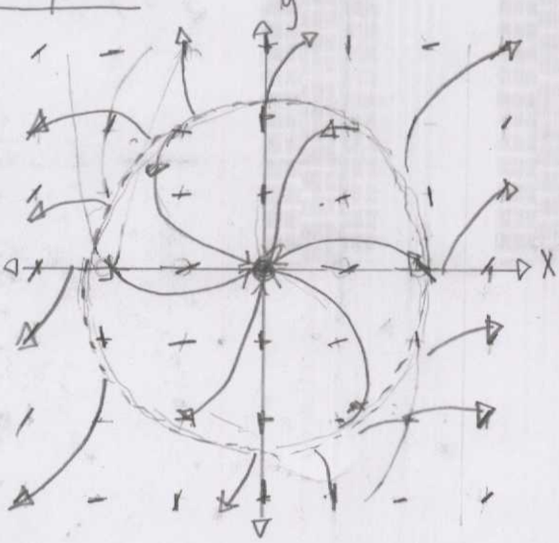
Chapter 7: Limit Cycles:

$\dot{r} = r^3 - 4r$ 7.1.1

$\dot{\theta} = 1$



$$\frac{dy}{dx} = \frac{dy}{dx}\frac{d\theta}{d\theta} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r\cos\theta + \sin\theta \frac{dr}{d\theta}}{-r\sin\theta + \cos\theta \frac{dr}{d\theta}}$$

Where $r = \sqrt{x^2+y^2}$; $\theta = \arctan \frac{y}{x}$

and $\frac{dr}{d\theta} = (\sqrt{x^2+y^2})^3 - 4\sqrt{x^2+y^2}$

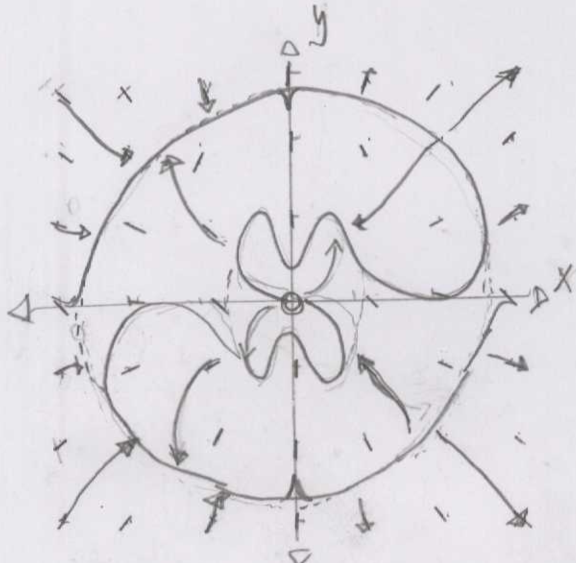
$\dot{r} = r(1-r^2)(9-r^2)$ 7.1.2

$\dot{\theta} = 1$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + \sin\theta \frac{dr}{d\theta}}{-r\cos\theta + \cos\theta \frac{dr}{d\theta}}$$

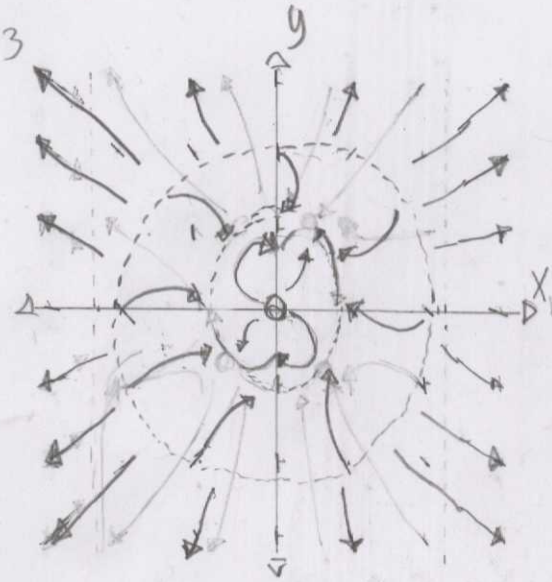
Where $r = \sqrt{x^2+y^2}$; $\theta = \arctan\left(\frac{y}{x}\right)$

and $\frac{dr}{d\theta} = r(1-r^2)(9-r^2)$



$\dot{r} = r(1-r^2)(4-r^2)$ 7.1.3

$\dot{\theta} = 2 - r^2$



$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + \sin \frac{dr}{d\theta}}{r\sin\theta + \cos\theta \frac{dr}{d\theta}}$$

Where $r = \sqrt{x^2+y^2}$; $\theta = \arctan\left(\frac{y}{x}\right)$

and $\frac{dr}{d\theta} = \frac{r(1-r^2)(4-r^2)}{2-r^2}$

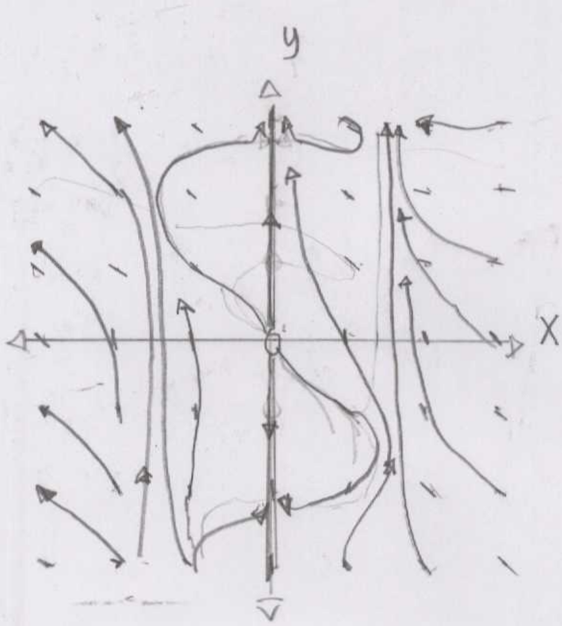
$\dot{r} = r\sin r$ 7.1.4

$\dot{\theta} = 1$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + \sin\theta \frac{dr}{d\theta}}{-r\cos\theta + \cos\theta \frac{dr}{d\theta}}$$

Where $r = \sqrt{x^2+y^2}$; $\theta = \arctan\left(\frac{y}{x}\right)$

and $\frac{dr}{d\theta} = r\sin r$



$\mathring{r} = r(1-r^2)$ 7.1.5. $x = r\cos\theta;\ y = r\sin\theta;$

$\mathring{\theta} = 1$

$$\mathring{x} = \frac{d}{dt} r\cos\theta = \mathring{r}\cos\theta - r\sin\theta\,\mathring{\theta}$$

$$= r(1-r^2)\cos\theta - r\sin\theta = x(1-x^2-y^2) - y$$

$$= x - x^3 - xy^2 - y = x - y - x(x^2+y^2)$$

$$\mathring{y} = \frac{d}{dt} r\sin\theta = \mathring{r}\sin\theta + r\cos\theta\,\mathring{\theta}$$

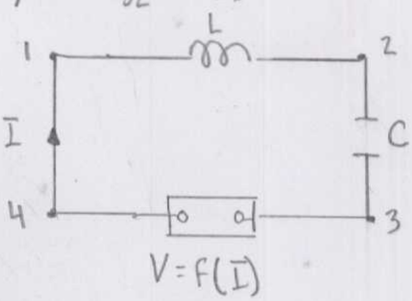
$$= r(1-r^2)\sin\theta - x = y(1-x^2-y^2) + x$$

$$= x + y - y(x^2+y^2)$$

$\mathring{V} = -I/C$ 7.1.6.

$V = L\mathring{I} + f(I)$

a) $V = V_{32} = -V_{23}$



$$V_{41} - V_{12} - V_{23} - V_{34} = 0$$

$$f(I)\ V_I - V_L - V_C - V_{F_C} = 0$$

$$V_I = V_L + V_C + V_F$$

$$= L\frac{dI}{dt} + \frac{I}{C} + f(I)$$

$$= L\frac{dI}{dt} + f(I) - \frac{I}{C}$$

$$= V + \mathring{V}$$

b. If $x = \sqrt{L}\,I;\ w = \sqrt{C}\,V;\ \tau = t/\sqrt{LC}$, and $F(x) = f(x/\sqrt{L})$ and $x = I$

then $\mathring{V} = -I/C = \dfrac{\frac{dw}{\sqrt{C}}}{d\tau\sqrt{LC}} = -\dfrac{x}{\sqrt{L}}\left(\dfrac{1}{C}\right);\ \dfrac{dw}{d\tau} = -x$

and $V = L\mathring{I} + f(I)$, $\dfrac{w}{\sqrt{C}} = L\dfrac{dx}{dt}\left(\dfrac{1}{\sqrt{L}}\right) + f(x/\sqrt{L})$

$$w = \sqrt{LC}\,\frac{dx}{d\tau\sqrt{LC}} + \sqrt{C}\,f(x/\sqrt{L})$$

$$\frac{dx}{d\tau} = w - \mu F(x)\ ;\text{ where } \mu = \sqrt{C}$$

$\mathring{r} = r(4-r^2)$ 7.1.7.

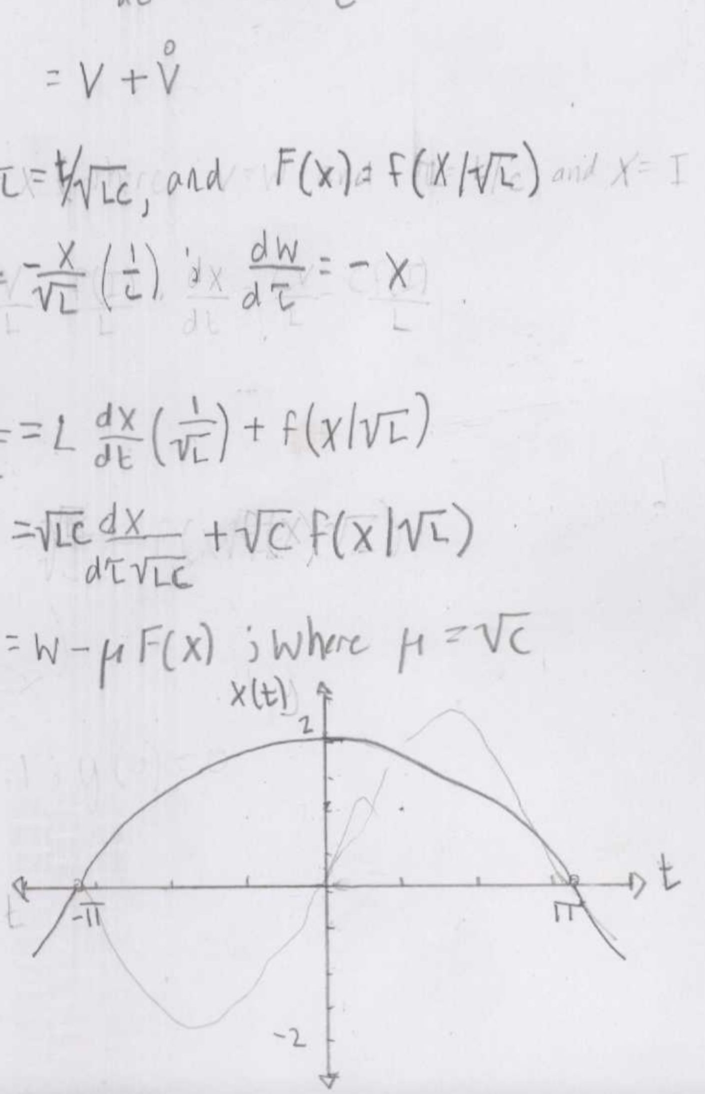
$\mathring{\theta} = 1$

$x(t) = r(t)\cos\theta(t);$

$x(0) = 0.1;\ y(0) = 0$

$r(t) = \displaystyle\int \frac{dr}{r(4-r^2)} = -2\ln(4-r^2) + C$

$\theta(t) = t\ ;\ r(t) = \sqrt{4 - Ce^{t/2}}$



$\ddot{x} + a\dot{x}(x^2+\dot{x}^2-1) + x = 0$

7.1.3:

a. $\dot{u} = \dot{x} = v$

$\dot{v} = \ddot{x} = -a\dot{x}(x^2+\dot{x}^2-1) - x = -av(u^2+v^2-1) - u$

Fixed Points: $\dot{u} = \dot{x} = 0$

$\dot{v} = -av(u^2+v^2-1) - u = 0$

$(u^*, v^*) = (0, 0)$

$\dot{\vec{U}} = A \cdot \vec{U}; \; A = \begin{pmatrix} 0 & 0 \\ -2vu-1 & -a(u^2+3v^2-1) \end{pmatrix}$

$\lambda_1 = 0; \; \lambda_2 = -a(u^2+3v^2-1)$

$\Delta = 0; \; \tau = -a(u^2+3v^2-1)$

$\tau^2 - 4\Delta > 0$ "Non-isolated Fixed Point"

b. $\dot{r} = \frac{(u\dot{u} + v\dot{v})}{r} = \frac{(r\cos\theta \cdot r\sin\theta + r\sin\theta \cdot (-av(u^2+v^2-1)-u))}{r}$

$= \frac{(r^2\cos\theta\sin\theta - ar^2\sin\theta\sin\theta(r^2\cos^2\theta + r^2\sin^2\theta - 1) - r^2\cos\theta\sin\theta}{r}$

$= -ar\sin^2\theta(r^2-1)$

$\dot{\theta} = \frac{(\dot{v}u - \dot{u}v)}{r^2} = \frac{(-a \cdot r\sin\theta(r^2-1) - r\cos\theta)r\cos\theta - r^2\sin^2\theta}{r^2}$

$= \frac{-ar^2\sin\theta\cos\theta(r^2-1) - r^2\cos^2\theta - r^2\sin^2\theta}{r^2}$

$= -a\sin\theta\cos\theta(r^2-1) - 1$

Amplitude: $-ar = -a\sqrt{u^2+v^2} = -a\sqrt{x^2+\dot{x}^2}$

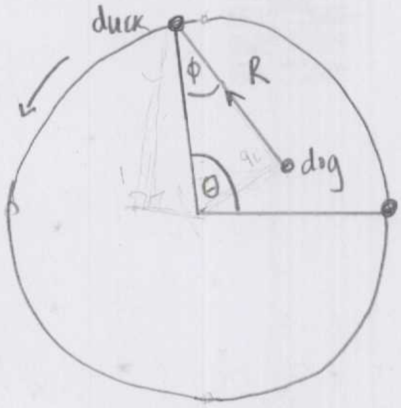
Period: $\frac{2\pi}{|T|} = 2\pi$

c. Stable limit cycle because larger a values generate a standard and periodic trajectory.

d. The limit cycle is unique because solutions containing a values and c initial conditions have many solutions, for similar initial conditions.

$\frac{dR}{d\theta}$, $\frac{d\phi}{d\theta}$

7.1.9. a.

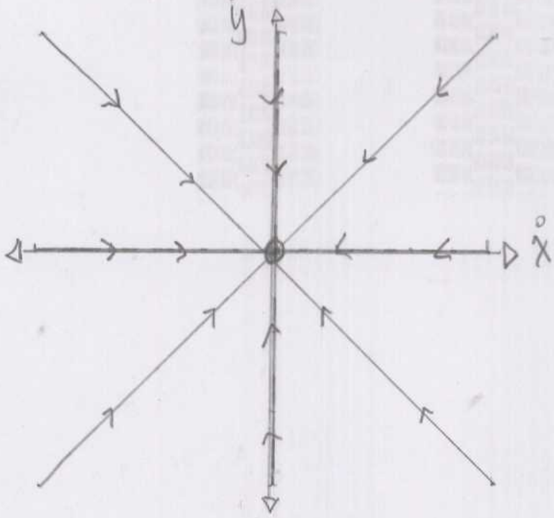


Duck: $\langle x, y \rangle = \langle r\cos k\theta, r\sin k\theta \rangle$

Dog: $\langle x, y \rangle =$

$V = x^2 + y^2$ 7.2.1: $\dot{x} = -\nabla V$; $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = -\begin{bmatrix}dx\\ dy\end{bmatrix} V$; $\dot{x} = -\int x^2 = -\frac{x^3}{3}$

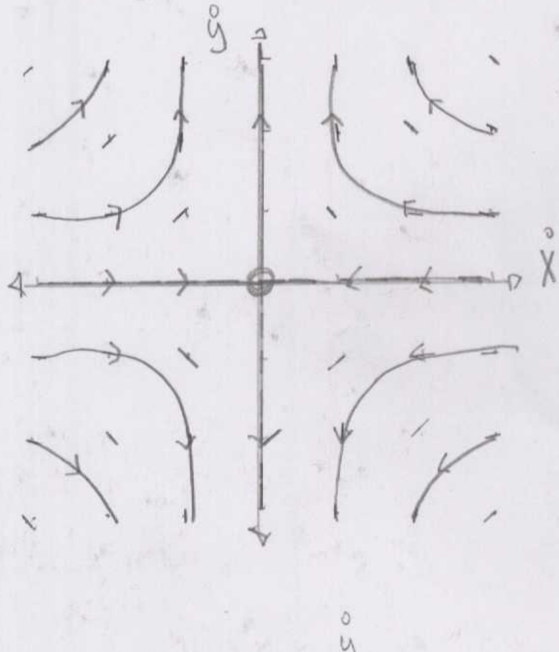
$\dot{y} = -\int y^2 = -\frac{y^3}{3}.$



$V = x^2 - y^2$ 7.2.2. $\dot{x} = -\nabla V$; $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = -\begin{bmatrix}dx\\ dy\end{bmatrix} V$; $\dot{x} = -\int x^2 dx = -\frac{x^3}{3} + c$

$\dot{y} = +\int y^2 dx = +\frac{y^3}{3} + c$

$V = e^x \sin y$ 7.2.3. $\dot{x} = -\nabla V$



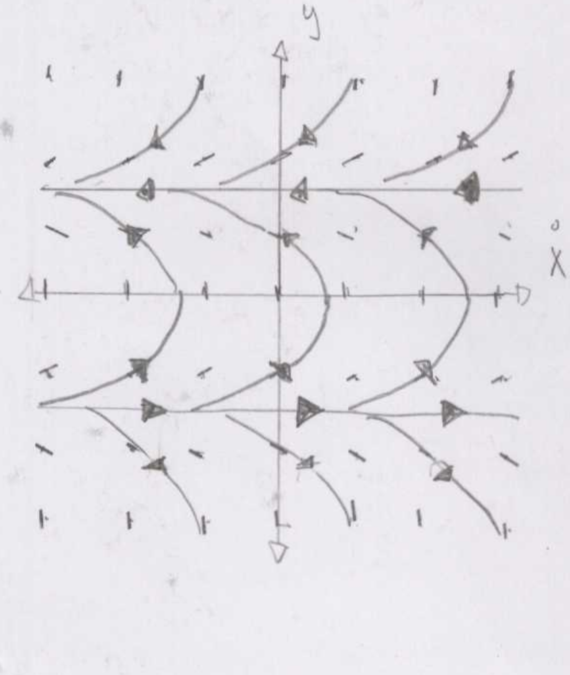
$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = -\begin{bmatrix}dx\\ dy\end{bmatrix} V$$

$$\dot{x} = -\int e^x \sin y \, dx$$

$$= -e^x \sin y + c$$

$$\dot{y} = -\int e^x \sin y \, dy$$

$$= e^x \cos y + c$$



7.2.4. Gradient System: When a system can be written as $\dot{x} = -\nabla V$, for a continuously differentiable, singled-valued scalar function

Line: A continuous function without curvature

Circle: A continuous and bounded function by an equidistant center.

Line: X; $\dot{x} = -\nabla V = -1$; A gradient system.

Circle: $\sqrt{x^2+y^2}$; $\dot{X} = -\nabla V = -\frac{X}{\sqrt{x^2+y^2}}$; A gradient system for X

$\dot{X} = f(X,y)$
$\dot{y} = g(x,y)$

7.2.5. a. $-\nabla V = \left(-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}\right)$; $-\frac{\partial V}{\partial x} = \dot{x} = f(X,y)$; $-\frac{\partial V}{\partial y} = \dot{y} = g(x,y)$

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

b. A sufficient condition is $p \to q$ and $\neg p \to \neg q$,

so $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ is sufficient by $V = -\int f(x,y)\,dx = -\int g(x,y)\,dy$.

$\dot{x} = y^2 + y\cos x$
$\dot{y} = 2xy + \sin x$

7.2.6. a. $V = \int \dot{x}\,dx + \int \dot{y}\,dy = xy^2 + y\sin x + xy^2 + y\sin x$

$$= 2(xy^2 + y\sin x)$$

$\dot{x} = 3x^2 - 1 - e^{2y}$
$\dot{y} = -2xe^{2y}$

b. $V = \int \dot{x}\,dx + \int \dot{y}\,dy = x^3 - x - xe^{2y} - xe^{2y}$

$$= x(x^2 - 1 - 2e^{2y})$$

$\dot{x} = y + 2xy$
$\dot{y} = x + x^2 - y^2$

7.2.7 a. If $\dot{x} = f(x,y)$ and $\dot{y} = g(x,y)$,

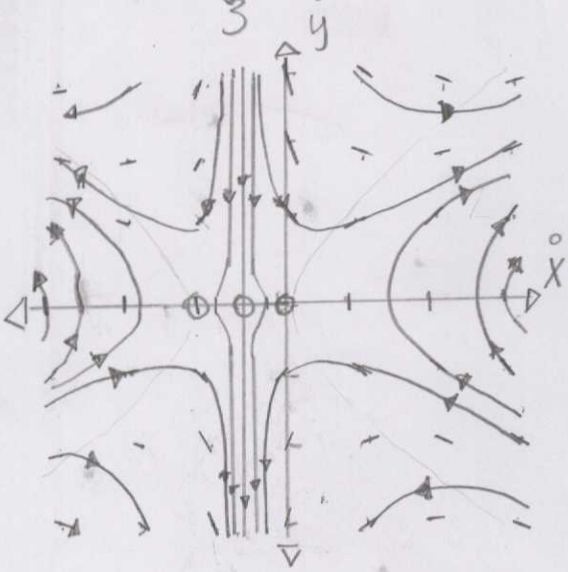
then $\frac{\partial f}{\partial y} = 1 + 2x$ and $\frac{\partial g}{\partial x} = 1 + 2x$

and $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$

b $V=\int \dot{x}\,dx+\int \dot{y}\,dy = xy+xy^2+xy+x^2y-\frac{y^3}{3}$

$= 2xy+xy^2+x^2y-\frac{y^3}{3}$

c.



