Chapter 11: Fractals:

$X_1 = 0.X_{11}X_{12}X_{13}X_{14}\ldots$
$X_2 = 0.X_{21}X_{22}X_{23}X_{24}\ldots$
$X_3 = 0.X_{31}X_{32}X_{33}X_{34}\ldots$
$\vdots$

11.1.1. Example 11.1.4 demonstrates the uncountable rational numbers by a "diagnol argument." A rational number is a fraction with traditional numerator and denominator. When digits never end, such as an extensive root, then the "diagnol argument" fails.

$2\mathbb{N}-1$;
$X = \{x \mid 2n-1, n \in \mathbb{N}\}$;

11.1.2 $X = \{1, 3, 5, \ldots\}$, specifically odd numbers, are countable. A one-to-one correspondence maps two sets into pairs, also counts.

$2 \longleftrightarrow 3, 3 \longleftrightarrow 5, 4 \longleftrightarrow 7, \ldots, N \longleftrightarrow 2N-1$

$\mathbb{R} - \mathbb{Q}$;
$\mathbb{R}/\mathbb{Q}$

11.1.3. The irrational numbers are uncountable because pi's infinite digits. A "diagnol argument" is endless, at infinite end. Also, a one-to-one correspondence seems difficult with Euler's number and special roots because the irregular sets.

11.1.4. A repeating decimal found by:

(Magnitude #1 - Magnitude #2)α = Value at Magnitude #1 - Value at Magnitude #2.

α = Repeating decimal, such as:

$2.727272\overline{72}\ldots$

The neverending digits limit Cantor's "diagnol argument" because the finite diagnol (n) never terminates.

$X = \{X \mid (p, q, r)\}$
$Y = \{Y \mid (i, j, k, l, m)\}$

11.1.5. By induction: Base case ($n=3$) - Total Points $= n^3 - 1 = 3^3 - 1 = 26$ points

Next case ($n+2$) - Total Points $= (n+2)^3 - 1 = 98$ points

Infinite Case ($n+\infty$) - Total Points $= (\infty)^3 - 1 = \infty$ points

Each set is countable by a larger cube with a larger side length.

- Other solutions suppose exact vs. inexact counting.
- A large solution set in structure theory prove shape dialation, subdivision, enumeration, infinite rays and infinite cones, for countable points.
- Alternative arguments in $\mathbb{R}^3$ are not only shapes, but polynomials and convex functions for countable points.

$x_{n+1} = 10x_n \pmod 1$

11.1.6. Fixed Points: $x_n^* \in \mathbb{Z}/10$

a) Stability: $|f'(x^*)| = |10x \pmod 1| = 10 > 1 =$ Unstable

Proof about Countability: A one-to-one correspondence exists between $\mathbb{Z}/10$ and $\mathbb{Q}/10$, $\mathbb{Z}/10$ and $\mathbb{R}/10$, also $\mathbb{Z}/10$ and $\mathbb{C}/10$.

Each periodic orbit is unstable and countable.

b) An aperiodic behavior exhibits sensitive dependence on initial conditions, such as a value between the integers i.e. a decimal.

Proof about countability:

Contradiction: A countable set between the integers, over 10 such as 0 to 0.1, have a list $\{x_1, x_2, x_3 \ldots\}$

$x_1 = 0.x_{11}\, x_{12}\, x_{13}\, x_{14} \ldots$

$x_2 = 0.x_{21}\, x_{22}\, x_{23}\, x_{24} \ldots$

$x_3 = 0.x_{31}\, x_{32}\, x_{33}\, x_{34} \ldots$

$\vdots$

Where x_{ij} denotes the jth digit of the real number

1) The first digit isn't x_1 the second x_2, then x_{nn} digit not be x_n.

A last digit is uncountable.

c) An "eventually-fixed point" is countable because the finites nature in the function $x_{n+1} = x_n$ for all $n > N$.

