Chapter 2:

$\dot{x} = \sin x$ 2.1.1. $\dot{x} = 0 = \sin x$; $\boxed{x = n\pi}$ 2.1.2 $\boxed{\left(n+\frac{1}{2}\right)\pi \text{ where } n \text{ is even.}}$

2.1.3. a) $\ddot{x} = \cos x \sin x$ b) $\frac{1}{2}\sin(2x) = \cos(x)\sin(x)$; $\boxed{\ddot{x} = \frac{1}{2}\sin(2x) ; \; x = \left(n+\frac{1}{4}\right)\pi ; n \in \mathbb{Z}}$

2.1.4. a) $x_0 = \pi/4$; $t = \ln\left|(\csc x_0 + \cot x_0)\right| \left|(\csc x + \cot x)\right|$

$$e^t = \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} = \frac{\csc \pi/4 + \cot \pi/4}{\csc x + \cot x} = \frac{\frac{2}{\sqrt{2}} + 1}{\csc x + \cot x} = \frac{\sqrt{2}+1}{\csc x + \cot x}$$

$$\frac{1}{\frac{1}{\sin x} + \frac{\cos x}{\sin x}} = \frac{\sin x}{1+\cos x} = \frac{\sin\left(2\frac{x}{2}\right)}{1+\cos\left(2\frac{x}{2}\right)} = \frac{\cancel{2}\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)}{\cancel{1}+\cancel{2}\cos^{\cancel{2}}\left(\frac{x}{2}\right)\cancel{-1}} = \tan\left(\frac{x}{2}\right) = \frac{e^t}{\sqrt{2}+1}$$

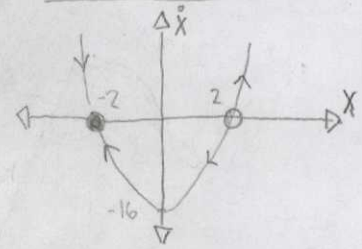
$$x(t) = 2\tan^{-1}\left(\frac{e^t}{\sqrt{2}+1}\right) ; \lim_{t\to\infty} x(t) = 2\tan^{-1}(\infty) = \boxed{2 \cdot \frac{\pi}{2} = \pi}$$

b) $\boxed{x(t) = 2\tan^{-1}\left(\frac{e^t}{\csc x_0 + \cot x_0}\right)}$

2.1.5 a) A mechanical analog of $\dot{x} = \sin x$ is the undamped pendulum having an x_0 of the maximal point

b) Unstable points are described by a positive slope (source) and stable points (sink), a negative slope. The function $\dot{x} = \sin x$ at $x^* = 0$ is unstable, while $x^* = \pi$, is stable.

$\dot{x} = 4x^2 - 16$ 2.2.1 Vector Field:



Fixed Points:	Stability:
$x = 2$	source (unstable)
$x = -2$	sink (stable)

Solving for x(t):

$$\frac{dx}{(x^2-4)} = 4t$$

$$\int \frac{A}{(x-2)}dx + \int \frac{B}{(x+2)}dx = 4t$$

$A(x+2) + B(x-2) = 1$

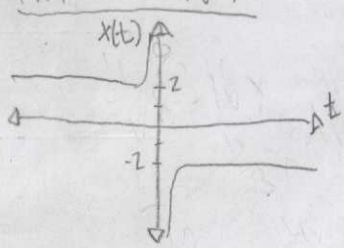
$A = 1/4 \; @ \; x = 2$

$B = -1/4 \; @ \; x = -2$

$$\ln\frac{|x-2|}{|x+2|} = 16t + c$$

$$\boxed{x(t) = \frac{2(e^{16t+c}+1)}{1-e^{16t+c}}}$$

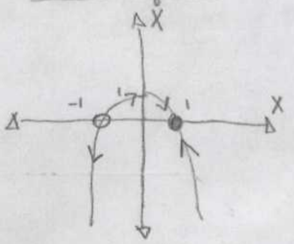
Plot of x(t):



Analytical Solution of x(t):

$$x(t) = \frac{2\left(e^{16t} \cdot \left(\frac{x_0-2}{2+x_0}\right) + 1\right)}{1 - e^{16t}\left(\frac{x_0-2}{2+x_0}\right)}$$

$\dot{x} = 1 - x^{14}$ 2.2.2 Vector Field:



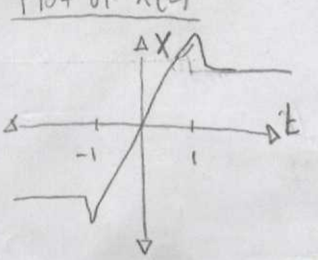
Fixed Points:	Stability:
$x = 1$	source (unstable)
$x = -1$	sink (stable)

Solving for x(t)

$$t = \int \frac{dx}{1-x^{14}}$$

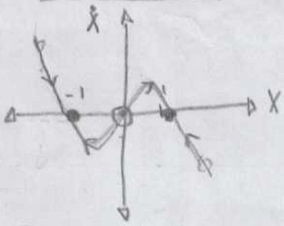
Unsolvable

Plot of x(t):

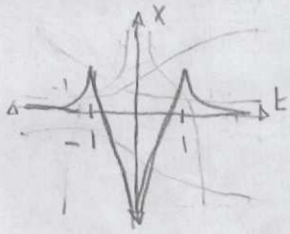


Analytical Solution of x(t):

$\dot{x} = x - x^3$ 2.2.3 Vector Field:



Plot of $x(t)$:



Fixed Points:	Stability:
$x = -1$	Stable (sink)
$x = 0$	unstable (source)
$x = 1$	stable (sink)

Analytical Solution of $x(t)$:

$x(t) =$

Solving for $x(t)$:

$$t = \int \frac{dx}{x - x^3} = \int \frac{dx}{x(1-x^2)}$$

$$= -\frac{1}{2}\int \frac{du}{(1-u)u} = -\frac{1}{2}\int \frac{A}{1-u}du - \frac{1}{2}\int \frac{B}{u}du$$

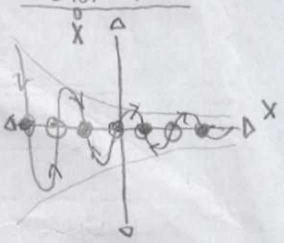
$$= -\frac{1}{2}\ln\left|\frac{1-u}{u}\right| = -\frac{1}{2}\ln\left|\frac{x^2}{1-x^2}\right|$$

$$= \frac{1}{2}\ln\left|\frac{1-x^2}{x^2}\right| + C$$

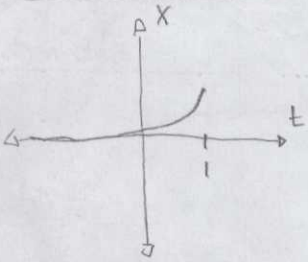
$$(e^{2t}+1)x^2 = 1$$

$$\boxed{x = \sqrt{\frac{1}{e^{2t+C}+1}}}$$

$\dot{x} = e^{-x}\sin x$ 2.2.4 Vector Field:



Plot of $x(t)$:



Fixed Points:	Stability:
$x = 2n\pi$	Source (unstable)
$x = (2n+1)\pi$	Sink (Stable)

Solving for $x(t)$:

$$t = \int \frac{e^x}{\sin x}dx = \int dx + \int \cot(x)\,dx$$

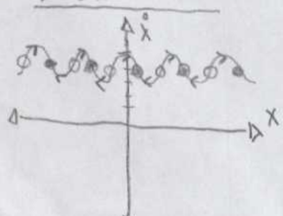
$$= 1 + \ln(\sin x) + C$$

$$\boxed{x(t) = \arcsin^{-1}(C \cdot e^{t-1})}$$

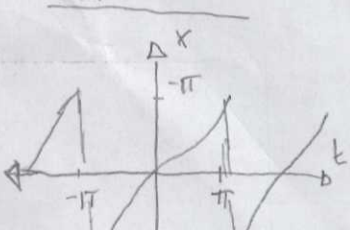
Analytical solution of $x(t)$

$$\boxed{x(t) = \arcsin^{-1}(Ce^{t} - 1) \text{ where } C = -(1+\ln(\sin x_0))}$$

$\dot{x} = 1 + \frac{1}{2}\cos x$ 2.2.5. Vector Field:



Plot of $x(t)$:



Fixed Points	Stability:
$x = (4n+1)\frac{\pi}{2}$	Sink (stable)
$x = (4n-1)\frac{\pi}{2}$	source (unstable)

Analytical solution of $x(t)$:

$$\frac{-\frac{2}{\sqrt{3}}\arctan\left(\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right)}{\sqrt{3}} + C$$

Solving for $x(t)$:

$$t = \int \frac{1}{1 + \frac{1}{2}\cos x}dx$$

$$= \int \frac{dx}{\frac{1}{2} + \cos^2\left(\frac{x}{2}\right)} = \int \frac{\sec^2\left(\frac{x}{2}\right)dx}{\frac{\sec^2\left(\frac{x}{2}\right)}{2} + 1}$$

$$\boxed{= \int \frac{\sec^2\left(\frac{x}{2}\right)dx}{3 + \tan^2(\theta/2)}}$$

$$u = \frac{\tan\left(\frac{x}{2}\right)}{\sqrt{3}};\ \frac{du}{dx} = \frac{\sec^2\left(\frac{x}{2}\right)}{2\sqrt{3}}$$

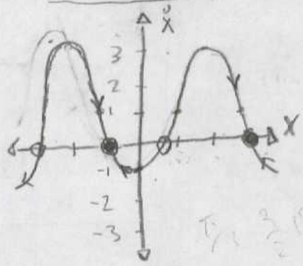
$$= \int \frac{2\sqrt{3}}{3u^2+3}du = \frac{2}{\sqrt{3}}\int \frac{du}{u^2+1}$$

$$= \frac{2}{\sqrt{3}}\arctan(u) + C$$

$$\boxed{\frac{\frac{2}{\sqrt{3}}\arctan\left(\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right)}{\sqrt{3}} + C}$$

$\dot{x} = 1 - 2\cos x$ 2.2.6.

Vector Field



Fixed Points

$x = (n+\frac{1}{2})\pi$; n = even

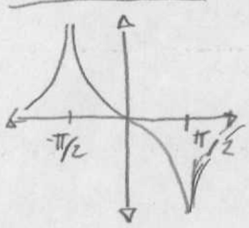
$x = (n+\frac{1}{2})\pi$; n = odd

Stability

sink (stable)

source (unstable)

Plot of x(t)



Analytical Solution of x(t)

$$x(t) = \frac{\ln}{\sqrt{3}}\left|\frac{3\tan(\frac{x}{2}) - \sqrt{3}}{3\tan(\frac{x}{2}) + \sqrt{3}}\right|$$

Solving for x(t)

$$t = \int \frac{dx}{1-2\cos x} \; ; \; \frac{1-\tan^2(\frac{x}{2})}{1+\tan^2(\frac{x}{2})} = \cos x$$

$$= -\int \frac{dx}{2\left[\frac{1-\tan^2(\frac{x}{2})}{1+\tan^2(\frac{x}{2})}\right] - 1}$$

$$= -\int \frac{dx}{2\left[\frac{1-u^2}{1+u^2}\right] - 1} \; ; \; u = \tan\left(\frac{x}{2}\right) \; ; \; \frac{du}{dx} = \frac{\sec^2(\frac{x}{2})}{2}$$

$$= -\int \frac{du}{2\sec^2(\frac{x}{2})\left[\frac{1-u^2}{1+u^2}\right] - 1}$$

$$= -\int \frac{\cancel{2}\, du}{[u^2+1]\left(2\left[\frac{1-u^2}{1+u^2}\right] - 1\right)}$$

$$= -\int \frac{2\,du}{2-2u^2-u^2-1}$$

$$= -\int \frac{2\,du}{-3u^2-1}$$

$$= -2\int\left(\frac{3}{1}\right)\frac{du}{(3u-\sqrt{3})(3u+\sqrt{3})}$$

$$= -6\left[\int\frac{A\,du}{(3u-\sqrt{3})} + \int\frac{B\,du}{(3u+\sqrt{3})}\right]$$

$$= -6\left[\frac{\sqrt{3}}{2}\int\frac{du}{(3u-\sqrt{3})} - \frac{\sqrt{3}}{2}\int\frac{du}{(3u+\sqrt{3})}\right]$$

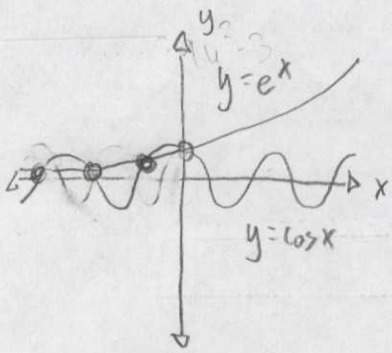
$$= \frac{\ln(3u+\sqrt{3})}{\sqrt{3}} - \frac{\ln(3u+\sqrt{3})}{\sqrt{3}}$$

$$= \frac{\ln(3\tan(\frac{x}{2})-\sqrt{3})}{\sqrt{3}} - \frac{\ln(3\tan(\frac{x}{2})+\sqrt{3})}{\sqrt{3}}$$

$$\boxed{= \frac{\ln}{\sqrt{3}}\left|\frac{3\tan(\frac{x}{2})-\sqrt{3}}{3\tan(\frac{x}{2})+\sqrt{3}}\right| + C}$$

x_0

$\dot{x} = e^x - \cos x$ 2.2.7.



Points of Stability

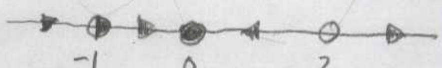
$e^x = \cos x$

$x_1 = 0$ source (unstable).

$x_2 = -1.29$ sink (stable).

$x_3 = -4.72$ source (unstable)

$\dot{x} = f(x)$ 2.2.8



$f(x) = -(x+1)(1-x)^3$

Slope zero Negative Positive

2.2.9

$X_0 = 2$ ~
$X_0 = 1$ ~
$X_0 = 0.5$ ~
$X_0 = -1$ ~

$f(x) = x(1-x)$

Fixed points

@ x = 1

@ x = 0

$\dot{x} = f(x)$ 2.2.10

a. A periodic function having solutions $n\pi$.

b. A periodic function with $n\pi$ solutions.

c. $f(x) = x^5$

d. $f(x) = x^2 + 1$

e. $f(x) = x^{10}$

$\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}$ 2.2.11. $Q(0) = 0$; $t = RC\int\frac{dQ}{V_0C - Q} = -RC\ln V_0C - Q + C$; $C = RC\ln V_0 C$; $\boxed{t = RC\ln\frac{V_0C}{V_0C - Q}}$

$Q = V_0C(1 - e^{-t/RC})$

$\dot{Q} = g(V) - \frac{Q}{RC}$ 2.2.12

$V = g(V) - V_{cap.} = V_0 - \frac{Q}{C}$; $-g(V) + RI + \frac{Q}{C} = 0$; $-g(V) + R\dot{I} + \frac{Q}{C} = -g(V) + R\dot{Q} + \frac{Q}{C} = 0$

$\dot{Q} = g(V) - \frac{Q}{RC}$;

Fixed Points: $g(V) = Q/RC$

Stability: Source (unstable)

The nonlinearity of the resistor has a relationship to resistance.

