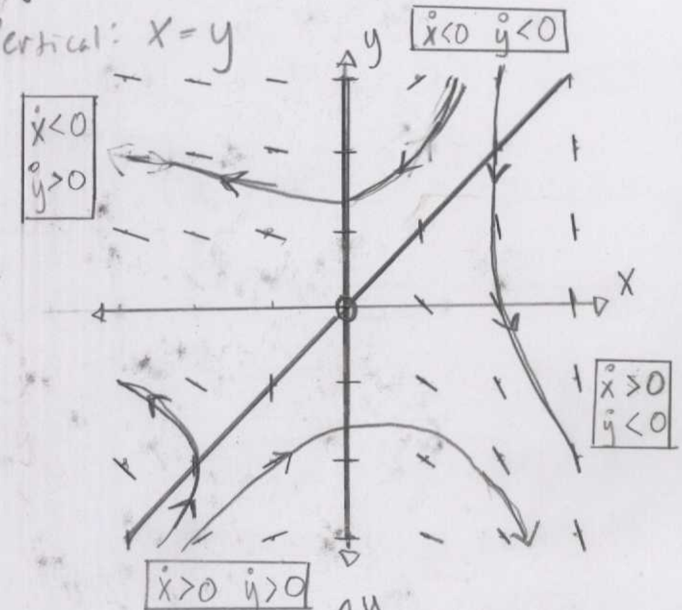
$\mathring{R} = 0$ 5.3.6.
$\mathring{J} = aR + bJ$.

b \ a	(+)	(−)
(+)	Unstable and Fixed Relationship	$a^2 > b^2$: Stable $a^2 < b^2$: Unstable $a = b$: Isolated
(−)	$a^2 > b^2$: Unstable $a^2 < b^2$: Stable $a = b$: Isolated.	Stable and Fixed Relationship

Chapter 6: Phase Plane:

$\mathring{x} = x - y$ 6.1.1: $\boxed{\text{Fixed Points}}$: $\mathring{x} = 0 = x - y$; $x = y$; $\mathring{y} = 0 = 1 - e^x$; $x = 0$; $(x^*, y^*) = (0,0)$
$\mathring{y} = 1 - e^x$ $\boxed{\text{Nullclines}}$: Horizontal: $x = 0$; Vertical: $x = y$



$\mathring{x} = x - x^3$ 6.1.2: $\boxed{\text{Fixed Points}}$: $\mathring{x} = 0 = x - x^3$
$\mathring{y} = -y$

$x^* = 1, 0, -1$

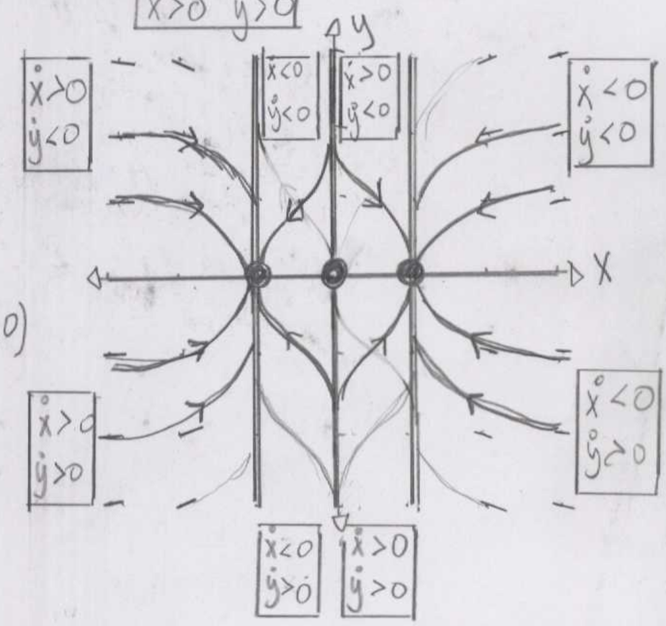
$\mathring{y} = 0 = -y$

$y^* = 0$

$(x^*, y^*) = (1,0), (0,0), (-1,0)$

$\boxed{\text{Nullclines}}$: Horizontal: $y = 0$

Vertical: $x = 0, 1, -1$



$\dot{x} = x(x-y)$
$\dot{y} = y(2x-y)$

6.1.3. $\boxed{\text{Fixed Points}}$: $\dot{x} = 0 = x(x-y)$

$\dot{y} = 0 = y(2x-y)$

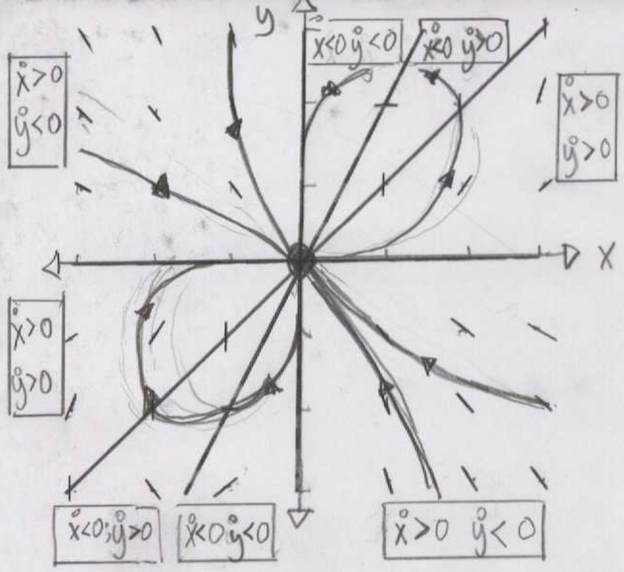
$(x^*, y^*) = (0,0)$

$\boxed{\text{Nullclines}}$: Horizontal: $y = 2x$

$y = 0$

Vertical: $y = x$

$x = 0$



$\dot{x} = y$
$\dot{y} = x(1+y)-1$

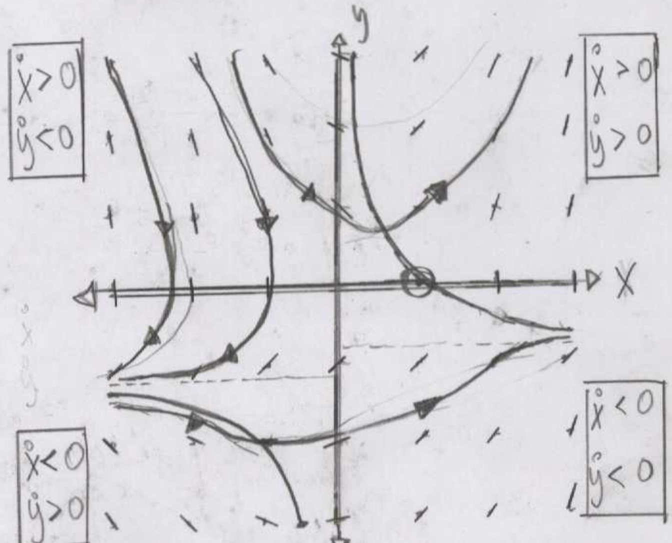
6.1.4: $\boxed{\text{Fixed Points}}$: $\dot{x} = 0 = y$

$\dot{y} = 0 = x(1+y)-1$

$(x^*, y^*) = (1,0)$

$\boxed{\text{Nullclines}}$: Horizontal: $y = \frac{1}{x} + 1$

Vertical: $y = 0$



$\dot{x} = x(2-x-y)$
$\dot{y} = x - y$

6.1.5. $\boxed{\text{Fixed Points}}$: $\dot{x} = 0 = x(2-x-y)$

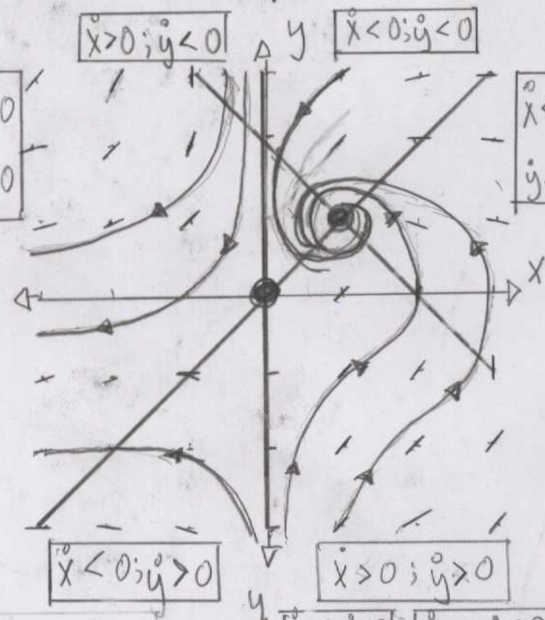
$\dot{y} = 0 = x - y$

$(x^*, y^*) = (0,0), (1,1)$

$\boxed{\text{Nullcline}}$: Horizontal: $y = x$

Vertical: $x = 0$

$y = 2 - x$



$\dot{x} = x^2 - y$
$\dot{y} = x - y$

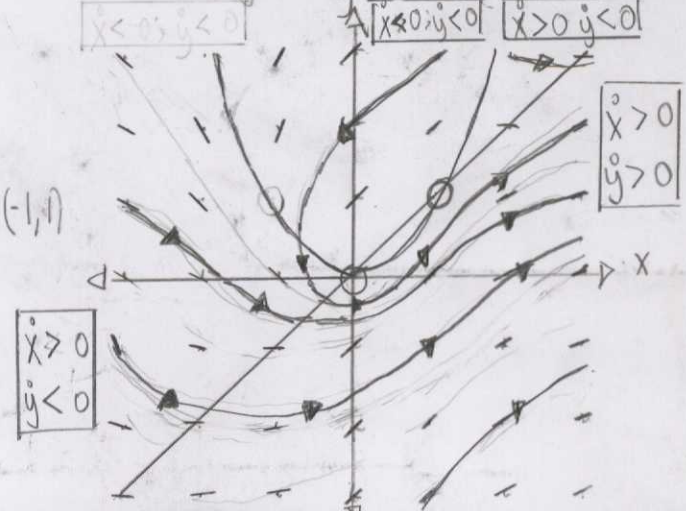
6.1.6. $\boxed{\text{Fixed Points}}$: $\dot{x} = 0 = x^2 - y$

$\dot{y} = 0 = x - y$

$(x^*, y^*) = (0,0), (1,1), (-1,1)$

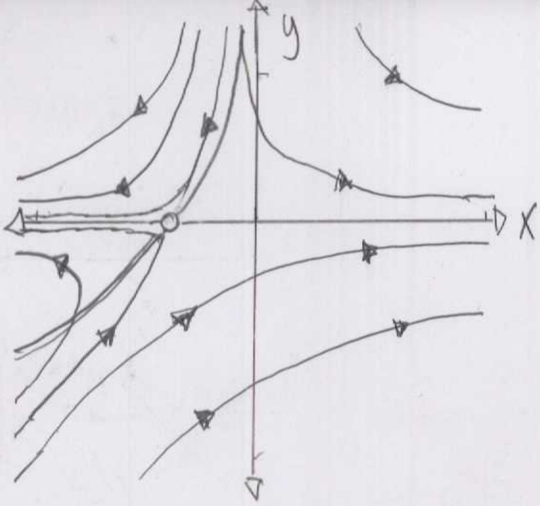
$\boxed{\text{Nullcline}}$: Horizontal: $y = x$

Vertical: $y = x^2$



$\dot{x} = x + e^{-y}$ 6.1.7

$\dot{y} = -y$



$\dot{x} = y$

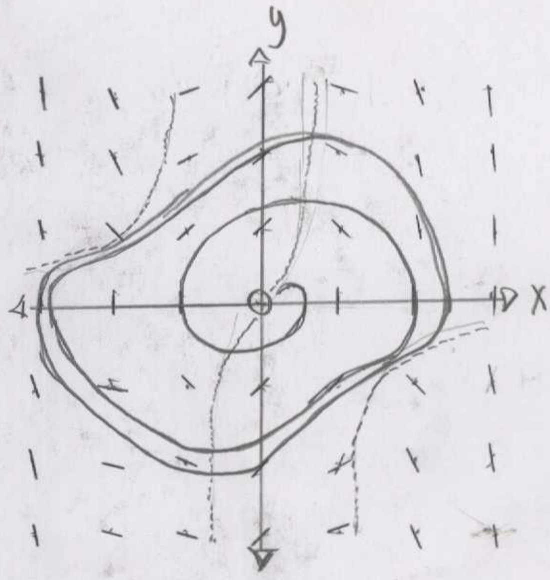
$\dot{y} = -x + y(1-x^2)$ 6.1.8 (Van der Poll oscillator)

$\boxed{\text{Fixed points}}$ $\dot{x} = 0 = y$

$\dot{y} = 0 = -x + y(1-x^2)$

$(x^*, y^*) = (0,0)$

$\boxed{\text{Nullclines}}$ $y = \frac{x}{1-x^2}$



$\dot{x} = 2xy$ 6.1.9. (Dipole Fixed Point)

$\dot{y} = y^2 - x^2$

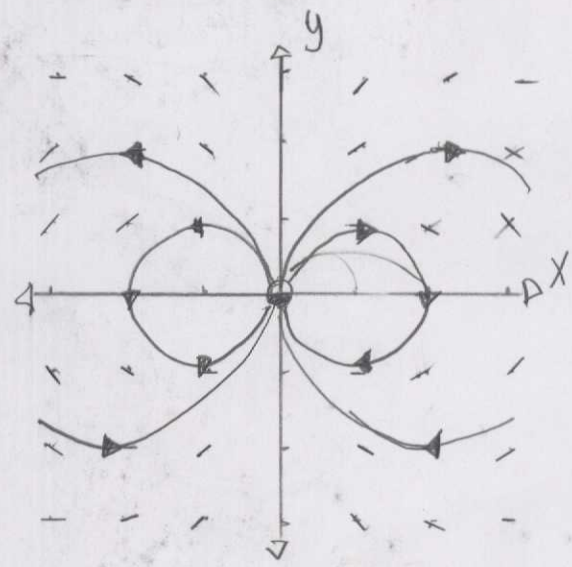
$\boxed{\text{Fixed Point}}$ $\dot{x} = 0 = 2xy$

$\dot{y} = 0 = y^2 - x^2$

$(x^*, y^*) = (0,0)$

$\boxed{\text{Nullcline}}$ $y = x;\ y = 0;\ x = 0.$

$y = -x$



$\dot{x} = y + y^2$ 6.1.10

$\dot{y} = -\frac{1}{2}x + \frac{1}{5}y - xy + \frac{6}{5}y^2$ (Two-eyed Monster)

$\boxed{\text{Fixed Points}}$

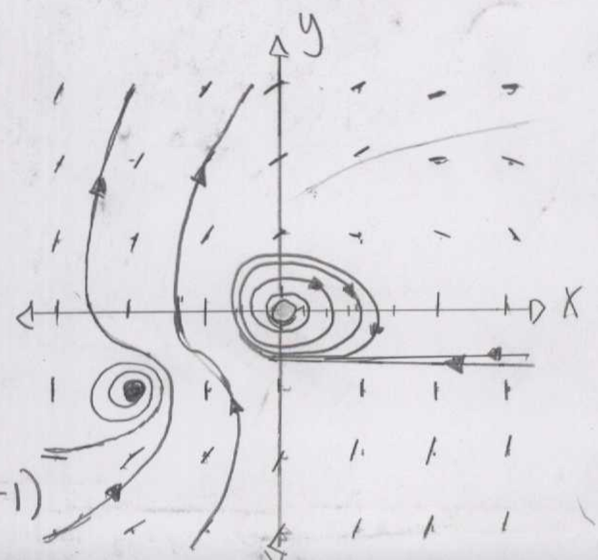
$\dot{x} = 0 = y + y^2$

$\dot{y} = 0 = -\frac{1}{2}x + \frac{1}{5}y - xy + \frac{6}{5}y^2$

$(x^*, y^*) = (0,0), (-2,-1)$

$\boxed{\text{Nullclines}}$ $y = 0;\ x = 0$

$y = \frac{1}{12}\left(\sqrt{25x^2 + 50x + 1} + 5x - 1\right)$



$\dot{x} = y + y^2$

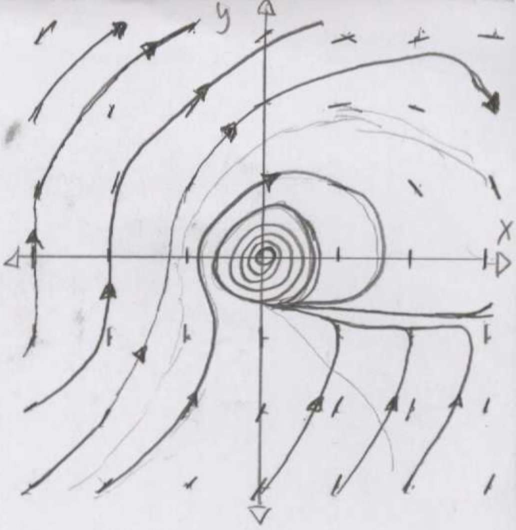
$\dot{y} = -x + \frac{1}{5}y - xy + \frac{6}{5}y^2$

6.1.11 $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = y + y^2$

$\dot{y} = 0$

$= -x + \frac{1}{5}y - xy + \frac{6}{5}y^2$

$(x^*, y^*) = (0,0), (4,4)$

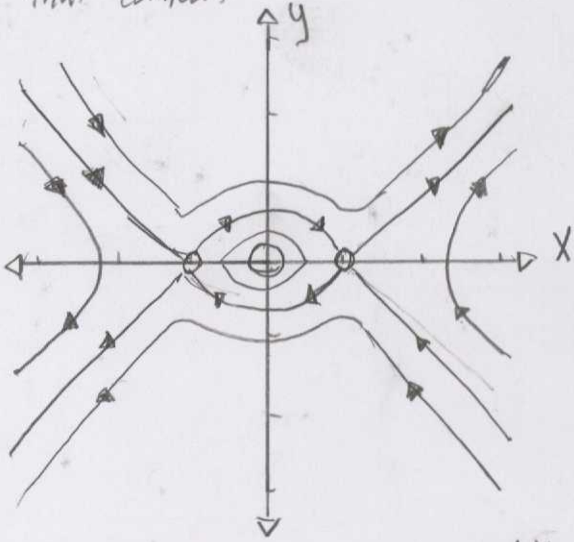


$\boxed{\text{Nullcline}}$

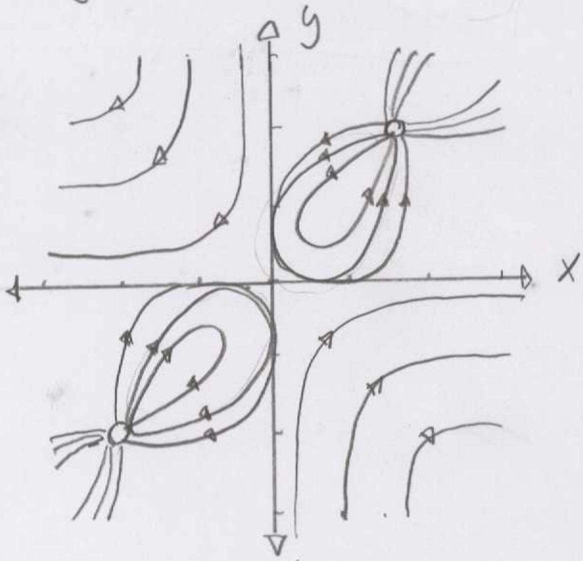
$$y = \frac{1}{12}\left(\pm\sqrt{25x^2 + 110x + 1} + 5x - 1\right)$$

6.1.12.

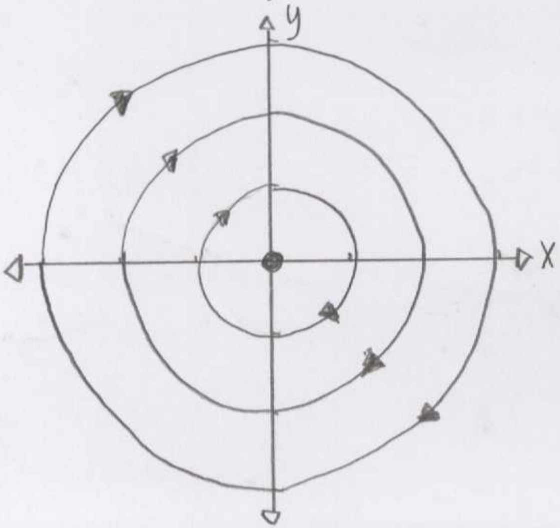
a a single trajectory that connects the saddles.



b. there is no trajectory that connects the saddles



6.1.13: A phase portrait with three closed orbits and one fixed point.



$\dot{x} = x + e^{-y}$ 6.1.14.