$x_{n+1} = 2x_n \pmod 1$ 11.1.7. <u>Fixed Points:</u> $x \in \mathbb{Z}/2$

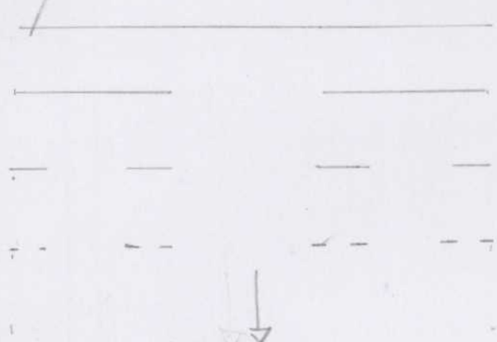
<u>Periodic Orbits:</u> Countable periodic orbits are true because one-to-one correspondence between $\mathbb{Z}/2$ and $\mathbb{R}/2$, $\mathbb{Q}/2$, and $\mathbb{C}/2$.

<u>Aperiodic Orbits:</u> The set for aperiodic orbits is not countable. Initial conditions are highly sensitive between fixed points, also involve irrational numbers that Cantors "diagnol arguments" fail to solve.

$S_0 = [0,1]$

$C = S_\infty$

11.2.1.



$S_0 = [0,1]$

$S_1 = [0,1/3] \wedge [2/3,1]$

$S_2 = [0,1/9] \wedge [2/9,3/9] \wedge [6/9,7/9] \wedge [8/9,1]$

$S_3 = [0,1/27] \wedge [2/27,3/27] \wedge [\circ\circ\circ] \wedge [26/27,1]$

$\vdots$

"Cantors Set" S_∞

Set	Length	Length Removed
S_0	$(2/3)^0$	$1-(2/3)^0$
S_1	$(2/3)^1$	$1-(2/3)^1$
S_2	$(2/3)^2$	$1-(2/3)^2$
S_3	$(2/3)^3$	$1-(2/3)^3$
S_n	$(2/3)^n$	$1-(2/3)^n$

$$\text{Length Removed} = 1 - S_n = 1 - \lim_{n\to\infty}(2/3)^n = 1$$

$\mathbb{Q}$ = Rationals

11.2.2 $\varepsilon_1 = 1/1$; $\varepsilon_2 = 1/2$; $\varepsilon_3 = 1/4 \cdots \varepsilon_n = 1/2^n$

$$\varepsilon_\infty = \lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} 1/2^n = 0$$

$\mathbb{R}$ = Reals

11.2.3.

Set	Length	Length Removed
S_0	$\varepsilon(\frac{1}{2})^0$	$1-\varepsilon(1/2)^0$
S_1	$\varepsilon(\frac{1}{2})^1$	$1-\varepsilon(1/2)^1$
S_2	$\varepsilon(\frac{1}{2})^2$	$1-\varepsilon(1/2)^2$
S_n	$\varepsilon(\frac{1}{2})^3$	$1-\varepsilon(1/2)^3$

An upper limit solution, bounds and counts the lower limit set.

$$0 \le \text{const} \cdot \left(\bigcup_{n=1}^{\infty} \mathbb{R}\right) = 1 + \varepsilon \sum_{1}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{\varepsilon}{\log(4)}$$

$X = \{x \mid 0 < x < 1 \wedge x \in \mathbb{R} - \mathbb{Q}\}$ 11.2.4.

a) $X = \frac{a_0}{10^0} + \frac{a_1}{10^1} + \frac{a_2}{10^2} + \cdots$ because base-10. Also, $a_0 = 0$, since not including zero or one.

$= \frac{a_1}{10^1} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \cdots$

$= 0.a_1 a_2 a_3 \ldots$ where each constant defines a digit or magnitude in base-10 counting. They change.

The measure is $L_n = \left(\frac{1}{10}\right)^n = 0$.

b) Irrational numbers between 0 and 1 are incountable because base-10 decimals end at infinity. A proof by conjecture is the "diagnol argument."

c) Disconnected subset - When a set representation is never a union of two or more disjoint non-empty open subsets.