$m\dot{v} = mg - kv^2$ 2.3.13 ; where m = mass, g = acceleration, $k>0$ = air resistance

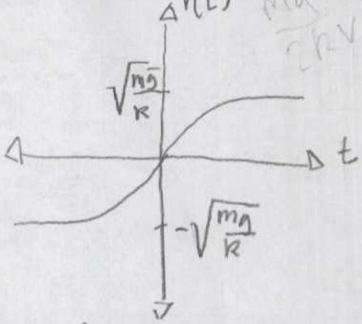
a) $\displaystyle\int \frac{dv}{g - \frac{k}{m}v^2} = \frac{1}{g}\int \frac{dv}{1 - \frac{k}{mg}v^2} = \frac{1}{g}\left[\int \frac{dv}{1-\sqrt{\frac{k}{mg}}v} + \int \frac{dv}{1+\sqrt{\frac{k}{mg}}v}\right] = \sqrt{\frac{mg}{k}}\frac{1}{2g}\left[\ln\left|1+\sqrt{\frac{k}{mg}}v\right| - \ln\left|1-\sqrt{\frac{k}{mg}}v\right|\right]$

$\displaystyle t = \frac{1}{2}\sqrt{\frac{m}{kg}}\ln\left|\frac{1+\sqrt{\frac{k}{mg}}v}{1-\sqrt{\frac{k}{mg}}v}\right| \quad \sqrt{\frac{kg}{m}}t = \frac{\ln}{2}\left|\frac{1+\sqrt{\frac{k}{mg}}v}{1-\sqrt{\frac{k}{mg}}v}\right| = \boxed{\tanh^{-1}\left(\sqrt{\frac{k}{mg}}v\right)}$

b) $\boxed{\displaystyle\lim_{t\to\infty} v(t) = \sqrt{\frac{mg}{k}} = \text{"terminal velocity"}}$

$\displaystyle v(t) = \sqrt{\frac{mg}{k}}\tanh\sqrt{\frac{kg}{m}}t$

c)



d. $\displaystyle V_{avg} = \frac{(31{,}400 - 2100)\text{ft}}{116\text{ sec}} = 252\frac{\text{ft}}{\text{sec.}}$

e. $\displaystyle \dot{s} = \frac{ds}{dt} = v = V\tanh\sqrt{\frac{kg}{m}}t \; ; \; s(t) = V\int \tanh\sqrt{\frac{kg}{m}}t\,dt$

$\displaystyle = V\int \frac{\sinh\sqrt{\frac{kg}{m}}t}{\cosh\sqrt{\frac{kg}{m}}t}dt$

$\displaystyle = V\int \frac{1}{u}dt$

$\displaystyle = \frac{m}{k}\ln\cosh\frac{gt}{v}$

$\displaystyle = \frac{V^2}{g}\ln\cosh\frac{gt}{V}$

$\displaystyle 29{,}300 = \frac{V^2}{32.2\text{ft/sec}^2}\ln\cosh\cdot\frac{32.2\text{ft/sec}^2}{V}116\text{sec}$

$\displaystyle e^{9.43\times10^5/V^2} = \frac{e^{\frac{3735\text{ ft/sec}}{V}} + e^{-\frac{3735\text{ft/sec}}{V}}}{2}$

$\boxed{V = 266\text{ ft/sec.}}$

$V \approx V_{avg} = 252\text{ft/sec}$

$\displaystyle \frac{gt}{v} = \frac{32.2\text{ft/sec}^2\cdot116\text{sec}}{252\text{ft/sec}} = 14.8$

$\displaystyle \frac{V^2}{g}\ln\cosh\frac{gt}{V} \approx \frac{V^2}{g}\left[\frac{gt}{V} - \ln 2\right] = 265\text{ ft/sec}$

$\dot{N} = rN(1-N/K)$ 2.3.1 a. $\displaystyle t = \frac{1}{r}\int\frac{dN}{N(1-N/K)}$

$\displaystyle = \frac{1}{r}\left[\int\frac{A\,dN}{N} + \int\frac{B\,dN}{(1-N/K)}\right]$

$\displaystyle = \frac{1}{r}\left[\int\frac{1\,dN}{N} + \frac{1}{K}\int\frac{dN}{(1-N/K)}\right]$

$\displaystyle = \frac{1}{r}\ln N - \ln|1-N/K| + C$

$\displaystyle = \frac{1}{r}\ln\frac{N}{1-N/K} + C$

$\displaystyle Ce^{-rt} = \frac{1-N/K}{N} \; ; \; N\left(1+\frac{Ce^{-rt}}{K}\right) = 1$

$\displaystyle N = \frac{1}{\left(1+\frac{Ce^{-rt}}{K}\right)} = \boxed{\frac{N_0}{e^{-rt}}}$

General Solution:

$\displaystyle x = Ce^{-rt} + \frac{1}{K}$

b. $x = 1/N$; $dN = -\frac{dx}{x^2}$; $\dot{x} = -rx\left(1-\frac{1}{Kx}\right) = \frac{r}{K} - rx$; $\dot{x} + rx - \frac{r}{K} = \dot{x} + r\left(x - \frac{1}{K}\right) = 0$

$\boxed{\dot{x} = r\left(\frac{1}{K} - x\right)}$

$\displaystyle N(t) = \frac{K}{KCe^{-rt} + 1}$

$\displaystyle N = \frac{N_0}{e^{-rt}}$

$\dot{x} = k_1ax - k_{-1}x^2$ 2.3.2

$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X$

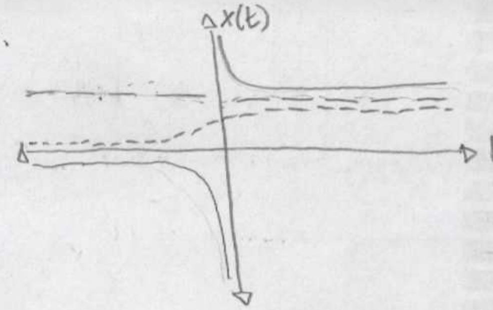
a. $\displaystyle t = \int\frac{dx}{k_1ax - k_{-1}x^2} = \frac{1}{k_1a}\int\frac{dx}{x - \frac{k_{-1}}{k_1a}x^2} = \frac{1}{k_1a}\int\frac{dx}{x\left(1-\frac{k_{-1}}{k_1a}x\right)}$

$\displaystyle = \frac{1}{k_1a}\left[\int\frac{A\,dx}{x} + \int\frac{B\,dx}{\left(1-\frac{k_{-1}}{k_1a}x\right)}\right] = \frac{1}{k_1a}\left[\int\frac{1\,dx}{x} + \frac{k_{-1}}{k_1a}\int\frac{dx}{1-\frac{k_{-1}}{k_1a}x}\right]$

$\displaystyle = \frac{1}{k_1a}\left[\ln|x| - \ln\left|1-\frac{k_{-1}}{k_1a}x\right|\right] = \frac{1}{k_1a}\ln\left|\frac{x}{1-\frac{k_{-1}}{k_1a}x}\right| + c \; ; \; \left(Ce^{-t/k_1a} + \frac{k_{-1}}{k_1a}\right)x = 1$

$\displaystyle x(t) = \frac{1}{\frac{k_{-1}}{k_1a} + Ce^{-t/k_1a}} \; ; \; C = \frac{1}{x_0} - \frac{k_{-1}}{k_1a}$; $\boxed{\text{Fixed points of Stability: } x = \frac{k_1a}{k_{-1}} \text{ source unstable}}$

b.



— $= X_0 > \frac{k_1 a}{k_{-1}}$

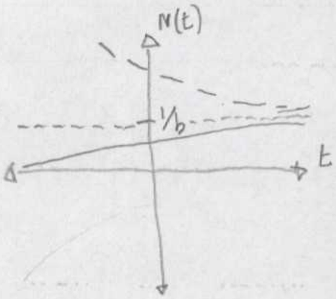
-- $= X_0 = \frac{k_1 a}{k_{-1}}$

---- $= X_0 < \frac{k_1 a}{k_{-1}}$

"Gompertz Law"

$\dot N = -aN\ln(bN)$ 2.3.3a) $t = -\frac{1}{a}\int \frac{dN}{N\ln(bN)} = -\frac{b}{a}\int\frac{du}{u} = -\frac{b}{a}\ln[\ln bN]$; $N(t) = \frac{Ce^{e^{-\frac{ab}{b}}}}{b} = \boxed{\frac{Ce^{-\frac{ate}{b}}}{b}}$; a = rate constant, b = Max amount of cells.

b)



— $= C < b$

---- $= C = 1$

-- $= C > b$

$\frac{\dot N}{N} = r - a(N-b)^2$ 2.3.4. a). $\lim_{N\to 0}\frac{\dot N}{N} = \lim_{N\to 0} r - a(N-b)^2 = r - ab^2 = \infty$; $r = \infty$;

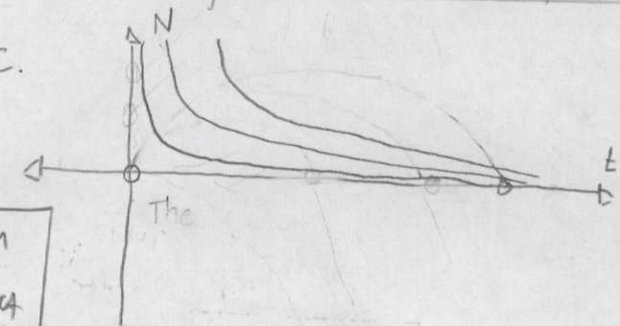
$\lim_{N\to\infty}\frac{\dot N}{N} = \lim_{N\to 0} r - a(N-b)^2 = r - \infty = 0$; $r = \infty$:

Each case of competition model at infinite population or extremely small populations that food amount or rate of consumption are insignificant to the competitors.

b). Fixed points of Stability:

$N = 0$	Source (unstable)
$N = \sqrt{\frac{r}{a}} + b$	Sink (stable)

C.



d) The solutions of the logistic equation $y = Ce^{-rt} + \frac{1}{k}$ are similar, if not exact to the Allele Effect.

