{4} (3 \sin(t) \cos(3t) + \sin(3t) \cos(t)) (1 - a^2 \cos^2(t))$$

$$= \frac{1}{4} [\cos(t) (8aw_2 - 2a^5 \sin(t) (3 \sin(t) + \sin(3t)))]$$

$$+ (3a^2 \cos^2(t) - 3) (3 \sin(t) \cos(3t)) + \sin(3t) \cos(t)$$

Linear Equation of Constant Coefficients

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \dots + a_{n-1} t^1 + a_n t^0 = f(t)$$

Solving the Homogeneous Equation

$$a_0 \lambda^{(n)} + a_1 \lambda^{(n-1)} + \dots + a_{n-1} \lambda^1 + a_n \lambda^0 = 0$$

$$4(\lambda^2 + 1) = 0 \Rightarrow \lambda_{1,2} = \pm i \Rightarrow k = 1$$

Solution to a Homogeneous Equation: Summand

$$t = \sum_{n=1}^{\infty} R_n(t) e^{kt} \sin \beta t + Q_{k1}(t) e^{kt} \cos \beta t$$

$$X = A_1 \cos t + B_1 \sin t \quad X(0) = 0 \quad \dot{X}(0) = 0$$

Lagrange's Method of Variation of Parameters

System of Equations: $f(t) X_1 + C_1'(t) X_2 = 0$

$$X_2 = t[A_2 \cos t + B_2 \sin t] \quad C_1'(t) X_1 + C_1(t) X_2^2 = \frac{f(t)}{a_0}$$

$$\ddot{X}_2 = (-2B_2 - 2A_2) \sin t \quad \text{where } X_1 = \cos(t) \text{ and } X_2 = \sin(t)$$

$$\ddot{X}_2 + X_2 = 2B_2 \cos t - 2A_2 \sin t \quad X_1' = -\sin(t) \quad X_2' = \cos(t) \quad + (3a^2 \cos^2(t) - 3)(3 \sin t \cos 3t)$$

$$a_0 X'' = 2$$

$$F(t) = \frac{1}{2} [\cos(t) (8aw_2 - 2a^5 \sin(t) (3 \sin(t) + \sin(3t)))]$$

$$2B_2 = \frac{1}{4} + 2(3a^2 \cos^2(t) - 3)(3 \sin(t) \cos(3t)) + \sin(3t) \cos(t)$$

$$B_2 = 0 \quad A_2 = -\frac{1}{8} [\tan(t) ((8aw_2 - 2a^5 \sin(t) (3 \sin(t) + \sin(3t)))] + (3a \cos t - 3)(3 \cos 3t) + 3 \sin 3t \cot t$$

Finding $C'(t)$, $C_1'(t)$ by Cramers Rule.

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = (\cos(t) - 2a^5 \sin(t))(\sin(3t) + 3\sin(t))$$

$$W_1 = \begin{vmatrix} X_1(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = \begin{vmatrix} 0 & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = 0$$

$$\frac{\sin(t)}{\cos(t)} = \frac{-\sin(t)}{\cos(t)} \left[\frac{\cos(t)[8aw_2 - 2a^5 \sin(t)(\sin(3t) + 3\sin(t))] + 2\cos(t)\sin(3t) + 6[3a^2 \cos^2(t) - 3]\sin(t)\cos(3t)}{2} \right]$$

$$W_2 = \begin{vmatrix} \cos(t) & 0 \\ -\sin(t) & \frac{\cos(t)[8aw_2 - 2a^5 \sin(t)(\sin(3t) + 3\sin(t))] + 2\cos(t)\sin(3t) + 6[3a^2 \cos^2(t) - 3]\sin(t)\cos(3t)}{2} \end{vmatrix}$$

$$= \cos(t) \left[\frac{\cos(t)[8aw_2 - 2a^5 \sin(t)(\sin(3t) + 3\sin(t))] + 2\cos(t)\sin(3t) + 6[3a^2 \cos^2(t) - 3]\sin(t)\cos(3t)}{2} \right]$$

$$C'(t) = \frac{W_1}{W} = W_1 \quad ; \quad C_1'(t) = \frac{W_2}{W} = W_2$$

$$C(t) = \int C'(t) dt = \dots ; \quad C_1(t) = \int C_1'(t) dt = \dots$$

$$\text{Solving for } w_2: \quad \ddot{x}_2 + x_2 = (4w_2 + \frac{1}{4}) \cos(t) + \dots$$

$$w_2 = -\frac{1}{16}$$

$$\text{Solving for } w: \quad w = (1 + \varepsilon w_1 + \varepsilon^2 w_2 + \dots)$$

$$= (1 - \frac{1}{16} \varepsilon^2 + \dots)$$

$$\ddot{x} + x + \varepsilon x^2 = 0 \quad 7.b.22. \quad x(0) = a; \quad \dot{x}(0) = 0$$