In the problem, $X = \{x \mid (0 < x < 1) \wedge x \in \mathbb{R} - \mathbb{Q}\}$ is diconnected from the other irrationals, $\mathbb{R} - \mathbb{Q}$.

d) Isolated Point - a subset with an element and without neighboring elements.

Irrational numbers between zero and one have an addressable representation. The elements are isolated from another element.

11.2.5

a) Base-3 expansion of 1/2:

$x = 1/2 = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^2} + \cdots = \sum_{1}^{\infty} \frac{a_i}{3^i}$

$a_1 = 1; \; x = \frac{1}{2} = \frac{1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots$

$$\frac{3}{2} = 1 = \frac{1}{2} = \frac{a_2}{3^2} + \frac{a_3}{3^2} + \cdots$$

$$a_2 = 1$$

$$\frac{3}{2} - 1 = \frac{1}{2} = \frac{a_3}{3} + \cdots$$

$$a_3 = 1$$

$$(0.5)_{10} = (0.\overline{1})_3$$

b) <u>One-to-one correspondence : $c \in C$ and $x \in [0,1]$</u>

$$\left(x = \cancel{\frac{a_1}{10^0}} + \frac{a_2}{10^1} + \frac{a_3}{10^2} + \cdots\right) = \left(c = \cancel{\frac{b_1}{3^0}} + \frac{b_2}{3^1} + \frac{b_3}{3^2} + \cdots\right)$$

$$\cancel{\frac{a_1}{10^0} = \frac{b_1}{3^0}}; \quad \frac{a_2}{10^1} = \frac{b_2}{3^1}; \quad \frac{a_3}{10^2} = \frac{b_3}{3^2}; \ \ldots$$

Where $x = 0.a_2 a_3 a_4$ and $c = 0.b_2 b_3 b_4$.

c) Endpoint - an extreme value or unattainable segment in a set.

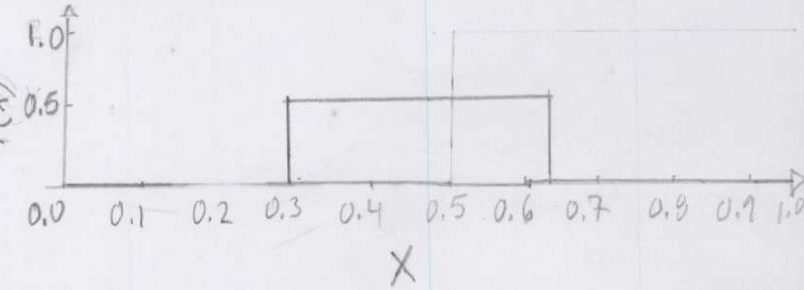
A base-3 counting system, Cantor's set, has endpoints located at ternary values.

$S_0 = [0, 1]$
$S_\infty = C$

11.2.6.

a) $S_0 = [0, 1]$; $P_0 \in \text{Random}(S_0) < X$
$= 1$

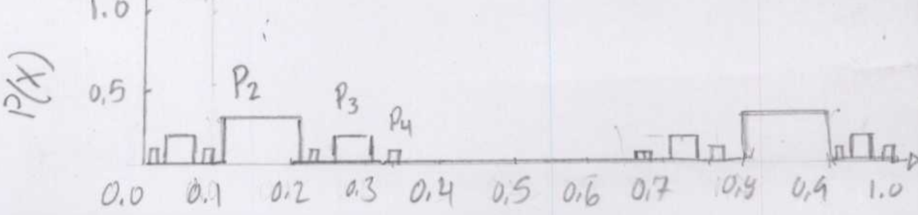
b) $S_1 = [(0, 1/3), (2/3, 1)]$; $P_0 \in \text{Random}(S_1) < X$



c) $P_2 \in \text{Random}(S_2) = 1/4 < X$

$P_3 \in \text{Random}(S_3) = 1/8 < X$

$P_4 \in \text{Random}(S_4) = 1/16 < X$



d) $P_\infty(x)$ is ~~s~~ continuous and nearly zero.
The graph is almost a line at $P(x)=0$.