$\dot X = aX$
$\dot Y = bY$ 2.3.5 a) $x(t) = \frac{X(t)}{[X(t)+Y(t)]} = \frac{e^{at}}{e^{at}+e^{bt}}$; $\boxed{\lim_{t\to\infty} x(t) \approx 1}$

b) $\dot x(t) = \frac{ae^{at}(e^{at}+e^{bt}) - e^{at}(ae^{at}+be^{bt})}{(e^{at}+e^{bt})^2}$

$= \frac{e^{at}[a-b]e^{bt}}{(e^{at}+e^{bt})^2} = \frac{[a-b]e^{at}}{(e^{at}+e^{bt})}\left(\frac{e^{bt}+e^{at}-e^{at}}{e^{at}+e^{bt}}\right)$

$\boxed{= x[a-b](1-x)}$

$t = \int\frac{du}{(b+u)(r-au^2)} = \int\frac{A}{b+u}du + \int\frac{Bu+C}{r-au^2}du$

$A(r-au^2) + (Bu+C)(b+u) = 1$

@ $u = \sqrt{\frac{r}{a}}$; $(B\sqrt{\frac{r}{a}} + C)(b + \sqrt{\frac{r}{a}}) = 1$

$B = \sqrt{\frac{a}{a}}$; $C = \sqrt{\frac{r}{a}}$

@ $u = -b$ $A = \frac{1}{r-(ab)^2}$

$= \frac{1}{r-(ab)^2}\int\frac{du}{b+u} + \frac{1}{\sqrt{ra}}\int\frac{ab\cdot u + r}{r - au^2}du$ Partial Fractions (x2)

$= \frac{\ln N}{r-(ab)^2} + \frac{b}{4}\sqrt{\frac{a}{r}}\tanh^{-1}([N-b]^2) + \frac{1}{\sqrt{a}}\tanh^{-1}\left(\sqrt{\frac{a}{r}}(N-b)\right) + C$

$\dot{x}=(1-x)P_{yx}-xP_{xy}$ 2.3.6 a. $x=0$

$P_{yx}=sx^a;\ P_{xy}=(1-s)(1-x)^a$ $\quad x=1$

$$\boxed{x=\frac{\sqrt[a-1]{\frac{(1-s)}{s}}}{1+\sqrt[a-1]{\frac{(1-s)}{s}}}}$$

b. A plot of $s(1-x)x^a$ and $-(1-s)x(1-x)^a$ demonstrate $-(1-s)x(1-x)^a > s(1-x)x^a$ For $x=0$ and $x=0$, indicating, each fixed point is stable.

c. For $x=\dfrac{\sqrt[a-1]{\frac{(1-s)}{s}}}{1+\sqrt[a-1]{\frac{(1-s)}{s}}}$ the plot of $s(1-x)x^a > (1-s)x(1-x)^a$, suggesting a source.

$\dot{x}=x(1-x)$ 2.4.1 $\dot{x}=f(x)=f(x^*+x)=f(x^*)+xf'(x^*)+O(x^2)$
$=xf'(x^*)+O(x^2)$
$\boxed{=x(1-2x)}$

|| $x=0;\ f'(x^*)=1$: Unstable (source)
$x=1;\ f'(x^*)=-1$: Stable (sink)

$\dot{x}=x(1-x)(2-x)$ 2.4.2 $\dot{x}=f(x)=f(x^*+x)=\boxed{xf'(x^*)}$
$\boxed{=x\,2x(1-x)}$

|| $x=0$ $f'(x^*)=0$ Half-stable
$x=1$ $f'(x^*)=0$ Half-stable
$x=2$ $f'(x^*)=-4$ sink (stable)

$\dot{x}=\tan x$ 2.4.3 $\dot{x}=f(x)=f(x^*+x)=\boxed{x\sec^2(x)}$ || $x=n\pi$ $f(x)=(+)$ source (unstable)

$\dot{x}=x^2(6-x)$ 2.4.4 $\dot{x}=f(x)=f(x^*+x)=\boxed{x[12x-3x^2]}$ || $x=0$ $f'(x)=0$ Halfstable
$x=6$ $f'(x)=-36$ sink (stable)

$\dot{x}=1-e^{-x^2}$ 2.4.5 $\dot{x}=f(x)=f(x^*+x)=\boxed{x[2e^{-x^2}]}$ || $x=0$ $f'(x^*)=0$ Half-stable

$\dot{x}=\ln x$ 2.4.6 $\dot{x}=f(x)=f(x^*+x)=\boxed{1}$ || $x=1$ $f'(x^*)=1$ Source (unstable)

$\dot{x}=ax-x^3$ 2.4.7 $\dot{x}=f(x)=f(x^*+x)=\boxed{x[a-3x^2]}$

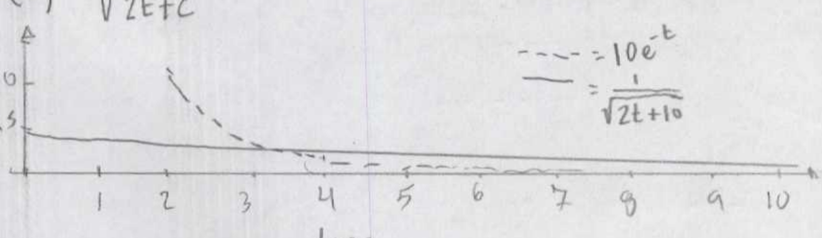
a	(+)	(−)	(0)
$x=0$	source	sink	Half-stable
$x=+\sqrt{a}$	sink	source	source
$x=-\sqrt{a}$	source	sink	Half-stable

$\dot{N}=-aN\ln(bN)$ 2.4.8 $\dot{N}=f(N)=f(N+N^*)=\boxed{-\frac{aN}{b}[1+b\ln(bN)]}$

$N=0$: Source (unstable)
$N=\frac{1}{b}$: sink (stable)

$\dot{x}=-x^3$ 2.4.9 a. $t=-\int\frac{dx}{x^3}=\frac{1}{2x^2}+c$; $x(t)=\sqrt{\frac{1}{2t+c}}$

$\lim_{x\to 0} t=\frac{1}{0}+c=\boxed{\infty}$



b. if $x_0=10$

$t=-\int\frac{1}{x}=-\ln x$;

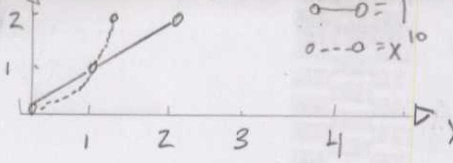
$\boxed{x(t)=x_0e^{-t}=10e^{-t}}$

$\dot{x}=-x^c$ 2.5.1 a. $c=0$

b. $dx=-dt$; $x(t)=-t$; if $t=0$ is considered finite time, then yes.

$t=-\int\frac{dx}{x^c}=-\frac{x^{1-c}}{1+c}$; $t(x=1)-t(x=0)=-\frac{1^{1-c}}{1-c}+\frac{0^{1-c}}{1-c}=\boxed{\frac{1}{c+1}}$

$\dot{x} = 1 + x^{10}$; 2.5.2.