① Define a new time $\tau = wt$

$$\omega^2 \ddot{x} + \omega x + \varepsilon x^2 = 0$$

$$\lambda^2 + 1 = 0 ; \lambda_{1,2} = \pm i ; K=S=1 ; \lambda = \kappa + \beta i$$

Solution to a Homogeneous Equations: Summand

$$t = \sum R_{n-1}(t) e^{xt} \sin \beta t + Q_{k-1}(t) e^{xt} \cos \beta t$$

$$X(t) = A_1 \sin(t) + B_1 \cos(t)$$

Solving the Particular Equation: Method for Undetermined Coefficients

$$X_1 = t^s e^{xt} (R_m(t) \cos \beta t + T_m(t) \sin \beta t)$$

$$X_1 = t [A_2 \cos t + B_2 \sin t]$$

$$\dot{X}_1 = (-B_2 - 2A_2) \sin(t) + (2B_2 - A_2 t) \cos(t)$$

$$\ddot{X}_1 + X_1 = 2B_2 \cos(t) - 2A_2 \sin(t) = a \cos(t) (w_1 + w_2) + a^2 \cos^2(t)$$

$$\text{Initial conditions: } X(0) = a \Rightarrow \dot{X}(0) = 0$$

$$2B_2 - 2B_2 = a(w_1 + w_2) + a^2 \Rightarrow -2A_2 = 0$$

$$B_2 = \frac{a}{2}(w_1 + w_2) + \frac{a^2}{2} ; A_2 = 0$$

General Solution = Particular + Homogeneous

$$X_1(t) = [a(w_1 + w_2) + a^2] \cos(t) t + A_1 \cos t + B_2 \sin t$$

$$\text{Initial conditions: } X(0) = a ; \dot{X}(0) = 0$$

$$X_1(0) = a = A_1 \quad ; \quad A_1 = a \quad ; \quad A_1 = a(1 - (w_1 + w_2)) - a^2$$

$$\dot{X}_1(0) = 0 = B_2 \quad ; \quad B_2 = 0$$

$$X_1(t) = [a(w_1 + w_2) + a^2 + a] \cos(t) (w_2) - a^2 \cos(t)$$

⑥ Solving for X_2 :

$$\ddot{X}_2 + X_2 = -2[X_0 X_1 + X_0 w_1 w_2 + \dot{X}_1 (w_1 + w_2)] - X_0 (w_1^2 + w_2^2) - X_1 (w_1 + w_2)$$

$$= -2[\alpha \cos^2(t) [(w_1 + w_2) + a + 1] + \alpha w_1 w_2 \cos(t)]$$

$$+ \alpha [(w_1 + w_2) + a + 1] \cos(t) (w_1 + w_2)$$

$$+ \alpha \cos(t) (w_1^2 + w_2^2) - \alpha [(w_1 + w_2) + a + 1] \cos(t) (w_1 + w_2)$$

$$= \alpha \cos(t) [2\alpha \cos(t) (a + w_1 + w_2 + 1) - (w_1 + w_2)(3\alpha + 2w_1 + 2w_2 + 3)]$$

$$+ (3\alpha + 2w_1 + 2w_2 + 3) \cos(t) (w_1 + w_2)$$

② Assign a new perturbation:

$$X(\tau, \varepsilon) = X_0(\tau) + \varepsilon X_1(\tau) + \varepsilon^2 X_2(\tau) + \dots$$

$$\omega = 1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \dots$$

③ Solve the perturbation equation:

$$\omega^2 X'' + \omega X + \varepsilon X^2 = 0$$

$$= (1 + \varepsilon \omega_1 + \varepsilon \omega_2)^2 (X_0'' + \varepsilon X_1'' + \varepsilon^2 X_2'') + (1 + \varepsilon \omega_1 + \varepsilon \omega_2)(X_0 + \varepsilon X_1 + \varepsilon^2 X_2)$$
$$+ (\varepsilon(X_0 + \varepsilon X_1 + \varepsilon^2 X_2))^2$$

$$O(1): \ddot{X}_0 + X_0 = 0$$

$$O(\varepsilon): \ddot{X}_0^2 + X_0(\omega_1 + \omega_2) + 2\dot{X}_0(\omega_1 + \omega_2) + X_1 + \dot{X}_1 = 0$$

$$\ddot{X}_1 + X_1 = -2\dot{X}_0(\omega_1 + \omega_2) - X_0(\omega_1 + \omega_2) + X_0^2$$

$$O(\varepsilon^2): \ddot{X}_2 + X_2 = -2X_0X_1 - \dot{X}_0\omega_1^2 - 2\dot{X}_0\omega_1\omega_2$$
$$- \dot{X}_0\omega_2^2 - X_1(\omega_1 + \omega_2) - 2\dot{X}_1(\omega_1 + \omega_2)$$
$$= -2[X_0X_1 + X_0\omega_1\omega_2 + \dot{X}_1(\omega_1 + \omega_2)]$$
$$- \dot{X}_0(\omega_1^2 + \omega_2^2) - X_1(\omega_1 + \omega_2)$$

