

## Chapter 2:

$$\dot{x} = \sin x \quad 2.1.1. \quad \dot{x} = 0 = \sin x ; \boxed{x = n\pi} \quad 2.1.2. \quad \left( n + \frac{1}{2} \right) \pi \text{ where } n \text{ is even.}$$

$$2.1.3. \quad a) \quad \ddot{x} = \cos x \sin x \quad b) \quad \frac{1}{2} \sin(2x) = \cos(x) \sin(x); \quad \ddot{x} = \frac{1}{2} \sin(2x); \quad X = \left( n + \frac{1}{4} \right) \pi; \quad n \in \mathbb{Z}$$

$$2.1.4. \quad x_0 = \pi/4; \quad t = \ln |(\csc x_0 + \cot x_0)| / (\csc x + \cot x)$$

$$\begin{aligned} e^t &= \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} = \frac{\csc \pi/4 + \cot \pi/4}{\csc x + \cot x} = \frac{\frac{2}{\sqrt{2}} + i}{\csc x + \cot x} = \frac{\sqrt{2} + i}{\csc x + \cot x} \\ \frac{1}{\sin x + \frac{\csc x}{\sin x}} &= \frac{\sin x}{1 + \cos x} = \frac{\sin(\frac{x}{2})}{1 + \cos(\frac{x}{2})} = \frac{2 \cos(\frac{x}{2}) \sin(\frac{x}{2})}{1 + 2 \cos^2(\frac{x}{2}) - 1} = \tan(\frac{x}{2}) = \frac{e^t}{\sqrt{2} + 1} \\ x(t) &= 2 \tan^{-1} \left( \frac{e^t}{\sqrt{2} + 1} \right); \quad \lim_{t \rightarrow \infty} x(t) = 2 \tan^{-1}(\infty) = \frac{2 \cdot \frac{\pi}{2}}{2} = \pi \end{aligned}$$

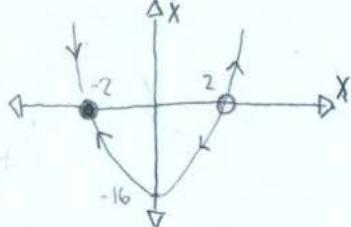
$$b) \quad \boxed{x(t) = 2 \tan^{-1} \left( \frac{e^t}{\csc x_0 + \cot x_0} \right)}$$

2.1.5a) A mechanical analog of  $\dot{x} = \sin x$  is the undamped pendulum having an  $x_0$  of the maximal point

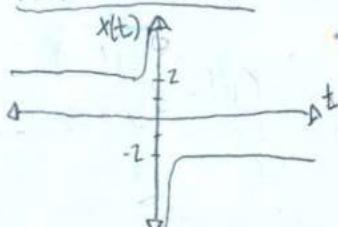
b) Unstable points are described by a positive slope (source) and stable points (sink), a negative slope. The function  $\dot{x} = \sin x$  at  $x^* = 0$  is unstable, while  $x^* = \pi$ , is stable.

$$\dot{x} = 4x^2 - 16 \quad 2.2.1$$

Vector Field:



Plot of x(t):



Fixed Points:

$$x=2$$

Stability:

Source (unstable)

$$x=-2$$

Sink (stable)

Solving for x(t):

$$\frac{dx}{(x^2 - 4)} = 4t$$

$$\int \frac{A}{(x-2)} dx + \int \frac{B}{(x+2)} dx = 4t$$

$$A(x+2) + B(x-2) = 1$$

$$A = 1/4 @ x=2$$

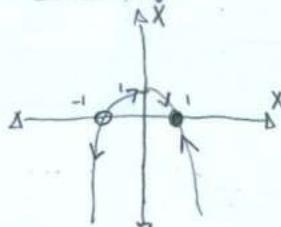
$$B = -1/4 @ x=-2$$

$$\ln|x-2| = 16t + C$$

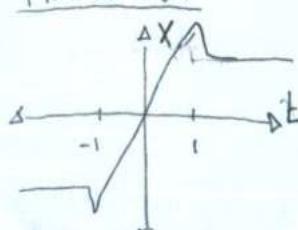
$$x(t) = \frac{2(e^{16t+C} + 1)}{1 - e^{16t+C}}$$

$$\dot{x} = 1 - x^{14} \quad 2.2.2$$

Vector Field:



Plot of x(t):



Fixed Points:

$$x=1$$

Stability:

Source (unstable)

$$x=-1$$

Sink (stable)

Solving for x(t):

$$t = \int \frac{dx}{1 - x^{14}} (e^t + 1)$$

Unsolvable  $e^t + 2$

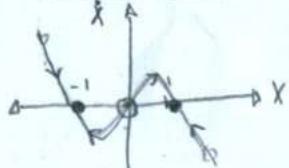
Analytical Solution of x(t):

$$x_0 - 2 = (2 + \lambda_0) e^{-t}$$

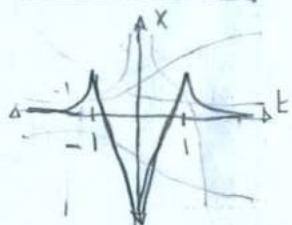
$$\ln \frac{x_0 - 2}{2} = -t$$

$$\dot{x} = x - x^3 \quad 2.2.3$$

### Vector Field

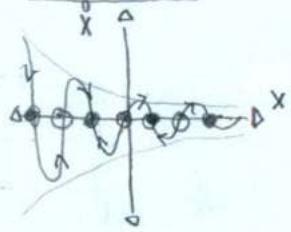


### Plot of x(t):

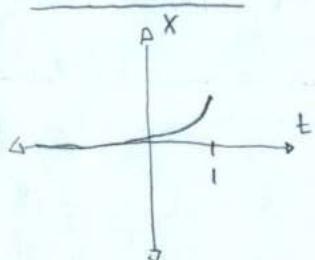


$$\dot{x} = e^{-x} \sin x \quad 2.2.4$$

### Vector Field:

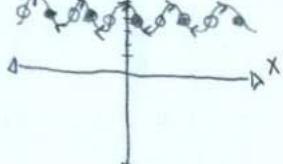


### Plot of x(t):

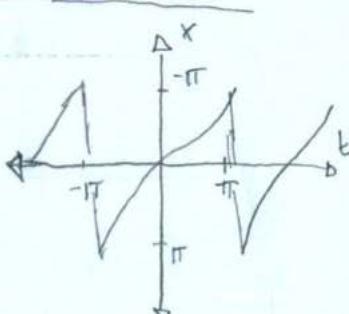


$$\dot{x} = 1 + \frac{1}{2} \cos x \quad 2.2.5$$

### Vector Field:



### Plot of x(t):



### Fixed Points: Stability:

$x = -1$	stable (sink)
$x = 0$	unstable (source)
$x = 1$	stable (sink)

### Solving for $x(t)$ :

$$t = \int \frac{dx}{x - x^3} = \int \frac{dx}{x(1-x^2)}$$

$$= \frac{1}{2} \int \frac{du}{(1-u)u} = \frac{1}{2} \int \frac{A}{1-u} du - \frac{1}{2} \int \frac{B}{u} du$$

$$= -\frac{1}{2} \ln \left| \frac{1-u}{u} \right| = -\frac{1}{2} \ln \left| \frac{x^2}{1-x^2} \right|$$

$$= \frac{1}{2} \ln \left| \frac{1-x^2}{x^2} \right| + C$$

$$(e^{2t} + 1)x^2 = 1 \quad \boxed{x = \sqrt{\frac{1}{e^{2t} + 1}}}$$

### Analytical Solution of $x(t)$ :

$$x(t) = \frac{e^{2t}}{e^{2t} + 1}$$

### Fixed Points:

### Stability:

$$x = 2n\pi$$

$n = 2k$

$$x = (2n+1)\pi$$

### Solving for $x(t)$ :

$$t = \int \frac{dx}{\sin x} = \int dx + \int \cot(x) dx$$

$$= 1 + \ln(\sin x) + C$$

$$\boxed{x(t) = \arcsin(C e^{t-\pi})}$$

### Analytical solution of $x(t)$

$$\boxed{x(t) = \arcsin(C e^{t-\pi}) \text{ where } C = -(1 + \ln(\sin x_0))}$$

### Fixed Points

$$x = (4n+1)\frac{\pi}{2}$$

sink (stable)

$$x = (4n-1)\frac{\pi}{2}$$

source (unstable)

### Solving for $x(t)$ :

$$t = \int \frac{1}{1 + \frac{1}{2} \cos x} dx$$

$$= \int \frac{dx}{\frac{1}{2} + \cos^2(\frac{x}{2})} = \int \frac{\sec^2(\frac{x}{2}) dx}{\frac{3}{2} + \tan^2(\frac{x}{2}) + 1}$$

$$= \int \frac{\sec^2(\frac{x}{2}) dx}{3 + \tan^2(\frac{x}{2})}$$

### Analytical solution of $x(t)$

$$= \frac{2}{\sqrt{3}} \operatorname{arctan} \left( \frac{\tan(\frac{x}{2})}{\sqrt{3}} \right) + C$$

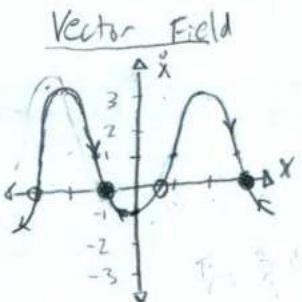
$$u = \frac{\tan(\frac{x}{2})}{\sqrt{3}}; \frac{du}{dx} = \frac{\sec^2(\frac{x}{2})}{2\sqrt{3}}$$

$$= \int \frac{2\sqrt{3}}{3u^2 + 3} du = \frac{2}{\sqrt{3}} \int \frac{du}{u^2 + 1}$$

$$= \frac{2}{\sqrt{3}} \operatorname{arctan}(u) + C$$

$$\boxed{x(t) = \frac{2}{\sqrt{3}} \operatorname{arctan} \left( \frac{\tan(\frac{x}{2})}{\sqrt{3}} \right) + C}$$

$$\dot{x} = 1 - 2\cos x \quad 2.2.6.$$

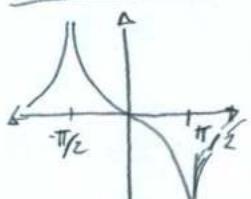


Fixed Points      Stability

$$x = (n + \frac{1}{2})\pi; n = \text{even} \quad \text{sink (stable)}$$

$$x = (n + \frac{1}{2})\pi; n = \text{odd} \quad \text{source (unstable)}$$

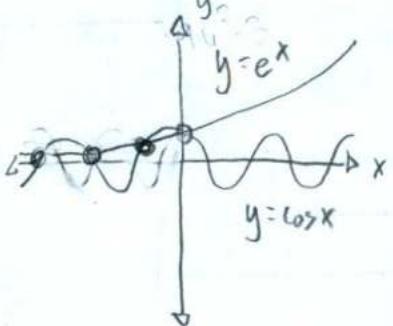
Plot of x(t)



Analytical Solution of x(t)

$$x(t) = \frac{\ln}{\sqrt{3}} \left| \frac{3 + \tan(\frac{x}{2}) - \sqrt{3}}{3 + \tan(\frac{x}{2}) + \sqrt{3}} \right|$$

$$\dot{x} = e^x - \cos x \quad 2.2.7.$$



Points of stability

$$e^x = \cos x$$

$$x = 0 \quad \text{source (unstable)}$$

$$x_1 = -1.29 \quad \text{sink (stable)}$$

$$x_2 = -4.72 \quad \text{source (unstable)}$$

$$\dot{x} = f(x)$$

$$2.2.8$$



slope zero Negative Positive

Fixed points

$$@ x = 1$$

$$@ x = 0$$

$$2.2.9$$

$$\begin{array}{|c|c|} \hline x_0 = 2 & x \\ \hline x_0 = 1 & \sim \\ \hline x_0 = 0.5 & \sim \\ \hline x_0 = -1 & \sim \\ \hline \end{array}$$

$$f(x) = x(1-x)$$

$$\dot{x} = f(x)$$

2.2.10 a. A periodic function having solutions  $\pi\pi$

b. A periodic function with  $\pi\pi$  solutions

$$c. |f(x)| \approx x^5$$

$$d. |f(x)| \approx x^2 + 1$$

$$e. |f(x)| \approx x^{10}$$

$$\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC} \quad 2.2.11.$$

$$\dot{Q}(0) = 0; \quad t = RC \int \frac{dQ}{V_0 C - Q} = -RC \ln V_0 C - Q + C; \quad C = R C \ln V_0 C; \quad Q = R C \ln V_0 C$$

$$Q = g(v) - \frac{Q}{RC}; \quad V = g(v) - V_{cap} = V_0 - \frac{Q}{C}; \quad -g(v) + RI + \frac{Q}{C} = 0; \quad -g(v) + RI + \frac{Q}{C} = -g(v) + R I + \frac{Q}{C} = 0$$

$$Q = g(v) - \frac{Q}{RC}; \quad V = g(v) - V_{cap}$$

$$\text{Fixed Points: } g(v) = \frac{Q}{RC}$$

Stability: source (unstable)

$$\text{Solving for } x(t)$$

$$\dot{x} = \int \frac{dx}{1 - 2\cos x} \Rightarrow \frac{1 - \tan^2(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})} = \cos x$$

$$= \int \frac{dx}{2 \left[ \frac{1 - u^2}{1 + u^2} - 1 \right]} =$$

$$= - \int \frac{du}{2 \left[ \frac{1 - u^2}{1 + u^2} - 1 \right]; \quad u = \tan(\frac{x}{2}) \Rightarrow \frac{du}{dx} = \sec^2(\frac{x}{2})}$$

$$= - \int \frac{du}{2 \sec^2(\frac{x}{2}) \left[ \frac{1 - u^2}{1 + u^2} - 1 \right]}$$

$$= - \int \frac{2u du}{\left[ u^2 + 1 \right] \left[ 2 \left[ \frac{1 - u^2}{1 + u^2} - 1 \right] \right]}$$

$$= - \int \frac{2 du}{2 - 2u^2 - u^2 - 1}$$

$$= - \int \frac{2 du}{-3u^2 - 1}$$

$$= -2 \left( \frac{3}{1} \right) \frac{du}{(3u - \sqrt{3})(3u + \sqrt{3})}$$

$$= -6 \left[ \int \frac{A du}{(3u - \sqrt{3})} + \int \frac{B du}{(3u + \sqrt{3})} \right]$$

$$= -6 \left[ \frac{\sqrt{3}}{2} \int \frac{du}{(3u - \sqrt{3})} - \frac{\sqrt{3}}{2} \int \frac{du}{(3u + \sqrt{3})} \right]$$

$$= \frac{\ln(3u + \sqrt{3})}{\sqrt{3}} - \frac{\ln(3u - \sqrt{3})}{\sqrt{3}}$$

$$= \frac{\ln(3 + \tan(\frac{x}{2}) + \sqrt{3})}{\sqrt{3}} - \frac{\ln(3 + \tan(\frac{x}{2}) - \sqrt{3})}{\sqrt{3}}$$

$$= \ln \frac{3 + \tan(\frac{x}{2}) - \sqrt{3}}{3 + \tan(\frac{x}{2}) + \sqrt{3}} + C$$

$$x_0 = V_0 C (1 - e^{-t/RC})$$

$$t = RC \ln \frac{V_0 C}{V_0 C - Q}$$

The nonlinearity of the resistor has a relationship to resistance

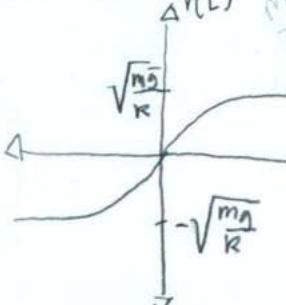
$m\ddot{v} = mg - kv^2$  2.3.13 - where  $m$  = mass,  $g$  = acceleration,  $K > 0$  = air resistance

$$a) \int \frac{dv}{g - \frac{k}{m}v^2} = \frac{1}{g} \int \frac{dv}{1 - \frac{k}{mg}v^2} = \frac{1}{g} \left[ \int \frac{dv}{1 - \sqrt{\frac{k}{mg}}v} + \int \frac{dv}{1 + \sqrt{\frac{k}{mg}}v} \right] = \sqrt{\frac{mg}{k}} \frac{1}{2g} \left[ \ln \left| 1 + \sqrt{\frac{k}{mg}}v \right| - \ln \left| 1 - \sqrt{\frac{k}{mg}}v \right| \right]$$

$$t = \frac{1}{2} \sqrt{\frac{m}{kg}} \ln \left| \frac{1 + \sqrt{\frac{k}{mg}}v}{1 - \sqrt{\frac{k}{mg}}v} \right| \quad \text{let } u = \sqrt{\frac{k}{mg}}v \quad \sqrt{\frac{k}{mg}}t = \ln \left| \frac{1 + \sqrt{\frac{k}{mg}}v}{1 - \sqrt{\frac{k}{mg}}v} \right| = \tanh^{-1} \left( \sqrt{\frac{k}{mg}}v \right)$$

$$b) \lim_{t \rightarrow \infty} v(t) = \sqrt{\frac{mg}{k}} = \text{"terminal velocity"} \quad v(t) = \sqrt{\frac{mg}{k}} \tanh \sqrt{\frac{kg}{m}} t$$

$$c) \text{Graph of } v(t) \text{ vs } t: \quad d. V_{avg} = \frac{(31,400 - 2100) \text{ ft}}{116 \text{ sec}} = 252 \text{ ft/sec}$$



$$c. s = \int \frac{ds}{dt} = v = V \tanh \sqrt{\frac{kg}{m}} t \quad ; \quad s(t) = V \int \tanh \sqrt{\frac{kg}{m}} t dt$$

$$29,300 = V^2 \quad \text{In} \cosh \cdot \frac{32.2 \text{ ft/sec}^2}{V} \cdot \frac{116 \text{ sec}}{V} = V \int \sinh \sqrt{\frac{kg}{m}} t dt$$

$$e^{3735 \text{ ft/sec}^2 / V} = e^{-\frac{3735 \text{ ft/sec}^2}{V} + \frac{3735 \text{ ft/sec}^2}{V}} = V \int \frac{1}{u} dt$$

$$V = 266 \text{ ft/sec.}$$

$$V = V_{avg} = 252 \text{ ft/sec}$$

$$\frac{gt}{V} = \frac{32.2 \text{ ft/sec}^2 \cdot 116 \text{ sec}}{252 \text{ ft/sec}} = 14.8$$

$$\frac{V^2}{g} \ln \cosh \frac{gt}{V} \approx \frac{V^2}{g} \left[ \frac{gt}{V} - \ln 2 \right] = 265 \text{ ft/sec}$$

$$N = \frac{1}{(1 + Ce^{-rt})^K} = \frac{No}{e^{-rt}}$$

$$\text{General Solution: } X = Ee^{-rt}$$

$$X = Ce^{-rt} + \frac{1}{K}$$

$$b. X = 1/N; \dot{X} = -rX(1 - \frac{1}{KX}) = \frac{r}{K} - rX; \dot{X} + rX - \frac{r}{K} = \dot{X} + r(X - \frac{1}{K}) = 0$$

$$\dot{X} = r(\frac{1}{K} - X)$$

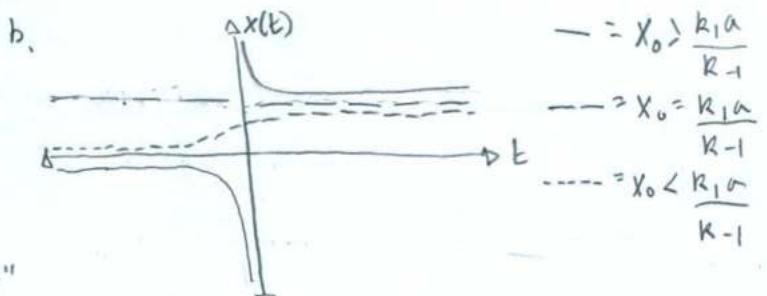
$$N(t) = \frac{K}{Ce^{-rt} + 1}$$

$$N = \frac{No}{e^{-rt}} \quad \text{XX} + \frac{X}{K} = \frac{X}{K} \quad X = C(\frac{1}{K} + X)$$

$$a. t = \int \frac{dx}{k_1 ax - k_2 x^2} = \frac{1}{R_1 a} \int \frac{dx}{x - \frac{k_2}{k_1 a} x^2} = \frac{1}{R_1 a} \int \frac{dx}{x(1 - \frac{k_2}{R_1 a} x)} \\ = \frac{1}{R_1 a} \left[ \int \frac{A dx}{x} + \int \frac{B dx}{(1 - \frac{k_2}{R_1 a} x)} \right] = \frac{1}{R_1 a} \left[ \int \frac{1 dx}{x} + \frac{k_2}{R_1 a} \int \frac{dx}{1 - \frac{k_2}{R_1 a} x} \right]$$

$$= \frac{1}{R_1 a} \left[ \ln|x| - \ln \left| 1 - \frac{k_2}{R_1 a} x \right| \right] = \frac{1}{R_1 a} \ln \left| \frac{x}{1 - \frac{k_2}{R_1 a} x} \right| + C; \left( C e^{-t/R_1 a} + \frac{k_2}{R_1 a} \right) x = 1$$

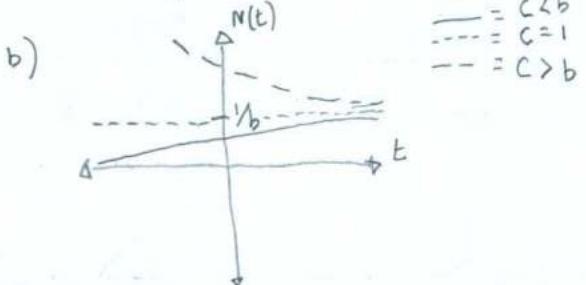
$$X(t) = \frac{1}{\frac{k_2}{R_1 a} + C e^{-t/R_1 a}}; C = \frac{1}{x_0} - \frac{k_2}{R_1 a}; \text{ Fix w. points of stability: } X = \frac{k_2}{R_1 a} \text{ source unstable}$$



"Gompertz Law"

$$\dot{N} = -aN \ln(bN) \quad 2.3.3a) \quad t = -\frac{1}{a} \int \frac{dN}{N \ln(bN)} = -\frac{b}{a} \int \frac{du}{u} = -\frac{b}{a} \ln[\ln bN] ; N(t) = C e^{\frac{-ab}{b} \frac{t}{a}} \quad ; \quad a = \text{rate constant}$$

$b = \text{Max amount of cells.}$



$$\frac{\dot{N}}{N} = r - a(N-b)^2 \quad 2.3.4. a) \quad \lim_{N \rightarrow 0} \frac{\dot{N}}{N} = \lim_{N \rightarrow 0} r - a(N-b)^2 = r - ab^2 = \infty ; r = \infty$$

Each case of competition model at infinite population

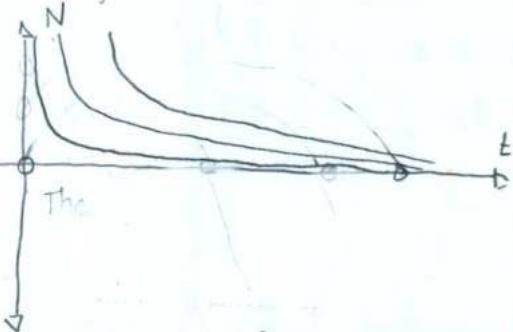
$$\lim_{N \rightarrow \infty} \frac{\dot{N}}{N} = \lim_{N \rightarrow \infty} r - a(N-b)^2 = r - \infty = 0 ; r = \infty$$

on extremely small populations that food amount or rate of consumption are insignificant to the competition.

b) Fixed points of stability:

$N=0$	source (unstable)	c.
$N=\sqrt{\frac{r}{a}}+b$	sink (stable)	

d) The solutions of the logistic equation  $y = Ce^{-rt} + \frac{1}{r}$  are similar, if not exact to the Allele Effect.



$$t = \int \frac{du}{(b+u)(r-au^2)} = \int \frac{A}{b+u} du + \int \frac{B u + C}{r-au^2} du$$

$$A(r-au^2) + (Bu+C)(b+u) = 1$$

$$@ u = \sqrt{\frac{r}{a}} ; (B\sqrt{\frac{r}{a}} + C)(b + \sqrt{\frac{r}{a}}) = 1$$

$$B = \sqrt{\frac{r}{a}} ; C = \sqrt{\frac{r}{a}}$$

$$@ u = -b \quad A = \frac{1}{r-(ab)^2}$$

$$= \frac{1}{r-(ab)^2} \int \frac{du}{b+u} + \frac{1}{\sqrt{r}} \int \frac{ab \cdot u + r}{r-au^2} du \quad \begin{matrix} \text{Partial} \\ \text{Fractions} \end{matrix} \quad (x^2)$$

$$= \frac{\ln N}{r-(ab)^2} + \frac{b}{4\sqrt{r}} \tanh^{-1} \left( \frac{2u+b}{\sqrt{r}(N+b)} \right) + \frac{1}{\sqrt{r}} \tanh^{-1} \left( \frac{b}{\sqrt{r}(N+b)} \right) + C$$

$$\dot{x} = (1-x)P_{xx} - xP_{xy} \quad 2.3.6. \text{ a. } x=0$$

$$P_{yx} = s x^a; P_{xy} = (1-s)(1-x)^a$$

$$x=1$$

$$x = \frac{a-1}{s} \sqrt{\frac{(1-s)}{s}} \\ \frac{1}{1 + \sqrt{\frac{(1-s)}{s}}}$$

b. A plot of  $s(1-x)x^a$  and  $-(1-s)x(1-x)^a$  demonstrate  $-(1-s)x(1-x)^a > s(1-x)x^a$  for  $x=0$  and  $x=1$ , indicating each fixed point is stable.

c. For  $x = \frac{a-1}{s} \sqrt{\frac{(1-s)}{s}} \frac{1}{1 + \sqrt{\frac{(1-s)}{s}}}$  the plot of  $s(1-x)x^a > (1-s)x(1-x)^a$  suggesting a source.

$$\dot{x} = x(1-x)$$

$$2.4.1 \quad \dot{x} = f(x) = F(x^* + x) = F(x^*) + xF'(x^*) + O(x^2) \\ = xF'(x^*) + O(x^2) \\ = x(1-2x)$$

$$x=0; F'(x^*) = 1 : \text{Unstable (source)} \\ x=1; F'(x^*) = -1 : \text{Stable (sink)}$$

$$\dot{x} = x(1-x)(2-x)$$

$$2.4.2 \quad \dot{x} = f(x) = F(x^* + x) + xF'(x^*) \\ = x(2x(1-x))$$

$$x=0 \quad F'(x^*) = 0 \quad \text{Half-stable}$$

$$x=1 \quad F'(x^*) = 0 \quad \text{Half-stable}$$

$$x=2 \quad F'(x^*) = -4 \quad \text{sink (stable)}$$

$$x=\pi \quad F(x) = (+) \text{ source (unstable)}$$

$$x=0 \quad F'(x) = 0 \quad \text{Half-stable}$$

$$x=6 \quad F'(x) = -36 \quad \text{sink (stable)}$$

$$x=0 \quad F'(x^*) = 0 \quad \text{Half-stable}$$

$$x=1 \quad F'(x^*) = 1 \quad \text{source (unstable)}$$

$$x < 0 \quad (+) \quad (-) \quad (0)$$

$$x > 0 \quad \text{source} \quad \text{sink} \quad \text{Half-stable}$$

$$x = \sqrt{\alpha} \quad \text{sink} \quad \text{source} \quad \text{source}$$

$$x = -\sqrt{\alpha} \quad \text{source} \quad \text{sink} \quad \text{Half-stable}$$

$$N=0 \quad : \text{source (unstable)}$$

$$N = \frac{1}{b} \quad : \text{sink (stable)}$$

$$2.4.6 \quad \dot{N} = f(N) = F(N + N^*) = -\frac{aN}{b} [1 + b \ln(bN)]$$

$$\dot{x} = -x^3 \quad 2.4.9 \text{ a. } t = \int \frac{dx}{x^3} = \frac{1}{2x^2} + C; X(t) = \sqrt{\frac{1}{2t+C}}$$

$$\lim_{x \rightarrow 0} t = \frac{1}{0} + C = \frac{1}{x(0)}$$

$$\text{b. if } x_0 = 10$$

$$t = -\int \frac{1}{x} dx = -\ln x;$$

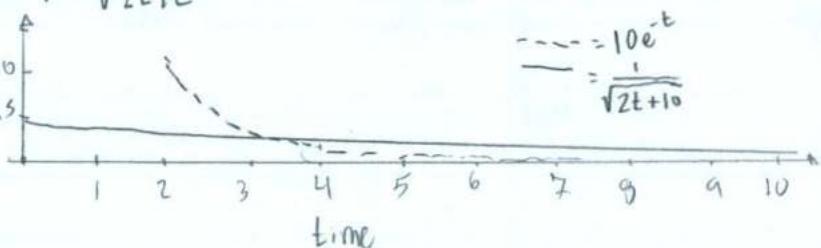
$$X(t) = x_0 e^{-t} = 10e^{-t}$$

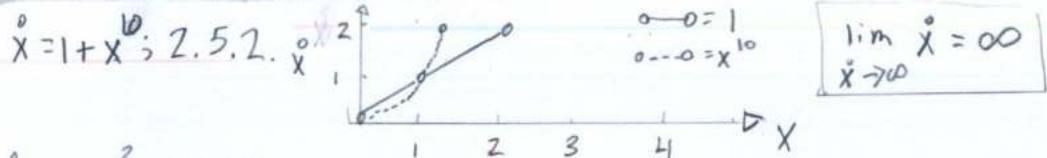
$$\dot{x} = -x^0$$

$$2.5.1 \text{ a. } c=0$$

b.  $dx = -dt; x(t) = -t$ ; if  $t=0$  is considered finite time, then yes.

$$t = - \int \frac{dx}{x^c} = \frac{-x^{1-c}}{1-c}; t(x=1) - t(x=0) = -\frac{1}{1-c} + \frac{0}{1-c} = \frac{1}{c+1}$$





$$\dot{x} = rx + x^3 \quad 2.5.3$$

$$\begin{aligned} t &= \int \frac{dx}{x(r+x^2)} = \int \frac{A}{x} dx + \int \frac{Bx+C}{r+x^2} dx ; A(rx^2) + (Bx+C)(x) = 1 \\ &= \frac{1}{r} \ln x - \frac{1}{2r} \ln r + x^2 = \frac{1}{r} \ln \frac{x}{\sqrt{r+x^2}} \\ x^2 e^{-2rb} &= (r+x^2) ; x = \sqrt{\frac{r}{1+Ce^{-2rb}}} \end{aligned}$$

If  $x_0 \neq 0$ ;  $\lim_{t \rightarrow \infty} x(t) = \infty$ .

$$\dot{x} = x^3 \quad 2.5.4. x(0) = 0 ; t = \int \frac{dx}{x^{1/3}} = \frac{3}{2} x^{2/3} ; x(t) = \sqrt{\frac{2}{3}} t - \frac{2}{3} C^3$$

$$\dot{x} = |x|^{p/q} \quad 2.5.5. x(0) = 0 ; a) t = \frac{q}{p+q} (x)^{\frac{p+q}{q}} ; x(t) = \left( \frac{p+q}{q} (t+C) \right)^{q/p+q} ; c = \text{many solutions at zero because of root.}$$

$$x(t) = \left( \frac{p+q}{q} (t+C) \right)^{q/p+q} ;$$

$$b) x(t) = \left( \frac{p+q}{q} (t+C) \right)^{q/p+q} ; \text{ if } p > q ; x(0) = \left( \frac{p+q}{q} (0+C) \right)^{q/p+q} = 0 ; C=0$$

$h(t)$ : height    2.5.6 a) Newton's first law that for every force there exist an equal and opposite counter force.

$$b) \frac{1}{2}mv^2 = mgh \Rightarrow v^2 = 2gh$$

$$c) h(t) = -\sqrt{\frac{a}{2g}} t \quad d) h(0) = 0 ; t = -\sqrt{\frac{2h}{a}} ; h(t) = -\sqrt{\frac{a}{2A}} t$$

$$\alpha V(t) = Ah(t)$$

$$m\ddot{x} = -kx \quad 2.6.1$$

The text states, there are no periodic solutions to  $\dot{x} = f(x)$  because undamped systems do not oscillate, and damped oscillations do not occur for first order systems. Strogatz' statement does not fit the equation of 2.6.1.

$$\dot{x} = f(x) \quad 2.6.2 \quad \int_t^{t+\tau} f(x) \frac{dx}{dt} dt = \int_t^{t+\tau} f(x) \dot{x}(t) dt = \int_t^{t+\tau} f(x) \dot{x}(t+\tau) d(t+\tau)$$

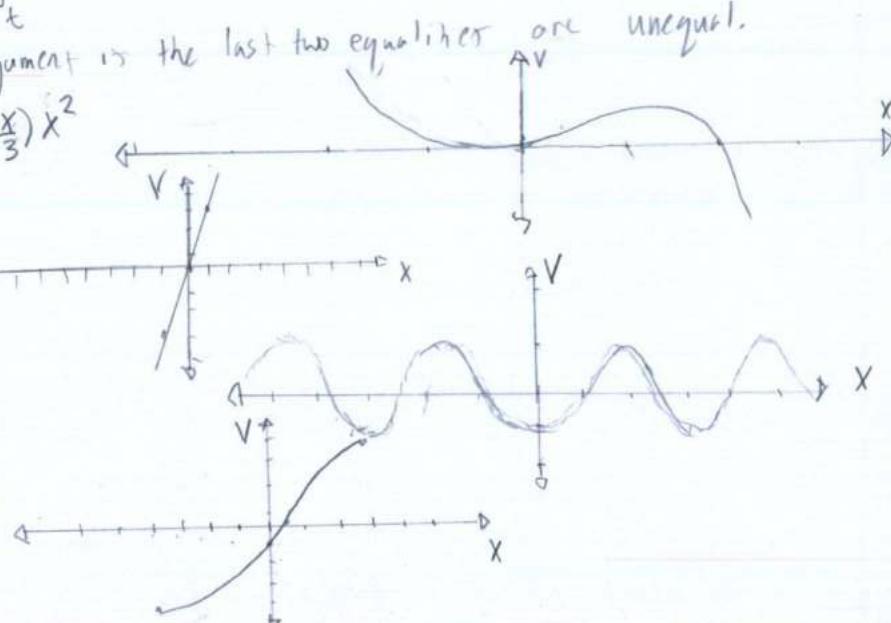
$x(t) = x(t+\tau)$  The contradiction of the argument is the last two equations are unequal.

$$\dot{x} = x(1-x) \quad 2.7.1 \quad \frac{dV}{dx} = \dot{x} = x(1-x) ; V = (1-\frac{x}{3})x^2$$

$$\dot{x} = 3 \quad 2.7.2 \quad \frac{dV}{dx} = \dot{x} = 3 ; V = 3x$$

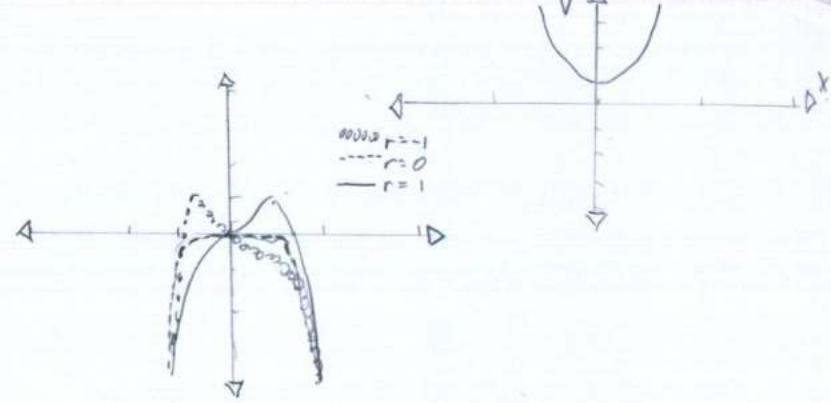
$$\dot{x} = \sin x \quad 2.7.3 \quad \frac{dV}{dx} = \dot{x} = \sin x ; V = -\cos(x)$$

$$\dot{x} = 2 + \sin x \quad 2.7.4 \quad \frac{dV}{dx} = \dot{x} = 2 + \sin x ; V = 2x - \cos(x)$$



$$\dot{x} = -\sinh x \quad 2.7.5. \quad \frac{dx}{dt} = -\sinh x; \quad y = -\cosh(x)$$

$$\dot{x} = r + x - x^3 \quad 2.7.6. \quad \frac{dx}{dt} = r + x - x^3; \quad V = rx + \frac{x^2}{2} - \frac{x^4}{4}$$



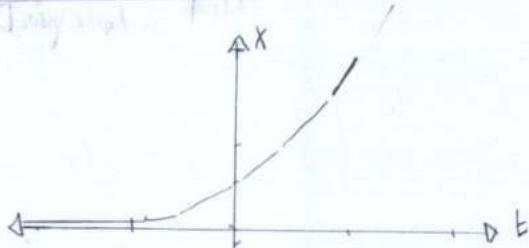
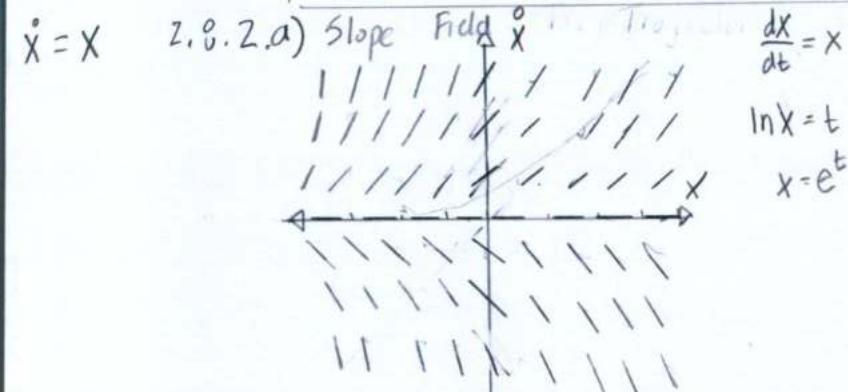
$$\dot{x} = f(x)$$

$$2.7.7. \quad \frac{dx}{dt} = \dot{x} = f(x) \Rightarrow V = \frac{df(x)}{dx} dx + C$$

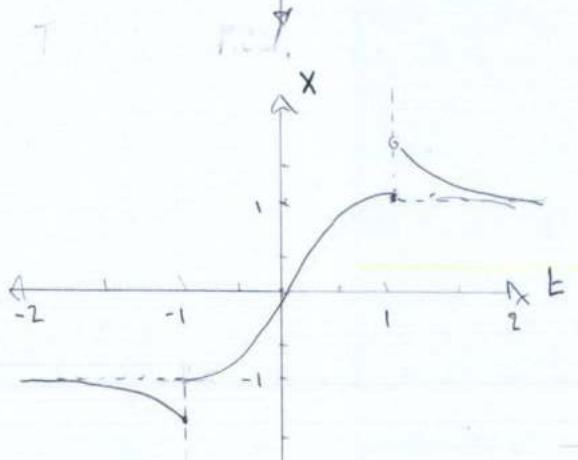
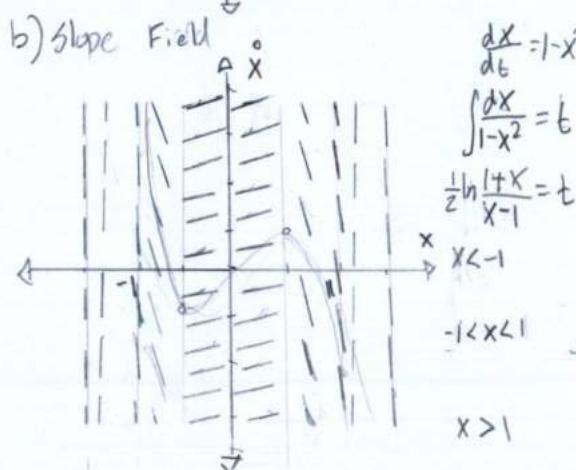
$$F(x) = \frac{d(V-C)}{dx}$$

The solution  $x(t)$  cannot oscillate because of the existence and uniqueness of  $f(x)$ , and the solutions for  $f(x)=0$ , that  $V=C$  or  $C=0$ ; withstanding,  $\frac{d(V-C)}{dx} = \frac{dx}{dt} \Rightarrow$  then the solution  $x(t)$  also corresponds to a nonperiodic function.

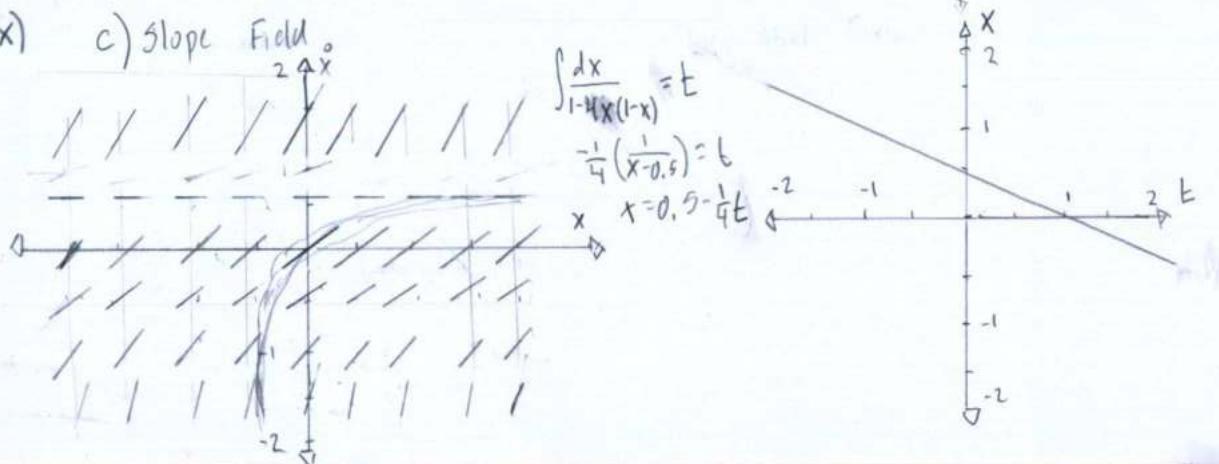
$$\dot{x} = x(1-x) \quad 2.8.1 \quad \text{The horizontal lines are to be expected in Figure 2.8.2 because of the slope being zero at } x=1.$$



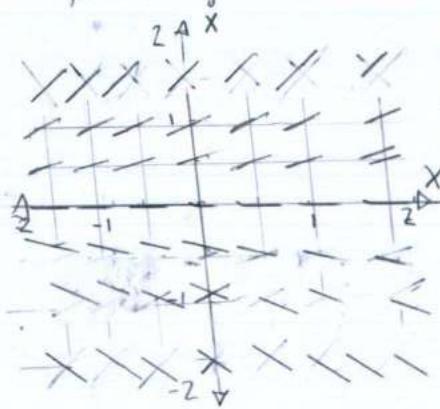
$$\dot{x} = 1 - x^2$$



$$\dot{x} = 1 - 4x(1-x)$$



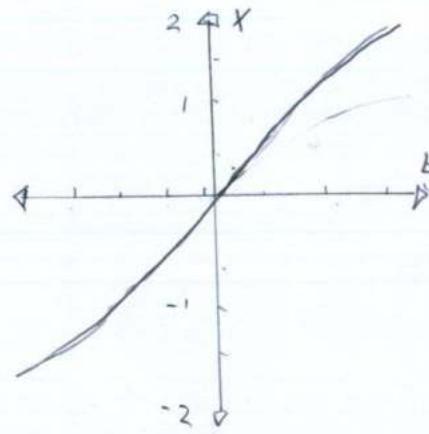
$$\dot{x} = \sin x \quad 2.82 \text{ d) Slope Field}$$



$$\frac{dx}{dt} = \sin x$$

$$\int \cosh x = b$$

$$x = \sin^{-1}(t)$$



$$\dot{x} = -x \quad x(0) = 1 \quad 2.8.3. \text{ a) } x(t) = C e^{-t}; \quad C = 1; \quad x(t) = e^{-t}$$

$$\text{b) } \Delta t = 1; \quad x(t_0 + \Delta t) \approx x_0 + f(x_0) \Delta t; \quad x(t_0 + \Delta t) = 0 + e^{-1} \cdot 10^0 = 0.3679$$

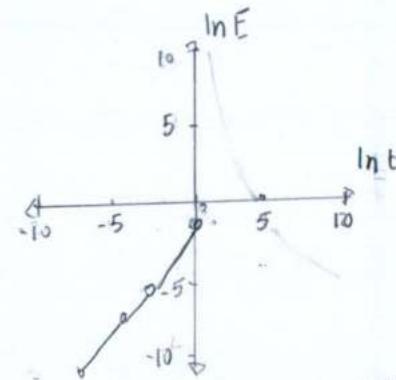
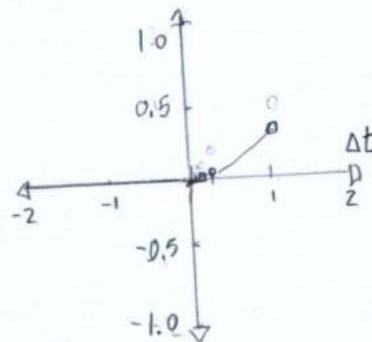
$$\Delta t = 10^{-1} \quad n = 1 \quad x_n + e^{-x_n} 10^{-1} = 0.36341$$

$$n = 2 \quad x_n + e^{-x_n} 10^{-2} = 0.36697$$

$$n = 3 \quad x_n + e^{-x_n} 10^{-3} = 0.36795$$

$$n = 4 \quad x_n + e^{-x_n} 10^{-4} = 0.36787$$

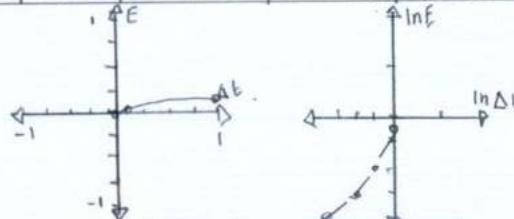
$$\text{c) } E = |\hat{x}(1) - x(1)|$$



The results of  $E = |\hat{x}(1) - x(1)|$  vs  $\Delta t$  represent errors of Euler's method. While the plot of  $\ln E$  vs  $\ln t$  characterizes nothing informative.

$$\dot{x} = -x; \quad x(0) = 1 \quad 2.8.4. \quad x(t) = e^{-t};$$

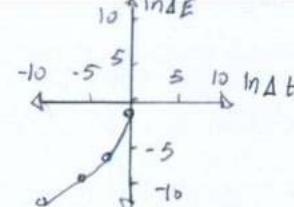
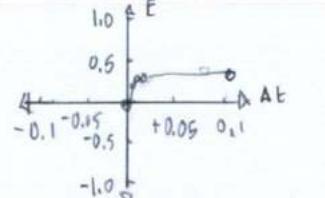
n	$\Delta t$	$f(x)$	$x_n = x_{n-1} + f(x_{n-1}) \Delta t$	$E =  \hat{x}(1) - x(1) $	$\ln E$
0	$10^0$		0.36788	0.00	-1.00
1	$10^{-1}$	$\exp(x_{n-1})$	0.33527	0.03269	-3.42
2	$10^{-2}$		0.36577	0.0211	-6.16
3	$10^{-3}$		0.36773	0.0002	-9.93
4	$10^{-4}$		0.36773	0.0000	-10.78



Improved Euler's Method

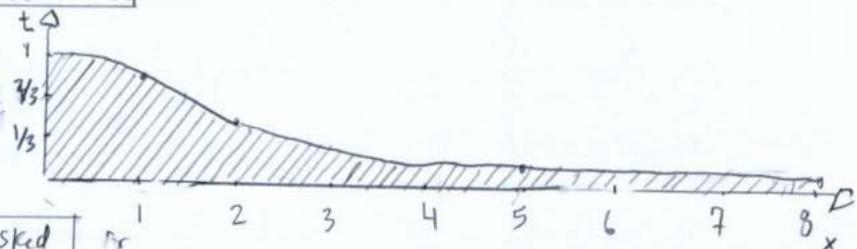
$$\dot{x} = -x; \quad x(0) = 1 \quad 2.8.5. \quad x(t) = e^{-t}$$

n	$\Delta t$	$f(x)$	$x_n = x_{n-1} + \frac{1}{6}(R_1 + 2R_2 + 2R_3 + R_4)$	$E =  \hat{x}(1) - x(1) $	$\ln E$
0	$10^0$				
1	$10^{-1}$	$\exp(x_{n-1})$			
2	$10^{-2}$				
3	$10^{-3}$				
4	$10^{-4}$				



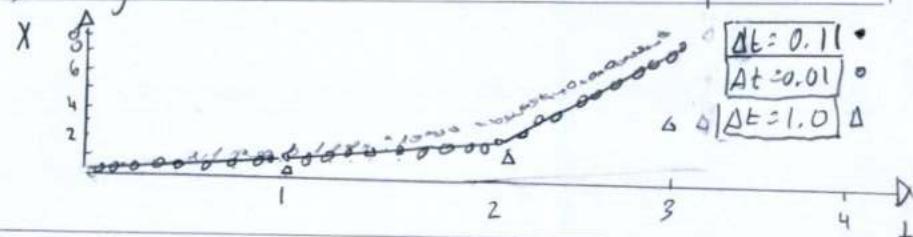
The Euler method aided the analysis of numerical methods; including, precision. Euler's Improved Method approached the solution of  $f(x) = e^{-x}$  with less round-off error. Runge-Kutta's Routine provided the least round-off errors with  $10^{-20}$  across the spreadsheet, and necessitated high-precision.

$$x = x + e^{-x} \quad \text{2.8.6. a: } t = \int \frac{1}{x e^{-x}} dx = \int \frac{e^x}{x e^{x+1}} dx$$



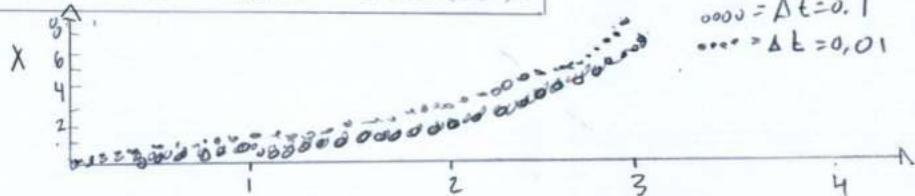
I noticed the book asked for  $x(t)$  (and not  $t(x)$ ).

This led me to investigate a Numerical method of integration; withstanding, Runge-Kutta Routine aided with the plot of  $x(t)$ .



b. At  $t=0$ , analytical arguments provided an  $x=1.011$ .

c. Stepsizes of  $\Delta t = 0.1$  and  $0.01$  had different results, including, inaccuracies above and below both estimates.



d) See part a.

$$x_1 = x_0 + f(x_0) \Delta t \quad \text{2.8.7 a) } x(t_1) = x(t_0 + \Delta t)$$

Taylor Series:

$$x(t + \Delta t) = \sum_{n=0}^{\infty} \frac{x^{(n)}(t)}{n!} (\Delta t)^n = x(t) + x'(t) \cdot \Delta t + O(\Delta t^2) + O(t)$$

$$f(t + \Delta t) = f(t) + f'(t) \cdot \Delta t + O(\Delta t^2) = [x_0 + f'(t) \cdot \Delta t]$$

$$\text{b) } |x(t_1) - x_1| = |x(t_1) - x(t_0) - x'(t_0) \cdot \Delta t + O(\Delta t^2)| = |O(\Delta t^2)| = \frac{|x''(t) \Delta t^2|}{2!} = C(\Delta t^2)$$

$$C = \frac{|x''(t)|}{2!}$$

Taylor Series: 2.8.8.  $\dot{x} = x + e^{-x}$  :  $|x(t_0) - x_0| = |x(t_0) - x(t_0) - x'(t_0)\Delta t - \frac{x''(t_0)\Delta t^2}{2}| = \frac{|x''(t_0)\Delta t^2|}{2}$

$$f(x+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)h^n}{n!}$$

$$= O(\Delta t^2)$$

$\dot{x} = x + e^{-x}$  2.8.9. Bunge-Kutta:  $x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$  where  $k_1 = f(x_n)\Delta t$

$$x(t + \Delta t) = x(t_0) + x'(t_0)\Delta t + \frac{x''(t_0)\Delta t^2}{2} + O(\Delta t^3)$$

$$k_1 = f(x_n)\Delta t = x'(t_0)\Delta t$$

$$k_2 = f(x_n + \frac{1}{2}k_1)\Delta t$$

$$k_3 = f(x_n + \frac{1}{2}k_2)\Delta t$$

$$k_4 = f(x_n + k_3)\Delta t$$

$$R_1 = f(x_n)\Delta t = f(x_n) + f'(x_n)\frac{1}{2}k_1 + O\left[\left(\frac{1}{2}k_1\right)^2\right]$$

$$R_2 = f(x_n + \frac{1}{2}k_2)\Delta t = f(x_n) + f'(x_n)\frac{1}{2}k_2 + O\left[\left(\frac{1}{2}k_2\right)^2\right]$$

$$= f(x_n) + f'(x_n)\frac{1}{2}\left[f(x_n) + f'(x_n)\frac{1}{2}k_1 + O\left[\left(\frac{1}{2}k_1\right)^2\right]\right] + O\left[\left(\frac{1}{2}k_2\right)^2\right]$$

$$R_3 = f(x_n + k_3)\Delta t = f(x_n) + f'(x_n) \cdot R_2 + O[R_2^2]$$

$$= f(x_n) + f'(x_n)\left[f(x_n) + f'(x_n)\cdot\frac{1}{2}[f(x_n) + f'(x_n)\frac{1}{2}k_1 + O\left[\left(\frac{1}{2}k_1\right)^2\right]] + O\left[\left(\frac{1}{2}k_2\right)^2\right]\right]$$

$$+ O[R_3^2]$$

$$x_{n+1} = x_n + \frac{1}{6}(R_1 + 2R_2 + 2R_3 + R_4) = x_n + \frac{1}{6}(x'(t_0)\Delta t + 2x'(t_0) + x''(t_0)k_1$$

$$+ 2x'(t_0) + x''(t_0)[x'(t_0) + x''(t_0)x'(t)\Delta t])$$

$$+ 2x'(t_0) + x''(t_0)[x(t) + \frac{x'(t)}{2}[x'(t_0) + x''(t_0)x'(t)]])$$

$$|x(t_1) - x_1| = |x(t_0 + \Delta t) - x_{n+1}| = O(\Delta t^5)$$

Chapter 3:

$\dot{x} = 1 + rx + x^2$  3.1.1. Vector Field:

$$x = \frac{r \pm \sqrt{r^2 - 4}}{2}$$

$$r = \frac{r \pm \sqrt{(r-2)(r+2)}}{2}$$

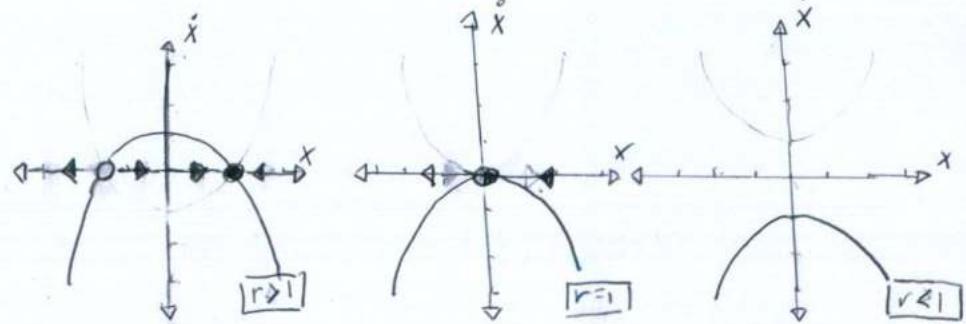
r	Bifurcations
$r > 2$	Two
-2	One
0	zero
2	One
$r < 2$	Two

Bifurcation Diagram:

$\dot{x} = r - \cosh x$  3.1.2. Vector Field

$$r = \cosh(x)$$

r	Bifurcations
<1	ZERO
=1	One
>1	Two



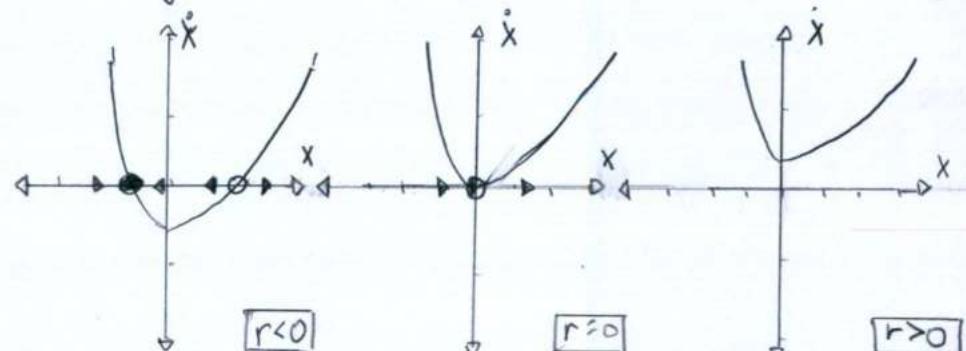
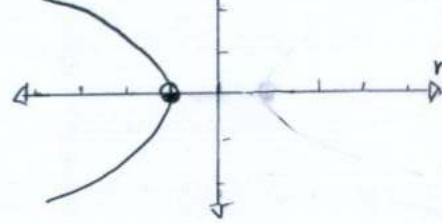
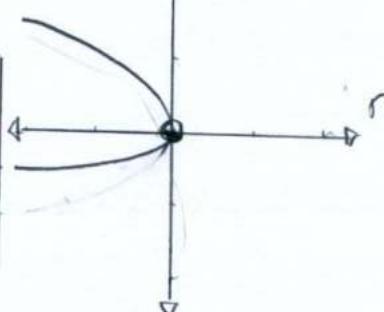
$\dot{x} = r + x - \ln(1+x)$  3.1.3

Vector Field

r	Bifurcation
>0	Zero
=0	One
<0	Two

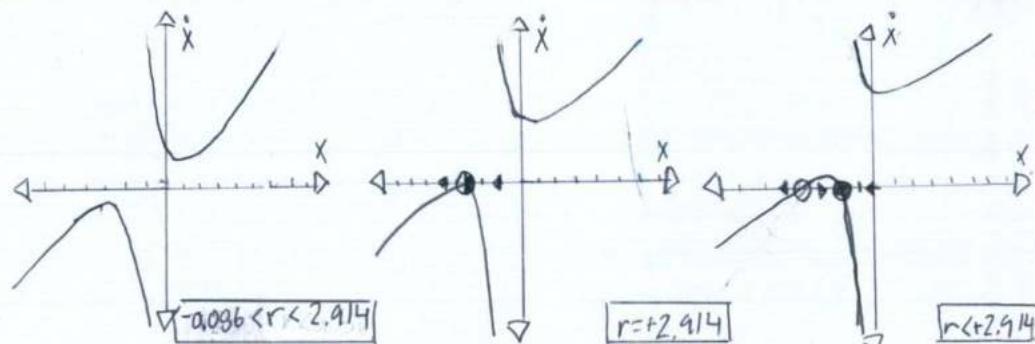
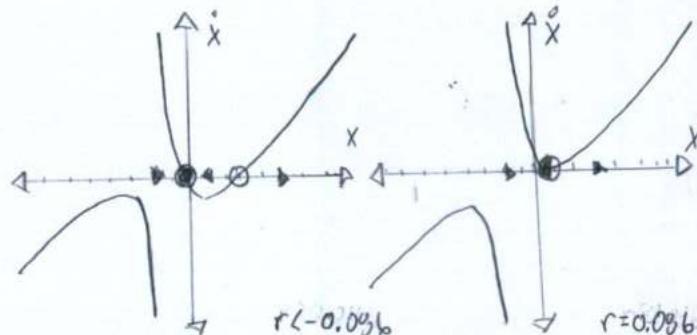
Bifurcation Diagram:

Bifurcation Diagram

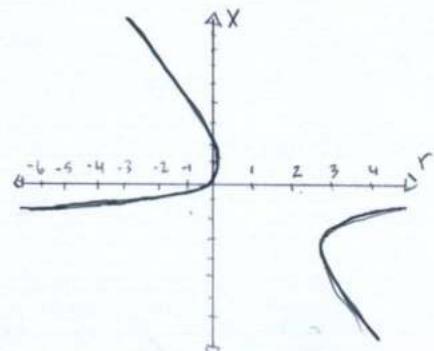


$\dot{x} = r + \frac{1}{2}x - x/(1+x)$  3.1.4. Vector Field

r	Bifurcations
$\approx -0.096$	Two
$= -0.096$	One
$-0.096 < r < 2.914$	Zero
$= +2.914$	One
$< +2.914$	Two.

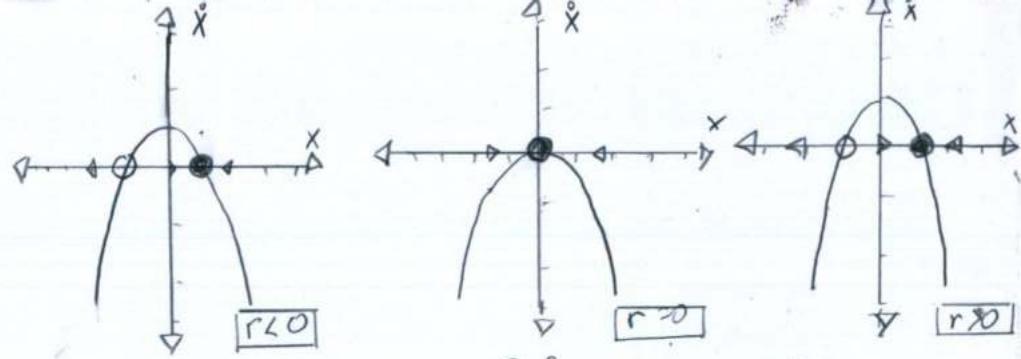


Bifurcation Diagram:



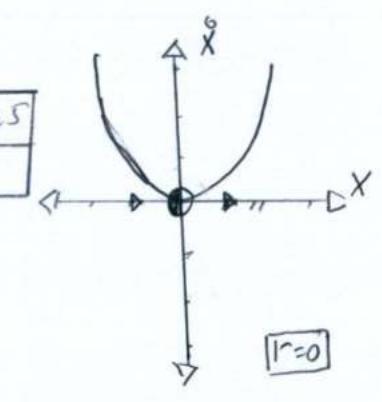
$\dot{x} = r^2 - x^2$  3.1.5. a) Vector Field:

r	Bifurcations
>0	Two
=0	One
<0	Two

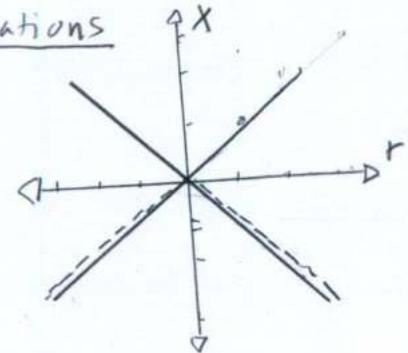


$\dot{x} = r^2 + x^2$  b) Vector Field:

r	Bifurcations
0	One



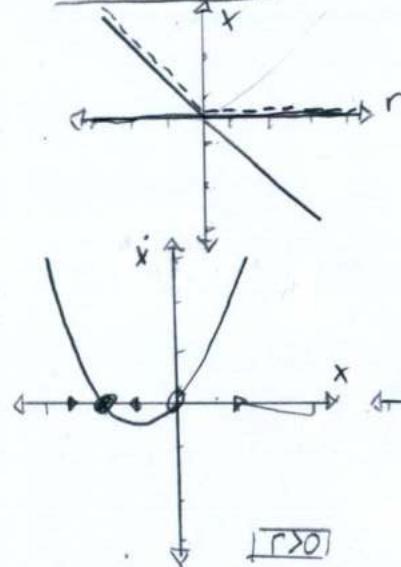
Bifurcations



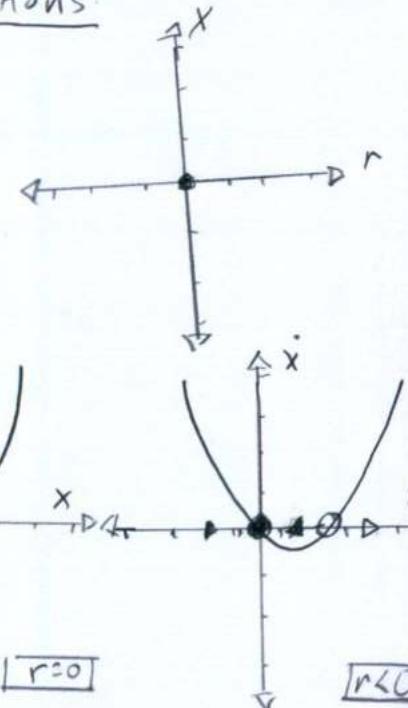
$\dot{x} = rx + x^2$  3.2.1 Vector Field:

r	Bifurcations
>0	Two
=0	One
<0	Two

Bifurcations

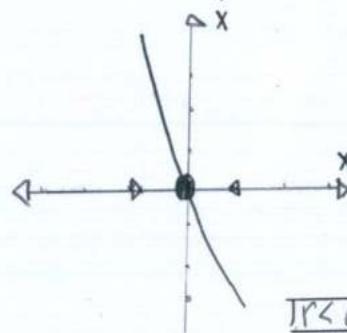


Bifurcations

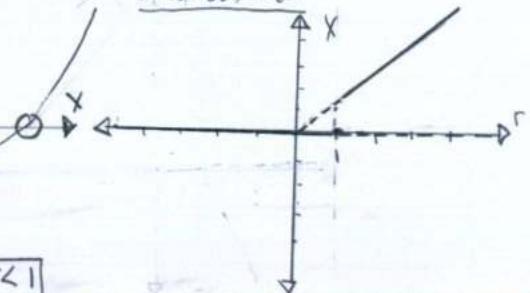


$\dot{x} = rx - \ln(1+x)$  3.2.2. Vector Field:

r	Bifurcations
<0	One
0 < r < 1	Two
r > 1	One
r > 1	Two

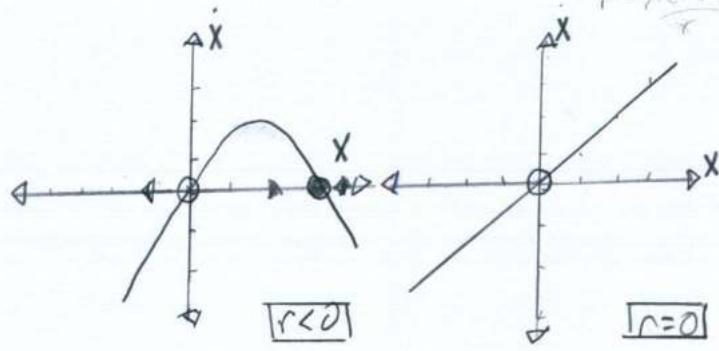


Bifurcations



$$\dot{x} = x - rx(1-x)$$

r	Bifurcations
$\leq 0$	TWO
$= 0$	ONE
$0 < r < 1$	ONE
$r \geq 1$	TWO
$> 1$	TWO

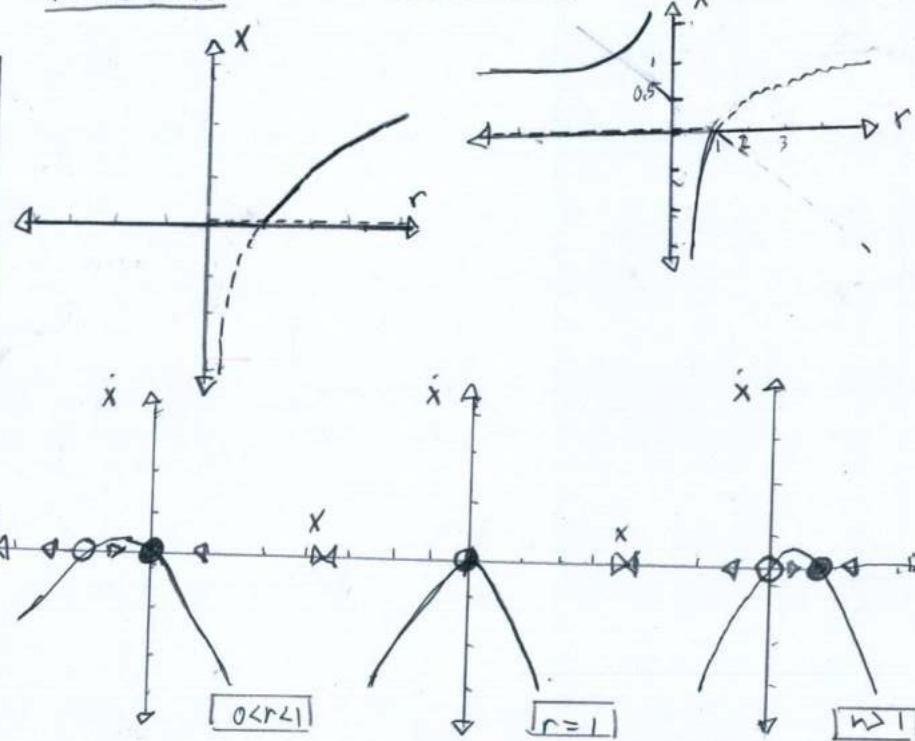


$$\dot{x} = x(r - e^x)$$

### Bifurcations:

r	Bifurcations
$\leq 0$	one
$0 < r < 1$	TWO
$0 \leq r < 1$	one
$r \geq 1$	TWO

### Bifurcations:



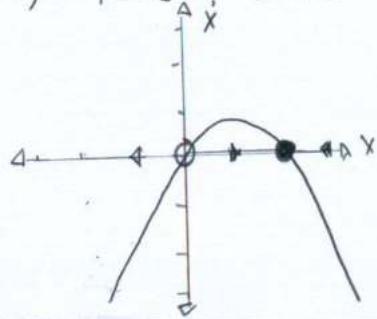
$$\dot{x} = c_1 x - c_2 x^2$$

$$a + x \xrightarrow{k_1} z \xrightarrow{k_2} x$$

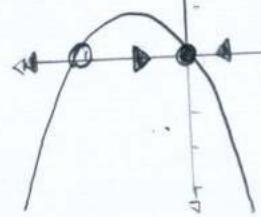
$$\frac{d[A]}{dt} = -k_1 [A][X] + k_2 [Z]^2 ; \quad \frac{d[X]}{dt} = (k_1 [A] - k_2 [Z]) [X] - k_{-1} [X]^2$$

$$\frac{d[B]}{dt} = -k_2 [B][X] ; \quad \frac{d[C]}{dt} = k_2 [B][X]$$

$$b) k_1 [A] > k_2 [B]$$



$$k_1 [A] < k_2 [B]$$



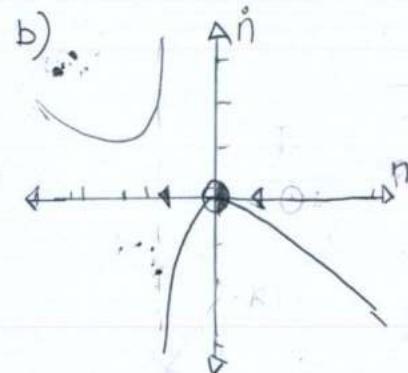
Chemically, a rate of change  $\frac{d[x]}{dt}$  that approaches zero, then remains zero is of greater stability than a rate of change which increases from zero.

$$\dot{n} = G_n N - Rn \quad 3.3.1 \text{ a) Suppose } \dot{N} > 0, \text{ then } \dot{N} \approx 0, \text{ "Adiabatic Elimination"}$$

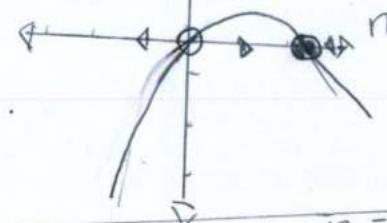
$$\dot{N} = -G_n N - FN + P$$

$$G_n N + FN = P ; \quad \dot{N} = -FN + P - RN$$

$$N = \frac{P}{G_n + F} ; \quad \dot{N} = -F \left[ \frac{P}{G_n + F} \right] + P - RN$$



$$P \times \frac{RN [G_n + F]}{1 - PF} = P_c$$



c) A transcritical bifurcation occurs at  $n=0$  because of the stability change for the fixed points.

d)  $G, n, P, F > 0, N=0$ , a constant amount of excited photons

$$\dot{E} = K(P-E)$$

$$\dot{P} = \gamma_1(ED-P)$$

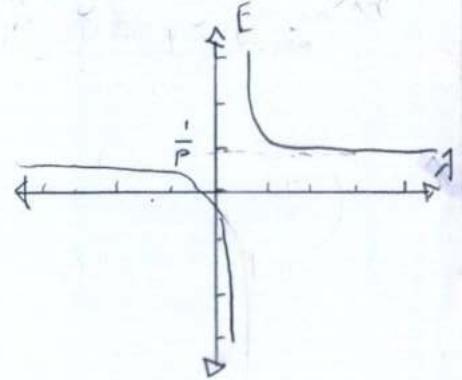
$$\dot{D} = \gamma_2(\lambda + 1 - D - \lambda EP)$$

3.3.2 a) Assume  $\dot{P} \approx 0, \dot{D} \approx 0, P = ED, D = \lambda + 1 - \lambda EP$

$$\dot{E} = K(ED - E) = K(E(\lambda + 1 - \lambda EP) - E)$$

b) Fixed Points:  $E=0, \frac{1}{P}$

c) Bifurcation Diagram:

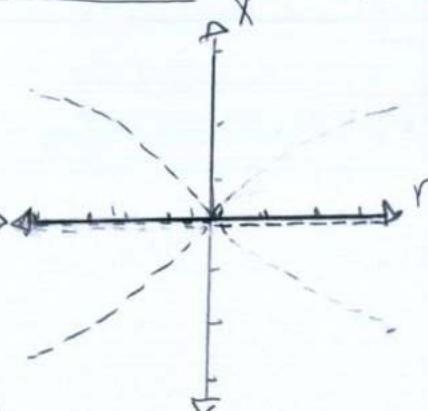
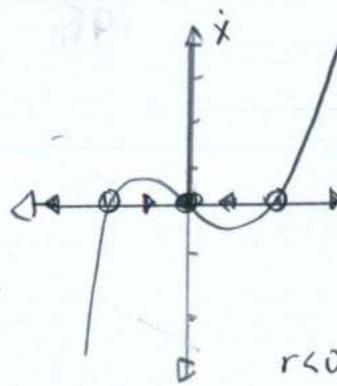


$\dot{x} = rx + 4x^3 \quad 3.4.1 \text{ Vector Field:}$

$$r = -4x^2$$

r	Bifurcations
< 0	Three
$\geq 0$	One

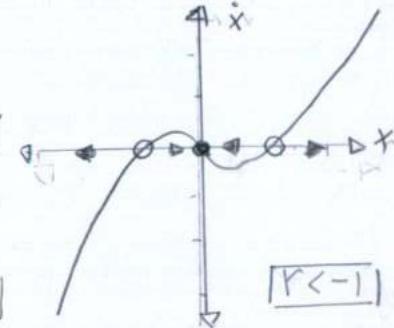
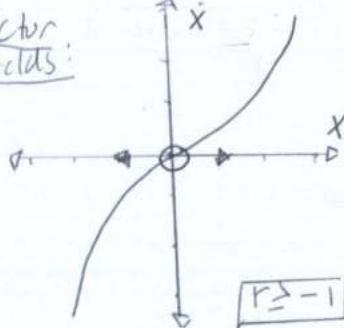
Bifurcations: Subcritical



$$\dot{x} = rx - \sinh x \quad 3.4.2.$$

$r$	Bifurcations
$\geq 1$	One
$< 1$	Three

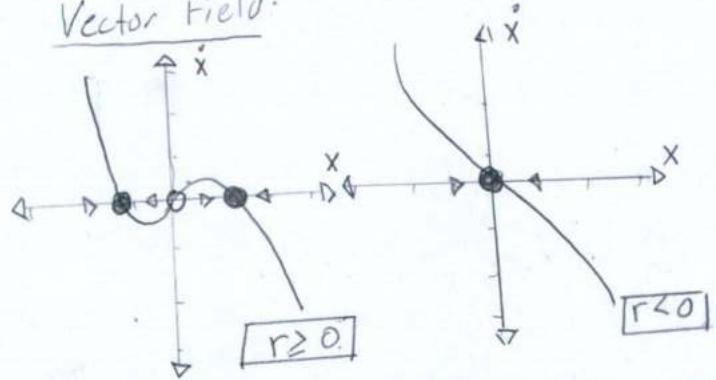
Vector Fields:



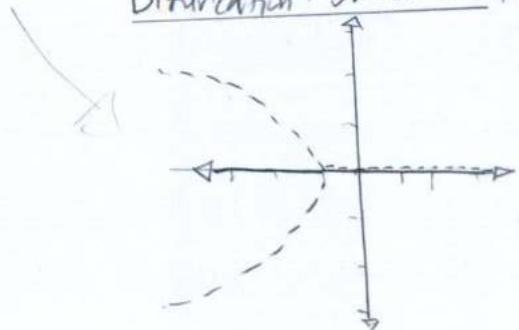
$$\dot{x} = rx - 4x^3 \quad 3.4.3$$

$r$	Bifurcations
$\geq 0$	Three
$\leq 0$	One

Vector Field:



Bifurcation: subcritical

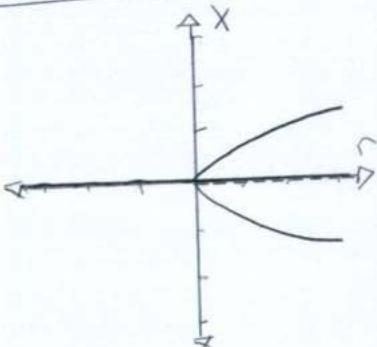
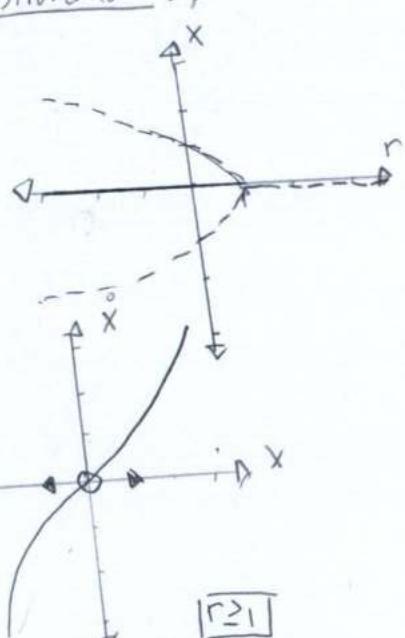


$$\dot{x} = x + \frac{rx}{1+x^2} \quad 3.4.4$$

$r$	Bifurcations
$< 1$	Three
$\geq 1$	One

Bifurcations: supercritical

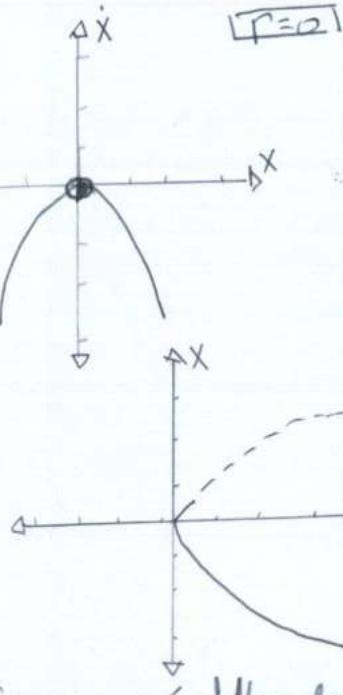
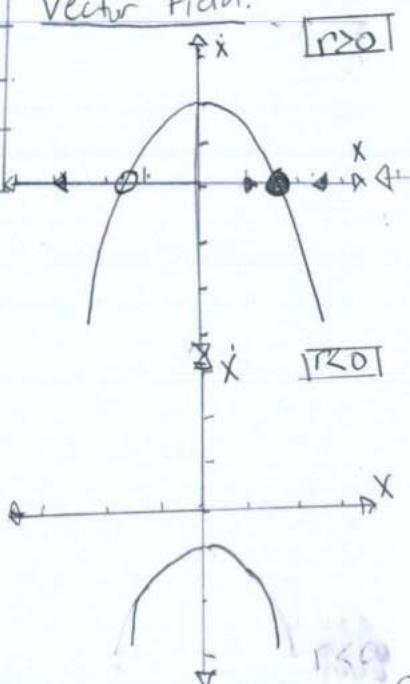
Bifurcation: supercritical



$$\dot{x} = r - 3x^2 \quad 3.4.5$$

$r$	Bifurcations
$\geq 0$	Two
$= 0$	One
$< 0$	Zero

Vector Field:



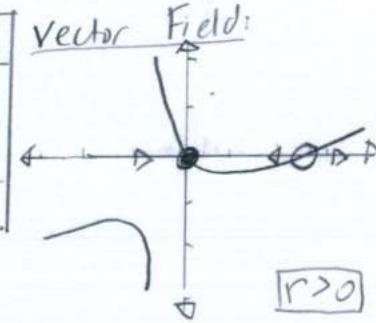
Bifurcation: Saddle-node

$$\dot{x} = rx - \frac{x}{1+x}$$

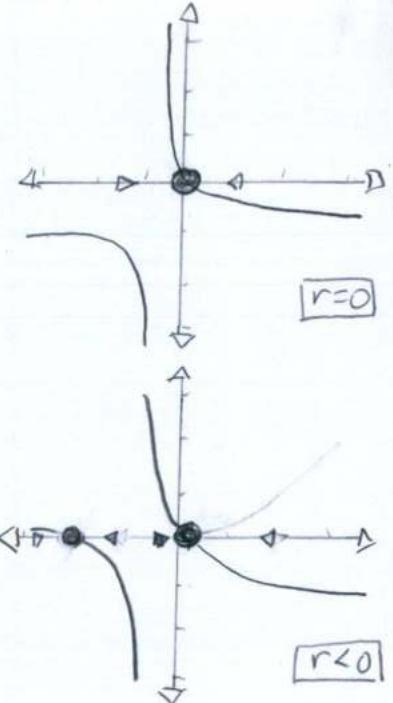
3.4.6.

r	Bifurcations
>0	Two
=b	One
<0	Two

Vector Field:

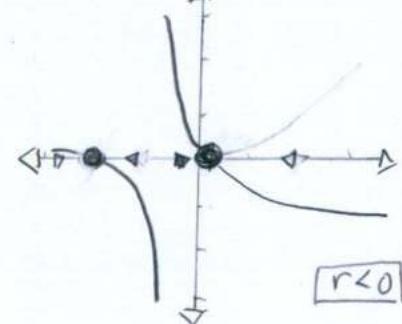
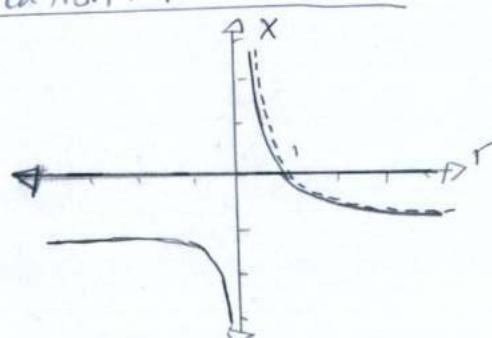


[r > 0]



[r = 0]

Bifurcation: Transcritical



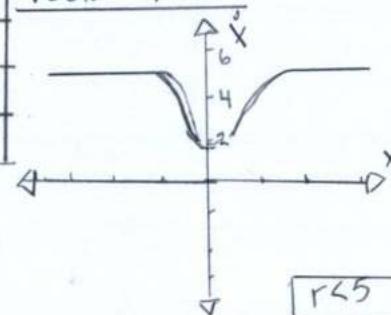
[r < 0]

$$\dot{x} = 5 - re^{-x^2}$$

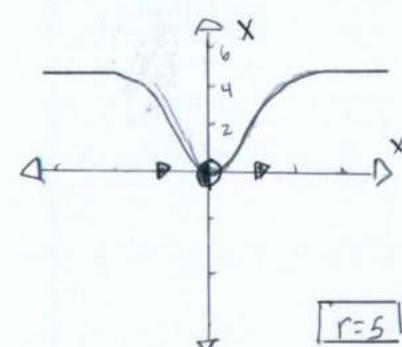
3.4.7

r	Bifurcations
<5	zero
=5	one
>5	two

Vector Field:



[r < 5]

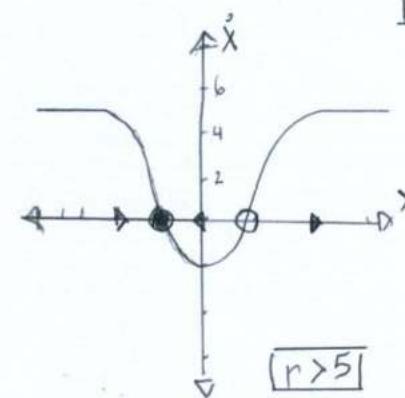


[r = 5]

$$\dot{x} = rx - \frac{x}{1+x^2}$$

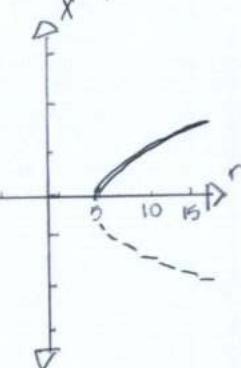
3.4.8,

r	Bifurcations
$\leq 0$	one
$0 < r < 1$	three
$\geq 1$	one

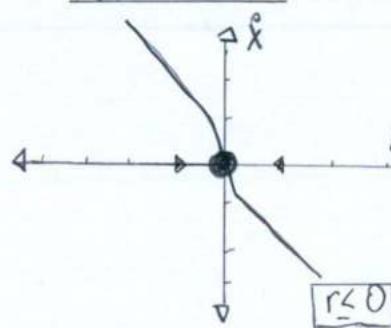


[r > 5]

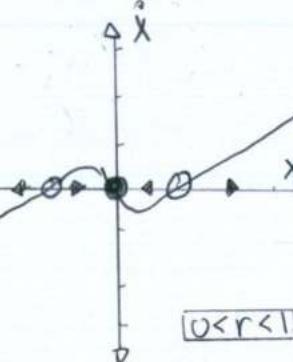
Bifurcation: Saddle-node



Vector Field:



[r < 0]



[0 < r < 1]

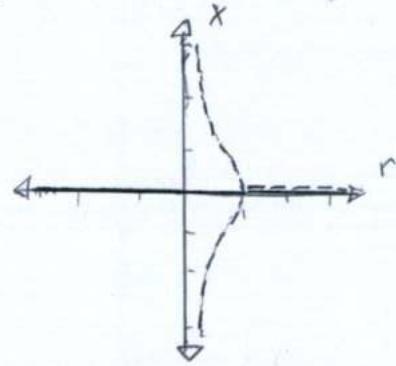


[r >= 1]

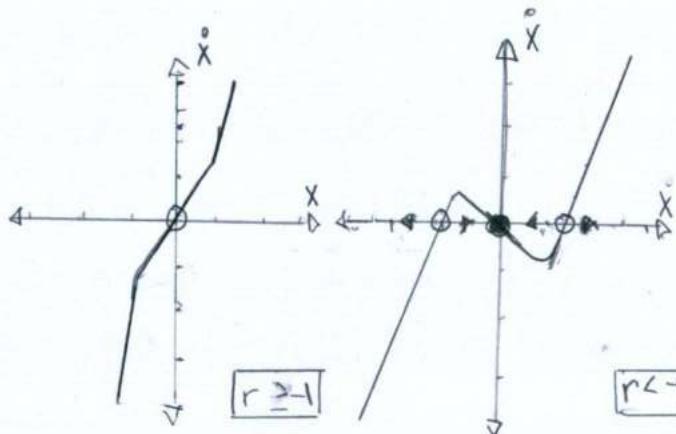
$$\dot{x} = x + \tanh(rx) \quad 3.4.9$$

$r$	Bifurcations
$\leq -1$	one
$> -1$	three

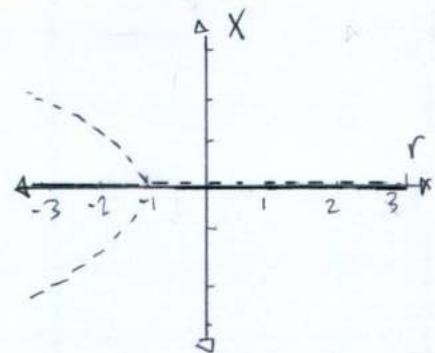
Bifurcation: Transcritical



Vector Field:



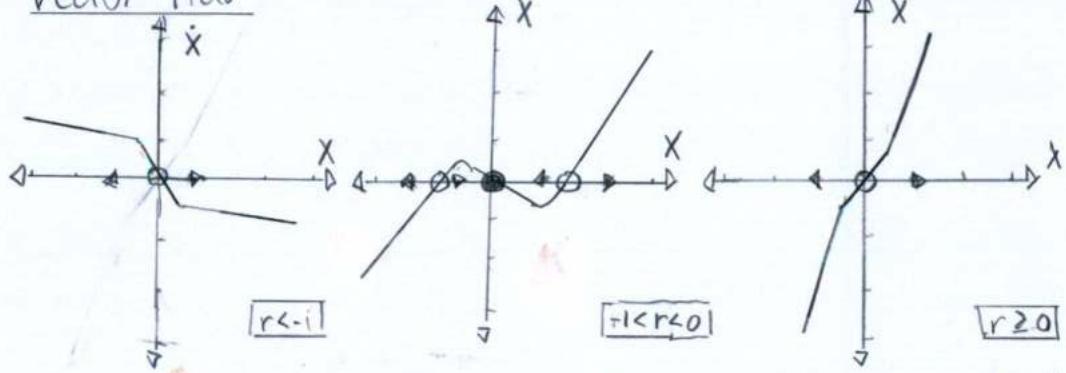
Bifurcation: Subcritical



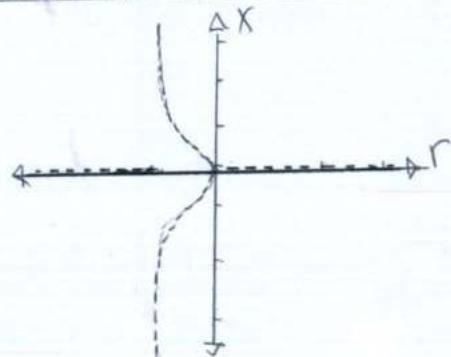
$$\dot{x} = rx + \frac{x^3}{1+x^2} \quad 3.4.10.$$

$r$	Bifurcations
$< -1$	one
$-1 < r < 0$	three
$\geq 0$	one

Vector Field:



Bifurcation: Subcritical Pitchfork



$$\dot{x} = rx - \sin x \quad 3.4.11$$

a) If  $r=0$ , then  $\dot{x} = -\sin x$

Fixed points: Stable  $\approx (2k+1)\pi$

Unstable  $\approx 2k\pi$

Where  $k \in \mathbb{Z}$

b) If  $r > 1$ ,  $\dot{x}=0$  is unstable

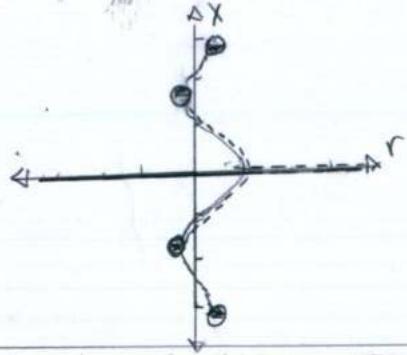
c) As  $r \rightarrow \infty \rightarrow 0$ , then a subcritical pitchfork best describes the bifurcation.

$$d) \dot{x} = rx - \sin(x); \quad r = \frac{\sin(x)}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + O(x^4)$$

$$x = \pm (6(1-r))^{1/2}$$

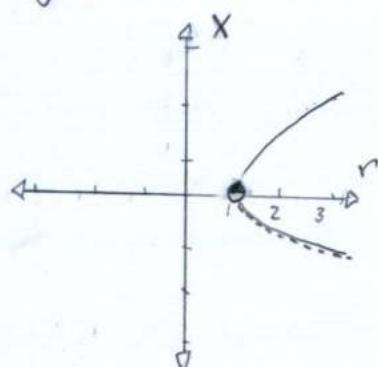
e) As  $r \rightarrow -\infty \rightarrow 0$ , then a supercritical pitchfork occurs across the function  $\dot{x} = rx - \sin(x)$ .

f) Bifurcations:

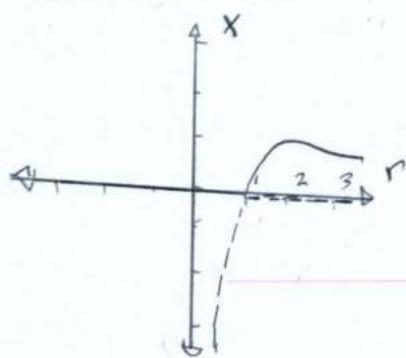


$\dot{x} = f(x, r)$  3.4.12 A "quadrification" function is  $x = \frac{1}{2}(3 \pm \sqrt{1+4\sqrt{r}})$  where  
 $\dot{x} = f(x, r) = (x-2)^2(x-1)^2 - r$ . This function has even polynomial  
multiplicities to describe zero bifurcations  $r < 0$  and  
four when  $r > 0$ ,

$\dot{x} = r - x - e^{-x}$  3.4.13 a) Best guess of roots:  $r=1, x=0$

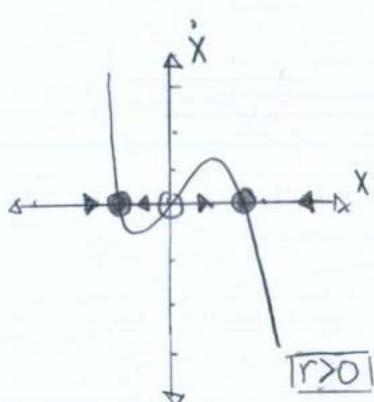


$\dot{x} = 1 - x - e^{-rx}$  b) Best Guess of roots:  $r=1, x=0$



$\dot{x} = rx + x^3 - x^5$  3.4.14 a)  $\dot{x} = 0 = r + 3x^2 - 5x^4$  - or -  $r = x^2(x^2 - 1)$

b) Vector Field:



c)  $|r_c \geq 0|$

$$\dot{x} = rx + x^3 - x^5 \quad 3.4.15. \quad -\frac{dV(x)}{dx} = \dot{x} = 0 \Rightarrow -r - x^2 + x^4 = 0$$

where  $a = x^2$

$$a_1, a_2 = \frac{+1 \pm \sqrt{(+1)^2 - 4(1)(-r)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1 + 4r}}{2}$$

$$= \frac{1}{4} \pm \sqrt{\frac{1 + 4r}{16}} = \left(\frac{1}{4} \pm \sqrt{\frac{1 + 4r}{16}}\right)$$

$$a = x^2 = 2\left(\frac{1}{4} \pm \sqrt{\frac{1 + 4r}{16}}\right)$$

$$x_1 = +\sqrt{\frac{1 + \sqrt{1 + 4r}}{2}}; \quad x_2 = -\sqrt{\frac{1 + \sqrt{1 + 4r}}{2}}$$

$$x_3 = +\sqrt{\frac{1 - \sqrt{1 + 4r}}{2}}; \quad x_4 = -\sqrt{\frac{1 - \sqrt{1 + 4r}}{2}}$$

$$x_5 = 0$$

$$V(x) = -r\frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6}$$

$$V(x_1) = V(x_2) = V(x_3) = V(x_4) = V(x_5) = 0$$

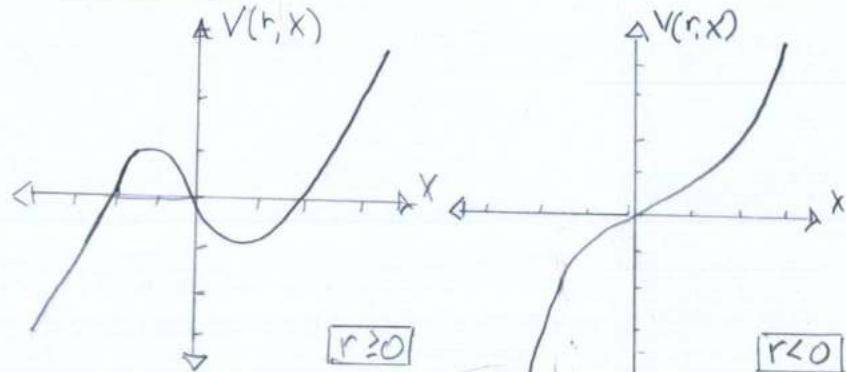
$$@ V(x_1) = -r\left(\frac{1 + \sqrt{1 + 4r}}{2}\right) + \frac{1}{4}\left(\frac{1 + \sqrt{1 + 4r}}{2}\right)^2 - \frac{1}{6}\left(\frac{1 + \sqrt{1 + 4r}}{2}\right)^3 = 0$$

$$r = -\frac{3}{16}$$

$$V(r, x) = \frac{x^3}{3} - rx$$

Potential Field

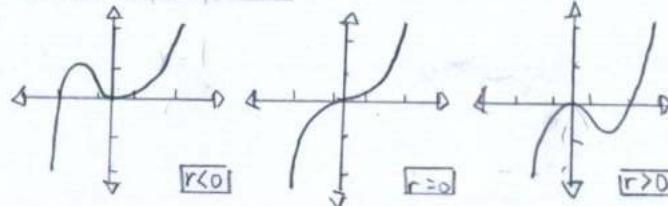
r	Bifurcations
$\geq 0$	Three
$< 0$	One



$$\dot{x} = rx - x^2$$

$$b) -\frac{dV}{dx} = \dot{x} = rx - x^2; \quad V(r, x) = \frac{x^3}{3} - \frac{rx^2}{2}$$

Potential Field:



r	Bifurcations
$< 0$	Two
$= 0$	One
$> 0$	Two

$$x = rx + x^3 - x^5 \quad c) \quad -\frac{dV}{dx} = rx + x^3 - x^5 \Rightarrow V(r, x) = \frac{x^6}{6} - \frac{x^4}{4} - rx^2$$

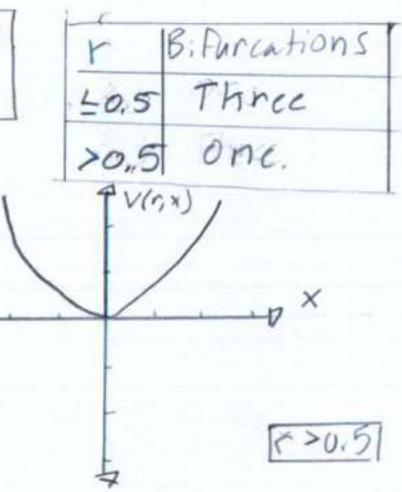
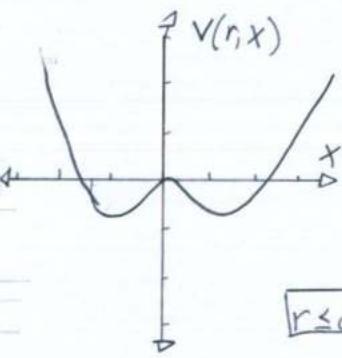
Potential S:

$$0 = r + x^2 - x^4$$

$$= r + a - a^2$$

$$= \frac{-1 \pm \sqrt{1+4r}}{2}$$

$$x = \frac{-1 \pm \sqrt{1+4r}}{2}$$



$r > 0.5$

$b\dot{\phi} = mg \sin \phi + mr\omega^2 \sin \phi \cos \phi$

3.5.1. A better representation of  $b\dot{\phi} = mg \sin \phi + mr\omega^2 \sin \phi \cos \phi$  is  $b\dot{\phi} = mg \sin \phi \left(\frac{r\omega^2}{g} \cos \phi - 1\right)$ , which best represents the maximum angle of  $\phi = \pi/2$ . If the pendulum approaches a fixed point during rotation, then  $b\dot{\phi} = 0 \Rightarrow \frac{r\omega^2}{g} \cos \phi = 1 \Rightarrow \cos \phi = \frac{g}{r\omega^2}$ ; and,  $\frac{g}{r\omega^2}$  requires a positive value above zero.

$$\begin{aligned} \frac{d\phi}{dt} &= F(\phi) \\ &= -\sin \phi + 8 \sin \phi \cos \phi \\ &= \sin \phi (8 \cos \phi - 1) \end{aligned}$$

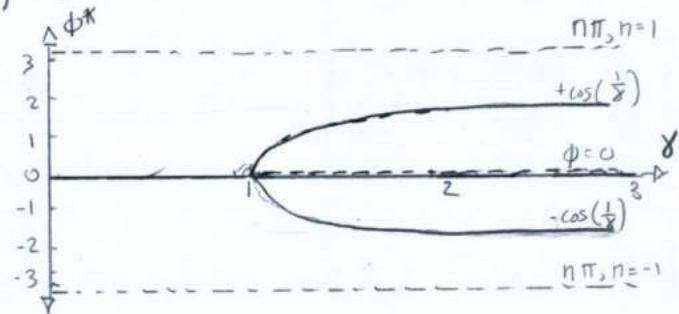
3.5.2  $F(\phi) = \sin \phi (8 \cos \phi - 1)$

$$f'(\phi) = 8[\cos^2 \phi - \sin^2 \phi - 1] = 8[\cos 2\theta - 1]$$

$$f''(\phi) = -16[\sin 2\theta]$$

$$\phi^* = n\pi; f'(\phi^*) = 0; \text{Half-Node}$$

$$\phi^* = \cos^{-1}\left(\frac{1}{8}\right)$$



$$\cos \phi \approx 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} \quad \text{and} \quad \frac{d\phi}{dt} = \phi \left( 8 \left[ 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} \right] - 1 \right)$$

$$= 8\phi - \frac{8\phi^3}{2!} + \frac{8\phi^5}{4!}$$

Where  $\frac{d\phi}{dt} = A\phi - B\phi^3 + O(\phi^5); A = 8, B = \frac{8}{2}, O(\phi^5) = \frac{8\phi^5}{4!}$

$$m\ddot{x} = -F_{\text{spring}} - F_{\text{fric}}$$

3.5.4.  $m\ddot{x} = -k \cdot l \cos \phi - k L_0 \cos \phi - b \dot{\phi}$

$$= -k(l - L_0) \cos \phi - b \dot{\phi} = -k(\sqrt{x^2 + h^2} - L_0) \frac{x}{l} - b \dot{\phi}$$

$$= -k(\sqrt{h^2 + x^2} - L_0) \frac{x}{\sqrt{h^2 + x^2}} - b \dot{\phi}$$

$$= -k \left( 1 - \frac{L_0}{\sqrt{h^2 + x^2}} \right) x - b \dot{\phi}$$

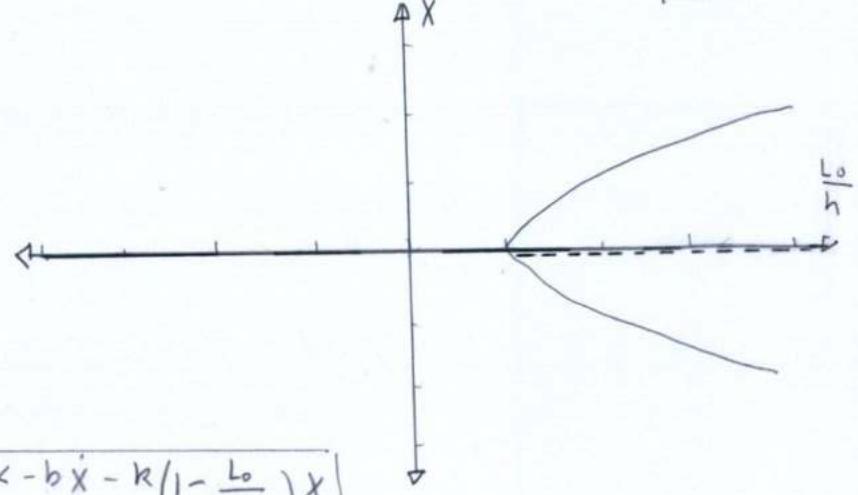
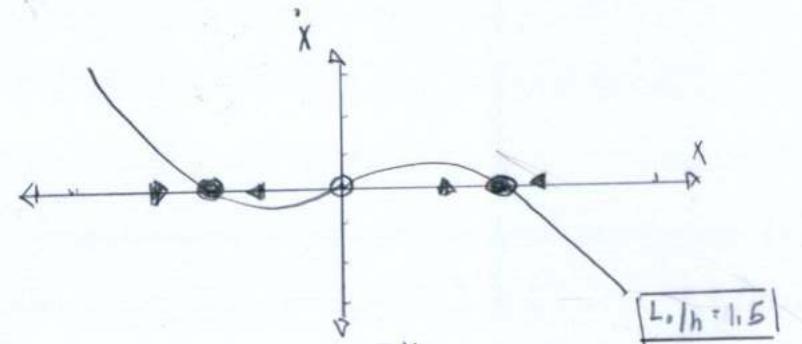
$$b. m\ddot{x} + b\dot{x} + R \left(1 - \frac{L_0}{\sqrt{x^2 + h^2}}\right) x = 0$$

if  $\dot{x}=0$ ,  $x^* = \sqrt{L_0^2 - h^2}, 0$

c. If  $m=0$ ,  $b\dot{x} + R \left(1 - \frac{L_0}{\sqrt{x^2 + h^2}}\right) x = 0$ , then

$$x^* = \sqrt{L_0^2 - h^2}, 0$$

Bifurcation Diagram



d. If  $m \neq 0$ , then  $m\ddot{x} \ll -b\dot{x} - R \left(1 - \frac{L_0}{\sqrt{x^2 + h^2}}\right) x$

$\epsilon \frac{d^2\phi}{dT^2} + \frac{d\phi}{dT} = f(\phi)$  3.5.5-a)  $\frac{d\phi}{dT} = f(\phi)$ ;  $T_{fast}$  is estimated to be:

$$\epsilon^{1-2k} \frac{d^2\phi}{dT^2} + \epsilon^{-k} \frac{d\phi}{dT} = f(\phi)$$

where  $k=1$ ,  $\epsilon^{1-2k} = \epsilon^{-k} \gg 1$

$$k=\frac{1}{2}, \epsilon^{1-2k} = 1 \gg e^{-k}$$

$$k=0, \epsilon^{-k} = 1 \gg e^{1-2k}$$

$$T = \epsilon \frac{b}{mg} = \frac{m^2 g n}{b^2} \frac{b}{mg} = \frac{m n}{g}$$

b) If  $T = \epsilon \zeta$ , then  $\epsilon \frac{d^2\phi}{dT^2} + \frac{d\phi}{dT} = \epsilon \frac{d^2\phi}{d(\epsilon \zeta)^2} + \frac{1}{\epsilon} \frac{d\phi}{d\zeta} = f(\phi)$

$$\frac{d^2\phi}{d\zeta^2} + \frac{d\phi}{d\zeta} = \epsilon f(\phi) \quad "Rescaled"$$

$$c. T_{Ass} = \epsilon T_{Slow}$$

$$\epsilon \ddot{x} + \dot{x} + x = 0 \quad 3.5.6. \quad x(0) = 1; \dot{x}(0) = 0$$

a) General solution:  $x(t) = C_1 e^{\lambda t} + C_2 e^{\lambda b t}$ ;  $\lambda = \frac{-1 \pm \sqrt{1-4\epsilon}}{2\epsilon}$

$$\dot{x}(t) = \lambda C_1 e^{\lambda t} + \lambda C_2 e^{\lambda b t}; \quad x(t) = C_1 e^{\frac{(-1+\sqrt{1-4\epsilon})t}{2\epsilon}} + C_2 e^{\frac{(-1-\sqrt{1-4\epsilon})t}{2\epsilon}}$$

$$x(0) = C_1 + C_2 = 1$$

$$\dot{x}(t) = C_1 \left( \frac{(-1+\sqrt{1-4\epsilon})}{2\epsilon} \right) e^{\frac{(-1+\sqrt{1-4\epsilon})t}{2\epsilon}} + C_2 \left( \frac{(-1-\sqrt{1-4\epsilon})}{2\epsilon} \right) e^{\frac{(-1-\sqrt{1-4\epsilon})t}{2\epsilon}}$$

$$\ddot{x}(0) = C_1 \left( -\frac{(1+\sqrt{1-4\epsilon})}{2\epsilon} \right) + C_2 \left( -\frac{(1-\sqrt{1-4\epsilon})}{2\epsilon} \right)$$

$$= -\frac{C_1 \sqrt{1-4\epsilon}}{2\epsilon} + \frac{C_1}{2\epsilon} + (1-C_1) \left( -\frac{1-\sqrt{1-4\epsilon}}{2\epsilon} \right)$$

$$= -\frac{C_1 \sqrt{1-4\epsilon}}{2\epsilon} + \frac{C_1}{2\epsilon} + \frac{(1+\sqrt{1-4\epsilon})}{2\epsilon}$$

$$= \frac{C_1 (1-\sqrt{1-4\epsilon})}{2\epsilon}$$

$$= -\frac{C_1 \sqrt{1-4\epsilon}}{2\epsilon} + \frac{C_1}{2\epsilon} + \frac{(1+\sqrt{1-4\epsilon})}{2\epsilon}$$

$$+ \frac{C_1}{2\epsilon} + \frac{C_1 \sqrt{1-4\epsilon}}{2\epsilon} = 0$$

$$= \frac{C_1}{2\epsilon} ; = \frac{(1+\sqrt{1-4\epsilon})}{2}$$

Therefore,  $x(t) = \left( \frac{1+\sqrt{1-4\epsilon}}{2} \right) \left( e^{\frac{(1+\sqrt{1-4\epsilon})t}{2\epsilon}} \right)$

$$+ \left( 1 - \frac{(1+\sqrt{1-4\epsilon})}{2} \right) \left( e^{\frac{(1-\sqrt{1-4\epsilon})t}{2\epsilon}} \right)$$

b.  $\epsilon \ddot{x} + \dot{x} + x : \epsilon \frac{d^2x}{dt^2} + \frac{dx}{dt} + x = 0$

$$\frac{\epsilon}{T^2} \frac{d^2x}{dT^2} + \frac{1}{T} \frac{dx}{dT} = -x$$

$$\frac{1}{T} \frac{d^2x}{dT^2} + \frac{1}{T} \frac{dx}{dT} = x; \quad \ddot{x} + \dot{x} = TX = \epsilon x; \quad \boxed{\ddot{x} + \dot{x} - \epsilon x = 0}$$

where  $T = \frac{t}{T} = e^{\frac{t}{T}}$

$$\dot{N} = rN(1 - N/K) \quad 3.5.7. \text{ a) } N(0) = N_0$$

Parameter	Dimensions
$r$	Per time (rate)
$K$	Same as $N$ (amount)
$N_0$	Same as $N$ (amount)

b)  $\frac{dN}{dt} = rN(1 - N/K)$ ; If  $\frac{N}{K} = x$ , then  $dN = Kdx$

$$\frac{dx}{dt} = rx(1-x); \text{ If } t = \frac{\tau}{r}, \text{ then } dt = d\tau$$

$$\boxed{\frac{dx}{d\tau} = x(1-x)}$$

c)  $u = x$ ;  $\frac{du}{d\tau} = u(1-u)$ ;  $u(0) = u_0$

$$\int \frac{du}{u(1-u)} = d\tau; \int \frac{A}{u} du + \int \frac{B}{(1-u)} du = \int \frac{du}{u} + \int \frac{du}{(1-u)} = \ln \frac{u}{1-u} = \tau + C$$

$$\frac{1-u}{u} = C e^{-\tau}$$

d) An advantage of the dimensionless functions are lower degrees of freedom during analysis. The graphical representations do not have further axis to plot, and the functions are closer to the basic functions of precalculus.

$$u = \frac{1}{1 + Ce^{-\tau}}$$

$$u(0) = u_0 = \frac{1}{1 + C}$$

$$C = \frac{1 - u_0}{u_0}$$

$$\boxed{u(\tau) = \frac{1}{1 + \left(\frac{1 - u_0}{u_0}\right)e^{-\tau}}}$$

3.5.8. Prove  $\frac{dx}{d\tau} = rx + x^3 - x^5$ , where  $x = \frac{u}{U}$

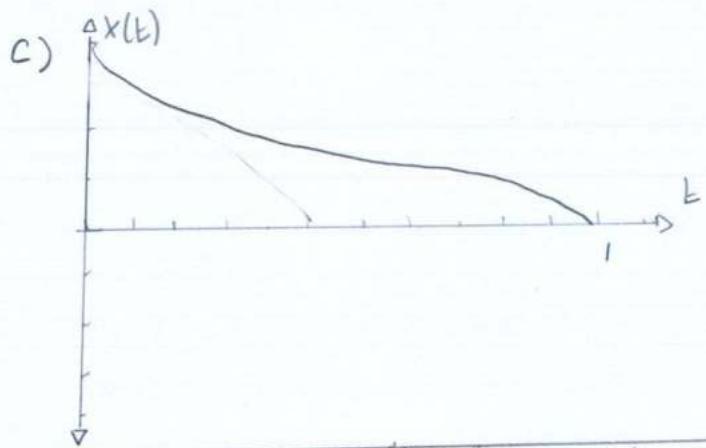
$$\tau = \frac{t}{T}$$

$$\frac{x}{T} \frac{dx}{d\tau} = aUx + bU^2x^3 - cU^4x^5$$

$$\frac{dx}{d\tau} = Tax + TbU^2x^3 - TcU^4x^5; \quad a = \frac{r}{T}, \quad b = \frac{1}{TU^2}, \quad c = \frac{1}{TU^4}$$

$$\boxed{\frac{dx}{d\tau} = rx + x^3 - x^5}$$

3.6.1. Figure 3.6.3b corresponds to Figure 3.6.1b; specifically, the relationship between  $y = h$ , and  $y = rx - x^3$ . The dotted lines support a single bifurcation to two bifurcations at  $h_c$ , then three when  $h > h_c$ . To answer the question, Figure 3.6.3b has information of  $h < 0$  and  $h > 0$ .



d) If  $\epsilon \ll 1$ , then  $\epsilon \ddot{x} + \dot{x} + x \approx \dot{x} + x$  and is a similar model to the boundary conditions.

e) Mechanical System An extremely viscous solution for an oscillating Newtonian device.

Electrical System An electrical system of the form  $v = Ri + L \frac{dv}{dt} + \frac{1}{c} \int i dt$

where  $\epsilon = \frac{1}{c} \ll 1$ .

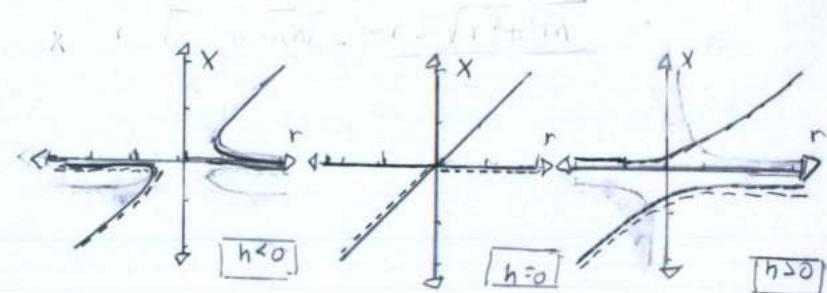


$$\dot{x} = h + rx - x^2 \quad 3.6.2. a)$$

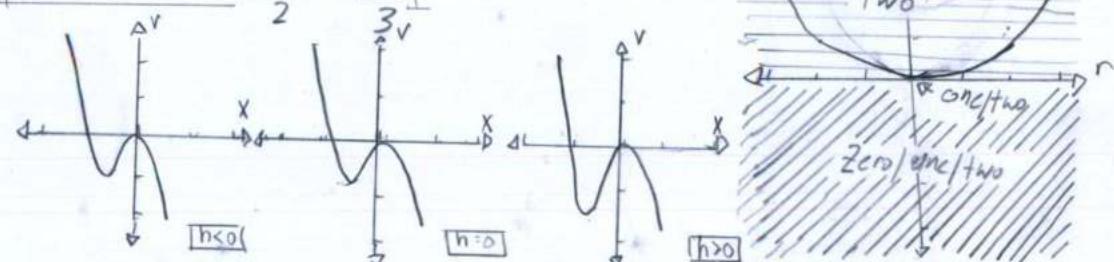
$h$	Bifurcations
$<0$	zero/One/two
$=0$	One/Two
$>0$	Two

$$2(-1)$$

b)  $(r, h)$  Plane



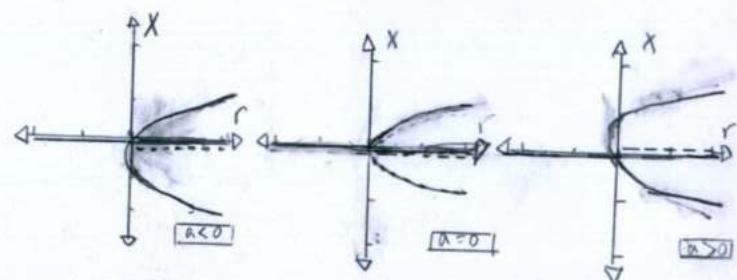
$$c) \frac{d}{dx}(rx - x^2) = r - 2x; x_{\max} = \frac{r}{2}; \frac{r^2}{2} - \frac{r^2}{4} = \frac{r^2}{4} = h_c$$



$$\dot{x} = rx + ax^2 - x^3$$

3.6.3 a)

$a$	Bifurcations
$<0$	one/two/three
$=0$	one/three
$>0$	one/two/three



b)  $(r, a)$  plane

$$\frac{d}{dx}(rx + ax^2 - x^3) = r + 2ax - 3x^2 = 0;$$

$$r + ax - x^2; a = \frac{x^2 - r}{x}$$

3.6.4 A small imperfection to a saddle-node bifurcation shifts the cusp either left or right.

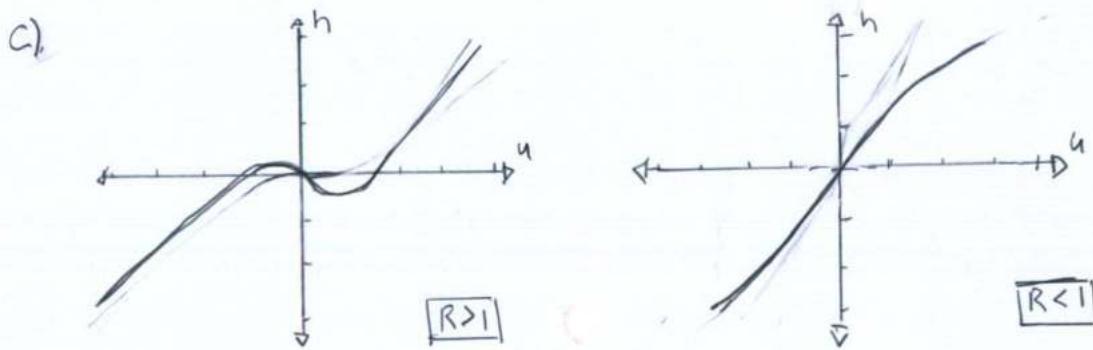
$$mg \sin \theta = Kx \left(1 - \frac{L}{\sqrt{x^2 + a^2}}\right) \quad 3.6.5 \text{ a) } F = -F_{\text{spring}} = F_g, x = F_g / mg \sin \theta = K(x - x \sin \theta) = K(x - x \cdot \frac{L}{\sqrt{x^2 + a^2}})$$

$$= Kx \left(1 - \frac{L}{\sqrt{x^2 + a^2}}\right)$$

b) Prove  $1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}}$

If  $1 - \frac{mg \sin \theta}{Kx} = \frac{L}{a \sqrt{\left(\frac{x}{a}\right)^2 + 1}}$ , then  $u = \frac{x}{a}$ ,  $R = \frac{L}{a \sin \theta}$ ,  $h = \frac{mg \sin \theta}{K a}$ .

and  $1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}}$



The variable  $h$ , as a function of  $u$ , has a single equilibrium point for both  $R > 1$  and  $R < 1$ .

d) If  $r = R - 1$ ,  $1 - \frac{h}{u} = \frac{r+1}{\sqrt{1+u^2}}$ ;  $u - h = \frac{(r+1)u}{\sqrt{1+u^2}}$ ;  $u\sqrt{1+u^2} - h\sqrt{1+u^2} = (r+1)u$

$$u\left(1 + \frac{1}{2}u^2 + O(u^4)\right) - h\left(1 + \frac{1}{2}u^2 + O(u^4)\right) = (r+1)u$$

$$u + \frac{u^3}{2} - h - \frac{h}{2}u^2 = ru + hu$$

$$h + ru + \frac{h}{2}u^2 - \frac{1}{2}u^3 \approx 0$$

e)  $h\left(1 + \frac{u^2}{2}\right) = \frac{1}{2}u^3 - ru$

$$\frac{d}{du} h\left(1 + \frac{u^2}{2}\right) = \frac{d}{du}\left(\frac{1}{2}u^3 - ru\right); \quad hu = \frac{3}{2}u^2 - r; \quad r_{\max} = \frac{3}{2}u^2 - hu$$

$$h\left(1 + \frac{u^2}{2}\right) = \frac{1}{2}u^3 - \left(\frac{3}{2}u^2 - hu\right)u; \quad h + \frac{hu^2}{2} = \frac{1}{2}u^3 - \frac{3}{2}u^2 + hu^2$$

$$h\left(1 - \frac{1}{2}u^2\right) = -u^3; \quad h = \frac{2u^3}{u^2 - 2}$$

$$r_{\max} = \frac{3}{2}u^2 - hu = \frac{3}{2}u^2 - \left(\frac{2u^3}{u^2 - 2}\right)u$$

$$= \frac{3}{2}u^2 - \frac{2u^4}{u^2 - 2}$$

$$= \frac{u^4 + 3u^2}{2(1-u^2)} \quad [= R-1]$$

f)  $1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}}$ ;  $\frac{d}{du}\left(1 - \frac{h}{u}\right) = \frac{d}{du}\left(\frac{R}{\sqrt{1+u^2}}\right); \quad \frac{h}{u^2} = -\frac{1}{2} \frac{R(2u)}{(1+u^2)^{3/2}}$

$$2 \cdot h(1+u^2)^{3/2} = -R \cdot u^3; \quad R = -\frac{h(1+u^2)^{3/2}}{u^3}$$

$$1 - \frac{h}{u} = \frac{-h(1+u^2)^{3/2}}{u^3 \sqrt{1+u^2}} = -\frac{h(1+u^2)}{u^3}; \quad u - h = -\frac{h(1+u^2)}{u^2}$$

$$h^3 - hu^2 = -h(1+u^2) ; \quad h^3 = -h - hu^2 + hu^2 ; \quad h = -u^3$$

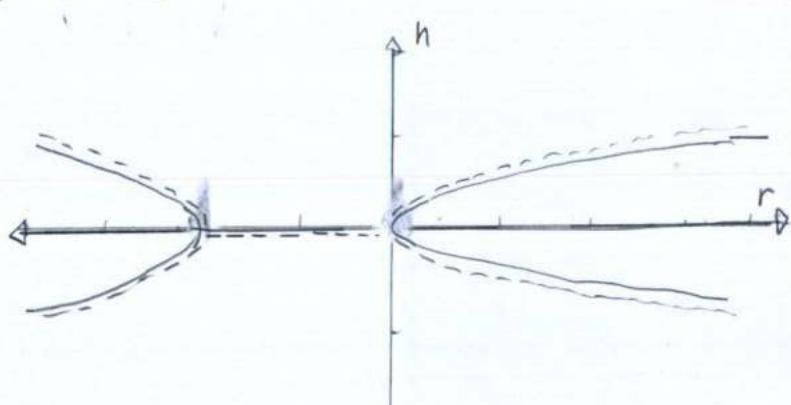
$$R = \frac{-h(1+u^2)^{3/2}}{u^3} = (1+u^2)^{3/2}$$

$$\lim_{u \rightarrow 0} h = -u^3 \approx \frac{2u^3}{u^2 - 2} \approx u^3$$

$$\lim_{u \rightarrow 0} R = (1+u^2)^{3/2} \approx \frac{u^4 + 3u^2}{2(1-u^2)} + 1 = r+1$$

$$g) R = (1+u^2)^{3/2} = r+1 ; \quad r(u) = (1+u^2)^{3/2} - 1 ; \quad u = \sqrt{(r+1)^{2/3} - 1}$$

$$h = -u^3 = \pm (\sqrt{(r+1)^{2/3} - 1})^3$$



$$h) h = -u^3 = -\left(\frac{x}{a}\right)^3 = \frac{m g \sin \theta}{k a}$$

$$R = \left(1 + \left(\frac{x}{a}\right)^2\right)^{3/2} = \frac{La}{a}$$

The bifurcation plot represents the points of stability for the oscillating system.

$$\tau \dot{A} = EA - gA^3$$

3.6.b.  $A(t)$  = Amplitude;  $\tau$  = typical timescale;  $E$  = dimensionless parameter

$$\tau \dot{A} = EA - gA^3 - KA^3$$

3.6.b. Supercritical:  $g > 0$ , subcritical:  $g < 0$ ,  $K > 0$

"Landau Equation"

a) Landau's Equation describes the change of amplitude for a fluid system

$$b), \tau \dot{A} = EA - gA^3 - KA^5 ; \text{ if } g=0, \text{ then } \tau \dot{A} = EA - KA^3 ; \quad A = \sqrt{\frac{E}{K}}$$

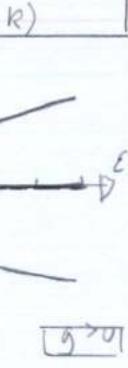
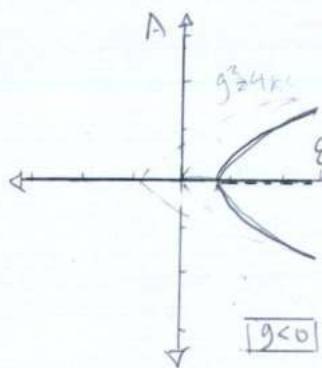
The function  $A(E)$  is a tricritical bifurcation because

$A=0$  is a solution; in addition to,  $A = +\sqrt{\frac{E}{K}}$ , and  $A = -\sqrt{\frac{E}{K}}$

$$c) \tau \dot{A} = h + EA - gA^3 - KA^5 ; \text{ An approximation } h \approx 0, \quad 0 = EA - gA^3 - KA^5$$

$$= E - gA^2 - KA^4$$

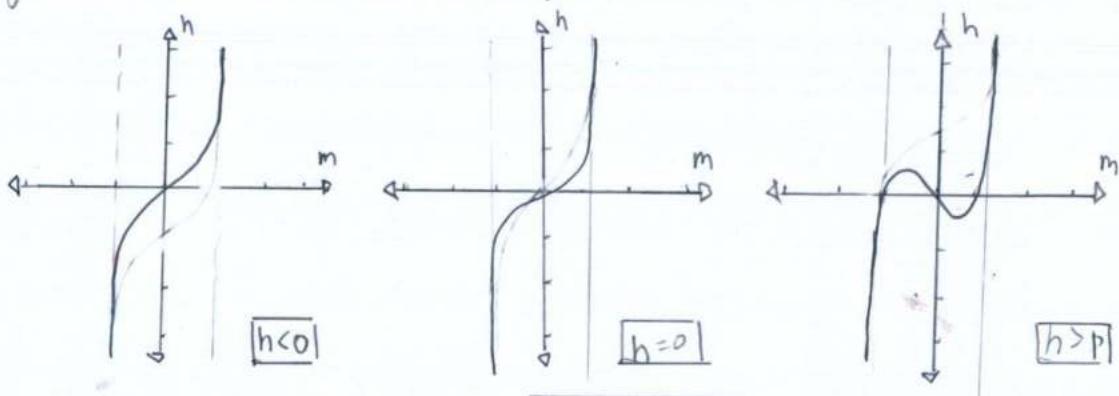
$$\text{Where } A^2 = b ; \quad 0 = E - gb - kb^2 ; \quad A = \sqrt{\frac{g \pm \sqrt{g^2 - 4(-K)(E)}}{2(-K)}}$$



d) The graphs appearance represent the relationship of amplitude vs. time, and if  $\epsilon$  is large, then the first order term approaches the steady state condition more rapidly.

$$m = \left| \frac{1}{N} \sum_{i=1}^N s_i \right| \quad 3.6.7 \text{ a)}$$

$$h = T \tanh^{-1}(m) - J_n m$$



$$b) h = T \tanh^{-1}(m) - J_n m ; \text{ If } h=0, \text{ then }$$

$$T_c = \frac{J_n m}{\tanh^{-1}(m)}$$

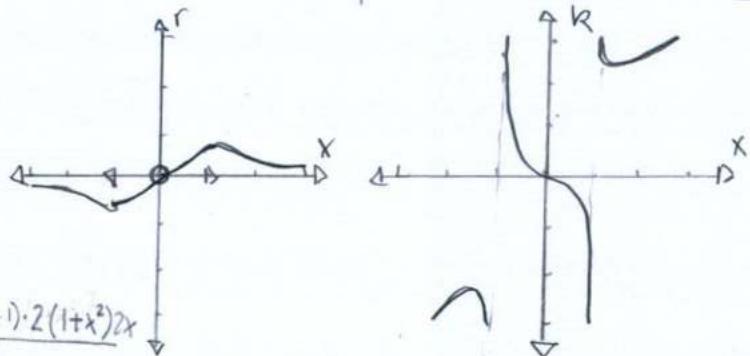
$$\frac{dx}{dt} = rx(1-\frac{x}{K}) - \frac{x^2}{1+x^2} \quad 3.7.1, @x^2=0 ; 0 < rx - (\frac{1}{K} + \frac{1}{1+x^2})x^2 : (\frac{1}{K} + \frac{1}{1+x^2})x < r ; \boxed{0 < r \text{ is positive and unstable.}}$$

$$r = \frac{2x^3}{(1+x^2)^2}$$

$$3.7.2. \quad \begin{array}{l} \lim_{x \rightarrow 1} r = \frac{2}{4} \\ \lim_{x \rightarrow \infty} r = 0 \end{array}$$

$$K = \frac{2x^3}{x^2 - 1}$$

$$\begin{array}{l} \lim_{x \rightarrow 1} K = \infty \\ \lim_{x \rightarrow \infty} K = \infty \end{array}$$



$$b) r = \frac{(x^2-1)K}{(1+x^2)^2} ; \frac{dr}{dx} = \frac{2x(1+x^2)^2 - (x^2-1) \cdot 2(1+x^2)2x}{(1+x^2)^4}$$

$$= \frac{2x(1+x^2)^2 - 4x^3(1+x^2) + 4x(1+x^2)}{(1+x^2)^4} = 0$$

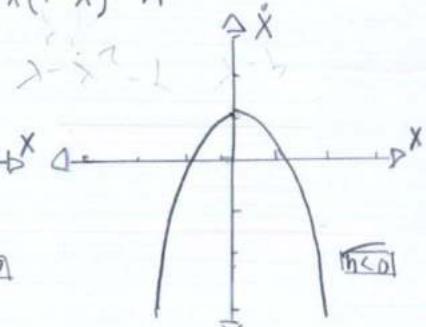
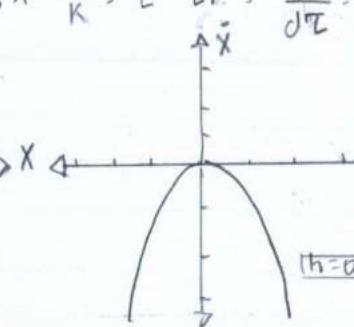
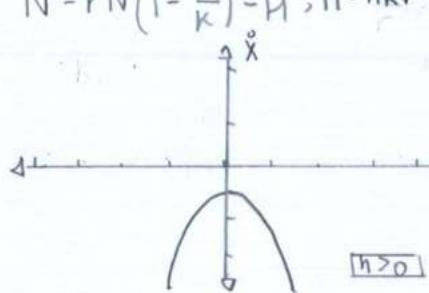
$$= 2x(1+x^2) - 4x^3 + 4x = 2(1+x^2) - 4x^2 + 4 = (1+x^2) - 2x^2 + 2 = 0$$

$$3 - x^2 = 0 ; x = \sqrt{3}$$

$$r_{max} = \frac{(3-1)K}{(1+3)^2} = \frac{1}{9} K_{max} ; r_{max} = \frac{2 \cdot 3^{3/2}}{(1+3)^2} = 0.6495 ; K_{max} = 5.1961$$

$$\frac{dX}{dt} = X(1-X) - h \quad 3.7.3. \quad N = rN(1-\frac{N}{K}) - H ; h = HKr ; X = \frac{N}{K} ; t = tr ; \frac{dX}{dt} = X(1-X) - h$$

a)



$$\therefore 0 = -x^2 + x - h ; x = \frac{-1 \pm \sqrt{1-4(-1)(-h)}}{2(-1)} = \frac{1 \pm \sqrt{1-4h}}{2} ; \boxed{h_c = 0}$$

d) The long-term behavior of the fish population

is to reduce the total population as population rises!

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) - H \frac{N}{A+N}$$

3.7.4. a) The variable  $A$  could represent the amount of fish in a school, and if  $A$  is large, then less fish are harvested.

b)  $x = \frac{N}{K}; T = \frac{1}{r}; h = HRK; a = A$

c)  $\frac{dx}{dT} = x(1-x) - h \frac{x}{a+x} = 0; x(1-x)(a+x) = (x-x^2)(a+x) = ax + x^2 - x^2a - x^3$

$$0 = (a-h)x^2 + (1-a)x^3 - x^3$$

$$0 = (a-h)x^2 + (1-a)x^3 - x^3$$

$$x_1 = 0, x_{2,3} = \frac{-(1-a) \pm \sqrt{(1-a)^2 - 4(a-h)}}{2(-1)}$$

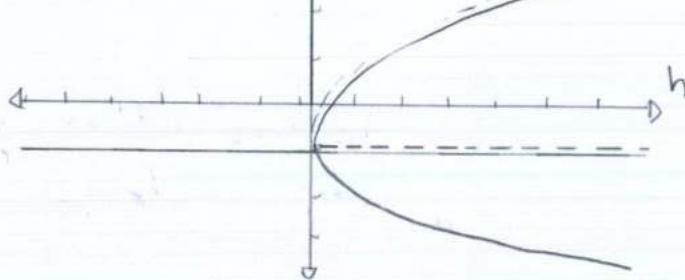
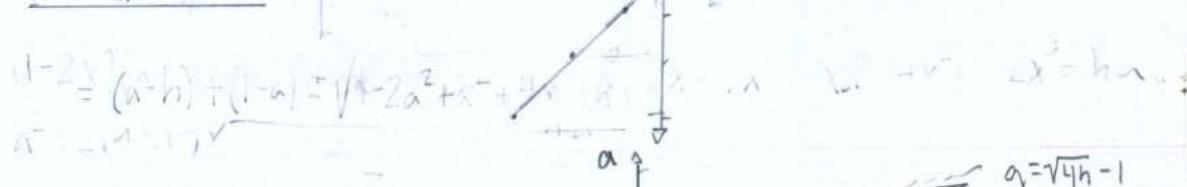
$$= \frac{(1-a) \pm \sqrt{(1-a)^2 + 4(a-h)}}{2}$$

Fixed Point	$a > h + u$	$a < h$
$x=0$	unstable	stable
$\frac{(1-a) + \sqrt{(1-a)^2 + 4(a-h)}}{2}$	stable	/
$\frac{(1-a) - \sqrt{(1-a)^2 + 4(a-h)}}{2}$	stable	/

d) At  $x=0$ , when  $h=a$ , the half-node indicates a transcritical bifurcation is about to occur when  $h$  becomes less than  $a$ .

e) The graph shows a supercritical bifurcation for  $h=a$ .  
 $h = (a+1)^2$ .

f)  $a = \frac{h}{x-1} - x$



$$\dot{g} = R_1 S_0 - k_2 g + \frac{k_3 g^2}{R_4 + g^2} \quad 3.7.5.$$

a)  $\frac{k_4}{k_3} \frac{dg}{dt} = \frac{R_4 \cdot R_1}{R_3} S_0 - \frac{k_4^2 R_2}{R_3} g + \frac{\left(\frac{g^2}{k_4}\right)}{1 + \left(\frac{g}{k_4}\right)^2}; X = \frac{g}{k_4}; r = \frac{k_4}{R_2}; S = \frac{R_1}{R_3}; S_0$

$$\frac{dX}{dT} = S - rX + \frac{X^2}{1 + X^2}$$

$$T = \left(\frac{k_4}{R_2}\right) T$$

b)  $0 = -rX + \frac{X^2}{1 + X^2}; rX = \frac{X^2}{1 + X^2}; r(1 + X^2) = X; rX^2 - X + r = 0$

$$X_{1,2} = \frac{1 \pm \sqrt{1 + 4r^2}}{2r}$$

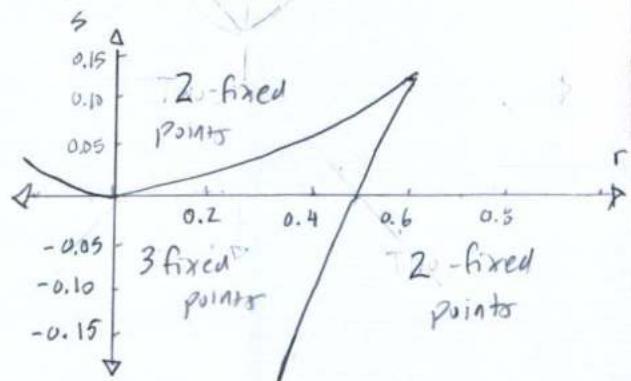
C)  $g(0)=0$ ;  $\frac{dg}{dt} = k_1 s_0 - k_2(0) + \frac{k_3(0)}{k_4 e^{t(0)}} = k_1 s_0$ ;  $g = k_1 s_0 t$ ;  $g(t)$  increases with additional  $s_0$ .

IF  $s_0$  is large, then gene production has higher likelihood of rising.

d)  $\frac{d}{dx}(s - rx + \frac{x^2}{1+x^2}) = -r + \frac{2x}{(1+x^2)^2} = 0$ ;  $r = \frac{2x}{(1+x^2)^2}$ ;  $s = \left(\frac{2x}{(1+x^2)^2}\right) + \frac{x^2}{1+x^2} = 0$

e) Parametric plot of  $(r, s)$

$$S = \frac{x^2(1-x^2)}{(x^2+1)^2}$$



$$\begin{aligned} \dot{x} &= -Rxy \\ \dot{y} &= Kxy - ly \\ \dot{z} &= ly \end{aligned}$$

3.7.b.  $x(t)$  = number of healthy people

$y(t)$  = number of sick people

$z(t)$  = number of dead people.

a)  $\dot{N} = \dot{x} + \dot{y} + \dot{z} = -Rxy + Kxy - ly + ly = 0$ ; therefore  $N = x + y + z$ .

b)  $\dot{x} = -Rxy$ ;  $\dot{z} = ly$ ;  $\dot{x} = -Rx \frac{dz}{dt} \left(\frac{1}{t}\right)$ ;  $\ln x = -\frac{Kz}{l} + C$ ;  $x(t) = C e^{-\frac{Kz}{l}} = X_0 e^{-\frac{Kz}{l}}$

c)  $\dot{z} = ly = l[N - x - z] = l[N - z - X_0 e^{-\frac{Kz}{l}}]$

d)  $u = \frac{Kz}{l}$ ;  $b = \frac{l}{KX_0}$ ;  $a = \frac{lN}{Kz - lX_0}$ ;  $T = \frac{l}{KX_0} t$

e) IF  $R, l, N$ , and  $X_0$  are positive, then both  $a$  and  $b$  are positive.

$b$	$a = 1$	$> 1$
$= 0$	$@ u=0$ , unstable	$@ u<0$ , unstable
$> 0$	$@ u=0$ , unstable; $@ u>0$ , stable	$@ u<0$ , unstable; $@ u>0$ , stable

g)  $\ddot{u} = -b\dot{u} + ue^{-u} = 0$ ;  $u = -\ln(b)$ ;  $\dot{u} = a - b\ln(b) + b^{-2}$

$\ddot{z} = b\dot{y} = b(Rxy - ly) = b(Rx - l)y = 0$ ;  $y = Rx$ ;  $y = Ce^{\frac{R}{l}t}$

h)  $b < 1$ ;  $\ddot{u} = -b + ue^{-u}$ ; Through plotting of  $b$  and  $ue^{-u}$  at time zero,  $b > ue^{-u}$ ; thus,  $\dot{u}$  is increasing.

$t_{peak}$ :  $\ddot{u} = -b + ue^{-u} = 0$ ;  $\dot{u} = e^{-u} - u^2 e^{-u} = 0$ ;  $u = 1$

$u = a - bu - e^{-u}$  @  $u=1$ ;  $u = a - b - \frac{1}{e}$ ;  $u = (a - b - \frac{1}{e})T = 1$ ;  $T = \frac{1}{a - b - \frac{1}{e}} (\frac{l}{R X_0})$

$\lim_{u \rightarrow \infty} \dot{u} = \lim_{u \rightarrow \infty} [a - bu - e^{-u}] = -b \cdot \infty - \frac{1}{e^\infty} = -\infty$

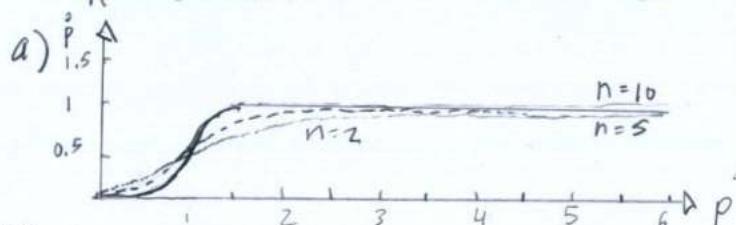
i) If  $b > 1$ ,  $\dot{u} = u - bu - e^{-u}$ ;  $\ddot{u} = -bu + u e^{-u}$ .  $\ddot{u} \geq 0$  does not exist. a logical maximum/minimum/inflection for an epidemic with peak at zero.

j) The variable  $b$  is assigned as  $\frac{k}{R\chi_0}$ . If  $b=1$ , then  $\frac{\partial}{\partial x_0} = 1$ . A threshold condition is when the rate of dying persons is greater than the rate of infection.

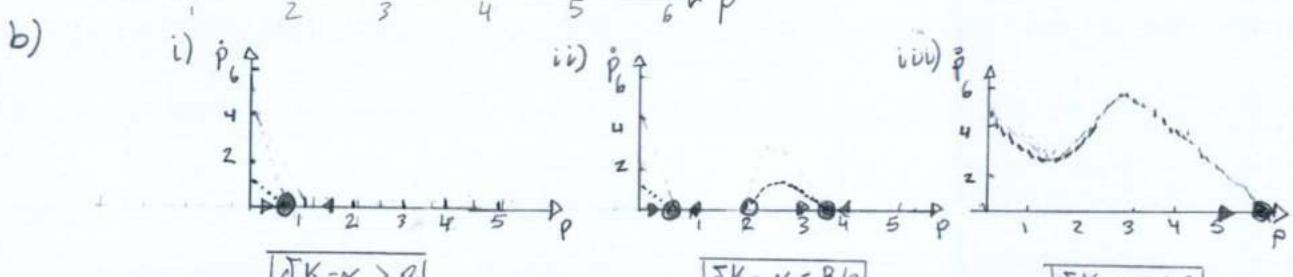
k) Autoimmuno deficiency is a disease following human immunodeficiency virus. The delayed onset from infection is time-dependent, showing that a model likely requires a time-dependent term or relationship.

$$\dot{p} = \kappa + \frac{\beta p^n}{K^n + p^n} - \delta p \quad 3.7.7 \quad \kappa = \text{Basal Transcription Rate}; \beta = \text{Maximal Transcription Rate}$$

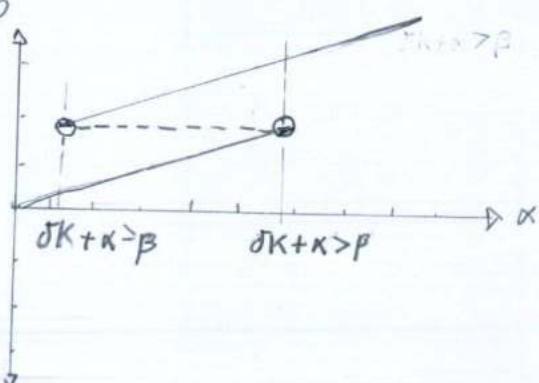
$K = \text{Activation Coefficient}; \delta = \text{Decay Rate of Protein.}$



The shape of the function is a Sigmoid about the point  $(1, 0.5)$  for  $K=1, b=1$ .



c) Assume  $\delta K > \beta$ ,  $\kappa = -\frac{p^n}{K^n + p^n} + \delta p$  at  $x \geq 0$



d) When protein levels are dependent upon  $\kappa$ , then up till  $\kappa > \delta K$ , protein production rate decreases until zero. While  $\kappa > \delta K$  is above a threshold  $\delta K - \beta > \kappa$ , there is active production of further protein, proving concentration regions of protein production.

$$\dot{A}_p = K_p S A + \beta \frac{A_p^n}{K^n + A_p^n} - K_d A_p ; \quad A = \text{unphosphorylated} ; \quad A_p = \text{phosphorylated} ; \quad A_T = A + A_p$$

concentration concentration.

$K_p$  = phosphorylation rate;  $K_d$  = dephosphorylation rate.

$$\text{Assume } K = A_T / 2 ; \quad \beta = K_d A_T$$

$$3.7.8a) \quad X = A_p / K ; \quad T = K_d t ; \quad S = K_p S / K_d ; \quad b = \beta / (K_d K)$$

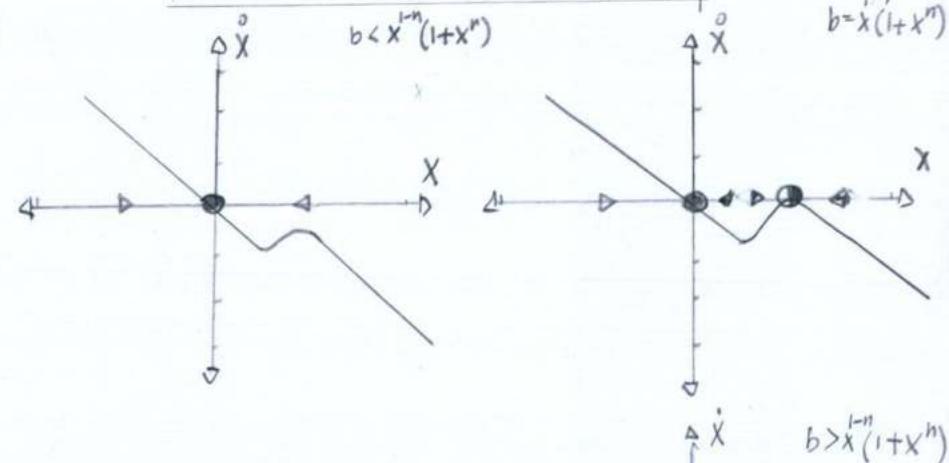
$$K_X \frac{dX}{dT} = K_d S A + K_d K \cdot b \cdot \frac{K^n X^n}{K^n + K^n X^n} - K_d K X$$

$$\frac{dX}{dT} = \frac{S A}{K} + b \frac{X^n}{1 + X^n} - X = \frac{S (A_T - A_p)}{K} + b \frac{X^n}{1 + X^n} - X$$

$$= \frac{S (2K - KX)}{K} + b \frac{X^n}{1 + X^n} - X$$

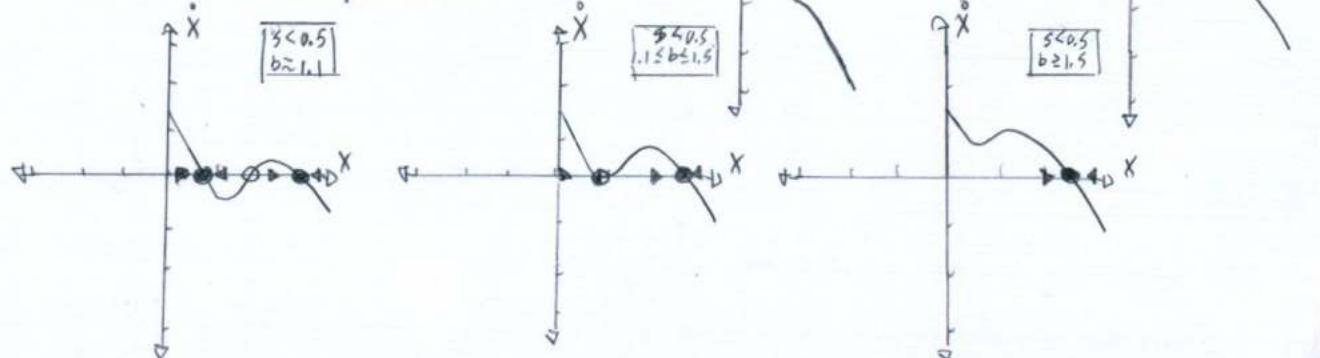
$$= S (2 - X) + b \frac{X^n}{1 + X^n} - X$$

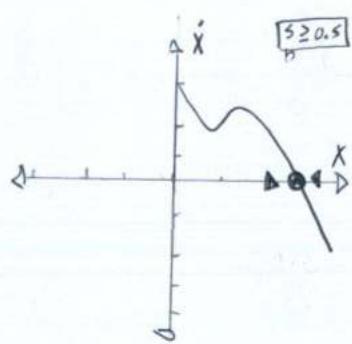
b) If  $S = 0$ , then



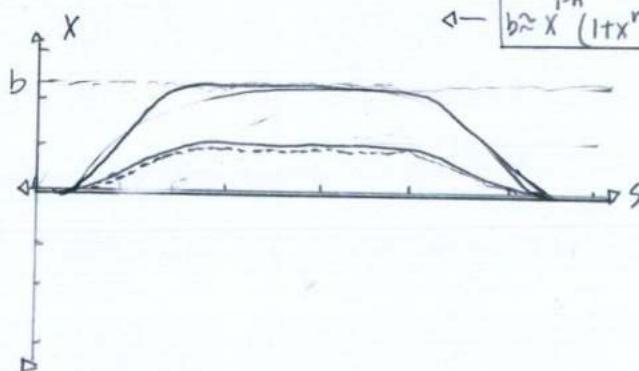
c) If  $S > 0$ , then a variety of bifurcations are produced.

$b \setminus S$	$\leq 0.5$	$\geq 0.5$
$b \leq 1$	1, Stable	
$b \geq 1$	2, Stable Half node	
$b \approx 1.1$	3, Stable unstable Stable	1, Stable
$1.1 \leq b \leq 1.5$	2, Half node Stable	
$b \geq 1.5$	1, Stable	





d)



Translation of bifurcation plot occurs near the left box.

$$b \approx x^{1-n} / (1+x^n)$$

Incorrect 11

## Chapter 4: Flows on the circle

$\dot{\theta} = \sin(a\theta)$  4.1.1. The real values of  $a$ , which give a well-defined vector field, on a circle for the function,  $\dot{\theta} = \sin(a\theta)$ , are fixed to  $n\pi$ , where  $n \in \mathbb{Z}$ .

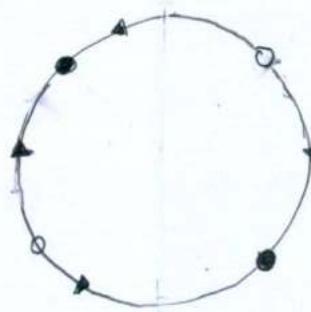
$\dot{\theta} = 1 + 2\cos\theta$  4.1.2. [Fixed points]  $\theta = \cos^{-1}(-\frac{1}{2})$

[Phase Portrait]

$$= \frac{2}{3}\pi, \frac{5}{3}\pi, \dots, (n + \frac{2}{3})\pi \text{ "stable"}$$

$$= \frac{4}{3}\pi, \frac{7}{3}\pi, \dots, (n + \frac{4}{3})\pi \text{ "unstable"}$$

where  $n \in \mathbb{Z}$



$\dot{\theta} = \sin 2\theta$  4.1.3. [Fixed Points]  $\theta = \frac{\sin^{-1}(0)}{2}$

[Phase Portrait]

$$= 0\pi, 1\pi, 2\pi, \dots, (n\pi) \text{ "unstable"}$$

$$= \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \dots, (n + \frac{1}{2})\pi \text{ "stable"}$$

where  $n \in \mathbb{Z}$



$\dot{\theta} = \sin^3\theta$  4.1.4 [Fixed points]  $\theta = \sin^{-1}(0)$

$$= 0\pi, 2\pi, \dots, (2n)\pi \text{ "unstable"}$$

$$= 1\pi, 3\pi, \dots, (2n+1)\pi \text{ "stable"}$$

where  $n \in \mathbb{Z}$

[Phase Portrait]



$$\dot{\theta} = \sin \theta + \cos \theta$$

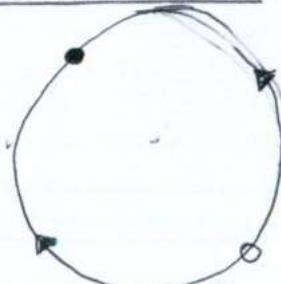
$$4.1.5. \dot{\theta} = -\cos \theta$$

$$\theta = \frac{3}{4}\pi, \frac{11}{4}\pi, \frac{19}{4}\pi, \dots, (n + \frac{3}{4})\pi \text{ "stable"}$$

$$= \frac{7}{3}\pi, \frac{15}{3}\pi, \frac{23}{3}\pi, \dots, (n + \frac{7}{3})\pi \text{ "unstable"}$$

where  $n \in \mathbb{Z}$

Phase Portrait



$$\dot{\theta} = 3 + \cos 2\theta$$

$$4.1.6. \dot{\theta} = \frac{\cos'(3)}{2}$$

$\theta = \text{undefined}$

Phase Portrait



$$\dot{\theta} = \sin k\theta$$

$$4.1.7. \dot{\theta} = \sin k\theta$$

$$\theta = \frac{\sin^{-1} 0}{k}$$

Where  $k \in \mathbb{N}$

Phase Portrait



$$\dot{\theta} = \cos \theta$$

$$4.1.8. a) \frac{dV}{d\theta} = \frac{d\theta}{dt} = \dot{\theta} = \cos \theta \Rightarrow V(\theta) = -\sin \theta$$

$$\theta = \sin^{-1}(0)$$

$$= 0, 2\pi, 4\pi, \dots, (2n)\pi \text{ "stable"}$$

$$= \pi, 3\pi, 5\pi, \dots, (2n+1)\pi \text{ "unstable"}$$

b)  $\dot{\theta} = 1$ ;  $V(\theta) = -\theta$  [The non-uniqueness of  $V(\theta)$  does not imply regularity for a vector field on a circle.]

c)  $\dot{\theta} = f(\theta)$  has a single-valued potential for periodic functions with periodic solutions of  $2\pi$  intervals.

4.1.9. Exercise 2.6.2 provided a contradiction that

$$\int_t^{t+\tau} f(x) \dot{x}(t) dt \neq \int_t^{t+\tau} f(x) \dot{x}(t+\tau) d(t+\tau)$$

Exercise 2.7.7 described a potential which could not oscillate because of the existence and uniqueness of  $f(x) = \frac{d(V-c)}{dx}$ .

Each of these arguments do not carry over to periodic solutions because another solution could be similar within  $2n\pi$  ( $n \in \mathbb{Z}$ ) intervals.

$$T_{\text{lap}} = \frac{2\pi}{\omega_1 - \omega_2} = \left[ \frac{1}{T_1} - \frac{1}{T_2} \right]^{-1} \quad \text{H.2.1. } T_1 = 3 \text{ sec}, T_2 = 4 \text{ sec.}$$

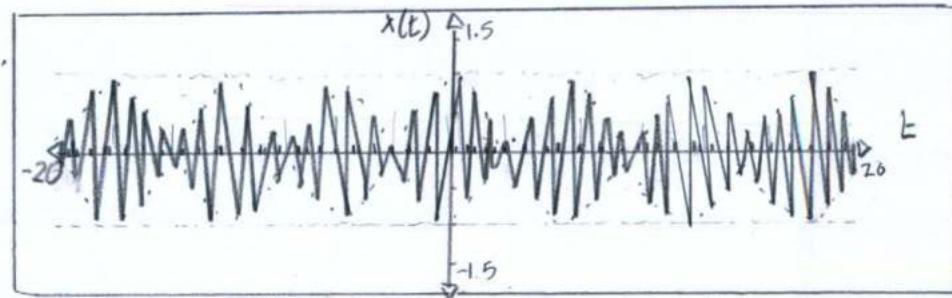
Common sense method:

# Rings	0	1	2	3	4	5
Bell #1	0	3	6	9	12	15
Bell #2	0	4	8	12	16	20

Bell #1 would ring four times  
 while Bell #2 three before  
 ringing together again.

Example 4.2.1 method:  $T_{\text{lap}} = \left[ \frac{1}{3 \text{ sec}} - \frac{1}{4 \text{ sec}} \right]^{-1} = [12 \text{ sec}]$

$x(t) = \sin 8t + \sin 9t \quad 4.2.2.$



a)  $T_{\text{lap}} = \left[ \frac{1}{8} - \frac{1}{9} \right]^{-1} = [72]$

b)  $x(t) = \sin 8t + \sin 9t = 2 \sin\left(\frac{17}{2}t\right) \cos\left(\frac{1}{2}t\right)$

4.2.3. 12:00pm is when long-hand angle is equal to short-hand.  
 Common sense method: short-hand period [ $T_1$ ] = 12 hours

long-hand period [ $T_2$ ] = 1 hour

$$T_{\text{lap}} = \left[ \frac{1}{1} - \frac{1}{12} \right]^{-1} = \frac{12}{11} \text{ hour}$$

Alternative method:  $x(t) = \sin(12t) + \sin(t) = 2 \sin\left(\frac{11}{2}t\right) \cos\left(\frac{13}{2}t\right)$

$$T_{\text{bottleneck}} = \int_{-\infty}^{\infty} \frac{dx}{r+x^2} \quad 4.3.1. \quad x = \sqrt{r} \tan \theta \quad ; \quad 1 + \tan^2 \theta = \sec^2 \theta$$

$$T = \int_{-\infty}^{\infty} \frac{dx}{r+r \tan^2 \theta} = \frac{1}{r} \int_{-\infty}^{\infty} \frac{dx}{1+\tan^2 \theta} = \frac{1}{r} \int_{-\infty}^{\infty} \frac{dx}{\sec^2 \theta}$$

$$= \frac{1}{r} \int_{-\infty}^{\infty} \frac{\sqrt{r} \sec^2 \theta}{\sec^2 \theta} d\theta = \frac{1}{\sqrt{r}} \theta \Big|_{-\infty}^{\infty} = \frac{1}{\sqrt{r}} \arctan\left(\frac{x}{\sqrt{r}}\right) \Big|_{-\infty}^{\infty} = \frac{\pi}{\sqrt{r}}$$

sum frequency ↑  
 difference frequency ↓

$$T = \int_{-\pi}^{\pi} \frac{d\theta}{\omega - \omega \sin \theta} \quad 4.3.2.$$

a)  $u = \tan \frac{\theta}{2}; du = \frac{1}{2} \sec^2 \frac{\theta}{2} d\theta; d\theta = \frac{2 du}{\sec^2 \frac{\theta}{2}} = \frac{2 du}{\sec^2 \frac{\theta}{2}}$

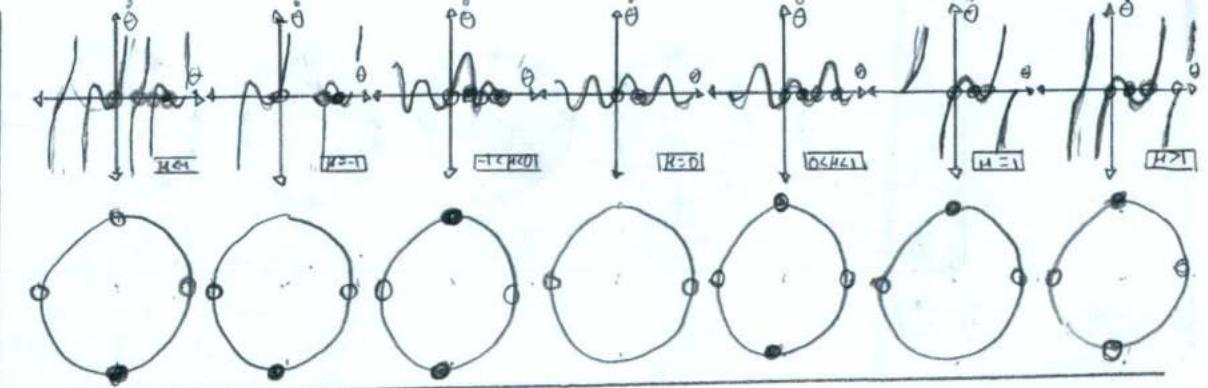
$$\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = 2 \frac{u}{\sqrt{1+u^2}} \frac{1}{\sqrt{1+u^2}} = \frac{2u}{1+u^2}$$

c)  $\lim_{\theta \rightarrow \pm\pi} \sin \theta = \lim_{u \rightarrow ?} \frac{2u}{1+u^2}; \text{ for } u \neq 0, u \rightarrow \pm\infty$

d)  $T = \frac{1}{\omega} \int_{-\infty}^{\infty} \frac{1}{1 - \frac{\omega}{\omega} \left[ \frac{2u}{1+u^2} \right]} \cdot \frac{2 \cdot du}{\sec^2 \left[ \arctan^{-1}(2u) \right]} = \frac{1}{\omega} \int_{-\infty}^{\infty} \frac{1}{1 - \frac{\omega}{\omega} \left[ \frac{2u}{1+u^2} \right]} \cdot \frac{2 du}{\left[ \frac{1}{1+u^2} - \frac{u^2}{1+u^2} \right]^2}$

e)  $T = \frac{1}{\omega} \int_{-\infty}^{\infty} \frac{1}{1 - \frac{\omega}{\omega} \left[ \frac{2u}{1+u^2} \right]} \cdot \frac{2 du}{\left( \frac{1}{1+u^2} - \frac{u^2}{1+u^2} \right)^2} = \frac{2u}{\left( \frac{1}{1+u^2} - \frac{u^2}{1+u^2} \right)^2} \cdot \frac{du}{\omega}$





$$r^{a-b} \frac{du}{d\tau} = r + r^{2a} u^2 \quad 4.3.9. \quad T_{\text{bifurcation}} \sim O(r^{-1/2})$$

a)  $O(r^n)$ ;  $x = r^a u$ , where  $u \sim O(1)$ ,  $t = r^b \tau$ , with  $\tau \sim O(1)$

$$\dot{x} = r + x^2 = r + (r^a u)^2 = r + r^{2a} u^2; \quad r^{a-b} \frac{du}{d\tau} = r + r^{2a} u^2$$

$$b) \boxed{r^{a-b} = r = r^{2a}; \quad a = \frac{1}{2}; \quad b = -\frac{1}{2}}$$

$$4.3.10. \quad x = r^a u; \quad t = r^b \tau; \quad r^{a-b} \frac{du}{d\tau} = r + r^{2a} u^2; \quad a = \frac{1}{2}; \quad b = -\frac{1}{2};$$

$$\frac{du}{d\tau} = 1 + r^{n-n} u^n; \quad r^b \tau = \tau$$

$$mL^2 \ddot{\theta} + b\dot{\theta} + mgL \sin \theta = T \quad 4.4.1. \quad \theta = 0 \text{ or } \theta \ll 1; \quad t = T\tau; \quad \frac{mL^2 d^2\theta}{T^2 d^2\tau} + \frac{b}{T} \frac{d\theta}{d\tau} + mgL \sin \theta = T$$

$$\frac{L^2}{gT^2} \frac{d^2\theta}{d^2\tau} + \frac{b}{mgL} \frac{d\theta}{d\tau} + \sin \theta = \frac{T}{mgL}$$

$$\frac{b}{mgL} = 1; \quad T = \frac{b}{mgL}$$

$$\frac{m^2 g L^3}{b^2} \frac{d^3\theta}{d\tau^2} + \frac{d\theta}{d\tau} + \sin(\theta) = \frac{T}{mgL}$$

$$\boxed{m^2 g L^3 \ll b^2}$$

$$\dot{\theta} = \gamma - \sin \theta$$

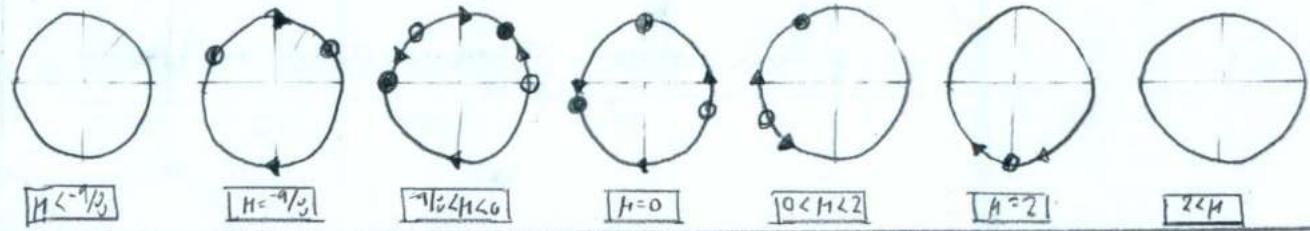
$$4.4.2. \quad \int \frac{d\theta}{w - \sin \theta} = dt; \quad t = - \int \frac{d\theta}{\frac{2\tan(\frac{\theta}{2})}{\tan^2(\frac{\theta}{2})+1} - a} = -2 \int \frac{du}{au^2 - 2u + a}; \quad \text{where } u = \tan\left(\frac{\theta}{2}\right)$$

$$du = \frac{\sec^2\left(\frac{\theta}{2}\right)d\theta}{2}$$

$$= -2 \int \frac{du}{\left(\sqrt{a}u - \frac{1}{\sqrt{a}}\right)^2 + a - \frac{1}{a}}; \quad \text{where } v = \frac{au-1}{\sqrt{a}\sqrt{a-1/a}} \quad d\theta = \frac{1}{u^2+1} \quad d\theta = \frac{1}{u^2+1}$$

$$dv = \frac{\sqrt{a}\sqrt{a-1/a}}{\sqrt{a-1/a}} du$$

$$\boxed{= \frac{-2}{\sqrt{a}\sqrt{a-1/a}} \int_{-\pi}^{\pi} \frac{dv}{v^2+1} = \frac{\arctan\left(\frac{a\tan(\frac{\theta}{2})-1}{\sqrt{a}\sqrt{a-1/a}}\right)}{\sqrt{a}\sqrt{a-1/a}} + C}$$



$$\dot{\theta} = \frac{\sin \theta}{\mu + \sin \theta}$$

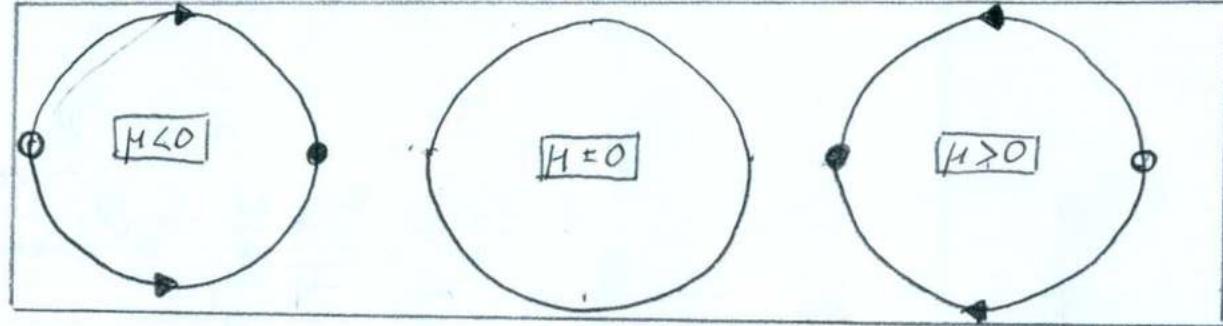
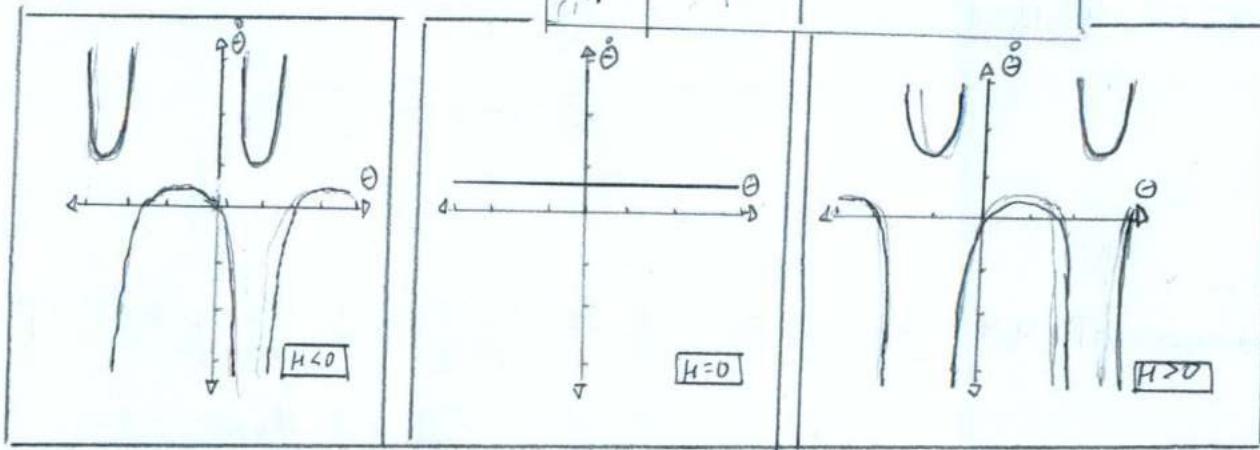
$$4.3.7 \quad \dot{\theta} = \frac{\sin \theta}{\mu + \sin \theta}$$

$$\theta = \mu + \sin \theta$$

$$H = \sin \theta$$

$$\theta = \sin^{-1}(-\mu)$$

$\theta$	$H$	Bifurcations
0	< 0	Two
$\pi$	= 0	
$\pi/2$	$> 0$	Zero
0	$> 1$	Two
$\pi$		



$$\dot{\theta} = \frac{\sin 2\theta}{1 + \mu \sin \theta}$$

$$4.3.8, \quad \dot{\theta} = \frac{\sin 2\theta}{1 + \mu \sin \theta}$$

$$\theta = 1 + \mu \sin \theta$$

$$\theta = \sin^{-1}(-\frac{1}{\mu})$$

$\theta$	$H$	Bifurcations
0	$\pi/2$	< -1
$\pi$	$3\pi/2$	= -1
0	$\pi$	Three
$\pi/2$	$3\pi/2$	$-1 < \mu < 0$
0	$\pi/2$	Four
$\pi/2$	$\pi$	$-\mu < 1 < 0$
$\pi/2$	$3\pi/2$	Three
0	$\pi/2$	$0 < \mu < 1$
$\pi/2$	$\pi$	Four
0	$\pi/2$	$= 1$
$\pi/2$	$\pi$	Three
0	$\pi/2$	$> 1$
$\pi/2$	$3\pi/2$	Four

$$\dot{\theta} = \mu + \cos\theta + \cos 2\theta \quad 4.3.5. \quad 0 = (\mu - 1) + \cos\theta + 2\cos^2\theta$$

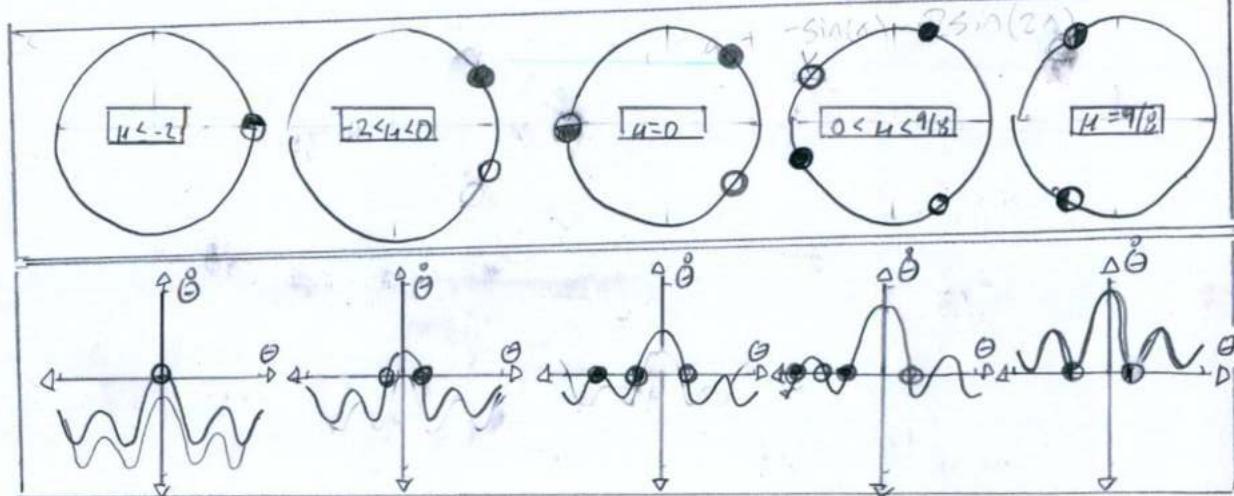
$$\cos\theta = \frac{-1 \pm \sqrt{1 - 4(\mu - 1)}}{2(2)}$$

$$= \frac{-1 \pm \sqrt{9 - 8\mu}}{4}$$

$$\theta = \cos^{-1}\left(\frac{-1 \pm \sqrt{9 - 8\mu}}{4}\right)$$

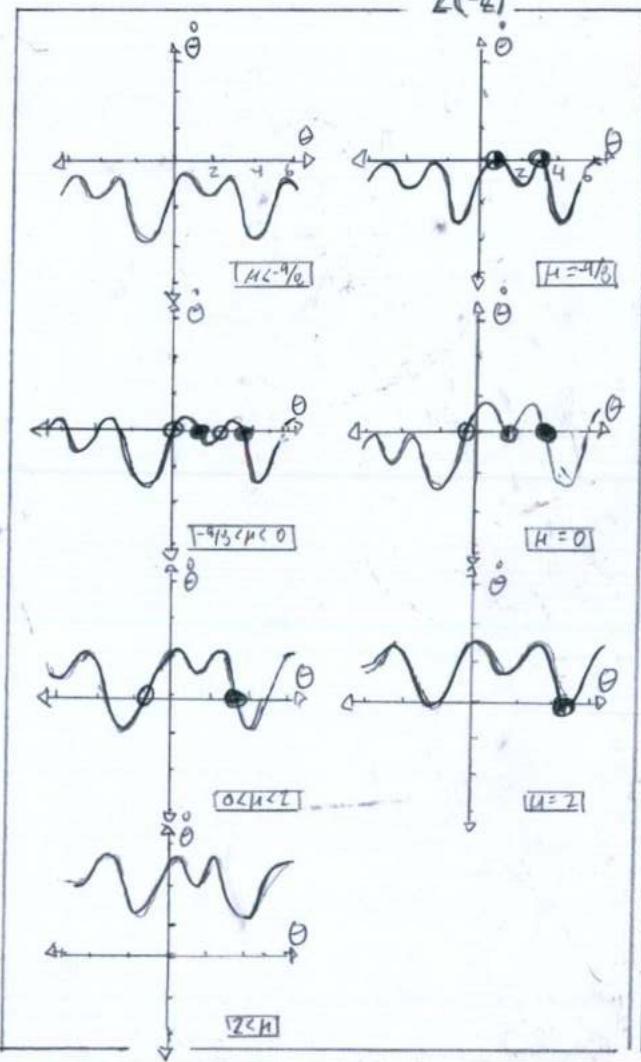
$\checkmark$

$\mu$	Bifurcations
$\leq -2$	- One
$-2 < \mu < 0$	Two
$\mu = 0$	Three
$0 < \mu < 9/8$	Four
$9/8$	Two



$$\dot{\theta} = \mu + \sin\theta + \cos 2\theta \quad 4.3.6. \quad \dot{\theta} = (\mu + 1) + \sin\theta - 2\sin^2\theta$$

$$\sin\theta = \frac{-1 \pm \sqrt{1 - 4(-2)(\mu + 1)}}{2(-2)}$$

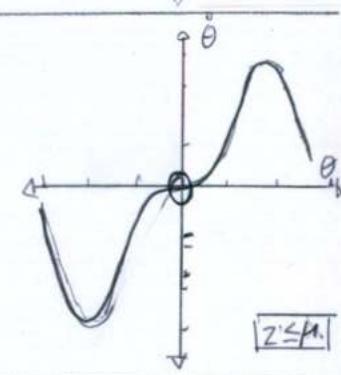
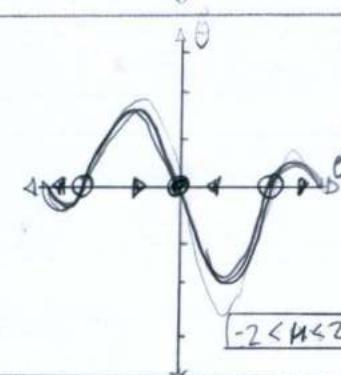
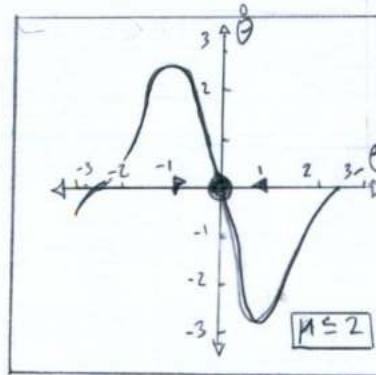
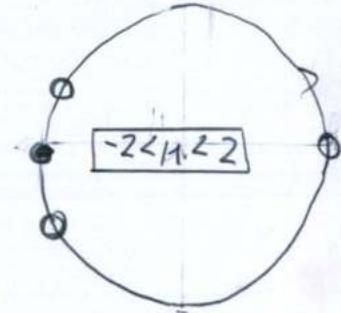
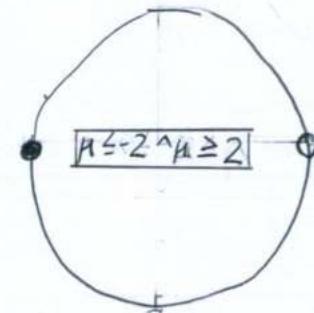


$\theta$	$\mu$	Bifurcations
$\pi/2$	$\mu < -9/8$	NA
$\arcsin(1/4)$	$\mu = -9/8$	Zero
$\arcsin(\pi - 1/4)$	$= -9/8$	Two
$\pi/2$	$\mu = 0$	Three
$\arcsin(1/4)$	$-\frac{9}{8} < \mu < 0$	Four
$2\pi$	$0 < \mu < 2$	Two
$\pi/2$	$\mu = 2$	One
$\frac{7}{6}\pi$	$\theta = 0$	Three
$\frac{7}{6}\pi < \theta < \frac{11}{6}\pi$	$0 < \mu < 2$	Two
$3\pi/2$	$\mu = 2$	One
NA	$2 < \mu$	Zero

$$\dot{\theta} = \mu \sin \theta - \sin 2\theta \quad 4.3.3. \quad \mu = \frac{\sin 2\theta}{\sin \theta} = 2 \cos \theta; \quad x = \cos^2(\frac{\mu}{2})$$

Phase Portrait: Saddle-node Bifurcation

$\mu$	Bifurcations
$\leq -2$	Two
$-2 \leq \mu \leq 2$	Four
$\geq 2$	Two



$$\dot{\theta} = \frac{\sin \theta}{\mu + \cos \theta}$$

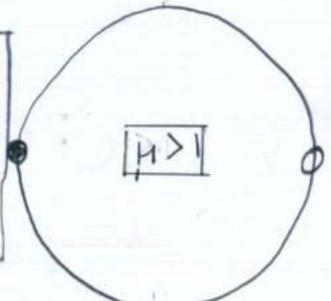
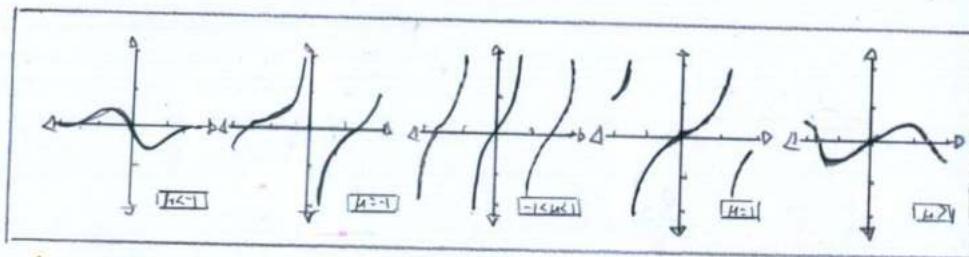
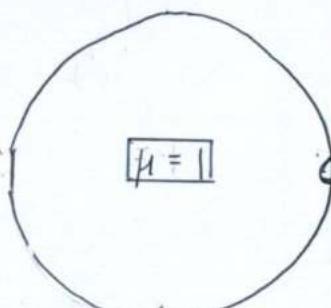
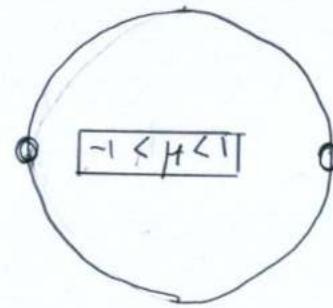
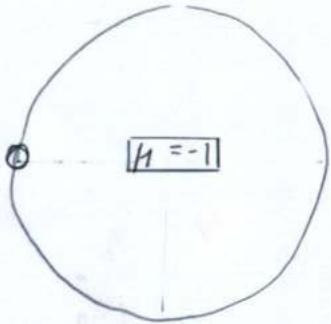
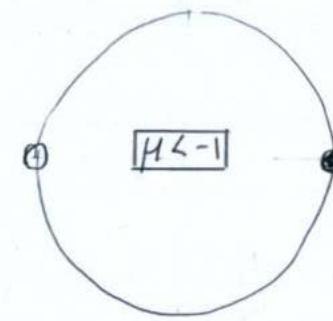
$$4.3.4. \quad \dot{\theta}(\mu + \cos \theta) = \sin \theta$$

$$\mu = -\cos \theta$$

$$\theta = \cos^{-1}(-\mu)$$

$\mu$	Bifurcations
$< -1$	Two
$= -1$	One
$-1 < \mu < 1$	Two
$= 1$	One
$> 1$	Two

Phase Portrait: Transcritical Bifurcation



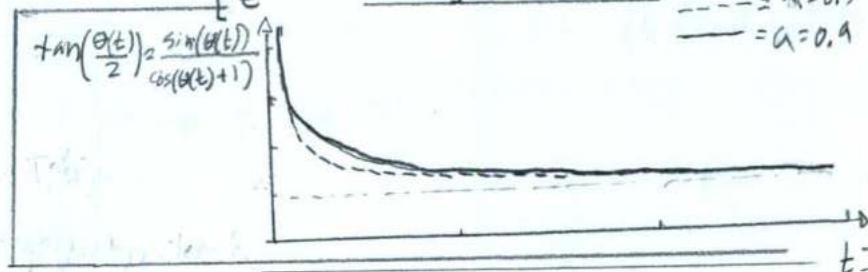
$$t = \ln \left( \frac{\left| \frac{a \sin(\theta)}{\cos(\theta) + 1} + \frac{-2\sqrt{1-a^2}-2}{2} \right|}{\left| \frac{a \sin(\theta)}{\cos(\theta) + 1} + \frac{2\sqrt{1-a^2}-2}{2} \right|} \right) / \sqrt{1-a^2}$$

$$e^{\sqrt{1-a^2} \cdot t} \circ \left| \frac{a \sin(\theta)}{\cos(\theta) + 1} + \frac{2\sqrt{1-a^2}-2}{2} \right| = \left| \frac{a \sin(\theta)}{\cos(\theta) + 1} - \frac{2\sqrt{1-a^2}-2}{2} \right|$$

$$\frac{a \sin(\theta)}{\cos(\theta) + 1} \left[ e^{\sqrt{1-a^2} \cdot t} - 1 \right] = -\frac{2\sqrt{1-a^2}-2}{2} \left[ e^{\sqrt{1-a^2} \cdot t} + 1 \right]$$

$$\frac{\sin(\theta)}{\cos(\theta) + 1} = \frac{(-2\sqrt{1-a^2}-2)}{2a} \cdot \frac{\left[ e^{\sqrt{1-a^2} \cdot t} + 1 \right]}{\left[ e^{\sqrt{1-a^2} \cdot t} - 1 \right]}$$

A graph of  $\sin(\theta(t))$  vs  $t$  represented as  $\tan\left(\frac{\theta(t)}{2}\right)$ .

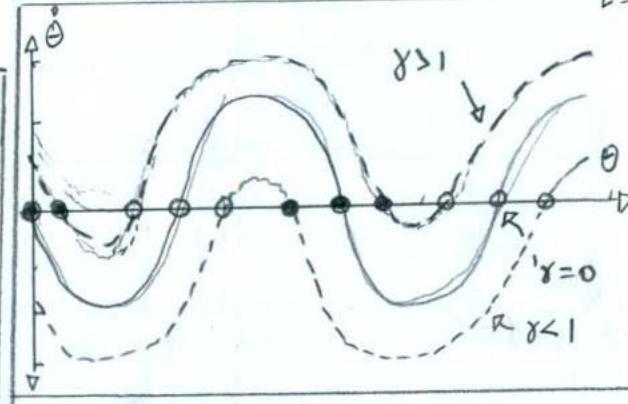


#### 4.4.3. $\ddot{\theta} = \gamma - \sin(\theta(t))$

Similar to Question 4.4.3,  $\gamma$

is the relationship of  $T/mgL$ .

If torque ( $T$ ) is greater than mass  $\times$  gravity  $\times$  length, then motion is directed; moreover, if torque ( $T$ ) is zero, then there is no direction.



$$b\ddot{\theta} + mgL \sin\theta = T - k\theta$$

a.  $\boxed{\theta = \left\{ 0, \frac{\pi k}{R}, \frac{2\pi k}{R}, \dots, \frac{n\pi k}{R} \right\}}$

b.  $b\ddot{\theta} + mgL \sin\theta = T - k\theta$

$$\frac{b}{mgL} \ddot{\theta} + \sin\theta = \frac{T - k\theta}{mgL}; \text{ if } T = mgL t; \gamma = \frac{T}{b} \quad ; \quad \boxed{H = \frac{1}{2} \frac{b^2}{mgL} \dot{\theta}^2 + \frac{1}{2} \frac{k^2}{mgL} \theta^2}$$

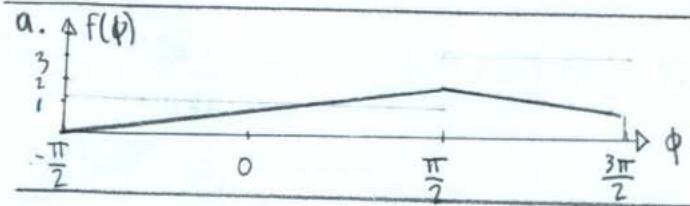
$$\ddot{\theta} + \sin\theta = \gamma + H\dot{\theta}; \quad \dot{\theta}' = \gamma - H\theta - \sin\theta$$

c. As the pendulum angle increases, the dampening lowers rate of angle change ( $\dot{\theta}'$ ).

d. As  $H$  is varied from 0 to  $\infty$ , then  $\dot{\theta}'$  is eqn to zero at  $\frac{\gamma - \sin\theta}{H}$ . The bifurcation type is subcritical.

$$\dot{\Theta} = \omega + A f(\Theta - \theta) \quad 4.5.1. \quad f(\phi) = \begin{cases} \phi & -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2} \\ \pi - \phi & \frac{\pi}{2} \leq \phi \leq \frac{3\pi}{2} \end{cases}$$

$$\dot{\Theta} = \Omega$$



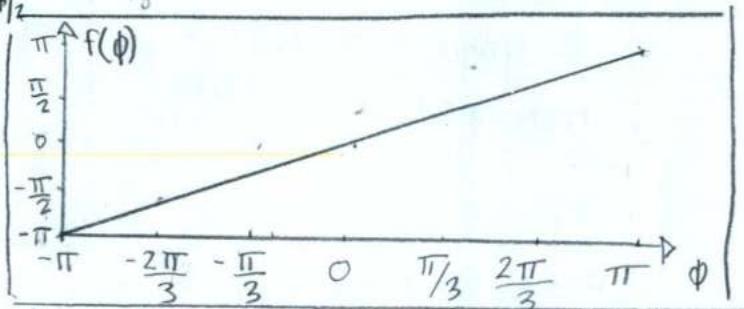
b. Range of Entrainment:  $-\frac{\pi}{2} \leq f(\phi) \leq \frac{\pi}{2}$

c.  $\phi^* = \dot{\Theta} - \Theta = \Omega - \omega - A \left[ \frac{\pi}{2} - |\Omega - \omega| \right] \leq \pi/2$

d.  $T_{\text{drift}} = \int_{0/\pi}^{2\pi} \frac{d\phi}{\Omega - \omega - A[\pi - \phi]} = \frac{1}{A} \int_{-\pi/2}^{\pi/2} \frac{du}{u} = \frac{2}{A} \ln \left( \frac{\Omega - \omega - A\pi/2}{\Omega - \omega + A\pi/2} \right) = \frac{2}{A} \ln \left( \frac{\Omega - \omega + A\pi/2}{\Omega - \omega - A\pi/2} \right)$

$$\dot{\Theta} = \omega + Af(\Theta - \theta) \quad 4.5.2$$

$f(\phi) = \phi \quad -\pi < \phi < \pi$



Range of Entrainment:  $-\pi < f(\phi) < \pi$

$\phi^* = \dot{\Theta} - \Theta = \Omega - \omega - A[\pi]; \quad |\Omega - \omega| \leq \pi$

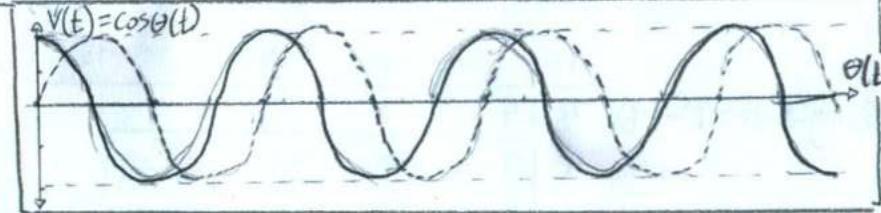
$T_{\text{drift}} = \int_{0/\pi}^{2\pi} \frac{d\phi}{\Omega - \omega - A[\phi]} = \frac{1}{A} \int_{\pi}^{\pi} \frac{du}{u} = \frac{1}{A} \ln \left( \frac{\Omega - \omega + A[\pi]}{\Omega - \omega - A[\pi]} \right)$

$$\dot{\Theta} = \mu + \sin \Theta$$

4.5.3. a) The 'rest state' and 'threshold' are described by the

nucleus ability to remain at rest or fire, respectively.

b)  $V(t) = \cos \Theta(t)$



4.6.1  $\beta = 0$

a.  $\phi' = \frac{I}{I_c} - \sin \phi(t)$

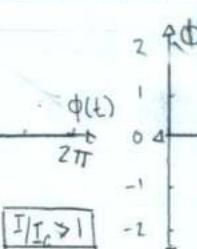
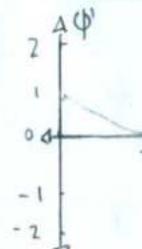
$$t = \int \frac{d\phi}{I/I_c - \sin \phi}$$

$$= - \int \frac{d\phi}{\sin \phi - I/I_c}$$

Method #1:  $u = \tan(\frac{\phi}{2})$ ;  $du = \sec^2(\frac{\phi}{2}) d\phi$

Method #2:

$$\begin{aligned} \sin \phi &= 2 \sin \frac{\phi}{2} \cos \frac{\phi}{2} \\ &= 2 \frac{u}{\sqrt{1+u^2}} \frac{1}{\sqrt{1+u^2}} = \frac{2u}{1+u^2} \end{aligned}$$



$|I/I_c \gg 1|$

$|I/I_c \gg 1|$

$$= -\frac{I_c}{I} \int \frac{du}{\frac{2u}{1+u^2} \left( \frac{I}{I_c} - 1 \right)} = \frac{2}{I} \int \frac{du}{\frac{2u}{1+u^2} \left( \frac{I}{I_c} - 1 \right) [u^2 + 1]}$$

$$= -\frac{2I_c}{I} \int \frac{du}{2u \left( \frac{I}{I_c} - u^2 - 1 \right)} = \frac{2I_c}{I} \int \frac{du}{(u - \frac{I}{I_c})^2 - (\frac{I}{I_c})^2 + 1}$$

$$= 2 \int \frac{du}{v^2 - 1} \quad \text{where } v = \frac{(u - I/I_c)^2}{\sqrt{1 - I/I_c}}$$

$$= 2 \frac{I}{I_c} \int \frac{\sqrt{1 - (I/I_c)^2}}{(1 - (I/I_c)^2) V^2 - (I/I_c)^2 + 1} dV = \frac{2 I/I_c}{\sqrt{1 - (I/I_c)^2}} \int \frac{dV}{V^2 + 1} = \frac{1}{\sqrt{1 - (I/I_c)^2}} \arctan(V)$$

$$= 2 \left( \frac{I}{I_c} \right) \frac{\arctan \left( \frac{V - I/I_c}{\sqrt{1 - (I/I_c)^2}} \right)}{\sqrt{1 - (I/I_c)^2}} = 2 \frac{I}{I_c} \frac{\arctan \left( \frac{\tan \frac{\phi}{2} - I/I_c}{\sqrt{1 - (I/I_c)^2}} \right)}{\sqrt{1 - (I/I_c)^2}} + C ; \text{ where } C=0$$

$$t = \frac{I}{I_c} \ln \left( \frac{1 - \frac{\tan \frac{\phi}{2} + I/I_c}{\sqrt{1 - (I/I_c)^2}}}{1 + \frac{\tan \frac{\phi}{2} - I/I_c}{\sqrt{1 - (I/I_c)^2}}} \right) \quad \checkmark \sqrt{(I/I_c)^2 - 1}$$

$$e^{(\frac{I_c}{I})\sqrt{(\frac{I}{I_c})^2 - 1} \cdot t} \circ \left( 1 + \frac{\tan \frac{\phi}{2} - I/I_c}{\sqrt{1 - (I/I_c)^2}} \right) = 1 - \frac{\tan \frac{\phi}{2} + I/I_c}{\sqrt{1 - (I/I_c)^2}}$$

$$\frac{\tan \frac{\phi}{2} - I/I_c}{\sqrt{1 - (I/I_c)^2}} \left[ e^{\frac{I_c}{I} \sqrt{(\frac{I}{I_c})^2 - 1} \cdot t} + 1 \right] = 1 - e^{\frac{I_c}{I} \sqrt{(\frac{I}{I_c})^2 - 1} \cdot t}$$

$$\sin(\frac{\phi}{2}) = \cos(\frac{\phi}{2}) \left[ \left( \frac{1 - e^{\frac{I_c}{I} \sqrt{(\frac{I}{I_c})^2 - 1} \cdot t}}{1 + e^{\frac{I_c}{I} \sqrt{(\frac{I}{I_c})^2 - 1} \cdot t}} \right) \sqrt{1 - (I/I_c)^2} + \frac{I}{I_c} \right]$$

$$\sin \phi = 2 \cos^2 \left( \frac{\phi}{2} \right) \left[ \sqrt{1 - (I/I_c)^2} \frac{\left( 1 - e^{\frac{I_c}{I} \sqrt{(\frac{I}{I_c})^2 - 1} \cdot t} \right)}{\left( 1 + e^{\frac{I_c}{I} \sqrt{(\frac{I}{I_c})^2 - 1} \cdot t} \right)} + \frac{I}{I_c} \right]$$

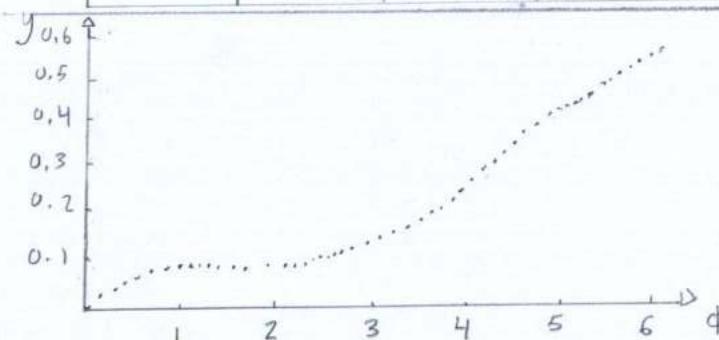
$$\dot{\phi} = \frac{I}{I_c} - \sin \phi$$

#### 4.6.2. Numerical Integration: Runge-Kutta 4th Order

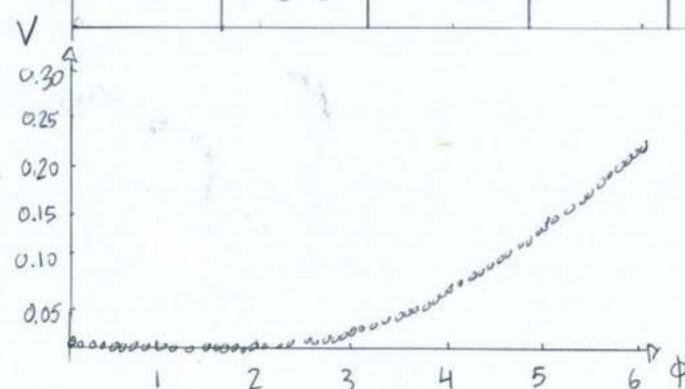
$\phi$	$k_1$	$k_2$	$k_3$	$k_4$
0.0	$\Delta h \cdot f(\phi)$	$\Delta h \cdot f(\phi + \frac{\Delta h}{2})$	$\Delta h \cdot f(\phi + \frac{\Delta h}{2})$	$\Delta h \cdot f(\phi + \Delta h)$
...	...	...	...	...
6.0	0.1379	0.1331	0.1331	0.1282

$$y_{n+1} = y_n + \frac{\Delta h}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

where  $\Delta h = 0.1$



$\phi$	$k_1$	$k_2$	$k_3$	$k_4$
0.0	$\Delta t \cdot \Delta h f(\phi)$	$\Delta t \cdot \Delta h f(\phi + \frac{\Delta h}{2})$	$\Delta t \cdot \Delta h f(\phi + \frac{\Delta h}{2})$	$\Delta t \cdot \Delta h f(\phi + \Delta h)$
...	0 0 0	0 0 0	0 0 0	0 0 0
6.0	0.63	-0.0103	-0.0103	-0.0098

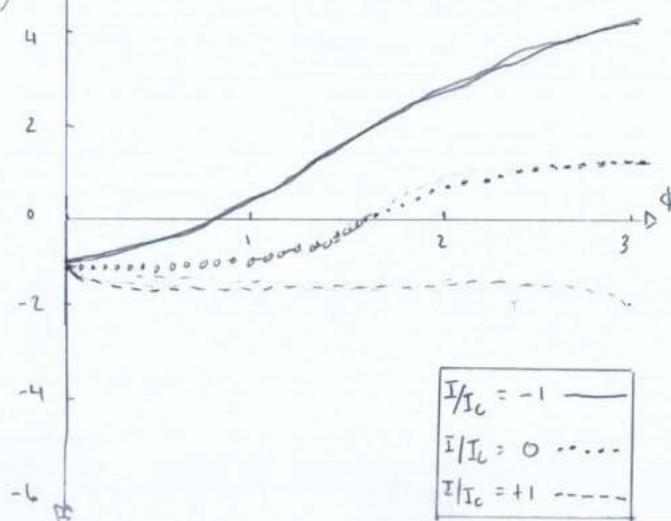


4.6.3.

$$a) V = -\dot{x} dx; V = -\dot{\phi} d\phi = -\left[\frac{I}{I_c} - \sin \phi\right] d\phi = -\left[\cos \phi + \frac{I}{I_c} \phi\right]$$

On a circle, solutions of  $2\pi$ -interval exist:  $\phi = \arcsin\left(\frac{I}{I_c}\right)$

b)



c) The increase of current ( $I$ ) lowers the potential ( $V$ ) per  $2\pi$  oscillation.

$$I_a = I_c \sin \phi_1 + V_1 / r$$

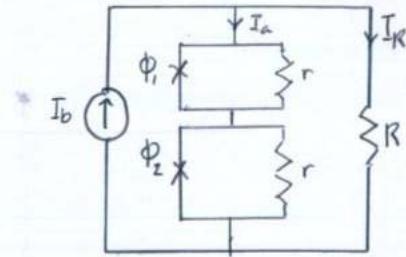
$$4.6.4. a) I_b = I_a + I_R$$

$$I_b = I_c \sin \phi_K + \frac{h}{2er} \dot{\phi}_K + \frac{h}{2eR} (\dot{\phi}_1 + \dot{\phi}_2)$$

b) Kirchoff's Law: Parallel Circuit

$$I_a = I_{a1} + I_{a2}, \quad I_a = I_{a2} + I_{aR}$$

$$= I_a \sin \phi_1 + \frac{V_1}{r}, \quad = I_a \sin \phi_2 + \frac{V_2}{r}$$



c) If  $k=1, 2$ , then

$$V_k = \begin{cases} \frac{h}{2er} \dot{\phi}_1 \\ \frac{h}{2eR} \dot{\phi}_2 \end{cases}$$

$$d) I_b = I_{a1} + I_{aR} + I_{a2} + I_R = I_c \sin \phi_K + \frac{h}{2er} \dot{\phi}_1 + \frac{h}{2eR} \dot{\phi}_2 + \frac{V_R}{R}$$

$$= I_c \sin \phi_K + \frac{h}{2er} [\dot{\phi}_1 + \dot{\phi}_2] + \frac{h}{2eR} \dot{\phi}_K$$

where  $K = 1, 2$ .

$$(e) I_b = I_c \sin(\phi_K) + \frac{h}{2er} \dot{\phi}_K + \frac{h}{2eR} \sum_{i=1}^N \dot{\phi}_i$$

$$= I_c \sin(\phi_K) + \frac{h}{2er} \dot{\phi}_K + \frac{Nr}{R} \left( I_c \sin(\phi_K) + \frac{h}{2er} \dot{\phi}_K - I_c \sum_{i=1}^N \sin(\phi_i) \right)$$

$$= \left( 1 + \frac{Nr}{R} \right) I_c \sin(\phi_K) + \left( \frac{h}{2er} + \frac{Nh}{2eR} \right) \dot{\phi}_K - \frac{r}{R} I_c \sum_{i=1}^N \sin(\phi_i)$$

$$\frac{h}{2er} \left( \frac{1}{R} + \frac{N}{2eR} \right) \dot{\phi}_K = I_b - \left( 1 + \frac{Nr}{R} \right) I_c \sin(\phi_K) + \frac{r}{R} I_c \sum_{i=1}^N \sin(\phi_i)$$

$$\frac{h(R+Nr)}{2erR} \dot{\phi}_K = I_b - \frac{R+Nr}{R} I_c \sin(\phi_K) + \frac{r}{R} I_c \sum_{i=1}^N \sin(\phi_i)$$

$$\dot{\phi}_K = \frac{2erR I_b}{h(R+Nr)} - \frac{2er}{h(R+Nr)} I_c \sin(\phi_K) + \frac{2er^2}{h(R+Nr)} I_c \sum_{i=1}^N \sin(\phi_i)$$

$$\Omega = \frac{2erR I_b}{h(R+Nr)}, \quad \alpha = -\frac{2er}{h} I_c, \quad K = \frac{2er^2 I_c}{h(R+Nr)}$$

$$4.6.5 \quad I_b = I_c \sin(\phi_K) + \frac{h}{2er} \dot{\phi}_K + \frac{h}{2eR} \sum \dot{\phi}_i$$

$$= I_c \sin(\phi_K) + \frac{h}{2er} \dot{\phi}_K + \frac{Nr}{R} \left( I_c \sin(\phi_K) + \frac{h}{2er} \dot{\phi}_K - I_c \sum_{i=1}^N \sin(\phi_i) \right)$$

$$= \left( 1 + \frac{Nr}{R} \right) I_c \sin(\phi_K) + \left( \frac{h}{2er} + \frac{Nh}{2eR} \right) \dot{\phi}_K - \frac{r}{R} I_c \sum_{i=1}^N \sin(\phi_i)$$

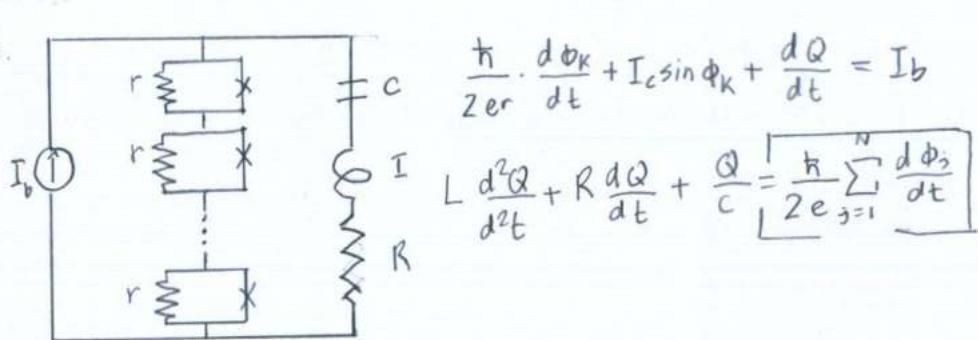
$$\frac{h}{2er} \left( \frac{1}{R} + \frac{N}{2eR} \right) \dot{\phi}_K = I_b - \frac{R+Nr}{R} I_c \sin(\phi_K) + \frac{r}{R} I_c \sum_{i=1}^N \sin(\phi_i)$$

$$\frac{h(R+Nr)}{2erR} \dot{\phi}_K = I_b - \frac{R+Nr}{R} I_c \sin(\phi_K) + \frac{r}{R} I_c \sum_{i=1}^N \sin(\phi_i)$$

$$\frac{h(R+Nr)}{2Nr^2 I_c} \dot{\phi}_K = \frac{R I_b}{Nr I_c} - \frac{R+Nr}{Nr} \sin(\phi_K) + \frac{1}{N} \sum_{i=1}^N \sin(\phi_i)$$

$$\frac{d\phi_K}{dt} = \Omega + \alpha \sin(\phi_K) + \frac{1}{Nr} \sum_{i=1}^N \sin(\phi_i); \quad \Omega = \frac{R I_b}{Nr I_c}; \quad \alpha = -\frac{(R+Nr)}{Nr}; \quad t = \frac{2Nr^2 I_c}{h(R+Nr)} t$$

$$\dot{\phi} = \Omega + a \sin \phi_k + K \sum_{j=1}^N \sin \phi_j \quad 4.6.6.$$



$$\frac{h}{2\pi r} \cdot \frac{d\phi_k}{dt} + I_c \sin \phi_k + \frac{dQ}{dt} = I_b$$

$$L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = \frac{h}{2\pi r} \sum_{j=1}^N \frac{d\phi_j}{dt}$$

## Chapter 5: Linear Systems

$$\ddot{x} = V \quad 5.1.1. \text{ a. } \frac{\ddot{x}}{V} = \frac{dx}{dv} = -\frac{V}{\omega^2 x}; \quad -\omega^2 x + C = V; \quad \boxed{\omega^2 x + V^2 = C}$$

$$\ddot{v} = -\omega^2 x$$

$$\text{b. Conservation of Energy: } \sum \frac{1}{2} m v^2 = E; \quad \frac{1}{2} m \omega^2 x^2 + \frac{1}{2} m v^2 = C$$

$$\boxed{KE_{rot} + KE_{lin} = KE_{tot}}$$

$$\ddot{x} = ax \quad 5.1.2 \quad \frac{\ddot{y}}{x} = \frac{dy}{dx} = -\frac{y}{ax} = -\frac{e^{-t}}{ae^{at}} = \frac{-1}{a e^{(a+1)t}}; \quad \lim_{t \rightarrow \infty} \frac{dy}{dx} = \lim_{t \rightarrow \infty} \frac{-1}{a e^{(a+1)t}} = \boxed{-\infty} \parallel y\text{-axis}$$

$$\ddot{y} = -y$$

$$\lim_{t \rightarrow -\infty} \frac{dy}{dx} = \lim_{t \rightarrow -\infty} \frac{-1}{a e^{(a+1)t}} = \boxed{0} \parallel x\text{-axis.}$$

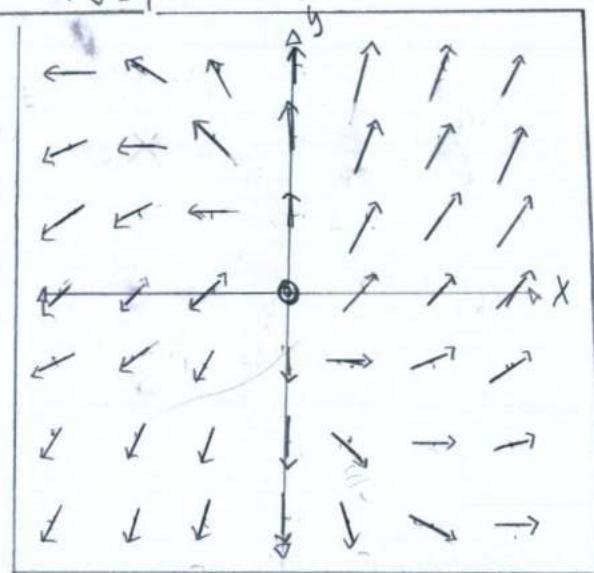
$$\begin{aligned} \ddot{x} &= -y & 5.1.3. \quad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} &= \begin{bmatrix} -1 \\ -1 \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \ddot{x} &= 3x - 2y & 5.1.4. \quad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} &= \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \ddot{y} &= 2y - x \end{aligned}$$

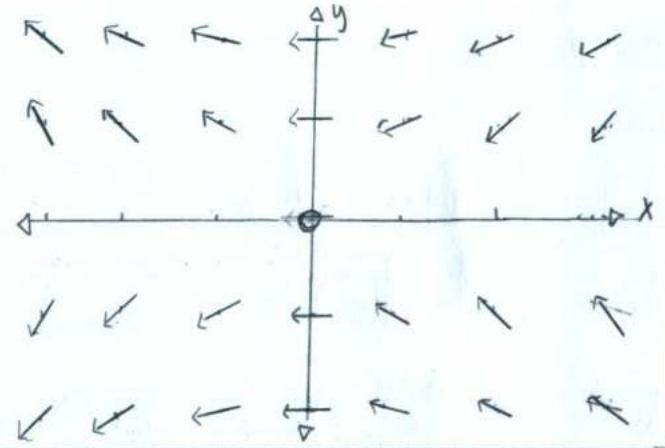
$$\begin{aligned} \ddot{x} &= 0 & 5.1.5. \quad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \ddot{y} &= x + y \end{aligned}$$

$$\begin{aligned} \ddot{x} &= x & 5.1.6. \quad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \ddot{y} &= 5x + y \end{aligned}$$

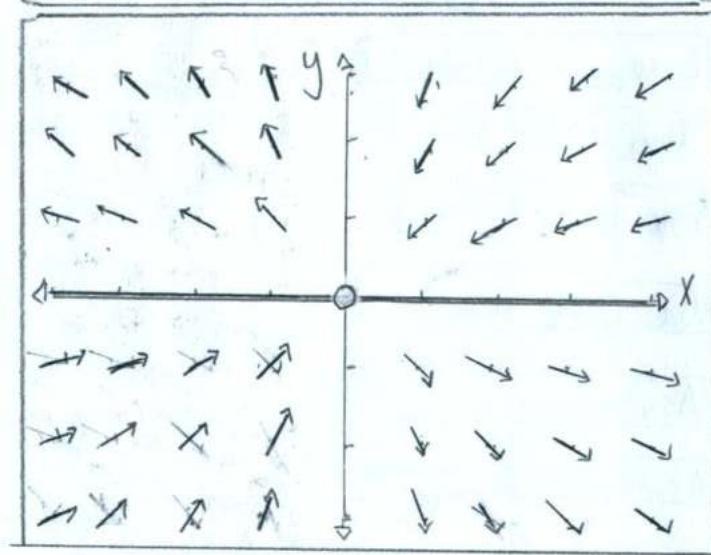
$$\begin{aligned} \ddot{x} &= x & 5.1.7. \quad \frac{dy}{dx} &= \frac{x+y}{x} \\ \ddot{y} &= x + y \end{aligned}$$



$$\begin{aligned} \dot{x} &= -2y \quad 5.1.9. \quad \frac{\dot{y}}{x} = \frac{dy}{dx} = \frac{-x}{-2y} = \frac{x}{2y} \\ \dot{y} &= -x \end{aligned}$$



$$\begin{aligned} \dot{x} &= -y \quad 5.1.9.a) \quad \frac{\dot{y}}{x} = \frac{dy}{dx} = \frac{y}{x} \\ \dot{y} &= -x \end{aligned}$$



b)  $\dot{x}\dot{x} = -xy ; \dot{y}\dot{y} = -xy$ ; therefore,  $\dot{x}\dot{x} = \dot{y}\dot{y} ; x\dot{x} - y\dot{y} = 0$   
 and  $\boxed{x dx - y dy = x^2 - y^2 = 0}$

c)  $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} ; \begin{pmatrix} -\lambda & -1 \\ -1 & \lambda \end{pmatrix} = \lambda^2 + 1 = (\lambda + 1)(\lambda - 1) = 0 ; \lambda_1 = 1, \lambda_2 = -1$

$\lambda_1 = 1 ; \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = 0$ ; Guess  $v_{11} = 1, v_{12} = -1$ ;  $x = C_1 e^t$ ;  $y = C_2 e^t$

$\lambda_2 = -1 ; \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = 0$ ; Guess  $v_{11} = 1, v_{12} = 1$ ;  $x = C_1 e^{-t}$ ;  $y = C_2 e^{-t}$

General Solution:  $x(t) = C_1 e^{-t} + C_1 e^t$ ;  $y(t) = C_2 e^{-t} + C_2 e^t$

$\lim_{t \rightarrow \infty} x(t) = -\infty$ ;  $\lim_{t \rightarrow -\infty} x(t) = \infty$  Unstable Manifold

$\lim_{t \rightarrow \infty} y(t) = \infty$ ;  $\lim_{t \rightarrow -\infty} y(t) = -\infty$  Stable Manifold

d)  $u = x + y$ ;  $\dot{u} = \dot{x} + \dot{y} = -y - x = -u$ ;  $u(t) = u_0 e^{-t}$   
 $v = x - y$ ;  $\dot{v} = \dot{x} - \dot{y} = x - y = v$ ;  $v(t) = v_0 e^t$

e)  $\lim_{t \rightarrow \infty} u(t) = 0$ ;  $\lim_{t \rightarrow -\infty} u(t) = \infty$ ;  $\lim_{t \rightarrow \infty} v(t) = \infty$ ;  $\lim_{t \rightarrow -\infty} v(t) = 0$ ;  
Stable Arbitrary Arbitrary Unstable

f) See part C.

5.1.10

a)  $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ ;  $\begin{bmatrix} -\lambda & 1 \\ -4 & -\lambda \end{bmatrix} = \lambda^2 + 14 = 0$ ;  $\lambda_1 = \pm 2i$ ;  $\lambda_2 = \pm 2i$

$\lambda_1 = 2i$ ;  $\begin{bmatrix} -2i & 1 \\ -4 & -2i \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$ ;  $-2iv_{11} + v_{12} = 0$ ;  $v_{11} = 0$ ;  $v_{22} = 2i$   
 $-4v_{11} - 2i \cdot v_{12} = 0$

$\lambda_2 = -2i$ ;  $\begin{bmatrix} 2i & 1 \\ -4 & 2i \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$ ;  $2v_{11} + v_{12} = 0$ ;  $v_{11} = 1$ ;  $v_{22} = 2i$   
 $-4v_{11} + 2v_{12} = 0$

Liapunov Stability

b) None of the Above

c) None of the Above

d) None of the Above

e) Asymptotically stable

f) Asymptotically stable

5.1.11. a)  $\|x(t) - x^*\| = \|C\cos(2t) + C\sin(2t) - x^*\| < C^2 = E$

$\|x(0) - x^*\| < C + \delta$

b)  $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$ ;  $\begin{bmatrix} -\lambda & 2 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 2 = 0$ ;  $\lambda_1 = +\sqrt{2}$ ;  $\lambda_2 = -\sqrt{2}$

$\lambda_1 = +\sqrt{2}$ ;  $\begin{bmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$ ;  $-\sqrt{2}v_{11} + 2v_{12} = 0$ ;  $v_{11} = 1$ ;  $v_{12} = \frac{1}{\sqrt{2}}$   
 $v_{11} - \sqrt{2}v_{12} = 0$

$\lambda_2 = -\sqrt{2}$ ;  $\begin{bmatrix} \sqrt{2} & 2 \\ 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$ ;  $\sqrt{2}v_{11} + 2v_{12} = 0$ ;  $v_{11} = 1$ ;  $v_{12} = -\frac{1}{\sqrt{2}}$

$x(t) = C_1 \cosh(t/\sqrt{2}) + C_2 \sinh(t/\sqrt{2})$ ;  $y(t) = X_0 \cosh(t) + C_4 \sinh(-t/\sqrt{2})$

$= X_0 \cosh(t) + \frac{Y_0}{\sqrt{2}} \sinh(t/\sqrt{2})$ ;  $y(t) = X_0 \cosh(t) - \frac{Y_0}{\sqrt{2}} \sinh(-t/\sqrt{2})$

$\|x(t) - x^*\| = \|X_0 \cosh(t) + \frac{Y_0}{\sqrt{2}} \sinh(t/\sqrt{2}) - 0\| = E$

$\|x(0) - x^*\| = \|X_0\| < \delta$  None of the above

c)  $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 2i \\ -2i & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ ;  $\lambda_1 = 1 + 2i$ ;  $\lambda_2 = 1 - 2i$

$\begin{bmatrix} -\lambda & 2i \\ -2i & -\lambda \end{bmatrix} = \lambda^2 - 1 - 4 = 0$ ;  $\lambda_1 = 1 + 2i$ ;  $\lambda_2 = 1 - 2i$   
 $\begin{bmatrix} -1 - 2i & 2i \\ -2i & -1 - 2i \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$ ;  $(-1 - 2i)v_{11} + 2iv_{12} = 0$ ;  $v_{11} = 1$   
 $(-1 - 2i)v_{11} - 2iv_{12} = 0$ ;  $v_{12} = \frac{1}{2}$   
 $\begin{bmatrix} 1 & 2i \\ -2i & 1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$ ;  $v_{11} = 1$ ;  $v_{12} = -\frac{1}{2}$   
 $x(t) = C_1 e^{(1+2i)t} + C_2 e^{(1-2i)t}$

c.  $\dot{x} = 0; x = l + C$ ;  $\dot{y} = x$ ; None of the Above

$$\begin{aligned} &= C \\ &= x_0 \\ &= x_0 t + C \\ &= x_0 t + y_0 \end{aligned}$$

d.  $\dot{x} = 0; x = l + C$ ;  $\dot{y} = x$ ; None of the above

$$\begin{aligned} &= C \\ &= x_0 \\ &= x_0 t + C \\ &= x_0 t + y_0 \end{aligned}$$

e.  $\dot{x} = -x$ ;  $\dot{y} = -5y$ ; Asymptotically Stable

$$\begin{aligned} x &= x_0 e^{-t} \\ \dot{x} &= y_0 e^{-5t} \\ \dot{y} &= y_0 e^{-5t} \end{aligned}$$

f.  $\dot{x} = x$ ;  $\dot{y} = y$ ; Asymptotically Stable

$$\begin{aligned} x &= e^t \\ y &= e^{+t} \end{aligned}$$

$\dot{x} = v; \dot{v} = -x$  5.1.12  $v$ -axis @  $(0, -v_0)$ ;  $x$ -axis @  $(x_1, 0)$ ;  $V(0) = -V_0 = V_0$ ;  $\dot{x}(x) = 0$

5.1.13 The "saddle point" is a category of bifurcation that is parabolic beyond a coordinate. A connection to real saddles is the "curved" shape where the rider sits.

$\dot{x} = 4x - y$  5.2.1 a.  $\dot{x} = A\vec{x}$ ;  $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ ;  $\begin{bmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{bmatrix} = (4-\lambda)(1-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = 0$

$$\lambda_1 = 2; \lambda_2 = 3$$

$$\lambda_1 = 2; \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \frac{2v_{11} - v_{12}}{v_{11}} = 0; v_{11} = 1 \Rightarrow v_{12} = 2; \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda_2 = 3; \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \frac{v_{21} - v_{22}}{v_{21}} = 0; v_{21} = 1 \Rightarrow v_{22} = 1; \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

b) General Solution:  $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{\lambda_1 t} \vec{v}_1 + C_2 e^{\lambda_2 t} \vec{v}_2 = C_1 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{aligned} x(t) &= C_1 e^{2t} + C_2 e^{3t} \\ y(t) &= 2C_1 e^{2t} + C_2 e^{3t} \end{aligned}$$

c) Stable

d)  $(x_0, y_0) = (3, 4) \Rightarrow 3 = C_1 + C_2 \Rightarrow 4 = 2C_1 + C_2$

$$\begin{aligned} C_1 &= 3 - C_2 \Rightarrow 4 = 2(3 - C_2) + C_2 = 6 - 2C_2 + C_2 \\ &= 6 - C_2 \Rightarrow C_2 = 2 \Rightarrow C_1 = 1 \end{aligned}$$

$$\begin{aligned} x(t) &= e^{2t} + 2e^{3t} \\ y(t) &= 2e^{2t} + 2e^{3t} \end{aligned}$$

$$\begin{aligned} \dot{x} &= x - y & 5.2.2. a) \quad X = Ax; \quad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}; \quad \begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2 \\ \dot{y} &= x + y & -\lambda_1 = 1-i; \quad \lambda_2 = 1+i \end{aligned}$$

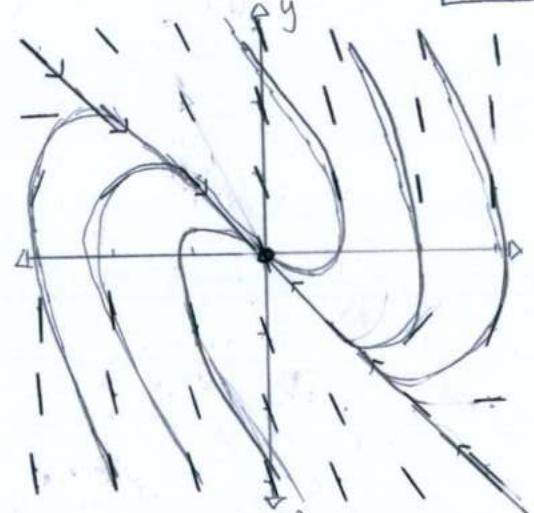
$$\lambda_1 = 1 - i \Rightarrow \begin{bmatrix} 1-(1-i) & -1 \\ 1 & 1-(1-i) \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad iV_{11} - V_{12} = 0; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\lambda_2 = 1 + i \Rightarrow \begin{bmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{bmatrix} \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad -iV_{21} - V_{22} = 0; \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

b. General Solution:  $\vec{X}(t) = V_1 e^{\lambda_1 t} + V_2 e^{\lambda_2 t}$

$$\begin{cases} x(t) = e^t \cdot 2 \cos(t) \\ y(t) = e^t \cdot 2i \sin(t) \end{cases}$$

$$\begin{aligned} \dot{x} &= y & 5.2.3. \quad \frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = -2 \cdot \frac{x}{y} - 3 \\ \dot{y} &= -2x - 3y \end{aligned}$$



$$x(t) = e^{(1-i)t} + e^{(1+i)t}$$

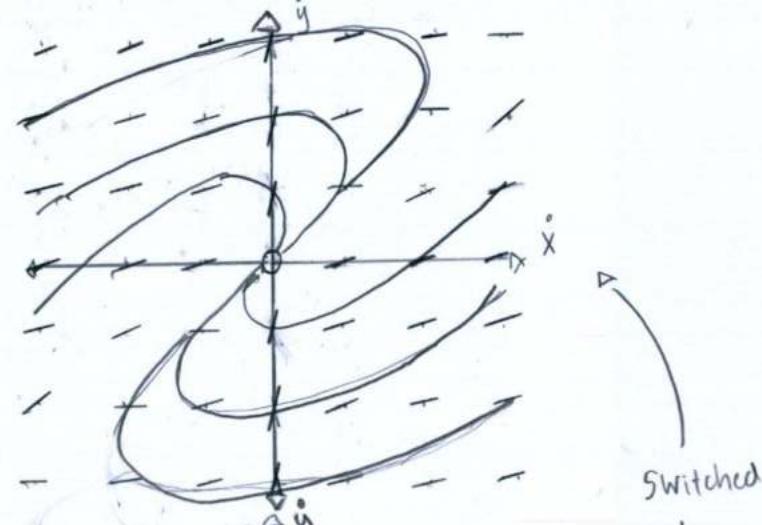
$$y(t) = e^{(1-i)t} + e^{(1+i)t}$$

$$+ e^{(1-i)t} + e^{(1+i)t}$$

$$- 2e^{(1-i)t} + 2$$

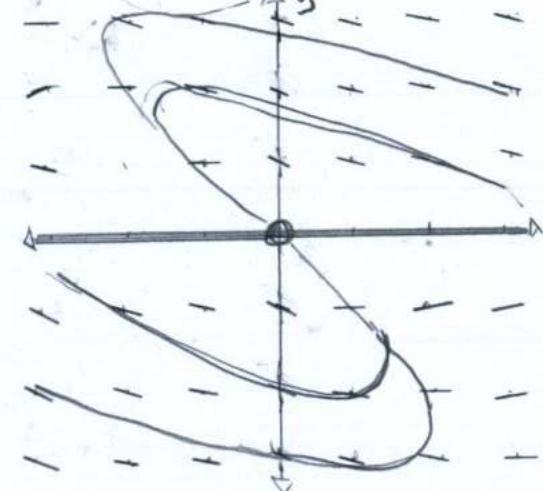
$$- 2e^{(1+i)t} + 2$$

$$\begin{aligned} \dot{x} &= 5x + 10y & 5.2.4 \quad \frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{-x-y}{5x+10y} \\ \dot{y} &= -x - y \end{aligned}$$



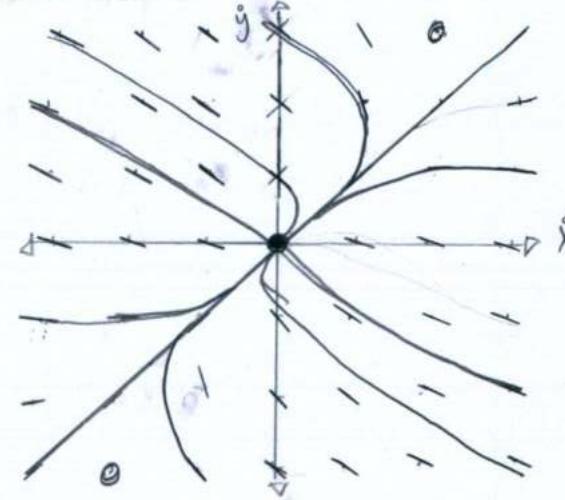
Switched

$$\begin{aligned} \dot{x} &= 3x - 4y & 5.2.5. \quad \frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{x-y}{3x-4y} \\ \dot{y} &= x - y \end{aligned}$$



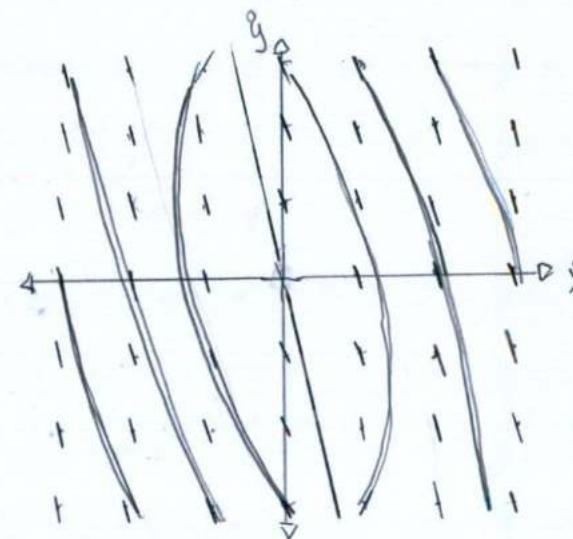
$$\begin{aligned}\dot{x} &= -3x + 2y \\ \dot{y} &= x - 2y\end{aligned}$$

5.2.6.  $\frac{\dot{y}}{x} = \frac{dy}{dx} = \frac{x - 2y}{-3x + 2y}$



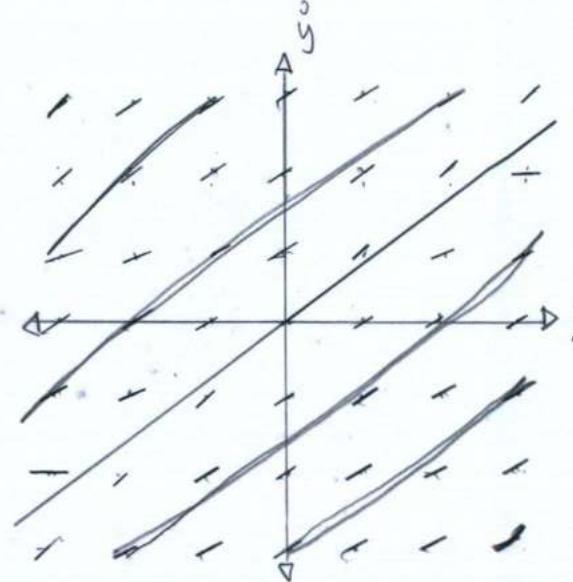
$$\begin{aligned}\dot{x} &= 5x + 2y \\ \dot{y} &= -17x - 5y\end{aligned}$$

5.2.7.  $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{-17x - 5y}{5x + 2y}$



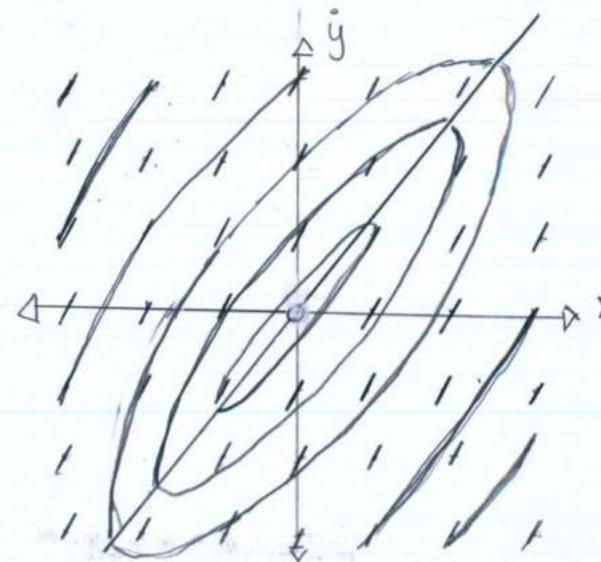
$$\begin{aligned}\dot{x} &= -3x + 4y \\ \dot{y} &= -2x + 3y\end{aligned}$$

5.2.8.  $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{-2x + 3y}{-3x + 4y}$



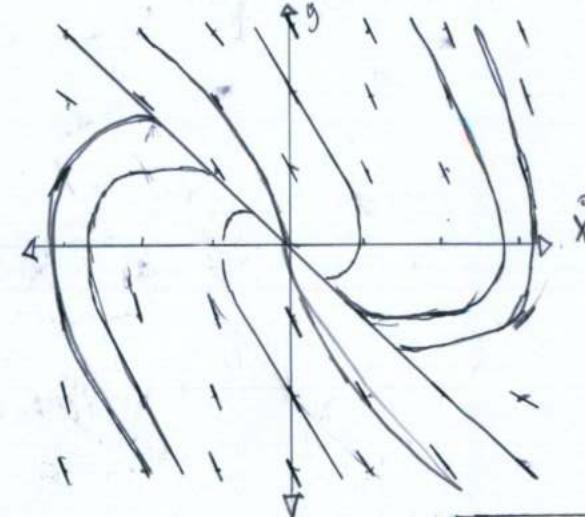
$$\begin{aligned}\dot{x} &= 4x - 3y \\ \dot{y} &= 8x - 6y\end{aligned}$$

5.2.9.  $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{8x - 6y}{4x - 3y}$



$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - 2y \end{aligned}$$

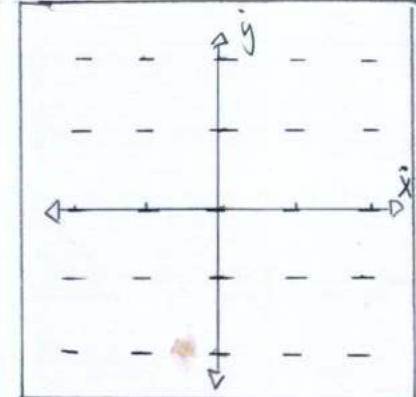
$$5.2.10. \frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{-x - 2y}{y}$$



$$A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \quad 5.2.11. A^{-1} \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix} = \lambda^2 = 0 \Rightarrow \lambda = 0$$

$$\dot{x} = Ax \Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \lambda & b \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}; \quad \dot{x} = \lambda x; \quad \dot{y} = 0; \quad \frac{\dot{y}}{\dot{x}} = 0$$

The book shows a typeset to the correct solution.



$$LI + RI + \frac{I}{C} = 0 \quad 5.2.12$$

$$a) x = I \quad \dot{y} = \dot{I}$$

$$\dot{x} = \dot{I} \quad \dot{y} = \ddot{I} = -RI - \frac{I}{C} = -Ry - \frac{x}{C} \quad ; \quad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{C} - R & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$b) R = 0; \quad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}; \quad \begin{bmatrix} -\lambda & 1 \\ -\frac{1}{C} & -\lambda \end{bmatrix} = \lambda^2 + \frac{1}{C} = 0; \quad \lambda_{1,2} = \pm \frac{i}{\sqrt{C}}$$

$$\lambda_{1,2} = \pm \frac{i}{\sqrt{C}}; \quad \begin{bmatrix} -i/\sqrt{C} & 1 \\ -1/\sqrt{C} & -i/\sqrt{C} \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = 0; \quad \frac{V_{11}}{C} + \frac{iV_{12}}{\sqrt{C}} = 0; \quad V_{11} = -i\sqrt{C}; \quad V_{12} = 1$$

$$\text{General Solution: } e^{\lambda_{1,2}t} = \cos(\frac{t}{\sqrt{C}}) + i\sin(\frac{t}{\sqrt{C}})$$

$$e^{\lambda_{1,2}t} \cdot \vec{V} = \begin{bmatrix} -\sqrt{C}\sin(t/\sqrt{C}) - i\sqrt{C}\cos(t/\sqrt{C}) \\ \cos(t/\sqrt{C}) + i\sin(t/\sqrt{C}) \end{bmatrix}$$

$$\begin{aligned} X &= \begin{bmatrix} x = C_1 \sin(t/\sqrt{C}) - C_2 \cos(t/\sqrt{C}) \\ y = C_1 \cos(t/\sqrt{C}) + C_2 \sin(t/\sqrt{C}) \end{bmatrix} \end{aligned}$$

Neutrally Stable

$$R > 0; \quad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{C} - R & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}; \quad \begin{bmatrix} -\lambda & 1 \\ -\frac{1}{C} - R - \lambda & 0 \end{bmatrix} = \lambda(R + \lambda) + \frac{1}{C} = 0; \quad \lambda_{1,2} = \frac{-R \pm \sqrt{R^2 - 4(1)(1/C)}}{2(1)}$$

$$\lambda_{1,2} = \frac{-R \pm \sqrt{R^2 - 4(1)(1/C)}}{2}; \quad \begin{bmatrix} \frac{-R - \sqrt{R^2 - 4/C}}{2} & 1 \\ -1/C & \frac{-R + \sqrt{R^2 - 4/C}}{2} \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = 0$$

$$\frac{R - \sqrt{R^2 - 4LC}}{2} \cdot V_{11} + V_{12} = 0 ; \quad V_{11} = 1 ; \quad V_{12} = -\frac{R + \sqrt{R^2 - 4LC}}{2}$$

$$\lambda_2 = \frac{-R - \sqrt{R^2 - 4LC}}{2} ; \quad \begin{bmatrix} \frac{R + \sqrt{R^2 - 4LC}}{2} & 1 \\ -1/C & -\frac{R + \sqrt{R^2 - 4LC}}{2} \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = 0$$

$$\frac{R + \sqrt{R^2 - 4LC}}{2} \cdot V_{11} + V_{12} = 0 ; \quad V_{11} = 1 ; \quad V_{12} = \frac{+R + \sqrt{R^2 - 4LC}}{2}$$

General Solution:  $X_i = C_i e^{\lambda_i t} / V_i$

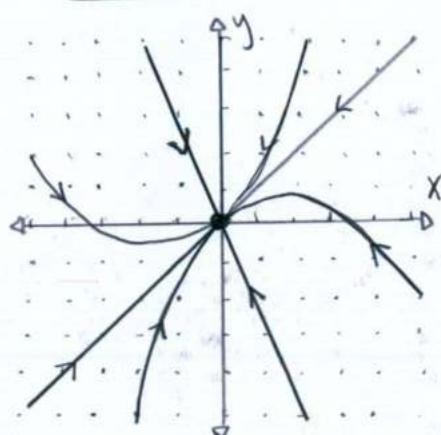
$$X_1 = C_1 e^{-\frac{R + \sqrt{R^2 - 4LC}}{2} t} \begin{bmatrix} 1 \\ -\frac{R + \sqrt{R^2 - 4LC}}{2} \end{bmatrix}$$

$$X_2 = C_2 e^{-\frac{R - \sqrt{R^2 - 4LC}}{2} t} \begin{bmatrix} 1 \\ \frac{R + \sqrt{R^2 - 4LC}}{2} \end{bmatrix}$$

$$\bar{X} = X_1 + X_2 = \begin{bmatrix} X(t) = C_1 e^{-\frac{R + \sqrt{R^2 - 4LC}}{2} t} + C_2 e^{-\frac{R - \sqrt{R^2 - 4LC}}{2} t} \\ Y(t) = C_1 e^{-\frac{R + \sqrt{R^2 - 4LC}}{2} t} \cdot \left( -\frac{R + \sqrt{R^2 - 4LC}}{2} \right) + C_2 e^{-\frac{R - \sqrt{R^2 - 4LC}}{2} t} \cdot \left( \frac{R + \sqrt{R^2 - 4LC}}{2} \right) \end{bmatrix}$$

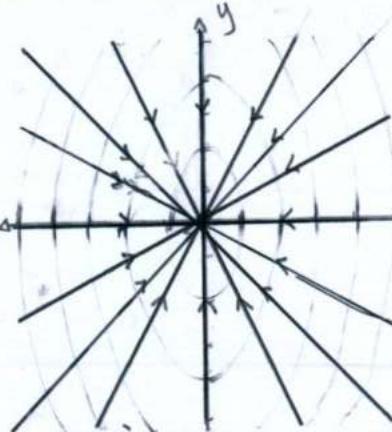
A asymptotically stable

C.  $R^2C - 4L > 0$



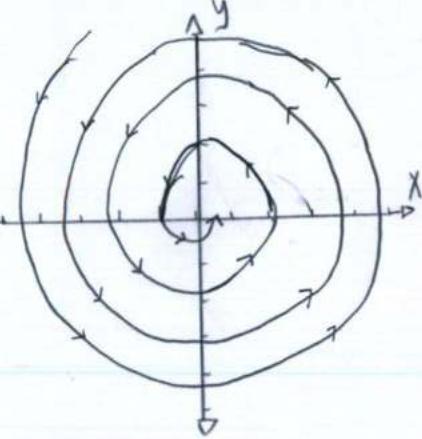
Unstable Node

$R^2C - 4L = 0$



Star, Degenerate Node

$R^2C - 4L < 0$



Unstable Spiral

$$R^2C - 4L > 0 \Rightarrow \lambda_2 = 0$$

$$R^2C > 4L + C_2 \frac{C}{R}$$

$$C > \frac{R^2}{4L} \in C'$$

$$C > 4L$$

$$m\ddot{x} + b\dot{x} + kx = 0 \quad 5.2.13:$$

$$i = x \quad j = \dot{x}$$

$$\dot{i} = \dot{x} \quad \dot{j} = \ddot{x} = -\frac{b}{m}\dot{x} - \frac{k}{m}x = -\frac{b}{m}j - \frac{k}{m}i$$

$$\begin{bmatrix} i \\ \dot{i} \\ j \\ \dot{j} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{b}{m} & -\frac{k}{m} \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix}$$

$$b. \quad \vec{I} = A\vec{z}; \quad \begin{bmatrix} -\lambda & 1 \\ -\frac{b}{m} & -\frac{k}{m} - \lambda \end{bmatrix} = \lambda^2 + \frac{k}{m}\lambda + \frac{b}{m} = 0; \quad \lambda_{1,2} = \frac{-\frac{k}{m} \pm \sqrt{\left(\frac{k}{m}\right)^2 - 4\left(\frac{b}{m}\right)}}{2(1)}$$

$$\lambda_1 = \frac{-\frac{k}{m} + \sqrt{\left(\frac{k}{m}\right)^2 - 4\left(\frac{b}{m}\right)}}{2}; \quad \begin{bmatrix} \frac{k}{m} - \sqrt{\left(\frac{k}{m}\right)^2 - 4\left(\frac{b}{m}\right)} & 1 \\ -\frac{b}{m} & -\frac{k}{m} - \sqrt{\left(\frac{k}{m}\right)^2 - 4\left(\frac{b}{m}\right)} \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$\left(\frac{k}{m} - \sqrt{\left(\frac{k}{m}\right)^2 - 4\left(\frac{b}{m}\right)}\right)v_{11} + v_{12} = 0$$

$$v_{11} = 1; \quad v_{12} = -\frac{k}{m} + \sqrt{\left(\frac{k}{m}\right)^2 - 4\left(\frac{b}{m}\right)}$$

$$\lambda_2 = \frac{-\frac{k}{m} - \sqrt{\left(\frac{k}{m}\right)^2 - 4\left(\frac{b}{m}\right)}}{2}; \quad \begin{bmatrix} \frac{k}{m} + \sqrt{\left(\frac{k}{m}\right)^2 - 4\left(\frac{b}{m}\right)} & 1 \\ -\frac{b}{m} & -\frac{k}{m} + \sqrt{\left(\frac{k}{m}\right)^2 - 4\left(\frac{b}{m}\right)} \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0$$

$$\left(\frac{k}{m} + \sqrt{\left(\frac{k}{m}\right)^2 - 4\left(\frac{b}{m}\right)}\right)v_{21} + v_{22} = 0$$

$$v_{21} = 1; \quad v_{22} = -\frac{\left(\frac{k}{m}\right) - \sqrt{\left(\frac{k}{m}\right)^2 - 4\left(\frac{b}{m}\right)}}{2}$$

$$-\frac{\left(\frac{k}{m}\right) + \sqrt{\left(\frac{k}{m}\right)^2 - 4\left(\frac{b}{m}\right)}}{2} t - \frac{\left(\frac{k}{m}\right) - \sqrt{\left(\frac{k}{m}\right)^2 - 4\left(\frac{b}{m}\right)}}{2} t$$

$$x(t) = C_1 e$$

$$+ C_2 e$$

$$y(t) = C_1 \cdot \frac{\left(\frac{k}{m}\right) + \sqrt{\left(\frac{k}{m}\right)^2 - 4\left(\frac{b}{m}\right)}}{2} \cdot e^{-\frac{\left(\frac{k}{m}\right) + \sqrt{\left(\frac{k}{m}\right)^2 - 4\left(\frac{b}{m}\right)}}{2} t} + C_2 \cdot \frac{\left(\frac{k}{m}\right) - \sqrt{\left(\frac{k}{m}\right)^2 - 4\left(\frac{b}{m}\right)}}{2} \cdot e^{-\frac{\left(\frac{k}{m}\right) - \sqrt{\left(\frac{k}{m}\right)^2 - 4\left(\frac{b}{m}\right)}}{2} t}$$

Unstable Spiral:  $\sqrt{\left(\frac{k}{m}\right)^2 - 4\left(\frac{b}{m}\right)} < 0$

Star, Degenerate Node:  $\sqrt{\left(\frac{k}{m}\right)^2 - 4\left(\frac{b}{m}\right)} = 0$

Unstable Node:  $\sqrt{\left(\frac{k}{m}\right)^2 - 4\left(\frac{b}{m}\right)} > 0$

C. Star, Degenerate Node is critically damped. An unstable spiral is underdamped. While an unstable node is an unstable node.

$\dot{x} = Ax$

5.2.14 $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ;	$\lambda_2 < \lambda_1 < 0$ : Stable Node $\lambda_1, \lambda_2 = K \pm iw < 0$ : Stable Spiral $\lambda_1 = \lambda_2 = \lambda$ : Star Node, Degenerate Node $\lambda_1, \lambda_2 = K + iw > 0$ : Unstable Spiral $\lambda_2, \lambda_1 > 0$ : Unstable Node
---	---

$$A = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

$$\lambda_{1,2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

If  $(a+d)^2 - 4(ad-bc) < 0$ , then  $\lambda_1, \lambda_2$  are imaginary.

If  $(a+d) > 0$ , then  $\lambda_1, \lambda_2$  are an unstable spiral.  
else,  $\lambda_1, \lambda_2$  are a stable spiral.

Else

$$\text{Doub } \lambda_1 = \frac{(a+d) + \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

$$\text{Doub } \lambda_2 = \frac{(a+d) - \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

If  $(\lambda_1 = \lambda_2)$ , then Star Node, Degenerate Node

Else if  $(\lambda_1 \& \lambda_2 > 0)$ , then Unstable Node

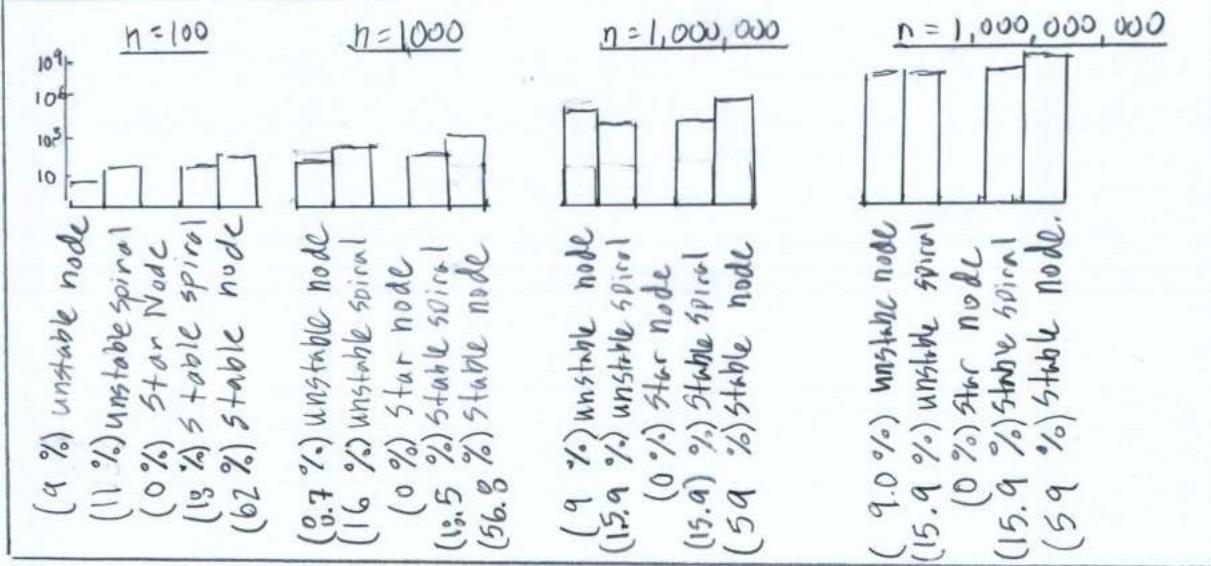
else, Stable Node.

Pseudo-code

```
#include <iostream>
#include <random>
using namespace std;

int main() {
    int s_node = 0, u_node = 0, star = 0, s_spiral = 0, u_spiral = 0, trials = 1000;
    double l1, l2, a, b, c, d;
    std::default_random_engine generator;
    std::uniform_real_distribution<-1, 1> distribution;
    if (((a+d)-4*(ad-bc)) < 0) {
        if ((a+d) > 0) u_spiral += 1;
        else s_spiral += 1;
    } else {
        l1 = ((a+d) + sqrt(pow((a+d), 2) - 4 * (ad - bc))) / 2;
        l2 = ((a+d) - sqrt(pow((a+d), 2) - 4 * (ad - bc))) / 2;
        if (l1 == l2) star += 1;
        else if ((l1 && l2 > 0)) u_node += 1;
        else s_node += 1;
    }
}
```

Real-code



An unstable spiral approaches the limit of 9%; while, stable spirals the most common, at 59%.

A normal distribution produced greater proportions of the stable phase plots (stable node [83%], stable spiral [14.8%], ...). than the uniform distribution modelled.

$R = aR + bJ$  5.3.1.  $R = \text{Romeo's love/hate}; J = \text{Juliet's love/hate}; a = b = \text{romantic style}.$

$$\begin{aligned} R &= J \\ \dot{J} &= -R + J \end{aligned}$$

5.3.2.  $a. \begin{bmatrix} \dot{R} \\ \dot{J} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}$  "cautious romance"

b.  $(R, J) = (0, 0)$  = Neverending love/hate  
= Stable Node

c.  $R(0) = I; J(0) = O;$

$$\begin{bmatrix} 0-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix} = -\lambda(1-\lambda) + 1 = \lambda^2 - \lambda + 1 = 0$$

$$\lambda_{1,2} = \frac{+1 \pm \sqrt{1-4(1)(1)}}{2} = \frac{1 \pm \sqrt{3}i}{2}$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{3}i}{2}; \begin{bmatrix} \frac{1 \pm \sqrt{3}i}{2} & 1 \\ -1 & 1 \mp \sqrt{3}i \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \frac{-1 \mp \sqrt{3}i}{2} V_{11} + V_{12} = 0; \vec{V}_{1,2} = \begin{bmatrix} 1 \\ \mp 1 \mp \sqrt{3}i \end{bmatrix}$$

General Solution:  $\vec{x} = \begin{bmatrix} R(t) = e^{\frac{t}{2}} \cdot 2 \left[ C_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}t\right) \right] \\ J(t) = e^{\frac{t}{2}} \cdot (1 + \sqrt{3}i) \left[ C_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}t\right) \right] \end{bmatrix}$

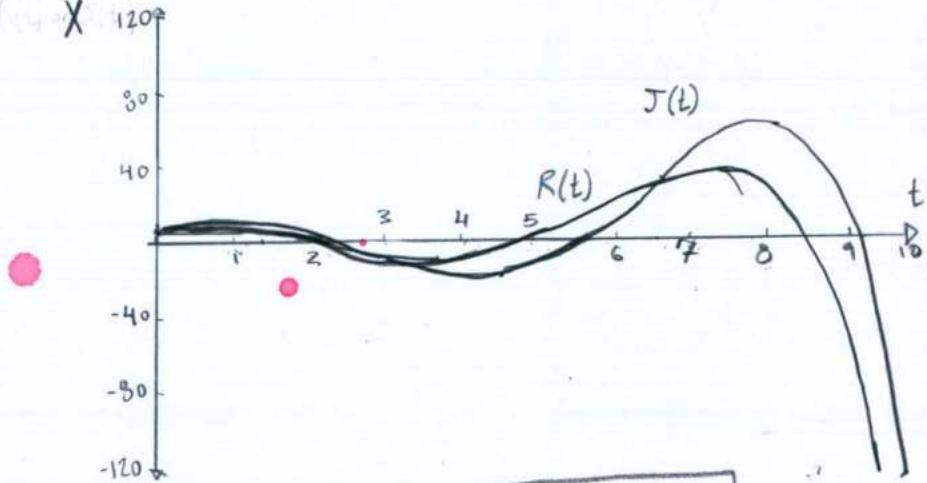
$$\vec{x} = \begin{bmatrix} R(t) = 2e^{\frac{t}{2}} \left[ C_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}t\right) \right] \\ J(t) = (1 + \sqrt{3})e^{\frac{t}{2}} \cdot C_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + (1 - \sqrt{3})e^{\frac{t}{2}} \cdot C_2 \sin\left(\frac{\sqrt{3}}{2}t\right) \end{bmatrix}$$

$$R(0) = 1; C_1 = \frac{1}{2}$$

$$J(0) = 0; 0 = C_1 + \sqrt{3}C_2 \Rightarrow C_2 = \frac{-1}{2\sqrt{3}} = \frac{1 + \sqrt{3}i}{2} C_2$$

Final Solution:

$$\vec{X} = \begin{bmatrix} R(t) = e^{t/2} \left[ \cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}t\right) \right] \\ J(t) = e^{t/2} \left[ \frac{(1+\sqrt{3})}{2} \cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{(1-\sqrt{3})}{2\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}t\right) \right] \end{bmatrix}$$



$$\begin{array}{l} R = aJ \\ \dot{J} = bR \end{array} \quad 5.3.3.$$

$b \backslash a$	(+)	(-)
(+)	$a^2 > b^2$ : Stable $a^2 < b^2$ : Unstable $a=b$ : Mutual	Stable center of neverending love & hate
(-)	Stable center of neverending love & hate	$a^2 > b^2$ : Stable $a^2 < b^2$ : Unstable $a=b$ : Mutual

Romeo and Juliet live happily when Juliet's love is of greater amounts.

$$\begin{array}{l} R = aR + bJ \\ \dot{J} = -bR - aJ \end{array} \quad 5.3.4.$$

$b \backslash a$	(+)	(-)
(+)	$a^2 > b^2$ : Unstable $a^2 \leq b^2$ : Stable $a=b$ : Star, degen node	$a^2 > b^2$ : Unstable $a^2 \leq b^2$ : Stable $a=b$ : Star, degen node
(-)	$a^2 > b^2$ : Unstable $a=b$ : star, degen node	$a^2 > b^2$ : Unstable $a^2 \leq b^2$ : Stable $a=b$ : star, degen node

Yes, opposites attract when the proportion of Romeo's love is larger.

$$\begin{array}{l} R = aR + bJ \\ \dot{J} = bR + aJ \end{array} \quad 5.3.5.$$

$b \backslash a$	(+)	(-)
(+)	$a^2 > b^2$ : Unstable Node $a^2 \leq b^2$ : Unstable Saddle	$a^2 > b^2$ : Unstable Node $a^2 \leq b^2$ : Unstable Saddle
(-)	$a^2 > b^2$ : Unstable Node $a^2 \leq b^2$ : Unstable Saddle	$a^2 > b^2$ : Unstable Node $a^2 \leq b^2$ : Unstable Saddle

The marriage of Romeo and Juliet of exact clone demonstrate an unstable relationship for all time.

$$\dot{R} = 0$$

5.3.6.

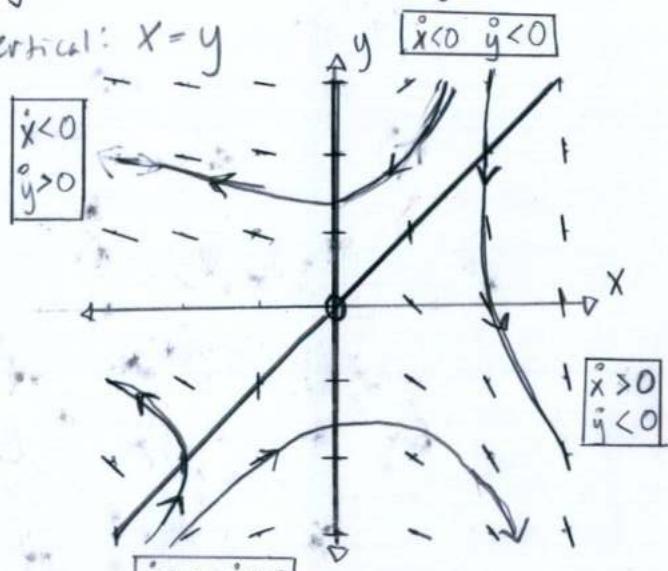
$$\dot{J} = aR + bJ.$$

$\begin{array}{c} a \\ \diagdown \\ b \end{array}$	(+)	(-)
(+)	Unstable and Fixed Relationship	$a^2 > b^2$ : Stable $a^2 < b^2$ : Unstable $a=b$ : Isolated
(-)	$a^2 > b^2$ : Unstable $a^2 < b^2$ : Stable $a=b$ : Isolated.	Stable and Fixed Relationship

## Chapter 6: Phase Plane:

$$\begin{aligned} \dot{x} &= x - y & 6.1.1: \boxed{\text{Fixed Points}}: \quad \dot{x} = 0 = x - y; \quad x = y; \quad \dot{y} = 0 = 1 - e^x; \quad x = 0; \quad (x^*, y^*) = (0, 0) \\ \dot{y} &= 1 - e^x \end{aligned}$$

**Nullclines:** Horizontal:  $x = 0$ ; Vertical:  $x = y$



$$\begin{aligned} \dot{x} &= x - x^3 & 6.1.2: \boxed{\text{Fixed Points}}: \quad \dot{x} = 0 = x - x^3 \\ \dot{y} &= -y \end{aligned}$$

$$x^* = 1, 0, -1$$

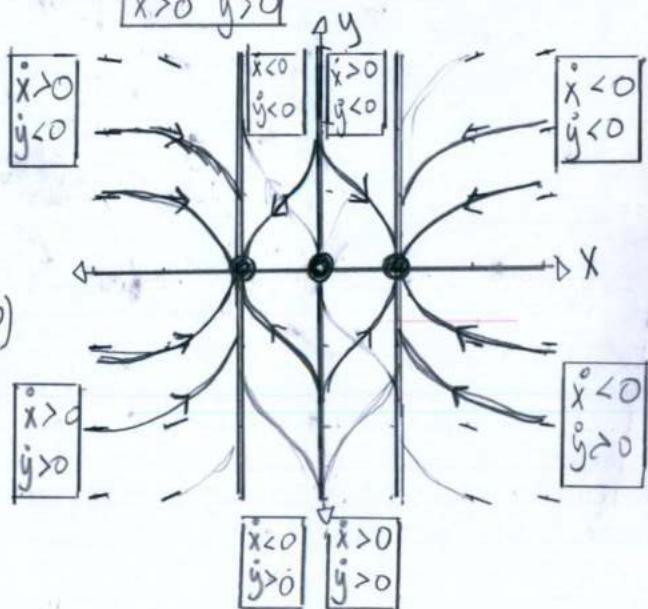
$$\dot{y} = 0 = -y$$

$$y^* = 0$$

$$(x^*, y^*) = (1, 0), (0, 0), (-1, 0)$$

**Nullclines:** Horizontal:  $y = 0$

Vertical:  $x = 0, 1, -1$



$$\dot{x} = x(x-y)$$

### 6.1.3. Fixed Points:

$$\dot{x} = 0 = x(x-y)$$

$$\dot{y} = 0 = y(2x-y)$$

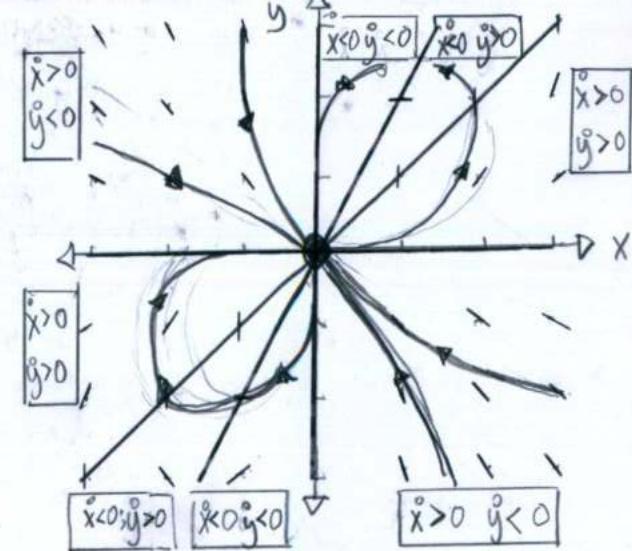
$$(x^*, y^*) = (0, 0)$$

**Nullclines:** Horizontal:  $y = 2x$

$$y = 0$$

Vertical:  $y = x$

$$x = 0$$



$$\dot{x} = y$$

### 6.1.4: Fixed Points:

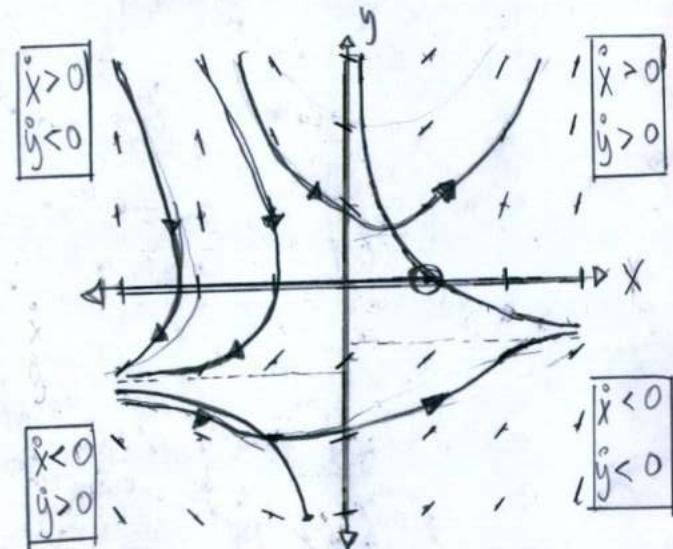
$$\dot{x} = 0 = y$$

$$\dot{y} = 0 = x(1+y)-1$$

$$(x^*, y^*) = (1, 0)$$

**Nullclines:** Horizontal:  $y = \frac{1}{x} + 1$

Vertical:  $y = 0$



$$\dot{x} = x(2-x-y)$$

### 6.1.5. Fixed Points:

$$\dot{x} = 0 = x(2-x-y)$$

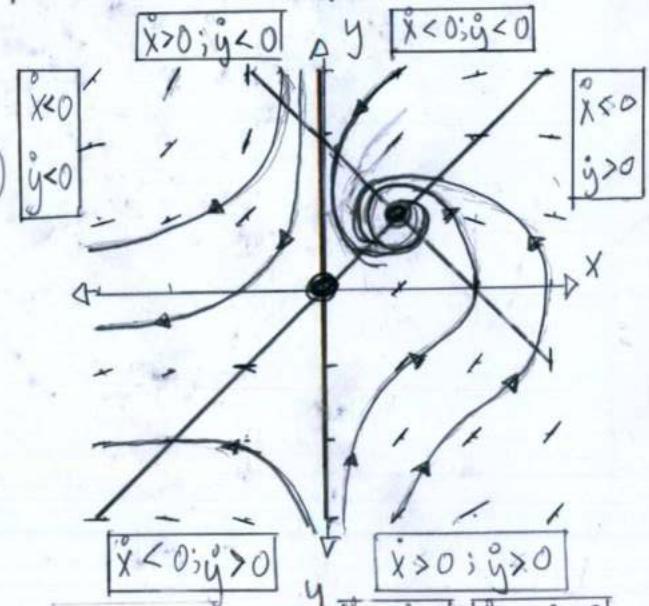
$$\dot{y} = 0 = x-y$$

$$(x^*, y^*) = (0, 0), (1, 1)$$

**Nullcline:** Horizontal:  $y = x$

Vertical:  $x = 0$

$$y = 2-x$$



$$\dot{x} = x^2 - y$$

### 6.1.6. Fixed Points:

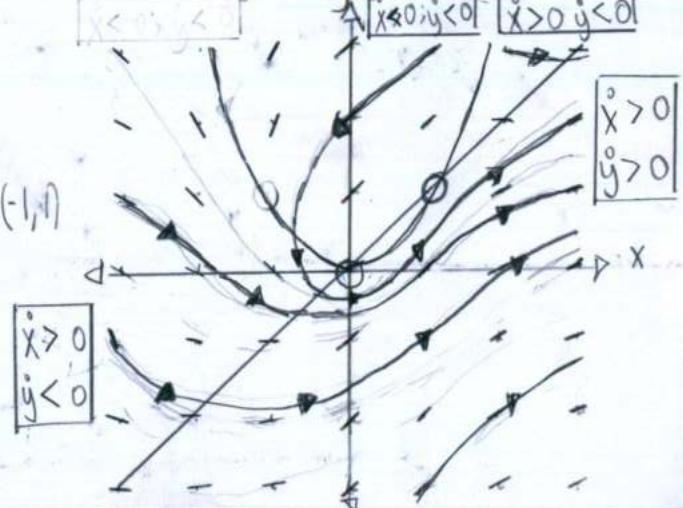
$$\dot{x} = 0 = x^2 - y$$

$$\dot{y} = 0 = x-y$$

$$(x^*, y^*) = (0, 0), (1, 1), (-1, 1)$$

**Nullcline:** Horizontal:  $y = x$

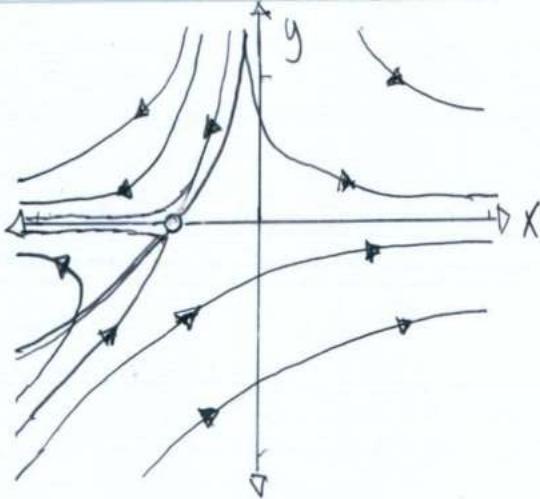
Vertical:  $y = x^2$



$$\dot{x} = x + e^{-y}$$

6.1.7

$$\dot{y} = -y$$



$$\dot{x} = y$$

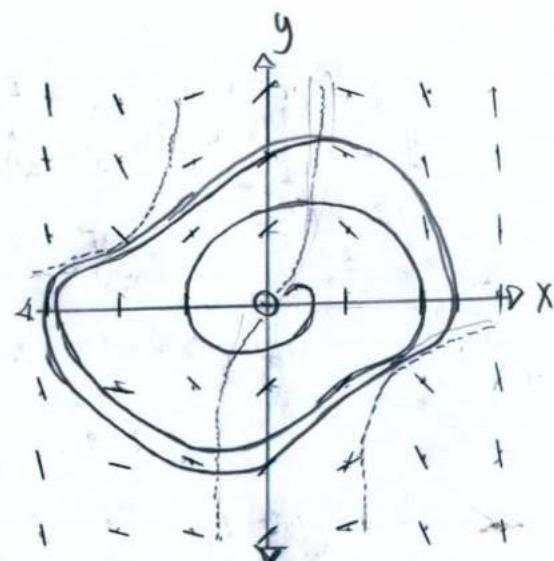
$\dot{y} = -x + y(1-x^2)$  6.1.8 (Van der Pol oscillator)

**Fixed points**  $\dot{x} = 0 = y$

$$\dot{y} = 0 = -x + y(1-x^2)$$

$$(x^*, y^*) = (0, 0)$$

**Nullclines**  $y = \frac{x}{1-x^2}$



$\dot{x} = 2xy$  6.1.9. (Dipole Fixed Point)

$$\dot{y} = y^2 - x^2$$

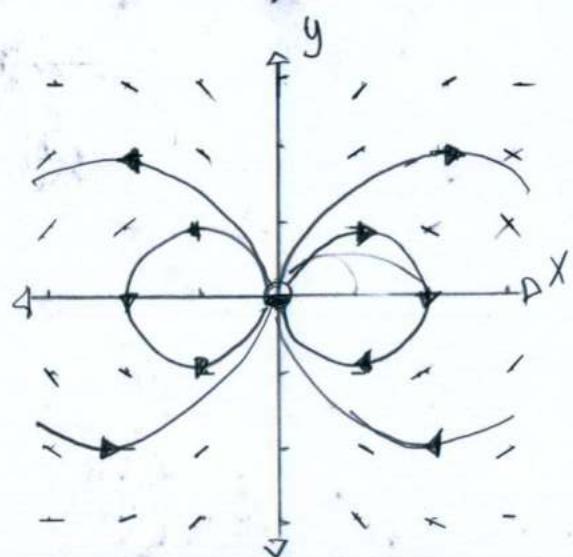
**Fixed Point**  $\dot{x} = 0 = 2xy$

$$\dot{y} = 0 = y^2 - x^2$$

$$(x^*, y^*) = (0, 0)$$

**Nullcline**  $y = x; y = 0; x = 0.$

$$y = -x$$



$$\dot{x} = y + y^2$$

6.1.10

$\dot{y} = -\frac{1}{2}x + \frac{1}{5}y - xy + \frac{6}{5}y^2$  (Two-eyed Monster)

**Fixed Points**

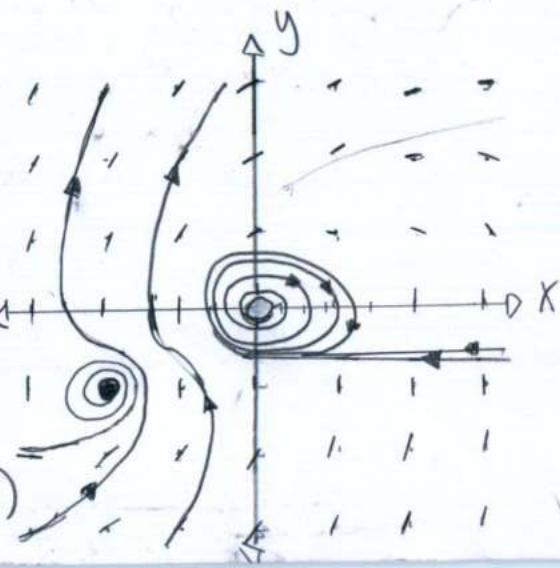
$$x = 0 = y + y^2$$

$$\dot{y} = 0 = -\frac{1}{2}x + \frac{1}{5}y - xy + \frac{6}{5}y^2$$

$$(x^*, y^*) = (0, 0), (-2, -1)$$

**Nullclines**  $y = 0; x = 0$

$$y = \frac{1}{12}(\sqrt{25x^2 + 50x + 1} + 5x - 1)$$



$$\dot{x} = y + y^2$$

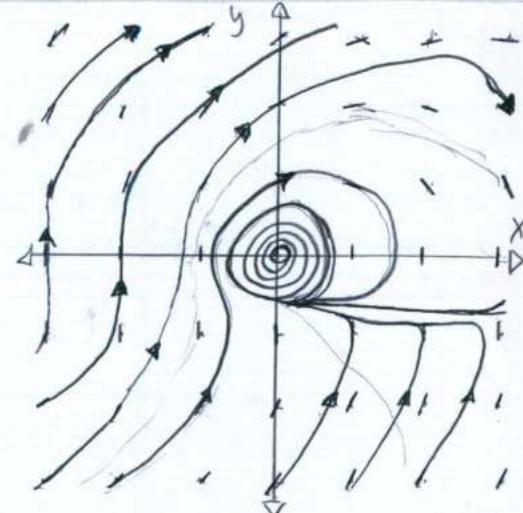
$$\dot{y} = -x + \frac{1}{5}y - xy + \frac{6}{5}y^2$$

6.1.11 [Fixed Points]  $\dot{x} = 0 = y + y^2$

$$\dot{y} = 0$$

$$= -x + \frac{1}{5}y - xy + \frac{6}{5}y^2$$

$$(x^*, y^*) = (0, 0), (4, 4)$$

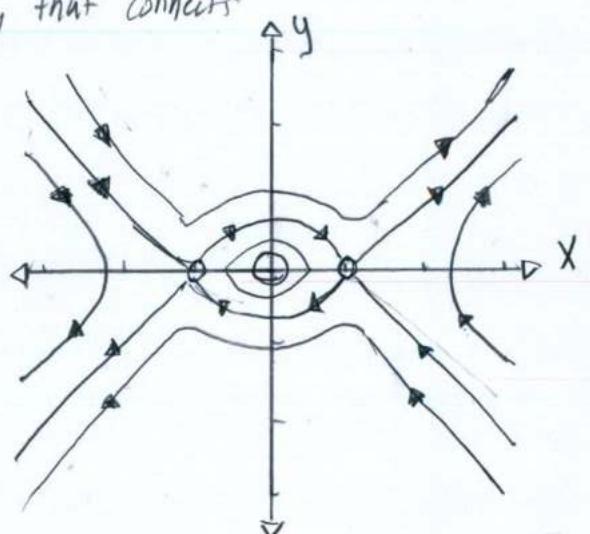


[Nullcline]

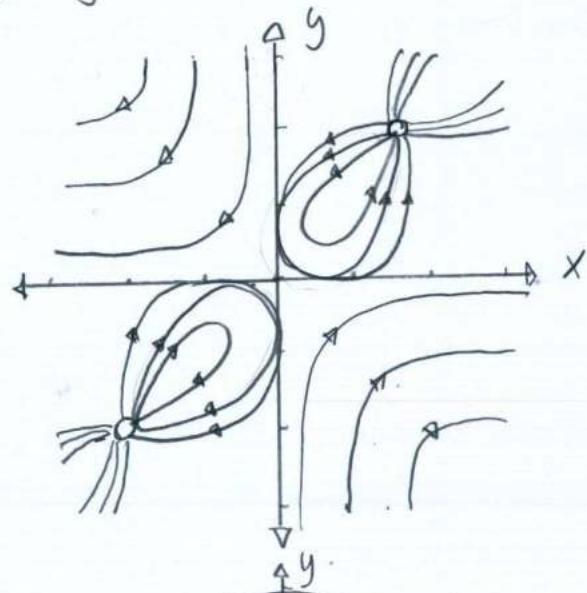
$$y = \frac{1}{12}(-1 \pm \sqrt{25x^2 + 110x + 1 + 5x^2})$$

6.1.12.

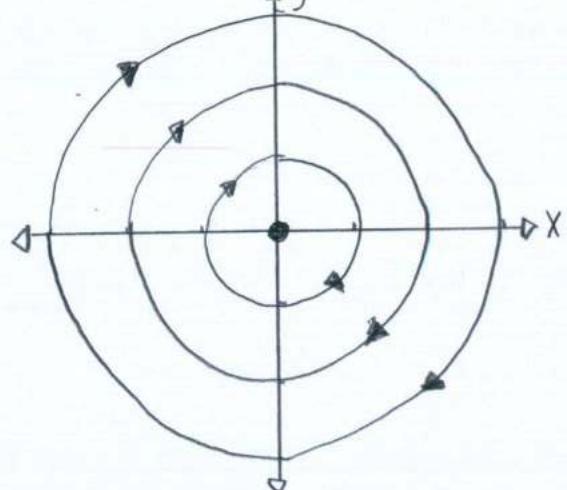
a. a single trajectory that connects the saddles.



b. there is no trajectory that connects the saddles



6.1.13: A phase portrait with three closed orbits and one fixed point.



$$\dot{x} = x + e^{-y}$$

$$\dot{y} = -y$$

6.1.14.

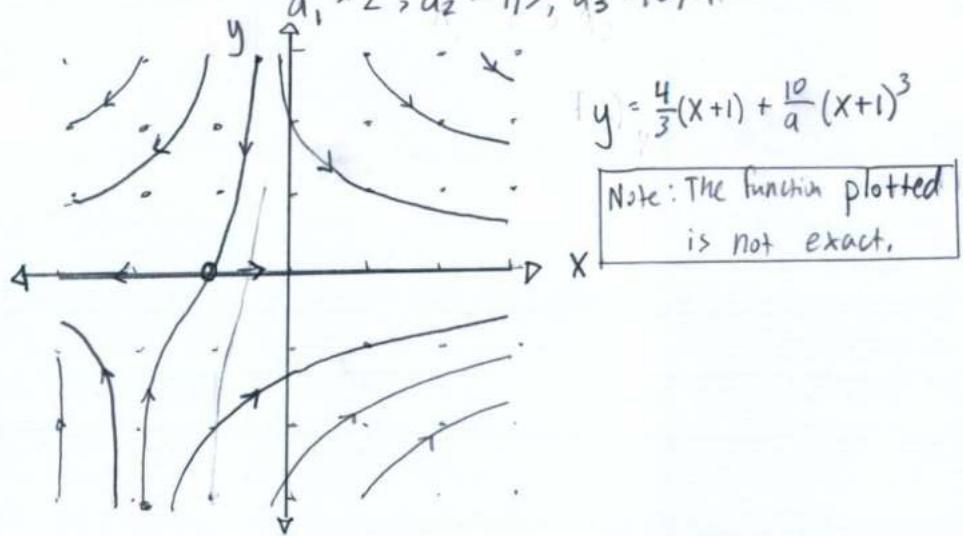
a.  $u = x+1$ ;  $\frac{dy}{du} = \frac{dy}{dt} \frac{du}{dt} = -y$ ;  $\frac{du}{dt} = \frac{dx}{dt} = u-1+(1-y+\frac{y^2}{2}-\dots)$

$$\frac{dy}{du} = -\frac{y}{u-y+y^2/2-y^3/6+\dots}$$

$$y = \frac{a_1}{a_1-1} + \frac{a_1^3-2a_2}{2(a_1-1)^2} u + \frac{2a_1^4+a_1^5-13a_1^2a_2+12(a_2^2+a_3-a_1a_3)}{12(a_1-1)^3} u^3 \\ = a_1 + 2a_2 u + 3a_3 u^3 + \dots$$

$$a_1 = 2; a_2 = 4/3; a_3 = 10/9.$$

b.



6.2.1: Yes, trajectories do not intersect, however, may seem so for low resolution plots.

$$\dot{x} = y$$

$$6.2.2: a. D: x^2+y^2 < 4$$

Poincaré-Bendixson Theorem: no fixed points and a bounded region, then the trajectory is a closed orbit, and approaches the closed orbit.

Bounded Region - D:  $x^2+y^2 < 4$

Fixed points: Zero, outside of the center

Existence and Uniqueness is satisfied for a Closed orbit.

b. If  $y(t) = \cos(t)$ , then  $\dot{y} = 0 = -x + (1-(x^2+\cos(t)^2))y @ t=0$

$$\text{Identity: } x^2+y^2=1$$

$$\text{then, } x=0 @ t=0$$

$$x(t) = \sin(t)$$

C.  $x(0) = \frac{1}{2} ; y(0) = 0$  ;  $x(t)^2 + y(t)^2$  must be less than one because  
 a larger value forces  $y$  to become  
negative and not a closed orbit.

$$\begin{aligned}\dot{x} &= x - y \\ \dot{y} &= x^2 - 4\end{aligned}$$

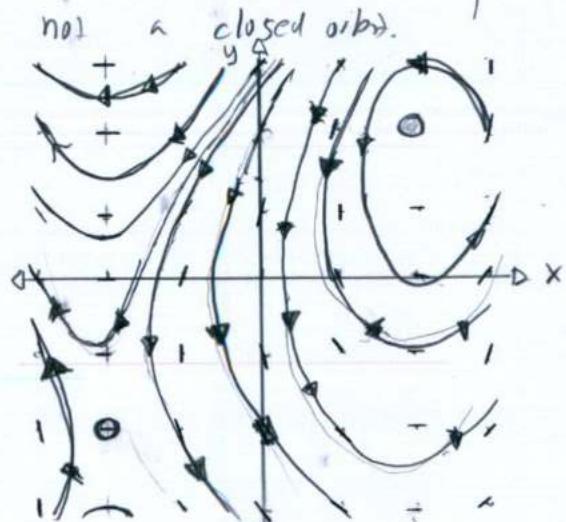
### 6.3.1 Fixed points

$$\dot{x} = 0 = x - y$$

$$\dot{y} = 0 = x^2 - 4$$

$$(x^*, y^*) = (2, 2), (-2, -2), (0, 0)$$

"unstable" "unstable"



$$\begin{aligned}\dot{x} &= \sin y \\ \dot{y} &= x - x^3\end{aligned}$$

### 6.3.2 Fixed points

$$\dot{x} = 0 = \sin y$$

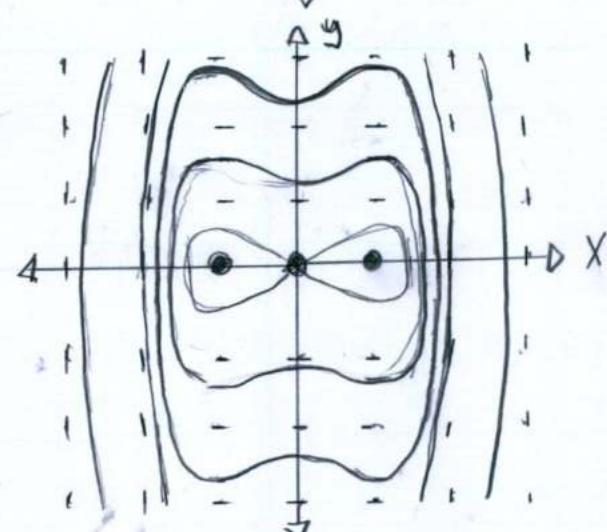
$$\dot{y} = 0 = x - x^3$$

$$(x^*, y^*) = (1, n\pi)$$

$(-1, n\pi)$

$$(0, n\pi)$$

"stable"



$$\begin{aligned}\dot{x} &= 1 + y + e^{-x} \\ \dot{y} &= x^3 - y\end{aligned}$$

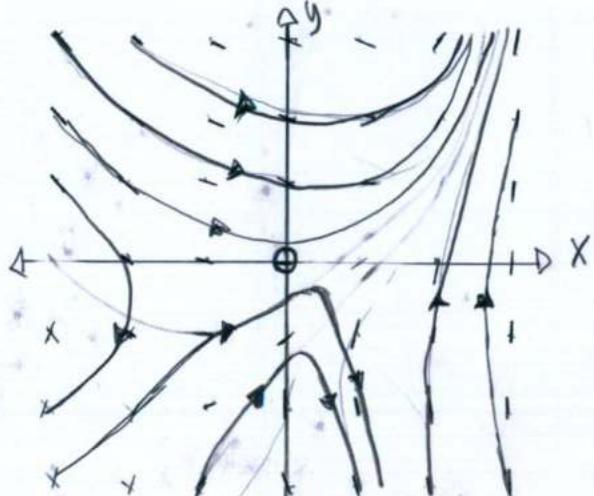
### 6.3.3 Fixed Points

$$\dot{x} = 1 + y - e^{-x} = 0$$

$$\dot{y} = x^3 - y = 0$$

$$(x^*, y^*) = (0, 0)$$

"unstable"



$$\begin{aligned}\dot{x} &= y + x - x^3 \\ \dot{y} &= -y\end{aligned}$$

### 6.3.4 Fixed Points

$$\dot{x} = 0 = y + x - x^3$$

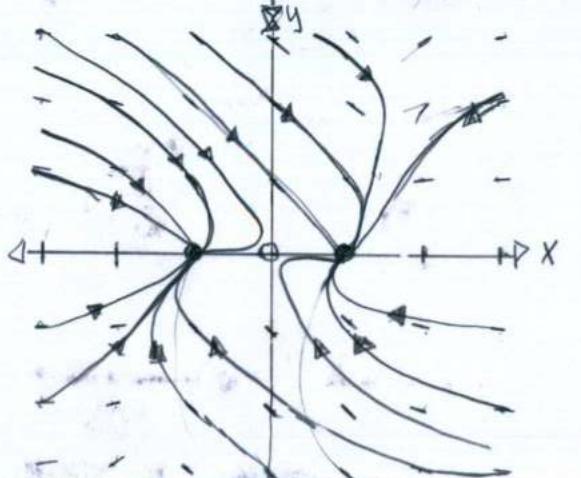
$$\dot{y} = 0 = -y$$

$$(x^*, y^*) = (1, 0), (-1, 0)$$

"stable"

$$(0, 0)$$

"unstable"





$$\dot{x} = \sin y$$

6.3.5. Fixed Points:  $\dot{x} = 0 = \sin y$

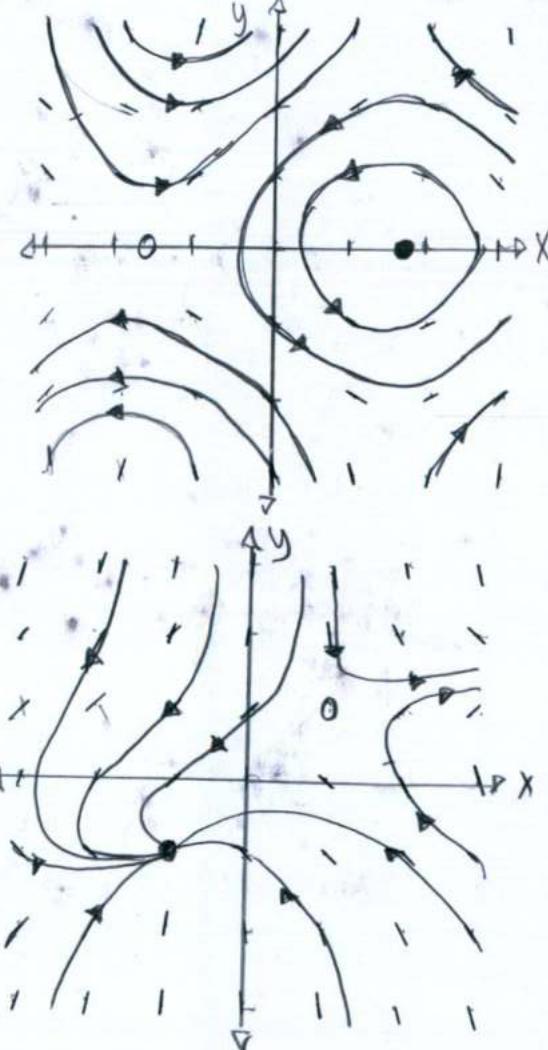
$$\dot{y} = 0 = \cos x$$

$$\dot{y} = 0 = \cos x$$

$$(x^*, y^*) = ((n + \frac{1}{2})\pi, n\pi)$$

$n$  is odd "stable"

$n$  is even "unstable"



$$\dot{x} = xy - 1$$

$$\dot{y} = x - y^3$$

6.3.6. Fixed Points:  $\dot{x} = 0 = xy - 1$

$$\dot{y} = 0 = x - y^3$$

$$(x^*, y^*) = (1, 1), (-1, -1)$$

"unstable" "stable"

6.3.7. The phase portraits of problems 6.3.1-6.3.6 are computer generated.

$$\ddot{x} = \frac{Gm_2}{(x-a)^2} - \frac{Gm_1}{x^2}$$

$$6.3.8. a. g = \ddot{x} = \frac{Gm}{r^2}; \quad m_1 \xrightarrow{a} m_2$$

$$(x) \qquad (x-a)$$

$$\ddot{x}_1 = \frac{Gm_1}{x^2} \quad \ddot{x}_2 = \frac{Gm_2}{(x-a)^2}; \quad \ddot{x} = \ddot{x}_2 - \ddot{x}_1 = \frac{Gm_2}{(x-a)^2} - \frac{Gm_1}{x^2}$$

b. Equilibrium Position:  $\ddot{x} = 0 = \frac{Gm_2}{(x-a)^2} - \frac{Gm_1}{x^2}; \quad m_1(x-a)^2 = m_2 \cdot x^2$   
 $m_1(x^2 - 2xa + a^2) = m_2 \cdot x^2$

$$(m_1 - m_2)x^2 - 2xa + a^2 = 0$$

$$x = \frac{2a \pm \sqrt{4a^2 - 4a^2(m_1 - m_2)}}{2(m_1 - m_2)}$$

When  $m_1 \neq m_2$   
 "stable"

$$\begin{aligned} \dot{x} &= y^3 - 4x \\ \dot{y} &= y^3 - y - 3x \end{aligned} \quad \text{6.3.9. Fixed points: } \begin{aligned} \dot{x} = 0 &= y^3 - 4x \quad ; (x^*, y^*) = (0, 0), (2, 2), (-2, -2) \\ \dot{y} = 0 &= y^3 - y - 3x \end{aligned}$$

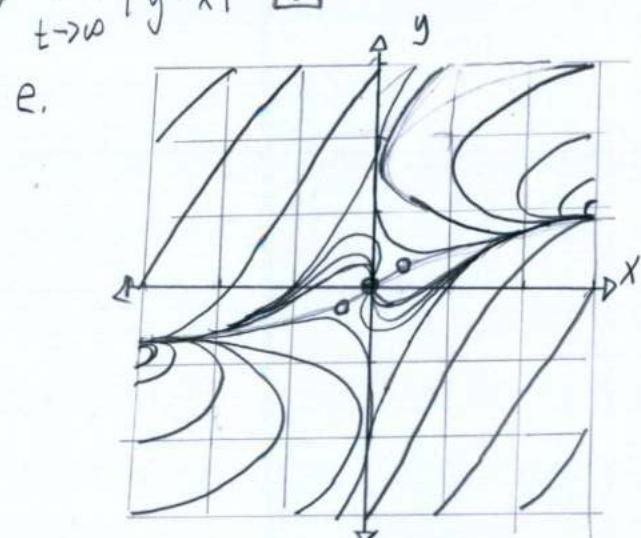
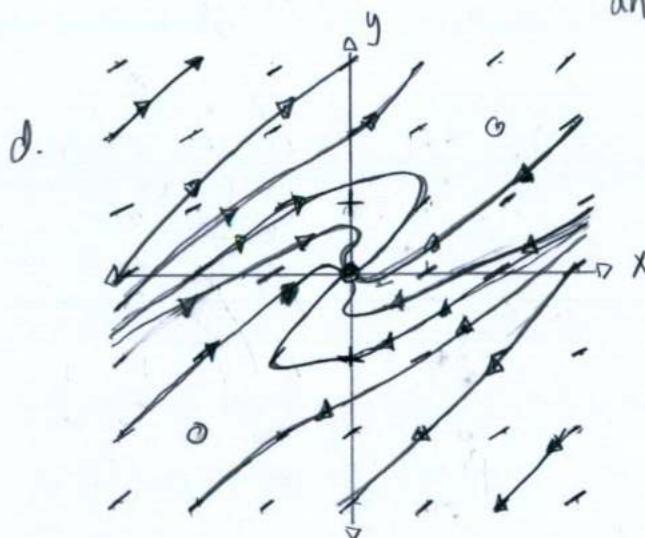
"stable" "unstable" "unstable"

b. If  $x=y$ , then  $\dot{x} = x^3 - 4x$  and  $\dot{y} = x^3 - y - 3x$ , so  $\frac{dy}{dx} = 1$

$$c. \lim_{t \rightarrow \infty} |\dot{x} - \dot{y}| = \lim_{t \rightarrow \infty} |y^3 - 4x - y^3 + y + 3x| = \lim_{t \rightarrow \infty} |y - x|$$

$$\text{If } u = y - x, \text{ then } y - x = Ce^{-t}, \text{ then } \lim_{t \rightarrow \infty} |Ce^{-t}| = 0$$

$$\text{and } \lim_{t \rightarrow \infty} |y - x| = 0$$



$$\begin{aligned} \dot{x} &= xy \\ \dot{y} &= x^2 - y \end{aligned} \quad \text{6.3.10}$$

$$a. u = x - x^*; v = y - y^*; \dot{u} = \dot{x} = f(x^* + u, y^* + v) = f(x^*, y^*) + u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + \dots$$

$$\dot{v} = \dot{y} = g(x^* + u, y^* + v) = g(x^*, y^*) + u \frac{\partial g}{\partial x} + v \frac{\partial g}{\partial y} + \dots$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} ; A = \begin{bmatrix} y & x \\ 2x & -1 \end{bmatrix}$$

Fixed Point  $(0, 0)$ ;  $A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ , so the origin is a non-isolated fixed point because  $\Delta = 0$ .

$$b. \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = 0; (A - \lambda)U = \dot{U} = 0; (A - \lambda) = \begin{bmatrix} y - \lambda & x \\ 2x & -1 - \lambda \end{bmatrix} = (y - \lambda)(-1 - \lambda) - 2x^2 = 0$$

$$\lambda = (y - 1) \pm \sqrt{0x^2 + (y - 1)^2}$$

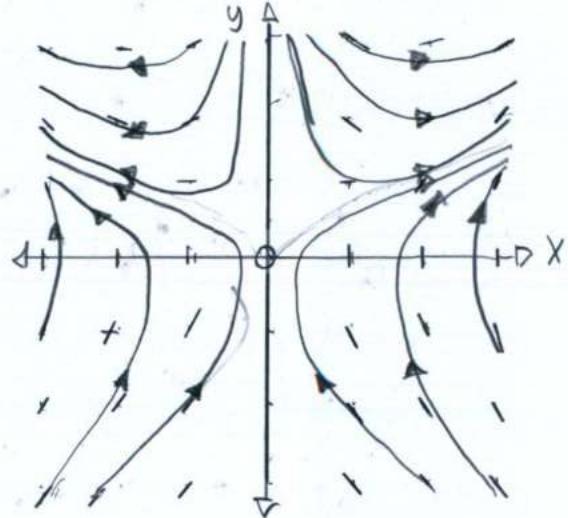
Z  
Thus,  $\Delta = \lambda_1 \lambda_2 \neq 0$  and the center is an isolated fixed point.

C. Nullclines

$$y = \pm \sqrt{x}$$

$$y = 0 \Rightarrow x = 0$$

"Saddle Point"



| d. See Part C |

6.3.11. a.  $\dot{r} = -r$   
 $\dot{\theta} = \frac{1}{\ln r}$

$$r(t) = C e^{-t}; \theta(t) = \ln \frac{\ln C}{\ln C - t} + D$$

Given  $(r_0, \theta_0)$ ; then  $r(t) = r_0 e^{-t}$

$$\theta(t) = \ln \frac{\ln r_0}{|\ln r_0 - t|} + \theta_0$$

b.  $\lim_{t \rightarrow \infty} |\theta(t)| = \lim_{t \rightarrow \infty} \left| \ln \frac{\ln r_0}{|\ln r_0 - t|} + \theta_0 \right| \neq \infty$

$$\lim_{t \rightarrow \infty} r(t) = \lim_{t \rightarrow \infty} r_0 e^{-t} = 0$$

c.  $\dot{r} = -\sqrt{x^2 + y^2}; \dot{\theta} = \frac{1}{\ln \sqrt{x^2 + y^2}}$

d.  $\dot{r} = \frac{d}{dt} \sqrt{x^2 + y^2} = \frac{x \dot{x} + y \dot{y}}{\sqrt{x^2 + y^2}} = -r = -\sqrt{x^2 + y^2}$

$$x \dot{x} + y \dot{y} = -x^2 - y^2$$

$$\dot{\theta} = \frac{d}{dt} \arctan \left( \frac{y}{x} \right) = \frac{x \dot{y} - y \dot{x}}{x^2 + y^2} = \frac{1}{\ln(x^2 + y^2)}$$

$$x \dot{y} - y \dot{x} = \frac{2(x^2 + y^2)}{\ln(x^2 + y^2)}$$

$$x(x \dot{x} - y \dot{y}) - y(x \dot{y} - y \dot{x}) = (x^2 + y^2) \dot{x} = -x(x^2 + y^2) - y \frac{2(x^2 + y^2)}{\ln(x^2 + y^2)}$$

$$\dot{x} = -x - \frac{2y}{\ln(x^2 + y^2)}$$

$$x(x \dot{y} - y \dot{x}) + y(x \dot{x} + y \dot{y}) = (x^2 + y^2) \dot{y} = -y(x^2 + y^2) + x \frac{2(x^2 + y^2)}{\ln(x^2 + y^2)}$$

$$\dot{y} = -y + \frac{2x}{\ln(x^2 + y^2)}$$

$$d. \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}; \quad A = \begin{pmatrix} -1 + \frac{4xy}{(x^2+y^2)\ln^2(x^2+y^2)} & \frac{4y^2}{(x^2+y^2)\ln^2(x^2+y^2)} - \frac{2}{\ln(x^2+y^2)} \\ \frac{2}{\ln(x^2+y^2)} - \frac{4x^2}{(x^2+y^2)\ln^2(x^2+y^2)} & \frac{-4xy}{(x^2+y^2)\ln^2(x^2+y^2)} - 1 \end{pmatrix}$$

$$A_{(0,0)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \boxed{\dot{x} = -x, \dot{y} = -y}$$

$$\Theta = \tan^{-1}\left(\frac{y}{x}\right) \quad 6.3.12. \quad \dot{\theta} = \frac{d}{dt} \tan^{-1}\left(\frac{y}{x}\right) = \frac{\frac{1}{x} \left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^2} = \frac{\dot{y}x - \dot{x}y}{x^2 + y^2} = \boxed{\frac{x\dot{y} - y\dot{x}}{r^2}}$$

$$\ddot{x} = -y - x^3$$

$$6.3.13. \text{ Linearization: } u = x - x^*; \quad v = y - y^*$$

$$\ddot{x} = \ddot{u} = f(x, y) = f(u + x^*, v + y^*) = f(x^*, y^*) + u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + \dots$$

$$\ddot{y} = \ddot{v} = g(x, y) = g(u + x^*, v + y^*) = g(x^*, y^*) + u \frac{\partial g}{\partial x} + v \frac{\partial g}{\partial y} + \dots$$

$$\begin{pmatrix} \ddot{u} \\ \ddot{v} \end{pmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}; \quad A = \begin{bmatrix} -3x^2 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A_{(0,0)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\Delta = 0; \quad \tau = 0; \quad \boxed{\text{center}}$$

$$\text{Eigenvalues: } \ddot{U} = A\ddot{U}; \quad \lambda U = A U; \quad (A - \lambda)U = 0;$$

$$\ddot{U} = 0; \quad (A - \lambda) = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0 \quad \lambda_{1,2} = \pm i$$

Thus,  $\Delta = -1, \tau = 0$ , so the center is a spiral,  
also supported by  $\tau^2 - 4\Delta > 0$ .

$$\ddot{x} = -y + ax^2 \quad 6.3.14. \quad a > 0; \quad \text{Fixed points: } \dot{x} = 0 = -y + ax^2;$$

$$\dot{y} = x + ay^3; \quad \dot{y} = 0 = x + ay^3;$$

$$(x^*, y^*) = (0, 0)$$

$$\text{Linearization: } u = x - x^*; \quad v = y - y^*$$

$$\ddot{x} = \ddot{u} = f(x, y) = f(u + x^*, v + y^*) = f(x^*, y^*) + u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + \dots$$

$$\ddot{y} = \ddot{v} = g(x, y) = g(u + x^*, v + y^*) = g(x^*, y^*) + u \frac{\partial g}{\partial x} + v \frac{\partial g}{\partial y} + \dots$$

$$\begin{pmatrix} \ddot{u} \\ \ddot{v} \end{pmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}; \quad A = \begin{bmatrix} 2ax & -1 \\ 1 & 3ay^2 \end{bmatrix};$$

$$A_{(0,0)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}; \Delta=0, \tau=0; \text{center}$$

Eigenvalues:  $\dot{U} = AU; \lambda U = AU; (A - \lambda)U = 0$

$$(A - \lambda) = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

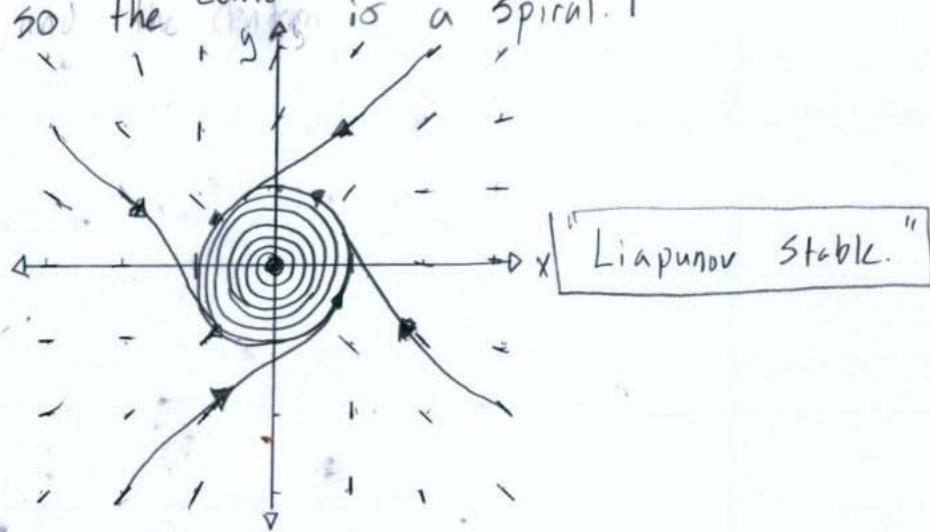
$$\lambda_{1,2} = \pm i$$

Thus, eigenvalues demonstrate  $\Delta = -1, \tau = 0, \tau^2 - 4\Delta > 0,$

so the center is a spiral.

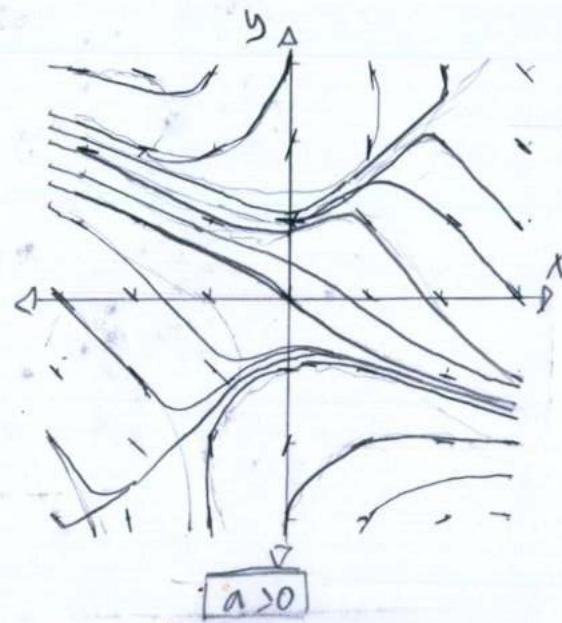
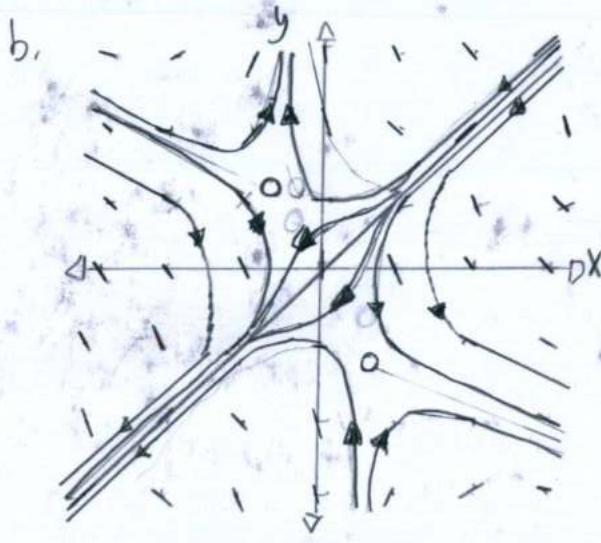
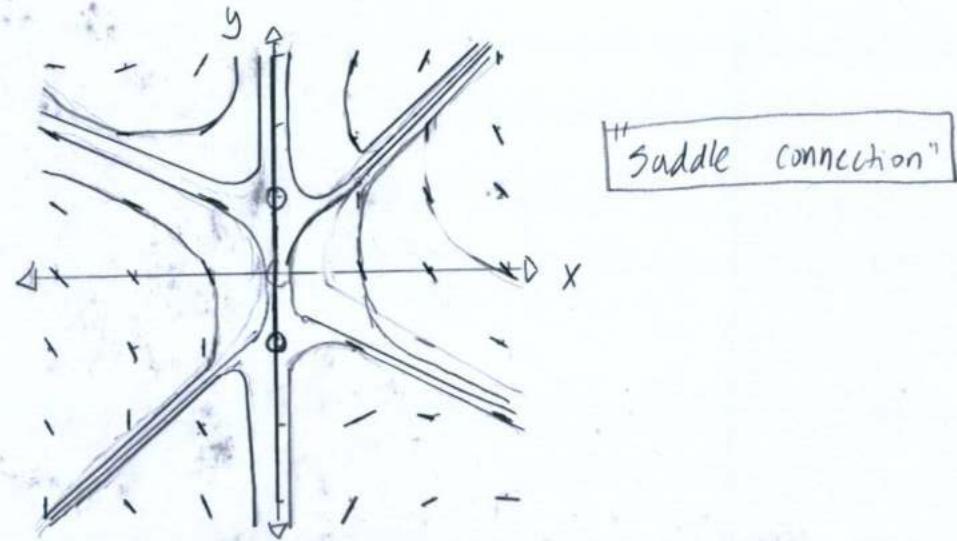
$$\dot{r} = r(1 - r^2) \quad 6.3.15$$

$$\dot{\theta} = 1 - \cos \theta$$



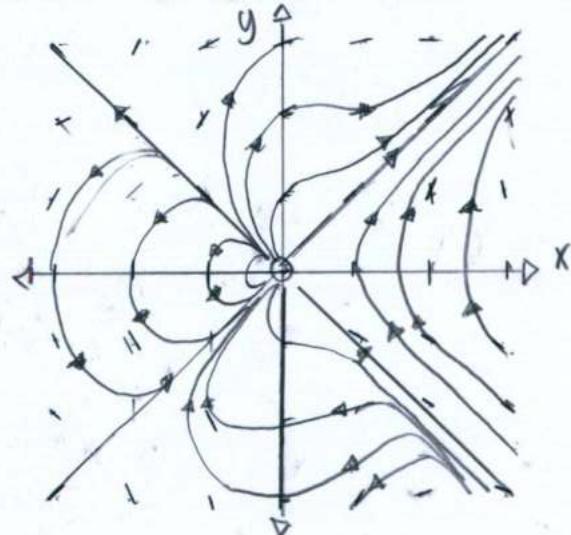
$$\dot{x} = a + x^2 - xy \quad 6.3.16$$

$$\dot{y} = y^2 - x^2 - 1$$



$$\dot{x} = xy - x^2y + y^3 \quad 6.3.17.$$

$$\dot{y} = y^2 + x^3 - xy^2$$



$$\dot{x} = x(3-x-y) \quad 6.4.1$$

$$\dot{y} = y(2-x-y)$$

$$\dot{x} = 0 = x(3-x-y)$$

$$\dot{y} = 0 = y(2-x-y)$$

$$(x^*, y^*) = (0, 0) \text{ "unstable"}$$

Nullclines

$$x = 0$$

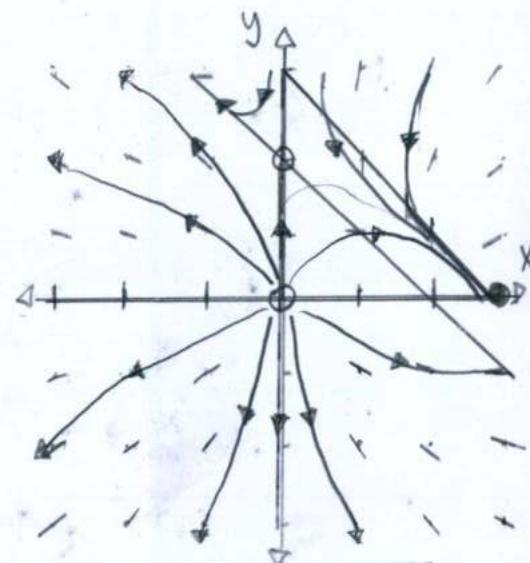
$$y = 0$$

$$y = 2-x ; y = 3-x$$

$$(3, 0) \text{ "stable"}$$

$$(0, 2) \text{ "unstable"}$$

Basin of Attraction  $x \geq 0 \wedge y \geq 0$



$$\dot{x} = x(3-2x-y) \quad 6.4.2$$

$$\dot{y} = y(2-x-y)$$

$$\dot{x} = 0 = x(3-2x-y)$$

$$\dot{y} = 0 = y(2-x-y)$$

$$(x^*, y^*) = (0, 0) \text{ "unstable"}$$

$$(0, 2) \text{ "unstable"}$$

$$(1, 1) \text{ "stable"}$$

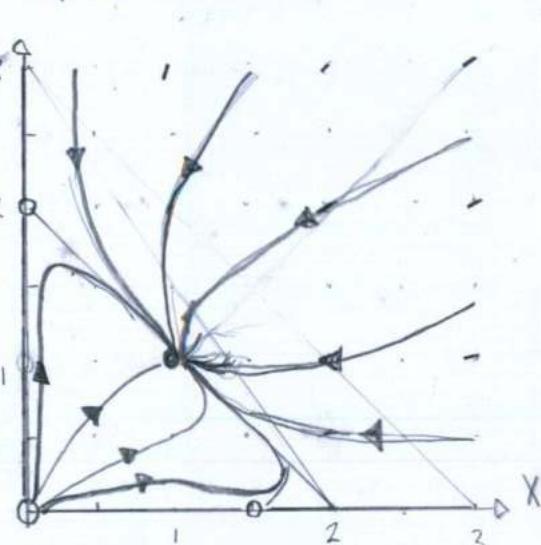
$$(3/2, 0) \text{ "unstable"}$$

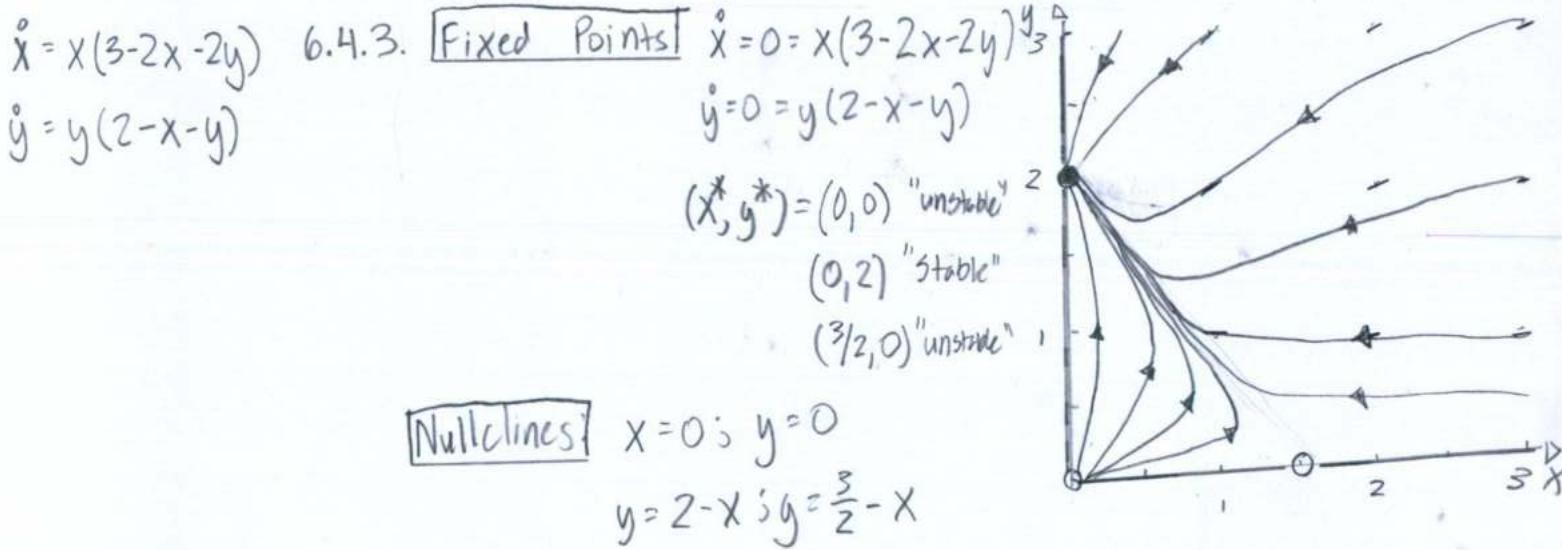
Nullclines  $x = 0 ; y = 0$

$$y = 3-2x$$

$$y = 2-x$$

Basin of Attraction  $(x \geq 0) \wedge (y \geq 0)$





**Nullclines:**  $x=0; y=0$   
 $y=2-x; y=\frac{3}{2}-x$

**Basin of Attraction:**  $(x > 0) \wedge (y > 0)$

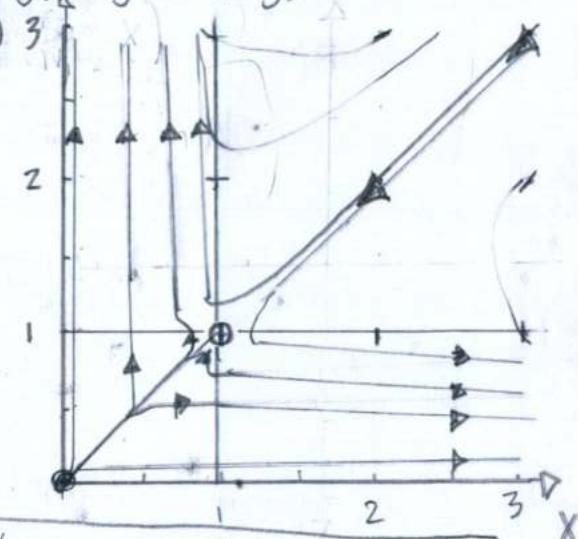
- $\dot{N}_1 = r_1 N_1 - b_1 N_1 N_2$  6.4.4.  
 $\dot{N}_2 = r_2 N_2 - b_2 N_1 N_2$
- a. The  $N_1$  and  $N_2$  model is less realistic because population for rabbits and sheep decreases from an interaction.

- b. Unable to complete problem without  $r_1 = r_2 = b = b_2 = 1$

$$x = N_1; y = N_2; t = T; \dot{x} = x(1-y); \dot{y} = y(1-x)$$

c. **Fixed Points:**  $\dot{x} = 0 = x(1-y)$   
 $\dot{y} = 0 = y(1-x)$   
 $(x^*, y^*) = (0,0), (1,1)$

**Nullclines:**  $y=1; x=1$   
 $y=0; x=0$



- d. See part c. in order to denote sheep or rabbit populations approach infinity when rabbit per sheep is less than 1 or sheep per rabbit is less than 1.

e.)  $\frac{dx}{dy} = \frac{x}{y} \frac{(1-y)}{(1-x)}; \int \frac{(1-x)}{x} dx = \int \frac{(1-y)}{y} dy; \ln x - x = \ln y - y + C$

when  $p=1$

$\dot{N}_1 = r_1 N_1 (1 - N_1/K_1) - b_1 N_1 N_2$  6.4.5.  $\frac{dN_1}{dt} \left(\frac{1}{K_1}\right) = r_1 N_1 \left(\frac{1}{K_1}\right) \left(1 - \frac{N_1}{K_1}\right) - b_1 N_1 N_2 \left(\frac{1}{K_1}\right); x = N_1/K_1$

$\dot{N}_2 = r_2 N_2 - b_2 N_1 N_2$   
 $\frac{dx}{dt} = r_1 x (1-x) - b_1 x N_2; \frac{dx}{dt} \left(\frac{1}{r_1}\right) = x(1-x) - \frac{b_1}{r_1} x N_2$

$p = \frac{b_1}{r_1}; T = tr; N_2 = y$

$$\boxed{x' = x(1-x) - p \cdot x \cdot y}$$

$$\dot{y} = \frac{r_2}{r_1} N_2 - \frac{b_2}{r_1} N_1 N_2 = y' = R y - \rho_2 x y^2$$

where  $R = \frac{r_2}{r_1}$ ;  $\rho_2 = \frac{b_2}{r_1} K_1$

Fixed Points

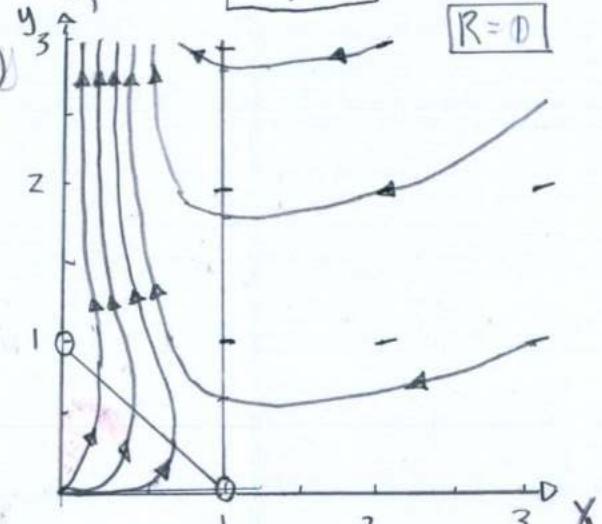
$$x' = 0 = x(1 - x - \rho_1 y)$$

$$y' = 0 = y(R - \rho_2 x)$$

$$(x^*, y^*) = (0, 0), (1, 0)$$

$$\text{If } R=0, (0, y)$$

$$\text{and } \rho_1 = \rho_2 = 1$$



$$\dot{N}_1 = r_1 N_1 (1 - N_1/K_1) - b_1 N_1 N_2$$

$$\dot{N}_2 = r_2 N_2 (1 - N_2/K_2) - b_2 N_1 N_2$$

6.4.6.

$$\text{a. } \dot{N}_1 \left( \frac{1}{K_1} \right) = r_1 \frac{N_1}{K_1} (1 - N_1/K_1) - b_1 \frac{N_1}{K_1} N_2$$

$$x = \frac{N_1}{K_1}$$

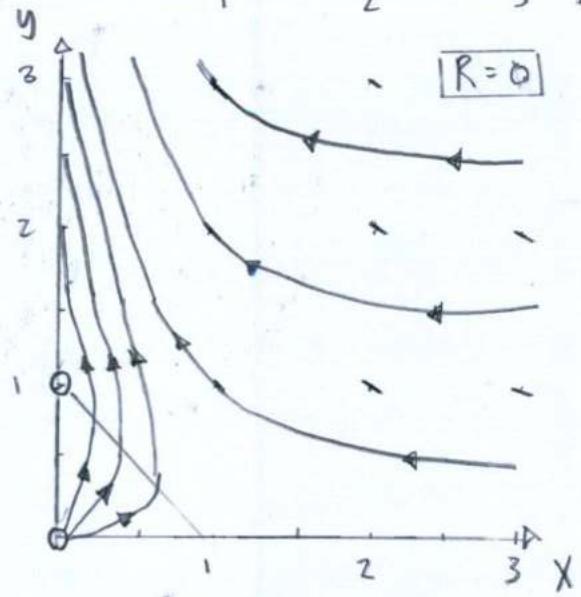
$$\frac{dx}{dt} = r_1 x (1 - x) - b_1 x N_2$$

$$\dot{N}_2 \left( \frac{1}{K_2} \right) = r_2 \frac{N_2}{K_2} (1 - N_2/K_2) - b_2 N_1 \frac{N_2}{K_2}$$

$$y = N_2/K_2$$

$$\frac{dy}{dt} = r_2 y (1 - y) - b_2 N_1 y$$

$$t = \tau \cdot r_1; R = r_2/r_1; \rho_1 = (\frac{b_1}{r_1}) K_2; \rho_2 = (\frac{b_2}{r_1}) K_1$$

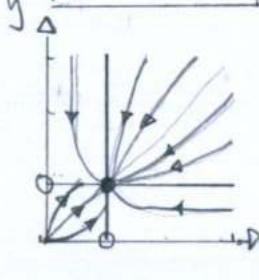


$$\dot{x} = x(1 - x - \rho_1 y); \dot{y} = y(1 - y - \rho_2 x) \text{ when } R=1$$

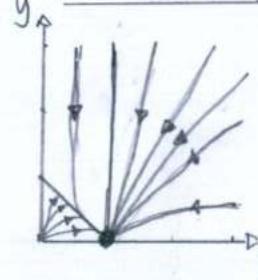
A total of six dimensionless groups suffice!

b.

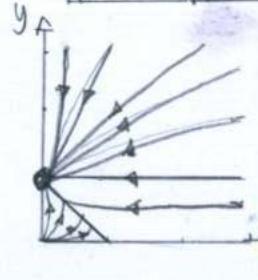
$$\boxed{\rho_1=0 \quad \rho_2=0}$$



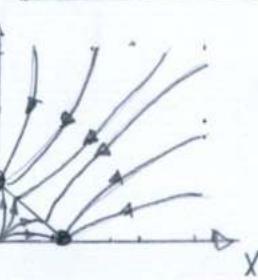
$$\boxed{\rho_1=0 \quad \rho_2=1}$$



$$\boxed{\rho_1=1 \quad \rho_2=0}$$



$$\boxed{\rho_1=1 \quad \rho_2=1}$$



C. The species coexist when  $\rho_1 = \rho_2 = 0$ . This parameter describes the interaction between the rabbits and sheep as noncompetitive.

$$\dot{n}_1 = G_1 N n_1 - K_1 n_1 \quad 6.4.7. \quad N(t) = N_0 - \lambda_1 n_1 - \lambda_2 n_2$$

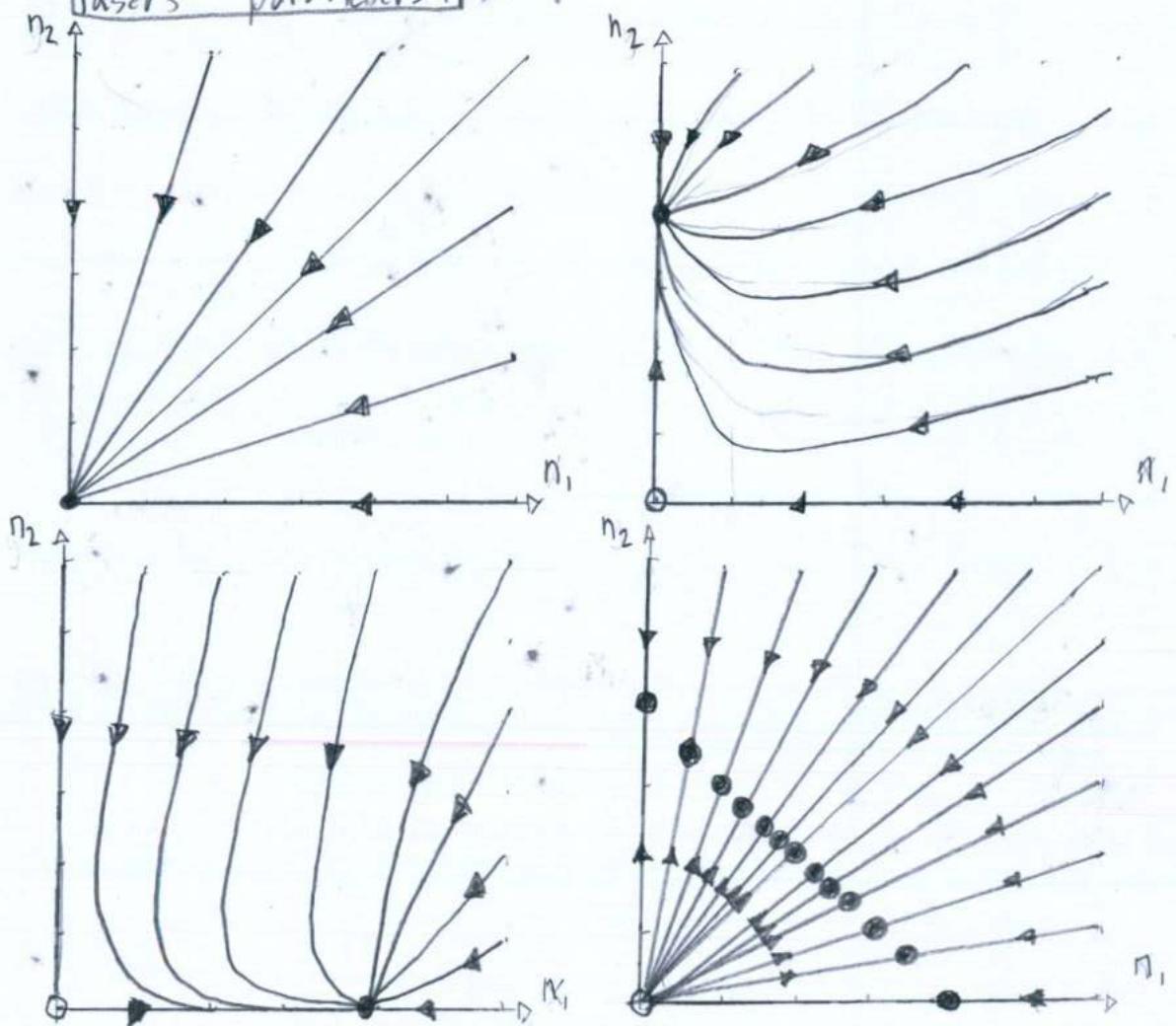
$$\dot{n}_2 = G_2 N n_2 - K_2 n_2 \quad a. \quad A = \begin{pmatrix} \frac{dn_1}{dn_1} & \frac{dn_1}{dn_2} \\ \frac{dn_2}{dn_1} & \frac{dn_2}{dn_2} \end{pmatrix} = \begin{pmatrix} G_1 N - K_1 & 0 \\ 0 & G_2 N - K_2 \end{pmatrix}$$

$$\Delta = (G_1 N - K_1)(G_2 N - K_2) \Rightarrow \tau = (G_1 + G_2)N - (K_1 + K_2)$$

$\tau^2 - 4\Delta > 0$ ; Unstable Node

b. The other fixed points are  $G_1 N = K_1$  and  $G_2 N = K_2$

c. Four phase portraits appear by varying the parameters.

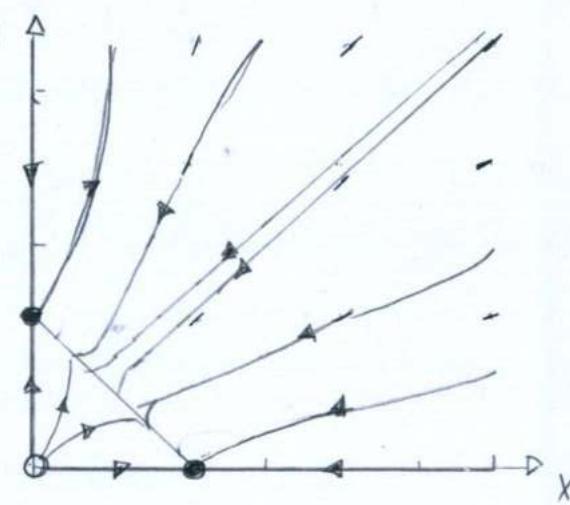


$$\begin{aligned} \dot{x} &= ax^c - \phi x & 6.4.8. a. \text{ If } x_0 + y_0 = 1, \dot{x} + \dot{y} = ax^c - (ax^c + by^c)x + by^c - (ax^c + by^c)y \\ \dot{y} &= by^c - \phi y & = a(1-x-y)x^c + b(1-x-y)y^c \\ \phi &\equiv ax^c + by^c & \text{then } \boxed{\dot{x} + \dot{y} = 0 \text{ and } x(t) + y(t) = 1} \end{aligned}$$

$$b. \lim_{x \rightarrow \infty} \frac{\dot{y}}{x} = \frac{by^c - \phi y}{ax^c - \phi x} = \frac{by^c - (ax^c + by^c)y}{ax^c - (ax^c + by^c)x} \cong \frac{-ax^c}{-ax^{c+1}} = \frac{1}{x} \stackrel{x \rightarrow \infty}{=} 0$$

$$\lim_{y \rightarrow \infty} \frac{\dot{y}}{x} = \frac{by^c - \phi y}{ax^c - \phi x} = \frac{by^c - (ax^c + by^c)y}{ax^c - (ax^c + by^c)x} \cong \frac{-by^{c+1}}{-by^c x} \stackrel{y \rightarrow \infty}{=} \infty$$

c. If  $c=1$ ,



d. If  $c > 1$ , then radial nullclines become generated.

e. If  $c < 1$ , then monotonically decreasing nullclines become generated.

$\dot{I} = I - \kappa C$  6.4.9.  $I \geq 0$ : National Income;  $C \geq 0$ : Rate of Consumer Spending.

$G \geq 0$ : Rate of Government Spending.

$1 < \kappa < \infty$  and  $1 \leq \beta < \infty$

a. Fixed Points:  $\dot{I} = 0 = I - \kappa C$ ;  $\dot{C} = 0 = \beta(I - C - G)$

$$(I^*, C^*) = \left( \frac{\kappa G}{\kappa - 1}, \frac{G}{\kappa - 1} \right)$$

$$\dot{I} = A \cdot I; A = \begin{pmatrix} \frac{\partial I}{\partial I} & \frac{\partial I}{\partial C} \\ \frac{\partial C}{\partial I} & \frac{\partial C}{\partial C} \end{pmatrix} = \begin{pmatrix} 1 - \kappa & 0 \\ \beta & -\beta \end{pmatrix}$$

$$\text{If } \beta = 1, A = \begin{pmatrix} 1 - \kappa & 0 \\ 1 & -1 \end{pmatrix}, \Delta = -(1-\kappa); \tau = 0; \tau^2 - 4\Delta = 4(1-\kappa)$$

A center node

$$b. G = G_0 + K I ; K > 0 ; I \geq 0, C \geq 0$$

$$\boxed{\text{Fixed Point}} \quad (I^*, C^*) = \left( \frac{K G_0}{K(1-K)-1}, \frac{G_0}{K(1-K)-1} \right)$$

If  $K < K_c = 1 - \frac{1}{\alpha}$ , then  $I & C > 0$

$$\overset{o}{I} = A I \therefore A = \begin{pmatrix} 1 & -\alpha \\ \beta(1-\alpha) & -\beta \end{pmatrix}; (A - \lambda I) = \begin{pmatrix} 1-\lambda & -\alpha \\ \beta(1-\alpha) & -\beta-\lambda \end{pmatrix}$$

$$(1-\lambda)(-\beta-\lambda) + \alpha \cdot \beta(1-\alpha) = 0$$

$$\lambda_{1,2} = \frac{-(\beta-1) \pm \sqrt{(\beta-1)^2 - 4(\alpha(1-\alpha)-1)\beta}}{2}$$

$$A \vec{v}_1 = \begin{pmatrix} (1+\beta) + \sqrt{(\beta-1)^2 - 4(\alpha(1-\alpha)-1)\beta} & -\alpha \\ \beta(1-\alpha) & -(1+\beta) + \sqrt{(\beta-1)^2 - 4(\alpha(1-\alpha)-1)\beta} \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$$

$$[(1+\beta) + \sqrt{(\beta-1)^2 - 4(\alpha(1-\alpha)-1)\beta}] v_1 - \alpha v_2 = 0$$

$$v_{11} = 1; v_{12} = \frac{(1+\beta) + \sqrt{\beta^2 + (2-4\alpha(1-\alpha))\beta}}{\alpha}$$

$$\boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ (1+\beta) + \sqrt{\beta^2 + (2-4\alpha(1-\alpha))\beta} \end{pmatrix}}$$

$$A \vec{v}_2 = \begin{pmatrix} (1+\beta) - \sqrt{(\beta-1)^2 - 4(\alpha(1-\alpha)-1)\beta} & -\alpha \\ \beta(1-\alpha) & -(1+\beta) - \sqrt{(\beta-1)^2 - 4(\alpha(1-\alpha)-1)\beta} \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$$

$$[(1+\beta) - \sqrt{(\beta-1)^2 - 4(\alpha(1-\alpha)-1)\beta}] v_1 - \alpha v_2 = 0$$

$$v_{21} = 1; v_{22} = \frac{(1+\beta) - \sqrt{\beta^2 + (2-4\alpha(1-\alpha))\beta}}{\alpha}$$

$$\boxed{\vec{v}_2 = \begin{pmatrix} 1 \\ (1+\beta) - \sqrt{\beta^2 + (2-4\alpha(1-\alpha))\beta} \end{pmatrix}}$$

When  $k > k_0$ , the economy gravitates to the positive eigen direction

$$C) G = G_0 + kI^2 \Rightarrow \dot{I} = I - kC = 0; \dot{C} = \beta(I - C - G_0 - k_0 I^2) = 0$$

$$\boxed{\text{Fixed Points}} \quad O = \beta(I(1 - \frac{1}{\lambda}) - G_0 - k_0 I^2) = -k_0 I^2 + I(1 - \frac{1}{\lambda}) - G_0$$

$$(I^*, C^*) = \left( \frac{(1 - \frac{1}{\lambda}) + \sqrt{(1 - \frac{1}{\lambda})^2 + 4k_0 G_0}}{2k_0}, \frac{(1 - \frac{1}{\lambda}) + \sqrt{(1 - \frac{1}{\lambda})^2 - 4k_0 G_0}}{2k_0 \lambda} \right)$$

$$\left( \frac{(1 - \frac{1}{\lambda}) - \sqrt{(1 - \frac{1}{\lambda})^2 - 4k_0 G_0}}{2k_0}, \frac{(1 - \frac{1}{\lambda}) - \sqrt{(1 - \frac{1}{\lambda})^2 - 4k_0 G_0}}{2k_0 \lambda} \right)$$

IF  $G_0 < \frac{(\lambda - 1)^2}{4\lambda^2 k_0}$ , then two positive fixed points exist

IF  $G_0 = \frac{(\lambda - 1)^2}{4\lambda^2 k_0}$ , then one fixed point exists in quadrant #1

IF  $G_0 > \frac{(\lambda - 1)^2}{4\lambda^2 k_0}$ , then zero fixed point exist because of the imaginary radical.

$$\overset{\circ}{X}_i = X_i \left( X_{i-1} - \sum_{j=1}^n X_j X_{j-1} \right) \quad \text{a. If } n=2, \quad \overset{\circ}{X}_1 = X_1 (X_0 - \sum_{j=1}^n X_j X_0) = X_1 (X_2 - \sum_{j=1}^2 X_1 X_2) = X_1 (X_2 - 2X_1 X_2)$$

$$\overset{\circ}{X}_2 = X_2 (X_1 - \sum_{j=1}^n X_2 X_1) = \boxed{X_2 (X_1 - 2X_2 X_1)}$$

$$\text{b. } \overset{\circ}{X}_1 = 0 = X_1 (X_2 - 2X_1 X_2); \quad \overset{\circ}{X}_2 = 0 = X_2 (X_1 - 2X_2 X_1)$$

$$(X_1^*, X_2^*) = (Y_2, Y_2); \quad A = \begin{pmatrix} X_2 - 4X_1 X_2 & 2X_1 (1 - 2X_1) \\ X_2 (1 - 2X_2) & X_1 - 4X_1 X_2 \end{pmatrix}$$

$$A_{(Y_2, Y_2)} = \begin{pmatrix} -1/2 & 0 \\ 0 & -1/2 \end{pmatrix}; \quad \Delta = \frac{1}{4}, \quad \Gamma = -1$$

$$\Gamma^2 - 4\Delta = 0$$

Degenerate and Stable Node

$$\text{c. } U = X_1 + X_2; \quad \dot{U} = \overset{\circ}{X}_1 + \overset{\circ}{X}_2 = X_1 (X_2 - 2X_1 X_2) + X_2 (X_1 - 2X_1 X_2)$$

$$= X_1 X_2 - 2X_1^2 X_2 + X_1 X_2 - 2X_1 X_2^2$$

$$= 2X_1 X_2 (1 - X_1 - X_2) = 2X_1 X_2 (1 - U)$$

$$U(t) = 1 - e^{-2X_1 X_2 t}$$

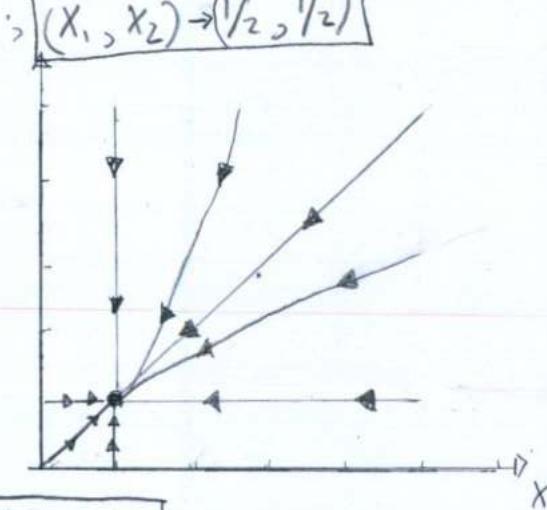
$$\boxed{\lim_{t \rightarrow \infty} U(t) = 1}$$

$$\begin{aligned}
 d. \quad & V = X_1 - X_2 ; \quad \dot{V} = \dot{X}_1 - \dot{X}_2 ; \quad \ddot{V} = X_1(X_2 - 2X_1X_2) - X_2(X_1 - 2X_1X_2) \\
 & = -2X_1X_2(X_1 - X_2) = -2X_1X_2 \cdot V \\
 & V(t) = e^{-2X_1X_2 t} \\
 & \boxed{\lim_{t \rightarrow \infty} V(t) = 0}
 \end{aligned}$$

$$e. \quad \lim_{t \rightarrow \infty} [u(t) + v(t)] = 1 = 2X_1 ; \quad X_1 = 1/2$$

$$\lim_{t \rightarrow \infty} [u(t) - v(t)] = 1 = 2X_2 ; \quad X_2 = 1/2 \quad \boxed{(X_1, X_2) \rightarrow (1/2, 1/2)}$$

f. A large  $n$  value generates a plot which seems to converge to zero, but actually, converges to a positive value close to zero. This argument implies RNA remain at low concentrations indefinitely.



$$\begin{aligned}
 \dot{x} &= rxz \\
 \dot{y} &= ryz \\
 \dot{z} &= -rxz - ryz
 \end{aligned}$$

$$6.4.11 \quad a. \quad \dot{z} = -\dot{x} - \dot{y} ; \quad 0 = \dot{x} + \dot{y} + \dot{z} ; \quad \boxed{1 = x + y + z}$$

b. The limit of the function is bounded by the invariance equation in part a.

Fixed Points:  $\dot{x} = 0 = rxz ; \dot{y} = ryz ; \dot{z} = -rxz - ryz$   
 $(x^*, y^*, z^*) = (x, y, 0), (0, 0, z)$

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial z} \\ \frac{\partial \dot{z}}{\partial x} & \frac{\partial \dot{z}}{\partial y} & \frac{\partial \dot{z}}{\partial z} \end{pmatrix} = \begin{pmatrix} rz & 0 & rx \\ 0 & rz & ry \\ -rz & -rz & -rx - ry \end{pmatrix}$$

$$A_{(x, y, 0)} = \begin{pmatrix} 0 & 0 & rx \\ 0 & 0 & ry \\ 0 & 0 & -rx - ry \end{pmatrix} ; \quad A_{(0, 0, z)} = \begin{pmatrix} rz & 0 & 0 \\ 0 & rz & 0 \\ -rz & -rz & 0 \end{pmatrix}$$

$$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = -rx - ry \quad \lambda_1 = \lambda_2 = 0, \lambda_3 = rz$$

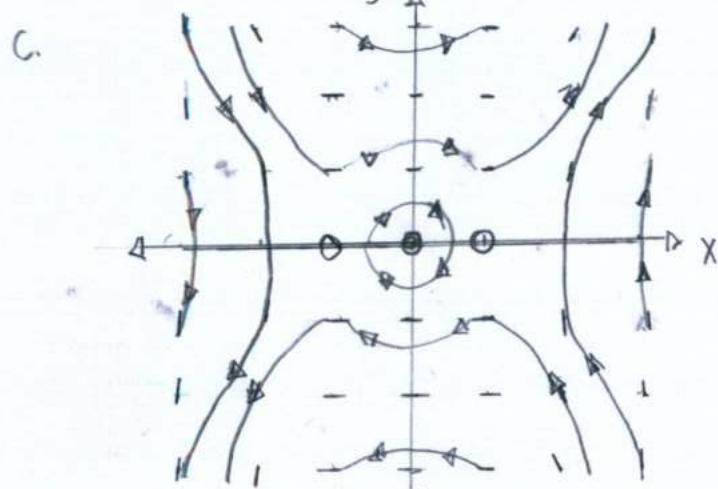
The eigenvectors point in the direction of  $\lambda_3$  for each fixed point.

C. An interpretation from the political terms is  
 $r < 0$ , the centrist pull the extremists to the  
 centrist, while  $r > 0$ , the extremist separate the  
 centrists.

$$\ddot{x} = x^3 - x \quad 6.5.1.a. \quad \dot{x} = y; \quad \dot{y} = x^3 - x; \quad A = \begin{pmatrix} 0 & 1 \\ 3x-1 & 0 \end{pmatrix};$$

Fixed Points:  $\dot{x} = 0 = y; \dot{y} = 0 = x^3 - x; (x^*, y^*) = (-1, 0)$  "center"  
 $(0, 0)$  "saddle"  
 $(1, 0)$  "center"

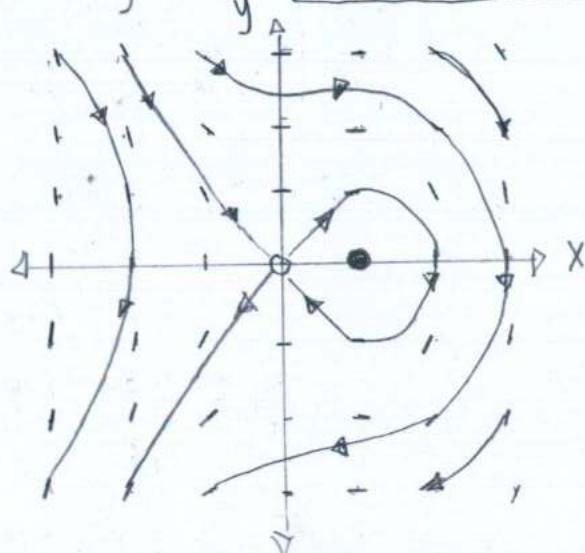
b.  $E = \frac{1}{2}\dot{x}^2 - \int x^3 - x dx = \frac{1}{2}y^2 - \frac{x^4}{4} + \frac{x^2}{2} + C$



$$\ddot{x} = x - x^2 \quad 6.5.2.a. \quad \dot{x} = y; \quad \dot{y} = x - x^2; \quad A = \begin{pmatrix} 0 & 1 \\ 1-2x & 0 \end{pmatrix}$$

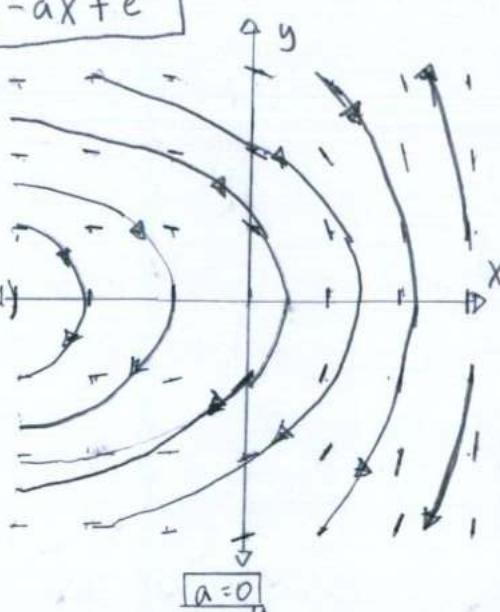
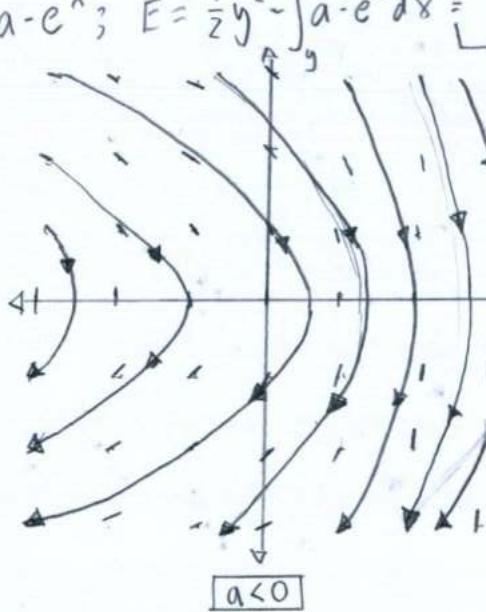
Fixed Points:  $\dot{x} = 0 = y; \dot{y} = 0 = x - x^2; (x^*, y^*) = (1, 0)$  "center"  
 $(0, 0)$  "saddle"

b.  $E = \frac{1}{2}\dot{x}^2 - \int (x - x^2) dx = \frac{1}{2}y^2 - \frac{x^2}{2} + \frac{x^3}{3} + C$



$$C. F = \frac{1}{2}y^2 - \frac{x^2}{2} + \frac{x^3}{3} + C.$$

$\ddot{x} = a - e^x$  6.5.3.  $\dot{x} = y; \dot{y} = a - e^x; E = \frac{1}{2}y^2 - \int a - e^x dx = \frac{1}{2}y^2 - ax + e^x$

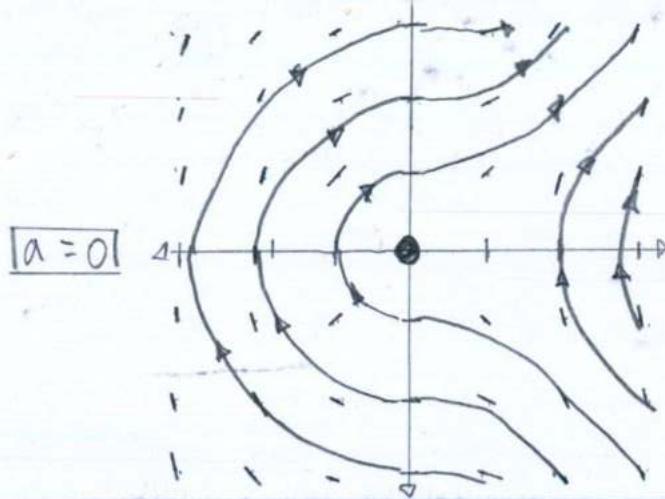
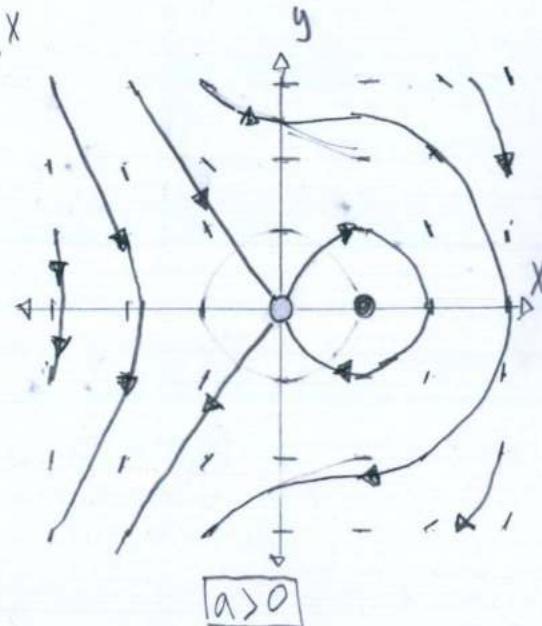
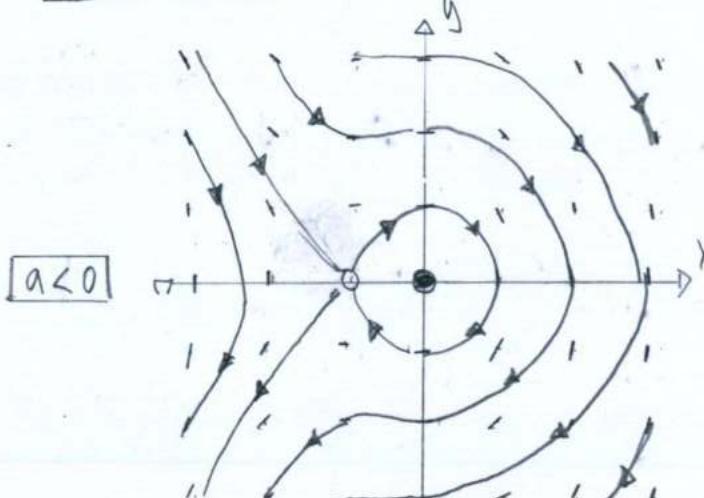


$\ddot{x} = ax - x^2$  6.5.4.  $\dot{x} = y; \dot{y} = ax - x^2$

Conserved Quantity:

$$E = \frac{1}{2}y^2 - \int ax - x^2 dx$$

$$= \frac{1}{2}y^2 - \frac{ax^2}{2} + \frac{x^3}{3} + C$$



$$\ddot{x} = (x-a)(x^2-a) \quad 6.5.5 \quad \dot{x} = y; \quad \ddot{y} = (x-a)(x^2-a)$$

<u>Fixed Points</u>	$\dot{x} = 0 = y$	If $a=1$ , then one fixed point exists in quadrants #1 and #4.
	$\dot{y} = 0 = (x-a)(x^2-a)$	
	$(x^*, y^*) = (a, 0)$	
	$(\sqrt{a}, 0)$	If $0 < a < 1$ or $a > 1$ , then two fixed points exist in quadrants #1 and #4
	$(-\sqrt{a}, 0)$	

$$\dot{x} = -kxy \quad 6.5.6. \text{ a. } \boxed{\text{Fixed Points}} \quad \dot{x} = 0 = -kxy; \quad \dot{y} = 0 = kxy - ly$$

$$\dot{y} = kxy - ly \quad (x^*, y^*) = (0, 0); \quad A = \begin{pmatrix} -ky & -RX \\ ky & RX - l \end{pmatrix}$$

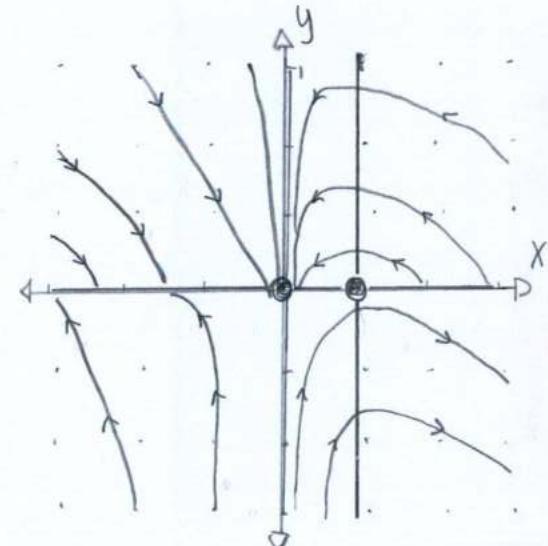
$(0/k, 0)$  "center"  
 $(\frac{l}{R}, 0)$

b. Nullclines

$$x = 0$$

$$y = 0$$

$$x = \frac{l}{k}$$



c.  $\boxed{\frac{dy}{dx} = -1 + \frac{l}{Rx}; \quad y = -x + \frac{l}{k} \ln x + C}$

d. See part c

e. A population is sick from infection. When  $y_0 \geq 0$ .

$$\frac{d^2u}{d\theta^2} + u = \alpha + \varepsilon u^2 \quad 6.5.7 \quad u = 1/r;$$

a.  $\boxed{V^2 + u = \alpha + \varepsilon u^2}$   
 Where  $V = du/d\theta$ .

b. Fixed Points:  $\overset{\circ}{u} = 0 = \overset{\circ}{v}$   
 $\overset{\circ}{v} = 0 = K + \epsilon u^2 - u$   
 $(\overset{\circ}{u}, \overset{\circ}{v}) = \left( \frac{1 + \sqrt{1 - 4K\epsilon}}{2\epsilon}, 0 \right), \left( \frac{1 - \sqrt{1 - 4K\epsilon}}{2\epsilon}, 0 \right)$

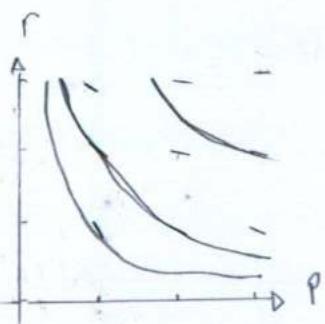
c.  $A = \begin{pmatrix} 0 & 1 \\ 2\epsilon u - 1 & 0 \end{pmatrix}; \lambda_{1A,B} = \pm i\sqrt{1 - 4\alpha\epsilon}; \lambda_{2A,B} = \pm \sqrt{1 - 4\alpha\epsilon}$   
 "Saddle point"      "Linear Center"

d.  $\frac{1}{r} = u = \frac{1 - \sqrt{1 - 4K\epsilon}}{2\epsilon}; r = \frac{2\epsilon}{1 - \sqrt{1 - 4K\epsilon}}$

$H = \frac{p^2}{2m} + \frac{kx^2}{2}$  6.5.8  $\dot{q} = \frac{p}{m}; \dot{p} = -kx$  ;  $H = \frac{p^2}{2m} + \frac{kx^2}{2}$   
 "Momentum"      "Force"      "Kinetic Energy"      "Potential Energy"

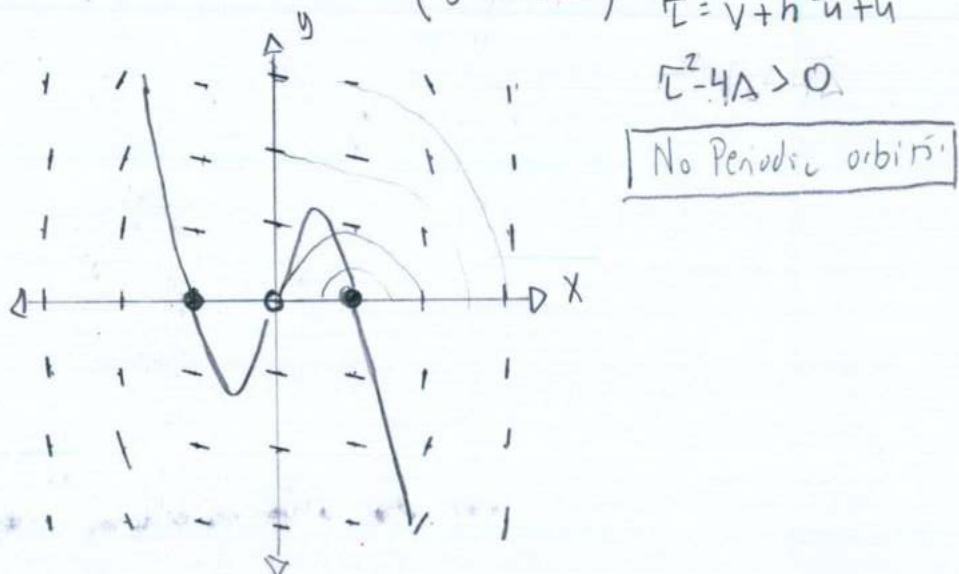
$H(x, p)$  6.5.9  $\dot{H} = \frac{p}{m} \dot{p} + kx \dot{x} = \frac{p}{m}(-kx) + kx\left(\frac{p}{m}\right) = 0$

a. The Hamiltonian plot is similar to the potential plot of  $1/r^2$  where  $K=1$  and  $h = 6.626 \times 10^{-34} \text{ J}\cdot\text{s}$ .



- b.  $E = -k^2/2h^2 < E < 0$        $E = 0$        $E > 0$
- |                          |                        |                          |
|--------------------------|------------------------|--------------------------|
| ◦ Slope is negative      | Slope is zero          | Slope is positive        |
| ◦ Momentum is decreasing | ◦ Momentum is constant | ◦ Momentum is increasing |
| ◦ Radius is increasing   | ◦ Radius is increasing | ◦ Radius is increasing   |

c. If  $K < 0$ , then  $A = \begin{pmatrix} V & 0 \\ 0 & h^2 u + u \end{pmatrix}; \Delta = V(h^2 u + u); T = V + h^2 u + u$



$$\begin{aligned}\dot{x} &= xy \\ \dot{y} &= -x^2\end{aligned}$$

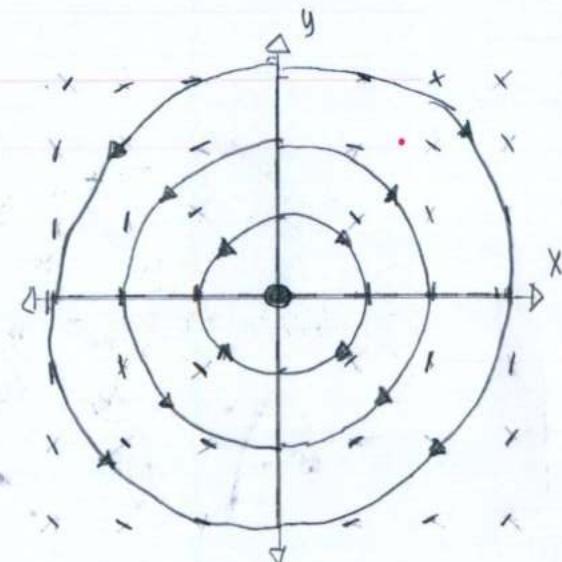
6.5.12.

a.  $E = x^2 + y^2$ ;  $E' = 2x\dot{x} + 2y\dot{y} = 2x^2y - 2y^2x = 0$

b.  $(x^*, y^*) = (0, 0)$ ;  $A = \begin{pmatrix} y & x \\ -2x & 0 \end{pmatrix}$ ;  $A_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$

$(0, y)$ ;  $A_{(0,y)} = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} = 0$

c. See part B; Non-isolated Fixed Point.



$$\ddot{x} + x + \varepsilon x^3 = 0$$

6.5.13.

a.  $E = \frac{1}{2}\dot{x}^2 - \int(-x - \varepsilon x^3)dx$

$$= \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{\varepsilon}{4}x^4 - U(x)$$

$$\begin{vmatrix} E_{xx} & E_{x\dot{x}} \\ E_{\dot{x}x} & E_{\dot{x}\dot{x}} \end{vmatrix} = \begin{vmatrix} 1+3\varepsilon x^2 & \dot{x} + x + \varepsilon x^3 \\ 0 & 1 \end{vmatrix} = 1 \Rightarrow \text{a continuous derivative exists.}$$

b. If  $\varepsilon < 0$ , a hyperbola trajectory is the closed orbit about  $(0,0)$ .

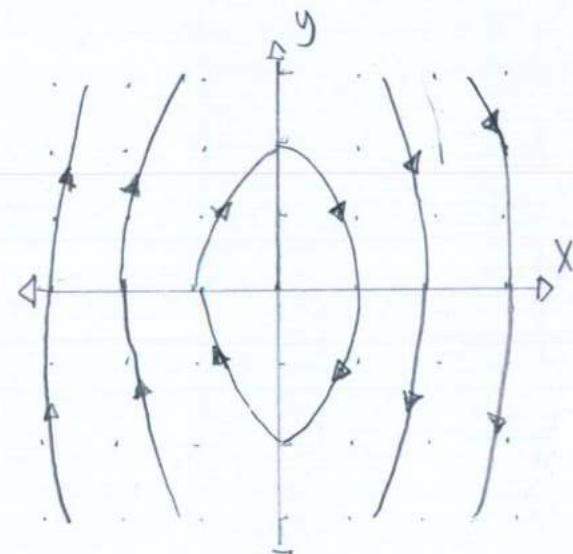
at the center  
i.e. nonlinear center,

$$\dot{x} = y$$

$$\dot{y} = u$$

$$\dot{y} = -x - \varepsilon x^3$$

For from the origin when  $\varepsilon > 0$ ,  
this phase plot appears.



$$\dot{v} = -\sin\theta \cdot Dv^2$$

6.5.14

$$\dot{v}\theta = -\cos\theta + v^2$$

a. If  $D=0$ , then

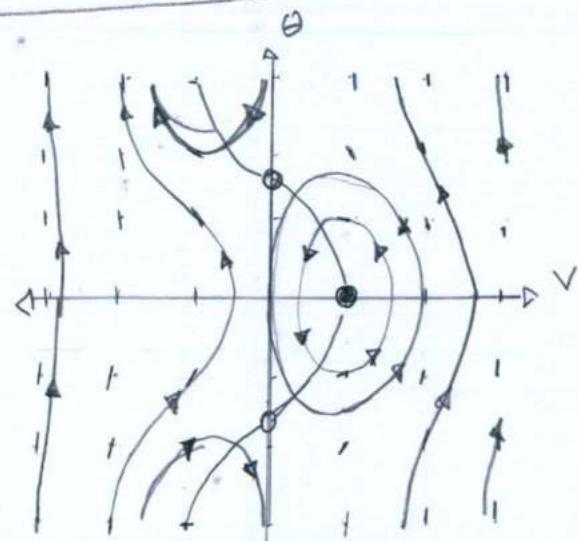
$$\dot{v} = -\sin\theta$$

$$\dot{v}\theta = -\cos\theta + v^2$$

$$E = \frac{1}{2}v^2 - \int v\dot{\theta} dv = \frac{1}{2}mv^2 + vc\cos\theta - \frac{v^3}{3} = 0$$

$$\frac{1}{2}v^2 - 3v\cos\theta + v^3 = 0; \quad \frac{dv}{d\theta} = v - 3\cos\theta + 3v^2; \quad \text{Fixed Points: } (v^*, \theta^*) = (1, 0), (0, 0)$$

The potential energy  $V(v, \theta) = -3\cos\theta + 3v^2$  has a single fixed point at  $(0, 2n\pi)$ .



b. If  $D > 0$ , then as the glider approaches  $v \rightarrow \infty$ , then the angle becomes more positive and the effect of lift propels the glider upward.

$$mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2 \sin\phi \cos\phi$$

6.5.15

$$a. b=0$$

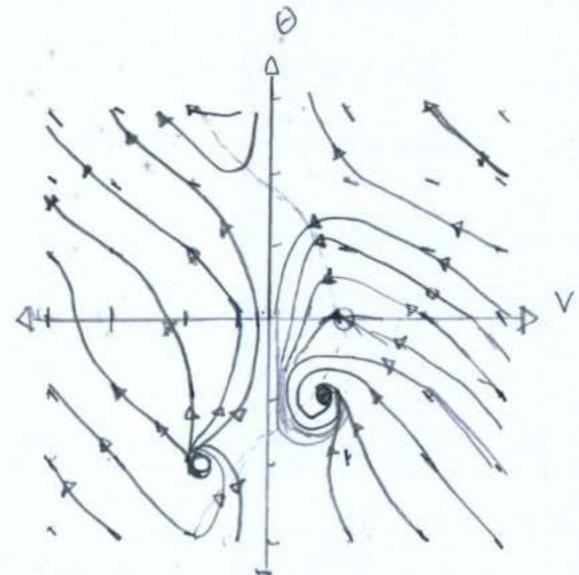
$$mr\ddot{\phi} = -mg\sin\phi + mr\omega^2 \sin\phi \cos\phi$$

If  $\gamma = r\omega^2/g$ , then

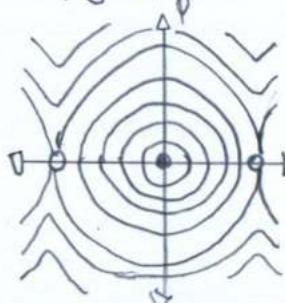
$$\ddot{\phi}\left(\frac{1}{g}\right) = -\sin\phi + \gamma \sin\phi \cos\phi$$

$$\ddot{\phi}\left(\frac{1}{g}\right) = \sin\phi (\cos\phi - \gamma^{-1})$$

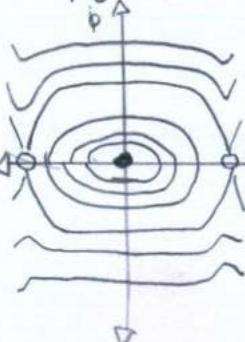
$$\text{If } \Gamma = \omega t, \text{ then } \boxed{\ddot{\phi} = \sin\phi (\cos\phi - \gamma^{-1})}$$



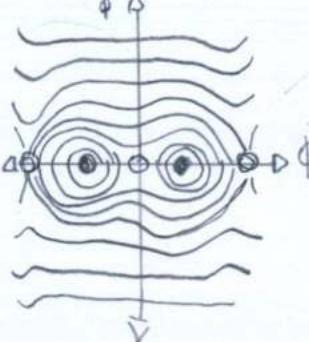
$$b. \gamma > 1$$



$$\gamma = 1$$



$$0 < \gamma < 1$$



$$v = d$$

$$\phi = \nu$$

$$\psi$$

C. The graphs  $1/\gamma > 1$  and  $1/\gamma = 1$  suggest a periodic stable point when the hoop spins, while  $1/\gamma = 1$ , a bead that doesn't stay in one place, and spins around the hoop.

$$6.5.16. mr\ddot{\phi} = -mg \sin\phi + mr\omega^2 \sin\phi \cos\phi$$

$$\ddot{\phi} = -\frac{g}{r} \sin\phi + \omega^2 \sin\phi \cos\phi = \sin\phi (\omega^2 \cos\phi - \frac{g}{r})$$

$$0 = \sin\phi (\omega^2 \cos\phi - \frac{g}{r}) ; \boxed{\phi = \pm \frac{\pi}{2} ; \arcsin \frac{g}{r\omega^2}}$$

$$6.5.17. mr\ddot{\phi} = -mg \sin\phi + mr\omega^2 \sin\phi \cos\phi$$

$$E = KE + PE = \frac{1}{2}\dot{\phi}^2 - \int \sin\phi (\omega^2 \cos\phi - \frac{g}{r}) d\phi$$

$$= \frac{1}{2}\dot{\phi}^2 + \cos(\phi)(\omega^2 \cos\phi - \frac{2g}{rw^2})$$

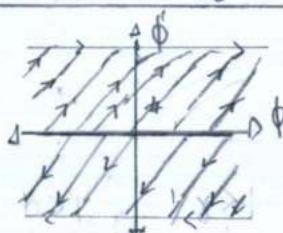
$$\dots = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\omega^2 \cos^2\phi - \frac{g}{r} \cos\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\omega^2(1 - \sin^2\phi) - mgr(1 - \cos\phi) - mgr$$

$$= (KE_{\text{Trans}} - KE_{\text{Rot}}) + PE$$

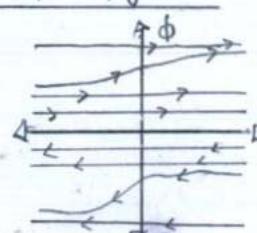
In terms of separation of motion, the bead hoop problem has translational and rotational energy.

$$6.5.18. mr\ddot{\phi} = -b\dot{\phi} - mg \sin\phi + mr\omega^2 \sin\phi \cos\phi$$

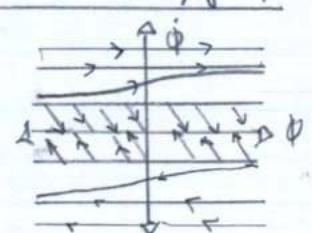
$$0 < b < 1 \quad 1/\gamma > 1$$



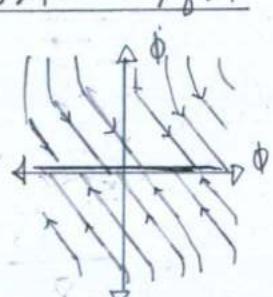
$$0 < b < 1 \quad 1/\gamma = 1$$



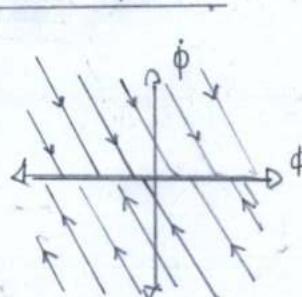
$$0 < b < 1 \quad 0 < 1/\gamma < 1$$



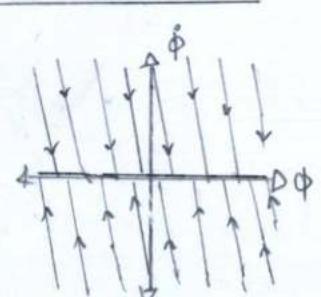
$$b > 1 \quad 1/\gamma > 1$$



$$b > 1 \quad 1/\gamma = 1$$



$$b > 1 \quad 0 < 1/\gamma < 1$$



$$\dot{R} = aR - bRF$$

### 6.5.19. Lotka-Volterra Predator-Prey Model

a. Term

$aR$ : Growth of the rabbit population

$-bRF$ : Decrease of the rabbit population by interacting foxes

$-cF$ : Decrease of the fox population

$dRF$ : Growth of the fox population by eating rabbits.

An unrealistic assumption is foxes do not decrease when rabbits are not present.

b.  $\dot{R} = R(a - bF)$ ;  $\dot{R}\left(\frac{1}{a}\right) = \frac{R}{a}(1 - \frac{b}{a}F)$ ;  $X = \frac{d}{c}R$ ;  $y = \frac{b}{a}F$ ;  $T = at$

$$\dot{F} = F(dR - c)$$
;  $\dot{y} = \frac{cy}{a}(x-1)$ ;  $\boxed{\dot{y} = hy(x-1)}$ ;  $\dot{x} = x(1-y)$

c.  $\dot{x} = 0 = x(1-y)$ ;  $\dot{y} = 0 = hy(x-1)$ ;  $(x^*, y^*) = \boxed{(0,0)}$   
 $\boxed{(1,1)}$

d.  $A = \begin{pmatrix} 1-y & -xy \\ hy & h(x-1) \end{pmatrix}$

$$A_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}; \Delta = \mu; T = 1 + \mu; T^2 - 4\Delta > 0$$

"Unstable Node"

$$A_{(1,1)} = \begin{pmatrix} 0 & -1 \\ \mu & 0 \end{pmatrix}; \Delta = -\mu; T = 0; T^2 - 4\Delta > 0$$

"Center" = cycle.

### 6.5.20.

a. The terms found in  $\dot{P}$ ,  $\dot{R}$ , and  $\dot{S}$  relate the existence of paper, rock, and scissors, but also, a relationship when each type of species is present at any given time.

b.  $P + R + S = PR - PS + RS - RP + SP - SR = 0$

c.  $E_1(P, R, S) = P + R + S$ ;  $E_2(R, S) = PRS$

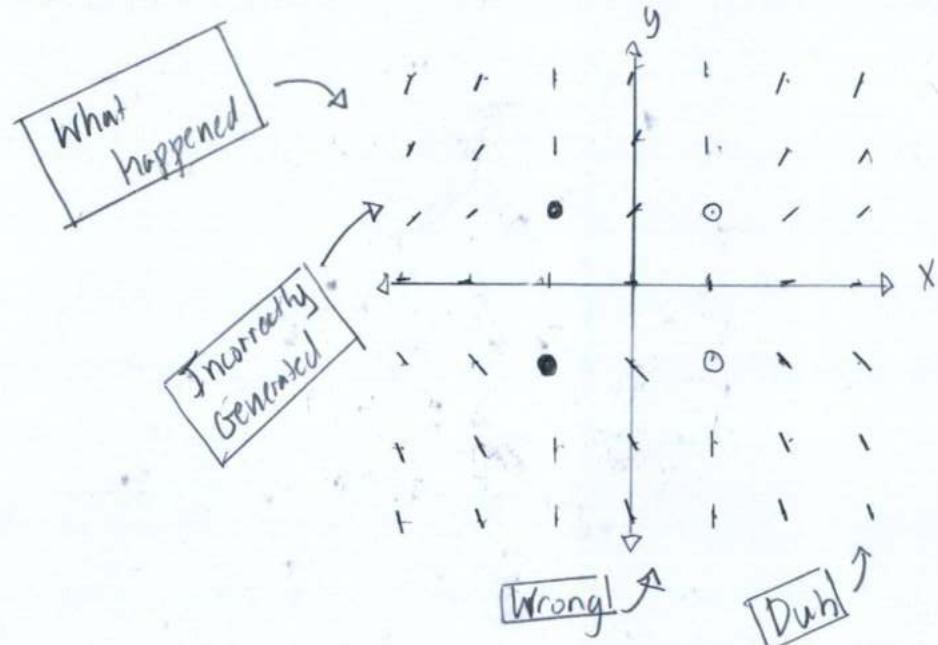
"Plane"      "Multiplane"

As  $t \rightarrow \infty$ , then a discrete solution exists of integer values between planes or amounts of  $P, R, S$ .

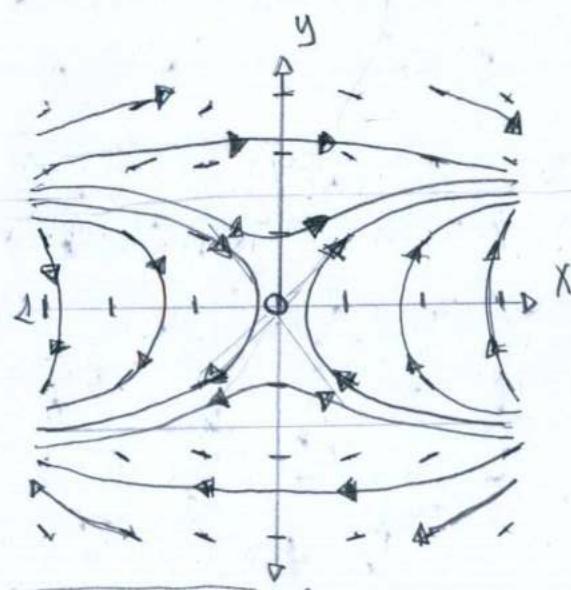
$$\begin{aligned}\dot{x} &= y(1-x^2) \quad 6.6.1. \text{ Reversible if } t \rightarrow -t, x \rightarrow -x, y \rightarrow -y \\ \dot{y} &= 1-y^2\end{aligned}$$

Fixed Points:  $\dot{x} = y(1-x^2) = 0$

$$\dot{y} = 1-y^2 = 0 \quad ; (x^*, y^*) = (\pm 1, \pm 1)$$



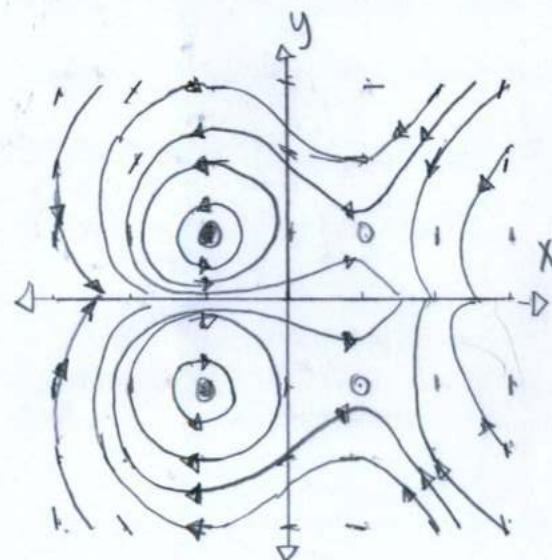
$$\begin{aligned}\dot{x} &= y \\ 6.6.2. \quad \dot{y} &= x \cos y\end{aligned}$$



Fixed Points:  $x=0=y$

$$\dot{y} = 0 = x \cos y = x \cos(-y) = -x \cos(-y)$$

$$(x^*, y^*) = (0, 0)$$



$$\begin{aligned}\dot{x} &= \sin y \\ \dot{y} &= \sin x\end{aligned}$$

6.6.3.

a.  $\frac{dy}{dx} = \left(\frac{-1}{-1}\right) \frac{dy}{dx} = \frac{-\sin x}{-\sin y} = \frac{\sin x}{\sin y}$

b. Fixed Points:  $\dot{x} = 0 = \sin y \quad ; (x^*, y^*) = (-n\pi, n\pi) \quad (2n+1)\frac{\pi}{2}$

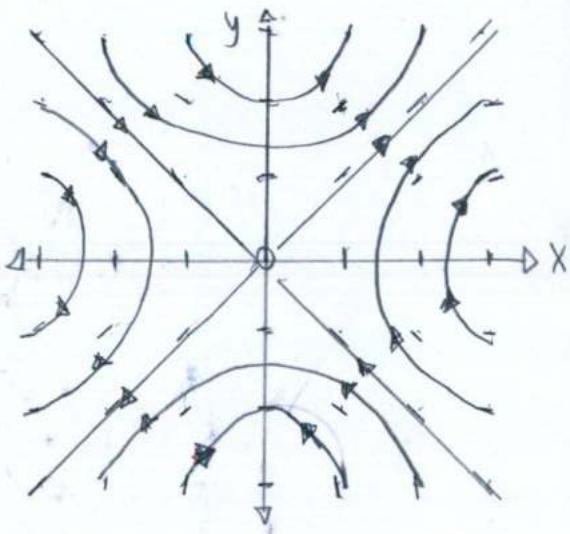
$$\dot{y} = 0 = \sin x$$

Where  $n \in \mathbb{R}$

If  $n$  is even, stable node, else unstable node.

C.  $\dot{x} = \sin y$ ;  $\sin y = \sin x$ ;  $y = \pm x$

d.  $\rightarrow$



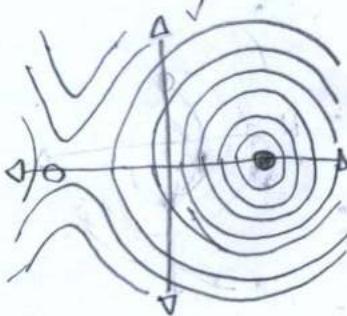
$$\ddot{x} + (\dot{x})^2 + x = 3$$

6.6.4.  $\dot{u} = \dot{x} = v$

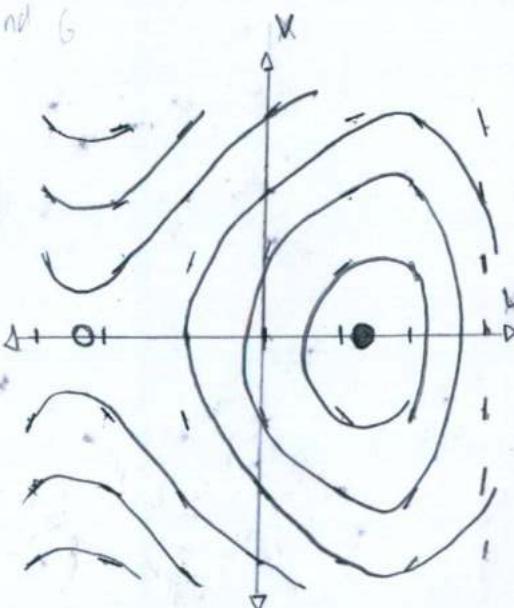
a.  $\dot{v} = \ddot{x} = 3 - \dot{x}^2 + x = 3 - u^2 + u$

**Fixed Points**  $(u^*, v^*) = (0, \frac{-1}{2} \pm \frac{\sqrt{13}}{2})$

**Guess:**



Hand 6



$$\dot{x} = y - y^3$$

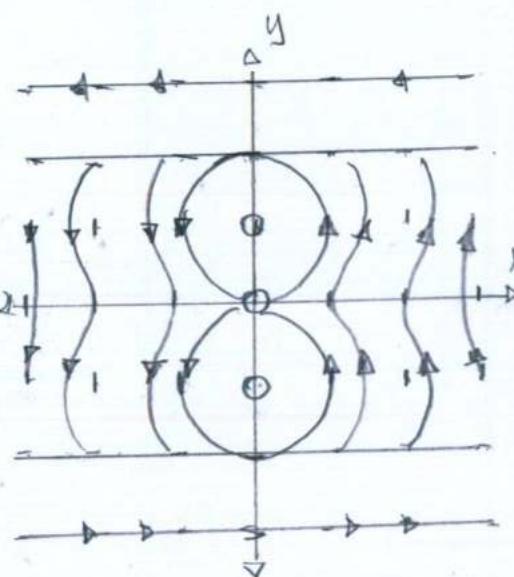
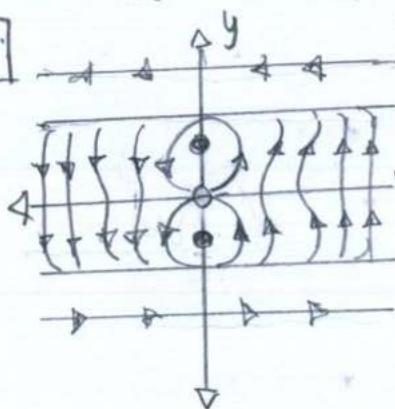
$$\dot{y} = x \cos y$$

b. **Fixed Points**  $\dot{x} = 0 = y - y^3$

$$\dot{y} = 0 = x \cos y$$

$$(x^*, y^*) = (0, 0), (\pm 1, 0)$$

**Guess:**



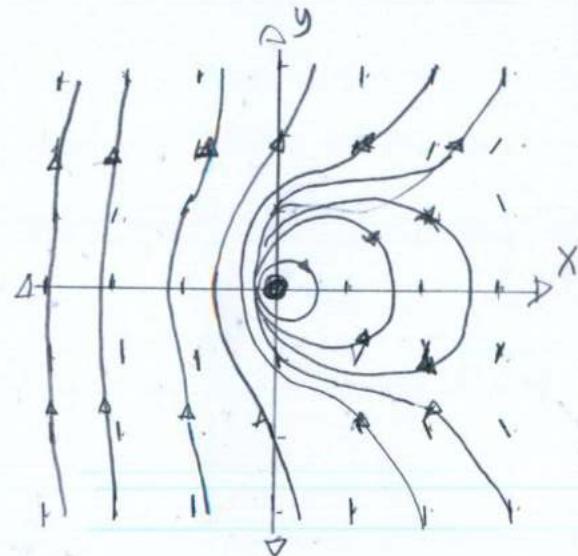
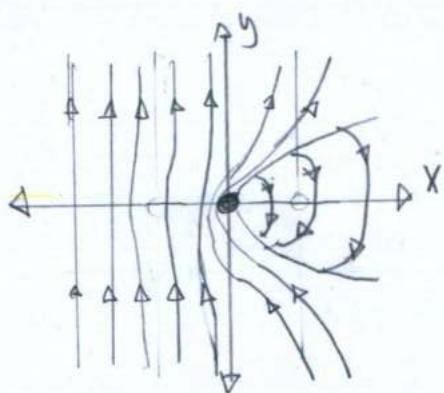
$$\begin{aligned}\dot{x} &= \sin y \\ \dot{y} &= y^2 - x\end{aligned}$$

### 6.6.4 Fixed Points

$$\begin{aligned}\dot{x} &= 0 = \sin y \\ \dot{y} &= 0 = y^2 - x\end{aligned}$$

$$(x^*, y^*) = (0, 0), (1, \pm 1)$$

Guess:



$\ddot{x} + f(\dot{x}) + g(x) = 0$  6.6.5,  $f$  is an even function;  $g$  is an odd function  
 $f$  &  $g$  are smooth

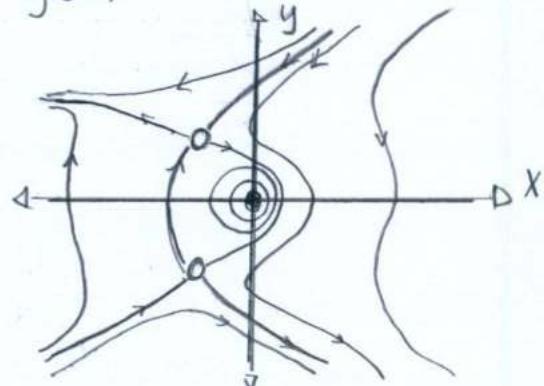
a.  $\ddot{x} + F(-\dot{x}) + g(-x) = -\ddot{x} - f(\dot{x}) - g(x) = \ddot{x} + f(\dot{x}) + g(x)$

b.  $\dot{u} = \dot{x} = v$  Definition of a reversible system  
 $\dot{v} = \ddot{x} = -f(\dot{x}) - g(x)$ ; is no stable nodes or spirals.

$$\begin{aligned}\dot{x} &= y - y^3 \\ \dot{y} &= -x - y^2\end{aligned}$$

6.6.6.

a.  $\dot{x} = 0; \dot{y} = 0$



b. Quadrant #1:  $\dot{x} < 0; \dot{y} < 0$

Quadrant #2: Mixed

Quadrant #3: Mixed

Quadrant #4:  $\dot{x} > 0; \dot{y} < 0$

c.  $A = \begin{pmatrix} 0 & 1-2y^2 \\ -1 & -2y \end{pmatrix}; A_{(-1, \pm 1)} = \begin{pmatrix} 0 & -1 \\ -1 & \pm 2 \end{pmatrix}; \Delta = (1-\sqrt{2})(1+\sqrt{2})$

$$T = 2$$

$$\Gamma^2 - 4\Delta > 0$$

$$\lambda_1 = (1-\sqrt{2}); \vec{V}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = (1+\sqrt{2}); \vec{V}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

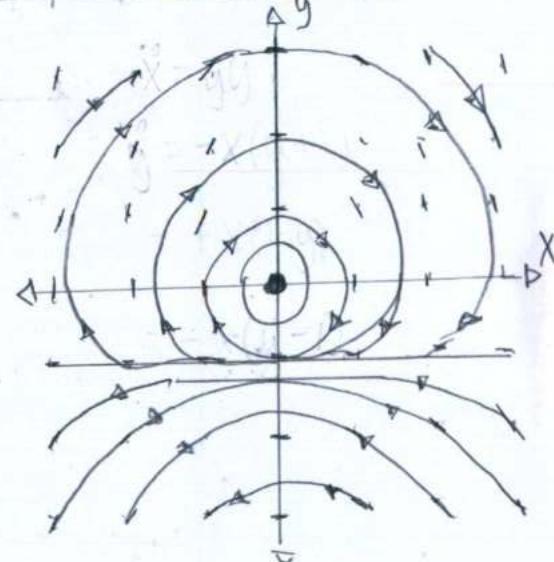
d. If Quadrant #2 and #3 are mixed sign  
 then a possible trajectory through  $x < 0$  may exist.  
 A heteroclinic trajectory that does cross from  $(-1, -1)$   
 to  $(1, 1)$  is present because of the reversible function.

e. Other examples of a heteroclinic trajectory  
 relate to the third fixed point. See part b.

$$\ddot{x} + x\dot{x} + x = 0 \quad 6.6.7. \quad \ddot{x} = y$$

$$\dot{y} = -x(\ddot{x} + 1) = -x(y + 1)$$

$$\boxed{\text{Reversibility}} \quad -\ddot{x} - x \cdot (-\dot{x}) - x \\ = \ddot{x} + x\dot{x} + x \\ = 0$$



$$\ddot{x} = \frac{\sqrt{2}}{4} x(x-1) \sin \phi \quad 6.6.8. \quad \text{a. } \boxed{\text{Reversibility}}: x \rightarrow -x; \phi \rightarrow -\phi$$

$$\dot{\phi} = \frac{1}{2} \left[ \beta - \frac{1}{\sqrt{2}} \cos \phi - \frac{1}{8\sqrt{2}} x \cos \phi \right]$$

$$\ddot{x} = \frac{\sqrt{2}}{4} - x(-x-1) \sin(-\phi)$$

$$= -\frac{\sqrt{2}}{4} x(1-x) \sin(\phi)$$

$$= \frac{\sqrt{2}}{4} x(x-1) \sin \phi$$

$$\dot{\phi} = \frac{1}{2} \left[ \beta - \frac{1}{\sqrt{2}} \cos(-\phi) - \frac{1}{8\sqrt{2}} (-x) \cos(-\phi) \right]$$

$$= \frac{1}{2} \left[ \beta - \frac{1}{\sqrt{2}} \cos(\phi) + \frac{1}{8\sqrt{2}} x \cos \phi \right]$$

$$\text{b. } \ddot{x} = 0 = \frac{\sqrt{2}}{4} x(x-1) \sin \phi =$$

$$\dot{\phi} = \frac{1}{2} \left[ \beta - \frac{1}{\sqrt{2}} \cos \phi - \frac{1}{8\sqrt{2}} x \cos \phi \right] = 0$$

$$(x^*, \phi^*) = (0, 2\pi n - \cos^{-1}(\sqrt{2}\beta)), (0, 2\pi n + \cos^{-1}(\sqrt{2}\beta))$$

$$(1, 2\pi n - \cos^{-1}\left(\frac{8\sqrt{2}\beta}{x+y}\right)), (0, 0)$$

$$(1, 2\pi n + \cos^{-1}\left(\frac{8\sqrt{2}\beta}{x+y}\right))$$

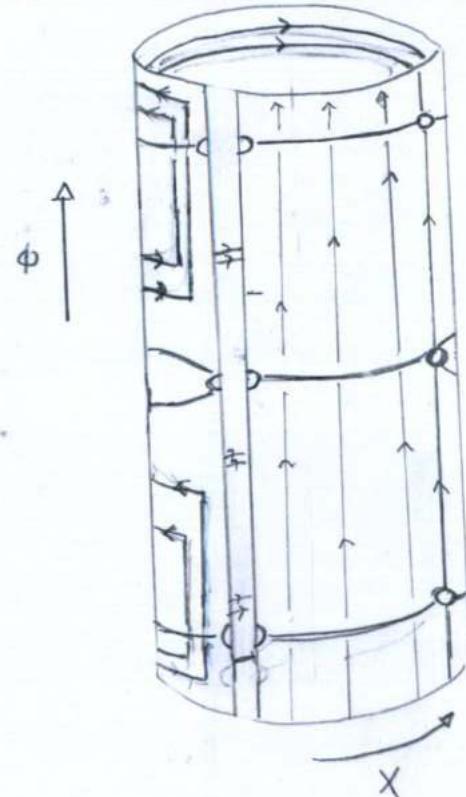
A homoclinic orbit is a nullcline.

$$\dot{x} = 0 = \frac{\sqrt{2}}{4} x(x-1) \sin \phi ; \dot{\phi} = 0 = \frac{1}{2} [\beta - \frac{1}{\sqrt{2}} \cos \phi - \frac{1}{y\sqrt{2}} x \cos \phi]$$

$$x = -4\sqrt{2} \sec(\phi) (\sqrt{2} \cos(\phi) - 2\beta)$$

c.  $\lim_{\beta \rightarrow \frac{1}{\sqrt{2}}} 2\pi n - \cos(2\sqrt{\beta}) = 2\pi n$  ;  
 then  $(x^*, \phi^*) = (0, 2\pi n)$   
 and the node on the line  
 $\phi=0$  is closer to  $(0, 0)$ ,  
 and the cylinder becomes  
 a smaller diameter shape.  
 with less closed orbits.

d. See Part C: cylinder



$$\frac{d\phi_k}{dt} = \Omega + a \sin \phi_k + \frac{1}{N} \sum_j^N \sin \phi_j$$

6. 6. 9.

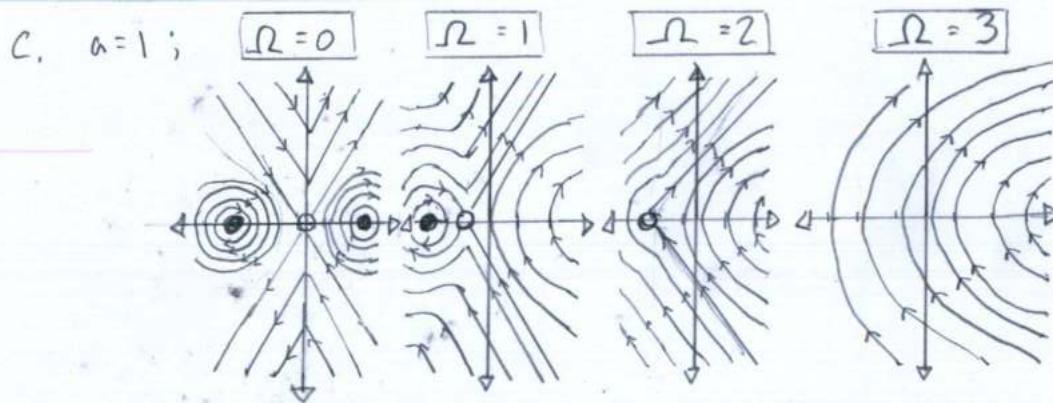
$$\begin{aligned} a. \quad \theta_k &= \phi_k - \frac{\pi}{2} ; \frac{d\theta}{dt} = \Omega + a \cos \theta_k + \frac{1}{N} \sum \cos \theta_k \\ &= \Omega + a \cos(-\theta_k) + \frac{1}{N} \sum \cos(-\theta_k) \end{aligned}$$

$$\begin{aligned} b. \quad \boxed{\text{Fixed Points:}} \quad \dot{\theta} &= 0 = \Omega + a \cos \theta_k + \frac{1}{N} \sum \cos \theta_k \\ \Omega &= -a \cos \theta_k - \frac{1}{N} \sum \cos \theta_k \\ &= -\cos \theta_k (a+1) \end{aligned}$$

$$-\cos \theta_k = \left| \frac{\Omega}{a+1} \right|$$

If  $\left| \frac{\Omega}{a+1} \right| < 1$ , then  $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

If  $\left| \frac{\Omega}{a+1} \right| > 1$ , then no. fixed point  
 is generated because  
 $\cos \theta$  is never greater  
 than 1.



$$\dot{x} = -y - x^2 \quad 6.6.10 \boxed{\text{Fixed Points}} \quad \dot{x} = 0 = -y - x^2 \Rightarrow (x^*, y^*) = (0, 0)$$

$$\dot{y} = x$$

$$A = \begin{pmatrix} -2x & -1 \\ 1 & 0 \end{pmatrix}; A_{(0,0)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\Delta = -1; \quad \Gamma = 0; \quad \Gamma^2 - 4\Delta > 0$$

"Saddle Point"

No, a nonlinear center is an isolated fixed point with closed orbits.

$$\dot{\theta} = \cot \phi \cos \theta \quad 6.6.11. \text{ a. } \boxed{\text{Reversibility}} \quad t \rightarrow -t; \theta \rightarrow -\theta$$

$$\dot{\phi} = (\cos^2 \phi + A \sin^2 \phi) \sin \theta$$

$$\dot{\theta} = \cot(\phi) \cos(-\theta)$$

$$= \cot(\phi) \cdot \cos(\theta)$$

$$t \rightarrow -t; \phi \rightarrow -\phi$$

$$\dot{\phi} = [\cos^2(-\phi) + A \sin^2(-\phi)] \cdot \sin(\theta)$$

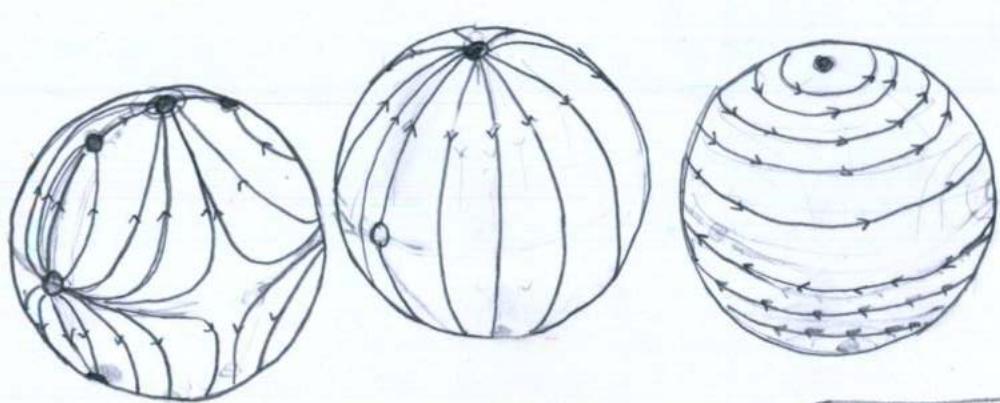
$$= [\cos^2(\phi) + A \sin^2(\phi)] \cdot \sin(\theta)$$

b.

$$\boxed{A = -1}$$

$$\boxed{A = 0}$$

$$\boxed{A = 1}$$



c. As  $t \rightarrow \infty$ , each case of shear flow trajectory to a stable node. This implies rotation of a body does not freely rotate in medium.

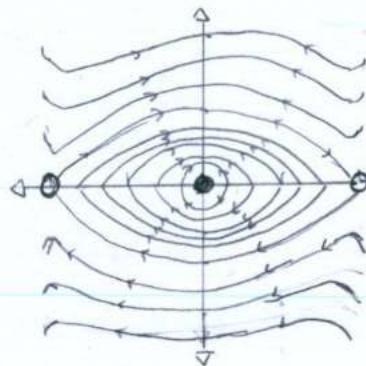
$$\ddot{\theta} + b\dot{\theta} + \sin\theta = 0$$

### 6.7.1. Fixed Points

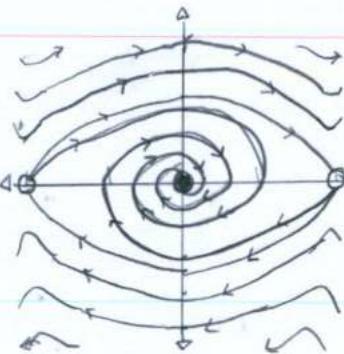
$$\dot{x} = \dot{\theta} = y$$

$$\dot{y} = \ddot{\theta} = -(by + \sin x)$$

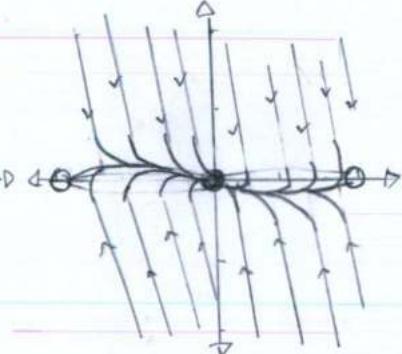
$$b=0$$



$$0 < b \leq 1$$



$$1 < b$$



$$\ddot{\theta} + \sin\theta = \gamma$$

### 6.7.2 a. Fixed Points

$$\dot{x} = \dot{\theta} = y = 0$$

$$\dot{y} = \ddot{\theta} = \gamma - \sin\theta = \gamma - \sin x$$

$$(x^*, y^*) = (\arcsin \gamma, 0)$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}; A = \begin{pmatrix} 0 & 1 \\ \gamma - \cos x & 0 \end{pmatrix}$$

$$\Delta = \cos x - \gamma; \Gamma = 0; \Gamma^2 - 4\Delta > 0$$

If  $\gamma = 0$ ,  $(x^*, y^*)$  is a center

If  $0 < \gamma < 1$ ,  $(x^*, y^*)$  is a center

If  $\gamma > 1$ ,  $(x^*, y^*)$  is a saddle point.

### b. Nullclines

$$y = \gamma - \sin x$$

$$E = \frac{1}{2}\dot{x}^2 - \int \gamma - \sin x \, dx$$

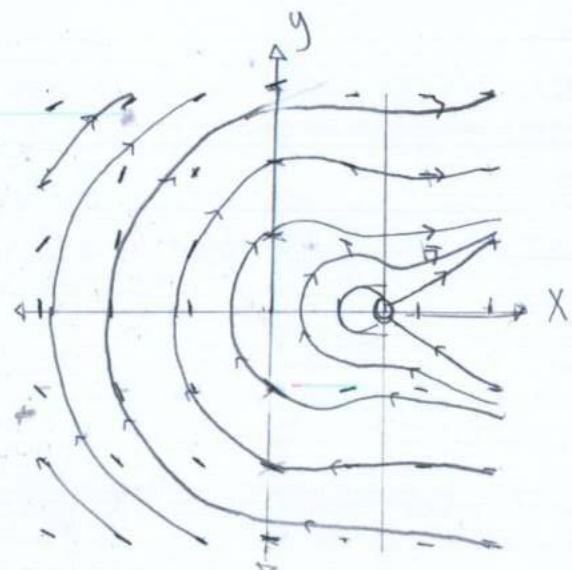
$$= \frac{1}{2}\dot{x}^2 - \gamma x - \cos x$$

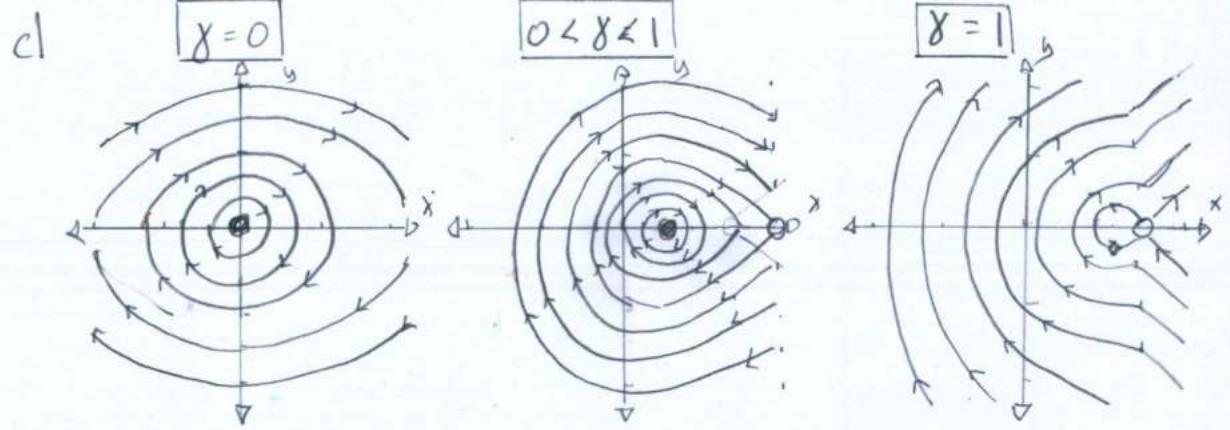
The system is not conservative because of no closed loops

### Reversibility

$$\dot{x} = -y + y$$

The system is not reversible.





e.

$\gamma = 0$ ; $\dot{x} = 0 = y$ ; $\dot{y} = 0 = -\sin x$ ; $y = \theta = -\sin \theta$
$0 < \gamma < 1$ ; $\dot{x} = 0 = y$ ; $\dot{y} = 0 = \gamma - \sin x$ ; $y = \theta = \gamma - \sin \theta$
$\gamma = 1$ ; $\dot{\theta} = 1 - \sin \theta$

$$\ddot{\theta} + (1 + \alpha \cos \theta) \dot{\theta} + \sin \theta = 0$$

6.7.3  $\dot{x} = \dot{\theta} = y$

$$\dot{y} = -(1 + \alpha \cos x) y - \sin x$$

Fixed Points:  $(x^*, y^*) = (0, 0)$

Reversible

Yes

No

Conservative

$$\ddot{\theta} + \sin \theta = 0 \quad 6.7.4$$

a.  $PE = mgh = mgL(1 - \cos(\theta))$ ;  $KE = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{\theta})^2$

$$E = PE + KE = mgL(1 - \cos(\theta)) + \frac{1}{2}m(\dot{\theta})^2 = 0$$

$$\dot{\theta}^2 = -2gL(1 - \cos(\theta)); \text{ If } \theta = \alpha = \text{max height} \text{ so } \dot{\theta}^2 = 0$$

$$= 2(\cos(\theta) - \cos(\alpha))$$

$$T = 4 \int_0^\alpha dt = 4 \int_0^\alpha \frac{d\theta}{\dot{\theta}} = 4 \int_0^\alpha \frac{d\theta}{\sqrt{2(\cos \theta - \cos \alpha)}}$$

b. Half-Angle Formula:  $\cos(2A) = 1 - 2\sin^2 A$  where  $A = \frac{\theta}{2}$  or  $\frac{\alpha}{2}$

$$T = 4 \int_0^{\alpha/2} \frac{d\theta}{\sqrt{4(\sin^2 \frac{1}{2}\alpha - \sin^2 \frac{1}{2}\theta)}} \quad \text{where } \alpha = \text{max height}$$

c. Half-Angle Formula:  $(\sin \frac{1}{2}\alpha) \sin \phi = \sin \frac{1}{2}\theta$

$$\frac{1}{2} \sin \frac{1}{2}\alpha \cos \phi \frac{d\phi}{d\theta} = \frac{1}{2} \cos \frac{\theta}{2}$$

$$d\theta = \frac{\cos \theta/2}{\sin \alpha/2 \cos \phi} d\phi$$

$$T = 4 \int_0^{\pi/2} \frac{d\phi}{\cos \theta/2} \Rightarrow H = \frac{1}{2} \times \left[ \int_0^{\pi/2} \frac{d\phi}{(1-m \sin^2 \phi)^{1/2}} \right]$$

"Elliptic Integral"

d. Binomial Series  $\frac{1}{(1-x)^{1/2}} = 1 + \frac{1}{2}x + \dots$

$$T = 4 \int_0^{\pi/2} \left[ \left( 1 + \frac{1}{2}m \sin^2 \phi + \dots \right) d\phi \right]; m = \sin^2 \frac{x}{2}$$

$$= 2\pi \left[ 1 + \frac{1}{16}x^2 + \dots \right]$$

6.7.5.

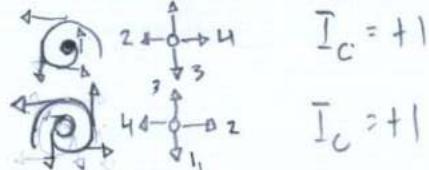
Numerical Integration of  $T = 4 \times \sum_{i=0}^{10} \sum_{j=0}^9 \left( 1 + \frac{1}{2} [\sin^2 \frac{10i}{2}] \sin^2 \frac{10j}{2} + \dots \right)$

$i \backslash j$	0	1	2	3	4	5	6	7	8	9
0	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00
1	4.00	5.69	4.54	4.78	5.53	4.03	7.00	4.34	5.02	5.33
2	4.00	4.54	4.10	4.25	4.49	4.01	4.50	4.11	4.33	4.43
3	4.00	4.76	4.25	4.36	4.76	4.01	4.93	4.16	4.48	4.61
4	4.00	5.53	4.49	4.70	5.39	4.63	5.63	4.31	4.93	5.21
5	4.00	4.03	4.01	4.01	4.03	4.00	4.03	4.01	4.02	4.03
6	4.00	5.80	4.58	4.93	5.63	4.03	5.91	4.36	5.09	5.41
7	4.00	4.34	4.11	4.16	4.31	4.01	4.36	4.17	4.20	4.27
8	4.00	5.02	4.33	4.47	4.93	4.02	5.08	4.20	4.62	4.80
9	4.00	5.33	4.43	4.61	5.21	4.03	5.41	4.29	4.80	5.05
10	4.00	4.13	4.04	4.06	4.11	4.00	4.13	4.03	4.08	4.10
11	4.00	5.84	4.59	4.95	5.67	4.04	5.95	4.25	5.11	5.45
12	4.00	4.17	4.05	4.09	4.15	4.00	4.14	4.22	4.10	4.13
13	4.00	5.26	4.40	4.58	5.14	4.02	5.33	4.26	4.76	4.91
14	4.00	5.10	4.35	4.51	5.00	4.02	5.17	4.22	4.67	4.87
15	4.00	4.28	4.09	4.13	4.23	4.01	4.29	4.06	4.67	4.22
16	4.00	5.82	4.58	4.84	5.65	4.03	5.93	4.36	5.10	5.43
17	4.00	4.06	4.02	4.05	4.05	4.00	4.06	4.01	4.03	4.04
18	4.00	5.47	4.47	4.68	5.33	4.03	5.36	4.29	4.84	5.16

$$T = 852.06$$

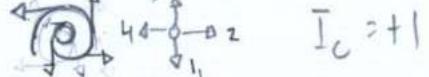
6.8.1

a. Stable spiral



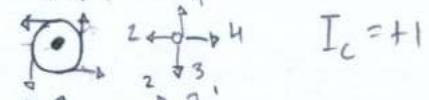
$$I_C = +1$$

b. Unstable spiral



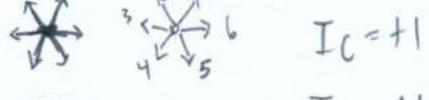
$$I_C = +1$$

c. center



$$I_C = +1$$

d. star



$$I_C = +1$$

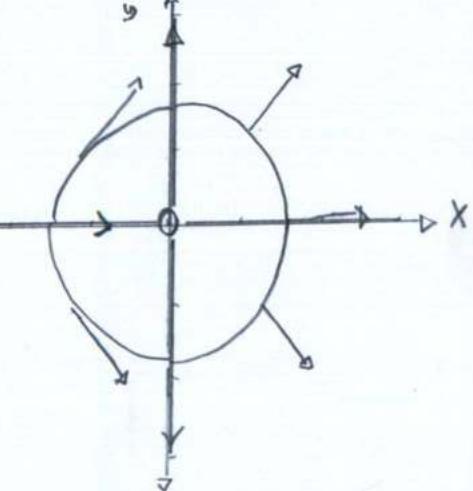
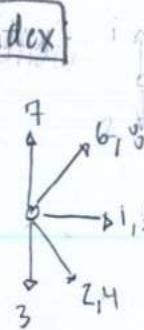
e. Degenerate Node.



$$I_C = +1$$

$$\begin{aligned} \dot{x} &= x^2 & 6.8.2 \quad \boxed{\text{Fixed Points}}: \quad \dot{x} = 0 = x^2 & ; A = \begin{pmatrix} 2x & 0 \\ 0 & 1 \end{pmatrix}; \Delta = 0; \tau = 1; \tau^2 - 4\Delta > 0 \\ \dot{y} &= y & \dot{y} = 0 = y & \text{"Non-isolated Fixed Points"} \end{aligned}$$

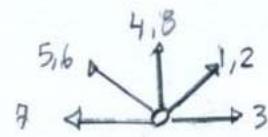
$$(\dot{x}, \dot{y}) = (0, 0) ; \boxed{\text{Index}}$$



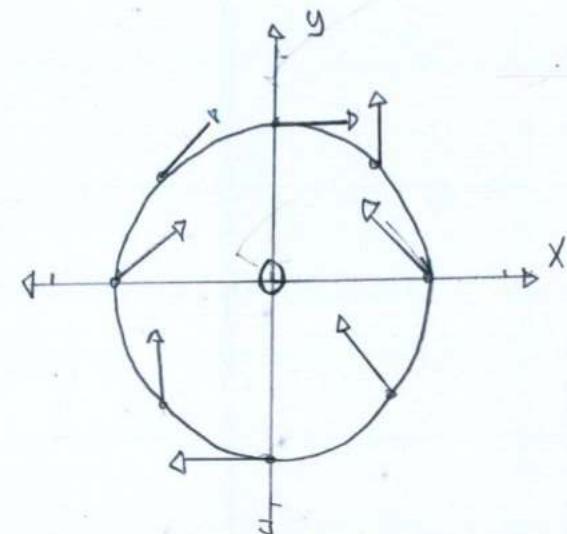
$$I_c = 0$$

$$\begin{aligned} \dot{x} &= y - x & 6.8.3, \quad \boxed{\text{Fixed Points}}: \quad \dot{x} = 0 = y - x & ; A = \begin{pmatrix} -1 & 1 \\ 2x & 0 \end{pmatrix}; \Delta = 0; \tau = -1; \tau^2 - 4\Delta > 0 \\ \dot{y} &= x^2 & \dot{y} = 0 = x^2 & \text{"Spiral sink" } \end{aligned}$$

$$(\dot{x}, \dot{y}) = (0, 0) ; \boxed{\text{Index}}$$

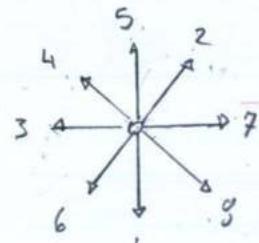


$$I_c = 0$$

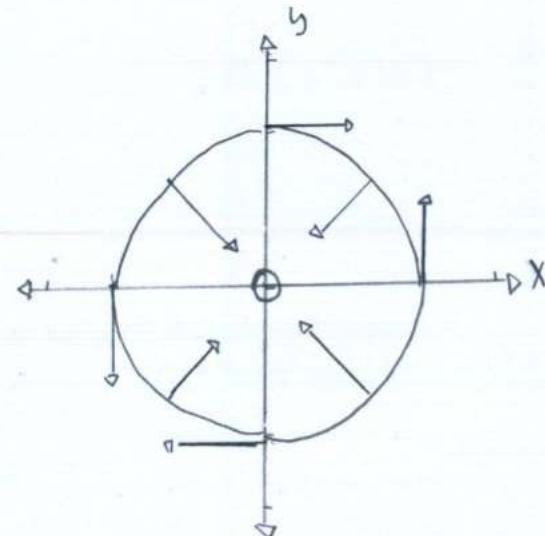


$$\begin{aligned} \dot{x} &= y^3 & 6.8.4 \quad \boxed{\text{Fixed Points}}: \quad \dot{x} = 0 = y^3 & ; A = \begin{pmatrix} 0 & 3y^2 \\ 1 & 0 \end{pmatrix}; \Delta = 0; \tau = 0; \tau^2 - 4\Delta = 0 \\ \dot{y} &= x & \dot{y} = 0 = x & \text{"Saddle" } \end{aligned}$$

$$(\dot{x}, \dot{y}) = (0, 0) ; \boxed{\text{Index}}$$



$$I_c = 0$$

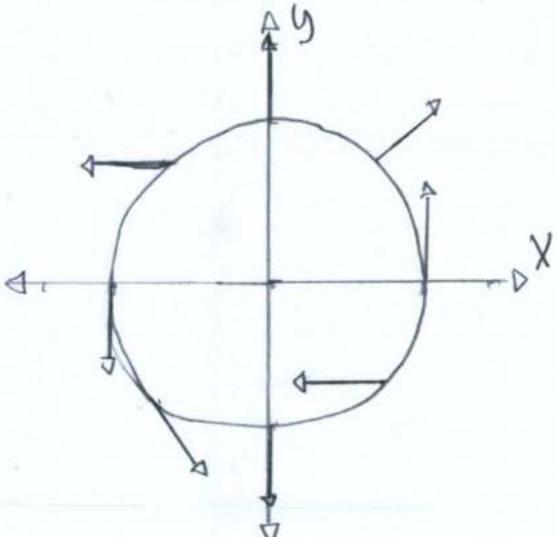
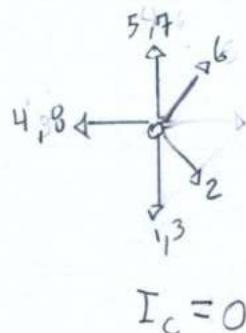


$$\begin{aligned} \dot{x} &= xy \\ \dot{y} &= x+ty \end{aligned}$$

6.8.5 [Fixed Points]  $\dot{x}=0=x$   
 $\dot{y}=0=y+tx$

$$A = \begin{pmatrix} y & x \\ 1 & t \end{pmatrix}; \Delta = 0; \text{tr} = 0; \quad t^2 - 4\Delta \quad \text{"unstable saddle"}$$

$$(x^*, y^*) = (0, 0) \boxed{\text{Index}}$$



6.8.6. Node [N]:  $I_c = +1$        $N+S+C = +1 = 1+S = +1$

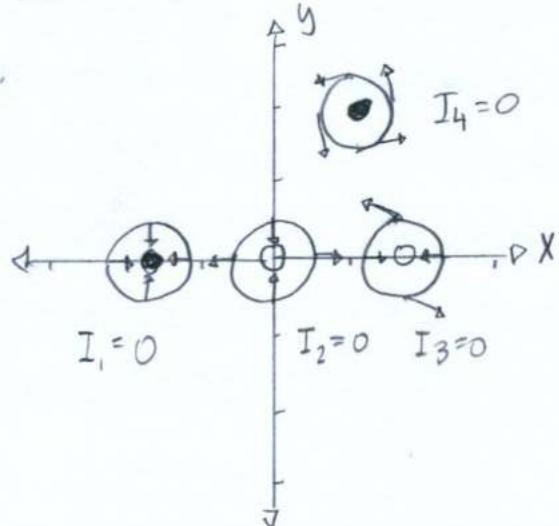
Spiral [F]:  $I_c = +1$

Center [C]:  $I_c = \pm 1$

Saddle [S]:  $I_c = 0$

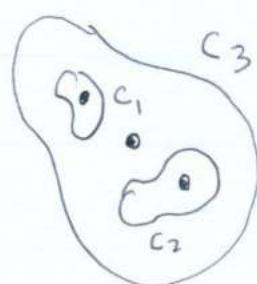
$$\dot{x} = x(4-y-x^2) \quad 6.8.7.$$

$$\dot{y} = y(x-1)$$



The indices of each fixed point are zero ( $I_c=0$ ), thus, no closed orbits exist.

6.8.8. a.



b.  $I_c = I_1 + I_2 + I_3 > 0$ ; A fixed point exists in the closed orbit.

6.8.1.  $C_1, C_2$  = closed trajectories

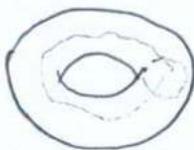
If  $C_1$  is clockwise,  $I_C < 0$

If  $C_2$  is counterclockwise,  $I_C > 0$

A fixed point in  $C_2$  is true because  $I_C > 0$

6.8.10

Torus



$$I_C > 0$$

cylinder



$$I_C = 0$$

sphere



$$I_C > 0$$

Theorem 6.8.2 is reasonable for closed orbit shapes.

$$\overset{\circ}{z} = z^k$$

$$\overset{\circ}{z} = (\bar{z})^k$$

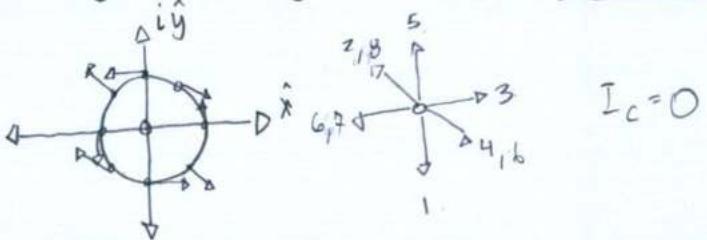
6.8.11

$$a. R=1 ; \overset{\circ}{z} = z = x+iy ; \langle x, y \rangle$$

$$k=2 ; \overset{\circ}{z} = z^2 = (x+iy)^2 = x^2 - y^2 - 2ixy ; \boxed{\langle x^2 - y^2, -2xy \rangle}$$

$$k=3 ; \overset{\circ}{z} = z^3 = (x+iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3) ; \boxed{\langle x^3 - 3xy^2, 3x^2y - y^3 \rangle}$$

$$b. z^k = (0, 0) ;$$



$$I_C = 0$$

c. The expansion is similar to a Binomial.

$$\left\langle \sum_{k=1}^{2R \leq n} \binom{n}{2k} x^{n-2k} \cdot (-1)^k y^{2k}, \sum_{k=1}^{2R+1 \leq n} \binom{n}{2k+1} x^{n-2k} (-1)^k y^{2k} \right\rangle$$

$$\dot{x} = a + x^2$$

6.8.12

a. Fixed Points

$$\overset{\circ}{x} = 0 = a + x^2$$

$$\overset{\circ}{y} = 0 = -y$$

$$(x^*, y^*) = (\pm i\sqrt{a}, 0)$$

$$A = \begin{pmatrix} 2x & 0 \\ 0 & -1 \end{pmatrix} ; \Delta_1 = -2i\sqrt{a} ; \tau_1 = 2i\sqrt{a} = \frac{1}{4} \Delta + \text{Imaginary}$$
$$\Delta_2 = 2i\sqrt{a} ; \tau_2 = -2i\sqrt{a} - 1$$

Fixed points in  $\mathbb{R}^3$  are non-existent.

b.  $I_C = I_1 + I_2 = 0$  because the imaginary fixed points  $(\pm i\sqrt{a}, 0)$  are symmetric.

c.  $\dot{x} = f(x, a)$  where  $x \in \mathbb{R}^2$  a conserved index is <sup>optionally</sup> independent of  $a_3$  and is the sum of two indices.

$$\dot{x} = F(x, y) \quad 6.8.13 \quad \phi = \tan^{-1}(\dot{y}/\dot{x})$$

$$\dot{y} = g(x, y)$$

$$a. \frac{d}{dy} \tan^{-1} \frac{\dot{y}}{\dot{x}} = \frac{1}{\dot{y}^2 + 1}; \quad d\phi = \frac{1}{(\frac{\dot{y}}{\dot{x}})^2 + 1} \cdot \left(\frac{\dot{y}}{\dot{x}}\right)' = \frac{\dot{x}^2}{\dot{y}^2 + \dot{x}^2} \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2}$$

$$= \frac{fdg - gdf}{f^2 + g^2}$$

$$b. I_C = \frac{d}{d\phi} \tan^{-1}(\phi) = \frac{1}{2\pi} \oint \frac{fdg - gdf}{f^2 + g^2} \quad \text{where } \phi = \frac{\dot{y}}{\dot{x}}$$

$$\dot{x} = x \cos \alpha - y \sin \alpha \quad 6.8.14.$$

$$a. \boxed{\text{Fixed Points}}: (x^*, y^*) = (0, 0)$$

$$\dot{y} = x \sin \alpha + y \cos \alpha$$

$$A = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}; \quad \Delta = \cos \alpha \sin \alpha$$

$$\zeta^2 - 4\Delta < 0 \quad \text{because } \sin \alpha > 0$$

$\boxed{\text{"Unstable Spiral"}}$

$$b. I_C = \frac{1}{2\pi} \oint \frac{(x \cos \alpha - y \sin \alpha)(x \sin \alpha + y \cos \alpha) - (x \sin \alpha + y \cos \alpha)(x \cos \alpha - y \sin \alpha)}{(x \cos \alpha - y \sin \alpha)^2 + (x \sin \alpha + y \cos \alpha)^2} \, d\phi$$

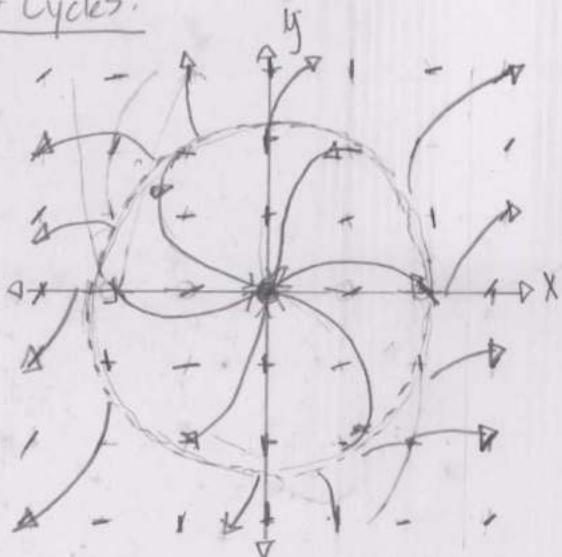
$$= \frac{1}{2\pi} \oint 1 = \boxed{1}$$

$$c. \boxed{I_C = 1}$$

## Chapter 7: Limit Cycles:

$$\dot{r} = r^3 - 4r \quad 7.1.1$$

$$\dot{\theta} = 1$$



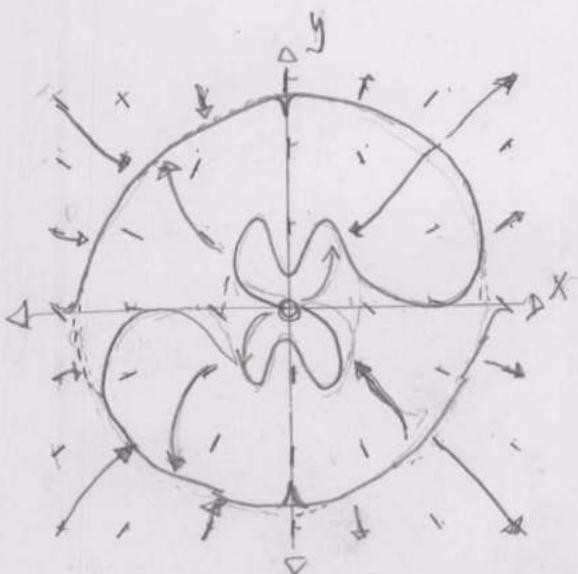
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r \cos \theta + \sin \theta \frac{dr}{d\theta}}{-r \sin \theta + \cos \theta \frac{dr}{d\theta}}$$

where  $r = \sqrt{x^2 + y^2}$ ;  $\theta = \arctan \frac{y}{x}$   
and  $\frac{dr}{d\theta} = (\sqrt{x^2 + y^2})^3 - 4 \sqrt{x^2 + y^2}$

$$\dot{r} = r(1 - r^2)(9 - r^2) \quad 7.1.2$$

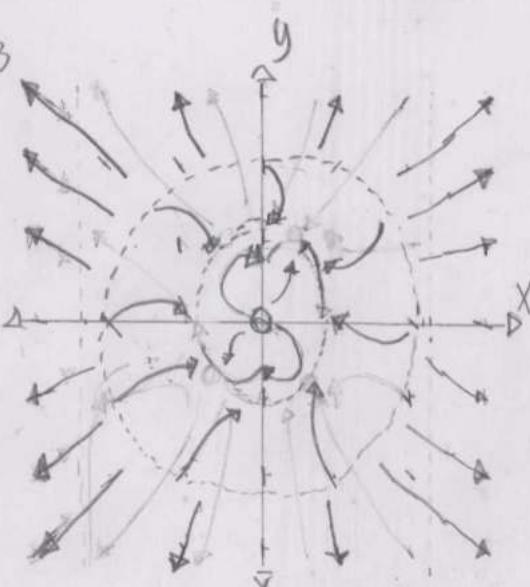
$$\dot{\theta} = 1$$

where  $r = \sqrt{x^2 + y^2}$ ;  $\theta = \arctan \left(\frac{y}{x}\right)$   
and  $\frac{dr}{d\theta} = r(1 - r^2)(9 - r^2)$



$$\dot{r} = r(1 - r^2)(4 - r^2) \quad 7.1.3$$

$$\dot{\theta} = 2 - r^2$$



$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r \cos \theta + \sin \theta \frac{dr}{d\theta}}{r \sin \theta + \cos \theta \frac{dr}{d\theta}}$$

where  $r = \sqrt{x^2 + y^2}$ ;  $\theta = \arctan \left(\frac{y}{x}\right)$   
and  $\frac{dr}{d\theta} = \frac{r(1 - r^2)(4 - r^2)}{2 - r^2}$

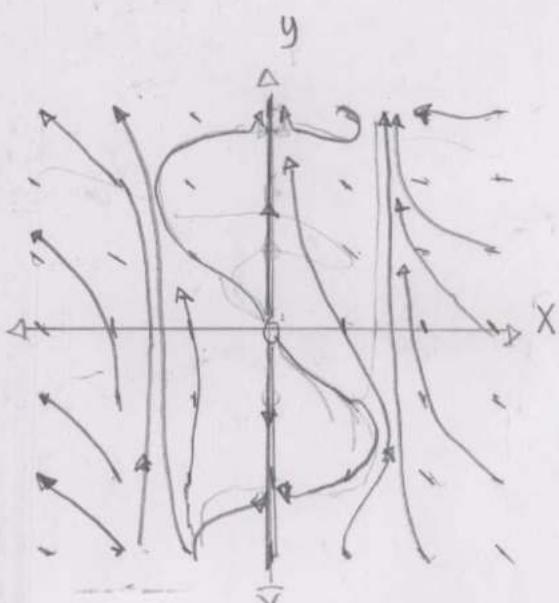
$$\dot{r} = r \sin n\theta \quad 7.1.4$$

$$\dot{\theta} = 1$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r \cos \theta + \sin \theta \frac{dr}{d\theta}}{-r \cos \theta + \cos \theta \frac{dr}{d\theta}}$$

where  $r = \sqrt{x^2 + y^2}$ ;  $\theta = \arctan \left(\frac{y}{x}\right)$

and  $\frac{dr}{d\theta} = r \sin n\theta$



$$\dot{r} = r(1-r^2) \quad 7.1.5. \quad x = r\cos\theta; \quad y = r\sin\theta;$$

$$\dot{\theta} = 1$$

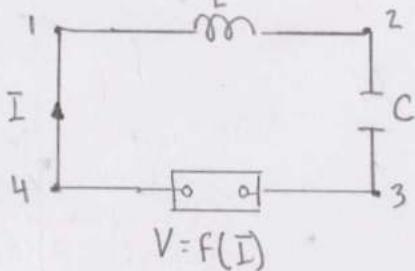
$$\begin{aligned}\dot{x} &= \frac{d}{dt} r\cos\theta = \dot{r}\cos\theta - r\sin\theta\dot{\theta} \\ &= r(1-r^2)\cos\theta - r\sin\theta = X(1-x^2-y^2) - y \\ &= x - x^3 - xy^2 - y = x - y - x(x^2+y^2)\end{aligned}$$

$$\begin{aligned}\dot{y} &= \frac{d}{dt} r\sin\theta = \dot{r}\sin\theta + r\cos\theta\dot{\theta} \\ &= r(1-r^2)\sin\theta - x = y(1-x^2-y^2) + x \\ &= x + y - y(x^2+y^2)\end{aligned}$$

$$\begin{aligned}\dot{V} &= -I/C \\ V &= L\dot{I} + f(I)\end{aligned}$$

7.1.6.

a)  $V = V_{32} = -V_{23}$



$$V_{41} - V_{12} - V_{23} - V_{34} = 0$$

~~$$V_I = V_L - V_C - V_F = 0$$~~

$$\begin{aligned}V_I &= V_L + V_C + V_F \\ &= L \frac{dI}{dt} + \frac{I}{C} + f(I) \\ &= L \frac{dI}{dt} + f(I) - \frac{I}{C} \\ &= V + \dot{V}\end{aligned}$$

b. If  $X = \sqrt{L}I$ ;  $W = \sqrt{C}V$ ;  $\tau = \sqrt{LC}$ , and  $F(x) = f(x/\sqrt{L})$  and  $\dot{X} = I$

then  $\dot{V} = -I/C = \frac{dW}{\sqrt{C}} = -\frac{X}{\sqrt{L}}\left(\frac{1}{\tau}\right)$  so  $\frac{dW}{d\tau} = -X$

and,  $V = L\dot{I} + f(I)$ ,  $\frac{W}{\sqrt{C}} = L \frac{dX}{dt}\left(\frac{1}{\sqrt{L}}\right) + f(X/\sqrt{L})$

~~$$W = \sqrt{LC} \frac{dx}{dt} + \sqrt{C}f(x/\sqrt{L})$$~~

$$\frac{dX}{d\tau} = W - \mu F(X) \text{ ; where } \mu = \sqrt{C}$$

$$\dot{r} = r(4-r^2)$$

$$\dot{\theta} = 1$$

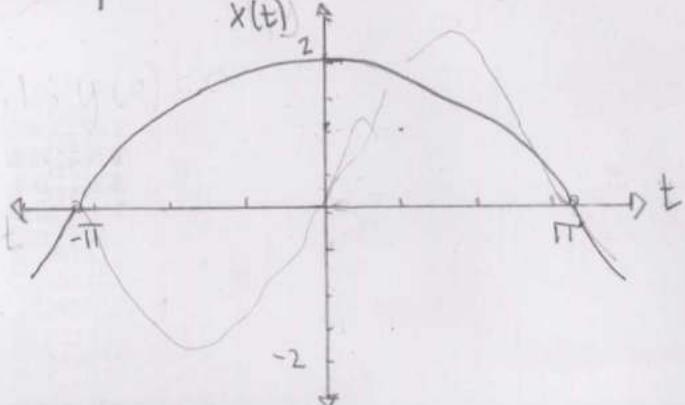
7.1.7.

$$x(t) = r(t)\cos\theta(t); \quad x(0) = 0.1$$

$$x(0) = 0.1; \quad y(0) = 0$$

$$r(t) = \int \frac{dr}{r(4-r^2)} = -2\ln(4-r^2) + C$$

$$\theta(t) = t; \quad r(t) = \sqrt{4 - Ce^{-t/2}}$$



$$\ddot{x} + \alpha \dot{x} (x^2 + \dot{x}^2 - 1) + x = 0$$

7.1.3:

$$a. \quad \dot{u} = \dot{x} = v$$

$$\dot{v} = \ddot{x} = -\alpha \dot{x} (x^2 + \dot{x}^2 - 1) - x = -\alpha v (u^2 + v^2 - 1) - u$$

$$\text{Fixed Points: } \dot{u} = \dot{x} = 0$$

$$\dot{v} = -\alpha v (u^2 + v^2 - 1) - u = 0$$

$$(u^*, v^*) = (0, 0)$$

$$\vec{U} = A \cdot \vec{v}; \quad A = \begin{pmatrix} 0 & 0 \\ -2vu-1 & -\alpha(u^2+3v^2-1) \end{pmatrix}$$

$$\lambda_1 = 0; \quad \lambda_2 = -\alpha(u^2 + 3v^2 - 1)$$

$$\Delta = 0; \quad \Gamma = -\alpha(u^2 + 3v^2 - 1)$$

$\Gamma^2 - 4\Delta > 0$  "Non-isolated  
Fixed Point"

$$b. \quad \dot{r} = \frac{(u\dot{u} + v\dot{v})}{r} = \frac{(r \cos \theta \dot{r} \sin \theta + r \sin \theta \dot{r} (-\alpha v (u^2 + v^2 - 1) - u))}{r}$$

$$V(u, v) = \frac{v(u - uv)}{r} = \frac{(r^2 \cos \theta \sin \theta - \alpha r \sin^2 \theta \sin \theta (r^2 \cos^2 \theta + r^2 \sin^2 \theta - 1) - r^2 \cos \theta \sin \theta)}{r}$$

$$= -\alpha r \sin^2 \theta (r^2 - 1)$$

$$\dot{\theta} = \frac{(v\dot{u} - u\dot{v})}{r^2} = \frac{(-\alpha r \sin \theta (r^2 - 1) - r \cos \theta) r \cos \theta - r^2 \sin \theta}{r^2}$$

$$= \frac{-\alpha r^2 \sin \theta \cos \theta (r^2 - 1) - r^2 \cos^2 \theta - r^2 \sin^2 \theta}{r^2}$$

$$= -\alpha \sin \theta \cos \theta (r^2 - 1)$$

$$\text{Amplitude: } -\alpha r = -\alpha \sqrt{u^2 + v^2} = -\alpha \sqrt{x^2 + \dot{x}^2}$$

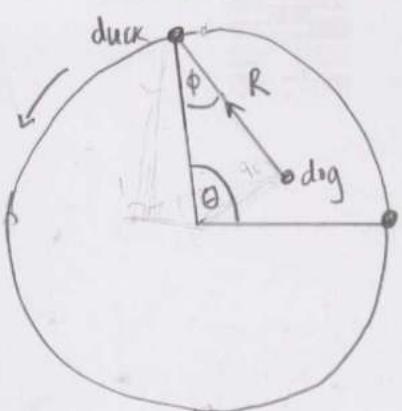
$$\text{Period: } \frac{2\pi}{|\dot{\theta}|} = 2\pi + C_1$$

c. Stable limit cycle because larger  $\alpha$  values generate a standard and periodic trajectory.

a. The limit cycle is unique because solutions containing a values and initial conditions have many solutions, for similar initial conditions.

$$\frac{dR}{d\theta}, \frac{d\phi}{d\theta}$$

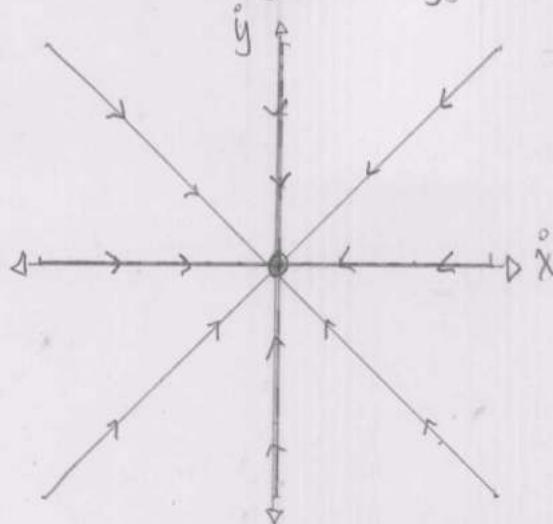
7.1.9. a.



$$\text{Duck: } \langle x, y \rangle = \langle r \cos k\theta, r \sin k\theta \rangle$$

$$\text{Dog: } \langle x, y \rangle =$$

$$V = x^2 + y^2 \quad 7.2.1: \dot{\vec{x}} = -\nabla V; \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = - \begin{bmatrix} dx \\ dy \end{bmatrix} V; \quad \dot{x} = - \int x^2 = -\frac{x^3}{3}$$



$$\dot{y} = - \int y^2 = -\frac{y^3}{3}$$

$$V = x^2 - y^2 \quad 7.2.2: \dot{\vec{x}} = -\nabla V; \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = - \begin{bmatrix} dx \\ dy \end{bmatrix} V; \quad \dot{x} = - \int x^2 dx = -\frac{x^3}{3} + C$$

$$\dot{y} = + \int y^2 dx = +\frac{y^3}{3} + C$$

$$V = e^x \sin y \quad 7.2.3: \dot{\vec{x}} = -\nabla V$$

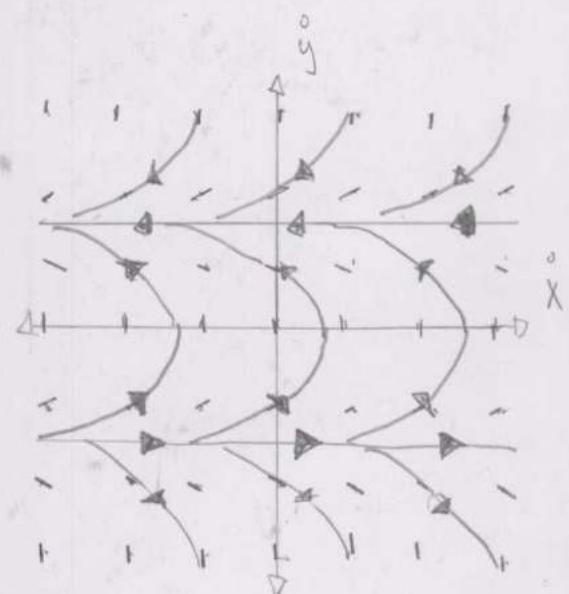
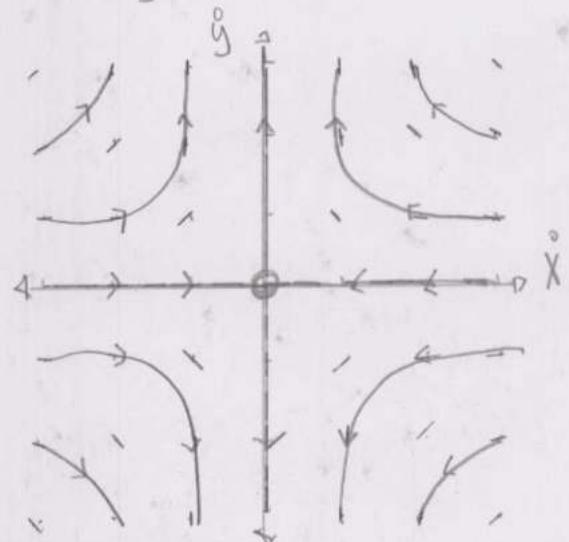
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = - \begin{bmatrix} dx \\ dy \end{bmatrix} V$$

$$\dot{x} = - \int e^x \sin y dx$$

$$= -e^x \sin y + C$$

$$\dot{y} = - \int e^x \sin y dy$$

$$= e^x \cos y + C$$



7.2.4. Gradient System: When a system can be written as  $\dot{x} = -\nabla V$ , for a continuously differentiable, single-valued scalar function

Line: A continuous function without curvature

Circle: A continuous and bounded function by an equidistant center.

Line:  $\dot{x}$ ;  $\dot{x} = -\nabla V = -1$ ; A gradient system.

Circle:  $\sqrt{x^2 + y^2}$ ;  $\dot{x} = -\nabla V = -\frac{X}{\sqrt{x^2 + y^2}}$  gradient system.

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned} \quad 7.2.5. \text{ a. } -\nabla V = \left( -\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y} \right), \quad \frac{\partial V}{\partial x} = \dot{x} = f(x, y) \quad ; \quad -\frac{\partial V}{\partial y} = \dot{y} = g(x, y)$$

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

b. A sufficient condition is  $p \rightarrow q$  and  $\neg p \rightarrow \neg q$ ,

so  $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$  is sufficient by  $V = - \int f(x, y) dx = - \int g(x, y) dy$ .

$$\begin{aligned} \dot{x} &= y^2 + y \cos x \\ \dot{y} &= 2xy + \sin x \end{aligned} \quad 7.2.6. \quad \text{a. } V = \int \dot{x} dx + \int \dot{y} dy = xy^2 + y \sin x + xy^2 + y \sin x \\ &= 2(xy^2 + y \sin x)$$

$$\begin{aligned} \dot{x} &= 3x^2 - 1 - e^{2y} \\ \dot{y} &= -2xe^{2y} \end{aligned} \quad \text{b. } V = \int \dot{x} dx + \int \dot{y} dy = x^3 - x - xe^{2y} + xe^{2y} \\ &= x(x^2 - 1 - 2e^{2y})$$

$$\begin{aligned} \dot{x} &= y + 2xy \\ \dot{y} &= x + x^2 - y^2 \end{aligned} \quad 7.2.7. \quad \text{a. If } \dot{x} = f(x, y) \text{ and } \dot{y} = g(x, y), \text{ then}$$

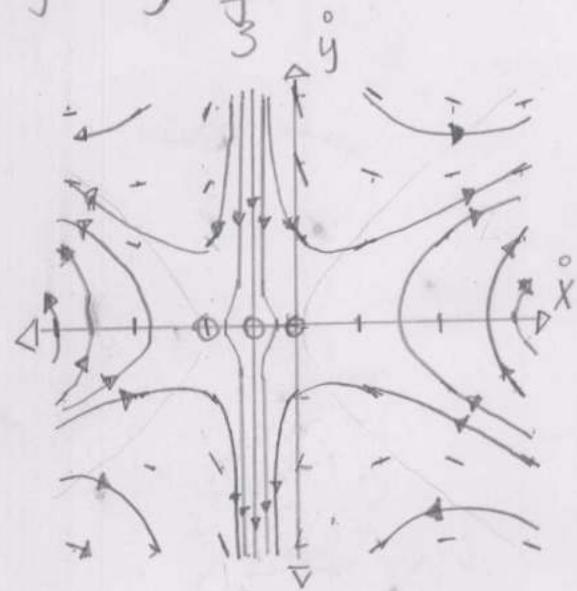
$$\text{then } \frac{\partial f}{\partial y} = 1 + 2x \text{ and } \frac{\partial g}{\partial x} = 1 + 2x$$

$$\text{and } \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

$$b. V = \int \dot{x} dx + \int \dot{y} dy = xy + xy^2 + xy + x^2y - \frac{y^3}{3}$$

$$= 2xy + xy^2 + x^2y - \frac{y^3}{3}$$

c.



7.2.8. If  $\frac{df}{dy} = \frac{dg}{dx}$  at an equipotential, then  $\frac{dy}{dx} = \frac{dg}{df}$ .

The solution  $\frac{dy}{dx} = \frac{dg}{df}$  is zero when  $dg \neq df$   
and one when  $dg = df$ , very similar to  
orthogonal slopes ( $dy/dx$ ).

$$\begin{aligned}\dot{x} &= y + x^2y \\ \dot{y} &= -x + 2xy\end{aligned}$$

7.2.9.

$$a. V = \int \dot{x} dx + \int \dot{y} dy = - \int y + x^2y dx + \int -x + 2xy dy$$

$$= -xy - \frac{x^3y}{3} + \frac{x^2}{2} = x^2y ; \frac{d\dot{x}}{dy} = \frac{d\dot{y}}{dx}$$

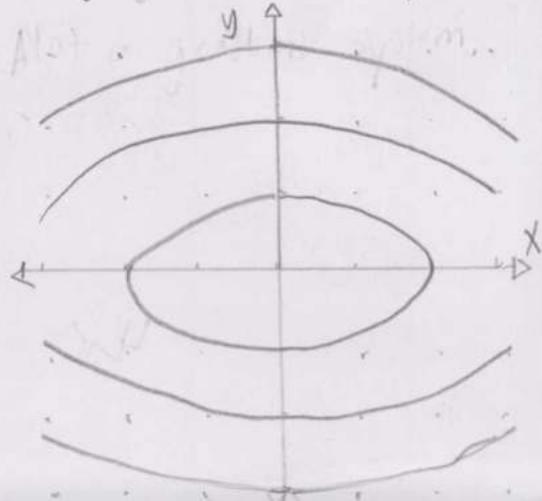
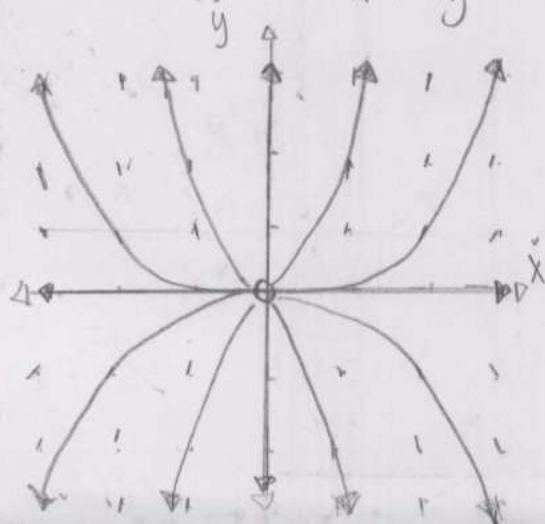
Not a gradient system.

$$\dot{x} = 2x$$

$$\dot{y} = 8y$$

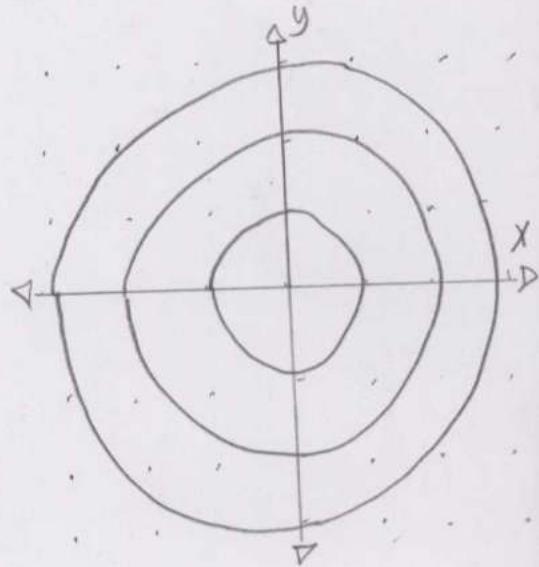
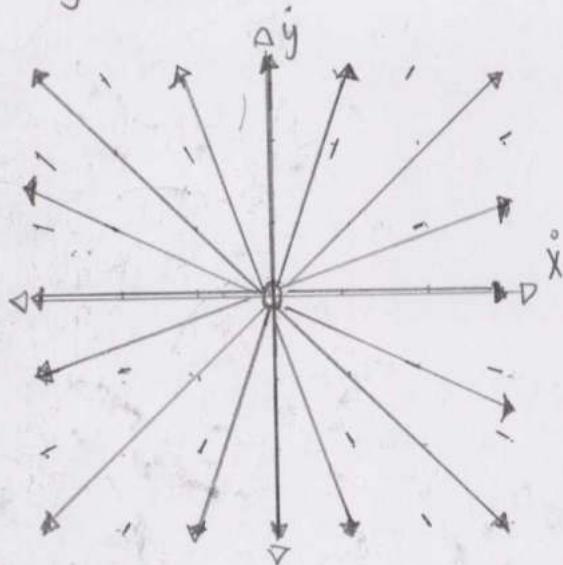
$$b. V = - \int \dot{x} dx - \int \dot{y} dy = - \int 2x dx - \int 8y dy$$

$$= -x^2 + 4y^2 ; x(0) \times \frac{d\dot{x}}{dy} = \frac{d\dot{y}}{dx} \text{ Gradient System}$$



$$\begin{aligned}\dot{x} &= -2x e^{x^2+y^2} & c. V(x,y) &= - \int \dot{x} dx - \int \dot{y} dy = +2 \int x e^{x^2+y^2} dx + 2 \int y e^{x^2+y^2} dy \\ \dot{y} &= -2y e^{x^2+y^2} & &= e^{x^2+y^2} + e^{x^2+y^2} = 2e^{x^2+y^2}\end{aligned}$$

$\frac{dx}{dy} (= \frac{dy}{dx})$ : Gradient system.



$$\begin{aligned}\dot{x} &= y - x^3 & 7.2.10. V &= - \int \dot{x} dx - \int \dot{y} dy = - \int y - x^3 dx + \int x + y^3 dy \\ \dot{y} &= -x - y^3 & &= -xy + \frac{x^4}{4} + xy + \frac{y^4}{4} = \frac{x^4}{4} + \frac{y^4}{4} \\ & & a = \frac{1}{4}, b = \frac{1}{4}\end{aligned}$$

The potential function has a suitable  $a$ , and  $b$ , so this function is Liapunov stable with no closed orbits.

$$V = ax^2 + 2bxy + cy^2 \quad 7.2.11. \quad \frac{\partial^2 V}{\partial x^2} \frac{\partial^2 V}{\partial y^2} - \left( \frac{\partial^2 V}{\partial x \partial y} \right)^2 = (2a)(2c) - (2b)^2 = 4(ac - b^2) \quad @ (0,0)$$

Positive definite is a strictly positive, meaning strictly positive when  $(ac - b^2) > 0$ .

$$\begin{aligned}\dot{x} &= -x + 2y^3 - 2y^4 & 7.2.12. V &= - \int \dot{x} dx - \int \dot{y} dy = \frac{x^2}{2} - 2y^3 x + 2y^4 x + xy + \frac{y^2}{2} - \frac{xy^2}{2} \\ \dot{y} &= -x - y + xy\end{aligned}$$

Fixed Points  $(x^*, y^*) = (0,0), (-2,0), (-2,1), (-1,0)$

No periodic solutions

$$\dot{N}_1 = r_1 N_1 (1 - N_1/K_1) - b_1 N_1 N_2$$

$$\dot{N}_2 = r_2 N_2 (1 - N_2/K_2) - b_2 N_1 N_2$$

$$7.2.13 \quad g = (N_1 N_2)^{-1}$$

Dulac's Criterion: If  $\dot{x} = f(x)$  is continuous, a real-valued function  $g(x)$  such that  $\nabla \cdot (g \dot{x})$  has one sign throughout  $\mathbb{R}$  in a closed orbit.

$$\begin{aligned}\nabla \cdot (g \dot{x}) &= \frac{\partial}{\partial N_1} (g \dot{N}_1) + \frac{\partial}{\partial N_2} (g \dot{N}_2) \\ &= \frac{\partial}{\partial N_1} \left[ \frac{r_1}{N_2} (1 - N_1/K_1) - b_1 \right] - \frac{\partial}{\partial N_2} \left[ \frac{r_2}{N_1} (1 - N_2/K_2) - b_2 \right] \\ &= \frac{r_2}{N_1 K_2} - \frac{r_1}{N_2 K_1}\end{aligned}$$

$< 0$

$$\begin{aligned}\dot{x} &= x^2 - y - 1 \\ \dot{y} &= y(x-2)\end{aligned}$$

$$7.2.14. \text{ a. Fixed Points: } \dot{x} = 0 = x^2 - y - 1$$

$$\dot{y} = 0 = y(x-2)$$

$$(x^*, y^*) = (-1, 0); A_{(-1,0)} = \begin{pmatrix} -2 & -1 \\ 0 & -2 \end{pmatrix}; \text{ Stable Node}$$

$$(1, 0); A_{(1,0)} = \begin{pmatrix} 2 & -1 \\ 0 & -2 \end{pmatrix}; \text{ Saddle Point}$$

$$(2, 3); A_{(2,3)} = \begin{pmatrix} 4 & -1 \\ 3 & -2 \end{pmatrix}; \text{ Saddle Point}$$

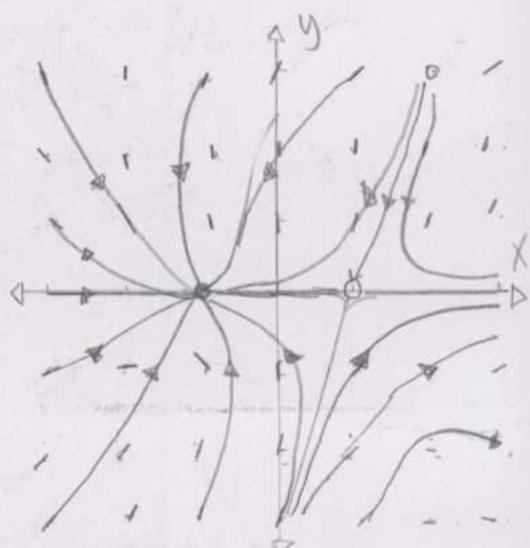
$$\text{b. } (-1, 0) \rightarrow (1, 0); \frac{dy}{dx} = 0$$

$$(1, 0) \rightarrow (2, 3); \frac{dy}{dx} = x^2 - 1$$

$$(2, 3) \rightarrow (-1, 0); \frac{dy}{dx} \approx -1$$

Closed orbits are nonexistent because of constant trajectory between fixed points.

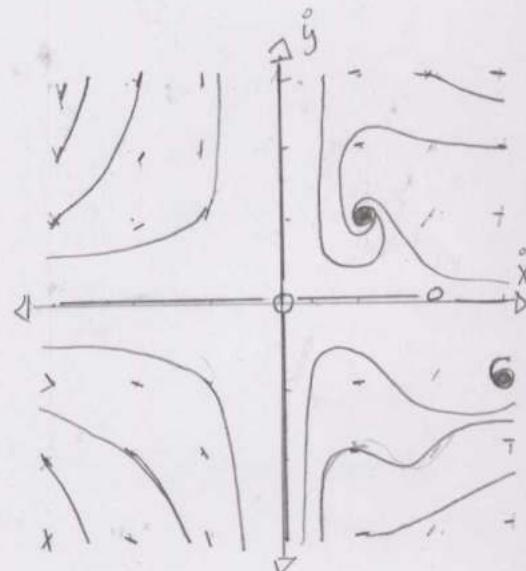
c.



$$\begin{aligned} \dot{x} &= x(2-x-y) & 7.2.15. \text{ Fixed Points: } \dot{x} = 0 = x(2-x-y) ; A = \begin{pmatrix} 2-2x-y & -x \\ 4y-2yx & 4x-x^2-3 \end{pmatrix} \\ \dot{y} &= y(4x-x^2-3) & \dot{y} = 0 = y(4x-x^2-3) \\ (x^*, y^*) &: (0, 0) : A_{(0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}; \text{Saddle Point} \\ (1,1) &: A_{(1,1)} = \begin{pmatrix} -1 & 2^{-1} \\ 2 & 0 \end{pmatrix}; \text{Stable spiral} \\ (2,0) &: A_{(2,0)} = \begin{pmatrix} -2 & -2 \\ 0 & 1 \end{pmatrix}; \text{Saddle Point} \\ (3,-1) &: A_{(3,-1)} = \begin{pmatrix} 9 & -3 \\ 2 & 0 \end{pmatrix}; \text{Stable spiral} \end{aligned}$$

### b. Phase Portrait

7.2.16. If  $R$  is a set of closed orbits, then  $\iint_R \nabla \cdot (g\dot{x}) dA = \oint_C g\dot{x} dx$



7.2.17. If  $A$  is an annulus, then  
Greens theorem holds true.  
Dulth's Criterion fails for  
multiple holes because the  
closed orbit (line integral) has only one path.

$$\dot{x} = rx\left(1 - \frac{x}{2}\right) - \frac{2x}{1+x}y \quad 7.2.18 \text{ If } g(x,y) = \frac{1+x}{x}y^{k-1}; \text{ then } \dot{x} = rx\left(1 - \frac{x}{2}\right) - 2g(x,y)^{-1}$$

$$\dot{y} = -y + \frac{2x}{1+x}y$$

$$\dot{y} = -y + 2g(x,y)^{-1}$$

$$\text{where } k = 0$$

$$\nabla \cdot (g\dot{x}) = \frac{\partial}{\partial x}(g\dot{x}) + \frac{\partial}{\partial y}(g\dot{y})$$

$$= \frac{r(1-2x)}{2y} > 0 ; \text{No closed orbits in the positive quadrants}$$

$$\dot{R} = -R + A_s + kS e^{-s} \quad 7.219.$$

$$\dot{S} = -S + A_r + kR e^{-R}$$

Term:	Meaning:
$-R$	Rhett's decreasing love for Scarlett.
$+A_s$	Scarlett's love for Rhett.
$+kS e^{-s}$	Scarlett's decaying love for Rhett.
$-S$	Scarlett's decreasing love for Rhett.
$+A_r$	Rhett's love.
$+kR e^{-R}$	Rhett's decaying love for Scarlett.

b.  $\dot{R} = 0 = -R + A_s + kS e^{-s}; (R^*, S^*) = (A_s + kS e^{-s}, A_r + kR e^{-R})$   
 $\dot{S} = 0 = -S + A_r + kR e^{-R};$  which are greater than zero.

c.  $\nabla \cdot (g \vec{x}) = \frac{\partial}{\partial R}(g \dot{R}) + \frac{\partial}{\partial S}(g \dot{S})$

$$= -1 - 1 = -2 < 0; \text{ where } g = 1.$$

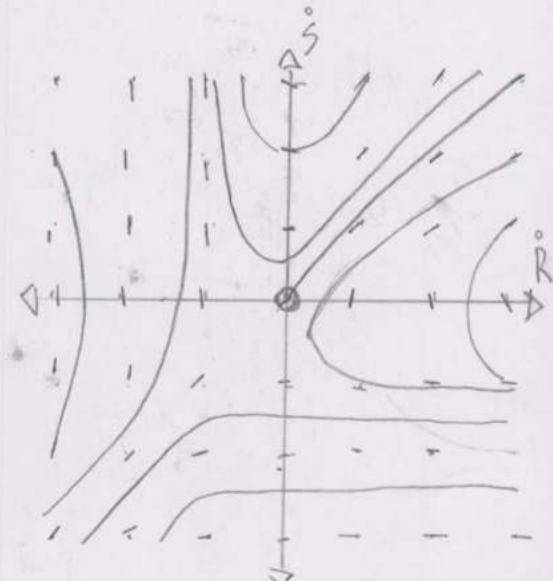
Since  $x, y > 0$ , there are no periodic solutions in the first quadrant.

d. Phase Portrait;  $A_s = 1, 2$

$$A_r = 1$$

$$k = 15$$

$$R(0) = S(0) = 0$$



$$\begin{aligned} \dot{x} &= x - y - x(x^2 + 5y^2) & 7.3.1 \text{ a. } A = \begin{pmatrix} 1-x^2-5y^2-2x^2 & -1-10xy \\ 1-2xy & 1-x^2-y^2-2y^2 \end{pmatrix} \\ \dot{y} &= x + y - y(x^2 + y^2) \\ &= \begin{pmatrix} 1-3x^2-5y^2 & -1-10xy \\ 1-2xy & 1-x^2-3y^2 \end{pmatrix} \end{aligned}$$

$$A_{(0,0)} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}; \Delta = 2; \tau = 2; \tau^2 - 4\Delta < 0; \text{ Unstable Spiral}$$

$$\begin{aligned}
 b. \quad \dot{r} &= \frac{x\dot{x} + y\dot{y}}{r} = \frac{r\cos\theta[x - y - x(x^2 + 5y^2)] + r\sin\theta[x + y - y(x^2 + y^2)]}{r} \\
 &= r\cos\theta[\cos\theta - \sin\theta - \cos\theta(r^2\cos^2\theta + 5r^2\sin^2\theta)] \\
 &\quad + r\sin\theta[\cos\theta + \sin\theta - \sin\theta(r^2)] \\
 &= r[\cos\theta(\cos\theta - \sin\theta - r^2\cos\theta(\cos^2\theta + 5\sin^2\theta)) \\
 &\quad + \sin\theta(\cos\theta + \sin\theta - r^2\sin^2\theta)] \\
 \dot{\theta} &= \frac{(x\dot{y} - y\dot{x})}{r^2} = \frac{\cos\theta(x + y - y(r^2)) - \sin\theta(x - y - x(x^2 + 5y^2))}{r^2} \\
 &= \cos\theta(\cos\theta + \sin\theta - r^2\sin\theta) - \sin\theta(\cos\theta - \sin\theta - r^2\cos\theta(\cos^2\theta + 5\sin^2\theta)) \\
 &= 4r^2\sin^3(\theta)\cos(\theta) + 1
 \end{aligned}$$

$$\begin{aligned}
 c. \quad r_{\min, \text{outward}} & \quad \dot{r} > 0 ; r[\cos\theta(\cos\theta - \sin\theta - r^2\cos\theta(\cos^2\theta + 5\sin^2\theta)) \\
 & \quad + \sin\theta(\cos\theta + \sin\theta - r^2\sin^2\theta)] > 0 \\
 r_{\min, \text{outward}} & \cong -\sqrt{\frac{\sin^2\theta - \sin\theta + \cos\theta}{\sin^3\theta + 3\cos\theta - 2\cos\theta\cos(2\theta)}}
 \end{aligned}$$

$$\begin{aligned}
 d. \quad r_{\max, \text{inward}} & \quad \dot{r} < 0 ; r[\cos\theta(\cos\theta - \sin\theta - r^2\cos\theta(\cos^2\theta + 5\sin^2\theta)) \\
 & \quad + \sin\theta(\cos\theta + \sin\theta - r^2\sin^2\theta)] < 0
 \end{aligned}$$

$$r_{\max, \text{inward}} \cong \sqrt{\frac{(\sin\theta - 1)\sin\theta + \cos\theta}{\sin^3\theta + \cos\theta(3 - 2\cos 2\theta)}}$$

$$e. \quad r_{\min, \text{outward}} < r < r_{\max, \text{inward}}$$

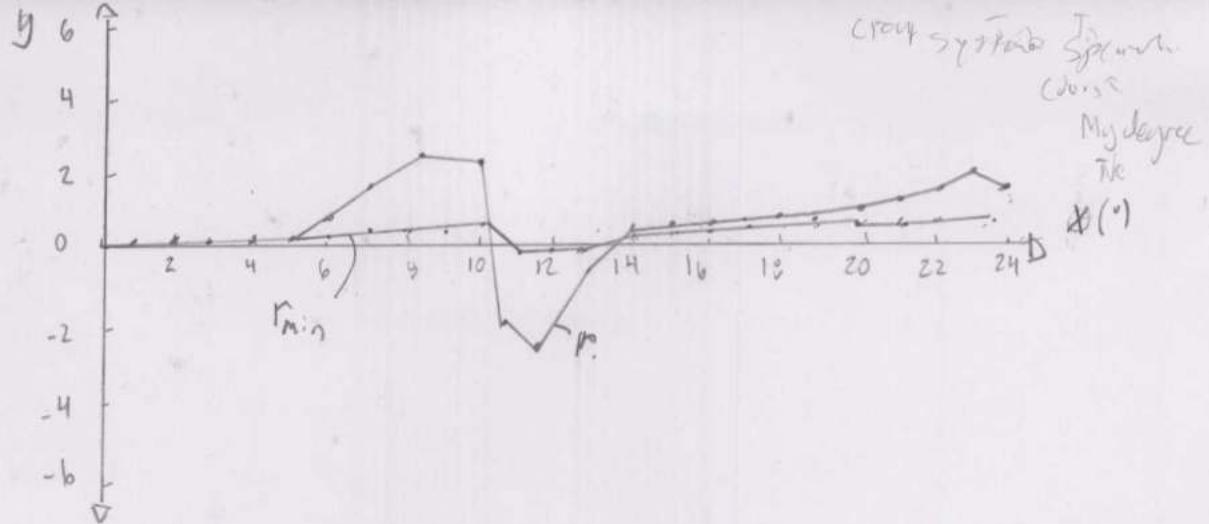
$$\Theta \cong 2(n\pi - 4\pi/10); \text{ This solution is close to the books } \Theta = \frac{3\pi}{2}$$

### 7.3.2 Runge Kutta Method [4th Order]

$$\begin{array}{|c|c|c|c|} \hline
 r_0 & \Theta_0 & k_1(r_0, \Theta_0 + \Theta_0\Delta h/2) & k_1(r_0 + r_0\Delta h/2, \Theta_0) \\
 \hline
 0.1 & & k_2(r_0 + k_1\Delta h/2, \Theta_0) & k_2(r_0 + k_1\Delta h/2, \Theta_0) \\
 \hline
 \end{array} \dots \dots$$

$$\begin{array}{|c|c|c|c|} \hline
 0.0 & k_3(r_0, \Theta_0 + k_1\Delta h/2) & k_3(r_0 + k_2\Delta h, \Theta_0) & k_3(r_0, \Theta_0 + k_2\Delta h) \\
 \hline
 \end{array} \dots \dots$$

$$r_n = r_{n-1} + \frac{\Delta h}{6}(r_0 + 2k_1 + 2k_2 + k_3) ; \quad \Theta_n = \Theta_{n-1} + \frac{\Delta h}{6}(\Theta_0 + 2k_1 + 2k_2 + k_3)$$



$$\begin{aligned}\dot{x} &= x - y - x^3 \\ \dot{y} &= x + y - y^3\end{aligned}$$

7.3.3.  $r = \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{r} = \frac{r \cos \theta (x - y - x^3) + r \sin \theta (x + y - y^3)}{r}$

$$= \frac{x \cos \theta (r \cos \theta - r \sin \theta - (r \cos \theta)^3) + y \sin \theta (r \cos \theta + r \sin \theta - (r \sin \theta)^3)}{r}$$

$$= r - r(r \cos^4(\theta) + \sin^4(\theta))$$

$r - r^3 < r < r - r^3/2$  Poincaré-Bendixson Theorem states  
at least one periodic solution.

$$\dot{x} = x(1 - 4x^2 - y^2) - \frac{1}{2}y(1+x)$$

$$\dot{y} = y(1 - 4x^2 - y^2) + 2x(1+x)$$

7.3.4. a.  $A = \begin{pmatrix} -12x^2 - y^2 - 1 & (-2x - 1)y - 1/2 \\ x(4 - 3y) + 2 & -4x^2 - 3y^2 + 1 \end{pmatrix}$

$$A_{(0,0)} = \begin{pmatrix} -1 & -1/2 \\ 2 & 1 \end{pmatrix}; \lambda_1 = 0, \lambda_2 = 0; \Delta = 0; \tau = 0; \tau^2 - 4\Delta = 0 \text{ "center"}$$

Although graph indicates an unstable spiral.

b.  $V = (1 - 4x^2 - y^2)^2; \dot{V} = 2(1 - 4x^2 - y^2)(1 - 4x^2 - y^2)'$

$$\lim_{t \rightarrow \infty} \dot{V} = 0; 1 - 4x^2 - y^2 = 0; 4x^2 + y^2 = 1$$

$$\begin{aligned}\dot{x} &= -x - y + x(x^2 + 2y^2) \\ \dot{y} &= x - y + y(x^2 + 2y^2)\end{aligned}$$

7.3.5  $r = \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{r} = \frac{r \cos \theta (-x - y + x(x^2 + 2y^2)) + r \sin \theta (x - y + y(x^2 + 2y^2))}{r}$

$$= \frac{r \cos \theta (-r \cos \theta - r \sin \theta + r \cos \theta ((r \cos \theta)^2 + 2(r \sin \theta)^2))}{r} \\ + \frac{r \sin \theta (r \cos \theta - r \sin \theta + r \sin \theta ((r \cos \theta)^2 + 2(r \sin \theta)^2))}{r}$$

$$= r^3 (\sin^2(x) + 1) \left( \frac{1}{2} \sin(2x) + \cos^2(x) \right) - r$$

$$\theta = \frac{38}{100} r^3 - r < r < \frac{152}{100} r^3 - r$$

A periodic solution exists by Poincare-Bendixson Theorem.

$$\ddot{x} + F(x, \dot{x})\dot{x} + x = 0$$

7.3.6.  $F(x, \dot{x}) < 0$ ,  $r \leq a$ , else,  $F(x, \dot{x}) > 0$  if  $(r \geq b)$  where  $r^2 = x^2 + \dot{x}^2$

a.  $\ddot{u} = \ddot{x} = V$  A physical interpretation of  
 $\ddot{V} = \ddot{\dot{x}} = -F(x, \dot{x})\dot{x} - x$   $F(x, \dot{x})$  is an additive force  
 to acceleration, which increases  
 or decreases.

$$b. r = \sqrt{x^2 + \dot{x}^2} = \sqrt{(r \cos \theta)^2 + ((\dot{r} \cos \theta) - (r \sin \theta \dot{\theta}))^2}$$

$$a < \sqrt{(r \cos \theta)^2 + ((\dot{r} \cos \theta) - (r \sin \theta \dot{\theta}))^2} < b$$

$$\ddot{x} = y + ax(1-2b-r^2)$$

$$\ddot{y} = -x + ay(1-r^2)$$

7.3.7. a.  $(0 < a \leq 1, 0 \leq b < 1/2)$  and  $r^2 = x^2 + y^2$

$$\ddot{r} = \frac{x\ddot{x} + y\ddot{y}}{r} = \frac{r \cos \theta (y + ax(1-2b-r^2)) + r \sin \theta (-x + ay(1-r^2))}{r}$$

$$= r \cos \theta (r \sin \theta + a \cos \theta (1-2b-r^2)) + r \sin \theta (-r \cos \theta + a \sin \theta (1-r^2))$$

$$\ddot{\theta} = \frac{x\ddot{y} - y\ddot{x}}{r^2} = \frac{r \cos \theta (-x + ay(1-r^2)) + r \sin \theta (y + ax(1-2b-r^2))}{r^2}$$

$$= \cos \theta (-\cos \theta + a \sin \theta (1-r^2)) - \sin \theta (\sin \theta + a \cos \theta (1-2b-r^2))$$

$$= ab \sin(2\theta) - b \sin(2\theta) - 1$$

b. A region of trapping  $a \pi (1-r^2) \leq \ddot{r} \leq a \pi (1-2b-r^2)$   
 exists as an annular cycle,  $T(a, b)$ .

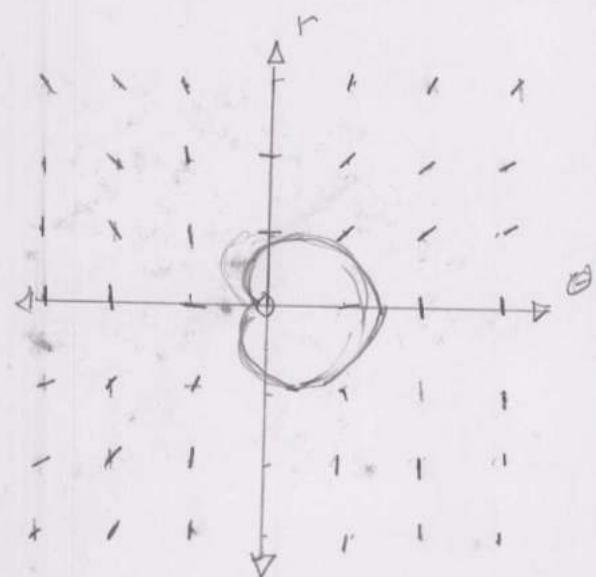
c) If  $b=0$ , then  $ar(1-r^2) \leq r \leq ar(1-r^2)$ , so  $r$   
must be  $ar(1-r^2)$ .

$$\dot{r} = r(1-r^2) + \mu r \cos \theta$$

$$\ddot{\theta} = 1 \quad 7.3.8. \quad \dot{r} = 0 = r(1-r^2) + \mu r \cos \theta$$

$$\mu = \frac{r(r^2-1)}{r \cos \theta}$$

If  $r=0, 1$ , or  $-1$ , then the closed orbit known as the cardioid becomes absent, but a circular orbit remains.



7.3.9.

$$a. \quad r(\theta) = 1 + \mu r_1(\theta) + O(\mu^2)$$

$$\frac{dr}{d\theta} = r(1-r^2) + \mu r \cos \theta = \mu r_1'(\theta)$$

$$\mu r_1'(\theta) = (1 + \mu r_1(\theta))(1 - (1 + \mu r_1(\theta))^2) + \mu(1 + \mu r_1(\theta)) \cos \theta$$

$r_1'(\theta) = -2r_1(\theta) + \cos(\theta)$  First-order linear differential equation.

$$\cos(\theta) = r_1'(\theta) + 2r_1(\theta)$$

$$\cos(\theta) d\theta = r_1'(\theta) + 2r_1(\theta) d\theta$$

$$b. \quad (\cos(\theta) - 2r_1') d\theta - dr(\theta) = 0 \quad \text{Exact Differential Equation.}$$

$$\mu N(r_1, \theta) d\theta + M(r_1, \theta) dr = 0$$

$N'(r_1, \theta) \neq M'(r_1, \theta)$ , so a multiplication is necessary.

$$(2N'(r_1, \theta) = \frac{dN}{dr} = 2) ; \quad M'(r_1, \theta) = \frac{dM}{d\theta} = 0$$

The assumption;  $\frac{dN}{dr} = \frac{dM}{d\theta}$ ;  $\frac{dN}{dr} - \frac{dM}{d\theta} = 0 = \mu \left( \frac{dN}{dr} - \frac{dM}{d\theta} \right)$

$$\text{Also, } M(r_1, \theta) = \mu(r); \quad \frac{dM}{d\theta} = 0; \quad \text{so } \frac{1}{\mu} \frac{d\mu}{dr} = \frac{1}{M} \left( \frac{dN}{dr} - \frac{dM}{d\theta} \right)$$

$$\int \frac{d\mu}{\mu} = \int \frac{1}{M} \left( \frac{dN}{dr} - \frac{dM}{d\theta} \right) d\theta; \quad \ln(\mu) = 2\theta; \quad \mu = e^{2\theta}$$

Multiplying the equation by  $e^{2\theta}$

$$(\cos(\theta) - 2r)e^{2\theta} d\theta - e^{2\theta} dr = 0$$

$$N(r, \theta) d\theta - M(r, \theta) dr = 0$$

$$N'(r, \theta) = M'(r, \theta) = -2e^{2\theta} \quad \text{Exact equation.}$$

$$dF(r, \theta) = N(r, \theta) d\theta + M(r, \theta) dr$$

$$\begin{aligned} F(r, \theta) &= \int N(r, \theta) dr = \int e^{2\theta} \cos(\theta) - 2e^{2\theta} \cdot r d\theta \\ &= \frac{e^{2\theta} \sin(\theta)}{5} + \frac{2e^{2\theta} \cos(\theta)}{5} - e^{2\theta} r + C \end{aligned}$$

$$r = \frac{\sin(\theta)}{5} + \frac{2\cos(\theta)}{5} + \frac{C}{e^{2\theta}}$$

$$r(\theta) = 1 + \mu \left( \frac{\sin(\theta)}{5} + \frac{2\cos(\theta)}{5} \right)$$

b.  $\frac{dr}{d\theta} = 0 = \frac{\mu \cos \theta}{5} - \frac{2 \sin \theta}{5} \quad @ \theta = \arctan\left(\frac{1}{2}\right) + n\pi$

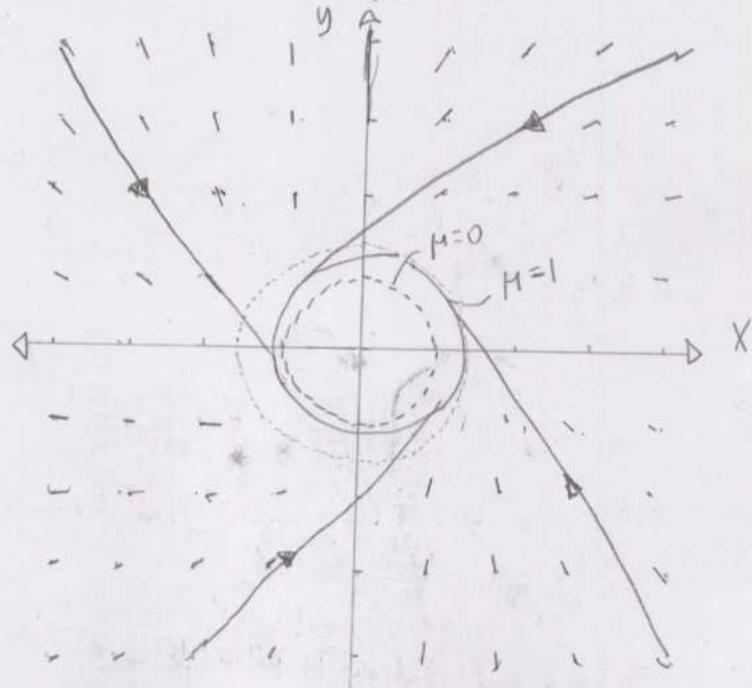
$$r(\arctan(\frac{1}{2})) = 1 + \mu \left( \frac{1}{5} \frac{1}{\sqrt{5}} + \frac{2}{5} \frac{2}{\sqrt{5}} \right) = 1 + \frac{\mu}{\sqrt{5}}$$

or

$$r(\arctan(\frac{1}{2}) + \pi) = 1 + \mu \left( \frac{1}{5} \frac{-1}{\sqrt{5}} + \frac{2}{5} \frac{-2}{\sqrt{5}} \right) = 1 - \frac{\mu}{\sqrt{5}}$$

$$\sqrt{1-\mu} < 1 - \frac{\mu}{\sqrt{5}} < r < 1 + \frac{\mu}{\sqrt{5}} < \sqrt{1+\mu}$$

c.



$$\dot{x} = Ax - r^2 x \quad 7.3.10. \quad r = \|x\|; \quad A \in \mathbb{R}; \quad \lambda_{1,2} = \alpha \pm i\omega$$

$$\Delta = (\alpha^2 + \omega^2); \quad \Gamma = 2\alpha; \quad \Gamma^2 - 4\Delta < 0$$

"Unstable Spiral"

Fixed Points:  $\dot{x} = 0 = Ax - r^2 x; \quad x = 0$  if  $\alpha < 0$  "stable fixed point"

$$x = 0,$$

if  $\alpha > 0$  "unstable spiral"

$$\dot{r} = r(1-r)[r^2 \sin^2 \theta + (r^2 \cos^2 \theta - 1)^2]$$

$$\dot{\theta} = r^2 \sin^2 \theta + (r^2 \cos^2 \theta - 1)^2$$

7.3.11.  $\dot{x} = f(x)$  is a vector field on  $\mathbb{R}^2$

Cycle graph: an invariant set containing a finite number of fixed points connected by a finite number of trajectories, all oriented clockwise or counter-clockwise.

a. Phase Portrait

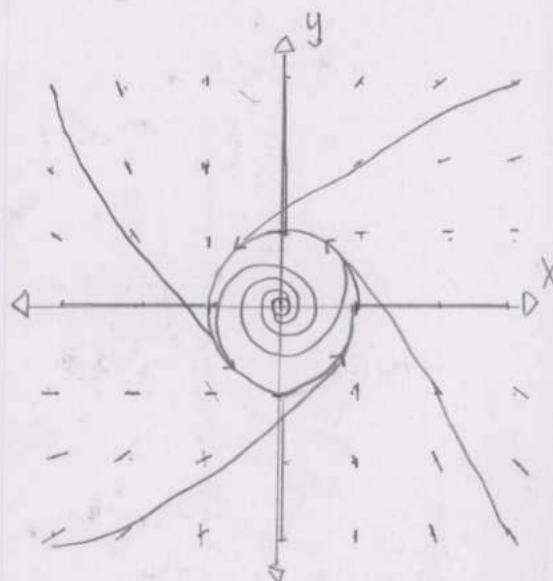
$$\dot{x} = \frac{r}{r} x - y \dot{\theta}$$

$$= r(1-r)[r^2 \sin^2 \theta + (r^2 \cos^2 \theta - 1)^2] x$$

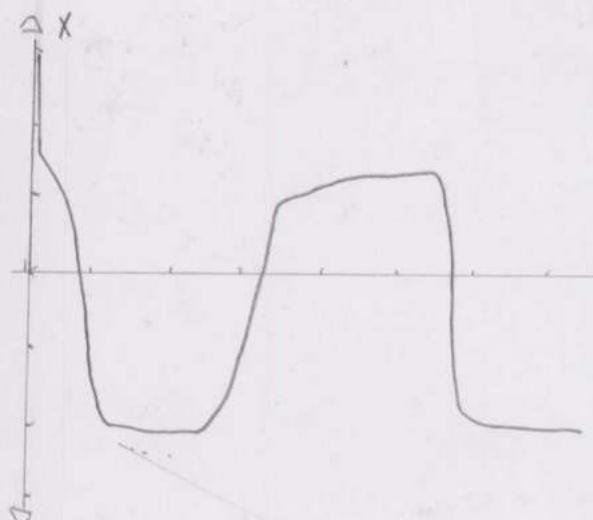
$$-y[r^2 \sin^2 \theta + (r^2 \cos^2 \theta - 1)^2]$$

$$\text{where } r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan(\frac{y}{x})$$



Runge-Kutta performance became absent because of problem set statement "Sketch".



$$\dot{P} = P[(aR - S) - (a-1)(PR + RS + PS)]$$

$$\overset{\circ}{E}_1(P, RS) = P + R + S$$

$$\dot{R} = R[(aS - P) - (a-1)(PR + RS + PS)]$$

$$\overset{\circ}{E}_2(P, R, S) = PRS.$$

$$\dot{S} = S[(aP - R) - (a-1)(PR + RS + PS)]$$

7.3.12.

a.  $\overset{\circ}{E}_1 = (1 - \overset{\circ}{E}_1)(a-1)(PR + RS + PS)$ ;  $P + R + S = 1$ ;  $\overset{\circ}{E}_1 = 1$

If  $P = -1$ ;  $\overset{\circ}{E}_1 = (1 - \overset{\circ}{E}_1)(a-1)(RS - PR - PS) = 0$

$$= (1 - \overset{\circ}{E}_1)(a-1)(PR + RS + PS) = 0$$

If  $R = -1$ ;  $\overset{\circ}{E}_1 = (1 - \overset{\circ}{E}_1)(a-1)(PS - PR - RS) = 0$

$$= (1 - \overset{\circ}{E}_1)(a-1)(PR + RS + PS)$$

If  $S = -1$ ;  $\overset{\circ}{E}_1 = (1 - \overset{\circ}{E}_1)(a-1)(PS - PR - RS)$

$$= (1 - \overset{\circ}{E}_1)(a-1)(PR + RS + PS)$$

"Ellipsoid"

b.  $P, R, S \geq 0$  &  $P + R + S = 1$

If  $(P = 1/2 \parallel R = 1/2 \parallel S = 1/2)$ , then  $\overset{\circ}{E}_1 = 0$  "sphere"

c.  $\overset{\circ}{E}_1 = 0 = (1 - \overset{\circ}{E}_1)(a-1)(PS + RS + PR)$

$$(P^*, R^*, S^*) = (P, -(P+S), -(P+R))$$

$$= (-S+R), R, -(P+R))$$

$$= (-S+P), -(S+R), S)$$

d.  $\frac{d\overset{\circ}{E}_2}{dt} = \frac{d}{dt}(PRS) = R(S \cdot P + P \cdot S) + P \cdot S \cdot \overset{\circ}{R}$

$$= R(S) = \frac{PRS(a-1)}{Z} [(P-R)^2 + (R-S)^2 + (S-P)^2]$$

e.  $(P^*, R^*, S^*) = (1/3, 1/3, 1/3)$

$$\overset{\circ}{E}_2 = \frac{(1/3)^3(a-1)}{2} [0^2 + 0^2 + 0^2] = 0$$

$$(P^*, R^*, S^*) = (3, 0, 0); \quad \overset{\circ}{E}_2 = 0$$

f. If  $\alpha < 1$ , then the model  $\dot{F}_2$  trajectory is decreasing  
else  $\alpha = 1$ , then  $F_2$  model does not change.

g. If  $\alpha < 1$ , then  $\dot{F}_2 < 0$  is less than zero and trajectory to fixed center.

$$\ddot{X} + \mu(X^2 - 1) \dot{X} + \tanh X = 0$$

$$7.4.1. \quad \dot{u} = \dot{X} = v$$

$$\text{Fixed Points: } \dot{u} = 0 = \dot{X} = v$$

$$\dot{v} = \mu(1-u^2)v - \tanh u$$

$$\dot{v} = 0 = \mu(1-u^2)v - \tanh u$$

$$(u^*, v^*) = (0, 0)$$

①  $u$  and  $v$  are continuously differentiable

$$② \quad \ddot{u} = 1; \quad \dot{v} = \mu(1-2u\dot{u})v + \mu(1-u^2)\dot{v}v - (1-\tanh^2 u) \quad \mu(1-u^2)$$

③  $\dot{v}(-u, -v) = -\dot{v}(+u, +v)$  is an odd function.

$$\dot{v}(-u, -v) = -\mu(1-(-u)^2)v - \tanh(-u)$$

$$= -[\mu(1-u^2)v - \tanh(u)]$$

④  $\dot{v}(u, v) > 0$  for  $u, v > 0$

$$\dot{v}(u, v) = \underbrace{\mu(1-u^2)v}_{(+)} - \underbrace{\tanh u}_{(+)} \quad \text{Unstable Fixed point } \Delta < 0$$

$$(+ \text{ till } u^2 = 1) (+)(+) ; \quad \mu(1-u^2)v > \tanh u \quad (+)$$

⑤  $\dot{u}(-v) = \dot{u}(v)$  is an even function

$$\dot{u}(-v) = -v = 0; \quad 0 = \dot{u}(v) \text{ at fixed point}$$

⑥ The odd function  $F(v) = \int_v^X u(k) dk$  is positive at  $v=a$ , and negative for  $0 < v < a$ , is positive and nondecreasing for  $v > 0$ , and  $F(v) \rightarrow \infty$  as  $v \rightarrow \infty$

A stable limit cycle.

$$\ddot{x} + \mu(x^4 - 1)\dot{x} + x = 0 \quad 7.4.2. \quad f(x) = \ddot{x} + \mu(x^4 - 1)x \quad ; \quad g(x) = x$$

a. ①  $f(x)$  and  $g(x)$  are continuously differentiable

$$F(x) = \ddot{x} + \mu(4x^3) + \mu(x^4 - 1) \quad ; \quad \dot{g}(x) = 1$$

②  $g(-x) = -g(x)$  is an odd function

$$g(-x) = -g(x) \text{ at } x=0$$

③  $f(-x) = f(x)$  is an even function

$$f(-x) = -\ddot{x} - \mu[-(-x)^4 - 1]x = -f(x) \text{ at } x=0$$

④  $g(x) > 0$  for  $x > 0$

$$⑤ F(x) = \int_0^x f(u) du = \int_0^x u + \mu(u^2 - 1) du = \frac{u^2}{2} + \mu\left(\frac{u^3}{3} - u\right)$$

$F(x)$  has a positive root at  $x = \sqrt{3}$

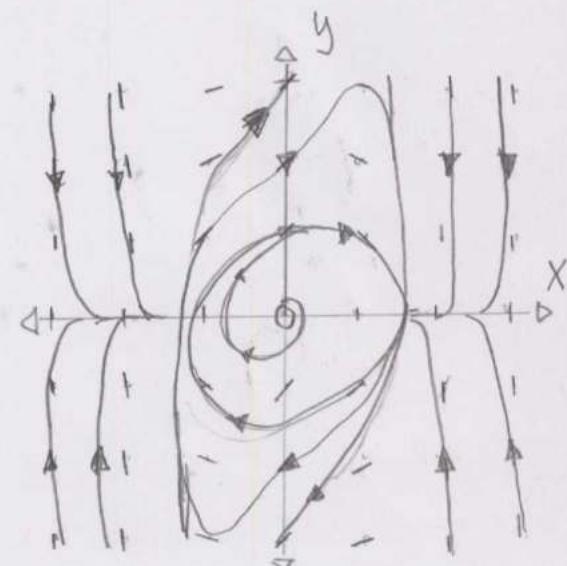
$F(x)$  is negative from  $0 < x < \sqrt{3}$

$F(x)$  is positive and nondecreasing for  $\sqrt{3} < x$

and  $F(x) \rightarrow \infty$  as  $x \rightarrow \infty$

b. Phase Portrait

c. If  $\mu < 1$ , then the function has an unstable periodic cycle in the opposite direction.



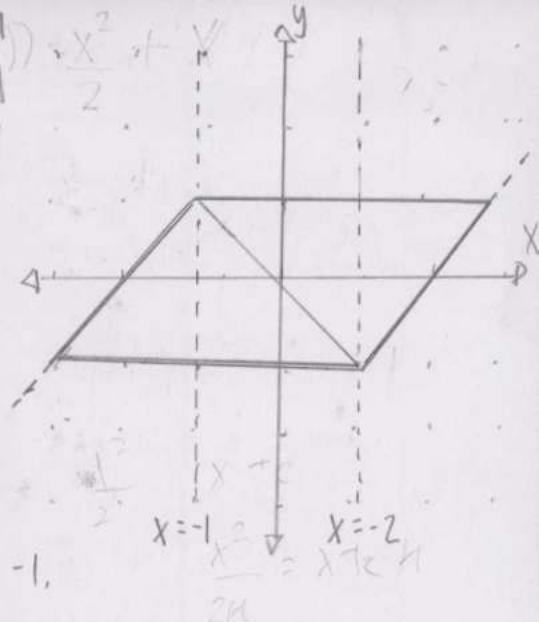
$$X_A = 2 \quad 7.5.1. \quad \text{Van der Pol Oscillator: } \ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

$$\mu > 1; \quad \ddot{x} + \mu(x^2 - 1)\dot{x} + x = \frac{d}{dt}\left(\dot{x} + \mu\left(\frac{x^3}{3} - x\right) + \frac{x^2}{2}\right)$$

$$V: x = -H(t) \quad ; \quad F(x) = \frac{x^3}{3} - x; \quad W = \dot{x} + \mu F(x) + \frac{x^2}{2}$$

$$\dot{x} = W - \mu F(x) - \frac{x^2}{2}; \quad \dot{W} = 0$$

$$F(x) = \int F(x) dx = \begin{cases} x+2 & x \leq -1 \\ -x & -1 \leq x \leq 1 \\ x-2 & x \geq 1 \end{cases}$$



b. Nullclines in graph.

c. If  $\mu \gg 1$ , then nullcline minimum

is  $y=-1$  and maximum ( $y=1$ )

d. See graph about the limit

cycle about  $F'(x)=-1$ ,  $x_F(-1)=1$  or  $-1$ .

d.  $|\dot{x}| \sim O(\mu) \gg 1$  and  $|\dot{y}| \sim O(\mu^{-1}) \ll 1$

The period of the nullcline is  $T \approx \mu \int_{-1}^2 \frac{y - F(x)}{-F'(x)} dx$

$$\approx \mu \int_{-1}^2 \frac{-x}{2\mu^2} + \frac{x^2}{2} dx$$

$$\approx -\frac{x^3}{6} + \frac{x^2}{2} \Big|_{-1}^2$$

$$\approx -\frac{(2^3-1)}{6} + \left[ \frac{4}{2} + \frac{1}{2} \right] \mu$$

$$\approx -\frac{7}{6} + \frac{5}{2} \mu$$

$\ddot{x} + \mu(x^2-1)\dot{x} + x = a$  7.5.6.

a. Fixed Points:  $\ddot{x} + \mu(x^2-1)\dot{x} + (x-a) = \frac{d}{dt} \left[ \dot{x} + \mu F(x) + \frac{x^2}{2} - ax \right]$

$$F(x) = \frac{x^3}{3} - x ; \omega = \dot{x} + \mu F(x) + \frac{x^2}{2} - ax$$

$$\ddot{x} = \omega - \mu F(x) - \frac{x^2}{2} + ax ; \ddot{\omega} = 0$$

$$\text{If } y = -\frac{\ddot{x}}{\mu} + ax + \omega ; \dot{x} = \mu [y - F(x)]$$

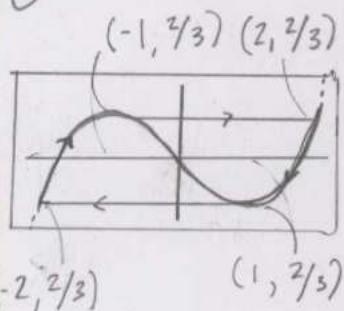
$$\dot{y} = \frac{a - x}{\mu}$$

$$(x^*, y^*) = (a, F(x)) = (a, \frac{x^3}{3} - x)$$

If  $y = \frac{\omega}{\mu}$ , then  $\dot{x} = \mu[y - F(x)]$  and  $\dot{y} = 0$

Nullcline minimum:  $F'(x) = x^2 - 1$ ;  $x = \pm 1$

Nullcline intersection:  $F(-1) = \frac{2}{3}$ ,  $x = -1, 2$



7.5.2. Nullclines:  $\dot{x} = 0 = y$

$$\dot{y} = 0 = -x - \mu(x^2 - 1)$$

$$(x^*, y^*) = \left( \frac{+1 \pm \sqrt{1 + 4\mu^2}}{-2\mu}, 0 \right)$$

A Liénard plane provides advantages, such as separation of time scales ( $\propto \mu$ ), from the fixed point.

$\ddot{x} + k(x^2 - 4)\dot{x} + x = 1$  7.5.2

$$7.5.3. \quad \ddot{x} + k(x^2 - 4)\dot{x} + x - 1 = \frac{d}{dt} \left[ \dot{x} + k \left( \frac{x^3}{3} - 4x \right) + \frac{x^2}{2} - x \right]$$

$$F(x) = \frac{x^3}{3} - 4x; \omega = \dot{x} + kF(x) + \frac{x^2}{2} - x$$

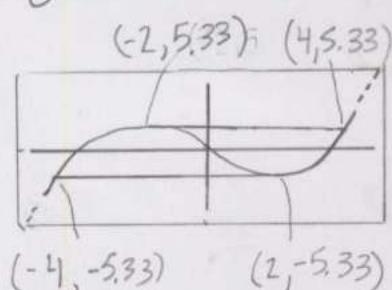
$$\dot{x} = \omega - kF(x) - \frac{x^2}{2} + x; \dot{\omega} = 0$$

$$\text{If } y = \frac{\omega}{k}, \text{ then } \dot{x} = k(y - F(x)) - \frac{x^2}{2} + x$$

$$\dot{y} = 0$$

Nullcline minimum:  $F'(x) = x^2 - 4$ ;  $x = \pm 2$

Nullcline Intersection:  $F(-2) = \frac{16}{3}$ ;  $x = 30$



$\ddot{x} + \mu f(x)\dot{x} + x = 0$  7.5.4.  $f(x) = -1$  for  $|x| < 1$ ;  $f(x) = 1$  for  $|x| \geq 1$

a.  $\ddot{x} + \mu f(x)\dot{x} + x = \frac{d}{dt} \left[ \dot{x} + \mu \int f(x) dx + \frac{x^2}{2} \right]$

$$= F(x) = \int f(x) dx; \omega = \dot{x} + \mu F(x) + \frac{x^2}{2}$$

$$\dot{x} = \omega - \mu F(x) - \frac{x^2}{2}; \dot{\omega} = 0$$

$$\text{If } y = -\frac{x^2}{2\mu}, \text{ then } \dot{x} = \mu(y - F(x)); \dot{y} = -\frac{x}{\mu}$$

b. Nullclines in the Liénard Plane.

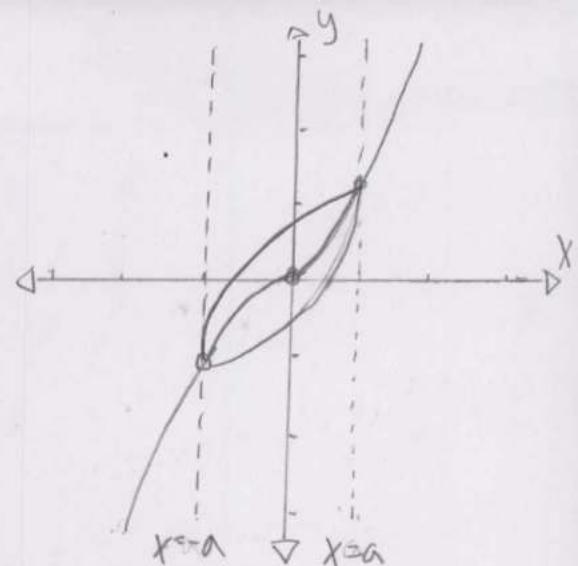
The center seems stable not corresponding to the book.

c. Nullcline Minimum:  $F'(X) = X^2 - 1$   
 $X = \pm 1$

Nullcline Intersection:  $F(-1) = \frac{2}{3}$

$|a| < 0$  because  $\frac{dy}{dx} = 0$

d. See plot.



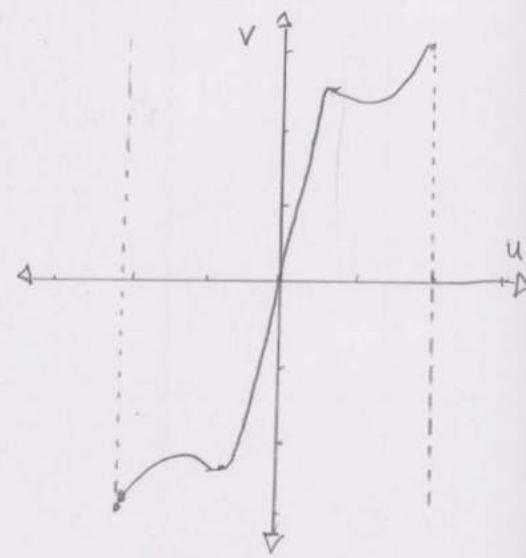
$$\ddot{U} = b(V-U)(X+U^2) - U$$

$$\dot{V} = C - U$$

7.5.7. a. Nullclines on graph.

b.  $|C_1| = |C_2|$

c. A fixed point exists when  $C$  is beyond the inflection points.



$$X(t, \varepsilon) = (1 - \varepsilon^2)^{-\frac{1}{2}} e^{-\varepsilon t} \sin |(1 - \varepsilon^2)^{\frac{1}{2}} t|$$

$$X(t, \varepsilon) = \sin t - \varepsilon t \sin t + O(\varepsilon^2)$$

$$7.6.1 \quad X(t, \varepsilon) = (1 - \varepsilon^2)^{\frac{1}{2}} \circ e^{-\varepsilon t} \circ \sin |(1 - \varepsilon^2)^{\frac{1}{2}} t|$$

Identities:  $(1+x)^a = 1 + ax + \frac{1}{2}(a-1)ax^2 + \dots$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$X(t, \varepsilon) = \left[ 1 + \frac{1}{2}\varepsilon^2 + \dots \right] \left[ 1 - \varepsilon t + \dots \right] \sin \left[ \left( 1 - \frac{1}{2}\varepsilon^2 + \dots \right) t \right]$$

$$= \left[ 1 - \varepsilon t + \frac{1}{2}\varepsilon^2 - \frac{\varepsilon^3 t}{2} + \dots \right] \sin \left[ \left( 1 - \frac{1}{2}\varepsilon^2 + \dots \right) t \right]$$

$$= \sin [t - O(\varepsilon^2)] - \varepsilon t \sin [t - O(\varepsilon^2)] + O(\varepsilon^2)$$

$$\cong \sin [t] - \varepsilon t \sin [t] + O(\varepsilon^2)$$

$$X + \dot{X} + \varepsilon X = 0 \quad 7.6.2. \quad X(0) = 1; \quad \dot{X}(0) = 0$$

$$\ddot{U} = \ddot{X} = V$$

$$\dot{V} = -(1 + \varepsilon)X = -(1 + \varepsilon)U$$

$$\vec{x} = Ax = \begin{pmatrix} 0 & 1 \\ -(1+\varepsilon) & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} ; 0 = Ax = \lambda ; Ax - \lambda = 0$$

$$\begin{pmatrix} -\lambda & 1 \\ -(1+\varepsilon) & -\lambda \end{pmatrix} = 0 ; \lambda_1 = +\sqrt{1+\varepsilon} ; \lambda_2 = -\sqrt{1-\varepsilon}$$

$$\lambda_1 = +\sqrt{1+\varepsilon} ; \begin{pmatrix} -(\sqrt{1+\varepsilon}) & 1 \\ -(1+\varepsilon) & -i(\sqrt{1+\varepsilon}) \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} = 0$$

$$-i(\sqrt{1+\varepsilon}) v_{11} + v_{12} = 0 ; \vec{v}_1 = \begin{bmatrix} 1 \\ i\sqrt{1+\varepsilon} \end{bmatrix}$$

$$\lambda_2 = -\sqrt{1+\varepsilon} ; \begin{pmatrix} \sqrt{1+\varepsilon} & 1 \\ -(1+\varepsilon) & i\sqrt{1+\varepsilon} \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = 0$$

$$i\sqrt{1+\varepsilon} v_{21} + v_{22} = 0 ; \vec{v}_2 = \begin{bmatrix} 1 \\ -i\sqrt{1+\varepsilon} \end{bmatrix}$$

$$\vec{x} = \vec{v}_1 c_1 e^{-\lambda_1 t} + \vec{v}_2 c_2 e^{-\lambda_2 t}$$

$$= \begin{bmatrix} 1 \\ i\sqrt{1+\varepsilon} \end{bmatrix} c_1 e^{-\sqrt{1+\varepsilon}t} + \begin{bmatrix} 1 \\ -i\sqrt{1+\varepsilon} \end{bmatrix} c_2 e^{i\sqrt{1+\varepsilon}t}$$

$$u(t) = c_1 e^{-\sqrt{1+\varepsilon}t} + c_2 e^{i\sqrt{1+\varepsilon}t}$$

$$v(t) = \sqrt{1+\varepsilon} c_1 e^{-\sqrt{1+\varepsilon}t} - \sqrt{1+\varepsilon} c_2 e^{i\sqrt{1+\varepsilon}t}$$

$$x(0) = 0 = c_1 + c_2 ; v(x)(0) = 0 = -c_1 \sqrt{1+\varepsilon} + c_2 \sqrt{1+\varepsilon}$$

$$c_1 = c_2 ;$$

$$u(t) = e^{-\sqrt{1+\varepsilon}t} + e^{i\sqrt{1+\varepsilon}t}$$

$$v(t) = \sqrt{1+\varepsilon} e^{-\sqrt{1+\varepsilon}t} - \sqrt{1+\varepsilon} e^{i\sqrt{1+\varepsilon}t}$$

$$b. X(t, \varepsilon) = X_0 + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + O(\varepsilon^2)$$

$$0 = (X_0 + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + O(\varepsilon^2)) + \varepsilon \frac{d}{dt} [X_0 + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + O(\varepsilon^2)]$$

$$+ \varepsilon^2 \frac{d^2}{dt^2} [X_0 + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + O(\varepsilon^2)]$$

$$O = (X_0 + \varepsilon X_1(t) + \varepsilon^2 X_2(t)) + \varepsilon [\ddot{X}_0 + \varepsilon \dot{X}_1(t) + \varepsilon^2 \ddot{X}_2(t)] \\ + \varepsilon^2 [\ddot{\dot{X}}_0 + \varepsilon \ddot{X}_1(t) + \varepsilon^2 \ddot{\dot{X}}_2(t)] \\ = X_0 + \varepsilon (X_1(t) + \dot{X}_0) + \varepsilon^2 (X_2(t) + \dot{X}_1(t) + \ddot{X}_0) + \dots$$

$$O(1) = X_0$$

$$O(\varepsilon) = X_1(t) + \dot{X}_0$$

$$O(\varepsilon^2) = X_2(t) + \dot{X}_1(t) + \ddot{X}_0$$

c. Secular terms consist of functions which go to ~~use~~ infinity as  $t$  approaches infinity, but the function has many ~~because~~ because

$$\ddot{X} + X = \varepsilon$$

$$7.6.3. X(0) = 1; \dot{X}(0) = 0$$

$$u = \dot{X} = V$$

$$\dot{V} = \varepsilon - X = \varepsilon - u$$

$$\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}; \lambda_1 = +i; \lambda_2 = -i$$

$$\lambda_1 = +i$$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; -iV_{11} + V_{12} = 0 \\ \vec{V}_1 = \begin{bmatrix} 1 \\ +i \end{bmatrix}$$

$$\lambda_2 = -i$$

$$\begin{bmatrix} +i & 1 \\ -1 & +i \end{bmatrix} \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; iV_{21} + V_{22} = 0 \\ \vec{V}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\vec{X} = \begin{bmatrix} u \\ v \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ +i \end{bmatrix} e^{-it} + C_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{it}$$

$$u(t) = C_1 e^{-it} + C_2 e^{it} = X(t)$$

$$v(t) = C_1 i e^{-it} + C_2 i e^{it}$$

$$u(0) = 1 = C_1 \sin(0) + C_2 \cos(0) + \mathcal{E} [C_1 \cos(0) + i C_2 \sin(0)]$$

$$\dot{u}(0) = 0 = C_1 \cos(0) - C_2 \sin(0) + i [C_1 \sin(0) - i C_2 \cos(0)]$$

$$1 = C_2 + \mathcal{E}$$

$$0 = C_1 - i C_2 \quad C_1 =$$

$$u(t) = (1 - \mathcal{E}) e^{it} + \mathcal{E}$$

$$v(t) = (1 - \mathcal{E}) i e^{it}$$

b.  $X(t, \mathcal{E}) = X_0(t) + \mathcal{E} X_1(t) + \mathcal{E}^2 X_2(t) + O(\mathcal{E}^3)$

$$O = X_0 + \mathcal{E} X_1(t) + \mathcal{E}^2 X_2(t) + \frac{d}{dt} \mathcal{E} [X_0(t) + \mathcal{E} X_1(t) + \mathcal{E}^2 X_2(t)]$$

$$+ \frac{d^2}{dt^2} \mathcal{E}^2 [X_0(t) + \mathcal{E} X_1(t) + \mathcal{E}^2 X_2(t)]$$

$$= X_0 + \mathcal{E} [X_1(t) + \dot{X}_0(t)] + \mathcal{E}^2 [X_2(t) + \dot{X}_1(t) + \ddot{X}_0(t)] + \dots$$

$$O(1) = X_0$$

$$O(\mathcal{E}) = X_1(t) + \dot{X}_0(t)$$

$$O(\mathcal{E}^2) = X_2(t) + \dot{X}_1(t) + \ddot{X}_0(t)$$

c. Secular terms not present because this is a function of  $t$ , this is periodic in the real-space.

$$\ddot{x} + x + \mathcal{E} h(x, \dot{x}) = 0$$

7.6.4.  $h(x, \dot{x}) = x$ ;  $\ddot{x} + x + \mathcal{E} x = 0$

Averaged equation:  $r' = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) \sin(\theta) d\theta \equiv \langle h \sin \theta \rangle$

$$r\phi' = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) \cos(\theta) d\theta \equiv \langle h \cos \theta \rangle$$

$$r' = \langle h \sin \theta \rangle = \langle r \cos \theta \sin \theta \rangle = \frac{r}{2\pi} \int_0^{2\pi} \cos \theta \sin \theta d\theta = r \frac{\sin^2 \theta}{2} \Big|_0^{2\pi} = 0$$

$$r\phi' = \langle h \cos \theta \rangle = \langle r \cos^2 \theta \rangle = \frac{r}{2\pi} \int_0^{2\pi} \cos^2 \theta d\theta = \frac{r}{2\pi} \int_0^{2\pi} \frac{\cos(2\theta) + 1}{2} d\theta$$

$$= \frac{r}{2\pi} \left[ \frac{\sin(2\theta)}{4} + \frac{1}{2} \theta \right]_0^{2\pi} = \frac{r}{2}$$

Initial Conditions:  $r(t) = \sqrt{x(t)^2 + \dot{x}(0)^2}$ ;  $\phi(t) \approx \arctan\left(\frac{\dot{x}(t)}{x(t)}\right)$

$$r(0) = \sqrt{a^2 + 0^2} = a; \phi(0) = \arctan(0) = 0$$

Amplitude/Frequency: Amplitude:  $r(T) = \text{constant}$

$$\begin{aligned} \text{Frequency: } \omega &= \frac{d\theta}{dt} = 1 + \frac{d\phi}{dT} \frac{dT}{dt} = 1 + \varepsilon \dot{\phi} \\ &= 1 + \frac{\varepsilon r(T)}{2} \\ &= 1 + \frac{\varepsilon}{2} \end{aligned}$$

$$\begin{aligned} \text{Solution: } x_0 &= r(T) \cos(\tau + \phi(T)) \\ &= a \cos(\tau + \frac{1}{2}) \\ &\sim x(t, \varepsilon) \end{aligned}$$

$$h(x, \dot{x}) = x \ddot{x} - 7.6.5. \ddot{x} + x + \varepsilon x \dot{x}^2 = 0$$

$$\begin{aligned} \text{Averaged Equation: } r' &= \frac{1}{2\pi} \int_0^{2\pi} h(\theta) \sin(\theta) d\theta = \langle h \sin \theta \rangle \\ &= \frac{1}{2\pi} \int_0^{2\pi} r \cos \theta (-r \sin \theta)^2 \sin \theta d\theta \\ &= -\frac{r^2}{2\pi} \int_0^{2\pi} \cos \theta \sin^3 \theta d\theta = -\frac{r^2}{2\pi} \int_0^{2\pi} u^3 du \\ &= +\frac{r^3}{2\pi} \left. \frac{\sin^2 \theta}{4} \right|_0^{2\pi} = 0 \end{aligned}$$

$$\begin{aligned} r\phi' &= \frac{1}{2\pi} \int_0^{2\pi} h(\theta) \cos(\theta) d\theta = \langle h \cos \theta \rangle \\ &= \frac{1}{2\pi} \int_0^{2\pi} r \cos \theta (-r \sin \theta)^2 \cos \theta d\theta \\ &= \frac{r^3}{2\pi} \int_0^{2\pi} \cos^2 \theta \sin^2 \theta d\theta = \frac{r^3}{2\pi} \int_0^{2\pi} (\cos \theta \sin \theta)^2 d\theta \\ &= \frac{r^3}{2\pi} \int_0^{2\pi} \frac{\sin^2(2\theta)}{4} d\theta = \frac{r^3}{2\pi} \int_0^{2\pi} \frac{1 - \cos(4\theta)}{8} d\theta \end{aligned}$$

$$= \frac{r^3}{16\pi} \left[ \theta - \frac{\sin(4\theta)}{4} \right]_0^{2\pi} = \frac{r^3}{16\pi} [2\pi] = \frac{r^3}{8}$$

Initial conditions:  $r(t) = \sqrt{x(t)^2 + \dot{x}(t)^2}$ ;  $\phi(t) = \arctan(\dot{x}(t)/x(t))$

$$= a \quad = 0$$

Amplitude/Frequency: Amplitude:  $r(T) = \text{constant}$

Frequency:  $\omega = 1 + \frac{d\phi}{dT} = 1 + \varepsilon \dot{\phi}$   
 $= 1 + \frac{\varepsilon a^2}{8}; \phi(T) = \frac{\varepsilon a^2}{8} T$

Solution:  $x_0 = r(T) \cos(\theta + \phi(T))$   
 $= a \cdot \cos\left(\theta + \frac{\varepsilon a^2}{8} T\right)$

$$x(t, \varepsilon) = a \cos\left(\left(\frac{\varepsilon a^2}{8} + 1\right)t\right)$$

$$h(x, \dot{x}) = x \ddot{x} - 7.6.6. \ddot{x} + x + x^2 \dot{x} = 0$$

Averaged Equations:  $r' = \frac{1}{2\pi} \int_0^{2\pi} r \cos \theta d\theta$   
 $= \frac{-r^2}{2\pi} \int_0^{2\pi} \cos^2 \theta \sin \theta d\theta$   
 $= -\frac{r^2}{2\pi} \int_0^{2\pi} \frac{u^2}{2} du = \frac{-r^2}{2\pi} \frac{\cos^2(\theta)}{2} \Big|_0^{2\pi} = \frac{-r^2}{2\pi} \left[\frac{1}{2} - \frac{1}{2}\right]$   
 $= 0$

$$\dot{r}\phi' = \frac{1}{2\pi} \int_0^{2\pi} r \sin \theta d\theta$$
  
 $= \frac{-r^2}{2\pi} \int_0^{2\pi} \cos \theta \sin^2 \theta d\theta = \frac{-r^2}{2\pi} \int_0^{2\pi} \frac{u^2}{2} du$   
 $= -\frac{r^2}{2\pi} \frac{\sin^3 \theta}{6} \Big|_0^{2\pi} = 0$

Initial conditions:  $r(t) = \sqrt{x(t)^2 + \dot{x}(t)^2}$ ;  $\phi(t) = \arctan(\dot{x}(t)/x(t))$

$$= a \quad = 0$$

Amplitude/Frequency: Amplitude:  $r(T) = \text{constant}$

Frequency:  $\omega = 1 + \phi \frac{d\tau}{dt} = 1 + \varepsilon \dot{\phi} = 1; \phi(T) = 0$

$$\begin{aligned}
&= \frac{-r}{2\pi} \left[ r^4 \left( \frac{3\theta}{8} + \frac{3\cos\theta\sin\theta}{8} + \frac{\cos^3\theta\sin\theta}{4} \right) - \right. \\
&\quad \left. - \frac{5}{6} \left[ \frac{3\theta}{8} + \frac{3\cos\theta\sin\theta}{8} + \frac{\cos^3\theta\sin\theta}{4} \right] \right]_{0}^{2\pi} \\
&= \frac{-r}{2\pi} \left[ r^4 \left( \frac{6\pi}{8} - \frac{5}{6} \left( \frac{6\pi}{8} \right) \right) - \pi \right] = \frac{-r}{2\pi} \left( r^4 \left( \frac{\pi}{3} \right) - \pi \right)
\end{aligned}$$

$$= \frac{r}{2} - \frac{r^5}{16} = \frac{r}{16} (8 - r^4)$$

$$\begin{aligned}
r\dot{\phi} &= \frac{1}{2\pi} \int h \cos\theta d\theta = \frac{1}{2\pi} \int_0^{2\pi} (x^4 - 1) \dot{x} \cos\theta d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} (r^4 \cos^4\theta - 1) (-r \sin\theta) \cos\theta d\theta \\
&= \frac{-r}{2\pi} \left[ r^4 \int_0^{2\pi} \cos^5\theta \sin\theta d\theta + \int_0^{2\pi} \sin\theta \cos\theta d\theta \right] \\
&= \frac{-r}{2\pi} \left[ r^4 \int_0^{2\pi} u^5 du + \int_0^{2\pi} v dv \right] = \frac{-r}{2\pi} \left[ r^4 \frac{\cos^6\theta}{6} + \frac{\sin^2\theta}{2} \right]_{0}^{2\pi} \\
&> \frac{-r}{2\pi} \left[ \frac{0}{6} - \frac{0}{2} \right] = 0
\end{aligned}$$

Initial conditions:

$$r(t) = \sqrt{x(t)^2 + \dot{x}(t)^2} ; \phi(t) = \arctan \left( \frac{\dot{x}(t)}{x(t)} \right) = 0$$

Amplitude/Frequency: Amplitude:  $16 \int \frac{dr}{r(r^3 - r^4)} = T + C$

$$16 \int \frac{dr}{r^5(r^3/r^4 - 1)} = 16 \int \frac{u^5}{32} \frac{1}{r^5(u-1)} du$$

$$= \frac{1}{2} \ln \left( \frac{8}{r^4 - 1} \right) = T + C$$

$$r(T) = \sqrt[4]{\frac{8}{e^{T/2} + 1 + C}}$$

Frequency:  $\omega = 1 + \varepsilon \dot{\phi} = 1 + O(\varepsilon^2)$

$$\phi(T) = O(\varepsilon^2)$$

$$\text{Solution: } x_0 = r(\tau) \cos(\tau + \phi(\tau)) \\ = 0$$

$$x(t, \varepsilon) = 0$$

$$h(x, \dot{x}) = (x^4 - 1) \dot{x}$$

$$7.6.7. \ddot{x} + x + (x^4 - 1) \dot{x} = 0$$

$$\begin{aligned} \text{Averaged Equations: } r' &= \frac{1}{2\pi} \int_0^{2\pi} h \sin \theta d\theta = \langle h \sin \theta \rangle \\ &= \frac{1}{2\pi} \int_0^{2\pi} (x^4 - 1) \dot{x} \sin \theta d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} ([r \cos \theta]^4 - 1) (-r \sin \theta) \sin \theta d\theta \\ &= \frac{-r}{2\pi} \left[ \int_0^4 r^4 \cos^4 \theta \sin^2 \theta d\theta - \int_0^{2\pi} \sin^2 \theta d\theta \right] \\ &= \frac{-r}{2\pi} \left[ r^4 \int_0^4 \cos^4(\theta) (1 - \cos^2(\theta)) d\theta - \int_0^{2\pi} \sin^2 \theta d\theta \right] \end{aligned}$$

Reduction formula:

$$\begin{aligned} \int \cos^n(x) dx &= \frac{n-1}{n} \int \cos^{n-2}(x) dx + \frac{\cos^{n-1}(x) \sin(x)}{n} \\ \int \sin^n(x) dx &= \frac{n-1}{n} \int \sin^{n-2}(x) dx + \frac{\cos(x) \sin^{n-1}(x)}{n} \\ &= \frac{-r}{2\pi} \left[ r^4 \left( \int_0^{2\pi} \cos^4(\theta) d\theta - \int_0^{2\pi} \cos^6(\theta) d\theta \right) - \int_0^{2\pi} \sin^2(\theta) d\theta \right] \\ \int \cos^4(\theta) d\theta &= \frac{3}{4} \int_0^{2\pi} \cos^2(\theta) d\theta + \frac{\cos^3(\theta) \sin(\theta)}{4} \\ &= \frac{-r}{2\pi} \left[ \frac{3}{4} \left( \frac{\theta}{2} + \frac{\cos(\theta) \sin(\theta)}{2} \right) + \frac{\cos^5(\theta) \sin(\theta)}{4} \right] \\ \int \cos^6(\theta) d\theta &= \frac{5}{6} \int_0^{2\pi} \cos^4(\theta) d\theta + \frac{\cos^5(\theta) \sin(\theta)}{6} \\ &= \frac{5}{6} \left[ \frac{3\theta}{8} + \frac{3\cos(\theta) \sin(\theta)}{8} + \frac{\cos^3(\theta) \sin(\theta)}{4} \right] \\ \int \sin^2(\theta) d\theta &= \left[ \frac{\theta}{2} + \frac{\cos(\theta) \sin(\theta)}{2} \right] \end{aligned}$$

$$\text{Solution: } x_0 = r(\tau) \cos(\tau + \phi(\tau)) \\ = \sqrt[4]{\frac{8}{e^{\tau/2} + 1 + C}} \cos(\tau + O(\varepsilon^2))$$

$$x(t, \varepsilon) = \sqrt[4]{\frac{8}{e^{t/2} - 1 + \frac{8}{a^4}}} \cos(\tau + O(\varepsilon^2))$$

$$h(x, \dot{x}) = (|x| - 1) \dot{x}$$

7.6.8 Averaged Equations:

$$r' = \frac{1}{2\pi} \int_0^{2\pi} h \sin \theta d\theta = \langle h \sin \theta \rangle$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (|x| - 1) \dot{x} \sin \theta d\theta = \frac{1}{2\pi} \int_0^{2\pi} (|r \cos \theta| - 1)(-r \sin \theta) \sin \theta d\theta$$

$$= \frac{-r}{2\pi} \left( \int_0^{2\pi} |r \cos \theta| \sin^2 \theta d\theta + \int_0^{2\pi} \sin^2 \theta d\theta \right)$$

$$= \frac{-r}{2\pi} \left( r \frac{\cos \theta}{|\cos \theta|} \int_0^{2\pi} \cos \theta \sin^2 \theta d\theta + \left( \frac{\theta}{2} - \frac{\cos(\theta) \sin(\theta)}{2} \right) \Big|_0^{2\pi} \right)$$

$$= \frac{-r}{2\pi} \left( r \frac{\cos \theta}{|\cos \theta|} \frac{\sin^3(\theta)}{3} + \frac{\theta}{2} - \frac{\cos(\theta) \sin(\theta)}{2} \right) \Big|_0^{2\pi}$$

$$= \frac{-r}{2\pi} [-\pi] = \frac{r}{2}$$

$$r'\phi' = \frac{1}{2\pi} \int_0^{2\pi} h \cos \theta d\theta = \langle h \cos \theta \rangle$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (|x| - 1) \dot{x} \cos \theta d\theta = \langle h \cos \theta \rangle$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (|r \cos \theta| - 1)(-r \sin \theta) \cos \theta d\theta$$

$$= \frac{-r}{2\pi} \left( r \int_0^{2\pi} |\cos \theta| \cos \theta \sin \theta d\theta - \int_0^{2\pi} \sin \theta \cos \theta d\theta \right)$$

$$= \frac{-r}{2\pi} \left( r \frac{\cos \theta}{|\cos \theta|} \int_0^{2\pi} \cos^2 \theta \sin \theta d\theta - \frac{\sin^2 \theta}{2} \Big|_0^{2\pi} \right)$$

$$= \frac{-r}{2\pi} \left( r \frac{\cos \theta}{|\cos \theta|} \frac{\cos^3 \theta}{3} \Big|_0^{2\pi} \right) = 0$$

$$\text{Initial Conditions: } r(t) = \sqrt{x(t)^2 + \dot{x}(t)^2} = a$$

$$\phi(t) = \arctan(\dot{x}(t)/x(t)) = 0$$

Amplitude / Frequency : Amplitude  $r(T) = a$   
 Frequency  $\omega = 1 + \epsilon \phi$ ;  $\phi(T) = 0$

$$\text{Solution: } x_0 = r(T) \cos(\pi + T) = a \cos(t)$$

$$x(t, \epsilon) = a \cos(t)$$

$$h(x, \dot{x}) = (x^2 - 1) \dot{x}^3 \quad 7, 6, 9, \ddot{x} + x + (x^2 - 1) \dot{x}^3 = 0$$

$$\text{Average Equations: } r' = \frac{1}{2\pi} \int_0^{2\pi} h(x, \dot{x}) \sin(\theta) d\theta = \langle h \sin \theta \rangle$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (x^2 - 1) \dot{x}^3 \sin(\theta) d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (r^2 \cos^2 \theta - 1) (-r^3) (\sin^3 \theta) \sin(\theta) d\theta$$

$$= -\frac{r^3}{2\pi} \left[ \int_0^{2\pi} r^2 \cos^2 \theta \sin^4 \theta d\theta - \int_0^{2\pi} \sin^4 \theta d\theta \right]$$

Reduction Formula:

$$\int_0^{2\pi} \cos^n(x) dx = \frac{n-1}{n} \int_0^{2\pi} \cos^{n-2}(x) dx + \frac{\cos^{n-1}(x) \sin(x)}{n}$$

$$\int_0^{2\pi} \sin^n(x) dx = \frac{n-1}{n} \int_0^{2\pi} \sin^{n-2}(x) dx + \frac{\cos(x) \sin^{n-1}(x)}{n}$$

$$\int_0^{2\pi} \sin^2(\theta) d\theta = \frac{\theta}{2} + \frac{\cos(\theta) \sin(\theta)}{2}$$

$$-\int_0^{2\pi} \sin^6(\theta) d\theta = -\frac{5}{6} \int_0^{2\pi} \sin^4(\theta) d\theta + \frac{\cos(\theta) \sin^5(\theta)}{6}$$

$$\int_0^{2\pi} \sin^4(\theta) d\theta = \frac{5}{4} \int_0^{2\pi} \sin^2(\theta) d\theta + \frac{\cos(\theta) \sin^3(\theta)}{4}$$

$$\int_0^{2\pi} \sin^2(\theta) d\theta = \frac{\theta}{2} + \frac{\cos(\theta) \sin(\theta)}{2}$$

$$= \frac{-r^3}{2\pi} \left[ r^2 \left[ \frac{\theta}{2} + \frac{\cos(\theta)\sin(\theta)}{2} \right] - \frac{5}{6} \left[ \frac{3}{4} \left[ \frac{\theta}{2} + \frac{\cos(\theta)\sin(\theta)}{2} \right] + \frac{\cos(\theta)\sin^3(\theta)}{4} \right] \right.$$

$$\left. + \frac{\cos(\theta)\sin^5(\theta)}{6} \right] - \frac{3}{4} \left[ \left[ \frac{\theta}{2} + \frac{\cos(\theta)\sin(\theta)}{2} \right] + \frac{\cos(\theta)\sin^3(\theta)}{6} \right] \Big|_0^{2\pi}$$

$$= \frac{-r^3}{2\pi} \left[ r^2 \left[ \pi - \frac{15}{24}\pi \right] - \frac{3}{4}\pi \right]$$

$$= \frac{-r^3}{2\pi} \left[ \frac{3}{8}\pi r^2 - \frac{3}{4}\pi \right] = \frac{r^3}{16} \left( \frac{6}{b} + \frac{r^2}{b} \right) \pi$$

$$r\phi' = \frac{1}{2\pi} \int_0^{2\pi} h \cos(\theta) d\theta = \langle h \cos \theta \rangle$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (X^2 - 1) \dot{X}^3 \cos(\theta) d\theta = \frac{1}{2\pi} \int_0^{2\pi} (r^2 \cos^2 \theta - 1) (-r^3) \sin^3 \theta \cos(\theta) d\theta$$

$$= \frac{-r^3}{2\pi} \left[ r^2 \int_0^{2\pi} \cos^3 \theta \sin^3 \theta d\theta - \int_0^{2\pi} \sin^3 \theta \cos \theta d\theta \right]$$

$$= -\frac{r^3}{2\pi} \left[ r^2 \int_0^{2\pi} \cos(\theta) (1 - \sin^2 \theta) \sin^3 \theta d\theta - \frac{\sin^4 \theta}{4} \right]_0^{2\pi}$$

$$= -\frac{r^3}{2\pi} \left[ r^2 \left[ \int_0^{2\pi} \cos \theta \sin^3 \theta d\theta - \int_0^{2\pi} \cos \theta \sin^5 \theta d\theta \right] \right]$$

$$= -\frac{r^3}{2\pi} \left[ r^2 \left[ \frac{\sin^4 \theta}{4} - \frac{\sin^6 \theta}{6} \right] \right]_0^{2\pi} = 0$$

$$\text{Initial Conditions: } r(t) = \sqrt{x(t)^2 + \dot{x}(t)^2} = a$$

$$\phi(t) = \arctan(\dot{x}(t)/x(t)) = 0$$

$$\text{Amplitude/Frequency: Amplitude: } T = \int \frac{dr}{r^3(r^2 - b)}$$

$$= -16 \int \frac{dr}{r^3(r^2 - b)}$$

$$= -16 \int \left( \frac{A}{r} + \frac{B}{r^2} + \frac{C}{r^3} + \frac{Dr + E}{(r^2 - b)} \right) dr$$

$$A = -\frac{1}{5b}; B = 0; C = -\frac{1}{6}; D = \frac{1}{36}; E = 0$$

$$= -16 \int \frac{r}{36(r^2 - b)} = \frac{1}{36} - \frac{1}{6r} dr$$

$$= \frac{2}{9} \ln(r^2 - 6) + \frac{4}{9} \ln(|r|) - \frac{12}{9} \frac{1}{r^2} = T + C.$$

Frequency:  $\omega = 1 + \varepsilon \phi'$ ;  $\phi(T) = \phi_0$ .

Solution:  $x_0 = r(T) \cos(\pi T + \phi(T))$

$$x_0 = (\sqrt{6} \cos(\pi T) + \phi_0)$$

$$x(b, \varepsilon) = \sqrt{6} \cos(\pi + \phi_0)$$

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

$$\sin \theta \cos^2 \theta = \frac{\sin \theta + \sin 3\theta}{4}$$

$$\begin{aligned} 7.6.10. \quad \sin \theta \cos^2 \theta &= \left[ \frac{e^{i\theta} + e^{-i\theta}}{2} \right] \left[ \frac{e^{i\theta} + e^{-i\theta}}{2} \right]^2 \\ &= \frac{1}{8} \left[ e^{i\theta} - e^{-i\theta} + e^{3i\theta} - e^{-3i\theta} \right] \\ &= \frac{\sin \theta + \sin 3\theta}{4}. \end{aligned}$$

$$2\pi x_1 + x_1 = \left[ -2r^3 + r - \frac{1}{4}r^3 \right] \sin(\pi + \phi)$$

$$+ \left[ -2r\phi' \right] \cos(\pi + \phi) - \frac{1}{4}r^3 \sin 3(\pi + \phi)$$

$$7.6.11 \quad X(0) = 2; \quad \dot{X}(0) = 0$$

$$r(T) = \sqrt{x(T)^2 + \dot{x}(T)^2} = 2; \quad \phi(T) = \arctan \left( \frac{\dot{x}(t)}{x(t)} \right) = 0$$

$$2\pi x_1 + x_1 = [2 - 2] \sin(\pi + \phi) + [0] \cos(\pi + \phi) - \frac{1}{2} \sin 3(\pi + \phi)$$

$$2\pi x_1 + x_1 = -\frac{1}{2} \sin 3(\pi + \phi)$$

$$\underbrace{2\pi x_1 + 2x_1}_{2x_1} = -\sin 3(\pi + \phi)$$

Linear Equation with Constant Coefficients.

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = f(y)$$

$$a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1} \lambda + a_n = 0$$

$$2\lambda^2 + 2 = 0$$

$$2(\lambda^2 + 1) = 0$$

$$\lambda_{1,2} = \pm i ; R=1 ; IC = C_1 \sin(y) + C_2 \cos(y)$$

Solution to a Homogeneous Differential Equation:

$$y = \sum P_{k-1}(y) e^{ky} \sin by + Q_{k-1}(y) e^{ky} \cos by$$

$$\text{where } \lambda = k \pm bi$$

$$P_{k-1}(y) e^{ky} = C_1$$

$$Q_{k-1}(y) e^{ky} = C_2$$

Particular Solution:  $2\partial_{tt}x_1 + 2x_1 = -\sin 3(t+T)$

Assume  $x_1 = A \sin 3t$  then  $\partial_{tt}x_1 = 0$

$$2A = -\sin 3(t+T)$$

$$A = -\frac{1}{2} \sin 3(t+T)$$

General solution:  $X = \text{Homogeneous} + \text{Particular}$

$$= C_1 \sin y + C_2 \cos y - \frac{\sin 3(t+T)}{2}$$

$$\langle f(\theta) \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta \quad 7.b.51$$

a.  $\langle \cos k\theta \sin m\theta \rangle$  Method #1: even  $\times$  odd = odd;  $\int_{-\infty}^{\infty} \text{odd} = \int_0^{\infty} \text{odd} = 0$

Method #2: Product to sum of Two angles

$$\cos \phi \sin \theta = \frac{\sin(\theta + \phi) - \sin(\theta - \phi)}{2}$$

$$\int \cos k\theta \sin m\theta d\theta = \int \frac{\sin((k+m)\theta) - \sin((k-m)\theta)}{2} d\theta$$

$\langle \cos k\theta \cos m\theta \rangle$  Method #1: even  $\times$  even = even  
odd  $\times$  odd = even

Method #2: Product to sum of Two Angles

$$\cos \theta \cos \phi = \frac{\cos(\theta - \phi) + \cos(\theta + \phi)}{2}$$

$$= \int \frac{\cos((k-m)\theta) + \cos((k+m)\theta)}{2} d\theta$$

$\neq 0$  for  $k \neq m$ .

$$\langle \cos^2 k\theta \rangle = \langle \sin^2 k\theta \rangle = \frac{1}{2}; \text{ Method #1: even} \times \text{even} = \text{even}$$

$$\int \text{even} = \text{constant.}$$

Method #2: Product to Sum of Two Angles

$$\cos\theta \cos m\theta = \frac{\cos(\theta-m) + \cos(\theta+m)}{2}$$

$$\int \cos^2 k\theta = \int \frac{1}{2} + \int \frac{\cos 2k\theta}{2} d\theta$$

$$= \frac{1}{2}$$

$$b. h(\theta) = \sum_{k=0}^{\infty} a_k \cos k\theta + \sum_{k=1}^{\infty} b_k \sin k\theta$$

$$\cos m\theta h(\theta) = \sum_{k=0}^{\infty} a_k \cos k\theta \cos m\theta + \sum_{k=1}^{\infty} b_k \sin k\theta \cos m\theta$$

$$\frac{\cos m\theta h(\theta)}{2\pi} = \frac{1}{2\pi} \sum_{k=0}^{2\pi} a_k \cos k\theta \cos m\theta + \frac{1}{2\pi} \sum_{k=1}^{2\pi} b_k \sin k\theta \cos m\theta$$

$$\langle h(\theta) \cos m\theta \rangle = \frac{1}{2} a_m$$

$$c. h(\theta) = \sum_{k=0}^{\infty} a_k \cos k\theta + \sum_{k=0}^{\infty} b_k \sin k\theta$$

$$\begin{aligned} \langle h(\theta) \sin k\theta \rangle &= \sum_{k=0}^{2\pi} a_k \cos k\theta \sin k\theta + \sum_{k=0}^{2\pi} b_k \sin k\theta \sin k\theta \\ &= \frac{1}{2} b_k \end{aligned}$$

$$\begin{aligned} \langle h(\theta) \rangle &= \sum_{k=0}^{2\pi} a_k \cos k\theta + \sum_{k=0}^{2\pi} b_k \sin k\theta \\ &= a_0 \end{aligned}$$

$$\ddot{x} + x + \varepsilon x^3$$

$$7. 6.13 \quad x(0) = a; \quad \dot{x}(0) = 0$$

a.  $h = x^3; \quad r' = \frac{1}{2\pi} \int_0^{2\pi} h \sin \theta d\theta = \frac{1}{2\pi} \int_0^{2\pi} x^3 \sin \theta d\theta$

$$= \frac{1}{2\pi} \int_0^{2\pi} (r \cos \theta)^3 \sin \theta d\theta = \frac{r^3}{2\pi} \int_0^{2\pi} \cos^3 \theta \sin \theta d\theta$$

$$= \frac{r^3}{2\pi} \left[ \frac{\cos^3 \theta}{3} \right]_0^{2\pi} = 0$$

$$T = r(T) = \text{constant.}$$

b.  $r\phi' = \frac{1}{2\pi} \int_0^{2\pi} h \cos \theta d\theta - \int_0^{2\pi} (r \cos \theta)^3 \cos \theta d\theta$

$$= \frac{r^3}{2\pi} \int_0^{2\pi} \cos^3 \theta \cos \theta d\theta = r^3 \int_0^{2\pi} \cos^4 \theta d\theta$$

Reduction Formula:

$$\int \cos^n \theta d\theta = \frac{n-1}{n} \int \cos^{n-2} \theta d\theta + \frac{\cos^{n-1} \theta \sin \theta}{n}$$

$$\int \cos^4 \theta d\theta = \frac{3}{4} \int \cos^2 \theta d\theta + \frac{\cos^3 \theta \sin \theta}{4}$$

$$\int \cos^2 \theta d\theta = \frac{\theta}{2} + \frac{\cos \theta \sin \theta}{2}$$

$$r\phi' = \frac{r^3}{2\pi} \left[ \frac{3}{4} \left[ \frac{\theta}{2} + \frac{\cos \theta \sin \theta}{2} \right] + \frac{\cos^3 \theta \sin \theta}{4} \right]_0^{2\pi}$$

$$= \frac{r^3}{2\pi} \left[ \frac{3}{4} \pi \right] = \frac{3}{8} r^3$$

$$\phi' = \frac{3}{8} r^2; \quad \omega = 1 + \varepsilon \phi' = 1 + \frac{3}{8} r^2 \varepsilon = 1 + \frac{3}{8} a^2 \varepsilon$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{1 + \frac{3}{8}\epsilon a^2} = 2\pi \left(1 - \frac{3}{8}\epsilon a^2 + \left(\frac{3}{8}\epsilon a^2\right)^2 + \dots\right)$$

b. Note: The book answer is a power series about kinetic energy, while the Fourier series arrived to the same answer.

$$\ddot{x} + \sin x = 0$$

7.6.15.

$$a. \ddot{x} = -\sin(x); F = ma = m\ddot{x} = -kx = -\sin(x)$$

$$\approx -(x - \frac{1}{6}x^3)$$

$$= -(1 - \frac{1}{6}x^2)x$$

$$= -kx$$

$$r\phi' = \frac{1}{2\pi} \int_0^{2\pi} h \cos \theta d\theta = \frac{1}{2\pi} \int_0^{2\pi} r^3 \cos^3 \theta \cos \theta d\theta = \frac{3}{8} r^3 \left[ \frac{\sin^2 \theta}{2} \right]_0^{2\pi} = 0$$

$$\omega = 1 + \epsilon \phi' = 1 + \epsilon \frac{3}{8} r^2 \approx 1 - \frac{1}{16} a^2$$

$$b. T = \frac{2\pi}{\omega} = \frac{2\pi}{1 - \frac{1}{16} a^2} = 2\pi \left(1 + \frac{1}{16} a^2 + \dots\right) \text{ "Agreement"}$$

$$\ddot{x} + \epsilon \dot{x}(x^2 - 1) + x = 0 \quad 7.6.16. \text{ Green's Theorem : } \oint_C \mathbf{v} \cdot \mathbf{n} d\ell = \iint_A \nabla \cdot \mathbf{v} dA$$

$$\mathbf{v} = \ddot{x} = (\ddot{x}, \dot{y})$$

$$\dot{v} = -x - \epsilon \dot{x}(x^2 - 1) = -x - \epsilon v(x^2 - 1)$$

$$\iint_A \nabla \cdot \mathbf{v} dA = \iint_A \frac{\partial v}{\partial x} (-x - \epsilon v(x^2 - 1)) dA = \iint_A -\epsilon(x^2 - 1) dA$$

$$= \int_0^{2\pi} \int_0^a -\epsilon(r^2 \cos^2 \theta - 1) r dr d\theta = \int_0^{2\pi} \left[ -\frac{\epsilon}{4} r^4 \cos^2 \theta + \frac{\epsilon r^2}{2} \right]_0^a d\theta$$

$$= \int_0^{2\pi} \left[ -\frac{\epsilon}{4} a^4 \cos^4 \theta + \frac{\epsilon a^2}{2} \right] d\theta = -\frac{\epsilon a^4 \pi}{4} + \epsilon a^2 \pi$$

$$\begin{aligned}
r' &= \frac{1}{2\pi} \int_0^{2\pi} r(\gamma + \cos(2\theta)) \cos \theta \sin \theta d\theta = \frac{1}{3} \left[ \frac{\sin^3 \theta}{3} \cos 2\phi \right]_0^{2\pi} \\
&= \frac{r}{2\pi} \left[ \int_0^{2\pi} \gamma \sin \theta \cos \theta d\theta + \int_0^{2\pi} \cos(2\theta - 2\phi) \cos \theta \sin \theta d\theta \right] \\
&= \frac{r}{2\pi} \left[ \int_0^{2\pi} \cos 2\theta \cos 2\phi \cos \theta \sin \theta d\theta + \int_0^{2\pi} \sin 2\theta \sin 2\phi \cos \theta \sin \theta d\theta \right] \\
&= \frac{r}{2\pi} \left[ \int_0^{2\pi} (2\cos^2 \theta - 1) \cos 2\phi \cos \theta \sin \theta d\theta + \int_0^{2\pi} 2\cos \theta \sin \theta \cos \theta \sin \theta d\theta \right] \\
&= \frac{r}{2\pi} \left[ \frac{2\pi}{4} \sin 2\phi \right] = \frac{r}{4} \sin 2\phi
\end{aligned}$$

b. If  $r=0$ ,  $\phi = 0 = \frac{r}{2} (\gamma + \frac{\cos 2\phi}{2})$   $\int \cos 2\phi = 2\gamma$

$$\begin{aligned}
r' &= 0 = \frac{r}{2} \sin(\arccos(-2\gamma)) \\
\frac{r}{2\pi} \left[ \frac{2\pi}{4} \right] &= \frac{r}{2} \sqrt{1 - (-2\gamma)^2} = \frac{r}{2} \sqrt{1 - 4\gamma^2}
\end{aligned}$$

When  $\gamma < \frac{1}{2}$ , then  $r' > 0$

$\gamma \geq \frac{1}{2}$ , then  $r' = 0$

$\gamma > \frac{1}{2}$ , then  $r' \propto i$

Therefore  $\gamma < \frac{1}{2}$  is a critical value.

c.  $r' = \frac{dr}{dT} = \frac{r}{4} \sqrt{1 - 4\gamma^2}$  @  $r=0$ , then  $T = \int \frac{4 dr}{r \sqrt{1 - 4\gamma^2}}$

$$\begin{aligned}
T &= \frac{4}{\sqrt{1 - 4\gamma^2}} \ln r \\
&\quad + \frac{4}{\sqrt{1 - 4\gamma^2}} T/4
\end{aligned}$$

and  $r(T) = e^T$

$$\text{where } R = \frac{\sqrt{1 - 4\gamma^2}}{4}$$

$$\int_V \mathbf{v} \cdot \mathbf{n} dL = \int_0^{2\pi} \langle a \sin(t), -a \cos(t) - \epsilon a \sin(t)(a^2 \cos^2(t) - 1), \langle -a \sin(t), a \cos(t) \rangle \rangle dt$$

$$= \int_0^{2\pi} -\epsilon a^4 \sin(t) \cos^3(t) + \epsilon a^3 \sin(t) \cos(t)$$

$$= \frac{a^4}{4} \epsilon \cos^4(t) - \frac{a^2}{2} \epsilon \cos^2(t) \Big|_0^{2\pi} = 0$$

$$\epsilon a^2 \pi \left(1 - \frac{1}{4} a^2\right) = 0 \Rightarrow a = 2$$

$$\ddot{x} + (1 + \epsilon \gamma + \epsilon \cos 2t) \sin x = 0$$

7.6.17.  $x$  = Angle Between swing and Downward Vertical

- $1 + \epsilon \gamma + \epsilon \cos 2t$  = Effect of gravity and periodic pumping



$$x = 0; \dot{x} = 0$$

$$a. \ddot{x} + (1 + \epsilon \gamma + \epsilon \cos 2t) x = 0$$

$$r \dot{\phi}^2 = \frac{r}{2\pi} \int_0^{2\pi} ((\gamma + \cos(2t)) \cos^2 \theta) d\theta$$

$$= \frac{r}{2\pi} \int_0^{2\pi} (\gamma \cos^2 \theta) d\theta + \frac{r}{2\pi} \int_0^{2\pi} \cos(2\theta - 2\phi) \cos^2 \theta d\theta$$

$$= \frac{r}{2\pi} \left[ \gamma \int_0^{2\pi} \cos^2 \theta d\theta + \int_0^{2\pi} (\cos 2\theta \cos 2\phi + \sin 2\theta \sin 2\phi) \cos^2 \theta d\theta \right]$$

$$= -\frac{r}{2\pi} \left[ \gamma \int_0^{2\pi} \cos 2\theta \cos^2 \theta d\theta + \int_0^{2\pi} \sin 2\theta \sin 2\phi \cos^2 \theta d\theta \right]$$

$$= -\frac{r}{2\pi} \left[ \gamma \int_0^{2\pi} (2 \cos^2 \theta - 1) \cos^2 \theta \cos 2\phi d\theta + \int_0^{2\pi} 2 \cos \theta \sin \theta \sin 2\phi \cos^2 \theta d\theta \right]$$

$$= \frac{r}{2\pi} \left[ \gamma \int_0^{2\pi} \left( \frac{6}{4} \cos^2 \theta - 1 \right) \cos^2 \theta \cos 2\phi d\theta + \int_0^{2\pi} 2 \cos \theta \sin \theta \sin 2\phi \cos^2 \theta d\theta \right]$$

$$= \frac{r}{2\pi} \left[ \gamma \int_0^{2\pi} \left( \frac{6}{4} \cos^2 \theta - 1 \right) \cos^2 \theta \cos 2\phi d\theta + \int_0^{2\pi} 2 \cos \theta \sin \theta \sin 2\phi \cos^2 \theta d\theta \right] = \frac{r}{2} \left[ \gamma + \frac{\cos 2\phi}{2} \right]$$

$$d. \frac{dr}{d\phi} = \frac{r'}{\phi'} = \frac{\left(\frac{r}{4} \sin(2\phi)\right)}{\left(\frac{1}{2}\left[\gamma + \frac{\cos 2\phi}{2}\right]\right)} = \frac{r \sin(2\phi)}{2\gamma + \cos(2\phi)}$$

$$\int \frac{dr}{r} = \int \frac{\sin(2\phi)}{2\gamma + \cos(2\phi)} d\phi$$

$$\ln(r) = -\frac{1}{2} \ln(2\gamma + \cos(2\phi))$$

$r(\phi) = \frac{C}{\sqrt{2\gamma + \cos(2\phi)}}$  is a periodic function and closed orbit.

e. A physical interpretation of the form

$\ddot{x} + \alpha x = 0$  is synonymous with Hooke's law, a pendulum, or swing. In this problem,

$$k = \frac{\sqrt{1-4\gamma^2}}{4} = r(\gamma + \cos 2t) \cos \theta$$

$$\ddot{x} + (\alpha + \epsilon \cos t)x = 0$$

7.6.13 Mathieu Equation;  $\alpha \approx 1$ ;  $T = \epsilon^2 t$

$$\ddot{x} + (\alpha + \epsilon \cos t)x = (x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \dots)$$

$$+ (\alpha + \epsilon \cos t)(x_0''(t) + \epsilon x_1''(t) + \epsilon^2 x_2''(t))$$

$$= x_0 + \epsilon [x_1 + \frac{1}{2} \epsilon^2 x_2] + \dots$$

$$+ \epsilon [x_0''(t) + \alpha x_1(t) + \cos t x_0(t)]$$

$$+ \epsilon^2 [x_1''(t) + x_2(t) + \alpha x_1(t) + \cos t x_0(t)]$$

$$+ O(\epsilon^3)$$

$$O(1); \ddot{x}_0(t) + \alpha x_0(t) = 0$$

$$O(\epsilon); \ddot{x}_1(t) + \alpha x_1(t) + \cos(t) x_0(t) = 0$$

$$O(\epsilon^2); \ddot{x}_2(t) + \alpha x_2(t) + \cos(t) x_1(t) = 0$$

$$\ddot{x} + x + \varepsilon x^3 = 0 \quad 7.6.19, \quad x(0) = a; \quad \dot{x}(0) = 0; \quad \boxed{\text{Poincaré-Lindstedt Method}}$$

$$\begin{aligned} a. \quad \tau &= \omega t; \quad \frac{d^2x}{d\tau^2} + x + \varepsilon x^3 = \omega \frac{d^2x}{d\tau^2} + x + \varepsilon x^3 \\ &= \omega x'' + x + \varepsilon x^3 \\ &= 0 \end{aligned}$$

$$b. \quad x(\tau, \varepsilon) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + O(\varepsilon^3)$$

$$\omega = 1 + \varepsilon w_1 + \varepsilon^2 w_2 + O(\varepsilon^3)$$

$$(1 + \varepsilon w_1 + \varepsilon^2 w_2)^2 (x_0''(\tau) + \varepsilon x_1''(\tau) + \varepsilon^2 x_2''(\tau))$$

$$+ (x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau)) + (x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau))^3 = 0$$

$$(1 + 2\varepsilon w_1 + 2\varepsilon^2 w_2 + \dots) (x_0''(\tau) + \varepsilon x_1''(\tau) + \varepsilon^2 x_2''(\tau) + \dots)$$

$$+ (x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \dots) + \varepsilon (x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon x_2(\tau))^3 = 0$$

$$O(1) = x_0''(\tau) + x_0(\tau) = 0$$

$$O(\varepsilon) = x_1''(\tau) + 2\varepsilon w_1 x_0''(\tau) + x_1(\tau) + \varepsilon x_0^3(\tau) = 0$$

$$c. \quad x_0(0) = a; \quad \dot{x}_0(0) = 0; \quad x_n(0) = \dot{x}_n(0) = 0$$

From the blurb,  $x(0) = a; \quad \dot{x}(0) = 0; \quad \ddot{x}_0 = (a); \quad \dot{x}_0'(0) = 0$

$$d. \quad x_0''(\tau) + x_0(\tau) = 0; \quad x_0 = a \cos(\tau)$$

$$e. \quad x_1''(\tau) + x_1(\tau) = -2\omega w_1 x_0''(\tau) - x_0^3(\tau) + C$$

$$= +2\omega w_1 \cos(\tau) - a^3 \cos^3(\tau) = 2\omega w_1 \cos(\tau)$$

$$= 2\omega w_1 \cos(\tau) - a^3 \left[ \frac{1}{4} (3 \cos(\tau) + \cos(3\tau)) \right]$$

$$= (2aw_1 - \frac{3}{4}a^3)\cos(\tau) - \frac{a^3}{4}\cos(3\tau) ; w_1 = a^2$$

F.  $X_1''(\tau) + X_1(\tau) = (2aw_1 - \frac{3}{4}a^3)\cos(\tau) - \frac{a^3}{4}\cos(3\tau)$

$$X_1''(\tau) + X_1 = -\frac{1}{4}a^3\cos(3\tau) + 3aw_1\cos(\tau) - 3a^3\cos(\tau)$$

$$4X_1''(\tau) + 4X_1 = -a^3\cos(3\tau)$$

Linear Equation of Constant Coefficients. "Homogeneous"

$$a_0\lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n = 0$$

$$4(\lambda^2 + 1) = 0 ; \lambda^2 + 1 = 0 ; \lambda_{1,2} = \pm i$$

Solution of Homogeneous Equations

$$t = \sum P_{k-1}(t) e^{\lambda t} \sin \beta t + Q_{k-1}(t) e^{\lambda t} \cos \beta t$$

$$\lambda = \alpha \pm \beta i ; X = C_1 \sin(t) + C_2 \cos(t)$$

Method of Undetermined Coefficients.

$$X_1 = E e^{\alpha t} (R_m(t) \cos \beta t + T_m(t) \sin \beta t) = -a^3 \cos(3t)$$

Logics ;  $S=0 ; \alpha=0 ; \beta=3$

$$X = B \sin(3t) + A \cos(3t)$$

$$X'' = -9B \sin(3t) - 9A \cos(3t)$$

$$-32B \sin(3t) - 32A \cos(3t) = -a^3 \cos(3t)$$

$$A = \frac{a^3}{32} ; B = 0 ; X = \frac{a^3}{32} \cos(3t)$$

General Solution

$$X = \frac{a^3}{32} \cos(3t) + C_1 \sin(t) + C_2 \cos(t)$$

$$X(0) = \frac{a^3}{32} + C_2 = 0 ;$$

$$= \frac{a^3}{32} \cos(3t) + C_1 \sin(t) - \frac{a^3}{32} \cos(t)$$

$$X(t, \varepsilon) = a \cos t + \varepsilon a^3 \left[ -\frac{3}{8} t \sin t + \frac{1}{32} (\cos 3t - \cos t) \right] + O(\varepsilon^2)$$

$$\text{7.6.20, } \ddot{X} + X + \varepsilon X^3 = (\ddot{X}_0 + \varepsilon \ddot{X}_1 + \varepsilon^2 \ddot{X}_2) + (\dot{X}_0 + \varepsilon \dot{X}_1 + \varepsilon^2 \dot{X}_2) \\ + \varepsilon (X_0 + \varepsilon X_1 + \varepsilon^2 X_2)$$

$$= (\ddot{X}_0 + X_0) + \varepsilon (\ddot{X}_1 + X_1 + X_0^3) + \varepsilon^2 (\ddot{X}_2 + X_2 + 3X_0^2 X_1) + O(\varepsilon^3)$$

$$+ \varepsilon^2 (\ddot{X}_2 + X_2 + 3X_0^2 X_1) + O(\varepsilon^3)$$

$$O(1) \ddot{X}_0 + X_0 = 0 \quad -a^3 \cos(3t) - \frac{3a^3 \cos(t)}{4} + 2a \cos(t) \cos(3t)$$

$$O(\varepsilon) : \ddot{X}_1 + X_1 + X_0^3 = 0 \quad -3a^3 \cos(t) \cos(3t)$$

$$O(\varepsilon^2) : \ddot{X}_2 + X_2 + 3X_0^2 X_1 = 0$$

$$C_2(y) = -a \cos(3t) \cos(y) - \frac{(a^3 - 3a^3 \cos(3t)) \cos(3t)}{4} + C_2$$

Solving for  $X_0$ ;  $X_0 + X_0 = 0 \Rightarrow X_0 = a \cos(t)$

Solving for  $X_1$ ;  $\ddot{X}_1 + X_1 = -a^3 \cos^3(t)$

Linear Equation of constant coefficients

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \dots + a_{n-1} t' + a_n t = f(t)$$

Solving the Homogeneous Equation

$$a_0 \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda' + a_n \lambda = 0$$

$$(\lambda^2 + 1) = 0 ; \lambda_{1,2} = \pm i$$

Solution of a Homogeneous Equation; Summand

$$t = \sum P_{k-1}(t) e^{\lambda t} \sin \beta t + Q_{k-1}(t) e^{\lambda t} \cos \beta t$$

$$\text{where } \lambda = \kappa + \beta i ; \kappa = 0 ; \beta = 1 ; k = 1$$

$$\text{so } X = A \sin(t) + B \cos(t)$$

Method for Undetermined Coefficients: Particular Solution

$$X_i = t^s e^{\lambda t} (R_m(t) \cos \beta t + T_m(t) \sin \beta t)$$

If  $s=R=1$ ,  $X=t(B \sin t + A \cos t)$

$$\ddot{X} = (-Bt - 2A) \sin t + (2B - At) \cos t$$

$$\ddot{X}_1 + X_1 = 2B \cos t - 2A \sin t = -a^3 \cos t$$

$$2B = -a^3 ; A = 0$$

$$-2A = 0 ; B = -\frac{a^3}{2}$$

$$X_1 = -\frac{a^3 t \sin t}{2}$$

[General Solution = Particular + Homogeneous Equation]

$$X(t) = -\frac{a^3 t \sin t}{2} + C_1 \sin t + C_2 \cos t$$

with initial conditions:  $X(0) = \alpha$ ;  $\dot{X}(0) = 0$

$$X_1(t) = a \cos t + \frac{a^3 t}{2} - 3a^2 \sin t$$

$$\text{Solving for } X_2: \ddot{X}_2 + X_2 = -3X_0 X_1$$

Linear - Equation with constant coefficients

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \dots + a_{n-1} t' + a_n t = f(t)$$

Solving the Homogeneous Equation

$$a_0 \lambda^n + a_1 \lambda^{(n-1)} + \dots + a_{n-1} \lambda' + a_n \lambda = 0$$

$$(\lambda^2 + 1) = 0 ; \lambda_{1,2} = \pm i ; K = 1$$

Solution of a Homogeneous Equation: Summand

$$t = \sum P_{K-1}(t) e^{kt} \sin \beta t + Q_{K-1}(t) e^{kt} \cos \beta t$$

Where  $\lambda = K + \beta i$ ;  $K = 0$ ;  $\beta = 1$ ;  $K = 1$

$$\text{so } X = A \sin t + B \cos t$$

Method for Undetermined Coefficients: Particular Solution

$$X_p = t^s e^{xt} (R_m(t) \cos \beta t + T_m(t) \sin \beta t)$$

$$= t (B \sin t + A \cos t)$$

$$\ddot{X} = (-\beta t - 2A) \sin t + (2B - At) \cos t$$

$$\ddot{X}_2 + X_2 = 2B \cos(t) - 2A \sin(t) = -3a^3 \cos^3(t)$$

$$2B = -3a^3 \cos^2(t); A = 0$$

$$B = \frac{-3a^3 \cos^2(t)}{2}$$

$$X_2 = \frac{-3\varepsilon a^3 t \cos^3(t)}{2}$$

General Solution = Particular + Homogeneous

$$X_2(t, \varepsilon) = \frac{-3\varepsilon a^3 t \cos^3(t)}{2} + A \sin(t) + B \cos(t)$$

With initial conditions;  $B = a + \frac{3\varepsilon a^3 t}{2}$

$$\ddot{X}_2(b, \varepsilon) = \frac{+9\varepsilon a^3 t \cos^2(t) \sin(t)}{2} + A \cos(t) - B \sin(t)$$

$$A = 0$$

$$\ddot{X}_2(b, \varepsilon) = \frac{-3\varepsilon a^3 t \cos^3(t)}{2} + \left(a + \frac{3\varepsilon a^3 t}{2}\right) \cos(t)$$

$$= \frac{-3\varepsilon a^3 t}{2} \left[ \frac{1}{4} [3 \cos t + \cos 3t] \right] + \left(a + \frac{3\varepsilon a^3 t}{2}\right) \cos(t)$$

$$= a \cos t + \varepsilon a^3 \left[ \frac{3}{2} t \cos(t) + \frac{3}{8} t [\cos 3t - 3 \cos t] \right]$$

This solution is not an exact answer  
to why the Duffing oscillator has a  
frequency dependent on amplitude.

$$\ddot{X} + \varepsilon(X^2 - 1)\dot{X} + X = 0 \quad 7.6.21. \quad \boxed{\text{Poincaré-Lindstedt Method.}}$$

(1) Define a new time  $\Pi = \omega t$

$$\omega^2 X'' + \varepsilon \omega(X^2 - 1) + X = 0$$

$$X_1(0) = 0 ; \dot{X}_1(0) = 0 ; \frac{3}{4} + \frac{\alpha^2}{4} \sin(3\pi t)$$

$$\ddot{X}_1 + X_1 = 0 = 2\omega, \omega = 3 \text{ rad/s}, \tan \lambda = 0$$

$$\ddot{X}_1 + X_1 = \frac{\pm \alpha^3}{4} [\sin(t) - \sin(3t)]$$

Solving the Linear Equation with Constant Coefficients

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \dots + a_{n-1} t' + a_n t = f(t)$$

Solving the Homogeneous Equation

$$a_0 \lambda^{(n)} + a_1 \lambda^{(n-1)} + \dots + a_{n-1} \lambda' + a_n \lambda = 0$$

$$(\lambda^2 + 1) = 0 ; \lambda = \pm i \Rightarrow K \text{ cost}$$

Solution to a Homogeneous Equation

$$t_h = \sum P_{k=1}^n (t) e^{\lambda k t} \sin \beta t + Q_{k=1}^n (t) e^{\lambda k t} \cos \beta t$$

$$\text{where } \lambda = \kappa + \beta i ; \kappa = 0 ; \beta = 1 ; K = 1$$

$$\text{so } X = A \sin(t) + B \cos(t)$$

Solving the Particular Equation: Method for Undetermined Coefficients

$$X_p = t^2 e^{\kappa t} (R_m(t) \cos \beta t + T_m(t) \sin \beta t)$$

$$X_p = t [A_2 \cos t + B_2 \sin t]$$

$$\ddot{X}_p = (-B_2 t - 2A_2) \sin t + (2B_2 - A_2 t) \cos t$$

$$\ddot{X}_p + X_p = 2B_2 \cos t - 2A_2 \sin t = -\frac{\alpha^3}{4} [\sin(t) - \sin(3t)]$$

$$2B_2 = 0 ; -2A_2 = \frac{\alpha^3}{8} \left[ 1 - \frac{2 \sin(t) \sin(3t)}{\cos(2t) - 1} \right]$$

General Solution = Particular + Homogeneous

$$X_p(t) = A_2 \sin(t) + B_2 \cos(t) - \frac{\alpha^3}{4} \sin(3t)$$

With initial conditions ;  $X_p(0) = 0 ; \dot{X}_p(0) = 0$

$$A_2 = -\frac{\alpha^3}{4} ; B_2 = 0$$

$$X_p(t) = -\frac{\alpha^3}{4} [\sin(t) + \sin(3t)]$$

② Assign a new  $X(T, \varepsilon) = X_0(T) + \varepsilon X_1(T) + \varepsilon^2 X_2(T) + \dots$

$$\omega = 1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \dots$$

③ solve the perturbation equation

$$\omega^2 \ddot{X}'' + \varepsilon \omega (\dot{X}^2 - 1) \dot{X}' + \dot{X}$$

$$= (\omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \dots)^2 (\ddot{X}_0 + \varepsilon \ddot{X}_1 + \varepsilon^2 \ddot{X}_2 + \dots)$$

$$+ \varepsilon \cdot (1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \dots) ([\dot{X}_0 + \varepsilon \dot{X}_1 + \varepsilon^2 \dot{X}_2]^2 - 1) \circ$$

$$(\ddot{X}_0 + \varepsilon \ddot{X}_1 + \varepsilon^2 \ddot{X}_2) + (\dot{X}_0 + \varepsilon \dot{X}_1 + \varepsilon^2 \dot{X}_2)$$

$$O(1): \ddot{X}_0 + X_0 = 0$$

$$O(\varepsilon): 2\omega_1 \ddot{X}_0 + \ddot{X}_1 + X_0 \dot{X}_0 - \dot{X}_0 \dot{X}_1 + X_1 \dot{X}_1 = 0$$

$$O(\varepsilon^2): 2\ddot{X}_0 \omega_2 + 2\ddot{X}_1 \omega_1^2 + 2\ddot{X}_1 \omega_1 + X_0 \ddot{X}_0 \omega_1 + \ddot{X}_0 \omega_1^2$$
$$+ X_0^2 \dot{X}_1 + 2X_0 \dot{X}_0 \dot{X}_1 - \dot{X}_1 \omega_1 - \dot{X}_1 \dot{X}_2 + \dot{X}_2 \dot{X}_1 = 0$$

$$\ddot{X}_2 + X_2 = -( \omega_1^2 + 2\omega_2 ) \ddot{X}_0 - 2\ddot{X}_1 \omega_1^2 - 2X_0 \dot{X}_0 \dot{X}_1$$
$$+ (1 - X_0^2)(\dot{X}_1 + \omega_1 X_0)$$

Solving for  $X_0$ :

$$\ddot{X}_0 + X_0 = 0; X(0) = \alpha; \dot{X}(0) = 0$$

$$X_0(t) = \alpha \cos(t)$$

Solving for  $X_1$ :

$$\ddot{X}_1 + X_1 = -2\omega_1 \ddot{X}_0 + (X_0^2 - X_0) \dot{X}_0$$

$$\text{Linear System: } +2\omega_1 \alpha \cos(t) - (1 - \alpha^2 \cos^2(t)) \alpha \sin(t)$$

$$a_0 t^{(n)} + a_1 t^{(n-1)} = \alpha [2\omega_1 \cos(t) - \frac{\alpha^2}{4} [3\sin(t) - \sin(3t)]]$$

Solving for  $X_2$ :

$$\begin{aligned}
 \ddot{X}_2 + X_2 &= -(w_1^2 + 2w_2) \ddot{X}_0 - 2\dot{X}_0 w_2^2 - 2X_0 \dot{X}_0 \dot{X}_1 + (1 - X_0^2)(\dot{X}_1 + w_1 \dot{X}_0) \\
 &= +2w_2 a \cos(t) + 2a^2 \cos(t) \sin(t) \left[ -\frac{1}{4} a^3 [3 \sin(t) + \sin(3t)] \right] \\
 &\quad + \frac{3}{4} (3 \sin(t) \cos(3t) + \sin(3t) \cos(t)) (1 - a^2 \cos^2(t)) \\
 &= \frac{1}{4} [\cos(t) (8aw_2 - 2a^5 \sin(t) (3 \sin(t) + \sin(3t))) \\
 &\quad + (3a^2 \cos^2(t) - 3)(3 \sin(t) \cos(3t)) + \sin(3t) \cos(t)]
 \end{aligned}$$

### Linear Equation of Constant Coefficients

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \dots + a_{n-1} t' + a_n t = f(t)$$

### Solving the Homogeneous Equation

$$a_0 \lambda^{(n)} + a_1 \lambda^{(n-1)} + \dots + a_{n-1} \lambda' + a_n \lambda = 0$$

$$4(\lambda^2 + 1) = 0 \Rightarrow \lambda = \pm i \Rightarrow k = 1$$

### Solution to a Homogeneous Equation: Summand

$$t = \sum_{n=1} R(t) e^{xt} \sin \beta t + Q_{k_1}(t) e^{xt} \cos \beta t$$

$$X = A \cos t + B \sin t \quad X(0) = 0 \quad \dot{X}(0) = 0$$

### Lagrange's Method of Variation of Parameters

System of Equations:  $C'(t) X_1 + C'_1(t) X_2 = 0$

$$X_2 = t[A_2 \cos \beta t + B_2 \sin \beta t] \quad C'(t) X_1 + C'_1(t) X_2 = \frac{f(t)}{a_0}$$

$$\dot{X}_1 = (-2B_2 - 2A) \sin t \quad \text{where } X_1 = \cos(t) \text{ and } X_2 = \sin(t)$$

$$X_1' = -\sin(t) \quad X_2' = \cos(t)$$

$$a_0 X'' = 2 + (3a^2 \cos^2(t) - 3)(3 \sin(t) \cos(3t)) + \sin(3t) \cos(t)$$

$$F(t) = \frac{1}{2} [\cos(t) (8aw_2 - 2a^5 \sin(t) (3 \sin(t) + \sin(3t)))]$$

$$2B_2 = \frac{1}{4} + 2(3a^2 \cos^2(t) - 3)(3 \sin(t) \cos(3t)) + \sin(3t) \cos(t)$$

$$B_2 = \frac{1}{8} + \frac{1}{4} \tan(t) ((3aw_2 - 2a^5 \sin(t) (3 \sin(t) + \sin(3t))) + (2a^2 \cos(t) - 3)(3 \cos(3t)))$$

$$+ 3 \sin(3t) \cos(t)$$

Finding  $C'(t)$ ,  $C_1'(t)$  by Cramer's Rule:

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = 1 \quad [3aW_1 - 2a^5 \sin(t)(\sin(3t) + 3\sin(t))]$$

$$W_1 = \frac{1}{W} \begin{vmatrix} W & 0 \\ \cos(t)[8aW_2 - 2a^5 \sin(t)(\sin(3t) + 3\sin(t))] & 0 \end{vmatrix} = \frac{\cos(t)[8aW_2 - 2a^5 \sin(t)(\sin(3t) + 3\sin(t))] + 2\cos(t)\sin(3t) + 6[3a^2 \cos^2(t) - 3]\sin(t)\cos(3t)}{2}$$

$$\begin{vmatrix} \sin(t) \\ \cos(t) \end{vmatrix} = -\sin(t) \left[ \frac{\cos(t)[8aW_2 - 2a^5 \sin(t)(\sin(3t) + 3\sin(t))] + 2\cos(t)\sin(3t) + 6[3a^2 \cos^2(t) - 3]\sin(t)\cos(3t)}{2} \right]$$

$$W_2 = \begin{vmatrix} \cos(t) & 0 \\ -\sin(t) & \frac{\cos(t)[8aW_2 - 2a^5 \sin(t)(\sin(3t) + 3\sin(t))] + 2\cos(t)\sin(3t) + 6[3a^2 \cos^2(t) - 3]\sin(t)\cos(3t)}{2} \end{vmatrix} = \frac{\cos(t)[\cos(t)[8aW_2 - 2a^5 \sin(t)(\sin(3t) + 3\sin(t))] + 2\cos(t)\sin(3t) + 6[3a^2 \cos^2(t) - 3]\sin(t)\cos(3t)]}{2}$$

$$C'(t) = \frac{W_1}{W} = W_1 \quad ; \quad C_1'(t) = \frac{W_2}{W} = W_2$$

$$C(t) = \int C'(t) dt = \dots \quad ; \quad C_1(t) = \int C_1'(t) dt = \dots$$

$$\text{Solving for } \omega_2: \ddot{x}_2 + x_2 = (4\omega_2 + \frac{1}{4})\cos(\tau) + \dots$$

$$\omega_2 = -\frac{1}{16}$$

$$\text{Solving for } \omega: \omega = (1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \dots)$$

$$= (1 - \frac{1}{16} \varepsilon^2 + \dots)$$

$$\ddot{x} + x + \varepsilon x^2 = 0 \quad 7.6.22. \quad x(0) = a; \quad \dot{x}(0) = 0$$

① Define a new time  $\tau = wt$

$$\omega^2 x'' + \omega x + \varepsilon x^2 = 0$$

$$\lambda^2 + 1 = 0 ; \lambda_{1,2} = \pm i ; K = S = 1 ; \lambda = \alpha + \beta i$$

Solution to a Homogeneous Equations: Summand

$$t = \sum R_{n-1}(t) e^{xt} \sin \beta t + Q_{k-1}(t) e^{xt} \cos \beta t$$

$$X(t) = A_1 \sin(t) + B_1 \cos(t)$$

Solving the Particular Equation: Method for Undetermined Coefficients

$$X_i = t^s e^{xt} (R_m(t) \cos \beta t + T_m(t) \sin \beta t)$$

$$X_1 = t [A_2 \cos t + B_2 \sin t]$$

$$\ddot{X}_1 = (-B_2 t - 2A_2) \sin(t) + (2B_2 - A_2 t) \cos(t)$$

$$\ddot{X}_1 + X_1 = 2B_2 \cos(t) - 2A_2 \sin(t) = a \cos(t) (w_1 + w_2) + a^2 \cos^2(t)$$

$$\text{Initial conditions: } X(0) = a \Rightarrow \dot{X}(0) = 0$$

$$2B_2 - 2A_2 = a(w_1 + w_2) + a^2 \Rightarrow -2A_2 = 0$$

$$B_2 = \frac{\alpha}{2}(w_1 + w_2) + \frac{\alpha^2}{2} ; A_2 = 0$$

General Solution = Particular + Homogeneous

$$X_1(t) = [a(w_1 + w_2) + a^2] \cos(t) t + A_1 \cos t + B_2 \sin t$$

$$\text{Initial conditions: } X(0) = a ; \dot{X}(0) = 0$$

$$X_1(0) = a = A_1 \Rightarrow A_1 = a$$

$$\dot{X}_1(0) = 0 = B_2 \Rightarrow B_2 = 0$$

$$X_1(t) = [a(w_1 + w_2) + a^2 + a] \cos(t)$$

⑥ Solving for  $X_2$ :

$$\ddot{X}_2 + X_2 = -2[X_0 X_1 + X_0 w_1 w_2 + \dot{X}_1 (w_1 + w_2)] - \ddot{X}_0 (w_1^2 + w_2^2) - X_1 (w_1 + w_2)$$

$$= -2[\alpha \cos^2(t) [(w_1 + w_2) + a + 1] + \alpha w_1 w_2 \cos(t)]$$

$$+ \alpha [(w_1 + w_2) + a + 1] \cos(t) (w_1 + w_2)$$

$$+ \alpha \cos(t) (w_1^2 + w_2^2) - \alpha [(w_1 + w_2) + a + 1] \cos(t) (w_1 + w_2)$$

$$= \alpha \cos(t) [2\alpha \cos(t) (\alpha + w_1 + w_2 + 1) - (w_1 + w_2)(3\alpha + 2w_1 + 2w_2 + 3)]$$

② Assign a new perturbation:

$$X(\tau, \varepsilon) = X_0(\tau) + \varepsilon X_1(\tau) + \varepsilon^2 X_2(\tau) + \dots$$

$$\omega = 1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \dots$$

③ Solve the perturbation equation:

$$\omega^2 X'' + \omega X + \varepsilon X^2 = 0$$

$$= (1 + \varepsilon \omega_1 + \varepsilon \omega_2)^2 (\ddot{X}_0 + \varepsilon \ddot{X}_1 + \varepsilon^2 \ddot{X}_2) + (1 + \varepsilon \omega_1 + \varepsilon \omega_2)(X_0 + \varepsilon X_1 + \varepsilon^2 X_2)$$
$$+ \varepsilon(X_0 + \varepsilon X_1 + \varepsilon^2 X_2)^2$$

$$O(1): \ddot{X}_0 + X_0 = 0$$

$$O(\varepsilon): \ddot{X}_0^2 + X_0(\omega_1 + \omega_2) + 2\ddot{X}_0(\omega_1 + \omega_2) + X_1 + \dot{X}_1 = 0$$

$$\ddot{X}_1 + X_1 = -2\ddot{X}_0(\omega_1 + \omega_2) - X_0(\omega_1 + \omega_2) + X_0^2$$

$$O(\varepsilon^2): \ddot{X}_2 + X_2 = -2X_0 X_1 - \dot{X}_0 \omega_1^2 - 2\ddot{X}_0 \omega_1 \omega_2$$

$$- \ddot{X}_0 \omega_2^2 - X_1(\omega_1 + \omega_2) - 2\ddot{X}_1(\omega_1 + \omega_2)$$

$$= -2[X_0 X_1 + X_0 \omega_1 \omega_2 + \ddot{X}_1(\omega_1 + \omega_2)]$$

$$- \ddot{X}_0(\omega_1^2 + \omega_2^2) - X_1(\omega_1 + \omega_2)$$

④ Solving for  $X_0$ :

$$\ddot{X}_0 + X_0 = 0; \quad X(0) = \alpha; \quad \dot{X}(0) = 0$$

$$X_0(t) = \alpha \cos(t)$$

⑤ Solving for  $X_1$ :

$$\ddot{X}_1 + X_1 = 2\alpha \cos(t)(\omega_1 + \omega_2) - \alpha \cos(t)(\omega_1 + \omega_2) + \alpha^2 \cos^2(t)$$
$$= \alpha \cos(t)(\omega_1 + \omega_2) + \alpha^2 \cos^2(t)$$

Linear Equation of Constant Coefficients

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \dots + a_{n-1} t^1 + a_n t^0 = f(t)$$

Solving the Homogeneous Equation

$$a_0 \lambda^{(n)} + a_1 \lambda^{(n-1)} + \dots + a_{n-1} \lambda^1 + a_n \lambda^0 = 0$$

## Linear Equation of Constant Coefficients

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \dots + a_{n-1} t^1 + a_n t^0 = f(t)$$

### Solving the Homogeneous Equation

$$a_0 \lambda^{(n)} + a_1 \lambda^{(n-1)} + \dots + a_{n-1} \lambda^1 + a_n \lambda^0 = 0$$

$$(\lambda^2 + 1) = 0; \lambda_{1,2} = \pm i; K = S = 1$$

### Solution to the Homogeneous Equation: Summand

$$t = \sum R_m(t) e^{xt} \sin \beta t + Q_k(t) e^{xt} \cos \beta t$$

$$X(t) = A_1 \sin(t) + B_2 \cos(t)$$

### Solving the Particular Equation: Method for Undetermined Coefficients.

$$X_1 = t^s e^{xt} \cdot [R_m(t) \cos \beta t + T_m(t) \sin \beta t]$$

$$X_2 = t [A_2 \cos t + B_2 \sin t]$$

$$\overset{\circ}{X}_2 = (-B_2 t - 2A_2) \sin t + (2B_2 - A_2 t) \cos t$$

$$\overset{\circ\circ}{X}_2 + X_2 = 2B_2 \cos t - 2A_2 \sin t =$$

$$= a \cos t [2a \cos t (a + w_1 + w_2 + 1) - (w_1 + w_2)(3a + 2w_1 + 2w_2 + 3)]$$

Initial conditions:  $X(0) = a, \overset{\circ}{X}(0) = 0$

$$2B_2 = \frac{a}{2} [2a(a + w_1 + w_2 + 1) - (w_1 + w_2)(3a + 2w_1 + 2w_2 + 3)]$$

$$A_2 = 0$$

General Solution = Particular + Homogeneous

$$X(t) = A_1 \sin t + B_2 \cos t + \frac{a \cdot t}{2} [2a(a + w_1 + w_2 + 1) - (w_1 + w_2)(3a + 2w_1 + 2w_2 + 3)] \sin t$$

$$X_2(0) = a = B_2; \overset{\circ}{X}_2(0) = 0 = A_1$$

$$X_2(t) = a \cos t + \frac{a \cdot t}{2} [2a(a + w_1 + w_2 + 1) - (w_1 + w_2)(3a + 2w_1 + 2w_2 + 3)] \sin t$$

$$KE = \frac{1}{2} \varepsilon X^2 = \frac{1}{2} \varepsilon \left( a - \frac{t^2}{2} a - O(t^4) \right)^2 \approx \frac{1}{2} \varepsilon a^2 + b(t^2)^2 = w$$

$$\ddot{X} - \varepsilon X \ddot{X} + X = 0 \quad 7.6.23.$$

(1) Assign a new time  $\tau = \omega t$

$$\omega^2 \ddot{X} - \varepsilon X \omega \dot{X} + X = 0$$

(2) Assign a new perturbation equation

$$X(\tau, \varepsilon) = X_0(\tau) + \varepsilon X_1(\tau) + \varepsilon^2 X_2(\tau) + \dots$$

$$\omega = 1 + \varepsilon w_1 + \varepsilon^2 w_2 + \dots$$

(3) Solve the perturbation equation:

$$(1 + \varepsilon w_1 + \varepsilon^2 w_2 + \dots)^2 (X_0 + \varepsilon \dot{X}_1 + \varepsilon^2 \ddot{X}_2 + \dots)$$

$$+ \varepsilon (X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \dots) (\dot{X}_0 + \varepsilon \dot{X}_1 + \varepsilon^2 \dot{X}_2 + \dots)$$

$$+ ((1 + \varepsilon w_1 + \varepsilon^2 w_2 + \dots) + (X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \dots)) = 0$$

$$O(1) : \ddot{X}_0 + X_0 = 0$$

$$O(\varepsilon) : \ddot{X}_1 + X_1 = -2w_1 \dot{X}_0 + \dot{X}_0 \dot{X}_0$$

$$O(\varepsilon^2) : \ddot{X}_2 + X_2 = -w_1^2 X_0 + w_1 \dot{X}_0 \dot{X}_0 - 2w_1 \ddot{X}_1$$

$$- 2w_2 \dot{X}_0 + X_1 \dot{X}_0 + X_0 \dot{X}_1$$

④ Solving for  $X_0$ :

$$\ddot{X}_0 + X_0 = 0 \Rightarrow X(0) = 0 \Rightarrow \dot{X}(0) = 0$$

$$X_0(t) = a \cos t$$

⑤ Solving for  $X_1$ :

$$\ddot{X}_1 + X_1 = -2w_1 \dot{X}_0 + \dot{X}_0 \dot{X}_0$$

$$= -2w_1 a \cos t - a^2 \cos t \sin t$$

$$= [-2w_1 a - a^2 \sin t] \cos t$$

Linear equation with constant coefficients.

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \dots + a_{n-1} t^1 + a_n t^0 = f(t)$$

Solving the Homogeneous Equation

$$a_0 \lambda^{(n)} + a_1 \lambda^{(n-1)} + \dots + a_{n-1} \lambda^1 + a_n \lambda^0 = 0$$

$$(\lambda^2 + 1) = 0; \quad \lambda_{1,2} = \pm i; \quad K=S=1$$

[Solution to a Homogeneous Equation: Summand]

$$t = \sum R_{M_1}(t) e^{kt} \cos \beta t + Q_{M_1}(t) e^{kt} \sin \beta t$$

$$\lambda = k + \beta i; \quad k = 0; \quad \beta = 1$$

$$X = A_1 \cos t + B_1 \sin(t)$$

[Lagrange's Method of Variation of Parameters.]

(1) Build a system  $A_1' X_1 + B_1' X_2 = 0$

$$A_1' X_1' + B_1' X_2' = \frac{P(t)}{a_0}$$

$$\text{where } X_1 = \cos(t) \quad X_2 = \sin(t)$$

$$X_1' = -\sin(t) \quad X_2' = \cos(t)$$

$$a_0 X'' = 1; \quad P(t) = \cos(t)(-\alpha^2 \sin(t) - 2\omega w_1)$$

(2) Solve the System using Cramer's Rule.

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & \sin(t) \\ \cos(t)(-\alpha^2 \sin(t) - 2\omega w_1) & \cos(t) \end{vmatrix} = \cos(t) \sin(t)(-\alpha^2 \sin(t) - 2\omega w_1)$$

$$W_2 = \begin{vmatrix} \cos(t) & 0 \\ -\sin(t) & \cos(t)(-\alpha^2 \sin(t) - 2\omega w_1) \end{vmatrix} = \cos^2(t)(-\alpha^2 \sin(t) - 2\omega w_1)$$

$$A_1'(t) = \frac{W_1}{W} = \cos(t) \sin(t)(-\alpha^2 \sin(t) - 2\omega w_1)$$

$$B_1'(t) = \frac{W_2}{W} = \cos^2(t)(-\alpha^2 \sin(t) - 2\omega w_1)$$

$$A_1 = \int A_1'(t) dt = \frac{\alpha^2 \sin^3(t)}{3} + \omega w_1 \sin^2(t) + C_1$$

$$B_1 = \int B_1'(t) dt = -\frac{\omega w_1 \sin(2t)}{2} + \frac{\alpha^2 \cos^3(t)}{3} - \omega w_1 t + C_2$$

$$X = \frac{-\alpha w \sin(t) \sin(2t)}{2} + \frac{\alpha^2 \cos(t) \sin^3(t)}{3} + \alpha w \cos(t) \sin^2(t)$$

$$+ \frac{\alpha^2 \cos^3(t) \sin(t)}{3} - \alpha w t \sin(t) + C_1 \sin(t) + C_2 \cos(t)$$

Initial conditions:  $X(0) = 0; \dot{X}(0) = 0$

$$X(0) = 0 = C_2; \dot{X}(0) = 0 = 2\alpha^2 + 6C_1; C_1 = -\frac{1}{3}\alpha^2$$

$$X_1(t) = \left[ \frac{-\alpha w_1 \sin(2t)}{2} + \frac{\alpha^2 \cos(t) \sin^2(t)}{3} + \alpha w_1 \cos(t) \sin^2(t) \right] + \left[ \frac{\alpha^2 \cos^3(t)}{3} - \alpha w_1 t - \frac{\alpha^2}{3} \right] \sin(t) + \alpha \cos(t)$$

$$+ \left[ \frac{\alpha^2 \cos^3(t)}{3} - \alpha w_1 t - \frac{\alpha^2}{3} \right] \sin(t) + \alpha \cos(t)$$

$w_1 = 0$  because no secular terms

$$X_1(t) = \left[ \frac{\alpha^2 \cos(t) \sin^2(t)}{3} + \frac{\alpha^2 \cos^3(t)}{3} - \frac{\alpha^2}{3} \right] \sin(t)$$

$$\text{Identities: } \sin(2t) = 2\cos(t)\sin(t)$$

$$\sin^2(t) = \frac{1}{2}(1 - \cos(2t))$$

$$\cos^2(t) = 1 - \sin^2(t)$$

$$X_1(t) = \frac{1}{6}(-2\alpha^2 \sin(t) + \alpha^2 \sin(2t))$$

③ Solving for  $X_2$

$$\begin{aligned} \ddot{X}_2 + X_2 &= -w_1^2 \ddot{X}_0 + w_1 \ddot{X}_0 X_0 - 2w_1 \ddot{X}_1 - 2w_2 \ddot{X}_0 + X_1 \ddot{X}_0 + X_0 \ddot{X}_1 \\ &= -2w_2 \ddot{X}_0 + X_1 \ddot{X}_0 + X_0 \ddot{X}_1 \\ &= 2w_2 \alpha \cos(t) - \frac{\alpha \sin(t)}{6} (-2\alpha^2 \sin(t) + \alpha^2 \sin(2t)) \\ &\quad + \frac{\alpha \cos(t)}{3} (-2\alpha^2 \cos(t) + \alpha^2 \cos(2t)) \end{aligned}$$

Linear Equation of Constant Coefficients

$$a_0 t^n + a_1 t^{n-1} + \dots + a_{(n-1)} t^1 + a_{(n)} t^0 = f(t)$$

Solving the Homogeneous Equation

$$a_0 \lambda^n + a_1 \lambda^{n-1} + \dots + a_{(n-1)} \lambda^1 + a_{(n)} \lambda^0 = 0$$

$$(\lambda^2 + 1) = 0 ; \lambda_{1,2} = \pm i ; k = s = 1$$

Solution to a Homogeneous Equation: Summand

$$t = \sum e^{kt} [R_{m-1}(t) \cos \beta t + Q_{m-1}(t) \sin \beta t]$$

$$X_2(t) = A_1 \cos t + B_1 \sin t$$

Lagrange's Method of Variation of Parameters

$$A'_1 X_1 + B'_1 X_2 = 0$$

$$A'_1 X_1 + B'_1 X_2 = \frac{F(t)}{a_0} \quad \text{where } X_1 = \cos(y) \quad X_2 = \sin(y)$$

$$X_1' = -\sin(y) \quad X_2' = \cos(y)$$

$$X_2 + X_1 = 2B_1 \cos(t) - 2A_1 \sin(t) = a_0 X'' = 6 \quad \text{Initial condition}$$

$$f(t) = 12W_2 \cos(t) - \frac{a^3}{6} \sin(-2\sin t + \sin 2t) + 2a^3 \cos t (\cos t + \cos 2t)$$

$$\text{Initial condition } X_2(0) = 0, X_1(0) = 0$$

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} + a_1 - 2A_2 = 0$$

$$W_1 = \begin{vmatrix} B_2 & W + a \\ +2W_2 \cos t - \frac{a^3}{6} \sin(-2\sin t + \sin 2t) + \frac{a^3 \cos t}{3} (\cos t + \cos 2t) & \sin t \end{vmatrix}$$

$$W_2 = \begin{vmatrix} \cos t & 0 \\ -\sin t & -2W_2 \cos t - \frac{a^3}{6} \sin(-2\sin t + \sin 2t) + \frac{a^3 \cos t}{3} (\cos t + \cos 2t) \end{vmatrix}$$

$$A'_1 = \frac{W_1}{W} = \sin t \left[ \frac{a^3 \sin t}{6} (-2\sin t + \sin 2t) - \frac{a^3 \cos t}{3} (\cos t + \cos 2t) - 2W_2 \cos t \right]$$

$$B'_1 = \frac{W_2}{W} = \cos t \left[ 2W_2 \cos t - \frac{a^3 \sin t}{6} (-2\sin t + \sin 2t) + \frac{a^3 \cos t}{3} (\cos t + \cos 2t) \right]$$

$$A_1 = \int A'_1 dt = \sin^2 t \left[ \frac{a^3}{4} \sin^2 t - \frac{(6aw + a^3)}{6} \right] + \cos t \left[ \frac{a^3}{3} - \frac{2a^3 \cos^2(t)}{9} \right] + C_1$$

$$B_1 = \int B'_1 dt = \frac{a^3 \sin 4t}{32} + \frac{(48aw + 8a^3) \sin 2t}{96} + \frac{2a^3 \sin^3(t)}{9} - \frac{a^3 \sin(t)}{3} + awt + \frac{a^3 t}{24} + C_2$$

$$X_2(t) = \left[ \frac{a^3 \sin 4t}{32} + \frac{(48aw + 8a^3) \sin 2t}{96} + \frac{2a^3 \sin^3 t}{9} - \frac{a^3 \sin t}{3} + awt + \frac{a^3 t}{24} + C_1 \right] \sin t + \left[ \sin^2 t \left( \frac{a^3 \sin^2 t}{4} - \frac{(6aw + a^3)}{6} \right) + \cos t \left( \frac{a^3}{3} - \frac{2a^3 \cos^2 t}{9} \right) + C_2 \right] \cos t$$

Initial conditions:  $X_2(0) = 0$ ;  $\dot{X}_2(0) = 0$

$$X_2(0) = 0 = \frac{a^3}{3} - \frac{2a^3}{9} + C_2; C_2 = -\frac{a^3}{3} + \frac{2a^3}{9}$$

$$\dot{X}_2(0) = 0 = awt + \frac{a^3 t}{24} + C_1; C_1 = -awt - \frac{a^3 t}{24}$$

$$X_2(t) = \left[ \frac{a^3 \sin 4t}{32} + \frac{(48aw_2 + 8a^3) \sin 2t}{96} + \frac{2a^3 \sin^3 t}{9} - \frac{a^3 \sin t}{3} \right] \sin t + \left[ \sin^2 t \left( \frac{a^3 \sin^2 t}{4} - \frac{(6aw + a^3)}{6} \right) + \cos t \left( \frac{a^3}{3} - \frac{2a^3 \cos^2 t}{9} \right) - awt - \frac{a^3 t}{24} \right] \cos t$$

$$w_2 = -\frac{1}{9}a^2$$

$$\begin{aligned} w(a) &= 1 + \varepsilon w_1 + \varepsilon^2 w_2 + O(\varepsilon^3) \\ &= 1 - \frac{1}{8}\varepsilon^2 a^2 \end{aligned}$$

$$\ddot{x} + x - \varepsilon x^3 = 0 \quad 7.6.24. \quad x(0) = a; \quad \dot{x}(0) = 0$$

① Assign a new time:  $\tau = wt$

$$\omega^2 \ddot{x} + x - \varepsilon x^3 = 0$$

② Apply perturbation equations

$$x(\tau, \varepsilon) = X_0(\tau) + \varepsilon X_1(\tau) + \varepsilon^2 X_2(\tau) + O(\varepsilon^3)$$

$$\omega = 1 + \varepsilon w_1 + \varepsilon^2 w_2 + O(\varepsilon^3)$$

$$(1 + \varepsilon w_1 + \varepsilon^2 w_2 + \dots)^2 (\ddot{X}_0 + \varepsilon \ddot{X}_1 + \varepsilon^2 \ddot{X}_2) + (X_0 + \varepsilon X_1 + \varepsilon^2 X_2) - \varepsilon (X_0 + \varepsilon X_1 + \varepsilon^2 X_2)^3 = 0$$

$$O(1); \ddot{X}_0 + X_0 = 0$$

$$O(\varepsilon); \ddot{X}_1 + X_1 = -\dot{\omega}_1 \ddot{X}_0 + X_0^3 - X_1$$

$$O(\varepsilon^2); \ddot{X}_2 + X_2 = -\omega_1 \ddot{X}_1 - \omega_2 \ddot{X}_0 + 3X_1 X_0^2$$

(3) Solving for  $X_0$ :

$$\ddot{X}_0 + X_0 = 0; \quad X_0(0) = a; \quad X_0'(0) = 0; \quad X_0(t) = a \cos t$$

(4) Solving for  $X_1$ :

$$\ddot{X}_1 + X_1 = -\omega_1 \ddot{X}_0 + X_0^3 = +\omega_1 a \cos t + a^3 \cos^3 t \\ = (\omega_1 a - \frac{3}{4} a^3) \cos t - \frac{a^3}{4} \cos 3t$$

Linear Equation with constant coefficients |  $\omega_1 = a^2$

$$a_0 t^n + a_1 t^{n-1} + \dots + a_{n-1} t^1 + a_n t^0 = f(t)$$

Solving the Homogeneous Equation.

$$a_0 \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda^1 + a_n \lambda^0 = 0$$

$$(\lambda^2 + 1) = 0; \quad \lambda_{1,2} = \pm i; \quad K = S = 1$$

Solution to Homogeneous Equation: Summand.

$$t = \sum_{m=1}^M R_m(t) e^{\lambda_m t} \cos \beta_m t + Q_m(t) e^{\lambda_m t} \sin \beta_m t$$

$$X_1(t) = A_1 \cos t + B_1 \sin t$$

Solving the particular Equation: Method for Undetermined Coefficients.

$$X_1 = t^3 e^{\lambda t} [R_m(t) \cos \beta t + Q_m(t) \sin \beta t]$$

$$X_1 = t [A_1 \cos 3t + B_1 \sin 3t]$$

$$\ddot{X}_1 = \cos 3t (6B_1 - 9A_1 t) - 3 \sin 3t (2A_1 + 3B_1 t)$$

$$\ddot{X}_1 + X_1 = \cos(3t)(6B_1 - 8A_1 t) - 2 \sin(3t)(3A_1 + 4B_1 t) = -\frac{a^3}{4} \cos 3t$$

$$B_1 = -\frac{a^3}{24}$$

$$X_1(t) = -\frac{a^3}{24} t \sin 3t$$

General Solution: Homogeneous + Particular

$$X_1(t) = A_1 \cos t + B_1 \sin t + \frac{\alpha^3}{24} t \cos 3t$$

Initial Conditions:  $X_1(0) = 0; \dot{X}_1(0) = 0$

$$X_1(0) = 0 = A_1; \dot{X}_1(0) = 0 = B_1 + \frac{\alpha^3}{24}; B_1 = \frac{\alpha^3}{24}$$

$$X_1(t) = \frac{\alpha^3}{24} \sin t - \frac{\alpha^3}{24} t \cos 3t$$

⑤ Solving for  $X_2$ :

$$\ddot{X}_2 + X_2 = -\omega_1 \ddot{X}_1 - \omega_2 \ddot{X}_0 + 3X_1 X_0^2$$

$$= \frac{\alpha^5}{24} (-\sin t + 6\sin 3t + 9t \cos 3t) + \omega_2 a \cos t$$

$$+ \frac{3\alpha^5}{8} [\sin t - t \cos 3t] \cos^2 t$$

$$\omega_2 = 0$$

⑥ I forgot  $O(\epsilon^3)$ , which at  $\epsilon = 0$  modification necessitates

a time-shift  $\epsilon (1 + \epsilon \omega_1 t + \epsilon^2 \omega_2 t + \epsilon^3 \omega_3 t + \dots)$  and

an perturbation of  $1 + \epsilon X_1 + \epsilon^2 X_2 + \epsilon^3 X_3 + \dots$

Next time, for this example,  $\omega = 1 + \epsilon \omega_1 + \epsilon^2 \omega_2$

$$= 1 + \epsilon a^2.$$

$$\ddot{X} + X + \epsilon h(X, \dot{X}, t) = 0$$

$$7.6.25. X(t) = r(t) \cos(t + \phi(t))$$

$$\dot{X}(t) = -r(t) \sin(t + \phi(t))$$

$$\text{a. } \overset{2\pi}{\int} \epsilon h(X, \dot{X}, t) dt = \epsilon h \sin(t + \phi(t))$$

$$\overset{2\pi}{\int} \epsilon \dot{\phi} dt = \overset{2\pi}{\int} \epsilon h \cos(t + \phi(t)) dt$$

$$b. \langle \mathbf{r} \rangle(t) = \bar{\mathbf{r}}(t) = \frac{1}{2\pi} \int_{t-\pi}^{t+\pi} \mathbf{r}(\tau) d\tau$$

$$\frac{d\langle \mathbf{r} \rangle}{dt} = \frac{d\bar{\mathbf{r}}(t)}{dt} = \frac{1}{2\pi} \frac{d}{dt} (\mathbf{r}(t+\pi) - \mathbf{r}(t-\pi)) = \langle \frac{d\mathbf{r}}{dt} \rangle$$

$$c. \frac{d\langle \mathbf{r} \rangle}{dt} = \langle E h \sin(t+\phi) \rangle = E \langle h(x, \dot{x}, t) \sin t + \phi \rangle \\ = E \langle h(r \cos(t+\phi), -r \sin(t+\phi), t) \sin t + \phi \rangle.$$

$$d. \frac{d\bar{\mathbf{r}}}{dt} = E \langle h(r \cos(t+\phi), -r \sin(t+\phi), t) \sin(t+\phi) \rangle + O(\epsilon^2)$$

$$\frac{d\bar{x}}{dt} = E \langle h(r \cos(t+\phi), -r \sin(t+\phi), t) \sin(t+\phi) \rangle + O(\epsilon^2)$$

$$x = -Ex \sin^2 t \quad 7.6.26 \quad 0 \leq E \ll 1; \quad x = x_0 \text{ @ } t=0$$

$$a. \ddot{x} = \frac{d^2 x}{dt^2} = -Ex \sin^2 t; \quad x \ln x - x = -E \left[ \int \frac{(1 - \cos 2t)}{2} dt \right] dt \\ = -E \left[ \frac{t}{2} - \frac{\sin 2t}{4} + C \right] dt$$

An alternative solution

is a homogeneous plus,

particular, which has

continuous style.

$$= -E \left[ \frac{t^2}{4} + \frac{\cos 2t}{8} \right] + Ct + C$$

$$x = e^{-ET \left[ \frac{t^2}{4} \right]} \log \left[ \frac{t^2}{4} + \frac{\cos 2t}{8} \right]$$

$$x = e$$

$$b. \bar{x}(t) = \frac{1}{2\pi} \int_{t-\pi}^{t+\pi} x(\tau) d\tau = -E\pi \log \left[ \frac{1}{4} (t-\pi)^2 + \frac{\cos 2t}{8} \right]$$

$$x(t) = \bar{x}(t) + O(\epsilon) = \frac{e}{\pi} + O(\epsilon)$$

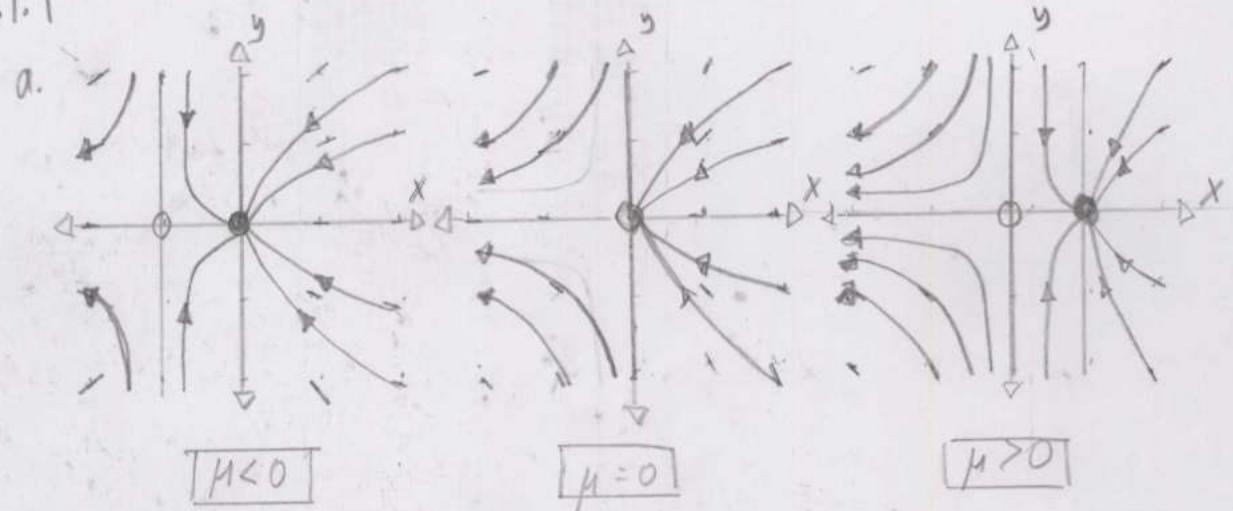
c. The error depends on the exactness and amount of terms.

A product-log function isn't the common method, either.

## Chapter 8: Bifurcations Revisited:

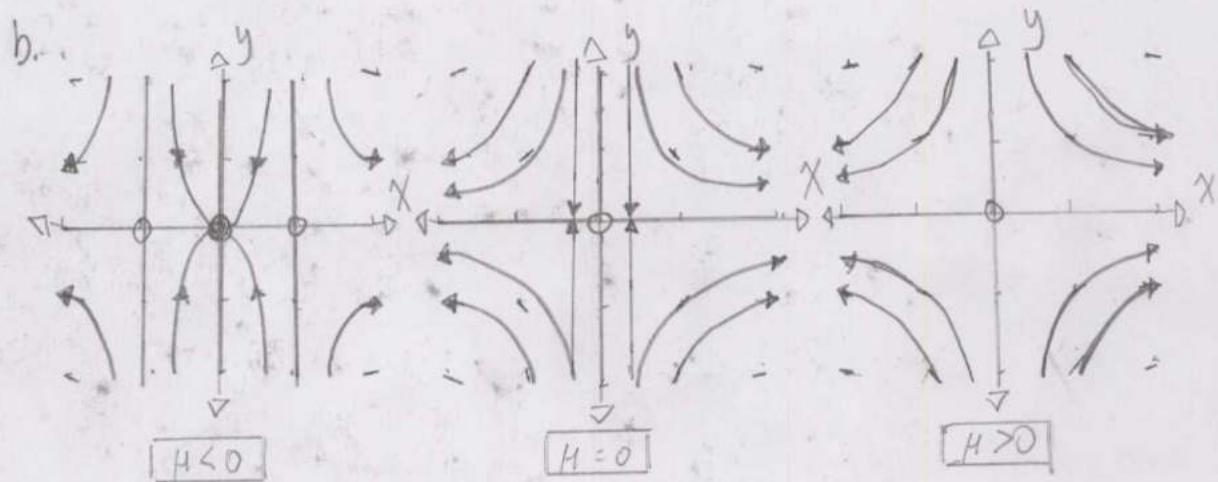
$$\dot{x} = \mu x - x^2 \quad 8.1.1$$

$$\dot{y} = -y$$



$$\dot{x} = \mu x + x^3$$

$$\dot{y} = -y$$



$$\begin{aligned} \dot{x} &= \mu - x^2 & 8.1.2. \text{ Eigenvalues: } \vec{x} = A\vec{x} = 0; A\vec{x} = 0 = \lambda\vec{x}; (A - \lambda)\vec{x} = 0 \\ \dot{y} &= -y \\ &= \begin{pmatrix} -2x - \lambda & 0 \\ 0 & -1 - \lambda \end{pmatrix} = (-2x - \lambda)(-1 - \lambda) = 0 \\ &\lambda_1 = -2x; \lambda_2 = -1 \end{aligned}$$

Fixed Points:  $\dot{x} = 0 = \mu - x^2$

$$\dot{y} = 0 = -y; (x^*, y^*) = (\sqrt{\mu}, 0)$$

Eigenvalues + Fixed Points:  $\lambda_1 = -2\sqrt{\mu}; \lim_{\mu \rightarrow 0} \lambda_1 = 0$

$$\dot{x} = \mu x - x^2 \quad 8.1.3. \text{ Eigenvalues: }$$

$$\dot{y} = -y$$

$$\vec{x} = A\vec{x} = 0; (A - \lambda)\vec{x} = \begin{pmatrix} \mu - 2x - \lambda & 0 \\ 0 & -1 - \lambda \end{pmatrix} \vec{x} = 0$$

$$\lambda_1 = \mu - 2x$$

$$\lambda_2 = -1$$

$$\underline{\text{Fixed Points:}} \quad \dot{x} = 0 = \mu x - x^2 \Rightarrow (x^*, y^*) = (\mu, 0)$$

$$\dot{y} = 0 = -y$$

$$\underline{\text{Eigenvalues + Fixed Points:}} \quad \lambda_1 = \mu - 2x; \lim_{\mu \rightarrow 0} \lambda_1 =$$

$$\lim_{\mu \rightarrow 0} \lambda_1 = \lim_{\mu \rightarrow 0} \mu - 2\mu = 0$$

$$\dot{x} = \mu x + x^3 \quad 8.1.4: \quad \underline{\text{Eigenvalues:}} \quad \vec{x} = A \vec{x} = 0; (A - \lambda) \vec{x} = \begin{pmatrix} \mu + 3x^2 - \lambda_1 & 0 \\ 0 & -1 - \lambda_2 \end{pmatrix} \vec{x}$$

$$\lambda_1 = \mu + 3x^2$$

$$\lambda_2 = -1$$

$$\underline{\text{Fixed Points:}} \quad \dot{x} = 0 = \mu x + x^3 \Rightarrow (x^*, y^*) = (\sqrt{\mu}, 0)$$

$$\dot{y} = 0 = -y$$

$$\underline{\text{Eigenvalues + Fixed Points:}} \quad \lambda_1 = \mu + 3(\sqrt{\mu})^2$$

$$\lim_{\mu \rightarrow 0} \lambda_1 = \lim_{\mu \rightarrow 0} \mu + 3(\sqrt{\mu})^2 \\ = \phi$$

8.1.5: True, since the zero-eigenvalue bifurcation is exemplified by saddle-nodes, transcritical, and pitchfork bifurcations that each have tangential intersections about their nullclines.

$$\dot{x} = y - 2x$$

$$\dot{y} = \mu + x^2 - y$$

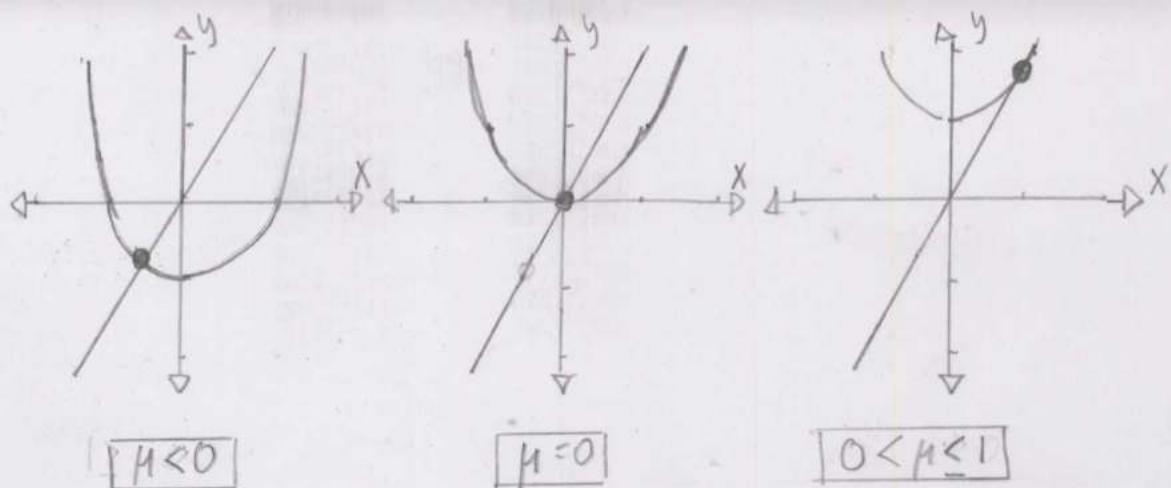
8.1.6:

a. Nullclines:  $\dot{x} = 0 = y - 2x$

$$\dot{y} = 0 = \mu + x^2 - y$$

$$(x^*, y^*) = (1 - \sqrt{1-\mu}, -2(\sqrt{1-\mu} - 1))$$

$$= (1 + \sqrt{1-\mu}, 2(\sqrt{1-\mu} + 1))$$



b. Saddle-node

c. See part A.

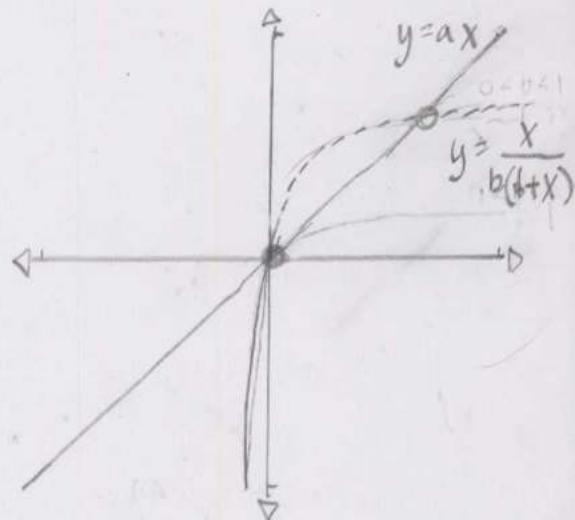
$$\begin{aligned} \dot{x} &= y - ax \\ \dot{y} &= -by + \frac{x}{1+x} \end{aligned}$$

8.1.7.  $y = ax$ ;  $y = \frac{x}{b(1+x)}$

$$ax = \frac{x}{b(1+x)}$$

$$x = 0, \frac{1}{ab} - 1$$

The book shows a ...  
Jacobion method. This  
is an equation, table  
or equation graph. Maybe  
an equation, table, and graph.



Bifurcation Amount	Conditions
2	$ab < 1$
1	$ab = 1$
2	$ab > 1$

Transcritical Bifurcation.

$$\varepsilon \frac{d^2\phi}{dt^2} = -\frac{d\phi}{dt} - \sin\phi + 8\sin\phi\cos\phi$$

$$0, 1, 0, \varepsilon > 0; 8 > 0;$$

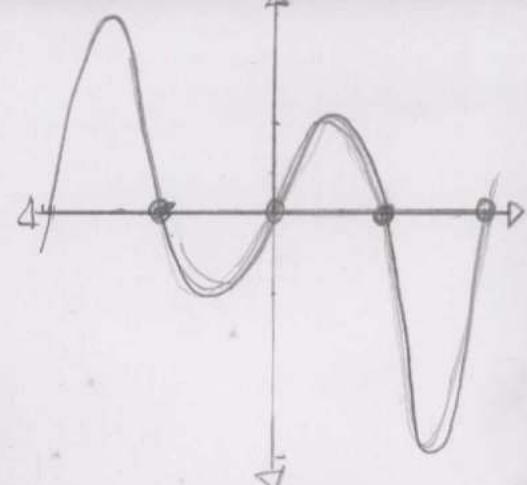
a.)  $\dot{x} = \frac{d\phi}{dt} = y$

$$\dot{y} = \frac{d^2\phi}{dt^2} = \frac{-y + \sin x (8\cos x - 1)}{\varepsilon}$$

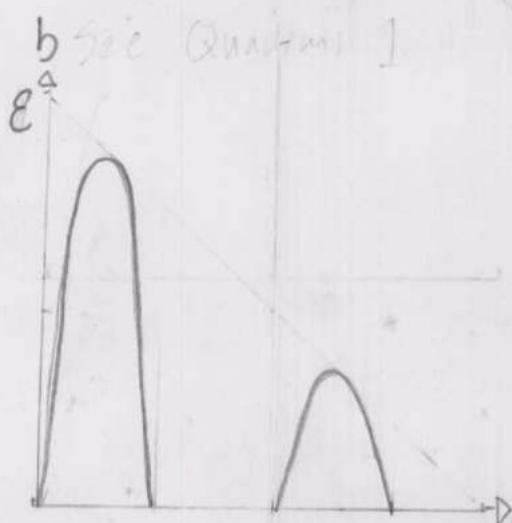
Bifurcations:  $\dot{x} = 0 = y$   
 $\dot{y} = 0 = -\frac{y + \sin x (8\cos x - 1)}{\varepsilon}$

$$y = \sin x (\gamma \cos x - 1)$$

$$x = 0, \arccos\left(\frac{1}{\gamma}\right)$$



b Sec Quadrant 1



Bifurcation Amount	Conditions
2	$0 \leq \gamma \leq 1$
4	$1 < \gamma$

"Pitchfork Bifurcation: Supercritical"

$$\ddot{x} + b\dot{x} - kx + x^3 = 0 \quad 8.1.9. \quad \dot{x} = y$$

$$\ddot{y} = -b\dot{x} + kx - x^3 = -by + kx - x^3$$

Bifurcations:  $\dot{x} = 0 = y$  Subcritical

$$\dot{y} = 0 = -b\dot{y} + kx - x^3$$

$$(x^*, y^*) = (0, 0)$$

$$(\pm \sqrt{k}, 0)$$

Unfixed  
Stable  
Point

Fixed  
Stable  
Point

(  
center

$$\dot{s} = r_s s \left(1 - \frac{r_E}{K_s} \frac{E}{E}\right)$$

8.1.10  $s(t)$  = Average Size of Trees

$E(t)$  = "Energy Reserve"

$B$  = Budworm Population

$r_s, r_E, K_s, E, P > 0$

- First term of  $\dot{s}$  is rate of increase of average tree size
- Second term of  $\dot{s}$  is rate of decrease of average tree size
- First term of  $\dot{E}$  is rate of increase of energy reserve
- Second term of  $\dot{E}$  is rate of decrease of energy reserve
- Third term of  $\dot{E}$  is rate of decrease of energy reserve from budworms.

b. Scaled budworm density  $M = \frac{P_B}{S}$ ;  $\alpha = \frac{E}{K_E}$ ;  $t = k_E T$

$$\dot{S} = r_E S' = r_S S \left(1 - \frac{S}{K_S} \frac{1}{K}\right); S' = \frac{r_S}{r_E} S \left(1 - \frac{S}{K_S} K\right);$$

$$\dot{E} = r_E E' = r_E E \left(1 - \frac{E}{K}\right) - \beta M; E' = E \left(1 - \frac{E}{K}\right) - \beta M$$

$$\text{If } R = \frac{r_S}{r_E}; C = \frac{S}{K_S}; C' = RC \left(1 - \frac{C}{K}\right)$$

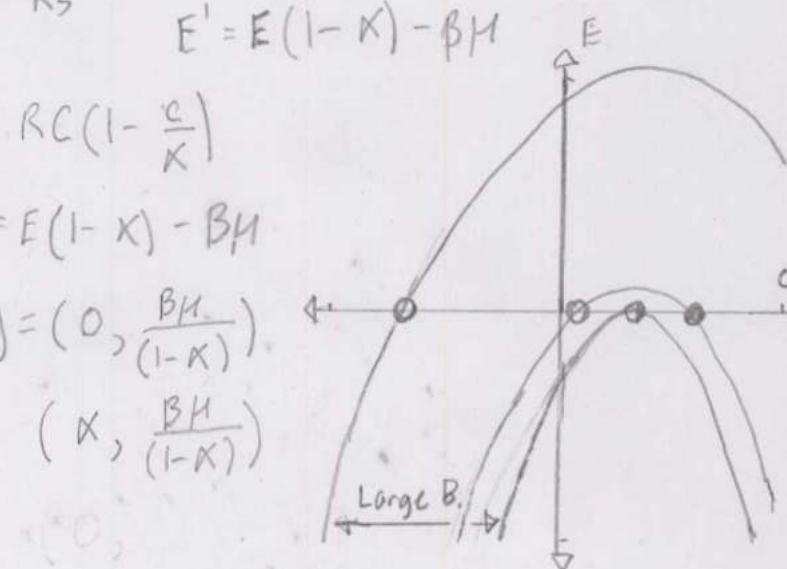
$$E' = E \left(1 - \frac{E}{K}\right) - \beta M$$

c. Nullclines:  $C' = 0 = RC \left(1 - \frac{C}{K}\right)$

$$E' = 0 = E \left(1 - \frac{E}{K}\right) - \beta M$$

$$(C^*, E^*) = (0, \frac{\beta M}{(1-K)})$$

$$(\frac{K}{2}, \frac{\beta M}{(1-K)})$$



d. See Part c.

$$\dot{u} = a(1-u) - uv^2 \quad \text{9.1.11. Bifurcations:}$$

$$\dot{v} = uv^2 - (a+k)v$$

$$\dot{u} = 0 = a(1-u) - uv^2$$

$$\dot{v} = 0 = uv^2 - (a+k)v$$

$$(u^*, v^*) = \left( \frac{a \pm \sqrt{a^2 - 4(a+k)^2}}{2a}, \frac{a + \sqrt{a^2 - 4(a+k)^2}}{2(a+k)} \right)$$

Bifurcation Amount	Conditions
0	$B > \frac{E(1-K)}{M}$
1	$B = \frac{E(1-K)}{M}$
2	$B < \frac{E(1-K)}{M}$

$$0 = a^2 - 4(a+k)^2; (a+k)^2 = \frac{a}{4}; k = -a \pm \frac{a}{2}v$$

$$\dot{\theta}_1 = K \sin(\theta_1 - \theta_2) - \sin \theta_1 \quad \text{9.1.12 a. Fixed Points:} \quad \dot{\theta}_1 = 0 = K \sin(\theta_1 - \theta_2) - \sin \theta_1,$$

$$\dot{\theta}_2 = K \sin(\theta_2 - \theta_1) - \sin \theta_2$$

$$(\theta_1^*, \theta_2^*) = (n_1\pi, n_2\pi) \quad n_1, n_2 \in \mathbb{Z}$$

b. Identity:  $\sin(a-b) = \cos b \sin a - \sin b \cos a$

$$\dot{\theta}_1 = 0 = K [\sin \theta_1 \cos \theta_2 - \sin \theta_2 \cos \theta_1] - \sin \theta_1,$$

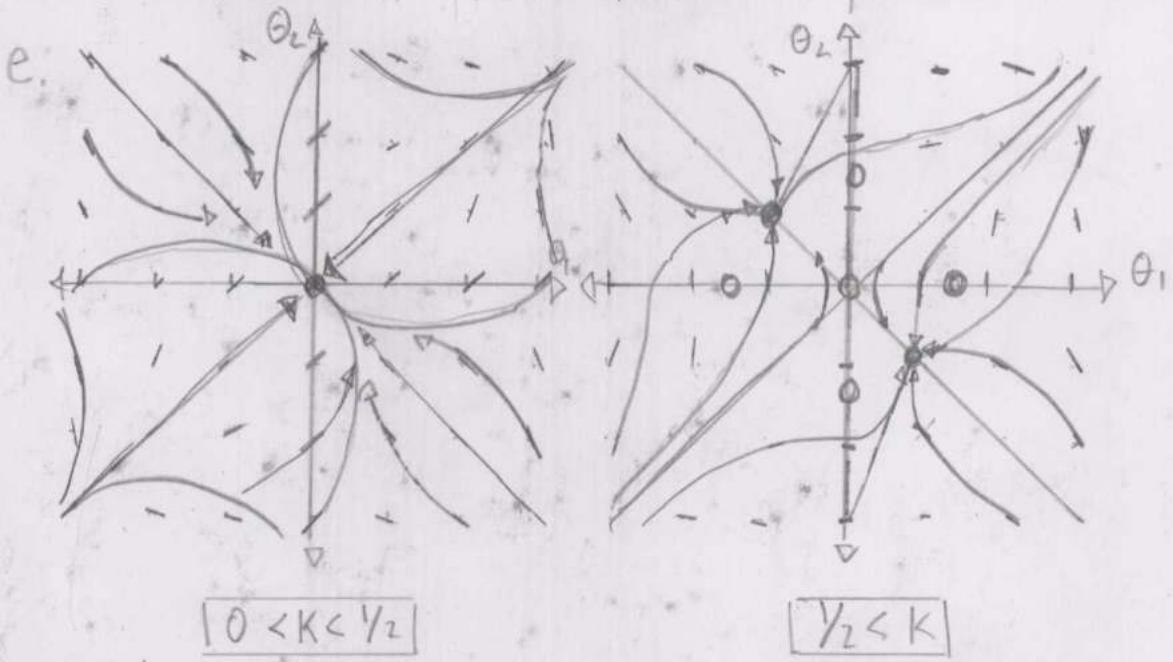
$$\dot{\theta}_2 = 0 = K [\sin \theta_2 \cos \theta_1 - \sin \theta_1 \cos \theta_2] - \sin \theta_2$$

$$0 = 2K [\sin \theta_1 \cos \theta_2 - \sin \theta_2 \cos \theta_1] + \sin \theta_2 - \sin \theta_1$$

$$K = \frac{1}{2} @ (\theta_1, \theta_2) = (n_1\pi, n_2\pi)$$

$$C. \dot{G} = -\frac{\partial V}{\partial \theta_1}; V(\theta_1, \theta_2) = K \cos(\theta_1 - \theta_2) - \cos \theta_1 \\ = K \cos(\theta_2 - \theta_1) - \cos \theta_1$$

d)  $\dot{\theta}_1$  and  $\dot{\theta}_2$  written as  $V(\theta_1, \theta_2)$  imply a Lyapunov function. Gradient flows have no periodic orbits.



$$\dot{n} = GnN - Kn \quad 8.1.B$$

$$\dot{N} = -GnN - fN + p$$

a.  $N(t) = \#$  of excited atoms

$n(t) = \#$  of photons in laser field

$G$  = Gain coefficient for Stimulated Emission

$K$  = Decay rate due to loss of photons  
by mirror transmission

$F$  = Decay rate for spontaneous emission

$p$  = pump strength

If  $\dot{n} = GnN - kn$ , then  $\frac{G^2}{K^2} \dot{n} = \frac{G^2 n N}{K^2} - \frac{G n}{K}$

$\dot{N} = -GnN - FN + p$ , then  $\frac{G^2}{K^2} \dot{N} = -\frac{G^2 n N}{K^2} - \frac{G}{K^2} (fN + p)$

If  $T = \frac{k^2}{G} t$ ,  $x = \frac{Gn}{K}$ ,  $y = \frac{GN}{K}$ ,  $a = \frac{f}{K}$ ,  $b = \frac{pG}{K^2}$

$$\dot{x} = x(y-1), \quad \dot{y} = -xy - ay + b$$

$$\text{b. Fixed Points: } \dot{x} = 0 = x(y-1)$$

$$\dot{y} = 0 = -xy - ay + b$$

$$(x^*, y^*) = (0, b/a)$$

$$= (b-a, 1)$$

$$\text{Classification: } \dot{\vec{x}} = A \vec{x}; \quad A = \begin{pmatrix} y-1 & -x \\ -y & -(x+a) \end{pmatrix}$$

$$(A_{(0,a/b)} - \lambda) = \begin{pmatrix} \frac{b}{a}-1-\lambda_1 & 0 \\ -b/a & -a-\lambda_2 \end{pmatrix} = 0$$

$$(A_{(b-a,1)}, -\lambda) = \begin{pmatrix} -\lambda_1 & b-a \\ -1 & -b-\lambda_2 \end{pmatrix} = (\lambda_1 = \frac{b}{a}-1; \lambda_2 = -a)$$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4(b-a)}}{2}$$

$$\Delta = \frac{(b^2 - (b-4)b - 4a^2)}{4}$$

$$\Delta = a(b-1)$$

$$\Gamma = \frac{b}{a}(-1-a)$$

$$\Gamma^2 - 4\Delta \geq 0 @ a > b$$

"Stable node"

$$\Gamma = -b$$

$$\Gamma^2 - 4\Delta > 0 @ b < \frac{4}{3}a$$

"Stable spiral when b > 0"

"Unstable node when b < 0"

$$\Gamma^2 - 4\Delta > 0 @ a < b$$

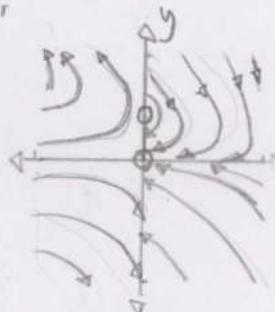
"Unstable node"

$$\Gamma^2 - 4\Delta < 0 @ b > \frac{4}{3}a$$

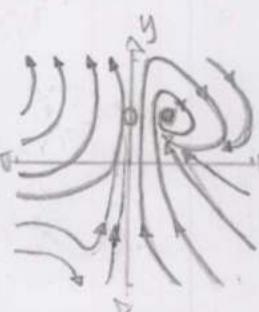
"Unstable spiral when b < 0"

"Stable node when b > 0"

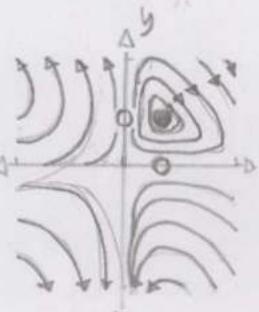
C<sub>r</sub>



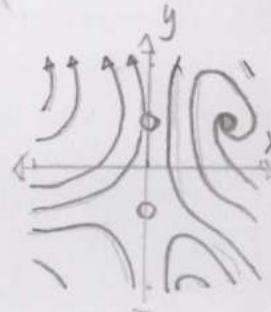
$$[a=0 \quad b=0]$$



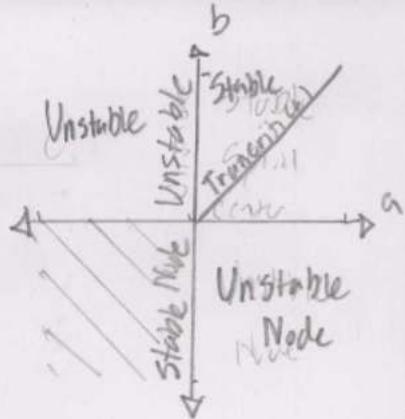
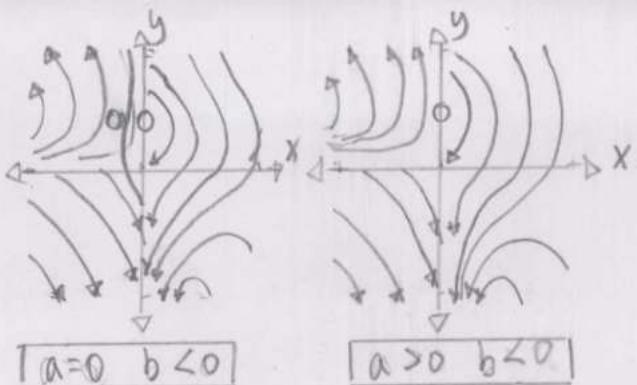
$$[a=0 \quad b>0]$$



$$[a>0 \quad b=0]$$



$$[a>0 \quad b>0]$$



d. Upper - Right page.

$$\dot{x}_1 = -x_1 + F(I - b x_2)$$

$$\dot{x}_2 = -x_2 + F(I - b x_1)$$

Q.1.14. If  $F(x) = 1/(1+e^{-x})$ : Gain Function

$I$ : Strength of the Stimulus

$b$ : Strength of the Mutual Antagonism

a. Nullclines:  $\dot{x}_1 = 0 = -x_1 + F(I - b x_2)$

$$= -x_1 + \frac{1}{1+e^{-I+b x_2}}$$

$$\dot{x}_1 = 0 = -x_2 + F(I - b x_1)$$

$$= -x_2 + \frac{1}{1+e^{-I+b x_1}}$$

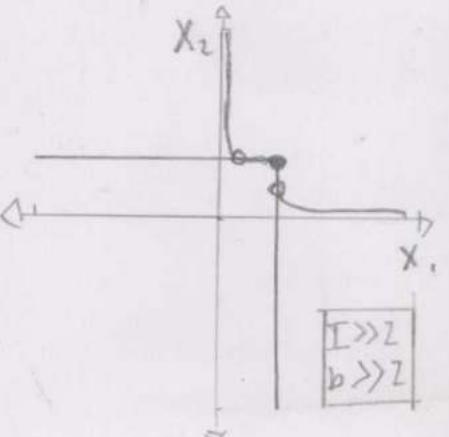
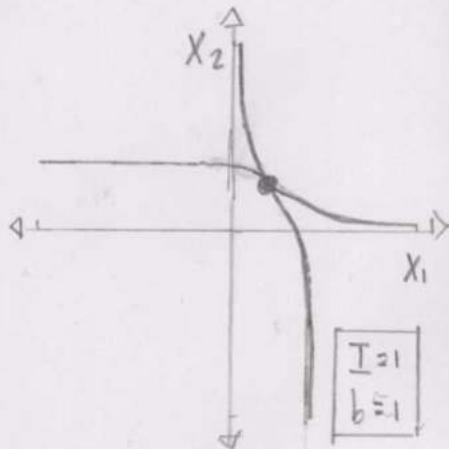
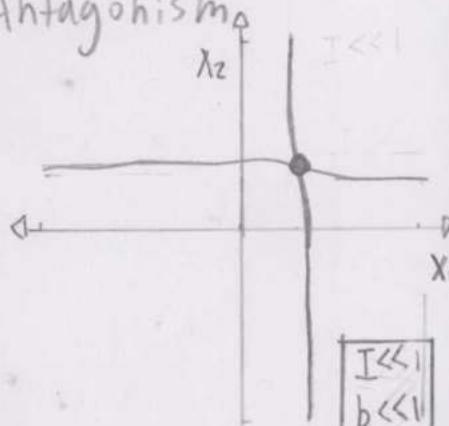
$$x_2 = I + \log\left(\frac{1-x_1}{x_1}\right)$$

$$x_2 = \frac{1}{1+e^{-I+b x_1}}$$

b. If  $x_1^* = x_2^* = x^*$ , then  $\dot{x}_1 = -x_1 + \frac{1}{1+e^{-I+b x^*}}$

$$\dot{x}_2 = -x_2 + \frac{1}{1+e^{-I+b x^*}}$$

$$\dot{x}_1 = \dot{x}_2 = x^*$$



$$C. \lim_{b \rightarrow \infty} x_1 = \lim_{b \rightarrow \infty} -x_2 + \frac{1}{1+e^{-\frac{x_1}{1+b}}} = -x_2$$

d. See part a; supercritical pitchfork because of the unstable center.

Q. 1.15.

$$\dot{n}_A = (p+n_A)n_{AB} - n_A n_B \quad \text{where } n_{AB} = 1 - (p+n_A) - n_B$$

$$\dot{n}_B = n_B n_{AB} - (p+n_A)n_B$$

a. The first term of  $\dot{n}_A$  fits a constant ( $p$ ) and changing ( $n_A$ ) population of A-B.

The second term of  $\dot{n}_A$  are the decreasing populations from A-B interaction.

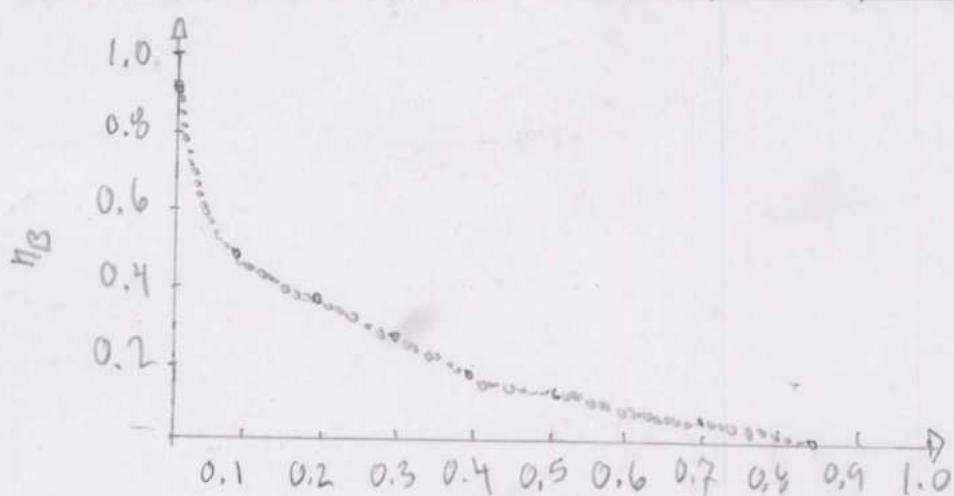
The first term of  $\dot{n}_B$  fits an increasing population from A-B interaction.

The second term of  $\dot{n}_B$  fits a constant ( $p$ ) and changing ( $n_A$ ) population of A-B.

$$b. n_B(0) = 1-p; n_A(0) = n_{AB}(0) = 0$$

Numerical Integration  $\Delta t = 0.001 \quad p = 0.15 \quad k_{x1} = f(x_n) \Delta t$

$n_A$	$n_B$	$n_{AB}$	$k_{x1}$	$k_{x2}$	$k_{x3}$	$k_{x4}$	$K_{x2} = f(x_n + \frac{1}{2}k_{x1}) \Delta t$	$K_{x3} = f(x_n + \frac{1}{2}k_{x2}) \Delta t$	$K_{x4} = f(x_n + k_{x3}) \Delta t$
0.85	0.85	0	-	-	-	-	-	-	-
$\vdots$	$\vdots$	$\vdots$							
0.85	0.00	0.0001	0.00	-0.0001	0.00	-0.0001	0.00	-0.0001	0.00



$$n_{x+1} = n_x + \frac{(k_1 + 2k_2 + 2k_3 + k_4)}{6}$$

Q. 2.1  $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = \alpha$   $\frac{n_A}{n_B}$   
Van der Pol oscillator:

Fixed Points:  $\dot{x} = y$

$$\dot{y} = -\mu(x^2 - 1)y + (\alpha - x)$$

$$(x^*, y^*) = (\alpha, 0)$$

$$\text{Eigenvalues: } (A - \lambda) \vec{x} = 0; A - \lambda = \begin{pmatrix} -\lambda & 1 \\ -2\mu xy - 1 & -\mu(x^2 - 1) - \lambda \end{pmatrix} = P \begin{pmatrix} -\lambda & 1 \\ -2\mu xy - 1 & -\mu(x^2 - 1) - \lambda \end{pmatrix}$$

$$= (\lambda)(\mu(x^2 - 1) + \lambda) + 2\mu xy - 1$$

$$A_{(a,0)} = \lambda(\mu(a^2 - 1) + \lambda) = 0$$

$$\lambda_1 = 0; \lambda_2 = -\mu(a^2 - 1)$$

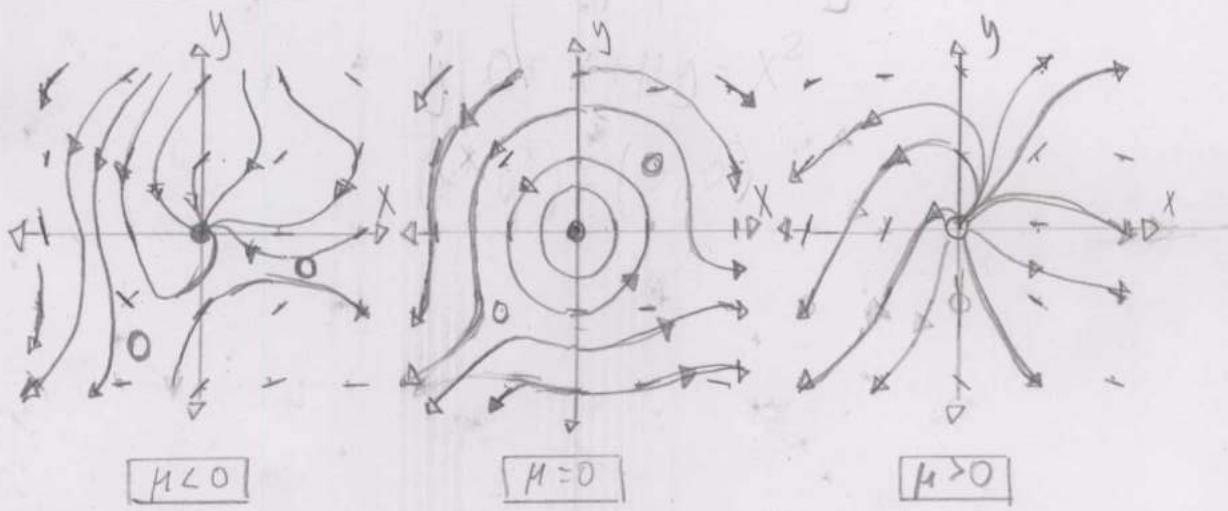
$$\Delta = 0; \Gamma = -\mu(a^2 - 1)$$

$$\Gamma^2 - 4\Delta > 0$$

Hopf Bifurcations: The phase plot changes when the sign of  $\mu$  becomes positive, zero, or negative, in addition to,  $a = \pm 1$ .

$$\dot{x} = -y + \mu x + xy^2 \quad 8.23.$$

$$\dot{y} = x + \mu y - x^2$$



Pitchfork; "Super-critical"

8.2.4.

$$a. r = \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2} = \sqrt{\frac{\dot{x}^2 + \dot{y}^2}{x^2 + y^2}} = \frac{(-y + \mu x + xy^2)y + x(x + \mu y - x^2)}{\sqrt{x^2 + y^2}}$$

$$\dot{\theta} = \frac{d}{dt} \arctan \frac{y}{x} = \frac{\dot{y}x - y\dot{x}}{x^2 + y^2} = \frac{x(x + \mu y - x^2) - y(-y + \mu x + xy^2)}{x^2 + y^2}$$

$$b. \text{ If } r \ll 1, \text{ then } \dot{\theta} \approx \frac{x^2 + y^2}{x^2 + y^2} = 1$$

$$\text{and } \dot{r} = \frac{x^2 - y^2 + 2\mu xy + xy^3 - x^3}{\sqrt{x^2 + y^2}}$$

$$x^2 - y^2 + 3xy^2 + 3y^3 + x^3 \\ x\left(\frac{x^2 - y^2}{2y^2} + \frac{3x^2}{3y^2}\right)$$

$$C, \dot{r} = 0 \approx \mu r + \frac{1}{3}r^3; r = \sqrt{-8H}$$

$$r = \sqrt{-8H}$$

IF  $\mu < 0$ , then  $r \in \mathbb{R}$ , and if  $\mu > 0$   
 $r \in \mathbb{I}$

$$\begin{aligned} \dot{x} &= y + \mu x \\ \dot{y} &= -x + \mu y - x^2 y \end{aligned}$$

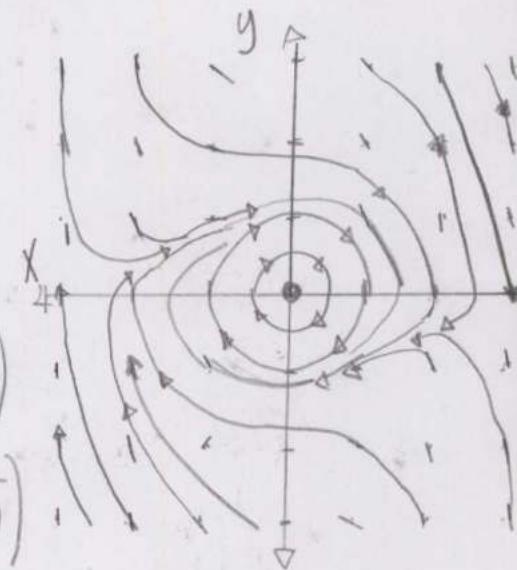
$$8.2.5. \text{ Fixed Points: } \dot{x} = 0 = y + \mu x$$

$$\dot{y} = 0 = -x + \mu y - x^2 y$$

$$(x^*, y^*) = (0, 0)$$

$$= \left( -\sqrt{\frac{\mu+1}{\mu}}, \sqrt{\mu(\mu^2+1)} \right)$$

$$= \left( \sqrt{\frac{\mu+1}{\mu}}, -\sqrt{\mu(\mu^2+1)} \right)$$



Pitchfork: Subcritical.

$$\begin{aligned} \dot{x} &= \mu x + y - x^3 \\ \dot{y} &= -x + \mu y - 2y^3 \end{aligned}$$

$$8.2.6. \text{ Fixed Points: } \dot{x} = 0 = \mu x + y - x^3$$

$$\dot{y} = 0 = -x + \mu y - 2y^3$$

$$(x^*, y^*) = (0, 0)$$

Pitchfork Supercritical

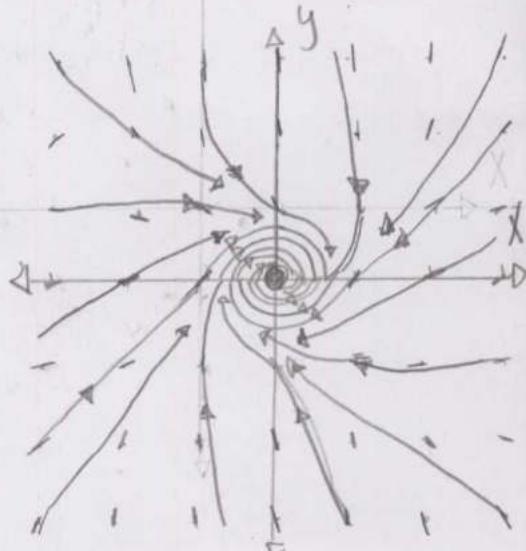
$$\begin{aligned} \dot{x} &= \mu x + y - x^2 \\ \dot{y} &= -x + \mu y - 2x^2 \end{aligned}$$

$$8.2.7. \text{ Fixed Points: } \dot{x} = 0 = \mu x + y - x^2$$

$$\dot{y} = 0 = -x + \mu y - 2x^2$$

$$(x^*, y^*) = (0, 0)$$

$$= \left( \frac{\mu^2 + 1}{m-2}, \frac{(2m+1)(m^2+1)}{(m-2)^2} \right)$$



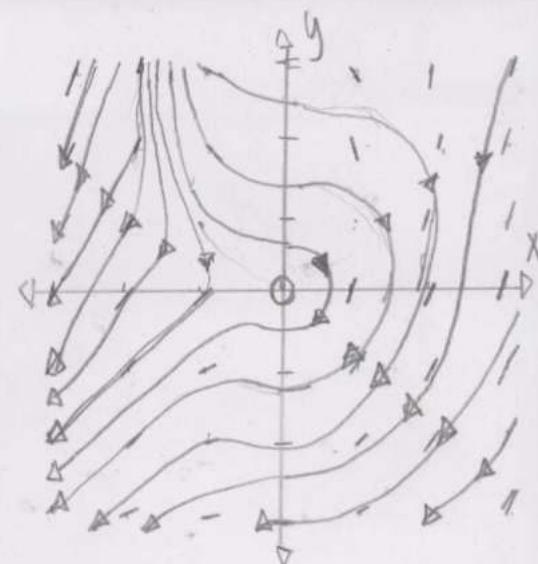
$$\dot{x} = x[x(1-x)-y] \quad \text{"Pitchfork: Supercritical"}$$

$$\dot{y} = y(x-a) \quad \text{a. } \underline{\text{Nullclines}}: \dot{x} = 0 \Rightarrow x[x(1-x)-y]$$

$$\dot{y} = 0 = y(x-a)$$

$$x=0; y=0$$

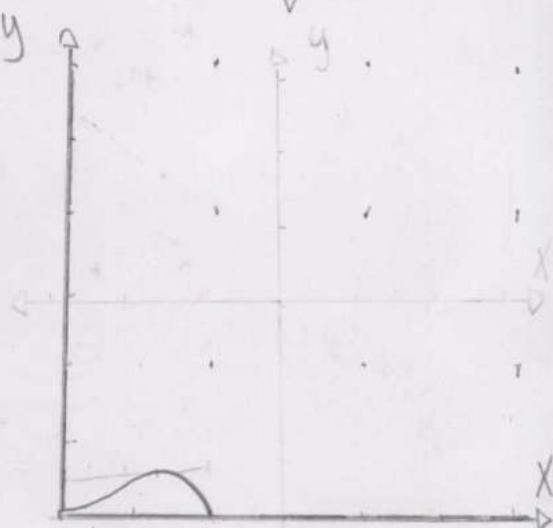
$$x=a; y=x^2(1-x)$$



$$\text{b. Fixed Points: } \dot{x}=0, \dot{y}=0$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2x-3x^2 & -x \\ y & x-a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A_{(0,0)} - \lambda = \begin{bmatrix} -\lambda & 0 \\ 0 & -(a+\lambda) \end{bmatrix}$$



$$\lambda_1 = 0, \lambda_2 = -a$$

$$\Delta = 0; \tau = -a; \tau^2 - 4\Delta > 0 \quad \text{"stable spiral"}$$

$$A_{(1,0)} - \lambda = \begin{bmatrix} -(\lambda+1) & -1 \\ 0 & 1-(a+\lambda) \end{bmatrix}$$

$$\lambda_1 = -1, \lambda_2 = a-1$$

$$\Delta = 1-a; \tau = a-2; \tau^2 - 4\Delta = a^2 - 6a$$

If  $a > 6$ , line of unstable fixed points

If  $a < 6$ , saddle node.  $a^2 - 6a$

If  $a = 6$ , unstable star / degenerate node.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2x-\lambda & -xa \\ a-x & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \lambda = \frac{1}{2}(a+\lambda \pm \sqrt{a^2 - 6a})$$

$$A_{(a, a-a^2)} - \lambda = \begin{bmatrix} 2a-3a^2-\lambda & -a \\ a^2-a^2 & -\lambda \end{bmatrix}$$

$$\lambda_1 = \frac{1}{2} (-\sqrt{9a-8} a^{3/2} - 3a^2 + 2a)$$

$$\lambda_2 = \frac{1}{2} (\sqrt{9a-8} a^{3/2} - 3a^2 + 2a)$$

$$\Delta = a^2 - a^3; \quad \Gamma = 2a - 3a^2$$

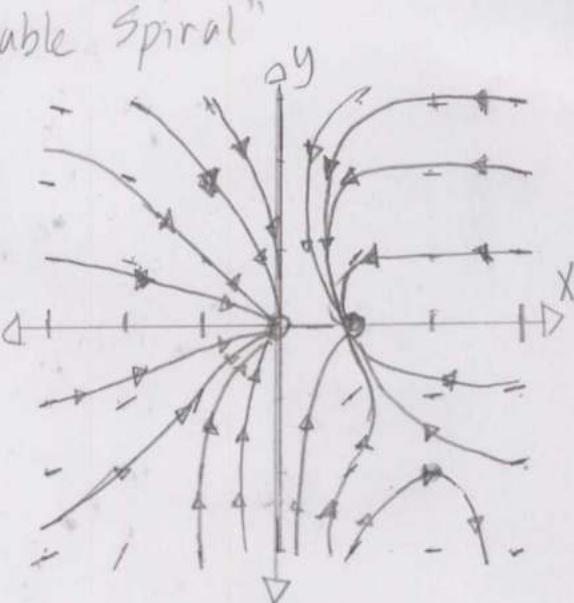
$\Gamma^2 - 4\Delta > 0$  "Stable Spiral"

c) IF  $a > 1$ , phase portrait.

d)

$a$	Bifurcations
$< 1/2$	2
$= 1/2$	3
$> 1/2$	3

Pitchfork: Supercritical Hopf



e) At Hopf Bifurcation  $(\frac{1}{2}, \frac{1}{4})$ ,

$$\lambda_1 = \frac{1}{2} (-\sqrt{9/2 - 8} (\frac{1}{2})^{3/2} - 3(\frac{1}{2})^2 + 2(\frac{1}{2}))$$

$$= \frac{1}{2} \left( -\frac{35}{100} \left( -\frac{7}{2} \right)^{1/2} - \left( \frac{3}{4} \right) + 1 \right)$$

$$= \frac{1}{8} + \frac{49}{80}$$

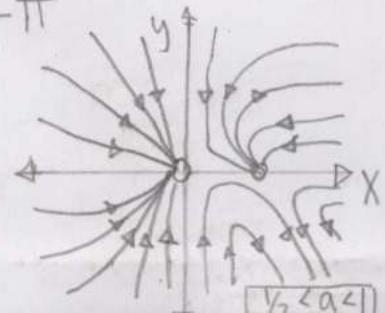
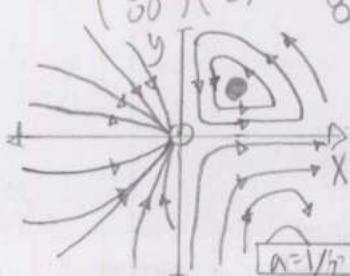
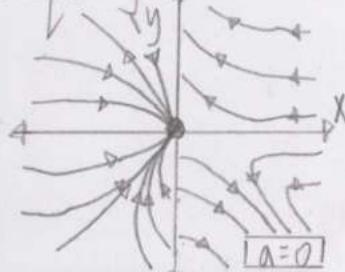
$$\lambda_2 = \frac{1}{2} (\sqrt{9/2 - 8} (\frac{1}{2})^{3/2} - 3(\frac{1}{2})^2 + 2(\frac{1}{2}))$$

$$= \frac{1}{2} \left( -\frac{35}{100} \left( \frac{7}{2} \right)^{1/2} - \frac{3}{4} + 1 \right)$$

$$= -\frac{1}{8} - \frac{49}{80}$$

Frequency  $= 2\pi \omega a = 2\pi \left( \frac{49}{80} \right) \left( \frac{1}{2} \right) = \frac{49}{80}\pi$

f)



$$\dot{x} = x \left( b - x - \frac{y}{1+x} \right) \quad 8.2.9, \quad x, y \geq 0; \quad a, b > 0$$

$$\dot{y} = y \left( \frac{x}{1+x} - ay \right)$$

a. Nullclines:  $\dot{x} = 0 = x \left( b - x - \frac{y}{1+x} \right)$

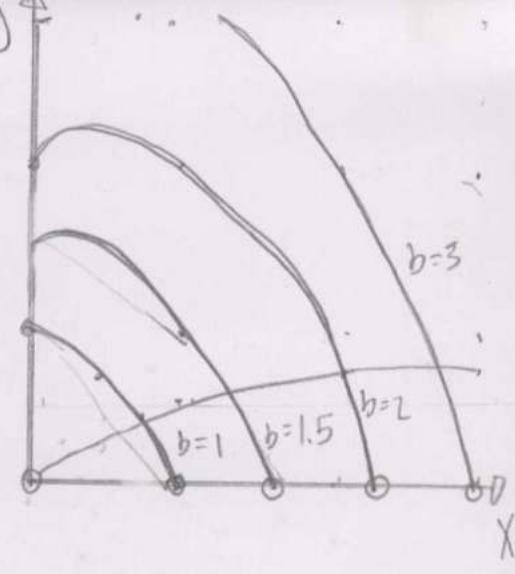
$$\dot{y} = 0 = y \left( \frac{x}{1+x} - ay \right)$$

$$y = 0; \quad x = 0$$

$$y = (1+x) * (b-x)$$

$$y = \frac{x}{a(1+x)}$$

part  
Feedback



b. A graphical argument for the fixed point

$x^* > 0, y^* > 0$  for all  $a, b > 0$  displayed in

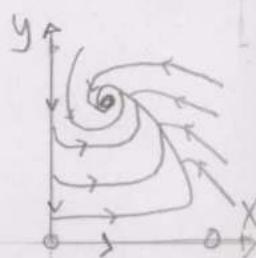
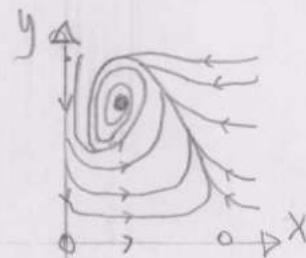
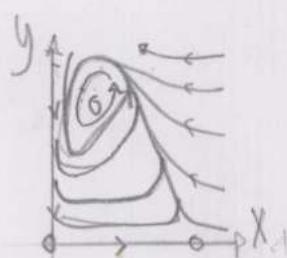
part a, becomes an integer-solution problem.

c. If  $a_c = \frac{4(b-2)}{b^2(b+2)}$ , then

and  $b < 2, b = 2$ , or  $b > 2$ , then

d. Phase Portrait:

$a = a_c$	Number of Bifurcations
$< \frac{4(b-2)}{b^2(b+2)}$	3 "unstable"
$= \frac{4(b-2)}{b^2(b+2)}$	3 "stable"
$> \frac{4(b-2)}{b^2(b+2)}$	3 "stable"



$$\dot{x} = B - x - \frac{xy}{1+qx^2} \quad 8.2.10,$$

$$\dot{y} = A - \frac{xy}{1+qx^2}$$