$\dot{y} = -y$

a. $u = x+1$; $\frac{dy}{du} = \frac{dy}{dt}\frac{du}{dt} = -y$; $\frac{du}{dt} = \frac{dx}{dt} = u - 1 + (1 - y + \frac{y^2}{2} - \cdots)$

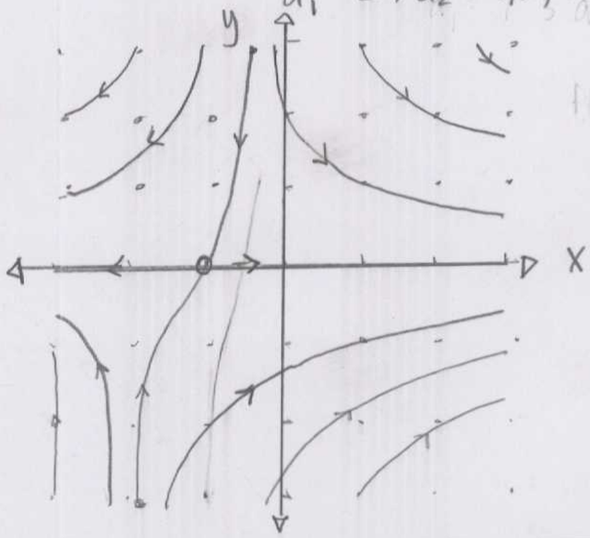
$$\frac{dy}{du} = -\frac{y}{u - y + y^2/2 - y^3/6 + \cdots}$$

$$x = \frac{a_1}{a_1 - 1} + \frac{a_1^3 - 2a_2}{2(a_1-1)^2}u + \frac{2a_1^4 + a_1^5 - 10a_1^2 a_2 + 12(a_2^2 + a_3 - a_1 a_3)}{12(a_1-1)^3}u^3$$

$$\boxed{1 = a_1 + 2a_2 u + 3a_3 u^3 + \cdots}$$

$a_1 = 2$; $a_2 = 4/3$; $a_3 = 10/9$.

b.



$y = \frac{4}{3}(x+1) + \frac{10}{9}(x+1)^3$

Note: The function plotted is not exact.

6.2.1: Yes, trajectories do not intersect, however, may seem so for low resolution plots.

$\dot{x} = y$

$\dot{y} = -x + (1 - x^2 - y^2)y$

6.2.2: a. $D: x^2 + y^2 < 4$

Poincare-Benedixon Theorem : no fixed points and a bounded region, then the trajectory is a closed orbit, and approaches the closed orbit.

Bounded Region - $D: x^2 + y^2 < 4$

Fixed points ; Zero, outside of the center

Existness and Uniqueness is satisfied for a Closed orbit.

b. If $y(t) = \cos(t)$, then $\dot{y} = 0 = -x + (1 - (x^2 + \cos(t)^2))y$ @ $t = 0$

Identity: $x^2 + y^2 = 1$

then, $x = 0$ @ $t = 0$

$$\boxed{x(t) = \sin(t)}$$

c. $x(0)=\frac{1}{2}$; $y(0)=0$; $x(t)^2+y(t)^2$ must be less than one because a larger value forces $\dot{y}$ to become negative and not a closed orbit.

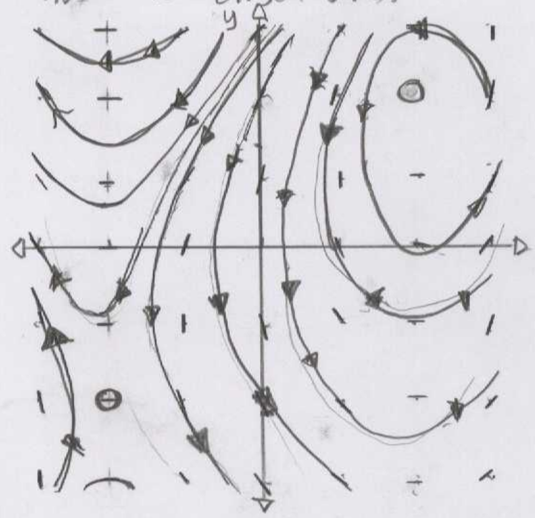
$\dot{x}=x-y$
$\dot{y}=x^2-4$

6.3.1 Fixed points $\dot{x}=0=x-y$

$\dot{y}=0=x^2-4$

$(x^*,y^*)=(2,2),(-2,-2),(0,0)$

"unstable" "unstable"



$\dot{x}=\sin y$
$\dot{y}=x-x^3$

6.3.2 Fixed points $\dot{x}=0=\sin y$

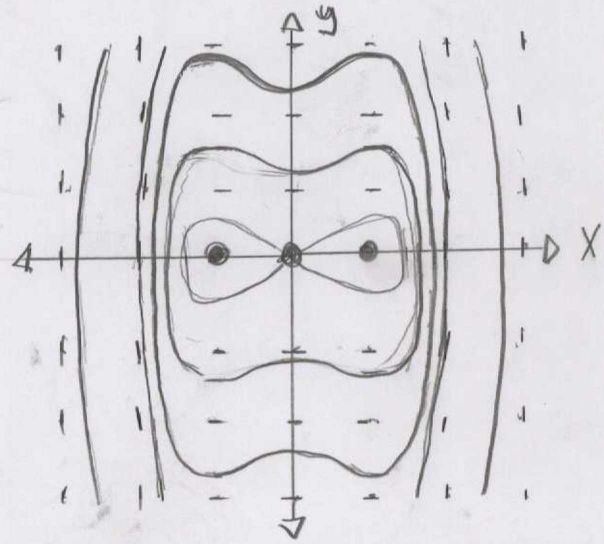
$\dot{y}=0=x-x^3$

$(x^*,y^*)=(1,n\pi)$

$(-1,n\pi)$

$(0,n\pi)$

"stable"



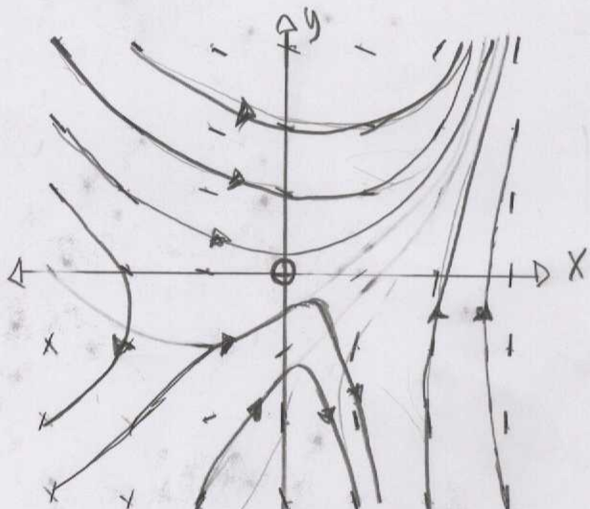
$\dot{x}=1+y+e^{-x}$
$\dot{y}=x^3-y$

6.3.3. Fixed Points $\dot{x}=1+y-e^{-x}=0$

$\dot{y}=x^3-y=0$

$(x^*,y^*)=(0,0)$

"unstable"



$\dot{x}=y+x-x^3$
$\dot{y}=-y$

6.3.4: Fixed Points $\dot{x}=0=y+x-x^3$

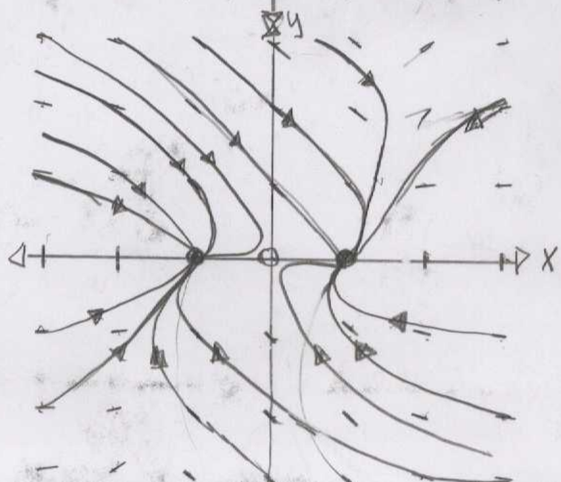
$\dot{y}=0=-y$

$(x^*,y^*)=(1,0),(-1,0)$

"stable"

$(0,0)$

"unstable"



$\dot{x} = \sin y$

$\dot{y} = \cos x$

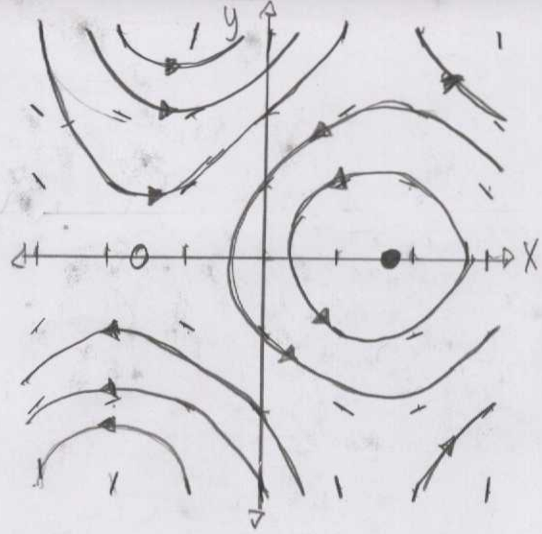
6.3.5. $\boxed{\text{Fixed Points}}$: $\dot{x} = 0 = \sin y$

$\dot{y} = 0 = \cos x$

$(x^*, y^*) = ((n+\tfrac{1}{2})\pi,\ n\pi)$

n is odd "stable"

n is even "unstable"



$\dot{x} = xy - 1$

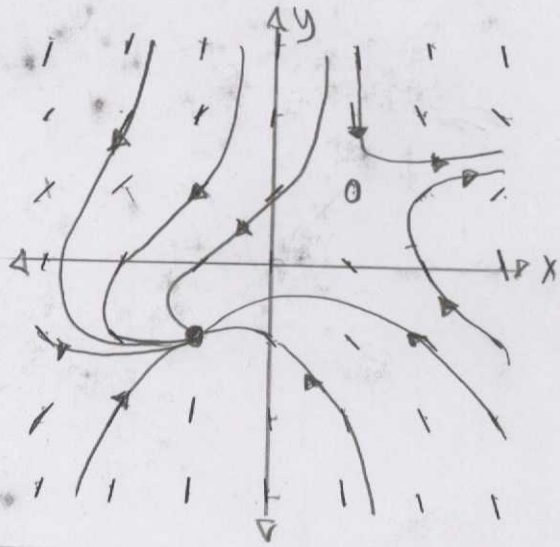
$\dot{y} = x - y^3$

6.3.6. $\boxed{\text{Fixed Points}}$: $\dot{x} = 0 = xy - 1$

$\dot{y} = 0 = x - y^3$

$(x^*, y^*) = (1,1), (-1,-1)$

"ustable" "stable"



6.3.7. The phase portraits of problems 6.3.1-6.3.6 are computer generated.

$\ddot{x} = \dfrac{Gm_2}{(x-a)^2} - \dfrac{Gm_1}{x^2}$

6.3.8. a. $g = \ddot{x} = \dfrac{Gm}{r^2}$; $(m_1) \xleftrightarrow{a} (m_2)$

(x) $(x-a)$

$\ddot{x}_1 = \dfrac{Gm_1}{x^2}$ $\ddot{x}_2 = \dfrac{Gm_2}{(x-a)^2}$; $\ddot{x} = \ddot{x}_2 - \ddot{x}_1 \overset{!}{=} \boxed{\dfrac{Gm_2}{(x-a)^2} - \dfrac{Gm_1}{x^2}}$

b. Equilibrium Position: $\ddot{x} = 0 = \dfrac{Gm_2}{(x-a)^2} - \dfrac{Gm_1}{x^2}$; $m_1(x-a)^2 = m_2 \cdot x^2$

$m_1(x^2 - 2xa + a^2) = m_2 \cdot x^2$

$(m_1 - m_2)x^2 - 2xa + a^2 = 0$

$x = \dfrac{2a \pm \sqrt{4a^2 - 4a^2(m_1 - m_2)}}{2(m_1 - m_2)}$

when $m_1 \neq m_2$

"stable"

$\dot x = y^3 - 4x$
$\dot y = y^3 - y - 3x$

6.3.9. Fixed Points: $\dot x = 0 = y^3 - 4x$; $(x^*, y^*) = (0,0), (2,2), (-2,-2)$
"stable" "unstable" "unstable"

a. $\dot y = 0 = y^3 - y - 3x$

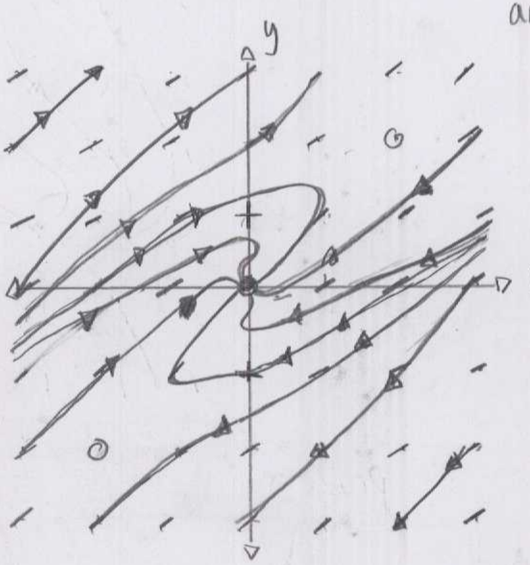
b. If $x = y$, then $\dot x = x^3 - 4x$ and $\dot y = x^3 - 4x$, so $\boxed{\frac{dy}{dx} = 1}$

c. $\lim_{t\to\infty} |\dot x - \dot y| = \lim_{t\to\infty} |y^3 - 4x - y^3 + y + 3x| \stackrel{!}{=} \boxed{\lim_{t\to\infty} |y - x|}$

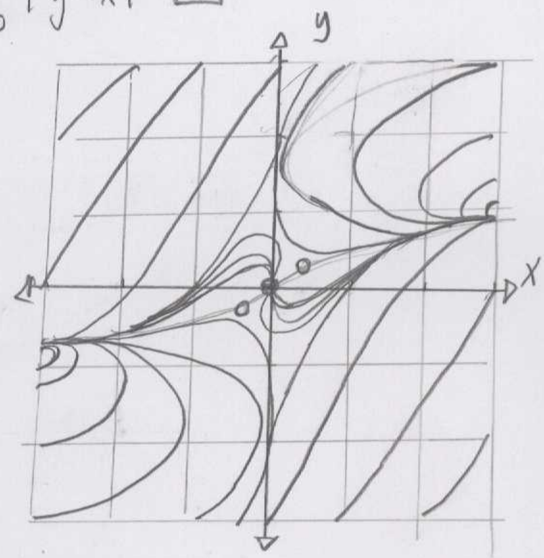
If $\dot u = y - x$, then $y - x = Ce^{-t}$, then $\lim_{t\to\infty} |Ce^{-t}| = \boxed{0}$

and $\lim_{t\to\infty} |y - x| = \boxed{0}$

d.



e.



$\dot x = xy$
$\dot y = x^2 - y$

6.3.10

a. $u = x - x^*$; $v = y - y^*$; $\dot u = \dot x = f(x^* + u, y^* + v) \doteq f(x^*, y^*) + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + \cdots$

$\dot v = \dot y = g(x^* + u, y^* + v) = g(x^*, y^*) + u\frac{\partial g}{\partial x} + v\frac{dg}{dy} + \cdots$

$$\begin{pmatrix} \dot u \\ \dot v \end{pmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} ; A = \begin{bmatrix} y & x \\ 2x & -1 \end{bmatrix}$$

Fixed Point $(0,0)$; $A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$, so the origin is a non-isolated fixed point because $\Delta = 0$.

b. $\begin{pmatrix} \dot u \\ \dot v \end{pmatrix} = 0$: $(A - \lambda)U = \dot U = 0$; $(A - \lambda) = \begin{bmatrix} y-\lambda & x \\ 2x & -1-\lambda \end{bmatrix} = (y-\lambda)(-1-\lambda) - 2x^2 = 0$

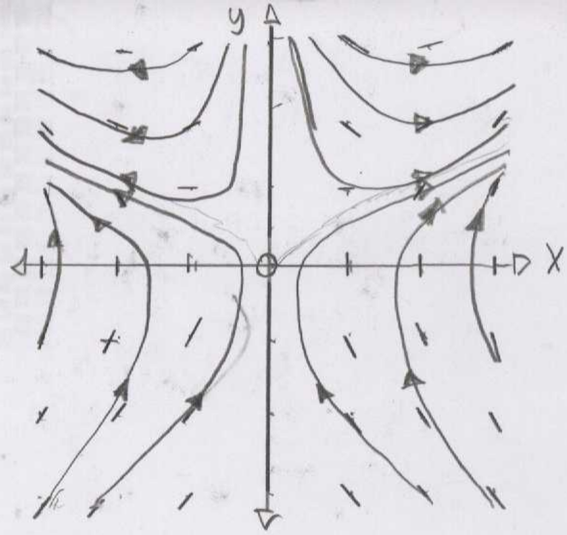
$$\lambda = \frac{(y-1) \pm \sqrt{8x^2 + (y-1)^2}}{2}$$

Thus, $\Delta = \lambda_1 \lambda_2 \neq 0$ and the center is an isolated fixed point.

c. $\boxed{\text{Nullclines}}$ $y = \pm\sqrt{x}$

$y = 0 ; x = 0$

"Saddle Point"



$\boxed{\text{d. See Part C}}$

$\dot{r} = -r$

$\dot{\theta} = \frac{1}{\ln r}$

6.3.11. a. $r(t) = Ce^{-t}$; $\theta(t) = \ln \frac{\ln C}{\ln C - t} + D$; Given (r_0, θ_0); then $r(t) = r_0 e^{-t}$

$$\boxed{\theta(t) = \ln \frac{\ln r_0}{|\ln r_0 - t|} + \theta_0}$$

b. $\lim_{t\to\infty} |\theta(t)| = \lim_{t\to\infty} \left| \ln \frac{\ln r_0}{|\ln r_0 - t|} + \theta_0 \right| \boxed{\neq \infty}$

$\lim_{t\to\infty} r(t) = \lim_{t\to\infty} r_0 e^{-b} \overset{!}{=} \boxed{0}$

c. $\dot{r} = -\sqrt{x^2 + y^2}$; $\dot{\theta} \overset{!}{=} \boxed{\frac{1}{\ln\sqrt{x^2 + y^2}}}$

d. $\dot{r} = \frac{d}{dt}\sqrt{x^2 + y^2} = \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2 + y^2}} = -r = -\sqrt{x^2 + y^2}$

$$x\dot{x} + y\dot{y} = -x^2 - y^2$$

$\dot{\theta} = \frac{d}{dt}\arctan\left(\frac{y}{x}\right) = \frac{x\dot{y} - \dot{x}y}{x^2 + y^2} = \frac{1}{\ln(\sqrt{x^2 + y^2})}$;

$$x\dot{y} - \dot{x}y = \frac{2(x^2 + y^2)}{\ln(x^2 + y^2)}$$

$$x(x\dot{x} - y\dot{y}) - y(x\dot{y} - y\dot{x}) = (x^2 + y^2)\dot{x} = -x(x^2 + y^2) - y\frac{2(x^2 + y^2)}{\ln(x^2 + y^2)}$$

$$\boxed{\dot{x} = -x - \frac{2y}{\ln(x^2 + y^2)}}$$

$$x(x\dot{y} - y\dot{x}) + y(x\dot{x} + y\dot{y}) = (x^2 + y^2)\dot{y} = -y(x^2 + y^2) + x\frac{2(x^2 + y^2)}{\ln(x^2 + y^2)}$$

$$\boxed{\dot{y} = -y + \frac{2x}{\ln(x^2 + y^2)}}$$

d. $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = A\begin{pmatrix}x\\ y\end{pmatrix}$; $A = \begin{pmatrix} -1 + \frac{4xy}{(x^2+y^2)\ln^2(x^2+y^2)} & \frac{4y^2}{(x^2+y^2)\ln^2(x^2+y^2)} - \frac{2}{\ln(x^2+y^2)} \\ \frac{2}{\ln(x^2+y^2)} - \frac{4x^2}{(x^2+y^2)\ln^2(x^2+y^2)} & \frac{-4xy}{(x^2+y^2)\ln^2(x^2+y^2)} - 1 \end{pmatrix}$

$$A_{(0,0)} = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} ; \boxed{\dot{x} = -x,\ \dot{y} = -y}$$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ 6.3.12. $\dot{\theta} = \frac{d}{dt}\tan^{-1}\left(\frac{y}{x}\right) = \frac{\frac{d}{dt}\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = \frac{\dot{y}x - \dot{x}y}{x^2+y^2} = \boxed{\frac{x\dot{y} - y\dot{x}}{r^2}}$