④ Solving for X_0 :

$$\ddot{X}_0 + X_0 = 0 ; \quad X(0) = a ; \quad \dot{X}(0) = 0$$

$$X_0(t) = a \cos(t)$$

⑤ Solving for X_1 :

$$\ddot{X}_1 + X_1 = 2a \cos(t)(\omega_1 + \omega_2) - a \cos(t)(\omega_1 + \omega_2) + a^2 \cos^2(t)$$
$$= a \cos(t)(\omega_1 + \omega_2) + a^2 \cos^2(t)$$

Linear Equation of Constant Coefficients

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \dots + a_{n-1} t^1 + a_n t^0 = f(t)$$

Solving the Homogeneous Equation

$$a_0 \lambda^{(n)} + a_1 \lambda^{(n-1)} + \dots + a_{n-1} \lambda^1 + a_n \lambda^0 = 0$$

Linear Equation Homogeneous Coefficients

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \dots + a_{n-1} t^1 + a_n t^0 = f(t)$$

Solving the Homogeneous Equation

$$a_0 \lambda^{(n)} + a_1 \lambda^{(n-1)} + \dots + a_{n-1} \lambda^1 + a_n \lambda^0 = 0$$

$$(\lambda^2 + 1) = 0; \lambda_{1,2} = \pm i; K = S = 1$$

Solution to the Homogeneous Equation: Summand

$$t = \sum R_m(t) e^{xt} \sin \beta t + Q_k(t) e^{xt} \cos \beta t$$

$$X(t) = A_1 \sin(t) + B_2 \cos(t)$$

Solving the Particular Equation: Method for Undetermined Coefficients.

$$X_1 = t \cdot e^{xt} \cdot [R_m(t) \cos \beta t + T_m(t) \sin \beta t]$$

$$X_2 = t [A_2 \cos t + B_2 \sin t]$$

$$\ddot{X}_2 = (-B_2 t - 2A_2) \sin t + (2B_2 - A_2 t) \cos t$$

$$\ddot{X}_2 + X_2 = 2B_2 \cos(t) - 2A_2 \sin(t) =$$

$$= a \cos t [2a \cos(t)(a + w_1 + w_2 + 1) - (w_1 + w_2)(3a + 2w_1 + 2w_2 + 3)]$$

Initial conditions: $X(0) = a$, $\dot{X}(0) = 0$

$$2B_2 = \frac{a}{2} [2a(a + w_1 + w_2 + 1) - (w_1 + w_2)(3a + 2w_1 + 2w_2 + 3)]$$

$$A_2 = 0$$

General Solution = Particular + Homogeneous

$$X(t) = A_1 \sin t + B_1 \cos t + \frac{a t}{2} [2a(a + w_1 + w_2 + 1) - (w_1 + w_2)(3a + 2w_1 + 2w_2 + 3)] \sin(t)$$

$$X_2(0) = a = B_1; \dot{X}_2(0) = 0 = A_1$$

$$X_2(t) = a \cos t + \frac{a t}{2} [2a(a + w_1 + w_2 + 1) - (w_1 + w_2)(3a + 2w_1 + 2w_2 + 3)] \sin(t)$$

$$KE = \frac{1}{2} \varepsilon X^2 = \frac{1}{2} \varepsilon \left(a - \frac{t^2}{2} a - O(t^4) \right)^2 \approx \frac{1}{2} \varepsilon a^2 + B(t^2)^2 = \omega$$

$$\ddot{x} - \varepsilon x \ddot{x} + x = 0 \quad 7.6.23.$$

(1) Assign a new time variable $\tau = \omega t$

$$\omega^2 \ddot{x} - \varepsilon x \omega \dot{x} + x = 0$$

(2) Assign the new perturbation equation

$$x(\tau, \varepsilon) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \dots$$

$$\omega = 1 + \varepsilon w_1 + \varepsilon^2 w_2 + \dots$$

(3) Solve the perturbation equation:

$$(1 + \varepsilon w_1 + \varepsilon^2 w_2 + \dots)^2 (x_0 + \varepsilon \dot{x}_1 + \varepsilon^2 \ddot{x}_2 + \dots)$$

$$+ \varepsilon (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots) (\dot{x}_0 + \varepsilon \dot{x}_1 + \varepsilon^2 \dot{x}_2 + \dots)$$

$$+ ((1 + \varepsilon w_1 + \varepsilon^2 w_2 + \dots) + (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots)) = 0$$

$$O(1) \ddot{x}_0 + x_0 = 0$$

$$O(\varepsilon) \ddot{x}_1 + x_1 = -2w_1 \dot{x}_0 + \dot{x}_0 \dot{x}_0$$

$$O(\varepsilon^2) \ddot{x}_2 + x_2 = -w_1^2 x_0 + w_1 \dot{x}_0 \dot{x}_0 - 2w_1 \ddot{x}_1$$

$$- 2w_2 \dot{x}_0 + x_1 \dot{x}_0 + x_0 \dot{x}_1$$

(4) Solving for x_0 :

$$\ddot{x}_0 + x_0 = 0 \Rightarrow x(0) = 0 \Rightarrow \dot{x}(0) = 0$$

$$x_0(t) = a \cos t$$

(5) Solving for x_1 :

$$\ddot{x}_1 + x_1 = -2w_1 \dot{x}_0 + \dot{x}_0 \dot{x}_0$$

$$= -2w_1 a \cos t - a^2 \cos t \sin t$$

$$= [-2w_1 a - a^2 \sin t] \cos t$$

Linear equation with constant coefficients.

