Taylor Series: $f(x+h) = \sum \frac{f^{n}(x)h^{n}}{n!}$

2.8.8. $\dot{x} = x + e^{-x}$; $|x(t_0) - t_0| = |x(t_0) - x(t_0) - x(t_0)'\Delta t - \frac{x''(t_0)\Delta t^2}{2!}| = \frac{x''(t_0)\Delta t^2}{2}$

$\boxed{= O(\Delta t^2)}$

$\dot{x} = x + e^{-x}$ 2.8.9 Runge-Kutta: $x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ Where $k_1 = f(x_n)\Delta t$

$k_2 = f(x_n + \frac{1}{2}k_1)\Delta t$

$k_3 = f(x_n + \frac{1}{2}k_2)\Delta t$

$k_4 = f(x_n + k_3)\Delta t$

$$x(t+\Delta t) = x(t_0) + x'(t_0)\Delta t + \frac{x''(t_0)\Delta t^2}{2!} + O(\Delta t^3)$$

$$k_1 = f(x_n)\Delta t = x'(t_0)\Delta t$$

$$k_2 = f(x_n + \tfrac{1}{2}k_1)\Delta t = f(x_n) + f'(x_n)\tfrac{1}{2}k_1 + O[(\tfrac{1}{2}k_1)^2]$$

$$\therefore \quad k_3 = f(x_n + \tfrac{1}{2}k_2)\Delta t = f(x_n) + f'(x_n)\tfrac{1}{2}k_2 + O[(\tfrac{1}{2}k_2)^2]$$

$$= f(x_n) + f'(x_n)\tfrac{1}{2}\left[f(x_n) + f'(x_n)\tfrac{1}{2}k_1 + O[(\tfrac{1}{2}k_1)^2]\right] + O[(\tfrac{1}{2}k_2)^2]$$

$$k_4 = f(x_n + k_3)\Delta t = f(x_n) + f'(x_n)\cdot k_3 + O[k_3^2]$$

$$= f(x_n) + f'(x_n)\left[f(x_n) + f'(x_n)\cdot\tfrac{1}{2}\left[f(x_n) + f'(x_n)\tfrac{1}{2}k_1 + O[(\tfrac{1}{2}k_1)^2]\right] + O[(\tfrac{1}{2}k_2)]\right] + O[k_3]$$

$$x_{n+1} = x_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = x_n + \tfrac{1}{6}\Big(x'(t_0)\Delta t + 2x'(t_0) + x''(t_0)k_1 + 2x'(t_0) + x''(t_0)\left[x'(t_0) + x''(t_0)x'(t)\Delta t\right] + 2x'(t_0) + x''(t_0)\left[x(t) + \tfrac{x'(t)}{2}\left[x'(t_0) + x''(t_0)\tfrac{x'(t)}{2}\right]\right]\Big)$$

$$\boxed{|x(t_1) - x_1| = |x(t_0 + \Delta t) - x_{n+1}| = O(\Delta t^5)}$$

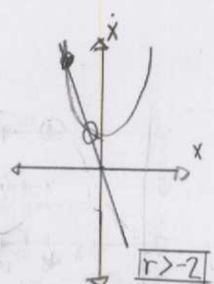
Chapter 3:

$\dot{x} = 1 + rx + x^2$ 3.1.1. Vector Field:

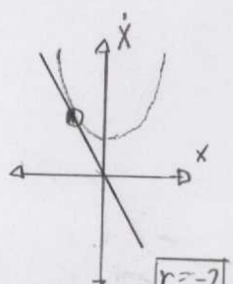
$$x = \frac{r \pm \sqrt{r^2 - 4}}{2}$$

$$r = \frac{r \pm \sqrt{(r-2)(r+2)}}{2}$$

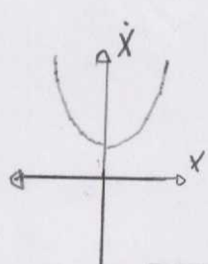
r	Bifurcations
>-2	Two
-2	One
0	zero
2	one
>2	Two.



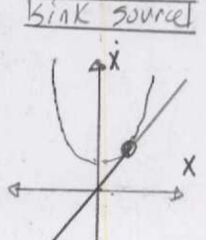
$\boxed{r>-2}$ $\boxed{\text{Sink Source}}$



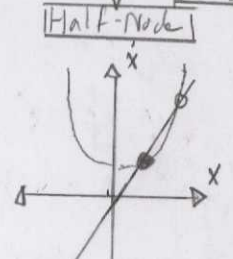
$\boxed{r=-2}$ $\boxed{\text{Half-Node}}$



$\boxed{-2<r<2}$ $\boxed{\text{None}}$

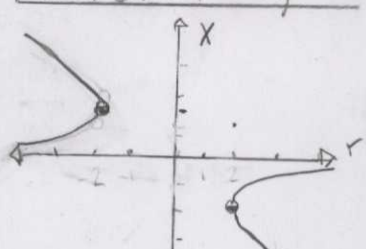


$\boxed{r=2}$ $\boxed{\text{Half-Node}}$



$\boxed{r>2}$ $\boxed{\text{Sink Source}}$

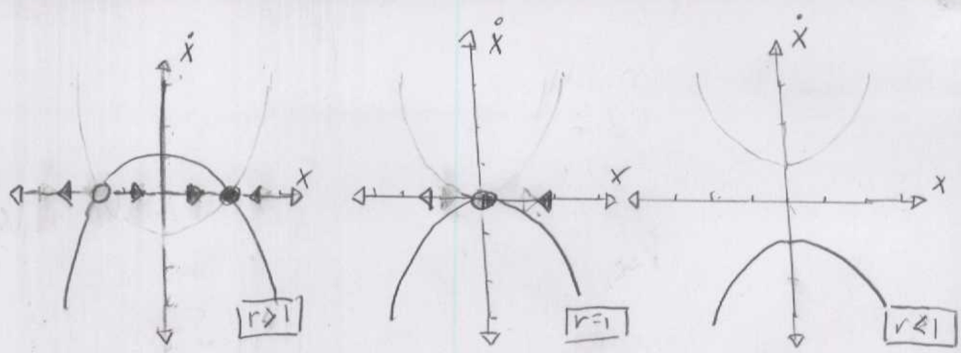
Bifurcation Diagram:



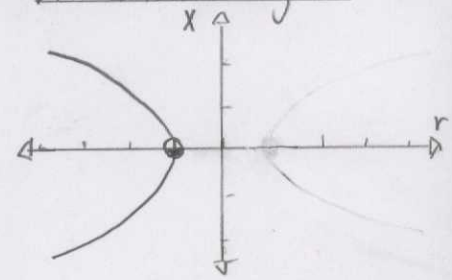
$\dot{x} = r - \cosh x$ 3.1.2. Vector Field

$r = \cosh(x)$

r	Bifurcations
<1	Zero
=1	One
>1	Two



Bifurcation Diagram.

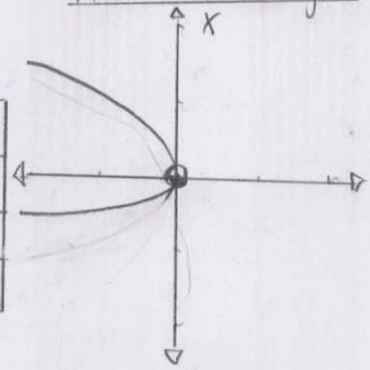


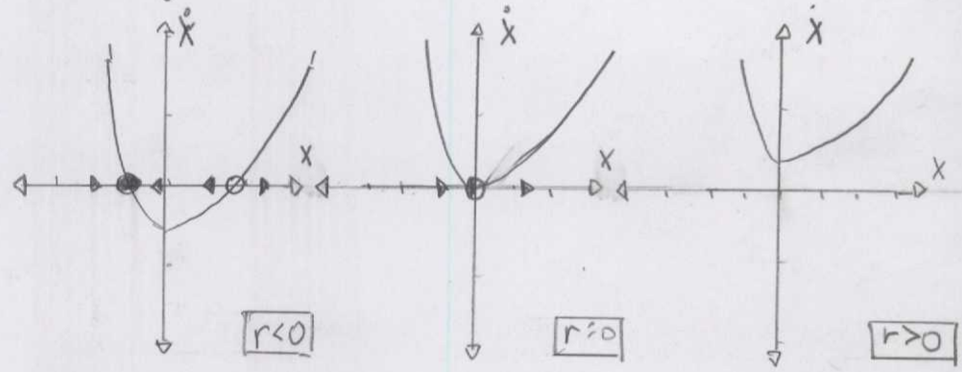
$\dot{x} = r + x - \ln(1+x)$ 3.1.3

Vector Field

r	Bifurcation
>0	Zero
=0	One
<0	Two

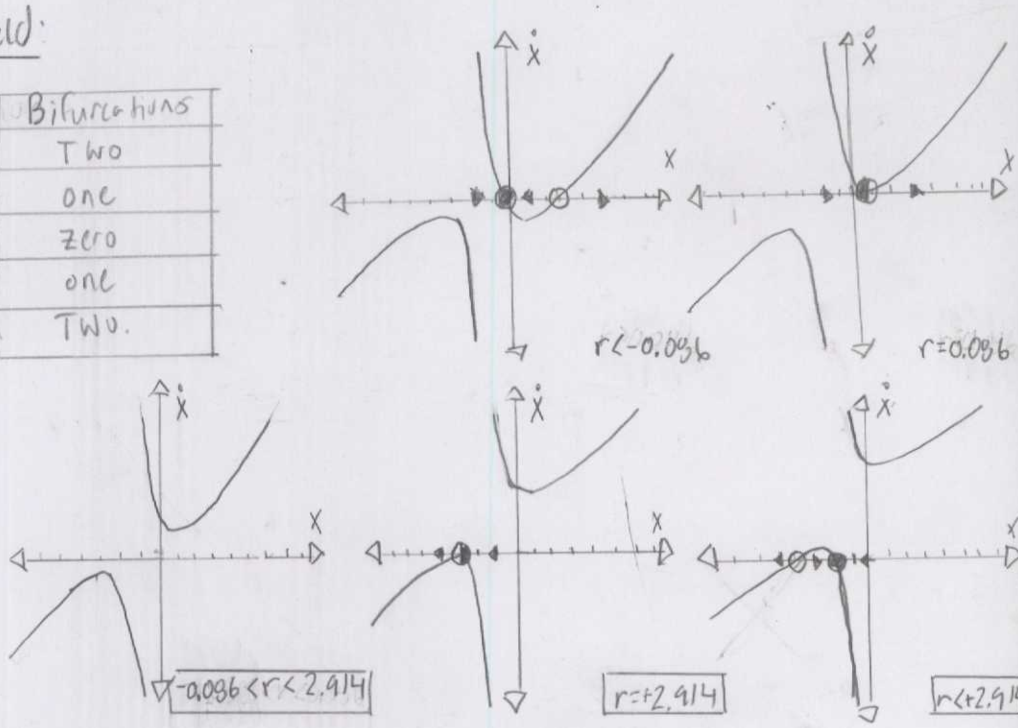
Bifurcation Diagram:



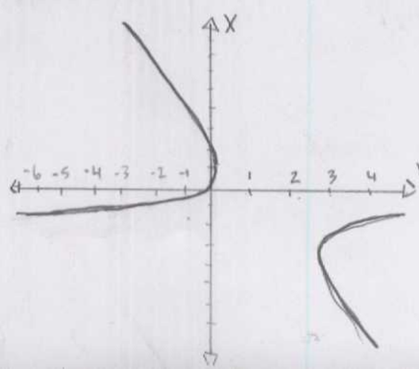


$\dot{x} = r + \frac{1}{2}x - x/(1+x)$ 3.1.4. Vector Field:

r	Bifurcations
< -0.096	Two
= -0.096	one
-0.096 < r < 2.914	zero
= +2.914	one
< +2.914	Two.

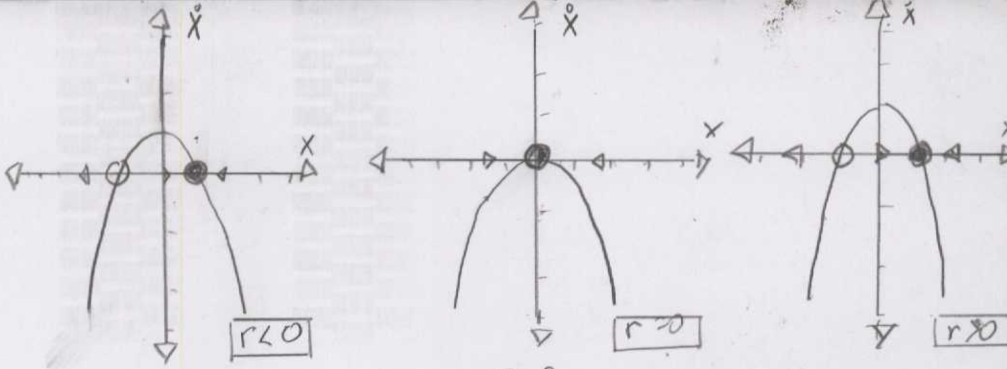


Bifurcation Diagram:



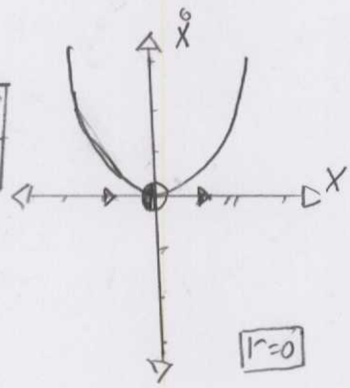
$\dot{x} = r^2 - x^2$ 3.1.5. a) Vector Field:

r	Bifurcations
>0	Two
=0	one
<0	Two

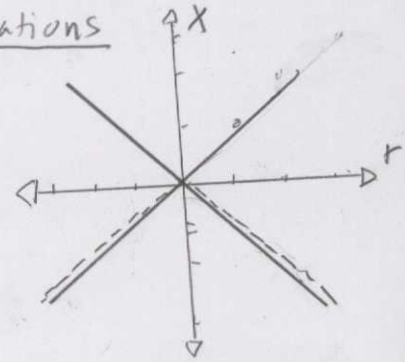


$\dot{x} = r^2 + x^2$ b) Vector Field:

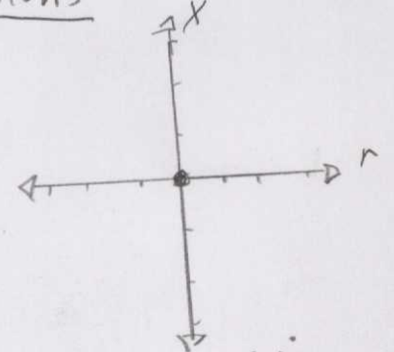
r	Bifurcations
0	One



Bifurcations



Bifurcations:

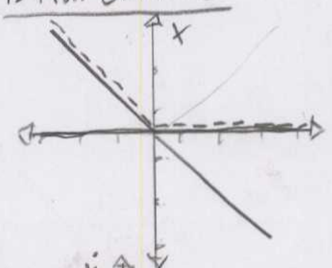


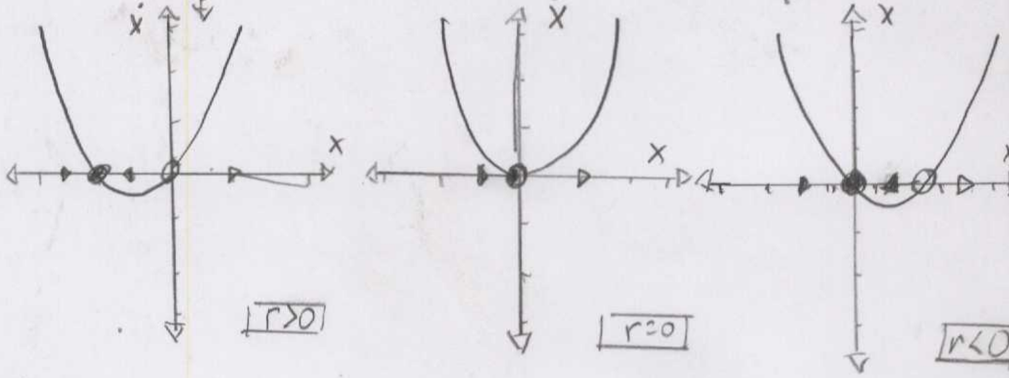
$\dot{x} = rx + x^2$ 3.2.1 Vector Field:

$(\bar{r}^2 + x^2)\, x$

r	Bifurcations
>0	Two
=0	one
<0	Two

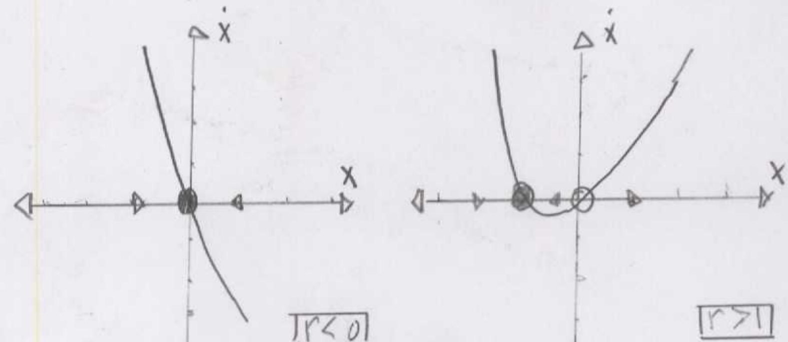
Bifurcations:

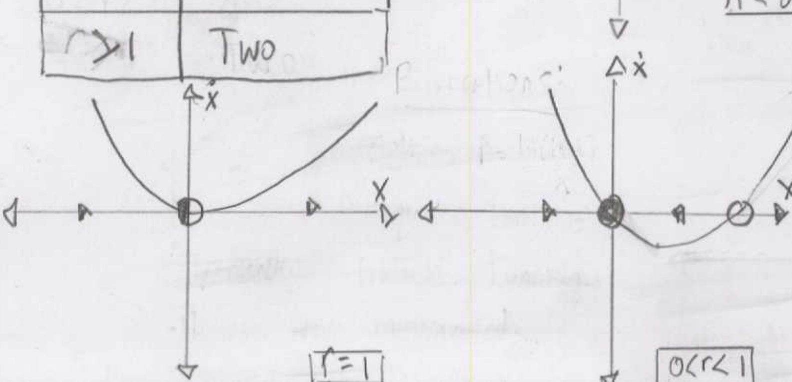




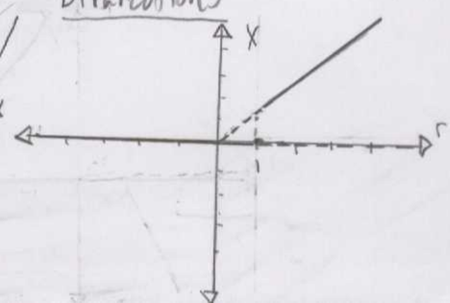
$\dot{x} = rx - \ln(1+x)$ 3.2.2. Vector Field:

r	Bifurcations
<0	One
0<r≤1	Two
r=1	One
r>1	Two





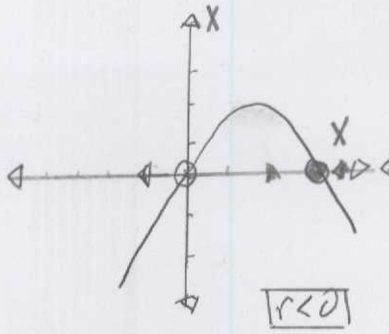
Bifurcations:



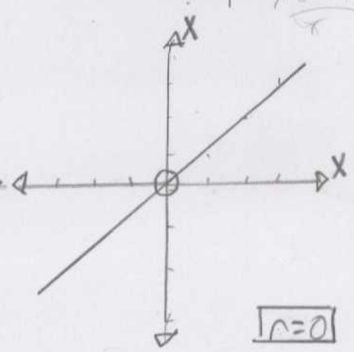
$\dot{x} = x - rx(1-x)$ 3.2.3. Vector Field:

$r = \frac{x(1-x)}{x}$

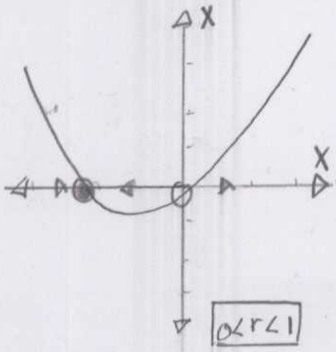
r	Bifurcations
≤ 0	Two
$=0$	one
$0<r<1$	Two
$r \geq 1$	One
>1	Two



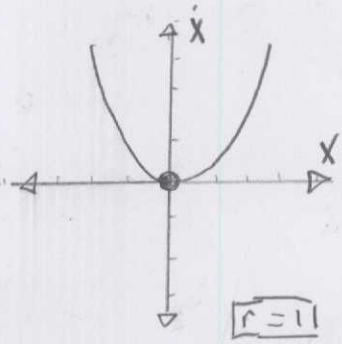
$r<0$



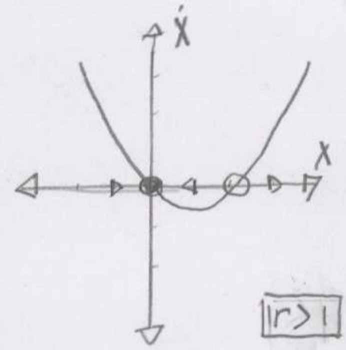
$r=0$



$0<r<1$



$r=1$

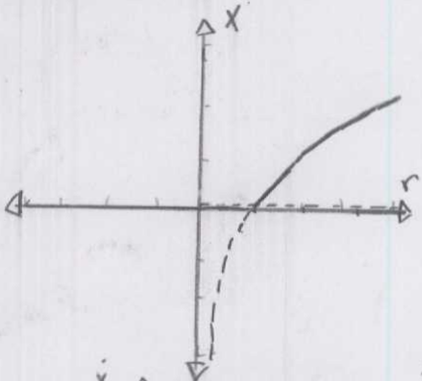


$r>1$

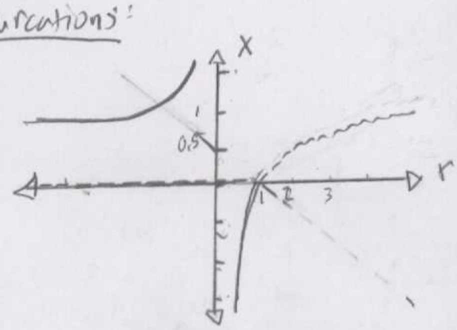
$\dot{x} = x(r - e^x)$ 3.2.4. Vector Field:

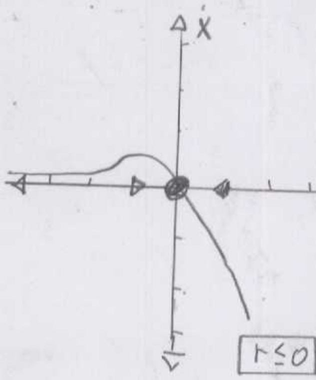
r	Bifurcations
≤ 0	one
$0<r<1$	Two
$0 \leq r \leq 1$	one
>1	Two

Bifurcations:

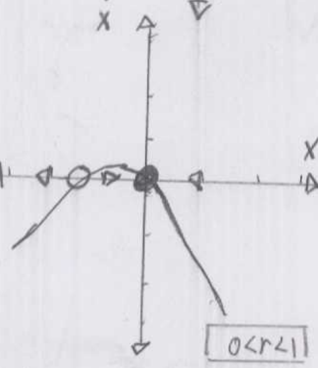


Bifurcations:

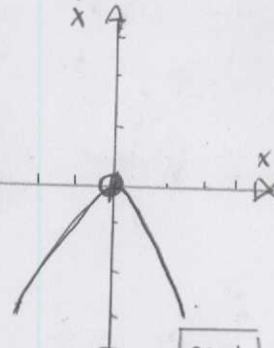




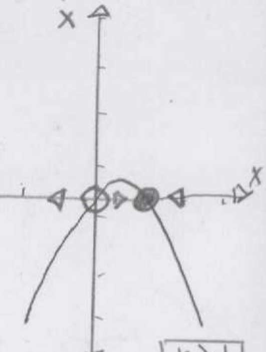
$r \leq 0$



$0<r<1$



$r=1$



$r>1$

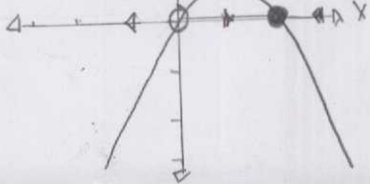
$\dot{x} = c_1 x - c_2 x^2$ 3.2.5 a) $\frac{d[A]}{dt} = -k_1[A][X] + k_2[X]^2$; $\boxed{\frac{d[X]}{dt} = (k_1[A] - k_2[B])[X] - k_{-1}[X]^2}$

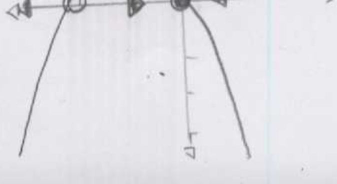
$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X$

$X + B \xrightarrow{k_2} C$

$\frac{d[B]}{dt} = -k_2[B][X]$; $\frac{d[C]}{dt} = k_2[B][X]$

b) $k_1[A] > k_2[B]$ $\qquad$ $k_1[A] < k_2[B]$





Chemically, a rate of change $\frac{d[X]}{dt}$ that approaches zero, then remains zero is of greater stability than a rate of change which increases from zero.

$\dot{n} = GnN - kn$

$\dot{N} = -GnN - fN + p$

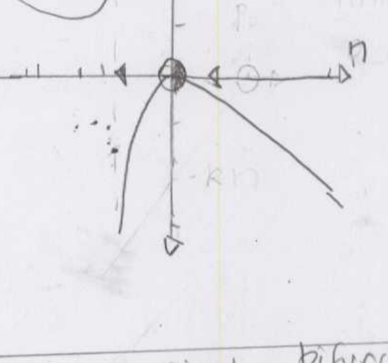
3.3.1 a) Suppose $\dot{N} > \dot{n}$, then $\dot{N} \approx 0$. "Adiabatic Elimination"

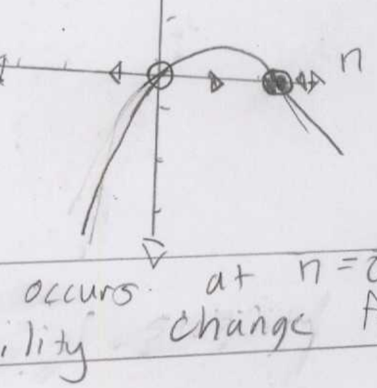
$GnN + fN = p$; $\dot{n} = -fN + p - kn$;

$N = \frac{p}{Gn + f}$; $\boxed{\dot{n} = -f\left[\frac{p}{Gn+f}\right] + p - kn}$

$\boxed{p > \frac{kn\,[Gn+f]}{1-pf} = p_c}$

b)





c) A transcritical bifurcation occurs at $n=0$ because of the stability change for the fixed point.

d) $G, n, p, f > 0$, $N = 0$, a constant amount of excited photons

$\dot{E} = K(P - E)$

$\dot{P} = \gamma_1(ED - P)$

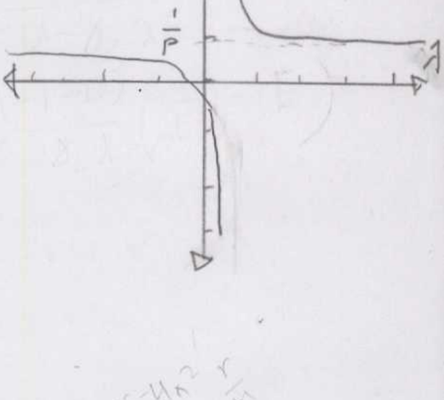
$\dot{D} = \gamma_2(\lambda + 1 - D - \lambda EP)$

3.3.2 a) Assume $\dot{P} \approx 0$, $\dot{D} \approx 0$; $P = ED$; $D = \lambda + 1 - \lambda EP$

$\dot{E} = K(ED - E) = K(E(\lambda + 1 - \lambda EP) - E)$

b) Fixed Points: $\boxed{E = 0, \frac{1}{P}}$

c) Bifurcation Diagram:



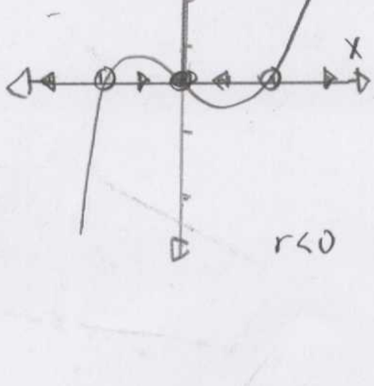
$\dot{x} = rx + 4x^3$

$r = -4x^2$

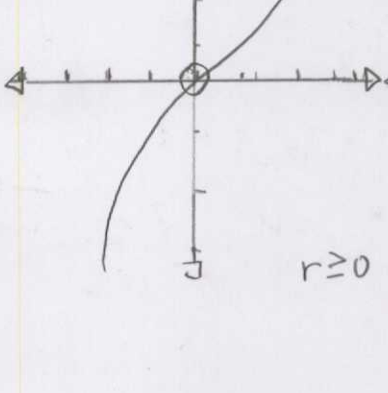
3.4.1 Vector Field:

r	Bifurcations
<0	Three
≥0	One

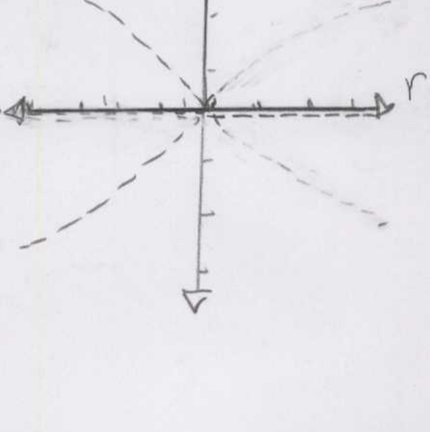
Bifurcations: Subcritical



r<0



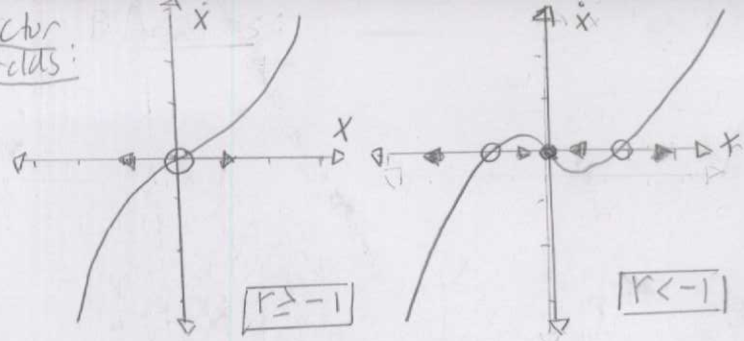
r≥0



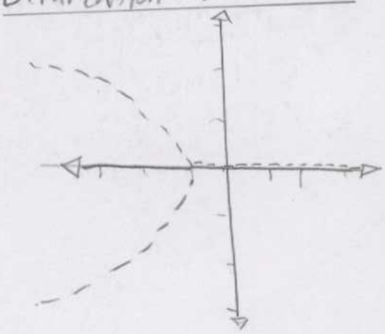
$\dot{x} = rx - \sinh x$ 3.4.2

r	Bifurcations
≥ -1	One
< -1	Three

Vector Fields:



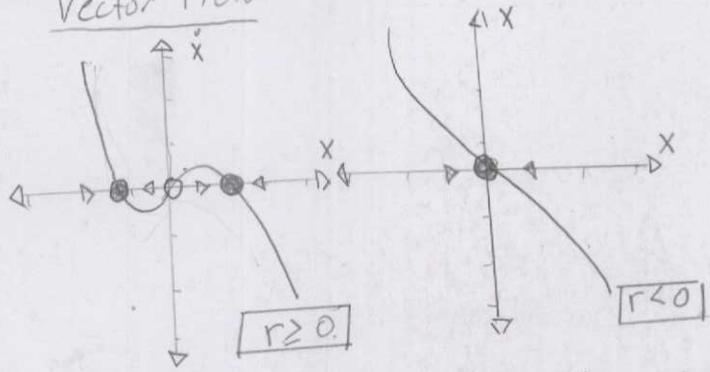
Bifurcation: subcritical



$\dot{x} = rx - 4x^3$ 3.4.3

r	Bifurcations
≥ 0	Three
≤ 0	One

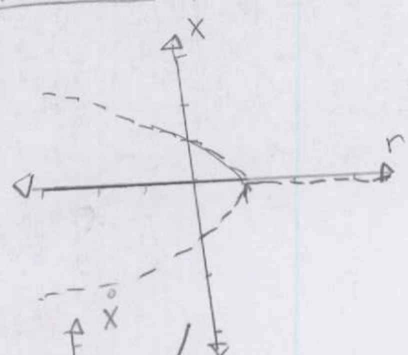
Vector Field:



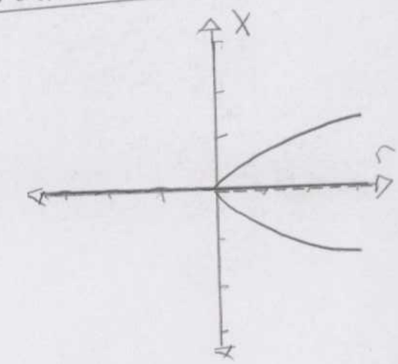
$\dot{x} = x + \frac{rx}{1+x^2}$ 3.4.4

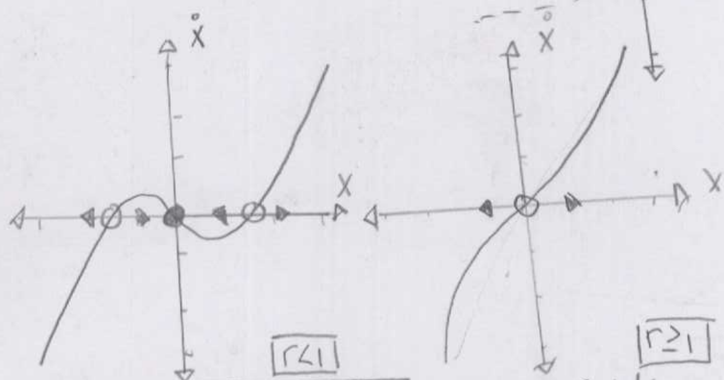
r	Bifurcations
< 1	Three
≥ 1	one

Bifurcations: Supercritical



Bifurcation: Supercritical

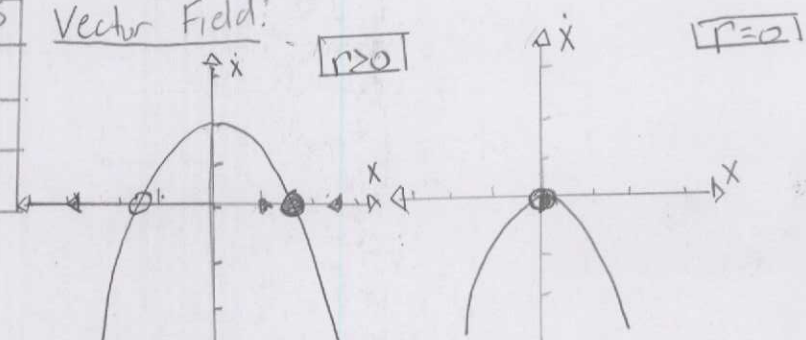


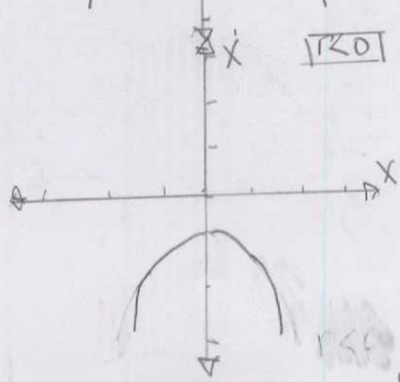


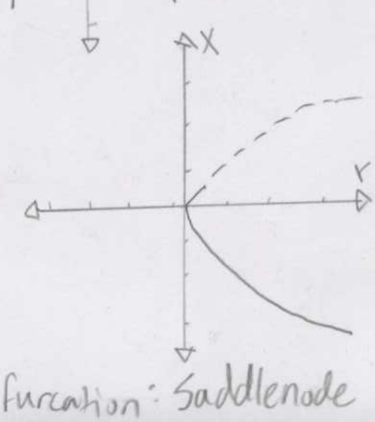
$\dot{x} = r - 3x^2$ 3.4.5

r	Bifurcations
> 0	Two
$= 0$	one
< 0	zero

Vector Field:





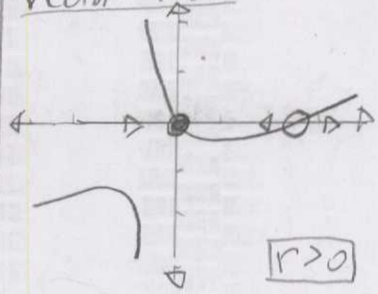


Bifurcation: Saddlenode

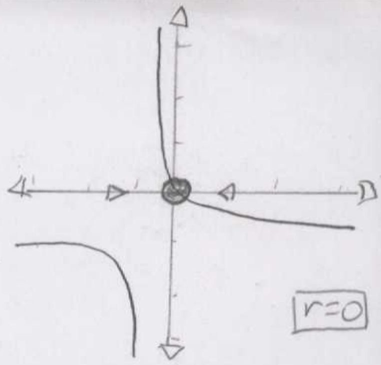
$\dot{x} = rx - \dfrac{x}{1+x}$ 3.4.6.

r	Bifurcations
≥ 0	Two
$= 0$	~~one~~ one
< 0	Two

Vector Field:

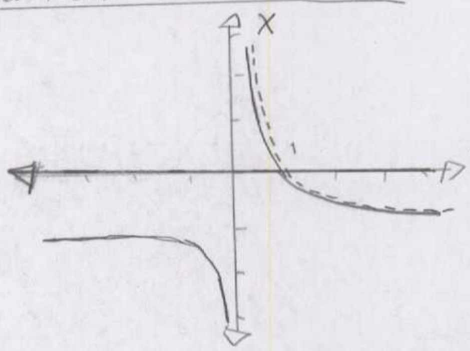


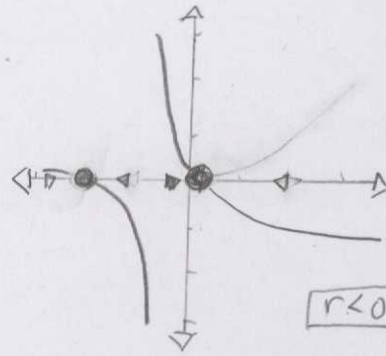
$\boxed{r>0}$



$\boxed{r=0}$

Bifurcation: Transcritical



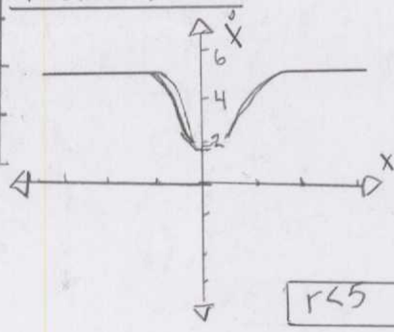


$\boxed{r<0}$

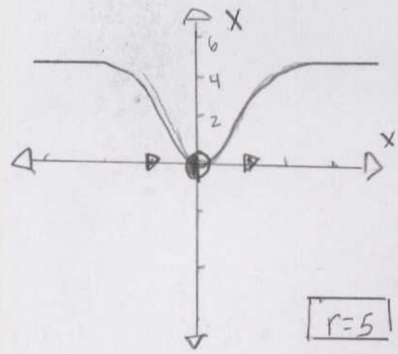
$\dot{x} = 5 - re^{-x^2}$ 3.4.7

r	Bifurcations
≤ 5	zero
$= 5$	one
> 5	Two

Vector Field:



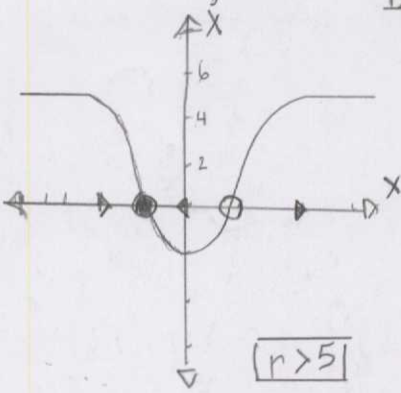
$\boxed{r<5}$



$\boxed{r=5}$

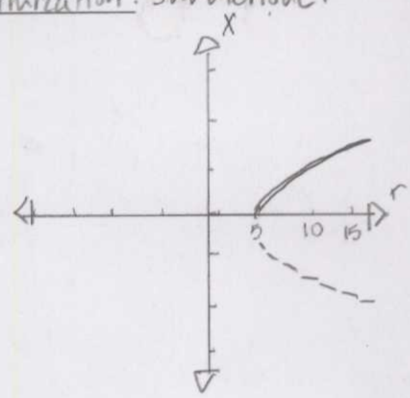
$\dot{x} = rx - \dfrac{x}{1+x^2}$ 3.4.8.

r	Bifurcations
≤ 0	one
$0<r<1$	three
≥ 1	one

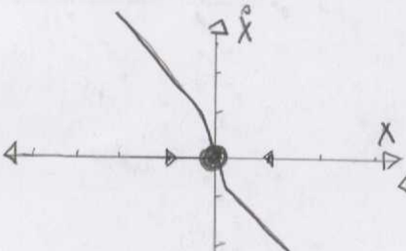


$\boxed{r>5}$

Bifurcation: Saddlenode.