$S_0 = [0,1]$
$S_\infty = C_{1/2}$

11.3.1

a) $S_0 = [0,1]$

$S_1 = [(0,1/4),(3/4,1)]$

$S_2 = [(0,1/16),(3/16,4/16),(12/16,13/16),(15/16,16/16)]$

⋮

S_∞ ;

$m = 1 \quad r = 1$
$m = 2 \quad r = 3$
$m = 4 \quad r = 8$
$05 + 15 + 35$

$m = \infty, \ r = \infty$

Similarity dimension (d): the exponent defined by $m = r^d$ or $d = \frac{\ln m}{\ln r}$, where m is the number of copies and r is a scale factor.

$$d = \frac{\ln(2)}{\ln(3)} = 0.63$$

b) The sets measure is zero because the zero length at $n=\infty$.

11.3.2. $d = \frac{\ln(m)}{\ln(r)} = \frac{\ln(m)}{\ln(\alpha)}$

11.3.3.

a) $d = \frac{\ln(m)}{\ln(r)} = \frac{\ln(4)}{\ln(7)} = 0.71$

b) $d = \frac{\ln(m)}{\ln(r)} = \frac{\ln(n)}{\ln(2n+1)} = \ln(n+1)$ where $n \in \mathbb{N}$

11.3.4. $d = \frac{\ln(m)}{\ln(r)} = \frac{\ln(5)}{\ln(10)} = 0.70$

11.3.5. $d = \frac{\ln(m)}{\ln(r)} = \frac{\ln(9)}{\ln(10)} = 0.95$

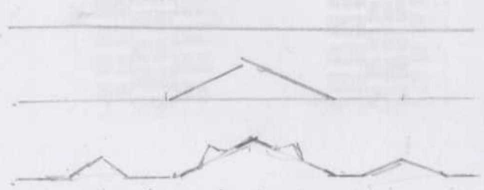
11.3.6. Cantor's set is all ternary numbers without any 1's in their ternary representation.

Also, the inner intervals are not isolated because $S_\infty \in S_5 \in S_4 \in S_3 \in S_2 \in S_1 \in S_0 \in Q$

von Koch snowflake.

11.3.7.

a.



S_0

S_1 $\xrightarrow{\text{Partial}}$ Star of David.

S_2.

b. See above

c. $L_0 = \left(\frac{4}{3}\right)^0 \cdot 1$

$L_1 = \left(\frac{4}{3}\right)^1 \cdot 1$

$L_2 = \left(4/3\right)^2 \cdot 1$

$\vdots$

$L_\infty = \lim_{n \to \infty} \left(\frac{4}{3}\right)^n = \infty$

d) $L_0 = 0$

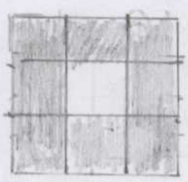
$L_1 = \left(\frac{1}{3}\right)^2 \left(\frac{\sqrt{3}}{4}\right) n^2$

$L_2 = \left(\frac{1}{3}\right)^4 \left(\frac{\sqrt{3}}{4}\right) 2^2 + \left(\frac{1}{3}\right)^2 \left(\frac{\sqrt{3}}{4}\right)$

$\vdots$

$L_\infty = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n} \left(\frac{\sqrt{3}}{4}\right) n^2 = \frac{45\sqrt{3}}{1024}$

e.) $d = \frac{\ln(m)}{\ln(r)} = \frac{\ln(4)}{\ln(2)} = 0.63$

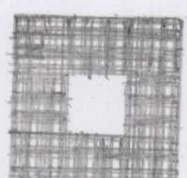


S_1

S_2

11.3.8.

a) S_3



b) $d = \frac{\ln(m)}{\ln(r)} = \frac{\ln(512)}{\ln(27)} = 1.89 \leftarrow$ Large value!!!

c.) Area = Length x Width ; S_0 (Area) = 1

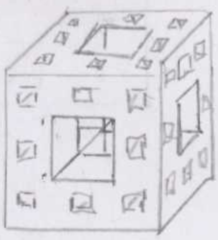
S_1 (Area) = 8/9

S_2 (Area) = 64/81

$S_n = (8/9)^n$

$S_\infty = 0$

11.3.9. $d = \frac{\ln(m)}{\ln(n)} = \frac{\ln(20)}{\ln(3)}$



Menger sponge

n-dimensional: $d = \frac{\ln(m)}{\ln(n)} = \frac{\ln(N2^{N-1}+2^{N})}{\ln(3)}$

11.3.10.
a),

Topological Cantor Set - a closed set with

1. "Totally disconnected" elements
2. No isolated points

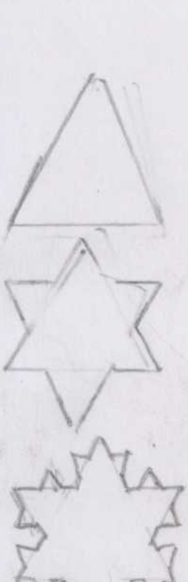
If $C = \bigcap_{n=0}^{\infty} C_n = C_0 \cap C_1 \cap C_2 \cap \cdots \cap C_n$

$B = \bigcap_{n=0}^{\infty} B_n = B_0 \cap B_1 \cap B_2 \cap \cdots \cap B_n$

$C_0 = B_0 = [0, 1)$

$B \in C$

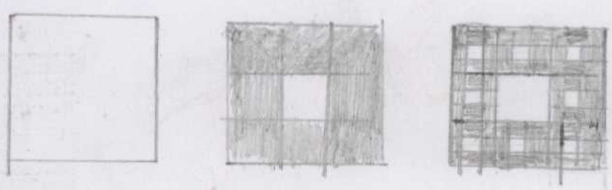
b) $B_0 = 1 : B_1 = \left(\frac{1}{2}\right)^1 ; B_2 = \left(\frac{1}{4}\right)^2 ; \lim_{n\to\infty} B_n = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$



11.4.1. Von Koch Snowflake:

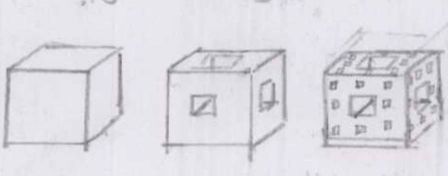
$d = \frac{\ln N}{\ln\left(\frac{1}{\varepsilon}\right)} = \frac{\ln 12}{\ln\left(1/(1/3\sqrt{2})\right)} = \frac{\ln 12}{\ln(3\sqrt{2})} = 1.72$

11.4.2. Sierpinski Carpet:

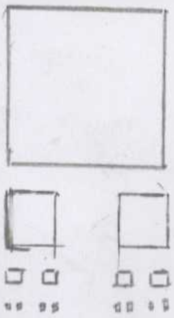


$d = \frac{\ln N}{\ln(1/\varepsilon)} = \frac{\ln(8^n)}{\ln\left(1/\left(\frac{1}{3}\right)^n\right)} = \frac{\ln 8}{\ln 3} = 1.89.$

11.4.3. Menger Sponge:



$d = \frac{\ln(N)}{\ln(1/\varepsilon)} = \frac{\ln(20)}{\ln(3)} = 2.72.$



11.4.4. The Cartesian Product of the middle-thirds Cantor set:

$$d=\frac{\ln(N)}{\ln(1/\varepsilon)}=\frac{\ln(2^2)}{\ln(1/(\frac{1}{3}))}=\frac{2\ln(2)}{\ln(3)}=1.26.$$

11.4.5. Menger Hypersponge: $d=\frac{\ln(N)}{\ln(1/\varepsilon)}=\frac{\ln(48^N)}{\ln(3^N)}=3.52$

$$f(x)=\begin{cases} rx & 0\le x\le 1/2 \\ r(1-x) & \frac{1}{2}\le x\le 1\end{cases}$$

11.4.6.

a) $x_{n+1}=f(x_n)$; $r>2$

If $f(x_0)>1$, then "escape."

Fixed Points: $0\le x\le 1/2$: $x_{n+1}=rx$

$x(1-r)=0$; $x^*=0$

$\frac{1}{2}\le x\le 1$: $x_{n+1}=r(1-x)$

$x(1+r)-r=0$; $x^*=\frac{r}{(1+r)}$

Stability: $x^*=0$; $|f'(0)|=r$, "unstable"

$x^*=\frac{r}{1+r}$; $\left|f'\left(\frac{r}{1+r}\right)\right|=r$ "unstable"

I calculate no "escape" because $f(x_0)_{max}=1$ and not greater than one.

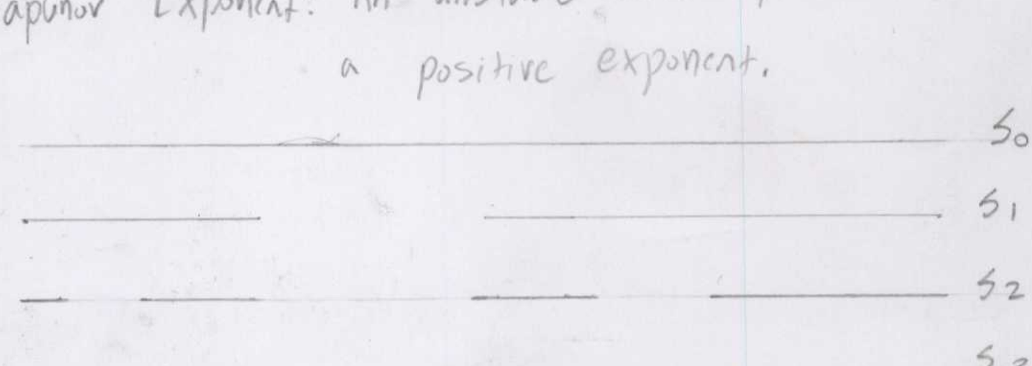
b) $x_0=\{x\,|\,[0,1]\}$

c) $d=\frac{\ln(N)}{\ln(1/\varepsilon)}=\frac{\ln(\infty)}{\ln\frac{1}{(\frac{1}{r})}}=\infty$

d) Liapunov Exponent: An unstable fixed point has a positive exponent.

11.4.7.

a.



b) $d = \ln \frac{N}{\ln 1/\varepsilon} = \frac{\ln(3^n)}{\ln\left(1/\left(\frac{1}{4}\right)^n\right)} = 0.79.$

c) S_∞ is not self-similar because the irregular construction.

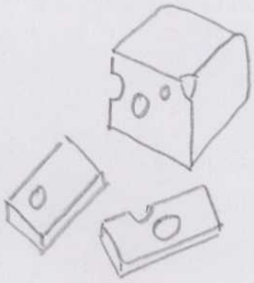
Random Fractal

11.4.8.

a) The box dimension(d) is $\frac{\ln(3^n)}{\ln\left(1/\left(\frac{1}{4}\right)^n\right)} = 0.79.$

A 50:50 coin generator no self-similar structure, but "boxable".

b) A first quarter selection makes one segment and not two, so the box dimension differs.



11.4.9. p^2 = unit square

m^2 = random square $\quad d = \frac{\ln(m)}{\ln(r)} = \frac{\ln(2m+1)}{\ln(p)} =$

$p > m+1$

$1/p$ = side

11.4.10 $d = \frac{\ln(m)}{\ln(r)} = \frac{\ln(2^n)}{\ln\left(1/\left(\frac{1}{2}\right)^n\right)} = 1$

11.5.1 Correlation Dimension: density of points near an attractor.

Pseudo-code:

```
//Necessary set of attracting functions
func1 = σ(y-x)
func2 = x(ρ-x)-y
func3 = xy-bz
// Initial conditions
ρ=28; σ=10; b=8/3; dt=0.1; x=0; y=0; z=0;
// Runge-Kutta 4th-order
k1x, k2x, k3x, k4x;
k1y, k2y, k3y, k4y;
k1z, k2z, k3z, k4z;
```

```
//Iterate and fill function values
int totalIterations =1000;
int i,j; int values [totalIterations/dH][2]
for(i=0; i<totalIterations/dH ; i++){
 k1x=σ(y-x);
 k1y= x(ρ-x)-y;
 k1z=xy-bz;
 k2x = σ(y+dH∘k1y/2 - x-dh∘k1x/2)∘dh
 k2y=[(x+k1x∘dh/2)(ρ-(x+k1x∘dh/2))-y-k1y∘dh/2]dh
 k2z =[(x+k1x∘dh/2)(y+k1y∘dh/2)-b(z+k1z∘dh/2)]dh
 k3x =σ(y+dh∘k2y/2-x-dh∘k2x/2)∘dh
 k3y =[(x+k2x∘dh/2)(ρ-(x+k2x∘dh/2))-y-k1y∘dh/2]dh
 k3z =[(x+k2x∘dh/2)(y+k2y∘dh/2)-b(z+k2z∘dh/2)]dh
 k4x =σ(y+dh∘k3y - x-dh∘k3x)∘dh
 k4y =[(x+k3x∘dh)(ρ-(x+k3x∘dh))-y-k3y∘dh]dh
 k4z =[(x+k3x∘dh)(y+k3y∘dh)-b(z+k3z∘dh)]∘dh
 x = x + dh/6(k1x+2k2x+2k3x+k4x)
 y = y + dh/6(k1y+2k2y+2k3y+k4y)
 z = z + dh/6(k1z+2k2z+2k3z+k4z)
 values[i][0]=x;
 values[i][1]=z;
}
//Grassberger and Procaccia (1983)
int point[2]={rand(), rand()}
int radius = 1;
int C=0;
```

```
for(i=0; i< totalIterations/dH; i++){
  if(sqrt([value[i][0] -point[0]]^2 + [value[i][1] -point[1]]^2) < 1){
    C+=1
  }
}
C/= totalIterations x total Iterations;
printf("%f", C);
//Real Code:
#include <iostream>
#include <cmath>
int main(){
 int i,j, exp=10, ro=100, sigma=10, radius=1, total, totalIter=10000;
 Float k1X, k2X, k3X, k4X, k1y, k2y, k3y, k4y, k1Z, k2Z,  \
     k3Z, k4Z, C=0, dh=0.1, b=8/3, X=-58.26 y=-3.3,  \
     Z=12.2, maxX=0, maxZ=0;
 total=totalIter/dH;
 Float values[total][2];
 for(i=0; i<total; i++){
   k1X=sigma(y-X)dh;
   k1y=(X(ro-Z)-y)dh;
   k1z=(Xy-bz)dh
   k2X=sigma(y+k1y∘dh/2 - X-k1x∘dh/2)dh
   k2y=(X+k1x∘dh/2)(ro-Z-k1z∘dh/2)-y-k1y∘dh/2)dh
   k2Z=((X+k1X∘dh/2)(y+k1y∘dh/2) - b(Z+k1z∘dh/2))dh
   k3x=sigma(y+k2y∘dh/2 - X - k2x∘dh/2)dh
   k3y=(X+k2x∘dh/2)(ro-Z-k2z∘dh/2)-y-k2y∘dh/2)∘dh
   k3z=((X+k2x∘dh/2)(y+k2y∘dh/2)-b(Z+k2z∘dh/2))dh
   k4X=sigma(y+k3y∘dh- X -k3X∘dh)dh
   k4y=(X+k3x∘dh)(ro-Z-k3z∘dh)-y-k3y∘dh)∘dh
   k4z=((X+k3X∘dh)(y+k3y∘dh) -b(Z+k3z∘dh))dh
   values[i][0]=X
   values[i][1]=Z
```