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \dots + a_{n-1} t^1 + a_n t^0 = f(t)$$

Solving the Homogeneous Equation

$$a_0 \lambda^{(n)} + a_1 \lambda^{(n-1)} + \dots + a_{n-1} \lambda^1 + a_n \lambda^0 = 0$$

$$(\lambda^2 + 1) = 0; \quad \lambda_{1,2} = \pm i; \quad K=S=1$$

Solution to a Homogeneous Equation: Summand

$$t = \sum R_{M_1}(t) e^{kt} \cos \beta t + Q_{M_1}(t) e^{kt} \sin \beta t$$

$$\lambda = k + \beta i; \quad k=0; \quad \beta=1$$

$$X = A_1 \cos t + B_1 \sin(t)$$

Lagrange's Method of Variation of Parameters.

(1) Build a system $A_1' X_1 + B_1' X_2 = 0$

$$A_1' X_1' + B_1' X_2' = \frac{P(t)}{a_0}$$

$$\text{where } X_1 = \cos(t) \quad X_2 = \sin(t)$$

$$X_1' = -\sin(t) \quad X_2' = \cos(t)$$

$$a_0 X'' = 1; \quad P(t) = \cos(t)(-\alpha^2 \sin(t) - 2\omega w_1)$$

(2) Solve the System using Cramer's Rule.

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & \sin(t) \\ \cos(t)(-\alpha^2 \sin(t) - 2\omega w_1) & \cos(t) \end{vmatrix} = \cos(t) \sin(t)(-\alpha^2 \sin(t) - 2\omega w_1)$$

$$W_2 = \begin{vmatrix} \cos(t) & 0 \\ -\sin(t) & \cos(t)(-\alpha^2 \sin(t) - 2\omega w_1) \end{vmatrix} = \cos^2(t)(-\alpha^2 \sin(t) - 2\omega w_1)$$

$$A_1'(t) = \frac{W_1}{W} = \cos(t) \sin(t)(-\alpha^2 \sin(t) - 2\omega w_1)$$

$$B_1'(t) = \frac{W_2}{W} = \cos^2(t)(-\alpha^2 \sin(t) - 2\omega w_1)$$

$$A_1 = \int A_1'(t) dt = \frac{\alpha^2 \sin^3(t)}{3} + \omega w_1 \sin^2(t) + C_1$$

$$B_1 = \int B_1'(t) dt = -\frac{\omega w_1 \sin(2t)}{2} + \frac{\alpha^2 \cos^3(t)}{3} - \omega w_1 t + C_2$$

$$X = -\frac{\alpha w \sin(t) \sin(2t)}{2} + \frac{\alpha^2 \cos(t) \sin^3(t)}{3} + \alpha w \cos(t) \sin^2(t)$$

$$+ \frac{\alpha^2 \cos^3(t) \sin(t)}{3} - \alpha w t \sin(t) + C_1 \sin(t) + C_2 \cos(t)$$

Initial conditions: $X(0) = 0; \dot{X}(0) = 0$

$$X(0) = 0 = C_2; \dot{X}(0) = 0 = 2\alpha^2 + 6C_1; C_1 = -\frac{1}{3}\alpha^2$$

$$X_1(t) = \left[-\frac{\alpha w_1 \sin(2t)}{2} + \frac{\alpha^2 \cos(t) \sin^2(t)}{3} + \alpha w_1 \cos(t) \sin^2(t) \right] \frac{1}{3} + \frac{\alpha^2 \cos^3(t)}{3}$$

$$+ \left[\frac{\alpha^2 \cos^3(t)}{3} - \alpha w_1 t - \frac{\alpha^2}{3} \right] \sin(t) + \alpha \cos(t)$$

$w_1 = 0$ because no secular terms

$$X_1(t) = \left[\frac{\alpha^2 \cos(t) \sin^2(t)}{3} + \frac{\alpha^2 \cos^3(t)}{3} - \frac{\alpha^2}{3} \right] \sin(t)$$

$$\text{Identities: } \sin(2t) = 2\cos(t)\sin(t)$$

$$\sin^2(t) = \frac{1}{2}(1 - \cos(2t))$$

$$\cos^2(t) = 1 - \sin^2(t)$$

$$X_1(t) = \frac{1}{6}(-2\alpha^2 \sin(t) + \alpha^2 \sin(2t))$$

③ Solving for X_2

$$\begin{aligned} \ddot{X}_2 + X_2 &= -\omega_1^2 \ddot{X}_0 + \omega_1 \dot{X}_0 - 2\omega_1 \ddot{X}_1 - 2\omega_2 \ddot{X}_0 + X_1 \ddot{X}_0 + X_0 \dot{X}_1 \\ &= -2\omega_2 \ddot{X}_0 + X_1 \ddot{X}_0 + X_0 \dot{X}_1 \\ &= 2\omega_2 \alpha \cos(t) - \frac{\alpha \sin(t)}{6} (-2\alpha^2 \sin(t) + \alpha^2 \sin(2t)) \\ &\quad + \frac{\alpha \cos(t)}{3} (-2\alpha^2 \cos(t) + \alpha^2 \cos(2t)) \end{aligned}$$