$\boxed{\lim_{\dot{x} \to \infty} \dot{x} = \infty}$

$\dot{x} = rx + x^3$ 2.5.3

$$t = \int \frac{dx}{x(r+x^2)} = \int \frac{A}{x} dx + \int \frac{Bx + C}{r + x^2} dx \; ; \; A(r+x^2) + (Bx+C)(x) = 1$$

$$x = 0 ; A = \frac{1}{r} ; 1 + \frac{x^2}{r} + Bx^2 + Cx = 1$$

$$(B + \tfrac{1}{r})x^2 + Cx = 0$$

$$(B + \tfrac{1}{r})x = -C$$

$$\boxed{x = 0 ; C = 0}$$

$$\boxed{x = 1 ; B = -\tfrac{1}{r}}$$

$$= \frac{1}{r}\ln x - \frac{1}{2r}\ln r + x^2 = \frac{1}{r}\ln \frac{x}{\sqrt{r+x^2}}$$

$$x^2 Ce^{-2rt} = (r + x^2) ; \; x = \sqrt{\frac{r}{1 + Ce^{-2rt}}}$$

if $x_0 \neq 0$; $\lim_{t \to \infty} x(t) = \infty$.

$\dot{x} = x^{1/3}$ 2.5.4. $x(0) = 0$; $t = \int \frac{dx}{x^{1/3}} = \frac{3}{2} x^{2/3}$; $\boxed{x(t) = \sqrt{(\tfrac{2}{3}t - \tfrac{2}{3}c)^3}}$

$\dot{x} = |x|^{p/q}$ 2.5.5. $x(0) = 0$; a) $t = \frac{q}{p+q}(x)^{\frac{p+q}{q}}$; $x(t) = \left(\frac{p+q}{q}(t + C)\right)^{q/p+q}$; c = many solutions at zero because of root.

$$\boxed{x(t) = \left(\frac{p+q}{q}(t+C)\right)^{q/p+q}}$$

b) $x(t) = \left(\frac{p+q}{q}(t+c)\right)^{q/p+q}$; if $p > q$; $x(0) = \left(\frac{p+q}{q}(0+c)\right)^{q/p+q} = 0$; $\boxed{C = 0}$

$h(t)$ = height
t = time
A = cross-sectional Area
$v(t)$ = velocity
$av(t) = A\dot{h}(t)$

2.5.6 a) Newton's first law that for every force there exist an equal and opposite counter force.

b) $\frac{1}{2}mv^2 = mgh$; $\boxed{v^2 = 2gh}$

c) $\boxed{\dot{h}(t) = -\sqrt{\tfrac{a}{A}\, 2gh}}$

d) $\boxed{h(0) = 0; \; t = -\sqrt{\tfrac{Ah}{2a}}\, 2\sqrt{h} \; ; \; h(t) = -\sqrt{\tfrac{a}{2A}}\, t}$

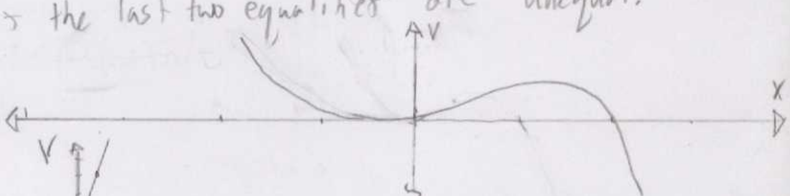
$m\ddot{x} = -kx$ 2.6.1 The text states, "there are no periodic solutions to $\dot{x} = F(x)$ because undamped systems do not oscillate, and, "damped oscillations do not occur for first order systems." Strogatz' Statement does not fit the equation of 2.61.

$\dot{x} = F(x)$
$x(t) = x(t+T)$

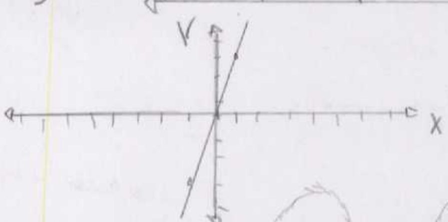
2.6. 2. $$\int_t^{t+T} F(x)\frac{dx}{dt}\,dt = \int_t^{t+T} F(x)\,\dot{x}(t)\,dt = \int_t^{t+T} F(x)\,\dot{x}(t+T)\,d(t+T)$$

The contridiction of the argument is the last two equalities are unequal.

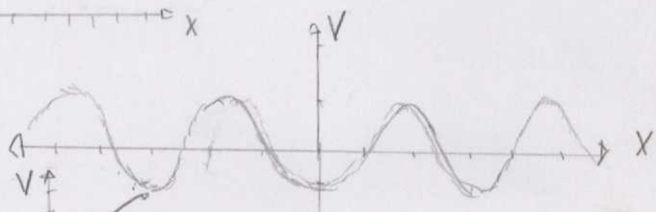
$\dot{x} = x(1-x)$ 2.7.1 $\frac{dV}{dx} = \dot{x} = x(1-x)$; $V = (1 - \frac{x}{3})x^2$



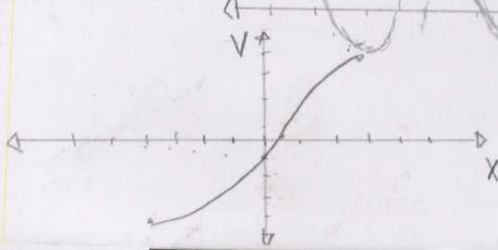
$\dot{x} = 3$ 2.7.2 $\frac{dV}{dx} = \dot{x} = 3$; $V = 3X$



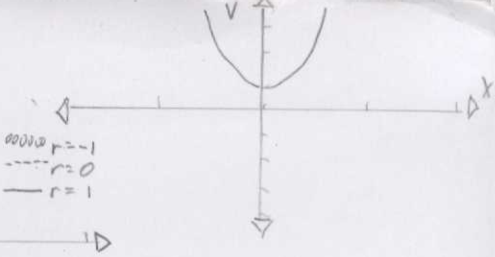
$\dot{x} = \sin x$ 2.7.3 $\frac{dV}{dx} = \dot{x} = \sin x$; $V = -\cos(x)$



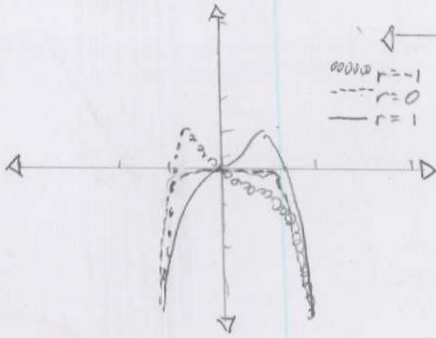
$\dot{x} = 2 + \sin x$ 2.7.4 $\frac{dV}{dx} = \dot{x} = 2 + \sin x$; $V = 2X - \cos(x)$



$\dot{x} = -\sinh x$ 2.7.5. $\frac{dV}{dx} = -\sinh x$; $V = -\cosh(x)$



$\dot{x} = r + x - x^3$ 2.7.6 $\frac{dV}{dx} = r + x - x^3$; $V = rx + \frac{x^2}{2} - \frac{x^4}{4}$



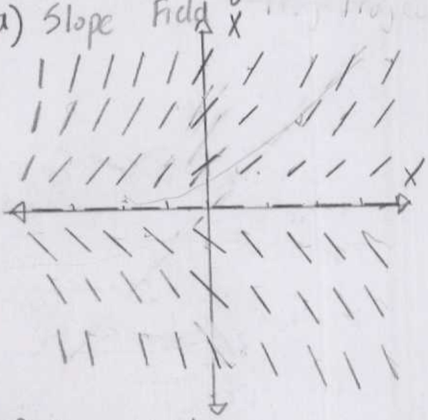
$\dot{x} = f(x)$ 2.7.7. $\frac{dV}{dx} = \dot{x} = f(x)$; $V = \frac{dF(x)}{dx} dx + C$

$$F(x) = \frac{d(V-C)}{dx} dx$$

The solution $x(t)$ cannot oscillate because of the existence and uniqueness of $f(x)$, and the solutions for $f(x) = 0$, that $V = C$ or $C = 0$; withstanding, $\frac{d(V-C)}{dx} = \frac{dx}{dt}$, then the solution $x(t)$ also corresponds to a nonperiodic function.

$\dot{x} = x(1-x)$ 2.8.1 The horizontal lines are to be expected in Figure 2.8.2 because of the slope being zero at $x = 1$.