$\dot{x} = -y - x^3$ 6.3.13. Linearization: $u = x - x^*$; $v = y - y^*$

$\dot{y} = x$

$$\dot{x} = \dot{u} = f(x,y) = f(u+x^*, v+y^*) = f(x^*,y^*) + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + \cdots$$

$$\dot{y} = \dot{v} = g(x,y) = g(u+x^*, v+y^*) = g(x^*,y^*) + u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y} + \cdots$$

$$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = \begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{bmatrix}\begin{pmatrix}u\\ v\end{pmatrix} ; A = \begin{bmatrix}-3x^2 & -1\\ 1 & 0\end{bmatrix}$$

$$A_{(0,0)} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

$$\Delta = 0 ; \tau = 0 ; \boxed{\text{center}}$$

Eigenvalues: $\dot{\vec{U}} = A\vec{U}$; $\lambda U = AU$; $(A - \lambda)U = 0$;

$$(A - \lambda) = \begin{bmatrix}-\lambda & -1\\ 1 & -\lambda\end{bmatrix} = \lambda^2 + 1 = 0$$

$$\lambda_{1,2} = \pm i$$

$\boxed{\text{Thus, } \Delta = -1,\ \tau = 0, \text{ so the center is a spiral, also supported by } \tau^2 - 4\Delta > 0.}$

$\dot{x} = -y + ax^2$ 6.3.14. $a > 0$; Fixed points: $\dot{x} = 0 = -y + ax^2$;

$\dot{y} = x + ay^3$

$$\dot{y} = 0 = x + ay^3 ;$$

$$(x^*, y^*) = (0,0)$$

Linearization: $u = x - x^*$; $v = y - y^*$

$$\dot{x} = \dot{u} = f(x,y) = f(u+x^*, v+y^*) = f(x^*,y^*) + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + \cdots$$

$$\dot{y} = \dot{v} = g(x,y) = g(u+x^*, v+y^*) = g(x^*,y^*) + u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y} + \cdots$$

$$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = \begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{bmatrix}\begin{pmatrix}u\\ v\end{pmatrix} ; A = \begin{bmatrix}2ax & -1\\ 1 & 3ay^2\end{bmatrix} ;$$

$$A_{(0,0)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}; \ \Delta = 0, \ \tau = 0; \text{ center}$$

Eigenvalues: $\dot{\mathbb{U}} = A\mathbb{U}$; $\lambda \mathbb{U} = A\mathbb{U}$; $(A-\lambda)\mathbb{U} = 0$

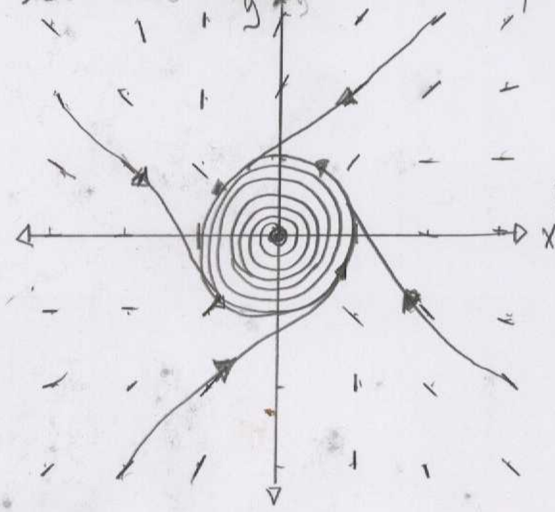
$$(A-\lambda) = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

$$\lambda_{1,2} = \pm i$$

Thus, eigenvalues demonstrate $\Delta = -1$, $\tau = 0$, $\tau^2 - 4\Delta > 0$, so the center is a spiral.

$\dot{r} = r(1-r^2)$ 6.3.15

$\dot{\theta} = 1 - \cos\theta$

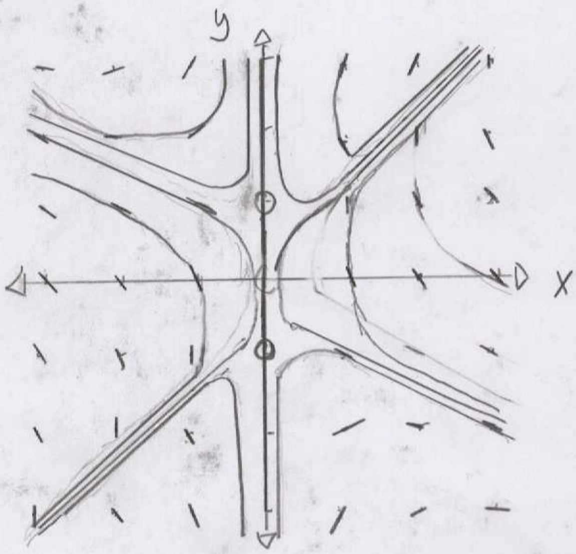


"Liapunov Stable."

$\dot{x} = a + x^2 - xy$ 6.3.16

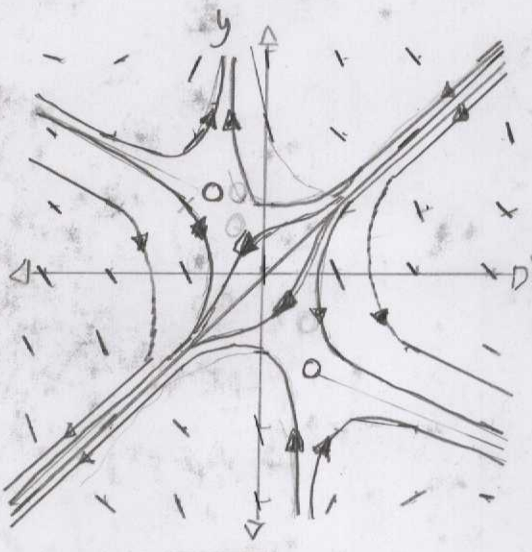
$\dot{y} = y^2 - x^2 - 1$

a.

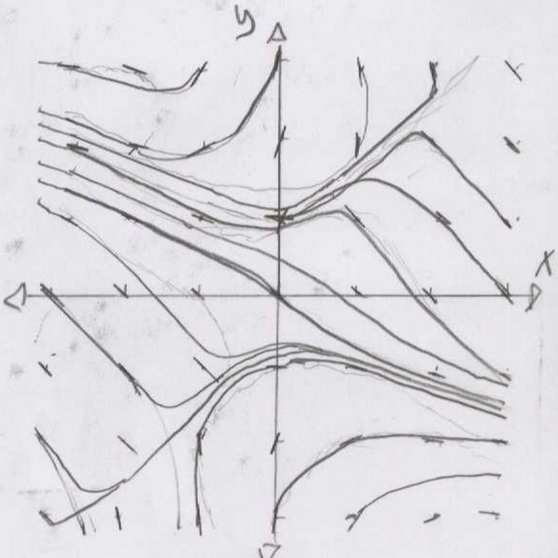


"Saddle connection"

b.



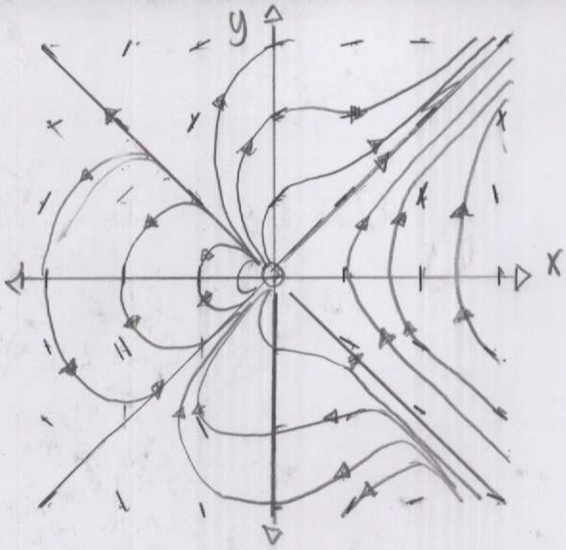
$\boxed{a<0}$



$\boxed{a>0}$

$\dot{x} = xy - x^2y + y^3$ 6.3.17.

$\dot{y} = y^2 + x^3 - xy^2$



$\dot{x} = x(3-x-y)$ 6.4.1 $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = x(3-x-y)$

$\dot{y} = y(2-x-y)$

$\dot{y} = 0 = y(2-x-y)$

$(x^*, y^*) = (0,0)$ "unstable"

$(3,0)$ "stable"

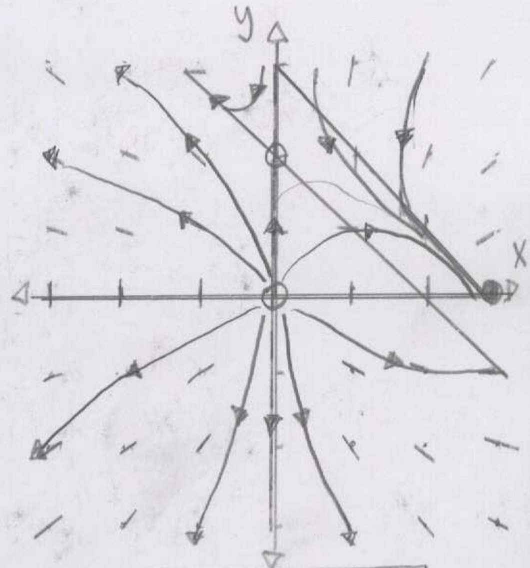
$(0,2)$ "unstable"

$\boxed{\text{Nullclines}}$ $x = 0$

$y = 0$

$y = 2-x$; $y = 3-x$

$\boxed{\text{Basin of Attraction}}$ $x \geq 0 \wedge y \geq 0$



$\boxed{\text{I forgot } (x \text{ and } y) \geq 0}$

$\dot{x} = x(3-2x-y)$ 6.4.2 $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = x(3-2x-y)$

$\dot{y} = y(2-x-y)$

$\dot{y} = 0 = y(2-x-y)$

$(x^*, y^*) = (0,0)$ "unstable"

$(0,2)$ "unstable"

$(1,1)$ "stable"

$(3/2, 0)$ "unstable"

$\boxed{\text{Nullclines}}$ $x = 0$; $y = 0$

$y = 3-2x$

$y = 2-x$

$\boxed{\text{Basin of Attraction}}$ $(x \geq 0) \wedge (y \geq 0)$

$\dot{x} = x(3-2x-2y)$
$\dot{y} = y(2-x-y)$

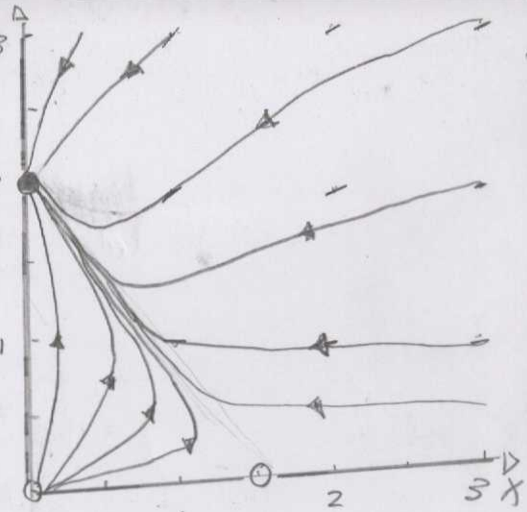
6.4.3. $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = x(3-2x-2y)$

$\dot{y} = 0 = y(2-x-y)$

$(x^*, y^*) = (0,0)$ "unstable"

$(0,2)$ "Stable"

$(3/2, 0)$ "unstable"



$\boxed{\text{Nullclines}}$ $x = 0$; $y = 0$

$y = 2 - x$; $y = \frac{3}{2} - x$

$\boxed{\text{Basin of Attraction}}$ $(x>0) \wedge (y>0)$

$\dot{N}_1 = r_1N_1 - bN_1N_2$
$\dot{N}_2 = r_2N_2 - b_2N_1N_2$

6.4.4.

a. The N_1 and N_2 model is less realistic because population for rabbits and sheep decreases from an interaction.

b. Unable to complete problem without $r_1 = r_2 = b = b_2 = 1$

$x = N_1$; $y = N_2$; $t = \tau$; $x' = x(1-y)$; $y' = x(1-y)$

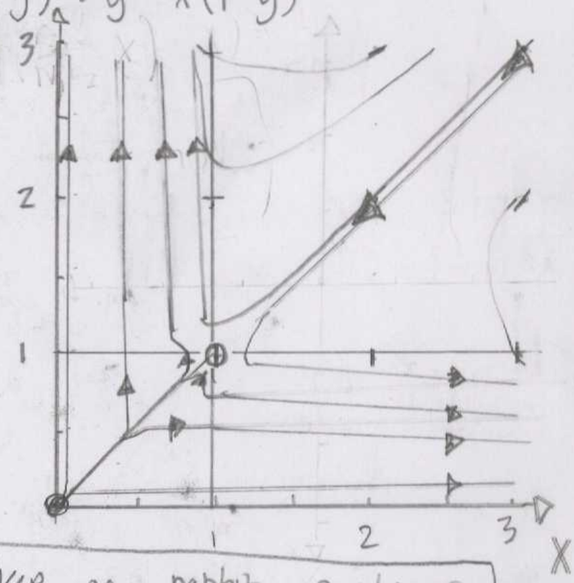
c. $\boxed{\text{Fixed Points}}$ $x' = 0 = x(1-y)$

$y' = 0 = y(1-x)$

$(x^*, y^*) = (0,0), (1,1)$

$\boxed{\text{Nullclines}}$ $y = 1$; $x = 1$

$y = 0$; $x = 0$



d. See part c in order to denote sheep or rabbit populations approach infinity when rabbit per sheep is less than 1 or sheep per rabbit is less than 1.

e. $\frac{dx}{dy} = \frac{x}{y}\frac{(1-y)}{(1-x)}$; $\int \frac{(1-x)}{x}\,dx = \int \frac{(1-y)}{y}\,dy$; $\ln x - x = \boxed{\ln y - y + C}$

when $\rho = 1$

$\dot{N}_1 = r_1N_1(1 - N_1/K_1) - b_1N_1N_2$
$\dot{N}_2 = r_2N_2 - b_2N_1N_2$

6.4.5. $\frac{dN_1}{dt}\left(\frac{1}{K_1}\right) = r_1N_1\left(\frac{1}{K_1}\right)(1 - N_1/K_1) - b_1N_1N_2\left(\frac{1}{K_1}\right)$; $x = N_1/K_1$

$\frac{dx}{dt} = r_1x(1-x) - bxN_2$; $\frac{dx}{dt}\left(\frac{1}{r_1}\right) = x(1-x) - \frac{b}{r_1}xN_2$

$\rho = \frac{b_1}{r_1}$; $\tau = tr$; $N_2 = y$

$$\boxed{x' = x(1-x) - \rho_1 x \cdot y}$$

$y' = \frac{r_2}{r_1} N_2 - \frac{b_2}{r_1} N_1 N_2 = y' = Ry - \rho_2 xy$

where $R = \frac{r_2}{r_1}$; $\rho_2 = \boxed{\frac{b_2}{r_1} K_1}$

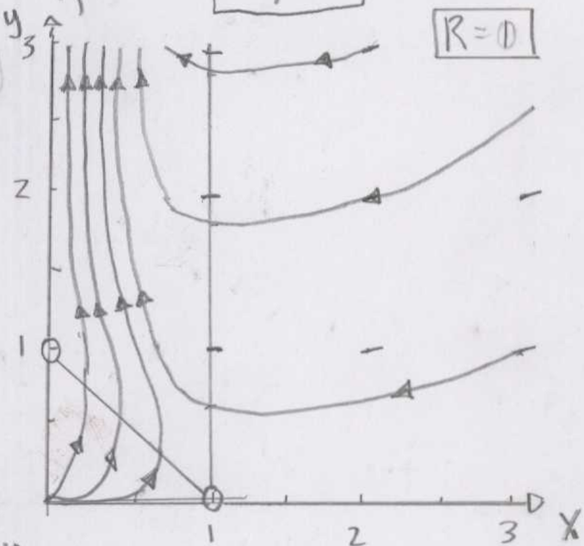
$\boxed{\text{Fixed Points}}$ $x' = 0 = x(1 - x - \rho_1 y)$

$y' = 0 = y(R - \rho_2 x)$

$(x^*, y^*) = (0,0), (1,0)$

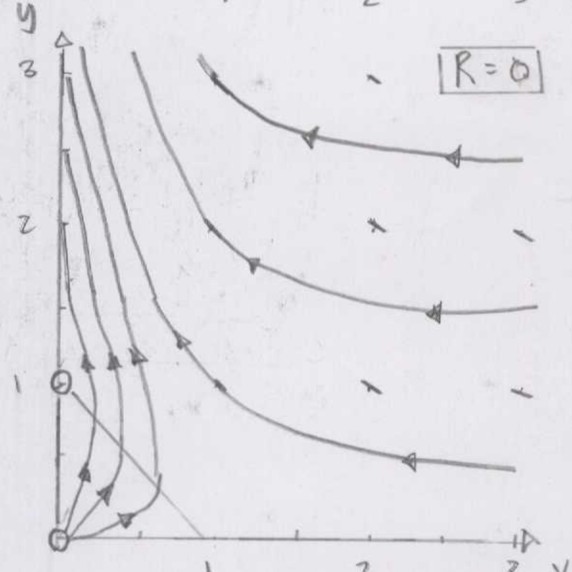
If $R = 0$, $(0, y)$

and $\rho_1 = \rho_2 = 1$



$\dot{N}_1 = r_1 N_1 (1 - N_1 / K_1) - b_1 N_1 N_2$

$\dot{N}_2 = r_2 N_2 (1 - N_2 / K_2) - b_2 N_1 N_2$



6.4.6.

a. $\dot{N}_1 \left(\frac{1}{K_1}\right) = r_1 \frac{N_1}{K_1} (1 - N_1/K_1) - b_1 \frac{N_1}{K_1} N_2$

$x = \frac{N_1}{K_1}$

$\frac{dx}{dt} = r_1 x (1 - x) - b_1 x N_2$

$\dot{N}_2 \left(\frac{1}{K_2}\right) = r_2 \frac{N_2}{K_2} \left(1 - \frac{N_2}{K_2}\right) - b_2 N_1 \frac{N_2}{K_2}$

$y = N_2 / K_2$

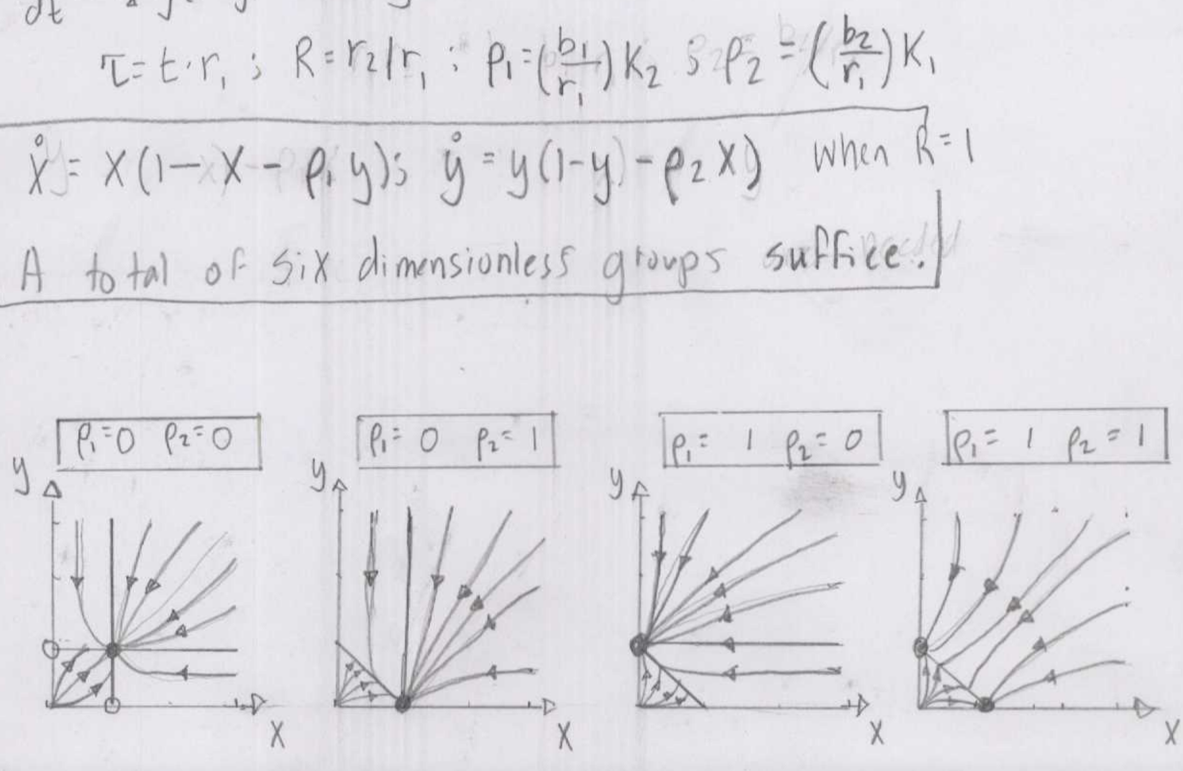
$\frac{dy}{dt} = r_2 y (1 - y) - b_2 N_1 y$

$\tau = t \cdot r_1$; $R = r_2 / r_1$; $\rho_1 = \left(\frac{b_1}{r_1}\right) K_2$; $\rho_2 = \left(\frac{b_2}{r_1}\right) K_1$

$\boxed{\dot{x} = x(1 - x - \rho_1 y); \ \dot{y} = y(1 - y - \rho_2 x) \text{ when } R = 1}$

$\boxed{\text{A total of six dimensionless groups suffice.}}$

b.



c. $\boxed{\text{The species coexist when } \rho_1 = \rho_2 = 0\text{. This parameter describes the interaction between the rabbits and sheep as noncompetitive.}}$

$\dot{n}_1 = G_1 N n_1 - K_1 n_1$

$\dot{n}_2 = G_2 N n_2 - K_2 n_2$

6.4.7. $N(t) = N_0 - \alpha_1 n_1 - \alpha_2 n_2$

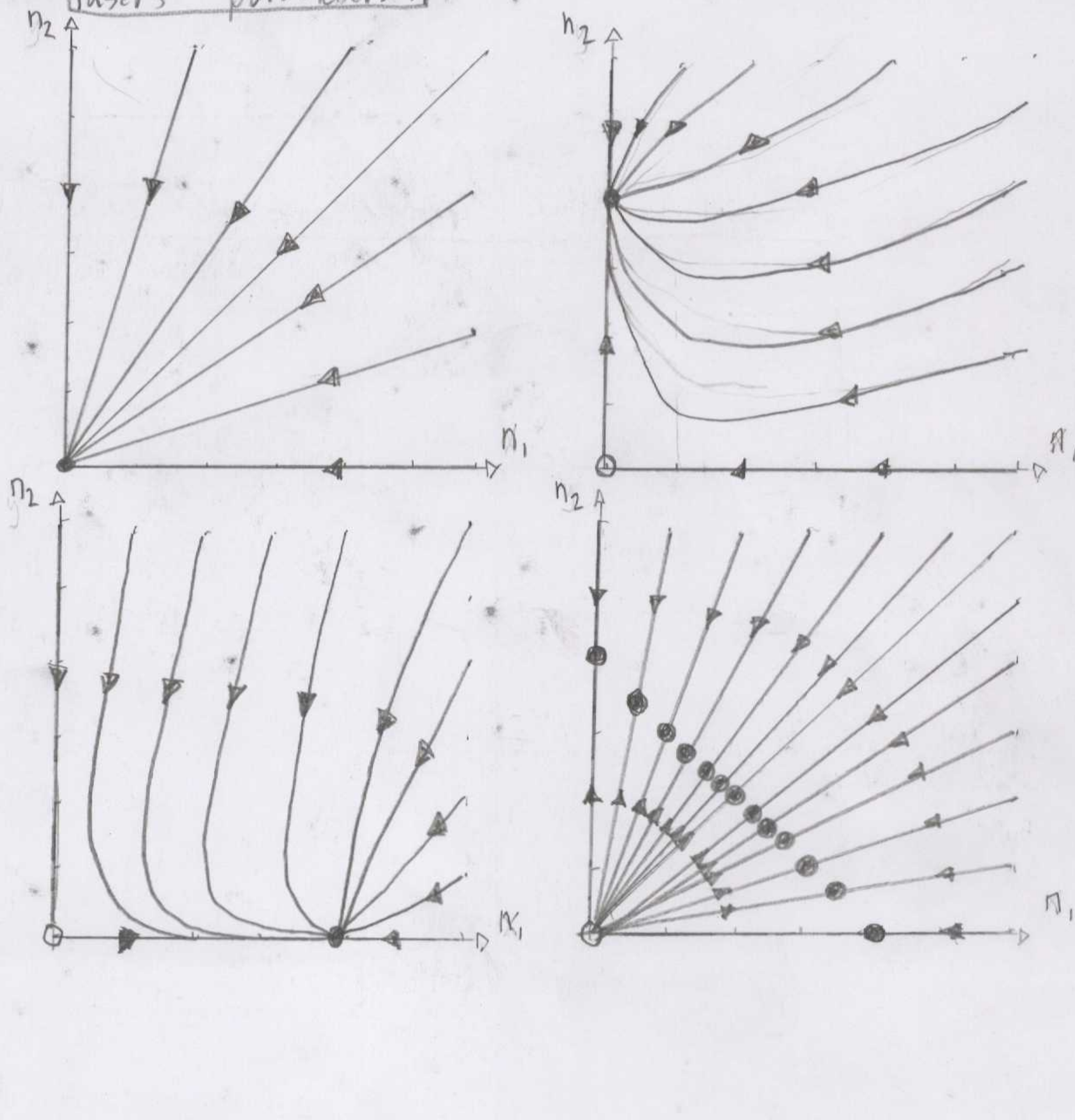
a.
$$A = \begin{pmatrix} \frac{d\dot{n}_1}{dn_1} & \frac{d\dot{n}_1}{dn_2} \\ \frac{d\dot{n}_2}{dn_1} & \frac{d\dot{n}_2}{dn_2} \end{pmatrix} = \begin{pmatrix} G_1 N - K_1 & 0 \\ 0 & G_2 N - K_2 \end{pmatrix}$$

$\Delta = (G_1 N - K_1)(G_2 N - K_2)$; $\tau = (G_1 + G_2)N - (K_1 + K_2)$

$\boxed{\tau^2 - 4\Delta > 0 \text{ ; Unstable Node}}$

b. $\boxed{\text{The other fixed points are } G_1 N = K_1 \text{ and } G_2 N = K_2}$

c. $\boxed{\text{Four phase portraits appear by varying the lasers parameters.}}$ If



$\dot{x} = ax^c - \phi x$

$\dot{y} = by^c - \phi y$

$\phi \equiv ax^c + by^c$

6.4.8. a. If $x_0 + y_0 = 1$, $\dot{x} + \dot{y} = ax^c - (ax^c + by^c)x + by^c - (ax^c + by^c)y$

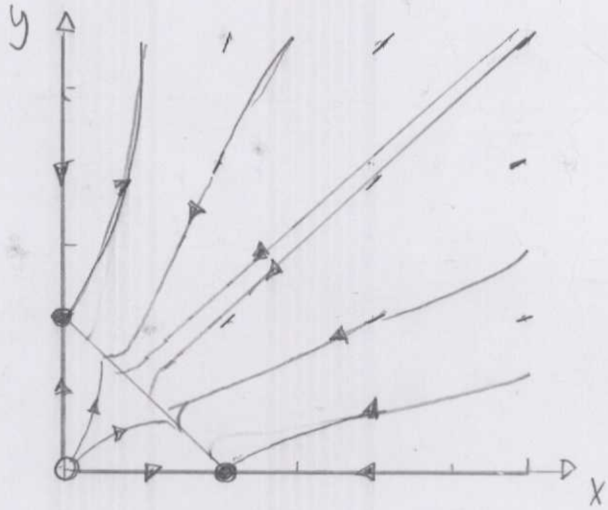
$$= a(1-x-y)x^c + b(1-x-y)y^c$$

then $\boxed{\dot{x} + \dot{y} = 0 \quad \text{and} \quad x(t) + y(t) = 1}$

b. $\lim_{x \to \infty} \dot{y}/\dot{x} = \frac{by^c - \phi y}{ax^c - \phi x} = \frac{by^c - (ax^c + by^c)y}{ax^c - (ax^c + by^c)x} \cong \frac{-ax^c}{-ax^{c+1}} = \frac{1}{x} = \boxed{0}$

$\lim_{y \to \infty} \dot{y}/\dot{x} = \frac{by^c - \phi y}{ax^c - \phi x} = \frac{by^c - (ax^c + by^c)y}{ax^c - (ax^c + by^c)x} \cong \frac{-by^{c+1}}{-by^c x} = \boxed{\infty}$

c. If $c = 1$,



d. $\boxed{\text{If } c > 1, \text{ then radial nullclines become generated.}}$

e. $\boxed{\text{If } c < 1, \text{ then monotonically decreasing nullclines become generated.}}$

$\dot{I} = I - \alpha C$

$\dot{C} = \beta(I - C - G)$

6.4.9. $I \geq 0$: National Income ; $C \geq 0$: Rate of Consumer Spending.

$G \geq 0$: Rate of Government Spending.

$1 < \alpha < \infty$ and $1 \leq \beta < \infty$

a. $\boxed{\text{Fixed Points}}$ $\dot{I} = 0 = I - \alpha C$; $\dot{C} = 0 = \beta(I - C - G)$

$$(I^*, C^*) = \left(\frac{\alpha G}{\alpha - 1}, \frac{G}{\alpha - 1}\right)$$

$$\dot{\mathbb{I}} = A \cdot \mathbb{I} \; ; \; A = \begin{pmatrix} \frac{\partial \dot{I}}{\partial I} & \frac{\partial \dot{I}}{\partial C} \\ \frac{\partial \dot{C}}{\partial I} & \frac{\partial \dot{C}}{\partial C} \end{pmatrix} = \begin{pmatrix} 1 & -\alpha \\ \beta & -\beta \end{pmatrix}$$

If $\beta = 1$, $A = \begin{pmatrix} 1 & -\alpha \\ 1 & -1 \end{pmatrix}$, $\Delta = -(1-a)$; $\tau = 0$; $\tau^2 - 4\Delta \Rightarrow 4(1-a)$

$\boxed{\text{A center node}}$

b. $G = G_0 + \kappa I ;\ \kappa > 0 ;\ I \geq 0,\ C \geq 0$

$\boxed{\text{Fixed Point}}$ $(I^*, C^*) = \left(\frac{\alpha G_0}{\alpha(1-\kappa)-1}, \frac{G_0}{\alpha(1-\kappa)-1} \right)$

If $\kappa < \kappa_c = 1 - \frac{1}{\alpha}$, then $I \ \& \ C > 0$

$\mathring{I} = A I ;\ A = \begin{pmatrix} 1 & -\alpha \\ \beta(1-\kappa) & -\beta \end{pmatrix} ;\ (A - \lambda I) = \begin{pmatrix} 1-\lambda & -\alpha \\ \beta(1-\kappa) & -\beta-\lambda \end{pmatrix}$

$$(1-\lambda)(-\beta-\lambda) + \alpha \cdot \beta(1-\kappa) = 0$$

$$\lambda_{1,2} = \frac{-(\beta-1) \pm \sqrt{(\beta-1)^2 - 4(\alpha(1-\kappa)-1)\beta}}{2}$$

$$A\vec{v}_1 = \begin{pmatrix} (1+\beta)+\sqrt{(\beta-1)^2-4(\alpha(1-\kappa)-1)\beta} & -\alpha \\ \beta(1-\kappa) & \frac{-(1+\beta)+\sqrt{(\beta-1)^2-4(\alpha(1-\kappa)-1)\beta}}{2} \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$$

$$\left[(1+\beta)+\sqrt{(\beta-1)^2-4(\alpha(1-\kappa)-1)\beta}\right] v_1 - \alpha_1 v_2 = 0$$

$$v_{11} = 1 ;\ v_{12} = \frac{(1+\beta)+\sqrt{\beta^2+(2-4\alpha(1-\kappa))\beta}}{\alpha}$$

$$\boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ \frac{(1+\beta)+\sqrt{\beta^2+(2-4\alpha(1-\kappa))\beta}}{\alpha} \end{pmatrix}}$$

$$A\vec{v}_2 = \begin{pmatrix} (1+\beta)-\sqrt{(\beta-1)^2-4(\alpha(1-\beta)-1)\beta} & -\alpha \\ \beta(1-\kappa) & \frac{-(1+\beta)-\sqrt{(\beta-1)^2-4(\alpha(1-\kappa)-1)\beta}}{2} \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$$

$$\left[(1+\beta)-\sqrt{(\beta-1)^2-4(\alpha(1-\kappa)-1)\beta}\right] v_1 - \alpha_1 v_2 = 0$$

$$v_{21} = 1 ;\ v_{22} = \frac{(1+\beta)-\sqrt{\beta^2+(2-4\alpha(1-\kappa))\beta}}{\alpha}$$

$$\boxed{\vec{v}_2 = \begin{pmatrix} 1 \\ \frac{(1+\beta)-\sqrt{\beta^2+(2-4\alpha(1-\kappa))\beta}}{\alpha} \end{pmatrix}}$$

When $k > k_c$, the economy gravitates to the positive eigendirection.

c) $G = G_0 + kI^2$; $\dot{I} = I - \alpha C = 0$; $\dot{C} = \beta(I - C - G_0 - k_0 I^2) = 0$

Fixed Points: $0 = \beta(I(1-1/\alpha) - G_0 - k_0 I^2) = -k_0 I^2 + I(1-1/\alpha) - G_0$

$$(I^*, C^*) = \left(\frac{+(1-1/\alpha) + \sqrt{(1-1/\alpha)^2 - 4k_0 G_0}}{2k_0}, \frac{(1-1/\alpha) + \sqrt{(1-1/\alpha)^2 - 4k_0 G_0}}{2k_0 \alpha}\right)$$

$$, \left(\frac{(1-1/\alpha) - \sqrt{(1-1/\alpha)^2 - 4k_0 G_0}}{2k_0}, \frac{(1-1/\alpha) - \sqrt{(1-1/\alpha)^2 - 4k_0 G_0}}{2k_0 \alpha}\right)$$

If $G_0 < \frac{(\alpha-1)^2}{4\alpha^2 k_0}$, then two positive fixed points exist

If $G_0 = \frac{(\alpha-1)^2}{4\alpha^2 k_0}$, then one fixed point exists in quadrant #1

If $G_0 > \frac{(\alpha-1)^2}{4\alpha^2 k_0}$, then zero fixed points exist because of the imaginary radical.

6.4.10. $\dot{x}_i = x_i\left(x_{i-1} - \sum_{j=1}^{n} x_j x_{j-1}\right)$

a. If $n=2$, $\dot{x}_1 = x_1(x_0 - \sum_{j=1}^{2} x_1 x_0) = x_1(x_2 - \sum_{j=1}^{2} x_1 x_2) = x_1(x_2 - 2x_1 x_2)$

$\dot{x}_2 = x_2(x_1 - \sum_{j=1}^{2} x_2 x_1) = \boxed{x_2(x_1 - 2x_2 x_1)}$

b. $\dot{x}_1 = 0 = x_1(x_2 - 2x_1 x_2)$; $\dot{x}_2 = 0 = x_2(x_1 - 2x_2 x_1)$

$(x_1^*, x_2^*) = (1/2, 1/2)$; $A = \begin{pmatrix} x_2 - 4x_1 x_2 & x_1(1-2x_1) \\ x_2(1-2x_2) & x_1 - 4x_1 x_2 \end{pmatrix}$

$$A_{(1/2,1/2)} = \begin{pmatrix} -1/2 & 0 \\ 0 & -1/2 \end{pmatrix} ; \Delta = \frac{1}{4}, \tau = -1$$

$\tau^2 - 4\Delta = 0$

Degenerate and stable Node

c. $u = x_1 + x_2$; $\dot{u} = \dot{x}_1 + \dot{x}_2 = x_1(x_2 - 2x_1 x_2) + x_2(x_1 - 2x_1 x_2)$

$= x_1 x_2 - 2x_1^2 x_2 + x_1 x_2 - 2x_1 x_2^2$

$= 2x_1 x_2(1 - x_1 - x_2) = 2x_1 x_2(1-u)$

$u(t) = 1 - e^{-2x_1 x_2 t}$

$\boxed{\lim_{t\to\infty} u(t) = 1}$

d. $v = x_1 - x_2$; $\dot{v} = \dot{x}_1 - \dot{x}_2$; $\dot{v} = x_1(x_2 - 2x_1x_2) - x_2(x_1 - 2x_1x_2)$

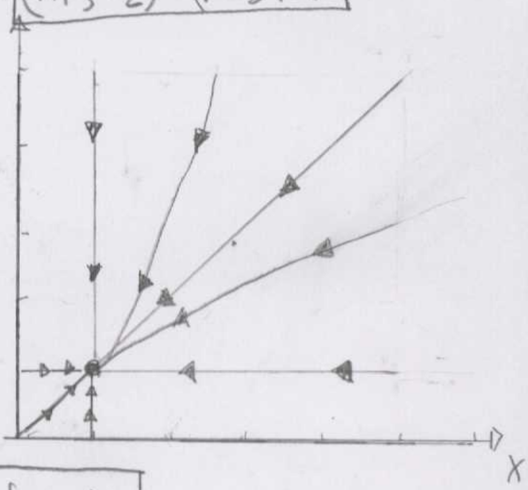
$$= -2x_1x_2(x_1 - x_2) = -2x_1x_2 \cdot v$$

$$v(t) = e^{-2x_1x_2 t}$$

$$\boxed{\lim_{t\to\infty} v(t) = 0}$$

e. $\lim_{t\to\infty} [u(t)+v(t)] = 1 = 2x_1$; $x_1 = 1/2$

$\lim_{t\to\infty} [u(t) - v(t)] = 1 = 2x_2$; $x_2 = 1/2$; $\boxed{(x_1, x_2) \to (1/2, 1/2)}$



f. A large n value generates a plot which seems to converge to zero, but actually, converges to a positive value close to zero. This argument implies RNA remain at low concentrations indefinitely.

$\dot{x} = rxz$
$\dot{y} = ryz$
$\dot{z} = -rxz - ryz$

6.4.11

a. $\dot{z} = -\dot{x} - \dot{y}$; $0 = \dot{x} + \dot{y} + \dot{z}$; $\boxed{1 = x + y + z}$

b. The limit of the function is bounded by the invariance equation in part a.

Fixed Points $\dot{x} = 0 = rxz$; $\dot{y} = ryz$; $\dot{z} = -rxz - ryz$

$$(x^*, y^*, z^*) = (x, y, 0), (0, 0, z)$$

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{dx} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{dz} \\ \frac{\partial \dot{z}}{\partial x} & \frac{\partial \dot{z}}{\partial y} & \frac{\partial \dot{z}}{\partial z} \end{pmatrix} = \begin{pmatrix} rz & 0 & rx \\ 0 & rz & ry \\ -rz & -rz & -rx-ry \end{pmatrix}$$

$$A_{(x,y,0)} = \begin{pmatrix} 0 & 0 & rx \\ 0 & 0 & ry \\ 0 & 0 & -rx-ry \end{pmatrix} ; A_{(0,0,z)} = \begin{pmatrix} rz & 0 & 0 \\ 0 & rz & 0 \\ -rz & -rz & 0 \end{pmatrix}$$

$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = -rx - ry$ $\quad \lambda_1 = \lambda_2 = 0, \lambda_3 = rz$

The eigenvectors point in the direction of λ_3 for each fixed point

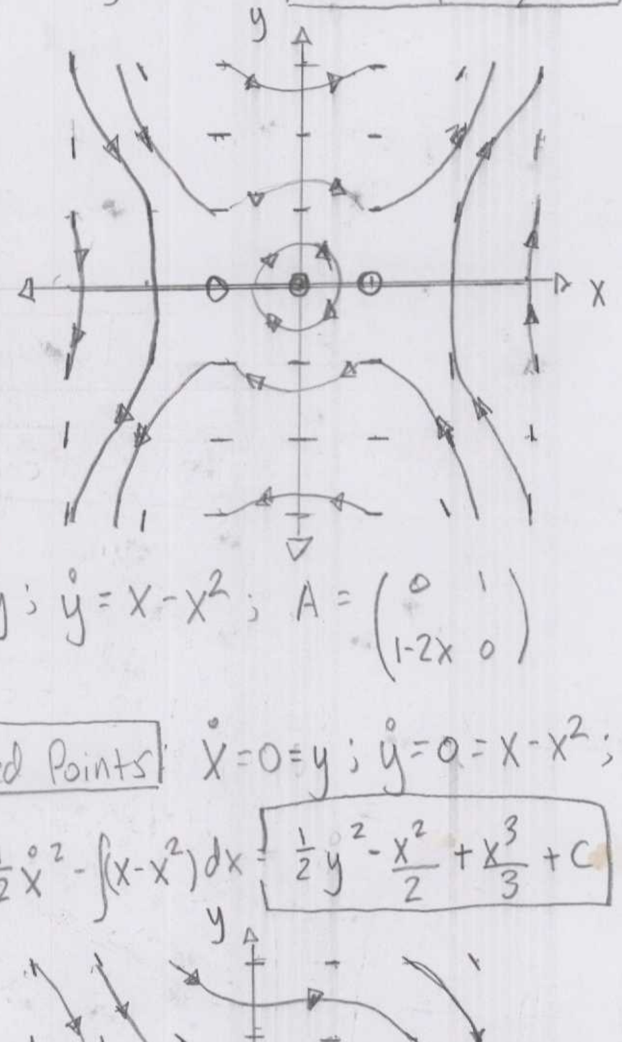
C. An interpretation from the political terms is $r<0$, the centrist pull the extremists to the centrist, while $r>0$, the extremist separate the centrists.

$\ddot{x} = x^3 - x$ 6.5.1.a. $\dot{x} = y$; $\dot{y} = x^3 - x$; $A = \begin{pmatrix} 0 & 1 \\ 3x-1 & 0 \end{pmatrix}$;

$\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = y$; $\dot{y} = 0 = x^3 - x$; $(x^*, y^*) = (-1, 0)$ "center"
$(0, 0)$ "saddle"
$(1, 0)$ "center"

b. $E = \frac{1}{2}\dot{x}^2 - \int x^3 - x\,dx = \boxed{\frac{1}{2}\dot{y}^2 - \frac{x^4}{4} + \frac{x^2}{2} + C}$

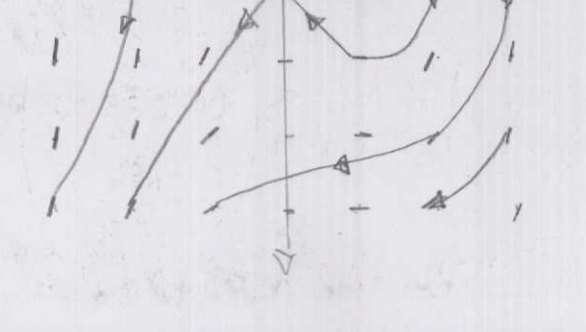
c.



$\ddot{x} = x - x^2$ 6.5.2 .a. $\dot{x} = y$; $\dot{y} = x - x^2$; $A = \begin{pmatrix} 0 & 1 \\ 1-2x & 0 \end{pmatrix}$

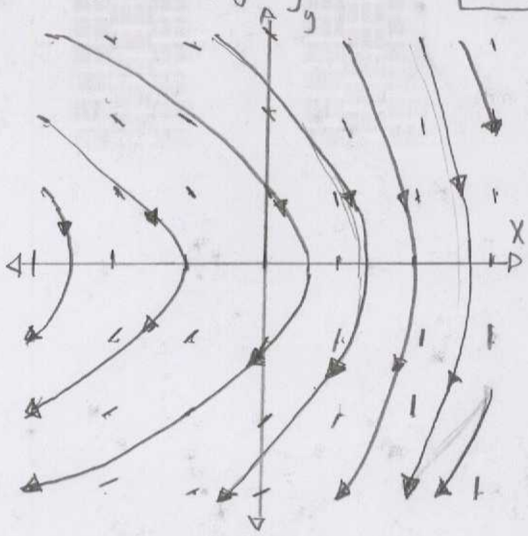
$\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = y$; $\dot{y} = 0 = x - x^2$; $(x^*, y^*) = (+1, 0)$ "center"
$(0, 0)$ "saddle"

b. $E = \frac{1}{2}\dot{x}^2 - \int (x - x^2)\,dx = \boxed{\frac{1}{2}\dot{y}^2 - \frac{x^2}{2} + \frac{x^3}{3} + C}$

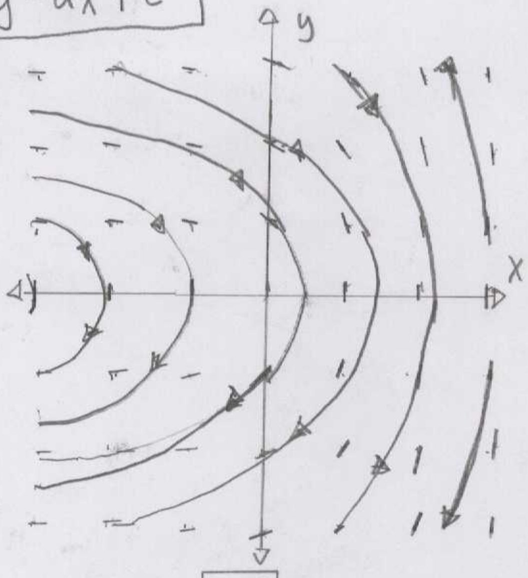


c. $\boxed{E = \frac{1}{2}y^2 - \frac{x^2}{2} + \frac{x^3}{3} + C.}$

$\ddot{x} = a - e^x$ 6.5.3. $\dot{x} = y$; $\dot{y} = a - e^x$; $E = \frac{1}{2}y^2 - \int a - e^x \, dx = \boxed{\frac{1}{2}y^2 - ax + e^x}$



$\boxed{a<0}$



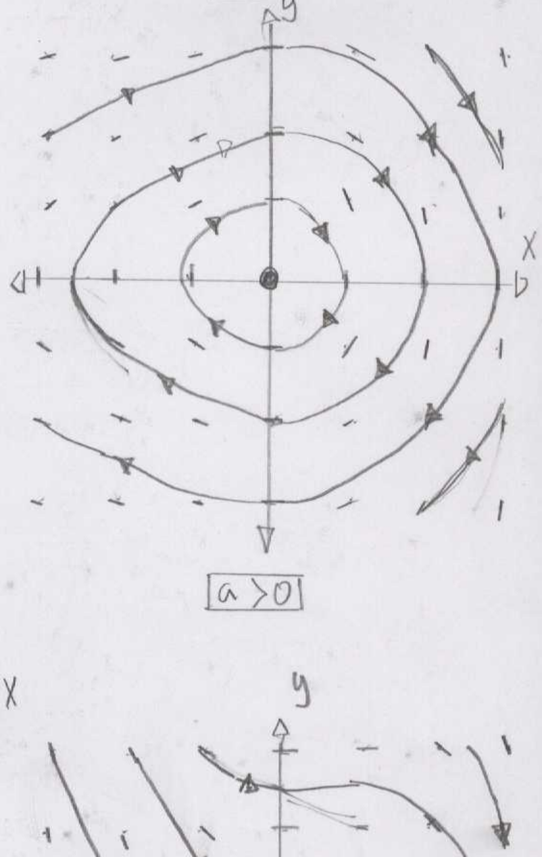
$\boxed{a=0}$

$\ddot{x} = ax - x^2$ 6.5.4. $\dot{x} = y$; $\dot{y} = ax - x^2$

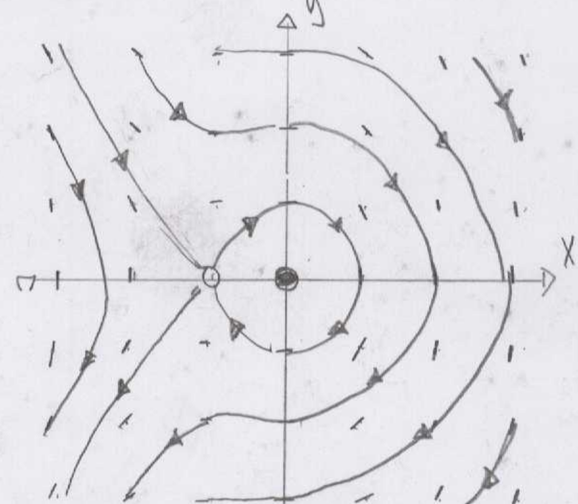
Conserved Quantity:

$E = \frac{1}{2}y^2 - \int ax - x^2 \, dx$

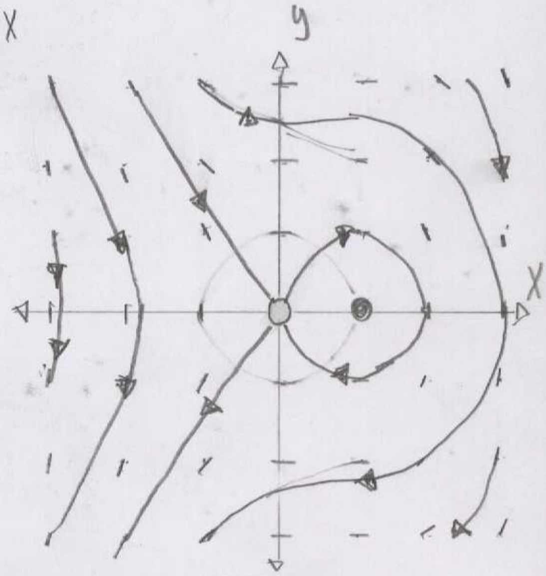
$= \boxed{\frac{1}{2}y^2 - \frac{ax^2}{2} + \frac{x^3}{3} + C}$



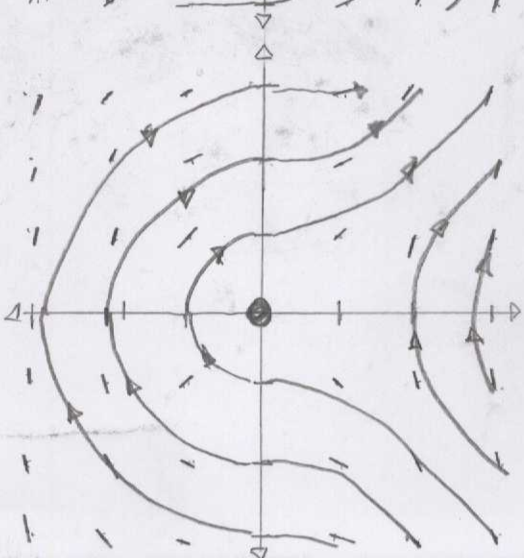
$\boxed{a>0}$



$\boxed{a<0}$



$\boxed{a>0}$



$\boxed{a=0}$

$\ddot{x} = (x-a)(x^2-a)$ 6.5.5 $\dot{x} = y$; $\dot{y} = (x-a)(x^2-a)$

$\boxed{\text{Fixed Points}}$: $\dot{x} = 0 = y$

$\dot{y} = 0 = (x-a)(x^2-a)$

$(x^*, y^*) = (a, 0)$

$(\sqrt{a}, 0)$

$(-\sqrt{a}, 0)$

If $a = 1$, then one fixed point exists in quadrant #1 and #4.

If $0 < a < 1$ or $a > 1$, then two fixed points exist in quadrants #1 and #4

$\dot{x} = -kxy$
$\dot{y} = kxy - ly$

6.5.6. a. $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = -kxy$; $\dot{y} = 0 = kxy - ly$

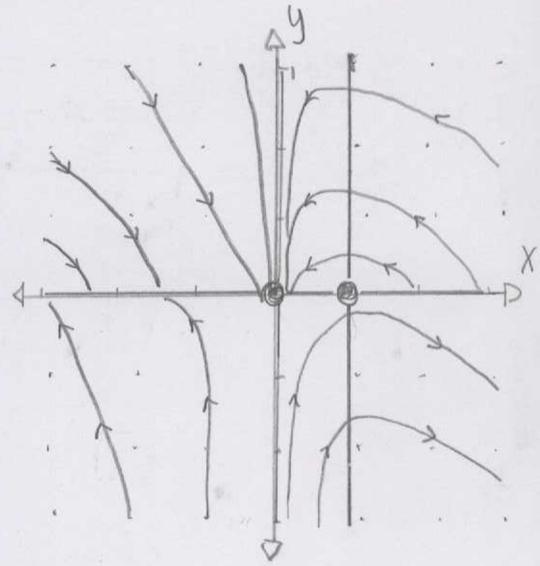
$(x^*, y^*) = (0,0)$ "center" ; $A = \begin{pmatrix} -ky & -kx \\ ky & kx - l \end{pmatrix}$

$(l/k, 0)$ "center"

b. $\boxed{\text{Nullclines}}$ $x = 0$

$y = 0$

$x = \frac{l}{k}$



c. $\boxed{\frac{dy}{dx} = -1 + l/kx \,;\; y = -x + \frac{l}{k} \ln x + c}$

d. $\boxed{\text{See part c}}$

e. $\boxed{\text{A population is sick from infection. When } y_0 \geq 0.}$

$\frac{d^2u}{d\theta^2} + u = \alpha + \varepsilon u^2$ 6.5.7 $u = 1/r$;

a. $\boxed{\begin{array}{l} v^2 + u = \alpha + \varepsilon u^2 \\ \text{where } v = du/d\theta. \end{array}}$

b. $\boxed{\text{Fixed Points}}$: $\mathring{u} = 0 = v$

$$\mathring{v} = 0 = \kappa + \varepsilon u^2 - u$$

$$(u^*, v^*) = \left(\frac{1 + \sqrt{1-4\kappa\varepsilon}}{2\varepsilon}, 0\right), \left(\frac{1 - \sqrt{1-4\kappa\varepsilon}}{2\varepsilon}, 0\right)$$

c. $A = \begin{pmatrix} 0 & 1 \\ 2\varepsilon u - 1 & 0 \end{pmatrix}$; $\lambda_{1A,B} = \pm i\sqrt{1-4\alpha\varepsilon}$; $\lambda_{2A,B} = \pm \sqrt{1-4\kappa\varepsilon}$

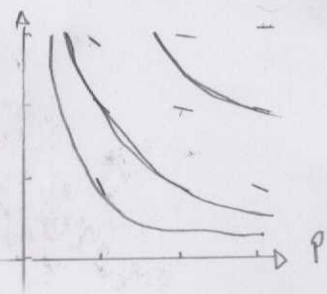
"Saddle point" $\boxed{\text{"linear "center"}}$

d. $\frac{1}{r} \pm u = \frac{1 - \sqrt{1-4\kappa\varepsilon}}{2\varepsilon}$; $\boxed{r = \frac{2\varepsilon}{1 - \sqrt{1-4\kappa\varepsilon}}}$

$H = \frac{p^2}{2m} + \frac{kx^2}{2}$ 6.5.8 $\underbrace{\mathring{q} = \frac{p}{m}}_{\text{"Momentum"}}$; $\underbrace{\mathring{p} = -kx}_{\text{"Force"}}$; $H = \underbrace{\frac{p^2}{2m}}_{\text{"Kinetic Energy"}} + \underbrace{\frac{kx^2}{2}}_{\text{"Potential Energy"}}$

$H(x,p)$ 6.5.9 $\mathring{H} = \frac{p}{m}\mathring{p} + kx\mathring{x} = \frac{p}{m}(-kx) + kx\left(\frac{p}{m}\right) = \boxed{0}$

$H(p,r) = \frac{p^2}{2} + \frac{h^2}{2r^2} - \frac{k}{r}$ 6.5.10 a. The Hamiltonian plot is similar to the potential plot of $1/r^2$ where $k = 1$ and $h = 6.626 \times 10^{-34}$ J·s.



b.

$-k^2/2h^2 < E < 0$	$E = 0$	$E > 0$
◦ Slope is negative	Slope is zero	Slope is positive
◦ Momentum is decreasing	◦ Momentum is constant	◦ Momentum is increasing
◦ radius is increasing	◦ Radius is increasing	◦ radius is increasing.

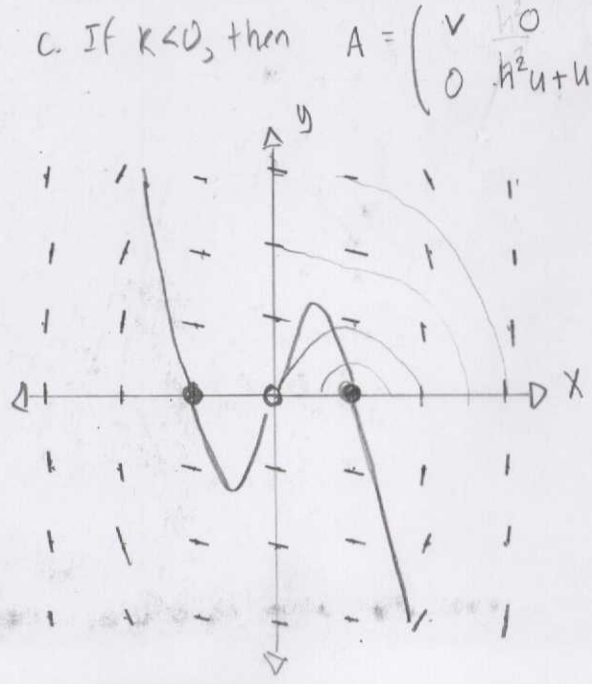
c. If $k < 0$, then $A = \begin{pmatrix} v & 0 \\ 0 & h^2 u + u \end{pmatrix}$; $\Delta = v(h^2 u + u)$

$\tau = v + h^2 u + u$

$\tau^2 - 4\Delta > 0$

$\boxed{\text{No Periodic orbits}}$

$\mathring{x} = y$
$\mathring{y} = -by + x - x^3$ 6.5.11



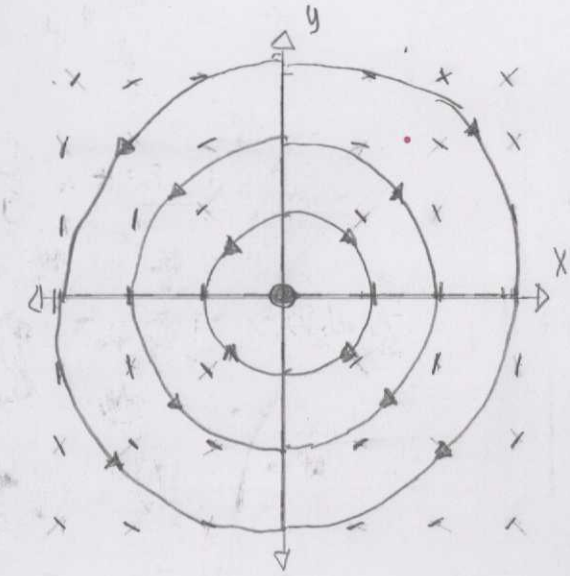
$\dot{x} = xy$ 6.5.12.

$\dot{y} = -x^2$

a. $E = x^2 + y^2$; $E' = 2x \cdot \dot{x} + 2y \cdot \dot{y} = 2x^2 y - 2y^2 x \stackrel{!}{=} \boxed{0}$

b. $(x^*, y^*) = (0,0)$; $A = \begin{pmatrix} y & x \\ -2x & 0 \end{pmatrix}$; $A_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \stackrel{!}{=} \boxed{0}$

$(0, y)$; $A_{(0,y)} = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} \stackrel{!}{=} \boxed{0}$



c. See part B ; Non-isolated. Fixed Point.

$\ddot{x} + x + \varepsilon x^3 = 0$ 6.5.13.

a. $E = \frac{1}{2}\dot{x}^2 - \int (-x - \varepsilon x^3)\, dx$

$= \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{\varepsilon}{4}x^4$

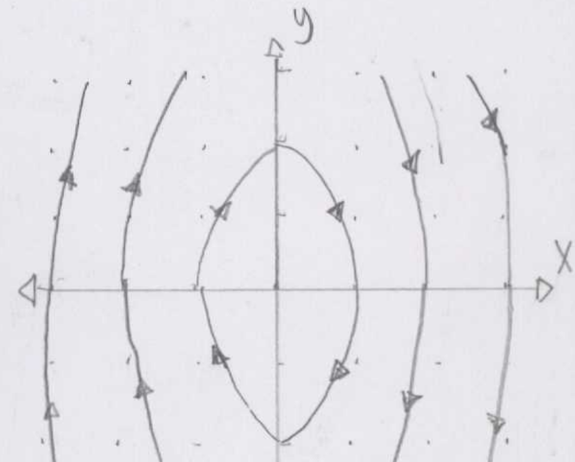
$\begin{vmatrix} E_{xx} & E_{x\dot{x}} \\ E_{\dot{x}x} & E_{\dot{x}\dot{x}} \end{vmatrix} = \begin{vmatrix} 1 + 3\varepsilon x^2 & \dot{x} + x + \varepsilon x^3 \\ \dot{x} + x + \varepsilon x^3 & 1 \end{vmatrix}_{(0,0)} \stackrel{!}{=} \boxed{1}$, a continuous derivative exists. at the center i.e. nonlinear center,

b. If $\varepsilon < 0$, a hyperbola trajectory is the closed orbit about $(0,0)$.

$\dot{x} = y$

$\dot{y} = -x - \varepsilon x^3$

For from the origin when $\varepsilon > 0$, this phase plot appears.