Linear Equation of Constant Coefficients

$$a_0 t^n + a_1 t^{n-1} + \dots + a_{(n-1)} t^1 + a_{(n)} t^0 = f(t)$$

Solving the Homogeneous Equation: Sum

$$a_0 \lambda^n + a_1 \lambda^{n-1} + \dots + a_{(n-1)} \lambda^1 + a_{(n)} \lambda^0 = 0$$

$$(\lambda^2 + 1) = 0; \quad \lambda_{1,2} = \pm i; \quad k = s = 1$$

Solution to a Homogeneous Equation: Summand

$$t = \sum e^{kt} [R_m(t) \cos \beta t + Q_m(t) \sin \beta t]$$

$$X_2(t) = A_1 \cos t + B_1 \sin t$$

Lagrange's Method of Variation of Parameters

$$A'_1 X_1 + B'_1 X_2 = 0$$

$$A'_1 X_1 + B'_1 X_2 = \frac{F(t)}{a_0} \quad \text{where } X_1 = \cos(y) \quad X_2 = \sin(y)$$

$$X_2 = (B_2 t - 2A_2) \sin t + (2B_2 - A_2) \cos t \quad X'_1 = -\sin(y) \quad X'_2 = \cos(y)$$

$$X_2 + X_1 = 2B_2 \cos(t) - 2A_2 \sin(t) = a_0 X_1 \cos(6t) - \frac{a^3 \sin(t)}{6} (-2 \sin(t) + \sin(2t))$$

$$f(t) = 12w_2 \cos(t) - \frac{a^3}{6} \sin(-2 \sin(t) + \sin(2t)) + 2a^3 \cos(t) (\cos t + \cos 2t)$$

$$\text{Initial Condition: } X_2(0) = 0; \quad X'_2(0) = 0$$

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} + a_0 - 2A_2 = 0$$

$$W_1 = \begin{vmatrix} B_2 & w_2 a & 0 & 0 \\ +2w_2 \cos t - \frac{a^3 \sin t}{6} (-2 \sin t + \sin 2t) & + \frac{a^3 \cos t}{3} (\cos t + \cos 2t) & \sin t & \cos t \end{vmatrix}$$

$$X_1(t) = A_1 \cos t + B_1 \sin t + 2w_2 a \cos t$$

$$W_2 = \begin{vmatrix} \cos t & 0 \\ -\sin t & 2w_2 \cos t - \frac{a^3 \sin t}{6} (-2 \sin t + \sin 2t) + \frac{a^3 \cos t}{3} (\cos t + \cos 2t) \end{vmatrix}$$

$$A'_1 = \frac{W_1}{W} = \sin t \left[\frac{a^3 \sin t}{6} (-2 \sin t + \sin 2t) - \frac{a^3 \cos t}{3} (\cos t + \cos 2t) - 2w_2 \cos t \right]$$

$$B'_1 = \frac{W_2}{W} = \cos t \left[2w_2 \cos t - \frac{a^3 \sin t}{6} (-2 \sin t + \sin 2t) + \frac{a^3 \cos t}{3} (\cos t + \cos 2t) \right]$$

$$A_1 = \int A'_1 dt = \sin^2 t \left[\frac{a^3}{4} \sin^2 t - \frac{(6aw + a^3)}{6} \right] + \cos t \left[\frac{a^3}{3} - \frac{2a^3 \cos^2(t)}{9} \right] + C_1$$

$$B_1 = \int B'_1 dt = \frac{a^3 \sin 4t}{32} + \frac{(48aw + 8a^3) \sin 2t}{96} + \frac{2a^3 \sin^3(t)}{9} - \frac{a^3 \sin(t)}{3} + awt + \frac{a^3 t}{24} + C_2$$

$$X_2(t) = \left[\frac{a^3 \sin 4t}{32} + \frac{(48aw + 8a^3) \sin 2t}{96} + \frac{2a^3 \sin^3(t)}{9} - \frac{a^3 \sin t}{3} + awt + \frac{a^3 t}{24} + C_1 \right] \sin t + \left[\sin^2 t \left(\frac{a^3 \sin^2 t}{4} - \frac{(6aw + a^3)}{6} \right) + \cos t \left(\frac{a^3}{3} - \frac{2a^3 \cos^2 t}{9} \right) + C_2 \right] \cos t$$