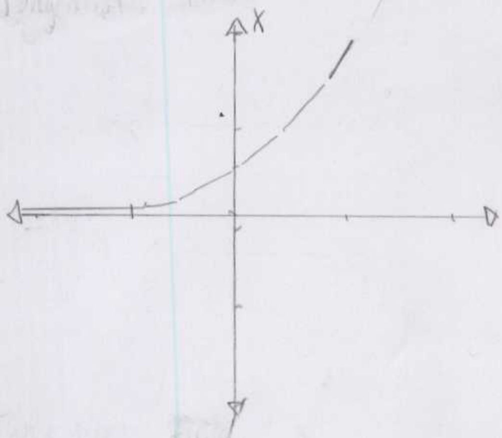
$\dot{x} = x$ 2.8.2.a) Slope Field



$\frac{dx}{dt} = x$

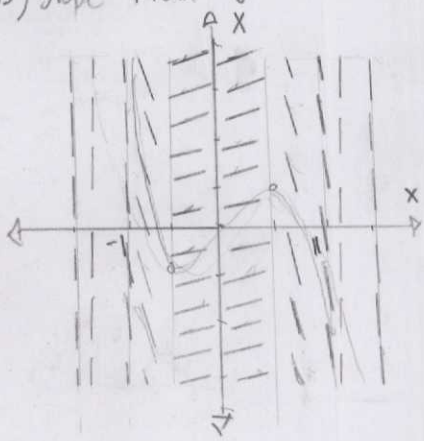
$\ln x = t$

$x = e^t$



$\dot{x} = 1 - x^2$

b) Slope Field



$\frac{dx}{dt} = 1 - x^2$

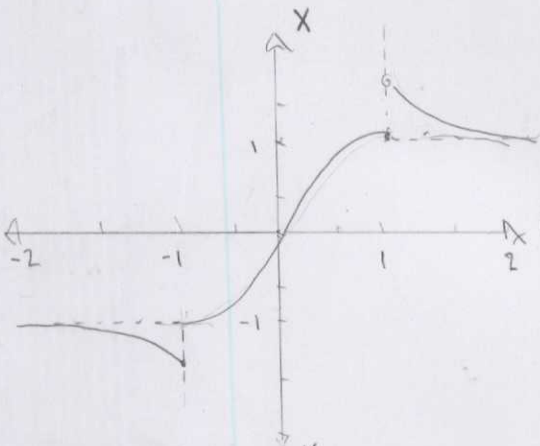
$\int \frac{dx}{1-x^2} = t$

$\frac{1}{2} \ln \frac{1+x}{x-1} = t$

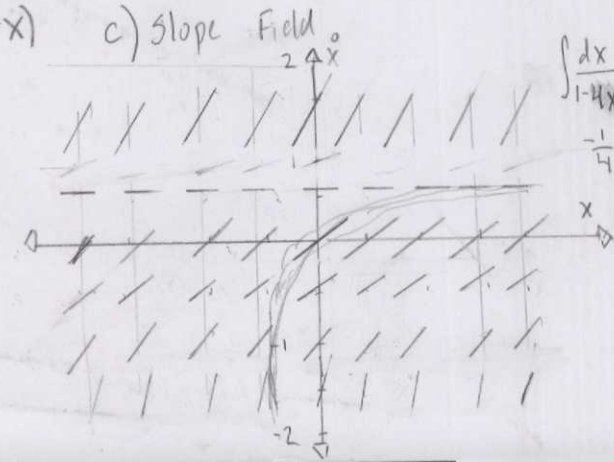
$x < -1$

$-1 < x < 1$

$x > 1$



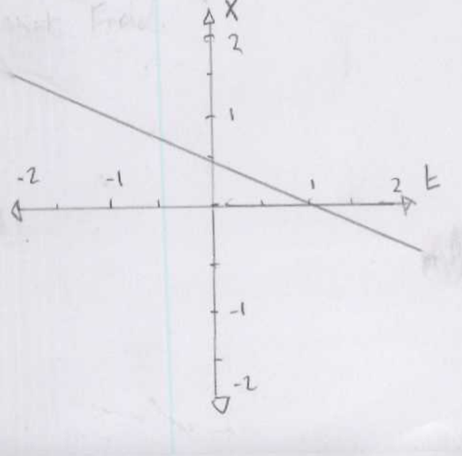
$\dot{x} = 1 - 4x(1-x)$ c) Slope Field



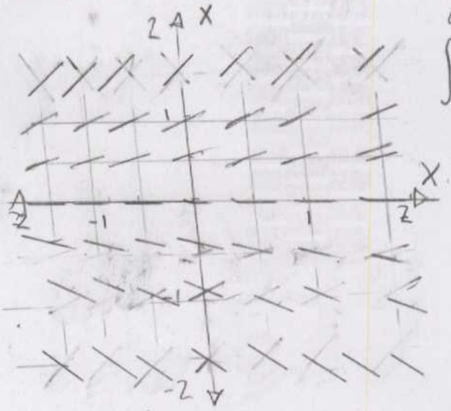
$\int \frac{dx}{1-4x(1-x)} = t$

$-\frac{1}{4}\left(\frac{1}{x-0.5}\right) = t$

$x = 0.5 - \frac{1}{4t}$



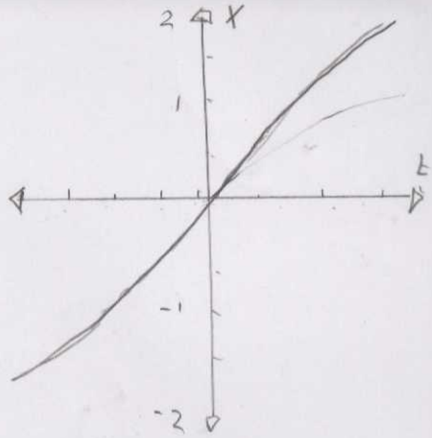
$\dot{x} = \sin x$ 2.32 d) Slope Field



$\frac{dx}{dt} = \sin x$

$\int \cosh x = t$

$x = \sinh^{-1}(t)$



Euler's Method

$\dot{x} = -x$; $x(0)=1$ 2.8.3. a) $x(t) = Ce^{-t}$; $C=1$; $x(t) = e^{-t}$ $\ln t = -t$

b) $\Delta t = 1$; $x(t_0 + \Delta t) \approx x_0 + f(x_0)\Delta t$; $x(t_0 + \Delta t) = 0 + e^{-1} 10^{0} = 0.3679$

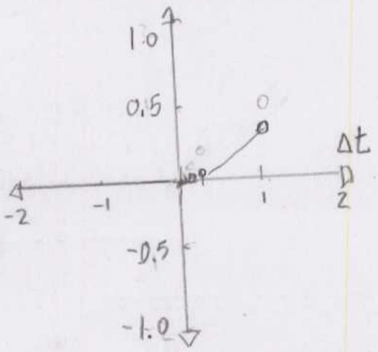
$\Delta t = 10^{-n}$ $n=1$ $x_n + e^{-x_n} 10^{-1} = 0.36341$

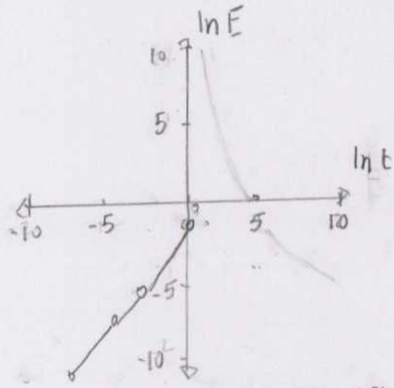
$n=2$ $x_n + e^{-x_n} 10^{-2} = 0.36697$

$n=3$ $x_n + e^{-x_n} 10^{-3} = 0.36785$

$n=4$ $x_n + e^{-x_n} 10^{-4} = 0.36787$

c) $E = |\hat{x}(1) - x(1)|$

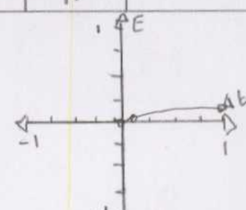


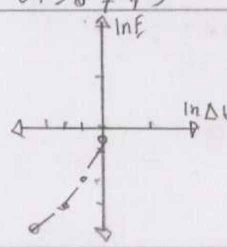


The results of $E = |\hat{x}(1) - x(1)|$ vs Δt represent error of Euler's method. While the plot of $\ln E$ vs $\ln t$ characterizes nothing informative.