7.2.8. If $\frac{df}{\partial y}=\frac{dg}{dx}$ at an equipotential, then $\frac{dy}{dx}=\frac{dg}{df}$;

The solution $\frac{dy}{dx}=\frac{dg}{df}$ is zero when $dg \neq df$ and one when $dy=df$, very similar to orthoganol slopes (dy/dx).

$\dot{x}=y+x^2y$
$\dot{y}=-x+2xy$

7.2.9.

a. $V=-\int \dot{x}\,dx+\int \dot{y}\,dy=-\int y+x^2y\,dx+\int -x+2xy\,dy$

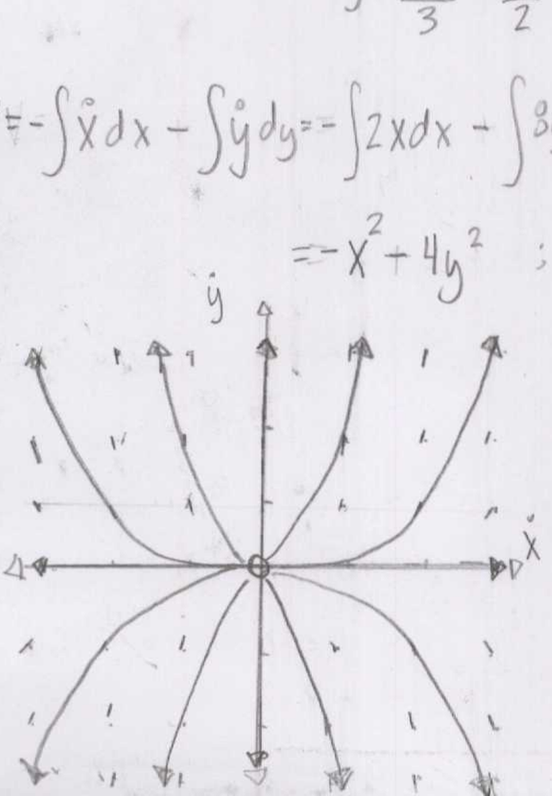
$=-xy-\frac{x^3y}{3}+\frac{x^2}{2}+x^2y$; $\frac{d\dot{x}}{dy}\neq\frac{d\dot{y}}{dx}$

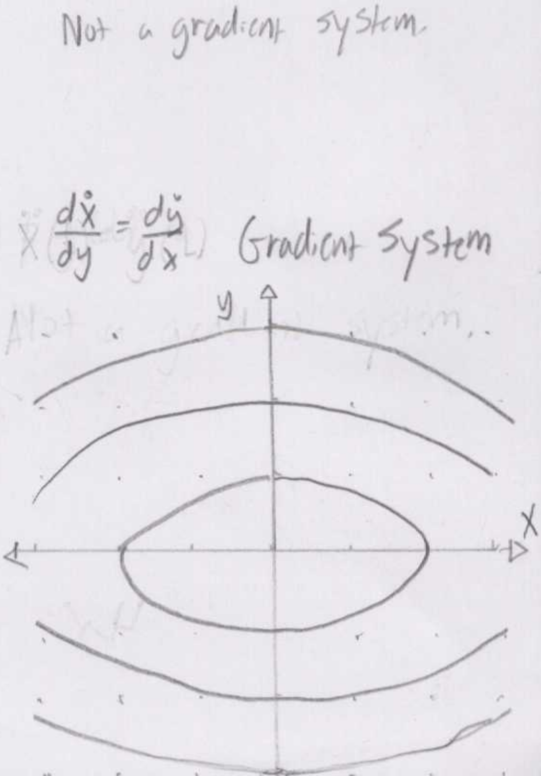
Not a gradient system.

$\dot{x}=2x$
$\dot{y}=8y$

b. $V=-\int \dot{x}\,dx-\int \dot{y}\,dy=-\int 2x\,dx-\int 8y\,dy$

$=-x^2+4y^2$; $\frac{d\dot{x}}{dy}=\frac{d\dot{y}}{dx}$ Gradient System



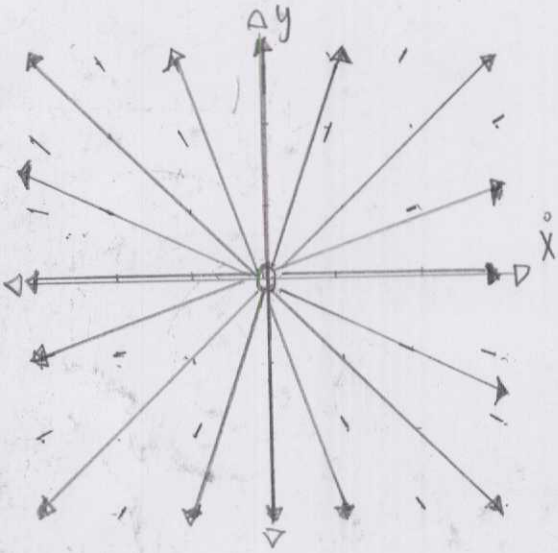


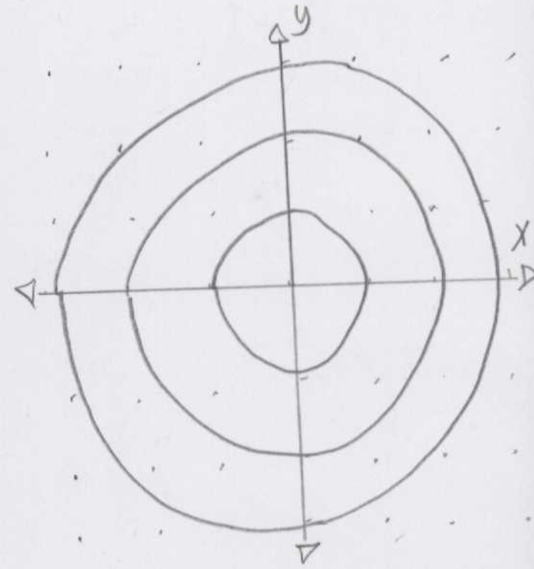
$\dot{x} = -2xe^{x^2+y^2}$ c. $V(x,y) = -\int \dot{x}\,dx - \int \dot{y}\,dy = +2\int xe^{x^2+y^2}dx + 2\int ye^{x^2+y^2}dy$

$\dot{y} = -2ye^{x^2+y^2}$

$$= e^{x^2+y^2} + e^{x^2+y^2} = 2e^{x^2+y^2}$$

$\frac{d\dot{x}}{dy} = \frac{d\dot{y}}{dx}$: Gradient system.





$\dot{x} = y - x^3$ 7.2.10. $V = -\int \dot{x}\,dx - \int \dot{y}\,dy = -\int y - x^3 dx + \int x + y^3 dy$

$\dot{y} = -x - y^3$

$$= -xy + \frac{x^4}{4} + xy + \frac{y^4}{4} = \frac{x^4}{4} + \frac{y^4}{4}$$

$$a = \frac{1}{4} ; b = \frac{1}{4}$$

The potential function has a suitable a, and b, so this function is Liapunov stable with no closed orbits.

$V = ax^2 + 2bxy + cy^2$ 7.2.11. $\frac{\partial^2 V}{\partial x^2}\frac{\partial^2 V}{\partial y^2} - \left(\frac{\partial^2 V}{\partial x \partial y}\right)^2 = (2a)(2c) - (2b)^2 = 4(ac - b^2)$ @ (0,0)

Positive definite is a strictly positive, meaning strictly positive when $(ac-b^2) > 0$.

$\dot{x} = -x + 2y^3 - 2y^4$ 7.2.12 $V = -\int \dot{x}\,dx - \int \dot{y}\,dy = \frac{x^2}{2} - 2y^3x + 2y^4x + xy + \frac{y^2}{2} - \frac{xy^2}{2}$

$\dot{y} = -x - y + xy$

Fixed Points $(x^*, y^*) = (0,0), (-2,0), (-2,1), (-1,0)$.

No periodic solutions

$\dot{N}_1 = r_1 N_1 (1 - N_1/K_1) - b_1 N_1 N_2$

$\dot{N}_2 = r_2 N_2 (1 - N_2/K_2) - b_2 N_1 N_2$

Dulac's Criterion: If $\dot{x} = f(x)$ is continuous, a real-valued function $g(x)$ such that $\nabla \cdot (g\dot{x})$ has one sign throughout R in a closed orbit.

7.2.13. $g = (N_1 N_2)^{-1}$

$$\nabla \cdot (g\dot{x}) = \frac{\partial}{\partial N_1}(g\dot{N}_1) + \frac{\partial}{\partial N_2}(g\dot{N}_2)$$

$$= \frac{\partial}{\partial N_1}\left[\frac{r_1}{N_2}(1 - N_1/K_1) - b_1\right] + \frac{\partial}{\partial N_2}\left[\frac{r_2}{N_1}(1 - N_2/K_2) - b_2\right]$$

$$= \frac{r_2}{N_1 K_2} - \frac{r_1}{N_2 K_1}$$

$$\leq 0$$

$\dot{x} = x^2 - y - 1$

$\dot{y} = y(x-2)$

7.2.14. a. Fixed Points: $\dot{x} = 0 = x^2 - y - 1$;

$\dot{y} = 0 = y(x-2)$

$2x \quad -1$

$y \quad -2$

$(x^*, y^*) = (-1, 0)$; $A_{(-1,0)} = \begin{pmatrix} -2 & -1 \\ 0 & -2 \end{pmatrix}$; Stable Node

$(1,0)$ $A_{(1,0)} = \begin{pmatrix} 2 & -1 \\ 0 & -2 \end{pmatrix}$; Saddle Point

$(2,3)$ $A_{(2,3)} = \begin{pmatrix} 4 & -1 \\ 3 & -2 \end{pmatrix}$; Saddle Point

b.

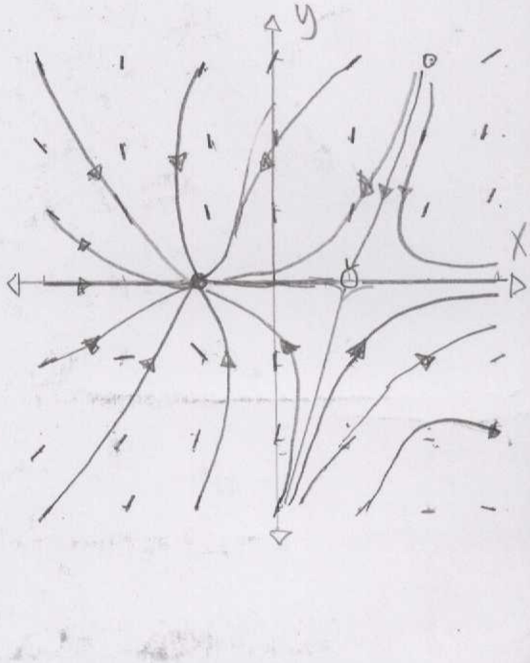
b. $(-1,0) \rightarrow (1,0)$: $\frac{dy}{dx} = 0$

$(1,0) \rightarrow (2,3)$: $\frac{dy}{dx} = x^2 - 1$

$(2,3) \rightarrow (-1,0)$: $\frac{dy}{dx} \cong -1$

Closed orbits are nonexistent because of constant trajectory between fixed points.

c.



$\dot{x} = x(2-x-y)$
$\dot{y} = y(4x-x^2-3)$

7.2.15. a. <u>Fixed Points:</u> $\dot{x} = 0 = x(2-x-y)$; $A = \begin{pmatrix} 2-2x-y & -x \\ 4y-2yx & 4x-x^2-3 \end{pmatrix}$

$\dot{y} = 0 = y(4x-x^2-3)$

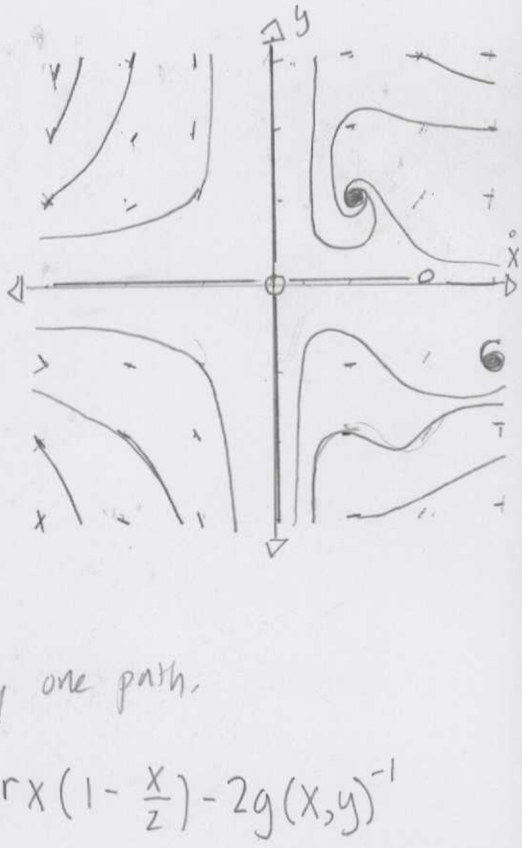
(x^*, y^*): $(0,0)$: $A_{(0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$; Saddle Point

$(1,1)$; $A_{(1,1)} = \begin{pmatrix} -1 & -1 \\ 2 & 0 \end{pmatrix}$; ~~Stable~~ spiral ~~Fixed Point~~

$(2,0)$: $A_{(2,0)} = \begin{pmatrix} -2 & -2 \\ 0 & 1 \end{pmatrix}$; Saddle Point

$(3,-1)$: $A_{(3,-1)} = \begin{pmatrix} 9 & -3 \\ 2 & 0 \end{pmatrix}$; stable spiral

b. <u>Phase Portrait</u>



7.2.16. If R is a set of closed orbits, then $\iint_A \nabla\cdot(g\dot{x})\,dA = \oint_C g\dot{x}\,n\,dl$

7.2.17. If A is an annular, then Greens theorem holds true. Dulac's Criterion fails for multiple holes because the closed orbit (line integral) has only one path.

$\dot{x} = rx(1-\frac{x}{2}) - \frac{2x}{1+x}y$ 7.2.18 If $g(x,y) = \frac{1+x}{x}y^{\alpha-1}$; then $\dot{x} = rx(1-\frac{x}{2}) - 2g(x,y)^{-1}$

$\dot{y} = -y + \frac{2x}{1+x}y$

$\dot{y} = -y + 2g(x,y)^{-1}$

where $x = 0$

$$\nabla\cdot(g\dot{x}) = \frac{\partial}{\partial x}(g\dot{x}) + \frac{\partial}{\partial y}(g\dot{y})$$

$$= \frac{r(1-2x)}{2y} > 0$$; No closed orbits in the positive quadrants

$\dot{R} = -R + A_s + kSe^{-S}$ 7.2.19.

$\dot{S} = -S + A_r + kRe^{-R}$

a.

Term:	Meaning:
$-R$	Rhett's decreasing love for Scarlett.
$+A_s$	Scarlett's love ~~sing love~~
$+kSe^{-S}$	Scarelett's decaying love for Rhett
$-S$	Scarlett's decreasing love for Rhett
$+A_r$	Rhett's love.
$+kRe^{-R}$	Rhett's decaying love for Scarlett.

b. $\dot{R} = 0 = -R + A_s + kSe^{-S}$; $(R^*, S^*) = (A_s + kSe^{-S}, A_r + kRe^{-R})$

$\dot{S} = 0 = -S + A_r + kRe^{-R}$; Which are greater than zero.

c. $\nabla \cdot (g\dot{x}) = \frac{\partial}{\partial R}(g\dot{R}) + \frac{\partial}{\partial S}(g\dot{S})$

$= -1 - 1 = -2 < 0$; Where $g = 1$.

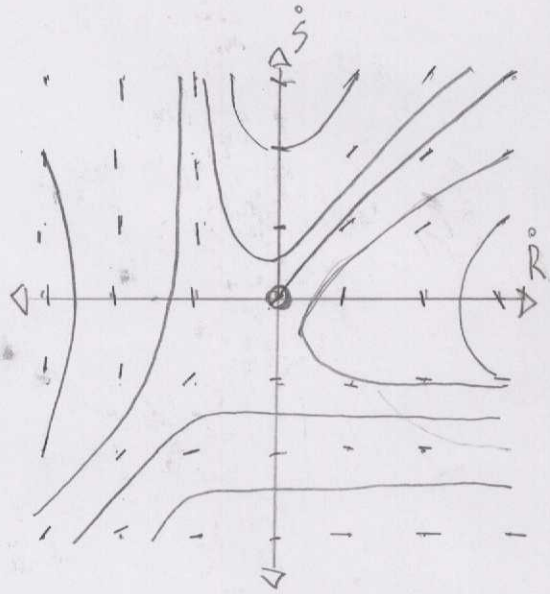
Since $x, y > 0$, there are no periodic solutions in the first quadrant.

d. Phase Portrait ; $A_s = 1.2$

$A_r = 1$

$k = 15$

$R(0) = S(0) = 0$



$\dot{x} = x - y - x(x^2 + 5y^2)$ 7.3.1 a. $A = \begin{pmatrix} 1 - x^2 - 5y^2 - 2x^2 & -1 - 10xy \\ 1 - 2xy & 1 - x^2 - y^2 - 2y^2 \end{pmatrix}$

$\dot{y} = x + y - y(x^2 + y^2)$

$= \begin{pmatrix} 1 - 3x^2 - 5y^2 & -1 - 10xy \\ 1 - 2xy & 1 - x^2 - 3y^2 \end{pmatrix}$

$A_{(0,0)} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$; $\Delta = 2$; $\tau = 2$; $\tau^2 - 4\Delta < 0$; Unstable Spiral

b. $\dot{r} = \dfrac{x\dot{x} + y\dot{y}}{r} = \dfrac{r\cos\theta[x - y - x(x^2+5y^2)] + r\sin\theta[x+y-y(x^2+y^2)]}{r}$

$$= r\cos\theta[\cos\theta - \sin\theta - \cos\theta(r^2\cos^2\theta + 5r^2\sin^2\theta)]$$
$$+ r\sin\theta[\cos\theta + \sin\theta - \sin\theta(r^2)]$$
$$= r[\cos\theta(\cos\theta - \sin\theta - r^2\cos\theta(\cos^2\theta + 5\sin^2\theta)$$
$$+ \sin\theta(\cos\theta + \sin\theta - r^2\sin^2\theta)]$$

$$\dot{\theta} = \frac{(x\dot{y} - y\dot{x})}{r^2} = \frac{\cos\theta(x+y-y(r^2)) - \sin\theta(x-y-x(x^2+5y^2))}{r}$$

$$= \cos\theta(\cos\theta + \sin\theta - r^2\sin\theta) - \sin\theta(\cos\theta - \sin\theta - r^2\cos\theta(\cos^2\theta + 5\sin^2\theta))$$

$$= 4r^2\sin^3(\theta)\cos(\theta) + 1$$

c. $r_{min,\,outward}$ $\quad \dot{r} > 0;\ r[\cos\theta(\cos\theta - \sin\theta - r^2\cos\theta(\cos^2\theta + 5\sin^2\theta))$
$$+ \sin\theta(\cos\theta + \sin\theta - r^2\sin^2\theta)] > 0$$

$$r_{min,\,outward} \cong -\sqrt{\frac{\sin^2\theta - \sin\theta\, 4\cos\theta}{\sin^3\theta + 3\cos\theta - 2\cos\theta\cos(2\theta)}}$$

d. $r_{max,\,inward}$ $\quad \dot{r} < 0;\ r[\cos\theta(\cos\theta - \sin\theta - r^2\cos\theta(\cos^2\theta + 5\sin^2\theta))$
$$+ \sin\theta(\cos\theta + \sin\theta - r^2\sin^2\theta)] < 0$$

$$r_{max,\,inward} \cong \sqrt{\frac{(\sin\theta - 1)\sin\theta + \cos\theta}{\sin^3\theta + \cos\theta(3 - 2\cos 2\theta)}}$$

e. $r_{min,\,outward} < r < r_{max,\,inward}$

$\theta \cong 2(n\pi - 4\pi/10)$; This solution is close to the books $\theta = \frac{3\pi}{2}$

7.3.2 Runge Kutta Method [4th Order]

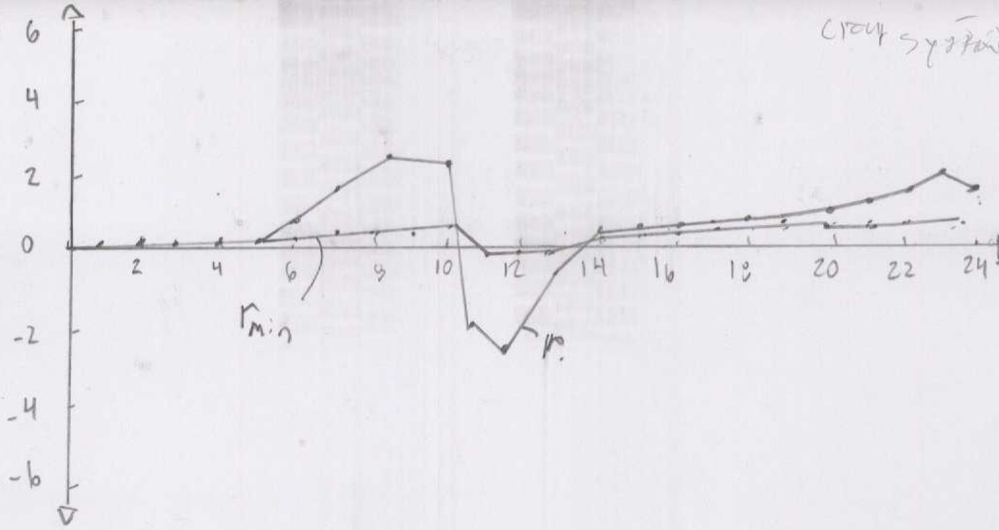
r_0	θ_0	$k_1(r_0, \theta_0 + \theta\Delta h/2)$	$k_1(r_0 + r\Delta h/2, \theta_0)$	$k_2(r_0 + k_1\Delta h/2, \theta_0)$

0.1

○○○

$k_2(r_0, \theta_0 + k_1\Delta h/2)$	$k_3(r_0 + k_2\Delta h, \theta_0)$	$k_3(r_0, \theta_0 + k_2\Delta h)$

$$r_n = r_{n-1} + \frac{\Delta h}{6}(r_0 + 2k_1 + 2k_2 + k_3);\quad \theta_n = \theta_{n-1} + \frac{\Delta h}{6}(\theta_0 + 2k_1 + 2k_2 + k_3)$$



Croup systems Speculum Course?