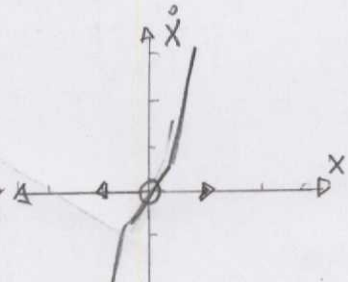


Vector Field:



$\boxed{r\leq 0}$

$\boxed{0<r<1}$

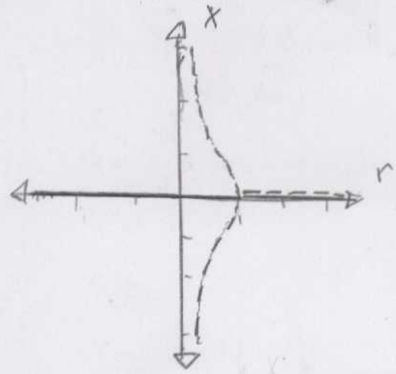


$\boxed{r\geq 1}$

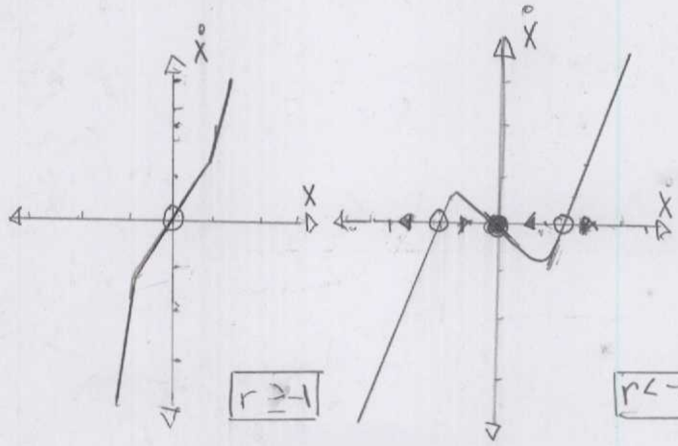
$\dot{x} = x + \tanh(rx)$ 3.4.9

r	Bifurcations
≤ -1	one
> -1	three

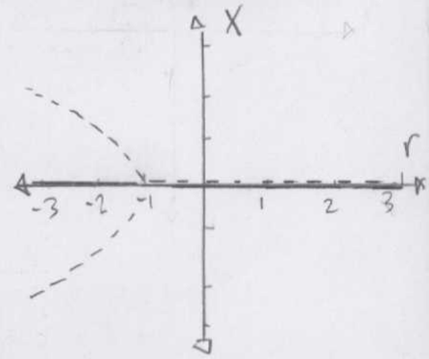
Bifurcation : Transcritical



Vector Field:



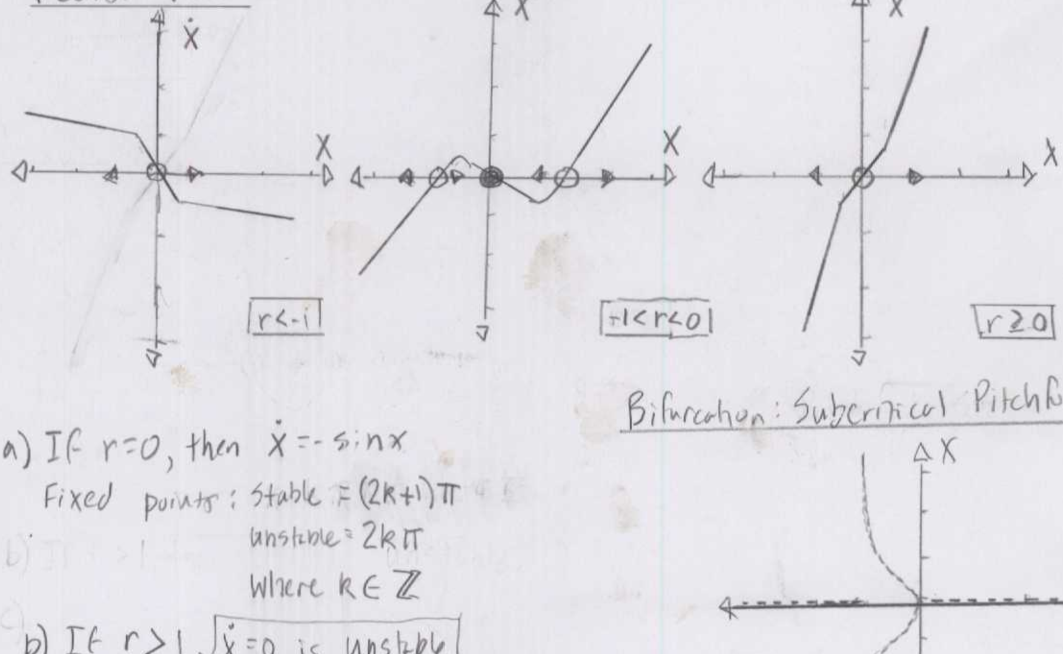
Bifurcation: Subcritical



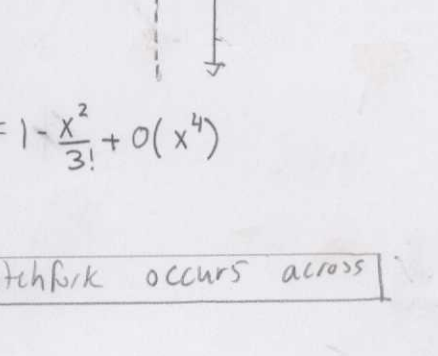
$\dot{x} = rx + \dfrac{x^3}{1+x^2}$ 3.4.10.

r	Bifurcations.
<-1	one
$-1<r<0$	three
≤ 0	one

Vector Field:



Bifurcation: Subcritical Pitchfork



$\dot{x} = rx - \sin x$ 3.4.11 a) If $r=0$, then $\dot{x} = -\sin x$

Fixed points : stable $= (2k+1)\pi$

unstable $= 2k\pi$

where $k \in \mathbb{Z}$

b) If $r>1$, $\boxed{\dot{x}=0 \text{ is unstable}}$

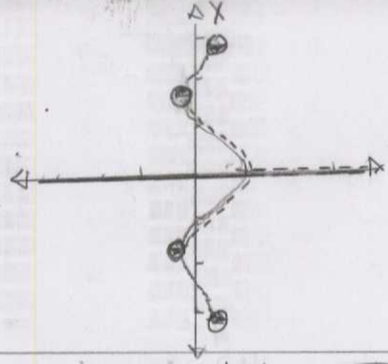
c) $\boxed{\text{As } r = \infty \to 0\text{, then a subcritical pitchfork best describes the bifurcation.}}$

d) $\dot{x} = rx - \sin(x)$; $r = \dfrac{\sin(x)}{x} = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n+1)!} = 1 - \dfrac{x^2}{3!} + O(x^4)$

$$\boxed{x = \pm(6[1-r])^{1/2}}$$

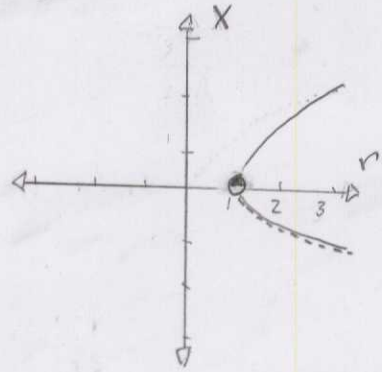
e) $\boxed{\text{As } r = -\infty \to 0\text{, then a supercritical pitchfork occurs across the function } \dot{x} = rx - \sin(x).}$

f) <u>Bifurcations:</u>

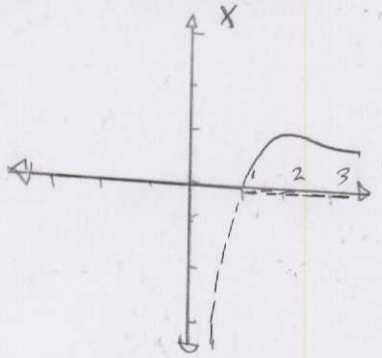


$\dot{x} = f(x,r)$ 3.4.12 $\boxed{\text{A "quadrification" function is } x = \frac{1}{2}(3 \pm \sqrt{1 \pm 4\sqrt{r}}) \text{ where}}$
$\boxed{\dot{x} = f(x,r) = (x-2)^2(x-1)^2 - r. \text{ This function has even polynomial}}$
$\boxed{\text{multiplicities to describe zero bifurcations } r<0 \text{ and}}$
$\boxed{\text{four when } r>0.}$

$\dot{x} = r - x - e^{-x}$ 3.4.13 a) Best guess of roots: $r=1$, $x=0$

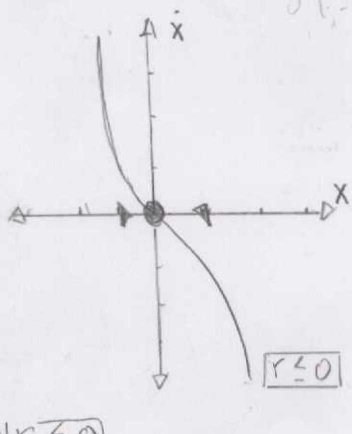


$\dot{x} = 1 - x - e^{-rx}$ b) Best Guess of roots: $r=1$, $x=0$

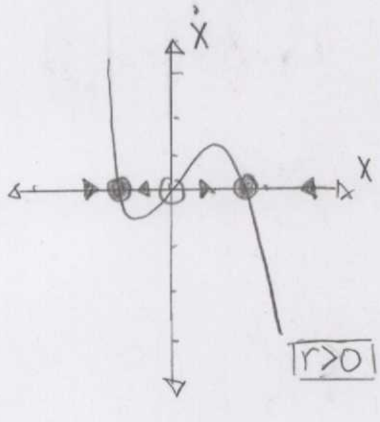


$\dot{x} = rx + x^3 - x^5$ 3.4.14 a) $\boxed{\ddot{x} = 0 = r + 3x^2 - 5x^4}$ -or- $\boxed{r = x^2(x^2-1)}$

b) <u>Vector Field:</u>



$\boxed{r \leq 0}$



$\boxed{r > 0}$

c) $\boxed{r_c \leq 0}$

$\dot{x} = rx + x^3 - x^5$ 3.4.15. $-\frac{dV(x)}{dx} = \dot{x} = 0 = -\frac{r}{2} - \frac{x^2}{4} + x^4$

Where $a = x^2$

$$a_1, a_2 = \frac{+\frac{1}{4} \pm \sqrt{(+1)^2 - 4(1)(-r)}}{2(1)}$$

$$= \left(\frac{-1 \pm \sqrt{1+4r}}{2}\right)$$

$$x_1 = +\sqrt{\frac{1+\sqrt{1+4r}}{2}} \; ; \; x_2 = -\sqrt{\frac{1+\sqrt{1+4r}}{2}}$$

$$x_3 = +\sqrt{\frac{1-\sqrt{1+4r}}{2}} \; ; \; x_4 = -\sqrt{\frac{1-\sqrt{1-4r}}{2}}$$

$$x_5 = 0$$

$$V(x) = -r\frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6}.$$

$$V(x_1) = V(x_2) = V(x_3) = V(x_4) = V(x_5) = 0$$

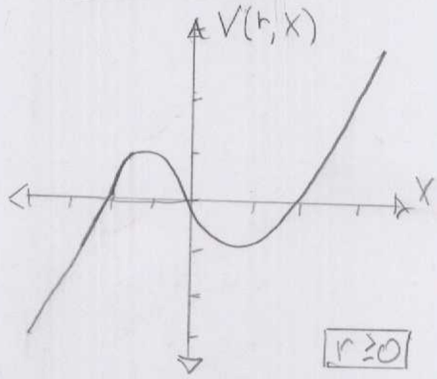
$$@\, V(x_1) = \frac{-r}{2}\left(\frac{1+\sqrt{1+4r}}{2}\right) + \frac{1}{4}\left(\frac{1+\sqrt{1+4r}}{2}\right)^2 - \frac{1}{6}\left(\frac{1+\sqrt{1+4r}}{2}\right)^3 \boxed{=0}$$

$r = -3/16$

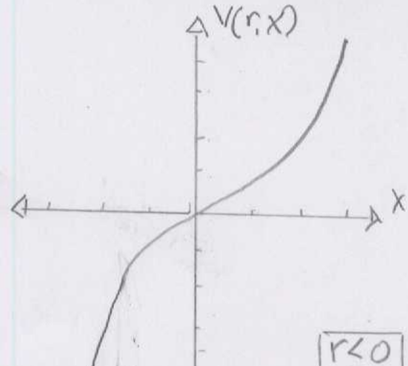
$\dot{x} = r - x^2$ 3.4.16 a) $-\frac{dV}{dx} = \dot{x} = r - x^2$; $\boxed{V(r,x) = \frac{x^3}{3} - rx}$

r	Bifurcations
≥ 0	Three
<0	One

Potential Field:



$\boxed{r \geq 0}$

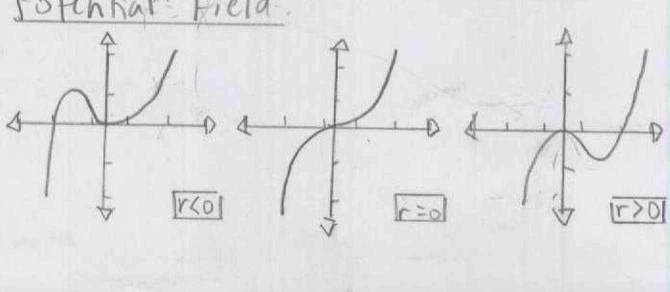


$\boxed{r<0}$

$\dot{x} = rx - x^2$

b) $\frac{-dV}{dx} = \dot{x} = rx - x^2$; $\boxed{V(r,x) = \frac{x^3}{3} - \frac{rx^2}{2}}$

Potential Field:



r	Bifurcations
<0	Two
$=0$	One
>0	Two.

$\dot{x} = rx + x^3 - x^5$ c) $-\frac{dV}{dx} = rx + x^3 - x^5$; $\boxed{V(r,x) = \frac{x^6}{6} - \frac{x^4}{4} - rx^2}$

r	Bifurcations
≤0.5	Three
>0.5	One.

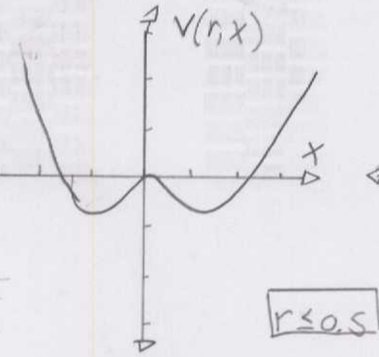
<u>Potentials:</u>

$0 = r + x^2 - x^4$

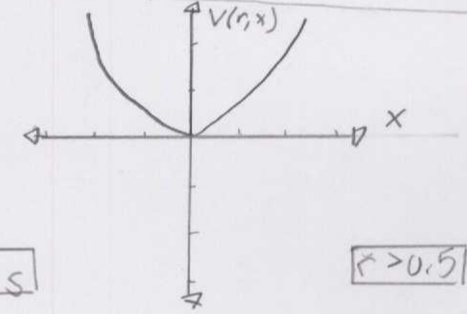
$= r + a - a^2$

$= \frac{-1 \pm \sqrt{1^2 - 4(-1)(r)}}{2(1)}$

$x^2 = \sqrt{\frac{-1 \pm \sqrt{1+4r}}{2}}$



$\boxed{r \leq 0.5}$



$\boxed{r > 0.5}$

$b\dot{\phi} = -mg\sin\phi + mr\omega^2 \sin\phi\cos\phi$ 3.5.1.

> A better representation of $b\dot{\phi} = mg\sin\phi + mr\omega^2 \sin\phi\cos\phi$ is $b\dot{\phi} = mg\sin\phi\left(\frac{r\omega^2}{g}\cos\phi - 1\right)$, which best represents the maximum angle of $\phi = \pi/2$. If the bead approaches a fixed point during rotation, then $b\dot{\phi} = 0$; $\frac{r\omega^2}{g}\cos\phi = 1$; $\cos\phi = \frac{g}{r\omega^2}$; and, $\frac{g}{r\omega^2}$ requires a positive value above zero.