$\dot{x} = -x$; $x(0)=1$ 2.8.4. $x(t) = e^{-t}$;

n	Δt	$f(x)$	$x_n = x_{n-1} + f(x_{n-1})\Delta t$	$E = \lvert\hat{x}(1) - x(1)\rvert$	$\ln E$
0	10^{0}		0.36788	0.00	-1.00
1	10^{-1}	$\exp(x_{n-1})$	0.33527	0.03261	-3.42
2	10^{-2}		0.36577	0.0211	-6.16
3	10^{-3}		0.36773	0.0002	-8.83
4	10^{-4}		0.36773	0.0000	-10.79

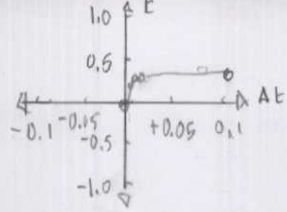


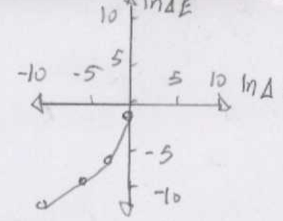


Improved Eulers Method

$\dot{x} = -x$; $x(0)=1$ 2.8.5. $x(t) = e^{-t}$

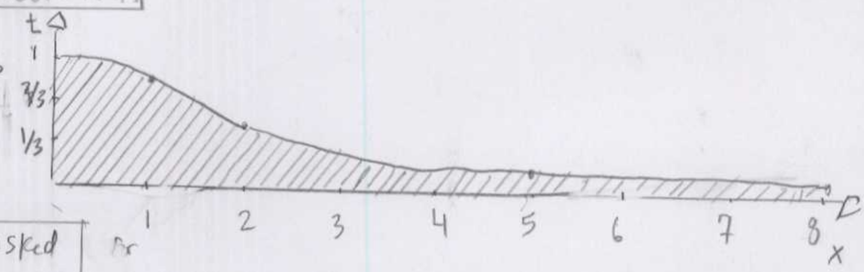
n	Δt	$f(x)$	$x_n = x_n + \frac{1}{6}(R_1 + 2R_2 + 2R_3 + R_4)$	$E = \lvert\hat{x}(1) - x(1)\rvert$	$\ln E$
0	10^{0}				
1	10^{-1}				
2	10^{-2}	$\exp(x_{n-1})$			
3	10^{-3}				
4	10^{-4}				



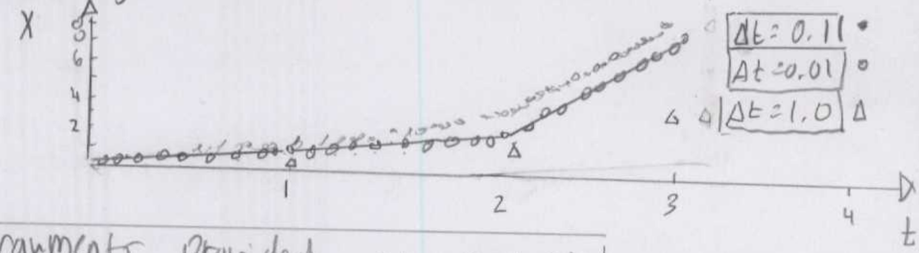


The Euler method aided the analysis of numerical methods; including, Precision. Euler's Improved Method approached the solution of $f(x) = e^{-x}$ with less round-off error. Runge-Kutta's Routine provided the least round-off errors with 10^{-20} across the spreadsheet, and necessitated high-precision.

$\dot{x} = x + e^{-x}$ 2.8.6. a: $t = \int \frac{1}{x + e^{-x}}\,dx = \int \frac{e^{x}}{e^{x}x + 1}\,dx$;

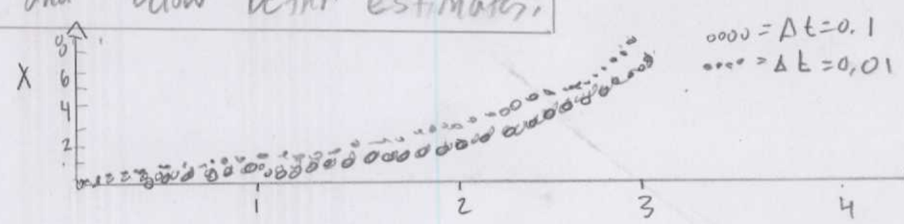


I noticed the book asked for $x(t)$ (and not $t(x)$).
This led me to investigate a Numerical method of integration; withstanding, Runge-Kutta Routine aided with the plot of $x(t)$.



b. At $t = 0$, analytical arguments provided an $x = 1.011$.

c. Stepsizes of $\Delta t = 0.1$ and 0.01 had different results, including inaccuracies above and below better estimates.



d) See part a.

$x_1 = x_0 + f(x_0)\Delta t$ 2.8.7 a) $x(t_1) = x(t_0 + \Delta t)$

Taylor Series:

$$x(t + \Delta t) = \sum_{n=0}^{\infty} \frac{x^{(n)}(t)}{n!}(\Delta t)^n = x(t) + x'(t)\cdot\Delta t + O(\Delta t^2)$$

$$f(t + \Delta t) = f(t) + f'(t)\cdot\Delta t + O(\Delta t^2) = \boxed{x_0 + f'(t)\cdot\Delta t}$$

b) $$|x(t_1) - x_1| = |x(t_1) - x(t_1) - x'(t)\cdot\Delta t - O(\Delta t^2)| = |O(\Delta t^2)| = \frac{x''(t)\Delta t^2}{2!} = C(\Delta t^2)$$

$$\boxed{C = \frac{x''(t)}{2!}}$$

Taylor Series: $f(x+h) = \sum \frac{f^{n}(x)h^{n}}{n!}$

$\dot{x} = x + e^{-x}$ 2.8.8. $\dot{x} = x + e^{-x}$; $|x(t_0) - t_0| = |x(t_0) - x(t_0) - x(t_0)'\Delta t - \frac{x''(t_0)\Delta t^2}{2!}| = \frac{x''(t_0)\Delta t^2}{2}$

$\boxed{= O(\Delta t^2)}$

$\dot{x} = x + e^{-x}$ 2.8.9 Runge-Kutta: $x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ Where $k_1 = f(x_n)\Delta t$

$k_2 = f(x_n + \frac{1}{2}k_1)\Delta t$

$k_3 = f(x_n + \frac{1}{2}k_2)\Delta t$

$k_4 = f(x_n + k_3)\Delta t$

$$x(t + \Delta t) = x(t_0) + x'(t_0)\Delta t + \frac{x''(t_0)\Delta t^2}{2!} + O(\Delta t^3)$$

$$k_1 = f(x_n)\Delta t = x'(t_0)\Delta t$$

$$k_2 = f(x_n + \tfrac{1}{2}k_1)\Delta t = f(x_n) + f'(x_n)\tfrac{1}{2}k_1 + O[(\tfrac{1}{2}k_1)^2]$$

$$k_3 = f(x_n + \tfrac{1}{2}k_2)\Delta t = f(x_n) + f'(x_n)\tfrac{1}{2}k_2 + O[(\tfrac{1}{2}k_2)^2]$$

$$= f(x_n) + f'(x_n)\tfrac{1}{2}\left[f(x_n) + f'(x_n)\tfrac{1}{2}k_1 + O[(\tfrac{1}{2}k_1)^2]\right] + O[(\tfrac{1}{2}k_2)^2]$$

$$k_4 = f(x_n + k_3)\Delta t = f(x_n) + f'(x_n)\cdot k_3 + O[k_3^2]$$

$$= f(x_n) + f'(x_n)\left[f(x_n) + f'(x_n)\cdot\tfrac{1}{2}[f(x_n) + f'(x_n)\tfrac{1}{2}k_1 + O[(\tfrac{1}{2}k_1)]] + O[(\tfrac{1}{2}k_2)]\right] + O[k_3]$$

$$x_{n+1} = x_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = x_n + \tfrac{1}{6}\big(x'(t_0)\Delta t + 2x'(t_0) + x''(t_0)k_1$$
$$+ 2x'(t_0) + x''(t_0)[x'(t_0) + x''(t_0)x'(t)\Delta t]$$
$$+ 2x'(t_0) + x''(t_0)[x(t) + \tfrac{x'(t)}{2}[x'(t_0) + x''(t_0)\tfrac{x'(t)}{2}]]\big)$$

$$\boxed{|x(t_1) - x_1| = |x(t_0 + \Delta t) - x_{n+1}| = O(\Delta t^5)}$$

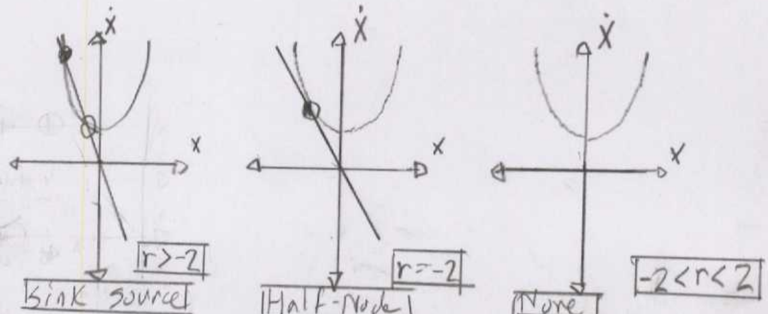
Chapter 3:

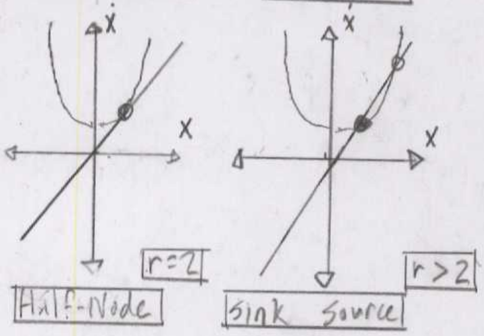
$\dot{x} = 1 + rx + x^2$ 3.1.1. Vector Field:

$$x = \frac{r \pm \sqrt{r^2 - 4}}{2}$$

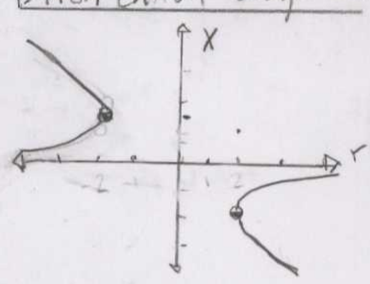
$$r = \frac{r \pm \sqrt{(r-2)(r+2)}}{2}$$

r	Bifurcations
>-2	Two
-2	One
0	zero
2	one
>2	Two.





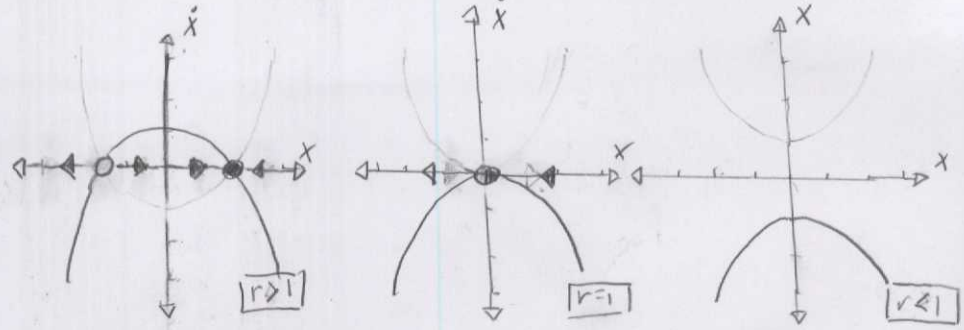
Bifurcation Diagram:



$\dot{x} = r - \cosh x$ 3.1.2. Vector Field

$r = \cosh(x)$

r	Bifurcations
<1	Zero
=1	One
>1	Two

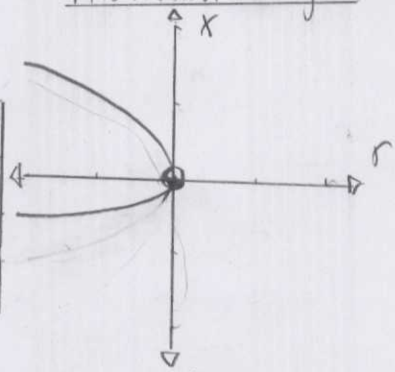


$\dot{x} = r + x - \ln(1+x)$ 3.1.3

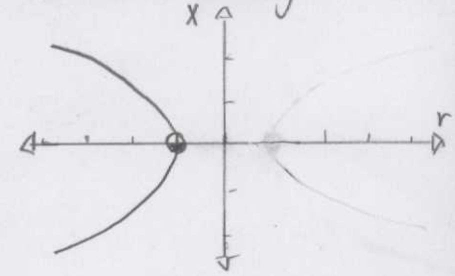
Vector Field

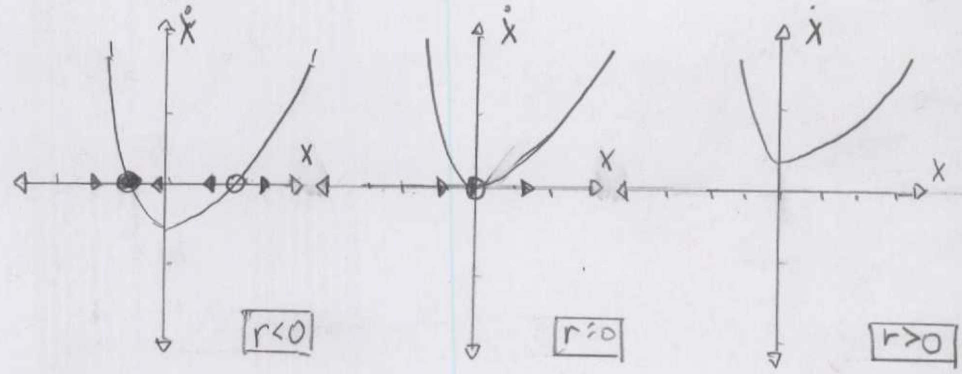
r	Bifurcation
>0	Zero
=0	One
<0	Two

Bifurcation Diagram:



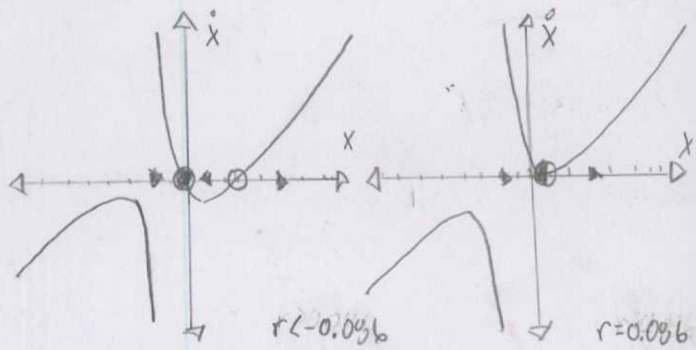
Bifurcation Diagram

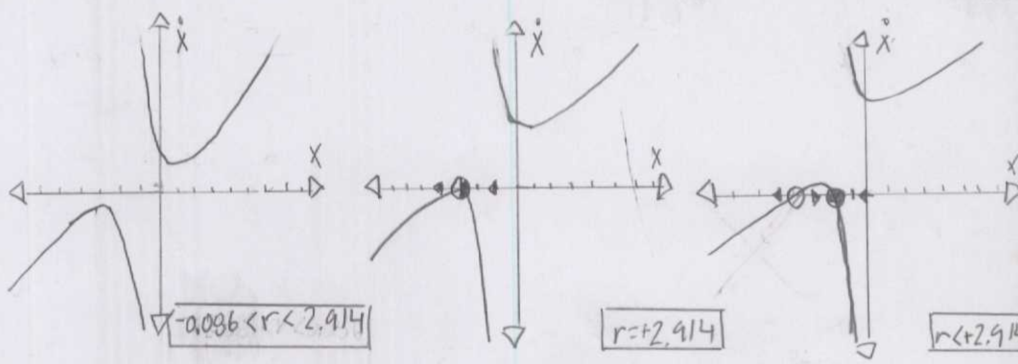




$\dot{x} = r + \frac{1}{2}x - x/(1+x)$ 3.1.4. Vector Field:

r	Bifurcations
< -0.096	Two
= -0.096	one
-0.096 < r < 2.914	zero
=+2.914	one
<+2.914	Two.





Bifurcation Diagram: