Chapter 10: One-Dimensional Maps:

$X_{n+1} = \sqrt{X_n}$

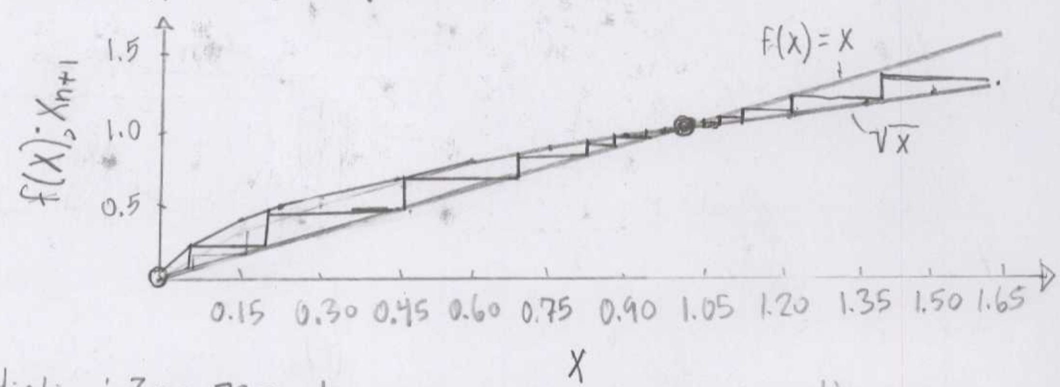
10.1.1. Prediction: Zero, Zero to one, one, and greater than one correspond with unique events on the calculator.

Fixed Points: $X_{n+1} = 0\sqrt{X}$

$X^* = 0, 1$

Stability: $X^* = 0$; $f'(0) = \infty$; unstable

$X^* = 1$; $f'(1) = 1/2$; stable



$X_{n+1} = X_n^3$

10.1.2. Prediction: Zero, zero to one, one, and greater than one are unique regions. Also, negative one to zero, negative one, and less than negative one show unique diagrams.

Fixed Points: $X_{n+1} = 0 = X_n^3$

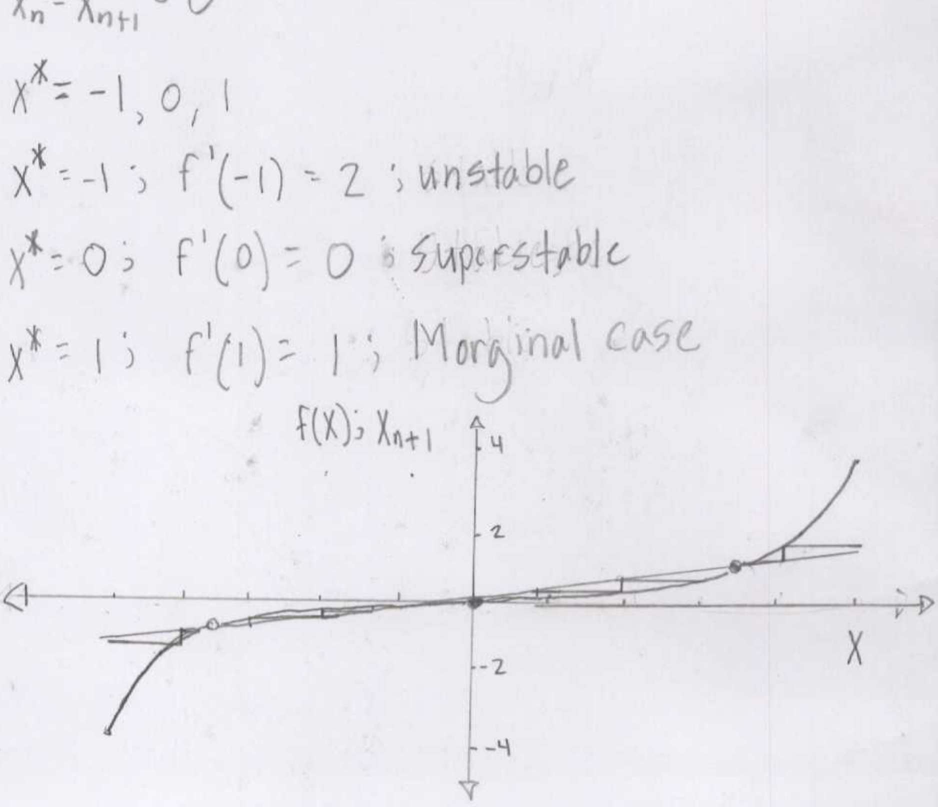
$X_n^3 - X_{n+1} = 0$

$X^* = -1, 0, 1$

Stability: $X^* = -1$; $f'(-1) = 2$; unstable

$X^* = 0$; $f'(0) = 0$; Superstable

$X^* = 1$; $f'(1) = 1$; Marginal case



$X_{n+1} = \exp X_n$

10.1.3. Prediction: Values below zero, at zero, and above zero dictate map's terminal behavior.

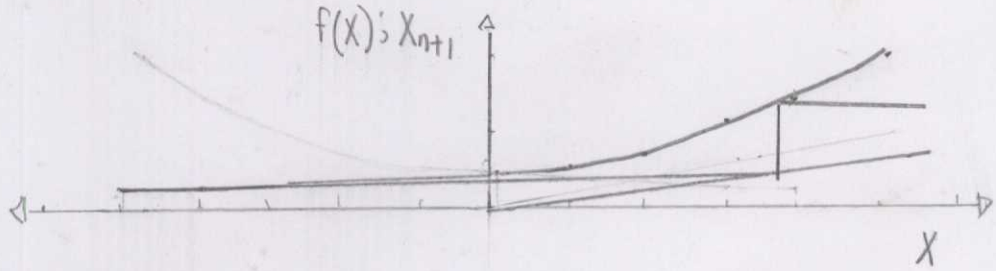
¡wrong!

Fixed Points: $X_{n+1} = 0 = \exp X_n$

$\exp X_n - X_{n+1} = 0$

$X^* = NaN$

Stability: Null



$X_{n+1} = \ln X_n$

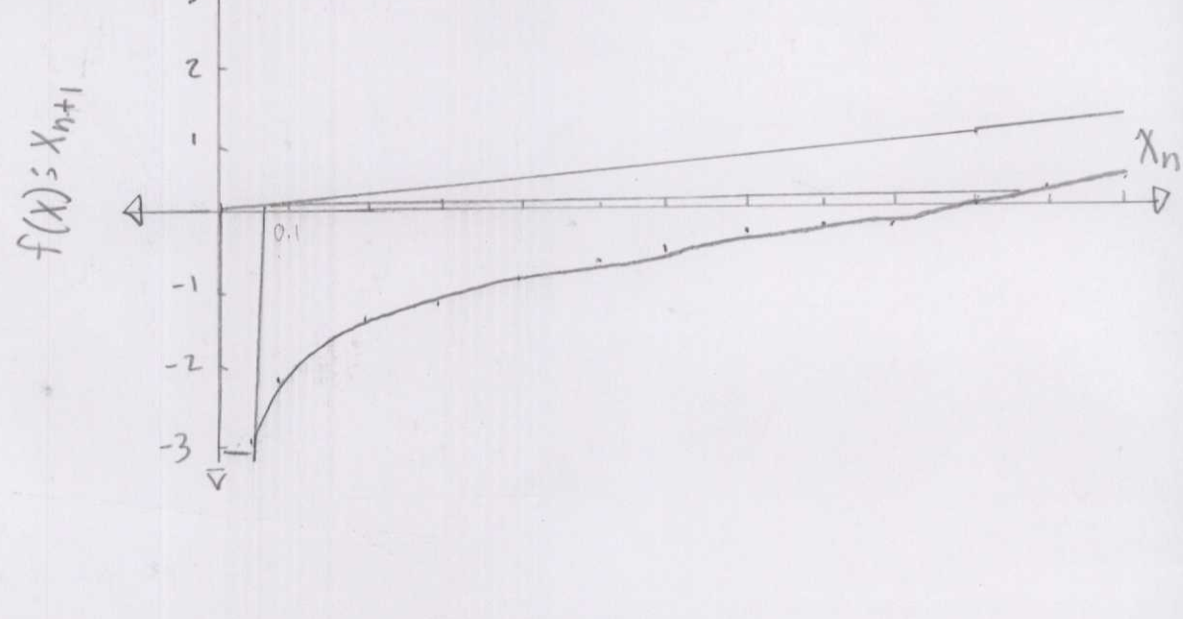
10.1.4. Prediction: The logarithm domain input values greater than zero, successive iterations fail up till natural logarithm base, e, euler's number, 2.718..., then an exponential output.

Fixed Points: $X_{n+1} = 0 = \ln X_n$

$\ln X_n - X_{n+1} = 0$

$X^* = NaN$

Stability: Null



$X_{n+1} = \cot X_n$

10.1.5. Prediction: Similar solutions appear from periodic functions, such as cotangent because $n\text{-}\pi$ relations.

Fixed Points: $X_{n+1} = 0 = \cot X_n$

$$\cot X_n - X_{n+1} = 0$$

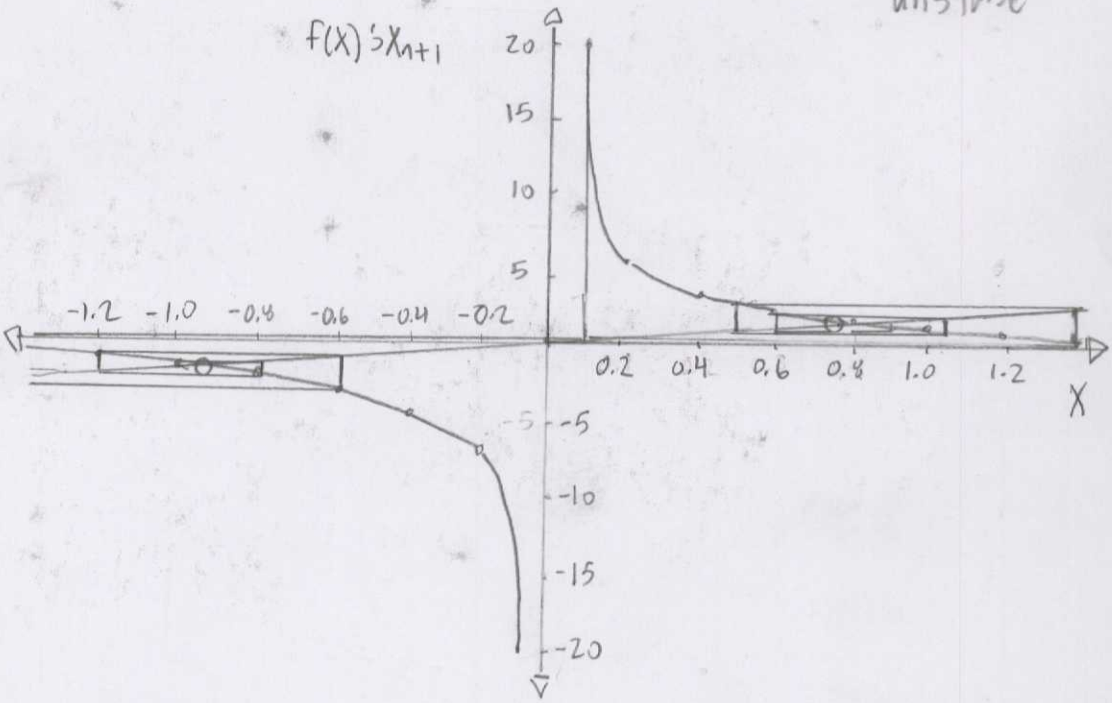
X^* when $\cot x = x$; $x^* = \pm 0.86, \pm 3.43, \ldots$

Stability: $x^* = +0.86$; $|f'(0.86)| = |-\csc^2(0.86)| = 1.74$ "unstable"

$x^* = -0.86$; $|f'(-0.86)| = |-\csc^2(-0.86)| = 1.74$ "unstable"

$x^* = 3.43$; $|f'(3.43)| = |-\csc^2(3.43)| = 12.36$ "unstable"

$x^* = -3.43$; $|f'(-3.43)| = |-\csc^2(-3.43)| = 12.36$ "unstable"



$X_{n+1} = \tan X_n$

10.1.6. Prediction: Since $\tan x$ equals zero at many points, the periodic nature has many solutions.

Fixed Points: $X_{n+1}^* = 0 = \tan X_n$

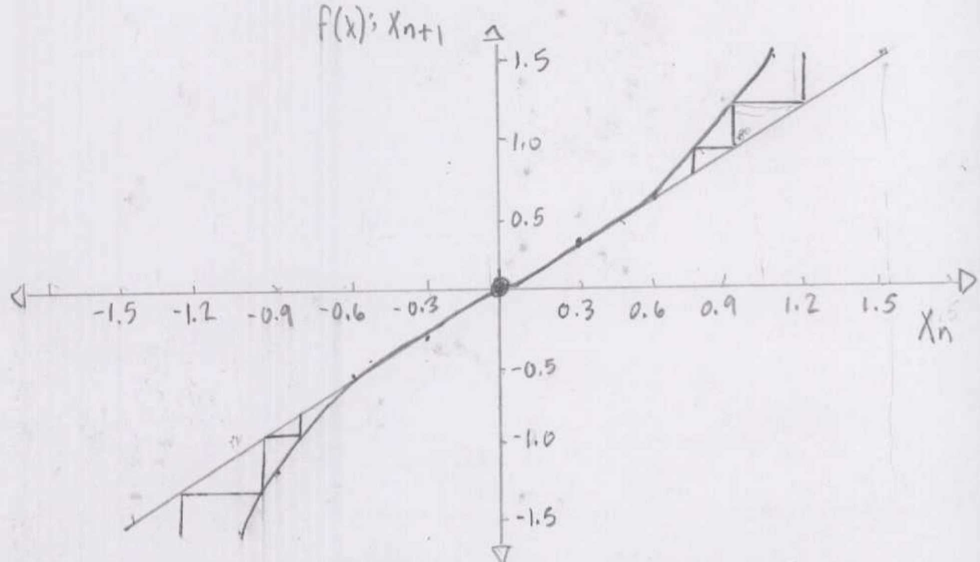
$$\tan X_n - X_{n+1} = 0$$

$$X^* = (\pm n\pi) \text{ where } n \in \mathbb{Z}$$

Stability: $x^* = 0$; $|f'(0)| = |1 - \tan^2(0)| = 1$ "Marginal case"

$x^* = -\pi$; $|f'(-\pi)| = |1 - \tan^2(-\pi)| = 1$ "Marginal case"

$x^* = \pi$; $|f'(\pi)| = |1 - \tan^2(\pi)| = 1$ "Marginal case"



$X_{n+1} = \sinh X_n$

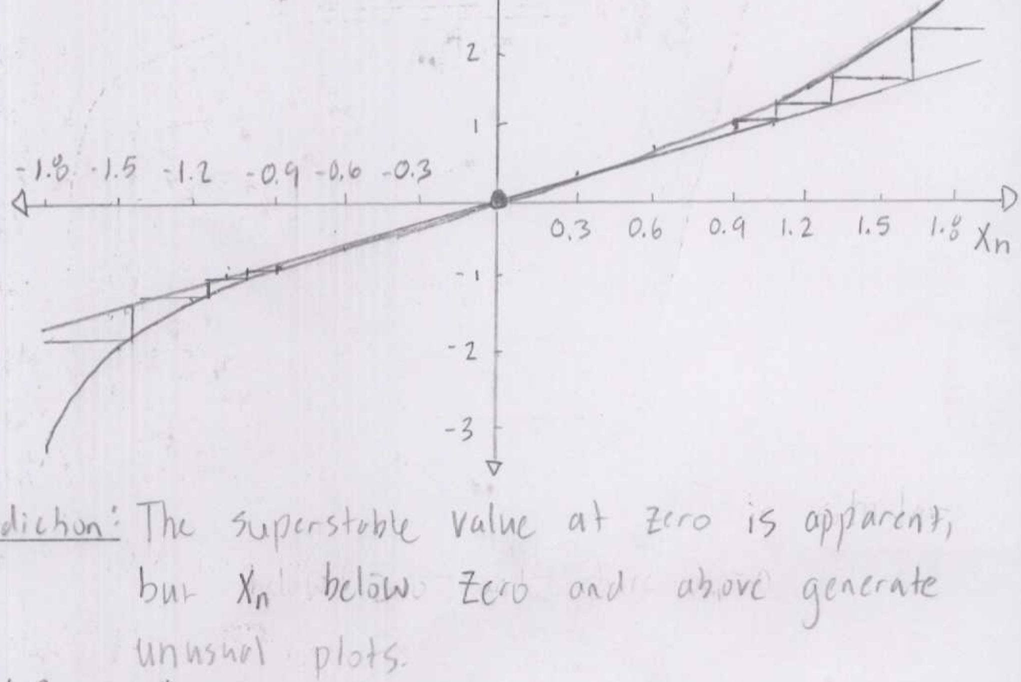
10.1.7. Prediction: Three important regions appear from $\sinh x$; less than zero, zero, and greater than zero.

Fixed points: $X_{n+1} = 0 = \sinh X_n$

$\sinh X_n - X_{n+1} = 0$

$x^* = 0$

Stability: $x^* = 0$; $|f'(0)| = |\cosh(0)| = 1$ "marginal case"



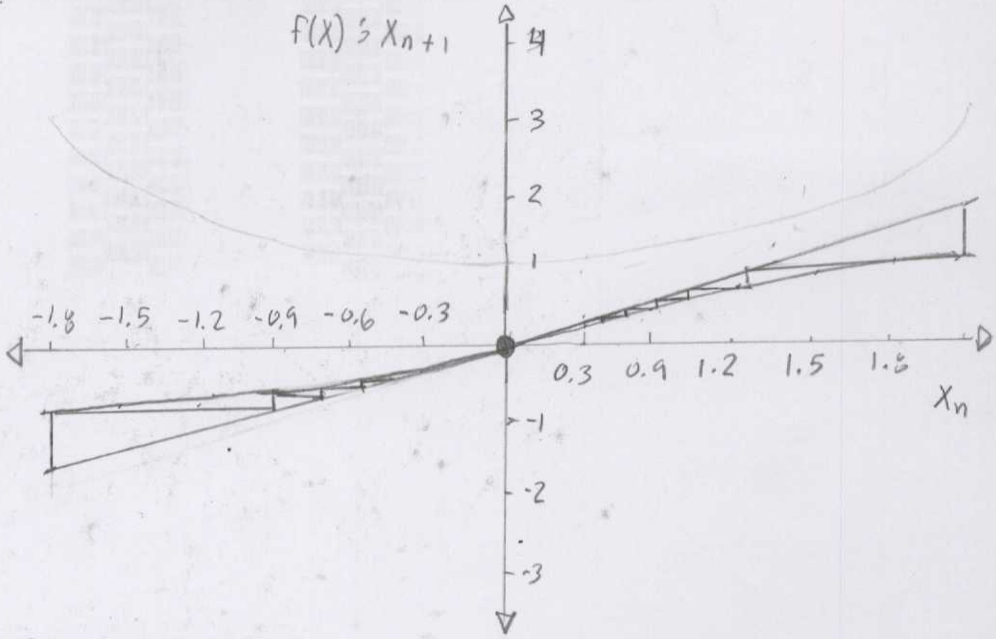
$X_{n+1} = \tanh X_n$

10.1.8 Prediction: The superstable value at zero is apparent, but X_n below zero and above generate unusual plots.

Fixed Points: $X^*_{n+1} = 0 = \tanh X_n$

$\tanh X_n - X_{n+1} = 0$; $x^* = 0$

Stability: $x^* = 0$; $|f'(0)| = |1 - \tanh(0)| = 1$ "marginal case"



$X_{n+1} = 2X_n/(1+X_n)$

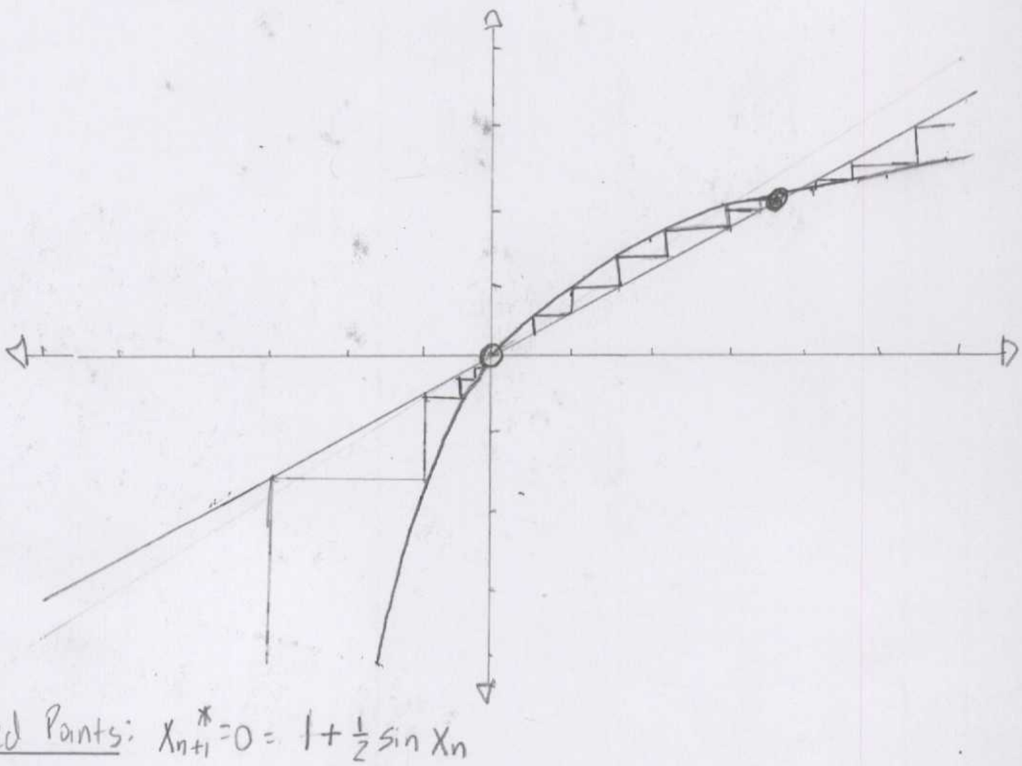
10.1.9. Fixed Points: $X_{n+1} = 0 = 2X_n/(1+X_n)$

$$\frac{2X_n}{(1+X_n)} - X_{n+1} = 0$$

$$X^* = 0, 1$$

Stability: $X^* = 0$; $|f'(0)| = \left|\frac{2-4(0)}{(1+0)^2}\right| = 2$; "unstable"

$X^* = 1$; $|f'(1)| = \left|\frac{2-4(1)}{(1+1)^2}\right| = 1$; "Marginal case"



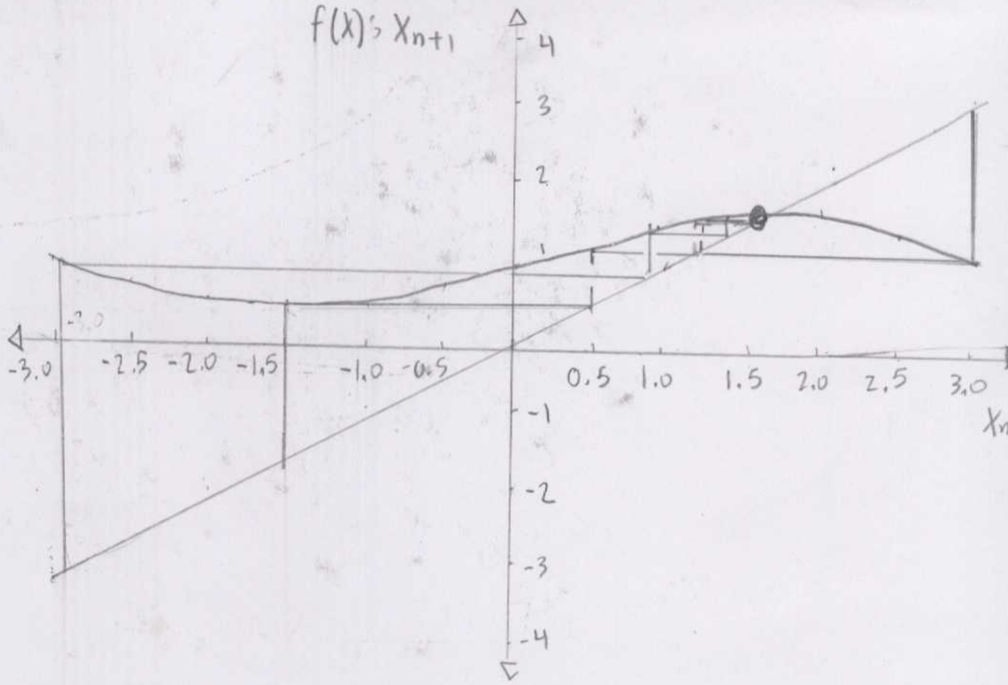
$X_{n+1} = 1 + \frac{1}{2}\sin X_n$

10.1.10 Fixed Points: $X^*_{n+1} = 0 = 1 + \frac{1}{2}\sin X_n$

$$1 + \frac{1}{2}\sin X_n - X = 0$$

$$X^* = 1.4987\ldots$$

Stability: $x^* = 1.4987..$; $|f'(1.4987)| = \left|\frac{\cos(1.4987)}{2}\right| = 0.03$ "stable"



$X_{n+1} = 3X_n - X_n^3$ 10.1.11.

a) Fixed Points: $X_{n+1} = 0 = 3X_n - X_n^3$

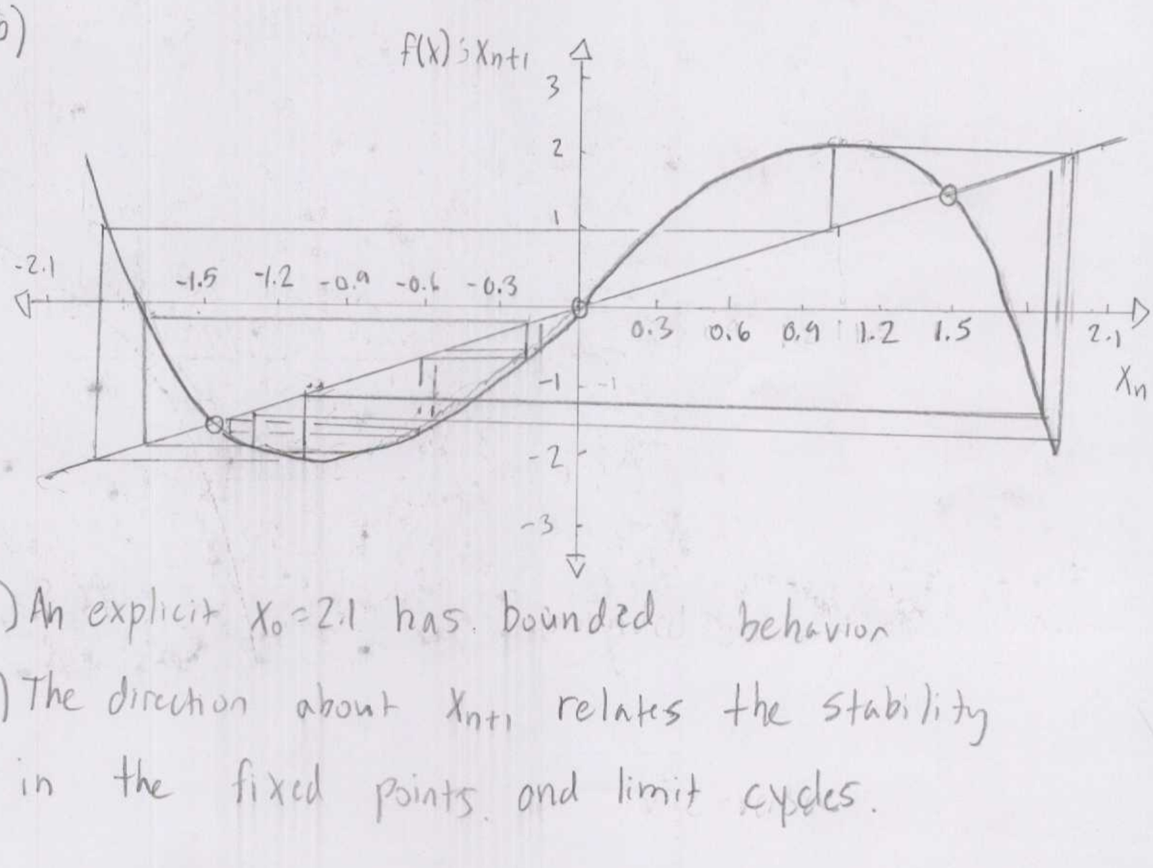
$$x^3 - 2x_n = 0 \qquad x^2-1$$

$$x^* = 0, \pm\sqrt{2}$$

Stability: $x^* = 0$; $|f'(0)| = |3 - 3(0)^2| = 3$ "unstable"

$x^* = \pm\sqrt{2}$; $|f'(\sqrt{2})| = |3 - 3(\sqrt{2})^2| = 3$ "unstable"

b)



c) An explicit $X_0 = 2.1$ has bounded behavior

d) The direction about X_{n+1} relates the stability in the fixed points and limit cycles.

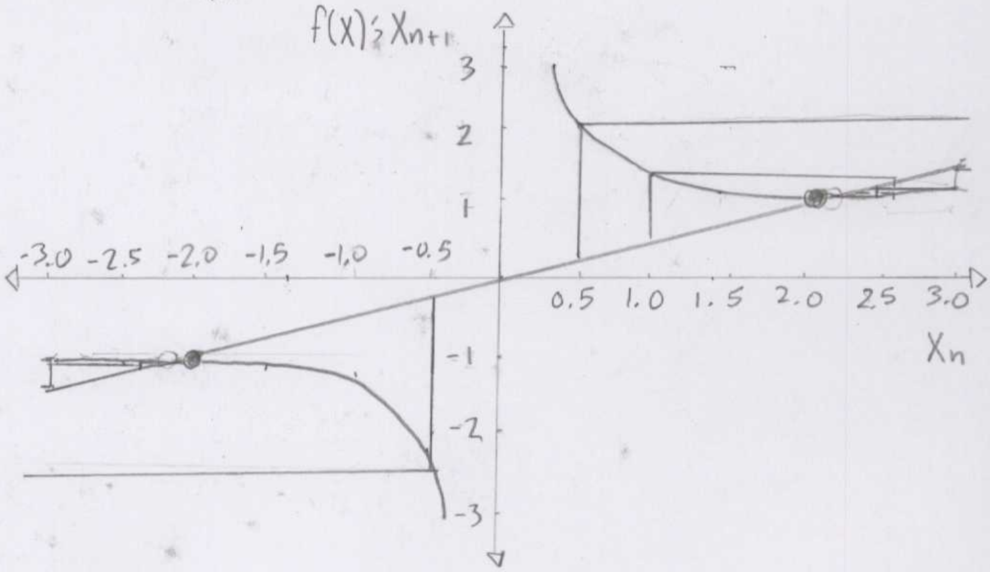
$f(x) = X_n - \frac{g(X_n)}{g'(X_n)}$

10.1.12.

a) $X_{n+1} = f(X_n)$; $g(x) = x^2 - 4$

$g'(x) = 2x$

$$X_{n+1} = X_n - \frac{X_n^2 - 4}{2X_n}$$



b) <u>Fixed Points:</u> $X_{n+1} = 0 = X_n - \frac{X_n^2 - 4}{2X_n}$

$$\frac{x^2 - 4}{2X_n} = 0$$

$$x^* = \pm 2$$

c) $x^* = 2$; $|f'(2)| = \left| 1 - \frac{4(2)^2 + 2(2^2 - 4)}{4(2)^2} \right| = 0$ "superstable"

$x^* = -2$; $|f'(-2)| = \left| 1 - \frac{4(-2)^2 + 2((-2)^2 - 4)}{4(-2)^2} \right| = 0$ "superstable"

d) Part a shows $X_0 = 1$.

10.1.13. Exercise 10.1.12, part c is an exact solution.

$X_{n+1} = -\sin X_n$

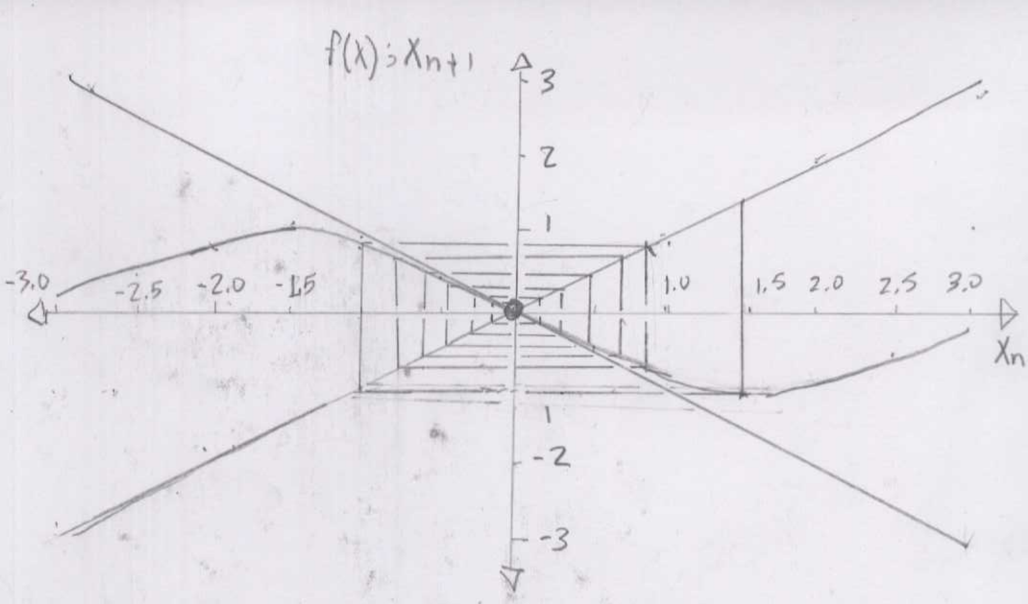
10.1.14. <u>Fixed Points:</u> $X_{n+1}^* = 0 = -\sin X_n$

$\sin x - x = 0$

$x^* = 0$

<u>Stability:</u> $x^* = 0$; $|f'(0)| = |-\cos(0)| = 1$; "Marginal case"

Note: The problem states $x^* = 0$ is stable....



$X_{n+1} = rX_n(1-X_n)$

10.2.1.

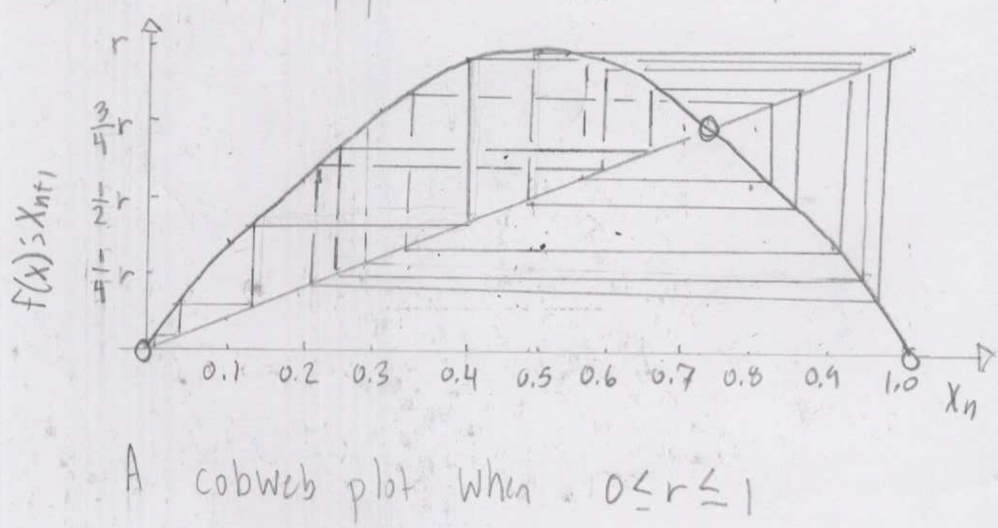
a) If $X_n > 1$, then subsequent iterations diverge toward $-\infty$.

<u>Proof by example:</u>

n	X_n	X_{n+1}
1	2	-2
2	-2	-6
3	-6	-42
4	-42	-1806
5	-1806	-326344
...	...	...

b) A restriction of $r \in [0,4]$ and $x \in [0,1]$ aids the biological model by max population at $r = 1$ and total population near $x = 1$.

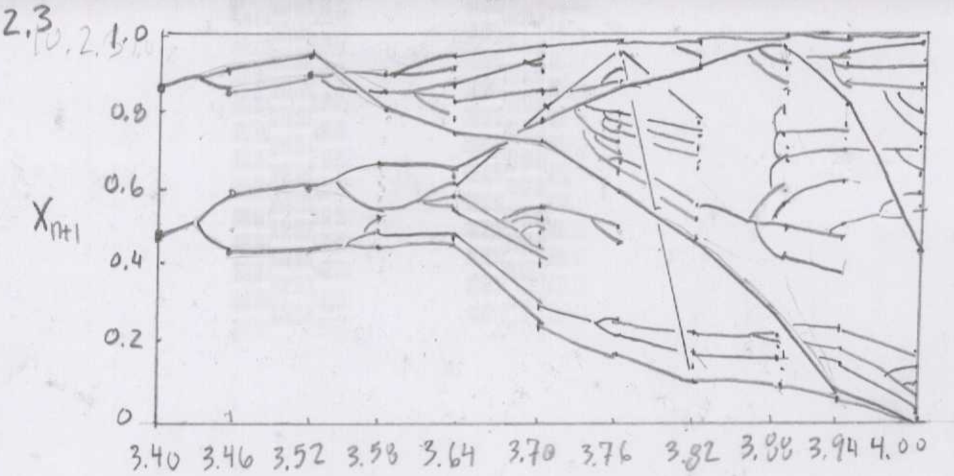
10.2.2.



A cobweb plot when $0 \leq r \leq 1$ approaches $x^* = 0$ because a small range is less than zero

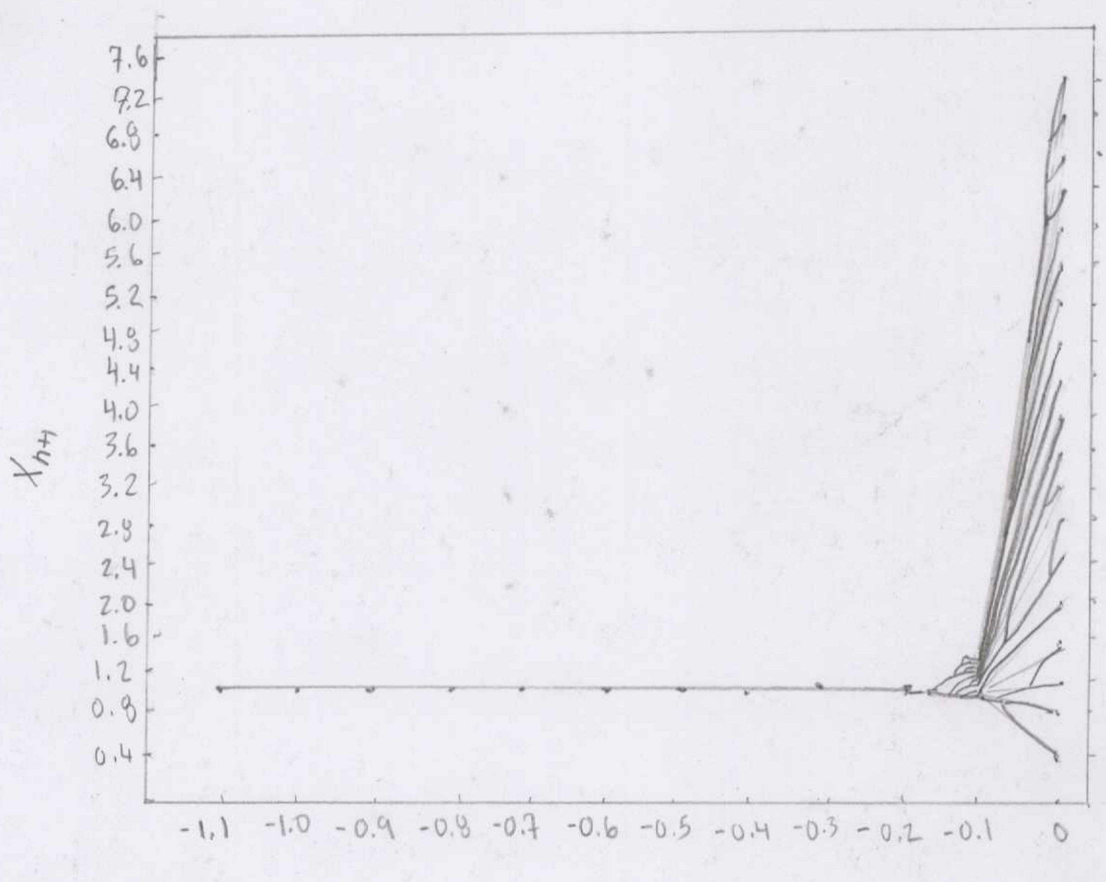
$X_{n+1} = rX_n(1-X_n)$

10.2.3



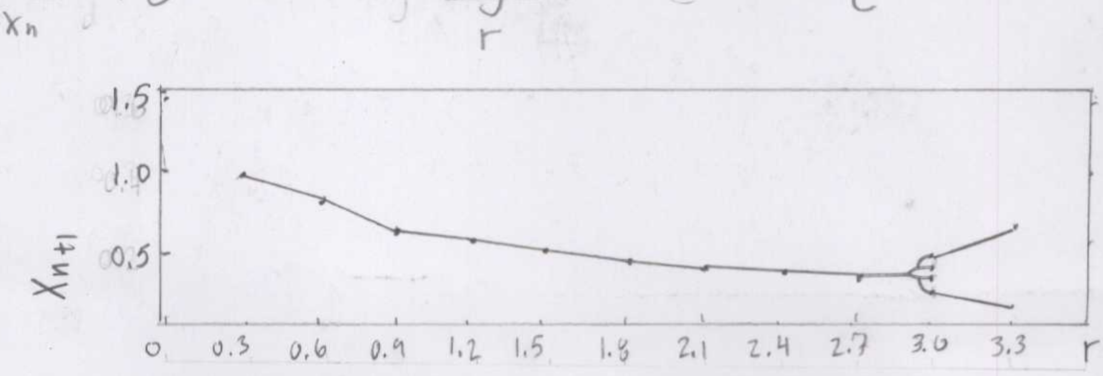
$X_{n+1} = X_n e^{-r(1-X_n)}$

10.2.4. I used $r=-1$ because the derivative's maximum at $x=r1$ with roots about r. The solution $|f'(1)| = |e^{r(1-1)}(r(1)+1)|$ is $r+1$ with stability less than zero but greater than negative one.



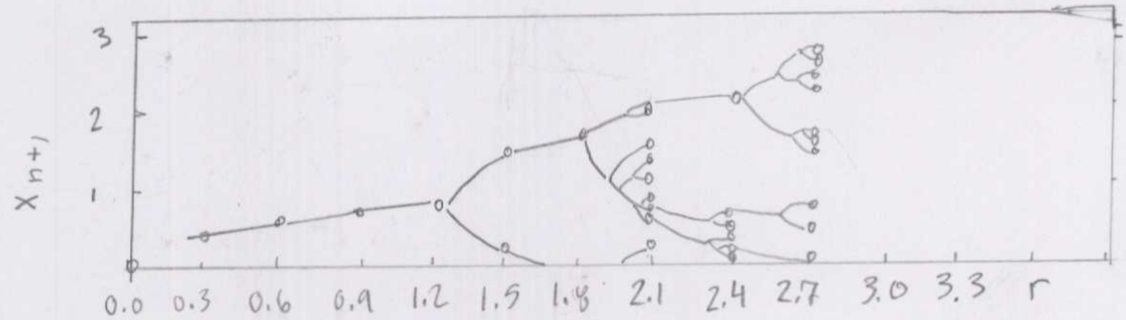
$X_{n+1} = e^{-rX_n}$

10.2.5 $\frac{dX_{n+1}}{dX_n} = -re^{-rX_n} = 0$; $X_n = \frac{\log r}{r}$; $r = e$; $X_n = \frac{1}{e}$



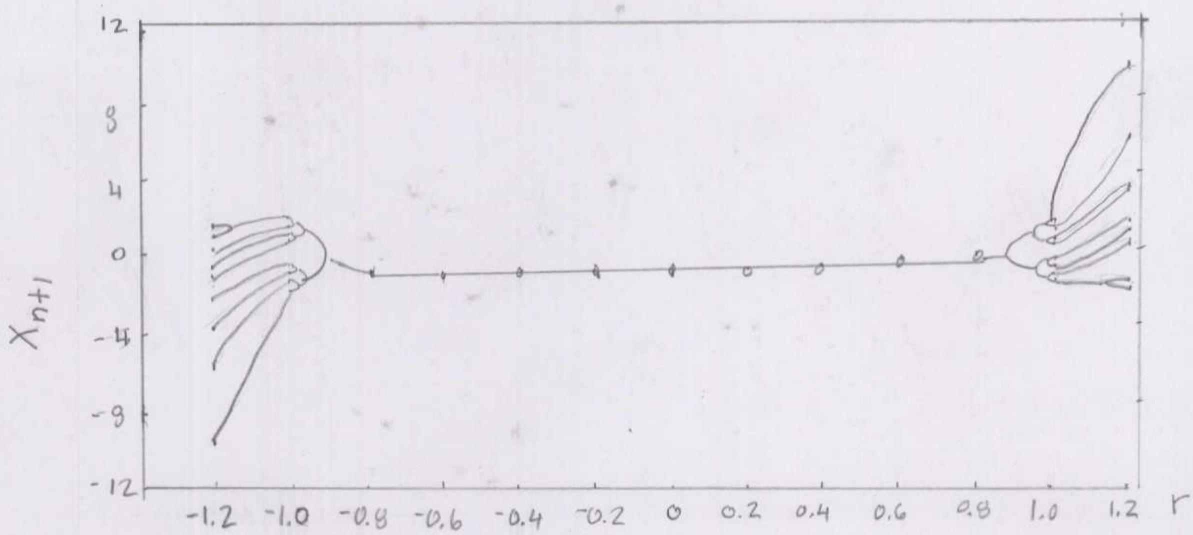
$x_{n+1} = r\cos x_n$

10.2.6. $\frac{d}{dx}x_{n+1} = -r\sin x_n = 0$; $x_n = n\pi$; $r = x$



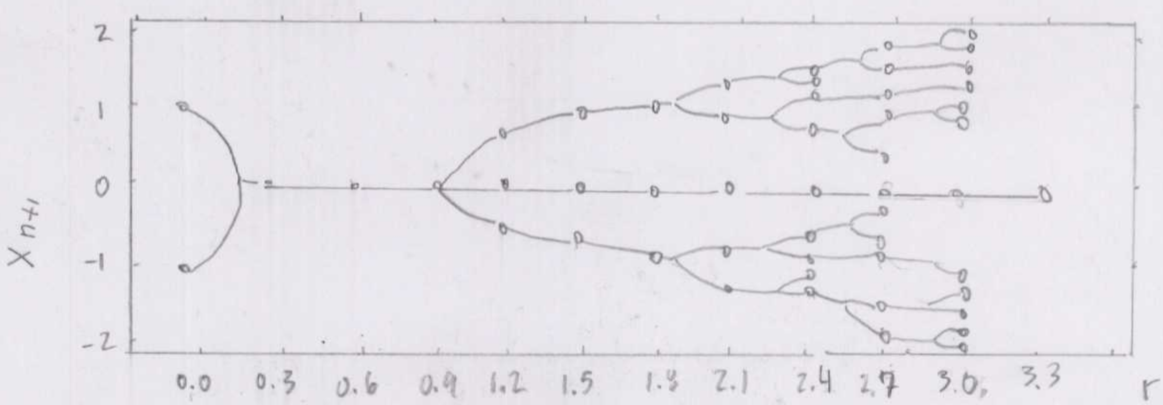
$x_{n+1} = r\tan x_n$

10.2.7. $\frac{d}{dx} r\tan x_n = r(1+\tan^2 x_n) = 0$; $x_n = \text{NaN}$; $r = 0$



$x_{n+1} = rx_n - x_n^3$

10.2.8. $\frac{d}{dx}x_{n+1} = r - 3x_n^2 = 0$; $x_n = \sqrt{\frac{r}{3}}$



$x_{n+1} = rx_n(1-x_n)$

10.3.1 $x_{n+1} = 0 = rx_n(1-x_n)$; $x_n = 0, 1$

Stability: $|f'(0)| = 0$ "superstable"

$|f'(1)| = 2$ "unstable"

$x_{n+1} = rx_n(1-x_n)$

10.3.2

a) $p = 1/2$; $q = 1/2$: $f(x)' = r - 2rx = 0$; $x = 1/2$

$f(f(x))' = -r^2(2x-1)(2r(x-1)x+1) = 0$; $x = 1/2$

b) $X_{max} = 1/2$ for $f(x)$, $f(f(x))$, and $f(f(f(x)))$ with $r = 1/2$

$X_{n+1} = rX_n/(1+X_n^2)$

10.3.3. Fixed Points: $f(X_n) = X_{n+1} = rX_n/(1+X_n^2) = 0$

$$X(1+X^2) - rX = 0$$

$$X = \pm\sqrt{r-1},\ 0$$

$$f'(X_n) = X_{n+1} = \frac{-r(X^2-1)}{(X^2+1)^2} = 0$$

Stability: $|f'(0)| = r$ "stable, marginal, or unstable"

$|f'(\pm\sqrt{r-1})| = \frac{2}{r} - 1$ "stable, marginal, or unstable"

The fixed points show direction toward the origin by an $r<1$, $r=1$, or $r>1$.

$X_{n+1} = X_n^2 + c$

10.3.4. a) Fixed Points: $0 = X^2 - X + c$

$$X^* = \frac{1 \pm \sqrt{1-4(c)}}{2}$$

Stability: $X^* = \frac{1+\sqrt{1-4c}}{2}$; $\left|f'\left(\frac{1+\sqrt{1-4c}}{2}\right)\right| = 1+\sqrt{1-4c}$

$X^* = \frac{1-\sqrt{1-4c}}{2}$; $\left|f'\left(\frac{1-\sqrt{1-4c}}{2}\right)\right| = 1-\sqrt{1-4c}$

c / Fixed Point	$< 1/4$	$= 1/4$
$1+\sqrt{1-4c}$	unstable	Marginal case
$1-\sqrt{1-4c}$	Stable	Super Stable

b) Bifurcations: $X = 1+\sqrt{1-4c}$

$$c = \frac{1-(X-1)^2}{4}$$

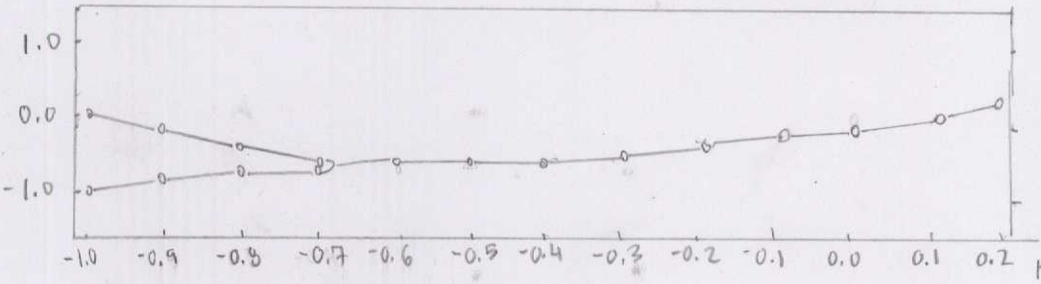
$X = 1-\sqrt{1-4c}$

$$c = -\frac{1+(X-1)^2}{4}$$

Saddlenode bifurcation because zero, one, then two fixed points arise from new c's.

c) A stable 2-cycle shows when $c < 1/4$ near the $x^* = 1 - \sqrt{1-4c}$ fixed point. Super stable at $c = 1/4$ with a similar maximum.

d)



$x_{n+1} = r x_n (1 - x_n)$

$y_{n+1} = y_n^2 + c$

10.3.5. $x_{n+1} = r x_n (1 - x_n) = r(y_n^2 + c)(1 - (y_n^2 + c))$

$= a y_{n+1} + b = a(y_n^2 + c) + b$

$r(ay + b)(1 - (ay + b)) = a(y_n^2 + c) + b$

$\underline{-a^2 r y^2} + ar(1-2b)y + rb(1-b) = \underline{ay^2} + ac + b$

$a = -\frac{1}{r}$; $b = \frac{1}{2}$; $c = \frac{r(2-r)}{4}$

$x_{n+1} = f(x_n)$

$f(x_n) = r x_n - x_n^3$

10.3.6. a) <u>Fixed Points:</u> $x_{n+1} = r x_n - x_n^3$

$x^3 + (1-r)x = 0$

$x^* = 0, \pm\sqrt{r-1}$

<u>Stability:</u> $x^* = 0$; $|f'(0)| = r$ "Super stable"

$x^* = \sqrt{r-1}$; $|f'(\sqrt{r-1})| = -2r + 3$

"Superstable, unstable"

$x^* = -\sqrt{r-1}$; $|f'(-\sqrt{r-1})| = -2r + 3$

"Superstable, unstable"

$r = 0$; $r = 3/2$; "Stable" occurs when $1 < r < \frac{3}{2}$

b) When $f(p) = q$ and $f(q) = p$, then $q = p = 0$ or

$q = p = \pm\sqrt{r-1}$

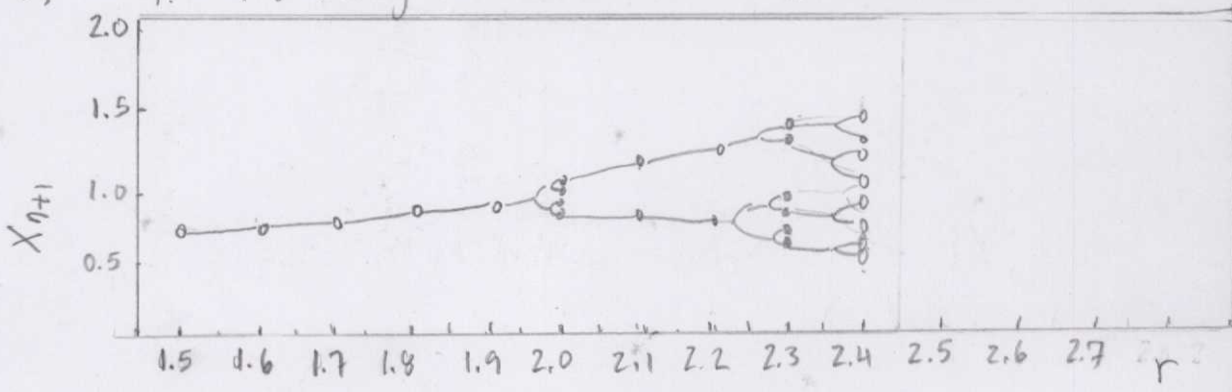
c) Stability $= |f'(x)|$

$q = p = 0$; $|f'(q)| = |f'(p)| = r$

$q = p = \pm\sqrt{r-1}$; $|f'(q)| = |f'(p)| = -2r+3$

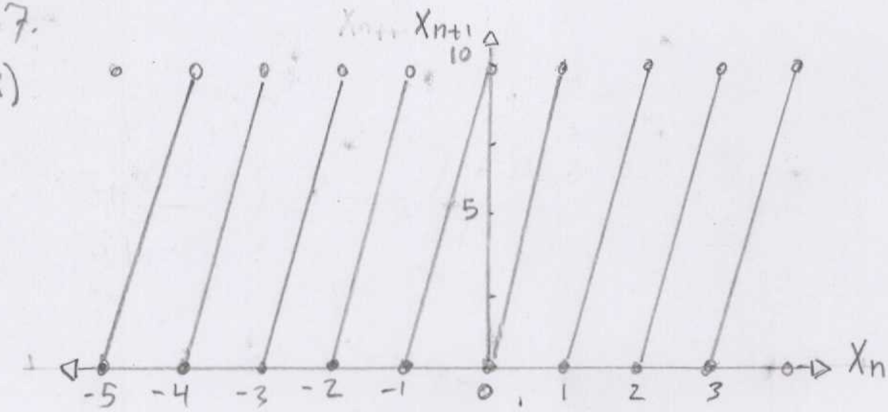
$r = 0$; "Unstable" ; $r = \frac{3}{2}$; "superstable"

d) $r = 1$; "marginal case" ; $1 < r < \frac{3}{2}$ "Stable"



$X_{n+1} = 10X_n \pmod{1}$ 10.3.7.

a)



b) Fixed Points: $X_{n+1} = 10X_n \pmod{1}$

$x^* = n \circ 1.00$ where $n \in \mathbb{R}$

c) The aperiodic orbits appear between n-values. e.g. irrational numbers. Many aperiodic solutions exist from zero to one, or one to two, etc,

d) Infinite many solutions are the irrational values from zero up till one.

e) By example:

n	1	2		1		1
X_n	0.01	0.1	∘∘∘	10^6	∘∘∘	1
X_{n+1}	0.1	1.0		0		1

$X_{n+1} = 10 X_n$

10.3.8. A dense orbit appears about initial conditions at irrational values. An irrational number shifted in base-10, $10 x_n \pmod 1$, never ends as with a decimal shift map.

$X_{n+1} = 2X_n \pmod 1$

10.3.9. An aperiodic map in the function is the dense map for irrational initial conditions. A periodic orbit is when $\frac{d}{dx} f^n(x) > 1$.

$X_n = \sin^2(\pi \theta_n)$

$X_{n+1} = 4X_n(1-X_n)$

10.3.10.

a) $\theta_{n+1} = 2\theta_n \pmod 1$; $X_n = \sin^2(\pi \theta_n)$

Identity: $\sin^2(a) = [2\sin(a)\cos(a)]$

$X_n = \sin^2(\pi\theta_n) = 2\sin(\pi\theta_n)\cos(\pi\theta_n)$

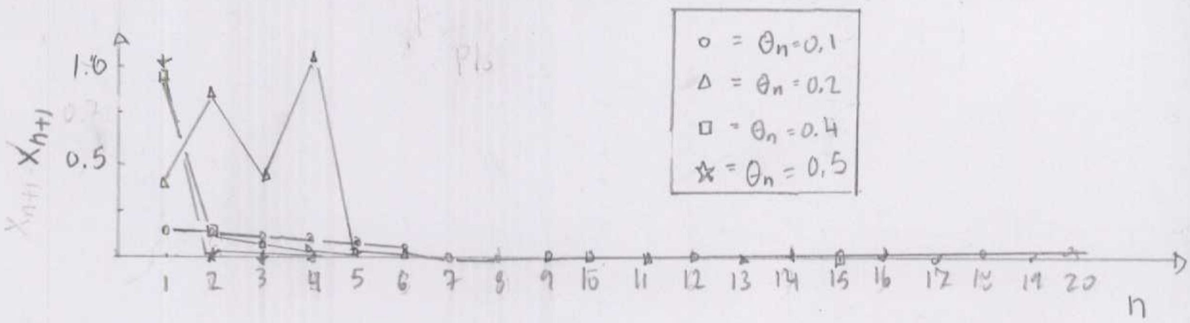
$= [2\sin(2\pi\theta_n \pmod 1)\cos(2\pi\theta_n \pmod 1)]^2$

If $\sqrt{X_n} = \sin(\pi\theta_n)$, then the equation becomes.

$= [2\sqrt{X_n}(1-\sqrt{X_n})]^2$

$= 4X_n(1-X_n)$

b) Plots of X_{n+1} vs. n.



$X_{n+1} = f(X_n)$

$f(x) = -(1+r)x - x^2 - 2x^3$

10.3.11

a) Fixed Points: $X_{n+1} = -(1+r)X - X^2 - 2X^3$

$-(2+r)X - X^2 - 2X^3 = 0$

Stability: $|f'(0)| = |-(1+r) - 2\cdot(0) - 6(0)^2| = r-1$ "superstable, stable, marginal, unstable"

b) Flip Bifurcation - a location when period doubling shows in the map.

b) $r<0$: $|f'(0)| = |r-1| >$ ~~Unst~~ "Unstable"

$r=0$: $|f'(0)| = |r-1| = 1$ "Marginal case"

$r>0$: $|f'(0)| = |r-1| < \infty$: "Superstable, Stable, marginal case, Unstable"

A bifurcation about the fixed point occurs at $r=0$.

c) <u>Taylor Series</u>: $f(x) = f(a)^{\circ} \cdot \frac{(x-a)^0}{0!} + f(a)^{\prime} \cdot \frac{(x-a)^1}{1!} + f(a)^{\prime\prime} \cdot \frac{(x-a)^2}{2!}$

$$= \sum_{n=0}^{\infty} f(a)^{(n)} \frac{(x-a)^n}{n!}$$

$$f(x) = -(1+r)x - x^2 - 2x^3$$

$$f^2(x) = -(1+r)[-(1+r)x - x^2 - 2x^3] - [-(1+r)x - x^2 - 2x^3]^2 - 2[-(1+r)x - x^2 - 2x^3]^3$$

$$f^2(x)' = (-r - 6x^2 - 2x - 1)(-6x^2(r + 2x^2 + x + 1)^2 + 2(r+1)x - r + 4x^3 + 2x^2 - 1)$$

$$f^2(x)'' = 12r^3x + r^2(240x^3 + 72x^2 + 36x - 2) + 2r(504x^5 + 360x^4 + 300x^3 + 48x^2 + 18 - x - 1) + 4x(288x^6 + 336x^5 + 378x^4 + 165x^3 + 70x^2 + 3x + 3)$$

$$f^2(x) = (r+1)^2 x - r(r+1)x^2$$

$r<-1$ $|f^2(0)'| = |(r+1)^2| \geq 1$ "Marginal Case or Unstable"

$r=-1$ $|f^2(0)'| = |(r+1)^2| = 0$ "Superstable"

$-1<r<0$ $|f^2(0)'| = |(r+1)^2| < 1$ "Stable"

$r=0$ $|f^2(0)'| = |(r+1)^2| = 1$ "Marginal case"

d) ~~A~~ $r<0$ has a 2-period behavior and above zero an instability when $r>0$, each around the flip bifurcation.

$X_{n+1} = rx(1-x)$

10.3.12

a) $R_n = \frac{R_n - R_{n-1}}{R_{n+1} - R_n}$

b)

R_2	3.44949
R_3	3.54409
R_4	3.56441
R_5	3.56876
R_6	3.56969
R_7	3.56989
R_8	3.56993

c) 4.65000 "Feigenbaum's constant"

10.3.13

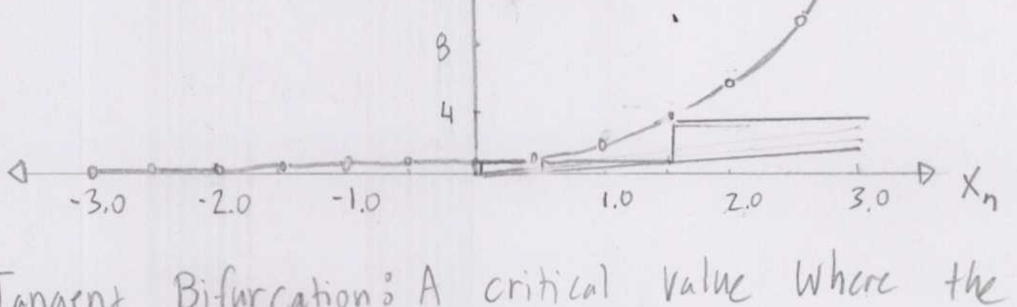
a) The dark curves represent bifurcations in the $f^P(x)$ function. A maximum value at $x_m = 1/2$ is the solution for $f^P(x)$'s maximum value.

b) r's value at the "intersecting" point shown in the orbit diagram, $R_n = \frac{R_n - R_{n-1}}{R_{n+1} - R_n} \approx 3.56$

$X_{n+1} = re^{X_n}$

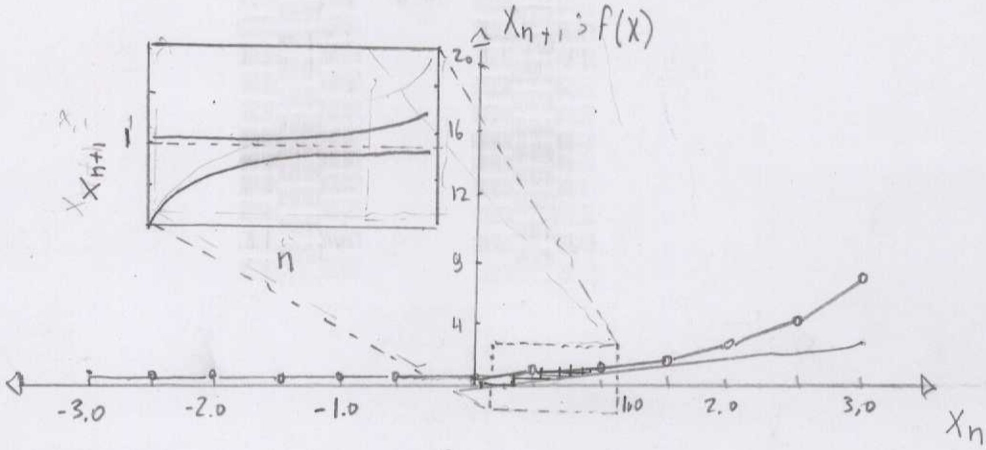
10.4.1.

a)



b) Tangent Bifurcation: A critical value where the stable and unstable coalesce.

c)



$$x_{n+1} = \frac{r x_n^2}{(1+x_n^2)}$$

10.4.2 Fixed Points: $x_{n+1} = \dfrac{r x_n^2}{(1+x_n^2)}$

$$x^* = \pm\sqrt{r-1}\ ,\ 0$$

Stability: $x^* = \sqrt{r-1}\ ;\ |f'(\sqrt{r-1})| = \left|\dfrac{r - r(r-1)}{(r-1+1)^2}\right|$

$$= \left|\frac{2-r^2}{r}\right|$$

$r<1$ "Unstable"

$r=1$ "Marginal case"

$r>1$ "Stable

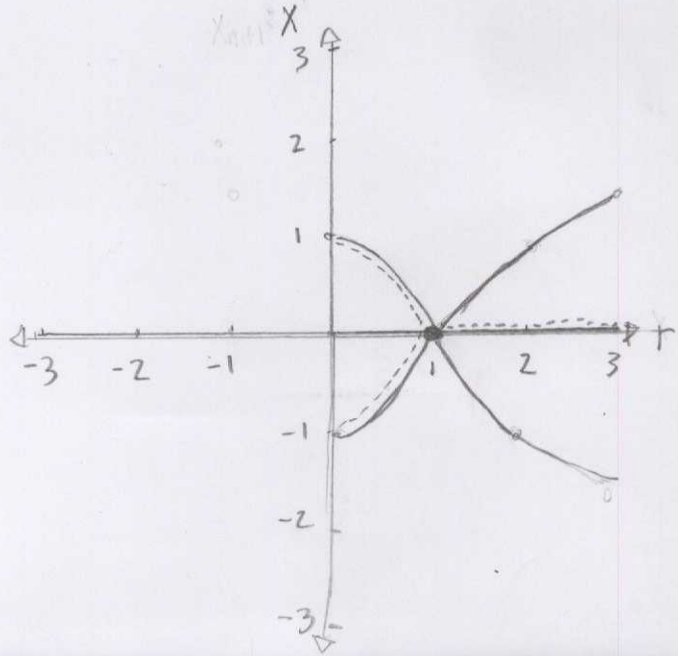
$x^* = 0\ ;\ |f'(0)| = |r|$

$r<1$ "Stable

$r=1$ "Marginal case"

$r>1$ "Unstable"

Bifurcations:



Intermittency - chaotic behavior below a period 3 window
The system's 3-period behavior is fact for an investigation about intermittency.

$X_{n+1} = 1 - rX_n^2$ 10.4.3. Cycles occur when $f^p(X_{n+1}) = 0$. The variable p describes p-cycles. A 3-cycle has similar outputs for each cycle of zero.

$$\frac{d}{dx} f^3(X_{n+1}) = \frac{d}{dx} f(f(f(X_{n+1})))$$

$$= f'(f(f(X_{n+1}))) \circ f'(f(X_{n+1})) \circ f'(X_{n+1})$$

$$= (-2r\, f(f(X_{n+1}))) \circ (-2r\,(f(X_{n+1}))) \circ -2r X_n$$

$$= -8r^3 f(f(X_{n+1})) \circ f(X_{n+1}) \circ X$$

$$= 0$$

$f(0) = 1$; Unable to be zero

$f(f(0)) = 1 - r$; Zero at $r = 1$

$f(f(f(0))) = 1 - r(1-r)^2$; Capable of zero at $r \cong 1.754$

The successive derivatives are zero with a cubic equation dependent upon r.

10.4.4. Logistic equation: $X_{n+1} = rX_n(1 - X_n)$

$f(f(f(X_n))) = 1 - r(1 - r^2(1 - rx^2)^2)^2 = 0$

Excel: $r = 3.82$, $0 < x < 1$

⋮

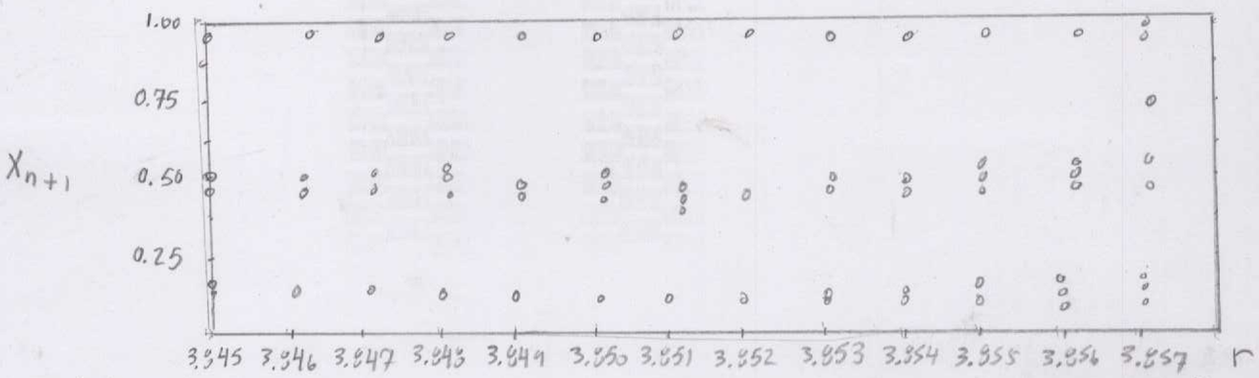
$r = 3.828$, $0 < x < 1$

⋮

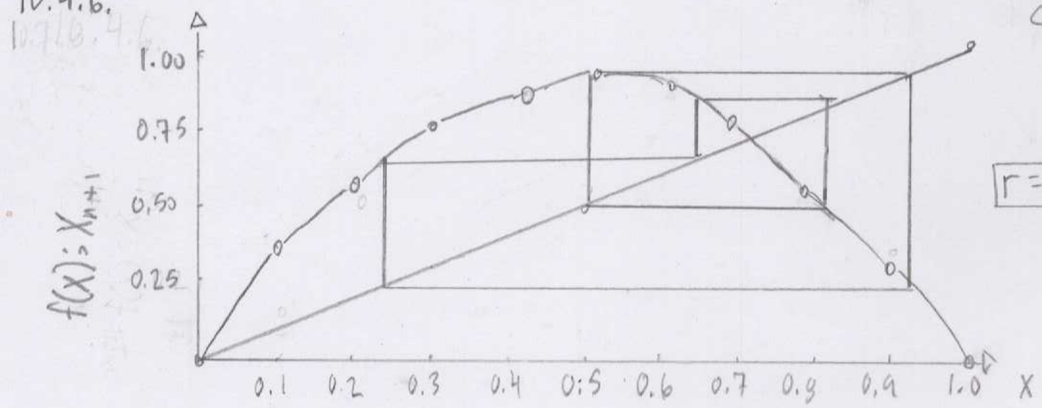
$r = 3.8284$, $0 < x < 1$

⋮

10.4.5



10.4.6.



$\boxed{r = 3.738915}$

The superstable cycle's period is five because the number of iterations before $X_0 = X_5 = 1/2$.

10.4.7.

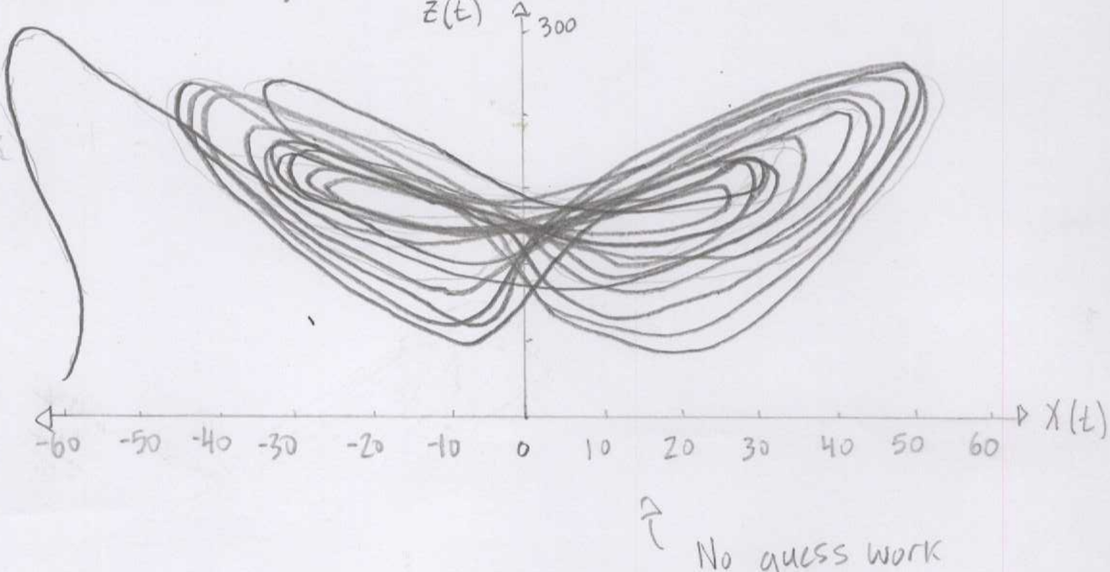
a) An $r > 1+\sqrt{5}$ generates the sequence RL in a logistic equation because p-cyles where $p > 2$ oscillates right, then left.

b. RLRR.

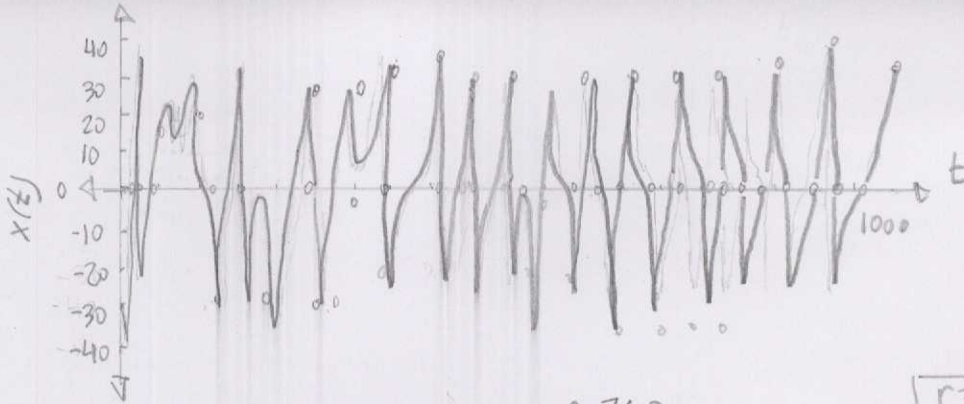
$\dot{x} = \sigma(y-x)$
$\dot{y} = rx - y - xz$
$\dot{z} = xy - bz$

10.4.8. $\sigma = 10$; $b = 8/3$; $r \approx 166$.

a)

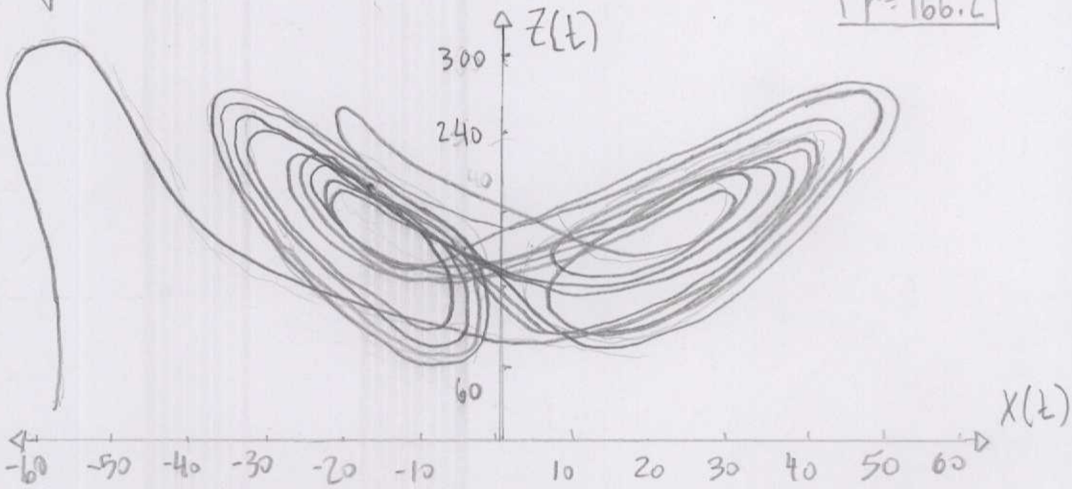


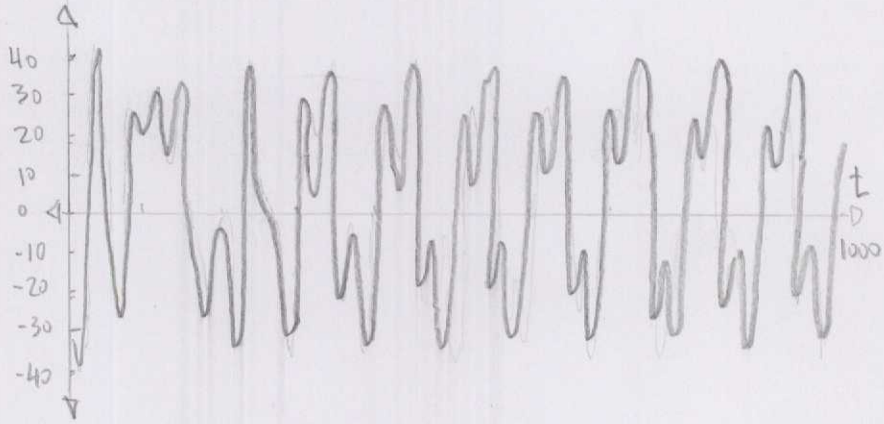
No guess work



r = 166.2

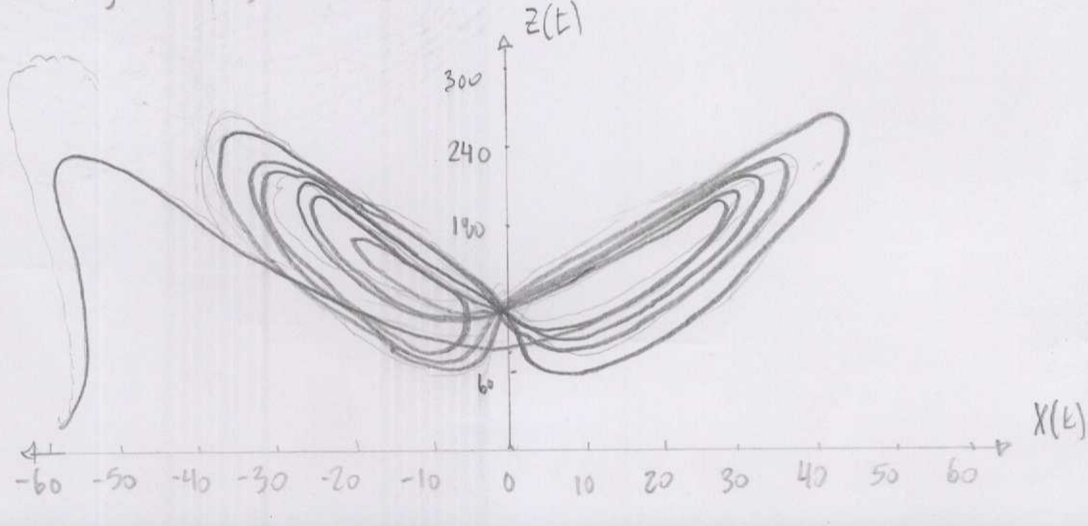
b)



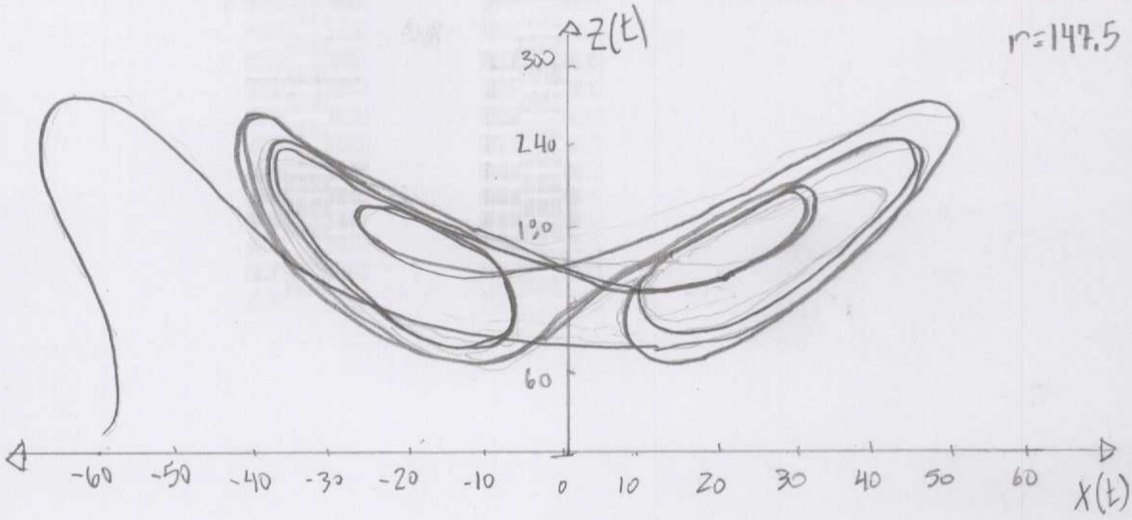


c) The regularity from $r = 166$ to $r = 166.2$ demonstrates a frequency with longer lifetimes.

10.4.9 $\sigma = 10$, $b = 8/3$, $r = 148.5$



$r = 147.5$



$f(a) = b$
$f(b) = c$
$f(c) = a$

10.4.10.

$r = 1+\sqrt{8} = 3.8284$; a, b, c are period-3 bifurcations.

$X_{n+1} = r X_n (1 - X_n)$;

Three equations representing a period-3 cycle:

$$f(a) = b = ra(1-a)$$

$$f(b) = c = rb(1-b) = f(f(a))$$

$$f(c) = a = rc(1-b) = f(f(f(a)))$$

Maximum for the period-3 cycle:

$$\frac{df(f(f(a)))}{da} = \frac{d(f^3(a))}{d(f^2(a))} \cdot \frac{d(f^2(a))}{d(f(a))} \cdot \frac{df(a)}{da}$$

$$= r^3(1-2a)(1-2b)(1-2c) \quad ①$$

$$= 1$$

A shifted maximum toward zero for F(a), f(b), f(c):

$$A = r(a - 1/2) \; ; \; B = r(b - 1/2) \; ; \; C = r(z - 1/2)$$

When the maximal equation becomes A, B, C:

$$r^3(1-2a)(1-2b)(1-2c) = 1$$

$$\therefore A \cdot B \cdot C = -1$$

A second equation appears from A, B, C:

$$\frac{r^2}{4} - \frac{r}{2} = A^2 + B = B^2 + C = C^2 + A$$

Expanding the previous equation with three relationships:

$$a = A+B+C \; ; \; b = AB+BC+CA \; ; \; C = ABC$$

$$8ABC = -1 \Rightarrow 3A^3 = 8a^3+27 = (2a+3)(4a^2-6a+9) = 0$$

Three Identities: $A^2+B^2+C^2 = a^2-2b$

$$A^3+B^3+C^3 = a^3-3ab+3c$$

$$(AB)^2+(BC)^2+(CA)^2 = b^2-2ca$$

A new R-equation: $R = \frac{1}{3}(a^2+a-2b)$

$$R^2 = (A^2+B)(B^2+C)$$

A function with a and b and c:

$$a^4-4a^3+14ab+a^2+b^2+6ac-4a^2b-3b-18c = 0$$

$$c = -1/8 \; ; \; b = \frac{16a^3-8a^2-9}{56a-24}$$

The equation with only a:

$$24(2a-1)(2a+3)(4a^2-6a+9)^2 = 0 \; ; \; a = 1/2$$

Solving for $b = -9/4$

Solving for $R = 7/4$

Solving for $r = 1+2\sqrt{2}$

$x_1 = a$ 10.4.11. $x_{n+1} = a^{x_n}$

$x_2 = a^a$ The hyperpower converges when $x_1 = a = e^{-e}, e, e^{e}$ or e^{-1}.

$x_3 = a^{(a^a)}$ A long term behavior becomes complex with a geometric or arithmetic relationship.

$x_{n+1} = rx_n$ 10.5.1. $\lambda = \frac{1}{n}\ln\left|\frac{\delta_n}{\delta_0}\right| = \frac{1}{n}\ln\left|\frac{r^n\delta_0}{\delta_0}\right| = \ln|r|$

$x_{n+1} = 10x_n$ 10.5.2 $\lambda = \frac{1}{n}\ln\left|\frac{\delta_n}{\delta_0}\right| = \frac{1}{n}\ln\left|\frac{10^{10}\delta_0}{\delta_0}\right| = \ln 10$

$f(x)=\begin{cases} rx & 0\le x\le 1/2 \\ r-rx & 1/2\le x\le 1\end{cases}$

10.5.3. $\lambda = \frac{1}{n}\ln\left|\frac{\delta_n}{\delta_0}\right| = \frac{1}{n}\ln\left|\frac{r^n\delta_0}{\delta_0}\right| = \ln r$

A liapunov Exponent with $\lambda = \ln r$ describes chaotic solutions for all $r>1$.

10.5.4 $\lambda_{logistic} = \frac{1}{n}\ln\left|\frac{\delta_n}{\delta_0}\right| = \frac{1}{n}\ln\left|\frac{[r^n(1-x)^n - r^n x^n]\delta_0}{\delta_0}\right|$

$= \frac{1}{n}\ln|r^n(1-2x)^n|$

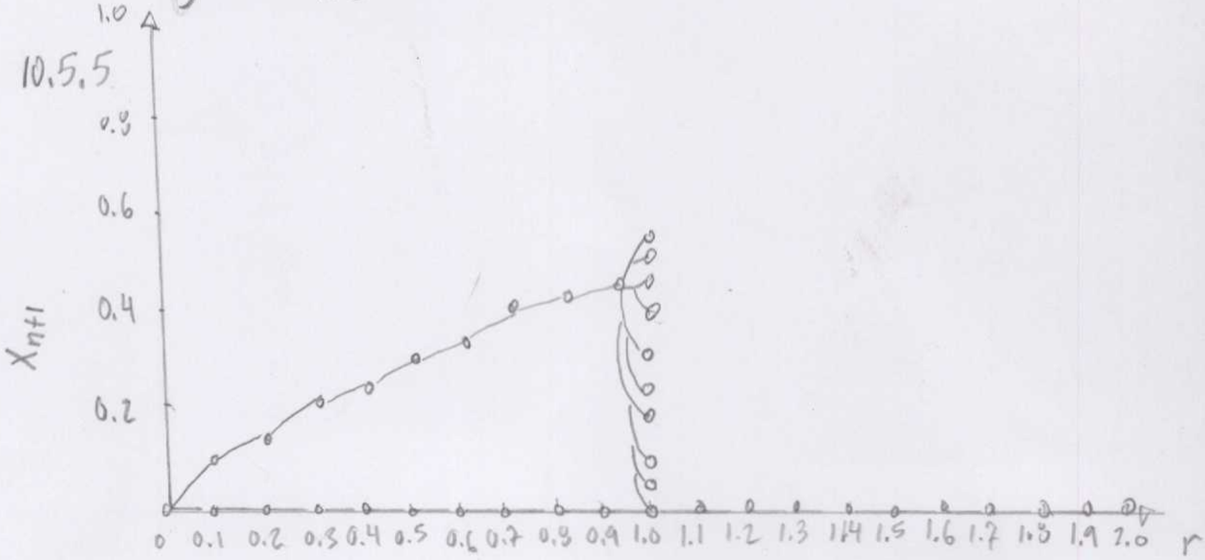
$= \ln|r(1-2x)|$

$\lambda_{tent} = \frac{1}{n}\ln\left|\frac{\delta_n}{\delta_0}\right| = \frac{1}{n}\ln\left|\frac{r^n\delta_0}{\delta_0}\right|$

$= \ln r$

When $r>0$, then the tent maps Liapunov constant is always greater than zero. The constant in the logistic equation alternates from positive into negative, chaotic into stable, when $3<r<4.0$

10.5.5



$X_{n+1} = r\sin\pi X_n$

10.5.6 <u>Fixed Points</u>: $X_{n+1} = 0 = r\sin\pi X_n$

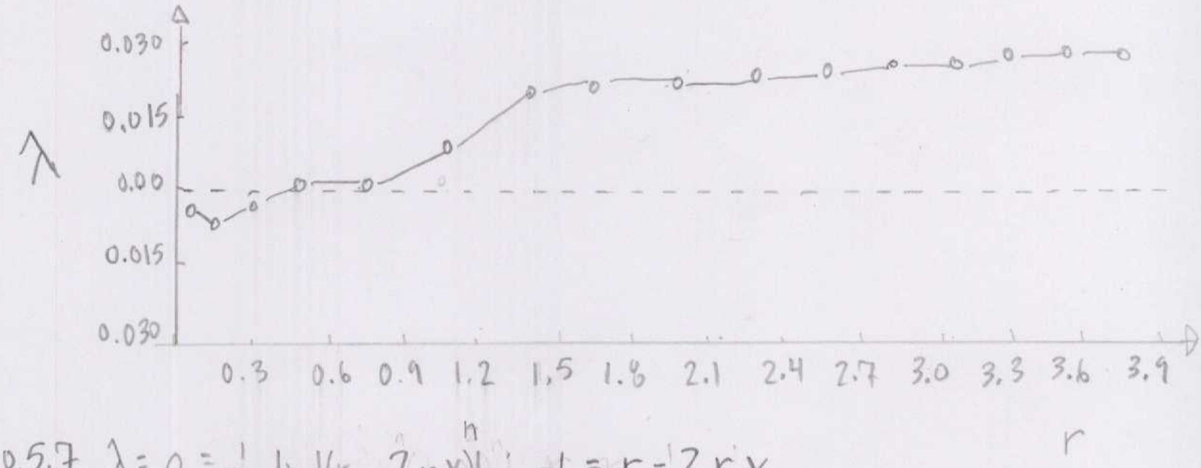
$r = \frac{X}{\sin\pi X}$

$X^* = 0$

Liapunov Exponent:

$$\lambda = \frac{1}{n}\ln\left|\frac{\delta_n}{\delta_0}\right| = \frac{1}{n}\ln\left|\frac{r^n\pi^n\cos^n\pi x_n\,\delta_0}{\delta_0}\right|$$

$$= \ln|r\pi\cos\pi x_n|$$

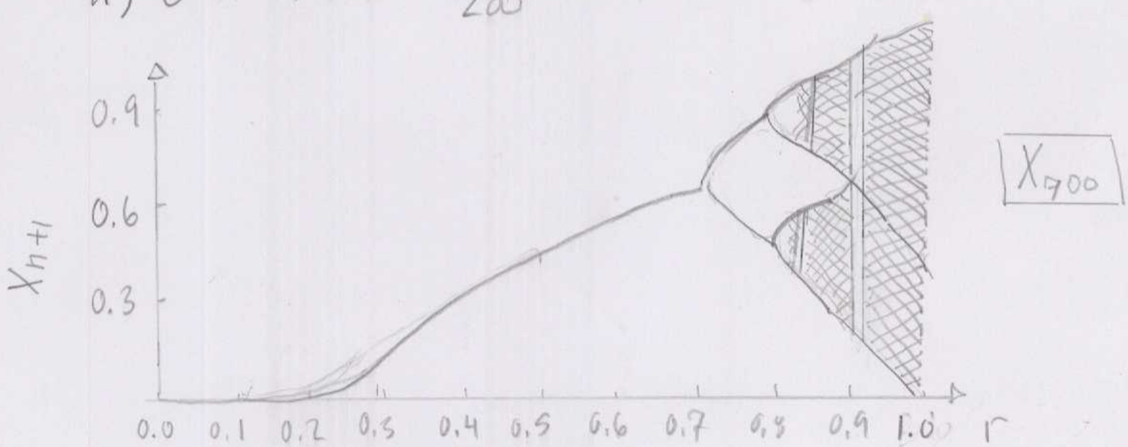


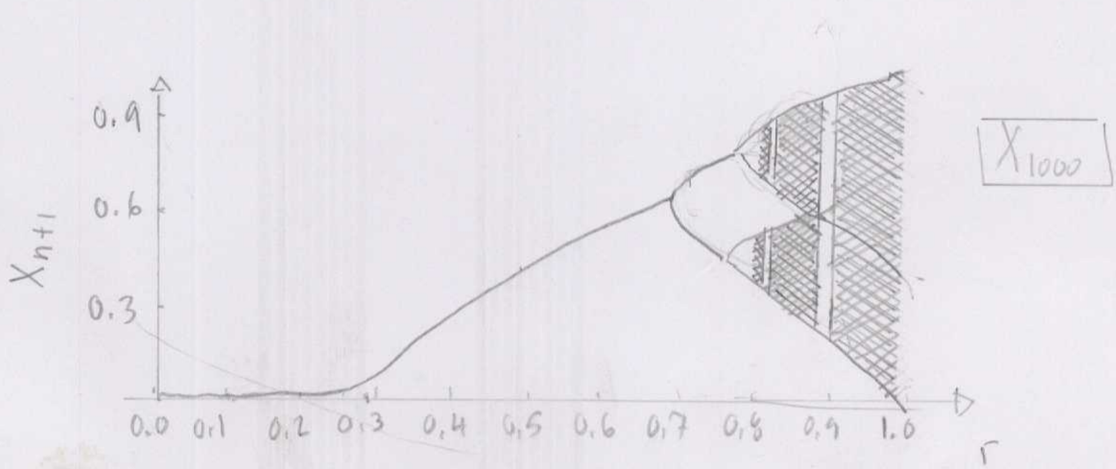
10.5.7 $\lambda = 0 = \frac{1}{n}\ln|(r-2rx)^n|$; $1 = r - 2rx$

$$r = \frac{1}{1-2x}$$

$x_{n+1} = r\sin\pi x_n$

10.6.1
a) $0<r<1$; $\Delta r = \frac{1}{200}$;





$r_1 \to r_6$

b)

r	Value
1	0.71967
2	0.83326
3	0.85860
4	0.86405
5	0.86522
6	0.86556

c) Feigenbaum Ratio $= \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = \frac{r_4 - r_3}{r_5 - r_4} = 4.65811$

% Error $= \frac{|4.65811 - 4.66921|}{4.6692} \times 100\% = 0.08\%$

10.6.2

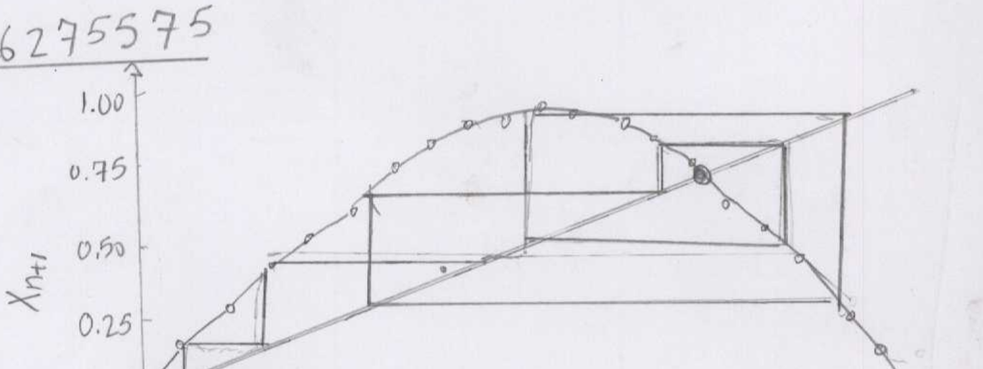
a) R_n computation is easier because R_n is found by graph, while r_n through calculating $f^n(x)$ where $n > 2$.

b) I comprehend no extreme difference. See 10.6.1.

10.6.3.

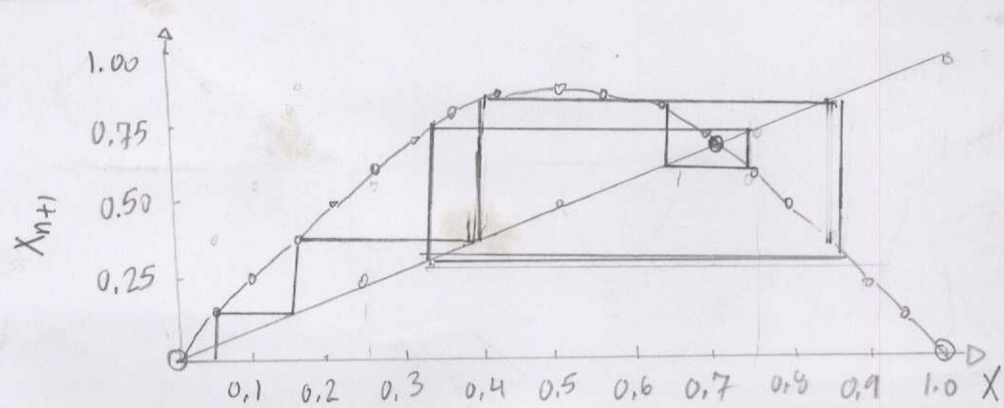
$X_{n+1} = rX_n(1 - X_n)$

a) $r = 3.6275575$



$X_{n+1} = r \sin \pi X_n$

$r = 0.8811406$



b) The 6-cycles match both graphically and functionally by:

Maximums:

$x_{n+1} = r x_n(1-x_n)$; $f'(x_n) = r(1-2x_n) = 0$

$x_n = 1/2$

$x_{n+1} = r\sin(\pi x_n)$; $f'(x_n) = r\pi\cos(\pi x_n) = 0$

$x_n = 1/2$.

Nodes:

$x^*_{n+1} = 0 = r x(1-x_n)$;

$x_{n+1} = 0 = r\sin(\pi x_n)$;

$x^* = 0, 1$

Fixed Points:

$x_{n+1} = r x_n(1-x_n)$; $0 = 3.62756x_n^2 - 2.62756x_n$

$x_n = 0.724332$

$x_{n+1} = r\sin(\pi x_n)$; $0 = r\sin(\pi x_n) - x_n$

$x_n = 0.704963$

10.6.4.

a) (Metropolis et. al 1973) $r = 3.9602701$ for a period-4 cycle

<u>RLL U-Sequence:</u>

$x_6 = 0.3$	R
$x_7 = 0.8$	L
$x_8 = 0.5$	L

<u>RLR U-Sequence:</u>

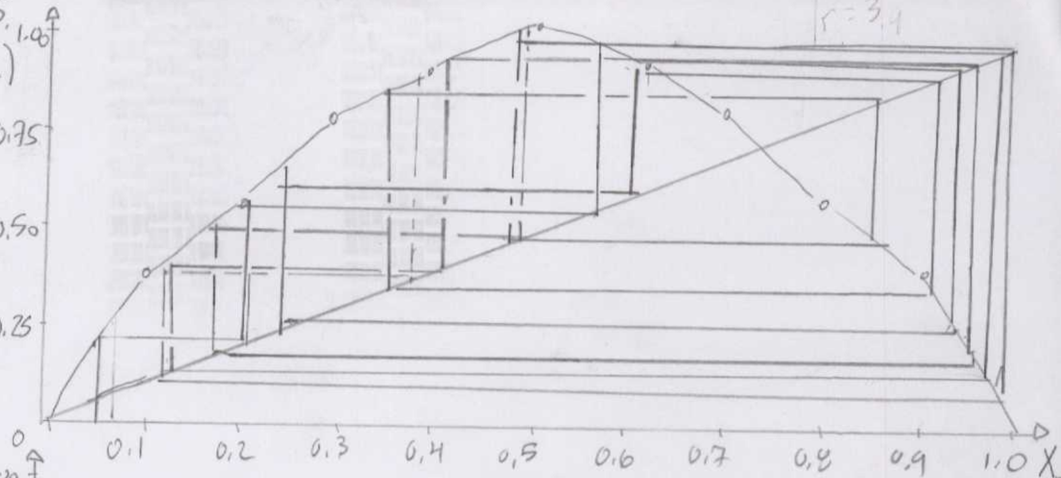
$x_0 = 0.6$	R	$x_3 = 0.6$	R
$x_1 = 0.9$	L	$x_4 = 0.9$	L
$x_2 = 0.2$	R	$x_5 = 0.3$	R

The RLL sequence seemed intermittent in the x_{n+1} analysis.

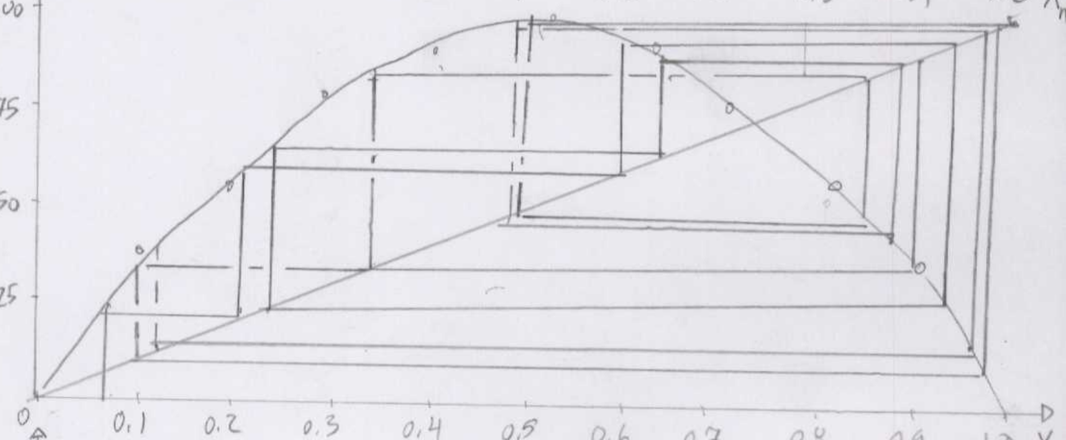
b) $x_i > x_2, x_5$ is a unique period-4 orbit with an alternating U-sequence.

10.6.5.
a)

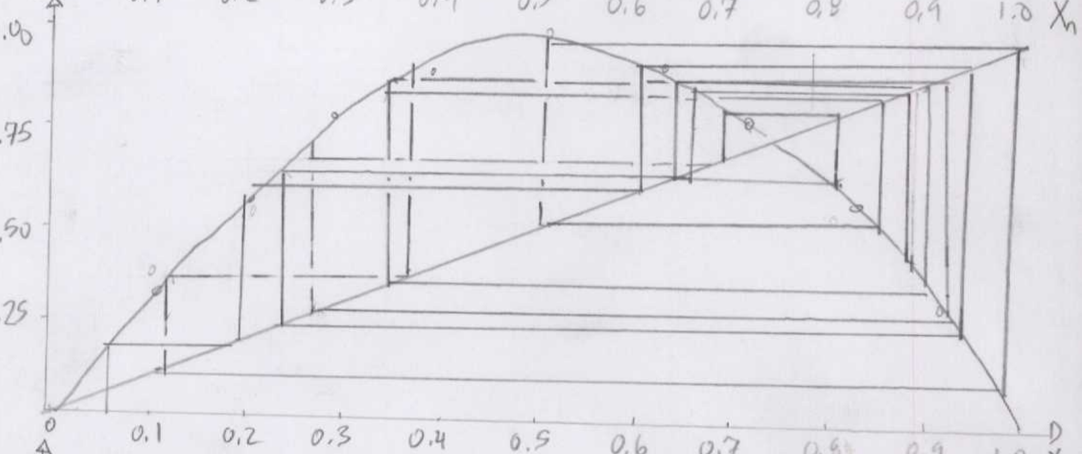
r = 3.9057065



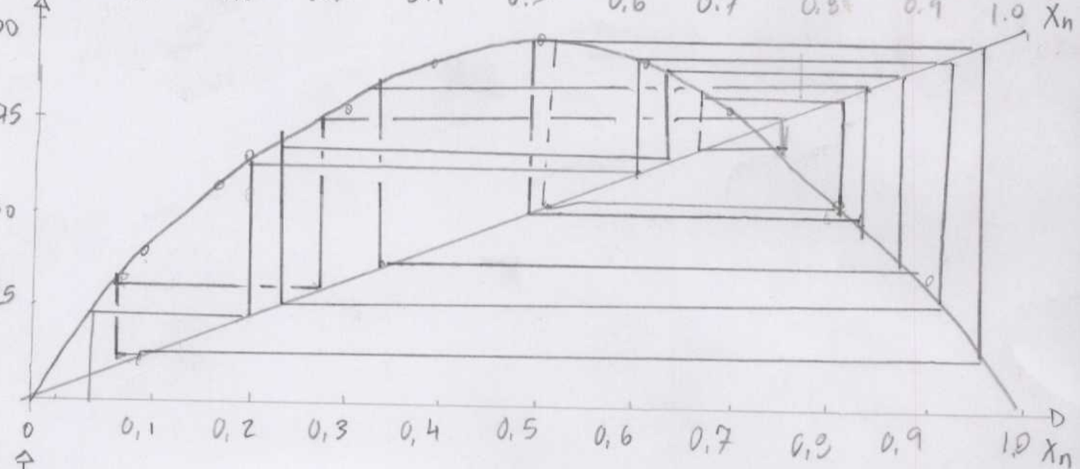
r = 3.9375364



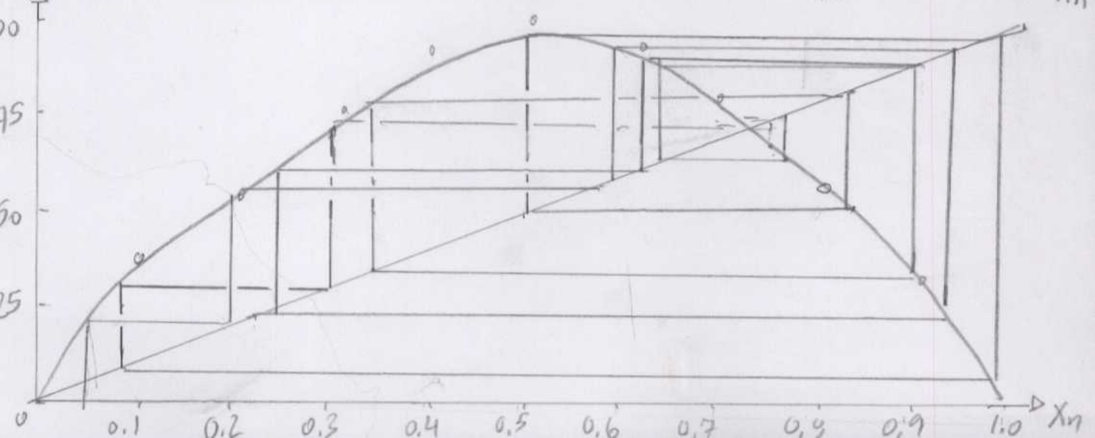
r = 3.9602701



r = 3.9777664



r = 3.9902670



b) Accuracy by a computer would improve results:

$x_{n+1} = r - x_n^2$

$R(y) = \sqrt{r-y}$

$L(y) = -\sqrt{r-y}$

10.6.6.

a) $RLLR(0) = R(L(L(R(0))))$

$= \sqrt{r + \sqrt{r + \sqrt{r - \sqrt{r + 0}}}}$

$= \sqrt{r + \sqrt{r + \sqrt{r - \sqrt{r}}}}$

b) $r_0 = 2$; $r_{n+1} = \sqrt{r_n + \sqrt{r_n + \sqrt{r_n - \sqrt{r_n}}}}$

$r_1 = 1.913891 \to r_2 = 1.88132 \to r_3 = 1.86877$

$\downarrow$

$r_4 = 1.86390$

$\downarrow$

$r_n \approx 1.860$

c) $R \to L \to L \to R$ around a stable node $r_{n+1} \approx 1.86$.

$g(x) = \alpha g^2(x/\alpha)$

10.7.1

a) $g(x) = 1 + c_2 x^2$; $g(x) = \alpha\left(1 + c_2\left(1 + c_2\left(\frac{x}{\alpha}\right)^2\right)^2\right)$

$= \alpha + \alpha c_2 + \frac{2c_2^2 x^2}{\alpha} + \frac{c_2^3 x^4}{\alpha^3}$

$1 + c_2 x^2 = \alpha(1 + c_2) + \frac{2c_2^2 x^2}{\alpha} + O(x^4)$

$1 = \alpha(1 + c_2)$; $1 = \frac{2c_2}{\alpha}$

$(\alpha, c) = (1, 0), \left(-1 \pm \sqrt{3}, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\right)$

b) $g(x) = 1 + c_2 x^2 + c_4 x^4$;

$= \alpha\left(1 + c_2\left(1 + c_2\left(\frac{x}{\alpha}\right)^2 + c_4\left(\frac{x}{\alpha}\right)^4\right)^2 + c_4\left(1 + c_2\left(\frac{x}{\alpha}\right)^2 + c_4\left(\frac{x}{\alpha}\right)^4\right)^4\right)$

$= \alpha(1 + c_2 + c_4) + \left(\frac{2c_2^2 + 4c_2 c_4}{\alpha}\right)x^2$

$+ \left(\frac{c_2^3 + 4c_4^2 + 6c_2^2 + 2c_2 c_4}{\alpha^3}\right)x^4 + O(x^6)$

$1 = \alpha(1 + c_2 + c_4)$; $1 = \frac{2c_2 + 4c_4}{\alpha}$; $c_4 = \left(\frac{c_2^3 + 4c_4^2 + 6c_2^2 c_4 + 2c_2 c_4}{\alpha^3}\right)$

$$(\alpha, c_2, c_4) \approx (-2.82, -1.30, -0.06)$$
$$(0.67, 0.63, -0.14)$$
$$(4.65, -3.89, 3.11)$$

$y_{n+1} = f(y_n)$

$x_n = \alpha y_n$

10.7.2. $x_n = \alpha y_n = \alpha f(y_n) = \alpha f\left(\frac{x}{\alpha}\right)$

$$x_{n+1} = \alpha f^2\left(\frac{x}{\alpha}\right)$$

10.7.3. Functional Equation: a function defined in terms of itself.

$$g(x) = g^2(x) = \mu g^2\left(\frac{x}{\mu}\right)$$

10.7.4. $g(x) \approx$ Parabolic. $= x^2$

$$g(x^*) = g(\alpha x^*) = \alpha^n f^{(2^n)}\left(\frac{x^*}{\alpha^n}\right)$$

$f(x,r) = r - x^2$

10.7.5.

a) $f(x, R_0) = R_0 - x^2$

$$f^2(x, R_1) = R_0 - [R_1 - x^2]^2$$

A solution into R_0 and R_1:

R_0: $R_0 - x^2 = R_0 - x^2$

$$R_0 = R_0$$
$$= 0$$

R_1: $R_p - x^2 = R_0 - [R_1 - x^2]^2$

$$x = \pm\sqrt{R_1}, \pm\sqrt{R_1 + 1}$$

$$R_1 = 1$$

$$f(x, R_0) = -x^2$$

$$\alpha f\left(\frac{x}{\alpha}, R_1\right) = \alpha\left[1 - \left[1 - \left(\frac{x}{\alpha}\right)^2\right]^2\right]$$

$$= -\frac{x^4}{\alpha^3} + \frac{2x^2}{\alpha}$$

b) When $\alpha = -2$, then:

$$f(x, R_0) = \alpha f^2(x/\alpha, R_1)$$

$$-x^2 = \frac{2x^2}{\alpha} - \frac{x^4}{\alpha^3} \approx \frac{2x^2}{\alpha}$$

$$-x^2 = -x^2 = \text{"Resemblance"}$$

$f(x,r) = r - x^2$

10.7.6. $$\alpha f^2\left(\frac{x}{\alpha}, R_1\right) = -\frac{x^4}{\alpha^3} + \frac{2x^2}{\alpha}$$

$$\alpha^2 f^4\left(\frac{x}{\alpha^2}, R_4\right) = -\alpha^2\left(R_4^8 - 4R_4^7 + 6R_4^6 - 6R_4^5 + 5R_4^4 - 2R_4^3 + R_4^2 - R_4\right) + \frac{8R_4^3(R_4^4 - 3R_4^3 + 3R_4^2 - 2R_4 + 1)x^2}{\alpha^2} + O\left(\frac{x}{\alpha^2}\right)$$

A Solution for R_4:

$$R_4 = -\alpha^2\left(R_4^8 - 4R_4^7 + 6R_4^6 - 6R_4^5 + 5R_4^4 - 2R_4^3 + R_4^2 - R_4\right)$$

$$-1 = \frac{8R_4^3(R_4^4 - 3R_4^3 + 3R_4^2 - 2R_4 + 1)}{\alpha^2}$$

$$(\alpha; R_4) = (0,0), (\pm 0.747144, -0.322988), (\pm 1.00087, 1.74495)$$

$f(x,r) = r - x^4$

10.7.7 $f(x, R_0) = R_0 - x^4$

Fixed Points: $f'(x, R_0) = R_0 - 4x^3 = 0$

$$x^* = \sqrt[3]{\frac{R_0}{4}}$$

When fixing to zero, $x^* = 0$, $R_0 = 0$.

A solution into R_1:

$$f^2(x, R_1) = R_1 - [R_1 - x^4]^4$$

$$0 = R_1 - R_1^4$$

$$R_1 = 1$$

α^2 $\frac{\alpha}{\alpha^2}$ α

$$f(x,R_0) = \alpha f^2\left(\frac{x}{\alpha}, R_1\right)$$

$$-x^4 = \alpha\left[1-\left[1-\left(\frac{x}{\alpha}\right)^4\right]^4\right]$$

$$= \frac{4x^4}{\alpha^3} - \frac{6x^8}{\alpha^7} + \frac{4x^{12}}{\alpha^{11}} - \frac{x^{16}}{\alpha^{15}}$$

$$-x^4 \approx \frac{4x^4}{\alpha^3} \quad ; \quad \alpha = \sqrt[3]{-2^2}$$

<u>Universal g-function:</u>

$$g(x) = \alpha g\left(g\left(\frac{x}{\alpha}\right)\right)$$

$$1 + c_1 x^4 = \alpha(c_1+1) + \frac{4c_1^2 x^4}{\alpha^3} + \frac{6c_1^3 x^8}{\alpha^7} + \frac{4c_1^4 x^{12}}{\alpha^{11}} + \frac{c_1^5 \cdot x^{16}}{\alpha^{15}}$$

$$1 = \alpha(c_1+1) \; ; \; c_1 = \frac{4c_1}{\alpha^3}$$

$$(\alpha, c_1) = (-1.835, -1.545), (0.862, 0.160)$$

<u>δ-estimation:</u> $\delta = \dfrac{R_n - R_{n-1}}{R_{n+1} - R_n}$

<u>R_2 calculation:</u> $f^3(x, R_2) = R_2 - [R_2 - [R_2 - x^4]^4]^4$

$$= R_2 - [R_2 - [R_2]^4]^4$$

$$= R_2 - R_2^4 + 4R_2^7 - 6R_2^{10} + 4R_2^{13} - R_2^{16}$$

$$R_2 = 1.229.$$

$$\delta = \frac{1-0}{1.229-1} = 4.366$$

$$\delta_{Actual} = 4.669$$

$$\text{Error} = \frac{|\delta - \delta_{Actual}|}{\delta_{Actual}} = 6.5\%$$

10.7.8

$x_{n+1} = f(x_n, r)$

$f(x_n, r) = -r + x - x^2$

a) If $r = 0$, then.

$f(x_n, r) = x - x^2$

<u>Tangent line</u>: $y = mx + b$

where $m = f(x_n, r)' = 1 - 2x$

when evaluated at $x = 0, m = 1$

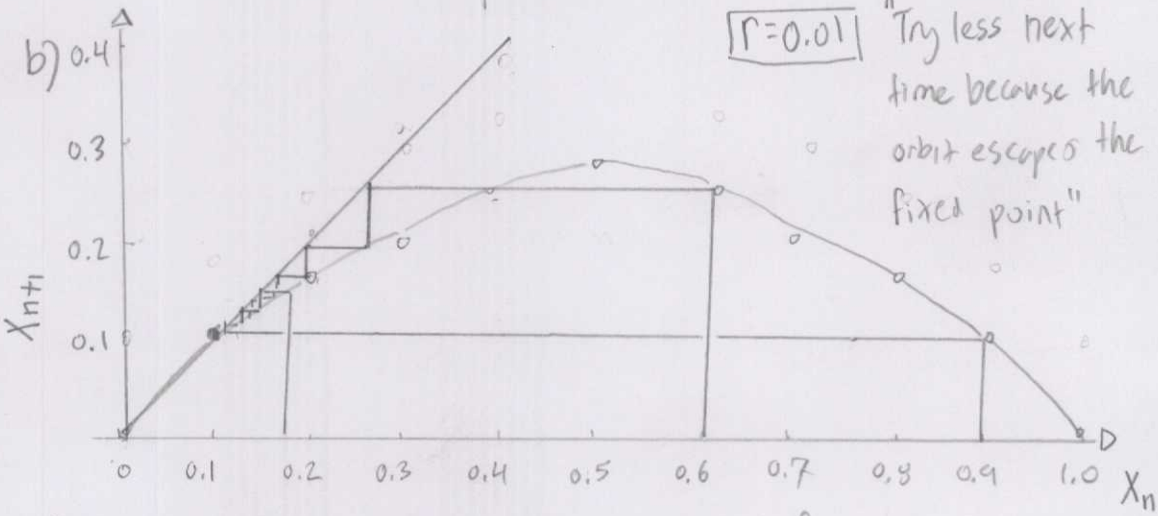
$y = x + b$

at the point $(0,0)$, $b = 0$

$y = x$

Essentially, a tangent line exists at $x = 0$ with n-periodic cycles at the same point.

b)



c) $N(r)$ = typical number of iterations of f

Universal Function: $\lim_{n\to\infty} \alpha \cdot F^{(2^n)}\left(\frac{x}{\alpha^n}, R_n\right)$

$r = n$ for the problem: $\lim_{r\to 0} \alpha^r \cdot F^{(2r)}\left(\frac{x}{\alpha^r}, R_n\right)$

$N(r) = \lim_{r\to 0} 2 \cdot 2^r \log \alpha^r \cdot f\left(\frac{x}{\alpha^r}, R_n\right)$

$N(r) = \lim_{r\to 0} 2^r \log f\left(\frac{x}{\alpha^r}, R_n\right) + r \log \alpha$

$\frac{1}{2} N(r) = 2 \log\left(f\left(\frac{x}{\alpha^2}, R_n\right)\right) + \log \alpha$

$= \log \alpha^2 f^2\left(\frac{x}{\alpha^2}, R_n\right)$

"The derivation iterations show logarithmic relation."

d) $f^2(x,R) = R + (R+x-x^2) - (R+x-x^2)^2$

$$= -R^2 - 2R + (2R+1)x - (2R+2)x^2 + O(x^4)$$

$$\frac{1}{2}N(r) = \log \alpha^2 f^2\left(\frac{x}{\alpha^2}, R_n\right)$$

$$N(4r) = \log \alpha^{4r} \circ f^{2^{4r}}\left(\frac{x}{\alpha^{2r}}, R_n\right) = \log\left[\alpha^2 \circ f^2\left(\frac{x}{\alpha^r}, R_n\right)\right]^{2r}$$

$$\frac{1}{2}N(r) \approx N(4r)$$

e) $N(r) = ar^b$ is a solution where $a = 2\log \alpha f\left(\frac{x}{\alpha^2}, R_n\right)$

$b = 1$

$g(x) = \alpha g^2(x/\alpha)$

10.7.9

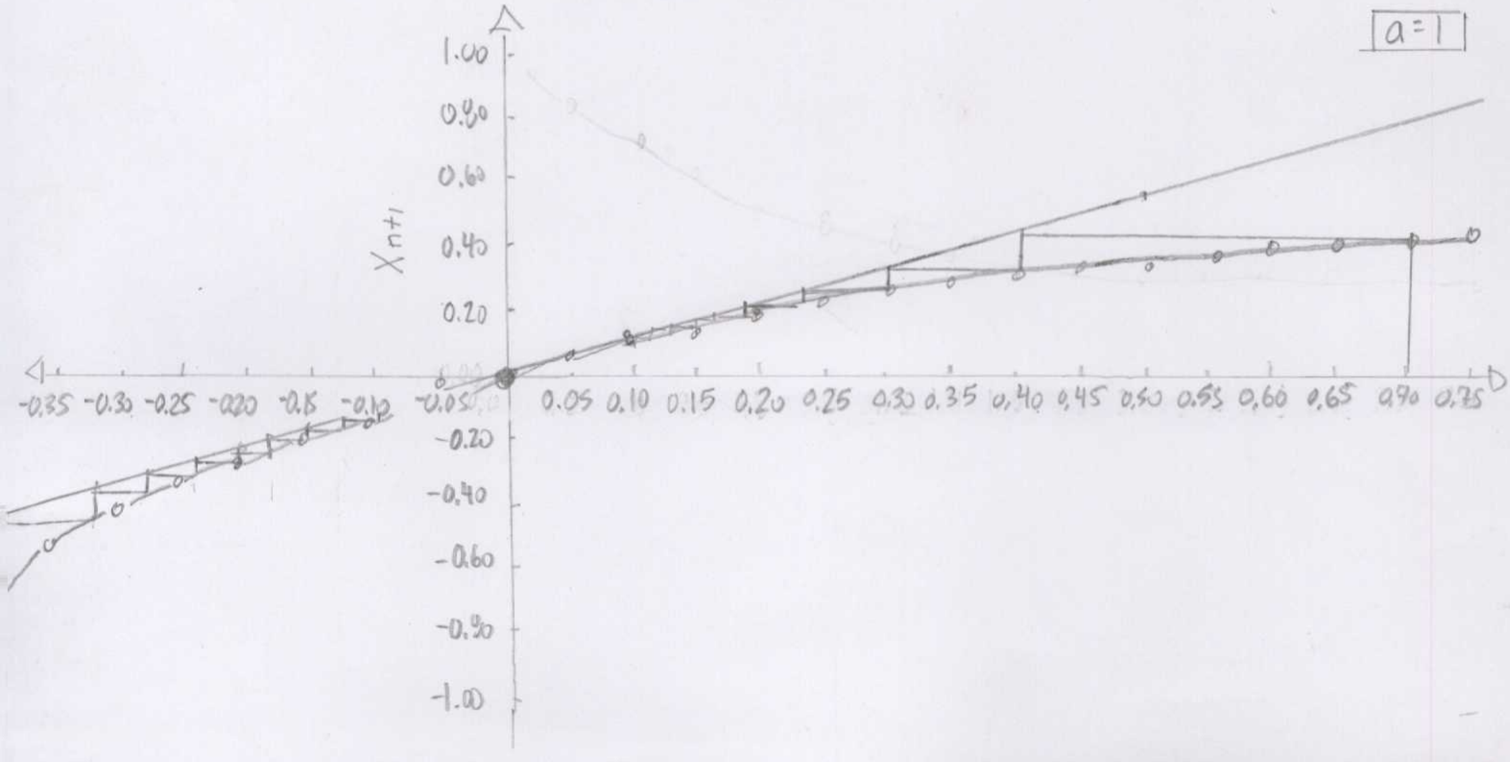
a) $\alpha = 2$, $g(x) = \dfrac{x}{1+ax}$

$$\alpha g^2\left(\frac{x}{\alpha}\right) = \alpha\left[\frac{\frac{(x/\alpha)}{1+a(x/\alpha)}}{1 + a\left(\frac{(x/\alpha)}{1+a(x/\alpha)}\right)}\right]$$

$$= \frac{x}{1+x}$$

$$g(0) = \alpha g^2(0) = 0$$

b)



$f(x) = -(1+\mu)x + x^2$ 10.7.10

$p+\eta_{n+1} = f^2(p+\eta_n)$

$$p = \frac{\mu + \sqrt{\mu^2 + 4\mu}}{2}$$

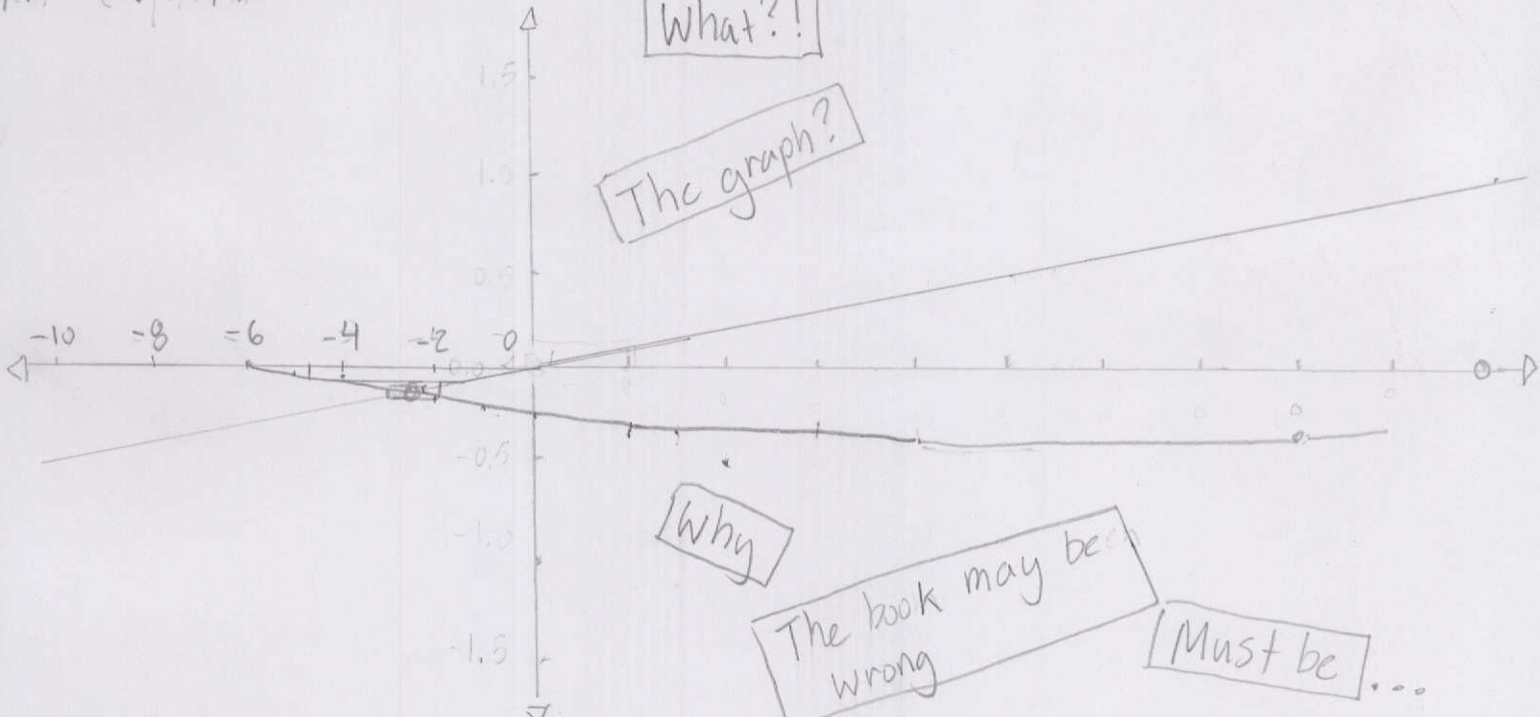
$$p+\eta_{n+1} = f^2(p+\eta_n)$$

$$= -(1+\mu)\left[-(1+\mu)[p+\eta_n] + [p+\eta_n]^2\right] + \left[-(1+\mu)[p+\eta_n] + [p+\eta_n]^2\right]^2$$

$$= \eta - 6p^2\eta + 4p^3\eta + 2\mu p + 2\mu p\eta - 6\mu p^2 + \mu^2 + 2\mu^2 p\eta + O(\eta^2) + O(\eta^0)$$

The missing zeroth order led no solution, rather author contact.

$\mu_K = -2\sqrt{6+\mu_{K-1}}$ 10.7.11.



What?!

The graph?

Why

The book may be wrong

Must be...

$\mu_K = -2 + \sqrt{6+\mu_{K-1}}$

Uhh oh

The plus sign