Initial conditions: $X_2(0) = 0; \dot{X}_2(0) = 0$

$$X_2(0) = 0 = \frac{a^3}{3} - \frac{2a^3}{9} + C_2; C_2 = -\frac{a^3}{3} + \frac{2a^3}{9}$$

$$\overset{\circ}{X}_2(0) = 0 = awt + \frac{a^3 t}{24} + C_1; C_1 = -awt - \frac{a^3 t}{24}$$

$$X_2(t) = \left[\frac{a^3 \sin 4t}{32} + \frac{(48aw_2 + 8a^3) \sin 2t}{96} + \frac{2a^3 \sin^3(t)}{9} - \frac{a^3 \sin t}{3} \right] \sin t + \left[\sin^2 t \left(\frac{a^3 \sin^2 t}{4} - \frac{(6aw_2 + a^3)}{6} \right) + \cos t \left(\frac{a^3}{3} - \frac{2a^3 \cos^2 t}{9} \right) - aw_2 t - \frac{a^3 t}{24} \right] \cos t$$

$$w_2 = -\frac{1}{3} a^2$$

$$\begin{aligned} w(a) &= 1 + \varepsilon w_1 + \varepsilon^2 w_2 + O(\varepsilon^3) \\ &= 1 - \frac{1}{3} \varepsilon^2 a^2 \end{aligned}$$

$$\ddot{X} + X - \varepsilon X^3 = 0 \quad 7.6.24, \quad X(0) = a; \quad \dot{X}(0) = 0$$

① Assign a new time: $\tau = wt$

$$\varepsilon^2 \ddot{X} + X - \varepsilon X^3 = 0$$

② Apply perturbation equations

$$x(\tau, \varepsilon) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + O(\varepsilon^3)$$

$$w = 1 + \varepsilon w_1 + \varepsilon^2 w_2 + O(\varepsilon^3)$$

$$(1 + \varepsilon w_1 + \varepsilon^2 w_2 + \dots)^2 (\ddot{x}_0 + \varepsilon \ddot{x}_1 + \varepsilon^2 \ddot{x}_2) + (x_0 + \varepsilon x_1 + \varepsilon^2 x_2) - \varepsilon (x_0 + \varepsilon x_1 + \varepsilon^2 x_2)^3 = 0$$

$$O(1) \ddot{X}_0 + X_0 = 0$$

$$O(\varepsilon) \ddot{X}_1 + X_1 = -\dot{\omega}_1 \ddot{X}_0 + \ddot{X}_0^3 - X_1$$

$$O(\varepsilon^2) \ddot{X}_2 + X_2 = -\dot{\omega}_1 \ddot{X}_1 - \dot{\omega}_2 \ddot{X}_0 + 3X_1 \dot{X}_0^2$$

(3) Solving for X_0 :

$$\ddot{X}_0 + X_0 = 0; \quad X_0(0) = a; \quad \dot{X}_0(0) = 0; \quad X_0(t) = a \cos t$$

(4) Solving for X_1 :

$$\ddot{X}_1 + X_1 = -\dot{\omega}_1 \ddot{X}_0 + \ddot{X}_0^3 = +\dot{\omega}_1 a \cos t + a^3 \cos^3 t$$

$$X_1 = (\dot{\omega}_1 a - \frac{3}{4} a^3) \cos t - \frac{a^3}{4} \cos 3t$$

| Linear Equation with constant coefficients | $\dot{\omega}_1 = a^2$

$$a_0 t^n + a_1 t^{n-1} + \dots + a_{n-1} t^1 + a_n t^0 = f(t)$$

| Solving the Homogeneous Equation. |

$$a_0 \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda^1 + a_n \lambda^0 = 0$$

$$(\lambda^2 + 1) = 0; \quad \lambda_{1,2} = \pm i; \quad R = S = 1; \quad \lambda = \alpha + i\beta$$

| Solution to Homogeneous Equation: Summand. |

$$t = \sum_{m=1}^M R_m(t) e^{\alpha t} \cos \beta t + Q_m(t) e^{\alpha t} \sin \beta t$$

$$X_1(t) = A_1 \cos t + B_1 \sin t$$

| Solving the particular Equation: Method for Undetermined Coefficients. |

$$X_1 = t^3 e^{\alpha t} [R_m(t) \cos \beta t + Q_m(t) \sin \beta t]$$

$$X_1 = t [A_1 \cos 3t + B_1 \sin 3t]$$

$$\ddot{X}_1 = \cos 3t (6B_1 - 9A_1 t) - 3 \sin 3t (2A_1 + 3B_1 t)$$

$$\ddot{X}_1 + X_1 = \cos(3t)(6B_1 - 8A_1 t) - 2 \sin(3t)(3A_1 + 4B_1 t) = -\frac{a^3}{4} \cos 3t$$

$$B_1 = -\frac{a^3}{24}$$

$$X_1(t) = -\frac{a^3}{24} t \sin 3t$$

General Solution: Homogeneous + Particular

$$x_1(t) = A_1 \cos t + B_1 \sin t + \frac{\alpha^3}{24} t \cos 3t$$

Initial Conditions: $x_1(0) = 0$; $\dot{x}_1(0) = 0$

$$x_1(0) = 0 = A_1; \quad \dot{x}_1(0) = 0 = B_1 + \frac{\alpha^3}{24}; \quad B_1 = -\frac{\alpha^3}{24}$$

$$x_1(t) = \frac{\alpha^3}{24} \sin t - \frac{\alpha^3}{24} t \cos 3t$$

⑤ Solving for x_2 :

$$\ddot{x}_2 + x_2 = -\omega_1 \ddot{x}_1 - \omega_2 \ddot{x}_0 + 3x_1 x_0^2$$

$$= \frac{\alpha^5}{24} (-\sin t + 6\sin 3t + 9t \cos 3t) + \omega_2 a \cos t$$

$$+ \frac{3\alpha^5}{8} [\sin t - t \cos 3t] \cos^2 t$$

$$\omega \omega_2 \neq 0$$

⑥ I forgot $O(\epsilon^3)$, which at 2 modification necessitates

a time-shift $\epsilon (1 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \epsilon^3 \omega_3 + \dots)$ and

an perturbation of $1 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \dots$

Next time, for this example, $\omega = 1 + \epsilon \omega_1 + \epsilon^2 \omega_2$

$$= 1 + \epsilon a^2.$$

$$\ddot{x} + x + \epsilon h(x, \dot{x}, t) = 0$$

$$7.6.25. \quad x(t) = r(t) \cos(t + \phi(t))$$

$$\dot{x}(t) = -r(t) \sin(t + \phi(t))$$

a. $r = \langle \epsilon h(x, \dot{x}, t) \rangle \approx \epsilon h \sin(t + \phi(t))$

$$r\dot{\phi} = \langle \epsilon h(x, \dot{x}, t) \rangle = \epsilon h \cos(t + \phi(t))$$

$$b. \langle r \rangle(t) = \bar{r}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} r(\tau) d\tau$$

$$\frac{d\langle r \rangle}{dt} = \frac{d\bar{r}(t)}{dt} = \frac{1}{2\pi} \frac{d}{dt} (r(t+\pi) - r(t-\pi)) = \langle \frac{dr}{dt} \rangle$$

$$c. \frac{d\langle r \rangle}{dt} = \langle E h \sin(t+\phi) \rangle = E \langle h(x, \dot{x}, t) \sin t + \phi \rangle$$

$$= E \langle h(r \cos(t+\phi), -r \sin(t+\phi), t) \sin t + \phi \rangle$$

$$d. \frac{d\bar{r}}{dt} = E \langle h(r \cos(t+\phi), -r \sin(t+\phi), t) \sin(t+\phi) \rangle + O(\epsilon^2)$$

$$\frac{d\bar{r}}{dt} = E \langle h(r \cos(t+\phi), -r \sin(t+\phi), t) \sin(t+\phi) \rangle + O(\epsilon^2)$$

- (1-x)

$$\ddot{x} = -E X \sin^2 t \quad 7.6.26 \quad 0 \leq E \ll 1 ; \quad x = x_0 \quad @ \quad t = 0$$

$$a. \ddot{x} = \frac{d^2x}{dt^2} = -E X \sin^2 t ; \quad x \ln x - x = -E \left[\int \left[\frac{(1-\cos 2t)}{2} dt \right] dt \right]$$

$$= -E \left[\frac{t}{2} - \frac{\sin 2t}{4} + C \right] dt$$

An alternative solution

$$\text{is by homogeneous, plus, particular, which has} \quad = -E \left[\frac{t^3}{4} + \frac{\cos 2t}{8} + Ct \right] + C$$

continuous style.

$$x = e^{-ET \int_0^\infty \left[\frac{t^2}{4} + \frac{\cos 2t}{8} \right] dt}$$

$$x = e$$

$$b. \bar{x}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(\tau) d\tau = -E \pi \log \left[\frac{1}{4} (t-\pi)^2 + \frac{\cos 2t}{8} \right]$$