$\frac{d\phi}{d\tau} = f(\phi)$

$= -\sin\phi + \gamma\sin\phi\cos\phi$

$= \sin\phi(\gamma\cos\phi - 1)$

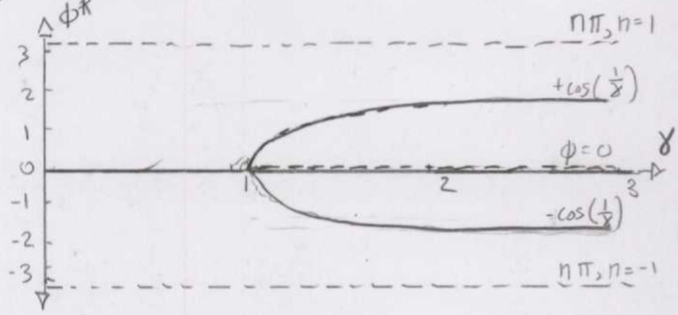
3.5.2 $f(\phi) = \sin\phi(\gamma\cos\phi - 1)$

$f'(\phi) = \gamma[\cos^2\phi - \sin^2\phi - 1] = \gamma[\cos 2\theta - 1]$

$f''(\phi) = -2\gamma[\sin 2\theta]$

$\phi^* = n\pi$; $f'(\phi) = 0$; Half-Node

$\phi^* = \cos^{-1}\left(\frac{1}{\gamma}\right)$



$\frac{d\phi}{d\tau} = f(\phi)$

$= -\sin\phi + \gamma\sin\phi\cos\phi$

$= \sin\phi(\gamma\cos\phi - 1)$

3.5.3. If $\phi \approx 0$,

then $\sin\phi \approx \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!}$

$\approx \phi$

$\cos\phi \approx 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!}$ and $\frac{d\phi}{d\tau} = \phi\left(\gamma\left[1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!}\right] - 1\right)$

$= \gamma\phi - \frac{\gamma\phi^3}{2!} + \frac{\gamma\phi^5}{4!}$

Where $\boxed{\frac{d\phi}{d\tau} = A\phi - B\phi^3 + O(\phi^3) ; A = \gamma, B = \frac{\gamma}{2}, O(\phi^5) = \frac{\gamma\phi^5}{4!}}$

$m\ddot{x} = -F_{spring} - F_{Fric}$ 3.5.4. $m\ddot{x} = -k \cdot l\cos\phi - kL_0\cos\phi - b\dot{\phi}$

$= -k(l - L_0)\cos\phi - b\dot{\phi} = -k(\sqrt{x^2+h^2} - L_0)\frac{x}{l} - b\dot{\phi}$

$= -k(\sqrt{h^2+h^2} - L_0)\frac{x}{\sqrt{h^2+x^2}} - b\dot{x}$

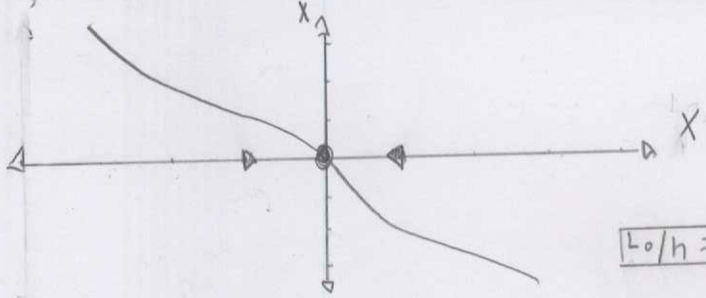
$\boxed{= -k\left(1 - \frac{L_0}{\sqrt{h^2+x^2}}\right)x - b\dot{x}}$

b. $m\ddot{x} + b\dot{x} + k\left(1 - \frac{L_0}{\sqrt{x^2+h^2}}\right)x = 0$

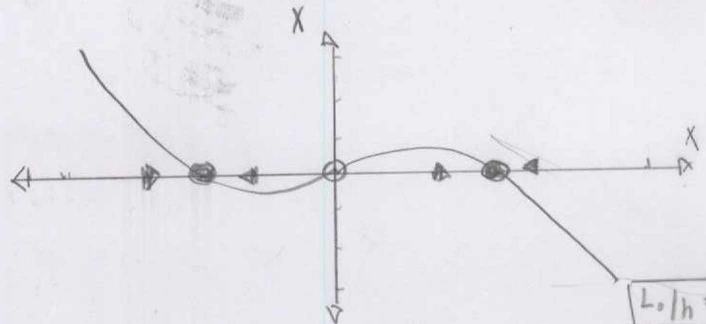
if $\dot{x} = 0$, $\boxed{x^* = \sqrt{L_0^2 - h^2},\ 0}$

c. If $m = 0$, $b\dot{x} + k\left(1 - \frac{L}{\sqrt{x^2+h^2}}\right)x = 0$, then $\boxed{x^* = \sqrt{L_0^2 - h^2},\ 0}$

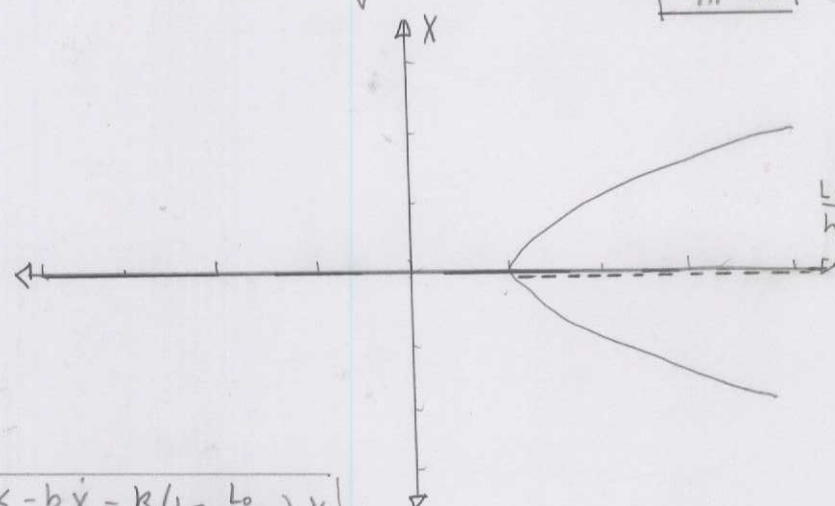
Bifurcation Diagram:



$\boxed{L_0/h = 0.5}$



$\boxed{L_0/h = 1.5}$



d. $\boxed{\text{If } m \neq 0, \text{ then } m\ddot{x} \ll -b\dot{x} - k\left(1 - \frac{L_0}{\sqrt{x^2+h^2}}\right)x}$

$\varepsilon \frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau} = f(\phi)$ 3.5.5. a) $\frac{d\phi}{d\tau} = f(\phi)$; T_{fast} is estimated to be:

$$\varepsilon^{1-2k} \frac{d^2\phi}{d\tau^2} + \varepsilon^{-k} \frac{d\phi}{d\tau} = f(\phi);$$

where $k = 1$, $\varepsilon^{1-2k} = \varepsilon^{-k} \gg 1$

$k = \frac{1}{2}$, $\varepsilon^{1-2k} = 1 \gg e^{-k}$

$k = 0$, $\varepsilon^{-k} = 1 \gg e^{1-2k}$

$$T = \varepsilon \frac{b}{mg} = \frac{m^2 g r}{b^2} \frac{b}{mg} = \boxed{\frac{mr}{g}}$$

b) If $\tau = \varepsilon \zeta$, then $\varepsilon \frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau} = \varepsilon \frac{d^2\phi}{d(\varepsilon\zeta)^2} + \frac{1}{\varepsilon}\frac{d\phi}{d\zeta} = f(\phi)$

$$\boxed{\frac{d^2\phi}{d\zeta^2} + \frac{d\phi}{d\zeta} = \epsilon f(\phi) \text{ "Rescaled"}}$$

c. $T_{fast} = \varepsilon T_{slow}$

$\varepsilon \ddot{x} + \dot{x} + x = 0$ 3.5.6. $x(0) = 1;\ \dot{x}(0) = 0$

a) General solution: $x(t) = C_1 e^{\lambda t}$ $\quad \varepsilon\lambda^2 + \lambda + 1 = 0$

$\dot{x}(t) = \lambda C_1 e^{\lambda t};\ \lambda = \dfrac{-1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon}$

$\ddot{x}(t) = \lambda^2 C_1 e^{\lambda t};\ x(t) = C_1 e^{(-1+\sqrt{1-4\varepsilon})x/2\varepsilon} + C_2 e^{(-1-\sqrt{1-4\varepsilon})x/2\varepsilon}$

$x(0) = C_1 + C_2 = 1$

$\dot{x}(t) = C_1\left(\dfrac{-1+\sqrt{1-4\varepsilon}}{2\varepsilon}\right) e^{(-1+\sqrt{1-4\varepsilon})x/2\varepsilon}$

$+ C_2\left(-\dfrac{(1-\sqrt{1-4\varepsilon})}{2\varepsilon}\right) e^{(-1-\sqrt{1-4\varepsilon})x/2\varepsilon}$

$\dot{x}(0) = C_1\left(-\dfrac{(1+\sqrt{1-4\varepsilon})}{2\varepsilon}\right) + C_2\left(\dfrac{-1-\sqrt{1-4\varepsilon}}{2\varepsilon}\right)$

$= \dfrac{-C_1\sqrt{1-4\varepsilon}}{2\varepsilon} + \dfrac{C_1}{2\varepsilon} + (1 - C_1)\left(\dfrac{-1-\sqrt{1-4\varepsilon}}{2\varepsilon}\right)$

$= -\dfrac{C_1\sqrt{1-4\varepsilon}}{2\varepsilon} + \dfrac{C_1}{2\varepsilon} + \dfrac{(1+\sqrt{1-4\varepsilon})}{2\varepsilon} +$

$+ \dfrac{C_1(1-\sqrt{1-4\varepsilon})}{2\varepsilon}$

$= -\dfrac{\cancel{C_1\sqrt{1-4\varepsilon}}}{2\varepsilon} + \dfrac{C_1}{2\varepsilon} + \dfrac{(1+\sqrt{1-4\varepsilon})}{2\varepsilon}$

$+ \dfrac{C_1}{2\varepsilon} + \dfrac{\cancel{C_1\sqrt{1-4\varepsilon}}}{2\varepsilon} = 0$

$= \dfrac{C_1}{\varepsilon};\ = \dfrac{(1+\sqrt{1-4\varepsilon})}{2}$

$$\boxed{\text{Therefore, } x(t) = \left(\frac{1+\sqrt{1-4\varepsilon}}{2}\right)\left(e^{(-1+\sqrt{1-4\varepsilon})x/2\varepsilon}\right) + \left(1 - \frac{(1+\sqrt{1-4\varepsilon})}{2}\right)\left(e^{(-1-\sqrt{1-4\varepsilon})x/2\varepsilon}\right)}$$

b. $\varepsilon\ddot{x} + \dot{x} + x : \varepsilon \dfrac{dx^2}{dt^2} + \dfrac{dx}{dt} + x = 0$

$\dfrac{\varepsilon}{T^2}\dfrac{d^2x}{d\tau^2} + \dfrac{1}{T}\dfrac{dx}{d\tau} = -x$ $\quad$ where $T = \dfrac{t}{\tau} = \varepsilon$

$\dfrac{1}{T}\dfrac{d^2x}{d\tau^2} + \dfrac{1}{T}\dfrac{dx}{d\tau} = x;\ \ddot{x} + \dot{x} = Tx = \varepsilon x;\ \boxed{\ddot{x} + \dot{x} - Tx = 0}$

$\mathring{N} = rN(1-N/K)$ 3.5.7. a) $N(0) = N_0$;

Parameter	Dimensions
r	Per time (rate
K	Same as N (amount)
N_0	Same as N (amount)

b) $\frac{dN}{dt} = rN(1-N/K)$; If $\frac{N}{K} = x$, then $dN = K\,dx$

$\frac{dx}{dt} = rx(1-x)$; If $t = \frac{\tau}{r}$, then $dt = \frac{d\tau}{r}$

$$\boxed{\frac{dx}{d\tau} = x(1-x)}$$

c) $u = x$; $\frac{du}{d\tau} = u(1-u)$; $u(0) = u_0$

$\int \frac{du}{u(1-u)} = d\tau$; $\int \frac{A}{u}du + \int \frac{B}{(1-u)}du = \int \frac{du}{u} + \int \frac{du}{(1-u)} = \ln \frac{u}{1-u} = \tau + c$

$A(1-u) + Bu = 1$

$\frac{1-u}{u} = Ce^{-\tau}$

$1 = u(1+Ce^{-\tau})$

$u = \frac{1}{1+Ce^{-\tau}}$

$u(0) = u_0 = \frac{1}{1+C}$

$C = \frac{1-u_0}{u_0}$

$$\boxed{u(\tau) = \frac{1}{1+\left(\frac{1-u_0}{u_0}\right)e^{-\tau}}}$$

d) An advantage of the dimensionless functions are lower degrees of freedom during analysis. The graphical representations do not have further axis to plot, and the functions are closer to the basic functions of precalculus.

$\mathring{u} = au + bu^3 - cu^5$ 3.5.8. Prove $\frac{dx}{d\tau} = rx + x^3 - x^5$, where $x = \frac{u}{U}$

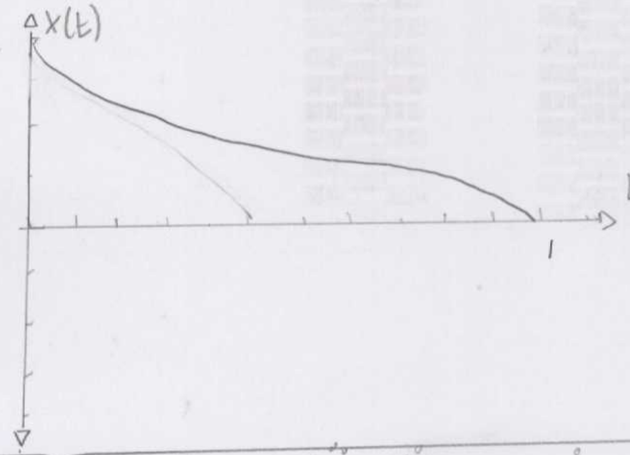
$\tau = \frac{t}{T}$

$\frac{\cancel{U}}{T}\frac{dx}{d\tau} = a\cancel{U}x + bU^{\cancel{3}\,2}x^3 - cU^{\cancel{5}\,4}x^5$

$\frac{dx}{d\tau} = Tax + TbU^2x^3 - TcU^4x^5$; $a = \frac{r}{T}$, $b = \frac{1}{TU^2}$, $c = \frac{1}{TU^4}$

$$\boxed{\frac{dx}{d\tau} = rx + x^3 - x^5}$$

3.6.1 Figure 3.6.3b corresponds to Figure 3.61b; specifically, the relationship between $y = h$, and $y = rx - x^3$. The dotted lines support a single bifurcation to two bifurcations at h_c, then three when $h > h_c$. To answer the question, Figure 3.6.3b has information of $h < 0$ and $h > 0$.

c) 

d) If $\varepsilon \ll 1$, then $\varepsilon \ddot{x} + \dot{x} + x \approx \dot{x} + x = 0$ and is a similar model to the boundary conditions.

e) Mechanical System: An extremely viscous solution for an oscillating Newtonian device.

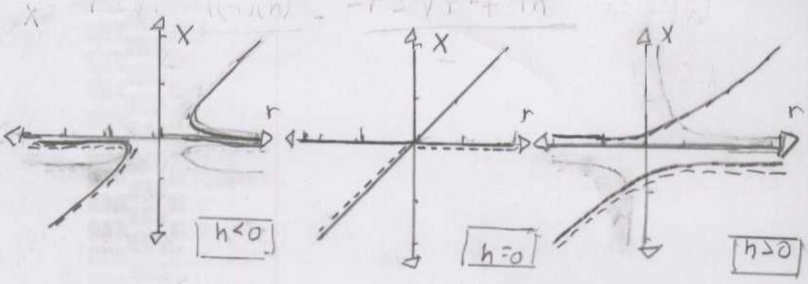
Electrical System: An electrical system of the form $v = Ri + L\frac{di}{dt} + \frac{1}{c}\int i \, dt$

where $\varepsilon = \frac{1}{c} \ll 1$.

$\dot{x} = h + rx - x^2$ 3.6.2. a)

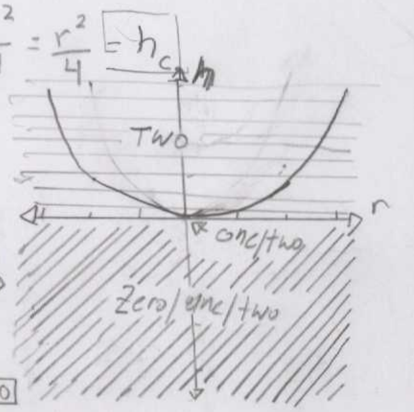
$r = \frac{-r \pm \sqrt{r^2 - 4(-1)(h)}}{2(-1)}$

h	Bifurcations
<0	zero/one/two
=0	one/Two
>0	Two

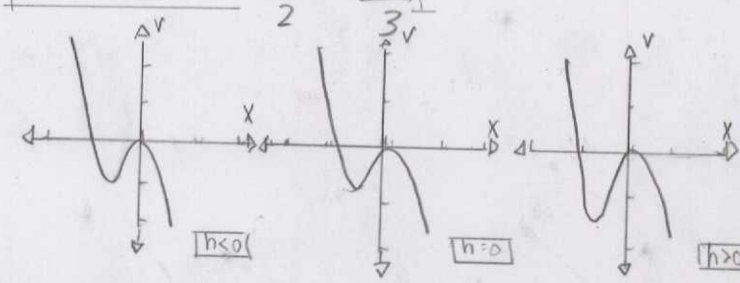


b) (r,h) Plane

$$\frac{d}{dx}(rx - x^2) = r - 2x \;;\; x_{max} = \frac{r}{2} \;;\; \frac{r^2}{2} - \frac{r^2}{4} = \frac{r^2}{4} \boxed{= h_c}$$

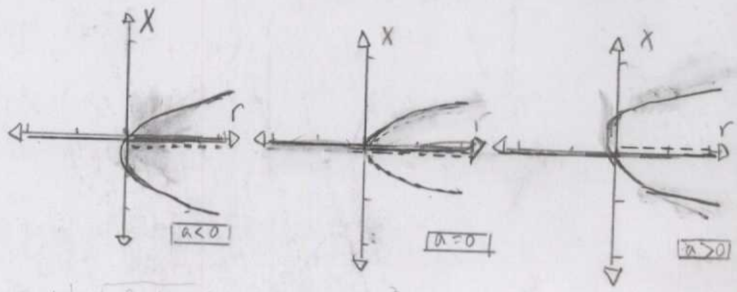


c) $\boxed{V(x,h,r) = hx + \frac{rx^2}{2} - \frac{x^3}{3}}$



$\dot{x} = rx + ax^2 - x^3$ 3.6.3 a)

a	Bifurcations
<0	one/two/three
=0	one/three
>0	one/two/three



b) (r,a) plane

$$\frac{d}{dx}(rx + ax^2 - x^3) = r + 2ax - 3x^2 = 0 \;;$$

$$r + ax - x^2 \;;\; \boxed{a = \frac{x^2 - r}{x}}$$

3.6.4 A small imperfection to a saddlenode bifurcation shifts the cusp either left or right.

$mg\sin\theta = kx\left(1 - \frac{L}{\sqrt{x^2+a^2}}\right)$ 3.6.5 a) $F = -F_{spring} = F_{g,x} = F_g \sin\theta = mg\sin\theta = k(x - x\sin\theta)$

$$= k\left(x - x \cdot \frac{L_0}{\sqrt{x^2+a^2}}\right)$$

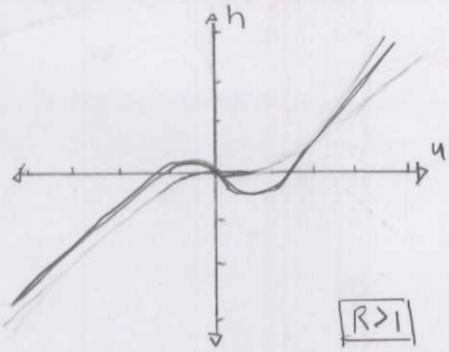
$$\boxed{= kx\left(1 - \frac{L_0}{\sqrt{x^2+a^2}}\right)}$$

b) Prove $1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}}$

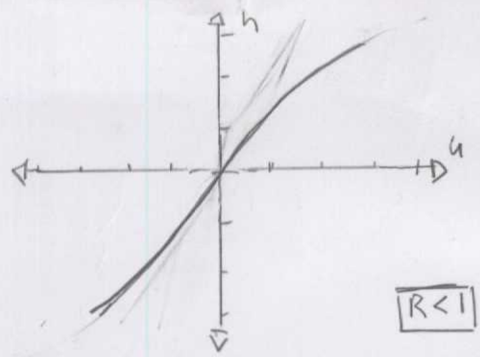
If $1 - \frac{mg\sin\theta}{kx} = \frac{L_0}{a\sqrt{\left(\frac{x}{a}\right)^2 + 1}}$, then $u = \frac{x}{a}$, $R = \frac{L_0}{a}$, $h = \frac{mg\sin(\theta)}{ka}$.

and $\boxed{1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}}}$

c)



$\boxed{R>1}$



$\boxed{R<1}$

The variable h, as a function of u, has a single equilibrium point for both $R>1$ and $R<1$.

d) If $r = R-1$, $1-\frac{h}{u} = \frac{r+1}{\sqrt{1+u^2}}$; $u - h = \frac{(r+1)u}{\sqrt{1+u^2}}$; $u\sqrt{1+u^2} - h\sqrt{1+u^2} = (r+1)u$

$$u\left(1+\tfrac{1}{2}u^2 + O(u^4)\right) - h\left(1+\tfrac{1}{2}u^2 + O(u^4)\right) = (r+1)u$$

$$\cancel{u} + \frac{u^3}{2} - h - \frac{h}{2}u^2 = ru + \cancel{u}$$

$$\boxed{h + ru + \frac{h}{2}u^2 - \frac{1}{2}u^3 \approx 0}$$

e) $h\left(1+\frac{u^2}{2}\right) = \frac{1}{2}u^3 - ru$

$\frac{d}{du}h\left(1+\frac{u^2}{2}\right) = \frac{d}{du}\left(\frac{1}{2}u^3 - ru\right)$; $hu = \frac{3}{2}u^2 - r$; $r_{max} = \frac{3}{2}u^2 - hu$

$h\left(1+\frac{u^2}{2}\right) = \frac{1}{2}u^3 - \left(\frac{3}{2}u^2 - hu\right)u$; $h + \frac{hu^2}{2} = \frac{1}{2}u^3 - \frac{3}{2}u^3 + hu^2$

$h\left(1-\frac{1}{2}u^2\right) = -u^3$; $h = \frac{2u^3}{u^2-2}$

$$r_{max} = \frac{3}{2}u^2 - hu = \frac{3}{2}u^2 - \left(\frac{2u^3}{u^2-2}\right)u$$

$$= \frac{3}{2}u^2 - \frac{2u^4}{u^2-2}$$

$$= \frac{u^4+3u^2}{2(1-u^2)} \boxed{= R-1}$$

f) $1-\frac{h}{u} = \frac{R}{\sqrt{1+u^2}}$; $\frac{d}{du}\left(1-\frac{h}{u}\right) = \frac{d}{du}\left(\frac{R}{\sqrt{1+u^2}}\right)$; $\frac{h}{u^2} = -\frac{1}{2}\frac{R(2u)}{(1+u^2)^{3/2}}$

$h(1+u^2)^{3/2} = -R\cdot u^3$; $R = \frac{-h(1+u^2)^{3/2}}{u^3}$

$1-\frac{h}{u} = \frac{-h(1+u^2)^{3/2}}{u^3\sqrt{1+u^2}} = -\frac{h(1+u^2)}{u^3}$; $u - h = -\frac{h(1+u^2)}{u^2}$

$u^3 - hu^2 = -h(1+u^2)$; $u^3 = -h - hu^2 + hu^2$; $h = -u^3$

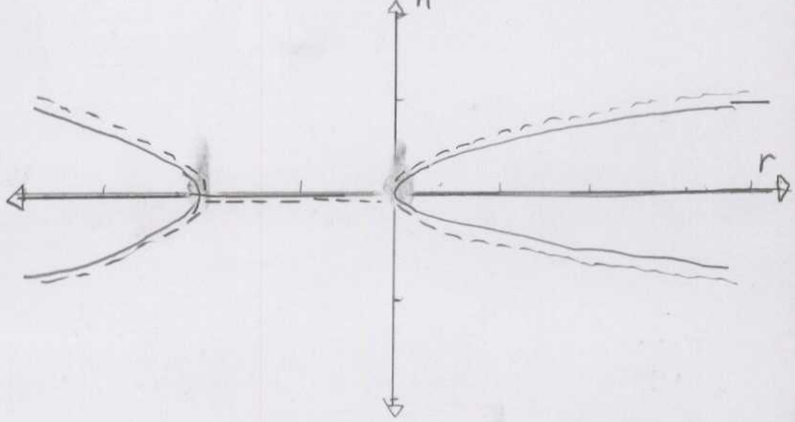
$$R = \frac{-h(1+u^2)^{3/2}}{u^3} = (1+u^2)^{3/2}$$

$$\boxed{\lim_{u\to 0} h = -u^3 \approx \frac{2u^3}{u^2-2} \approx u^3}$$

$$\boxed{\lim_{u\to 0} R = (1+u^2)^{3/2} \approx \frac{u^4+3u^2}{2(1-u^2)} + 1 = r+1}$$

g) $R = (1+u^2)^{3/2} = r+1$; $r(u) = (1+u^2)^{3/2} - 1$; $u = \sqrt{(r+1)^{2/3} - 1}$

$$\boxed{h = -u^3 = \pm\left(\sqrt{(r+1)^{2/3}-1}\right)^3}$$



h) $h = -u^3 = -\left(\frac{x}{a}\right)^3 = \frac{mg\sin\theta}{ka}$

$R = \left(1+\left(\frac{x}{a}\right)^2\right)^{3/2} = \frac{L_0}{a}$

The bifurcation plot represents the points of stability for the oscillating system.

$\tau\dot{A} = \varepsilon A - gA^3$

$\tau\dot{A} = \varepsilon A - gA^3 - kA^3$

3.6.6. $A(t)$ = Amplitude; τ = typical timescale; ε = dimensionless parameter

~~3.6.6.~~ Supercritical : $g>0$, subcritical: $g<0$, $K>0$

"Landaus Equation"

a) Landau's Equation describes the change of amplitude for a fluid system.

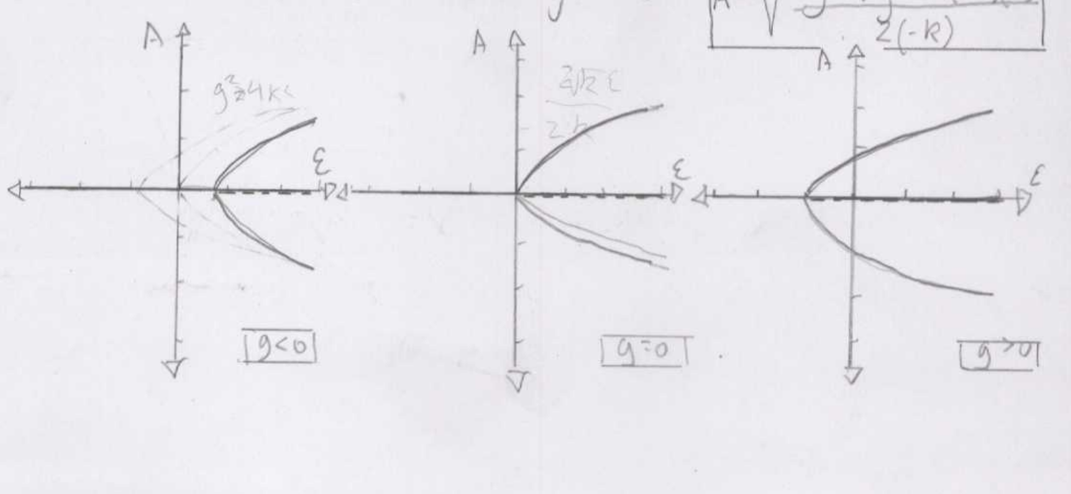
b). $\tau\dot{A} = \varepsilon A - gA^3 - kA^3$; if $g=0$, then $\tau\dot{A} = \varepsilon A - kA^3$; $A = \sqrt{\frac{\varepsilon}{k}}$

The function $A(\varepsilon)$ is a tricritical bifurcation because $A=0$ is a solution; in addition to, $A = +\sqrt{\frac{\varepsilon}{k}}$, and $A = -\sqrt{\frac{\varepsilon}{k}}$

c) $\tau\dot{A} = h + \varepsilon A - gA^3 - kA^5$; An approximation $h \approx 0$, $0 = \varepsilon A - gA^3 - kA^5$

$= \varepsilon - gA^2 - kA^4$

Where $A^2 = b$; $0 = \varepsilon - gb - kb^2$; $\boxed{A = \sqrt{\frac{g \pm \sqrt{g^2 - 4(-k)(\varepsilon)}}{2(-k)}}}$

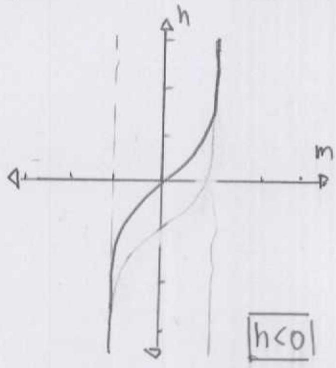


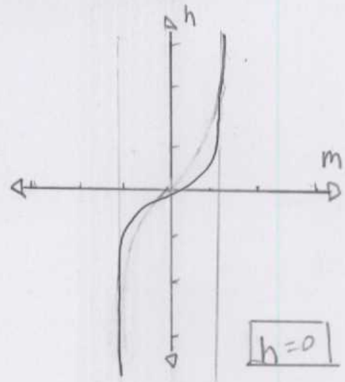
d) $\boxed{\text{The graphs appearance represent the relationship of amplitude vs. time and if } \epsilon \text{ is large, then the first order term approaches the steadystate condition more rapidly.}}$

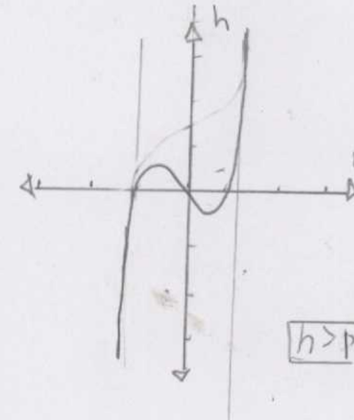
$m = \left|\frac{1}{N}\sum_{i=1}^{N} S_i\right|$

$h = T\tanh^{-1}(m) - Jnm$

3.6.7. a)







b) $h = T\tanh^{-1}(m) - Jnm$; If $h=0$, then $\boxed{T_c = \frac{Jnm}{\tanh^{-1}(m)}}$

$\frac{dx}{d\tau} = rx\left(1-\frac{x}{K}\right) - \frac{x^2}{1+x^2}$

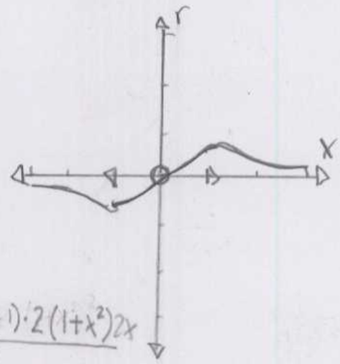
3.7.1. @ $x^*=0$; $0 < rx - \left(\frac{1}{K}+\frac{1}{1+x^2}\right)x^2$; $\left(\frac{1}{K}+\frac{1}{1+x^2}\right)x < r$; $\boxed{0<r \text{ is positive and unstable}}$

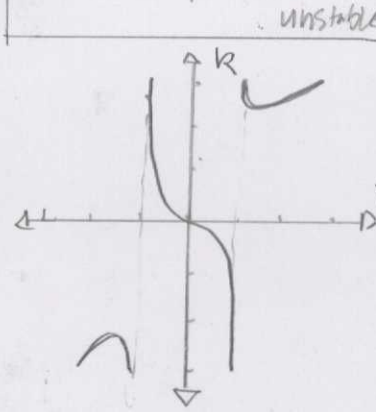
$r = \frac{2x^3}{(1+x^2)^2}$

$K = \frac{2x^3}{x^2-1}$

3.7.2. a)

$\lim_{x\to 1} r = \frac{2}{4}$	; $\lim_{x\to\infty} r = 0$
$\lim_{x\to 1} K = -\infty$	; $\lim_{x\to\infty} K = \infty$





b) $r = \frac{(x^2-1)K}{(1+x^2)^2}$; $\frac{dr}{dx} = \frac{2x(1+x^2)^2-(x^2+1)\cdot 2(1+x^2)2x}{(1+x^2)^4}$

$= \frac{2x(1+x^2)^2 - 4x^3(1+x^2) + 4x(1+x^2)}{(1+x^2)^4} = 0$

$= 2x(1+x^2) - 4x^3 + 4x = 2(1+x^2) - 4x^2 + 4 = (1+x^2) - 2x^2 + 2 = 0$

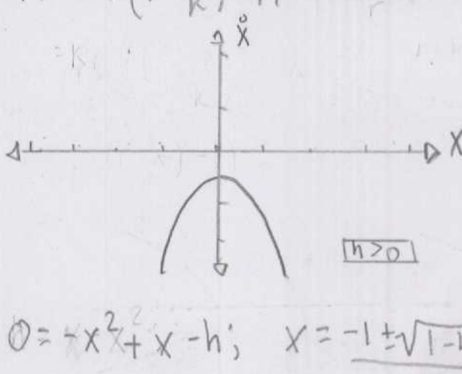
$3 = x^2$; $\boxed{x = \sqrt{3}}$

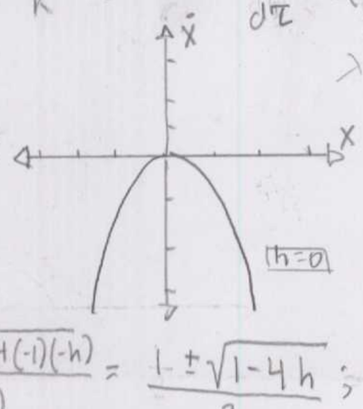
$r_{max} = \frac{(3-1)K}{(1+3)^2} = \frac{1}{9}K_{max}$; $\boxed{r_{max} = \frac{2\cdot 3^{3/2}}{(1+3)^2} = 0.6495}$; $\boxed{K_{max} = 5.1961}$

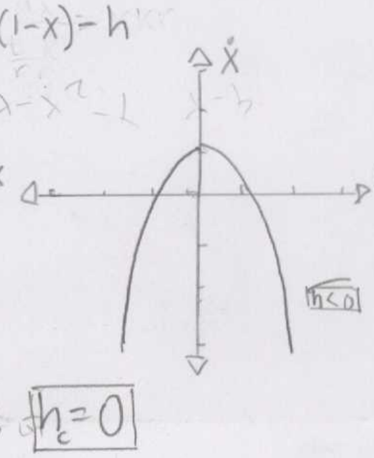
$\frac{dx}{d\tau} = x(1-x) - h$

3.7.3. a) $\dot{N} = rN\left(1-\frac{N}{K}\right) - H$; $h = HKr$; $x = \frac{N}{K}$; $\tau = tr$; $\frac{dx}{d\tau} = x(1-x) - h$

b)







c) $0 = -x^2 + x - h$; $x = \frac{-1 \pm \sqrt{1-4(-1)(-h)}}{2(-1)} = \frac{1 \pm \sqrt{1-4h}}{2}$; $\boxed{h_c = 0}$

d) $\boxed{\text{The long-term behavior of the fish population is to reduce the total population as population rises.}}$

$\dot{N} = rN\left(1-\frac{N}{K}\right) - H\frac{N}{A+N}$ 3.7.4. a) The variable A could represent the amount of fish in a school, and if A is large, then less fish are harvested.

b) $\boxed{x = \frac{N}{K};\ \tau = tr;\ h = HRK;\ a = A}$

c) $\frac{dx}{d\tau} = x(1-x) - h\frac{x}{a+x} = 0;\ x(1-x)(a+x) = (x-x^2)(a+x) = ax + x^2 - x^2 a - x^3$

$0 = (a-h)x + (1-a)x^2 - x^3$

$0 = (a-h) + (1-a)x - x^2$

$x_1 = 0,\ x_{2,3} = \frac{-(1-a) \pm \sqrt{(1-a)^2 - 4(-1)(a-h)}}{2(-1)}$

$= \frac{(1-a) \pm \sqrt{(1-a)^2 + 4(a-h)}}{2}$

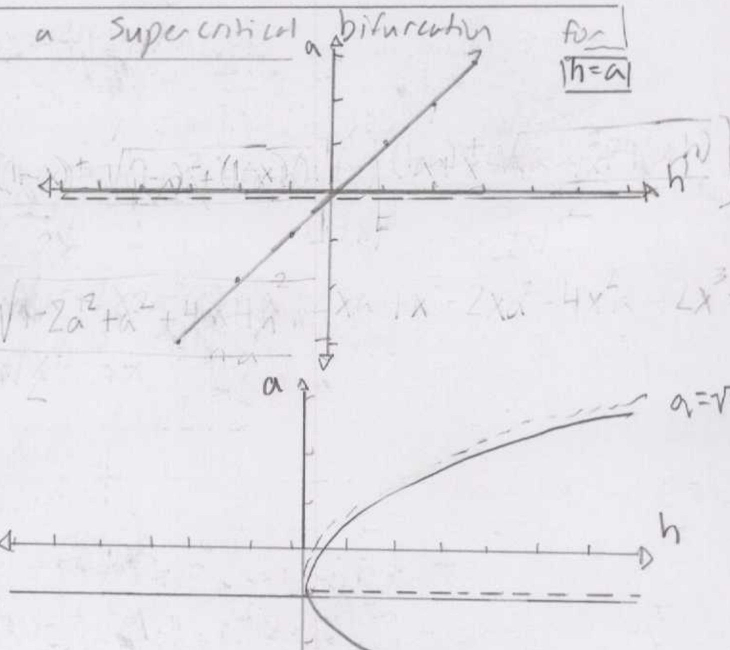
Fixed Point	Stability $a>h$	$a<h$
0	unstable	stable
$\frac{(1-a)+\sqrt{(1-a)^2+4(a-h)}}{2}$	stable	/
$\frac{(1-a)-\sqrt{(1-a)^2+4(a-h)}}{2}$	stable	/

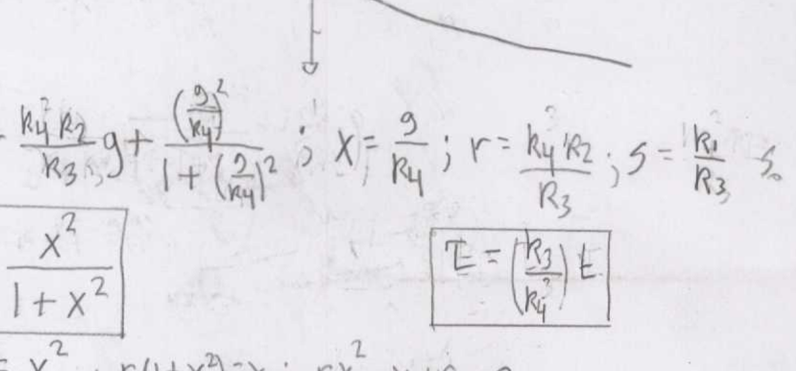
d) At $x=0$, when $h=a$, the half-node indicates a transcritical bifurcation is about to occur when h becomes less than a.

e) The graph shows a supercritical bifurcation for $\boxed{h=a}$

$h = \frac{(a+1)^2}{4}$.

f) $\boxed{a = \frac{h}{x-1} - x}$





$\dot{g} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2}$ 3.7.5.

a) $\frac{k_4}{k_3}\frac{dg}{dt} = \frac{k_4 \cdot k_1}{k_3} s_0 - \frac{k_4 k_2}{k_3} g + \frac{\left(\frac{g}{k_4}\right)^2}{1+\left(\frac{g}{k_4}\right)^2};\ x = \frac{g}{k_4};\ r = \frac{k_4 k_2}{k_3};\ s = \frac{k_1}{k_3} s_0$

$\boxed{\frac{dx}{d\tau} = s - rx + \frac{x^2}{1+x^2}}$ $\boxed{\tau = \left(\frac{k_3}{k_4^2}\right)t}$

b) $0 = -rx + \frac{x^2}{1+x^2};\ rx = \frac{x^2}{1+x^2};\ r(1+x^2) = x;\ rx^2 - x + r = 0$

$\boxed{x_{1,2} = \frac{1 \pm \sqrt{1-4r^2}}{2r}}$

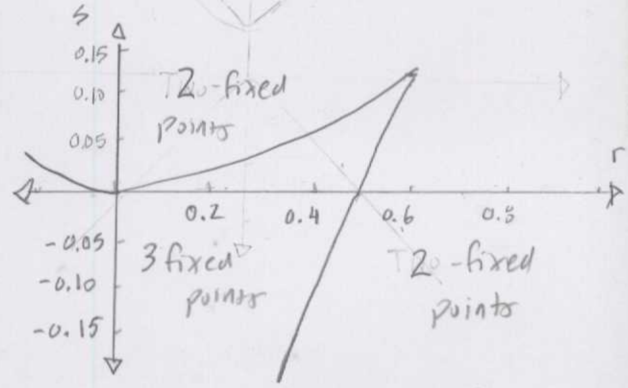
c) $g(0)=0$; $\frac{dg}{dt} = k_1 s_0 - k_2(0) + \frac{k_3(0)}{k_4^2+(0)^2} = k_1 s_0$; $g = k_1 s_0 t$; $g(t)$ increases with additional s_0.

$\boxed{\text{If } s_0 \text{ is large, then gene production has higher likelihood of rising.}}$

d) $\frac{d}{dx}\left(s - rx + \frac{x^2}{1+x^2}\right) = -r + \frac{2x}{(1+x^2)^2} = 0$; $r = \frac{2x}{(1+x^2)^2}$; $s - \left(\frac{2x}{(1+x^2)^2}\right) + \frac{x^2}{1+x^2} = 0$

$$\boxed{s = \frac{x^2(1-x^2)}{(x^2+1)^2}}$$

e) Parametric plot of (r, s)



$\dot{x} = -kxy$
$\dot{y} = kxy - ly$
$\dot{z} = ly$

3.7.6. $x(t)$ = number of healthy people
$y(t)$ = number of sick people
$z(t)$ = number of dead people.

a) $\dot{N} = \dot{x} + \dot{y} + \dot{z} = -kxy + kxy - ly + ly = 0$; therefore $N = x + y + z$.

b) $\dot{x} = -kxy$; $\dot{z} = ly$; $\dot{x} = -kx\frac{dz}{dt}\left(\frac{1}{l}\right)$; $\ln x = -\frac{kz}{l} + C$; $x(t) = Ce^{-kz/l} = \boxed{x_0 e^{-kz/l}}$

c) $\dot{z} = ly = l[N - x - z] = \boxed{l\left[N - z - x_0 e^{-kz/l}\right]}$

d) $\boxed{u = \frac{kz}{l};\ b = \frac{l}{kx_0};\ a = \frac{lN}{kx - lx_0};\ \tau = \frac{l}{kx_0}t}$

e) $\boxed{\text{If } k, l, N, \text{ and } x_0 \text{ are positive, then both } a \text{ and } b \text{ are positive.}}$

f)

b \ a		>1
$=0$	@$u=0$, unstable	@$u<0$, unstable
>0	@$u=0$, unstable @$u>0$, stable	@$u<0$, unstable @$u>0$, stable

g) $\ddot{u} = -b\dot{u} + \dot{u}e^{-u} = \boxed{0}$; $u = -\ln b$; $\dot{u} = a - b\ln b + b$
$\ddot{z} = l\dot{y} = l(kxy - ly) = l(kx - l)y = \boxed{0}$; $y = kx$; $y = Ce^{(kx-l)t}$

h) $b < 1$; $\ddot{u} = -b + ue^{-u}$; $\boxed{\text{Through plotting of } b \text{ and } ue^{-u}, \text{ at time zero, } b > ue^{-u}; \text{ thus, } \dot{u} \text{ is increasing.}}$

t_{peak}; $\ddot{u} = -b + ue^{-u} = 0$; $\dddot{u} = e^{-u} - u^2e^{-u} = 0$; $u^2 = 1$
$\dot{u} = a - bu - e^{-u}$ @ $u<1$; $\dot{u} = a - b - \frac{1}{e}$; $u = (a - b - \frac{1}{e})\tau = 1$; $\boxed{t = \frac{1}{a - b - \frac{1}{e}}\left(\frac{l}{kx_0}\right)}$

$\lim_{u\to\infty} \dot{u} = \lim_{u\to\infty}\left[a - bu - e^{-u}\right] = -b\cdot\infty - \frac{1}{e^{\infty}} = \boxed{-\infty}$.

i) $\boxed{\text{If } b>1,}\ \dot{u} = a - bu - e^{-u}$; $\ddot{u} = -b\dot{u} + \dot{u}e^{-u}$; $\dddot{u} \ldots \ddddot{u}$ does not contain a logical maximum/minimum/inflection for an epidemic with t_{peak} at zero.

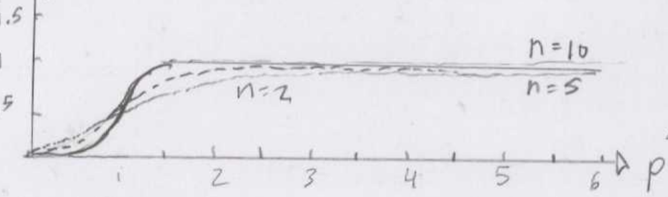
j) The variable b is assigned as $\frac{l}{kx_0}$. If $b=1$, then $\frac{l}{kx_0} = 1$.

A threshold condition is when the rate of dying persons is greater than the rate of infection.

k) Autoimmunodeficiency is a disease following human immunodeficiency virus. The delayed onset from infection is time-dependent, showing that a model likely requires a time-dependent term or relationship.

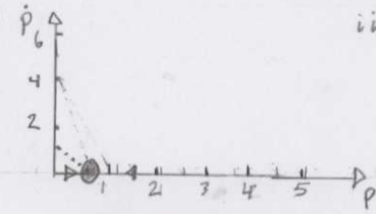
$\dot{p} = \alpha + \frac{\beta p^n}{K^n + p^n} - \delta p$ 3.7.7. α = Basal Transcription Rate; β = Maximal Transcription Rate

K = Activation Coefficient; δ = Decay Rate of Protein.

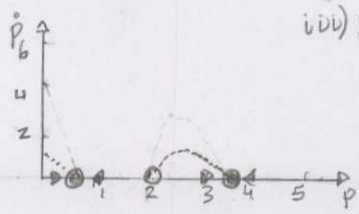
a) 

The shape of the function is a sigmoid about the point (1, 0.5) for K=1, b=1.

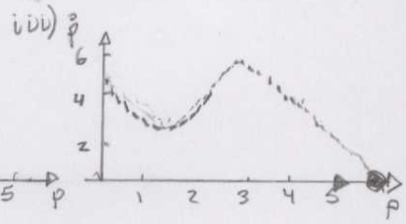
b)

i) 

$\boxed{\delta K - \alpha > \beta}$

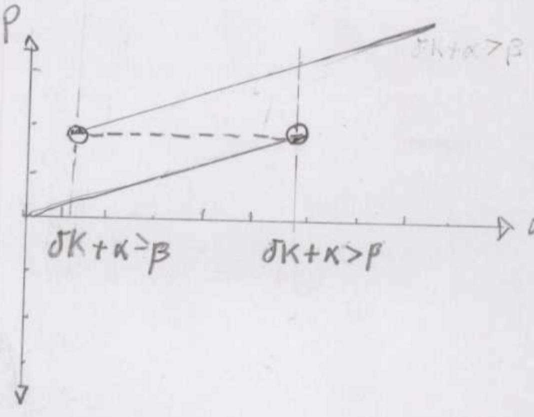
ii) 

$\boxed{\delta K - \alpha = \beta/2}$

iii) 

$\boxed{\delta K - \alpha < 0}$

c) Assume $\delta K > \beta$, $\alpha = -\frac{\beta^n}{K^n + p^n} + \delta p$ at $\alpha \geq 0$



d) When protein levels are dependent upon α, then up till $\alpha > \delta K$, Protein production rate decreases until zero. While ? is above a threshold $\delta K - \beta > \alpha$, there is active production of further protein. Proving concentration regions of protein production.

$\mathring{A}_p = K_p S A + \beta \frac{A_p^n}{K^n + A_p^n} - K_d A_p$; A = unphosphorylated concentration ; A_p = phosphorylated concentration ; $A_T = A + A_p$

K_p = phosphorylation rate ; K_d = dephosphorylation rate.

Assume $K = A_T/2$; $\beta = K_d A_T$

3.7.8a) $X = A_p/K$; $\tau = K_d t$; $s = K_p S/K_d$; $b = \beta/(K_d K)$

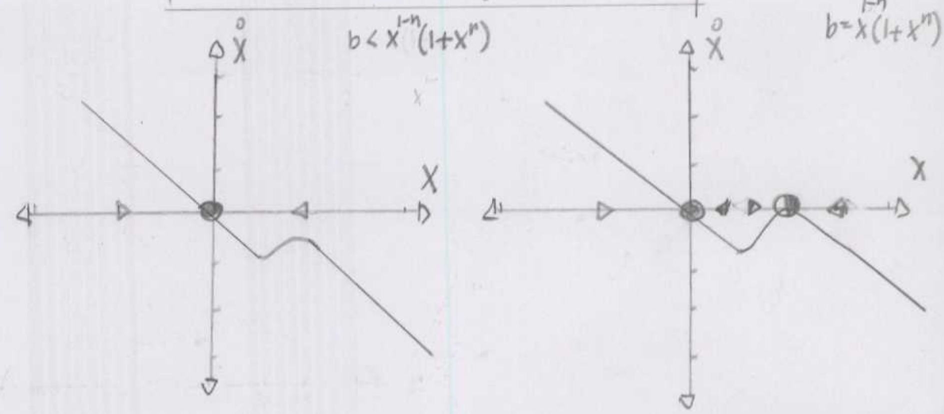
$$\cancel{K} \cancel{K_d} \frac{dx}{d\tau} = \cancel{K_d} s A + K_d \cdot K \cdot b \cdot \frac{K^n x^n}{K^n + K^n x^n} - \cancel{K_d} \cancel{K} x$$

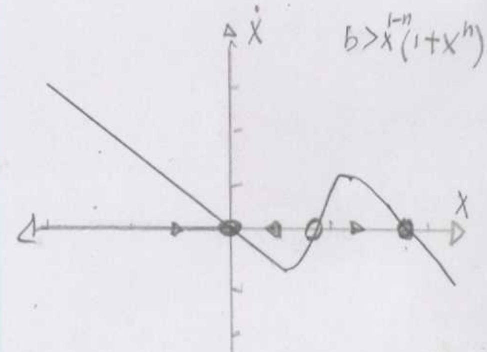
$$\frac{dx}{d\tau} = \frac{sA}{K} + b\frac{X^n}{1+X^n} - X = \frac{s(A_T - A_p)}{K} + b\frac{X^n}{1+x^n} - X$$

$$= \frac{s(2K - Kx)}{K} + b\frac{X^n}{1+X^n} - X$$

$$\boxed{= s(2-x) + b\frac{X^n}{1+X^n} - X}$$

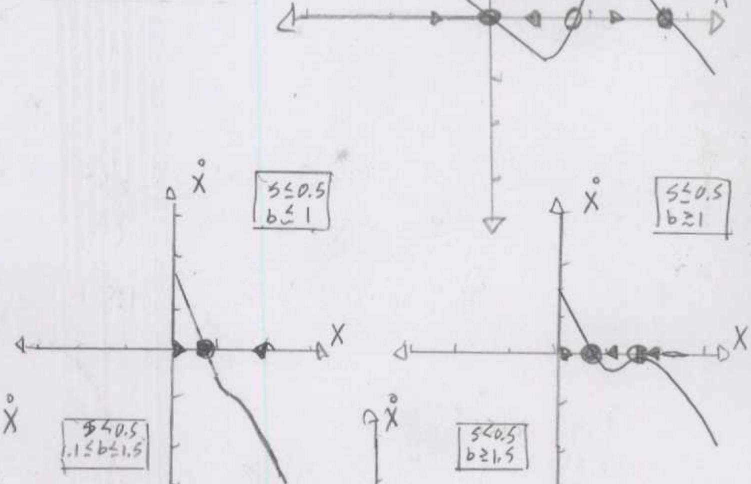
b) If $s=0$, then

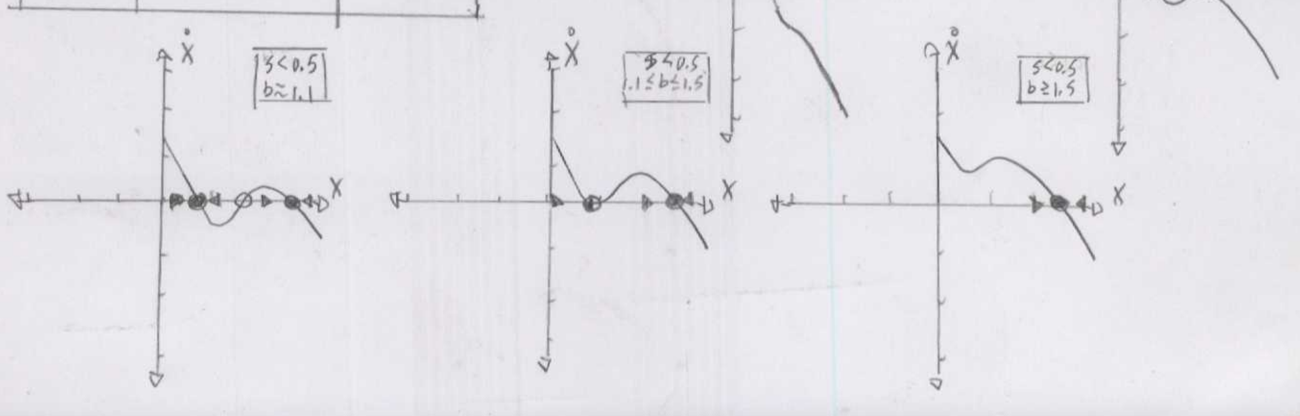


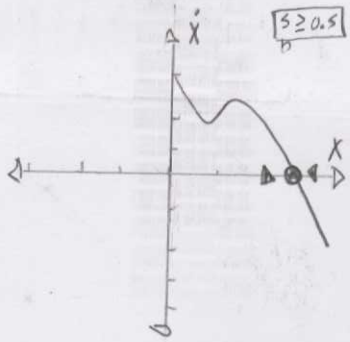


c) If $s > 0$, then a variety of bifurcations are produced.

b \ s	$\lesssim 0.5$	$\gtrsim 0.5$
$b \lesssim 1$	1, Stable	1, Stable
$b \gtrsim 1$	2, Stable Half node	
$b \cong 1.1$	3, Stable unstable Stable	
$1.1 \lesssim b \lesssim 1.5$	2, Half node Stable	
$b \gtrsim 1.5$	1, Stable	



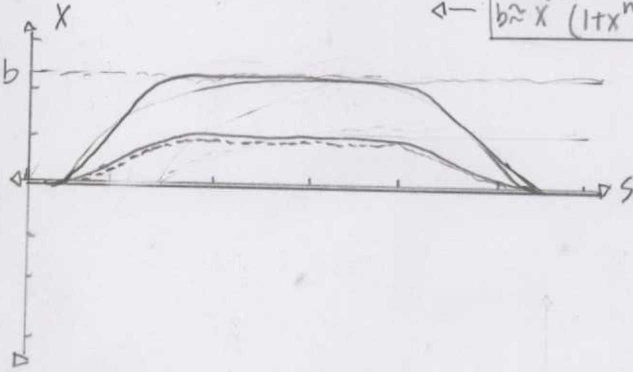




d)

Translation of bifurcation plot occurs

← $\boxed{b \approx x^{1-n}(1+x^n)}$ → near the left box.



Incorrect

Chapter 4: Flows on the circle

$\dot{\theta}=\sin(a\theta)$ 4.1.1. $\boxed{\text{The real values of } a \text{, which give a well-defined vector field, on a circle for the function } \dot{\theta}=\sin(a\theta) \text{, are fixed to } n\pi \text{ where } n \in \mathbb{Z}.}$

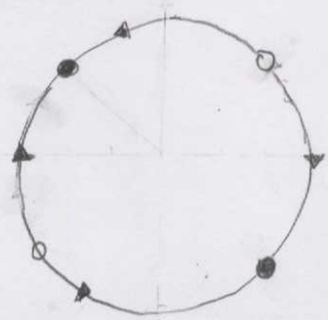
$\dot{\theta}=1+2\cos\theta$ 4.1.2. $\boxed{\text{Fixed points}}$ $\theta=\cos^{-1}\left(\frac{-1}{2}\right)$

$= \frac{2}{3}\pi, \frac{5}{3}\pi \dots \left(n+\frac{2}{3}\right)\pi$ "stable"

$= \frac{4}{3}\pi, \frac{7}{3}\pi \dots \left(n+\frac{4}{3}\right)\pi$ "unstable"

where $n \in \mathbb{Z}$

$\boxed{\text{Phase Portrait}}$



$\dot{\theta}=\sin 2\theta$ 4.1.3. $\boxed{\text{Fixed Points}}$ $\theta = \frac{\sin^{-1}(0)}{2}$

$= 0\pi, 1\pi, 2\pi \dots (n\pi)$ "unstable"

$= \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi \dots \left(n+\frac{1}{2}\right)\pi$ "stable"

where $n \in \mathbb{Z}$

$\boxed{\text{Phase Portrait}}$

$\dot{\theta}=\sin^3\theta$ 4.1.4 $\boxed{\text{Fixed points:}}$ $\theta=\sin^{-1}(0)$

$= 0\pi, 2\pi \dots (2n)\pi$ "unstable"

$= 1\pi, 3\pi \dots (2n+1)\pi$ "stable"

where $n \in \mathbb{Z}$

$\boxed{\text{Phase Portrait:}}$