$\dot{v} = -\sin\theta - Dv^2$ 6.5.14

$v\dot{\theta} = -\cos\theta + v^2$

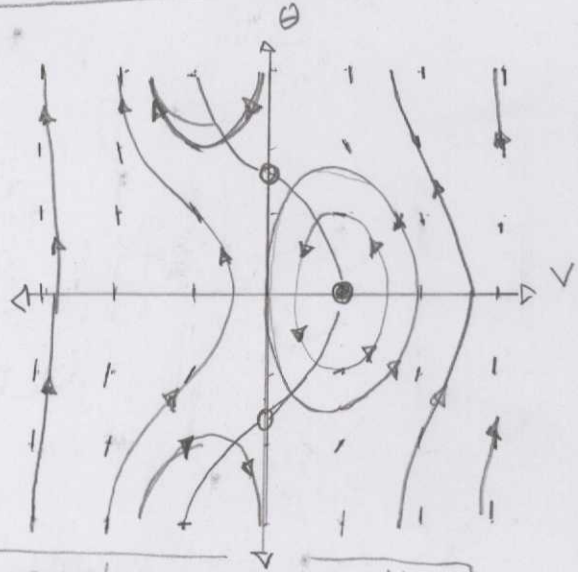
a. If $D = 0$, then $\dot{v} = -\sin\theta$

$v\dot{\theta} = -\cos\theta + v^2$

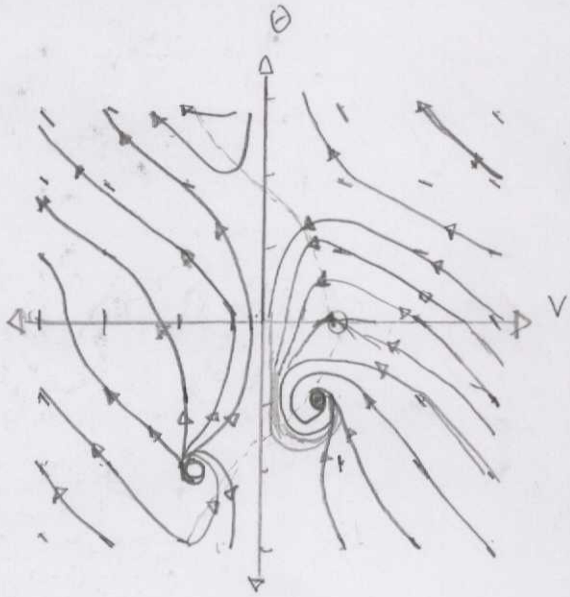
$E = \frac{1}{2}v^2 - \int v\dot{\theta}\, dv = \frac{1}{2}mv^2 + v\cos\theta - \frac{v^3}{3} \stackrel{!}{=} \boxed{0}$

$\frac{1}{2}v^2 - 3v\cos\theta + v^3 = 0$; $\frac{dE}{dv} = v - 3\cos\theta + 3v^2$; $\boxed{\text{Fixed Points:}}$ $(v^*, \theta^*) = (1, 0), (0, 0)$

$\boxed{\text{The potential energy } V(v,\theta) = -3\cos\theta + 3v^2 \text{ has a fixed point at } (0, 2n\pi).}$



b. $\boxed{\text{If } D>0\text{, then as the glider approaches } v\to\infty\text{, then the angle becomes more positive and the effect of lift propels the glider upward.}}$



$mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi$

6.5.15

a. $b = 0$

$mr\ddot{\phi} = -mg\sin\phi + mr\omega^2\sin\phi\cos\phi$

If $\gamma = r\omega^2/g$, then

$\ddot{\phi}\left(\frac{1}{g}\right) = -\sin\phi + \gamma\sin\phi\cos\phi$

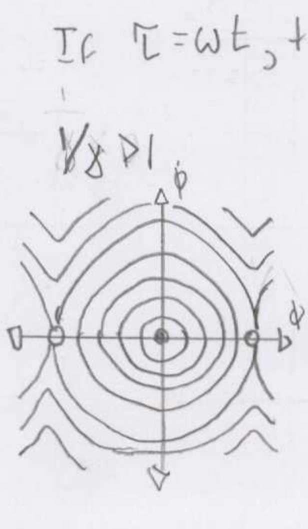
$\ddot{\phi}\left(\frac{1}{\gamma g}\right) = \sin\phi(\cos\phi - \gamma^{-1})$

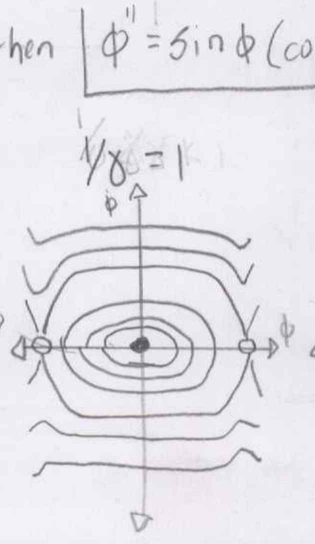
If $\tau = \omega t$, then $\boxed{\phi'' = \sin\phi(\cos\phi - \gamma^{-1})}$

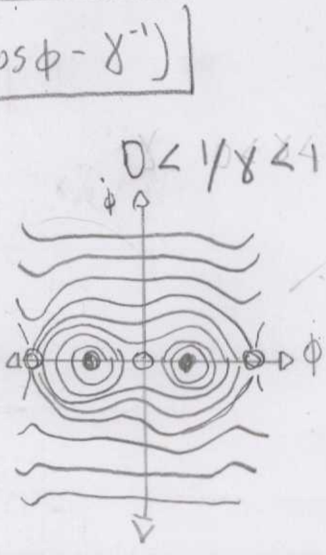
$\dot{v} = \phi$

$\dot{\phi} = v$

b. $1/\gamma > 1$ $\quad$ $1/\gamma = 1$ $\quad$ $0 < 1/\gamma < 1$







c. The graphs $1/\gamma > 1$ and $1/\gamma = 1$ suggest a periodic stable point when the hoop spins, while $1/\gamma = 1$, a bead that doesn't stay in one place, and spins around the hoop.

6.5.16. $m r \ddot{\phi} = -mg\sin\phi + m r \omega^2 \sin\phi\cos\phi$

$$\ddot{\phi} = -\frac{g}{r}\sin\phi + \omega^2\sin\phi\cos\phi = \sin\phi(\omega^2\cos\phi - g/r)$$

$$0 = \sin\phi(\omega^2\cos\phi - g/r) ; \boxed{\phi = \pm\frac{\pi}{2} ; \arccos\frac{g}{r\omega^2}}$$

6.5.17: $m r \ddot{\phi} = -mg\sin\phi + m r\omega^2\sin\phi\cos\phi$

$$E = KE + PE = \frac{1}{2}\dot{\phi}^2 - \int \sin\phi(\omega^2\cos\phi - g/r)\,d\phi$$

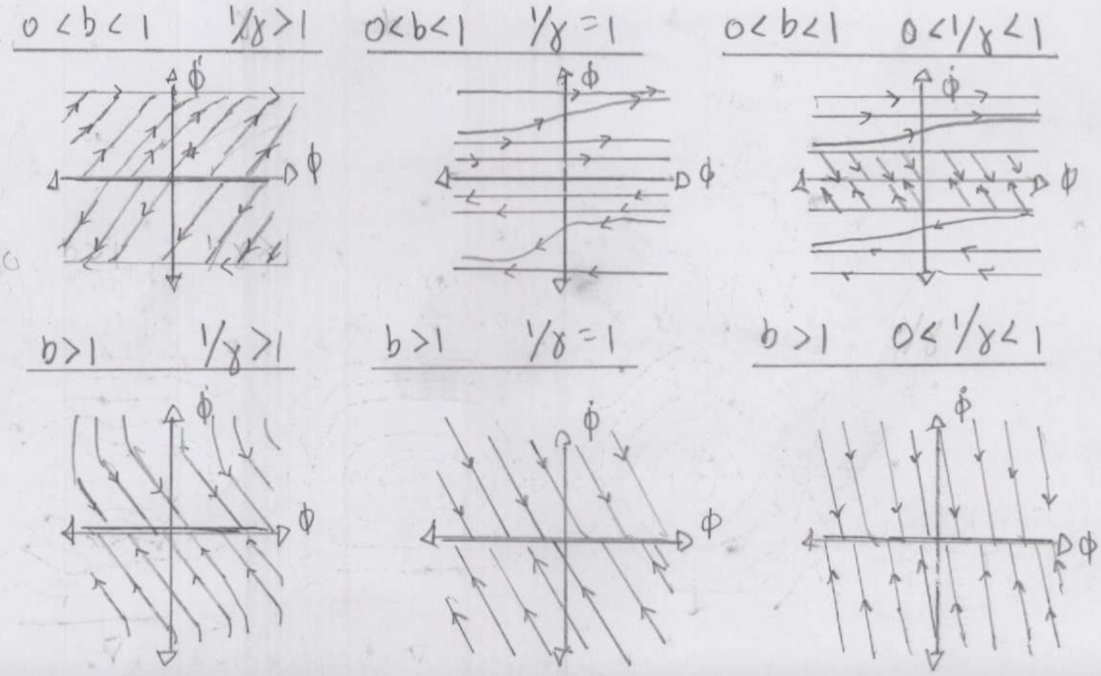
$$= \frac{1}{2}\dot{\phi}^2 + \cos(\phi)\left(\omega^2\cos\phi - \frac{2g}{r\omega^2}\right)$$

$$\ldots = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\omega^2\cos^2\phi - \frac{g}{r}\cos\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\omega^2(1-\sin^2\phi) - mgr(1-\cos\phi) - mgr$$

$$\boxed{= (KE_{Trans} - KE_{Rot}) + PE}$$

In terms of seperation of motion, the bead hoop problem has translational and rotational energy.

6.5.18. $m r \ddot{\phi} = -b\dot{\phi} - mg\sin\phi + m r\omega^2\sin\phi\cos\phi$



$\mathring{R} = aR - bRF$
$\mathring{F} = -cF + dRF$

6.5.19. Lotka-Volterra Predator-Prey Model

a. Term

aR : Growth of the rabbit population

$-bRF$: Decrease of the rabbit population by interacting foxes

$-CF$: Decrease of the fox population

dRF: Growth of the fox population by eating rabbits.

An unrealistic assumption is foxes do not decrease when rabbits are not present.

b. $\mathring{R} = R(a-bF)$; $\mathring{R}\left(\frac{1}{a}\right) = \frac{R}{a}\left(1-\frac{b}{a}F\right)$; $x = \frac{d}{c}R$; $y = \frac{b}{a}F$; $\tau = at$

$\mathring{F} = F(dR-c)$; $\mathring{y} = \frac{cy}{a}(x-1)$; $\boxed{\mathring{y} = \mu y(x-1) \;;\; \mathring{x} = x(1-y)}$

c. $\mathring{x} = 0 = x(1-y)$; $\mathring{y} = 0 = \mu y(x-1)$; $(x^*, y^*) = \boxed{(0,0) \quad (1,1)}$

d. $A = \begin{pmatrix} 1-y & -xy \\ \mu y & \mu(x-1) \end{pmatrix}$

$A_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}$; $\Delta = \mu$; $\tau = 1+\mu$; $\tau^2 - 4\Delta > 0$

"Unstable node"

$A_{(1,1)} = \begin{pmatrix} 0 & -1 \\ \mu & 0 \end{pmatrix}$; $\Delta = \mu$; $\tau = 0$; $\tau^2 - 4\Delta > 0$

"center" cycle.

$\mathring{P} = P(R-S)$
$\mathring{R} = R(S-P)$
$\mathring{S} = S(P-R)$

6.5.20.

a. The terms found in $\mathring{P}, \mathring{R}$, and $\mathring{S}$ relate the existence of paper, rock, and scissors, but also, a relationship when each type of species is present at any given time.

b. $P + R + S = PR - PS + RS - RP + SP - SR = \boxed{0}$

c. $E(P,R,S) = P + R + S$; $E_2(P,R,S) = PRS$

"Plane" "Multiplane"

As $t \to \infty$, then a discrete solution exists of integer values between planes or amounts of P, R, S.

$\dot{x} = y(1-x^2)$ 6.6.1. Reversible if $t \to -t$, $x \to -x$, $y \to -y$

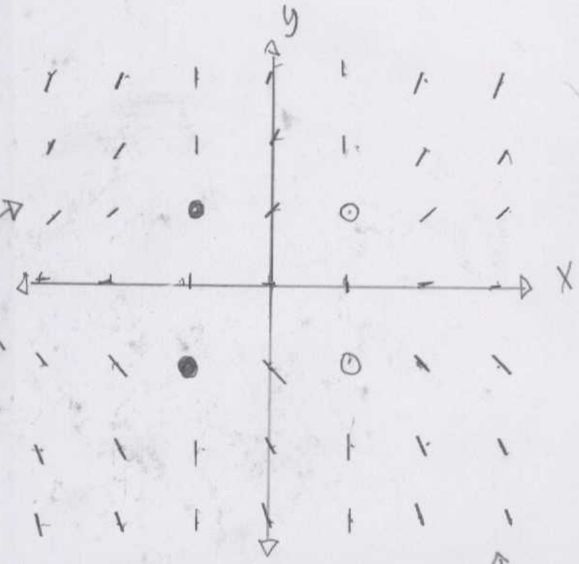
$\dot{y} = 1-y^2$

$\boxed{\text{Fixed Points:}}$ $\dot{x} = y(1-x^2) = 0$

$\dot{y} = 1 - y^2 = 0$; $(x^*, y^*) = (\pm 1, \pm 1)$

What happened

Incorrectly Generated

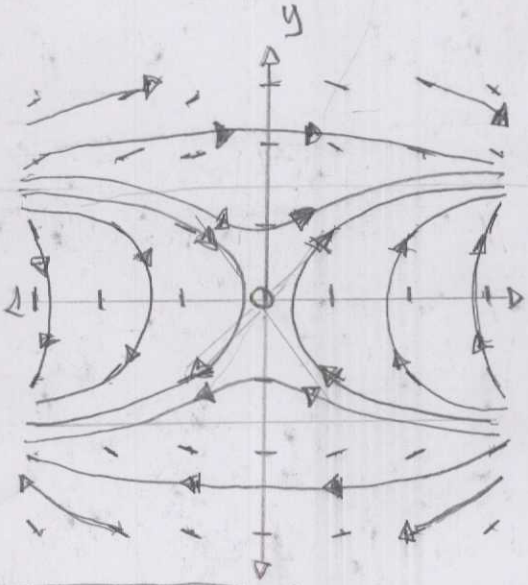


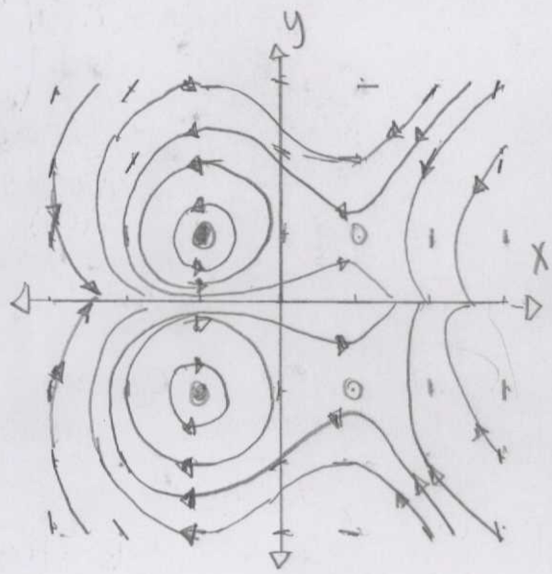
Wrong!

Duh

$\dot{x} = y$ 6.6.2.

$\dot{y} = x\cos y$





$\boxed{\text{Fixed Points:}}$ $\dot{x} = 0 = y$;

$\dot{y} = 0 = x\cos y = x\cos(-y) = -x\cos(-y)$

$(x^*, y^*) = (0, 0)$

$\dot{x} = \sin y$ 6.6.3.

$\dot{y} = \sin x$

a. $\frac{dy}{dx} = \left(\frac{-1}{-1}\right)\frac{dy}{dx} = \frac{-\sin x}{-\sin y} = \boxed{\frac{\sin x}{\sin y}}$

b. $\boxed{\text{Fixed Points:}}$ $\dot{x} = 0 = \sin y$; $(x^*, y^*) = (\pm n\pi, n\pi)$ $(2n+1)\frac{\pi}{2}$

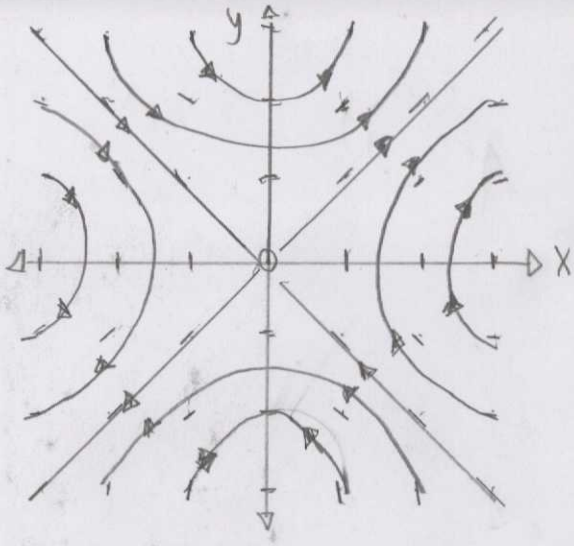
$\dot{y} = 0 = \sin x$

Where $n \in \mathbb{R}$

If n is even, stable node, else unstable node.

c. $\dot{x} = \sin y$; $\sin y = \sin x$; $\boxed{y = \pm x}$

$\dot{y} = \sin x$



d. →

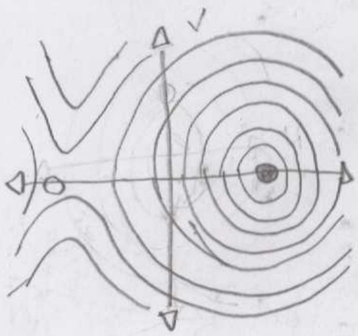
$\ddot{x} + (\dot{x})^2 + x = 3$

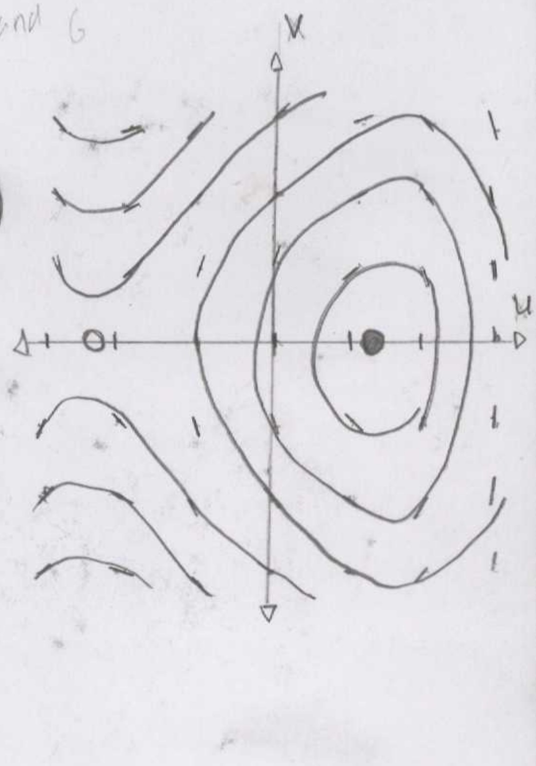
6.6.4. $\dot{u} = \dot{x} = v$

a. $\dot{v} = \ddot{x} = 3 - \dot{x}^2 + x = 3 - u^2 + u$

$\boxed{\text{Fixed Points}}$ $(u^*, v^*) = \left(0, \frac{-1}{2} \pm \frac{\sqrt{13}}{2}\right)$

$\boxed{\text{Guess}}$





$\dot{x} = y - y^3$

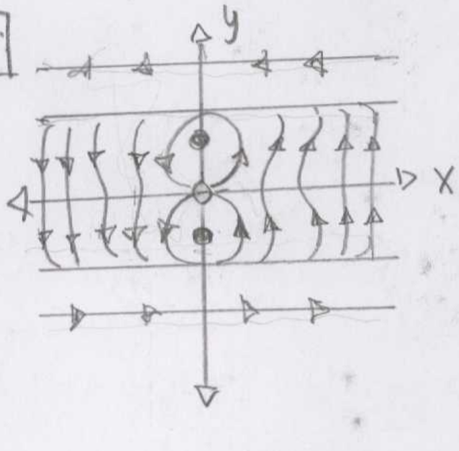
$\dot{y} = x\cos y$

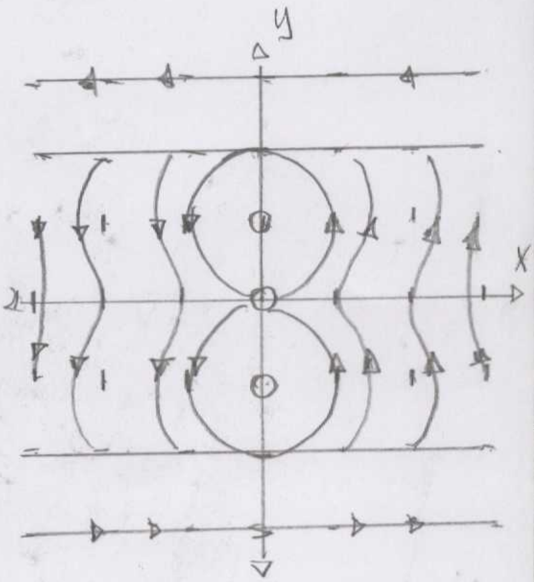
b. $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = y - y^3$

$\dot{y} = 0 = x\cos y$

$(x^*, y^*) = (0,0), (\pm x, 1)$

$\boxed{\text{Guess}}$



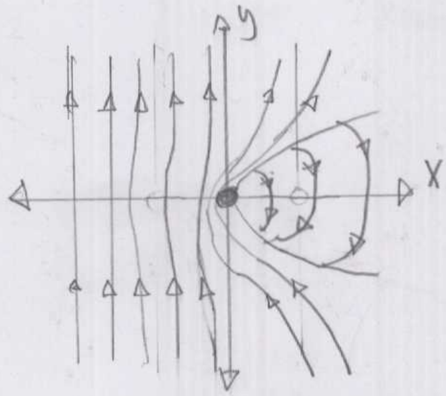


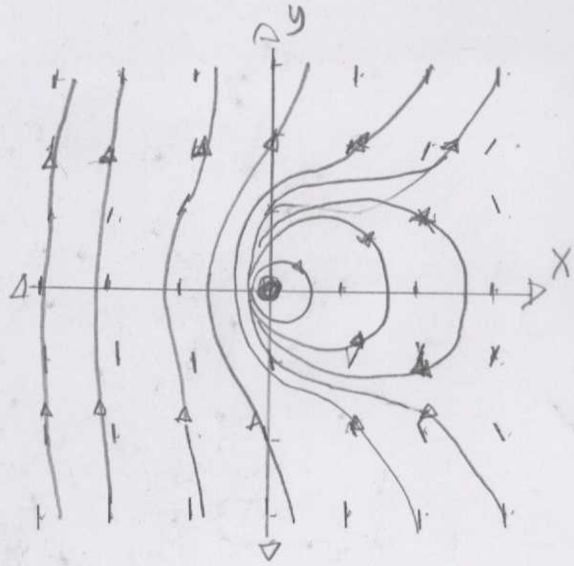
$\dot{x} = \sin y$ 6.6.C **Fixed Points** $\dot{x} = 0 = \sin y$

$\dot{y} = y^2 - x$ $\dot{y} = 0 = y^2 - x$

$(x^*, y^*) = (0,0), (1, \pm 1)$

Guess:





$\ddot{x} + f(\dot{x}) + g(x) = 0$ 6.6.5. f is an even function; g is an odd function
f & g are smooth

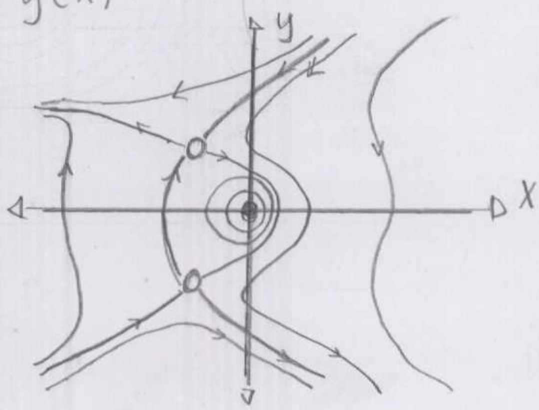
a. $-\ddot{x} + f(-\dot{x}) + g(-x) = -\ddot{x} - f(\dot{x}) - g(x) = \boxed{\ddot{x} + f(\dot{x}) + g(x)}$

b. $\dot{u} = \dot{x} = v$ Definition of a reversible system

$\dot{v} = \ddot{x} = -f(\dot{x}) - g(x)$; is no stable nodes or spirals.

$\dot{x} = y - y^3$ 6.6.6.

$\dot{y} = -x - y^2$ a. $\boxed{\dot{x} = 0; \dot{y} = 0}$



b. Quadrant #1: $\dot{x} < 0; \dot{y} < 0$ Quadrant #3: Mixed

Quadrant #2: Mixed Quadrant #4: $\dot{x} > 0; \dot{y} < 0$

c. $A = \begin{pmatrix} 0 & 1-2y^2 \\ -1 & -2y \end{pmatrix}; A_{(-1, \pm 1)} = \begin{pmatrix} 0 & -1 \\ -1 & \pm 2 \end{pmatrix}; \Delta = (1-\sqrt{2})(1+\sqrt{2})$

$\tau = 2$

$\tau^2 - 4\Delta > 0$

$\lambda_1 = (1 - \sqrt{2}); \boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

$\lambda_2 = (1 + \sqrt{2}); \boxed{v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

d. $(-1,-1)$; If Quadrant #2 and #3 are mixed sign then a possible trajectory through $x<0$ may exist. A heteroclinic trajectory that does cross from $(-1,-1)$ to $(-1,1)$ is present because of the reversible function.

e. Other examples of a heteroclinic trajectory relate to the third fixed point. See part b.

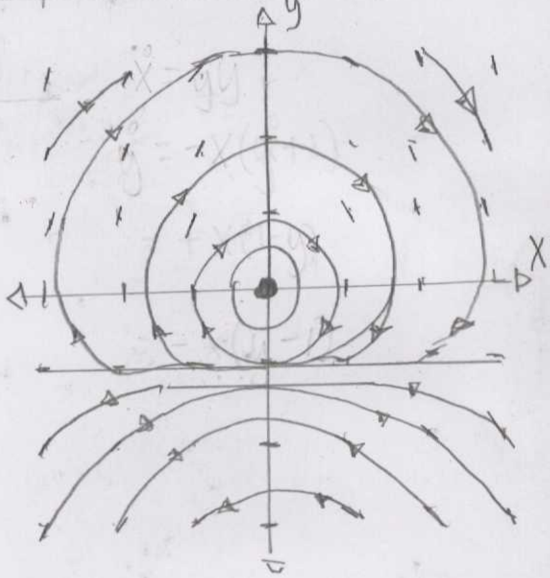
$\ddot{x}+x\dot{x}+x=0$ 6.6.7. $\dot{x}=y$

$\dot{y}=-x(\dot{x}+1)=-x(y+1)$

$\boxed{\text{Reversibility}}$ $-\ddot{x}-x\cdot(-\dot{x})-x$

$=\ddot{x}+x\dot{x}+x$

$=0$



$\dot{x}=\frac{\sqrt{2}}{4}x(x-1)\sin\phi$

$\dot{\phi}=\frac{1}{2}\left[\beta-\frac{1}{\sqrt{2}}\cos\phi-\frac{1}{8\sqrt{2}}x\cos\phi\right]$

6.6.8. a. $\boxed{\text{Reversibility}}$ $x\to -x$; $\phi\to -\phi$

$\dot{x}=\frac{\sqrt{2}}{4}-x(-x-1)\sin(-\phi)$

$=-\frac{\sqrt{2}}{4}x(1-x)\sin(\phi)$

$=\frac{\sqrt{2}}{4}x(x-1)\sin\phi$

$\dot{\phi}=\frac{1}{2}\left[\beta-\frac{1}{\sqrt{2}}\cos(-\phi)-\frac{1}{8\sqrt{2}}(-x)\cos(-\phi)\right]$

$=\frac{1}{2}\left[\beta-\frac{1}{\sqrt{2}}\cos(\phi)+\frac{1}{8\sqrt{2}}x\cos\phi\right]$

b. $\dot{x}=0=\frac{\sqrt{2}}{4}x(x-1)\sin\phi=$

$\dot{\phi}=\frac{1}{2}\left[\beta-\frac{1}{\sqrt{2}}\cos\phi-\frac{1}{8\sqrt{2}}x\cos\phi\right]=0$

$(x^*,\phi^*)=(0,2\pi n-\cos^{-1}(\sqrt{2}\cdot\beta))$, $(0,2\pi n+\cos^{-1}(\sqrt{2}\beta))$

$(1,2\pi n-\cos^{-1}\left(\frac{8\sqrt{2}\beta}{x+8}\right))$, $(0,0)$

$(1,2\pi n+\cos^{-1}\left(\frac{8\sqrt{2}\beta}{x+8}\right))$

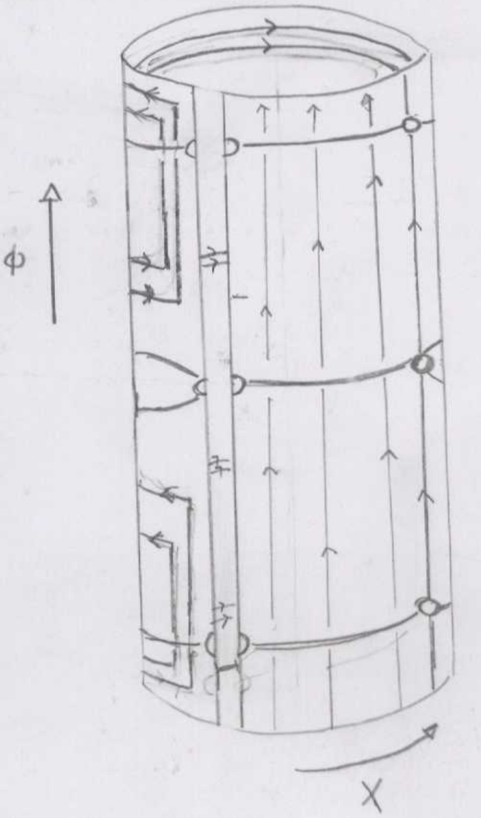
A homoclinic orbit is a nullcline,

$$\mathring{x} = 0 = \frac{\sqrt{2}}{4} x(x-1)\sin\phi \; ; \; \mathring{\phi} = 0 = \frac{1}{2}\left[\beta - \frac{1}{\sqrt{2}}\cos\phi - \frac{1}{8\sqrt{2}} x\cos\phi\right]$$

$$\boxed{x = -4\sqrt{2}\sec(\phi)(\sqrt{2}\cos(\phi) - 2\beta)}$$

c. $\lim_{\beta \to \frac{1}{\sqrt{2}}} 2\pi n - \cos^{-1}(2\sqrt{\beta}) = 2\pi n$;

, then $(x^*, \phi^*) = (0, 2\pi n)$ and the node on the line $\phi = 0$ is closer to $(0,0)$, and the cylinder becomes a smaller diameter shape. with less closed orbits.



d. $\boxed{\text{See Part c: cylinder}}$

$$\frac{d\phi_k}{d\tau} = \Omega + a\sin\phi_k + \frac{1}{N}\sum_j^N \sin\phi_j$$

6.6.9.

a. $\theta_K = \phi_K - \frac{\pi}{2}$; $\frac{d\theta}{d\tau} = \Omega + a\cos\theta_K + \frac{1}{N}\sum\cos\theta_K$

$$= \Omega + a\cos(-\theta_K) + \frac{1}{N}\sum\cos(-\theta_K)$$

b. $\boxed{\text{Fixed Points:}}$ $\mathring{\theta} = 0 = \Omega + a\cos\theta_K + \frac{1}{N}\sum\cos\theta_K$

$$\Omega = -a\cos\theta_K - \frac{1}{N}\sum\cos\theta_K$$

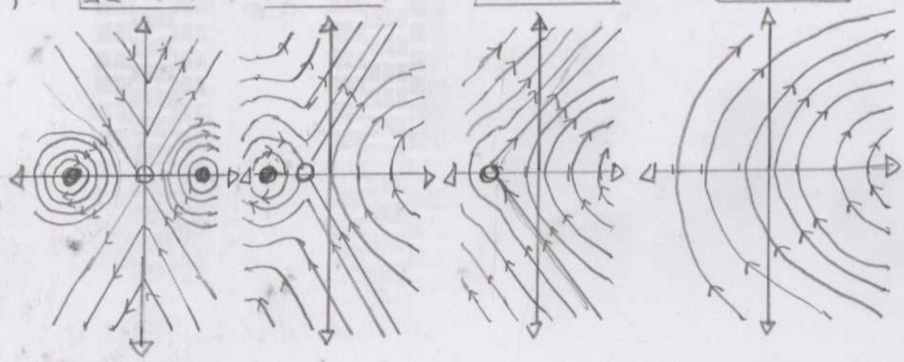
$$= -\cos\theta_K(a+1)$$

$$-\cos\theta_K = \left|\frac{\Omega}{a+1}\right|$$

If $\left|\frac{\Omega}{a+1}\right| < 1$, then $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

If $\left|\frac{\Omega}{a+1}\right| > 1$, then no fixed point is generated because $\cos\theta$ is never greater than 1.

c. $a=1$; $\boxed{\Omega=0}$ $\boxed{\Omega=1}$ $\boxed{\Omega=2}$ $\boxed{\Omega=3}$



$\dot{x}=-y-x^2$ 6.6.10 $\boxed{\text{Fixed Points}}$ $\dot{x}=0=-y-x^2$; $(x^*, y^*)=(0,0)$

$\dot{y}=x$ $\dot{y}=0=x$

$$A=\begin{pmatrix} -2x & -1 \\ 1 & 0 \end{pmatrix} ; A_{(0,0)}=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\Delta=-1$; $\tau=0$; $\tau^2-4\Delta>0$

"Saddle Point"

No, a nonlinear center is an isolated fixed point with closed orbits

$\dot{\theta}=\cot\phi\cos\theta$ 6.6.11. a. $\boxed{\text{Reversibility}}$ $t\to -t$; $\theta\to-\theta$

$\dot{\phi}=(\cos^2\phi+A\sin^2\phi)\sin\theta$

$\dot{\theta}=\cot(+\phi)\cos(-\theta)$

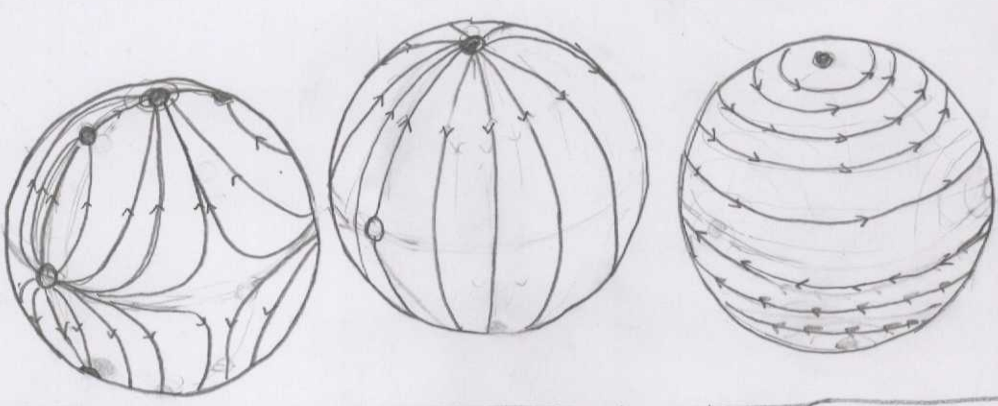
$=\cot(+\phi)\cdot\cos(\theta)$

$t\to -t$; $\phi\to-\phi$

$\dot{\phi}=[\cos^2(-\phi)+A\sin(-\phi)^2]\cdot\sin(\theta)$

$=[\cos^2(\phi)+A\sin^2(\phi)]\cdot\sin(\theta)$

b. $\boxed{A=-1}$ $\boxed{A=0}$ $\boxed{A=1}$

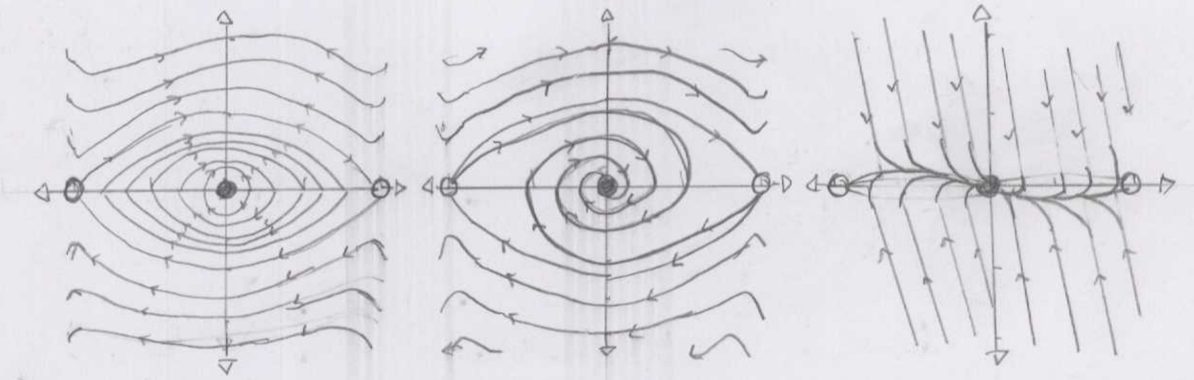


c. As $t\to\infty$, each case of shear flow trajects to a stable node. This implies rotation of a body does not freely rotate in medium.

$\ddot{\theta} + b\dot{\theta} + \sin\theta = 0$ 6.7.1. $\boxed{\text{Fixed Points}}$ $\dot{x} = \dot{\theta} = y$

$\dot{y} = \ddot{\theta} = -(by + \sin x)$

$\boxed{b=0}$ $\boxed{0 < b \leq 1}$ $\boxed{1 < b}$



$\ddot{\theta} + \sin\theta = \gamma$ 6.7.2 a. $\boxed{\text{Fixed Points}}$ $\dot{x} = \dot{\theta} = y = 0$

$\dot{y} = \ddot{\theta} = \gamma - \sin\theta = \gamma - \sin x$

$(x^*, y^*) = (\arcsin\gamma, 0)$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}; \quad A = \begin{pmatrix} 0 & 1 \\ \gamma - \cos x & 0 \end{pmatrix}$$

$\Delta = \cos x - \gamma$; $\tau = 0$; $\tau^2 - 4\Delta > 0$

If $\gamma = 0$, (x^*, y^*) is a center

If $0 < \gamma < 1$, (x^*, y^*) is a center

If $1 < \gamma$, (x^*, y^*) is a saddle point.

b. $\boxed{\text{Nullclines}}$ $y = \gamma - \sin x$

$E = \frac{1}{2}\dot{x}^2 - \int \gamma - \sin x \, dx$

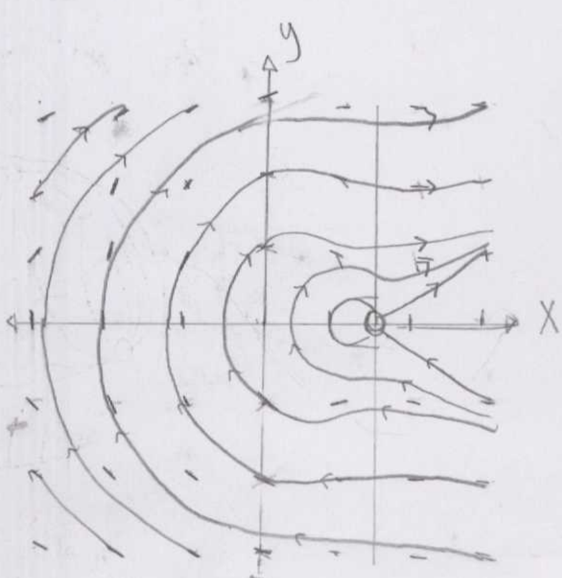
$= \frac{1}{2}y^2 - \gamma x - \cos x$

The system is not conservative because of no closed loops

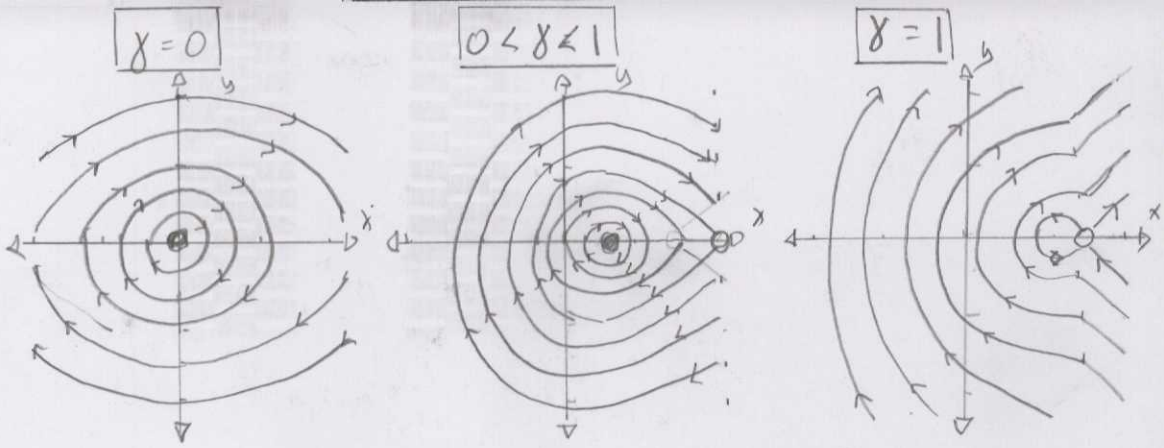
$\boxed{\text{Reversibility}}$

$\dot{x} = -y \neq y$

The system is not reversible.



d



e. $\gamma = 0$; $\dot{x} = 0 = y$; $\dot{y} = 0 = -\sin x$; $y = \dot{\theta} = -\sin\theta$

$0 < \gamma < 1$; $\dot{x} = 0 = y$; $\dot{y} = 0 = \gamma - \sin x$; $y = \dot{\theta} = \gamma - \sin\gamma$

$\gamma = 1$; ... $\dot{\theta} = \gamma - \sin\theta$

$\ddot{\theta} + (1 + a\cos\theta)\dot{\theta} + \sin\theta = 0$

6.7.3 $\dot{x} = \dot{\theta} = y$

$\dot{y} = -(1 + a\cos x)y - \sin x$

Fixed Points: $(x^*, y^*) = (0,0)$

	Yes	No
Reversible	☐	☑
Conservative	☐	☑

$\ddot{\theta} + \sin\theta = 0$ 6.7.4

a. $PE = mgh = mgL(1 - \cos(\theta))$; $KE = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{\theta})^2$

$E = PE + KE = mgL(1 - \cos(\theta)) + \frac{1}{2}m\dot{\theta}^2 = 0$

$\dot{\theta}^2 = -2gL(1 - \cos(\theta))$; If $\theta = \alpha$ = max height; $\dot{\theta}^2 = 0$

$$\boxed{= 2(\cos(\theta) - \cos(\alpha))}$$

$$T = 4\int_0^\alpha dt = 4\int_0^\alpha \frac{d\theta}{\dot{\theta}} = 4\int_0^\alpha \frac{d\theta}{\sqrt{2(\cos\theta - \cos\alpha)}}$$

b. Half-Angle Formula: $\cos(2A) = 1 - 2\sin^2 A$ where $A = \frac{\theta}{2}$ or $\frac{\alpha}{2}$

$$\boxed{T = 4\int_0^\alpha \frac{d\theta}{\left[4\left(\sin^2\frac{1}{2}\alpha - \sin^2\frac{1}{2}\theta\right)\right]^{1/2}}}$$

c. Half-Angle Formula: $(\sin\frac{1}{2}\alpha)\sin\phi = \sin\frac{1}{2}\theta$

$$\frac{1}{2}\sin\frac{1}{2}\alpha\cos\phi\frac{d\phi}{d\theta} = \frac{1}{2}\cos\frac{\theta}{2}$$

$$d\theta = \frac{\cos\theta/2}{\sin\alpha/2\,\cos\phi}\,d\phi$$

$$T = 4\int_0^{\pi/2} \frac{d\phi}{\cos \theta/2} = 4 \tfrac{1}{2} \times \boxed{\underbrace{\left[\int_0^{\pi/2} \frac{d\phi}{(1 - m\sin^2\phi)^{1/2}}\right]}_{\text{"Elliptic Integral"}}}$$

d. Binomial Series $\frac{1}{(1-x)^{1/2}} = 1 + \frac{1}{2}x + \cdots$

$$T = 4 \times \int_0^{\pi/2} \left(1 + \tfrac{1}{2} m \sin^2\phi + \cdots\right) d\phi \;\; ; \;\; m = \sin^2 \frac{x}{2}$$

$$\boxed{= 2\pi\left[1 + \tfrac{1}{16}x^2 + \cdots\right]}$$

6.7.5.

Numerical Integration of $T = 4 \times \sum_{i=0}^{18}\sum_{j=0}^{9}\left(1 + \frac{1}{2}\left[\sin^2\frac{10i}{2}\right]\sin^2\frac{10j}{2} + \cdots\right)$

i \ j	0	1	2	3	4	5	6	7	8	9
0	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00
1	4.00	5.69	4.54	4.78	5.53	4.03	5.80	4.34	5.02	5.33
2	4.00	4.54	4.18	4.25	4.49	4.01	4.59	4.11	4.33	4.43
3	4.00	4.78	4.25	4.36	4.70	4.01	4.83	4.16	4.47	4.61
4	4.00	5.53	4.49	4.70	5.39	4.63	5.63	4.31	4.93	5.21
5	4.00	4.03	4.01	4.01	4.03	4.00	4.03	4.01	4.02	4.03
6	4.00	5.80	4.59	4.83	5.63	4.03	5.91	4.36	5.09	5.41
7	4.00	4.34	4.11	4.16	4.31	4.01	4.36	4.07	4.20	4.27
8	4.00	5.02	4.33	4.47	4.93	4.02	5.08	4.20	4.62	4.80
9	4.00	5.33	4.43	4.61	5.21	4.03	5.41	4.27	4.80	5.05
10	4.00	4.13	4.04	4.06	4.11	4.00	4.13	4.03	4.08	4.10
11	4.00	5.84	4.59	4.85	5.67	4.04	5.95	4.25	5.11	5.45
12	4.00	4.17	4.05	4.08	4.15	4.00	4.19	4.22	4.10	4.13
13	4.00	5.26	4.40	4.58	5.14	4.02	5.33	4.25	4.76	4.99
14	4.00	5.10	4.35	4.51	5.00	4.02	5.17	4.22	4.67	4.87
15	4.00	4.28	4.09	4.13	4.25	4.01	4.29	4.06	4.67	4.22
16	4.00	5.92	4.58	4.84	5.65	4.03	5.93	4.36	5.10	5.43
17	4.00	4.06	4.02	4.03	4.05	4.00	4.06	4.01	4.03	4.04
18	4.00	5.47	4.47	4.68	5.33	4.03	5.56	4.29	4.84	5.16

$$\boxed{T = 852.06}$$

6.8.1

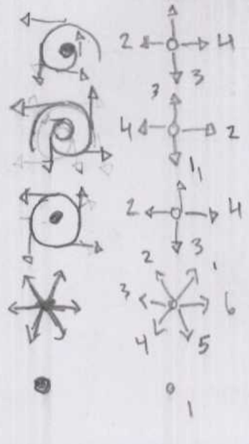
a. Stable Spiral $I_C = +1$

b. Unstable Spiral $I_C = +1$

c. center $I_C = +1$

d. Star $I_C = +1$

e. Degenerate Node. $I_C = +1$

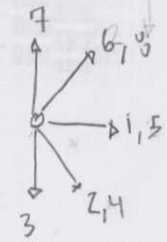


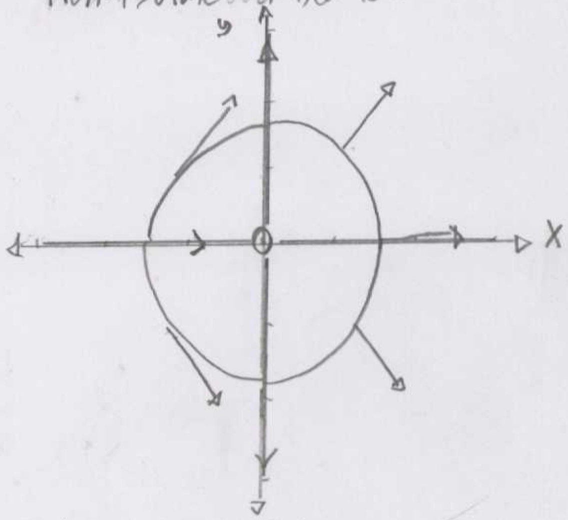
$\dot{x} = x^2$ 6.8.2 $\boxed{\text{Fixed Points}}$; $\dot{x} = 0 = x^2$; $A = \begin{pmatrix} 2x & 0 \\ 0 & 1 \end{pmatrix}$; $\Delta = 0$; $\tau = 1$; $\tau^2 - 4\Delta > 0$

$\dot{y} = y$ $\qquad \dot{y} = 0 = y$

"Non-isolated Fixed Points"

$(x^*, y^*) = (0,0)$; $\boxed{\text{Index}}$





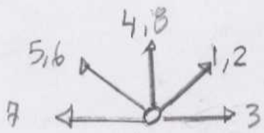
$I_c = 0$

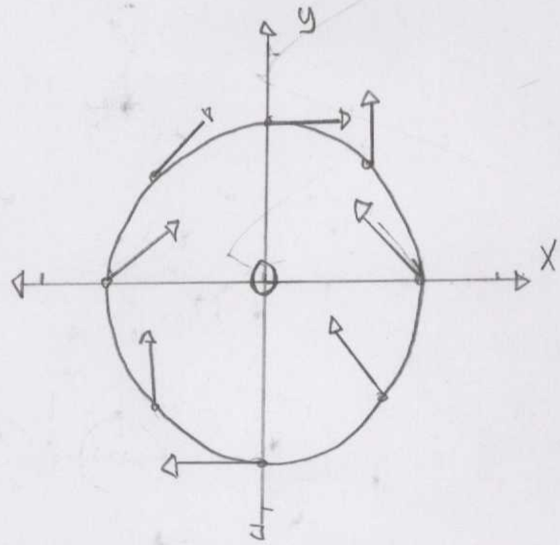
$\dot{x} = y - x$ 6.8.3. $\boxed{\text{Fixed Points}}$; $\dot{x} = 0 = y - x$; $A = \begin{pmatrix} -1 & 1 \\ 2x & 0 \end{pmatrix}$; $\Delta = 0$; $\tau = -1$; $\tau^2 - 4\Delta > 0$

$\dot{y} = x^2$ $\qquad \dot{y} = 0 = x^2$

"Spiral sink"

$(x^*, y^*) = (0,0)$; $\boxed{\text{Index}}$





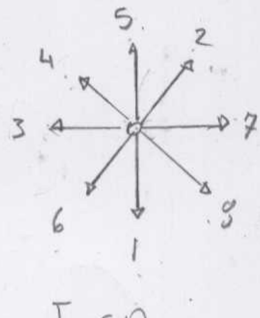
$I_c = 0$

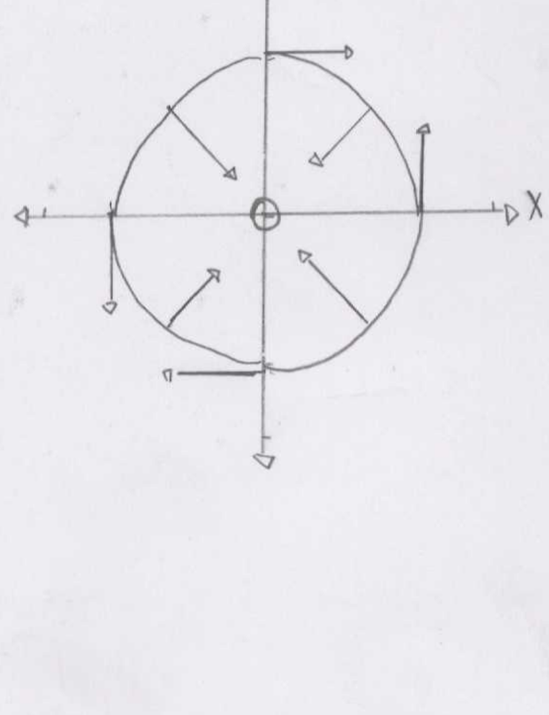
$\dot{x} = y^3$ 6.8.4 $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = y^3$ $\quad A = \begin{pmatrix} 0 & 3y^2 \\ 1 & 0 \end{pmatrix}$; $\Delta = 0$; $\tau = 0$; $\tau^2 - 4\Delta = 0$

$\dot{y} = x$ $\qquad \dot{y} = 0 = x$

"Saddle"

$(x^*, y^*) = (0,0)$ $\boxed{\text{Index}}$





$I_c = 0$

$\dot{x} = xy$

$\dot{y} = x + y$

6.8.5 $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = xy$

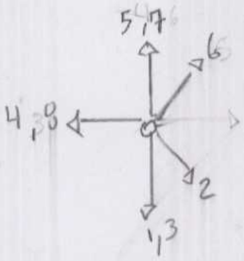
$\dot{x} =$

$\dot{y} = 0 = x + y$

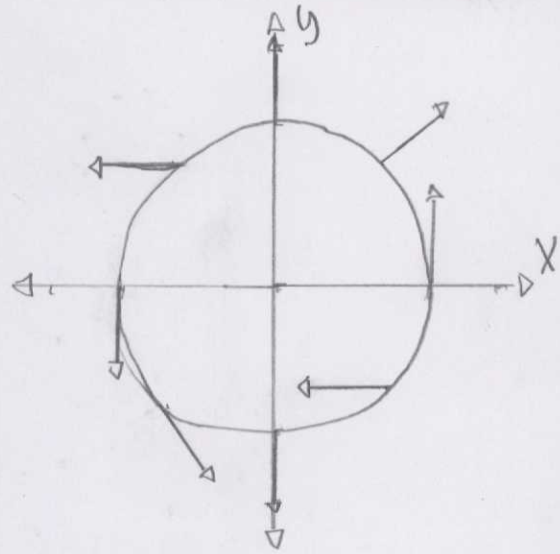
$(x^*, y^*) = (0,0)$ $\boxed{\text{Index}}$

$A = \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix}$; $\Delta = 0$; $\tau = 0$;

$\tau^2 - 4\Delta$ "unstable saddle"



$I_c = 0$



6.8.6. Node [N] : $I_c = +1$

Spiral [F] : $I_c = +1$

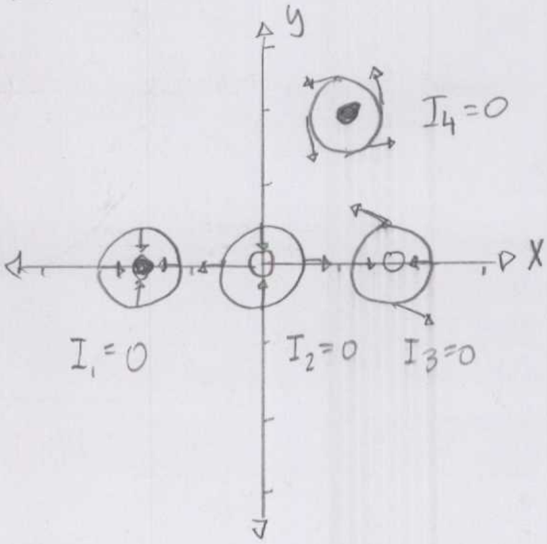
Center [C] : $I_c = +1$

Saddle [S] : $I_c = 0$

$N + S + C = +1 = 1 + S = +1$

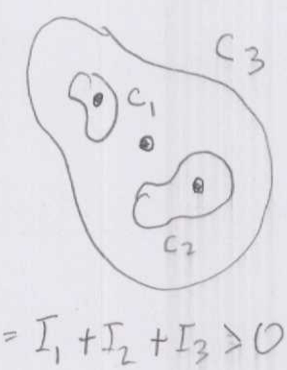
$\dot{x} = x(4 - y - x^2)$ 6.8.7.

$\dot{y} = y(x - 1)$



The indices of each fixed point are zero ($I_c = 0$), thus, no closed orbits exist.

6.8.8. a.



b. $I_c = I_1 + I_2 + I_3 > 0$; A fixed point exists in the closed orbit.

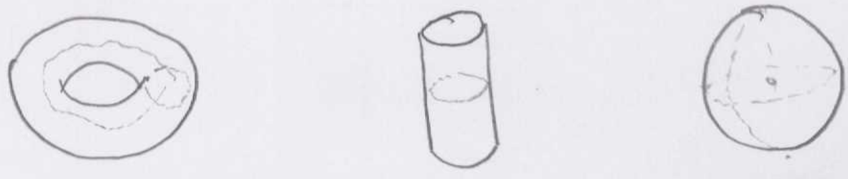
6.8.9. C_1, C_2 = closed Trajectories

If C_1 is clockwise, $I_c < 0$

If C_2 is counterclockwise, $I_c > 0$

$\boxed{\text{A fixed point in } C_2 \text{ is true because } I_c > 0}$

6.8.10 Torus | Cylinder | Sphere



$I_c > 0$ | $I_c = 0$ | $I_c > 0$

$\boxed{\text{Theorem 6.8.2 is reasonable for closed orbit shapes.}}$

$\dot{z} = z^k$

$\dot{z} = (\bar{z})^k$

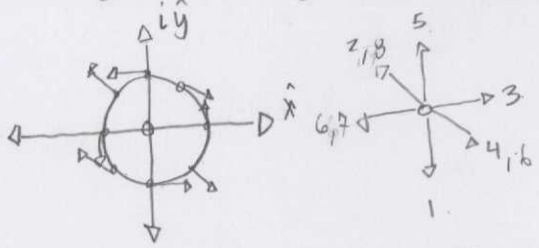
6.8.11

a. $k = 1$; $\dot{z} = z = x + iy$; $\langle x, y \rangle$

$k = 2$; $\dot{z} = z^2 = (x+iy)^2 = x^2 - y^2 - 2ixy$; $\boxed{\langle x^2 - y^2, -2xy \rangle}$

$k = 3$; $\dot{z} = z^3 = (x+iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$; $\boxed{\langle x^3 - 3xy^2, 3x^2y - y^3 \rangle}$

b. $z^* = (0,0)$;



$I_c = 0$

c. The expansion is similar to a Binomial.

$$\left\langle \sum_{k=1}^{2k \le n} \binom{n}{2k} x^{n-2k} \cdot (-1)^k \cdot y^{2k}, \sum_{k=1}^{2k+1 \le n} \binom{n}{2k+1} x^{n-2k} (-1)^k y^{2k} \right\rangle$$

$\dot{x} = a + x^2$

$\dot{y} = -y$

6.8.12.

a. $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = a + x^2$

$\dot{y} = 0 = -y$

$(x^*, y^*) = (\pm i\sqrt{a}, 0)$

$A = \begin{pmatrix} 2x & 0 \\ 0 & -1 \end{pmatrix}$; $\Delta_1 = -2i\sqrt{a}$; $\tau_1 = 2i\sqrt{a} - 1$; $\tau_1^2 - 4\Delta$ = Imaginary

$\Delta_2 = 2i\sqrt{a}$; $\tau_2 = -2i\sqrt{a} - 1$

$\boxed{\text{Fixed points in } \mathbb{R}^3 \text{ are non-existent.}}$

b. $I_c = I_1 + I_2 = \boxed{0}$ because the imaginary fixed points $(\pm i\sqrt{a}, 0)$ are symmetric.

c. $\dot{x} = F(x, a)$ where $x \in R^2$ a conserved index is independent of a, and is the sum of two indices.

$\dot{x} = f(x,y)$ 6.8.13 $\phi = \tan^{-1}(\dot{y}/\dot{x})$

$\dot{y} = g(x,y)$

a. $\frac{d}{dj}\tan^{-1} j = \frac{1}{j^2+1}$; $d\phi = \frac{1}{\left(\frac{\dot{y}}{\dot{x}}\right)^2+1} \cdot \left(\frac{\dot{y}}{\dot{x}}\right)' = \frac{\dot{x}^2}{\dot{y}^2+\dot{x}^2} \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2}$

$$\boxed{= \frac{f dg - g df}{f^2+g^2}}$$

b. $I_c = \frac{d}{d\phi}\tan^{-1}(\phi) = \boxed{\frac{1}{2\pi}\oint \frac{f dg \cdot g df}{f^2+g^2}}$ where $\phi = \frac{\dot{y}}{\dot{x}}$

$\dot{x} = x\cos\alpha - y\sin\alpha$ 6.8.14.

$\dot{y} = x\sin\alpha + y\cos\alpha$

a. $\boxed{\text{Fixed Points}}$ $(x^*, y^*) = (0,0)$

$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$; $\Delta = \cos\alpha\sin\alpha$

$\tau = \cos\alpha + \sin\alpha$

$\tau^2 - 4\Delta < 0$

$\boxed{\text{"Unstable Spiral"}}$

b. $I_c = \frac{1}{2\pi}\oint \frac{(x\cos\alpha - y\sin\alpha)(x\sin\alpha + y\cos\alpha)' - (x\sin\alpha + y\cos\alpha)(x\cos\alpha - y\sin\alpha)'}{(x\cos\alpha - y\sin\alpha)^2 + (x\sin\alpha + y\cos\alpha)^2} dc$

$= \frac{1}{2\pi}\oint 1 = \boxed{1}$

c. $\boxed{I_c = 1}$