$$x(t) = \bar{x}(t) + O(\epsilon) = \frac{e}{\pi} + O(\epsilon)$$

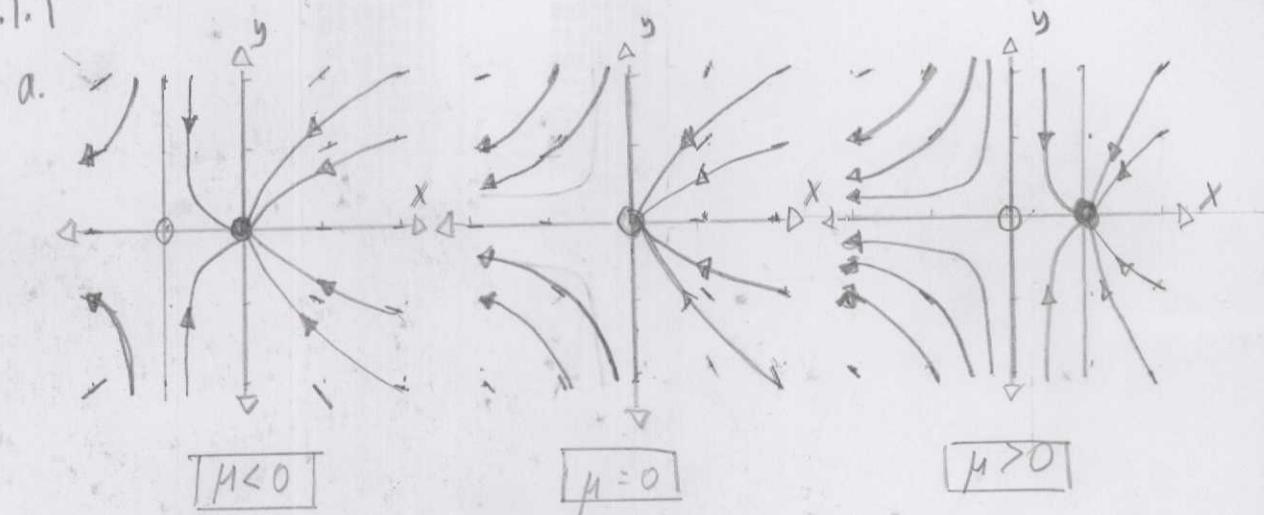
c. The error depends on the exactness and amount of terms.

A product-log function isn't the common method, either.

Chapter 8: Bifurcations Revisited!

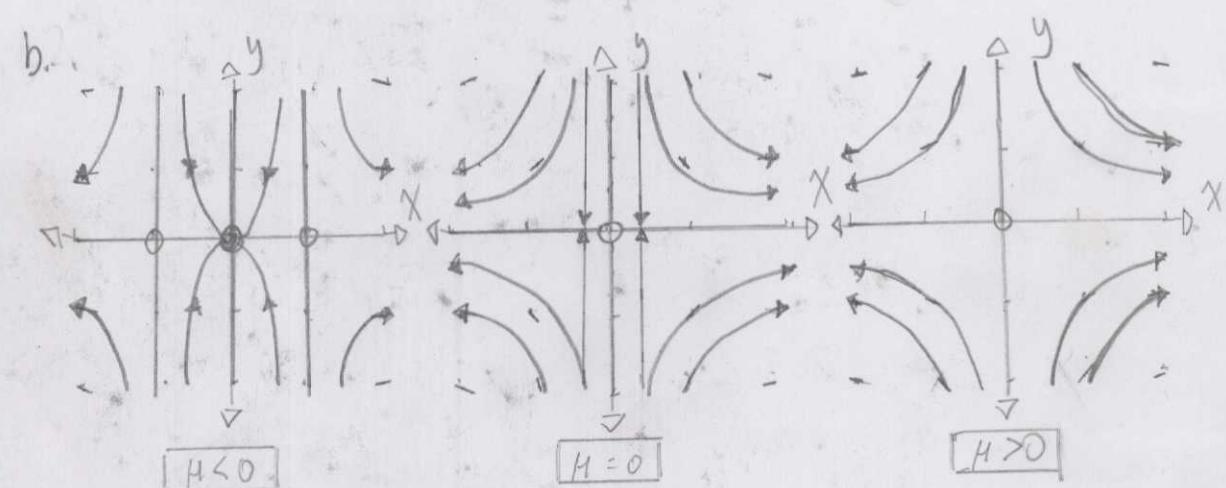
$$\dot{x} = \mu x - x^2 \quad 8.1.1$$

$$\dot{y} = -y$$



$$\dot{x} = \mu x + x^3$$

$$\dot{y} = -y$$



$$\dot{x} = \mu - x^2$$

8.1.2. Eigenvalues: $\vec{x}' = A\vec{x} = 0 ; A\vec{x} = 0 = \lambda\vec{x} ; (A - \lambda)\vec{x} = 0$

$$\dot{y} = -y$$

$$= \begin{pmatrix} -2x - \lambda & 0 \\ 0 & -1 - \lambda \end{pmatrix} = (-2x - \lambda)(-1 - \lambda) = 0$$

$$\lambda_1 = -2x ; \lambda_2 = -1$$

Fixed Points: $\dot{x} = 0 = \mu - x^2$

$$\dot{y} = 0 = -y ; (x^*, y^*) = (\sqrt{\mu}, 0)$$

Eigenvalues + Fixed Points: $\lambda_1 = -2\sqrt{\mu} ; \lim_{\mu \rightarrow 0} \lambda_1 = 0$

$$\dot{x} = \mu x - x^2 \quad 8.1.3.$$

$$\dot{y} = -y$$

Eigenvalues: $\vec{x}' = A\vec{x} = 0 ; (A - \lambda)\vec{x} = \begin{pmatrix} \mu - 2x - \lambda & 0 \\ 0 & -1 - \lambda \end{pmatrix}\vec{x} = 0$

$$\lambda_1 = \mu - 2x$$

$$\lambda_2 = -1$$