Chapter 2:

$\dot{x} = \sin x$ 2.1.1. $\dot{x} = 0 = \sin x$; $\boxed{x = n\pi}$ 2.1.2 $\boxed{(n+\frac{1}{2})\pi \text{ where } n \text{ is even.}}$

2.1.3. a) $\ddot{x} = \cos x \sin x$ b) $\frac{1}{2}\sin(2x) = \cos(x)\sin(x)$; $\boxed{\ddot{x} = \frac{1}{2}\sin(2x) ;\ x = (n+\frac{1}{4})\pi ;\ n \in \mathbb{Z}}$

2.1.4.a) $x_0 = \pi/4$; $t = \ln|(\csc x_0 + \cot x_0)|(\csc x + \cot x)|$

$$e^t = \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} = \frac{\csc \pi/4 + \cot \pi/4}{\csc x + \cot x} = \frac{\frac{2}{\sqrt{2}} + 1}{\csc x + \cot x} = \frac{\sqrt{2}+1}{\csc x + \cot x}$$

$$\frac{1}{\frac{1}{\sin x} + \frac{\cos x}{\sin x}} = \frac{\sin x}{1+\cos x} = \frac{\sin(2 \cdot \frac{x}{2})}{1+\cos(2\frac{x}{2})} = \frac{2\cos(\frac{x}{2})\sin(\frac{x}{2})}{\cancel{1}+2\cos^2(\frac{x}{2})\cancel{-1}} = \tan\left(\frac{x}{2}\right) = \frac{e^t}{\sqrt{2}+1}$$

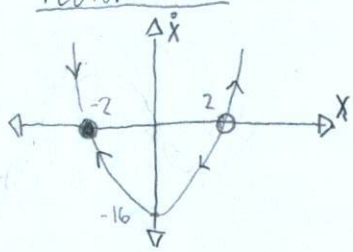
$$x(t) = 2\tan^{-1}\left(\frac{e^t}{\sqrt{2}+1}\right) ;\ \lim_{t\to\infty} x(t) = 2\tan^{-1}(\infty) = \boxed{2 \cdot \frac{\pi}{2} = \pi}$$

b) $\boxed{x(t) = 2\tan^{-1}\left(\frac{e^t}{\csc x_0 + \cot x_0}\right)}$

2.1.5a) A mechanical analog of $\dot{x} = \sin x$ is the undamped pendulum having an x_0 of the maximal point

b) Unstable points are described by a positive slope (source) and Stable points (sink), a negative slope. The function $\dot{x} = \sin x$ at $x^* = 0$ is unstable, while $x^* = \pi$, is stable.

$\dot{x} = 4x^2 - 16$ 2.2.1 Vector Field:



Fixed Points: $x = 2$ — Stability: source (unstable)

$x = -2$ — sink (stable)

Solving for $x(t)$:

$$\frac{dx}{(x^2-4)} = 4t$$

$$\int \frac{A}{(x-2)}dx + \int \frac{B}{(x+2)}dx = 4t$$

$A(x+2) + B(x-2) = 1$

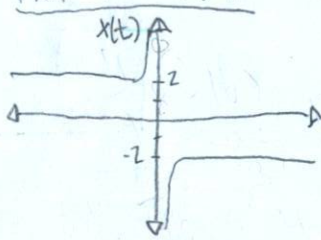
$A = 1/4$ @ $x = 2$

$B = -1/4$ @ $x = -2$

$$\ln\frac{|x-2|}{|x+2|} = 16t + c$$

$$\boxed{x(t) = \frac{2(e^{16t+c}+1)}{1-e^{16t+c}}}$$

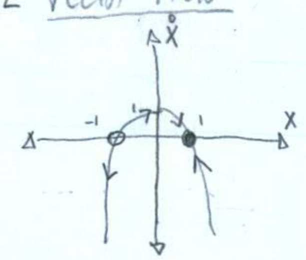
Plot of $x(t)$:



Analytical Solution of $x(t)$:

$$x(t) = \frac{2\left(e^{16t} \cdot \left(\frac{x_0-2}{2+x_0}\right)+1\right)}{1 - e^{16t}\left(\frac{x_0-2}{2+x_0}\right)}$$

$\dot{x} = 1 - x^{14}$ 2.2.2 Vector Field:



Fixed Points: $\boxed{x = 1 \quad \text{source (unstable)}}$

$\boxed{x = -1 \quad \text{sink (stable)}}$

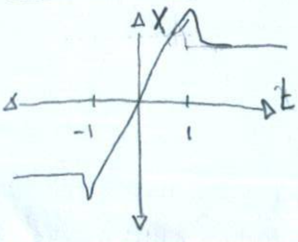
Stability:

Solving for $x(t)$:

$$t = \int \frac{dx}{1-x^{14}}$$

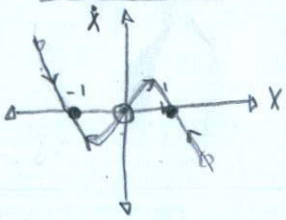
Unsolvable

Plot of $x(t)$:

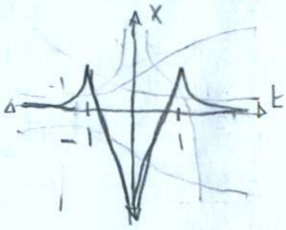


Analytical Solution of $x(t)$:

$\dot{x} = x - x^3$ 2.2.3 Vector Field:



Plot of $x(t)$:



Fixed Points: | Stability:

$x = -1$ stable (sink)

$x = 0$ unstable (source)

$x = 1$ stable (sink)

Analytical Solution of $x(t)$:

$x(t) =$

Solving for $x(t)$:

$$t = \int \frac{dx}{x - x^3} = \int \frac{dx}{x(1-x^2)}$$

$$= -\frac{1}{2}\int \frac{du}{(1-u)u} = -\frac{1}{2}\int \frac{A}{1-u}\,du - \frac{1}{2}\int \frac{B}{u}\,du$$

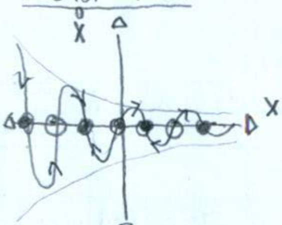
$$= -\frac{1}{2}\ln\left|\frac{1-u}{u}\right| = -\frac{1}{2}\ln\left|\frac{x^2}{1-x^2}\right|$$

$$= \frac{1}{2}\ln\left|\frac{1-x^2}{x^2}\right| + C$$

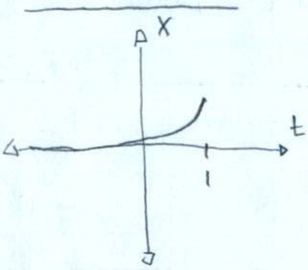
$$(e^{2t}+1)x^2 = 1$$

$$\boxed{x = \sqrt{\frac{1}{e^{2t+c}+1}}}$$

$\dot{x} = e^{-x}\sin x$ 2.2.4 Vector Field:



Plot of $x(t)$:



Fixed Points: | Stability:

$x = 2n\pi$ source (unstable)

$x = (2n+1)\pi$ sink (stable)

Solving for $x(t)$:

$$t = \int \frac{e^x}{\sin x}\,dx = \int dx + \int \cot(x)\,dx$$

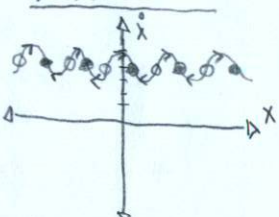
$$= 1 + \ln(\sin x) + C$$

$$\boxed{x(t) = \arcsin^{-1}\left(C \cdot e^{t-1}\right)}$$

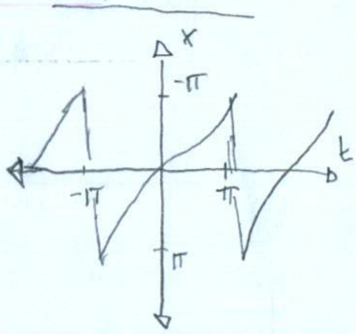
Analytical solution of $x(t)$

$$\boxed{x(t) = \arcsin^{-1}(Ce^{t} - 1) \text{ where } C = -(1 + \ln(\sin x_0))}$$

$\dot{x} = 1 + \frac{1}{2}\cos x$ 2.2.5. Vector Field:



Plot of $x(t)$:



Fixed Points | Stability:

$x = (4n+1)\frac{\pi}{2}$ sink (stable)

$x = (4n-1)\frac{\pi}{2}$ source (unstable)

Analytical solution of $x(t)$:

$$\frac{\frac{2}{\sqrt{3}}\arctan\left(\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right)}{\sqrt{3}} + C$$

Solving for $x(t)$:

$$t = \int \frac{1}{1 + \frac{1}{2}\cos x}\,dx$$

$$= \int \frac{dx}{\frac{1}{2} + \cos^2\left(\frac{x}{2}\right)} = \int \frac{\sec^2\left(\frac{x}{2}\right)dx}{\frac{\sec^2\left(\frac{x}{2}\right)}{2} + 1}$$

$$\boxed{= \int \frac{\sec^2\left(\frac{x}{2}\right)dx}{3 + \tan^2(\theta/2)}}$$

$$u = \frac{\tan\left(\frac{x}{2}\right)}{\sqrt{3}}; \quad \frac{du}{dx} = \frac{\sec^2\left(\frac{x}{2}\right)}{2\sqrt{3}}$$

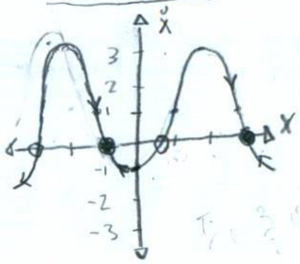
$$= \int \frac{2\sqrt{3}}{3u^2 + 3}\,du = \frac{2}{\sqrt{3}}\int \frac{du}{u^2+1}$$

$$= \frac{2}{\sqrt{3}}\arctan(u) + C$$

$$\boxed{\frac{\frac{2}{\sqrt{3}}\arctan\left(\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right)}{\sqrt{3}} + C}$$

$\dot{x} = 1 - 2\cos x$ 2.2.6.

Vector Field

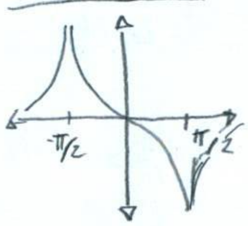


Fixed Points | Stability

$x = (n + \frac{1}{2})\pi$; n = even — sink (stable)

$x = (n + \frac{1}{2})\pi$; n = odd — source (unstable)

Plot of x(t)



Analytical Solution of x(t)

$$x(t) = \frac{\ln}{\sqrt{3}} \left| \frac{3\tan(\frac{x}{2}) - \sqrt{3}}{3\tan(\frac{x}{2}) + \sqrt{3}} \right|$$

Solving for x(t)

$$t = \int \frac{dx}{1 - 2\cos x} \quad ; \quad \frac{1 - \tan^2(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})} = \cos x$$

$$= -\int \frac{dx}{2\left[\frac{1 - \tan^2(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})}\right] - 1}$$

$$= -\int \frac{dx}{2\left[\frac{1 - u^2}{1 + u^2}\right] - 1} \quad ; \quad u = \tan\left(\frac{x}{2}\right) \quad \frac{du}{dx} = \frac{\sec^2(\frac{x}{2})}{2}$$

$$= -\int \frac{du}{2\sec^2(\frac{x}{2})\left[\frac{1-u^2}{1+u^2}\right] - 1}$$

$$= -\int \frac{2\,du}{[u^2 + 1]\left(2\left[\frac{1-u^2}{1+u^2}\right] - 1\right)}$$

$$= -\int \frac{2\,du}{2 - 2u^2 - u^2 - 1}$$

$$= -\int \frac{2\,du}{-3u^2 + 1}$$

$$= -2\int \left(\frac{3}{1}\right) \frac{du}{(3u - \sqrt{3})(3u + \sqrt{3})}$$

$$= -6\left[\int \frac{A\,du}{(3u - \sqrt{3})} + \int \frac{B\,du}{(3u + \sqrt{3})}\right]$$

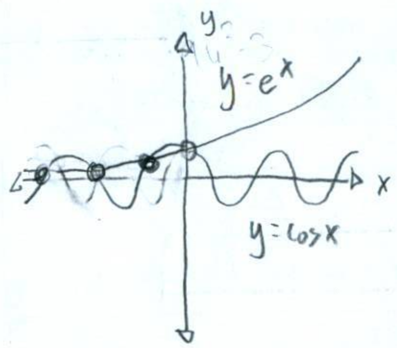
$$= -6\left[\frac{\sqrt{3}}{2}\int \frac{du}{(3u - \sqrt{3})} - \frac{\sqrt{3}}{2}\int \frac{du}{(3u + \sqrt{3})}\right]$$

$$= \frac{\ln(3u + \sqrt{3})}{\sqrt{3}} - \frac{\ln(3u + \sqrt{3})}{\sqrt{3}}$$

$$= \frac{\ln(3\tan(\frac{x}{2}) - \sqrt{3})}{\sqrt{3}} - \frac{\ln(3\tan(\frac{x}{2}) + \sqrt{3})}{\sqrt{3}}$$

$$\boxed{= \frac{\ln}{\sqrt{3}} \left| \frac{3\tan(\frac{x}{2}) - \sqrt{3}}{3\tan(\frac{x}{2}) + \sqrt{3}} \right| + C}$$

$\dot{x} = e^x - \cos x$ 2.2.7.



Points of Stability

$e^x = \cos x$

$x_1 = 0$ source (unstable)

$x_2 = -1.29$ sink (stable)

$x_3 = -4.72$ source (unstable)

$\dot{x} = f(x)$ 2.2.8

$f(x) = -(x+1)(1-x)^3$

slope zero Negative Positive

2.2.9.

$x_0 = 2$ ~
$x_0 = 1$ ~
$x_0 = 0.5$ ~
$x_0 = -1$ ~

$f(x) = x(1-x)$

Fixed points

@ x = 1

@ x = 0

$\dot{x} = f(x)$ 2.2.10

a. A periodic function having solutions $n\pi$

b. A periodic function with $n\pi$ solutions

c. $f(x) = x^5$

d. $f(x) = x^2 + 1$

e. $f(x) = x^{101}$

$\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}$ 2.2.11. $Q(0) = 0$; $t = RC\int \frac{dQ}{V_0C - Q} = -RC\ln V_0C - Q + C$; $C = RC\ln V_0C$; $\boxed{t = RC\ln \frac{V_0C}{V_0C - Q}}$

$Q = V_0C(1 - e^{-t/RC})$

$\dot{Q} = g(v) - \frac{Q}{RC}$ 2.2.12

$V = g(v) - V_{cap.} = V_0 - \frac{Q}{C}$; $-g(v) + RI + \frac{Q}{C} = 0$; $-g(v) + R\dot{I} + \frac{Q}{C} = -g(v) + R\dot{Q} + \frac{Q}{C} = 0$

$\dot{Q} = g(v) - \frac{Q}{RC}$;

Fixed Points: $g(v) = Q/RC$

The nonlinearity of the resistor has a relationship to resistance

Stability: Source (unstable)

$m\dot{v}=mg-kv^2$ 2.2.13 - ; where m = mass, g = acceleration, $k>0$ = air resistance

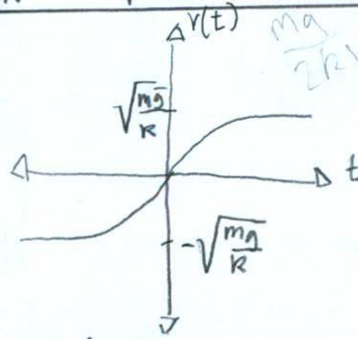
a) $\displaystyle\int\frac{dv}{g-\frac{k}{m}v^2}=\frac{1}{g}\int\frac{dv}{1-\frac{k}{mg}v^2}=\frac{1}{g}\left[\int\frac{dv}{1-\sqrt{\frac{k}{mg}}v}+\int\frac{dv}{1+\sqrt{\frac{k}{mg}}v}\right]=\sqrt{\frac{mg}{k}}\frac{1}{2g}\left[\ln\left|1+\sqrt{\frac{k}{mg}}v\right|-\ln\left|1-\sqrt{\frac{k}{mg}}v\right|\right]$

$\displaystyle t=\frac{1}{2}\sqrt{\frac{m}{kg}}\ln\left|\frac{1+\sqrt{\frac{k}{mg}}v}{1-\sqrt{\frac{k}{mg}}v}\right|;\ \sqrt{\frac{kg}{m}}t=\frac{\ln}{2}\left|\frac{1+\sqrt{\frac{k}{mg}}v}{1-\sqrt{\frac{k}{mg}}v}\right|=\boxed{\tanh^{-1}\left(\sqrt{\frac{k}{mg}}v\right)}$

b) $\boxed{\displaystyle\lim_{t\to\infty}v(t)=\sqrt{\frac{mg}{k}}=\text{"terminal velocity"}}$

$\displaystyle v(t)=\sqrt{\frac{mg}{k}}\tanh\sqrt{\frac{kg}{m}}t$

c)



d. $\displaystyle V_{avg}=\frac{(31{,}400-2100)\,ft}{116\,sec}=252\frac{ft}{sec.}$

e. $\displaystyle \dot{s}=\frac{ds}{dt}=v=V\tanh\sqrt{\frac{kg}{m}}t\ ;\ s(t)=V\int\tanh\sqrt{\frac{kg}{m}}t\,dt$

$\displaystyle =V\int\frac{\sinh\sqrt{\frac{kg}{m}}t}{\cosh\sqrt{\frac{kg}{m}}t}dt=V\int\frac{1}{u}dt=\frac{m}{k}\ln\cosh\frac{gt}{V}=\frac{V^2}{g}\ln\cosh\frac{gt}{V}$

$\displaystyle 29{,}300=\frac{V^2}{32.2ft/sec^2}\ln\cosh\cdot\frac{32.2ft/sec^2}{V}116sec$

$\displaystyle e^{9.43\times10^5/V^2}=\frac{e^{\frac{3735ft/sec}{V}}+e^{-\frac{3735ft/sec}{V}}}{2}$

$\boxed{V=266\,ft/sec.}$

$V\approx V_{avg}=252ft/sec$

$\displaystyle\frac{gt}{V}=\frac{32.2ft/sec^2\cdot116sec}{252ft/sec}=14.8$

$\displaystyle\frac{V^2}{g}\ln\cosh\frac{gt}{V}\approx\frac{V^2}{g}\left[\frac{gt}{V}-\ln2\right]=265\,ft/sec$

$\dot{N}=rN(1-N/K)$ 2.3.1 a. $\displaystyle t=\frac{1}{r}\int\frac{dN}{N(1-N/K)}$

$\displaystyle=\frac{1}{r}\left[\int\frac{A\,dN}{N}+\int\frac{B\,dN}{(1-N/K)}\right]$

$\displaystyle=\frac{1}{r}\left[\int\frac{1\,dN}{N}+\frac{1}{K}\int\frac{dN}{(1-N/K)}\right]$

$\displaystyle=\frac{1}{r}\ln N-\ln|1-N/K|+C$

$\displaystyle=\frac{1}{r}\ln\frac{N}{1-N/K}+C$

$\displaystyle Ce^{-rt}=\frac{1-N/K}{N};\ N\left(1+\frac{Ce^{-rt}}{K}\right)=1$

$\displaystyle N=\frac{1}{\left(1+\frac{Ce^{-rt}}{K}\right)}=\boxed{\frac{N_0}{1\,e^{-rt}}}$

General Solution: $\displaystyle x=Ce^{-rt}+\frac{1}{K}$

b. $x=1/N$; $\displaystyle dN=-\frac{dx}{x^2}$; $\displaystyle\dot{x}=-rx\left(1-\frac{1}{Kx}\right)=\frac{r}{K}-rx\ ;\ \dot{x}+rx-\frac{r}{K}=\dot{x}+r\left(x-\frac{1}{K}\right)=0$

$\boxed{\displaystyle\dot{x}=r\left(\frac{1}{K}-x\right)}$

$\displaystyle N(t)=\frac{K}{KCe^{-rt}+1}$

$\displaystyle N=\frac{N_0}{e^{-rt}}$

$\dot{x}=k_1ax-k_{-1}x^2$ 2.3.2

$A+X\underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}}2X$

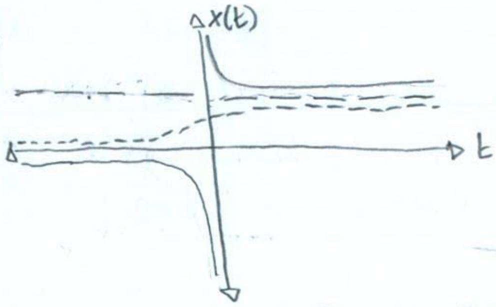
a. $\displaystyle t=\int\frac{dx}{k_1ax-k_{-1}x^2}=\frac{1}{k_1a}\int\frac{dx}{x-\frac{k_{-1}}{k_1a}x^2}=\frac{1}{k_1a}\int\frac{dx}{x\left(1-\frac{k_{-1}}{k_1a}x\right)}$

$\displaystyle=\frac{1}{k_1a}\left[\int\frac{A\,dx}{x}+\int\frac{B\,dx}{\left(1-\frac{k_{-1}}{k_1a}x\right)}\right]=\frac{1}{k_1a}\left[\int\frac{1\,dx}{x}+\frac{k_{-1}}{k_1a}\int\frac{dx}{1-\frac{k_{-1}}{k_1a}x}\right]$

$\displaystyle=\frac{1}{k_1a}\left[\ln|x|-\ln\left|1-\frac{k_{-1}}{k_1a}x\right|\right]=\frac{1}{k_1a}\ln\left|\frac{x}{1-\frac{k_{-1}}{k_1a}x}\right|+c\ ;\ \left(Ce^{-t/k_1a}+\frac{k_{-1}}{k_1a}\right)x=1$

$\displaystyle x(t)=\frac{1}{\frac{k_{-1}}{k_1a}+Ce^{-t/k_1a}}\ ;\ C=\frac{1}{x_0}-\frac{k_{-1}}{k_1a}\ ;$ $\boxed{\text{Fixed points of stability: } x=\frac{k_1a}{k_{-1}}\ \text{source unstable}}$

b.



—— $= X_0 > \frac{k_1 a}{k-1}$

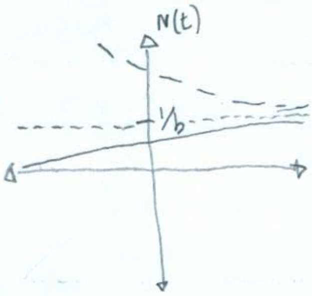
— — — $= X_0 = \frac{k_1 a}{k-1}$

- - - - $= X_0 < \frac{k_1 a}{k-1}$

"Gompertz Law"

$\dot{N} = -aN\ln(bN)$ 2.3.3 a) $t = -\frac{1}{a}\int \frac{dN}{N\ln(bN)} = -\frac{b}{a}\int \frac{du}{u} = -\frac{b}{a}\ln[\ln bN]$; $N(t) = \frac{Ce^{e^{-\frac{ab}{b}}}}{b} = \boxed{\frac{Ce^{-\frac{ate}{b}}}{b}}$; a = rate constant, b = Max amount of cells.

b)



—— = $C < b$
- - - - = $C = 1$
— — = $C > b$

$\frac{\dot{N}}{N} = r - a(N-b)^2$ 2.3.4. a). $\lim_{N\to 0} \frac{\dot{N}}{N} = \lim_{N\to b} r - a(N-b)^2 = r - ab^2 = \infty$; $r = \infty$;

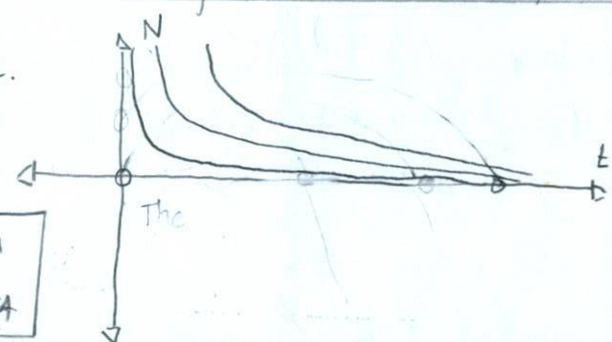
$\lim_{N\to\infty} \frac{\dot{N}}{N} = \lim_{N\to 0} r - a(N-b)^2 = r - \infty = 0$; $r = \infty$;

Each case of competition model at infinite population or extremely small populations that food amount or rate of consumption are insignificant to the competitors.

b). Fixed points of stability:

$N = 0$	Source (unstable)
$N = \sqrt{\frac{r}{a}} + b$	Sink (stable)

c.



d) The solutions of the logistic equation $y = Ce^{-rt} + \frac{1}{k}$ are similar, if not exact to the Allele Effect.

$t = \int \frac{du}{(b+u)(r-au^2)} = \int \frac{A}{b+u}du + \int \frac{Bu+C}{r-au^2}du$

$A(r-au^2) + (Bu+C)(b+u) = 1$

$@ u = \sqrt{\frac{r}{a}}$; $(B\sqrt{\frac{r}{a}} + C)(b + \sqrt{\frac{r}{a}}) = 1$

$B = \sqrt{\frac{a}{a}}$; $C = \sqrt{\frac{r}{a}}$

$@ u = -b$ $A = \frac{1}{r-(ab)^2}$

$= \frac{1}{r-(ab)^2}\int \frac{du}{b+u} + \frac{1}{\sqrt{ra}}\int \frac{ab\cdot u + r}{r - au^2}du$ Partial Fractions (x2)

$= \frac{\ln N}{r-(ab)^2} + \frac{b}{4}\sqrt{\frac{a}{r}}\tanh^{-1}([N-b]^2) + \frac{1}{\sqrt{a}}\tanh^{-1}\left(\sqrt{\frac{a}{r}}(N-b)\right) + C$

$\dot{X} = aX$
$\dot{Y} = bY$ 2.3.5 a) $x(t) = \frac{X(t)}{[X(t)+Y(t)]} = \frac{e^{at}}{e^{at}+e^{bt}}$; $\lim_{t\to\infty} x(t) \boxed{\approx 1}$

b) $\dot{x}(t) = \frac{ae^{at}(e^{at}+e^{bt}) - e^{at}(ae^{at}+be^{bt})}{(e^{at}+e^{bt})^2}$

$= \frac{e^{at}[a-b]e^{bt}}{(e^{ab}+e^{bt})^2} = \frac{[a-b]e^{at}}{(e^{at}+e^{bt})}\left(\frac{e^{bt}+e^{ab}-e^{at}}{e^{ab}+e^{bt}}\right)$

$\boxed{= x[a-b](1-x)}$

$\dot{x} = (1-x)P_{yx} - xP_{xy}$ 2.3.6 a. $x = 0$

$P_{yx} = sx^a$; $P_{xy} = (1-s)(1-x)^a$ $x = 1$

$$\boxed{x = \frac{\sqrt[a-1]{\frac{(1-s)}{s}}}{1+\sqrt[a-1]{\frac{(1-s)}{s}}}}$$

b. A plot of $s(1-x)x^a$ and $-(1-s)x(1-x)^a$ demonstrate $-(1-s)x(1-x)^a > s(1-x)x^a$ for $x=0$ and $x=0$, indicating, each fixed point is stable.

c. For $x = \frac{\sqrt[a-1]{\frac{(1-s)}{s}}}{1+\sqrt[a-1]{\frac{(1-s)}{s}}}$ the plot of $s(1-x)x^a > (1-s)x(1-x)^a$, suggesting a source.

$\dot{x} = x(1-x)$ 2.4.1 $\dot{x} = f(x) = f(x^*+x) = f(x^*) + xf'(x^*) + O(x^2)$

$= xf'(x^*) + O(x^2)$

$\boxed{= x(1-2x)}$

$x=0$; $f'(x^*) = 1$: Unstable (source)

$x=1$; $f'(x^*) = -1$: Stable (sink)

$\dot{x} = x(1-x)(2-x)$ 2.4.2 $\dot{x} = f(x) = f(x^*+x) = \boxed{xf'(x^*)}$

$\boxed{= x\cdot 2x(1-x)}$

$x=0$ $f'(x^*) = 0$ Half-stable

$x=1$ $f'(x^*) = 0$ Half-stable

$x=2$ $f'(x^*) = -4$ sink (stable)

$\dot{x} = \tan x$ 2.4.3 $\dot{x} = f(x) = f(x^*+x) = \boxed{x\sec^2(x)}$

$x = n\pi$ $f(x) = (+)$ source (unstable)

$\dot{x} = x^2(6-x)$ 2.4.4. $\dot{x} = f(x) = f(x^*+x) = \boxed{x[12x-3x^2]}$

$x=0$ $f'(x) = 0$ Half-stable

$x=6$ $f'(x) = -36$ sink (stable)

$\dot{x} = 1-e^{-x^2}$ 2.4.5. $\dot{x} = f(x) = f(x^*+x) = \boxed{x[2e^{-x^2}]}$

$x=0$ $f'(x^*) = 0$ Half-stable

$\dot{x} = \ln x$ 2.4.6 $\dot{x} = f(x) = f(x^*+x) = \boxed{1}$

$x=1$ $f'(x^*) = 1$ Source (unstable)

$\dot{x} = ax - x^3$ 2.4.7. $\dot{x} = f(x) = f(x^*+x) = \boxed{x[a-3x^2]}$

a	(+)	(−)	(0)
$x=0$	Source	sink	Half-stable
$x=+\sqrt{a}$	sink	Source	Source
$x=-\sqrt{a}$	Source	sink	Half-stable

$\dot{N} = -aN\ln(bN)$ 2.4.8 $\dot{N} = f(N) = f(N+N^*) = \boxed{-\frac{aN}{b}[1+b\ln(bN)]}$

$N=0$: Source (unstable)

$N=\frac{1}{b}$: sink (Stable)

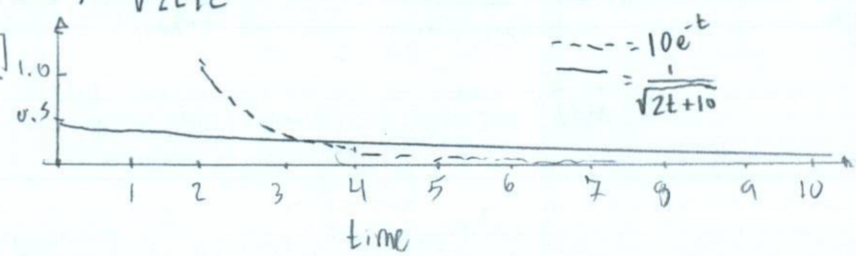
$\dot{x} = -x^3$ 2.4.9 a. $t = -\int \frac{dx}{x^3} = \frac{1}{2x^2} + c$; $x(t) = \sqrt{\frac{1}{2t+c}}$

$\lim_{x\to 0} t = \frac{1}{0} + c = \boxed{\infty}$

b. if $x_0 = 10$

$t = -\int \frac{1}{x} = -\ln x$;

$\boxed{x(t) = x_0 e^{-t} = 10e^{-t}}$

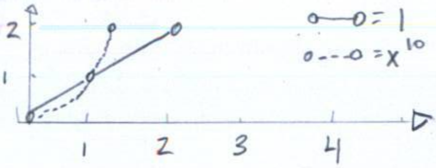


$\dot{x} = -x^c$ 2.5.1 a. $c = 0$

b. $dx = -dt$; $x(t) = -t$; if $t=0$ is considered finite time, then yes.

$t = -\int \frac{dx}{x^c} = \frac{-x^{1-c}}{1+c}$; $t(x=1) - t(x=0) = -\frac{1^{1-c}}{1-c} + \frac{0^{1-c}}{1-c} = \boxed{\frac{1}{c+1}}$

$\dot{x} = 1 + x^{10}$; 2.5.2.



$\boxed{\lim_{\dot{x} \to \infty} \dot{x} = \infty}$

$\dot{x} = rx + x^3$ 2.5.3

$t = \int \frac{dx}{x(r+x^2)} = \int \frac{A}{x} dx + \int \frac{Bx+C}{r+x^2} dx$; $A(r+x^2) + (Bx+C)(x) = 1$

$x=0$; $A = \frac{1}{r}$; $1 + \frac{x^2}{r} + Bx^2 + Cx = 1$

$= \frac{1}{r} \ln x - \frac{1}{2r} \ln r + x^2 = \frac{1}{r} \ln \frac{x}{\sqrt{r+x^2}}$

$(B + \frac{1}{r})x^2 + Cx = 0$

$x^2 Ce^{-2rt} = (r+x^2)$; $x = \sqrt{\frac{r}{1+Ce^{-2rt}}}$

$(B + \frac{1}{r})x = -C$

$\boxed{x = 0 ; C = 0}$

$\boxed{x = 1 ; B = -\frac{1}{r}}$

if $x_0 \neq 0$; $\lim_{t \to \infty} x(t) = \infty$.

$\dot{x} = x^{1/3}$ 2.5.4. $x(0) = 0$; $t = \int \frac{dx}{x^{1/3}} = \frac{3}{2} x^{2/3}$; $\boxed{x(t) = \sqrt{(\frac{2}{3}t - \frac{2}{3}c)^3}}$

$\dot{x} = |x|^{p/q}$ 2.5.5. $x(0) = 0$; a) $t = \frac{q}{p+q}(x)^{\frac{p+q}{q}}$; $x(t) = \left(\frac{p+q}{q}(t+C)\right)^{q/p+q}$; c = many solutions at zero because of root.

$\boxed{x(t) = \left(\frac{p+q}{q}(t+C)\right)^{q/p+q}}$;

b) $x(t) = \left(\frac{p+q}{q}(t+c)\right)^{q/p+q}$; if $p > q$; $x(0) = \left(\frac{p+q}{q}(0+c)\right)^{q/p+q} = 0$; $\boxed{C = 0}$

h(t) = height
t = time
A = cross-sectional Area
v(t) = velocity
$av(t) = A\dot{h}(t)$

2.5.6 a) Newton's first law that for every force there exists an equal and opposite counter force.

b) $\frac{1}{2}mv^2 = mgh$; $\boxed{v^2 = 2gh}$;

c) $\boxed{\dot{h}(t) = -\sqrt{\frac{a}{A} 2gh}}$

d) $\boxed{h(0) = 0 ; t = -\sqrt{\frac{Ah}{2a}} 2\sqrt{h} ; h(t) = -\sqrt{\frac{a}{2A}} t}$

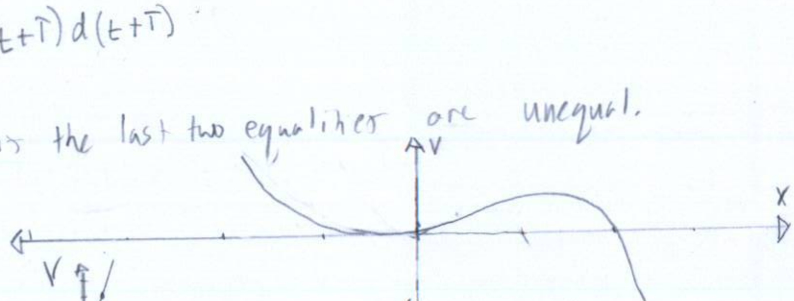
$m\ddot{x} = -kx$ 2.6.1 The text states, "there are no periodic solutions to $\dot{x} = f(x)$ because undamped systems do not oscillate, and "damped oscillations do not occur for first order systems." fit the equation of Strogatz' Statement does not 2.61.

$\dot{x} = f(x)$ 2.6.2 $\int_t^{t+T} f(x) \frac{dx}{dt} dt = \int_t^{t+T} f(x) \dot{x}(t) dt = \int_t^{t+T} f(x) \dot{x}(t+T) d(t+T)$

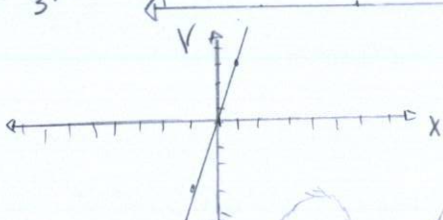
$x(t) = x(t+T)$

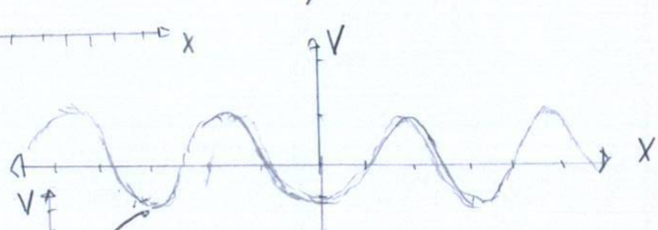
The contradiction of the argument is the last two equalities are unequal.

$\dot{x} = x(1-x)$ 2.7.1 $\frac{dV}{dx} = \dot{x} = x(1-x)$; $V = (1 - \frac{x}{3})x^2$



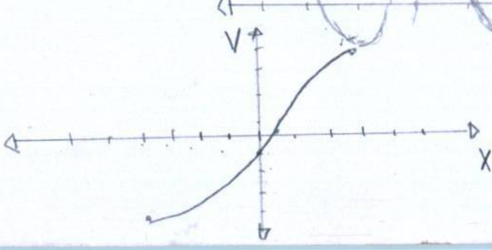
$\dot{x} = 3$ 2.7.2 $\frac{dV}{dx} = \dot{x} = 3$; $V = 3X$



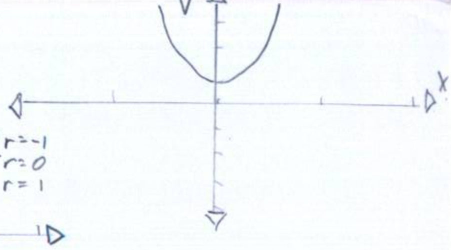


$\dot{x} = \sin x$ 2.7.3 $\frac{dV}{dx} = \dot{x} = \sin x$; $V = -\cos(x)$

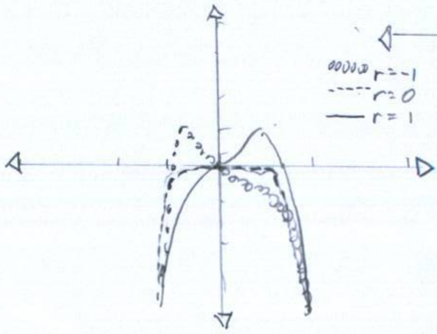
$\dot{x} = 2 + \sin x$ 2.7.4 $\frac{dV}{dx} = \dot{x} = 2 + \sin x$; $V = 2X - \cos(x)$



$\dot{x} = -\sinh x$ 2.7.5. $\frac{dV}{dx} = -\sinh x$; $V = -\cosh(x)$



$\dot{x} = r + x - x^3$ 2.7.6 $\frac{dV}{dx} = r + x - x^3$; $V = rx + \frac{x^2}{2} - \frac{x^4}{4}$



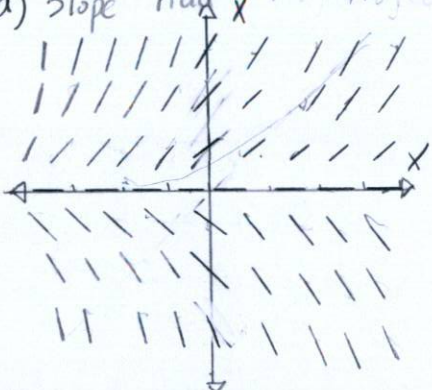
$\dot{x} = f(x)$ 2.7.7. $\frac{dV}{dx} = \dot{x} = f(x)$; $V = \frac{df(x)}{dx} dx + c$

$$f(x) = \frac{d(V-c)}{dx} dx$$

The solution $x(t)$ cannot oscillate because of the existence and uniqueness of $f(x)$, and the solutions for $f(x) = 0$, that $V = C$ or $C = 0$; withstanding, $\frac{c\, d(V-c)}{dx} = \frac{dx}{dt}$, then the solution $x(t)$ also corresponds to a nonperiodic function.

$\dot{x} = x(1-x)$ 2.8.1 The horizontal lines are to be expected in Figure 2.8.2 because of the slope being zero at $x = 1$.

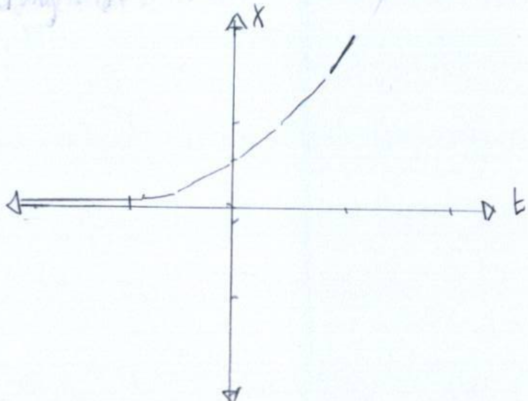
$\dot{x} = x$ 2.8.2.a) Slope Field



$\frac{dx}{dt} = x$

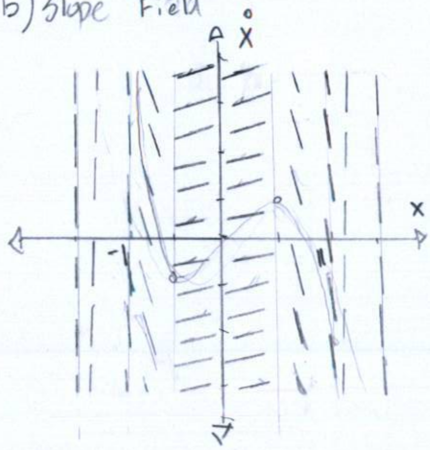
$\ln x = t$

$x = e^t$



$\dot{x} = 1 - x^2$

b) Slope Field



$\frac{dx}{dt} = 1 - x^2$

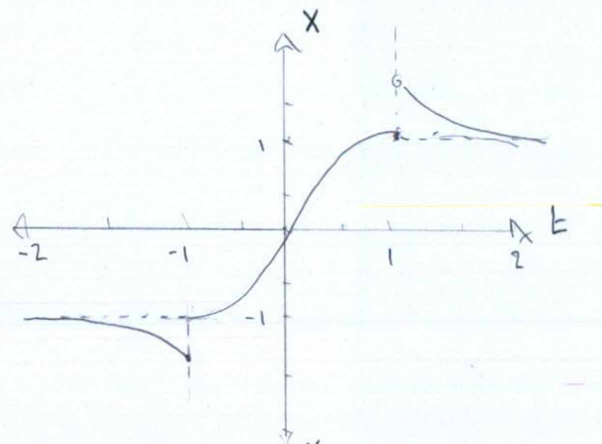
$\int \frac{dx}{1-x^2} = t$

$\frac{1}{2} \ln \frac{1+x}{x-1} = t$

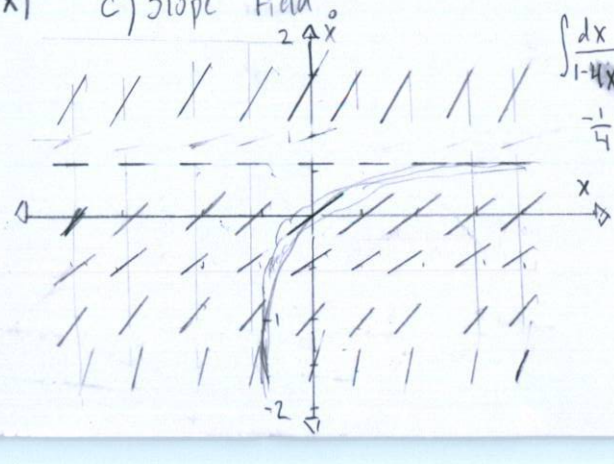
$x < -1$

$-1 < x < 1$

$x > 1$



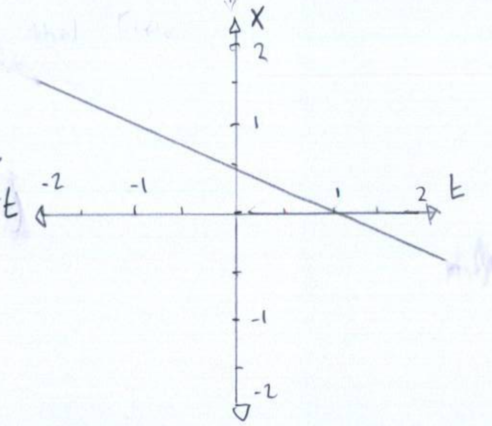
$\dot{x} = 1 - 4x(1-x)$ c) Slope Field



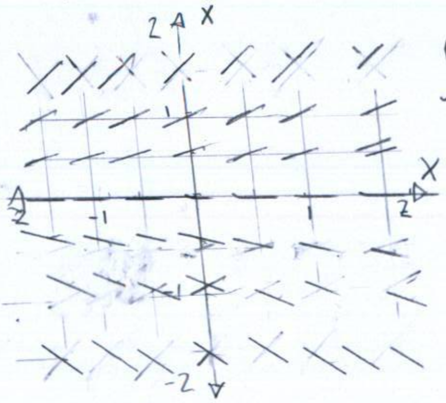
$\int \frac{dx}{1-4x(1-x)} = t$

$-\frac{1}{4}\left(\frac{1}{x-0.5}\right) = t$

$x = 0.5 - \frac{1}{4}t$



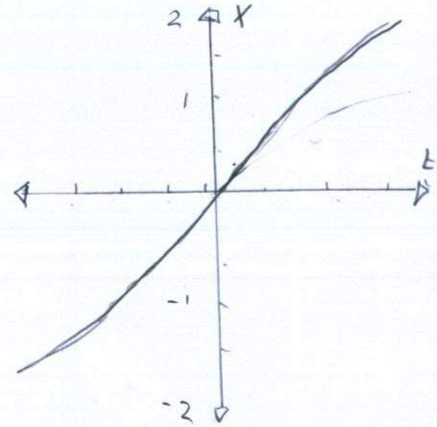
$\dot{x} = \sin x$ 2.8.2 d) Slope Field



$\frac{dx}{dt} = \sin x$

$\int \cosh x = t$

$x = \sinh^{-1}(t)$



$\dot{x} = -x ; x(0) = 1$ 2.8.3. a) $x(t) = Ce^{-t} ; C = 1 ; x(t) = e^{-t}$

Euler's Method

b) $\Delta t = 1 ; x(t_0 + \Delta t) \approx x_0 + f(x_0)\Delta t ; x(t_0 + \Delta t) = 0 + e^{-1} \cdot 10^0 = 0.3679$

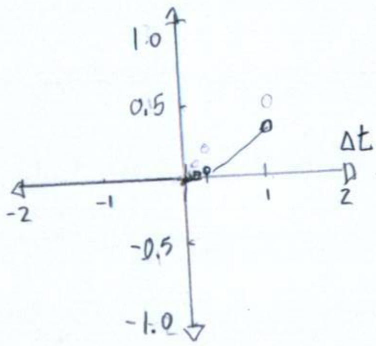
$\Delta t = 10^{-n}$ $n = 1$ $x_n + e^{-x_n} 10^{-1} = 0.36341$

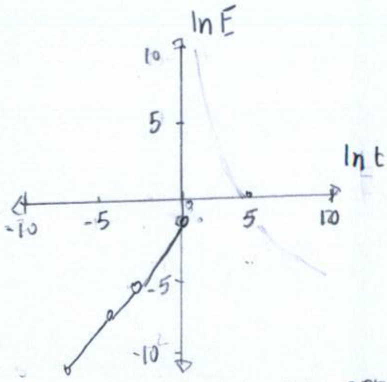
$n = 2$ $x_n + e^{-x_n} 10^{-2} = 0.36697$

$n = 3$ $x_n + e^{-x_n} 10^{-3} = 0.36785$

$n = 4$ $x_n + e^{-x_n} 10^{-4} = 0.36787$

c) $E = |\hat{x}(1) - x(1)|$

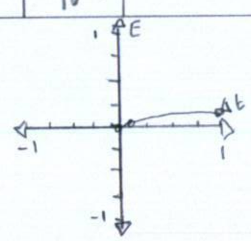


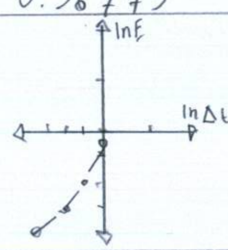


The results of $E = |\hat{x}(1) - x(1)|$ vs Δt represent error of Euler's method. While the plot of $\ln E$ vs $\ln t$ characterizes nothing informative.

$\dot{x} = -x ; x(0) = 1$ 2.8.4. $x(t) = e^{-t}$;

n	Δt	$f(x)$	$x_n = x_{n-1} + f(x_{n-1})\Delta t$	$E = \lvert\hat{x}(1) - x(1)\rvert$	$\ln E$
0	10^0		0.36788	0.00	-1.00
1	10^{-1}	$\exp(x_{n-1})$	0.33527	0.03261	-3.42
2	10^{-2}		0.36577	0.0211	-6.16
3	10^{-3}		0.36773	0.0002	-8.33
4	10^{-4}		0.36773	0.0000	-10.78

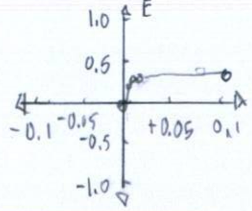


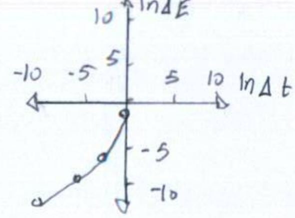


Improved Eulers Method

$\dot{x} = -x ; x(0) = 1$ 2.8.5. $x(t) = e^{-t}$

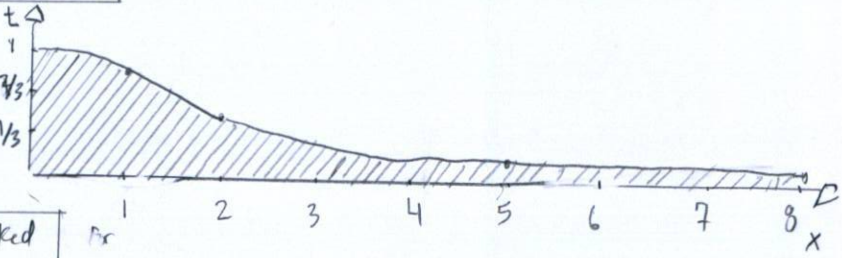
n	Δt	$f(x)$	$x_n = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$	$E = \lvert\hat{x}(1) - x(1)\rvert$	$\ln E$
0	10^0				
1	10^{-1}				
2	10^{-2}	$\exp(x_{n-1})$			
3	10^{-3}				
4	10^{-4}				



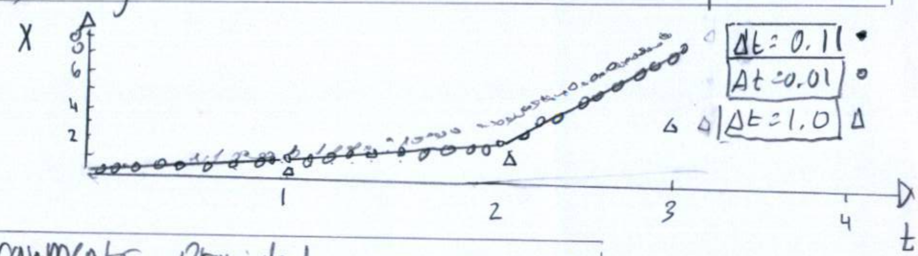


The Euler method aided the analysis of numerical methods; including, Precision. Euler's Improved Method approached the solution of $f(x)=e^{-x}$ with less round-off error, Runge-Kutta's Routine provided the least round-off errors with 10^{-20} across the spreadsheet, and necessitated high-precision.

$\dot{x}=x+e^{-x}$ 2.8.6. a: $t=\int \frac{1}{x+e^{-x}}dx = \int \frac{e^{x}}{xe^{x}+1}dx$;

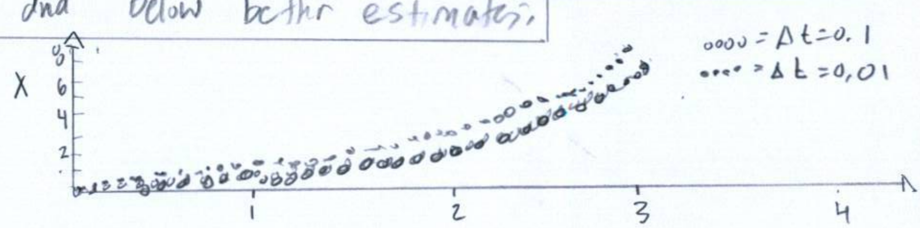


I noticed the book asked for x(t) (and not t(x)).
This led me to investigate a Numerical method of integration; Withstanding, Runge-Kutta Routine aided with the plot of x(t).



b. At $t=0$, analytical arguments provided an $x=1.011$.

c. Stepsizes of $\Delta t=0.1$ and 0.01 had different results, including inaccuracies above and below better estimates.



d) See part a.

$x_1=x_0+f(x_0)\Delta t$ 2.8.7 a) $x(t_1)=x(t_0+\Delta t)$

Taylor Series:

$$x(t+\Delta t)=\sum_{n=0}^{\infty}\frac{x^{(n)}(t)}{n!}(\Delta t)^{n} = x(t)+x'(t)\cdot\Delta t + O(\Delta t^2)$$

$$f(t+\Delta t)=f(t)+f'(t)\cdot\Delta t+O(\Delta t^2) = \boxed{x_0+f'(t)\cdot\Delta t}$$

b) $|x(t_1)-x_1| = |x(t_1)-x(t_1)-x'(t)\cdot\Delta t - O(\Delta t^2)| = |O(\Delta t^2)| = \frac{x''(t)\Delta t^2}{2!} = C(\Delta t^2)$

$$\boxed{C=\frac{x''(t)}{2!}}$$

Taylor Series: $f(x+h)=\sum \frac{f^n(x)h^n}{n!}$

2.8.8. $\dot{x}=x+e^{-x}$; $|x(t_0)-t_0| = |x(t_0)-x(t_0)-x(t_0)'\Delta t - \frac{x''(t_0)\Delta t^2}{2!}| = \frac{x''(t_0)\Delta t^2}{2}$

$\boxed{= O(\Delta t^2)}$

$\dot{x}=x+e^{-x}$

2.8.9 Runge-Kutta: $x_{n+1}=x_n+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$ Where $k_1=f(x_n)\Delta t$

$k_2=f(x_n+\frac{1}{2}k_1)\Delta t$

$k_3=f(x_n+\frac{1}{2}k_2)\Delta t$

$k_4=f(x_n+k_3)\Delta t$

$$x(t+\Delta t)=x(t_0)+x'(t_0)\Delta t+\frac{x''(t_0)\Delta t^2}{2!}+O(\Delta t^3)$$

$$k_1=f(x_n)\Delta t = x'(t_0)\Delta t$$

$$k_2=f(x_n+\tfrac{1}{2}k_1)\Delta t = f(x_n)+f'(x_n)\tfrac{1}{2}k_1+O[(\tfrac{1}{2}k_1)^2]$$

$$k_3=f(x_n+\tfrac{1}{2}k_2)\Delta t = f(x_0)+f'(x_n)\tfrac{1}{2}k_2+O[(\tfrac{1}{2}k_2)^2]$$
$$=f(x_n)+f'(x_n)\tfrac{1}{2}\left[f(x_n)+f'(x_n)\tfrac{1}{2}k_1+O[(\tfrac{1}{2}k_1)^2]\right]+O[(\tfrac{1}{2}k_2)^2]$$

$$k_4=f(x_n+k_3)\Delta t = f(x_n)+f'(x_n)\cdot k_3+O[k_3^2]$$
$$=f(x_n)+f'(x_n)\left[f(x_n)+f'(x_n)\cdot\tfrac{1}{2}[f(x_n)+f'(x_n)\tfrac{1}{2}k_1+O[(\tfrac{1}{2}k_1)]]+O[(\tfrac{1}{2}k_2)]\right]$$
$$+O[k_3]$$

$$x_{n+1}=x_n+\tfrac{1}{6}(k_1+2k_2+2k_3+k_4)=x_n+\tfrac{1}{6}(x'(t_0)\Delta t+2x'(t_0)+x''(t_0)k_1$$
$$+2x'(t_0)+x''(t_0)[x'(t_0)+x''(t_0)x'(t)\Delta t]$$
$$+2x'(t_0)+x''(t_0)[x(t)+\tfrac{x'(t)}{2}[x'(t_0)+x''(t_0)\tfrac{x'(t)}{2}]])$$

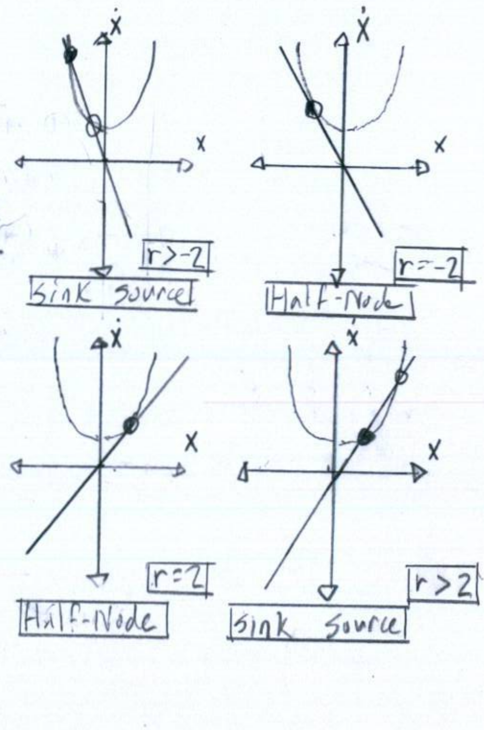
$$\boxed{|x(t_1)-x_1|=|x(t_0+\Delta t)-x_{n+1}|=O(\Delta t^5)}$$

Chapter 3:

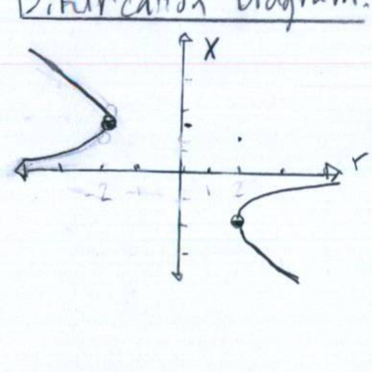
$\dot{x}=1+rx+x^2$ 3.1.1. Vector Field:

$$x=\frac{r\pm\sqrt{r^2-4}}{2}$$
$$=\frac{r\pm\sqrt{(r-2)(r+2)}}{2}$$

r	Bifurcations
>-2	Two
-2	One
0	zero
2	one
>2	Two



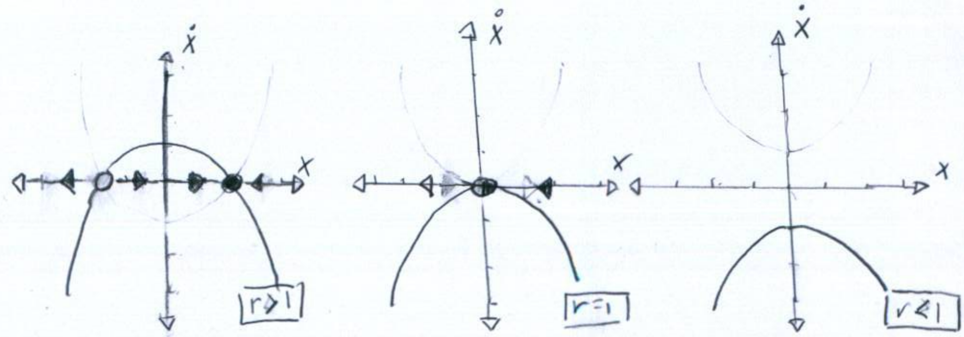
Bifurcation Diagram:



$\dot{x} = r - \cosh x$ 3.1.2. Vector Field

$r = \cosh(x)$

r	Bifurcations
<1	Zero
=1	One
>1	Two

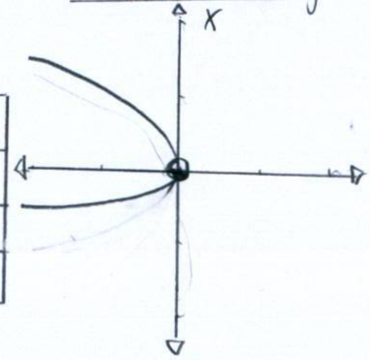


$\dot{x} = r + x - \ln(1+x)$ 3.1.3

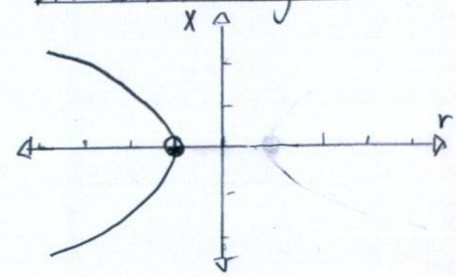
Vector Field

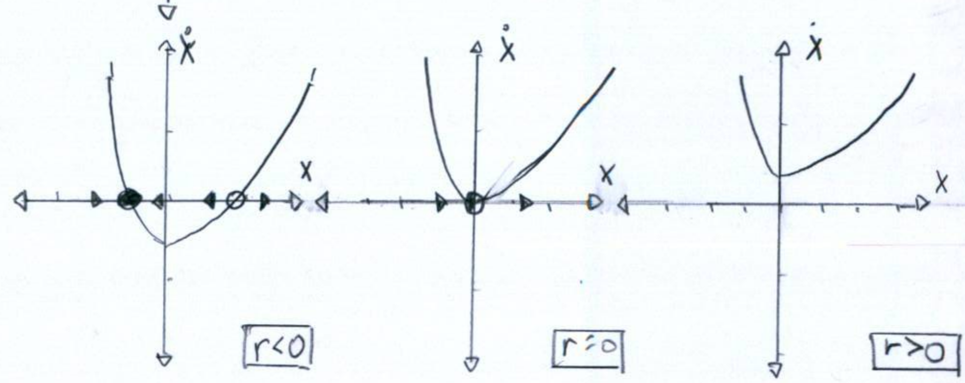
r	Bifurcation
>0	Zero
=0	One
<0	Two

Bifurcation Diagram:



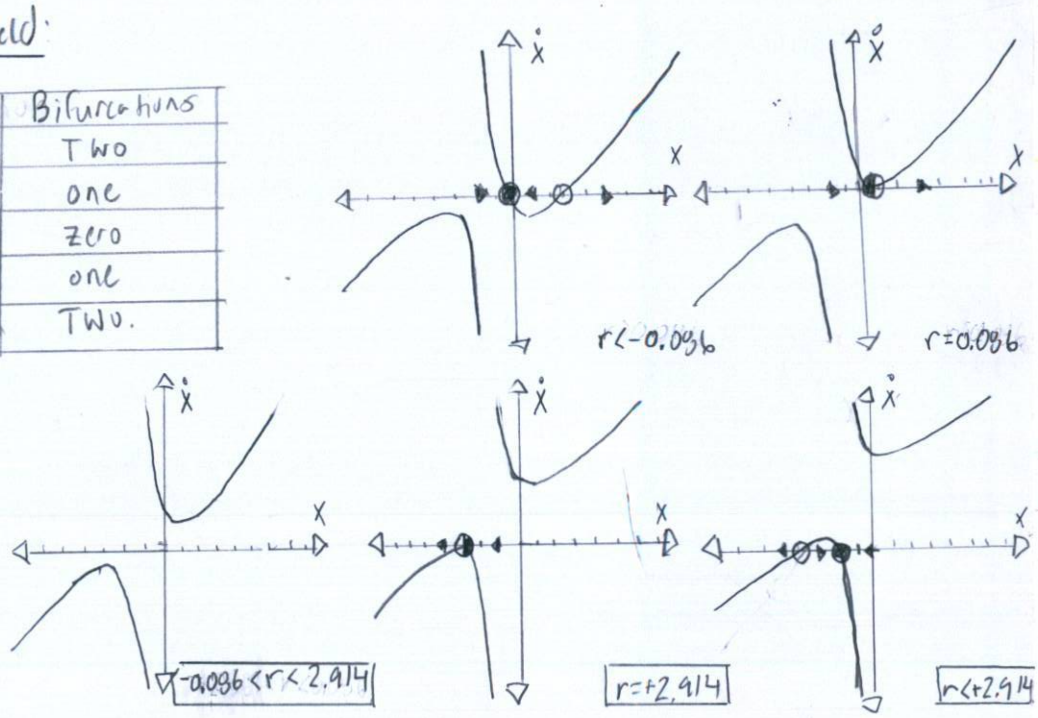
Bifurcation Diagram



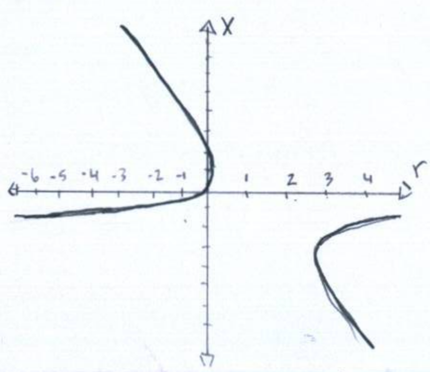


$\dot{x} = r + \frac{1}{2}x - x/(1+x)$ 3.1.4. Vector Field:

r	Bifurcations
< -0.096	Two
= -0.096	one
-0.096 < r < 2.914	zero
= +2.914	one
< +2.914	Two.

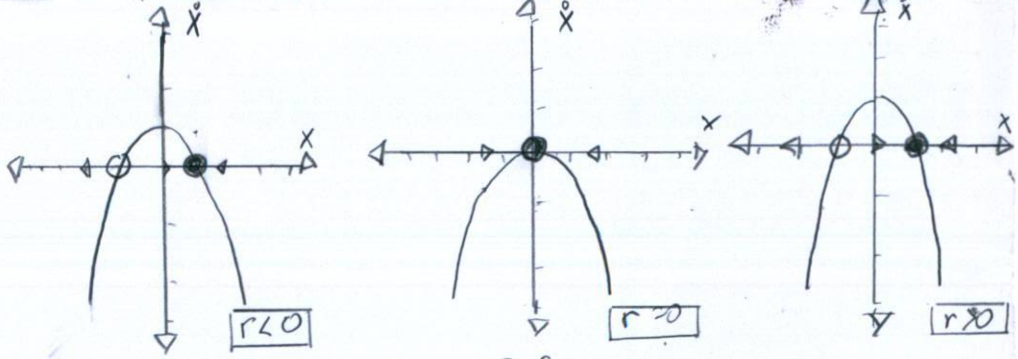


Bifurcation Diagram:



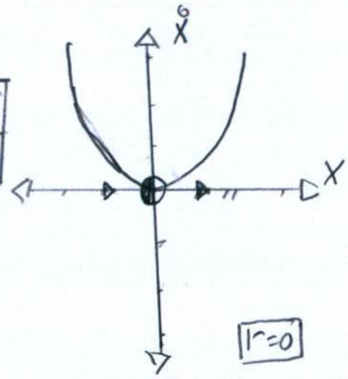
$\dot{x}=r^2-x^2$ 3.1.5. a) <u>Vector Field:</u>

r	Bifurcations
>0	Two
=0	one
<0	Two

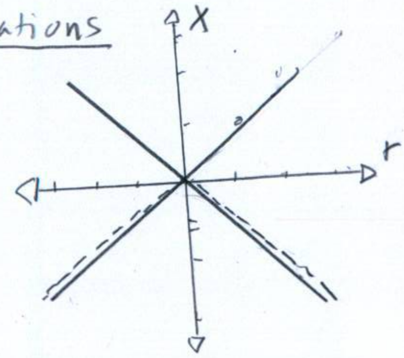


$\dot{x}=r^2+x^2$ b) <u>Vector Field:</u>

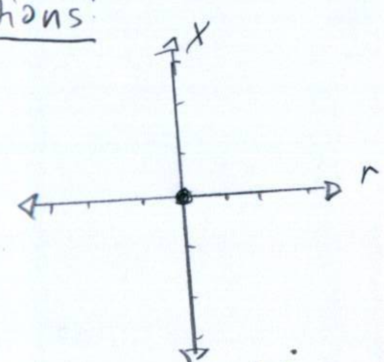
r	Bifurcations
0	one



<u>Bifurcations</u>



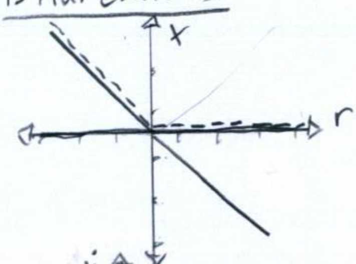
<u>Bifurcations:</u>

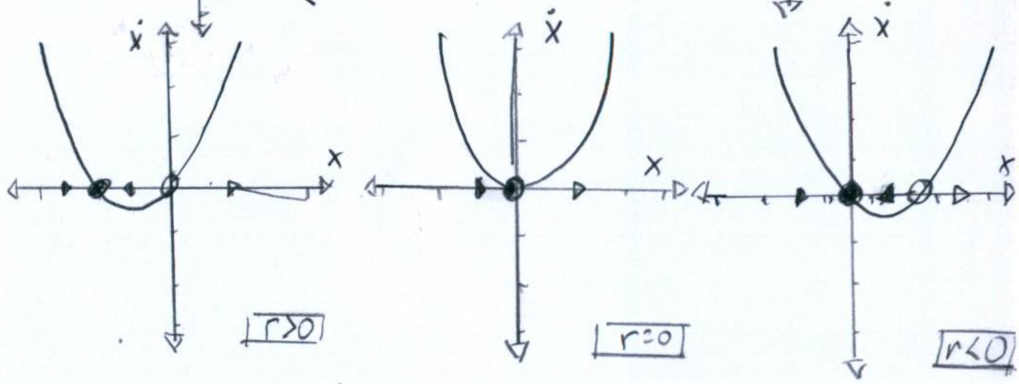


$\dot{x}=rx+x^2$ 3.2.1 <u>Vector Field:</u>

r	Bifurcations
>0	Two
=0	one
<0	Two

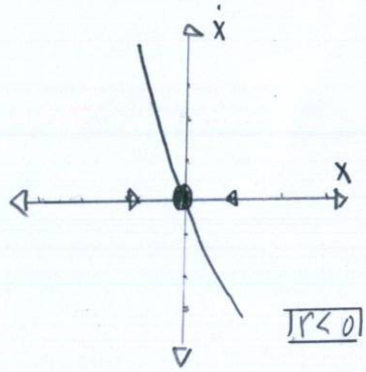
<u>Bifurcations:</u>

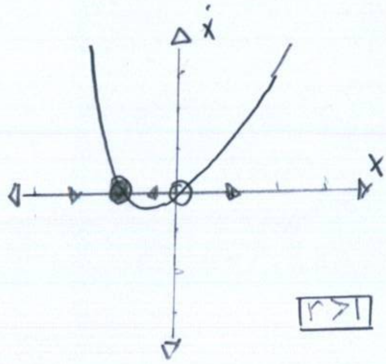


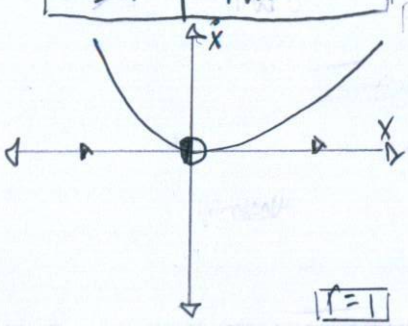


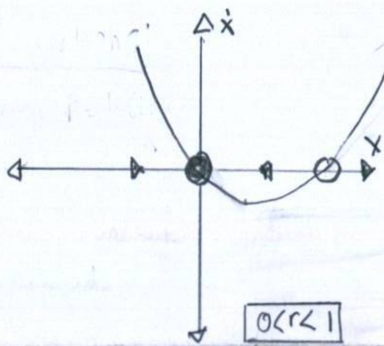
$\dot{x}=rx-\ln(1+x)$ 3.2.2. <u>Vector Field:</u>

r	Bifurcations
<0	one
0<r<1	Two
r=1	One
r>1	Two

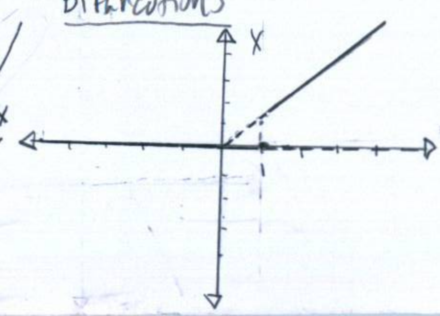








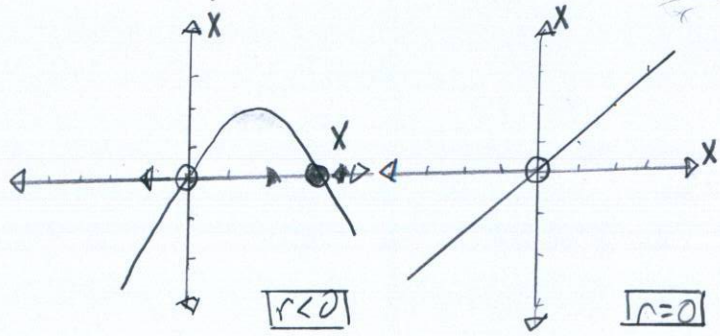
<u>Bifurcations</u>

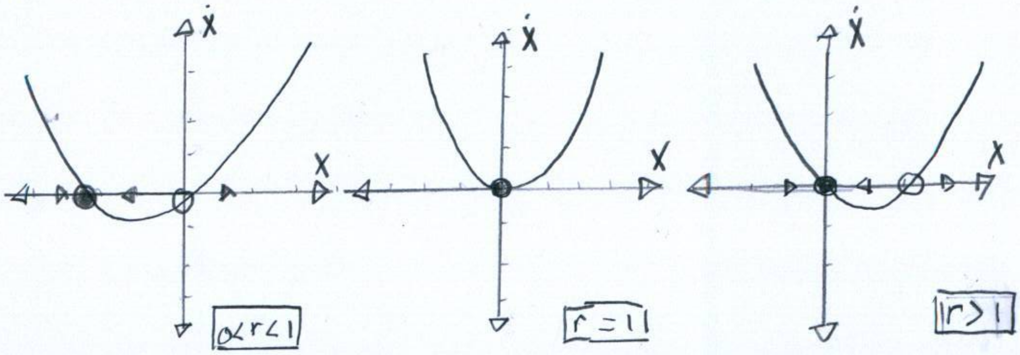


$\dot{x} = x - rx(1-x)$ 3.2.3. Vector Field:

r	Bifurcations
<0	Two
$=0$	one
$0<r<1$	Two
$r\geq 1$	One
>1	Two

$r = \frac{x(1-x)}{r}$

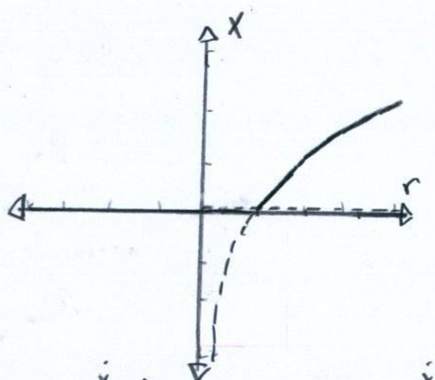




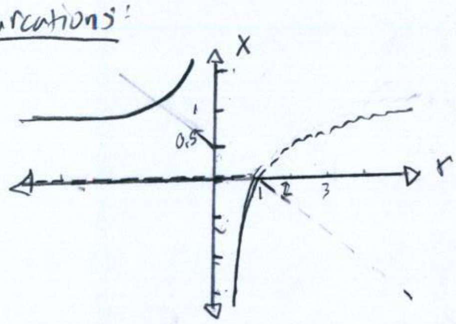
$\dot{x} = x(r - e^x)$ 3.2.4. Vector Field:

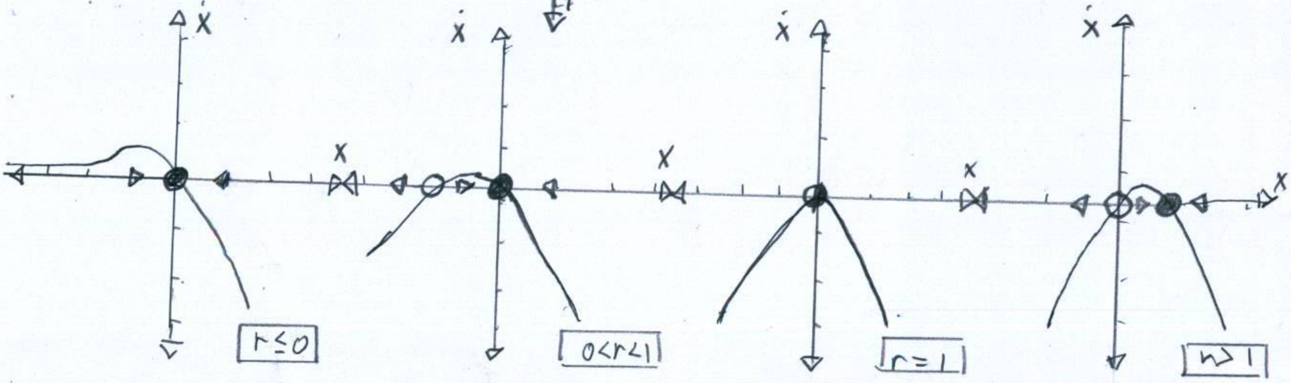
r	Bifurcations
≤ 0	one
$0<r<1$	Two
$0\leq r\leq 1$	one
$r>1$	Two

Bifurcations:



Bifurcations:





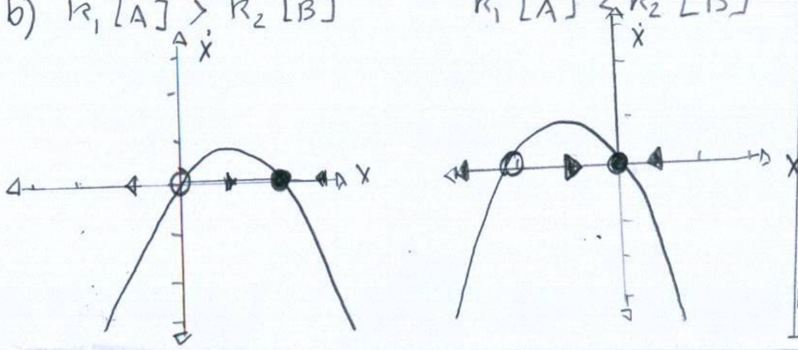
$\dot{x} = c_1 x - c_2 x^2$ 3.2.5 a) $\frac{d[A]}{dt} = -k_1[A][X] + k_2[X]^2$; $\boxed{\frac{d[X]}{dt} = (k_1[A] - k_2[B])[X] - k_{-1}[X]^2}$

$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X$

$X + B \xrightarrow{k_2} C$

$\frac{d[B]}{dt} = -k_2[B][X]$; $\frac{d[C]}{dt} = k_2[B][X]$

b) $k_1[A] > k_2[B]$ $\quad$ $k_1[A] < k_2[B]$



Chemically, a rate of change $\frac{d[x]}{dt}$ that approaches zero, then remains zero is of greater stability than a rate of change which increases from zero.

$\dot{n} = GnN - kn$

$\dot{N} = -GnN - fN + p$

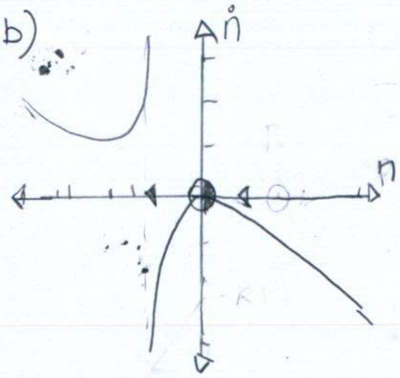
3.3.1 a) Suppose $\dot{N} > \dot{n}$, then $\dot{N} \approx 0$. "Adiabatic Elimination"

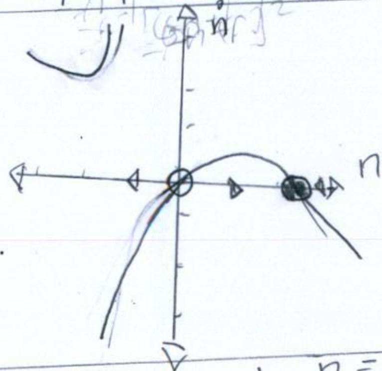
$GnN + fN = p$; $\dot{n} = -fN + p - kn$;

$N = \frac{p}{Gn + f}$; $\boxed{\dot{n} = -f\left[\frac{p}{Gn+f}\right] + p - kn}$

$\boxed{p > \frac{kn\,[Gn+f]}{1-pf} = p_c}$

b)





c) A transcritical bifurcation occurs at $n = 0$ because of the stability change for the fixed point.

d) $G, n, p, f > 0$, $N = 0$, a constant amount of excited photons

$\dot{E} = K(P - E)$

$\dot{P} = \gamma_1(ED - P)$

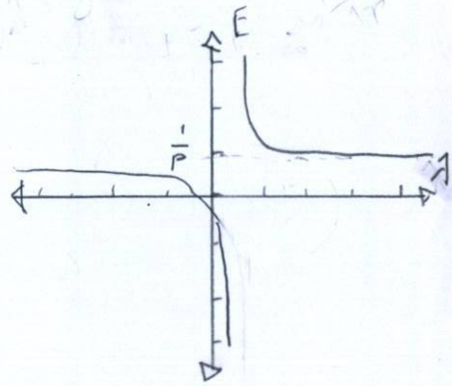
$\dot{D} = \gamma_2(\lambda + 1 - D - \lambda EP)$

3.3.2 a) Assume $\dot{P} \approx 0$, $\dot{D} \approx 0$; $P = ED$; $D = \lambda + 1 - \lambda EP$

$\dot{E} = K(ED - E) = K(E(\lambda + 1 - \lambda EP) - E)$

b) Fixed Points: $E = 0, \frac{1}{P}$

c) Bifurcation Diagram:

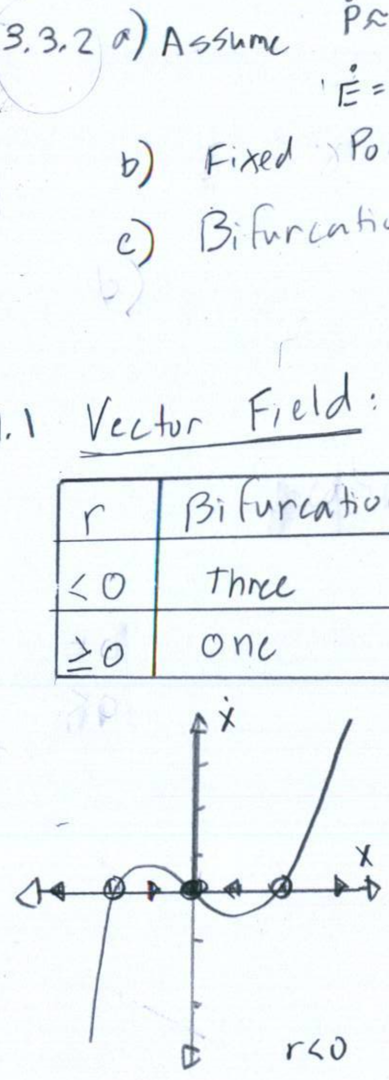


$\dot{x} = rx + 4x^3$

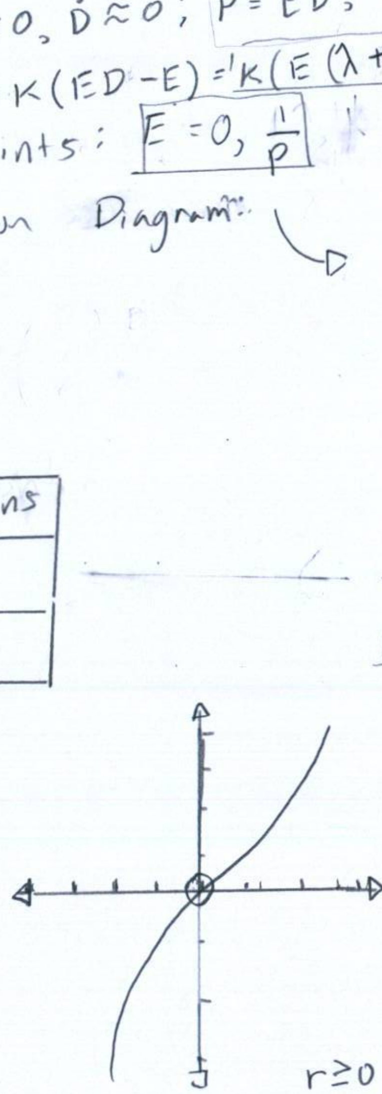
3.4.1 Vector Field:

r	Bifurcations
<0	Three
≥0	One

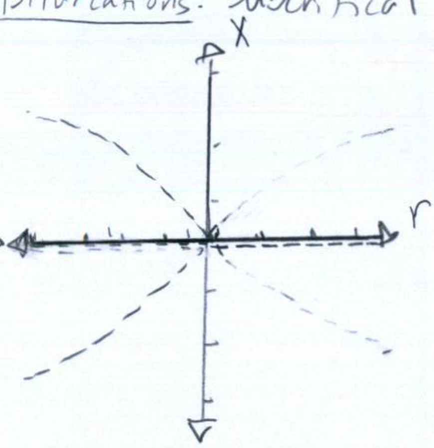
Bifurcations: Subcritical



r<0



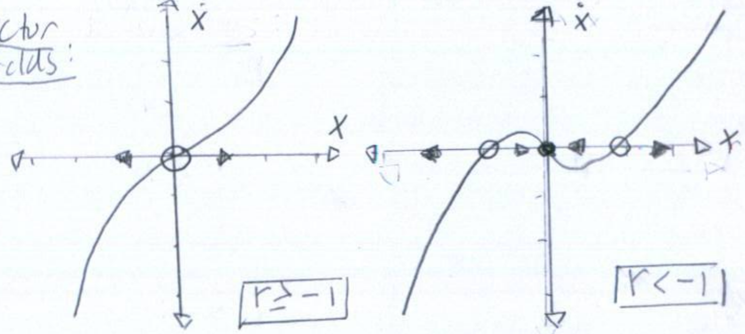
r≥0



$\dot{x} = rx - \sinh x$ 3.4.2

r	Bifurcations
≥ -1	One
< -1	Three

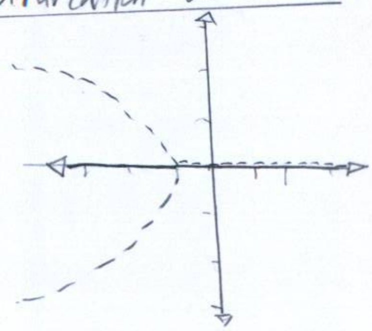
Vector Fields:



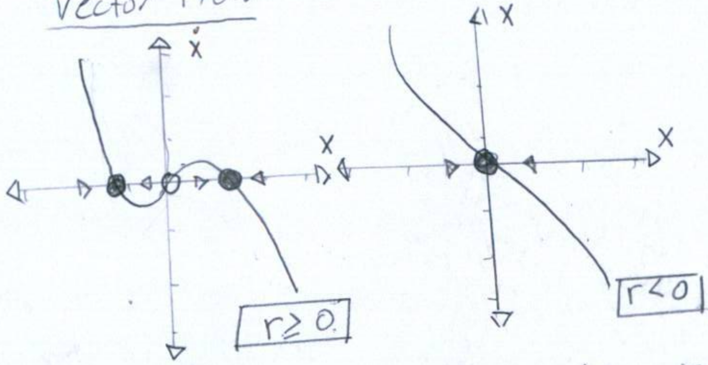
$\dot{x} = rx - 4x^3$ 3.4.3

r	Bifurcations
≥ 0	Three
≤ 0	one

Bifurcation: subcritical



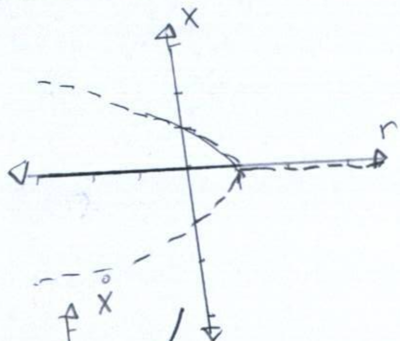
Vector Field:



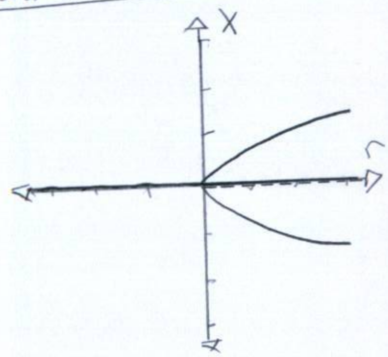
$\dot{x} = x + \frac{rx}{1+x^2}$ 3.4.4

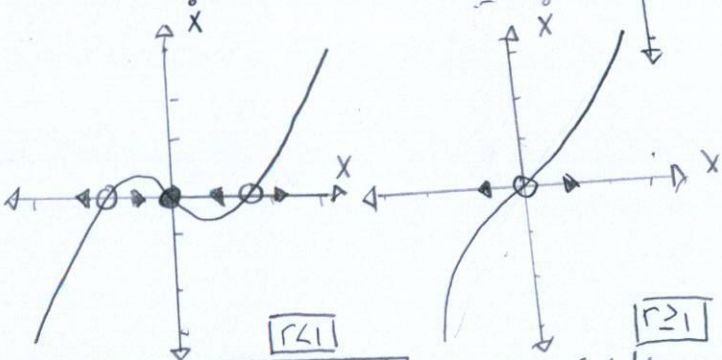
r	Bifurcations
< 1	Three
≥ 1	one

Bifurcations: Supercritical



Bifurcation: Supercritical.

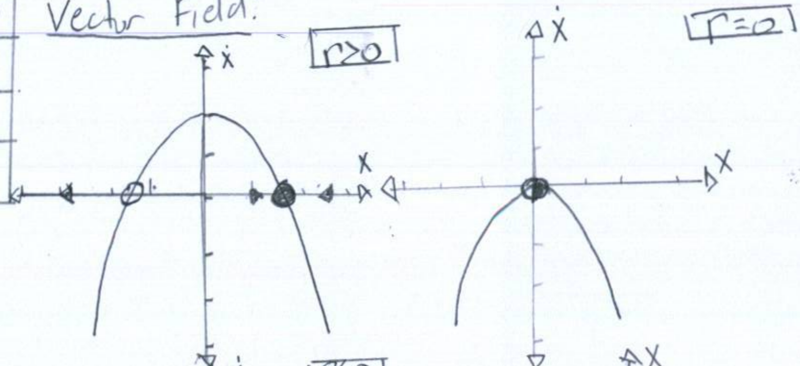


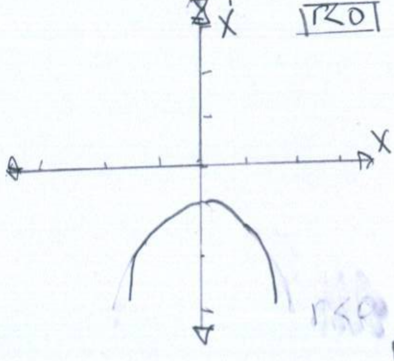


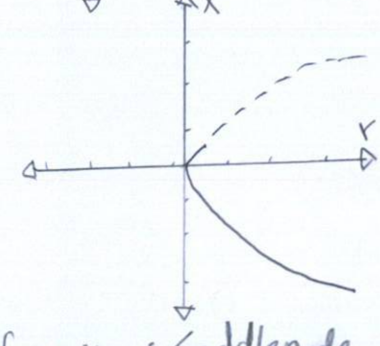
$\dot{x} = r - 3x^2$ 3.4.5

r	Bifurcations
≥ 0	Two
$= 0$	one
< 0	zero

Vector Field:





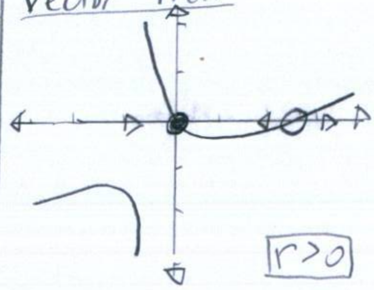


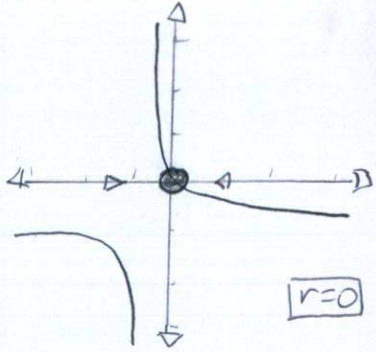
Bifurcation: Saddlenode

$\dot{x} = rx - \frac{x}{1+x}$ 3.4.6.

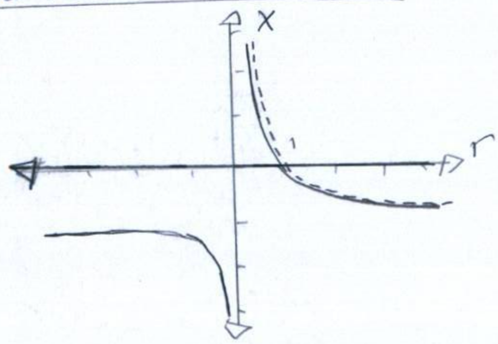
r	Bifurcations
>0	Two
=0	One
<0	Two

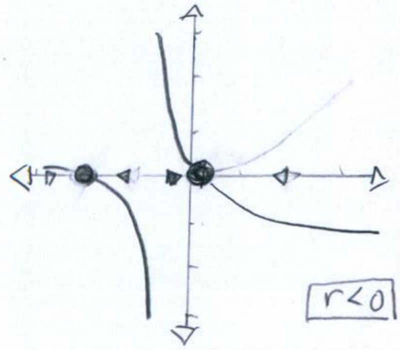
Vector Field:





Bifurcation: Transcritical

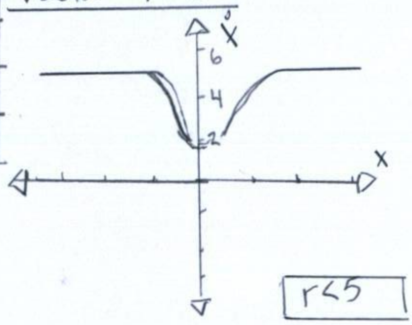


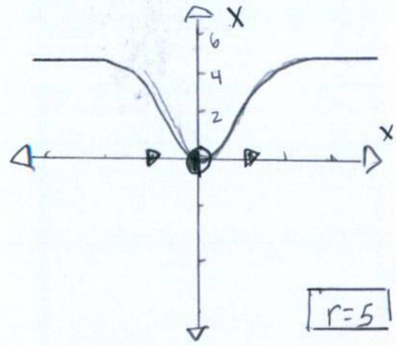


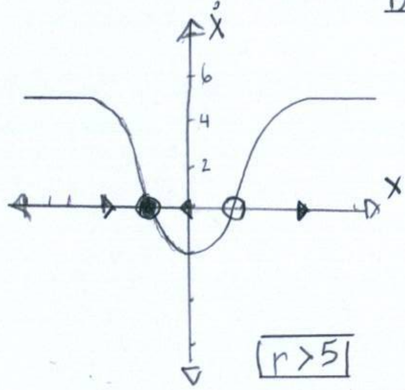
$\dot{x} = 5 - re^{-x^2}$ 3.4.7

r	Bifurcations
<5	zero
=5	one
>5	TWO

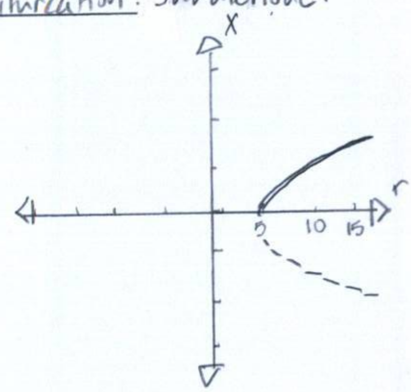
Vector Field:







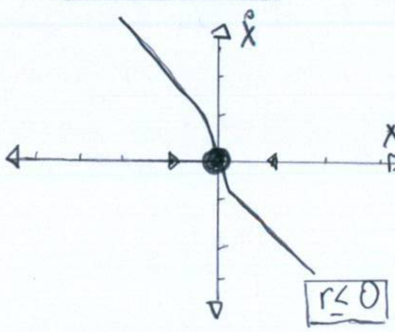
Bifurcation: Saddlenode.

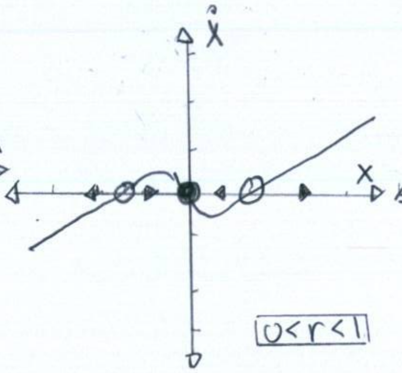


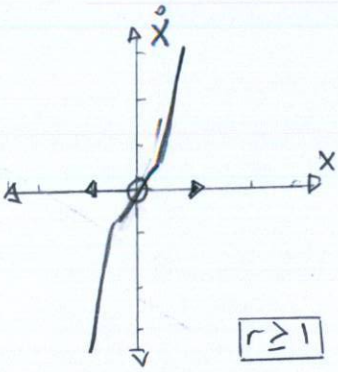
$\dot{x} = rx - \frac{x}{1+x^2}$ 3.4.8.

r	Bifurcations
≤0	one
0<r<1	three
≥1	one

Vector Field:



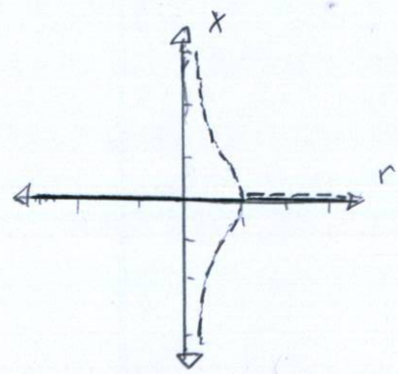




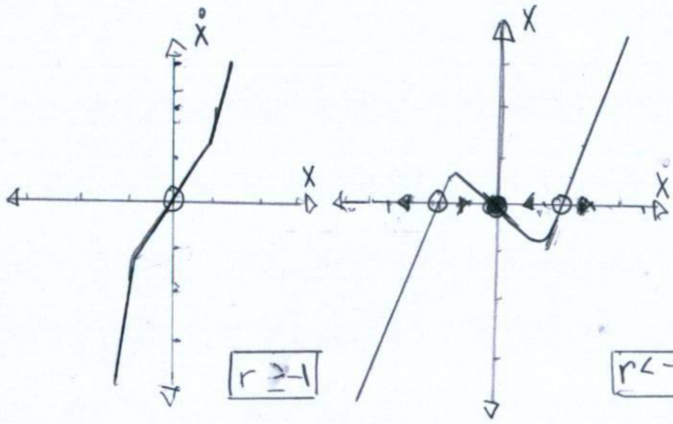
$\dot{x} = x + \tanh(rx)$ 3.4.9

r	Bifurcations
≤ -1	one
> -1	three

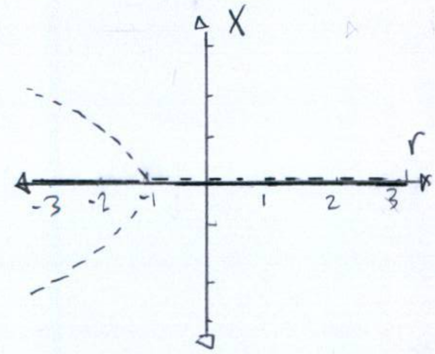
Bifurcation: Transcritical



Vector Field:



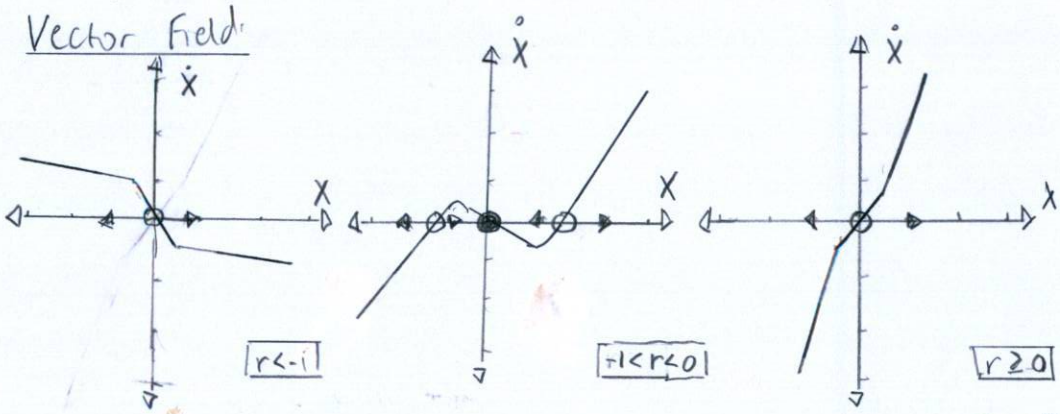
Bifurcation: subcritical



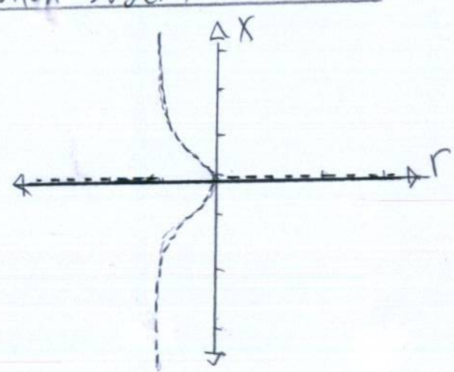
$\dot{x} = rx + \frac{x^3}{1+x^2}$ 3.4.10.

r	Bifurcations.
<-1	one
$-1<r<0$	three
≤ 0	one

Vector Field:



Bifurcation: Subcritical Pitchfork



$\dot{x} = rx - \sin x$ 3.4.11 a) If $r=0$, then $\dot{x} = -\sin x$

Fixed points: stable $= (2k+1)\pi$

unstable $= 2k\pi$

where $k \in \mathbb{Z}$

b) If $r > 1$, $\boxed{\dot{x} = 0 \text{ is unstable}}$

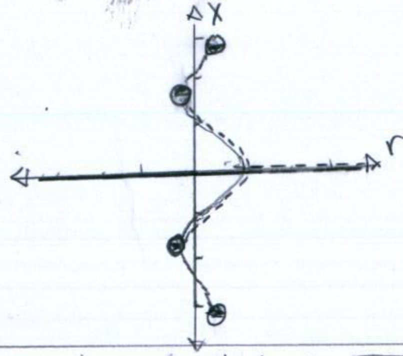
c) As $r = \infty \to 0$, then a subcritical pitchfork best describes the bifurcation.

d) $\dot{x} = rx - \sin(x)$; $r = \frac{\sin(x)}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + O(x^4)$

$$\boxed{x = \pm(6[1-r])^{1/2}}$$

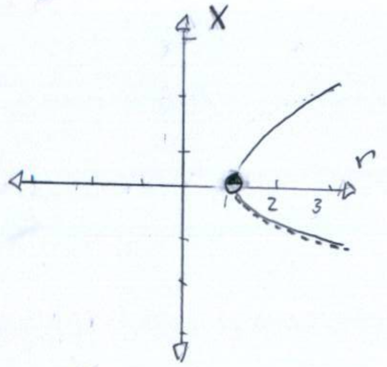
e) As $r = -\infty \to 0$, then a supercritical pitchfork occurs across the function $\dot{x} = rx - \sin(x)$.

f) Bifurcations:

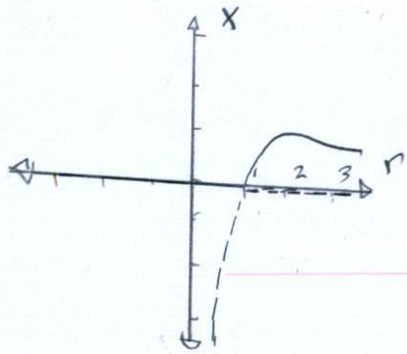


$\dot{x} = f(x,r)$ 3.4.12 A "quadrification" function is $x = \frac{1}{2}(3 \pm \sqrt{1 \pm 4\sqrt{r}})$ where $\dot{x} = f(x,r) = (x-2)^2(x-1)^2 - r$. This function has even polynomial multiplicities to describe zero bifurcations $r<0$ and four when $r>0$.

$\dot{x} = r - x - e^{-x}$ 3.4.13 a) Best guess of roots: $r=1$, $x=0$

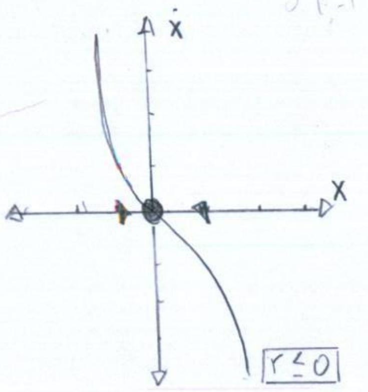


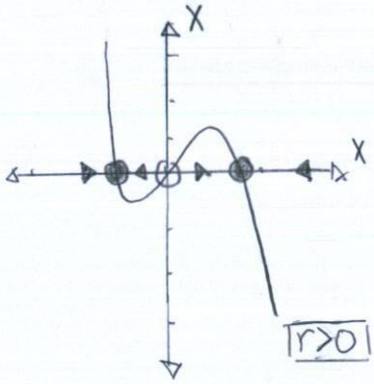
$\dot{x} = 1 - x - e^{-rx}$ b) Best Guess of roots: $r=1$, $x=0$



$\dot{x} = rx + x^3 - x^5$ 3.4.14 a) $\boxed{\ddot{x} = 0 = r + 3x^2 - 5x^4}$ -or- $\boxed{r = x^2(x^2-1)}$

b) Vector Field:





c) $\boxed{r_c \leq 0}$

$\dot{x} = rx + x^3 - x^5$ 3.4.15. $-\frac{dV(x)}{dx} = \dot{x} = 0 = -\frac{r}{2} - \frac{x^2}{4} + x^4$

where $a = x^2$

$$a_1, a_2 = \frac{+1 \pm \sqrt{(+1)^2 - 4(1)(-r)}}{2(1)}$$

$$= \left(\frac{1 \pm \sqrt{1+4r}}{2}\right)$$

$$x_1 = +\sqrt{\frac{1+\sqrt{1+4r}}{2}} \; ; \; x_2 = -\sqrt{\frac{1+\sqrt{1+4r}}{2}}$$

$$x_3 = +\sqrt{\frac{1-\sqrt{1+4r}}{2}} \; ; \; x_4 = -\sqrt{\frac{1-\sqrt{1-4r}}{2}}$$

$$x_5 = 0$$

$$V(x) = -r\frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6}.$$

$$V(x_1) = V(x_2) = V(x_3) = V(x_4) = V(x_5) = 0$$

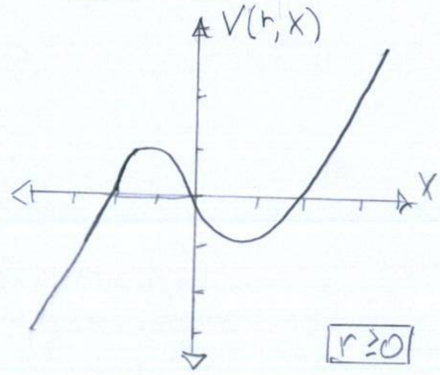
$$@\, V(x_1) = \frac{-r}{2}\left(\frac{1+\sqrt{1+4r}}{2}\right) + \frac{1}{4}\left(\frac{1+\sqrt{1+4r}}{2}\right)^2 - \frac{1}{6}\left(\frac{1+\sqrt{1+4r}}{2}\right)^3 \boxed{=0}$$

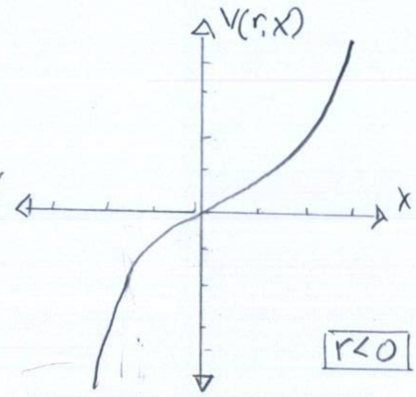
$r = -3/16$

$\dot{x} = r - x^2$ 3.4.16 a) $-\frac{dV}{dx} = \dot{x} = r - x^2$; $\boxed{V(r,x) = \frac{x^3}{3} - rx}$

r	Bifurcations
≥ 0	Three
<0	One

Potential Field

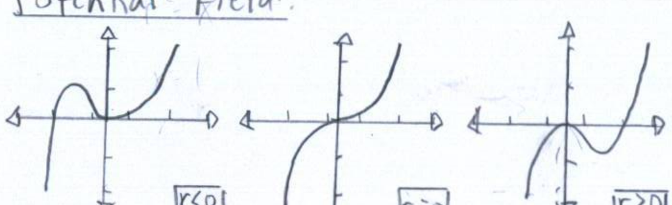




$\dot{x} = rx - x^2$

b) $-\frac{dV}{dx} = \dot{x} = rx - x^2$; $\boxed{V(r,x) = \frac{x^3}{3} - \frac{rx^2}{2}}$

Potential Field

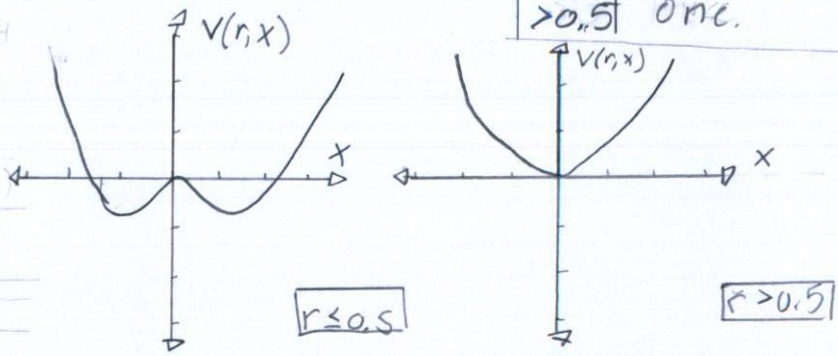


r	Bifurcations
<0	Two
$=0$	One
>0	Two.

$\dot{x} = rx + x^3 - x^5$ c) $-\frac{dV}{dx} = rx + x^3 - x^5$; $\boxed{V(r,x) = \frac{x^6}{6} - \frac{x^4}{4} - rx^2}$

r	Bifurcations
≤ 0.5	Three
>0.5	One.

<u>Potentials:</u>



$b\dot{\phi} = -mg\sin\phi + mr\omega^2 \sin\phi\cos\phi$ 3.5.1. A better representation of $b\dot{\phi} = mg\sin\phi + mr\omega^2\sin\phi\cos\phi$ is $b\dot{\phi} = mg\sin\phi\left(\frac{r\omega^2}{g}\cos\phi - 1\right)$, which best represents the maximum angle of $\phi = \pi/2$. If the bead approaches a fixed point during rotation, then $b\dot{\phi} = 0$; $\frac{r\omega^2}{g}\cos\phi = 1$; $\cos\phi = \frac{g}{r\omega^2}$; and, $\frac{g}{r\omega^2}$ requires a positive value above zero.

$$\frac{d\phi}{d\tau} = F(\phi) = -\sin\phi + \gamma\sin\phi\cos\phi = \sin\phi(\gamma\cos\phi - 1)$$

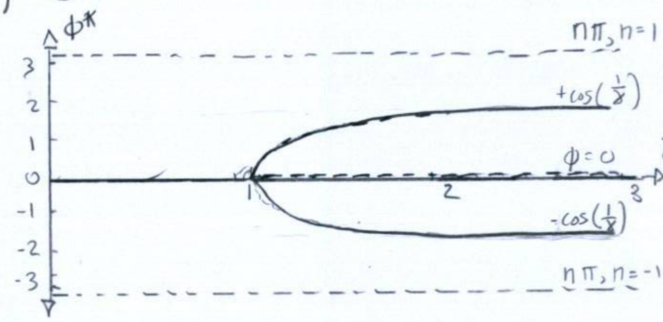
3.5.2 $F(\phi) = \sin\phi(\gamma\cos\phi - 1)$

$f'(\phi) = \gamma[\cos^2\phi - \sin^2\phi - 1] = \gamma[\cos 2\theta - 1]$

$f''(\phi) = -2\gamma[\sin 2\theta]$

$\phi^* = n\pi$; $f'(\phi) = 0$; Half-Node

$\phi^* = \cos^{-1}\left(\frac{1}{\gamma}\right)$



$$\frac{d\phi}{d\tau} = f(\phi) = -\sin\phi + \gamma\sin\phi\cos\phi = \sin\phi(\gamma\cos\phi - 1)$$

3.5.3. If $\phi \approx 0$, then $\sin\phi \approx \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!}$

$\approx \phi$

$\cos\phi \approx 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!}$ and $\frac{d\phi}{d\tau} = \phi\left(\gamma\left[1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!}\right] - 1\right)$

$= \gamma\phi - \frac{\gamma\phi^3}{2!} + \frac{\gamma\phi^5}{4!}$

Where $\boxed{\frac{d\phi}{d\tau} = A\phi - B\phi^3 + O(\phi^3) ; A = \gamma, B = \frac{\gamma}{2}, O(\phi^5) = \frac{\gamma\phi^5}{4!}}$

$m\ddot{x} = -F_{spring} - F_{Fric}$ 3.5.4. $m\ddot{x} = -k \cdot l\cos\phi - kL_0\cos\phi - b\dot{\phi}$

$= -k(l - L_0)\cos\phi - b\dot{\phi} = -k\left(\sqrt{x^2 + h^2} - L_0\right)\frac{x}{l} - b\dot{\phi}$

$= -k\left(\sqrt{h^2 + h^2} - L_0\right)\frac{x}{\sqrt{h^2 + x^2}} - b\dot{x}$

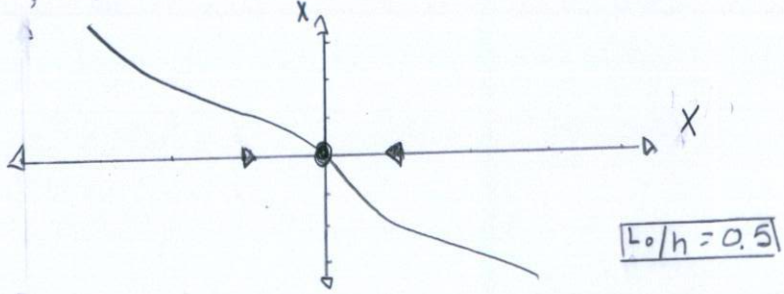
$$\boxed{= -k\left(1 - \frac{L_0}{\sqrt{h^2 + x^2}}\right)x - b\dot{x}}$$

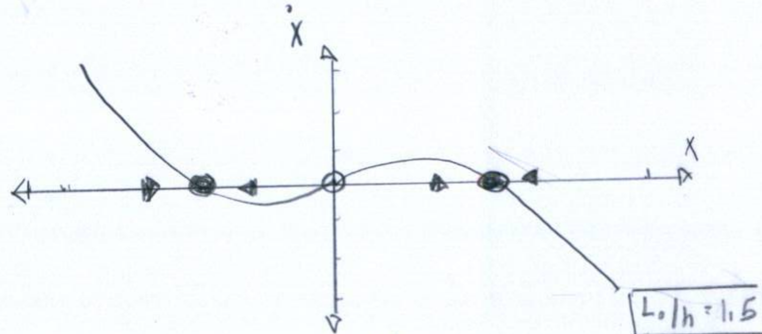
b. $m\ddot{x} + b\dot{x} + k\left(1 - \frac{L_0}{\sqrt{x^2+h^2}}\right)x = 0$

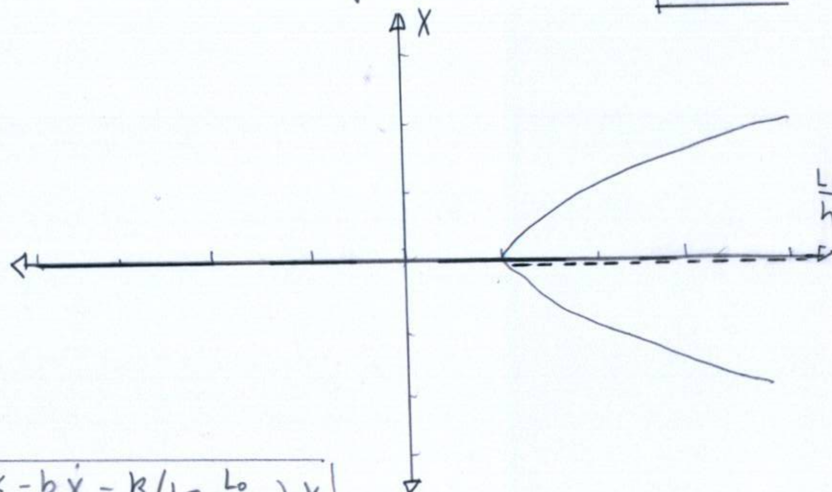
if $\dot{x} = 0$, $\boxed{x^* = \sqrt{L_0^2 - h^2},\ 0}$

c. If $m = 0$, $b\dot{x} + k\left(1 - \frac{L}{\sqrt{x^2+h^2}}\right)x = 0$, then $\boxed{x^* = \sqrt{L_0^2 - h^2},\ 0}$

Bifurcation Diagram







d. $\boxed{\text{If } m \neq 0, \text{ then } m\ddot{x} \ll -b\dot{x} - k\left(1 - \frac{L_0}{\sqrt{x^2+h^2}}\right)x}$

$\varepsilon \frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau} = f(\phi)$ 3.5.5. a) $\frac{d\phi}{d\tau} = f(\phi)$; T_{fast} is estimated to be:

$$\varepsilon^{1-2k} \frac{d^2\phi}{d\tau^2} + \varepsilon^{-k} \frac{d\phi}{d\tau} = f(\phi);$$

where $k = 1$, $\varepsilon^{1-2k} = \varepsilon^{-k} \gg 1$

$k = \frac{1}{2}$, $\varepsilon^{1-2k} = 1 \gg e^{-k}$

$k = 0$, $\varepsilon^{-k} = 1 \gg e^{1-2k}$

$$T = \varepsilon \frac{b}{mg} = \frac{m^2 gr}{b^2} \frac{b}{mg} = \boxed{\frac{mr}{g}}$$

b) If $\tau = \varepsilon \mathfrak{z}$, then $\varepsilon \frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau} = \varepsilon \frac{d^2\phi}{d(\varepsilon \mathfrak{z})^2} + \frac{1}{\varepsilon}\frac{d\phi}{d\mathfrak{z}} = f(\phi)$

$$\boxed{\frac{d^2\phi}{d\mathfrak{z}^2} + \frac{d\phi}{d\mathfrak{z}} = \epsilon f(\phi) \quad \text{"Rescaled"}}$$

c. $T_{fast} = \varepsilon T_{slow}$

$\varepsilon \ddot{x} + \dot{x} + x = 0$ 3.5.6 $x(0) = 1; \dot{x}(0) = 0$

a) General solution: $x(t) = C_1 e^{\lambda t}$ $\quad \varepsilon\lambda^2 + \lambda + 1 = 0$

$$\dot{x}(t) = \lambda C_1 e^{\lambda t}; \lambda = \frac{-1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon}$$

$$\ddot{x}(t) = \lambda^2 C_1 e^{\lambda t}; \; x(t) = C_1 e^{(-1+\sqrt{1-4\varepsilon})x/2\varepsilon} + C_2 e^{(-1-\sqrt{1-4\varepsilon})x/2\varepsilon}$$

$$x(0) = C_1 + C_2 = 1$$

$$\dot{x}(t) = C_1\left(\frac{-1+\sqrt{1-4\varepsilon}}{2\varepsilon}\right) e^{(-1+\sqrt{1-4\varepsilon})x/2\varepsilon}$$

$$+ C_2\left(-\frac{(1-\sqrt{1-4\varepsilon})}{2\varepsilon}\right) e^{(-1-\sqrt{1-4\varepsilon})x/2\varepsilon}$$

$$\dot{x}(0) = C_1\left(-\frac{(1+\sqrt{1-4\varepsilon})}{2\varepsilon}\right) + C_2\left(\frac{-1-\sqrt{1-4\varepsilon}}{2\varepsilon}\right)$$

$$= \frac{-C_1\sqrt{1-4\varepsilon}}{2\varepsilon} + \frac{C_1}{2\varepsilon} + (1-C_1)\left(\frac{-1-\sqrt{1-4\varepsilon}}{2\varepsilon}\right)$$

$$= \frac{-C_1\sqrt{1-4\varepsilon}}{2\varepsilon} + \frac{C_1}{2\varepsilon} + \frac{(1+\sqrt{1-4\varepsilon})}{2\varepsilon}$$

$$+ \frac{C_1(1-\sqrt{1-4\varepsilon})}{2\varepsilon}$$

$$= \cancel{\frac{-C_1\sqrt{1-4\varepsilon}}{2\varepsilon}} + \frac{C_1}{2\varepsilon} + \frac{(1+\sqrt{1-4\varepsilon})}{2\varepsilon}$$

$$+ \frac{C_1}{2\varepsilon} + \cancel{\frac{C_1\sqrt{1-4\varepsilon}}{2\varepsilon}} = 0$$

$$= \frac{C_1}{\varepsilon}; = \frac{(1+\sqrt{1-4\varepsilon})}{2}$$

$$\boxed{\text{Therefore, } x(t) = \left(\frac{1+\sqrt{1-4\varepsilon}}{2}\right)\left(e^{(-1+\sqrt{1-4\varepsilon})x/2\varepsilon}\right) + \left(1 - \frac{(1+\sqrt{1-4\varepsilon})}{2}\right)\left(e^{(-1-\sqrt{1-4\varepsilon})x/2\varepsilon}\right)}$$

b. $\varepsilon \ddot{x} + \dot{x} + x = \varepsilon \frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0$

$$\frac{\varepsilon}{T^2}\frac{d^2x}{d\tau^2} + \frac{1}{T}\frac{dx}{d\tau} = -x \quad \text{where } T = \frac{t}{\tau} = \varepsilon$$

$$\frac{1}{T}\frac{d^2x}{d\tau^2} + \frac{1}{T}\frac{dx}{d\tau} = x; \; \ddot{x} + \dot{x} = Tx = \varepsilon x; \; \boxed{\ddot{x} + \dot{x} - Tx = 0}$$

$\overset{\circ}{N} = rN(1 - N/K)$ 3.5.7. a) $N(0) = N_0$;

Parameter	Dimensions
r	Per time (rate
K	Same as N (amount)
N_0	Same as N (amount)

b) $\frac{dN}{dt} = rN(1 - N/K)$; If $\frac{N}{K} = x$, then $dN = K\,dx$

$\frac{dx}{d\tau} = rx(1-x)$; If $t = \frac{\tau}{r}$, then $dt = \frac{d\tau}{r}$

$$\boxed{\frac{dx}{d\tau} = x(1-x)}$$

c) $u = x$; $\frac{du}{d\tau} = u(1-u)$; $u(0) = u_0$

$$\int \frac{du}{u(1-u)} = d\tau;\ \int \frac{A}{u}\,du + \int \frac{B}{(1-u)}\,du = \int \frac{du}{u} + \int \frac{du}{(1-u)} = \ln \frac{u}{1-u} = \tau + c$$

$A(1-u) + Bu = 1$

$$\frac{1-u}{u} = Ce^{-\tau}$$

$$1 = u(1 + Ce^{-\tau})$$

$$u = \frac{1}{1 + Ce^{-\tau}}$$

$$u(0) = u_0 = \frac{1}{1 + C}$$

$$C = \frac{1 - u_0}{u_0}$$

$$\boxed{u(\tau) = \frac{1}{1 + \left(\frac{1-u_0}{u_0}\right)e^{-\tau}}}$$

d) An advantage of the dimensionless functions are lower degrees of freedom during analysis. The graphical representations do not have further axis to plot, and the functions are closer to the basic functions of precalculus.

$\dot{u} = au + bu^3 - cu^5$ 3.5.8. Prove $\frac{dx}{d\tau} = rx + x^3 - x^5$, where $x = \frac{u}{U}$

$\tau = \frac{t}{T}$

$$\frac{\cancel{U}}{T}\frac{dx}{d\tau} = a\cancel{U}x + bU^{\cancel{3}\,2}x^3 - cU^{\cancel{5}\,4}x^5$$

$$\frac{dx}{d\tau} = Tax + TbU^2x^3 - TcU^4x^5;\quad a = \frac{r}{T},\ b = \frac{1}{TU^2},\ c = \frac{1}{TU^4}$$

$$\boxed{\frac{dx}{d\tau} = rx + x^3 - x^5}$$

3.6.1. Figure 3.6.3b corresponds to Figure 3.61b; specifically, the relationship between $y = h$, and $y = rx - x^3$. The dotted lines support a single bifurcation to two bifurcations at h_c, then three when $h > h_c$. To answer the question, Figure 3.6.3b has information of $h < 0$ and $h > 0$.

c)

d) If $\varepsilon << 1$, then $\varepsilon \ddot{x} + \dot{x} + x \approx \dot{x} + x$ and is a similar model to the boundary conditions

e) Mechanical System: An extremely viscous solution for an oscillating Newtonian device.

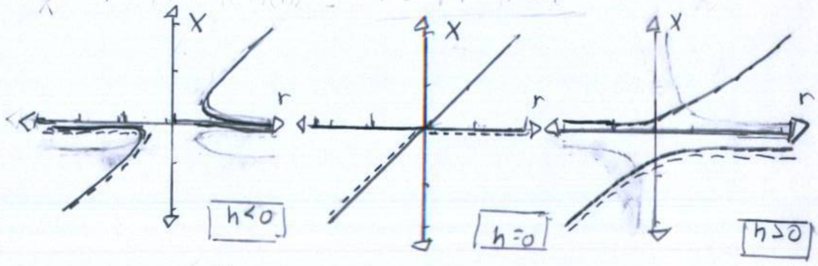
Electrical System: An electrical system of the form $v = Ri + L\frac{di}{dt} + \frac{1}{c}\int i\,dt$

where $\varepsilon = \frac{1}{c} << 1$.

$\dot{x} = h + rx - x^2$ 3.6.2. a)

$$x = \frac{-r \pm \sqrt{r^2 - 4(-1)(h)}}{2(-1)}$$

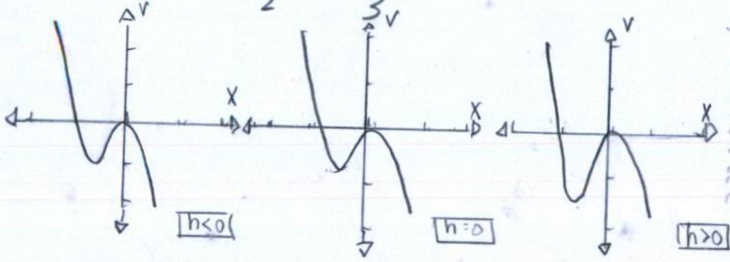
h	Bifurcations
<0	zero/one/two
=0	one/Two
>0	Two

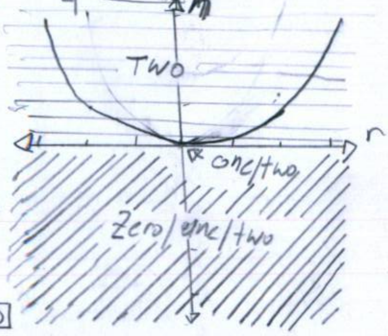


b) (r,h) Plane

$$\frac{d}{dx}(rx - x^2) = r - 2x \;;\; x_{max} = \frac{r}{2} \;;\; \frac{r^2}{2} - \frac{r^2}{4} = \frac{r^2}{4} = -h_c$$

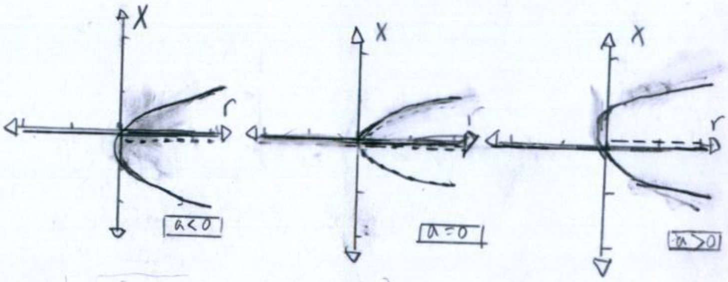
c) $\boxed{V(x,h,r) = hx + \frac{rx^2}{2} - \frac{x^3}{3}}$





$\dot{x} = rx + ax^2 - x^3$ 3.6.3 a)

a	Bifurcations
<0	one/two/three
=0	one/three
>0	one/two/three



b) (r,a) plane

$$\frac{d}{dx}(rx + ax^2 - x^3) = r + 2ax - 3x^2 = 0 \;;$$

$$r + ax - x^2 \;;\; \boxed{a = \frac{x^2 - r}{x}}$$

3.6.4 A small imperfection to a saddlenode bifurcation shifts the cusp either left or right.

$mg\sin\theta = kx\left(1 - \frac{L}{\sqrt{x^2+a^2}}\right)$ 3.6.5 a) $F = -F_{spring} = F_{g,x} = F_g \sin\theta = mg\sin\theta = k(x - x\sin\theta)$

$$= k\left(x - x \cdot \frac{L_0}{\sqrt{x^2+a^2}}\right)$$

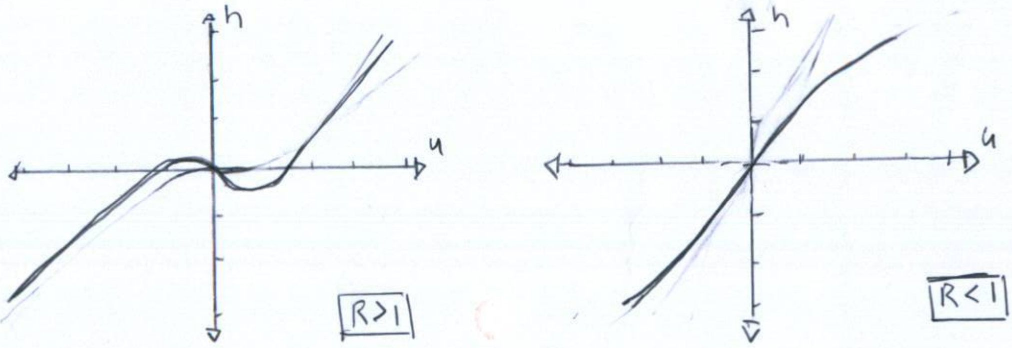
$$\boxed{= kx\left(1 - \frac{L_0}{\sqrt{x^2+a^2}}\right)}$$

b) Prove $1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}}$

If $1 - \frac{mg\sin\theta}{kx} = \frac{L_0}{a\sqrt{\left(\frac{x}{a}\right)^2 + 1}}$, then $u = \frac{x}{a}$, $R = \frac{L_0}{a}$, $h = \frac{mg\sin(\theta)}{ka}$

and $\boxed{1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}}}$

c).



The variable h, as a function of u, has a single equilibrium point for both R>1 and R<1.

d) If $r=R-1$, $1-\frac{h}{u}=\frac{r+1}{\sqrt{1+u^2}}$; $u-h=\frac{(r+1)u}{\sqrt{1+u^2}}$; $u\sqrt{1+u^2}-h\sqrt{1+u^2}=(r+1)u$

$$u\left(1+\tfrac{1}{2}u^2+O(u^4)\right)-h\left(1+\tfrac{1}{2}u^2+O(u^4)\right)=(r+1)u$$

$$\cancel{u}+\frac{u^3}{2}-h-\frac{h}{2}u^2=ru+\cancel{u}$$

$$\boxed{h+ru+\frac{h}{2}u^2-\frac{1}{2}u^3\approx 0}$$

e) $h\left(1+\frac{u^2}{2}\right)=\frac{1}{2}u^3-ru$

$\frac{d}{du}h\left(1+\frac{u^2}{2}\right)=\frac{d}{du}\left(\frac{1}{2}u^3-ru\right)$; $hu=\frac{3}{2}u^2-r$; $r_{max}\ \frac{3}{2}u^2-hu$

$h\left(1+\frac{u^2}{2}\right)=\frac{1}{2}u^3-\left(\frac{3}{2}u^2-hu\right)u$; $h+\frac{hu^2}{2}=\frac{1}{2}u^3-\frac{3}{2}u^3+hu^2$

$h\left(1-\frac{1}{2}u^2\right)=-u^3$; $h=\frac{2u^3}{u^2-2}$

$2-u^2$

$$r_{max}=\frac{3}{2}u^2-hu=\frac{3}{2}u^2-\left(\frac{2u^3}{u^2-2}\right)u$$

$$=\frac{3}{2}u^2-\frac{2u^4}{u^2-2}$$

$$=\frac{u^4+3u^2}{2(1-u^2)}\ \boxed{=R-1}$$

f) $1-\frac{h}{u}=\frac{R}{\sqrt{1+u^2}}$; $\frac{d}{du}\left(1-\frac{h}{u}\right)=\frac{d}{du}\left(\frac{R}{\sqrt{1+u^2}}\right)$; $\frac{h}{u^2}=-\frac{1}{2}\frac{R(2u)}{(1+u^2)^{3/2}}$

$2\cdot h(1+u^2)^{3/2}=-R\cdot u^3$; $R=-\frac{h(1+u^2)^{3/2}}{u^3}$

$1-\frac{h}{u}=\frac{-h(1+u^2)^{3/2}}{u^3\sqrt{1+u^2}}=-\frac{h(1+u^2)}{u^3}$; $u-h=-\frac{h(1+u^2)}{u^2}$

$u^3 - hu^2 = -h(1+u^2)$; $u^3 = -h - hu^2 + hu^2$; $h = -u^3$

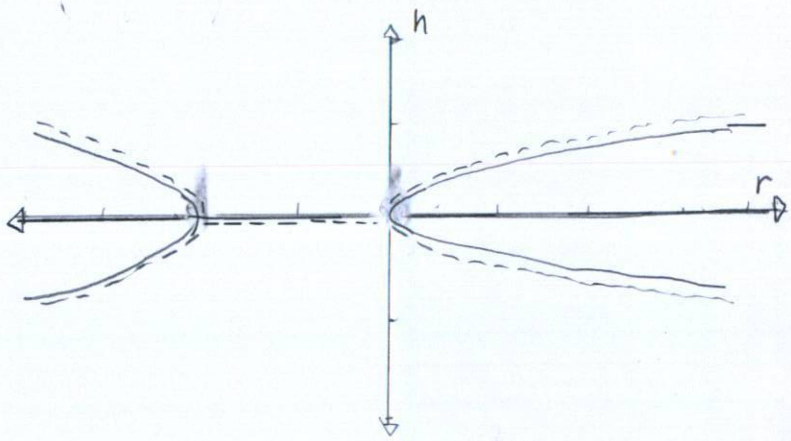
$$R = \frac{-h(1+u^2)^{3/2}}{u^3} = (1+u^2)^{3/2}$$

$$\boxed{\lim_{u\to 0} h = -u^3 \approx \frac{2u^3}{u^2-2} \approx u^3}$$

$$\boxed{\lim_{u\to 0} R = (1+u^2)^{3/2} \approx \frac{u^4+3u^2}{2(1-u^2)} + 1 = r+1}$$

g) $R = (1+u^2)^{3/2} = r+1$; $r(u) = (1+u^2)^{3/2} - 1$; $u = \sqrt{(r+1)^{2/3} - 1}$

$$\boxed{h = -u^3 = \pm\left(\sqrt{(r+1)^{2/3}-1}\right)^3}$$



h) $h = -u^3 = -\left(\frac{x}{a}\right)^3 = \frac{mg\sin\theta}{ka}$

$R = \left(1+\left(\frac{x}{a}\right)^2\right)^{3/2} = \frac{L_0}{a}$

The bifurcation plot represents the points of stability for the oscillating system.

$\tau\dot{A} = \varepsilon A - gA^3$

$\tau\dot{A} = \varepsilon A - gA^3 - kA^3$

3.6.6. $A(t)$ = Amplitude; τ = typical timescale; ε = dimensionless parameter

Supercritical: $g > 0$, subcritical: $g < 0$, $K > 0$

"Landaus Equation"

a) Landau's Equation describes the change of amplitude for a fluid system.

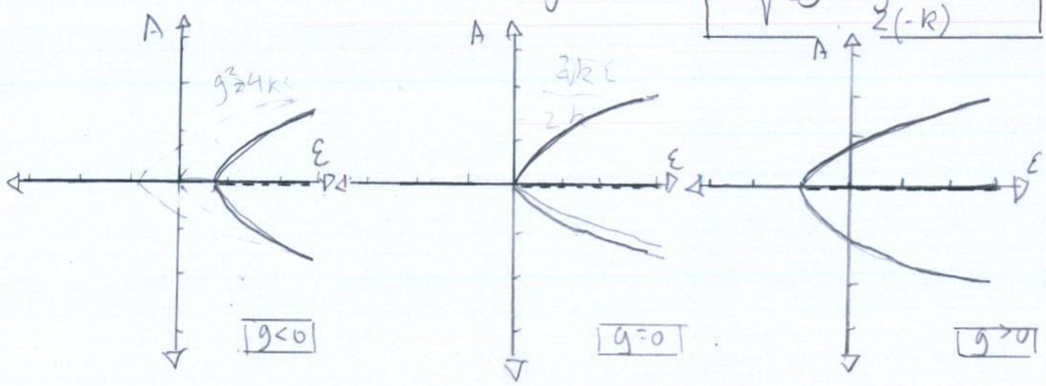
b). $\tau\dot{A} = \varepsilon A - gA^3 - kA^3$; if $g = 0$, then $\tau\dot{A} = \varepsilon A - kA^3$; $A = \sqrt{\frac{\varepsilon}{k}}$

The function $A(\varepsilon)$ is a tricritical bifurcation because $A = 0$ is a solution; in addition to, $A = +\sqrt{\frac{\varepsilon}{K}}$, and $A = -\sqrt{\frac{\varepsilon}{K}}$

c) $\tau\dot{A} = h + \varepsilon A - gA^3 - kA^5$; An approximation $h \approx 0$, $0 = \varepsilon A - gA^3 - kA^5$

$= \varepsilon - gA^2 - kA^4$

Where $A^2 = b$; $0 = \varepsilon - gb - kb^2$; $\boxed{A = \sqrt{\frac{g \pm \sqrt{g^2 - 4(-k)(\varepsilon)}}{2(-k)}}}$

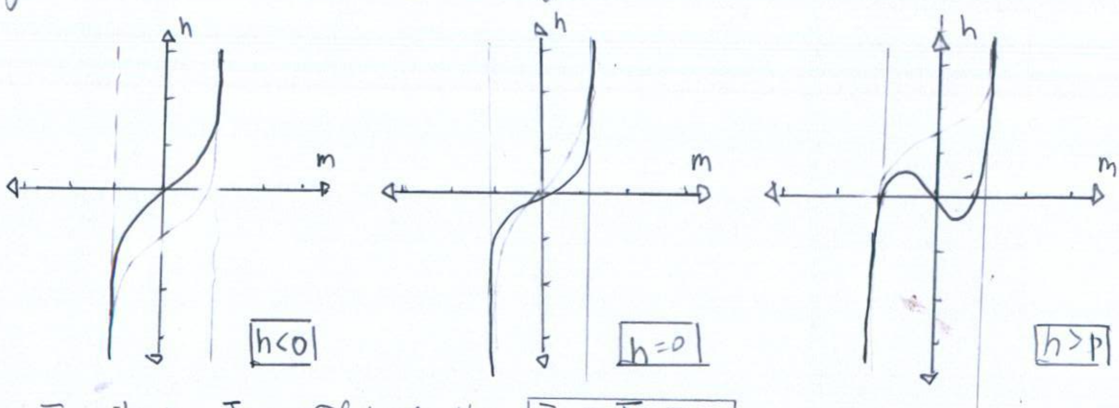


d) The graphs appearance represent the relationship of amplitude vs. time and if ϵ is large, then the first order term approaches the steadystate condition more rapidly.

$m = \left| \frac{1}{N} \sum_{i=1}^{N} S_i \right|$

$h = T \tanh^{-1}(m) - Jnm$

3.6.7. a)



b) $h = T\tanh^{-1}(m) - Jnm$; If $h=0$, then $\boxed{T_c = \frac{Jnm}{\tanh^{-1}(m)}}$

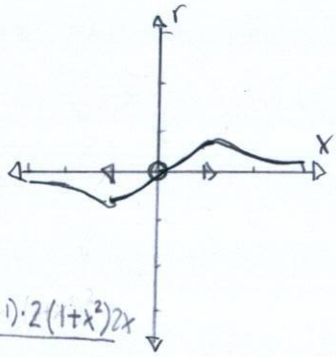
$\frac{dx}{d\tau} = rx\left(1-\frac{x}{K}\right) - \frac{x^2}{1+x^2}$ 3.7.1. @ $x^*=0$; $0 < rx - \left(\frac{1}{K}+\frac{1}{1+x^2}\right)x^2$; $\left(\frac{1}{K}+\frac{1}{1+x^2}\right)x < r$; $0<r$ is positive and unstable.

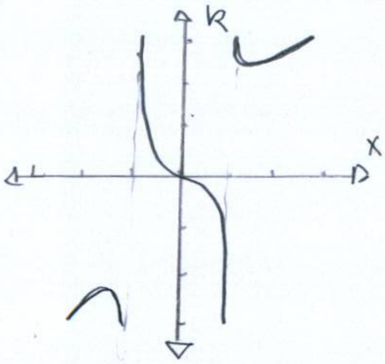
$r = \frac{2x^3}{(1+x^2)^2}$

$K = \frac{2x^3}{x^2-1}$

3.7.2. a) $\boxed{\lim_{x\to1} r = \frac{2}{4}}$; $\boxed{\lim_{x\to\infty} r = 0}$

$\boxed{\lim_{x\to1} K = -\infty}$; $\boxed{\lim_{x\to\infty} K = \infty}$





b) $r = \frac{(x^2-1)K}{(1+x^2)^2}$; $\frac{dr}{dx} = \frac{2x(1+x^2)^2 - (x^2+1)\cdot 2(1+x^2)2x}{(1+x^2)^4}$

$= \frac{2x(1+x^2)^2 - 4x^3(1+x^2) + 4x(1+x^2)}{(1+x^2)^4} = 0$

$= 2x(1+x^2) - 4x^3 + 4x = 2(1+x^2) - 4x^2 + 4 = (1+x^2) - 2x^2 + 2 = 0$

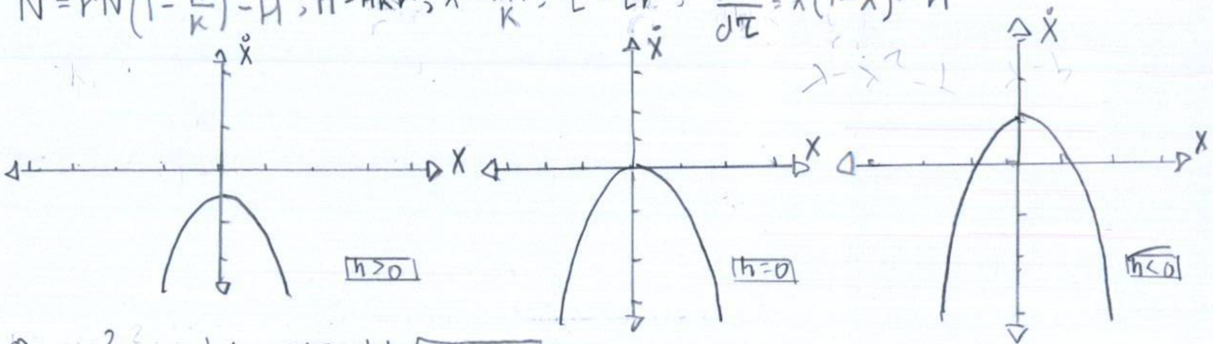
$3 = x^2$; $\boxed{x = \sqrt{3}}$

$r_{max} = \frac{(3-1)K}{(1+3)^2} = \frac{1}{8}K_{max}$; $\boxed{r_{max} = \frac{2\cdot 3^{3/2}}{(1+3)^2} = 0.6495}$; $\boxed{K_{max} = 5.1961}$

$\frac{dx}{d\tau} = x(1-x) - h$ 3.7.3. $\dot{N} = rN\left(1-\frac{N}{K}\right) - H$; $h = HKr$; $x = \frac{N}{K}$; $\tau = tr$; $\frac{dx}{d\tau} = x(1-x) - h$

a)

b)



c) $0 = -x^2 + x - h$; $x = \frac{-1 \pm \sqrt{1-4(-1)(-h)}}{2(-1)} = \frac{1 \pm \sqrt{1-4h}}{2}$; $\boxed{h_c = 0}$

d) The long-term behavior of the fish population is to reduce the total population as population rises.

$\dot{N} = rN\left(1-\frac{N}{K}\right) - H\frac{N}{A+N}$ 3.7.4. a) The variable A could represent the amount of fish in a school, and if A is large, then less fish are harvested.

b) $\boxed{x = \frac{N}{K};\ T = tr;\ h = HRK;\ a = A}$

c) $\frac{dx}{d\tau} = x(1-x) - h\frac{x}{a+x} = 0$; $x(1-x)(a+x) = (x-x^2)(a+x) = ax + x^2 - x^2a - x^3$

$$0 = (a-h)x + (1-a)x^2 - x^3$$

$$0 = (a-h) + (1-a)x - x^2$$

$$x_1 = 0,\ x_{2,3} = \frac{-(1-a) \pm \sqrt{(1-a)^2 - 4(-1)(a-h)}}{2(-1)}$$

$$= \frac{(1-a) \pm \sqrt{(1-a)^2 + 4(a-h)}}{2}$$

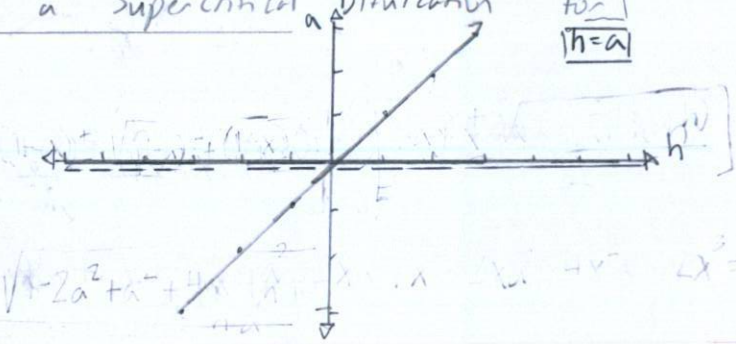
Fixed Point	$a > h$	$a < h$
0	unstable	stable
$\frac{(1-a)+\sqrt{(1-a)^2+4(a-h)}}{2}$	stable	/
$\frac{(1-a)-\sqrt{(1-a)^2+4(a-h)}}{2}$	stable	/

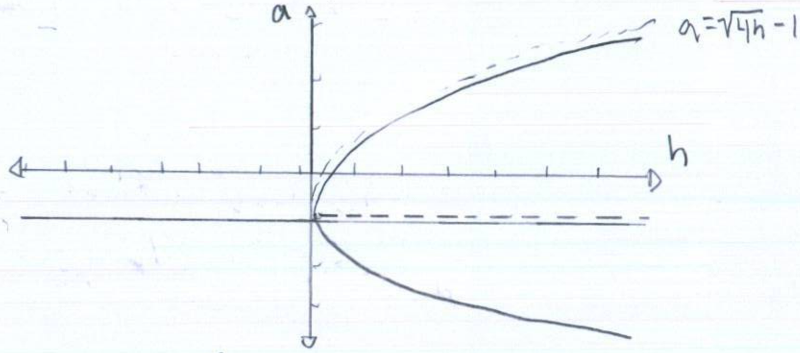
d) At $x=0$, when $h=a$, the half-node indicates a transcritical bifurcation is about to occur when h becomes less than a.

e) The graph shows a supercritical bifurcation for $h=a$

$h = \frac{(a+1)^2}{4}$.

f) $\boxed{a = \frac{h}{x-1} - x}$





$\dot{g} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2}$ 3.7.5.

a) $\frac{k_4}{k_3}\frac{dg}{dt} = \frac{k_4 \cdot k_1}{k_3}s_0 - \frac{k_4 k_2}{k_3}g + \frac{\left(\frac{g}{k_4}\right)^2}{1+\left(\frac{g}{k_4}\right)^2}$; $x = \frac{g}{k_4}$; $r = \frac{k_4 k_2}{k_3}$; $s = \frac{k_1}{k_3}s_0$

$$\boxed{\frac{dx}{d\tau} = s - rx + \frac{x^2}{1+x^2}}$$

$$\boxed{\tau = \left(\frac{k_3}{k_4}\right)t}$$

b) $0 = -rx + \frac{x^2}{1+x^2}$; $rx = \frac{x^2}{1+x^2}$; $r(1+x^2) = x$; $rx^2 - x + r = 0$

$$\boxed{x_{1,2} = \frac{1 \pm \sqrt{1-4r^2}}{2r}}$$

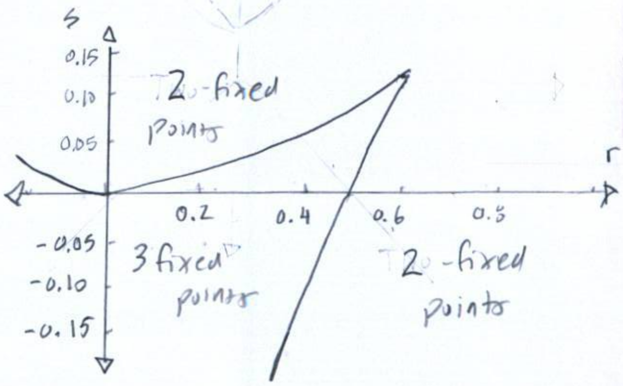
c) $g(0)=0$; $\frac{dg}{dt} = k_1 s_0 - k_2(0) + \frac{k_3(0)}{k_4^2+(0)^2} = k_1 s_0$; $g = k_1 s_0 t$; $g(t)$ increases with additional s_0.

$\boxed{\text{If } s_0 \text{ is large, then gene production has higher likelihood of rising.}}$

d) $\frac{d}{dx}\left(s - rx + \frac{x^2}{1+x^2}\right) = -r + \frac{2x}{(1+x^2)^2} = 0$; $r = \frac{2x}{(1+x^2)^2}$; $s - \left(\frac{2x}{(1+x^2)^2}\right) + \frac{x^2}{1+x^2} = 0$

$$\boxed{s = \frac{x^2(1-x^2)}{(x^2+1)^2}}$$

e) Parametric plot of (r, s)



$\dot{x} = -kxy$
$\dot{y} = kxy - ly$
$\dot{z} = ly$

3.7.6. $x(t)$ = number of healthy people
$y(t)$ = number of sick people
$z(t)$ = number of dead people.

a) $\dot{N} = \dot{x} + \dot{y} + \dot{z} = -kxy + kxy - ly + ly = 0$; therefore $N = x + y + z$.

b) $\dot{x} = -kxy$; $\dot{z} = ly$; $\dot{x} = -kx\frac{dz}{dt}\left(\frac{1}{l}\right)$; $\ln x = -\frac{kz}{l} + C$; $x(t) = Ce^{-kz/l} = \boxed{x_0 e^{-kz/l}}$

c) $\dot{z} = ly = l[N - x - z] = \boxed{l\left[N - z - x_0 e^{-kz/l}\right]}$

d) $\boxed{u = \frac{kz}{l} \; ; \; b = \frac{l}{kx_0} \; ; \; a = \frac{lN}{kx_0} \; ; \; \tau = \frac{l}{kx_0}t}$

e) $\boxed{\text{If } k, l, N, \text{ and } x_0 \text{ are positive, then both } a \text{ and } b \text{ are positive.}}$

f)

b \ a		>1
$=0$	@$u=0$, unstable	@$u<0$, unstable
>0	@$u=0$, unstable @$u>0$, stable	@$u<0$, unstable @$u>0$, stable

g) $\ddot{u} = -b\dot{u} + \dot{u}e^{-u} = \boxed{0}$; $u = -\ln b$; $\dot{u} = a - b\ln b + b$

$\ddot{z} = l\dot{y} = l(kxy - ly) = l(kx - l)y = \boxed{0}$; $y = kx$; $y = Ce^{\mp(kx-l)t}$

h) $b < 1$; $\ddot{u} = -b + ue^{-u}$; $\boxed{\text{Through plotting of } -b \text{ and } ue^{-u}\text{, at time zero, } b > ue^{-u} \text{ ; thus, } \dot{u} \text{ is increasing.}}$

t_{peak} ; $\ddot{u} = -b + ue^{-u} = 0$; $\dddot{u} = e^{-u} - u^2e^{-u} \stackrel{?}{=} 0$; $u \stackrel{?}{=} 1$

$\dot{u} = a - bu - e^{-u}$ @ $u=1$; $\dot{u} = a - b - \frac{1}{e}$; $u = (a - b - \frac{1}{e})\tau = 1$; $\boxed{t = \frac{1}{a - b - \frac{1}{e}}\left(\frac{l}{kx_0}\right)}$

$\lim_{u\to\infty} \dot{u} = \lim_{u\to\infty}\left[a - bu - e^{-u}\right] = -b\cdot\infty - \frac{1}{e^{\infty}} = \boxed{-\infty}$

i) If $b>1$, $\dot{u} = a - bu - e^{-u}$; $\ddot{u} = -b\dot{u} + ue^{-u}$ ⇒ $\dddot{u} \cdots \ddddot{u}$ does not contain a logical maximum/minimum/inflection for an epidemic with t_{peak} at zero.

j) The variable b is assigned as $\frac{l}{kx_0}$. If $b=1$, then $\frac{l}{kx_0} = 1$.

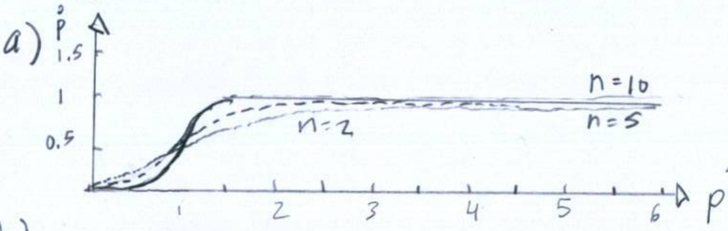
A threshold condition is when the rate of dying persons is greater than the rate of infection.

k) Autoimmunodeficiency is a disease following human immunodeficiency virus. The delayed onset from infection is time-dependent, showing that a model likely requires a time-dependent term or relationship.

$\dot{p} = \alpha + \frac{\beta p^n}{K^n + p^n} - \delta p$ 3.7.7 α = Basal Trancription Rate; β = Maximal Trancription Rate

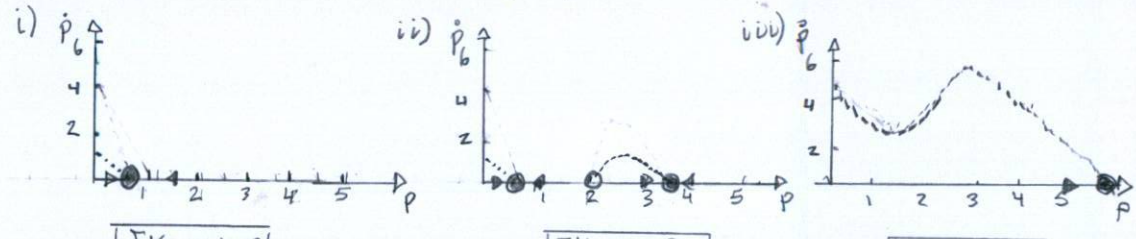
K = Activation Coefficient; δ = Decay Rate of Protein.

a)



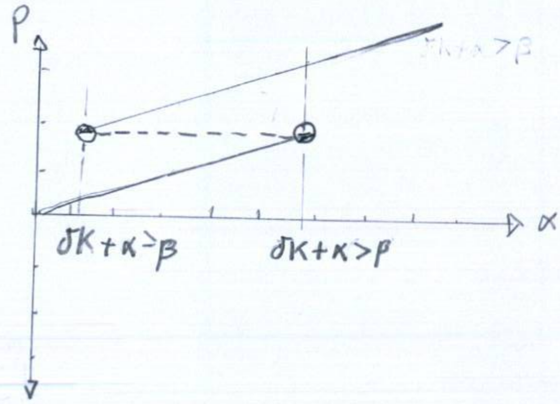
The shape of the function is a sigmoid about the point (1, 0.5) for $K=1$, $b=1$.

b)



$\delta K - \alpha > \beta$ | $\delta K - \alpha = \beta/2$ | $\delta K - \alpha < 0$

c) Assume $\delta K > \beta$, $\alpha = -\frac{\beta^n}{K^n + p^n} + \delta p$ at $\alpha \geq 0$



d) When protein levels are dependent upon α, then up till $\alpha > \delta K$, protein production rate decreases until zero. While above a threshold $\delta K - \beta > \alpha$, there is active production of further protein, proving concentration regions of protein production.

$\dot{A}_p = K_p S A + \beta \dfrac{A_p^n}{K^n + A_p^n} - K_d A_p$; A = unphosphorylated concentration ; A_p = phosphorylated concentration. ; $A_T = A + A_p$

K_p = phosphorylation rate ; K_d = dephosphorylation rate.

Assume $K = A_T/2$; $\beta = K_d A_T$

3.7.9a) $x = A_p/K$; $\tau = K_d t$; $s = K_p S/K_d$; $b = \beta/(K_d K)$

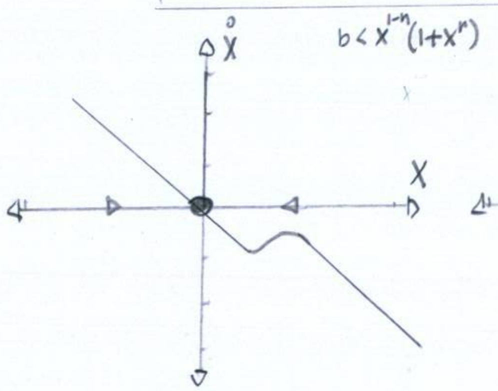
$$\cancel{K}\cancel{K_d}\frac{dx}{d\tau} = \cancel{K_d} s A + \cancel{K_d K} b \frac{K^n x^n}{\cancel{K^n} + K^n x^n} - \cancel{K_d}\cancel{K} x$$

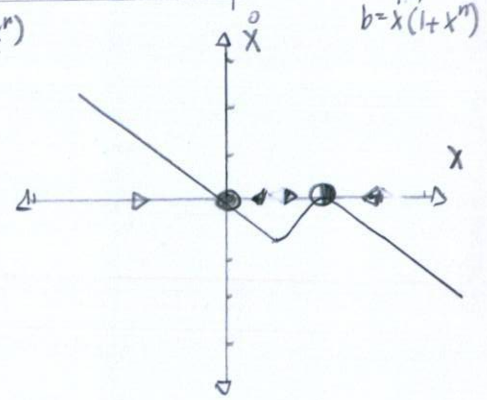
$$\frac{dx}{d\tau} = \frac{sA}{K} + b\frac{x^n}{1+x^n} - x = \frac{s(A_T - A_p)}{K} + b\frac{x^n}{1+x^n} - x$$

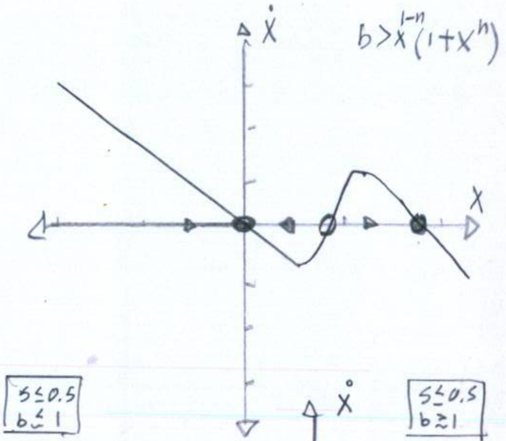
$$= \frac{s(2\cancel{K} - \cancel{K}x)}{\cancel{K}} + b\frac{x^n}{1+x^n} - x$$

$$\boxed{= s(2-x) + b\frac{x^n}{1+x^n} - x}$$

b) If $s=0$, then

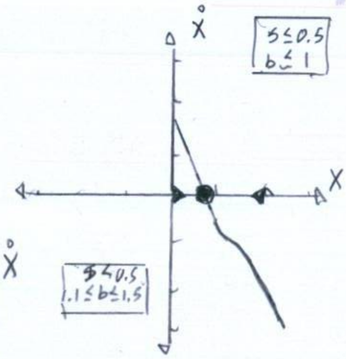


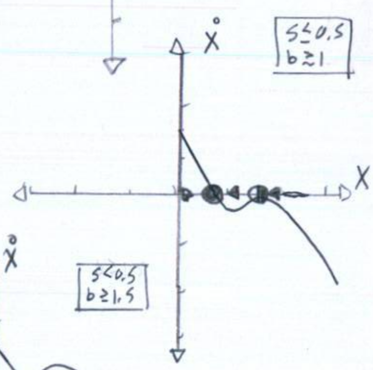


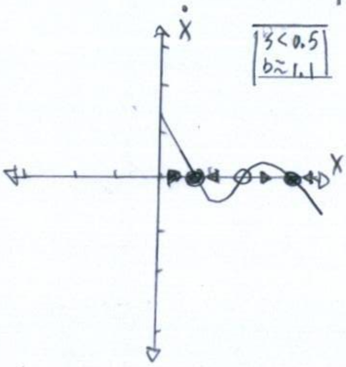


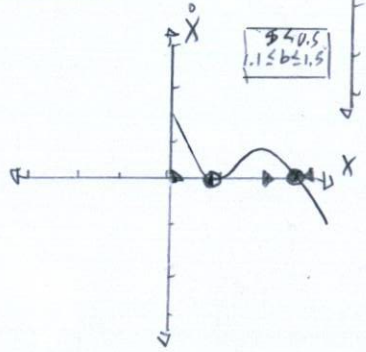
c) If $s > 0$, then a variety of bifurcations are produced.

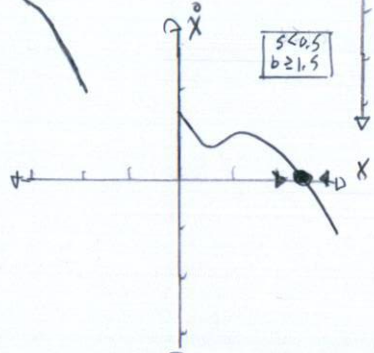
b \ s	$\lesssim 0.5$	$\gtrsim 0.5$
$b \lesssim 1$	1, stable	
$b \gtrsim 1$	2, stable Half node	
$b \cong 1.1$	3, stable unstable stable	1, stable
$1.1 \leq b \leq 1.5$	2, Half node Stable	
$b \gtrsim 1.5$	1, stable	

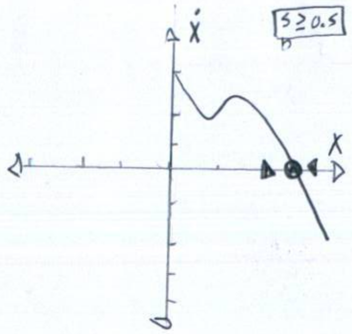






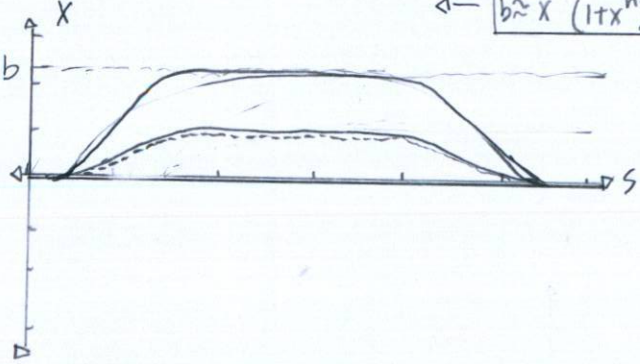






d)

Translation of bifurcation plot occurs $\leftarrow \boxed{b \approx x^{1-n}(1+x^n)} \rightarrow$ near the left box.



Incorrect

Chapter 4: Flows on the circle

$\dot{\theta} = \sin(a\theta)$ 4.1.1. The real values of a, which give a well-defined vector field, on a circle for the function, $\dot{\theta} = \sin(a\theta)$, are fixed to $n\pi$ where $n \in \mathbb{Z}$.

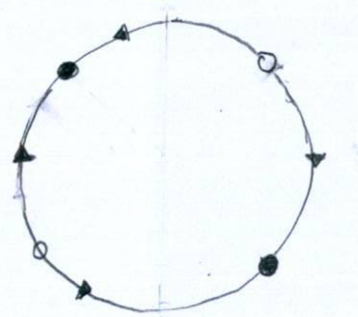
$\dot{\theta} = 1 + 2\cos\theta$ 4.1.2. $\boxed{\text{Fixed points}}$ $\theta = \cos^{-1}(-\frac{1}{2})$

$= \frac{2}{3}\pi, \frac{5}{3}\pi \cdots (n+\frac{2}{3})\pi$ "stable"

$= \frac{4}{3}\pi, \frac{7}{3}\pi \cdots (n+\frac{4}{3})\pi$ "unstable"

where $n \in \mathbb{Z}$

Phase Portrait



$\dot{\theta} = \sin 2\theta$ 4.1.3. $\boxed{\text{Fixed Points}}$ $\theta = \frac{\sin^{-1}(0)}{2}$

$= 0\pi, 1\pi, 2\pi \cdots (n\pi)$ "unstable"

$= \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi \cdots (n+\frac{1}{2})\pi$ "stable"

where $n \in \mathbb{Z}$

Phase Portrait

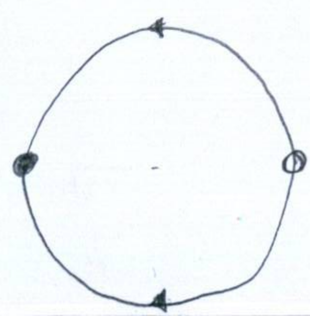
$\dot{\theta} = \sin^3\theta$ 4.1.4 $\boxed{\text{Fixed points}}$ $\theta = \sin^{-1}(0)$

$= 0\pi, 2\pi \ldots (2n)\pi$ "unstable"

$= 1\pi, 3\pi \ldots (2n+1)\pi$ "stable"

where $n \in \mathbb{Z}$

Phase Portrait:



$\dot{\theta} = \sin\theta + \cos\theta$

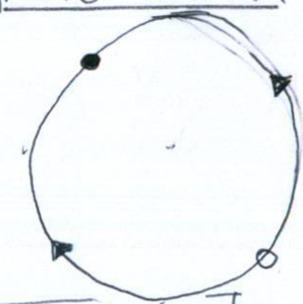
4.1.5. $\sin\theta = -\cos\theta$

$\theta = \frac{3}{4}\pi, \frac{11}{4}\pi, \frac{19}{4}\pi \ldots (n+\frac{3}{4})\pi$ "stable"

$= \frac{7}{8}\pi, \frac{15}{8}\pi, \frac{23}{8}\pi \ldots (n+\frac{7}{8})\pi$ "unstable"

where $n \in \mathbb{Z}$

Phase Portrait

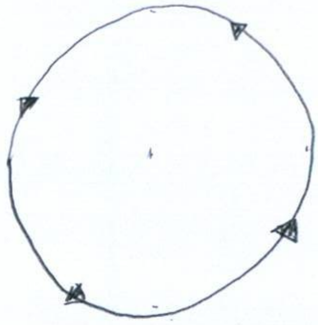


$\dot{\theta} = 3 + \cos 2\theta$

4.1.6. $\theta = \frac{\cos^{-1}(3)}{2}$

θ = undefined

Phase Portrait



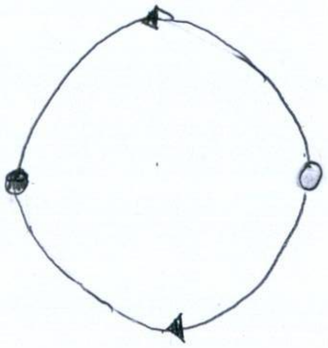
$\dot{\theta} = \sin k\theta$ where $k \in \mathbb{N}$

4.1.7. $\dot{\theta} = \sin k\theta$

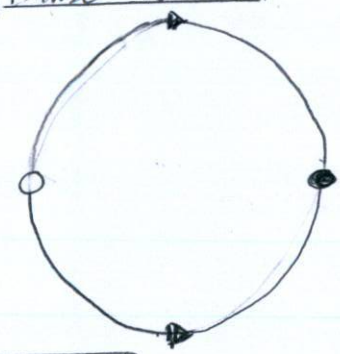
$\theta = \frac{\sin^{-1} 0}{k}$

where $k \in \mathbb{N}$

Phase Portrait



Phase Portrait



$\dot{\theta} = \cos\theta$

4.1.8. a) $\frac{-dV}{d\theta} = \frac{d\theta}{dt} = \dot{\theta} = \cos\theta \;;\; V(\theta) = -\sin\theta$

$\theta = \sin^{-1}(0)$

$= 0, 2\pi, 4\pi \ldots (2n)\pi$ "stable"

$= \pi, 3\pi, 5\pi \ldots (2n+1)\pi$ "unstable"

b) $\dot{\theta} = 1 \;;\; V(\theta) = -\theta$. The non-uniqueness of $V(\theta)$ does not imply regularity for a vector field on a circle.

c) $\dot{\theta} = f(\theta)$ has a single-valued potential for periodic functions with periodic solutions of 2π intervals.

4.1.9. Exercise 2.6.2 provided a contradiction that

$$\int_t^{t+T} f(x)\dot{x}(t)\,dt \neq \int_t^{t+T} f(x)\dot{x}(t+T)\,d(t+T)$$

Exercise 2.7.7 described a potential which could not oscillate because of the existence and uniqueness of $f(x) = \frac{d(V-c)}{dx}dx$.

Each of these arguments do not carry over to periodic solutions because another solution could be similar within $2n\pi$ ($n \in \mathbb{Z}$) intervals.

$T_{lap} = \frac{2\pi}{\omega_1 - \omega_2} = \left[\frac{1}{T_1} - \frac{1}{T_2}\right]^{-1}$ 4.2.1. $T_1 = 3\,sec$, $T_2 = 4\,sec$.

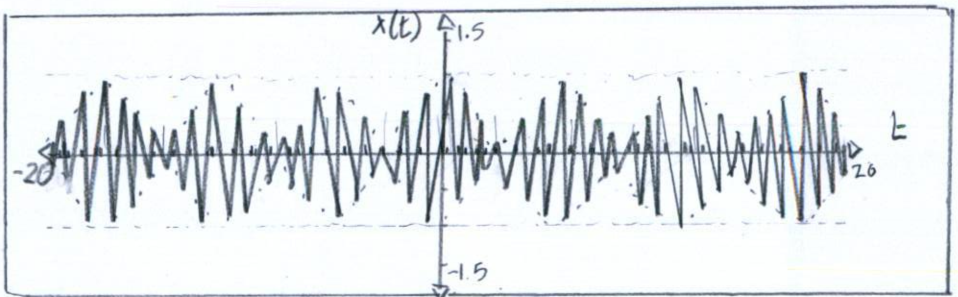
Common sense method:

# Rings	0	1	2	3	4	5
Bell #1	0	3	6	9	12	15
Bell #2	0	4	8	12	16	20

$\boxed{\text{Bell \#1 would ring four times while Bell \#2 three before ringing together again.}}$

Example 4.2.1 method: $T_{lap} = \left[\frac{1}{3sec} - \frac{1}{4sec}\right]^{-1} = \boxed{12\ sec}$.

$x(t) = \sin 8t + \sin 9t$ 4.2.2.



a) $T_{lap} = \left[\frac{1}{8} - \frac{1}{9}\right]^{-1} = \boxed{72}$

b) $x(t) = \sin 8t + \sin 9t = \boxed{2\sin\left(\frac{17}{2}t\right)\cos\left(\frac{1}{2}t\right)}$

4.2.3. 12:00pm is when long-hand angle is equal to short-hand.

$\boxed{\text{Common sense method}}$: short-hand period $[T_1] = 12$ hour

long-hand period $[T_2] = 1$ hour

$T_{lap} = \left[\frac{1}{1} - \frac{1}{12}\right]^{-1} = \frac{12}{11}$ hour

$\boxed{\text{Alternative method}}$: $x(t) = \sin(12t) + \sin(t) = 2\sin\left(\frac{11}{2}t\right)\cos\left(\frac{13}{2}t\right)$

Sum frequency — Difference frequency

$T_{bottleneck} = \int_{-\infty}^{\infty} \frac{dx}{r + x^2}$ 4.3.1. $x = \sqrt{r}\tan\theta$; $1 + \tan^2\theta = \sec^2\theta$

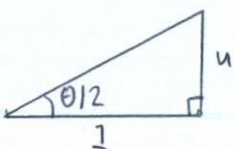
$$T = \int_{-\infty}^{\infty} \frac{dx}{r + r^2\tan^2\theta} = \frac{1}{r}\int_{-\infty}^{\infty} \frac{dx}{1 + \tan^2\theta} = \frac{1}{r}\int_{-\infty}^{\infty} \frac{dx}{\sec^2\theta}$$

$$= \frac{1}{r}\int_{-\infty}^{\infty} \frac{\sqrt{r}\sec^2\theta\, d\theta}{\sec^2\theta} = \frac{1}{\sqrt{r}}\,\theta\Big|_{-\infty}^{\infty} = \frac{1}{\sqrt{r}}\arctan\left(\frac{x}{\sqrt{r}}\right)\Big|_{-\infty}^{\infty} = \boxed{\frac{\pi}{\sqrt{r}}}$$

$T = \int_{-\pi}^{\pi} \frac{d\theta}{\omega - a\sin\theta}$ 4.3.2.

a) $u = \tan\frac{\theta}{2}$; $du = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta$; $d\theta = \frac{2\,du}{\sec^2\frac{\theta}{2}} \boxed{= \frac{2\cdot du}{\sec^2\left[\frac{\arctan^{-1}(2u)}{2}\right]}}$

b)



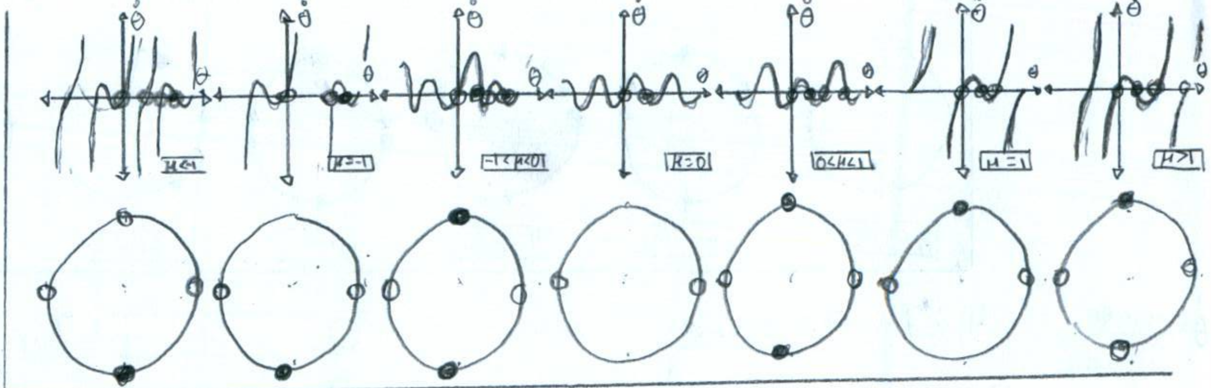
$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}} = \boxed{\frac{2u}{1+u^2}}$

c) $\lim_{\theta\to\pm\pi} \sin\theta = \boxed{\lim_{u\to ?} \frac{2u}{1+u^2}}$; for $u \neq 0$, $u \to \pm\infty$

d) $T = \frac{1}{\omega}\int_{-\infty}^{\infty} \frac{1}{1 - \frac{a}{\omega}\left[\frac{2u}{1+u^2}\right]} \cdot \frac{2\cdot du}{\sec^2\left[\frac{\arctan^{-1}(2u)}{2}\right]} = \boxed{\frac{1}{\omega}\int_{-\infty}^{\infty} \frac{1}{1 - \frac{a}{\omega}\left[\frac{2u}{1+u^2}\right]} \cdot \frac{2\,du}{\left[\frac{1}{1+u^2} - \frac{u^2}{1+u^2}\right]^2}}$

e) $T = \frac{1}{\omega}\int^{\infty} \frac{1}{1 - \frac{a}{\omega}[}$

$\frac{1}{(1+u^2)} - \frac{2u^2}{(1+u^2)} + \frac{u^4}{(1+u^2)}$



$r^{a-b}\frac{du}{d\tau}=r+r^{2a}u^2$ 4.3.9. $T_{bottleneck}\sim O(r^{-1/2})$

a) $O(r^a)$; $x=r^a u$, where $u\sim O(1)$. $t=r^b\tau$, with $\tau\sim O(1)$

$\dot{x}=r+x^2=r+(r^a u)^2=r+r^{2a}u^2$; $r^{a-b}\frac{du}{d\tau}=\boxed{r+r^{2a}u^2}$

b) $\boxed{r^{a-b}=r=r^{2a}\ ;\ a=\frac{1}{2}\ ;\ b=-\frac{1}{2}}$

$\dot{x}=r+x^{2n}$ 4.3.10. $x=r^a u$; $t=r^b\tau$; $r^{a-b}\frac{du}{d\tau}=r+r^{2an}u^{2an}$; $a=\frac{1}{2}$; $b=-\frac{1}{2}$;

$\frac{du}{d\tau}=1+r^n u^n$; $r^b\tau=$

$mL^2\ddot{\theta}+b\dot{\theta}+mgL\sin\theta=\Gamma$ 4.4.1. $\ddot{\theta}=0$ -or- $\ddot{\theta}<<1$; $t=T\tau$; $\frac{mL^2}{T^2}\frac{d^2\theta}{d^2\tau}+\frac{b}{T}\frac{d\theta}{d\tau}+mgL\sin\theta=\Gamma$

$$\frac{L}{gT^2}\frac{d^2\theta}{d^2\tau}+\frac{b}{mgL}\frac{d\theta}{d\tau}+\sin\theta=\frac{\Gamma}{mgL}$$

$$\frac{b}{mgL\Gamma}=1\ ;\ T=\frac{b}{mgL}$$

$$\frac{m^2gL^3}{b^2}\frac{d^2\theta}{d\tau^2}+\frac{d\theta}{d\tau}+\sin(\theta)=\frac{\Gamma}{mgL}$$

$$\boxed{m^2gL^3<<b^2}$$

$\theta'=\gamma-\sin\theta$ 4.4.2 $\int\frac{d\theta}{\omega-\sin\theta}=dt$; $t=-\int\frac{d\theta}{\frac{2\tan(\frac{\theta}{2})}{\tan^2(\frac{\theta}{2})+1}-a}=-2\int_{-\pi}\frac{du}{au^2-2u+a}$; where $u=\tan(\frac{\theta}{2})$

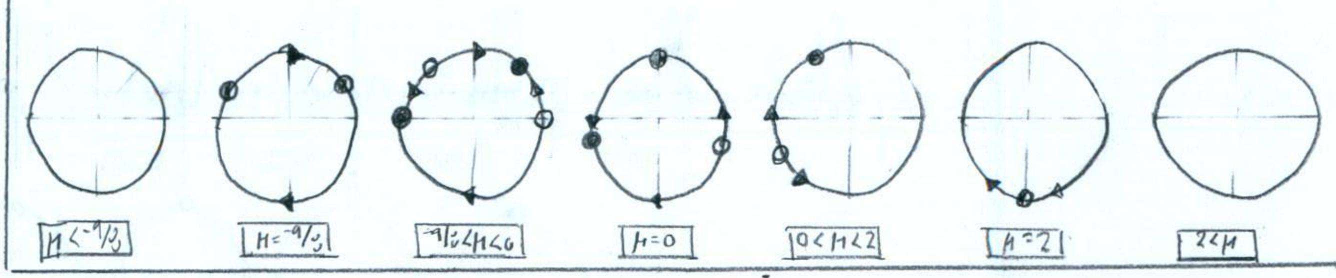
$du=\frac{\sec^2(\frac{\theta}{2})d\theta}{2}$

$d\theta=\frac{1}{u^2+1}$

$=-2\int_{-\pi}^{\pi}\frac{du}{(\sqrt{a}u-\frac{1}{\sqrt{a}})^2+a-\frac{1}{a}}$; where $v=\frac{au-1}{\sqrt{a}\sqrt{a-1/a}}$

$du=\frac{\sqrt{a-1/a}}{\sqrt{a}}dv$

$$\boxed{=\frac{-2}{\sqrt{a}\sqrt{a-\frac{1}{a}}}\int_{-\pi}^{\pi}\frac{dv}{v^2+1}=\frac{2\arctan\left(\frac{a\tan(\frac{\theta}{2})-1}{\sqrt{a}\sqrt{a-1/a}}\right)}{\sqrt{a}\sqrt{a-1/a}}+C}$$



$\dot{\theta} = \dfrac{\sin\theta}{\mu + \sin\theta}$

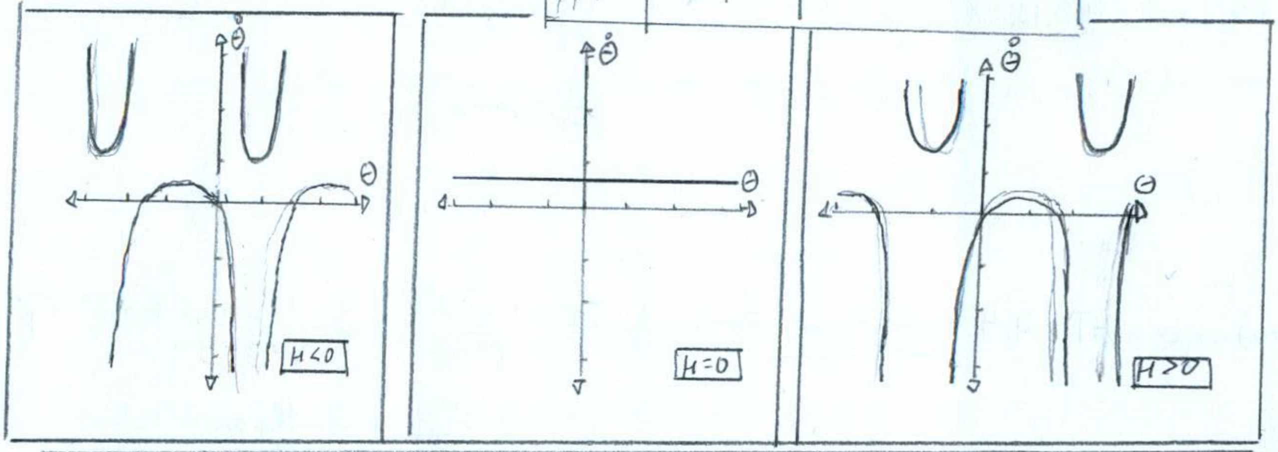
4.3.7 $0 = \dfrac{\sin\theta}{\mu + \sin\theta}$

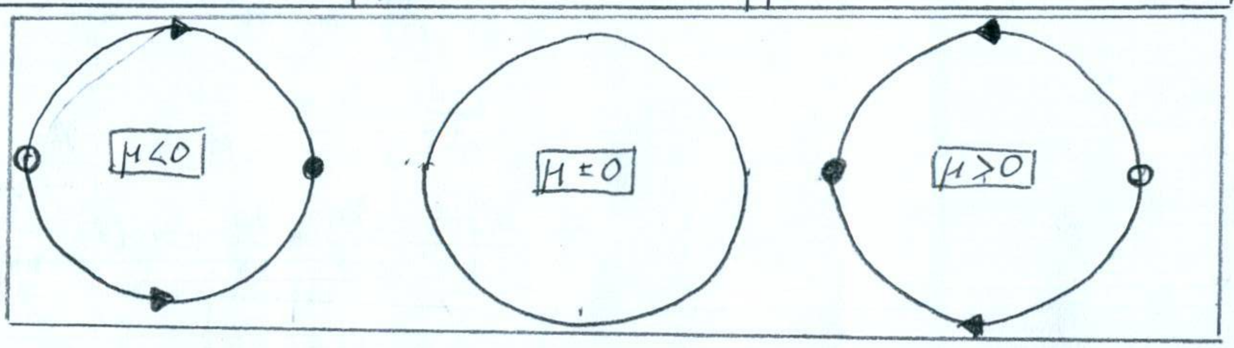
$0 = \mu + \sin\theta$

$\mu = -\sin\theta$

$\theta = \sin^{-1}(-\mu)$

Θ	μ	Bifurcations
0, π	<0	Two
N/A	$=0$	Zero
0, π	>0	Two





$\dot{\theta} = \dfrac{\sin 2\theta}{1 + \mu\sin\theta}$

4.3.8. $0 = \dfrac{\sin 2\theta}{1 + \mu\sin\theta}$

$0 = 1 + \mu\sin\theta$

$\boxed{\theta = \sin^{-1}\left(\dfrac{-1}{\mu}\right)}$

Θ	μ	Bifurcations
0, $\pi/2$, π, $3\pi/2$	<-1	Four
0, π, $3\pi/2$	$=-1$	Three
0, $\pi/2$, π, $3\pi/2$	$-1<\mu<0$	Four
0, $\pi/2$, π	$=0$	Three
0, $\pi/2$, π, $3\pi/2$	$0<\mu<1$	Four
0, $\pi/2$, π	$=1$	Three
0, $\pi/2$, π, $3\pi/2$	>1	Four

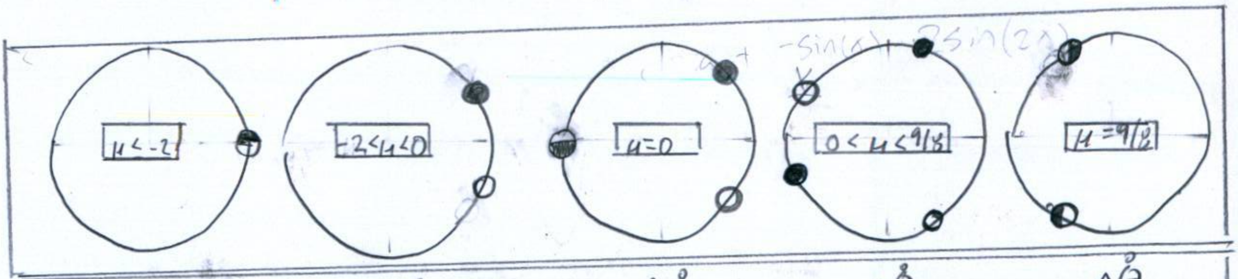
$\dot{\theta} = \mu + \cos\theta + \cos 2\theta$ 4.3.5. $0 = (\mu - 1) + \cos\theta + 2\cos^2\theta$

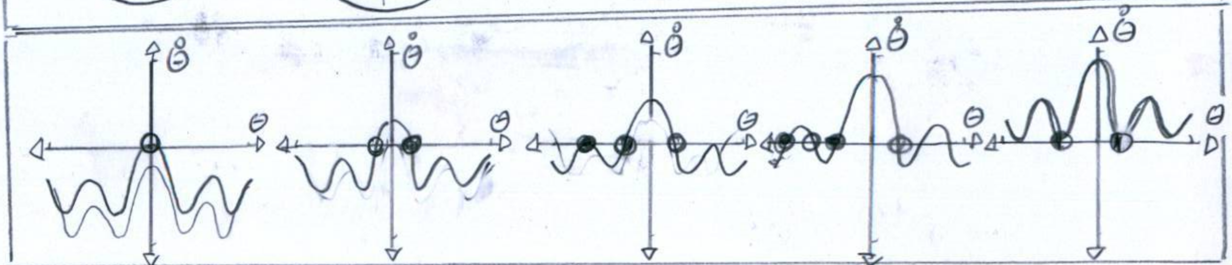
$$\cos\theta = \frac{-1 \pm \sqrt{1 - 4(2)(\mu - 1)}}{2(2)}$$

$$= \frac{-1 \pm \sqrt{9 - 8\mu}}{4}$$

$$\theta = \cos^{-1}\left(\frac{-1 \pm \sqrt{9 - 8\mu}}{4}\right)$$

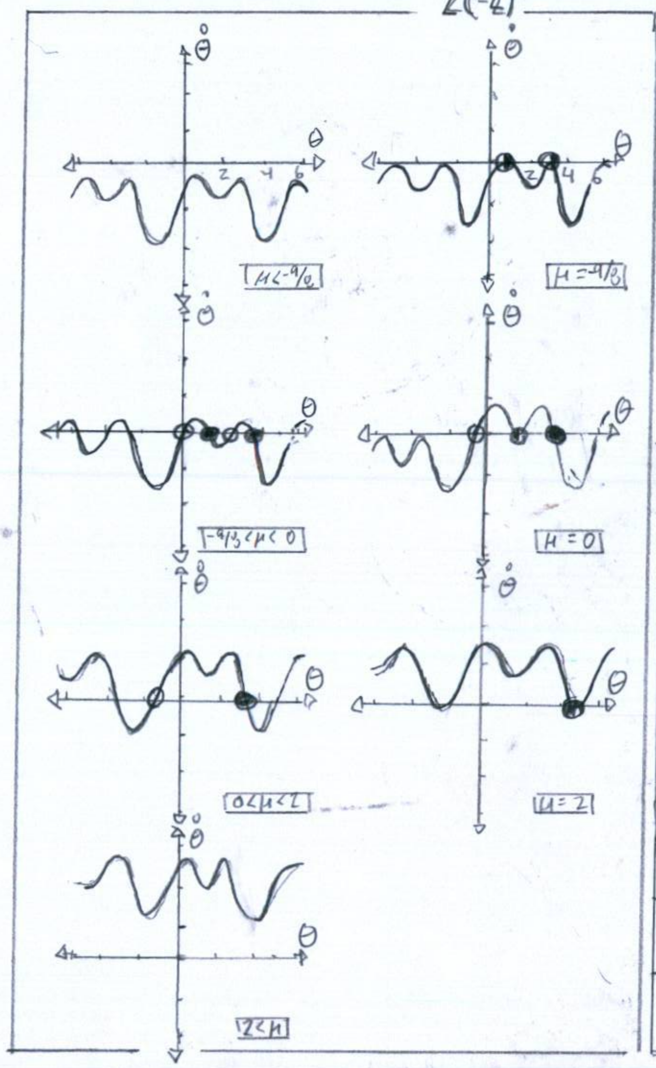
μ	Bifurcations
≤ -2	One
$-2 < \mu < 0$	Two
$= 0$	Three
$0 < \mu < 9/8$	Four
$9/8$	Two





$\dot{\theta} = \mu + \sin\theta + \cos 2\theta$ 4.3.6. $\dot{\theta} = (\mu + 1) + \sin\theta - 2\sin^2\theta$

$$\sin\theta = \frac{-1 \pm \sqrt{1 - 4(-2)(\mu + 1)}}{2(-2)}$$

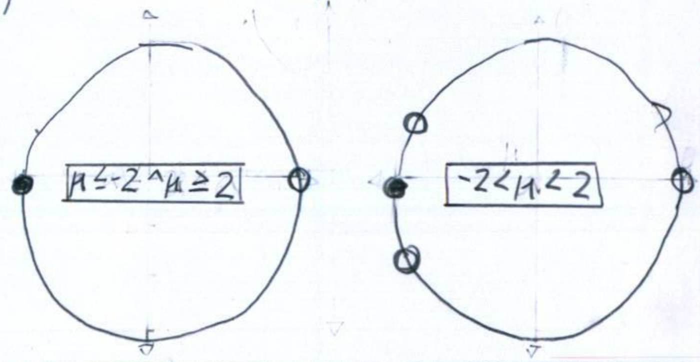


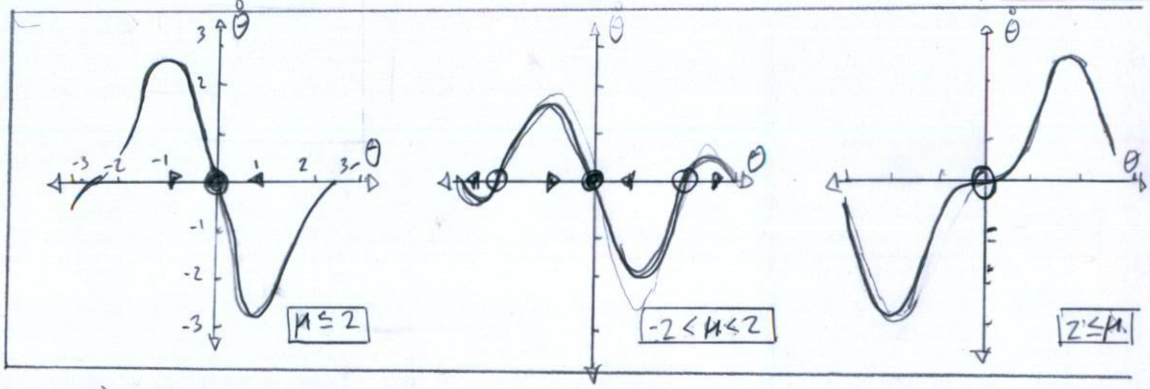
θ	μ	Bifurcations.
NA	$< -9/8$	Zero
$\arcsin(1/4)$, $\arcsin(\pi - 1/4)$	$= -9/8$	Two
$\arcsin(1/4)$ to 2π	$-\frac{9}{8} < \mu < 0$	Four
$\pi/2$, $\frac{7}{6}\pi$, $\frac{11}{6}\pi$	$= 0$	Three
$\frac{7}{6}\pi < \theta < \frac{11}{6}\pi$	$0 < \mu < 2$	Two
$3\pi/2$	$= 2$	One
NA	$2 < \mu$	Zero

$\dot{\theta} = \mu\sin\theta - \sin 2\theta$ 4.3.3. $\mu = \dfrac{\sin 2\theta}{\sin\theta} = 2\cos x$; $x = \cos^{-1}\left(\dfrac{\mu}{2}\right)$

Phase Portrait: Saddle-node Bifurcation

μ	Bifurcations
≤ -2	Two
$-2 \leq \mu \leq 2$	Four
≥ 2	Two





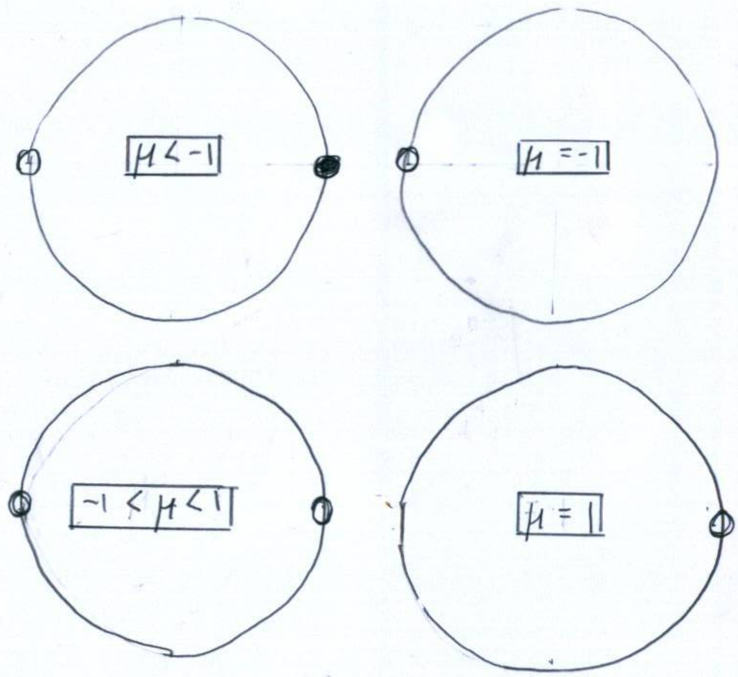
$\dot{\theta} = \dfrac{\sin\theta}{\mu + \cos\theta}$ 4.3.4. $0(\mu + \cos\theta) = \sin\theta$

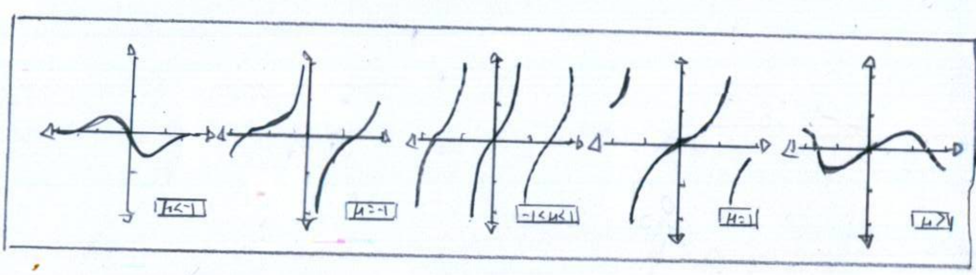
$\mu = -\cos\theta$

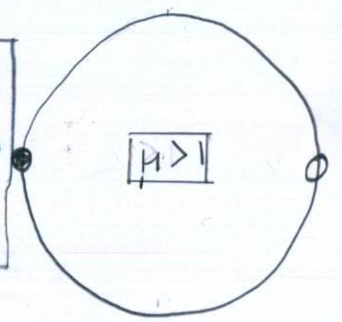
$\theta = \cos^{-1}(-\mu)$

Phase Portrait Transcritical Bifurcation

μ	Bifurcations
< -1	Two
$= -1$	one
$-1 \leq \mu < 1$	Two
$= 1$	one
> 1	Two







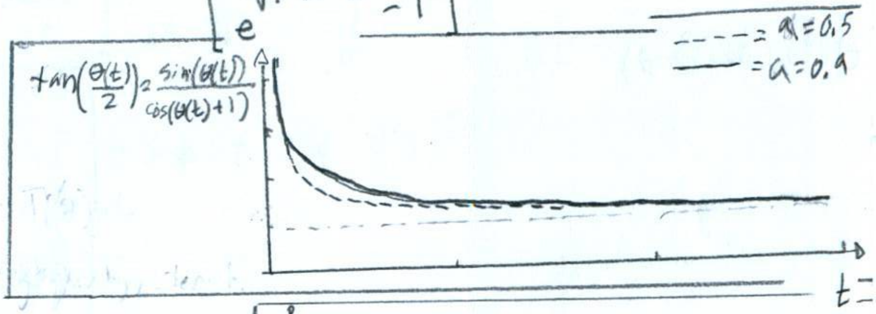
$$t = \ln\left(\frac{\left|\frac{a\sin(\theta)}{\cos\theta+1} + \frac{-2\sqrt{1-a^2}-2}{2}\right|}{\left|\frac{a\sin(\theta)}{\cos(\theta)+1} + \frac{2\sqrt{1-a^2}-2}{2}\right|}\right) \Big/ \sqrt{1-a^2}$$

$$e^{\sqrt{1-a^2}\cdot t} \circ \left|\frac{a\sin(\theta)}{\cos(\theta)+1} + \frac{2\sqrt{1-a^2}-2}{2}\right| = \left|\frac{a\sin(\theta)}{\cos(\theta)+1} - \frac{2\sqrt{1-a^2}-2}{2}\right|$$

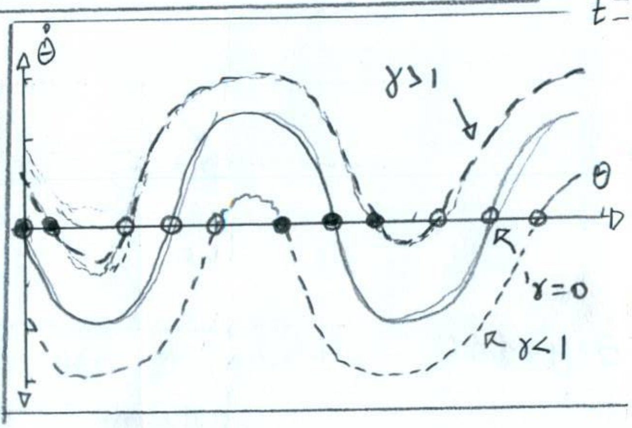
$$\frac{a\sin(\theta)}{\cos(\theta)+1}\left[e^{\sqrt{1-a^2}\,t} - 1\right] = \frac{-2\sqrt{1-a^2}-2}{2}\left[e^{\sqrt{1-a^2}\cdot t} + 1\right]$$

$$\frac{\sin(\theta)}{\cos(\theta)+1} = \frac{(-2\sqrt{1-a^2}-2)}{2a} \circ \frac{\left[e^{\sqrt{1-a^2}\,t}+1\right]}{\left[e^{\sqrt{1-a^2}\,t}-1\right]}$$

A graph of $\sin(\theta(t))$ vs t represented as $\tan\left(\frac{\theta(t)}{2}\right)$.



4.4.3. $\mathring{\theta} = \gamma - \sin(\theta(t))$



Similar to Question 4.4.3, γ is the relationship of T/mgL. If torque(T) is greater than mass x gravity x length, then motion is directed; moreover, if torque(T) is zero, then there is no direction.

$b\mathring{\theta} + mgL\sin\theta = T - k\theta$ 4.4.4.

a. $\boxed{\theta = \{0, \frac{\pi k}{T}, \frac{2\pi k}{T}, \dots, \frac{n\pi k}{T}\}}$

b. $b\mathring{\theta} + mgL\sin\theta = T - k\theta$

$\frac{b}{mgL}\mathring{\theta} + \sin\theta = \frac{T - k\theta}{mgL}$; if $\tau = \frac{mgL}{b}t$; $\gamma = \frac{T}{mgL}$ $\boxed{;\ \mu = +\frac{k}{mgL}}$

$\theta' + \sin\theta = \gamma + \mu\theta$; $\theta' = \gamma - \mu\theta - \sin\theta$

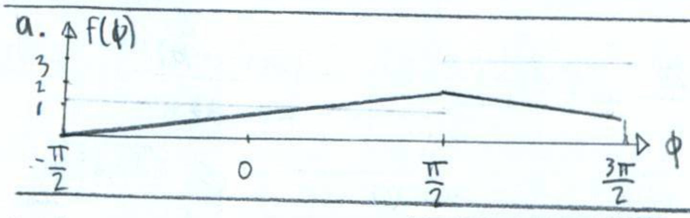
c. As the pendulum angle increases, the dampening lowers rate of angle change (θ').

d. As K is varied from 0 to ∞, then θ' is equal to zero at $\frac{\gamma - \sin\theta}{\mu}$. The bifurcation type is subcritical.

$\dot{\theta} = \omega + A f(\Theta - \theta)$ 4.5.1. $f(\phi) = \begin{cases} \phi & -\frac{\pi}{2} \le \phi \le \frac{\pi}{2} \\ \pi - \phi & \frac{\pi}{2} \le \phi \le \frac{3\pi}{2} \end{cases}$

$\dot{\Theta} = \Omega$

a.



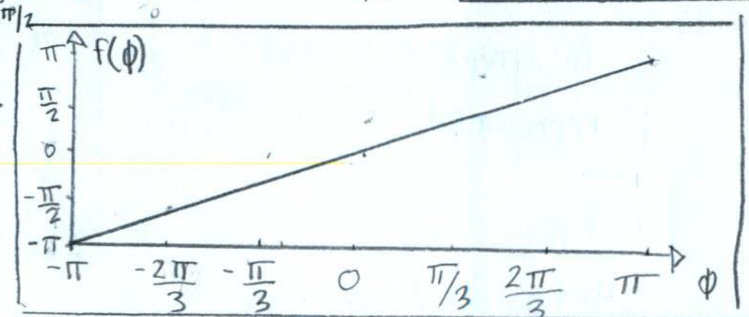
b. Range of Entrainment: $\boxed{-\frac{\pi}{2} \le f(\phi) \le \frac{\pi}{2}}$

c. $\phi^* = \dot{\Theta} - \dot{\theta} = \Omega - \omega - A\left[\frac{\pi}{2}\right] \cdot \boxed{|\Omega - \omega| \le \pi/2}$

d. $T_{drift} = \int_{0/2}^{2\pi} \frac{d\phi}{\Omega - \omega - A[\pi - \phi]} = \frac{1}{A}\int_{-\pi/2}^{\frac{3\pi}{2}} \frac{du}{u} = \frac{2}{A}\ln\left(\frac{\Omega - \omega - A\pi/2\pi}{\Omega - \omega + A\pi/2}\right) = \boxed{\frac{2}{A}\ln\left(\frac{\Omega - \omega - A\pi/2}{\Omega - \omega + A\pi/2}\right)}$

$\dot{\theta} = \omega + A f(\Theta - \theta)$ 4.5.2

$f(\phi) = \phi \quad -\pi < \phi < \pi$



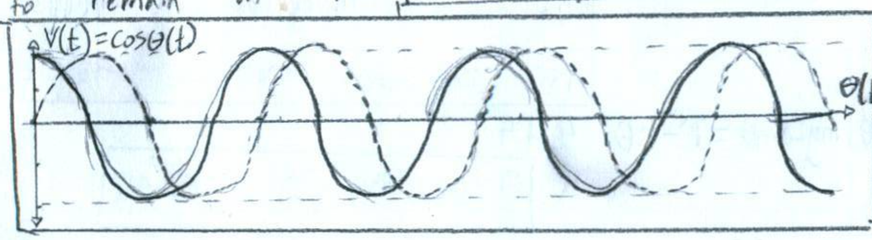
Range of Entrainment: $-\pi < f(\phi) < \pi$

$\phi^* = \dot{\Theta} - \dot{\theta} = \Omega - \omega - A[\pi] : |\Omega - \omega| \le \pi$

$T_{drift} = \int_0^{2\pi} \frac{d\phi}{\Omega - \omega - A[\phi]} = \frac{1}{A}\int_{\pi}^{\pi} \frac{du}{u} = \boxed{\frac{1}{A}\ln\left(\frac{\Omega - \omega + A[\pi]}{\Omega - \omega - A[\pi]}\right)}$

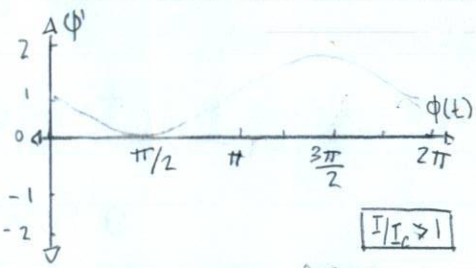
$\dot{\theta} = \mu + \sin\theta$ 4.5.3. a) The 'rest state' and 'threshold' are described by the neurons ability to remain at rest, or fire, respectively.

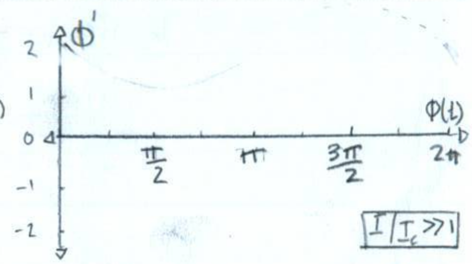
b) $V(t) = \cos\theta(t)$



4.6.1 $\beta = 0$

a. $\phi' = \frac{I}{I_c} - \sin\phi(t)$





$t = \int \frac{d\phi}{I/I_c - \sin\phi}$

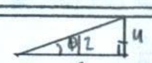
$= -\int \frac{d\phi}{\sin\phi - I/I_c}$

$= -\frac{I_c}{I}\int \frac{du}{\left[\frac{2u}{1+u^2}\left(\frac{I}{I_c}\right) - 1\right]} \frac{2}{\sec^2\left(\frac{\phi}{2}\right)} = -\frac{I_c}{I}\int \frac{du}{\left[\frac{2u}{1+u^2}\left(\frac{I}{I_c}\right) - 1\right]} \frac{2}{[u^2+1]}$

Method #1: $u = \tan\left(\frac{\phi}{2}\right)$; $du = \frac{\sec^2\left(\frac{\phi}{2}\right) d\phi}{2}$

$= -\frac{2I_c}{I}\int \frac{du}{2u\left(\frac{I}{I_c}\right) - u^2 - 1} = \boxed{+\frac{2I_c}{I}\int \frac{du}{\left(u - \frac{I}{I_c}\right)^2 - \left(\frac{I}{I_c}\right)^2 + 1}}$

Method #2:



$\sin\phi = 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}$

$= 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}} = \frac{2u}{1+u^2}$

where $v = \frac{(u - I/I_c)^2}{\sqrt{1 - I/I_c}}$

$$= 2\frac{I}{I_c}\int \frac{\sqrt{1-(I/I_c)^2}}{(1-(I/I_c)^2)v^2-(I/I_c)^2+1}\,dv = \frac{2I/I_c}{\sqrt{1-(I/I_c)^2}}\int\frac{dv}{v^2+1} = \frac{1}{\sqrt{1-(I/I_c)^2}}\arctan(v)$$

$$= 2\left(\frac{I}{I_c}\right)\frac{\arctan\left(\frac{u-I/I_c}{\sqrt{1-(I/I_c)^2}}\right)}{\sqrt{1-(I/I_c)^2}} = 2\frac{I}{I_c}\frac{\arctan\left(\frac{\tan\frac{\phi}{2}-I/I_c}{\sqrt{1-(I/I_c)^2}}\right)}{\sqrt{1-(I/I_c)^2}} + C \text{ ; where } C=0$$

$$t = \frac{I}{I_c}\ln\left(\frac{1-\frac{\tan\phi/2+I/I_c}{\sqrt{1-(I/I_c)^2}}}{1+\frac{\tan\phi/2-I/I_c}{\sqrt{1-(I/I_c)^2}}}\right)\Big/\sqrt{(I/I_c)^2-1}$$

$$e^{(I_c/I)\sqrt{(I/I_c)^2-1}\cdot t}\circ\left(1+\frac{\tan\phi/2-(I/I_c)}{\sqrt{1-(I/I_c)^2}}\right) = 1-\frac{\tan\phi/2+I/I_c}{\sqrt{1-(I/I_c)^2}}$$

$$\frac{\tan\phi/2-(I/I_c)}{\sqrt{1-(I/I_c)^2}}\left[e^{\frac{I_c}{I}\sqrt{(I/I_c)^2-1}\circ t}+1\right] = 1-e^{\frac{I_c}{I}\sqrt{(I/I_c)^2-1}\circ t}$$

$$\sin(\phi/2) = \cos\left(\frac{\phi}{2}\right)\left[\frac{\left(1-e^{I_c/I\sqrt{(I/I_c)^2-1}\cdot t}\right)}{\left(1+e^{I_c/I\sqrt{(I/I_c)^2-1}\circ t}\right)}\sqrt{1-(I/I_c)^2}+I/I_c\right]$$

$$\sin\phi = 2\cos^2\left(\frac{\phi}{2}\right)\left[\sqrt{1-(I/I_c)^2}\,\frac{\left(1-e^{I_c/I\sqrt{(I/I_c)^2-1}\cdot t}\right)}{\left(1+e^{I_c/I\sqrt{(I/I_c)^2-1}\cdot t}\right)}+I/I_c\right]$$

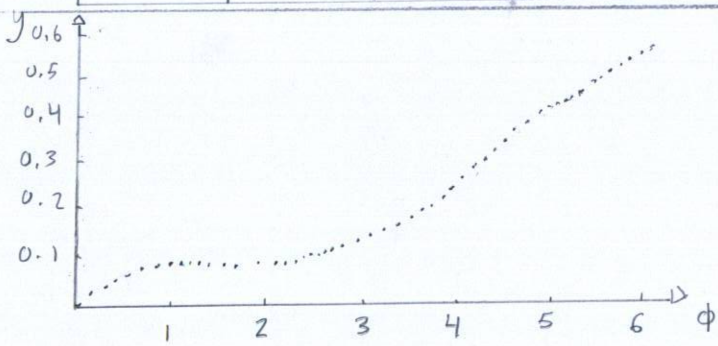
$\phi' = \frac{I}{I_c} - \sin\phi$

4.6.2. Numerical Integration : Runge-Kutta 4th Order

ϕ	k_1	k_2	k_3	k_4
0.0	$\Delta h \cdot f(\phi)$	$\Delta h \cdot f(\phi + \frac{\Delta h}{2})$	$\Delta h \cdot f(\phi + \frac{\Delta h}{2})$	$\Delta h \cdot f(\phi + \Delta h)$
...	...	...	...	...
6.0	0.1379	0.1331	0.1331	0.1282

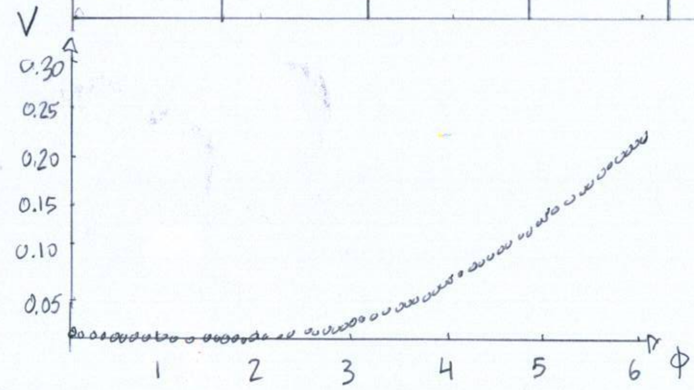
$y_{n+1} = y_n + \frac{\Delta h}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

where $\Delta h = 0.1$



ϕ	k_1	k_2	k_3	k_4
0.0	$\Delta t \cdot \Delta h f(\phi)$	$\Delta t \Delta h \cdot f(\phi + \frac{\Delta h}{2})$	$\Delta t \cdot \Delta h \cdot f(\phi + \frac{\Delta h}{2})$	$\Delta t \cdot \Delta h f(\phi + \Delta h)$
...	...	...	...	...
6.0	0.63	-0.0103	-0.0103	-0.0098

$V_{n+1} = V_n + \frac{\Delta h}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

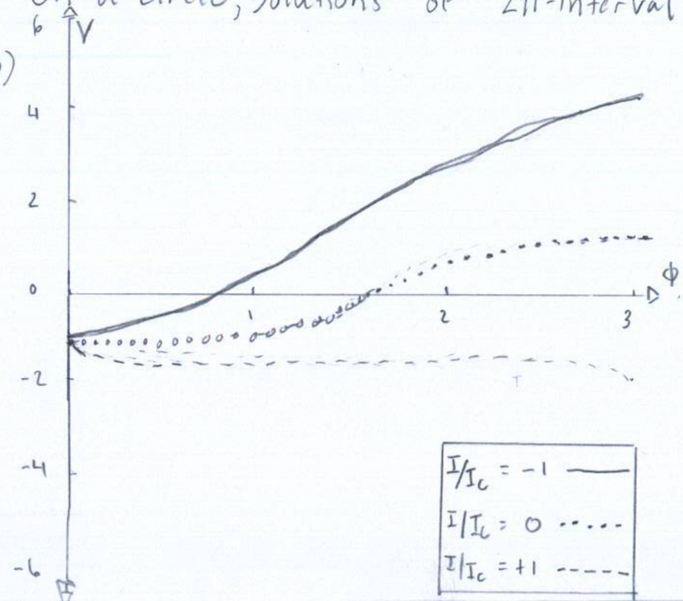


4.6.3.

a) $V = -\dot{x}\,dx$; $V = -\dot{\phi}\,d\phi = -[I/I_c - \sin\phi]\,d\phi = -[\cos\phi + \frac{I}{I_c}\phi]$

On a circle, solutions of 2π-interval exist : $\phi = \arcsin(I/I_c)$

b)

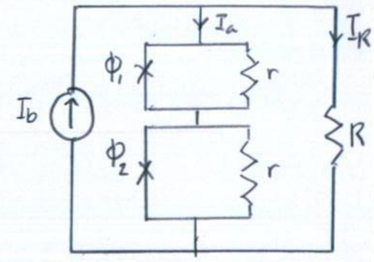


c) The increase of current (I) lowers the potential (V) per 2π oscillation.

$I_a = I_c \sin\phi_1 + V_1/r$

$I_b = I_c \sin\phi_K + \frac{\hbar}{2er}\dot{\phi}_K + \frac{\hbar}{2eR}(\dot{\phi}_1 + \dot{\phi}_2)$

4.6.4. a) $I_b = I_a + I_R$



b) Kirchoff's Law: Parallel Circuit

$$I_a = I_{a1} + I_{ar} \quad ; \quad I_a = I_{a2} + I_{ar}$$
$$= I_a \sin\phi_1 + \frac{V_1}{r} \qquad \boxed{I_a \sin\phi_2 + \frac{V_2}{r}}$$

c) If $k = 1, 2$, then $\boxed{V_k = \begin{cases} \frac{\hbar}{2eR}\dot{\phi}_1 \\ \frac{\hbar}{2eR}\dot{\phi}_2 \end{cases}}$

d) $I_b = I_{a1} + I_{1ar} + I_{2ar} + I_R = I_c \sin\phi_k + \frac{\hbar}{2eR}\dot{\phi}_1 + \frac{\hbar}{2eR}\dot{\phi}_2 + \frac{V_k}{R}$

$$\boxed{= I_c \sin\phi_k + \frac{\hbar}{2eR}[\dot{\phi}_1 + \dot{\phi}_2] + \frac{\hbar}{2eR}\dot{\phi}_k}$$

where $k = 1, 2$.

e) $I_b = I_c \sin(\phi_K) + \frac{\hbar}{2er}\dot{\phi}_K + \frac{\hbar}{2eR}\sum_{i=1}^{N}\dot{\phi}_i$

$$= I_c \sin(\phi_K) + \frac{h}{2er}\dot{\phi}_K + \frac{Nr}{R}\left(I_c \sin(\phi_K) + \frac{\hbar}{2er}\dot{\phi}_K - I_c\sum_{i=1}^{N}\sin(\phi_i)\right)$$

$$= \left(1 + \frac{Nr}{R}\right)I_c \sin(\phi_K) + \left(\frac{h}{2er} + \frac{N\hbar}{2eR}\right)\dot{\phi}_K - \frac{r}{R}I_c\sum_{i=1}^{2}\sin(\phi_j)$$

$$\hbar\left(\frac{1}{2er} + \frac{N}{2eR}\right)\dot{\phi}_K = I_b - \left(1 + \frac{Nr}{R}\right)I_c \sin(\phi_K) + \frac{r}{R}I_c\sum_{i=1}^{2}\sin(\phi_j)$$

$$\frac{\hbar(R+Nr)}{2erR}\dot{\phi}_K = I_b - \frac{R+Nr}{R}I_c \sin(\phi_K) + \frac{r}{R}I_c\sum_{i=1}^{2}\sin(\phi_j)$$

$$\dot{\phi}_K = \frac{2erR I_b}{\hbar(R+Nr)} - \frac{2er}{\hbar}I_c \sin(\phi_K) + \frac{2er^2}{\hbar(R+Nr)}I_c\sum_{i=1}^{2}\sin(\phi_j)$$

$$\boxed{\Omega = \frac{2erR I_b}{\hbar(R+Nr)} \; ; \; a = \frac{-2er}{\hbar}I_c \; ; \; K = \frac{2er^2 I_c}{\hbar(R+Nr)}}$$

4.6.5 $I_b = I_c \sin(\phi_K) + \frac{\hbar}{2er}\dot{\phi}_K + \frac{\hbar}{2eR}\sum\dot{\phi}_i$

$$= I_c \sin(\phi_K) + \frac{\hbar}{2er}\dot{\phi}_K + \frac{Nr}{R}\left(I_c \sin(\phi_K) + \frac{\hbar}{2er}\dot{\phi}_K - I_c\sum_{i=1}^{N}\sin(\phi_i)\right)$$

$$= \left(1 + \frac{Nr}{R}\right)I_c \sin\phi_K + \left(\frac{h}{2er} + \frac{N\hbar}{2eR}\right)\dot{\phi}_K - \frac{r}{R}I_c\sum_{i=1}^{2}\sin(\phi_j)$$

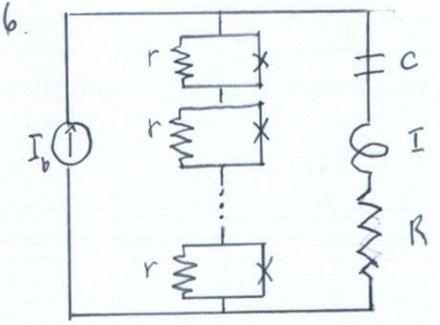
$$\hbar\left(\frac{1}{2er} + \frac{N}{2eR}\right)\dot{\phi}_K = I_b - \frac{R+Nr}{R}I_c \sin\phi_K + \frac{r}{R}I_c\sum_{i=1}^{2}\sin(\phi_j)$$

$$\frac{\hbar(R+Nr)}{2erR}\dot{\phi}_K = I_b - \frac{R+Nr}{R}I_c \sin\phi_K + \frac{r}{R}I_c\sum_{i=1}^{2}\sin(\phi_j)$$

$$\frac{\hbar(R+Nr)}{2Ner^2 I_c}\dot{\phi}_K = \frac{R I_b}{Nr I_c} - \frac{R+Nr}{Nr}\sin\phi_K + \frac{1}{N}\sum_{i=1}^{2}\sin\phi_j$$

$$\frac{d\phi_K}{d\tau} = \Omega + a\sin\phi_K + \frac{1}{N}\sum_{j=1}^{2}\sin\phi_j \; ; \; \Omega = \frac{R I_b}{Nr I_c} \; ; \; \boxed{a = \frac{-(R+Nr)}{Nr} \; ; \; \tau = \frac{2Ner^2 I_c}{\hbar(R+Nr)}t}$$

$\dot\phi = \Omega + a\sin\phi_k + K\sum_{j=1}^{2}\sin\phi_j$ 4.6.6



$$\frac{\hbar}{2er}\cdot\frac{d\phi_k}{dt} + I_c\sin\phi_k + \frac{dQ}{dt} = I_b$$

$$L\frac{d^2Q}{d^2t} + R\frac{dQ}{dt} + \frac{Q}{C} = \boxed{\frac{\hbar}{2e}\sum_{j=1}^{N}\frac{d\phi_j}{dt}}$$

Chapter 5: Linear Systems

$\dot x = v$
$\dot v = -\omega^2 x$

5.1.1. a. $\frac{\dot x}{\dot v} = \frac{dx}{dv} = \frac{v}{-\omega^2 x}$; $-\omega^2 x + C = v$; $\boxed{\omega^2x^2 + v^2 = C}$

b. Conservation of Energy: $\sum \frac{1}{2}mv^2 = E$; $\frac{1}{2}m\omega^2x^2 + \frac{1}{2}mv^2 = C$

$$\boxed{KE_{rot} + KE_{Lin} = KE_{TOT.}}$$

$\dot x = ax$
$\dot y = -y$

5.1.2 $\frac{\dot y}{\dot x} = \frac{dy}{dx} = \frac{-y}{ax} = \frac{-e^{-t}}{ae^{at}} = \frac{-1}{ae^{(a+1)t}}$; $\lim_{t\to\infty}\frac{dy}{dx} = \lim_{t\to\infty}\frac{-1}{ae^{(a+1)t}} = \boxed{-\infty}$ ∥ y-axis

$\lim_{t\to-\infty}\frac{dy}{dx} = \lim_{t\to-\infty}\frac{-1}{ae^{(a+1)t}} \boxed{= 0}$ ∥ x-axis.

$\dot x = -y$
$\dot y = -x$

5.1.3. $\begin{bmatrix}\dot x\\ \dot y\end{bmatrix} = \boxed{\begin{bmatrix}-1\\ -1\end{bmatrix}\begin{bmatrix}x & y\end{bmatrix}}$

$\dot x = 3x - 2y$
$\dot y = 2y - x$

5.1.4 $\begin{bmatrix}\dot x\\ \dot y\end{bmatrix} = \boxed{\begin{bmatrix}3 & -2\\ -1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}}$

$\dot x = 0$
$\dot y = x + y$

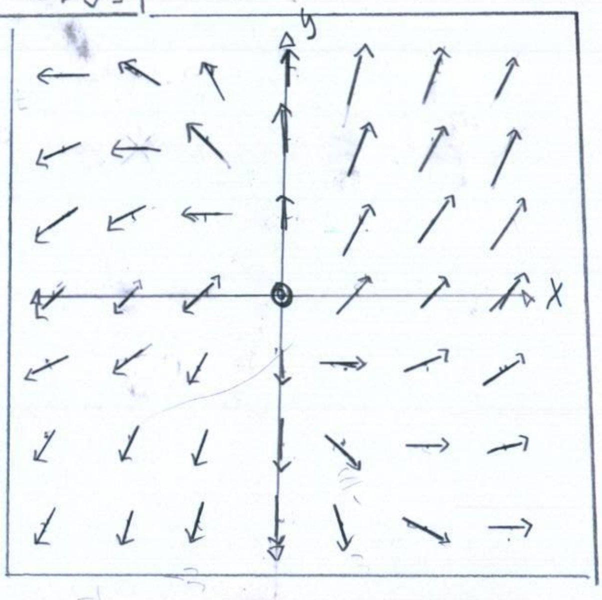
5.1.5. $\begin{bmatrix}\dot x\\ \dot y\end{bmatrix} = \boxed{\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}}$

$\dot x = x$
$\dot y = 5x + y$

5.1.6. $\begin{bmatrix}\dot x\\ \dot y\end{bmatrix} = \boxed{\begin{bmatrix}1 & 0\\ 5 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}}$

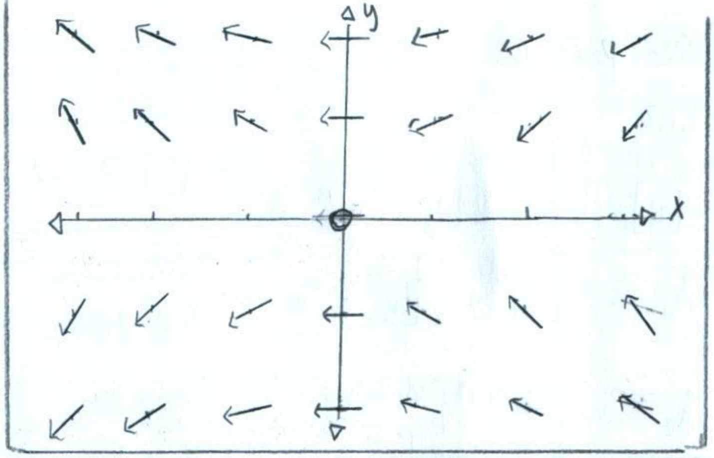
$\dot x = x$
$\dot y = x + y$

5.1.7. $\frac{dy}{dx} = \frac{x+y}{x}$



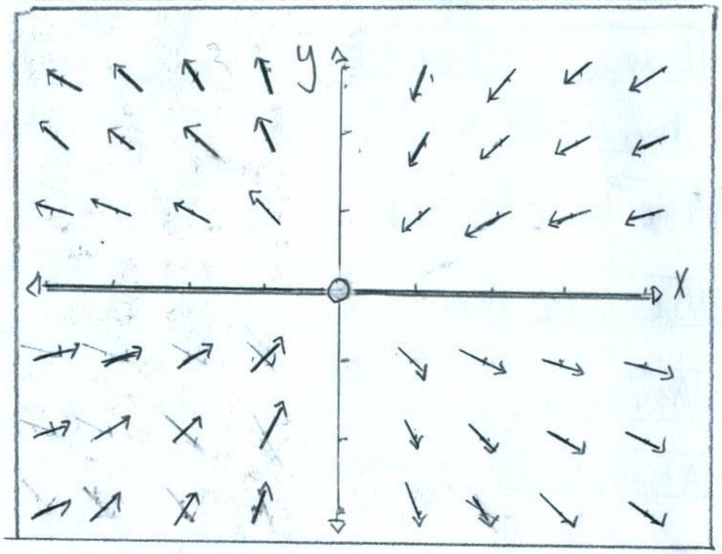
$\dot{x} = -2y$ 5.1.8. $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{-x}{-2y} = \frac{x}{2y}$.

$\dot{y} = -x$



$\dot{x} = -y$ 5.1.9 a) $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{y}{x}$

$\dot{y} = -x$



b) $x\dot{x} = -xy$; $y\dot{y} = -xy$; therefore, $x\dot{x} = y\dot{y}$; $x\dot{x} - y\dot{y} = 0$

and $\boxed{x\,dx - y\,dy = x^2 - y^2 = C}$

c) $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$; $\begin{pmatrix}-\lambda & -1\\ -1 & -\lambda\end{pmatrix} = \lambda^2 + 1 = (\lambda+1)(\lambda-1) = 0$; $\lambda_1 = 1$, $\lambda_2 = -1$

$\lambda_1 = 1$; $\begin{pmatrix}+1 & -1\\ -1 & +1\end{pmatrix}\begin{pmatrix}v_{11}\\ v_{12}\end{pmatrix} = 0$; Guess $v_{11} = 1$, $v_{12} = -1$; $x = -c_1 e^{t}$; $y = c_2 e^{t}$

$\lambda_2 = -1$; $\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}v_{11}\\ v_{12}\end{pmatrix} = 0$; Guess $v_{11} = 1$, $v_{12} = 1$; $x = c_1 e^{-t}$; $y = c_2 e^{-t}$

General Solution: $x(t) = c_1 e^{-t} - c_1 e^{t}$; $y(t) = c_2 e^{-t} + c_2 e^{t}$

$\lim_{t\to\infty} x(t) = -\infty$; $\lim_{t\to-\infty} x(t) = \infty$: $\boxed{\text{Unstable Manifold}}$

$\lim_{t\to\infty} y(t) = \infty$; $\lim_{t\to-\infty} y(t) = \infty$: $\boxed{\text{Stable Manifold}}$

d) $u = x + y$; $\dot{u} = \dot{x} + \dot{y} = -y - x = -u$; $\boxed{u(t) = u_0 e^{-t}}$

$v = x - y$; $\dot{v} = \dot{x} - \dot{y} = x - y = v$; $\boxed{v(t) = v_0 e^{t}}$

e) $\lim_{t\to\infty} u(t) = 0$; $\lim_{t\to-\infty} u(t) = \infty$; $\lim_{t\to\infty} v(t) = \infty$; $\lim_{t\to-\infty} v(t) = 0$;

$\boxed{\text{Stable}}$ $\boxed{\text{Arbitrary}}$ $\boxed{\text{Arbitrary}}$ $\boxed{\text{unstable}}$

f) See part C.

5.1.10

$\dot{x}=y;\ \dot{y}=-4x$

a) $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}=\begin{bmatrix}0 & 1\\ -4 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix};\ \begin{bmatrix}-\lambda & 1\\ -4 & -\lambda\end{bmatrix}=\lambda^2+4=0;\ \lambda_1=\pm 2i;\ \lambda_2=\pm 2i$

$\lambda_1=2i;\ \begin{bmatrix}-2i & 1\\ -4 & -2i\end{bmatrix}\begin{bmatrix}v_{11}\\ v_{12}\end{bmatrix}=0;\ \begin{matrix}-2iv_{11}+v_{12}=0\\ -4v_{11}-2i\,v_{12}=0\end{matrix};\ v_{11}=1;\ v_{22}=2i$

$\lambda_2=-2i;\ \begin{bmatrix}2i & 1\\ -4 & 2i\end{bmatrix}\begin{bmatrix}v_{11}\\ v_{12}\end{bmatrix}=0;\ \begin{matrix}2v_{11}+v_{12}=0\\ -4v_{11}+2v_{12}=0\end{matrix};\ v_{11}=1;\ v_{22}=2i$

$\boxed{\text{Liapunov Stable}}$

Identity: $e^{\lambda t}=\cos(t)+i\sin(t)$

$e^{\lambda t}V=e^{\lambda t}\begin{bmatrix}v_{11}\\ v_{12}\end{bmatrix}=\begin{bmatrix}\cos(2t)+i\sin(2t)\\ 2i(\cos(2t)+i\sin(2t))\end{bmatrix}$

$X_1=\begin{bmatrix}x=C_2\cos(2t)+C_1\sin(2t)\\ y=2C_2\cos(2t)-2C_1\sin(2t)\end{bmatrix}$

$X_i=C_1\mathrm{Re}(e^{\lambda t}V_1)+C_2\,\mathrm{Im}(e^{\lambda t}V_1)$

$\dot{x}=2y;\ \dot{y}=x$ b) $\boxed{\text{None of the Above}}$

$\dot{x}=0,\ \dot{y}=x$ c) $\boxed{\text{None of the Above}}$

$\dot{x}=0,\ \dot{y}=-y$ d) $\boxed{\text{None of the Above}}$

$\dot{x}=-x;\ \dot{y}=-5y$ e) $\boxed{\text{Asymptotically stable}}$

$\dot{x}=x;\ \dot{y}=y$ f) $\boxed{\text{Asymptotically stable}}$

5.1.11 a) $\|x(t)=C\cos(2t)+C\sin(2t)-x^*\|<C^2=\epsilon$

$\|x(0)=C-x^*\|<C+\delta$

b) $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}=\begin{bmatrix}0 & 2\\ 1 & 0\end{bmatrix}\begin{bmatrix}v_{11}\\ v_{12}\end{bmatrix};\ \begin{bmatrix}-\lambda & 2\\ 1 & -\lambda\end{bmatrix}=\lambda^2-2=0;\ \lambda_1=+\sqrt{2};\ \lambda_2=-\sqrt{2}$

$\lambda_1=+\sqrt{2};\ \begin{bmatrix}-\sqrt{2} & 2\\ 1 & -\sqrt{2}\end{bmatrix}\begin{bmatrix}v_{11}\\ v_{12}\end{bmatrix}=0;\ \begin{matrix}-\sqrt{2}v_{11}+2v_{12}=0;\\ v_{11}-\sqrt{2}v_{12}=0;\end{matrix}\ v_{11}=1;\ v_{12}=\frac{1}{\sqrt{2}}$

$\lambda_2=-\sqrt{2};\ \begin{bmatrix}\sqrt{2} & 2\\ 1 & \sqrt{2}\end{bmatrix}\begin{bmatrix}v_{11}\\ v_{12}\end{bmatrix}=0;\ \sqrt{2}v_{11}+2v_{12}=0;\ v_{11}=1;\ v_{12}=-\frac{1}{\sqrt{2}}$

$x(t)=C_1\cosh(t)+C_2\sinh(t/\sqrt{2});\ y(t)=x_0\cosh(t)+C_4\sinh(-t/\sqrt{2})$

$=x_0\cosh(t)+\frac{y_0}{\sqrt{2}}\sinh(t/\sqrt{2});\ y(t)=x_0\cosh(t)-\frac{y_0}{\sqrt{2}}\sinh(-t/\sqrt{2})$

$\|x(t)-x^*\|=\|x_0\cosh(t)+\frac{y_0}{\sqrt{2}}\sinh(t/\sqrt{2})-0\|\,!=\epsilon$

$\|x(0)-x^*\|=\|x_0\|<\delta$ $\boxed{\text{None of the above}}$

c. $\dot{x}=0$; $x = 1+C$; $\dot{y}=x$; $\boxed{\text{None of the Above}}$

$= C$; $y = x_0 t + C$

$= x_0$; $= x_0 t + y_0$

d. $\dot{x}=0$; $x = 1+C$; $\dot{y} = x$; $\boxed{\text{None of the above}}$

$= C$; $= x_0 t + C$

$= x_0$; $= x_0 t + y_0$

e. $\dot{x}=-x$; $\dot{y}=-5y$; $\boxed{\text{Asymptotically Stable}}$

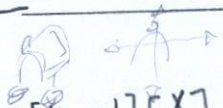
$x = x_0 e^{-t}$; $y = y_0 e^{-5t}$

f. $\dot{x}=x$; $\dot{y}=y$; $\boxed{\text{Asymptotically Stable}}$.

$x = e^{t}$; $y = e^{+t}$

$\dot{x}=v$; $\dot{v}=-x$ 5.1.12 v-axis @ $(0,-v_0)$; x-axis @ $(x,0)$; $\boxed{\dot{v}(0) = -v = v_0 \;;\; \dot{x}(x) = 0}$

5.1.13 The "saddle point" is a category of bifurcation that is parabolic beyond a coordinate. A connection to real saddles is the "curved" shape where the rider sits



$\dot{x}=4x-y$
$\dot{y}=2x+y$

5.2.1 a. $\dot{x} = A\vec{x}$; $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}4 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$; $\begin{bmatrix}4-\lambda & -1\\ 2 & 1-\lambda\end{bmatrix} = (4-\lambda)(1-\lambda)+2 = \lambda^2-5\lambda+6=0$

$\lambda_1 = 2$; $\lambda_2 = 3$

$\lambda_1 = 2$; $\begin{bmatrix}2 & -1\\ 2 & -1\end{bmatrix}\begin{bmatrix}v_{11}\\ v_{12}\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$; $2v_{11} - v_{12} = 0$; $v_{11} = 1$; $v_{12} = 2$; $\vec{v}_1 = \begin{bmatrix}1\\ 2\end{bmatrix}$

$\lambda_2 = 3$; $\begin{bmatrix}1 & -1\\ 2 & -2\end{bmatrix}\begin{bmatrix}v_{21}\\ v_{22}\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$; $v_{21} - v_{22} = 0$; $v_{21} = 1$; $v_{22} = 1$; $\vec{v}_2 = \begin{bmatrix}1\\ 1\end{bmatrix}$

b) General Solution: $X(t) = \begin{bmatrix}x(t)\\ y(t)\end{bmatrix} = C_1 e^{\lambda_1 t} V_1 + C_2 e^{\lambda_2 t} V_2 = C_1 e^{2t}\begin{bmatrix}1\\ 2\end{bmatrix} + C_2 e^{3t}\begin{bmatrix}1\\ 1\end{bmatrix}$

$$\boxed{x(t) = C_1 e^{2t} + C_2 e^{3t}}$$

$$\boxed{y(t) = 2C_1 e^{2t} + C_2 e^{3t}}$$

c) $\boxed{\text{Stable}}$

d) $(x_0, y_0) = (3,4)$; $3 = C_1 + C_2$; $4 = 2C_1 + C_2$

$C_1 = 3 - C_2$; $4 = 2(3-C_2) + C_2 = 6 - 2C_2 + C_2$

$= 6 - C_2$; $C_2 = 2$; $C_1 = 1$

$$\boxed{x(t) = e^{2t} + 2e^{3t}}$$

$$\boxed{y(t) = 2e^{2t} + 2e^{3t}}$$

$\dot{x} = x - y$
$\dot{y} = x + y$

5.2.2. a) $X = Ax$; $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$; $\begin{bmatrix}1-\lambda & -1\\ 1 & 1-\lambda\end{bmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2$

$\lambda_1 = 1 - i$; $\lambda_2 = 1 + i$

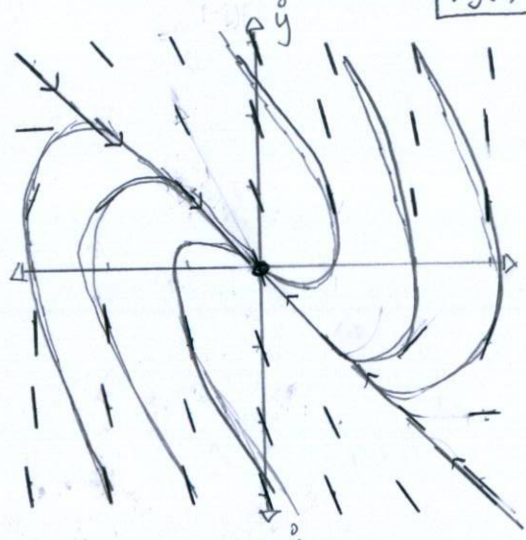
$\lambda_1 = 1 - i$; $\begin{bmatrix}1-(1-i) & -1\\ 1 & 1-(1-i)\end{bmatrix}\begin{bmatrix}V_{11}\\ V_{12}\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$; $iV_{11} - V_{12} = 0$; $\vec{V}_1 = \begin{bmatrix}1\\ i\end{bmatrix}$

$\lambda_2 = 1 + i$; $\begin{bmatrix}1-(1+i) & -1\\ 1 & 1-(1+i)\end{bmatrix}\begin{bmatrix}V_{21}\\ V_{22}\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$; $-iV_{21} - V_{22} = 0$; $\vec{V}_2 = \begin{bmatrix}1\\ -i\end{bmatrix}$

b. General Solution: $\vec{X}(t) = V_1 e^{\lambda_1 t} + V_2 e^{\lambda_2 t} = \begin{bmatrix} x(t) = e^{t} \circ 2\cos(t)\\ y(t) = e^{t} \circ 2i\sin(t)\end{bmatrix}$

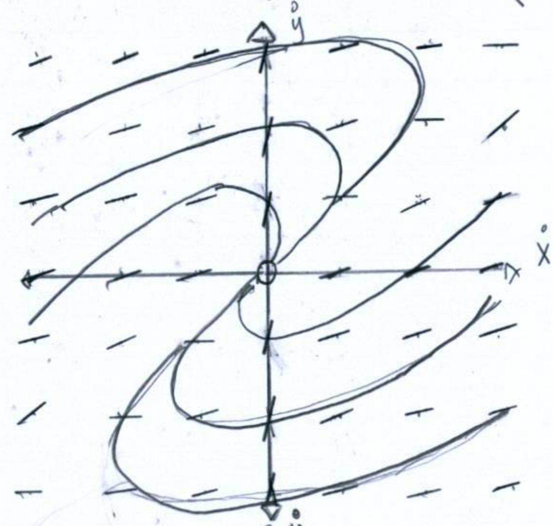
$\dot{x} = y$
$\dot{y} = -2x - 3y$

5.2.3. $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = -2 \cdot \frac{x}{y} - 3$



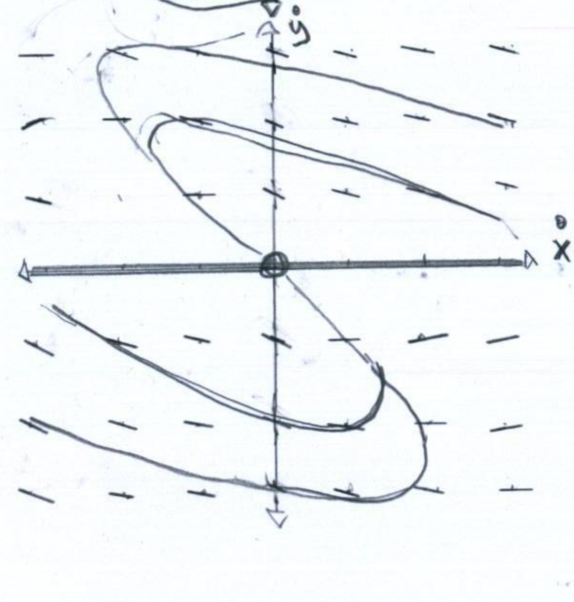
$\dot{x} = 5x + 10y$
$\dot{y} = -x - y$

5.2.4 $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{-x - y}{5x + 10y}$



$\dot{x} = 3x - 4y$
$\dot{y} = x - y$

5.2.5. $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{x - y}{3x - 4y}$

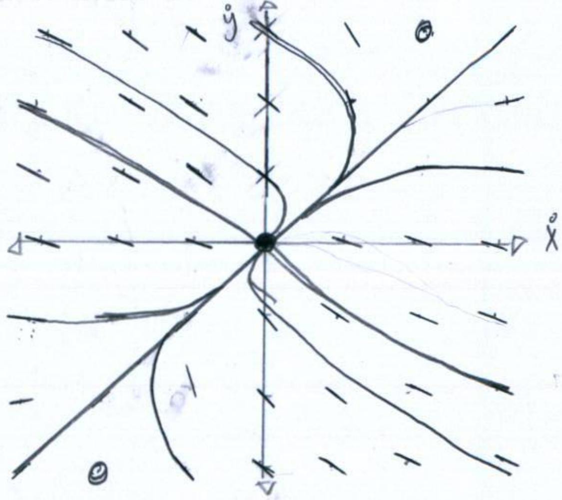


Switched

$\dot{x} = -3x + 2y$

$\dot{y} = x - 2y$

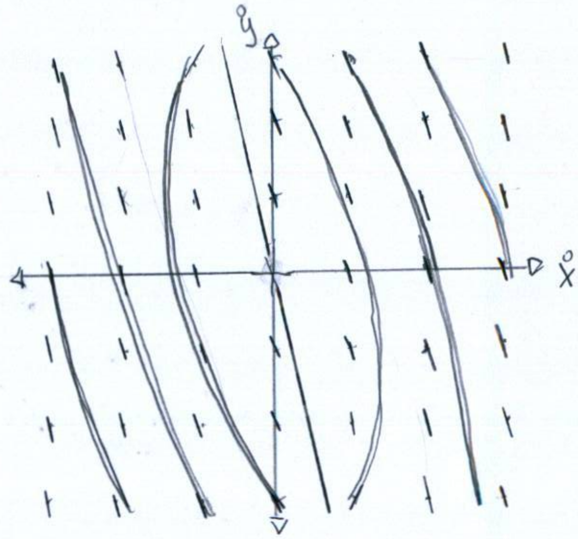
5.2.6. $\dfrac{\dot{y}}{\dot{x}} = \dfrac{dy}{dx} = \dfrac{x - 2y}{-3x + 2y}$



$\dot{x} = 5x + 2y$

$\dot{y} = -17x - 5y$

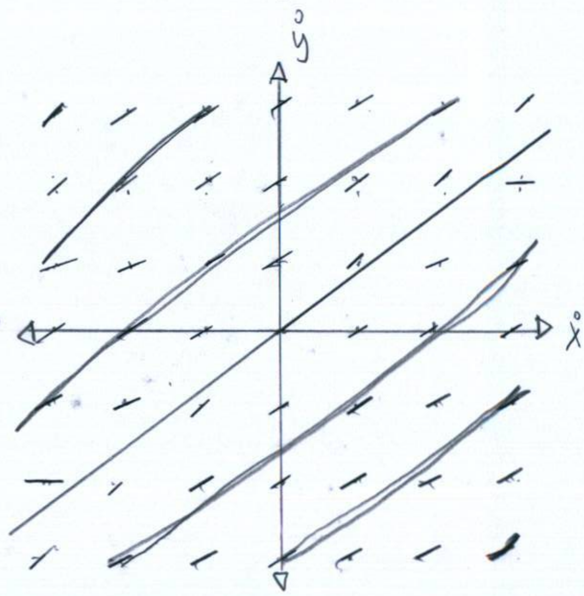
5.2.7. $\dfrac{\dot{y}}{\dot{x}} = \dfrac{dy}{dx} = \dfrac{-17x - 5y}{5x + 2y}$



$\dot{x} = -3x + 4y$

$\dot{y} = -2x + 3y$

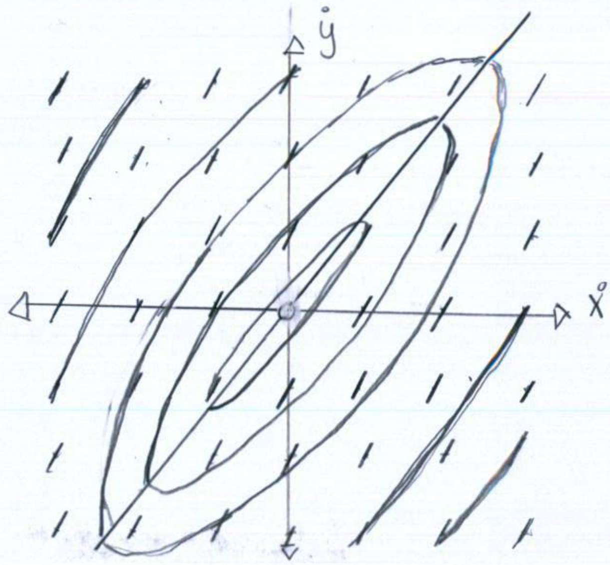
5.2.8. $\dfrac{\dot{y}}{\dot{x}} = \dfrac{dy}{dx} = \dfrac{-2x + 3y}{-3x + 4y}$



$\dot{x} = 4x - 3y$

$\dot{y} = 8x - 6y$

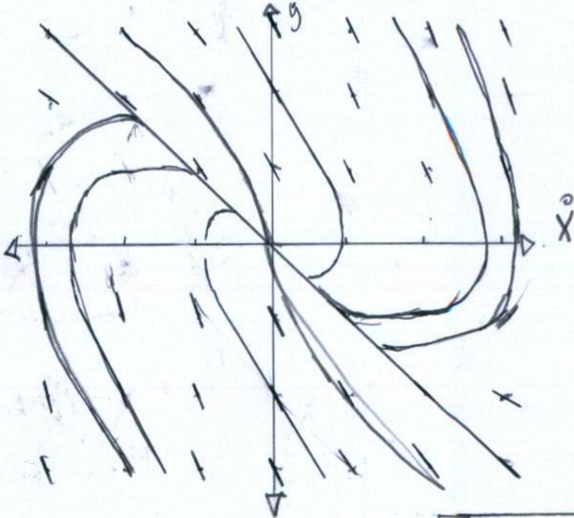
5.2.9. $\dfrac{\dot{y}}{\dot{x}} = \dfrac{dy}{dx} = \dfrac{8x - 6y}{4x - 3y}$



$\dot{x} = y$

$\dot{y} = -x - 2y$

5.2.10. $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{-x-2y}{y}$

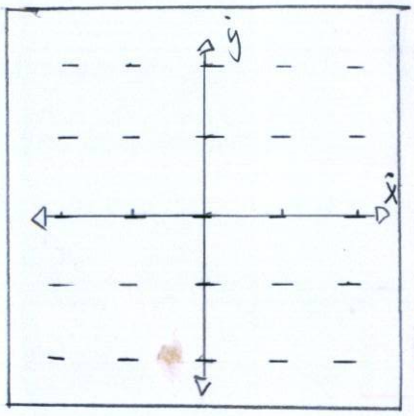


$A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}; \lambda = 0$

5.2.11 $A - \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix} = \lambda^2 = 0; \lambda = 0$

$\dot{x} = Ax; \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \lambda & b \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}; \dot{x} = by; \dot{y} = 0; \frac{\dot{y}}{\dot{x}} = 0$

The book shows a typeset to the correct solution.



$L\ddot{I} + R\dot{I} + \frac{I}{C} = 0$

5.2.12

a) $x = I \quad y = \dot{I}$

$\dot{x} = \dot{I} \quad \dot{y} = \ddot{I} = -R\dot{I} - \frac{I}{C} = -Ry - \frac{x}{c}$; $\boxed{\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{c} & -R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}$

b) $R = 0$; $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/c & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$; $\begin{bmatrix} -\lambda & 1 \\ -1/c & -\lambda \end{bmatrix} = \lambda^2 + \frac{1}{c} = 0$; $\lambda_{1,2} = \frac{i}{\sqrt{c}}$

$\lambda_{1,2} = \frac{i}{\sqrt{c}}$; $\begin{bmatrix} -i/c & 1 \\ -1/c & -i/\sqrt{c} \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$; $\frac{v_{11}}{c} + \frac{i v_{12}}{\sqrt{c}} = 0$; $v_{11} = -i\sqrt{c}$; $v_{12} = 1$

General Solution: $e^{\lambda_{1,2} t} = \cos\left(\frac{t}{\sqrt{c}}\right) + i \sin\left(\frac{t}{\sqrt{c}}\right)$

$$e^{\lambda_{1,2} t} \cdot \vec{v} = \begin{bmatrix} \sqrt{c} \sin(t/\sqrt{c}) - i\sqrt{c} \cos(t/\sqrt{c}) \\ \cos(t/\sqrt{c}) + i \sin(t/\sqrt{c}) \end{bmatrix}$$

$$X = \begin{bmatrix} x = C_1 \sin(t/\sqrt{c}) - C_2 \cos(t/\sqrt{c}) \\ y = C_1 \cos(t/\sqrt{c}) + C_2 \sin(t/\sqrt{c}) \end{bmatrix}$$

$\boxed{\text{Neutrally Stable}}$

$R > 0$; $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{c} & -R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, $\begin{bmatrix} -\lambda & 1 \\ -1/c & -R-\lambda \end{bmatrix} = \lambda(R+\lambda) + \frac{1}{c} = 0$; $\lambda_{1,2} = \frac{-R \pm \sqrt{R^2 - 4(1)(1/c)}}{2(1)}$

$\lambda_1 = \frac{-R + \sqrt{R^2 - 4(1)(1/c)}}{2}$; $\begin{bmatrix} \frac{R - \sqrt{R^2 - 4/c}}{2} & 1 \\ -1/c & \frac{-R - \sqrt{R^2 - 4/c}}{2} \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$

$$\frac{R-\sqrt{R^2-4/C}}{2}\cdot V_{11} + V_{12} = 0 \; ; \quad V_{11} = 1 \; ; \; V_{12} = -\frac{R+\sqrt{R^2-4/C}}{2}$$

$$\lambda_2 = \frac{-R-\sqrt{R^2-4/C}}{2} \; ; \quad \begin{bmatrix} \frac{R+\sqrt{R^2-4/C}}{2} & 1 \\ -1/C & \frac{-R+\sqrt{R^2-4/C}}{2} \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = 0$$

$$\frac{R+\sqrt{R^2-4/C}}{2}\cdot V_{11} + V_{12} = 0 \; ; \quad V_{11} = 1 \; ; \; V_{12} = \frac{+R+\sqrt{R^2-4/C}}{2}$$

General Solution: $X_i = C_i e^{\lambda_i t} V_i$

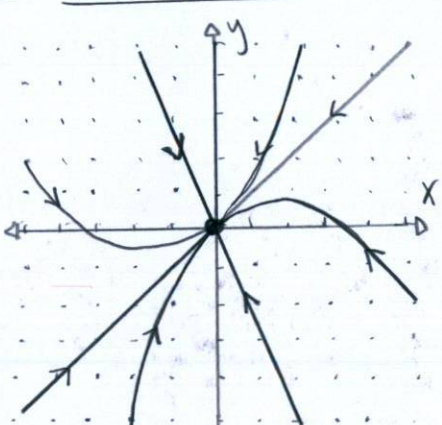
$$X_1 = C_1 e^{\frac{-R+\sqrt{R^2-4/C}}{2}t} \begin{bmatrix} 1 \\ \frac{-R+\sqrt{R^2-4/C}}{2} \end{bmatrix}$$

$$X_2 = C_2 \cdot e^{\frac{-R-\sqrt{R^2-4/C}}{2}t} \begin{bmatrix} 1 \\ \frac{R+\sqrt{R^2-4/C}}{2} \end{bmatrix}$$

$$\bar{X} = X_1 + X_2 = \begin{bmatrix} x(t) = C_1 e^{\frac{-R+\sqrt{R^2-4/C}}{2}t} + C_2 e^{\frac{-R-\sqrt{R^2-4/C}}{2}t} \\ y(t) = C_1 e^{\frac{-R+\sqrt{R^2-4/C}}{2}t}\cdot\left(\frac{-R+\sqrt{R^2-4/C}}{2}\right) + C_2 e^{\frac{-R-\sqrt{R^2-4/C}}{2}t}\cdot\left(\frac{R+\sqrt{R^2-4/C}}{2}\right) \end{bmatrix}$$

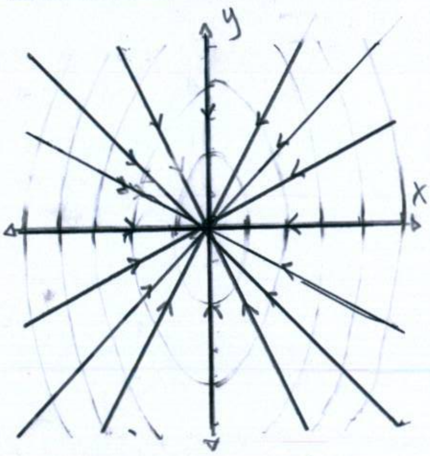
$\boxed{\text{Asymptotically Stable}}$

c. $R^2C-4L>0$



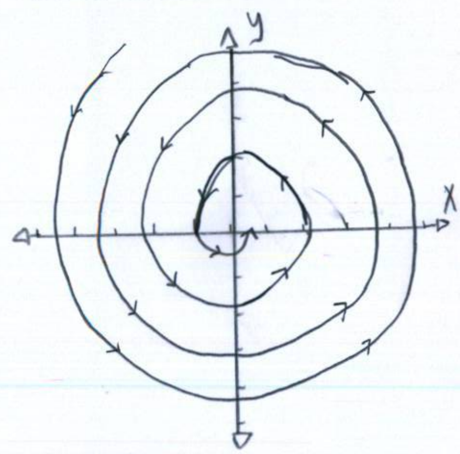
Unstable Node

$R^2C-4L=0$



Star, Degenerate Node

$R^2C-4L<0$



Unstable Spiral

$R^2C-4L>0$

$R^2C>4L$

$C>\frac{4L}{R}$

$R^2C>4L$

$m\ddot{x} + b\dot{x} + kx = 0$ 5.2.13:

a. $i = x \quad j = \dot{x}$

$\dot{i} = \dot{x} \quad \dot{j} = \ddot{x} = -\frac{b}{m}\dot{x} - \frac{k}{m}x = -\frac{b}{m}j - \frac{k}{m}i$; $\boxed{\begin{bmatrix} \dot{i} \\ \dot{j} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{b}{m} & -\frac{k}{m} \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix}}$

b. $\vec{I} = A\vec{i}$; $\begin{bmatrix} -\lambda & 1 \\ -\frac{b}{m} & -\frac{k}{m} - \lambda \end{bmatrix} = \lambda^2 + \frac{k}{m}\lambda + \frac{b}{m} = 0$; $\lambda_{1,2} = \frac{-\frac{k}{m} \pm \sqrt{\left(\frac{k}{m}\right)^2 - 4(1)(b/m)}}{2(1)}$

$\lambda_1 = \frac{-\frac{k}{m} + \sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)}}{2}$; $\begin{bmatrix} \frac{\frac{k}{m} - \sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)}}{2} & 1 \\ -\frac{b}{m} & \frac{-\frac{k}{m} - \sqrt{(k/m)^2 - 4(b/m)}}{2} \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = 0$

$\left(\frac{k}{m} - \sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)}\right) V_{11} + V_{12} = 0$

$V_{11} = 1$; $V_{12} = -\frac{k}{m} + \sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)}$

$\lambda_2 = \frac{-\frac{k}{m} - \sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)}}{2}$; $\begin{bmatrix} \frac{\frac{k}{m} + \sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)}}{2} & 1 \\ -\frac{b}{m} & \frac{-\frac{k}{m} + \sqrt{(k/m)^2 - 4(b/m)}}{2} \end{bmatrix} \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = 0$

$\frac{\left(\frac{k}{m}\right) + \sqrt{(k/m)^2 - 4(b/m)}}{2} V_{21} + V_{22} = 0$

$V_{21} = 1$; $V_{22} = \frac{-\left(\frac{k}{m}\right) - \sqrt{(k/m)^2 - 4(b/m)}}{2}$

$x(t) = C_1 e^{\frac{-\left(\frac{k}{m}\right) + \sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)}}{2} t} + C_2 e^{\frac{-\left(\frac{k}{m}\right) - \sqrt{(k/m)^2 - 4(b/m)}}{2} t}$

$y(t) = c_1 \circ \frac{\left(-\frac{k}{m}\right) + \sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)}}{2} \cdot e^{\frac{-\left(\frac{k}{m}\right) + \sqrt{(k/m)^2 - 4(b/m)}}{2} t} + C_2 \frac{\left(-\frac{k}{m}\right) - \sqrt{(k/m)^2 - 4(b/m)}}{2} \circ e^{\frac{-\left(\frac{k}{m}\right) - \sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)}}{2} t}$

Unstable Spiral : $\sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)} < 0$

Star, Degenerate Node : $\sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)} = 0$

Unstable Node ; $\sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)} > 0$

c. Star, Degenerate Node is critically damped. An unstable spiral is underdamped. While an unstable node is an unstable node.

$\dot{x} = Ax$ 5.2.14 $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$;

$\lambda_2 < \lambda_1 < 0$: Stable Node
$\lambda_{1,2} = \kappa \pm i\omega < 0$: Stable Spiral
$\lambda_1 = \lambda_2 = \lambda$: Star Node, Degenerate Node
$\lambda_{1,2} = \kappa \pm i\omega > 0$: Unstable Spiral
$\lambda_2 > \lambda_1 > 0$: Unstable Node

$$A = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

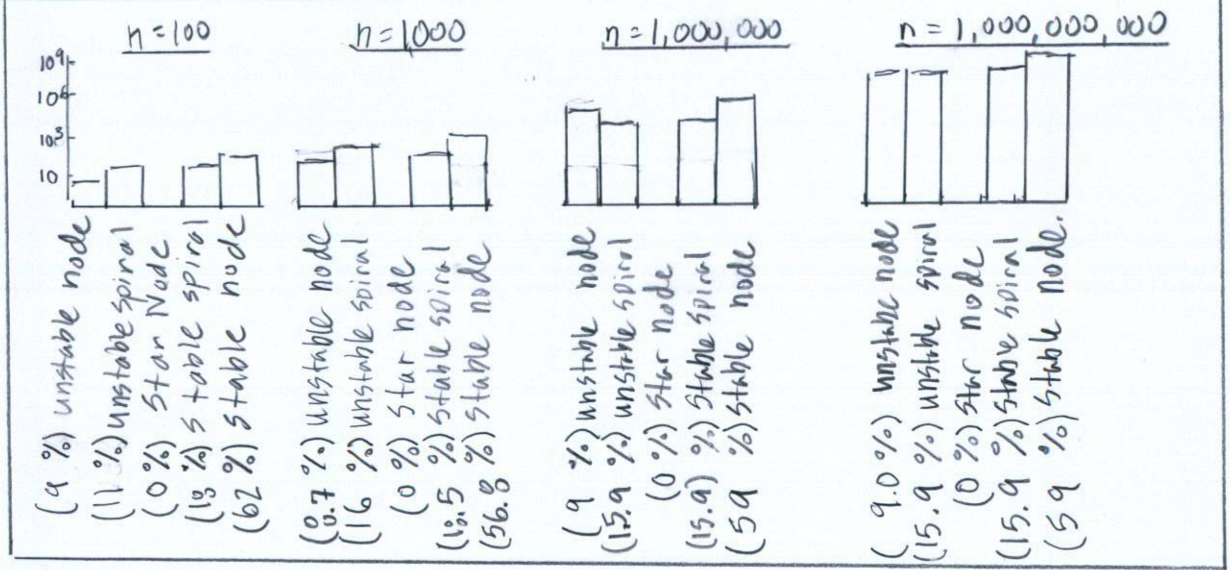
$$\lambda_{1,2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

Pseudo-code

If $(a+d)^2 - 4(ad-bc) < 0$, then λ_1 & λ_2 are imaginary.
  If $(a+d) > 0$, then λ_1, λ_2 are an unstable spiral.
  else, λ_1, λ_2 are a stable spiral.
Else
  Doub $\lambda_1 = \dfrac{(a+d) + \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$
  Doub $\lambda_2 = \dfrac{(a+d) - \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$
  If $(\lambda_1 = \lambda_2)$, then Star Node, Degenerate Node
  Else if$(\lambda_1$ & $\lambda_2 > 0)$, then unstable Node
  else, Stable node.

Real-code

```
#include <iostream>
#include <random>
using namespace std;
int main() {
  int i, s_node = 0, u_node = 0, star = 0, s_spiral = 0, u_spiral = 0, trials = 100;
  double l1, l2, a, b, c, d;
  std::default_random_engine generator;
  std::uniform_real_distribution (-1,1);
  if(((a+d)-4(ad-bc))<0){
     if((a+d)>0) u_spiral += 1;
     else s_spiral += 1;
  }else{
    l1 = ((a+d) + sqrt(pow((a+d),2) - 4(ad-bc)))/2;
    l2 = ((a+d) - sqrt(pow((a+d),2) - 4(ad-bc)))/2;
    if(l1 = l2) star += 1
    else if(l1 && l2 > 0) u_node += 1
    else s_node += 1;
  }
} }
```



An unstable spiral approaches the limit of 9%; while, stable spirals the most common, at 59%.
A normal distribution produced greater proportions of the stable phase plots (stable node [83%], stable spiral [14.8%], ...). than the uniform distribution modelled.

$\dot{R} = aR + bJ$ 5.3.1. R = Romeo's love/hate; J = Juliet's love/hate; a = b = romantic style.

$\dot{R} = J$

$\dot{J} = -R + J$

5.3.2.

a. $\begin{bmatrix} \dot{R} \\ \dot{J} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}$ "cautious romance"

b \ a	(+)	(−)
(+)	Star crossed romance	cautious romance
(−)	Indifferent romance.	Sensitive romance

b. (R, J) = (0, 0) = Neverending love/hate = Stable Node.

c. $R(0) = I;\ J(0) = 0;$

$$\begin{bmatrix} 0-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix} = -\lambda(1-\lambda) + 1 = \lambda^2 - \lambda + 1 = 0$$

$$\lambda_{1,2} = \frac{+1 \pm \sqrt{1-4(1)(1)}}{2} = \frac{1 \pm \sqrt{3}i}{2}$$

$$\lambda_{1,2} = \frac{1+\sqrt{3}i}{2};\ \begin{bmatrix} \frac{-1-\sqrt{3}i}{2} & 1 \\ -1 & \frac{-1-\sqrt{3}i}{2} \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix};\ \frac{-1+\sqrt{3}i}{2} V_{11} + V_{12} = 0;\ \vec{V}_{1,2} = \begin{bmatrix} 1 \\ \frac{+1+\sqrt{3}i}{2} \end{bmatrix}$$

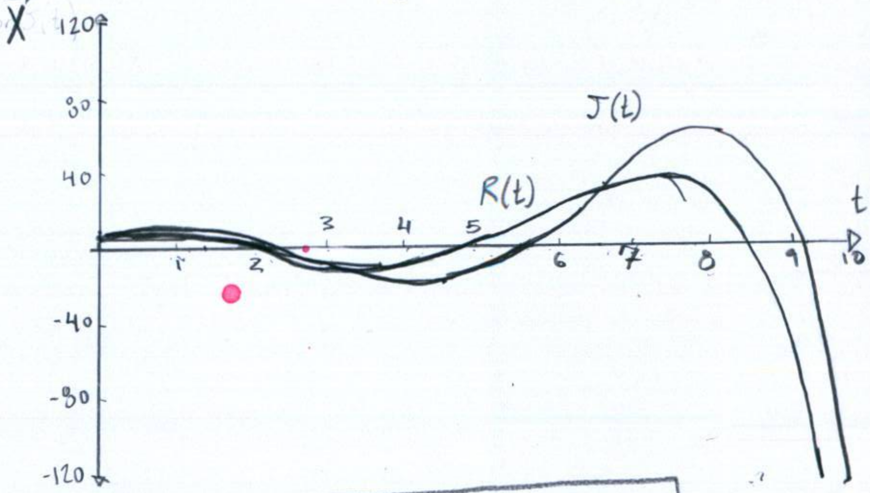
General Solution: $$\vec{X} = \begin{bmatrix} R(t) = e^{\frac{t}{2}} \cdot 2\left[C_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right] \\ J(t) = e^{\frac{t}{2}} \cdot (1+\sqrt{3}i) \cdot \left[C_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right] \end{bmatrix}$$

$$\vec{X} = \begin{bmatrix} R(t) = 2e^{\frac{t}{2}} \cdot \left[C_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right] \\ J(t) = (1+\sqrt{3})e^{\frac{t}{2}} C_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + (1-\sqrt{3})e^{\frac{t}{2}} C_2 \sin\left(\frac{\sqrt{3}}{2}t\right) \end{bmatrix}$$

$R(0) = 1;\ C_1 = \frac{1}{2}$

$J(0) = 0;\ 0 = C_1 + \sqrt{3}C_2;\ C_2 = \frac{-1}{2\sqrt{3}}$

Final Solution: $\vec{X} = \begin{bmatrix} R(t) = e^{t/2}\left[\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right] \\ J(t) = e^{t/2}\left[\frac{(1+\sqrt{3})}{2}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{(1-\sqrt{3})}{2\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right] \end{bmatrix}$



$\dot{R} = aJ$
$\dot{J} = bR$

5.3.3.

b \ a	(+)	(−)
(+)	$a^2>b^2$: Stable $a^2<b^2$: Unstable $a=b$: Mutual	Stable center of neverending love & hate
(−)	Stable center of neverending love & hate	$a^2>b^2$: Stable $a^2<b^2$: Unstable $a=b$: Mutual

Romeo and Juliet live happily when Juliet's love is of greater amounts.

$\dot{R} = aR+bJ$
$\dot{J} = -bR-aJ$

5.3.4.

b \ a	(+)	(−)
(+)	$a^2>b^2$: Unstable $a^2<b^2$: Stable $a=b$: Star, degen node	$a^2>b^2$: Unstable $a^2<b^2$: Stable $a=b$: Star, degen node
(−)	$a^2>b^2$: Unstable $a^2<b^2$: Stable $a=b$: Star, degen node	$a^2>b^2$: Unstable $a^2<b^2$: Stable $a=b$: Star, degen node

Yes, opposites attract when the proportion of Romeo's love is larger.

$\dot{R} = aR+bJ$
$\dot{J} = bR+aJ$

5.3.5.

b \ a	(+)	(−)
(+)	$a^2>b^2$: Unstable Node $a^2\leq b^2$: Unstable Saddle	$a^2>b^2$: Unstable Node $a^2\leq b^2$: Unstable Saddle
(−)	$a^2>b^2$: Unstable Node $a^2\leq b^2$: Unstable Saddle	$a^2>b^2$: Unstable Node $a^2\leq b^2$: Unstable Saddle

The marriage of Romeo and Juliet of exact clone demonstrate an unstable relationship for all time.

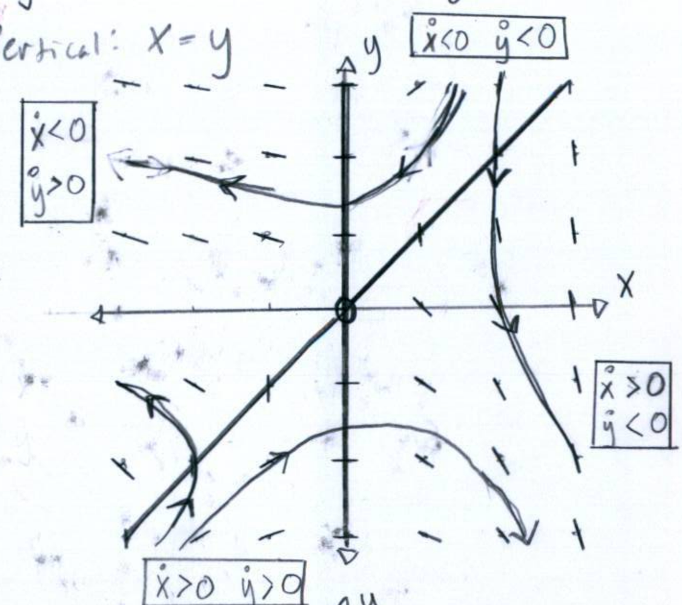
$\mathring{R} = 0$ 5.3.6.
$\mathring{J} = aR + bJ$

b \ a	(+)	(−)
(+)	Unstable and Fixed Relationship	$a^2 > b^2$: Stable $a^2 < b^2$: Unstable $a = b$: Isolated
(−)	$a^2 > b^2$: Unstable $a^2 < b^2$: Stable $a = b$: Isolated.	Stable and Fixed Relationship

Chapter 6: Phase Plane:

$\mathring{x} = x - y$ 6.1.1: $\boxed{\text{Fixed Points}}$: $\mathring{x} = 0 = x - y$; $x = y$; $\mathring{y} = 0 = 1 - e^x$; $x = 0$; $(x^*, y^*) = (0,0)$
$\mathring{y} = 1 - e^x$

$\boxed{\text{Nullclines}}$: Horizontal: $x = 0$; Vertical: $x = y$



$\mathring{x} = x - x^3$ 6.1.2: $\boxed{\text{Fixed Points}}$: $\mathring{x} = 0 = x - x^3$
$\mathring{y} = -y$

$x^* = 1, 0, -1$

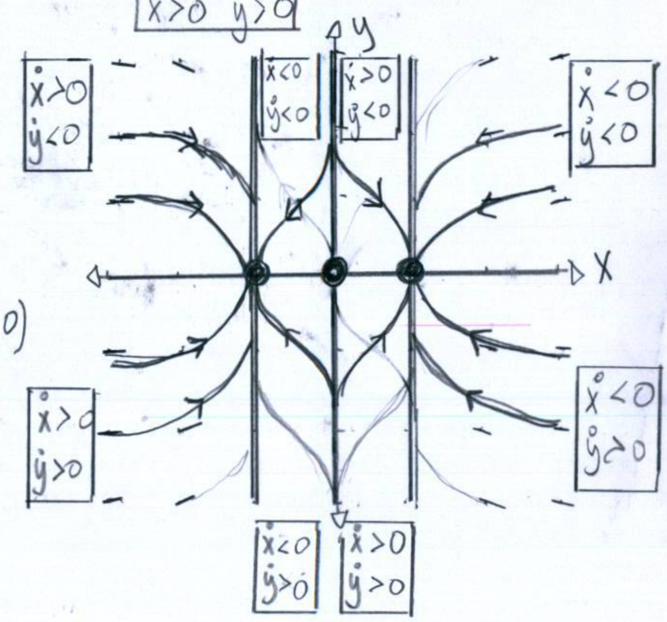
$\mathring{y} = 0 = -y$

$y^* = 0$

$(x^*, y^*) = (1,0), (0,0), (-1,0)$

$\boxed{\text{Nullclines}}$: Horizontal: $y = 0$

Vertical: $x = 0, 1, -1$



$\dot{x} = x(x-y)$

$\dot{y} = y(2x-y)$

6.1.3. $\boxed{\text{Fixed Points}}$: $\dot{x} = 0 = x(x-y)$

$\dot{y} = 0 = y(2x-y)$

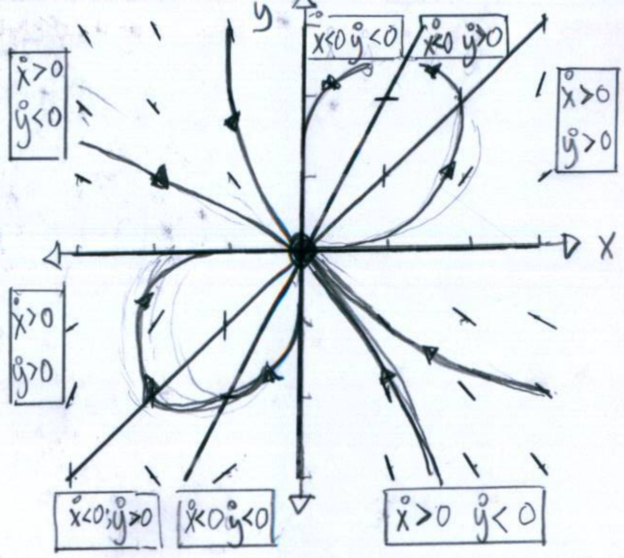
$(x^*, y^*) = (0,0)$

$\boxed{\text{Nullclines}}$: Horizontal: $y = 2x$

$y = 0$

Vertical: $y = x$

$x = 0$



$\dot{x} = y$

$\dot{y} = x(1+y) - 1$

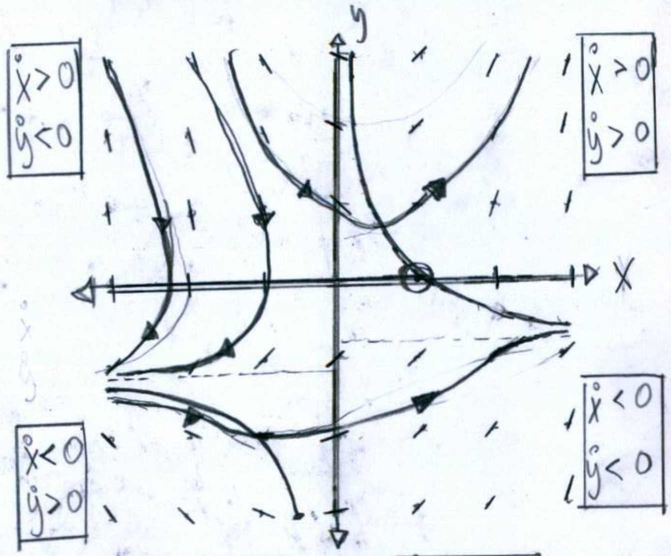
6.1.4: $\boxed{\text{Fixed Points}}$: $\dot{x} = 0 = y$

$\dot{y} = 0 = x(1+y) - 1$

$(x^*, y^*) = (1,0)$

$\boxed{\text{Nullclines}}$: Horizontal: $y = \frac{1}{x} + 1$

Vertical: $y = 0$



$\dot{x} = x(2-x-y)$ 6.1.5. $\boxed{\text{Fixed Points}}$: $\dot{x} = 0 = x(2-x-y)$

$\dot{y} = x - y$

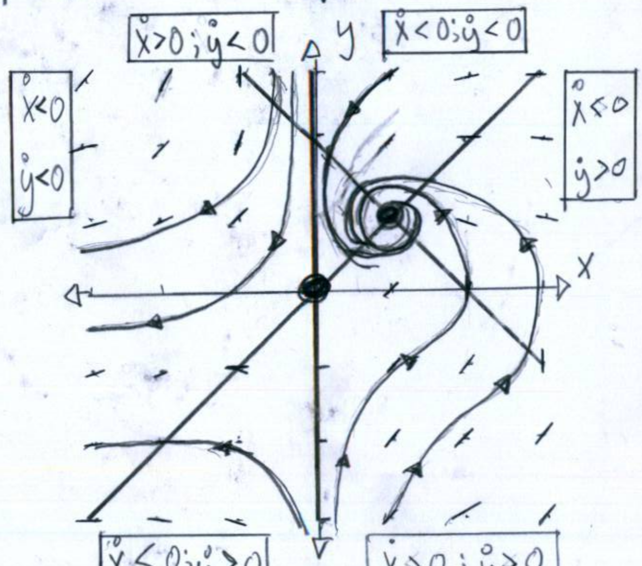
$\dot{y} = 0 = x - y$

$(x^*, y^*) = (0,0), (1,1)$

$\boxed{\text{Nullcline}}$: Horizontal: $y = x$

Vertical: $x = 0$

$y = 2 - x$



$\dot{x} = x^2 - y$ 6.1.6. $\boxed{\text{Fixed Points}}$: $\dot{x} = 0 = x^2 - y$

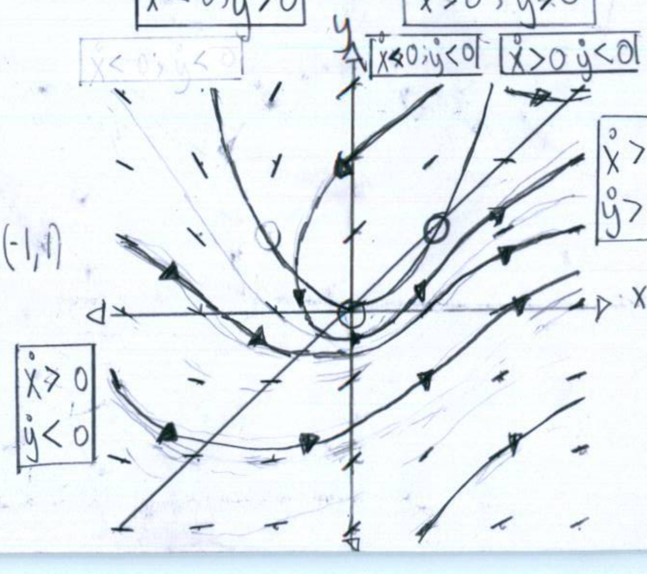
$\dot{y} = x - y$

$\dot{y} = 0 = x - y$

$(x^*, y^*) = (0,0), (1,1), (-1,1)$

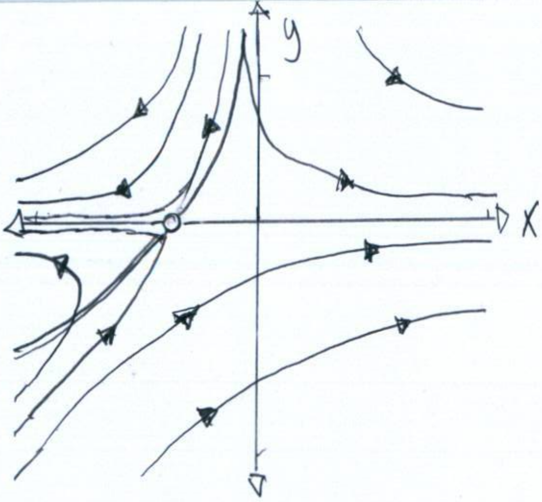
$\boxed{\text{Nullcline}}$: Horizontal: $y = x$

Vertical: $y = x^2$



$\dot{x} = x + e^{-y}$ 6.1.7

$\dot{y} = -y$



$\dot{x} = y$

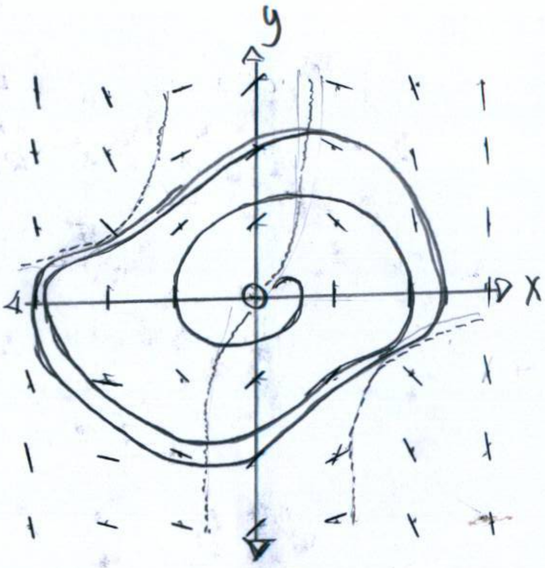
$\dot{y} = -x + y(1-x^2)$ 6.1.8 (Van der Poll oscillator)

$\boxed{\text{Fixed points}}$ $\dot{x} = 0 = y$

$\dot{y} = 0 = -x + y(1-x^2)$

$(x^*, y^*) = (0,0)$

$\boxed{\text{Nullclines}}$ $y = \dfrac{x}{1-x^2}$



$\dot{x} = 2xy$ 6.1.9. (Dipole Fixed Point)

$\dot{y} = y^2 - x^2$

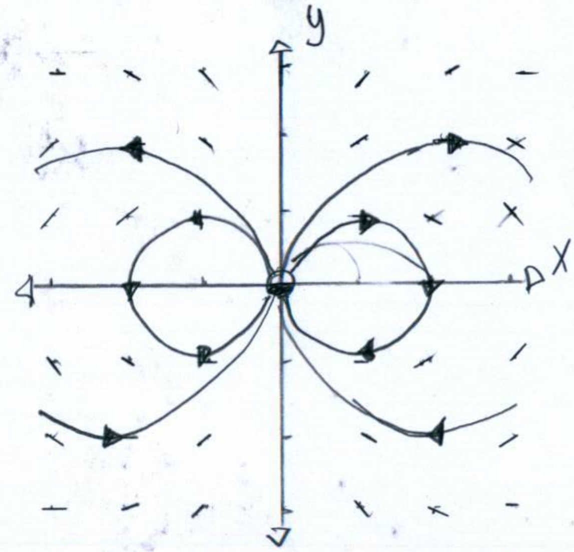
$\boxed{\text{Fixed Point}}$ $\dot{x} = 0 = 2xy$

$\dot{y} = 0 = y^2 - x^2$

$(x^*, y^*) = (0,0)$

$\boxed{\text{Nullcline}}$ $y = x;\ y = 0;\ x = 0.$

$y = -x$



$\dot{x} = y + y^2$ 6.1.10

$\dot{y} = -\frac{1}{2}x + \frac{1}{5}y - xy + \frac{6}{5}y^2$ (Two-eyed Monster)

$\boxed{\text{Fixed Points}}$

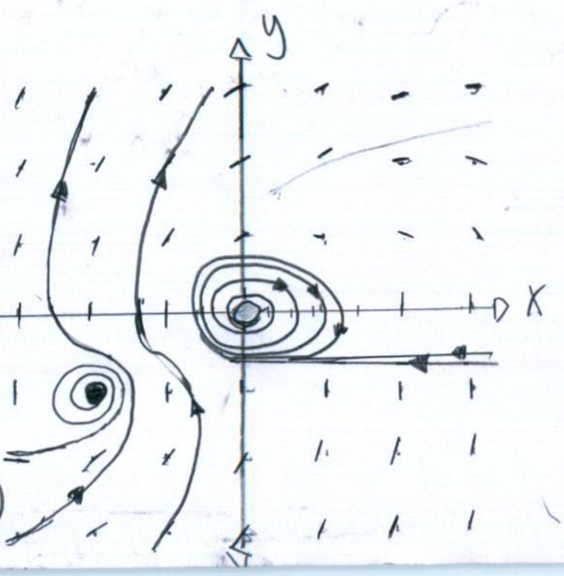
$\dot{x} = 0 = y + y^2$

$\dot{y} = 0 = -\frac{1}{2}x + \frac{1}{5}y - xy + \frac{6}{5}y^2$

$(x^*, y^*) = (0,0), (-2,-1)$

$\boxed{\text{Nullclines}}$ $y = 0;\ x = 0$

$y = \frac{1}{12}\left(\sqrt{25x^2 + 50x + 1} + 5x - 1\right)$



$\dot{x} = y + y^2$

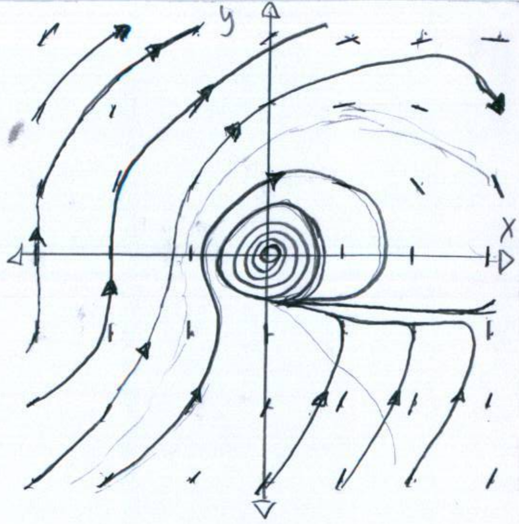
$\dot{y} = -x + \frac{1}{5}y - xy + \frac{6}{5}y^2$

6.1.11 $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = y + y^2$

$\dot{y} = 0$

$= -x + \frac{1}{5}y - xy + \frac{6}{5}y^2$

$(x^*, y^*) = (0,0), (4,4)$

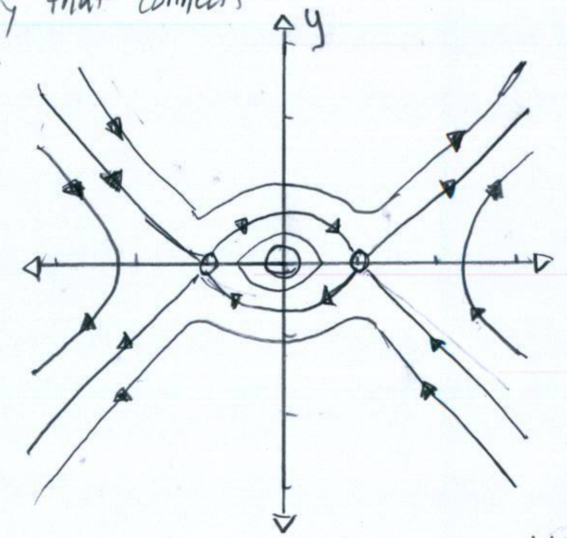


$\boxed{\text{Nullcline}}$

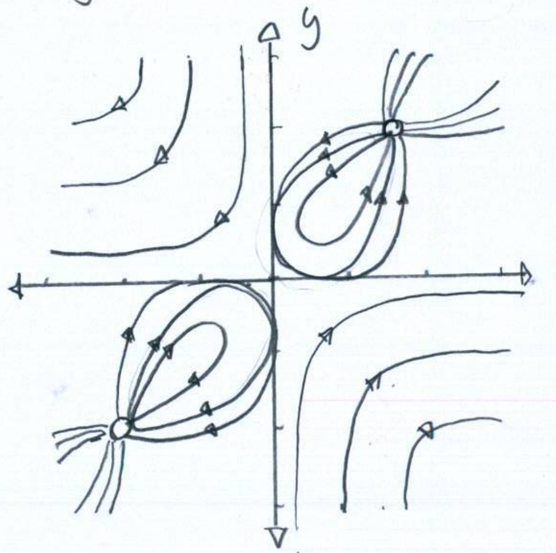
$y = \frac{1}{12}\left(\pm\sqrt{25x^2 + 110x + 1} + 5x - 1\right)$

6.1.12.

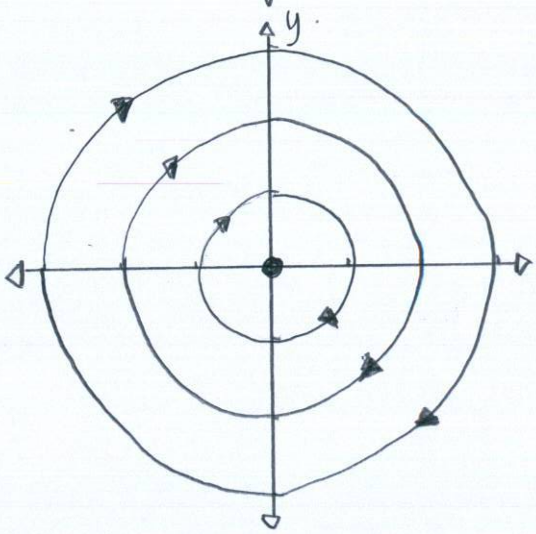
a a single trajectory that connects the saddles.



b. there is no trajectory that connects the saddles



6.1.13: A phase portrait with three closed orbits and one fixed point.



$\dot{x} = x + e^{-y}$ 6.1.14.

$\dot{y} = -y$

a. $u = x+1$; $\frac{dy}{du} = \frac{dy}{dt}\frac{du}{dt} = -y$; $\frac{du}{dt} = \frac{dx}{dt} = u - 1 + (1 - y + \frac{y^2}{2} - \cdots)$

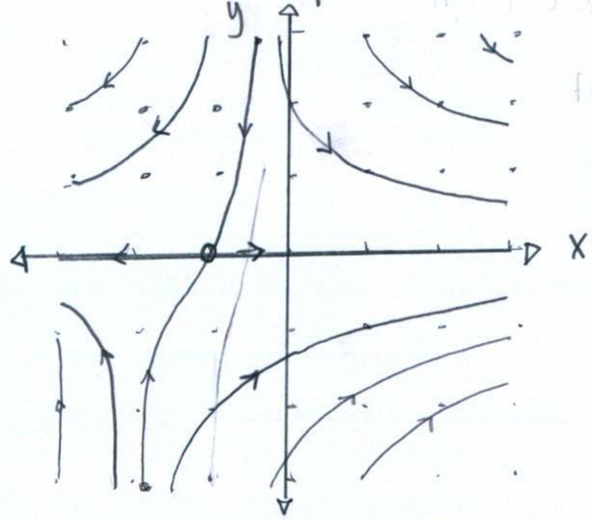
$$\frac{dy}{du} = -\frac{y}{u - y + y^2/2 - y^3/6 + \cdots}$$

$$x = \frac{a_1}{a_1 - 1} + \frac{a_1^3 - 2a_2}{2(a_1-1)^2}u + \frac{2a_1^4 + a_1^5 - 18a_1^2 a_2 + 12(a_2^2 + a_3 - a_1 a_3)}{12(a_1-1)^3}u^3$$

$$\boxed{= a_1 + 2a_2 u + 3a_3 u^3 + \cdots}$$

$a_1 = 2$; $a_2 = 4/3$; $a_3 = 10/9$

b.



$y = \frac{4}{3}(x+1) + \frac{10}{9}(x+1)^3$

Note: The function plotted is not exact.

6.2.1: Yes, trajectories do not intersect, however, may seem so for low resolution plots.

$\dot{x} = y$

$\dot{y} = -x + (1 - x^2 - y^2)y$

6.2.2: a. $D: x^2 + y^2 < 4$

Poincare - Benedixon Theorem : no fixed points and a bounded region, then the trajectory is a closed orbit, and approaches the closed orbit.

Bounded Region - $D: x^2 + y^2 < 4$

Fixed points ; Zero, outside of the center

Existness and Uniqueness is satisfied for a Closed orbit.

b. If $y(t) = \cos(t)$, then $\dot{y} = 0 = -x + (1 - (x^2 + \cos(t)^2))y$ @ $t=0$

Identity: $x^2 + y^2 = 1$

then, $x = 0$ @ $t = 0$

$$\boxed{x(t) = \sin(t)}$$

c. $x(0) = \frac{1}{2}$; $y(0) = 0$; $x(t)^2 + y(t)^2$ must be less than one because a larger value forces $\dot{y}$ to become negative and not a closed orbit.

$\dot{x} = x - y$
$\dot{y} = x^2 - 4$

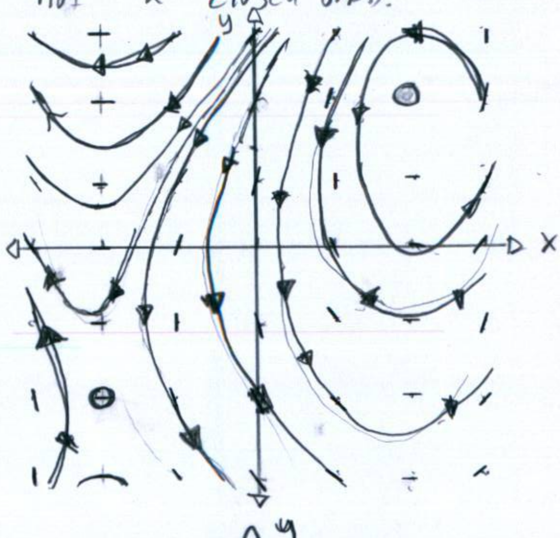
6.3.1 Fixed points

$\dot{x} = 0 = x - y$

$\dot{y} = 0 = x^2 - 4$

$(x^*, y^*) = (2,2), (-2,-2), (0,0)$

"unstable" "unstable"



$\dot{x} = \sin y$
$\dot{y} = x - x^3$

6.3.2 Fixed points

$\dot{x} = 0 = \sin y$

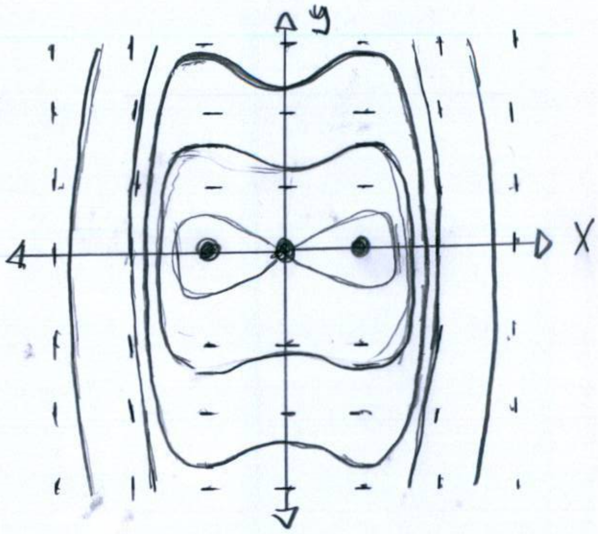
$\dot{y} = 0 = x - x^3$

$(x^*, y^*) = (1, n\pi)$

$(-1, n\pi)$

$(0, n\pi)$

"stable"



$\dot{x} = 1 + y + e^{-x}$
$\dot{y} = x^3 - y$

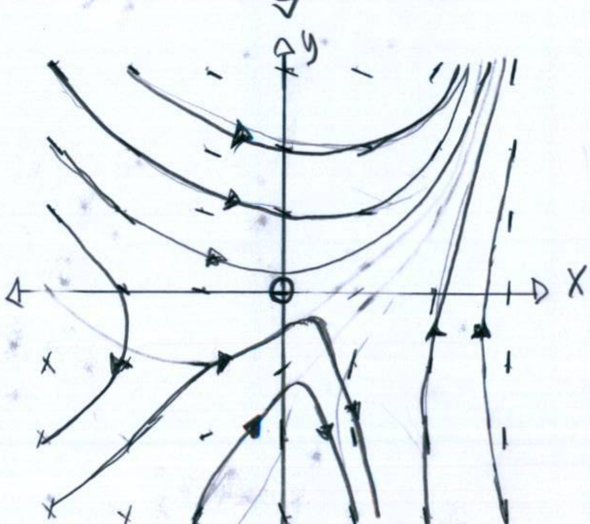
6.3.3 Fixed Points

$\dot{x} = 1 + y - e^{-x} = 0$

$\dot{y} = x^3 - y = 0$

$(x^*, y^*) = (0,0)$

"unstable"



$\dot{x} = y + x - x^3$
$\dot{y} = -y$

6.3.4: Fixed Points

$\dot{x} = 0 = y + x - x^3$

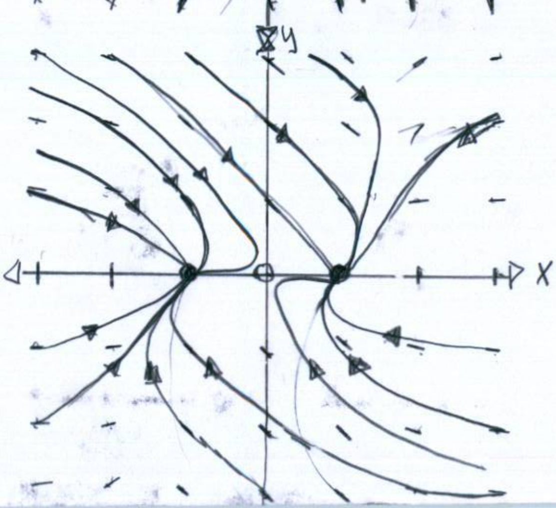
$\dot{y} = 0 = -y$

$(x^*, y^*) = (1,0), (-1,0)$

"stable"

$(0,0)$

"unstable"



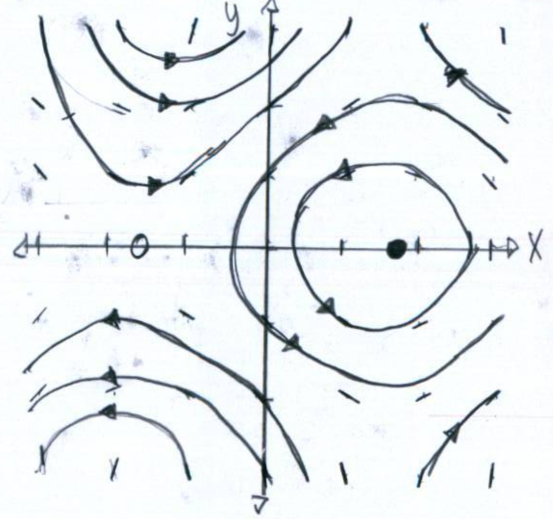
$\dot{x} = \sin y$
$\dot{y} = \cos x$

6.3.5. $\boxed{\text{Fixed Points}}$: $\dot{x} = 0 = \sin y$
$\dot{y} = 0 = \cos x$

$(x^*, y^*) = ((n+\frac{1}{2})\pi, n\pi)$

n is odd "stable"

n is even "unstable"

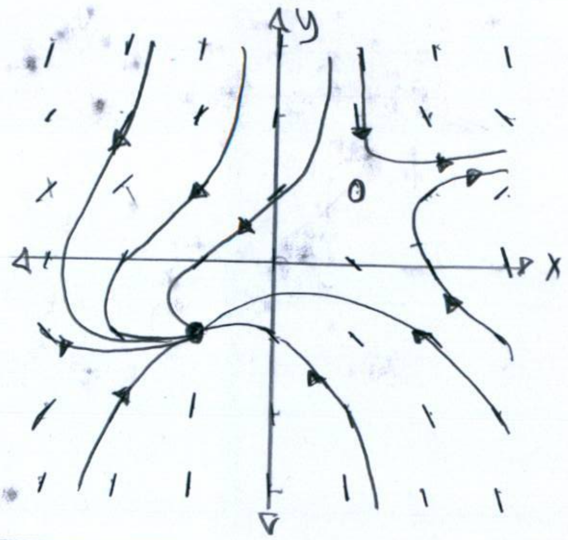


$\dot{x} = xy - 1$
$\dot{y} = x - y^3$

6.3.6. $\boxed{\text{Fixed Points}}$: $\dot{x} = 0 = xy - 1$
$\dot{y} = 0 = x - y^3$

$(x^*, y^*) = (1,1), (-1,-1)$

"ustable" "stable"



6.3.7. The phase portraits of problems 6.3.1-6.3.6 are computer generated.

$\ddot{x} = \frac{Gm_2}{(x-a)^2} - \frac{Gm_1}{x^2}$

6.3.8. a. $g = \ddot{x} = \frac{Gm}{r^2}$;

$m_1 \xleftrightarrow{a} m_2$

(x) (x-a)

$\ddot{x}_1 = \frac{Gm_1}{x^2}$ $\ddot{x}_2 = \frac{Gm_2}{(x-a)^2}$; $\ddot{x} = \ddot{x}_2 - \ddot{x}_1 = \boxed{\frac{Gm_2}{(x-a)^2} - \frac{Gm_1}{x^2}}$

b. Equilibrium Position: $\ddot{x} = 0 = \frac{Gm_2}{(x-a)^2} - \frac{Gm_1}{x^2}$; $m_1(x-a)^2 = m_2 \cdot x^2$

$m_1(x^2 - 2xa + a^2) = m_2 x^2$

$(m_1 - m_2)x^2 - 2xa + a^2 = 0$

$$x = \frac{2a \pm \sqrt{4a^2 - 4a^2(m_1 - m_2)}}{2(m_1 - m_2)}$$

$\boxed{\text{when } m_1 \neq m_2 \text{ "stable"}}$

$\dot{x} = y^3 - 4x$
$\dot{y} = y^3 - y - 3x$

6.3.9. $\boxed{\text{Fixed Points:}}$ $\dot{x} = 0 = y^3 - 4x$; $(x^*, y^*) = (0,0), (2,2), (-2,-2)$

"stable" "unstable" "unstable"

a. $\dot{y} = 0 = y^3 - y - 3x$

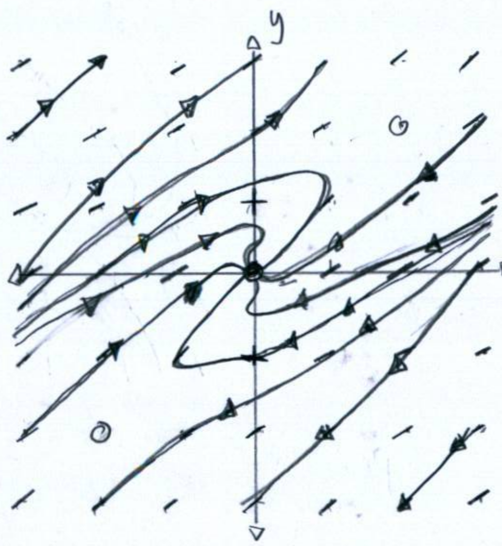
b. If $x = y$, then $\dot{x} = x^3 - 4x$ and $\dot{y} = x^3 - 4x$, so $\boxed{\frac{dy}{dx} = 1}$

c. $\lim_{t \to \infty} |\dot{x} - \dot{y}| = \lim_{t \to \infty} |y^3 - 4x - y^3 + y + 3x| \overset{!}{=} \boxed{\lim_{t \to \infty} |y - x|}$

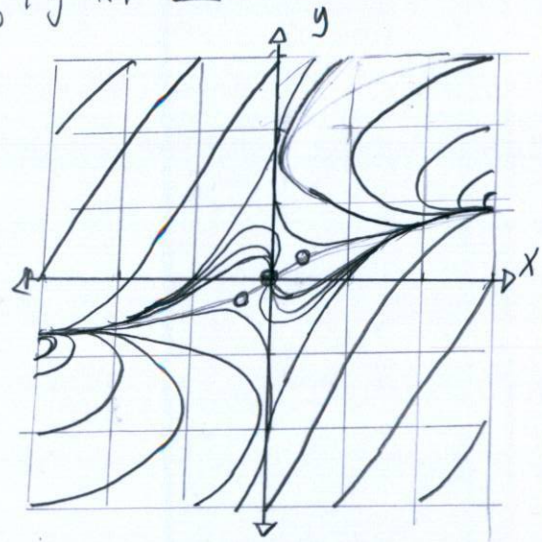
If $\dot{u} = y - x$, then $y - x = Ce^{-t}$, then $\lim_{t \to \infty} |Ce^{-t}| = \boxed{0}$

and $\lim_{t \to \infty} |y - x| = \boxed{0}$

d.



e.



$\dot{x} = xy$
$\dot{y} = x^2 - y$

6.3.10

a. $u = x - x^*$; $v = y - y^*$; $\dot{u} = \dot{x} = f(x^* + u, y^* + v) \doteq f(x^*, y^*) + u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + \cdots$

$\dot{v} = \dot{y} = g(x^* + u, y^* + v) = g(x^*, y^*) + u \frac{\partial g}{\partial x} + v \frac{dg}{dy} + \cdots$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} ; A = \begin{bmatrix} y & x \\ 2x & -1 \end{bmatrix}$$

$\boxed{\text{Fixed Point}}$ $(0,0)$; $A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$, so the origin is a non-isolated fixed point because $\Delta = 0$.

b. $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = 0 : (A - \lambda) U = \dot{U} = 0$; $(A - \lambda) = \begin{bmatrix} y - \lambda & x \\ 2x & -1-\lambda \end{bmatrix} = (y - \lambda)(-1 - \lambda) - 2x^2 = 0$

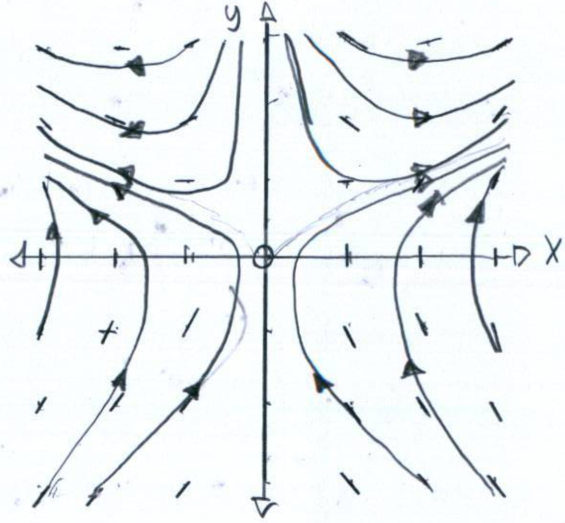
$$\lambda = \frac{(y-1) \pm \sqrt{8x^2 + (y-1)^2}}{2}$$

$\boxed{\text{Thus, } \Delta = \lambda_1 \lambda_2 \neq 0 \text{ and the center is an isolated fixed point.}}$

c. $\boxed{\text{Nullclines}}$ $y = \pm\sqrt{x}$

$y = 0; x = 0$

"Saddle Point"



$\boxed{\text{d. See Part C}}$

$\dot{r} = -r$

$\dot{\theta} = \frac{1}{\ln r}$

6.3.11. a. $r(t) = Ce^{-t}; \theta(t) = \ln \frac{\ln C}{\ln C - t} + D$; Given (r_0, θ_0); then $r(t) = r_0 e^{-t}$

$$\boxed{\theta(t) = \ln \frac{\ln r_0}{|\ln r_0 - t|} + \theta_0}$$

b. $\lim_{t\to\infty} |\theta(t)| = \lim_{t\to\infty} \left| \ln \frac{\ln r_0}{|\ln r_0 - t|} + \theta_0 \right| \boxed{= \infty}$

$\lim_{t\to\infty} r(t) = \lim_{t\to\infty} r_0 e^{-b} \boxed{= 0}$

c. $\dot{r} = -\sqrt{x^2 + y^2}$; $\boxed{\dot{\theta} = \frac{1}{\ln\sqrt{x^2+y^2}}}$

d. $\dot{r} = \frac{d}{dt}\sqrt{x^2+y^2} = \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2+y^2}} = -r = -\sqrt{x^2+y^2}$

$$x\dot{x} + y\dot{y} = -x^2 - y^2$$

$\dot{\theta} = \frac{d}{dt}\arctan\left(\frac{y}{x}\right) = \frac{x\dot{y} - \dot{x}y}{x^2+y^2} = \frac{1}{\ln(\sqrt{x^2+y^2})}$;

$$x\dot{y} - \dot{x}y = \frac{2(x^2+y^2)}{\ln(x^2+y^2)}$$

$$x(x\dot{x} - y\dot{y}) - y(x\dot{y} - y\dot{x}) = (x^2+y^2)\dot{x} = -x(x^2+y^2) - y\frac{2(x^2+y^2)}{\ln(x^2+y^2)}$$

$$\boxed{\dot{x} = -x - \frac{2y}{\ln(x^2+y^2)}}$$

$$x(x\dot{y} - y\dot{x}) + y(x\dot{x} + y\dot{y}) = (x^2+y^2)\dot{y} = -y(x^2+y^2) + x\frac{2(x^2+y^2)}{\ln(x^2+y^2)}$$

$$\boxed{\dot{y} = -y + \frac{2x}{\ln(x^2+y^2)}}$$

d. $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = A\begin{pmatrix}x\\ y\end{pmatrix}$; $A = \begin{pmatrix} -1+\frac{4xy}{(x^2+y^2)\ln^2(x^2+y^2)} & \frac{4y^2}{(x^2+y^2)\ln^2(x^2+y^2)} - \frac{2}{\ln(x^2+y^2)} \\ \frac{2}{\ln(x^2+y^2)} - \frac{4x^2}{(x^2+y^2)\ln^2(x^2+y^2)} & \frac{-4xy}{(x^2+y^2)\ln^2(x^2+y^2)} - 1 \end{pmatrix}$

$$A_{(0,0)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \boxed{\dot{x} = -x,\ \dot{y} = -y}$$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

6.3.12. $\dot{\theta} = \frac{d}{dt}\tan^{-1}\left(\frac{y}{x}\right) = \frac{\frac{d}{dt}\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = \frac{\dot{y}x - \dot{x}y}{x^2+y^2} \stackrel{!}{=} \boxed{\frac{x\dot{y} - y\dot{x}}{r^2}}$

$\dot{x} = -y - x^3$
$\dot{y} = x$

6.3.13. Linearization: $u = x - x^*$; $v = y - y^*$

$$\dot{x} = \dot{u} = f(x,y) = f(u+x^*, v+y^*) = f(x^*,y^*) + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + \cdots$$

$$\dot{y} = \dot{v} = g(x,y) = g(u+x^*, v+y^*) = g(x^*,y^*) + u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y} + \cdots$$

$$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}\begin{pmatrix}u\\ v\end{pmatrix}; A = \begin{bmatrix} -3x^2 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A_{(0,0)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$\Delta = 0$; $\tau = 0$; $\boxed{\text{center}}$

Eigenvalues: $\dot{\vec{U}} = A\vec{U}$; $\lambda U = AU$; $(A-\lambda)U = 0$;

$$(A-\lambda) = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

$$\lambda_{1,2} = \pm i$$

$\boxed{\text{Thus, } \Delta = -1,\ \tau = 0, \text{ so the center is a spiral, also supported by } \tau^2 - 4\Delta > 0.}$

$\dot{x} = -y + ax^2$
$\dot{y} = x + ay^3$

6.3.14. $a > 0$; Fixed points: $\dot{x} = 0 = -y + ax^2$;

$\dot{y} = 0 = x + ay^3$;

$(x^*, y^*) = (0,0)$

Linearization: $u = x - x^*$; $v = y - y^*$

$$\dot{x} = \dot{u} = f(x,y) = f(u+x^*, v+y^*) = f(x^*,y^*) + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + \cdots$$

$$\dot{y} = \dot{v} = g(x,y) = g(u+x^*, v+y^*) = g(x^*,y^*) + u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y} + \cdots$$

$$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}\begin{pmatrix}u\\ v\end{pmatrix}; A = \begin{bmatrix} 2ax & -1 \\ 1 & 3ay^2 \end{bmatrix};$$

$A_{(0,0)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; $\Delta = 0$, $\tau = 0$; center

Eigenvalues: $\dot{U} = AU$; $\lambda U = AU$; $(A-\lambda)U = 0$

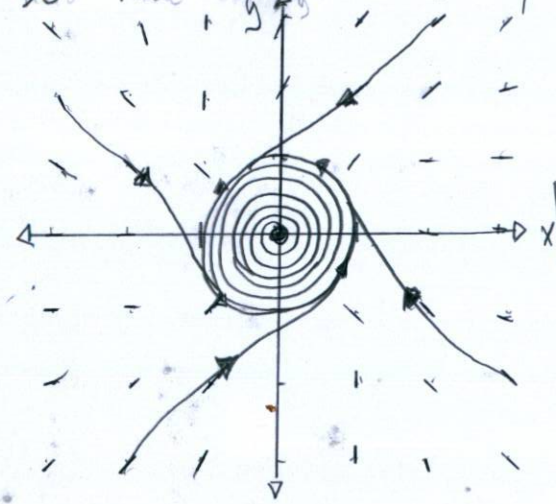
$$(A-\lambda) = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

$$\lambda_{1,2} = \pm i$$

Thus, eigenvalues demonstrate $\Delta = -1$, $\tau = 0$, $\tau^2 - 4\Delta > 0$, so the center is a spiral.

$\dot{r} = r(1-r^2)$ 6.3.15

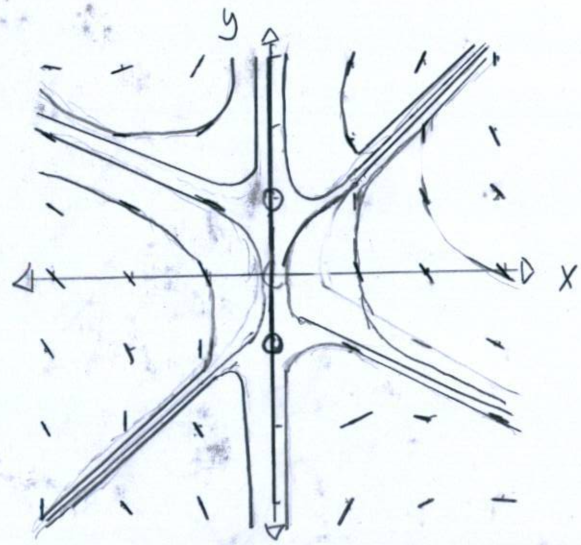
$\dot{\theta} = 1 - \cos\theta$



"Liapunov stable."

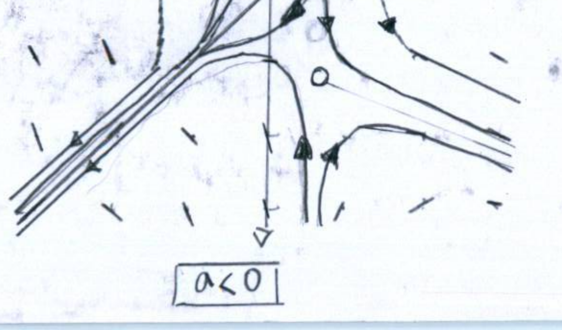
$\dot{x} = a + x^2 - xy$ 6.3.16

$\dot{y} = y^2 - x^2 - 1$ a.



"Saddle connection"

b.

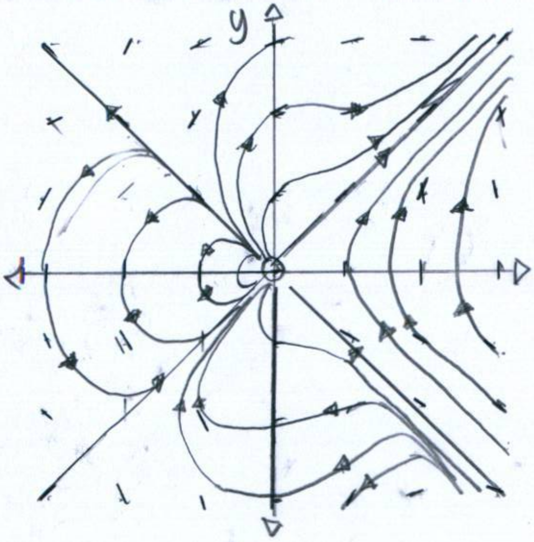


a<0

a>0

$\dot{x} = xy - x^2y + y^3$ 6.3.17.

$\dot{y} = y^2 + x^3 - xy^2$



$\dot{x} = x(3-x-y)$ 6.4.1 $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = x(3-x-y)$

$\dot{y} = y(2-x-y)$ $\qquad \dot{y} = 0 = y(2-x-y)$

$(x^*, y^*) = (0,0)$ "unstable"

$(3,0)$ "stable"

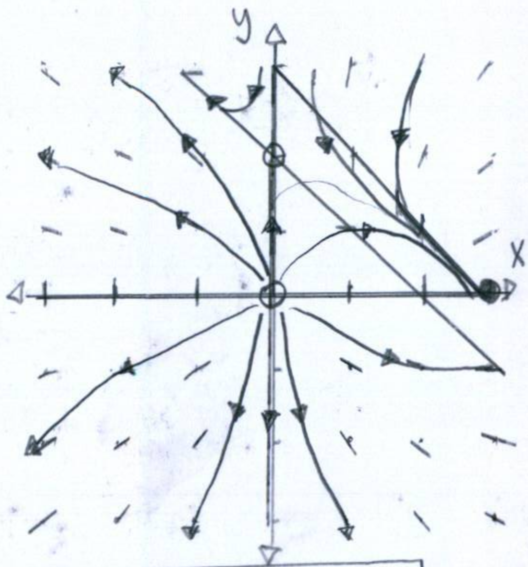
$(0,2)$ "unstable"

$\boxed{\text{Nullclines}}$ $x = 0$

$y = 0$

$y = 2 - x$; $y = 3 - x$

$\boxed{\text{Basin of Attraction}}$ $x \geq 0 \wedge y \geq 0$



$\boxed{\text{I forgot } (x \text{ and } y) \geq 0}$

$\dot{x} = x(3-2x-y)$ 6.4.2 $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = x(3-2x-y)$

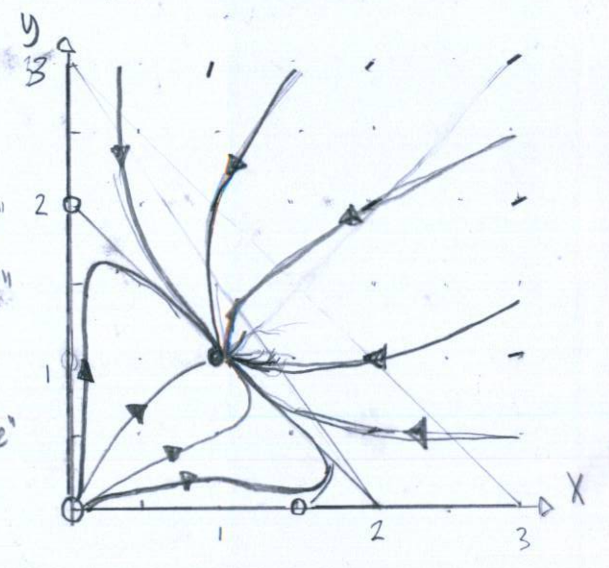
$\dot{y} = y(2-x-y)$ $\qquad \dot{y} = 0 = y(2-x-y)$

$(x^*, y^*) = (0,0)$ "unstable"

$(0,2)$ "unstable"

$(1,1)$ "stable"

$(3/2, 0)$ "unstable"



$\boxed{\text{Nullclines}}$ $x = 0$; $y = 0$

$y = 3 - 2x$

$y = 2 - x$

$\boxed{\text{Basin of Attraction}}$ $(x \geq 0) \wedge (y \geq 0)$

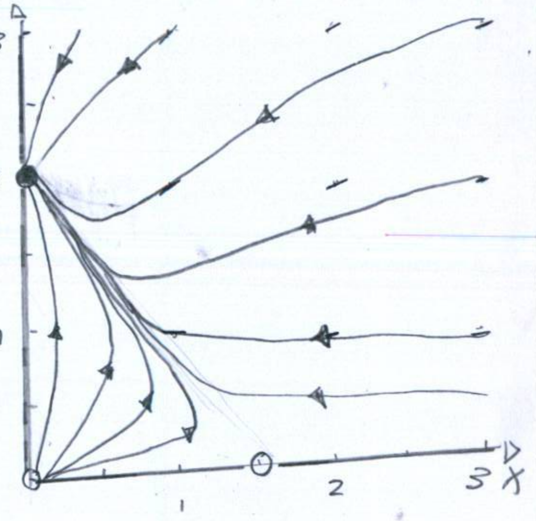
$\dot{x} = x(3-2x-2y)$ 6.4.3. $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = x(3-2x-2y)$

$\dot{y} = y(2-x-y)$ $\dot{y} = 0 = y(2-x-y)$

$(x^*, y^*) = (0,0)$ "unstable"

$(0,2)$ "stable"

$(3/2, 0)$ "unstable"



$\boxed{\text{Nullclines}}$ $x = 0 ; y = 0$

$y = 2 - x ; y = \frac{3}{2} - x$

$\boxed{\text{Basin of Attraction}}$ $(x>0) \wedge (y>0)$

$\dot{N}_1 = r_1 N_1 - b N_1 N_2$ 6.4.4.

$\dot{N}_2 = r_2 N_2 - b_2 N_1 N_2$

a. The N_1 and N_2 model is less realistic because population for rabbits and sheep decreases from an interaction.

b. Unable to complete problem without $r_1 = r_2 = b = b_2 = 1$

$x = N_1 ; y = N_2 ; t = \tau$; $x' = x(1-y)$; $y' = x(1-y)$

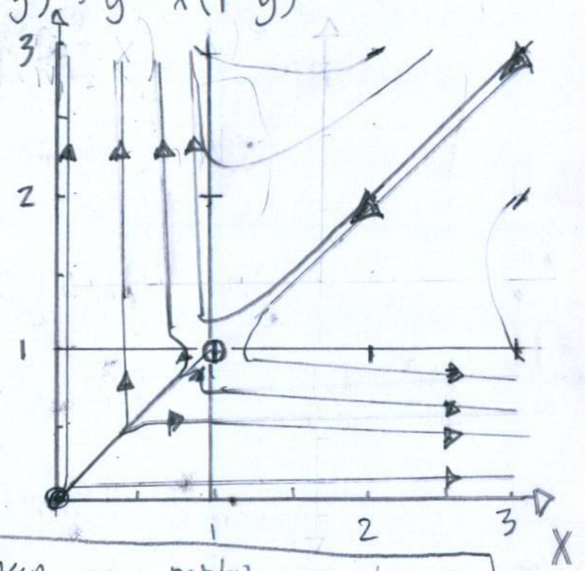
c. $\boxed{\text{Fixed Points}}$: $x' = 0 = x(1-y)$

$y' = 0 = y(1-x)$

$(x^*, y^*) = (0,0), (1,1)$

$\boxed{\text{Nullclines}}$ $y = 1 ; x = 1$

$y = 0 ; x = 0$



d. See part c in order to denote sheep or rabbit populations approach infinity when rabbit per sheep is less than 1 or sheep per rabbit is less than 1.

e. $\frac{dx}{dy} = \frac{x}{y}\frac{(1-y)}{(1-x)}$; $\int \frac{(1-x)}{x} dx = \int \frac{(1-y)}{y} dy$; $\ln x - x = \boxed{\ln y - y + C}$

when $\rho = 1$

$\dot{N}_1 = r_1 N_1 (1 - N_1/K_1) - b_1 N_1 N_2$ 6.4.5. $\frac{dN_1}{dt}\left(\frac{1}{K_1}\right) = r_1 N_1 \left(\frac{1}{K_1}\right)(1 - N_1/K_1) - b_1 N_1 N_2 \left(\frac{1}{K_1}\right)$; $x = N_1/K_1$

$\dot{N}_2 = r_2 N_2 - b_2 N_1 N_2$ $\frac{dx}{dt} = r_1 x(1-x) - b x N_2$; $\frac{dx}{dt}\left(\frac{1}{r_1}\right) = x(1-x) - \frac{b}{r_1} x N_2$

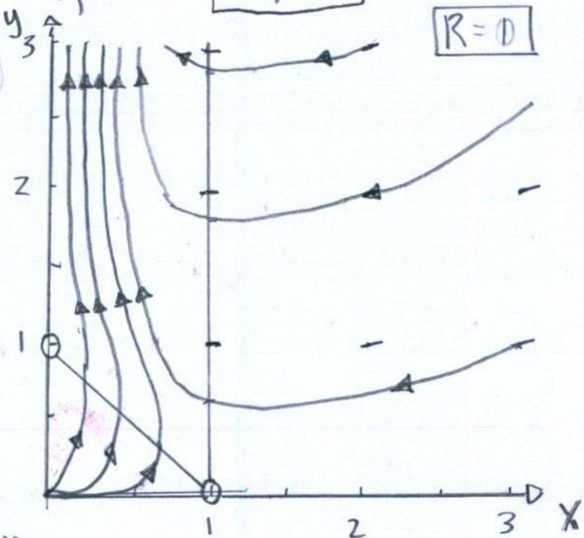
$\rho = \frac{b_1}{r_1}$; $\tau = t r$; $N_2 = y$

$$\boxed{x' = x(1-x) - \rho_1 x \cdot y}$$

$$y' = \frac{r_2}{r_1} N_2 - \frac{b_2}{r_1} N_1 N_2 = y' = Ry - \rho_2 xy$$

when $R = \frac{r_2}{r_1}$; $\boxed{\rho_2 = \frac{b_2}{r_1} K_1}$

$\boxed{R=1}$



$\boxed{\text{Fixed Points}}$ $x' = 0 = x(1 - x - \rho_1 y)$

$y' = 0 = y(R - \rho_2 x)$

$(x^*, y^*) = (0,0), (1,0)$

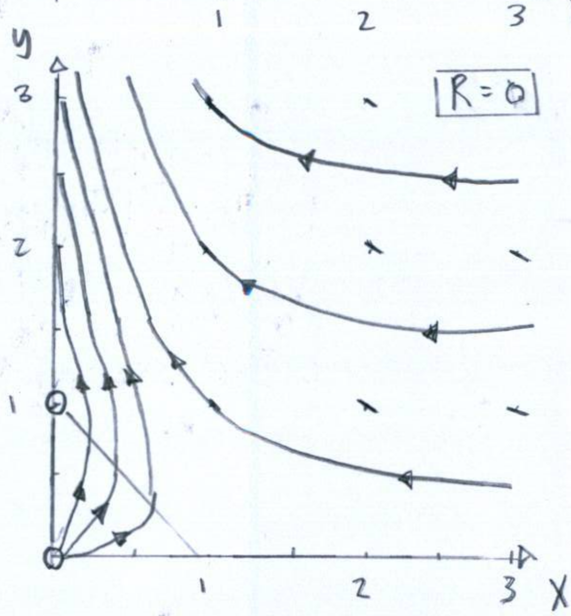
If $R = 0$, $(0, y)$

and $\rho_1 = \rho_2 = 1$

$\dot{N}_1 = r_1 N_1 (1 - N_1/K_1) - b_1 N_1 N_2$

$\dot{N}_2 = r_2 N_2 (1 - N_2/K_2) - b_2 N_1 N_2$

$\boxed{R=0}$



6.4.6.

a. $\dot{N}_1 \left(\frac{1}{K_1}\right) = r_1 \frac{N_1}{K_1}(1 - N_1/K_1) - b_1 \frac{N_1}{K_1} N_2$

$x = \frac{N_1}{K_1}$

$\frac{dx}{dt} = r_1 x(1-x) - b_1 x N_2$

$\dot{N}_2 \left(\frac{1}{K_2}\right) = r_2 \frac{N_2}{K_2}\left(1 - \frac{N_2}{K_2}\right) - b_2 N_1 \frac{N_2}{K_2}$

$y = N_2/K_2$

$\frac{dy}{dt} = r_2 y(1-y) - b_2 N_1 y$

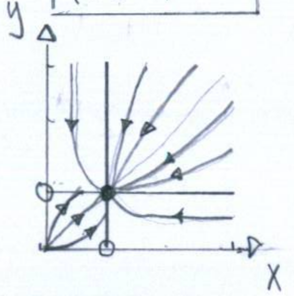
$\tau = t \cdot r_1$; $R = r_2/r_1$; $\rho_1 = \left(\frac{b_1}{r_1}\right) K_2$; $\rho_2 = \left(\frac{b_2}{r_1}\right) K_1$

$\boxed{\dot{x} = x(1 - x - \rho_1 y); \ \dot{y} = y(1 - y - \rho_2 x) \text{ when } R = 1}$

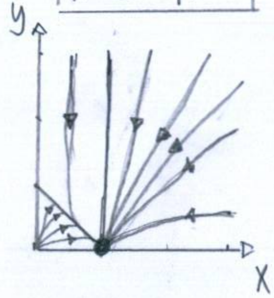
A total of six dimensionless groups suffice.

b.

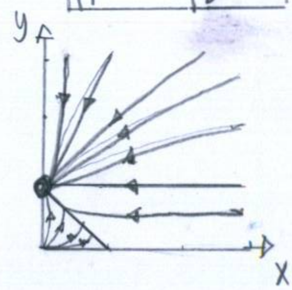
$\boxed{\rho_1 = 0 \quad \rho_2 = 0}$



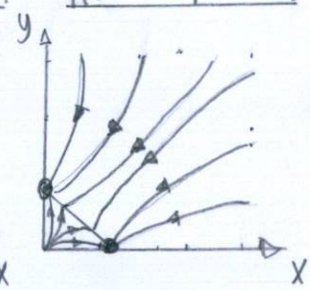
$\boxed{\rho_1 = 0 \quad \rho_2 = 1}$



$\boxed{\rho_1 = 1 \quad \rho_2 = 0}$



$\boxed{\rho_1 = 1 \quad \rho_2 = 1}$



c. $\boxed{\text{The species coexist when } \rho_1 = \rho_2 = 0. \text{ This parameter describes the interaction between the rabbits and sheep as noncompetitive.}}$

$\dot{n}_1 = G_1 N n_1 - K_1 n_1$

$\dot{n}_2 = G_2 N n_2 - K_2 n_2$

6.4.7. $N(t) = N_0 - \alpha_1 n_1 - \alpha_2 n_2$

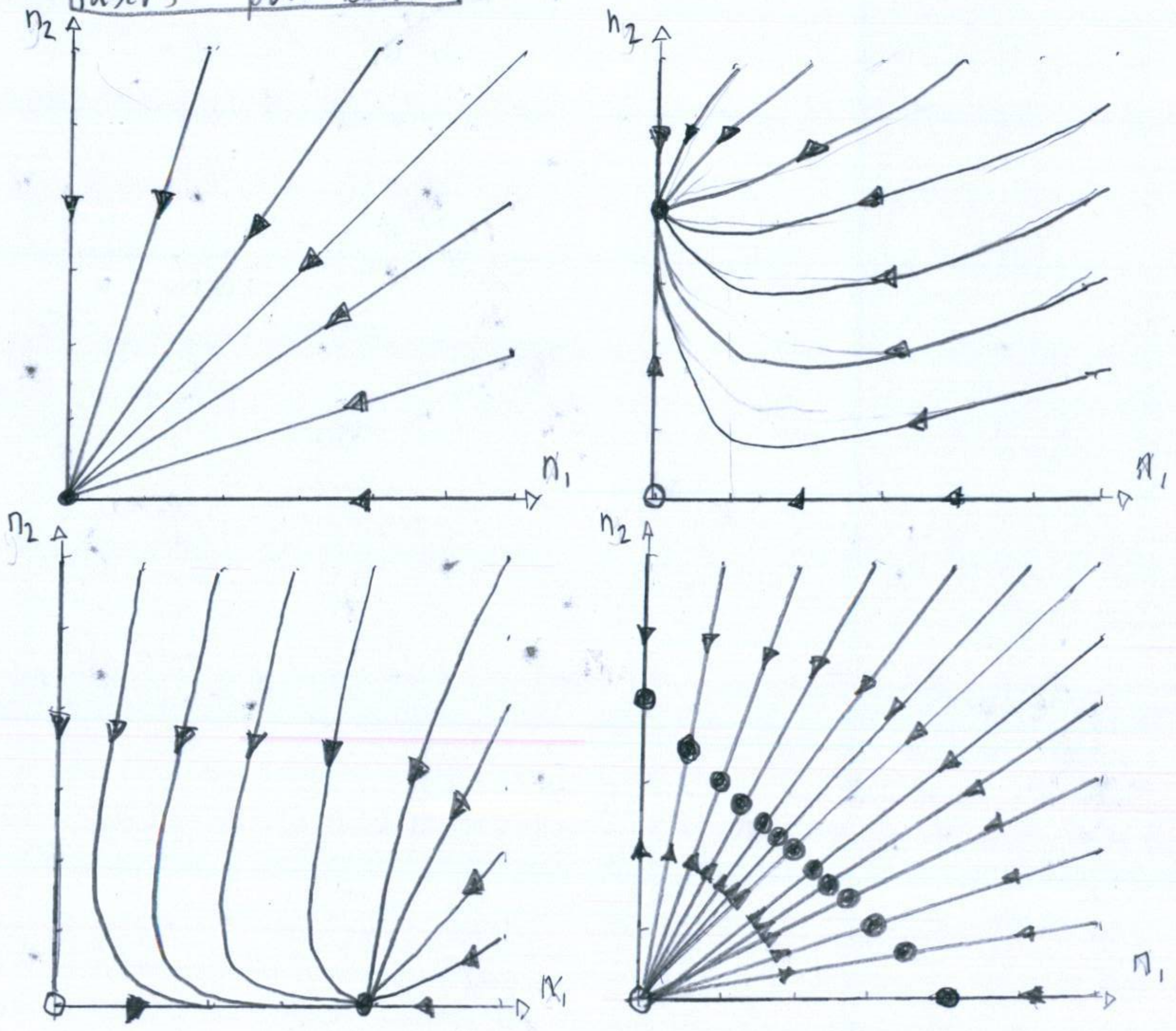
a.
$$A = \begin{pmatrix} \frac{d\dot{n}_1}{dn_1} & \frac{d\dot{n}_1}{dn_2} \\ \frac{d\dot{n}_2}{dn_1} & \frac{d\dot{n}_2}{dn_2} \end{pmatrix} = \begin{pmatrix} G_1 N - K_1 & 0 \\ 0 & G_2 N - K_2 \end{pmatrix}$$

$$\Delta = (G_1 N - K_1)(G_2 N - K_2) \; ; \; \tau = (G_1 + G_2)N - (K_1 + K_2)$$

$\boxed{\tau^2 - 4\Delta > 0 \; ; \text{Unstable Node}}$

b. $\boxed{\text{The other fixed points are } G_1 N = K_1 \text{ and } G_2 N = K_2}$

c. $\boxed{\text{Four phase portraits appear by varying the lasers parameters.}}$



$\dot{x} = ax^c - \phi x$

$\dot{y} = by^c - \phi y$

$\phi \equiv ax^c + by^c$

6.4.8. a. If $x_0 + y_0 = 1$, $\dot{x} + \dot{y} = ax^c - (ax^c + by^c)x + by^c - (ax^c + by^c)y$

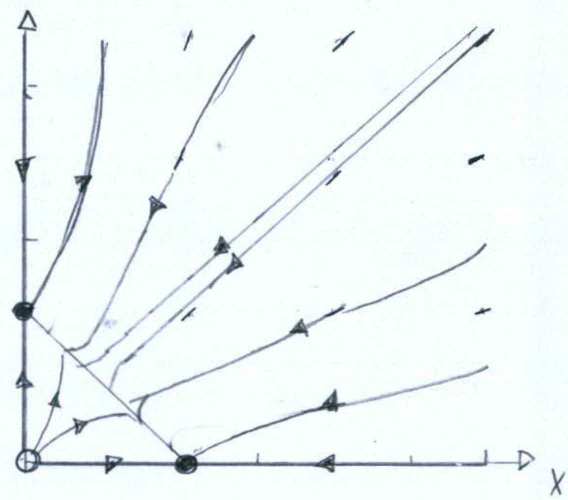
$$= a(1-x-y)x^c + b(1-x-y)y^c$$

then $\boxed{\dot{x} + \dot{y} = 0 \quad \text{and} \quad x(t) + y(t) = 1}$

b. $\lim_{x \to \infty} \dot{y}/\dot{x} = \dfrac{by^c - \phi y}{ax^c - \phi x} = \dfrac{by^c - (ax^c + by^c)y}{ax^c - (ax^c + by^c)x} \simeq \dfrac{-ax^c}{-ax^{c+1}} = \dfrac{1}{x} = \boxed{0}$

$\lim_{y \to \infty} \dot{y}/\dot{x} = \dfrac{by^c - \phi y}{ax^c - \phi x} = \dfrac{by^c - (ax^c + by^c)y}{ax^c - (ax^c + by^c)x} \simeq \dfrac{-by^{c+1}}{-by^c x} = \boxed{\infty}$

c. If $c = 1$,



d. $\boxed{\text{If } c > 1, \text{ then radial nullclines become generated.}}$

e. $\boxed{\text{If } c < 1, \text{ then monotonically decreasing nullclines become generated.}}$

$\dot{I} = I - \alpha C$

$\dot{C} = \beta(I - C - G)$

6.4.9. $I \geq 0$: National Income ; $C \geq 0$: Rate of Consumer Spending.

$G \geq 0$: Rate of Government Spending.

$1 < \alpha < \infty$ and $1 \leq \beta < \infty$

a. $\boxed{\text{Fixed Points}}$ $\dot{I} = 0 = I - \alpha C$; $\dot{C} = 0 = \beta(I - C - G)$

$$(I^*, C^*) = \left(\frac{\alpha G}{\alpha - 1}, \frac{G}{\alpha - 1}\right)$$

$$\dot{\mathbb{I}} = A \cdot \mathbb{I} \; ; \; A = \begin{pmatrix} \frac{\partial \dot{I}}{\partial I} & \frac{\partial \dot{I}}{\partial C} \\ \frac{\partial \dot{C}}{\partial I} & \frac{\partial \dot{C}}{\partial C} \end{pmatrix} = \begin{pmatrix} 1 & -\alpha \\ \beta & -\beta \end{pmatrix}$$

If $\beta = 1$, $A = \begin{pmatrix} 1 & -\alpha \\ 1 & -1 \end{pmatrix}$, $\Delta = -(1-a)$; $\tau = 0$; $\tau^2 - 4\Delta = 4(1-a)$

$\boxed{\text{A center node}}$

b. $G = G_0 + KI \,;\, K > 0 \,;\, I \geq 0,\ C \geq 0$

$\boxed{\text{Fixed Point}}$ $(I^*, C^*) = \left(\dfrac{\alpha G_0}{\alpha(1-K)-1}, \dfrac{G_0}{\alpha(1-K)-1} \right)$

If $K < K_c = 1 - \dfrac{1}{\alpha}$, then $I \,\&\, C > 0$

$$\mathring{I} = AI \,;\; A = \begin{pmatrix} 1 & -\alpha \\ \beta(1-K) & -\beta \end{pmatrix};\; (A - \lambda I) = \begin{pmatrix} 1-\lambda & -\alpha \\ \beta(1-K) & -\beta-\lambda \end{pmatrix}$$

$$(1-\lambda)(-\beta-\lambda) + \alpha \cdot \beta(1-K) = 0$$

$$\lambda_{1,2} = \frac{-(\beta-1) \pm \sqrt{(\beta-1)^2 - 4(\alpha(1-K)-1)\beta}}{2}$$

$$A\vec{v}_1 = \begin{pmatrix} \dfrac{(1+\beta)+\sqrt{(\beta-1)^2-4(\alpha(1-K)-1)\beta}}{2} & -\alpha \\ \beta(1-K) & \dfrac{-(1+\beta)+\sqrt{(\beta-1)^2-4(\alpha(1-K)-1)\beta}}{2} \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$$

$$= \left[(1+\beta)+\sqrt{(\beta-1)^2-4(\alpha(1-K)-1)\beta}\right] v_1 - \alpha_1 v_2 = 0$$

$$v_{11} = 1 \,;\; v_{12} = \frac{(1+\beta)+\sqrt{\beta^2+(2-4\alpha(1-K))\beta}}{\alpha}$$

$$\boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ \dfrac{(1+\beta)+\sqrt{\beta^2+(2-4\alpha(1-K))\beta}}{\alpha} \end{pmatrix}}$$

$$A\vec{v}_2 = \begin{pmatrix} (1+\beta)-\sqrt{(\beta-1)^2-4(\alpha(1-\beta)-1)\beta} & -\alpha \\ \beta(1-K) & \dfrac{-(1+\beta)-\sqrt{(\beta-1)^2-4(\alpha(1-K)-1)\beta}}{2} \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$$

$$\left[(1+\beta)-\sqrt{(\beta-1)^2-4(\alpha(1-K)-1)\beta}\right] v_1 - \alpha_1 v_2 = 0$$

$$v_{21} = 1 \,;\; v_{22} = \frac{(1+\beta)-\sqrt{\beta^2+(2-4\alpha(1-K))\beta}}{\alpha}$$

$$\boxed{\vec{v}_2 = \begin{pmatrix} 1 \\ \dfrac{(1+\beta)-\sqrt{\beta^2+(2-4\alpha(1-K))\beta}}{\alpha} \end{pmatrix}}$$

When $k > k_c$, the economy gravitates to the positive eigendirection.

c) $G = G_0 + kI^2$; $\dot{I} = I - \alpha C = 0$; $\dot{C} = \beta(I - C - G_0 - k_0 I^2) = 0$

Fixed Points: $0 = \beta(I(1 - 1/\alpha) - G_0 - k_0 I^2) = -k_0 I^2 + I(1 - 1/\alpha) - G_0$

$$(I^*, C^*) = \left(\frac{+(1-1/\alpha) + \sqrt{(1-1/\alpha)^2 + 4k_0 G_0}}{2k_0}, \frac{(1-1/\alpha) + \sqrt{(1-1/\alpha)^2 - 4k_0 G_0}}{2k_0 \alpha} \right)$$

$$, \left(\frac{(1-1/\alpha) - \sqrt{(1-1/\alpha)^2 - 4k_0 G_0}}{2k_0}, \frac{(1-1/\alpha) - \sqrt{(1-1/\alpha)^2 - 4k_0 G_0}}{2k_0 \alpha} \right)$$

If $G_0 < \frac{(\alpha-1)^2}{4\alpha^2 k_0}$, then two positive fixed points exist

If $G_0 = \frac{(\alpha-1)^2}{4\alpha^2 k_0}$, then one fixed point exists in quadrant #1

If $G_0 > \frac{(\alpha-1)^2}{4\alpha^2 k_0}$, then zero fixed points exist because of the imaginary radical.

$\dot{X}_i = X_i \left(X_{i-1} - \sum_{j=1}^{n} X_j X_{j-1} \right)$

6.4.10.

a. If $n = 2$, $\dot{X}_1 = X_1 (X_0 - \sum_{j=1}^{n} X_1 X_0) = X_1 (X_2 - \sum_{j=1}^{2} X_1 X_2) = X_1 (X_2 - 2X_1 X_2)$

$\dot{X}_2 = X_2 (X_1 - \sum_{j=1}^{2} X_2 X_1) = \boxed{X_2 (X_1 - 2X_2 X_1)}$

b. $\dot{X}_1 = 0 = X_1 (X_2 - 2X_1 X_2)$; $\dot{X}_2 = 0 = X_2 (X_1 - 2X_2 X_1)$

$(X_1^*, X_2^*) = (1/2, 1/2)$; $A = \begin{pmatrix} X_2 - 4X_1 X_2 & X_1(1 - 2X_1) \\ X_2(1 - 2X_2) & X_1 - 4X_1 X_2 \end{pmatrix}$

$A_{(1/2, 1/2)} = \begin{pmatrix} -1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$; $\Delta = \frac{1}{4}$, $\tau = -1$

$\tau^2 - 4\Delta = 0$

$\boxed{\text{Degenerate and stable Node}}$

c. $u = X_1 + X_2$; $\dot{u} = \dot{X}_1 + \dot{X}_2 = X_1 (X_2 - 2X_1 X_2) + X_2 (X_1 - 2X_1 X_2)$

$= X_1 X_2 - 2X_1^2 X_2 + X_1 X_2 - 2X_1 X_2^2$

$= 2X_1 X_2 (1 - X_1 - X_2) = 2X_1 X_2 (1 - u)$

$u(t) = 1 - e^{-2X_1 X_2 t}$

$\boxed{\lim_{t \to \infty} u(t) = 1}$

d. $v = x_1 - x_2$; $\dot{v} = \dot{x}_1 - \dot{x}_2$; $\dot{v} = x_1(x_2 - 2x_1x_2) - x_2(x_1 - 2x_1x_2)$

$$= -2x_1x_2(x_1 - x_2) = -2x_1x_2 \cdot v$$

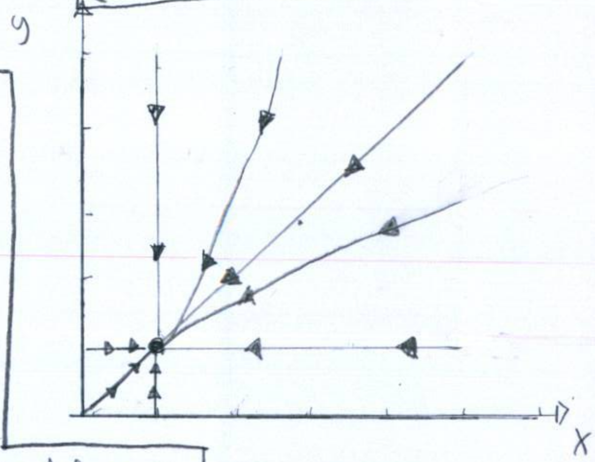
$$v(t) = e^{-2x_1x_2 t}$$

$$\boxed{\lim_{t\to\infty} v(t) = 0}$$

e. $\lim_{t\to\infty} [u(t) + v(t)] = 1 = 2x_1$; $x_1 = 1/2$

$\lim_{t\to\infty} [u(t) - v(t)] = 1 = 2x_2$; $x_2 = 1/2$; $\boxed{(x_1, x_2) \to (1/2, 1/2)}$

f. A large n value generates a plot which seems to converge to zero, but actually, converges to a positive value close to zero. This argument implies RNA remain at low concentrations indefinitely.



$\dot{x} = rxz$
$\dot{y} = ryz$
$\dot{z} = -rxz - ryz$

6.4.11

a. $\dot{z} = -\dot{x} - \dot{y}$; $0 = \dot{x} + \dot{y} + \dot{z}$; $\boxed{1 = x + y + z}$

b. The limit of the function is bounded by the invariance equation in part a.

Fixed Points: $\dot{x} = 0 = rxz$; $\dot{y} = ryz$; $\dot{z} = -rxz - ryz$

$$(x^*, y^*, z^*) = (x, y, 0), (0, 0, z)$$

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial z} \\ \frac{\partial \dot{z}}{\partial x} & \frac{\partial \dot{z}}{\partial y} & \frac{\partial \dot{z}}{\partial z} \end{pmatrix} = \begin{pmatrix} rz & 0 & rx \\ 0 & rz & ry \\ -rz & -rz & -rx-ry \end{pmatrix}$$

$$A_{(x,y,0)} = \begin{pmatrix} 0 & 0 & rx \\ 0 & 0 & ry \\ 0 & 0 & -rx-ry \end{pmatrix} ; A_{(0,0,z)} = \begin{pmatrix} rz & 0 & 0 \\ 0 & rz & 0 \\ -rz & -rz & 0 \end{pmatrix}$$

$\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = -rx - ry$ $\quad$ $\lambda_1 = \lambda_2 = 0,\ \lambda_3 = rz$

The eigenvectors point in the direction of λ_3 for each fixed point

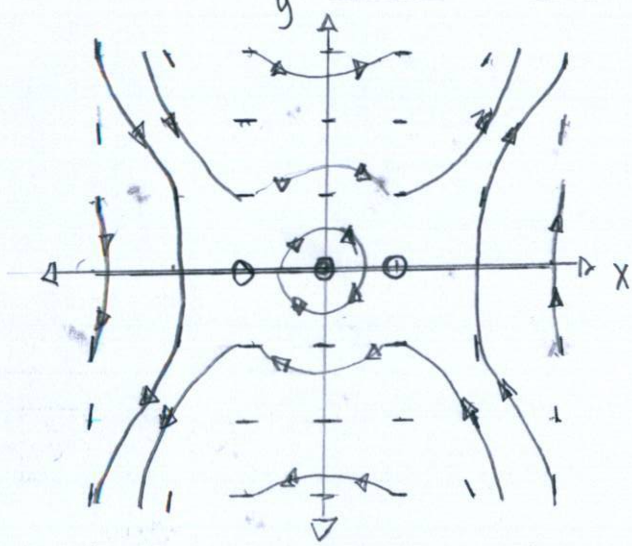
c. An interpretation from the political terms is $r<0$, the centrist pull the extremists to the centrist, while $r>0$, the extremist separate the centrists.

$\ddot{x}=x^3-x$ 6.5.1.a. $\dot{x}=y$; $\dot{y}=x^3-x$; $A=\begin{pmatrix}0 & 1\\ 3x-1 & 0\end{pmatrix}$;

$\boxed{\text{Fixed Points}}$ $\dot{x}=0=y$; $\dot{y}=0=x^3-x$; $(x^*,y^*)=(-1,0)$ "center"
$(0,0)$ "saddle"
$(1,0)$ "center"

b. $E=\frac{1}{2}\dot{x}^2-\int x^3-x\,dx=\boxed{\frac{1}{2}y^2-\frac{x^4}{4}+\frac{x^2}{2}+C}$

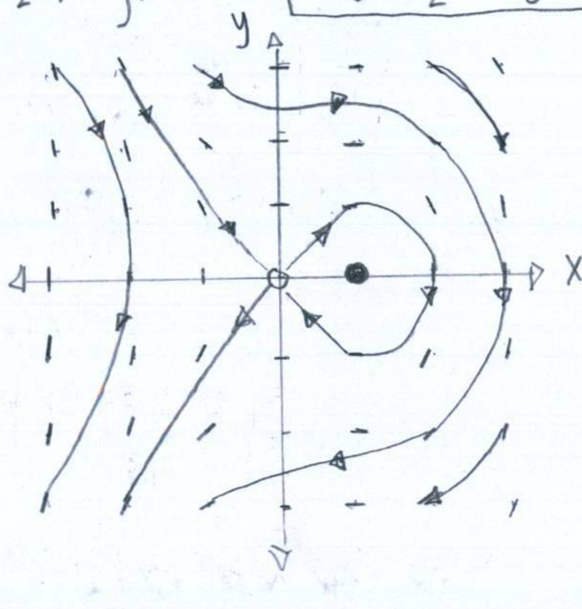
c.



$\ddot{x}=x-x^2$ 6.5.2 .a. $\dot{x}=y$; $\dot{y}=x-x^2$; $A=\begin{pmatrix}0 & 1\\ 1-2x & 0\end{pmatrix}$

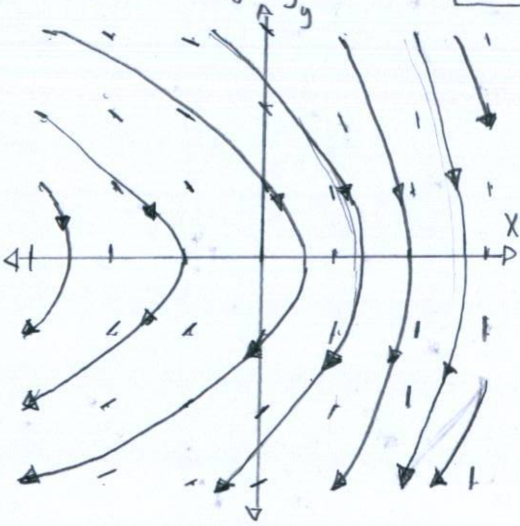
$\boxed{\text{Fixed Points}}$ $\dot{x}=0=y$; $\dot{y}=0=x-x^2$; $(x^*,y^*)=(+1,0)$ "center"
$(0,0)$ "saddle"

b. $E=\frac{1}{2}\dot{x}^2-\int(x-x^2)dx=\boxed{\frac{1}{2}y^2-\frac{x^2}{2}+\frac{x^3}{3}+C}$

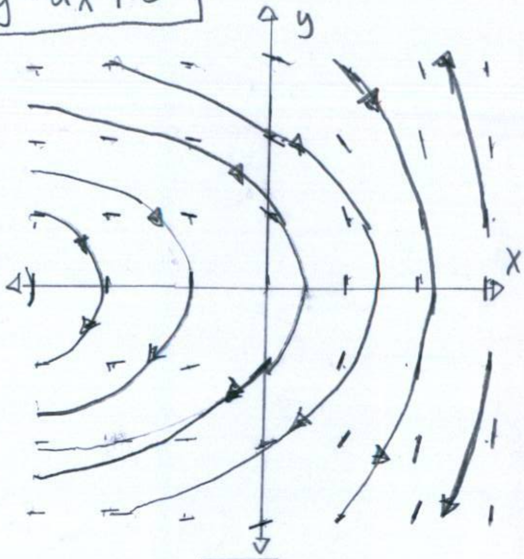


C. $\boxed{E = \frac{1}{2}y^2 - \frac{x^2}{2} + \frac{x^3}{3} + c.}$

$\ddot{x} = a - e^x$ 6.5.3. $\dot{x} = y$; $\dot{y} = a - e^x$; $E = \frac{1}{2}y^2 - \int_y a - e^x\,dx = \boxed{\frac{1}{2}y^2 - ax + e^x}$



$\boxed{a<0}$



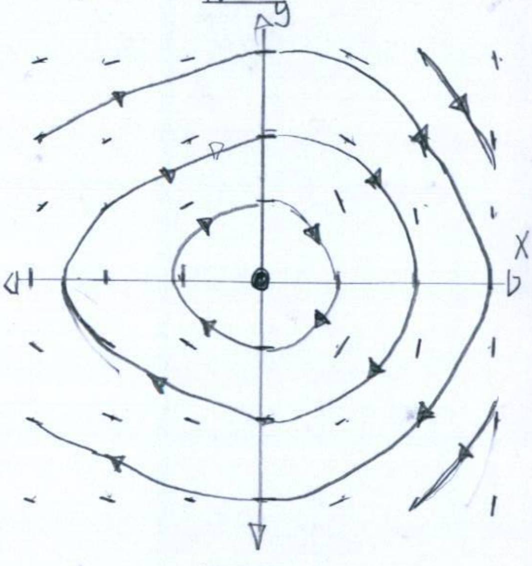
$\boxed{a=0}$

$\ddot{x} = ax - x^2$ 6.5.4. $\dot{x} = y$; $\dot{y} = ax - x^2$

Conserved Quantity:

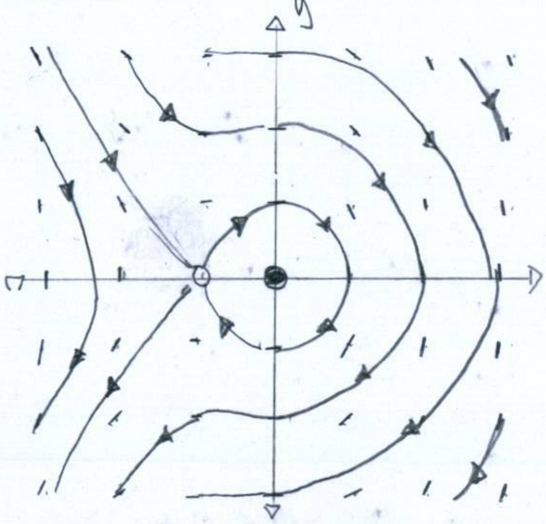
$$E = \frac{1}{2}y^2 - \int ax - x^2\,dx$$

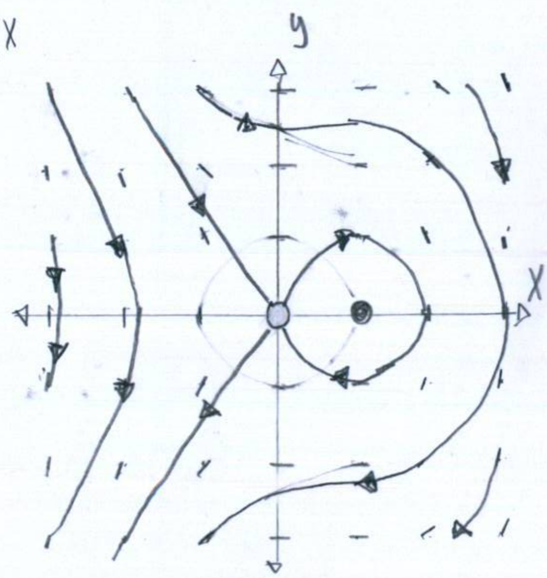
$$\boxed{= \frac{1}{2}y^2 - \frac{ax^2}{2} + \frac{x^3}{3} + C}$$



$\boxed{a>0}$

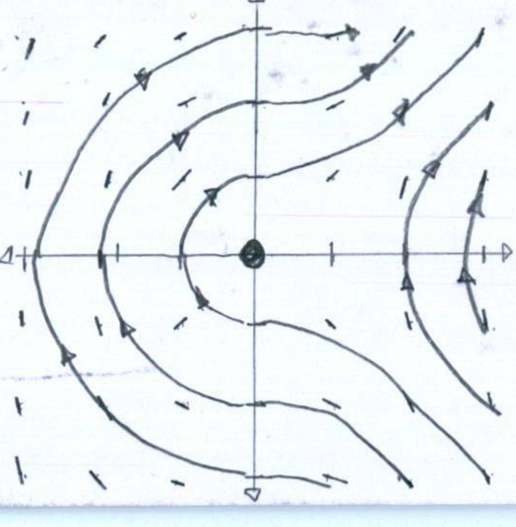
$\boxed{a<0}$





$\boxed{a>0}$

$\boxed{a=0}$



$\ddot{x} = (x-a)(x^2-a)$ 6.5.5 $\dot{x} = y$; $\dot{y} = (x-a)(x^2-a)$

$\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = y$

$\dot{y} = 0 = (x-a)(x^2-a)$

$(x^*, y^*) = (a, 0)$

$(\sqrt{a}, 0)$

$(-\sqrt{a}, 0)$

If $a = 1$, then one fixed point exists in quadrant #1 and #4.

If $0 < a < 1$ or $a > 1$, then two fixed points exist in quadrants #1 and #4

$\dot{x} = -kxy$

$\dot{y} = kxy - ly$

6.5.6. a. $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = -kxy$; $\dot{y} = 0 = kxy - ly$

$(x^*, y^*) = (0, 0)$; $A = \begin{pmatrix} -ky & -kx \\ ky & kx - l \end{pmatrix}$

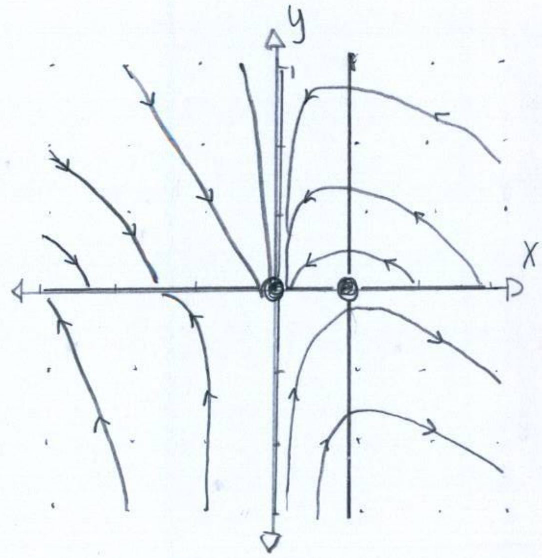
"center"

$(l/k, 0)$ "center"

b. $\boxed{\text{Nullclines}}$ $x = 0$

$y = 0$

$x = \frac{l}{k}$



c. $\boxed{\frac{dy}{dx} = -1 + l/kx \ ; \ y = -x + \frac{l}{k}\ln x + c}$

d. $\boxed{\text{See part c}}$

e. $\boxed{\text{A population is sick from infection when } y_0 \geq 0.}$

$\frac{d^2u}{d\theta^2} + u = \alpha + \varepsilon u^2$ 6.5.7 $u = 1/r$;

a. $\boxed{\begin{array}{l} v^2 + u = \alpha + \varepsilon u^2 \\ \text{where } v = du/d\theta. \end{array}}$

b. $\boxed{\text{Fixed Points}}$: $\mathring{u} = 0 = v$

$\mathring{v} = 0 = \kappa + \varepsilon u^2 - u$

$(u^*, v^*) = \left(\frac{1+\sqrt{1-4\kappa\varepsilon}}{2\varepsilon}, 0\right), \left(\frac{1-\sqrt{1-4\kappa\varepsilon}}{2\varepsilon}, 0\right)$

c. $A = \begin{pmatrix} 0 & 1 \\ 2\varepsilon u - 1 & 0 \end{pmatrix}$; $\lambda_{1A,B} = \pm i\sqrt{1-4\alpha\varepsilon}$; $\lambda_{2A,B} = \pm\sqrt{1-4\cdot\kappa\varepsilon}$

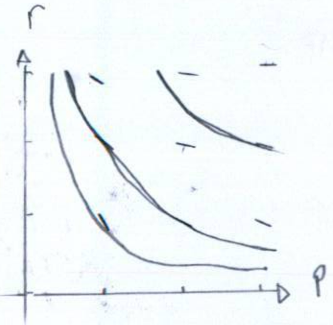
"Saddle point" $\boxed{\text{"linear "Center"}}$

d. $\frac{1}{r} = u = \frac{1-\sqrt{1-4\kappa\varepsilon}}{2\varepsilon}$; $\boxed{r = \frac{2\varepsilon}{1-\sqrt{1-4\kappa\varepsilon}}}$

$H = \frac{p^2}{2m} + \frac{kx^2}{2}$ 6.5.8 $\underbrace{\mathring{q} = \frac{p}{m}}_{\text{"Momentum"}}$; $\underbrace{\mathring{p} = -kx}_{\text{"Force"}}$; $H = \underbrace{\frac{p^2}{2m}}_{\text{"Kinetic Energy"}} + \underbrace{\frac{kx^2}{2}}_{\text{"Potential Energy"}}$

$H(x,p)$ 6.5.9 $\mathring{H} = \frac{p}{m}\mathring{p} + kx\mathring{x} = \frac{p}{m}(-kx) + kx\left(\frac{p}{m}\right) \boxed{= 0}$

$H(p,r) = \frac{p^2}{2} + \frac{h^2}{2r^2} - \frac{k}{r}$ 6.5.10 a. $\boxed{\text{The Hamiltonian plot is similar to the potential plot of } 1/r^2 \text{ where } k = 1 \text{ and } h = 6.626\times10^{-34}\,\text{J}\cdot\text{s}.}$



b.

$-k^2/2h^2 < E < 0$	$E = 0$	$E > 0$
• Slope is negative	Slope is zero	Slope is positive
• Momentum is decreasing	• Momentum is constant	• Momentum is increasing
• radius is increasing	• Radius is increasing	• radius is increasing.

c. If $k < 0$, then $A = \begin{pmatrix} v & 0 \\ 0 & h^2u + u \end{pmatrix}$; $\Delta = v(h^2u + u)$

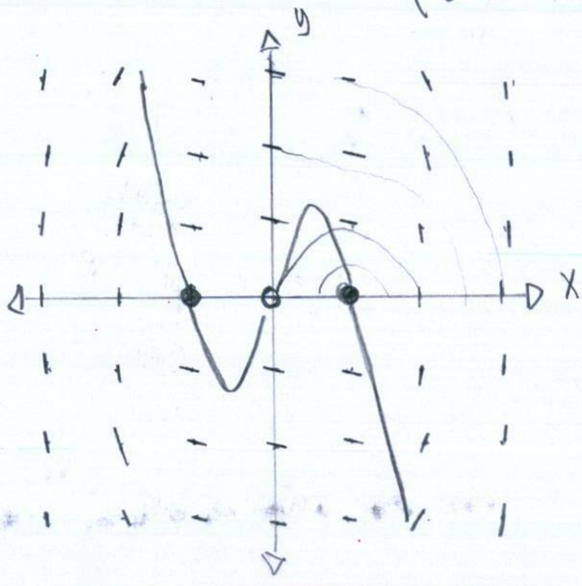
$\tau = v + h^2u + u$

$\tau^2 - 4\Delta > 0$

$\boxed{\text{No Periodic orbits}}$

$\mathring{x} = y$
$\mathring{y} = -by + x - x^3$

6.5.11



$\dot{x} = xy$

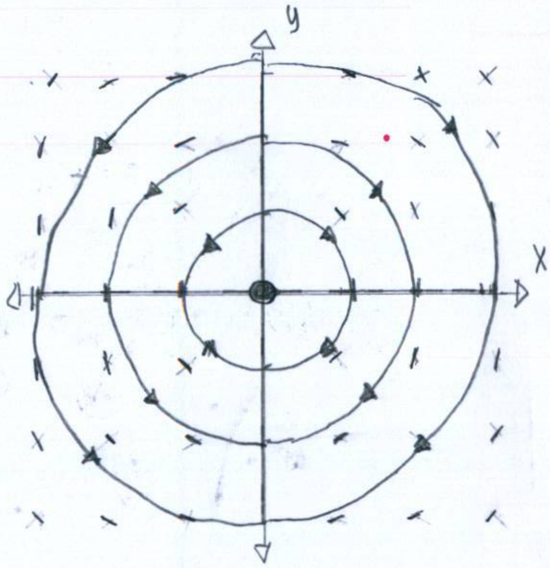
$\dot{y} = -x^2$

6.5.12.

a. $E = x^2 + y^2$; $E' = 2x\cdot\dot{x} + 2y\cdot\dot{y} = 2x^2y - 2y^2x \boxed{= 0}$

b. $(x^*, y^*) = (0,0)$; $A = \begin{pmatrix} y & x \\ -2x & 0 \end{pmatrix}$; $A_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \boxed{= 0}$

$(0, y)$; $A_{(0,y)} = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} \boxed{= 0}$



c. See part B; Non-isolated Fixed Point.

$\ddot{x} + x + \varepsilon x^3 = 0$

6.5.13.

a. $E = \frac{1}{2}\dot{x}^2 - \int(-x - \varepsilon x^3)\,dx$

$= \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{\varepsilon}{4}x^4$

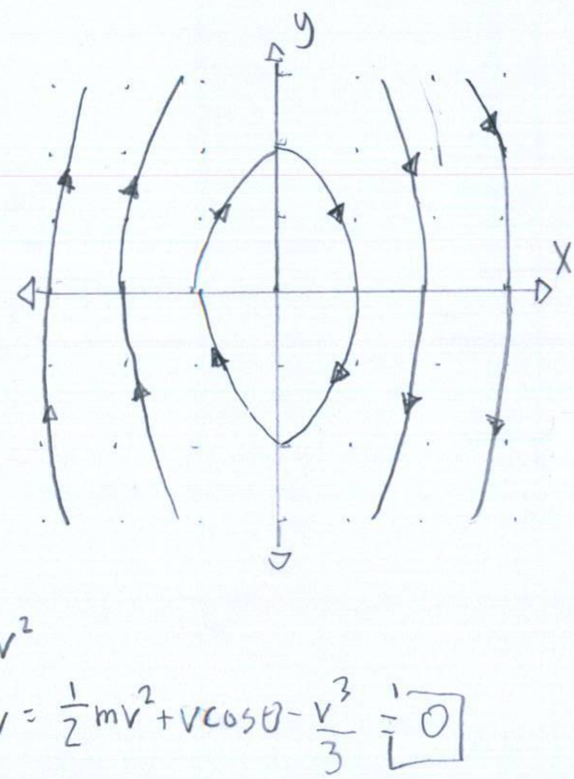
$\begin{vmatrix} E_{xx} & E_{x\dot{x}} \\ E_{\dot{x}x} & E_{\dot{x}\dot{x}} \end{vmatrix} = \begin{vmatrix} 1 + 3\varepsilon x^2 & \dot{x} + x + \varepsilon x^3 \\ \dot{x} + x + \varepsilon x^3 & 1 \end{vmatrix}_{(0,0)} \boxed{= 1}$, a continuous derivative exists at the center i.e. nonlinear center.

b. If $\varepsilon < 0$, a hyperbola trajectory is the closed orbit about $(0,0)$.

$\dot{x} = y$

$\dot{y} = -x - \varepsilon x^3$

For from the origin when $\varepsilon > 0$, this phase plot appears.



$\dot{v} = -\sin\theta - Dv^2$

$v\dot{\theta} = -\cos\theta + v^2$

6.5.14

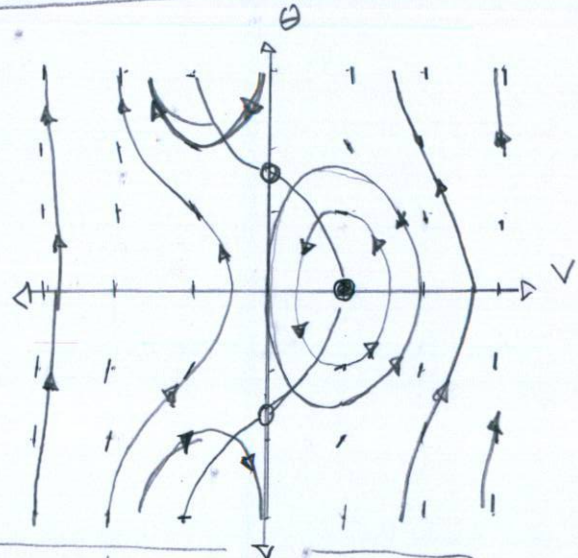
a. If $D = 0$, then $\dot{v} = -\sin\theta$

$v\dot{\theta} = -\cos\theta + v^2$

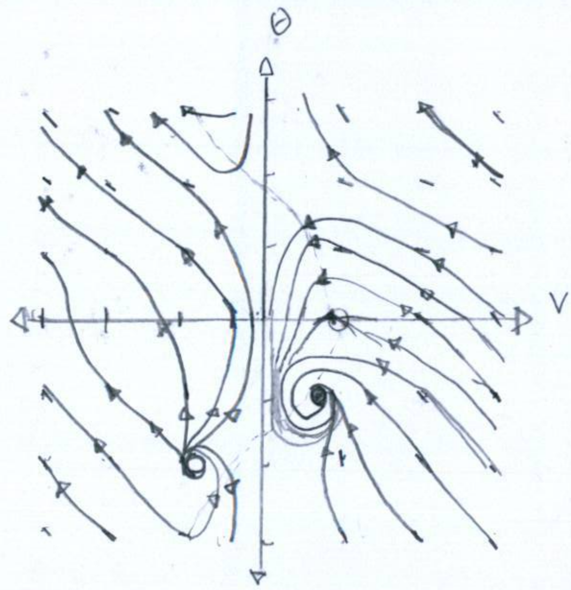
$E = \frac{1}{2}v^2 - \int v\dot{\theta}\,dv = \frac{1}{2}mv^2 + v\cos\theta - \frac{v^3}{3} \boxed{= 0}$

$\frac{1}{2}v^2 - 3v\cos\theta + v^3 = 0$; $\frac{dE}{dv} = v - 3\cos\theta + 3v^2$; $\boxed{\text{Fixed Points:}}$ $(v^*, \theta^*) = (1,0), (0,0)$

The potential energy $V(v,\theta) = -3\cos\theta + 3v^2$ has a fixed point at $(0, 2n\pi)$.



b. If $D>0$, then as the glider approaches $v \to \infty$, then the angle becomes more positive and the effect of lift propels the glider upward.



$mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi$

6.5.15

a. $b = 0$

$mr\ddot{\phi} = -mg\sin\phi + mr\omega^2\sin\phi\cos\phi$

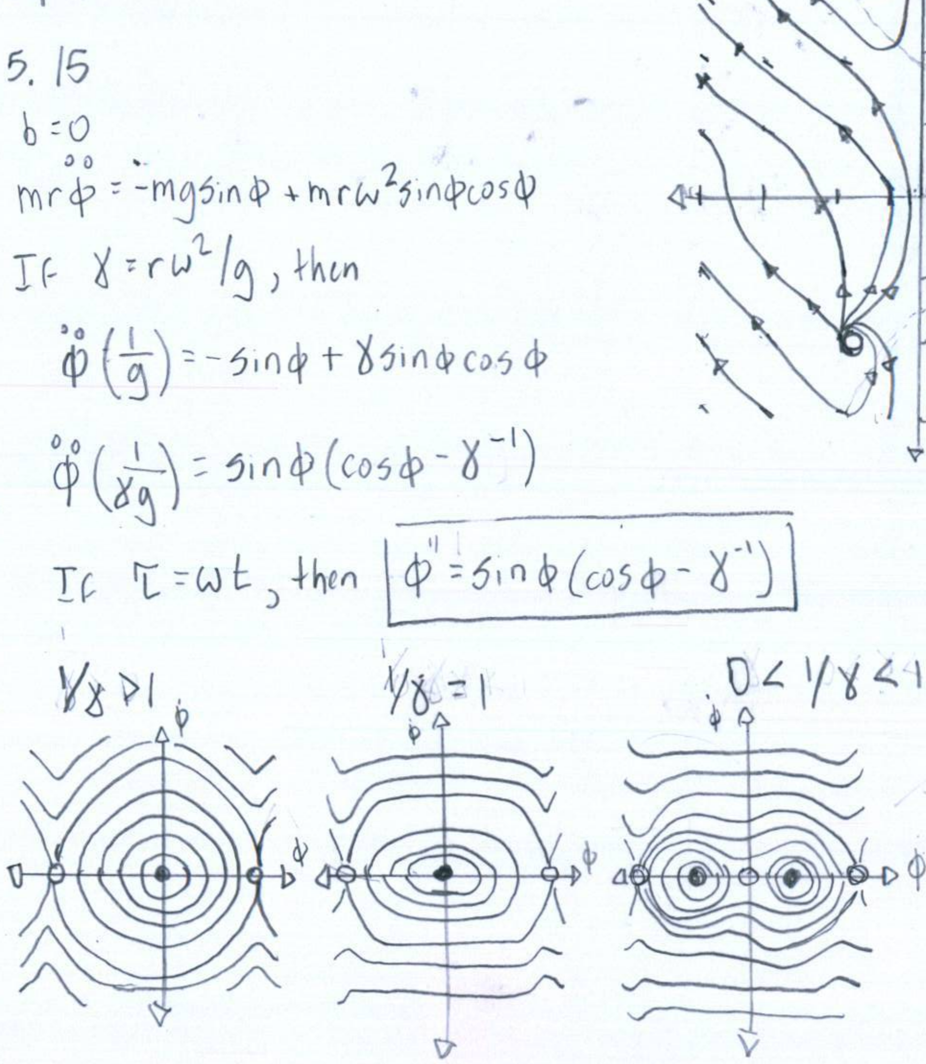
If $\gamma = r\omega^2/g$, then

$\ddot{\phi}\left(\frac{1}{g}\right) = -\sin\phi + \gamma\sin\phi\cos\phi$

$\ddot{\phi}\left(\frac{1}{\gamma g}\right) = \sin\phi(\cos\phi - \gamma^{-1})$

If $\tau = \omega t$, then $\boxed{\phi'' = \sin\phi(\cos\phi - \gamma^{-1})}$

b. $1/\gamma > 1$ $\quad$ $1/\gamma = 1$ $\quad$ $0 < 1/\gamma < 1$



c. The graphs $1/\gamma > 1$ and $1/\gamma = 1$ suggest a periodic stable point when the hoop spins, while $1/\gamma = 1$, a bead that doesn't stay in one place, and spins around the hoop.

6.5.16. $mr\ddot{\phi} = -mg\sin\phi + mr\omega^2 \sin\phi\cos\phi$

$$\ddot{\phi} = -\frac{g}{r}\sin\phi + \omega^2\sin\phi\cos\phi = \sin\phi(\omega^2\cos\phi - g/r)$$

$$0 = \sin\phi(\omega^2\cos\phi - g/r) \;;\; \boxed{\phi = \pm\frac{\pi}{2} \,;\, \arccos\frac{g}{r\omega^2}}$$

6.5.17: $mr\ddot{\phi} = -mg\sin\phi + mr\omega^2\sin\phi\cos\phi$

$$E = KE + PE = \frac{1}{2}\dot{\phi}^2 - \int \sin\phi(\omega^2\cos\phi - g/r)\,d\phi$$

$$= \frac{1}{2}\dot{\phi}^2 + \cos(\phi)\left(\omega^2\cos\phi - \frac{2g}{r\omega^2}\right)$$

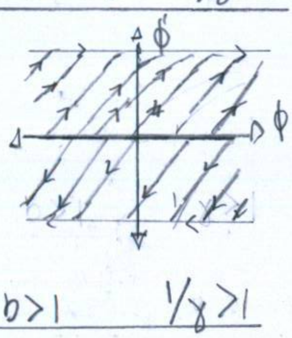
$$\ldots = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\omega^2\cos^2\phi - \frac{g}{r}\cos\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\omega^2(1-\sin^2\phi) - mgr(1-\cos\phi) - mgr$$

$$\boxed{= (KE_{Trans} - KE_{Rot}) + PE}$$

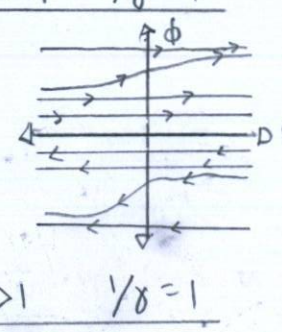
In terms of seperation of motion, the bead hoop problem has translational and rotational energy.

6.5.18. $mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi$

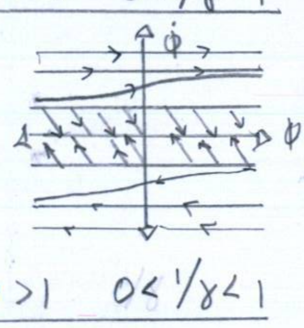
$0 < b < 1 \quad 1/\gamma > 1$



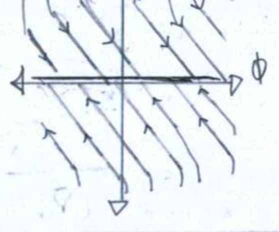
$0 < b < 1 \quad 1/\gamma = 1$



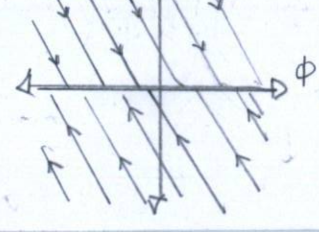
$0 < b < 1 \quad 0 < 1/\gamma < 1$



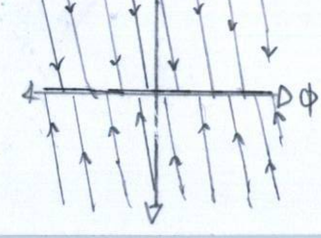
$b > 1 \quad 1/\gamma > 1$



$b > 1 \quad 1/\gamma = 1$



$b > 1 \quad 0 < 1/\gamma < 1$



$\mathring{R} = aR - bRF$
$\mathring{F} = -cF + dRF$

6.5.19. Lotka-Volterra Predator-Prey Model

a.

Term	
aR	Growth of the rabbit population
$-bRF$	Decrease of the rabbit population by interacting foxes
$-cF$	Decrease of the fox population
dRF	Growth of the fox population by eating rabbits.

An unrealistic assumption is foxes do not decrease when rabbits are not present.

b. $\mathring{R} = R(a - bF)$; $\mathring{R}\left(\frac{1}{a}\right) = \frac{R}{a}\left(1 - \frac{b}{a}F\right)$; $x = \frac{d}{c}R$; $y = \frac{b}{a}F$; $\tau = at$

$\mathring{F} = F(dR - c)$; $\mathring{y} = \frac{cy}{a}(x-1)$; $\boxed{\mathring{y} = \mu y(x-1) \; ; \; \mathring{x} = x(1-y)}$

c. $\mathring{x} = 0 = x(1-y)$; $\mathring{y} = 0 = \mu y(x-1)$; $(x^*, y^*) = \boxed{(0,0) \quad (1,1)}$

d. $A = \begin{pmatrix} 1-y & -xy \\ \mu y & \mu(x-1) \end{pmatrix}$

$A_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}$; $\Delta = \mu$; $\tau = 1 + \mu$; $\tau^2 - 4\Delta > 0$ $\boxed{\text{"Unstable node"}}$

$A_{(1,1)} = \begin{pmatrix} 0 & -1 \\ \mu & 0 \end{pmatrix}$; $\Delta = -\mu$; $\tau = 0$; $\tau^2 - 4\Delta > 0$ $\boxed{\text{"center" cycle.}}$

$\mathring{P} = P(R - S)$
$\mathring{R} = R(S - P)$
$\mathring{S} = S(P - R)$

6.5.20.

a. The terms found in $\mathring{P}$, $\mathring{R}$, and $\mathring{S}$ relate the existence of paper, rock, and scissors, but also, a relationship when each type of species is present at any given time.

b. $P + R + S = PR - PS + RS - RP + SP - SR = \boxed{0}$

c. $E(P, R, S) = P + R + S$; $E_2(R, R, S) = PRS$
"Plane" "Multiplane"

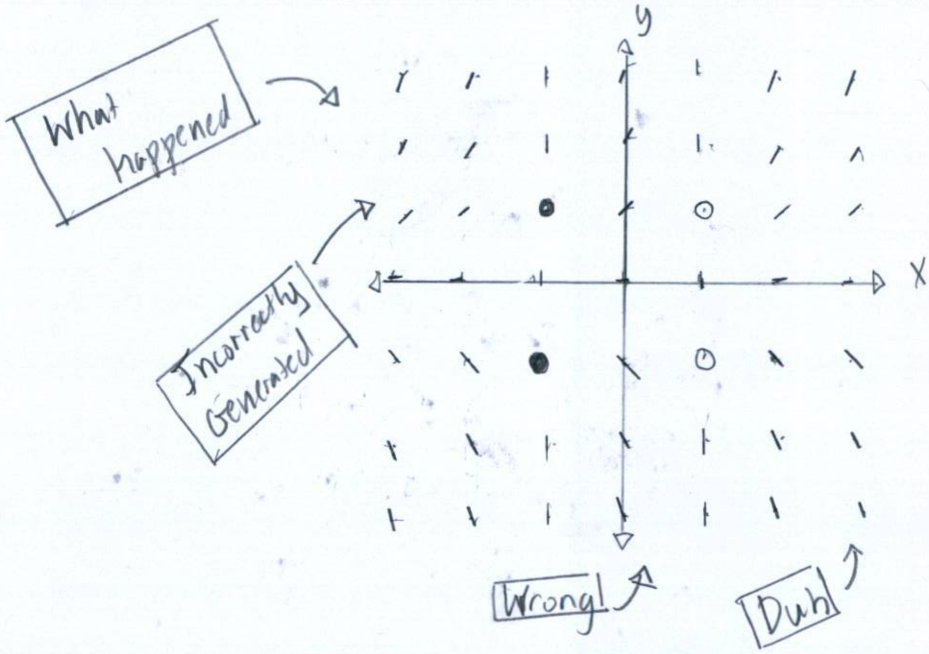
As $t \to \infty$, then a discrete solution exists of integer values between planes or amounts of P, R, S.

$\dot{x} = y(1-x^2)$ 6.6.1. Reversible if $t \to -t$, $x \to -x$, $y \to -y$

$\dot{y} = 1-y^2$

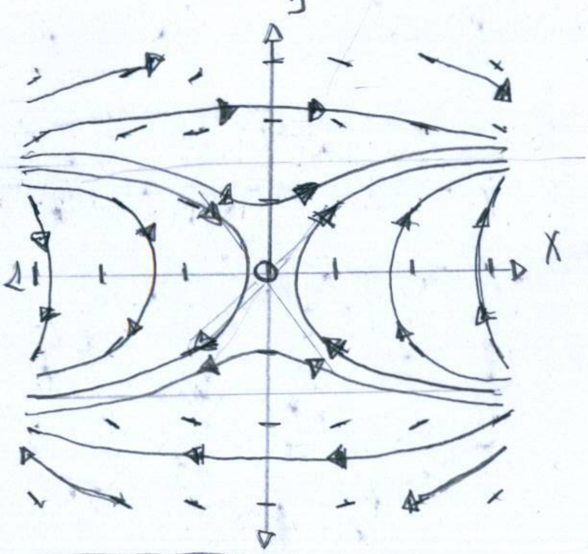
$\boxed{\text{Fixed Points:}}$ $\dot{x} = y(1-x^2) = 0$

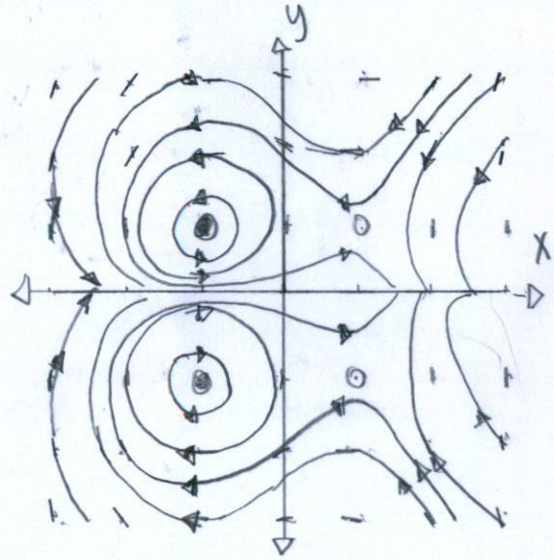
$\dot{y} = 1-y^2 = 0$; $(x^*, y^*) = (\pm 1, \pm 1)$



$\dot{x} = y$ 6.6.2.

$\dot{y} = x\cos y$





$\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = y$;

$\dot{y} = 0 = x\cos y = x\cos(-y) = -x\cos(-y)$

$(x^*, y^*) = (0,0)$

$\dot{x} = \sin y$ 6.6.3.

$\dot{y} = \sin x$

a. $\dfrac{dy}{dx} = \left(\dfrac{-1}{-1}\right)\dfrac{dy}{dx} = \dfrac{-\sin x}{-\sin y} \overset{!}{=} \boxed{\dfrac{\sin x}{\sin y}}$

b. $\boxed{\text{Fixed Points:}}$ $\dot{x} = 0 = \sin y$; $(x^*, y^*) = (\pm n\pi, n\pi)$

$\dot{y} = 0 = \sin x$

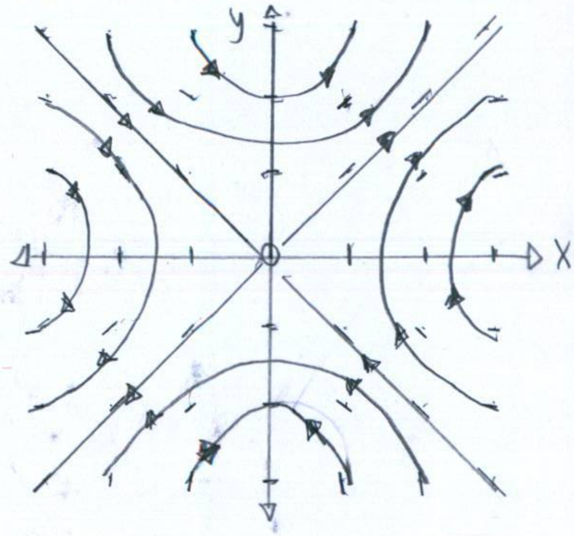
Where $n \in \mathbb{R}$

If n is even, stable node, else unstable node.

c. $\dot{x} = \sin y$; $\sin y = \sin x$; $\boxed{y = \pm x}$

$\dot{y} = \sin x$

d. $\rightarrow$



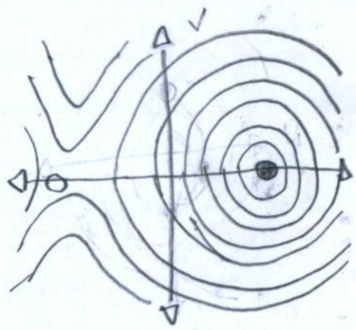
$\ddot{x} + (\dot{x})^2 + x = 3$

6.6.4. $\dot{u} = \dot{x} = v$

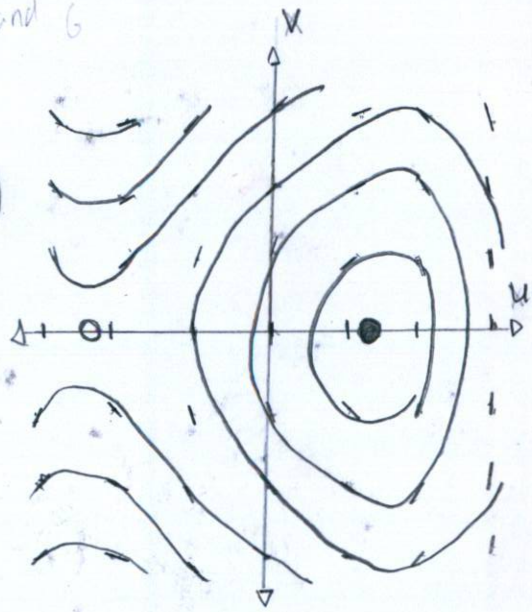
a. $\dot{v} = \ddot{x} = 3 - \dot{x}^2 + x = 3 - u^2 + u$

$\boxed{\text{Fixed Points}}$ $(u^*, v^*) = \left(0, \frac{-1}{2} \pm \frac{\sqrt{13}}{2}\right)$

$\boxed{\text{Guess}}$



Hand 6



$\dot{x} = y - y^3$

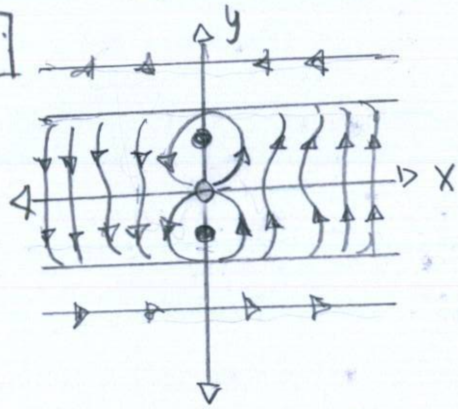
$\dot{y} = x\cos y$

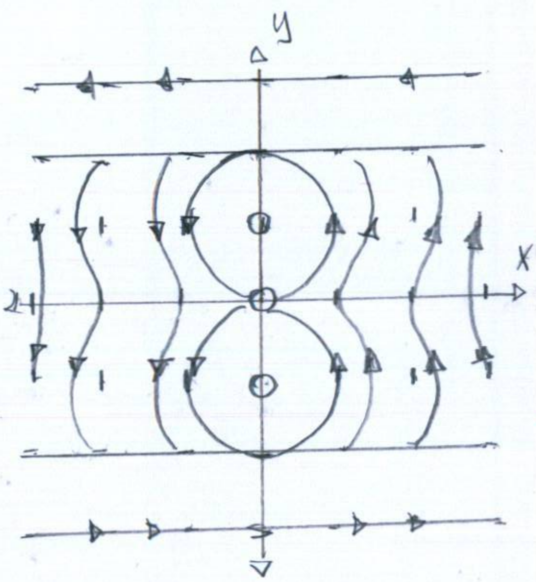
b. $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = y - y^3$

$\dot{y} = 0 = x\cos y$

$(x^*, y^*) = (0,0), (\pm x, 1)$

$\boxed{\text{Guess}}$





$\dot{x} = \sin y$

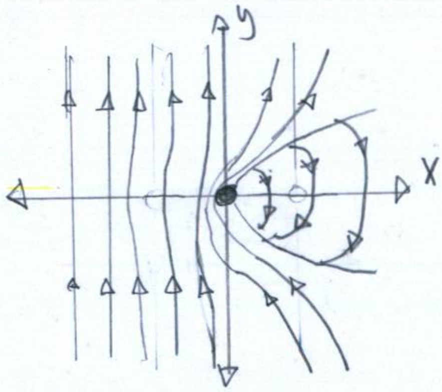
$\dot{y} = y^2 - x$

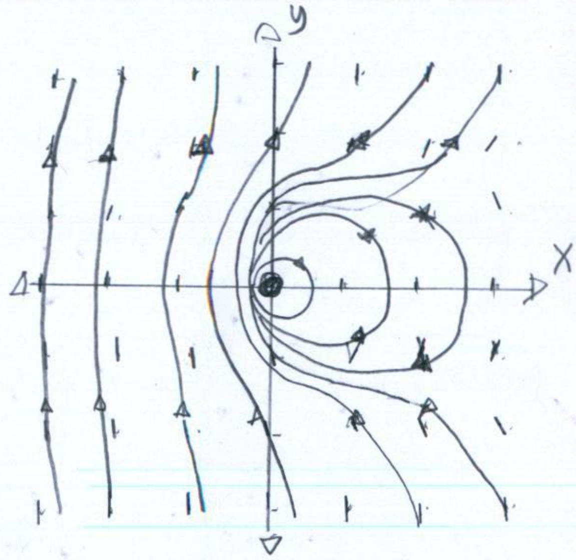
c. Fixed Points $\dot{x} = 0 = \sin y$

$\dot{y} = 0 = y^2 - x$

$(x^*, y^*) = (0,0), (1, \pm 1)$

Guess:





$\ddot{x} + f(\dot{x}) + g(x) = 0$ 6.6.5, f is an even function; g is an odd function

f & g are smooth

a. $-\ddot{x} + f(-\dot{x}) + g(-x) = -\ddot{x} - f(\dot{x}) - g(x) = \boxed{\ddot{x} + f(\dot{x}) + g(x)}$

b. $\dot{u} = \dot{x} = v$ Definition of a reversible system

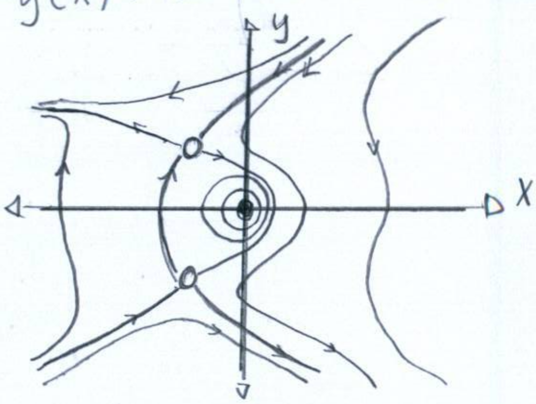
$\dot{v} = \ddot{x} = -f(\dot{x}) - g(x)$; is no stable nodes or spirals.

$\dot{x} = y - y^3$

$\dot{y} = -x - y^2$

6.6.6.

a. $\boxed{\dot{x} = 0; \dot{y} = 0}$



b. Quadrant #1: $\dot{x} < 0; \dot{y} < 0$ Quadrant #3: Mixed

Quadrant #2: Mixed Quadrant #4: $\dot{x} > 0; \dot{y} < 0$

c. $A = \begin{pmatrix} 0 & 1-2y^2 \\ -1 & -2y \end{pmatrix}$; $A_{(-1,\pm 1)} = \begin{pmatrix} 0 & -1 \\ -1 & \pm 2 \end{pmatrix}$; $\Delta = (1-\sqrt{2})(1+\sqrt{2})$

$\tau = 2$

$\tau^2 - 4\Delta > 0$

$\lambda_1 = (1-\sqrt{2})$; $\boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

$\lambda_2 = (1+\sqrt{2})$; $\boxed{v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

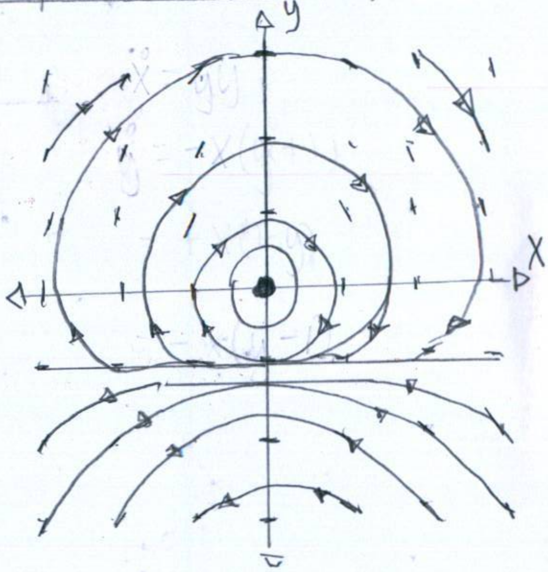
d. $(-1,-1)$; If Quadrant #2 and #3 are mixed sign then a possible trajectory through $x<0$ may exist. A heteroclinic trajectory that does cross from $(-1,-1)$ to $(-1,1)$ is present because of the reversible function.

e. Other examples of a heteroclinic trajectory relate to the third fixed point. See part b.

$\ddot{x}+x\dot{x}+x=0$ 6.6.7. $\dot{x}=y$

$$\dot{y}=-x(\dot{x}+1)=-x(y+1)$$

$\boxed{\text{Reversibility}}$
$$-\ddot{x}-x\cdot(-\dot{x})-x$$
$$=\ddot{x}+x\dot{x}+x$$
$$=0$$



$\dot{x}=\frac{\sqrt{2}}{4}x(x-1)\sin\phi$

$\dot{\phi}=\frac{1}{2}\left[\beta-\frac{1}{\sqrt{2}}\cos\phi-\frac{1}{8\sqrt{2}}x\cos\phi\right]$

6.6.8. a. $\boxed{\text{Reversibility}}$ $x\to -x$; $\phi\to -\phi$

$$\dot{x}=\frac{\sqrt{2}}{4}-x(-x-1)\sin(-\phi)$$
$$=-\frac{\sqrt{2}}{4}x(1-x)\sin(\phi)$$
$$=\frac{\sqrt{2}}{4}x(x-1)\sin\phi$$

$$\dot{\phi}=\frac{1}{2}\left[\beta-\frac{1}{\sqrt{2}}\cos(-\phi)-\frac{1}{8\sqrt{2}}(-x)\cos(-\phi)\right]$$
$$=\frac{1}{2}\left[\beta-\frac{1}{\sqrt{2}}\cos(\phi)+\frac{1}{8\sqrt{2}}x\cos\phi\right]$$

b. $\dot{x}=0=\frac{\sqrt{2}}{4}x(x-1)\sin\phi=$

$$\dot{\phi}=\frac{1}{2}\left[\beta-\frac{1}{\sqrt{2}}\cos\phi-\frac{1}{8\sqrt{2}}x\cos\phi\right]=0$$

$$(x^*,\phi^*)\overset{!}{=}\left(0,2\pi n-\cos^{-1}(\sqrt{2}\cdot\beta)\right),\ \left(0,2\pi n+\cos^{-1}(\sqrt{2}\beta)\right)$$
$$\left(1,2\pi n-\cos^{-1}\left(\frac{8\sqrt{2}\beta}{x+8}\right)\right),\ (0,0)$$
$$\left(1,2\pi n+\cos^{-1}\left(\frac{8\sqrt{2}\beta}{x+8}\right)\right)$$

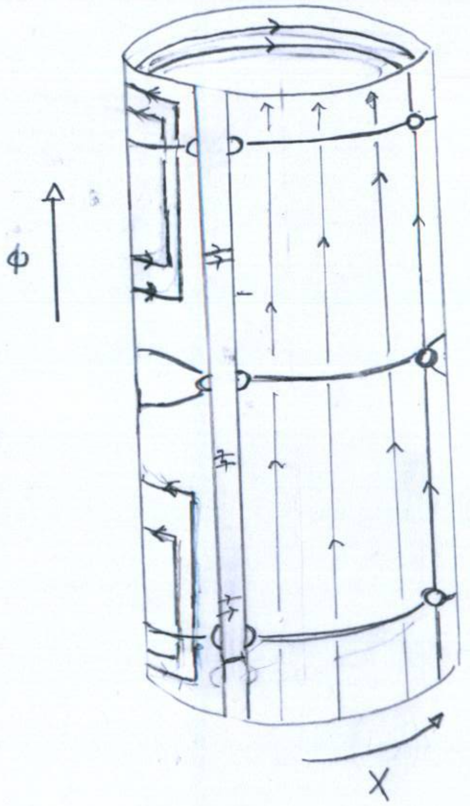
A homoclinic orbit is a nullcline,

$$\dot{x} = 0 = \frac{\sqrt{2}}{4} x(x-1)\sin\phi \; ; \; \dot{\phi} = 0 = \frac{1}{2}\left[\beta - \frac{1}{\sqrt{2}}\cos\phi - \frac{1}{8\sqrt{2}} x\cos\phi\right]$$

$$\boxed{x = -4\sqrt{2}\sec(\phi)(\sqrt{2}\cos(\phi) - 2\beta)}$$

c. $\lim_{\beta \to \frac{1}{\sqrt{2}}} 2\pi n - \cos^{-1}(2\sqrt{\beta}) = 2\pi n$;

, then $(x^*, \phi^*) = (0, 2\pi n)$ and the node on the line $\phi = 0$ is closer to $(0,0)$, and the cylinder becomes a smaller diameter shape. with less closed orbits.



d. See Part c: cylinder

$$\frac{d\phi_k}{d\tau} = \Omega + a\sin\phi_k + \frac{1}{N}\sum_j^N \sin\phi_j$$

6.6.9.

a. $\theta_K = \phi_K - \frac{\pi}{2}$; $\frac{d\theta}{d\tau} = \Omega + a\cos\theta_K + \frac{1}{N}\sum_j \cos\theta_K($

$$= \Omega + a\cos(-\theta_K) + \frac{1}{N}\sum_j \cos(-\theta_K)$$

b. Fixed Points: $\dot{\theta} = 0 = \Omega + a\cos\theta_K + \frac{1}{N}\sum_j \cos\theta_K$

$$\Omega = -a\cos\theta_K - \frac{1}{N}\sum_j \cos\theta_K$$

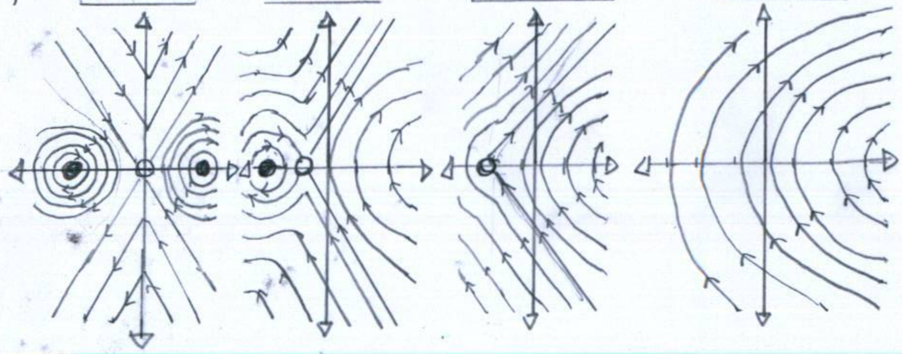
$$= -\cos\theta_K(a+1)$$

$$\cos\theta_K = \left|\frac{\Omega}{a+1}\right|$$

If $\left|\frac{\Omega}{a+1}\right| < 1$, then $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

If $\left|\frac{\Omega}{a+1}\right| > 1$, then no fixed point is generated because $\cos\theta$ is never greater than 1.

c. $a=1$; $\boxed{\Omega=0}$ $\boxed{\Omega=1}$ $\boxed{\Omega=2}$ $\boxed{\Omega=3}$



$\dot{x}=-y-x^2$ 6.6.10 $\boxed{\text{Fixed Points}}$ $\dot{x}=0=-y-x^2$; $(x^*, y^*)=(0,0)$

$\dot{y}=x$ $\dot{y}=0=x$

$$A=\begin{pmatrix} -2x & -1 \\ 1 & 0 \end{pmatrix} ; \quad A_{(0,0)}=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\Delta=-1$; $\tau=0$; $\tau^2-4\Delta>0$

"Saddle Point"

No, a nonlinear center is an isolated fixed point with closed orbits

$\dot{\theta}=\cot\phi\cos\theta$ 6.6.11. a. $\boxed{\text{Reversibility}}$ $t\to -t$; $\theta\to-\theta$

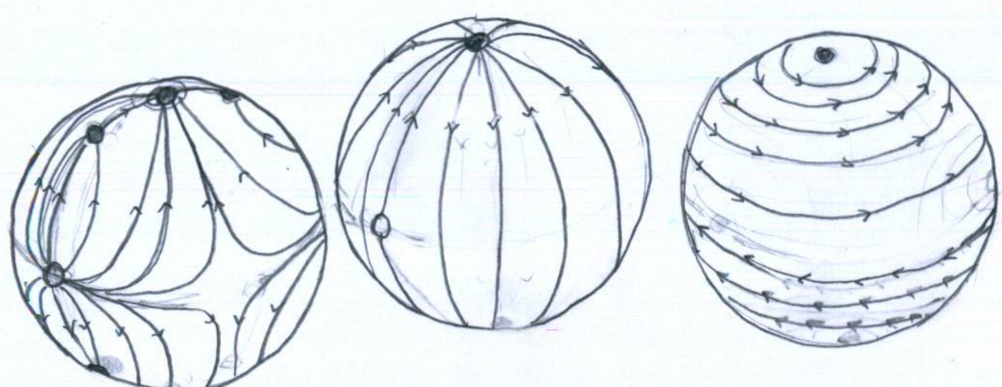
$\dot{\phi}=(\cos^2\phi+A\sin^2\phi)\sin\theta$

$$\dot{\theta}=\cot(+\phi)\cos(-\theta)$$
$$=\cot(+\phi)\cdot\cos(\theta)$$

$t\to -t$; $\phi\to-\phi$

$$\dot{\phi}=\left[\cos^2(-\phi)+A\sin(-\phi)^2\right]\cdot\sin(\theta)$$
$$=\left[\cos^2(\phi)+A\sin^2(\phi)\right]\cdot\sin(\theta)$$

b. $\boxed{A=-1}$ $\boxed{A=0}$ $\boxed{A=1}$

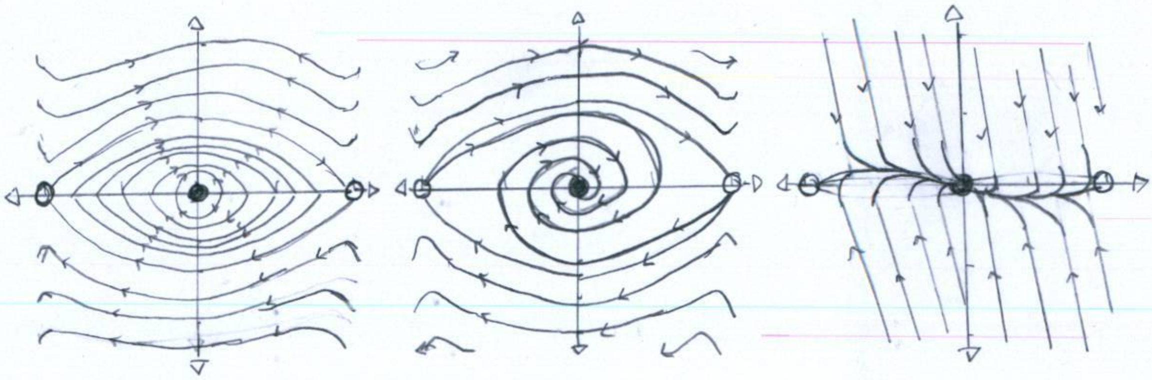


c. As $t\to\infty$, each case of shear flow trajects to a stable node. This implies rotation of a body does not freely rotate in medium.

$\ddot{\theta} + b\dot{\theta} + \sin\theta = 0$ 6.7.1. $\boxed{\text{Fixed Points}}$ $\dot{x} = \dot{\theta} = y$

$\dot{y} = \ddot{\theta} = -(by + \sin x)$

$\boxed{b=0}$ $\boxed{0<b\leq 1}$ $\boxed{1<b}$



$\ddot{\theta} + \sin\theta = \gamma$ 6.7.2 a. $\boxed{\text{Fixed Points}}$ $\dot{x} = \dot{\theta} = y = 0$

$\dot{y} = \ddot{\theta} = \gamma - \sin\theta = \gamma - \sin x$

$(x^*, y^*) = (\arcsin\gamma, 0)$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} ; \quad A = \begin{pmatrix} 0 & 1 \\ \gamma - \cos x & 0 \end{pmatrix}$$

$\Delta = \cos x - \gamma$; $\tau = 0$; $\tau^2 - 4\Delta > 0$

If $\gamma = 0$, (x^*, y^*) is a center

If $0 < \gamma < 1$, (x^*, y^*) is a center

If $1 < \gamma$, (x^*, y^*) is a saddle point.

b. $\boxed{\text{Nullclines}}$ $y = \gamma - \sin x$

$E = \frac{1}{2}\dot{x}^2 - \int \gamma - \sin x \, d\gamma$

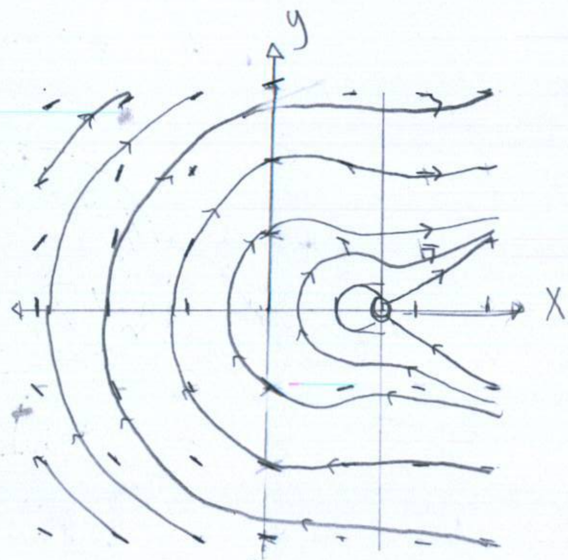
$= \frac{1}{2}y^2 - \gamma x - \cos x$

The system is not conservative because of no closed loops

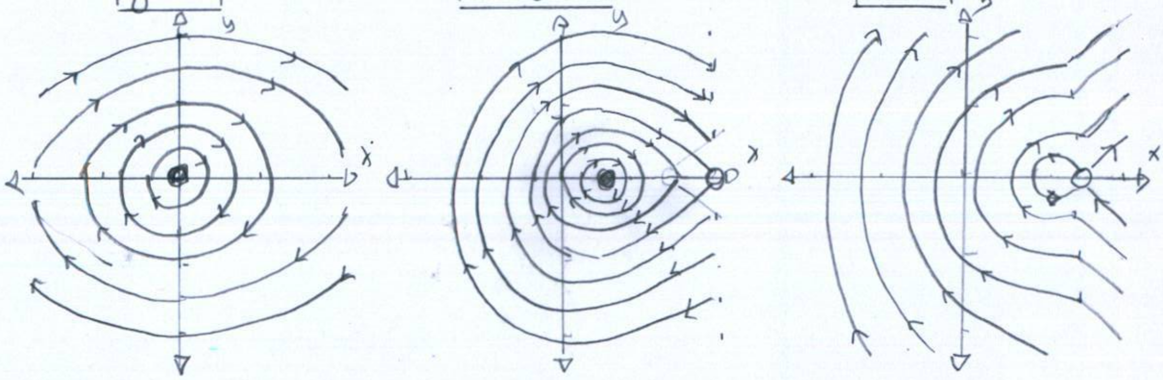
$\boxed{\text{Reversibility}}$

$\dot{x} = -y \neq y$

The system is not reversible.



d. $\boxed{\gamma = 0}$ $\boxed{0 < \gamma < 1}$ $\boxed{\gamma = 1}$



e. $\gamma = 0$; $\dot{x} = 0 = y$; $\dot{y} = 0 = -\sin x$; $y = \dot{\theta} = -\sin\theta$

$0 < \gamma < 1$; $\dot{x} = 0 = y$; $\dot{y} = 0 = \gamma - \sin x$; $y = \dot{\theta} = \gamma - \sin\gamma$

$\gamma = 1$; ... $\dot{\theta} = 1 - \sin\theta$

$\ddot{\theta} + (1 + a\cos\theta)\dot{\theta} + \sin\theta = 0$

6.7.3 $\dot{x} = \dot{\theta} = y$

$\dot{y} = -(1 + a\cos x)y - \sin x$

$\boxed{\text{Fixed Points}}$ $(x^*, y^*) = (0,0)$

	Yes	No
Reversible	☐	☑
Conservative	☐	☑

$\ddot{\theta} + \sin\theta = 0$ 6.7.4

a. $PE = mgh = mgL(1-\cos(\theta))$; $KE = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{\theta})^2$

$E = PE + KE = mgL(1-\cos(\theta)) + \frac{1}{2}m\dot{\theta}^2 = 0$

$\dot{\theta}^2 = -2gL(1-\cos(\theta))$; If $\theta = \alpha$ = max height ; $\dot{\theta}^2 = 0$

$\boxed{= 2(\cos(\theta) - \cos(\alpha))}$

$$T = 4\int_0^\alpha dt = 4\int_0^\alpha \frac{d\theta}{\dot{\theta}} = 4\int_0^\alpha \frac{d\theta}{\sqrt{2(\cos\theta - \cos\alpha)}}$$

b. Half-Angle Formula: $\cos(2A) = 1 - 2\sin^2 A$ where $A = \frac{\theta}{2}$ or $\frac{\alpha}{2}$

$$\boxed{T = 4\int_0^\alpha \frac{d\theta}{[4(\sin^2\frac{1}{2}\alpha - \sin^2\frac{1}{2}\theta)]^{1/2}}}$$

c. Half-Angle Formula: $(\sin\frac{1}{2}\alpha)\sin\phi = \sin\frac{1}{2}\theta$

$$\frac{1}{2}\sin\frac{1}{2}\alpha\cos\phi\frac{d\phi}{d\theta} = \frac{1}{2}\cos\frac{\theta}{2}$$

$$d\theta = \frac{\cos\theta/2}{\sin\alpha/2\cos\phi}d\phi$$

$$T = 4\int_0^{\pi/2}\frac{d\theta}{\cos\theta/2} = 4\,\frac{1}{2}\times\Bigg[\underbrace{\int_0^{\pi/2}\frac{d\phi}{(1-m\sin^2\phi)^{1/2}}}_{\text{"Elliptic Integral"}}\Bigg]$$

d. Binomial Series $\dfrac{1}{(1-x)^{1/2}} = 1 + \frac{1}{2}x + \cdots$

$$T = 4\times\int_0^{\pi/2}\left(1+\tfrac{1}{2}m\sin^2\phi+\cdots\right)d\phi \quad ; \quad m = \sin^2\frac{x}{2}$$

$$\boxed{= 2\pi\left[1+\frac{1}{16}x^2+\cdots\right]}$$

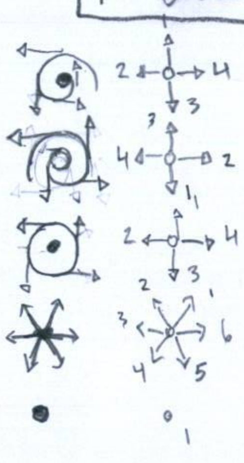
6.7.5.

Numerical Integration of $T = 4\times\sum_{i=0}^{18}\sum_{j=0}^{9}\left(1+\frac{1}{2}\left[\sin^2\frac{10i}{2}\right]\sin^2\frac{10j}{2}+\cdots\right)$

i \ j	0	1	2	3	4	5	6	7	8	9
0	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00
1	4.00	5.69	4.54	4.78	5.53	4.03	5.80	4.34	5.02	5.33
2	4.00	4.54	4.18	4.25	4.49	4.01	4.58	4.11	4.33	4.43
3	4.00	4.78	4.25	4.36	4.70	4.01	4.83	4.16	4.47	4.61
4	4.00	5.53	4.49	4.70	5.34	4.03	5.63	4.31	4.93	5.21
5	4.00	4.03	4.01	4.01	4.03	4.00	4.03	4.01	4.02	4.03
6	4.00	5.80	4.58	4.83	5.63	4.03	5.91	4.36	5.08	5.41
7	4.00	4.34	4.11	4.16	4.31	4.01	4.36	4.07	4.20	4.27
8	4.00	5.02	4.33	4.47	4.93	4.02	5.08	4.20	4.62	4.80
9	4.00	5.33	4.43	4.61	5.21	4.03	5.41	4.27	4.80	5.05
10	4.00	4.13	4.04	4.06	4.11	4.00	4.13	4.03	4.08	4.10
11	4.00	5.84	4.59	4.85	5.67	4.04	5.95	4.25	5.11	5.45
12	4.00	4.17	4.05	4.08	4.15	4.00	4.18	4.22	4.10	4.13
13	4.00	5.26	4.40	4.58	5.14	4.02	5.33	4.25	4.76	4.99
14	4.00	5.10	4.35	4.51	5.00	4.02	5.17	4.22	4.67	4.87
15	4.00	4.28	4.09	4.13	4.25	4.01	4.29	4.06	4.17	4.22
16	4.00	5.82	4.58	4.84	5.65	4.03	5.93	4.36	5.10	5.43
17	4.00	4.06	4.02	4.03	4.05	4.00	4.06	4.01	4.03	4.04
18	4.00	5.47	4.47	4.68	5.33	4.03	5.56	4.29	4.84	5.16

$$\boxed{T = 852.06}$$

6.8.1



a. Stable Spiral $I_c = +1$

b. Unstable spiral $I_c = +1$

c. center $I_c = +1$

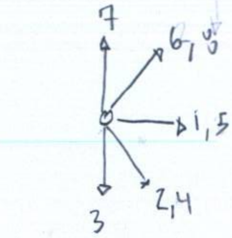
d. Star $I_c = +1$

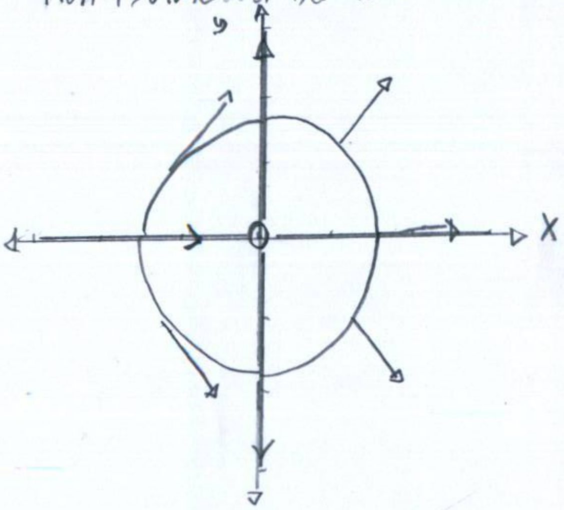
e. Degenerate Node. $I_c = +1$

$\dot{x} = x^2$ 6.8.2 $\boxed{\text{Fixed Points}}$: $\dot{x} = 0 = x^2$; $A = \begin{pmatrix} 2x & 0 \\ 0 & 1 \end{pmatrix}$: $\Delta = 0$; $\tau = 1$; $\tau^2 - 4\Delta > 0$
$\dot{y} = y$

$\dot{y} = 0 = y$ "Non-isolated Fixed Points"

$(x^*, y^*) = (0,0)$; $\boxed{\text{Index}}$





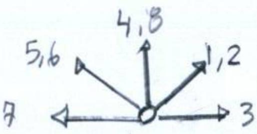
$I_c = 0$

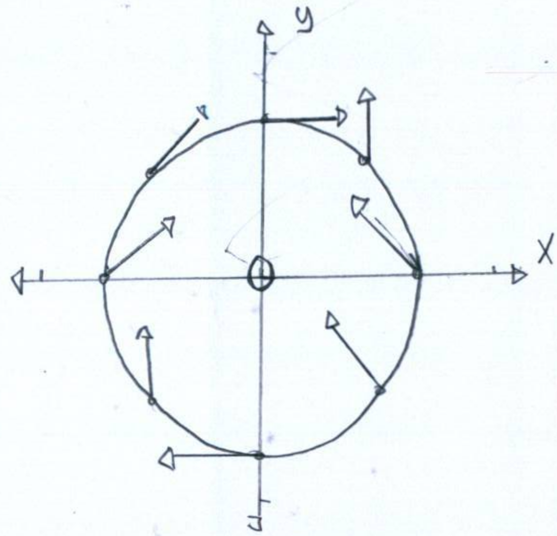
$\dot{x} = y - x$ 6.8.3. $\boxed{\text{Fixed Points}}$: $\dot{x} = 0 = y - x$; $A = \begin{pmatrix} -1 & 1 \\ 2x & 0 \end{pmatrix}$; $\Delta = 0$; $\tau = -1$; $\tau^2 - 4\Delta > 0$
$\dot{y} = x^2$

"Spiral sink"

$\dot{y} = 0 = x^2$

$(x^*, y^*) = (0,0)$; $\boxed{\text{Index}}$



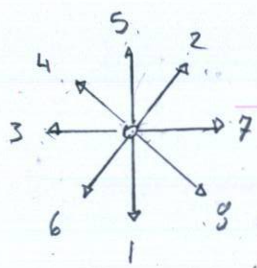


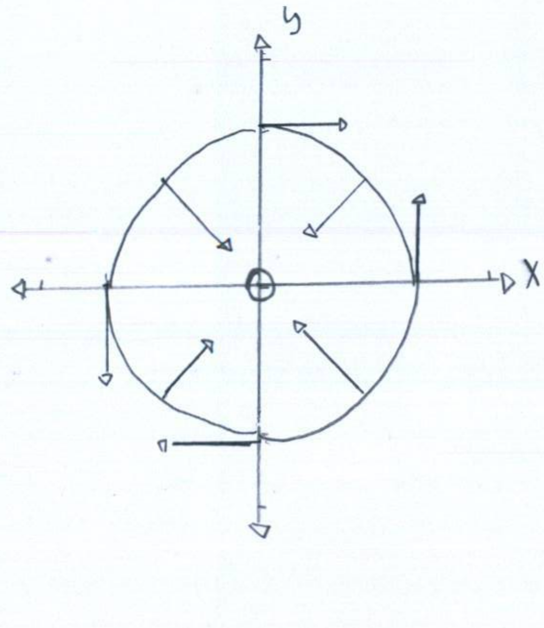
$I_c = 0$

$\dot{x} = y^3$ 6.8.4 $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = y^3$ $A = \begin{pmatrix} 0 & 3y^2 \\ 1 & 0 \end{pmatrix}$; $\Delta = 0$; $\tau = 0$; $\tau^2 - 4\Delta = 0$
$\dot{y} = x$

$\dot{y} = 0 = x$ "Saddle"

$(x^*, y^*) = (0,0)$ $\boxed{\text{Index}}$





$I_c = 0$

$\dot{x} = xy$

$\dot{y} = x+y$

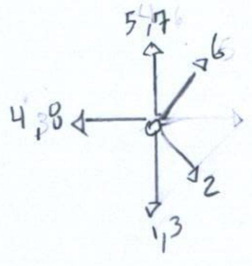
6.8.5 $\boxed{\text{Fixed Points}}$ $\dot{x}=0=xy$

$\dot{y}=0=x+y$

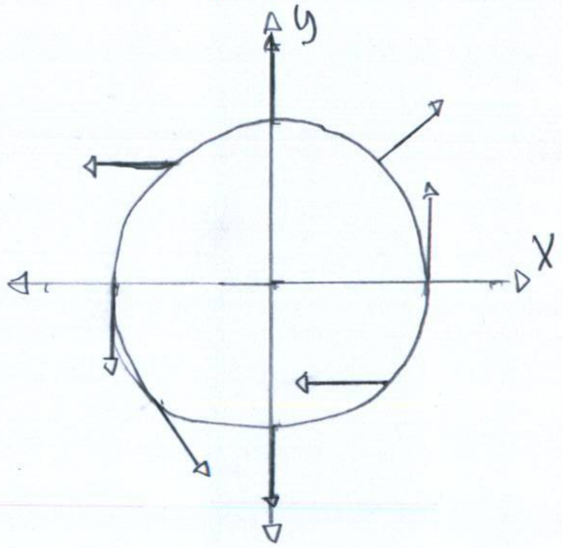
$(x^*, y^*) = (0,0)$ $\boxed{\text{Index}}$

$A = \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix}$; $\Delta = 0$; $\tau = 0$;

$\tau^2 - 4\Delta$ "unstable saddle"



$I_c = 0$



6.8.6. Node [N]: $I_c = +1$ $N+S+C = +1 = 1+S = +1$

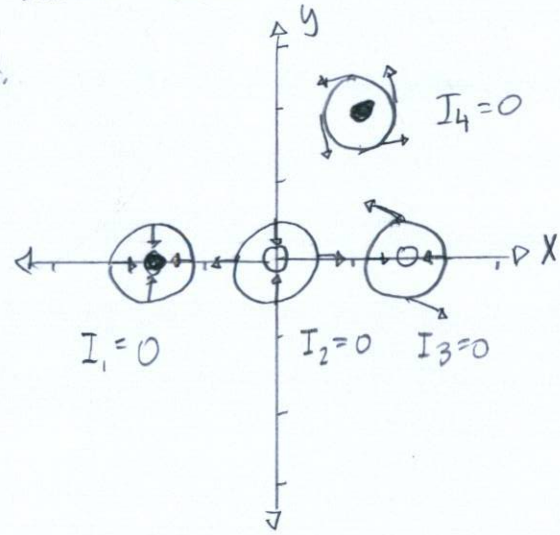
Spiral [F]: $I_c = +1$

Center [C]: $I_c = +1$

Saddle [S]: $I_c = 0$

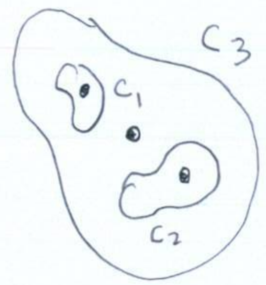
$\dot{x} = x(4-y-x^2)$ 6.8.7.

$\dot{y} = y(x-1)$



The indices of each fixed point are zero ($I_c = 0$), thus, no closed orbits exist.

6.8.8. a.



b. $I_c = I_1 + I_2 + I_3 > 0$; A fixed point exists in the closed orbit.

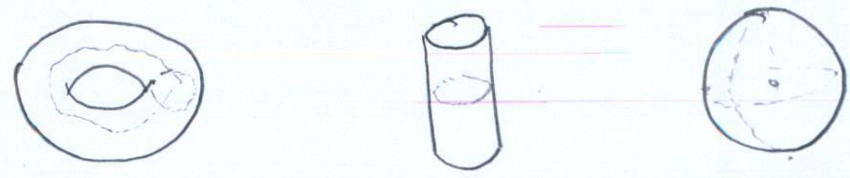
6.8.9. C_1, C_2 = Closed Trajectories

If C_1 is clockwise, $I_c < 0$

If C_2 is counterclockwise, $I_c > 0$

$\boxed{\text{A fixed point in } C_2 \text{ is true because } I_c > 0}$

6.8.10 Torus Cylinder Sphere



$I_c > 0$ $I_c = 0$ $I_c > 0$

$\boxed{\text{Theorem 6.8.2 is reasonable for closed orbit shapes.}}$

$\dot{z} = z^k$

$\dot{z} = (\bar{z})^k$

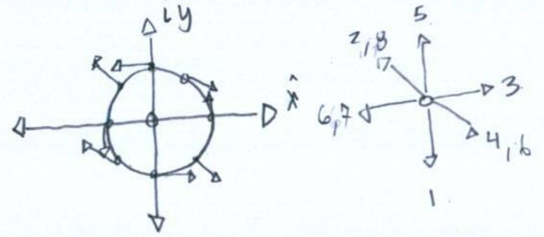
6.8.11

a. $k=1$; $\dot{z} = z = x+iy$; $\langle x, y \rangle$

$k=2$; $\dot{z} = z^2 = (x+iy)^2 = x^2-y^2-2ixy$; $\boxed{\langle x^2-y^2, -2xy \rangle}$

$k=3$; $\dot{z} = z^3 = (x+iy)^3 = x^3-3xy^2+i(3x^2y-y^3)$; $\boxed{\langle x^3-3xy^2, 3x^2y-y^3 \rangle}$

b. $z^* = (0,0)$;



$I_c = 0$

c. The expansion is similar to a Binomial.

$$\left\langle \sum_{k=1}^{2k\le n} \binom{n}{2k} x^{n-2k} \cdot (-1)^k \cdot y^{2k}, \sum_{k=1}^{2k+1\le n} \binom{n}{2k+1} x^{n-2k} (-1)^k y^{2k} \right\rangle$$

$\dot{x} = a + x^2$

$\dot{y} = -y$

6.8.12.

a. $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = a + x^2$

$\dot{y} = 0 = -y$

$(x^*, y^*) = (\pm i\sqrt{a}, 0)$

$$A = \begin{pmatrix} 2x & 0 \\ 0 & -1 \end{pmatrix}; \quad \Delta_1 = -2i\sqrt{a}; \ \tau_1 = 2i\sqrt{a} - 1; \ \tau^2 - 4\Delta = \text{Imaginary}$$

$$\Delta_2 = 2i\sqrt{a}; \ \tau_2 = -2i\sqrt{a} - 1$$

$\boxed{\text{Fixed points in } \mathbb{R}^3 \text{ are non-existent.}}$

b. $I_c = I_1 + I_2 = \boxed{0}$ because the imaginary fixed points $(\pm i\sqrt{a}, 0)$ are symmetric.

c. $\dot{x} = f(x, a)$ where $x \in \mathbb{R}^2$ a conserved index is independent of a, and is the sum of two indices.

$\dot{x} = f(x,y)$ 6.8.13 $\phi = \tan^{-1}(\dot{y}/\dot{x})$

$\dot{y} = g(x,y)$

a. $\frac{d}{dj}\tan^{-1} j = \frac{1}{j^2+1}$; $d\phi = \frac{1}{\left(\frac{\dot{y}}{\dot{x}}\right)^2+1} \cdot \left(\frac{\dot{y}}{\dot{x}}\right)' = \frac{\cancel{\dot{x}^2}}{\dot{y}^2 + \dot{x}^2} \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\cancel{\dot{x}^2}}$

$$\boxed{= \frac{f dg - g df}{f^2 + g^2}}$$

b. $I_c = \frac{d}{d\phi}\tan^{-1}(\phi) = \boxed{\frac{1}{2\pi}\oint \frac{f dg \cdot g df}{f^2+g^2}}$ where $\phi = \frac{\dot{y}}{\dot{x}}$

$\dot{x} = x\cos\alpha - y\sin\alpha$ 6.8.14.

$\dot{y} = x\sin\alpha + y\cos\alpha$

a. $\boxed{\text{Fixed Points}}$ $(x^*, y^*) = (0,0)$

$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$; $\Delta = \cos\alpha\sin\alpha$

$\tau = \cos\alpha + \sin\alpha$

$\tau^2 - 4\Delta < 0$

$\boxed{\text{"Unstable Spiral"}}$

b. $I_c = \frac{1}{2\pi}\oint \frac{(x\cos\alpha - y\sin\alpha)(x\sin\alpha + y\cos\alpha)' - (x\sin\alpha + y\cos\alpha)(x\cos\alpha - y\sin\alpha)'}{(x\cos\alpha - y\sin\alpha)^2 + (x\sin\alpha + y\cos\alpha)^2} dc$

$= \frac{1}{2\pi}\oint 1 = \boxed{1}$

c. $\boxed{I_c = 1}$