My degree Ne

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$\dot{x} = x - y - x^3$

$\dot{y} = x + y - y^3$

7.3.3. $$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \frac{r\cos\theta(x - y - x^3) + r\sin\theta(x + y - y^3)}{r}$$

$$= \frac{\cancel{r}\cos\theta\,(r\cos\theta - r\sin\theta - (r\cos\theta)^3) + \cancel{r}\sin\theta\,(r\cos\theta + r\sin\theta - (r\sin\theta)^3)}{\cancel{r}}$$

$$= r^2 - r^3(\cos^4(\theta) + \sin^4(\theta))$$

$r - r^3 < \dot{r} < r - r^3/2$ Poincare-Bendixon Theorem states at least one periodic solution.

$\dot{x} = x(1 - 4x^2 - y^2) - \frac{1}{2}y(1 + x)$

$\dot{y} = y(1 - 4x^2 - y^2) + 2x(1 + x)$

7.3.4. a. $$A = \begin{pmatrix} -12x^2 - y^2 - 1 & (-2x - 1)y - 1/2 \\ x(4 - 8y) + 2 & -4x^2 - 3y^2 + 1 \end{pmatrix}$$

$$A_{(0,0)} = \begin{pmatrix} -1 & -1/2 \\ 2 & 1 \end{pmatrix};\ \lambda_1 = 0;\ \lambda_2 = 0;\ \Delta = 0;\ \tau = 0;$$

$\tau^2 - 4\Delta = 0$ "center"

Although graph indicates an unstable spiral.

b. $V = (1 - 4x^2 - y^2)^2;\ \dot{V} = 2(1 - 4x^2 - y^2)(1 - 4x^2 - y^2)'$

$$\lim_{t \to \infty} \dot{V} = 0;\ 1 - 4x^2 - y^2 = 0;\ 4x^2 + y^2 = 1$$

$\dot{x} = -x - y + x(x^2 + 2y^2)$

$\dot{y} = x - y + y(x^2 + 2y^2)$

7.3.5 $$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \frac{r\cos\theta(-x - y + x(x^2 + 2y^2)) + r\sin\theta(x - y + y(x^2 + 2y^2))}{r}$$

$$= \frac{r\cos\theta(-r\cos\theta - r\sin\theta + r\cos\theta((r\cos\theta)^2 + 2(r\sin\theta)^2)) + r\sin\theta(r\cos\theta - r\sin\theta + r\sin\theta((r\cos\theta)^2 + 2(r\sin\theta)^2))}{r}$$

$= r^3(\sin^2(x)+1)\left(\frac{1}{2}\sin(2x) + \cos^2(x)\right) - r$

$\theta =$ $\frac{38}{100} r^3 - r < r < \frac{152}{100} r^3 - r$

A periodic solution exists by Poincare-Benedixon Theorem.

$\ddot{x} + F(x,\dot{x})\dot{x} + x = 0$

7.3.6. $F(x,\dot{x}) < 0$, $r \le a$, else $F(x,\dot{x}) > 0$ if $r \ge b$ where $r^2 = x^2 + \dot{x}^2$

a. $\dot{u} = \dot{x} = v$
$\dot{v} = \ddot{x} = -F(x,\dot{x})\dot{x} - x$

A physical interpretation of $F(x,\dot{x})$ is an additive force to acceleration, which increases or decreases.

b. $r = \sqrt{x^2 + \dot{x}^2} = \sqrt{(r\cos\theta)^2 + (\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})^2}$

$a < \sqrt{(r\cos\theta)^2 + (\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})^2} < b$

$\dot{x} = y + ax(1-2b-r^2)$
$\dot{y} = -x + ay(1-r^2)$

7.3.7.

a. $(0 < a \le 1, 0 \le b < 1/2)$ and $r^2 = x^2 + y^2$

$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \frac{r\cos\theta\,(y + ax(1-2b-r^2)) + r\sin\theta\,(-x + ay(1-r^2))}{r}$

$= r\cos\theta\,(r\sin\theta + ar\cos\theta(1-2b-r^2)) + r\sin\theta\,(-r\cos\theta + ar\sin\theta(1-r^2))$

$\dot{\theta} = \frac{x\dot{y} - \dot{x}y}{r^2} = \frac{r\cos\theta(-x + ay(1-r^2)) + r\sin\theta(y + ax(1-2b-r^2))}{r^2}$

$= \cos\theta(-\cos\theta + a\sin\theta(1-r^2)) - \sin\theta(\sin\theta + a\cos\theta(1-2b-r^2))$

$= ab\sin(2x) - 1$

b. A region of trapping $ar(1-r^2) \le \dot{r} \le ar(1-2b-r^2)$ exists as an annular cycle, $T(a,b)$.

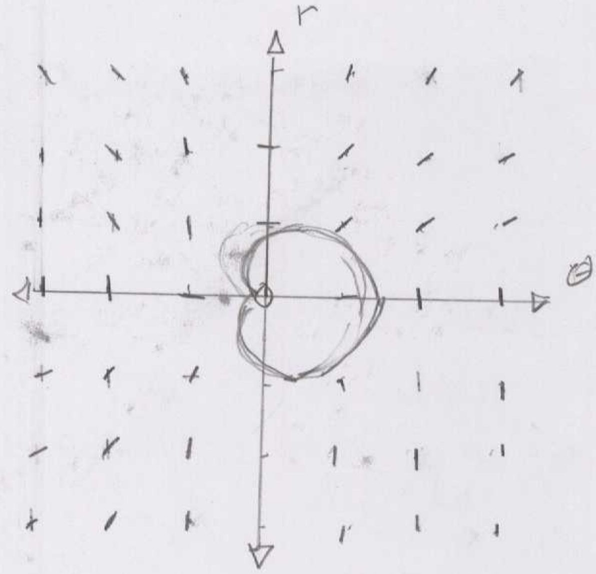
c) If $b=0$, then $ar(1-r^2) \le \dot{r} \le ar(1-r^2)$, so $\dot{r}$ must be $ar(1-r^2)$.

$\dot{r} = r(1-r^2) + \mu r\cos\theta$

$\dot{\theta} = 1$

7.3.8. $\dot{r} = 0 = r(1-r^2) + \mu r\cos\theta$

$$\mu = \frac{r(r^2-1)}{r\cos\theta}$$



If $r=0, 1,$ or -1, then the closed orbit known as the cardoid becomes absent, but a circular orbit remains.

7.3.9.

$\dot{r} = r(1-r^2) + \mu r\cos\theta$

$\dot{\theta} = 1$

a. $r(\theta) = 1 + \mu r_1(\theta) + O(\mu^2)$

$$\frac{dr}{d\theta} = r(1-r^2) + \mu r\cos\theta = \mu r_1'(\theta)$$

$$\mu r_1'(\theta) = (1+\mu r_1(\theta))(1-(1+\mu r_1(\theta))^2) + \mu(1+\mu r_1(\theta))\cos\theta$$

$r_1'(\theta) = -2r_1(\theta) + \cos(\theta)$ First-order linear differential equation.

$\cos(\theta) = r_1'(\theta) + 2r_1(\theta)$

$\cos(\theta)\,d\theta = r_1'(\theta) + 2r_1(\theta)\,d\theta$

b. $(\cos(\theta) - 2r')\,d\theta - dr(\theta) = 0$ Exact Differential Equation.

$\mu N(r,\theta)\,d\theta + M(r,\theta)\,dr = 0$

$N'(r,\theta) \neq M'(r,\theta)$, so a mutiplicative is necessary.

$(2)N'(r,\theta) = \frac{dN}{dr} = 2$; $M'(r,\theta) = \frac{dM}{d\theta} = 0$

The assumption; $\frac{dN}{dr} = \frac{dM}{d\theta}$; $\frac{dN}{dr} - \frac{dM}{d\theta} = 0 = \mu\left(\frac{dN}{dr} - \frac{dM}{d\theta}\right)$

Also, $\mu(r,\theta) = \mu(r)$; $\frac{d\mu}{d\theta} = 0$; so $\frac{1}{\mu}\frac{d\mu}{dr} = \frac{1}{M}\left(\frac{dN}{dr} - \frac{dM}{d\theta}\right)$

$$\int\frac{d\mu}{\mu} = \int\frac{1}{M}\left(\frac{dN}{dr} - \frac{dM}{d\theta}\right)d\theta;\ \ln(\mu) = 2\theta;\ \mu = e^{2\theta}$$

Multiplying the equation by $e^{2\theta}$

$(\cos(\theta)-2r)e^{2\theta}d\theta - e^{2\theta}dr = 0$

$N(r,\theta)\,d\theta - M(r,\theta)dr = 0$

$N'(r,\theta) = M'(r,\theta) = -2e^{2\theta}$ Exact equation.

$dF(r,\theta) = N(r,\theta)d\theta + M(r,\theta)dr$

$F(r,\theta) = \int N(r,\theta)\,dr = \int e^{2\theta}\cos(\theta) - 2e^{2\theta}\cdot r\,d\theta$

$= \frac{e^{2\theta}\sin(\theta)}{5} + \frac{2e^{2\theta}\cos(\theta)}{5} - e^{2\theta}r + C$

$r = \frac{\sin(\theta)}{5} + \frac{2\cos(\theta)}{5} + \frac{C}{e^{2\theta}}$

$r(\theta) = 1 + \mu\left(\frac{\sin(\theta)}{5} + \frac{2\cos(\theta)}{5}\right)$

b. $\frac{dr}{d\theta} = 0 = \frac{\mu\cos\theta}{5} - \frac{2\sin\theta}{5}$ @ $\theta = \arctan\left(\frac{1}{2}\right) + n\pi$

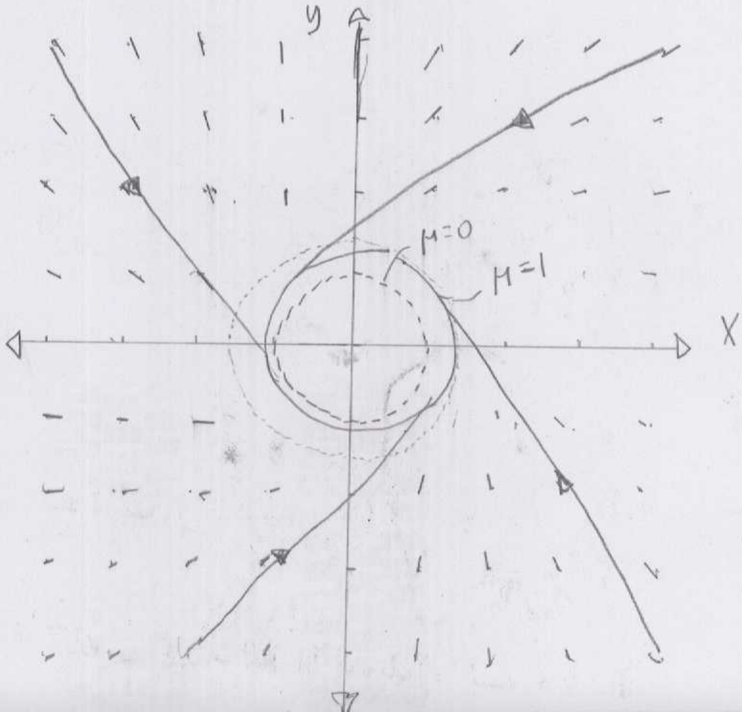
$r\left(\arctan\left(\frac{1}{2}\right)\right) = 1 + \mu\left(\frac{1}{5}\frac{1}{\sqrt{5}} + \frac{2}{5}\frac{2}{\sqrt{5}}\right) = 1 + \frac{\mu}{\sqrt{5}}$

or

$r\left(\arctan\left(\frac{1}{2}\right)+\pi\right) = 1 + \mu\left(\frac{1}{5}\frac{-1}{\sqrt{5}} + \frac{2}{5}\frac{-2}{\sqrt{5}}\right) = 1 - \frac{\mu}{\sqrt{5}}$

$\sqrt{1-\mu} < 1 - \frac{\mu}{\sqrt{5}} < r < 1 + \frac{\mu}{\sqrt{5}} < \sqrt{1+\mu}$

c.



$\dot{x} = Ax - r^2 x$ 7.3.10. $r = \|x\|$; $A \in \mathbb{R}$; $\lambda_{1,2} = \alpha \pm i\omega$

$\Delta = (\alpha^2 + \omega^2)$; $\tau = 2\alpha$; $\tau^2 - 4\Delta < 0$

"Unstable Spiral"

Fixed Points: $\dot{x} = 0 = Ax - r^2 x$; $x = 0$ if $\alpha < 0$ "stable fixed point"

if $\alpha > 0$ "Unstable spiral"

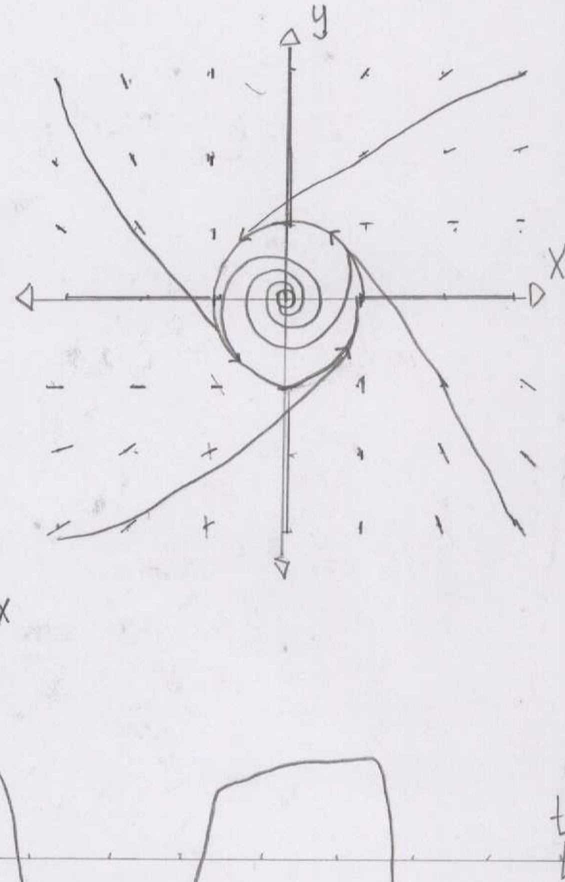
$\dot{r} = r(1-r)\left[r^2 \sin^2\theta + (r^2\cos^2\theta - 1)^2\right]$

$\dot{\theta} = r^2\sin^2\theta + (r^2\cos^2\theta - 1)^2$

7.3.11 $\dot{x} = f(x)$ is a vector field on $\mathbb{R}^2$

Cycle graph: an invariant set containing a finite number of fixed points connected by a finite number of trajectories, all oriented clockwise or counter clockwise.

a. Phase Portrait



b. $\dot{x} = \frac{\dot{r}x}{r} - y\dot{\theta}$

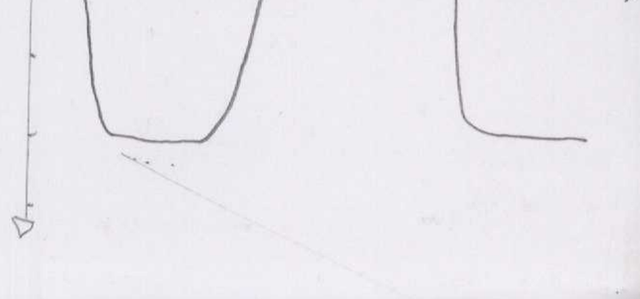
$$= \frac{r(1-r)\left[r^2\sin^2\theta + (r^2\cos^2\theta - 1)^2\right]x}{r}$$

$$- y\left[r^2\sin^2\theta + (r^2\cos^2\theta - 1)^2\right]$$

where $r = \sqrt{x^2 + y^2}$

$\theta = \operatorname{atan}(y/x)$

Runge-Kutta performance became absent because of problem sets Statement "Sketch".



$\mathring{P} = P[(aR-S)-(a-1)(PR+RS+PS)]$

$\dot{R} = R[(aS-P)-(a-1)(PR+RS+PS)]$

$\mathring{S} = S[(aP-R)-(a-1)(PR+RS+PS)]$

$E_1(P,R,S) = P+R+S$

$E_2(P,R,S) = PRS.$

7.3.12.

a. $\mathring{E}_1 = (1-E_1)(a-1)(PR+RS+PS)$; $P+R+S=1$; $E_1=1$

If $P=-1$ $\quad \mathring{E}_1 = (1-E_1)(a-1)(RS-PR-PS) = 0$

$= (1-E_1)(a-1)(PR+RS+PS)$

If $R=-1$ $\quad \mathring{E}_1 = (1-E_1)(a-1)(PS-PR-RS) = 0$

$= (1-E_1)(a-1)(PR+RS+PS)$

If $S=-1$ $\quad \mathring{E}_1 = (1-E_1)(a-1)(PS-PR-RS)$

$= (1-E_1)(a-1)(PR+RS+PS)$ "Ellipsoid"

b. $P,R,S \geq 0$ & $P+R+S=1$

If $(P=1/2 \,||\, R=1/2 \,||\, S=1/2)$, then $\mathring{E}_1 = 0$ "sphere"

c. $\mathring{E}_1 = 0 = (1-E_1)(a-1)(PS+RS+PR)$

$(P^*, R^*, S^*) : (P, -(P+S), -(P+R))$

$= (-(S+R), R, -(P+R))$

$= (-(S+P), -(S+R), S)$

d. $\frac{dE_2}{dt} = \frac{d}{dt}(PRS) = R(S \circ \mathring{P} + P \circ \mathring{S}) + P \circ S \circ \mathring{R}$

$= \frac{PRS\,(a-1)}{2}\left[(P-R)^2+(R-S)^2+(S-P)^2\right]$

e. $(P^*, R^*, S^*) = (1/3, 1/3, 1/3)$

$\mathring{E}_2 = \frac{(1/3)^3(a-1)}{2}\left[0^2+0^2+0^2\right] = 0$

$(P^*, R^*, S^*) = (3,0,0)$; $\mathring{E}_2 = 0$

f. If $a<1$, then the model $\dot{E}_2$ trajectory is decreasing else $a=1$, then E_2 model does not change.

g. If $a<1$, then $\dot{E}_2$ is ~~a~~ less than zero and trajects to ~~f~~ fixed center.

$$\ddot{x}+\mu(x^2-1)\dot{x}+\tanh x=0$$

7.4.1. $\dot{u}=\dot{x}=v$

$\dot{v}=\mu(1-u^2)v-\tanh u$

Fixed Points: $\dot{u}=0=\dot{x}=v$

$\dot{v}=0=\mu(1-u^2)v-\tanh u$

$(u^*,v^*):(0,0)$

① u and v are continuously diflentiable

~~② $\ddot{u}=1$; $\ddot{v}=\mu(1-2u\dot{u})v+\mu(1-u^2)\dot{v}-(1-\tanh^2 u)$~~

② $\dot{v}(-u,-v)=-\dot{v}(+u,+v)$ is an odd function.

$\dot{v}(-u,-v)=-\mu(1-(-u)^2)v-\tanh(-u)$

$=-[\mu(1-u^2)v-\tanh(u)]$

③ $\dot{v}(u,v)>0$ for $u,v>0$

$\dot{v}(u,v)=\underbrace{\mu(1-u^2)v}_{(+)\text{ till } u^2=1}-\underbrace{\tanh u}_{(+)}$; $\mu(1-u^2)v>\tanh u$ (+)

④ $\dot{u}(-v)=\dot{u}(v)$ is an even function

$\dot{u}(-v)=-v=0$; $0=\dot{u}(v)$ at fixed point

⑤ The odd function $F(v)=\int_0^v \dot{u}(k)dk$ is positive at $v=a$, and negative for $0<v<a$, is positive and nondecreasing for $v>0$, and $F(v)\to\infty$ as $v\to\infty$

A stable limit cycle.

$\ddot{x}+\mu(x^4-1)\dot{x}+x=0$ 7.4.2. $f(x)=\ddot{x}+\mu(x^4-1)x$; $g(x)=x$

a. ① $f(x)$ and $g(x)$ are continuously differentiable

$\dot{f}(x)=\ddot{x}+\mu(4x^4)+\mu(x^4-1)$; $\dot{g}(x)=1$

② $g(-x)=-g(x)$ is an odd function

$g(-x)=-g(x)$ at $x=0$

③ $f(-x)=f(x)$ is an even function

$f(-x)=-\ddot{x}-\mu[(-x)^4-1]\cdot x=-f(x)$ at $x=0$

④ $g(x)>0$ for $x>0$

⑤ $F(x)=\int_0^x f(u)du=\int_0^x \ddot{u}+\mu(u^2-1)\,du=\cancel{\ddot{x}}+\mu(\frac{x^3}{3}-x)$

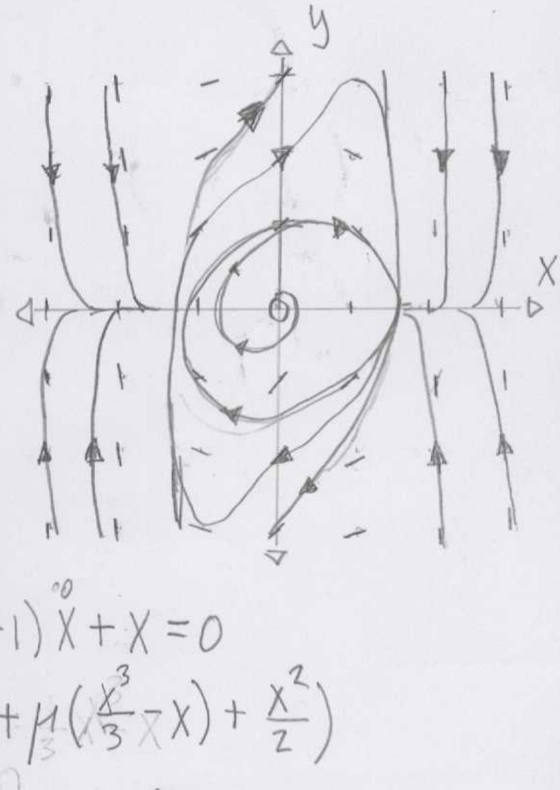
$F(x)$ has a positive root at $x=\sqrt{3}$

$F(x)$ is negative from $0<x<\sqrt{3}$

$F(x)$ is positive and nondecreasing for $\sqrt{3}<x$

and $F(x)\to\infty$ as $x\to\infty$

b. Phase Portrait



c. If $\mu<1$, then the function has an unstable periodic cycle in the opposite direction.

$x_A=2$

$x_B=1$

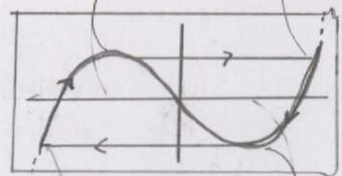
7.5.1. Van der Pol Oscillator: $\ddot{x}+\mu(x^2-1)\dot{x}+x=0$

$\mu>>1$; $\ddot{x}+\mu(x^2-1)\dot{x}+x=\frac{d}{dt}\left(\dot{x}+\mu(\frac{x^3}{3}-x)+\frac{x^2}{2}\right)$

$F(x)=\frac{x^3}{3}-x$; $w=\dot{x}+\mu F(x)+\frac{x^2}{2}$

$\dot{x}=w-\mu F(x)-\frac{x^2}{2}$; $\dot{w}=0$

If $y = \frac{w}{\mu}$, then $\dot{x} = \mu[y - F(x)]$ and $\dot{y} = 0$



Nullcline minimum: $F'(x) = x^2 - 1$; $x = \pm 1$

Nullcline intersection: $F(-1) = 2/3$, $x = -1, 2$

$\dot{x} = y$
$\dot{y} = -x - \mu(x^2 - 1)$

7.5.2. Nullclines: $\dot{x} = 0 = y$

$$\dot{y} = 0 = -x - \mu(x^2-1)$$

$$(x^*, y^*) = \left(\frac{+1 \pm \sqrt{1+4\mu^2}}{-2\mu}, 0\right)$$

A Liénard plane provides advantages, such as separation of time scales ($\propto \mu$) from the fixed point.

$\ddot{x} + k(x^2-4)\dot{x} + x = 1$ 7.5.2

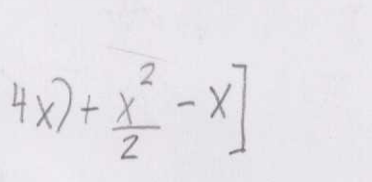
7.5.3. $\ddot{x} + k(x^2-4)\dot{x} + x - 1 = \frac{d}{dt}\left[\dot{x} + k\left(\frac{x^3}{3} - 4x\right) + \frac{x^2}{2} - x\right]$

$$F(x) = \frac{x^3}{3} - 4x \,;\, \omega = \dot{x} + kF(x) + \frac{x^2}{2} - x$$

$$\dot{x} = \omega - kF(x) - \frac{x^2}{2} + x \,;\, \dot{\omega} = 0$$

If $y = \frac{\omega}{k}$, then $\dot{x} = k(y - F(x)) - \frac{x^2}{2} + x$

$$\dot{y} = 0$$



Nullcline minimum: $F'(x) = x^2 - 4$; $x = \pm 2$

Nullcline Intersection: $F(-2) = \frac{16}{3}$; $x = 30$

$\ddot{x} + \mu f(x)\dot{x} + x = 0$ 7.5.4. $f(x) = -1$ for $|x| < 1$; $f(x) = 1$ for $|x| \geq 1$

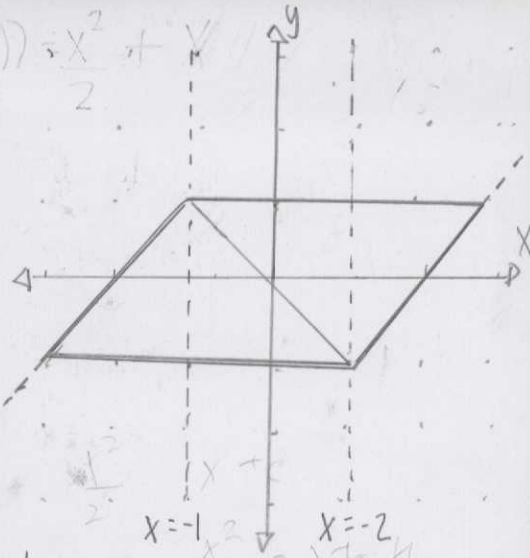
a. $\ddot{x} + \mu f(x)\dot{x} + x = \frac{d}{dt}\left[\dot{x} + \mu\int f(x)\,dx + \frac{x^2}{2}\right]$

$$F(x) = \int f(x)\,dx \,;\, \omega = \dot{x} + \mu F(x) + \frac{x^2}{2}$$

$$\dot{x} = \omega - \mu F(x) - \frac{x^2}{2} \,;\, \dot{\omega} = 0$$

If $y = \frac{-x^2}{2\mu}$, then $\dot{x} = \mu(y - F(x))$; $\dot{y} = \frac{-x}{\mu}$

$$F(x) = \int f(x)\,dx = \begin{cases} x+2 & x \le -1 \\ -x & |x| \le 1 \\ x-2 & x \ge 1 \end{cases}$$



b. Nullclines in graph.

c. If $\mu \gg 1$, then nullcline minimum is $y = -1$ and maximum ($y = 1$).

d. See graph about the limit cycle about $F'(x) = -1$, $F(-1) = 1$ or -1.

d. $|\dot{x}| \sim O(\mu) \gg 1$ and $|\dot{y}| \sim O(\mu^{-1}) \ll 1$

The period of the nullcline is $T \approx \mu \int_{-1}^{2} \frac{y - F(x)}{-\mu}\,dx$

$$\approx \mu \int_{-1}^{2} \frac{-x^2}{2\mu^2} + \cancel{x}\,dx$$

$$\approx \left. \frac{-x^3}{6} + \frac{x^2}{2}\mu \right|_{-1}^{2}$$

$$\approx \frac{-(2^3-1)}{6} + \left[\frac{4}{2} + \frac{1}{2}\right]\mu$$

$$\approx \frac{-7}{6} + \frac{5}{2}\mu$$

$\ddot{x} + \mu(x^2-1)\dot{x} + x = a$ 7.5.6.

a. Fixed Points: $\ddot{x} + \mu(x^2-1)\dot{x} + (x-a) = \frac{d}{dt}\left[\dot{x} + \mu F(x) + \frac{x^2}{2} - ax\right]$

$$F(x) = \frac{x^3}{3} - x \;;\; w = \dot{x} + \mu F(x) + \frac{x^2}{2} - ax$$

$$\dot{x} = w - \mu F(x) - \frac{x^2}{2} + ax \;;\; \dot{w} = 0$$

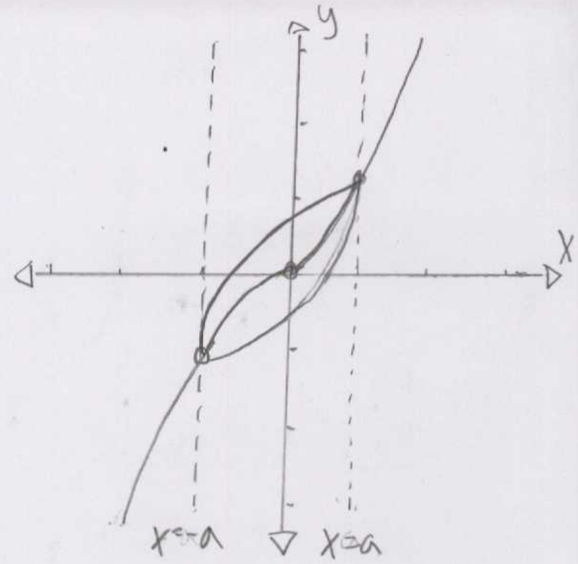
If $y = \dfrac{-\frac{x^2}{2} + ax + w}{\mu}$, $\dot{x} = \mu[y - F(x)]$

$$\dot{y} = \frac{a - x}{\mu}$$

$$(x^*, y^*) = (a, F(x)) = \left(a, \frac{x^3}{3} - x\right)$$

b. Nullclines in the Lienard Plane.

The center seems stable not corresponding to the book.



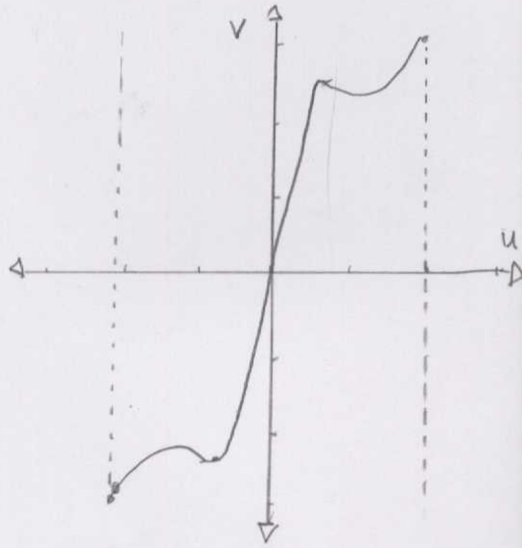
c. Nullcline Minimum: $F'(X) = X^2 - 1$

$X = \pm 1$

Nullcline Intersection: $F(-1) = \frac{2}{3}$

$|a| < 0$ because $dy/dx = 0$

d. See plot.



$\mathring{u} = b(v-u)(\alpha + u^2) - u$

$\mathring{v} = c - u$

7.5.7. a. Nullclines on graph.

b. $|C_1| = |C_2|$

c. A fixed point exists when c is beyond the inflection points.

$X(t,\varepsilon) = (1-\varepsilon^2)^{-1/2} e^{-\varepsilon t} \sin|(1-\varepsilon^2)^{1/2} t|$

$X(t,\varepsilon) = \sin t - \varepsilon t \sin t + O(\varepsilon^2)$

7.6.1 $X(t,\varepsilon) = (1-\varepsilon^2)^{-1/2} \circ e^{-\varepsilon t} \circ \sin|(1-\varepsilon^2)^{1/2} t|$

Identities: $(1+x)^a = 1 + aX + \frac{1}{2}(a-1)aX^2 + \cdots$

$e^X = 1 + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots$

$X(t,\varepsilon) = [1 + \frac{1}{2}\varepsilon^2 + \cdots][1 - \varepsilon t + \cdots] \sin[(1 - \frac{1}{2}\varepsilon^2 + \cdots)t]$

$= [1 - \varepsilon t + \frac{1}{2}\varepsilon^2 + \frac{\varepsilon^3 t}{2} + \cdots] \sin[(1 - \frac{1}{2}\varepsilon^2 + \cdots)t]$

$= \sin[t - O(\varepsilon^2)] - \varepsilon t \sin[t - O(\varepsilon^2)] + O(\varepsilon^2)$

$\cong \sin[t] - \varepsilon t \sin[t] + O(\varepsilon^2)$

$\ddot{X} + X + \varepsilon X = 0$ 7.6.2. $X(0) = 1$; $\dot{X}(0) = 0$

$\mathring{u} = \mathring{x} = v$

$\mathring{v} = -(1+\varepsilon)X = -(1+\varepsilon)u$

$$\vec{\dot{x}} = Ax = \begin{pmatrix} 0 & 1 \\ -(1+\varepsilon) & 0 \end{pmatrix}\begin{bmatrix} u \\ v \end{bmatrix} ; \quad 0 = Ax = \lambda ; \quad Ax - \lambda = 0$$

$$\begin{pmatrix} -\lambda & 1 \\ -(1+\varepsilon) & -\lambda \end{pmatrix} = 0 ; \quad \lambda_1 = +i\sqrt{1+\varepsilon} ; \quad \lambda_2 = -i\sqrt{1-\varepsilon}$$

$$\lambda_1 = +i\sqrt{1+\varepsilon} ; \quad \begin{pmatrix} -i(\sqrt{1+\varepsilon}) & 1 \\ -(1+\varepsilon) & -i(\sqrt{1+\varepsilon}) \end{pmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$-i(\sqrt{1+\varepsilon})v_{11} + v_{12} = 0 ; \quad \vec{V}_1 = \begin{bmatrix} 1 \\ i\sqrt{1+\varepsilon} \end{bmatrix}$$

$$\lambda_2 = -i\sqrt{1+\varepsilon} ; \quad \begin{pmatrix} i\sqrt{1+\varepsilon} & 1 \\ -(1+\varepsilon) & i\sqrt{1+\varepsilon} \end{pmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0$$

$$i\sqrt{1+\varepsilon}\, v_{21} + v_{22} = 0 ; \quad \vec{V}_2 = \begin{bmatrix} 1 \\ -i\sqrt{1+\varepsilon} \end{bmatrix}$$

$$\vec{x} = \vec{V}_1 c_1 e^{-\lambda_1 t} + \vec{V}_2 c_2 e^{-\lambda t}$$

$$= \begin{bmatrix} 1 \\ \sqrt{1+\varepsilon} \end{bmatrix} c_1 e^{-\sqrt{1+\varepsilon}\,t} + \begin{bmatrix} 1 \\ -\sqrt{1+\varepsilon} \end{bmatrix} c_2 e^{\sqrt{1+\varepsilon}\,t}$$

$$u(t) = c_1 e^{-\sqrt{1+\varepsilon}\,t} + c_2 e^{\sqrt{1+\varepsilon}\,t}$$

$$v(t) = \sqrt{1+\varepsilon}\, c_1 e^{-\sqrt{1+\varepsilon}\,t} - \sqrt{1+\varepsilon}\, c_2 e^{\sqrt{1+\varepsilon}\,t}$$

$$x(0) = 0 = c_1 - c_2 ; \quad \dot{x}(0) = 0 = -c_1\sqrt{1+\varepsilon} + c_2\sqrt{1+\varepsilon}$$

$$c_1 = c_2 ;$$

$$u(t) = e^{-\sqrt{1+\varepsilon}\,t} + e^{\sqrt{1+\varepsilon}\,t}$$

$$v(t) = \sqrt{1+\varepsilon}\, e^{-\sqrt{1+\varepsilon}\,t} - \sqrt{1+\varepsilon}\, e^{-\sqrt{1+\varepsilon}\,t}$$

b. $X(t,\varepsilon) = X_0 + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + O(\varepsilon^2)$

$$0 = (x_0 + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + O(\varepsilon^2)) + \varepsilon \frac{d}{dt}\left[X_0 + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + O(\varepsilon^2)\right]$$

$$+ \varepsilon^2 \frac{d^2}{d^2t}\left[X_0 + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + O(\varepsilon^2)\right]$$

$$0 = (X_0 + \varepsilon X_1(t) + \varepsilon^2 X_2(t)) + \varepsilon[\mathring{X}_0 + \varepsilon \mathring{X}_1(t) + \varepsilon^2 \mathring{X}_2(t))$$
$$+ \varepsilon^2[\ddot{X}_0 + \varepsilon \ddot{X}_1(t) + \varepsilon^2 \ddot{X}_2(t)]$$

$$= X_0 + \varepsilon(X_1(t) + \mathring{X}_0) + \varepsilon^2(X_2(t) + \mathring{X}_1(t) + \ddot{X}_0) + \cdots$$

$O(1) = X_0$

$O(\varepsilon) = X_1(t) + \mathring{X}_0$

$O(\varepsilon^2) = X_2(t) + \mathring{X}_1(t) + \ddot{X}_0$

c. Secular terms consist of functions which go to infinity as t approaches infinity, but the function has many.

$\ddot{X} + X = \varepsilon$

7.6.3. $X(0) = 1$; $\dot{X}(0) = 0$

a. $\mathring{u} = \dot{x} = v$

$\mathring{v} = \varepsilon - x = \varepsilon - u$

$\vec{X}' = A\vec{x} + f(t)$; $Ax + f(t) = \lambda x$; $(A - \lambda)x + f(t) = 0$

$$\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} ; \lambda_1 = +i ; \lambda_2 = -i$$

$\lambda_1 = +i$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ; \quad -iV_{11} + V_{12} = 0$$
$$\vec{V}_1 = \begin{bmatrix} 1 \\ +i \end{bmatrix}$$

$\lambda_2 = -i$

$$\begin{bmatrix} +i & 1 \\ -1 & +i \end{bmatrix}\begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ; \quad iV_{21} + V_{22} = 0$$
$$\vec{V}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\vec{X} = \begin{bmatrix} u \\ v \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ +i \end{bmatrix} e^{-it} + C_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{it}$$

$u(t) = C_1 e^{-it} + C_2 e^{it} = x(t)$

$v(t) = C_1 i e^{-it} + C_2 i e^{it}$

$u(0) = 1 = C_1 \sin(0) + C_2 \cos(0) + \varepsilon C_1 \cos(0) + iC_2 \sin(0)$

$\dot{u}(0) = 0 = C_1 \cos(0) - C_2 \sin(0) + iC_1 \cos(0) - iC_2 \cos(0)$

$1 = C_2 + \varepsilon_1 \quad ; \quad C_2 = iC_1$

$0 = C_1 - iC_2 \quad C_1 =$

$u(t) = (1 - \varepsilon) e^{it} + \varepsilon$

$v(t) = (1 - \varepsilon) i e^{it}$

b. $x(t, \varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + O(\varepsilon^3)$

$$0 = x_0 + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \frac{d}{dt}\varepsilon\left[x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t)\right] + \frac{d^2}{dt^2}\varepsilon^2\left[x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t)\right]$$

$$= x_0 + \varepsilon\left[x_1(t) + \dot{x}_0(t)\right] + \varepsilon^2\left[x_2(t) + \dot{x}_1(t) + \ddot{x}_0(t)\right] + \cdots$$

$O(1) = x_0$

$O(\varepsilon) = x_1(t) + \dot{x}_0(t)$

$O(\varepsilon^2) = x_2(t) + \dot{x}_1(t) + \ddot{x}_0(t)$

c. Secular terms not present because this is a function of t, this is periodic in the real-space.

$\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0$

7.6.4. $h(x, \dot{x}) = x \; ; \; \ddot{x} + x + \varepsilon x = 0$

Averaged equation: $r' = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\sin(\theta)\, d\theta \equiv \langle h \sin\theta \rangle$

$$r\phi' = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\cos(\theta)\, d\theta \equiv \langle h\cos\theta \rangle$$

$$r' = \langle h\sin\theta \rangle = \langle r\cos\theta\sin\theta \rangle = \frac{r}{2\pi}\int_0^{2\pi} \cos\theta\sin\theta\, d\theta = r\frac{\sin^2\theta}{2}\Big|_0^{2\pi} = 0$$

$$r\phi' = \langle h\cos\theta \rangle = \langle r\cos^2\theta \rangle = \frac{r}{2\pi}\int_0^{2\pi} \cos^2\theta\, d\theta = \frac{r}{2\pi}\int_0^{2\pi} \frac{\cos(2\theta)+1}{2}\, d\theta$$

$$= \frac{r}{2\pi}\left[\frac{\sin(2\theta)}{4} + \frac{1}{2}\theta\right]_0^{2\pi} = \frac{r}{2}$$

Initial conditions: $r(t) = \sqrt{x(t)^2 + \dot{x}(0)^2}$; $\phi(t) \approx \arctan\left(\dot{x}(t) / x(t) \right)$

$$r(0) = \sqrt{a^2 + 0^2} = a \; ; \; \phi(0) = \arctan(0) = 0$$

Amplitude/Frequency: Amplitude: $r(T) = \text{constant}$

Frequency: $$\omega = \frac{d\theta}{dt} = 1 + \frac{d\phi}{dT}\frac{dT}{dt} = 1 + \varepsilon\phi'$$
$$= 1 + \varepsilon \frac{r(T)}{2}$$
$$= 1 + \frac{\varepsilon}{2}$$

Solution: $$x_0 = r(T)\cos(\tau + \phi(T))$$
$$= a\cos(\tau + \tfrac{1}{2})$$
$$\approx x(t, \varepsilon)$$

$h(x, \dot{x}) = x\dot{x}^2$ 7.6.5. $\ddot{x} + x + \varepsilon x \dot{x}^2 = 0$

Averaged Equation: $$r' = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\sin(\theta)\,d\theta = \langle h \sin\theta \rangle$$
$$= \frac{1}{2\pi}\int_0^{2\pi} r\cos\theta(-r\sin\theta)^2 \sin\theta \, d\theta$$
$$= \frac{-r^2}{2\pi}\int_0^{2\pi} \cos\theta \sin^3\theta \, d\theta = \frac{-r^2}{2\pi}\int_0^{2\pi} u^3 \, du$$
$$= \frac{+r^3}{2\pi}\frac{\sin\theta}{4}\Big|_0^{2\pi} = 0$$

$$r\phi' = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\cos(\theta)\,d\theta = \langle h\cos\theta \rangle$$
$$= \frac{1}{2\pi}\int_0^{2\pi} r\cos\theta(-r\sin\theta)^2\cos\theta \, d\theta$$
$$= \frac{r^3}{2\pi}\int_0^{2\pi} \cos^2\theta\sin^2\theta \, d\theta = \frac{r^3}{2\pi}\int_0^{2\pi} (\cos\theta \sin\theta)^2 d\theta$$
$$= \frac{r^3}{2\pi}\int_0^{2\pi} \frac{\sin^2(2\theta)}{4} d\theta = \frac{r^3}{2\pi}\int_0^{2\pi} \frac{1 - \cos(4\theta)}{8} d\theta$$

$$= \frac{r^3}{16\pi}\left[\theta - \frac{\sin(4\theta)}{4}\right]_0^{2\pi} = \frac{r^3}{16\pi}[2\pi] = \frac{r^3}{8}.$$

Initial conditions: $r(t) = \sqrt{x(t)^2 + \dot{x}(t)^2}$; $\phi(t) = \arctan(\dot{x}(t)/x(t))$

$= a$ $\qquad = 0$

Amplitude / Frequency: Amplitude: $r(T) = \text{constant}$

Frequency: $w = 1 + \frac{d\phi}{dT}\frac{dT}{dt} = 1 + \varepsilon\phi'$

$= 1 + \frac{\varepsilon a^2}{8}$; $\phi(T) = \frac{\varepsilon a^2}{8}T$

Solution: $x_0 = r(T)\cos(\tau + \phi(T))$

$= a\cdot\cos(\tau + \frac{\varepsilon a^2}{8}T)$

$x(t,\varepsilon) = a\cos((\frac{\varepsilon a^2}{8} + 1)t)$

$h(x,\dot{x}) = x\dot{x}$ 7.6.6. $\ddot{x} + x + x^2\dot{x} = 0$

Averaged Equations: $r' = \frac{1}{2\pi}\int_0^{2\pi} h\cos\theta\, d\theta$

$= \frac{-r^2}{2\pi}\int_0^{2\pi} \cos^2\theta \sin\theta\, d\theta$

$= -\frac{r^2}{2\pi}\int_0^{2\pi} \frac{u^2}{2} du = \frac{-r^2}{2\pi}\frac{\cos^2(\theta)}{2}\Big|_0^{2\pi} = \frac{-r^2}{2\pi}\left[\frac{1}{2} - \frac{1}{2}\right]$

$= 0$

$r\phi' = \frac{1}{2\pi}\int_0^{2\pi} h\sin\theta\, d\theta$

$= \frac{-r^2}{2\pi}\int_0^{2\pi}\cos\theta\sin^2\theta\, d\theta = \frac{-r^2}{2\pi}\int_0^{2\pi}\frac{u^2}{2} du$

$= \frac{-r^2}{2\pi}\frac{\sin^3\theta}{6}\Big|_0^{2\pi} = 0$

Initial Conditions: $r(t) = \sqrt{x(t)^2 + \dot{x}(t)^2}$; $\phi(t) = \arctan(\dot{x}(t)/x(t))$

$= a$ $\qquad = 0$

Amplitude / Frequency: Amplitude: $r(T) = \text{constant}$

Frequency: $\omega = 1 + \phi'\frac{dT}{dt} = 1 + \varepsilon\phi' = 1$; $\phi(T) = 0$

$$= \frac{-r}{2\pi}\Bigg[r^4\left(\frac{3\theta}{8} + \frac{3\cos\theta\sin(\theta)}{8} + \frac{\cos^3(\theta)\sin(\theta)}{4}\right.$$

$$\left. - \frac{5}{6}\left[\frac{3\theta}{8} + \frac{3\cos\theta\sin\theta}{8} + \frac{\cos^3\theta\sin\theta}{4}\right]\right)$$

$$- \frac{\theta}{2} - \frac{\cos\theta\sin\theta}{2}\Bigg]_0^{2\pi}$$

$$= \frac{-r}{2\pi}\left[r^4\left(\frac{6\pi}{8} - \frac{5}{6}\left(\frac{6\pi}{8}\right)\right) - \pi\right] = \frac{-r}{2\pi}\left(r^4\left(\frac{\pi}{8}\right) - \pi\right)$$

$$= \frac{r}{2} - \frac{r^5}{16} = \frac{r}{16}(8 - r^4)$$

$$r\phi' = \frac{1}{2\pi}\int h\cos\theta\, d\theta = \frac{1}{2\pi}\int_0^{2\pi}(x^4 - 1)\dot{x}\cos\theta\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(r^4\cos^4\theta - 1)(-r\sin\theta)\cos\theta\, d\theta$$

$$= \frac{-r}{2\pi}\left[r^4\int_0^{2\pi}\cos^5\theta\sin\theta\, d\theta + \int_0^{2\pi}\sin\theta\cos\theta\, d\theta\right]$$

$$= \frac{-r}{2\pi}\left[r^4\int_0^{2\pi}-u^5\, du + \int_0^{2\pi}v\, dv\right] = \frac{-r}{2\pi}\left[r^4\frac{\cos^6\theta}{6} + \frac{\sin^2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{-r}{2\pi}[0] = 0$$

Initial Conditions:

$$r(t) = \underbrace{\sqrt{x(t)^2 + \dot{x}(t)^2}}_{=a} \; ; \; \phi(t) = \underbrace{\arctan\left(\dot{x}(t)/x(t)\right)}_{=0}$$

Amplitude/Frequency: Amplitude: $16\int \frac{dr}{r(8 - r^4)} = T + C$

$$16\int \frac{dr}{r^5(8/r^4 - 1)} = 16\int \frac{\cancel{r^5}}{32\,\cancel{r^5}}\frac{1}{u}\, du$$

$$= \frac{1}{2}\ln\left(8/r^4 - 1\right) = T + C \qquad e^{T/2} + 1$$

$$r(T) = \sqrt[4]{\frac{8}{e^{T/2} + 1 + C}}$$

Frequency: $\omega = 1 + \varepsilon\phi' = 1 + O(\varepsilon^2)$

$$\phi(T) = O(\varepsilon^2)$$

Solution: $x_0 = r(T)\cos(\tau + \phi(T))$

$= 0$

$x(t,\varepsilon) = 0$

$h(x,\dot{x}) = (x^4-1)\dot{x}$

7.6.7. $\ddot{x} + x + (x^4-1)\dot{x} = 0$

Averaged Equations: $r' = \frac{1}{2\pi}\int_0^{2\pi} h\sin\theta\, d\theta = \langle h\sin\theta\rangle$

$$= \frac{1}{2\pi}\int_0^{2\pi}(x^4-1)\dot{x}\sin\theta\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left([r\cos\theta]^4-1\right)(-r\sin\theta)\sin\theta\, d\theta$$

$$= \frac{-r}{2\pi}\left[r^4\int_0^{2\pi}\cos^4\theta\sin^2\theta\, d\theta - \int_0^{2\pi}\sin^2\theta\, d\theta\right]$$

$$= \frac{-r}{2\pi}\left[r^4\int\cos^4(\theta)(1-\cos^2(\theta))\, d\theta - \int_0^{2\pi}\sin^2\theta\, d\theta\right]$$

Reduction Formula:

$$\int\cos^n(x)\,dx = \frac{n-1}{n}\int\cos^{n-2}(x)\,dx + \frac{\cos^{n-1}(x)\sin(x)}{n}$$

$$\int\sin^n(x)\,dx = \frac{n-1}{n}\int\sin^{n-2}(x)\,dx + \frac{\cos(x)\sin^{n-1}(x)}{n}$$

$$= \frac{-r}{2\pi}\left[r^4\left(\int_0^{2\pi}\cos^4(\theta)\,d\theta - \int_0^{2\pi}\cos^6(\theta)\,d\theta\right) - \int_0^{2\pi}\sin^2(\theta)\,d\theta\right]$$

$$\int_0^{2\pi}\cos^4(\theta) = \frac{3}{4}\int_0^{2\pi}\cos^2(\theta)\,d\theta + \frac{\cos^3(\theta)\sin\theta}{4}$$

$$= \frac{3}{4}\left(\frac{\theta}{2} + \frac{\cos(\theta)\sin(\theta)}{2}\right) + \frac{\cos^3(\theta)\sin(\theta)}{4}$$

$$\int_0^{2\pi}\cos^6(\theta)\,d\theta = \frac{5}{6}\int_0^{2\pi}\cos^4(\theta)\,d\theta + \frac{\cos^5(\theta)\sin(\theta)}{6}$$

$$= \frac{5}{6}\left[\frac{3\theta}{8} + \frac{3\cos(\theta)\sin(\theta)}{8} + \frac{\cos^3(\theta)\sin(\theta)}{4}\right]$$

$$\int_0^{2\pi}\sin^2(\theta)\,d\theta = \frac{\theta}{2} + \frac{\cos(\theta)\sin(\theta)}{2}$$

Solution: $x_0 = r(\tau)\cos(\tau + \phi(\tau))$

$$= \sqrt[4]{\frac{8}{e^{\tau/2} + 1 + C}}\cos(\tau + O(\varepsilon^2))$$

$$x(t,\varepsilon) = \sqrt[4]{\frac{8}{e^{\tau/2} - 1 + \frac{8}{a^4}}}\cos(\tau + O(\varepsilon^2))$$

$h(x,\dot{x}) = (|x|-1)\dot{x}$ 7.6.8 Averaged Equations:

$$r' = \frac{1}{2\pi}\int_0^{2\pi} h\sin\theta\, d\theta = \langle h\sin\theta\rangle$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(|x|-1)\dot{x}\sin\theta\, d\theta = \frac{1}{2\pi}\int_0^{2\pi}(|r\cos\theta|-1)(-r\sin\theta)\sin\theta\, d\theta$$

$$= \frac{-r}{2\pi}\left(r\int_0^{2\pi}|\cos\theta|\sin^2\theta\, d\theta + \int_0^{2\pi}\sin^2\theta\, d\theta\right)$$

$$= \frac{-r}{2\pi}\left(r\frac{\cos\theta}{|\cos\theta|}\int_0^{2\pi}\cos\theta\sin^2\theta\, d\theta + \left(\frac{\theta}{2} - \frac{\cos(\theta)\sin(\theta)}{2}\right)\Bigg|_0^{2\pi}\right)$$

$$= \frac{-r}{2\pi}\left(r\frac{\cos\theta}{|\cos\theta|}\frac{\sin^3(\theta)}{3} + \frac{\theta}{2} - \frac{\cos(\theta)\sin(\theta)}{2}\right)\Bigg|_0^{2\pi}$$

$$= \frac{-r}{2\pi}\left[-\pi\right] = \frac{r}{2}$$

$$r\phi' = \frac{1}{2\pi}\int_0^{2\pi} h\cos\theta\, d\theta = \langle h\cos\theta\rangle$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(|x|-1)\dot{x}\cos\theta\, d\theta = \langle h\cos\theta\rangle$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(|r\cos\theta|-1)(-r\sin\theta)\cos\theta\, d\theta$$

$$= \frac{-r}{2\pi}\left(r\int_0^{2\pi}|\cos\theta|\cos\theta\sin\theta\, d\theta - \int_0^{2\pi}\sin\theta\cos\theta\, d\theta\right)$$

$$= \frac{-r}{2\pi}\left(r\frac{\cos\theta}{|\cos\theta|}\int_0^{2\pi}\cos^2\theta\sin\theta\, d\theta - \frac{\sin^2\theta}{2}\Bigg|_0^{2\pi}\right)$$

$$= \frac{-r}{2\pi}\left(-r\frac{\cos\theta}{|\cos\theta|}\frac{\cos^3\theta}{3}\Bigg|_0^{2\pi}\right) = 0$$

Initial Conditions: $r(t) = \sqrt{x(t)^2 + \mathring{x}(t)^2} = a$

$\phi(t) = \arctan\left(\mathring{x}(t)/x(t)\right) = 0$

Amplitude / Frequency: Amplitude $r(T) = a$

Frequency $\omega = 1 + \varepsilon\dot{\phi}$; $\phi(T) = 0$

Solution: $x_0 = r(T)\cos(\tau + T) = a\cos(t)$

$x(t,\varepsilon) = a\cos(t)$

$h(x,\dot{x}) = (x^2-1)\dot{x}^3$ 7.6.9. $\ddot{x} + x + (x^2-1)\mathring{x}^3 = 0$

Average Equations: $r' = \frac{1}{2\pi}\int_0^{2\pi} h(x,\mathring{x})\sin(\theta)\,d\theta = \langle h\sin\theta\rangle$

$$= \frac{1}{2\pi}\int_0^{2\pi} (x^2-1)\mathring{x}^3\sin(\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (r^2\cos^2\theta - 1)(-r^3)(\sin^3\theta)\sin(\theta)\,d\theta$$

$$= \frac{-r^3}{2\pi}\left[r^2\int_0^{2\pi}\cos^2\theta\sin^4\theta\,d\theta - \int_0^{2\pi}\sin^4\theta\,d\theta\right]$$

Reduction Formula:

$$\int_0 \cos^n(x)\,dx = \frac{n-1}{n}\int\cos^{n-2}(x)\,dx + \frac{\cos^{n-1}(x)\sin(x)}{n}$$

$$\int\sin^n(x)\,dx = \frac{n-1}{n}\int\sin^{n-2}(x)\,dx + \frac{\cos(x)\sin^{n-1}(x)}{n}$$

$$\int_0^{2\pi}\sin^2(\theta)\,d\theta = \frac{\theta}{2} + \frac{\cos(\theta)\sin(\theta)}{2}$$

$$-\int_0^{2\pi}\sin^6(\theta)\,d\theta = \frac{-5}{6}\int_0^{2\pi}\sin^4(\theta)\,d\theta + \frac{\cos(\theta)\sin^5(\theta)}{6}$$

$$\int_0^{2\pi}\sin^4(\theta)\,d\theta = \frac{3}{4}\int_0^{2\pi}\sin^2\theta\,d\theta + \frac{\cos(\theta)\sin^3(\theta)}{4}$$

$$\int_0^{2\pi}\sin^2\theta\,d\theta = \frac{\theta}{2} + \frac{\cos(\theta)\sin(\theta)}{2}$$

$$= \frac{-r^3}{2\pi}\left[r^2\left[\left[\frac{\theta}{2} + \frac{\cos(\theta)\sin(\theta)}{2}\right] - \frac{5}{6}\left[\frac{3}{4}\left[\frac{\theta}{2} + \frac{\cos(\theta)\sin(\theta)}{2}\right] + \frac{\cos(\theta)\sin^3(\theta)}{4}\right] + \frac{\cos(\theta)\sin^5(\theta)}{6}\right] - \frac{3}{4}\left[\left[\frac{\theta}{2} + \frac{\cos\theta\sin\theta}{2}\right] + \frac{\cos\theta\sin^5\theta}{6}\right]\right]_0^{2\pi}$$

$$= \frac{-r^3}{2\pi}\left[r^2\left[\pi - \frac{15}{24}\pi\right] - \frac{3}{4}\pi\right]$$

$$= -\frac{r^3}{2\pi}\left[\frac{3}{8}\pi r^2 - \frac{3}{4}\pi\right] = \frac{r^3}{16}\left(6 - r^2\right)\pi$$

$$r\phi' = \frac{1}{2\pi}\int_0^{2\pi} h\cos(\theta)\,d\theta = \langle h\cos\theta\rangle$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(x^2-1)\,\dot{x}^3\cos(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}(r^2\cos^2\theta - 1)(-r^3)\sin^3\theta\cos(\theta)\,d\theta$$

$$= \frac{-r^3}{2\pi}\left[r^2\int_0^{2\pi}\cos^3\theta\sin^3\theta\,d\theta - \int_0^{2\pi}\sin^3\theta\cos\theta\,d\theta\right]$$

$$= -\frac{r^3}{2\pi}\left[r^2\int_0^{2\pi}\cos(\theta)(1-\sin^2\theta)\sin^3\theta\,d\theta - \frac{\sin^4\theta}{4}\right]_0^{2\pi}$$

$$= -\frac{r^3}{2\pi}\left[r^2\left[\int_0^{2\pi}\cos\theta\sin^3\theta\,d\theta - \int_0^{2\pi}\cos\theta\sin^5\theta\,d\theta\right]\right]$$

$$= \frac{-r^3}{2\pi}\left[r^2\left[\frac{\sin^4\theta}{4} - \frac{\sin^6\theta}{6}\right]\right]_0^{2\pi} = 0$$

Initial Conditions: $r(t) = \sqrt{x(t)^2 + \dot{x}(t)^2} = a$

$\phi(t) = \arctan(\dot{x}(t)/x(t)) = 0$

Amplitude/Frequency: Amplitude: $T = \int \frac{dr}{r^3\left(\frac{3}{16}r^2 - \frac{3}{8}\pi\right)}$

$$= -16\int \frac{dr}{r^3(r^2 - 6)}$$

$$= -16\int\left(\frac{A}{r} + \frac{B}{r^2} + \frac{C}{r^3} + \frac{Dr + E}{(r^2 - 6)}\right)dr$$

$A = -\frac{1}{36}$; $B = 0$; $C = -\frac{1}{6}$; $D = \frac{1}{36}$; $E = 0$

$$= -16\int \frac{r}{36(r^2-6)} - \frac{1}{36r} - \frac{1}{6r}\,dr$$

$$= \frac{2}{9}\ln(|r^2-6|) + \frac{4}{9}\ln(|r|) - \frac{12}{9}\frac{1}{r^2} = T + C$$

Frequency: $\omega = 1 + \varepsilon\phi'$; $\phi(T) = \phi_0$

Solution: $X_0 = r(T)\cos(T + \phi(T))$

$$= \sqrt{6}\cos(T) + \phi_0)$$

$$X(t,\varepsilon) = \sqrt{6}\cos(\tau + \phi_0)$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

$$\sin\theta\cos^2\theta = \frac{\sin\theta + \sin 3\theta}{4}$$

7.6.10. $\sin\theta\cos^2\theta = \left[\frac{e^{i\theta} - e^{-i\theta}}{2}\right]\left[\frac{e^{i\theta} + e^{-i\theta}}{2}\right]^2$

$$= \frac{1}{8}\left[e^{i\theta} - e^{-i\theta} + e^{3i\theta} - e^{-3i\theta}\right]$$

$$= \frac{\sin\theta + \sin 3\theta}{4}.$$

$$\partial_{\tau\tau}X_1 + X_1 = \left[-2r' + r - \frac{1}{4}r^3\right]\sin(\tau+\phi)$$

$$+ \left[-2r\phi'\right]\cos(\tau+\phi) - \frac{1}{4}r^3\sin 3(\tau+\phi)$$

7.6.11 $X(0) = 2$; $\dot{X}(0) = 0$

$$r(T) = \sqrt{X(t)^2 + \dot{X}(t)^2} = 2;\quad \phi(T) = \arctan\left(\frac{\dot{X}(t)}{X(t)}\right) = 0$$

$$\partial_{\tau\tau}X_1 + X_1 = [2-2]\sin(\tau+\phi) + [0]\cos(\tau+\phi) - \frac{1}{2}\sin 3(\tau+\phi)$$

$$\partial_{\tau\tau}X_1 + X_1 = -\frac{1}{2}\sin 3(t+\phi)$$

$$\underbrace{2\partial_{\tau\tau}X_1 + 2X_1 = -\sin 3(t+\phi)}$$

Linear Equation with constant Coefficients.

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = f(y)$$

$$a_0\lambda^n + a_1\lambda^{(n-1)} + \cdots + a_{n-1}\lambda + a_n = 0$$

$$2\lambda^2 + 2 = 0$$

$$2(\lambda^2+1) = 0$$

$$\lambda_{1,2} = \pm i \,;\; k=1 \,;\; \tau = C_1 \sin(y) + C_2 \cos(y)$$

Solution to a Homogenous Differential Equation:

$$y = \sum_i P_{k-1}(y) e^{\alpha y} \sin \beta y + Q_{k-1}(y) e^{\alpha y} \cos \beta y$$

Where $\lambda = \alpha \pm \beta i$

$$P_{k-1}(y) e^{\alpha y} = C_1$$

$$Q_{k-1}(y) e^{\alpha y} = C_2$$

Particular Solution: $2\partial_{\tau\tau} X_1 + 2X_1 = -\sin 3(t+T)$

Assume: $X_1 = A$; then $\partial_{\tau\tau} X_1 = 0$

$$2A = -\sin 3(t+T)$$

$$A = -\tfrac{1}{2} \sin 3(t+T)$$

General Solution: X = Homogenous + Particular

$$= C_1 \sin \tau + C \cos \tau - \frac{\sin 3(t+T)}{2}$$

$$\langle f(\theta)\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\,d\theta$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\cos k\theta\,d\theta$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\sin k\theta\,d\theta$$

7.6.12

a. $\langle \cos k\theta \sin m\theta \rangle$ Method #1: even x odd = odd; $\int_{-\infty}^{\infty} \text{odd} = \int_0^{2\pi} \text{odd} = 0$

Method #2: Product to sum of Two angles

$$\cos\phi \sin\theta = \frac{\sin(\theta+\phi) - \sin(\theta-\phi)}{2}$$

$$\int \cos k\theta \sin m\theta \, d\theta = \int \frac{\sin((k+m)\theta) - \sin((k-m)\theta)}{2}\, d\theta$$

b. $\langle \cos k\theta \cos m\theta \rangle$ Method #1: even x even = even, odd x odd = even $\int \text{even} =$

Method #2: Product to sum of Two Angles

$$\cos\theta \cos\phi = \frac{\cos(\theta-\phi) + \cos(\theta+\phi)}{2}$$

$$= \int \frac{\cos((k-m)\theta) + \cos((k+m)\theta)}{2} \, d\theta$$

$\neq 0$ for $k \neq m$

c. $\langle \cos^2 k\theta \rangle = \langle \sin^2 k\theta \rangle = \frac{1}{2}$: Method #1 : even $\times$ even = even

$\int$ even = constant.

Method #2 : Product to Sum of Two Angles

$$\cos\theta\cos\phi = \frac{\cos(\theta-\phi) + \cos(\theta+\phi)}{2}$$

$$\int \cos^2 k\theta = \int \frac{1}{2} + \int \frac{\cos 2k\theta}{2} \, d\theta$$

$$= \frac{1}{2}$$

b. $h(\theta) = \sum_{k=0}^{\infty} a_k \cos k\theta + \sum_{k=1}^{\infty} b_k \sin k\theta$

$$\cos m\theta \, h(\theta) = \sum_{k=0}^{\infty} a_k \cos k\theta \cos m\theta + \sum_{k=1}^{\infty} b_k \sin k\theta \cos m\theta$$

$$\cos m\theta \, h(\theta) = \sum_{k=0}^{2\pi} a_k \cos k\theta \cos m\theta + \sum_{k=0}^{2\pi} b_k \sin k\theta \cos m\theta$$

$$\langle h(\theta) \cos m\theta \rangle = \frac{1}{2} a_k$$

c. $h(\theta) = \sum_{k=0}^{\infty} a_k \cos k\theta + \sum_{k=0}^{\infty} b_k \sin k\theta$

$$\langle h(\theta) \sin k\theta \rangle = \sum_{k=0}^{2\pi} a_k \cos k\theta \sin + \sum_{k=0}^{2\pi} b_k \sin k\theta \sin k\theta$$

$$= \frac{1}{2} b_k$$

$$\langle h(\theta) \rangle = \sum_{k=0}^{2\pi} a_k \cos k\theta + \sum_{k=0}^{2\pi} b_k \sin k\theta$$

$$= a_0$$

$\ddot{x}+x+\varepsilon x^3$ 7.6.13 $x(0)=a$; $\dot{x}(0)=0$

a, $h=x^3$; $r' = \frac{1}{2\pi}\int_0^{2\pi} h\sin\theta\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} x^3 \sin\theta\, d\theta$

$$= \frac{1}{2\pi}\int_0^{2\pi} (r\cos\theta)^3 \sin\theta\, d\theta = \frac{r^3}{2\pi}\int_0^{2\pi} \cos^3\theta \sin\theta\, d\theta$$

$$= \frac{r^3}{2\pi} \frac{\cos^3\theta}{3}\Big|_0^{2\pi} = 0$$

$T = r(T) =$ constant.

b, $r\phi' = \frac{1}{2\pi}\int_0^{2\pi} h\cos\theta\, d\theta - \int_0^{2\pi} (r^3\cos\theta)^3 \cos\theta\, d\theta$

$$= \frac{r^3}{2\pi}\int_0^{2\pi} \cos^3\theta \cdot \cos\theta\, d\theta = r^3 \int_0^{2\pi} \cos^4\theta\, d\theta$$

Reduction Formula:

$$\int \cos^n\theta\, d\theta = \frac{n-1}{n}\int \cos^{n-2}\theta\, d\theta + \frac{\cos^{n-1}\theta \sin\theta}{n}$$

$$\int \cos^4\theta\, d\theta = \frac{3}{4}\int \cos^2\theta\, d\theta + \frac{\cos^3\theta \sin\theta}{4}$$

$$\int \cos^2\theta\, d\theta = \frac{\theta}{2} + \frac{\cos\theta \sin\theta}{2}$$

$$r\phi' = \frac{r^3}{2\pi}\left[\frac{3}{4}\left[\frac{\theta}{2} + \frac{\cos\theta\sin\theta}{2}\right]_0 + \frac{\cos^3\theta\sin\theta}{4}\right]_0^{2\pi}$$

$$= \frac{r^3}{2\pi}\left[\frac{3}{4}\pi\right] = \frac{3}{8}r^3$$

$\phi' = \frac{3}{8}r^2$; $\omega = 1+\varepsilon\phi' = 1+\frac{3}{8}r^2\varepsilon = 1+\frac{3}{8}a^2\varepsilon$

$$T=\frac{2\pi}{\omega}=\frac{2\pi}{1+\frac{3}{8}\varepsilon a^2}=2\pi\left(1-\frac{3}{8}\varepsilon a^2+\left(\frac{3}{8}\varepsilon a^2\right)+\cdots\right)$$

b. Note: The book answer is a power series about kinetic energy, while the Fourier series arrived to the same answer.

$\ddot{x}+\sin x=0$ 7.6.15.

a. $\ddot{x}=-\sin(x)$; $F=ma=m\ddot{x}=-kx=-\sin(x)$

$$\cong-\left(x-\frac{1}{6}x^3\right)$$

$$=-\left(1-\frac{1}{6}x^2\right)x$$

$$=-kx$$

$$r\phi'=\frac{1}{2\pi}\int_0^{2\pi}h\cos\theta\, d\theta=\frac{1}{2\pi}\int_0^{2\pi}r^3\cos^3\theta\cos\theta\, d\theta=\frac{3}{8}r^3$$

$$\omega=1+\varepsilon\phi'=1+\varepsilon\frac{3}{8}r^2\approx1-\frac{1}{16}a^2$$

b. $T=\frac{2\pi}{\omega}=\frac{2\pi}{1-\frac{1}{16}a^2}=2\pi\left(1+\frac{1}{16}a^2+\cdots\right)$ "Agreement"

$\ddot{x}+\varepsilon\dot{x}(x^2-1)+x=0$ 7.6.16. Green's Theorem : $\oint_C v\cdot n\, dl=\iint_A \nabla\cdot v\, dA$

$$v=\dot{x}=(\dot{x},\dot{y})$$

$$\dot{v}=-x-\varepsilon\dot{x}(x^2-1)=-x-\varepsilon v(x^2-1)$$

$$\iint_A\nabla\cdot v\, dA=\iint_A\frac{\partial}{\partial v}\left(-x-\varepsilon v(x^2-1)\right)dA=\iint_A-\varepsilon(x^2-1)\, dA$$

$$=\int_0^{2\pi}\int_0^a-\varepsilon(r^2\cos^2\theta-1)\, r\, dr\, d\theta=\int_0^{2\pi}\left.-\frac{\varepsilon}{4}r^4\cos^2\theta+\frac{\varepsilon r^2}{2}\right|_0^a d\theta$$

$$=\int_0^{2\pi}-\frac{\varepsilon}{4}a^4\cos^4\theta+\frac{\varepsilon a^2}{2}\, d\theta=-\frac{\varepsilon a^4\pi}{4}+\varepsilon a^2\pi$$

$$r' = \frac{1}{2\pi}\int_0^{2\pi} r(\gamma + \cos(2t))\cos\theta\sin\theta\, d\theta$$

$$= \frac{r}{2\pi}\left[\int_0^{2\pi}\gamma\sin\theta\cos\theta\,d\theta + \int_0^{2\pi}\cos(2\theta-2\phi)\cos\theta\sin\theta\,d\theta\right]$$

$$= \frac{r}{2\pi}\left[\int_0^{2\pi}\cos 2\theta\cos 2\phi\cos\theta\sin\theta\,d\theta + \int_0^{2\pi}\sin 2\theta\sin 2\phi\cos\theta\sin\theta\,d\theta\right]$$

$$= \frac{r}{2\pi}\left[\int_0^{2\pi}(2\cos^2\theta-1)\cos 2\phi\cos\theta\sin\theta\,d\theta + \int_0^{2\pi}2\cos\theta\sin\theta\cos\theta\sin\theta\,d\theta\right]$$

$$= \frac{r}{2\pi}\left[\frac{2\pi}{4}\sin 2\phi\right] = \frac{r}{4}\sin 2\phi$$

b. If $r=0$, $\phi' = 0 = \frac{r}{2}\left(\gamma + \frac{\cos 2\phi}{2}\right)$; $-\cos 2\phi = 2\gamma$

$$r' = 0 = \frac{r}{2}\sin(\arccos(-2\gamma))$$

$$= \frac{r}{2}\sqrt{1-(-2\gamma)^2} = \frac{r}{2}\sqrt{1-4\gamma^2}$$

When $\gamma < 1/2$, then $r' > 0$

$\gamma = 1/2$, then $r' = 0$

$\gamma > 1/2$, then $r' \propto i$

Therefore $\gamma < 1/2$ is a critical value.

c. $r' = \frac{dr}{dT} = \frac{r}{4}\sqrt{1-4\gamma^2}$ @ $r=0$, then $T = \int \frac{4\,dr}{r\sqrt{1-4\gamma^2}}$

$$= \frac{4}{\sqrt{1-4\gamma^2}}\ln r$$

and $r(T) = e^{\sqrt{1-4\gamma^2}\,T/4}$

Where $R = \frac{\sqrt{1-4\gamma^2}}{4}$

$$\oint_V v\cdot n\, dl = \int_0^{2\pi} \langle a\sin(t), -a\cos(t) - \varepsilon a\sin(t)(a^2\cos^2(t)-1)\rangle \cdot \langle -a\sin(t), a\cos(t)\rangle\, dt$$

$$= \int_0^{2\pi} -\varepsilon a^4 \sin(t)\cos^3(t) + \varepsilon a^3 \sin(t)\cos(t)$$

$$= \frac{a^4}{4}\varepsilon\cos^4(t) - \frac{a^2}{2}\varepsilon\cos^2(t)\Big|_0^{2\pi} = 0$$

$$\varepsilon a^2\pi\left(1-\frac{1}{4}a^2\right)=0 \;;\; a=2$$

$$\ddot{x} + (1+\varepsilon\gamma+\varepsilon\cos 2t)\sin x = 0$$

7.6.17. x = Angle Between swing and Downward Vertical

- $1+\varepsilon\gamma+\varepsilon\cos 2t$ = Effect of gravity and periodic pumping

$$x=0 ;\; \dot{x}=0$$

a. $\ddot{x} + (1+\varepsilon\gamma+\varepsilon\cos 2t)x=0$

$$r\phi' = \frac{r}{2\pi}\int_0^{2\pi}\left((\gamma + \cos(2t)\right)\cos^2\theta\, d\theta$$

$$= \frac{r}{2\pi}\int_0^{2\pi}(\gamma\cos^2\theta)\, d\theta + \frac{r}{2\pi}\int_0^{2\pi}\cos(2\theta-2\phi)\cos^2\theta\, d\theta$$

$$= \frac{r}{2\pi}\left[\gamma\pi + \int_0^{2\pi}(\cos 2\theta\cos 2\phi + \sin 2\theta \sin 2\phi)\cos^2\theta\, d\theta\right]$$

$$= \frac{r}{2\pi}\left[\gamma\pi + \int_0^{2\pi}\cos 2\theta\cos^2\theta\cos 2\phi\, d\theta + \int_0^{2\pi}\sin 2\theta\sin 2\phi\cos^2\theta\, d\theta\right]$$

$$= \frac{r}{2\pi}\left[\gamma\pi + \int_0^{2\pi}(2\cos^2\theta-1)\cos^2\theta\cos 2\phi\, d\theta + \int_0^{2\pi}2\cos\theta\sin\theta\sin 2\phi\cos^2\theta\, d\theta\right]$$

$$= \frac{r}{2\pi}\left[\gamma\pi + \cos 2\phi\left[\frac{6\pi}{4}-\pi\right]\right] = \frac{r}{2}\left[\gamma + \frac{\cos 2\phi}{2}\right]$$

d. $\frac{dr}{d\phi} = \frac{r'}{\phi'} = \frac{\left(\frac{r}{4}\sin(2\phi)\right)}{\left(\frac{r}{2}\left[\gamma + \frac{\cos 2\phi}{2}\right]\right)} = \frac{r\sin(2\phi)}{2\gamma + \cos(2\phi)}$

$$\int \frac{dr}{r} = \int \frac{\sin(2\phi)}{2\gamma + \cos(2\phi)}\, d\phi$$

$$\ln(r) = -\frac{1}{2}\ln(2\gamma + \cos(2\phi))$$

$r(\phi) = \frac{c}{\sqrt{2\gamma + \cos(2\phi)}}$ is a periodic function and closed orbit.

e. A physical interpretation of the form $\ddot{x} + kx = 0$ is synonymous with Hooke's law, a pendulum, or swing. In this problem,

$$k = \frac{\sqrt{1-4\gamma^2}}{4} = r(\gamma + \cos 2t)\cos\theta$$

$\ddot{x} + (a + \varepsilon\cos t)x = 0$

7.6.18 Mathieu Equation ; $a \approx 1$; $T = \varepsilon^2 t$

$$\ddot{x} + (a + \varepsilon\cos t)x = \ddot{x}_0(\tau) + \varepsilon\ddot{x}_1(\tau) + \varepsilon^2\ddot{x}_2(\tau) + \cdots$$
$$+ (a + \varepsilon\cos t)(x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau))$$
$$= [\ddot{x}_0(\tau) + a x_0(\tau)]$$
$$+ \varepsilon[\ddot{x}_1(\tau) + a x_1(\tau) + \cos t\, x_0(\tau)]$$
$$+ \varepsilon^2[\ddot{x}_2(\tau) + a x_2(\tau) + \cos(t) x_1(\tau)]$$
$$+ O(\varepsilon^3)$$

$O(1)$: $\ddot{x}_0(\tau) + a x_0(\tau) = 0$

$O(\varepsilon)$: $\ddot{x}_1(\tau) + a x_1(\tau) + \cos(t) x_0(\tau) = 0$

$O(\varepsilon^2)$: $\ddot{x}_2(\tau) + a x_2(\tau) + \cos(t) x_1(\tau) = 0$

$\ddot{x}+x+\varepsilon x^3=0$ 7.6.19. $x(0)=a$; $\dot{x}(0)=0$; $\boxed{\text{Poincare - Lindstedt Method}}$

a. $\tau=\omega t$; $\frac{d^2x}{d^2t}+x+\varepsilon x^3=\omega\frac{d^2x}{d^2\tau}+x+\varepsilon x^3$

$=\omega x''+x+\varepsilon x^3$

$=0$

b. $x(\tau,\varepsilon)=x_0(\tau)+\varepsilon x_1(\tau)+\varepsilon^2 x_2(\tau)+O(\varepsilon^3)$

$\omega=1+\varepsilon\omega_1+\varepsilon^2\omega_2+O(\varepsilon^3)$

$(1+\varepsilon\omega_1+\varepsilon^2\omega_2)^2(x_0''(\tau)+\varepsilon x_1''(\tau)+\varepsilon^2 x_2''(\tau))$

$+(x_0(\tau)+\varepsilon x_1(\tau)+\varepsilon^2 x_2(\tau))+(x_0(\tau)+\varepsilon x_1(\tau)+\varepsilon^2 x_2(t))^3=0$

$(1+2\varepsilon\omega_1+2\varepsilon^2\omega_2+\cdots)(x_0''(\tau)+\varepsilon x_1''(\tau)+\varepsilon^2 x_2''(\tau)+\cdots)$

$+(x_0(\tau)+\varepsilon x_1(\tau)+\varepsilon^2 x_2(\tau)+\cdots)+\varepsilon(x_0(\tau)+\varepsilon x_1(\tau)+\varepsilon x_2(t))^3$

$=0$

$O(1)=x_0''(\tau)+x_0(\tau)=0$

$O(\varepsilon)=x_1''(\tau)+2\omega_1 x_0''(\tau)+x_1(\tau)+\cancel{\varepsilon}x_0^3(\tau)=0$

c. $x_0(0)=a$; $\dot{x}_0(0)=0$; $x_k(0)=\dot{x}_k(0)=0$

From the blurb, $x(0)=a$; $\dot{x}(0)=0$; $\dot{x}_0=(a)$; $x_0'(0)=0$

d. $x_0''(\tau)+x_0(\tau)=0$; $x_0=a\cos(\tau)$

e. $x_1''(\tau)+x_1(\tau)=-2\omega_1 x_0''(\tau)-x_0^3(\tau)$

$=+2a\omega_1\cos(\tau)-a^3\cos^3(\tau)=2a\omega_1\cos(\tau)$

$=2a\omega_1\cos(\tau)-a^3\left[\frac{1}{4}(3\cos(\tau)+\cos(3\tau))\right]$

$$= \left(2aw_1 - \frac{3}{4}a^3\right)\cos(\tau) - \frac{a^3}{4}\cos(3\tau) \; ; \; w_1 = a^2$$

F. $X_1''(\tau) + X_1(\tau) = \left(2aw_1 - \frac{3}{4}a^3\right)\cos(\tau) - \frac{a^3}{4}\cos(3\tau)$

$$X_1''(\tau) + X_1 = -\frac{1}{4}a^3\cos(3\tau)$$

$$4X_1''(\tau) + 4X_1 = -a^3\cos(3\tau)$$

$\boxed{\text{Linear Equation of Constant Coefficients.}}$ "Homogeneous"

$$a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

$$4(\lambda^2+1) = 0 \; ; \; \lambda^2 + 1 = 0 \; ; \; \lambda_{1,2} = \pm i$$

$\boxed{\text{Solution of Homogenous Equations}}$

$$t = \sum P_{k-1}(t)e^{\alpha t}\sin\beta t + Q_{k-1}(t)e^{\alpha t}\cos\beta t$$

$$\lambda = \alpha \pm \beta i \; ; \; X = C_1\sin(t) + C_2\cos(t)$$

$\boxed{\text{Method of Undetermined Coefficients.}}$

$$X_i = t^s e^{\alpha s}(R_m(t)\cos\beta t + T_m(t)\sin\beta t) = -a^3\cos(3t)$$

$$s = 0 \; ; \; \alpha = 0 \; ; \; \beta = 3$$

$$X = B\sin(3t) + A\sin(3t)$$

$$X'' = -9B\sin(3t) - 9A\sin(3t)$$

$$-32B\sin(3t) - 32A\cos(3t) = -a^3\cos(3t)$$

$$A = \frac{a^3}{32} \; ; \; B = 0 \; ; \; X_i = \frac{a^3}{32}\cos(3t)$$

$\boxed{\text{General Solution}}$

$$X = \frac{a^3}{32}\cos(3t) + C_1\sin(t) + C_2\cos(t)$$

$$X(0) = \frac{a^3}{32} + C_2 = 0 \; ;$$

$$X = \frac{a^3}{32}\cos(3t) - \frac{a^3}{32}\cos(t)$$

$X(t,\varepsilon) = a\cos t + \varepsilon a^3\left[-\frac{3}{8} t\sin t + \frac{1}{32}(\cos 3t - \cos t)\right] + O(\varepsilon^2)$

7.6.20. $\ddot{x} + x + \varepsilon x^3 = (\ddot{x}_0 + \varepsilon \ddot{x}_1 + \varepsilon^2 \ddot{x}_2) + (x_0 + \varepsilon x_1 + \varepsilon^2 x_2) + \varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2)^3$

$$= (\ddot{x}_0 + x_0) + \varepsilon(\ddot{x}_1 + x_1 + x_0^3) + \varepsilon^2(\ddot{x}_2 + x_2 + 3x_0^2 x_1) + O(\varepsilon^3)$$

$O(1)$: $\ddot{x}_0 + x_0 = 0$

$O(\varepsilon)$: $\ddot{x}_1 + x_1 + x_0^3 = 0$

$O(\varepsilon^2)$: $\ddot{x}_2 + x_2 + 3x_0^2 x_1 = 0$

Solving for x_0; $\ddot{x}_0 + x_0 = 0$; $x_0 = a\cos(t)$

Solving for x_1; $\ddot{x}_1 + x_1 = -a^3\cos^3(t)$

Linear Equation of constant coefficients

$a_0 t^{(n)} + a_1 t^{(n-1)} + \dots + a_{n-1} t' + a_n t = f(t)$

Solving the Homogenous Equation

$a_0 \lambda^n + a_1 \lambda^{(n-1)} + \dots + a_{n-1}\lambda' + a_n \lambda = 0$

$(\lambda^2 + 1) = 0$; $\lambda_{1,2} = \pm i$

Solution of a Homogenous Equation; Summand

$t = \sum P_{k-1}(t) e^{\alpha t}\sin\beta t + Q_{k-1}(t) e^{\alpha t}\cos\beta t$

where $\lambda = \alpha + \beta i$; $\alpha = 0$; $\beta = 1$; $k = 1$

so $X = A\sin(t) + B\cos(t)$

Method for Undetermined Coefficients: Particular Solution

$X_i = t^s e^{\alpha t}(R_m(t)\cos\beta t + T_m(t)\sin\beta t)$

If $s = k = 1$, $x = t(B\sin t + A\cos t)$

$$\ddot{x} = (-Bt - 2A)\sin t + (2B - At)\cos t$$

$$\ddot{x}_1 + x_1 = 2B\cos(t) - 2A\sin(t) = -a^3\cos t$$

$$2B = -a^3 \;;\; A = 0$$

$$-2A = 0 \;;\; B = -\frac{a^3}{2}$$

$$x_1 = -\frac{\varepsilon a^3 t \sin t}{2}$$

General Solution = Particular + Homogenous Equation

$$x_1(t) = -\frac{\varepsilon a^3 t\sin t}{2} + C_1\sin t + C_2\cos t$$

with initial conditions: $x(0) = a$; $\dot{x}(0) = 0$

$$x_1(t) = a\cos t$$

Solving for x_2: $\ddot{x}_2 + x_2 = -3x_0^2 x_1$

Linear Equation with constant coefficients

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \dots + a_{n-1} t' + a_n t = f(t)$$

Solving the Homogeneous Equation

$$a_0\lambda^n + a_1\lambda^{(n-1)} + \dots + a_{n-1}\lambda' + a_n\lambda = 0$$

$$(\lambda^2 + 1) = 0 \;;\; \lambda_{1,2} = \pm i \;;\; k = 1$$

Solution of a Homogeneous Equation: Summand

$$t = \sum P_{k-1}(t)e^{\alpha t}\sin\beta t + Q_{k-1}(t)e^{\alpha t}\cos\beta t$$

where $\lambda = \alpha + \beta i$; $\alpha = 0$; $\beta = 1$; $k = 1$

so $x = A\sin(t) + B\cos(t)$

Method for Undetermined Coefficients: Particular Solution

$$x_i = t^s e^{\alpha t}(R_m(t)\cos\beta t + T_m(t)\sin\beta t)$$

$$= t(B\sin t + A\cos t)$$

$$\ddot{X} = (-Bt-2A)\sin t + (2B-At)\cos t$$

$$\ddot{X}_2 + X_2 = 2B\cos(t) - 2A\sin(t) = -3a^3\cos^3(t)$$

$$2B = -3a^3\cos^2(t) \;;\; A=0$$

$$B = -\frac{3a^3\cos^2(t)}{2}$$

$$X_2 = -\frac{3\varepsilon a^3 t\cos^3(t)}{2}$$

$\boxed{\text{General Solution = Particular + Homogeneous}}$

$$X_2(t,\varepsilon) = -\frac{3\varepsilon^2 a^3 t\cos^3(t)}{2} + A\sin(t) + B\cos(t)$$

with initial conditions ; $B = a + \frac{3\varepsilon^2 a^3 t}{2}$

$$\ddot{X}_2(t,\varepsilon) = \frac{+9\varepsilon a^3 t\cos^2(t)\sin(t)}{2} + A\cos(t) - B\sin(t)$$

$$A = 0$$

$$\ddot{X}_2(t,\varepsilon) = -\frac{3\varepsilon^2 a^3 t\cos^3(t)}{2} + \left(a + \frac{3\varepsilon^2 a^3 t}{2}\right)\cos(t)$$

$$= \frac{-3\varepsilon^2 a^3 t}{2}\left[\frac{1}{4}\left[3\cos t + \cos 3t\right]\right] + \left(a + \frac{3\varepsilon a^2 t}{2}\right)\cos(t)$$

$$= a\cos t + \varepsilon a^3\left[\frac{3}{2}t\cos(t) + \frac{3}{8}t\left[\cos 3t - 3\cos t\right]\right]$$

This solution is not an exact answer to why the Duffing oscillator has a frequency dependent on amplitude.

$\ddot{X} + \varepsilon(X^2-1)\dot{X} + X = 0$ 7.6.21. $\boxed{\text{Poincare - Lindstedt Method.}}$

① Define a new time $\tau = \omega t$

$$\omega^2 x'' + \varepsilon\omega(x^2-1) + x = 0$$

$X_1(0) = 0 \; ; \; \dot{X}_1(0) = 0 \; ;$

$\ddot{X}_1 + X_1 = 0 = 2w_1 a \; ; \; w_1 = 0$

$\ddot{X}_1 + X_1 = \frac{+a^3}{4}\left[3\sin(t) - \sin(3t)\right]$

$\boxed{\text{Linear Equation with constant Coefficients}}$

$a_0 t^{(n)} + a_1 t^{(n-1)} + \cdots + a_{n-1} t' + a_n t = f(t)$

$\boxed{\text{Solving the Homogeneous Equation}}$

$a_0 \lambda^{(n)} + a_1 \lambda^{(n-1)} + \cdots + a_{n-1}\lambda' + a_n \lambda = 0$

$(\lambda^2 + 1) = 0 \; ; \; \lambda_{1,2} = \pm i \; ; \; K\beta i$

$\boxed{\text{Solution to a Homogeneous Equation : Summand}}$

$t_i = \sum P_{k-1}(t) e^{\alpha t} \sin\beta t + Q_{k-1}(t) e^{\alpha t} \cos\beta t$

where $\lambda = \alpha + \beta i \; ; \; \alpha = 0 \; ; \; \beta = 1 \; ; \; K = 1$

so $X = A_1 \sin(t) + B_1 \cos(t)$

$\boxed{\text{Solving the Particular Equation : Method for Undetermined Coefficients.}}$

$X_i = t^s e^{\alpha t}(R_m(t)\cos\beta t + T_m(t)\sin\beta t)$

$X_1 = t\left[A_2 \cos t + B_2 \sin t\right]$

$\ddot{X}_1 = (-B_2 t - 2A_1)\sin(t) + (2B_2 - A_1 t)\cos(t)$

$\ddot{X}_1 + X_1 = 2B_2\cos(t) - 2A_2\sin(t) = \frac{-a^3}{4}\left[3\sin(t) - \sin(3t)\right]$

$2B_2 = 0 \; ; \; -2A_2 = \frac{a^3}{8}\left[1 - \frac{2\sin(t)\sin(3t)}{\cos(2t) - 1}\right]$

$\boxed{\text{General Solution = Particular + Homogeneous}}$

$X_1(t) = A_1\sin(t) + B_1\cos(t) - \frac{a^3}{4}\sin(3t)$

With initial conditions : $X_1(0) = 0 \; ; \; \dot{X}_1(0) = 0$

$A_1 = \frac{-a^3}{4} \; ; \; B_1 = 0$

$X_1(t) = -\frac{a^3}{4}\left[3\sin(t) + \sin(3t)\right]$

② Assign a new $x(\tau,\varepsilon) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \cdots$

$$\omega = 1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \cdots$$

③ Solve the perturbation equation

$$\omega^2 x'' + \varepsilon\omega(x^2-1)x' + x$$

$$= (1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots)^2 (\ddot{x}_0 + \varepsilon \ddot{x}_1 + \varepsilon^2 \ddot{x}_2 + \cdots)$$

$$+ \varepsilon \cdot (1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots)([x_0 + \varepsilon x_1 + \varepsilon^2 x_2]^2 - 1) \circ$$

$$(\dot{x}_0 + \varepsilon \dot{x}_1 + \varepsilon^2 \dot{x}_2) + (x_0 + \varepsilon x_1 + \varepsilon^2 x_2)$$

$O(1)$: $\ddot{x}_0 + x_0 = 0$

$O(\varepsilon)$: $2\omega_1 \ddot{x}_0 + \ddot{x}_1 + x_0^2 \dot{x}_0 - \dot{x}_0 + x_1 = 0$

$O(\varepsilon^2)$: $2\ddot{x}_0 \omega_2 + 2\ddot{x}_0 \omega_1^2 + 2\ddot{x}_1 \omega_1 + x_0^2 \dot{x}_0 \omega_1 + \ddot{x}_0 \omega_1^2$

$$+ x_0^2 \dot{x}_1 + 2x_0 \dot{x}_0 x_1 - \dot{x}_1 \omega_1 - \dot{x}_1 + \ddot{x}_2 + x_2 = 0$$

$$\ddot{x}_2 + x_2 = -(\omega_1^2 + 2\omega_2) x_0'' - 2\ddot{x}_1 \omega_1^2 - 2x_0 \dot{x}_0 x_1$$

$$+ (1 - x_0^2)(x_1' + \omega_1 x_0')$$

Solving for x_0:

$$\ddot{x}_0 + x_0 = 0 \; ; \; x(0) = a \; ; \; \dot{x}(0) = 0$$

$$x_0(t) = a\cos(t)$$

Solving for x_1:

$$\ddot{x}_1 + x_1 = -2\omega_1 \ddot{x}_0 + (1 - x_0^2)\dot{x}_0$$

$$= +2\omega_1 a\cos(t) - (1 - a^2\cos^2(t))\, a\sin(t)$$

$$= a\left[2\omega_1 \cos(t) - \frac{a^2}{4}[3\sin(t) - \sin(3t)]\right]$$

Solving for X_2:

$-\frac{3}{4}(3\sin t\cos 3t+\sin 3t\cos t)$

$-\frac{3}{2}(3\cos(t)\cos 3t-5\sin t\sin 3t)$

$$\ddot{X}_2+X_2=-(\omega_1^2+2\omega_2)\ddot{X}_0-2\ddot{X}_1\omega_1^2-2X_0\dot{X}_0X_1+(1-X_0^2)(\dot{X}_1+\omega_1\ddot{X}_0)$$

$$=+2\omega_2 a\cos(t)+2a^2\cos(t)\sin(t)\left[\frac{-1}{4}a^3\left[3\sin(t)+\sin(3t)\right]\right]$$

$$-\frac{3}{4}\left(3\sin(t)\cos(3t)+\sin(3t)\cos(t)\right)\left(1-a^2\cos^2(t)\right)$$

$$=\frac{1}{4}\left[\cos(t)\left(8a\omega_2-2a^5\sin(t)\left(3\sin(t)+\sin(3t)\right)\right)\right]$$

$$+(3a^2\cos^2(t)-3)(3\sin(t)\cos(3t))+\sin(3t)\cos(t)$$

$\boxed{\text{Linear Equation of Constant Coefficients}}$

$$a_0t^{(n)}+a_1t^{(n-1)}+\dots+a_{n-1}t'+a_nt=f(t)$$

$\boxed{\text{Solving the Homogeneous Equation}}$

$$a_0\lambda^{(n)}+a_1\lambda^{(n-1)}+\dots+a_{n-1}\lambda'+a_n\lambda^{\circ}=0$$

$$4(\lambda^2+1)=0\ ;\ \lambda_{1,2}=\pm i\ ;\ K=1$$

$\boxed{\text{Solution to a Homogeneous Equation: Summand}}$

$$t=\sum_{n-1}R(t)e^{\alpha t}\sin\beta t+Q_{k1}(t)e^{\alpha t}\cos\beta t$$

$$X=A_1\cos t+B_1\sin t$$

$\boxed{\text{Lagranges Method of Variation of Parameters}}$

System of Equations: $C'(t)X_1+C_1'(t)X_2=0$

$$C'(t)X_1+C_1'(t)X_2^2=\frac{f(t)}{a_0}$$

where $X_1=\cos(t)$; $X_2=\sin(t)$

$$X_1'=-\sin(t)\ ;\ X_2'=\cos(t)$$

$$a_0X''=2$$

$$F(t)=\frac{1}{2}\left[\cos(t)\left(8a\omega_2-2a^5\sin(t)(3\sin(t)+\sin(3t))\right)\right]$$

$$+2(3a^2\cos^2(t)-3)(3\sin(t)\cos(3t))+\sin(3t)\cos(t)$$

Finding $C'(t)$, $C_1'(t)$ by Cramer's Rule:

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin(t) \\ \frac{\cos(t)[8aw_2 - 2a^5\sin(t)(\sin(3t)+3\sin(t))]}{2} + 2\cos(t)\sin(3t) + 6[3a^2\cos^2(t) - 3]\sin(t)\cos(3t) & \cos(t) \end{vmatrix}$$

$$= -\sin(t)\left[\frac{\cos(t)[8aw_2 - 2a^5\sin(t)(\sin(3t)+3\sin(t))}{2} + 2\cos(t)\sin(3t) + 6[3a^2\cos^2(t)-3]\sin(t)\cos(3t)\right]$$

$$W_2 = \begin{vmatrix} \cos(t) & 0 \\ -\sin(t) & \frac{\cos(t)[8aw_2 - 2a^5\sin(t)(\sin(3t)+3\sin(t))]}{2} + 2\cos(t)\sin(3t) + 6[3a^2\cos^2(t)-3]\sin(t)\cos(3t) \end{vmatrix}$$

$$= \cos(t)\left[\frac{\cos(t)[8aw_2 - 2a^5\sin(t)(\sin(3t)+3a\sin(t))]}{2} + 2\cos(t)\sin(3t) + 6[3a^2\cos(t) - 3]\sin(t)\cos(3t)\right]$$

$C'(t) = \frac{W_1}{W} = W_1$; $C_1'(t) = \frac{W_2}{W} = W_2$

$C(t) = \int C'(t)\,dt = \dots$; $C_1(t) = \int C_1'(t)\,dt = \dots$

Solving for w_2: $\ddot{X}_2 + X_2 = (4w_2 + \frac{1}{4})\cos(\tau) + \dots$

$$w_2 = -\frac{1}{16}$$

Solving for w: $\omega = (1 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots)$

$$= (1 - \frac{1}{16}\varepsilon^2 + \cdots)$$

$\ddot{X} + X + \varepsilon X^2 = 0$ 7.6.22. $x(0) = a$; $\dot{x}(0) = 0$

① Define a new time $\tau = \omega t$

$$\omega^2 X'' + \omega X + \varepsilon X^2 = 0$$

$\lambda^2+1=0$; $\lambda_{1,2}=\pm i$; $K=s=1$; $\lambda=\alpha+\beta i$

Solution to a Homogeneous Equations: Summand

$$t=\sum R_{n-1}(t)e^{\alpha t}\sin\beta t+Q_{K-1}(t)e^{\alpha t}\cos\beta t$$

$$X(t)=A_1\sin(t)+B_1\cos(t)$$

Solving the Particular Equation: Method for Undetermined Coefficients

$$X_i=t^s e^{\alpha t}(R_m(t)\cos\beta t+T_m(t)\sin\beta t)$$

$$X_1=t[A_2\cos t+B_2\sin t]$$

$$\ddot{X}_1=(-B_2t-2A_2)\sin(t)+(2B_2-A_2t)\cos(t)$$

$$\ddot{X}_1+X_1=2B_2\cos(t)-2A_2\sin(t)=a\cos(t)(W_1+W_2)+a^2\cos^2(t)$$

Initial conditions: $X(0)=a$; $\dot{X}(0)=0$

$$2B_2=a(W_1+W_2)+a^2 \quad ; -2A_2=0$$

$$B_2=\frac{a}{2}(W_1+W_2)+\frac{a^2}{2} \quad ; \quad A_2=0$$

General Solution = Particular + Homogeneous

$$X_1(t)=[a(W_1+W_2)+a^2]\cos(t)t+A_1\cos t+B_2\sin t$$

Initial conditions: $X(0)=a$; $\dot{X}(0)=0$

$$X_1(0)=a=A_1 \ ; \ A_1=a$$

$$\dot{X}_1(0)=0=B_2 \ ; \quad B_2=0$$

$$X_1(t)=[a(W_1+W_2)+a^2+a]\cos(t)$$

⑥ Solving for X_2:

$$\ddot{X}_2+X_2=-2[X_0X_1+X_0W_1W_2+\ddot{X}_1(W_1+W_2)]-\ddot{X}_0(W_1^2+W_2^2)-X_1(W_1+W_2)$$

$$=-2[a^2\cos^2(t)[(W_1+W_2)+a+1]+aW_1W_2\cos(t)$$

$$-a[(W_1+W_2)+a+1]\cos(t)(W_1+W_2)]$$

$$+a\cos(t)(W_1^2+W_2^2)-a[(W_1+W_2)+a+1]\cos(t)(W_1+W_2)$$

$$=a\cos(t)[2a\cos(t)(a+W_1+W_2+1)-(W_1+W_2)(3a+2W_1+2W_2+3)]$$

② Assign a new perturbation:

$$x(\tau,\varepsilon) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \cdots$$

$$\omega = 1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \cdots$$

③ Solve the perturbation equation:

$$\omega^2 x'' + \omega x + \varepsilon x^2 = 0$$

$$= (1+\varepsilon\omega_1+\varepsilon\omega_2)^2(\ddot{x}_0 + \varepsilon\ddot{x}_1 + \varepsilon^2\ddot{x}_2) + (1+\varepsilon\omega_1+\varepsilon\omega_2)(x_0+\varepsilon x_1+\varepsilon^2 x_2)$$

$$+ \varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2)^2$$

$$O(1): \ddot{x}_0 + x_0 = 0$$

$$O(\varepsilon): x_0^2 + x_0(\omega_1+\omega_2) + 2\ddot{x}_0(\omega_1+\omega_2) + x_1 + \ddot{x}_1 = 0$$

$$\ddot{x}_1 + x_1 = -2\ddot{x}_0(\omega_1+\omega_2) - x_0(\omega_1+\omega_2) + x_0^2$$

$$O(\varepsilon^2): \ddot{x}_2 + x_2 = -2x_0x_1 - \ddot{x}_0\omega_1^2 - 2\ddot{x}_0\omega_1\omega_2$$

$$- \ddot{x}_0\omega_2^2 - x_1(\omega_1+\omega_2) - 2\ddot{x}_1(\omega_1+\omega_2)$$

$$= -2\left[x_0x_1 + x_0\omega_1\omega_2 + \ddot{x}_1(\omega_1+\omega_2)\right]$$

$$- \ddot{x}_0(\omega_1^2+\omega_2^2) - x_1(\omega_1+\omega_2)$$

④ Solving for x_0:

$$\ddot{x}_0 + x_0 = 0; \quad x(0) = a; \ \dot{x}(0) = 0$$

$$x_0(t) = a\cos(t)$$

⑤ Solving for x_1:

$$\ddot{x}_1 + x_1 = 2a\cos(t)(\omega_1+\omega_2) - a\cos(t)(\omega_1+\omega_2) + a^2\cos^2(t)$$

$$= a\cos(t)(\omega_1+\omega_2) + a^2\cos^2(t)$$

Linear Equation of Constant coefficients

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \cdots + a_{n-1} t^1 + a_n t^0 = f(t)$$

Solving the Homogeneous Equation

$$a_0\lambda^{(n)} + a_1\lambda^{(n-1)} + \cdots + a_{n-1}\lambda^1 + a_n\lambda^0 = 0$$

$\boxed{\text{Linear Equation of Constant Coefficients}}$

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \circ\circ\circ + a_{n-1} t^{1} + a_n t^{0} = f(t)$$

$\boxed{\text{Solving the Homogeneous Equation}}$

$$a_0 \lambda^{(n)} + a_1 \lambda^{(n-1)} + \circ\circ\circ\ a_{n-1} \lambda^{1} + a_n \lambda^{0} = 0$$

$$(\lambda^2+1)=0;\ \lambda_{1,2} = \pm i;\ K=S=1$$

$\boxed{\text{Solution to the Homogeneous Equation: Summand}}$

$$t = \sum R_{m1}(t) e^{\alpha t} \sin\beta t + Q_{K1}(t) e^{\alpha t} \cos\beta t$$

$$X(t) = A_1 \sin(t) + B_2 \cos(t)$$

$\boxed{\text{Solving the Particular Equation: Method for Undetermined Coefficients.}}$

$$X_i = t^{s} \cdot e^{\alpha t} \cdot [R_m(t)\cos\beta t + T_m(t)\sin\beta t]$$

$$X_2 = t[A_2 \cos t + B_2 \sin t]$$

$$\ddot{X}_2 = (-B_2 t - 2A_2)\sin t + (2B_2 - A_2 t)\cos t$$

$$\ddot{X}_2 + X_2 = 2B_2 \cos(t) - 2A_2 \sin(t) =$$

$$= a\cos t\,[2a\cos(t)(a+W_1+W_2+1) - (W_1+W_2)(3a+2W_1+2W_2+3)]$$

Initial conditions: $X(0)=a;\ \dot{X}(0)=0$

$$2B_2 = \frac{a}{2}[2a(a+W_1+W_2+1) - (W_1+W_2)(3a+2W_1+2W_2+3)]$$

$$A_2 = 0$$

$\boxed{\text{General Solution = Particular + Homogeneous.}}$

$$X_2(t) = A_1 \sin t + B_1 \cos t + \frac{at}{2}[2a(a+W_1+W_2+1) - (W_1+W_2)(3a+2W_1+2W_2+3)]\sin(t)$$

$$X_2(0) = a = B_1;\ \dot{X}_2(0) = 0 = A_1$$

$$X_2(t) = a\cos t + \frac{at}{2}[2a(a+W_1+W_2+1) - (W_1+W_2)(3a+2W_1+2W_2+3)]\sin(t)$$

$$KE = \frac{1}{2}\varepsilon X^2 = \frac{1}{2}\varepsilon\left(a - \frac{t^2}{2}a - O(t^4)\right)^2 \approx \frac{1}{2}\varepsilon a^2 + O(t^2)$$

$\ddot{x} - \varepsilon x \dot{x} + x = 0$ 7.6.23. ① Assign a new time $\tau = \omega t$

$$\omega^2 x'' - \varepsilon x \omega x' + x = 0$$

② Assign a new perturbation

$$x(\tau, \varepsilon) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \cdots$$

$$\omega = 1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \cdots$$

③ Solve the perturbation equation:

$$(1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \cdots)^2 (\ddot{x}_0 + \varepsilon \ddot{x}_1 + \varepsilon^2 \ddot{x}_2 + \cdots)$$

$$+ \varepsilon (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)(\dot{x}_0 + \varepsilon \dot{x}_1 + \varepsilon^2 \dot{x}_2 + \cdots)$$

$$* (1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \cdots) + (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) = 0$$

$O(1)$: $\ddot{x}_0 + x_0 = 0$

$O(\varepsilon)$: $\ddot{x}_1 + x_1 = -2\omega_1 \ddot{x}_0 + \dot{x}_0 x_0$

$O(\varepsilon^2)$: $\ddot{x}_2 + x_2 = -\omega_1^2 \ddot{x}_0 + \omega_1 \dot{x}_0 x_0 - 2\omega_1 \ddot{x}_1$

$$-2\omega_2 \ddot{x}_0 + x_1 \dot{x}_0 + x_0 \dot{x}_1$$

④ Solving for x_0:

$\ddot{x}_0 + x_0 = 0$; $x(0) = a$; $\dot{x}(0) = 0$

$$x_0(t) = a \cos t$$

⑤ Solving for x_1:

$$\ddot{x}_1 + x_1 = -2\omega_1 x_0 + \dot{x}_0 x_0$$

$$= -2\omega_1 a \cos t - a^2 \cos t \sin t$$

$$= [-2\omega_1 a - a^2 \sin(t)] \cos t$$

$\boxed{\text{Linear equation with constant coefficients}}$

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \cdots + a_{(n-1)} t^1 + a_n t^0 = f(t)$$

$\boxed{\text{Solving the Homogenous Equation}}$

$$a_0 \lambda^{(n)} + a_1 \lambda^{(n-1)} + \cdots + a_{n-1} \lambda^1 + a_n \lambda^0 = 0$$

$(\lambda^2+1)=0$; $\lambda_{1,2}=\pm i$; $K=s=1$

Solution to a Homogeneous Equation: Summand.

$$t=\sum R_{M-1}(t)e^{\alpha t}\cos\beta t\ \ Q_{M-1}(t)e^{\alpha t}\sin\beta t$$

$\lambda=\alpha+\beta i$; $\alpha=0$; $\beta=1$

$$X=A_1\cos t+B_1\sin(t)$$

Lagrange's Method of variation of parameters.

① Build a system

$$A_1'X_1+B_1'X_2=0$$

$$A_1'X_1'+B_1'X_2'=\frac{P(t)}{a_0}$$

where $X_1=\cos(t)$ $X_2=\sin(t)$

$X_1'=-\sin(t)$ $X_2'=\cos(t)$

$a_0X''=1$; $F(t)=\cos(t)(-a^2\sin(t)-2aw_1)$

② Solve the System using Cramer's Rule.

$$W=\begin{vmatrix}\cos(t) & \sin(t)\\ -\sin(t) & \cos(t)\end{vmatrix}$$

$$W_1=\begin{vmatrix}0 & \sin(t)\\ \cos(t)(-a^2\sin(t)-2aw) & \cos(t)\end{vmatrix}=\cos(t)\sin(t)(-a^2\sin(t)-2aw_1)$$

$$W_2=\begin{vmatrix}\cos(t) & 0\\ -\sin(t) & \cos(t)(-a^2\sin(t)-2aw)\end{vmatrix}=\cos^2(t)(-a^2\sin(t)-2aw_1)$$

$$A_1'(t)=\frac{W_1}{W}=\cos(t)\sin(t)(-a^2\sin(t)-2aw_1)$$

$$B_1'(t)=\frac{W_2}{W}=\cos^2(t)(-a^2\sin(t)-2aw_1)$$

$$A_1=\int A_1'(t)dt=\frac{a^2\sin^3(t)}{3}+aw_1\sin^2(t)+C_1$$

$$B_1=\int B_1'(t)dt=-\frac{aw_1\sin(2t)}{2}+\frac{a^2\cos^3(t)}{3}-aw_1t+C_2$$

$$X = -\frac{aW\sin(t)\sin(2t)}{2} + \frac{a^2\cos(t)\sin^3(t)}{3} + aW\cos(t)\sin^2(t)$$

$$+\frac{a^2\cos^3(t)\sin(t)}{3} - aWt\sin(t) + C_1\sin(t) + C_2\cos(t)$$

Initial conditions: $X(0)=0$; $\dot{X}(0)=0$

$$X(0) = 0 = C_2;\quad \dot{X}(0) = 0 = 2a^2 + 6C_1;\quad C_1 = -\frac{1a^2}{3}$$

$$X_1(t) = \left[-\frac{aW_1\sin(2t)}{2} + \frac{a^2\cos(t)\sin^2(t)}{3} + aW_1\cos(t)\sin^2(t) + \frac{a^2\cos^3(t)}{3} - aW_1t - \frac{a^2}{3}\right]\sin(t)$$

$W_1 = 0$ because no secular terms

$$X_1(t) = \left[\frac{a^2\cos(t)\sin^2(t)}{3} + \frac{a^2\cos^3(t)}{3} - \frac{a^2}{3}\right]\sin(t)$$

Identities: $\sin(2t) = 2\cos(t)\sin(t)$

$$\sin^2(t) = \tfrac{1}{2}(1-\cos(2t))$$

$$\cos^2(t) = 1 - \sin^2(t)$$

$$X_1(t) = \frac{1}{6}\left(-2a^2\sin(t) + a^2\sin(2t)\right)$$

③ Solving for X_2

$$\ddot{X}_2 + X_2 = -\cancel{W_1^2\ddot{X}_0} + \cancel{W_1\dot{X}_0}X_0 - \cancel{2W_1\ddot{X}_1} - 2W_2\ddot{X}_0 + X_1\dot{X}_0 + X_0\dot{X}_1$$

$$= -2W_2\ddot{X}_0 + X_1\dot{X}_0 + X_0\dot{X}_1$$

$$= 2W_2 a\cos(t) - \frac{a\sin(t)}{6}\left(-2a^2\sin(t) + a^2\sin(2t)\right)$$

$$+ \frac{a\cos(t)}{3}\left(-2a^2\cos(t) + a^2\cos(2t)\right)$$

$\boxed{\text{Linear Equation of Constant Coefficients}}$

$$a_0 t^n + a_1 t^{(n-1)} + \dots + a_{(n-1)} t^1 + a_{(n)} t^0 = f(t)$$

$\boxed{\text{Solving the Homogeneous Equation}}$

$$a_0\lambda^n + a_1\lambda^{(n-1)} + \dots + a_{(n-1)}\lambda^1 + a_{(n)}\lambda^0 = 0$$

$(\lambda^2+1)=0$; $\lambda_{1,2}=\pm i$; $k=s=1$

Solution to a Homogeneous Equation: Summand

$$t=\sum e^{\alpha t}\left[R_{m-1}(t)\cos\beta t + Q_{m-1}(t)\sin\beta t\right]$$

$$X_2(t)=A_1\cos t + B_1\sin t$$

$\boxed{\text{Lagrange's Method of Variation of Parameters}}$

$$A_1' X_1 + B_1' X_2 = 0$$

$$A_1' X_1' + B_1' X_2' = \frac{f(t)}{a_0} \quad \text{where } X_1=\cos(y) \quad X_2=\sin(y)$$

$$X_1'=-\sin(y) \quad X_2'=\cos(y)$$

$$a_0 X''=6$$

$$f(t)=12w_2\cos(t)-a^3\sin t(-2\sin t+\sin 2t) + 2a^3\cos t(\cos t+\cos 2t)$$

$$W=\begin{vmatrix}\cos(t) & \sin(t)\\ -\sin(t) & \cos(t)\end{vmatrix}$$

$$W_1=\begin{vmatrix}0 & \sin t\\ +2w_2\cos t-\frac{a^3\sin t}{6}(-2\sin t+\sin 2t)+\frac{a^3\cos t}{3}(\cos t+\cos 2t) & \cos t\end{vmatrix}$$

$$W_2=\begin{vmatrix}\cos t & 0\\ -\sin t & 2w_2\cos t-\frac{a^3\sin t}{6}(-2\sin t+\sin 2t)+\frac{a^3\cos t}{3}(\cos t+\cos 2t)\end{vmatrix}$$

$$A_1'=\frac{W_1}{W}=\sin t\left[\frac{a^3\sin t}{6}(-2\sin t+\sin 2t)-\frac{a^3\cos t}{3}(\cos t+\cos 2t)-2w_2\cos t\right]$$

$$B_1'=\frac{W_2}{W}=\cos t\left[2w_2\cos t-\frac{a^3\sin t}{6}(-2\sin t+\sin 2t)+\frac{a^3\cos t}{3}(\cos t+\cos 2t)\right]$$

$$A_1=\int A_1'\,dt=\sin^2 t\left[\frac{a^3}{4}\sin^2 t-\frac{(6aw+a^3)}{6}\right]+\cos t\left[\frac{a^3}{3}-\frac{2a^3\cos^2(t)}{9}\right]+C_1$$

$$B_1=\int B_1'\,dt=\frac{a^3\sin 4t}{32}+\frac{(48aw+8a^3)\sin 2t}{96}+\frac{2a^3\sin^3(t)}{9}-\frac{a^3\sin(t)}{3}+awt+\frac{a^3 t}{24}+C_2$$

$$X_2(t) = \left[\frac{a^3 \sin 4t}{32} + \frac{(48aw + 8a^3)\sin 2t}{96} + \frac{2a^3 \sin^3(t)}{9} - \frac{a^3 \sin t}{3} + awt + \frac{a^3 t}{24} + C_1\right]\sin t + \left[\sin^2 t\left(\frac{a^3 \sin^2 t}{4} - \frac{(6aw + a^3)}{6}\right) + \cos t\left(\frac{a^3}{3} - \frac{2a^3\cos^2 t}{9}\right) + C_2\right]\cos t$$

Initial conditions: $X_2(0) = 0$; $\dot{X}_2(0) = 0$

$$X_2(0) = 0 = \frac{a^3}{3} - \frac{2a^3}{9} + C_2 \; ; \; C_2 = -\frac{a^3}{3} + \frac{2a^3}{9}$$

$$\dot{X}_2(0) = 0 = awt + \frac{a^3 t}{24} + C_1 \; ; \; C_1 = -awt - \frac{a^3 t}{24}$$

$$X_2(t) = \left[\frac{a^3 \sin 4t}{32} + \frac{(48aw_2 + 8a^3)\sin 2t}{96} + \frac{2a^3 \sin^3(t)}{9} - \frac{a^3 \sin t}{3}\right]\sin t + \left[\sin^2 t\left(\frac{a^3 \sin^2 t}{4} - \frac{(6aw_2 + a^3)}{6}\right) + \cos t\left(\frac{a^3}{3} - \frac{2a^3\cos^2 t}{9}\right) - aw_2 t - \frac{a^3 t}{24}\right]\cos t$$

$$w_2 = -\frac{1}{8}a^2$$

$$w(a) = 1 + \varepsilon w_1 + \varepsilon^2 w_2 + O(\varepsilon^3)$$
$$= 1 - \frac{1}{8}\varepsilon^2 a^2$$

$\ddot{X} + X - \epsilon X^3 = 0$ 7.6.24. $X(0) = a$; $\dot{X}(0) = 0$

① Assign a new time ; $\tau = wt$

$$w^2 \ddot{X} + X - \varepsilon X^3 = 0$$

② Apply perturbation equations

$$x(\tau, \varepsilon) = X_0(\tau) + \varepsilon X_1(\tau) + \varepsilon^2 X_2(\tau) + O(\varepsilon^3)$$

$$w = 1 + \varepsilon w_1 + \varepsilon^2 w_2 + O(\varepsilon^3)$$

$$(1 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots)^2(\ddot{X}_0 + \varepsilon \ddot{X}_1 + \varepsilon^2 \ddot{X}_2) + (X_0 + \varepsilon X_1 + \varepsilon^2 X_2) - \varepsilon(X_0 + \varepsilon X_1 + \varepsilon^2 X_2)^3 = 0$$

$O(1)$: $\ddot{X}_0 + X_0 = 0$

$O(\varepsilon)$: $\ddot{X}_1 + X_1 = -\omega_1 \ddot{X}_0 + X_0^3 \cancel{- X_1}$

$O(\varepsilon^2)$: $\ddot{X}_2 + X_2 = -\omega_1 \ddot{X}_1 - \omega_2 \ddot{X}_0 + 3 X_1 X_0^2$

(3) Solving for X_0:

$$\ddot{X}_0 + X_0 = 0 \;;\; X_0(0) = a \;;\; \dot{X}_0(0) = 0 : X_0(t) = a\cos t$$

(4) Solving for X_1:

$$\ddot{X}_1 + X_1 = -\omega_1 \ddot{X}_0 + X_0^3 = +\omega_1 a \cdot \cos t + a^3 \cos^3 t$$
$$= (\omega_1 a - \tfrac{3}{4} a^3)\cos t - \frac{a^3}{4}\cos 3t$$

$\boxed{\text{Linear Equation with constant coefficients}}$ $\omega_1 = a^2$

$$a_0 t^n + a_1 t^{(n-1)} + \cdots + a_{n-1} t^1 + a_n t^0 = f(t)$$

$\boxed{\text{Solving the Homogeneous Equation.}}$

$$a_0 \lambda^n + a_1 \lambda^{(n-1)} + \cdots + a_{n-1} t^1 + a_n t^0 = 0$$

$$(\lambda^2 + 1) = 0 \;;\; \lambda_{1,2} = \pm i \;;\; K = s = 1 \;;\; \lambda = \alpha + i\beta$$

$\boxed{\text{Solution to Homogeneous Equation: Summand.}}$

$$t = \sum^{1} R_{M-1}(t) e^{\alpha t}\cos\beta t + Q_{M-1}(t) e^{\alpha t}\sin\beta t$$

$$X_1(t) = A_1 \cos t + B_1 \sin t$$

$\boxed{\text{Solving the particular Equation: Method for Undetermined Coefficients.}}$

$$X_i = t^s \cdot e^{\alpha t} \cdot [R_M(t)\cos\beta t + Q_M(t)\sin\beta t]$$

$$X_1 = t[A_1 \cos 3t + B_2 \sin 3t]$$

$$\ddot{X}_1 = \cos 3t(6B_2 - 9A_2 t) - 3\sin 3t(2A_2 + 3B_2 t)$$

$$\ddot{X}_1 + X_1 = \cos(3t)(6B_2 - 8A_2 t) - 2\sin(3t)(3A_2 + 4B_2 t) = \frac{-a^3}{4}\cos 3t$$

$$B_2 = -\frac{a^3}{24}$$

$$X_1(t) = \frac{-a^3}{24} t \sin 3t$$

$\boxed{\text{General Solution: Homogeneous + Particular}}$

$$X_1(t) = A_1 \cos t + B_1 \sin t + \frac{a^3}{24} t \cos 3t$$

Initial Conditions: $X_1(0) = 0$; $\dot{X}_1(0) = 0$

$$X_1(0) = 0 = A_1 \; ; \; \dot{X}_1(0) = 0 = B_1 + \frac{a^3}{24} \; ; \; B_1 = \frac{a^3}{24}$$

$$X_1(t) = \frac{a^3}{24} \sin t - \frac{a^3}{24} t \cos 3t$$

⑤ Solving for X_2:

$$\ddot{X}_2 + X_2 = -\omega_1 \ddot{X}_1 - \omega_2 \ddot{X}_0 + 3 X_1 X_0^2$$

$$= \frac{a^5}{24}\left(-\sin t + 6 \sin 3t + 9 t \cos 3t\right) + \omega_2 a \cos t$$

$$+ \frac{3a^5}{8}\left[\sin t - t \cos 3t\right] \cos^2 t$$

$$\omega_2 = 0$$

⑥ I forgot $O(\varepsilon^3)$, which modification necessitates a time-shift $(1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \varepsilon^3\omega_3 + \cdots)$ and a perturbation of $1 + \epsilon X_1 + \epsilon^2 X_2 + \epsilon^3 X_3 + \cdots$.

Next time, for this example, $\omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2$
$= 1 + \epsilon a^2$.

$$\ddot{X} + X + \epsilon h(x, \dot{x}, t) = 0$$

7.6.25. $X(t) = r(t) \cos(t + \phi(t))$

$$\dot{X}(t) = -r(t) \sin(t + \phi(t))$$

a. $\dot{r} = \langle \epsilon h(x, \dot{x}, t) \rangle = \epsilon h \sin(t + \phi(t))$

$r\dot{\phi} = \langle \epsilon h(x, \dot{x}, t) \rangle = \epsilon h \cos(t + \phi(t))$

b. $\langle r\rangle(t) = \bar{r}(t) = \frac{1}{2\pi}\int_{t-\pi}^{t+\pi} r(\tau)\,d\tau$

$$\frac{d\langle r\rangle}{dt} = \frac{d\bar{r}(t)}{dt} = \frac{1}{2\pi}\frac{d}{dt}\left(r(t+\pi) - r(t-\pi)\right) = \left\langle \frac{dr}{dt}\right\rangle$$

c. $\frac{d\langle r\rangle}{dt} = \langle \epsilon h \sin(t+\phi)\rangle = \epsilon\langle h(x,\dot{x},t)\sin t+\phi\rangle$

$$= \epsilon\langle h(r\cos(t+\phi), -r\sin(t+\phi), t)\sin t+\phi\rangle$$

d. $\frac{d\bar{r}}{dt} = \epsilon\langle h(r\cos(t+\phi), -r\sin(t+\phi), t)\sin(t+\phi)\rangle + O(\epsilon^2)$

$$\bar{r}\frac{d\bar{\phi}}{dt} = \epsilon\langle h(r\cos(t+\phi), -r\sin(t+\phi), t)\sin(t+\phi)\rangle + O(\epsilon^2)$$

$\ddot{x} = -\epsilon x \sin^2 t$ 7.6.26 $0 \le \epsilon \ll 1$; $x = x_0$ @ $t=0$

a. $\ddot{x} = \frac{d^2x}{d^2t} = -\epsilon x\sin^2 t$; $x\ln x - x = -\epsilon\int\left[\int\frac{(1-\cos 2t)}{2}\,dt\right]dt$

$$= -\epsilon\int\left[\frac{t}{2} - \frac{\sin 2t}{4} + c\right]dt$$

$$= -\epsilon\left[\frac{t^2}{4} + \frac{\cos 2t}{8} + ct\right] \pm c$$

An alternative solution is a homogeneous, plus particular, which has continuous style.

$$x = e^{-\epsilon\prod_{1}^{\infty}\log\left[\frac{t^2}{4} + \frac{\cos 2t}{8}\right]}$$

b. $\bar{x}(t) = \frac{1}{2\pi}\int_{t-\pi}^{t+\pi} x(\tau)\,d\tau$

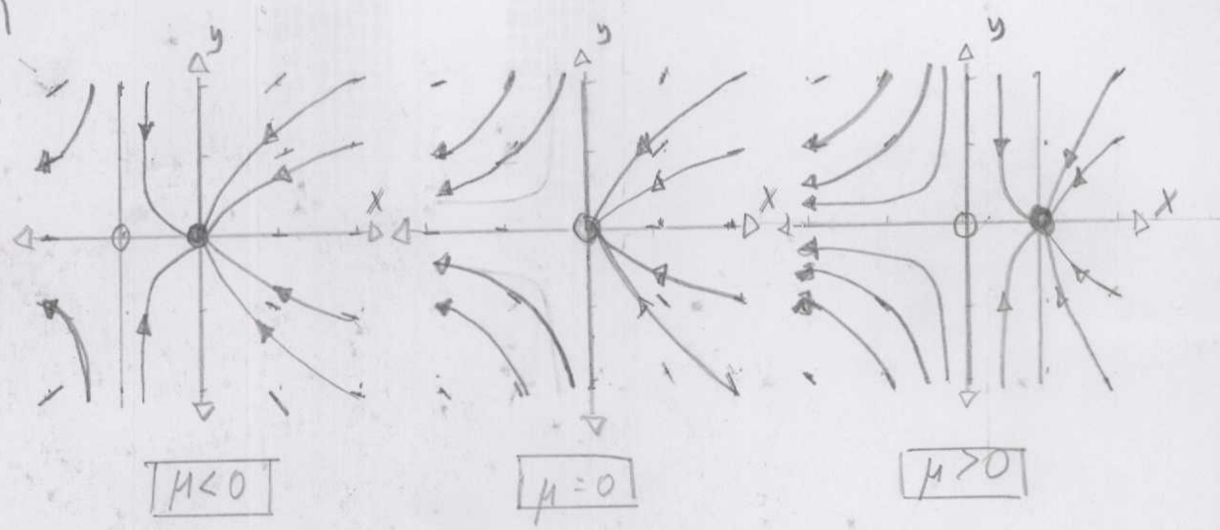
$$x(t) = \bar{x}(t) + O(\epsilon) = \frac{e^{-\epsilon\prod\log\left[\frac{1}{4}(t-\pi)^2 + \frac{\cos 2t}{8}\right]}}{\pi} + O(\epsilon)$$

c. The error depends on the exactness and amount of terms. A product-log function isn't the common method, either.

Chapter 8: Bifurcations Revisited!

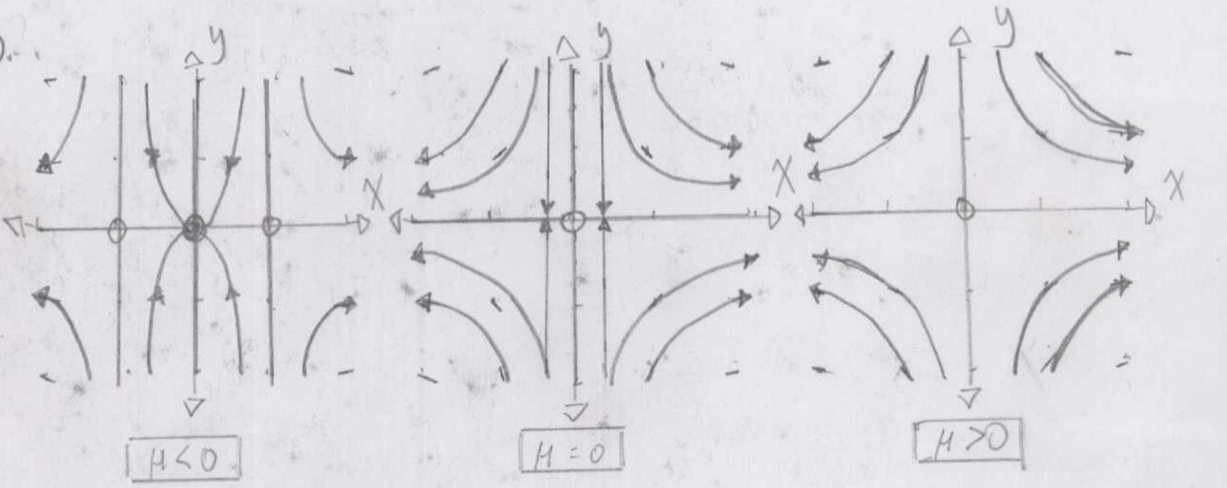
$\dot{x} = \mu x - x^2$ 8.1.1
$\dot{y} = -y$

a.



$\dot{x} = \mu x + x^3$
$\dot{y} = -y$

b.



$\dot{x} = \mu - x^2$
$\dot{y} = -y$

8.1.2. Eigenvalues: $\dot{\vec{x}} = A\vec{x} = 0$; $A\vec{x} = 0 = \lambda \vec{x}$; $(A-\lambda)\vec{x} = 0$

$$= \begin{pmatrix} -2x-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix} = (-2x-\lambda)(-1-\lambda) = 0$$

$$\lambda_1 = -2x ; \lambda_2 = -1$$

Fixed Points: $\dot{x} = 0 = \mu - x^2$
$\dot{y} = 0 = -y$; $(x^*, y^*) = (\sqrt{\mu}, 0)$

Eigenvalues + Fixed Points: $\lambda_1 = -2\sqrt{\mu}$; $\lim_{\mu \to 0} \lambda_1 = 0$

$\dot{x} = \mu x - x^2$
$\dot{y} = -y$

8.1.3. Eigenvalues: $\dot{\vec{x}} = A\vec{x} = 0$; $(A-\lambda)\vec{x} = \begin{pmatrix} \mu - 2x - \lambda_1 & 0 \\ 0 & -1-\lambda_2 \end{pmatrix} \vec{x} = 0$

$\lambda_1 = \mu - 2x$
$\lambda_2 = -1$