Chapter 2:

$\dot{x} = \sin x$ 2.1.1. $\dot{x} = 0 = \sin x$; $\boxed{x = n\pi}$ 2.1.2 $\boxed{(n+\frac{1}{2})\pi \text{ where } n \text{ is even.}}$

2.1.3. a) $\ddot{x} = \cos x \sin x$ b) $\frac{1}{2}\sin(2x) = \cos(x)\sin(x)$; $\boxed{\ddot{x} = \frac{1}{2}\sin(2x) ;\ x = (n+\frac{1}{4})\pi ; n \in \mathbb{Z}}$

2.1.4.a) $x_0 = \pi/4$; $t = \ln|(\csc x_0 + \cot x_0)|(\csc x + \cot x)|$

$$e^t = \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} = \frac{\csc \pi/4 + \cot \pi/4}{\csc x + \cot x} = \frac{\frac{2}{\sqrt{2}} + 1}{\csc x + \cot x} = \frac{\sqrt{2}+1}{\csc x + \cot x}$$

$$\frac{1}{\frac{1}{\sin x} + \frac{\cos x}{\sin x}} = \frac{\sin x}{1+\cos x} = \frac{\sin(2 \cdot \frac{x}{2})}{1+\cos(2\frac{x}{2})} = \frac{2\cos(\frac{x}{2})\sin(\frac{x}{2})}{\cancel{1} + \cancel{2}\cos^{\cancel{2}}(\frac{x}{2}) \cancel{-1}} = \tan\left(\frac{x}{2}\right) = \frac{e^t}{\sqrt{2}+1}$$

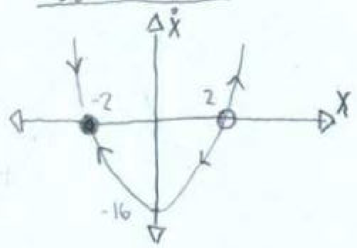
$$x(t) = 2\tan^{-1}\left(\frac{e^t}{\sqrt{2}+1}\right) ; \lim_{t\to\infty} x(t) = 2\tan^{-1}(\infty) = \boxed{2 \cdot \frac{\pi}{2} = \pi}$$

b) $\boxed{x(t) = 2\tan^{-1}\left(\frac{e^t}{\csc x_0 + \cot x_0}\right)}$

2.1.5a) A mechanical analog of $\dot{x} = \sin x$ is the undamped pendulum having an x_0 of the maximal point

b) Unstable points are described by a positive slope (source) and stable points (sink), a negative slope. The function $\dot{x} = \sin x$ at $x^* = 0$ is unstable, while $x^* = \pi$, is stable.

$\dot{x} = 4x^2 - 16$ 2.2.1 Vector Field:



Fixed Points: $x = 2$ Stability: source (unstable)

$x = -2$ sink (stable)

Solving for $x(t)$:

$$\frac{dx}{(x^2-4)} = 4t$$

$$\int \frac{A}{(x-2)}dx + \int \frac{B}{(x+2)}dx = 4t$$

$A(x+2) + B(x-2) = 1$

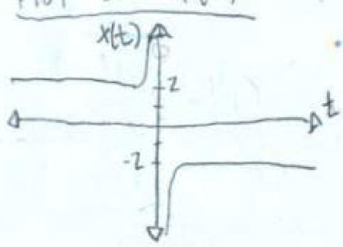
$A = 1/4$ @ $x = 2$

$B = -1/4$ @ $x = -2$

$$\ln\frac{|x-2|}{|x+2|} = 16t + c$$

$$\boxed{x(t) = \frac{2(e^{16t+c}+1)}{1-e^{16t+c}}}$$

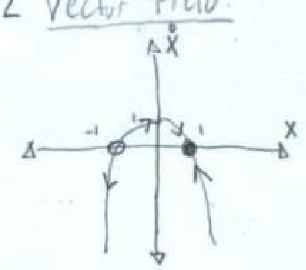
Plot of $x(t)$:



Analytical Solution of $x(t)$:

$$x(t) = \frac{2(e^{16t} \cdot \left(\frac{x_0-2}{2+x_0}\right)+1)}{1-e^{16t}\left(\frac{x_0-2}{2+x_0}\right)}$$

$\dot{x} = 1 - x^{14}$ 2.2.2 Vector Field:



Fixed Points: Stability:

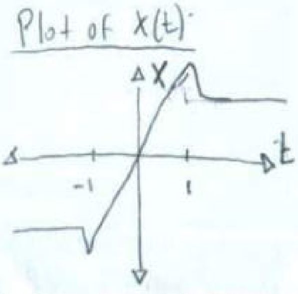
$x = 1$	source (unstable)
$x = -1$	sink (stable)

Solving for $x(t)$

$$t = \int \frac{dx}{1-x^{14}}$$

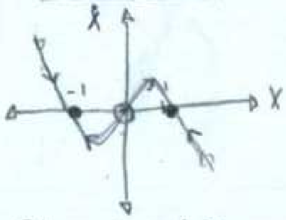
Unsolvable

Plot of $x(t)$:

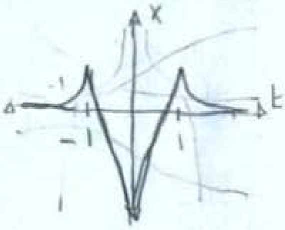


Analytical Solution of $x(t)$:

$\dot{x} = x - x^3$ 2.2.3 Vector Field:



Plot of $x(t)$:



Fixed Points: Stability:

$x=-1$ stable (sink)

$x=0$ unstable (source)

$x=1$ stable (sink)

Analytical Solution of $x(t)$:

$x(t) = $

Solving for $x(t)$:

$$t = \int \frac{dx}{x-x^3} = \int \frac{dx}{x(1-x^2)}$$

$$= -\frac{1}{2}\int \frac{du}{(1-u)u} = -\frac{1}{2}\int \frac{A}{1-u}du - \frac{1}{2}\int \frac{B}{u}du$$

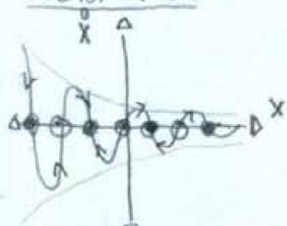
$$= -\frac{1}{2}\ln\left|\frac{1-u}{u}\right| = -\frac{1}{2}\ln\left|\frac{x^2}{1-x^2}\right|$$

$$= \frac{1}{2}\ln\left|\frac{1-x^2}{x^2}\right| + C$$

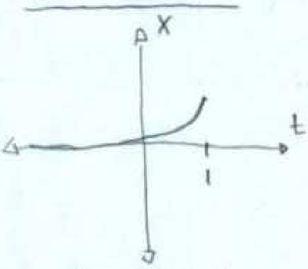
$$(e^{2t}+1)x^2 = 1$$

$$\boxed{x = \sqrt{\frac{1}{e^{2t+c}+1}}}$$

$\dot{x} = e^{-x}\sin x$ 2.2.4 Vector Field:



Plot of $x(t)$:



Fixed Points: Stability:

$x = 2n\pi$ Source (unstable)

$x = (2n+1)\pi$ sink (stable)

Analytical solution of $x(t)$

$$\boxed{x(t) = \arcsin^{-1}(Ce^{t}-1) \text{ where } C = -(1+\ln(\sin x_0))}$$

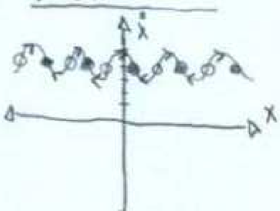
Solving for $x(t)$:

$$t = \int \frac{e^x}{\sin x}dx = \int dx + \int \cot(x)dx$$

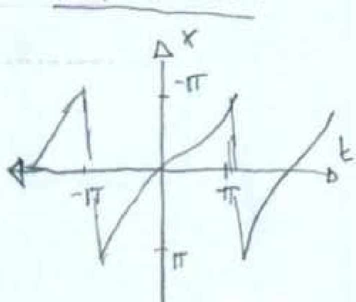
$$= 1 + \ln(\sin x) + C$$

$$\boxed{x(t) = \arcsin^{-1}(C \cdot e^{t-1})}$$

$\dot{x} = 1 + \frac{1}{2}\cos x$ 2.2.5. Vector Field:



Plot of $x(t)$:



Fixed Points Stability:

$x = (4n+1)\frac{\pi}{2}$ sink (stable)

$x = (4n-1)\frac{\pi}{2}$ source (unstable)

Analytical solution of $x(t)$:

$$\frac{\frac{2}{\sqrt{3}}\arctan\left(\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right)}{\sqrt{3}} + C$$

Solving for $x(t)$:

$$t = \int \frac{1}{1+\frac{1}{2}\cos x}dx$$

$$= \int \frac{dx}{\frac{1}{2}+\cos^2\left(\frac{x}{2}\right)} = \int \frac{\sec^2\left(\frac{x}{2}\right)dx}{\frac{\sec^2\left(\frac{x}{2}\right)}{2}+1}$$

$$\boxed{= \int \frac{\sec^2\left(\frac{x}{2}\right)dx}{3+\tan^2\theta/2}}$$

$$u = \frac{\tan\left(\frac{x}{2}\right)}{\sqrt{3}};\ \frac{du}{dx} = \frac{\sec^2\left(\frac{x}{2}\right)}{2\sqrt{3}}$$

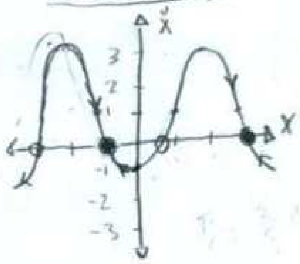
$$= \int \frac{2\sqrt{3}}{3u^2+3}du = \frac{2}{\sqrt{3}}\int \frac{du}{u^2+1}$$

$$= \frac{2}{\sqrt{3}}\arctan(u) + C$$

$$\boxed{\frac{\frac{2}{\sqrt{3}}\arctan\left(\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right)}{\sqrt{3}} + C}$$

$\dot{x} = 1 - 2\cos x$ 2.2.6.

Vector Field



Fixed Point	Stability
$x = (n+\frac{1}{2})\pi$; n = even	sink (stable)
$x = (n+\frac{1}{2})\pi$; n = odd	source (unstable)

Solving for $x(t)$

$$t = \int \frac{dx}{1-2\cos x} \;;\; \frac{1-\tan^2(\frac{x}{2})}{1+\tan^2(\frac{x}{2})} = \cos x$$

$$= -\int \frac{dx}{2\left[\frac{1-\tan^2(\frac{x}{2})}{1+\tan^2(\frac{x}{2})}\right] - 1}$$

$$= -\int \frac{dx}{2\left[\frac{1-u^2}{1+u^2}\right] - 1} \;;\; u = \tan\left(\frac{x}{2}\right) \;;\; \frac{du}{dx} = \frac{\sec^2(\frac{x}{2})}{2}$$

$$= -\int \frac{du}{2\sec^2(\frac{x}{2})\left[\frac{1-u^2}{1+u^2}\right] - 1}$$

$$= -\int \frac{2\,du}{[u^2+1]\left(2\left[\frac{1-u^2}{1+u^2}\right] - 1\right)}$$

$$= -\int \frac{2\,du}{2 - 2u^2 - u^2 - 1}$$

$$= -\int \frac{2\,du}{-3u^2 - 1}$$

$$= -2\int \left(\frac{3}{1}\right) \frac{du}{(3u-\sqrt{3})(3u+\sqrt{3})}$$

$$= -6\left[\int \frac{A\,du}{(3u-\sqrt{3})} + \int \frac{B\,du}{(3u+\sqrt{3})}\right]$$

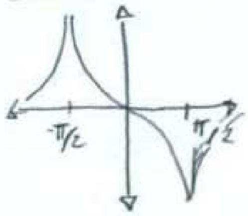
$$= -6\left[\frac{\sqrt{3}}{2}\int \frac{du}{(3u-\sqrt{3})} - \frac{\sqrt{3}}{2}\int \frac{du}{(3u+\sqrt{3})}\right]$$

$$= \frac{\ln(3u+\sqrt{3})}{\sqrt{3}} - \frac{\ln(3u+\sqrt{3})}{\sqrt{3}}$$

$$= \frac{\ln(3\tan(\frac{x}{2})-\sqrt{3})}{\sqrt{3}} - \frac{\ln(3\tan(\frac{x}{2})+\sqrt{3})}{\sqrt{3}}$$

$$\boxed{= \frac{\ln}{\sqrt{3}}\left|\frac{3\tan(\frac{x}{2})-\sqrt{3}}{3\tan(\frac{x}{2})+\sqrt{3}}\right| + C}$$

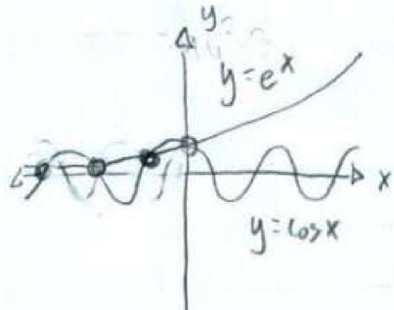
Plot of $x(t)$



Analytical Solution of $x(t)$

$$x(t) = \frac{\ln}{\sqrt{3}}\left|\frac{3\tan(\frac{x}{2})-\sqrt{3}}{3\tan(\frac{x}{2})+\sqrt{3}}\right|$$

$\dot{x} = e^x - \cos x$ 2.2.7.



Points of stability

$e^x = \cos x$

$x_1 = 0$ source (unstable)

$x_2 = -1.29$ sink (stable)

$x_3 = -4.72$ source (unstable)

$\dot{x} = f(x)$ 2.2.8

$f(x) = -(x+1)(1-x)^3$

slope zero Negative Positive

2.2.9

$x_0 = 2$ ~
$x_0 = 1$ ~
$x_0 = 0.5$ ~
$x_0 = -1$ ~

$f(x) = x(1-x)$

Fixed points

@ $x = 1$

@ $x = 0$

$\dot{x} = f(x)$ 2.2.10 a. A periodic function having solutions $n\pi$!

b. A periodic function with $n\pi$ solutions!

c. $f(x) = x^5$

d. $f(x) = x^2 + 1$

e. $f(x) = x^{n}$

$\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}$ 2.2.11. $Q(0) = 0$; $t = RC\int \frac{dQ}{V_0 C - Q} = -RC\ln V_0 C - Q + C$; $C = RC\ln V_0 C$; $\boxed{t = RC\ln\frac{V_0 C}{V_0 C - Q}}$

$Q = V_0 C(1 - e^{-t/RC})$

$\dot{Q} = g(V) - \frac{Q}{RC}$ 2.2.12 $V = g(V) - V_{cap.} = V_0 - \frac{Q}{C}$; $-g(V) + RI + \frac{Q}{C} = 0$; $-g(V) + R\dot{I} + \frac{Q}{C} = -g(V) + R\dot{Q} + \frac{Q}{C} = 0$

$\dot{Q} = g(V) - \frac{Q}{RC}$;

Fixed Points: $g(V) = Q/RC$

The nonlinearity of the resistor has a relationship to resistance

Stability: Source (unstable)

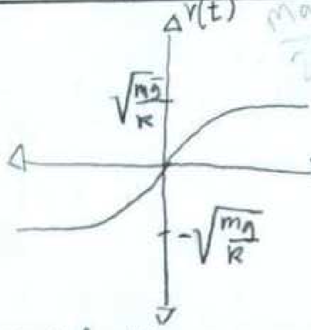
$m\dot{v}=mg-kv^2$ 2.2.13 ; where m = mass, g = acceleration, $k>0$ = air resistance

a) $\displaystyle\int \frac{dv}{g-\frac{k}{m}v^2} = \frac{1}{g}\int\frac{dv}{1-\frac{k}{mg}v^2} = \frac{1}{g}\left[\int\frac{dv}{1-\sqrt{\frac{k}{mg}}v}+\int\frac{dv}{1+\sqrt{\frac{k}{mg}}v}\right] = \sqrt{\frac{mg}{k}}\frac{1}{2g}\left[\ln\left|1+\sqrt{\tfrac{k}{mg}}v\right| - \ln\left|1-\sqrt{\tfrac{k}{mg}}v\right|\right]$

$\displaystyle t=\frac{1}{2}\sqrt{\frac{m}{kg}}\ln\left|\frac{1+\sqrt{\frac{k}{mg}}v}{1-\sqrt{\frac{k}{mg}}v}\right| ; \sqrt{\frac{kg}{m}}t = \frac{\ln}{2}\left|\frac{1+\sqrt{\frac{k}{mg}}v}{1-\sqrt{\frac{k}{mg}}v}\right| = \boxed{\tanh^{-1}\left(\sqrt{\tfrac{k}{mg}}v\right)}$

b) $\boxed{\displaystyle\lim_{t\to\infty} v(t)=\sqrt{\frac{mg}{k}} = \text{"terminal velocity"}}$ $\qquad v(t)=\sqrt{\frac{mg}{k}}\tanh\sqrt{\frac{kg}{m}}t$

c)



d. $\displaystyle V_{avg}=\frac{(31{,}400-2100)\,ft}{116\,sec}=252\,\frac{ft}{sec.}$

e. $\displaystyle \dot{s}=\frac{ds}{dt}=v=V\tanh\sqrt{\frac{kg}{m}}t ; \; s(t)=V\int\tanh\sqrt{\frac{kg}{m}}t\,dt$

$\displaystyle =V\int\frac{\sinh\sqrt{\frac{kg}{m}}t}{\cosh\sqrt{\frac{kg}{m}}t}dt = V\int\frac{1}{u}dt = \frac{m}{k}\ln\cosh\frac{gt}{v} = \frac{V^2}{g}\ln\cosh\frac{gt}{V}$

$\displaystyle 29{,}300=\frac{V^2}{32.2\,ft/sec^2}\ln\cosh\cdot\frac{32.2\,ft/sec^2}{V}116\,sec$

$\displaystyle e^{9.43\times10^5/V^2} = \frac{e^{\frac{3735\,ft/sec}{V}}+e^{-\frac{3735\,ft/sec}{V}}}{2}$

$\boxed{V=266\,ft/sec.}$

$V\approx V_{avg}=252\,ft/sec$

$\displaystyle \frac{gt}{v}=\frac{32.2\,ft/sec^2\cdot116\,sec}{252\,ft/sec}=14.8$

$\displaystyle \frac{V^2}{g}\ln\cosh\frac{gt}{V}\approx\frac{V^2}{g}\left[\frac{gt}{V}-\ln2\right]=265\,ft/sec$

$\dot{N}=rN(1-N/K)$ 2.3.1 a. $\displaystyle t=\frac{1}{r}\int\frac{dN}{N(1-N/K)}$

$\displaystyle =\frac{1}{r}\left[\int\frac{A\,dN}{N}+\int\frac{B\,dN}{(1-N/K)}\right]$

$\displaystyle =\frac{1}{r}\left[\int\frac{1\,dN}{N}+\frac{1}{K}\int\frac{dN}{(1-N/K)}\right]$

$\displaystyle =\frac{1}{r}\ln N-\ln|1-N/K|+C$

$\displaystyle =\frac{1}{r}\ln\frac{N}{1-N/K}+C$

$\displaystyle Ce^{-rt}=\frac{1-N/K}{N} ; \; N\left(1+\frac{Ce^{-rt}}{K}\right)=1$

$\displaystyle N=\frac{1}{\left(1+\frac{Ce^{-rt}}{K}\right)} = \boxed{\frac{N_0}{e^{-rt}}}$

General Solution: $\displaystyle x=Ce^{-rt}+\frac{1}{K}$

b. $x=1/N$; $dN=-\frac{dx}{x^2}$; $\dot{x}=-rx\left(1-\frac{1}{Kx}\right)=\frac{r}{K}-rx$; $\dot{x}+rx-\frac{r}{K}=\dot{x}+r\left(x-\frac{1}{K}\right)=0$

$\boxed{\dot{x}=r\left(\frac{1}{K}-x\right)}$

$\displaystyle N(t)=\frac{K}{KCe^{-rt}+1}$

$\displaystyle N=\frac{N_0}{e^{-rt}}$

$\dot{x}=k_1ax-k_{-1}x^2$ 2.3.2

$A+X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X$

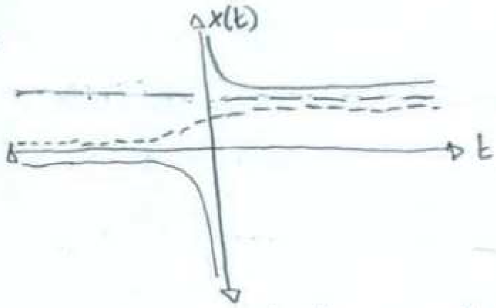
a. $\displaystyle t=\int\frac{dx}{k_1ax-k_{-1}x^2}=\frac{1}{k_1a}\int\frac{dx}{x-\frac{k_{-1}}{k_1a}x^2}=\frac{1}{k_1a}\int\frac{dx}{x\left(1-\frac{k_{-1}}{k_1a}x\right)}$

$\displaystyle =\frac{1}{k_1a}\left[\int\frac{A\,dx}{x}+\int\frac{B\,dx}{\left(1-\frac{k_{-1}}{k_1a}x\right)}\right]=\frac{1}{k_1a}\left[\int\frac{1\,dx}{x}+\frac{k_{-1}}{k_1a}\int\frac{dx}{1-\frac{k_{-1}}{k_1a}x}\right]$

$\displaystyle =\frac{1}{k_1a}\left[\ln|x|-\ln\left|1-\frac{k_{-1}}{k_1a}x\right|\right]=\frac{1}{k_1a}\ln\left|\frac{x}{1-\frac{k_{-1}}{k_1a}x}\right|+C ; \; \left(Ce^{-t/k_1a}+\frac{k_{-1}}{k_1a}\right)x=1$

$\displaystyle x(t)=\frac{1}{\frac{k_{-1}}{k_1a}+Ce^{-t/k_1a}} ; \; C=\frac{1}{x_0}-\frac{k_{-1}}{k_1a}$; $\boxed{\text{Fixed points of stability: } x=\frac{k_1a}{k_{-1}} \text{ source unstable}}$

b.



$— = X_0 > \frac{k_1 a}{k_{-1}}$

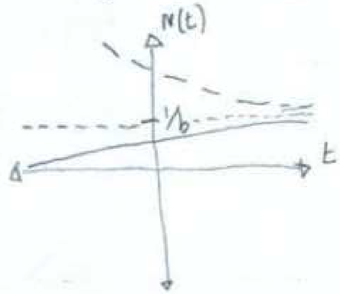
$-- = X_0 = \frac{k_1 a}{k_{-1}}$

$---- = X_0 < \frac{k_1 a}{k_{-1}}$

"Gompertz Law"

$\dot{N} = -aN\ln(bN)$ 2.3.3 a) $t = -\frac{1}{a}\int \frac{dN}{N\ln(bN)} = -\frac{b}{a}\int \frac{du}{u} = -\frac{b}{a}\ln[\ln bN]$; $N(t) = C\frac{e^{e^{-\frac{ab}{b}}}}{b} = \boxed{C\cdot\frac{e^{-\frac{ate}{b}}}{b}}$; a = rate constant, b = Max amount of cells.

b)



$— = C < b$

$---- = C = 1$

$-- = C > b$

$\frac{\dot{N}}{N} = r - a(N-b)^2$ 2.3.4. a). $\lim_{N\to 0} \frac{\dot{N}}{N} = \lim_{N\to b} r - a(N-b)^2 = r - ab^2 = \infty$; $r = \infty$

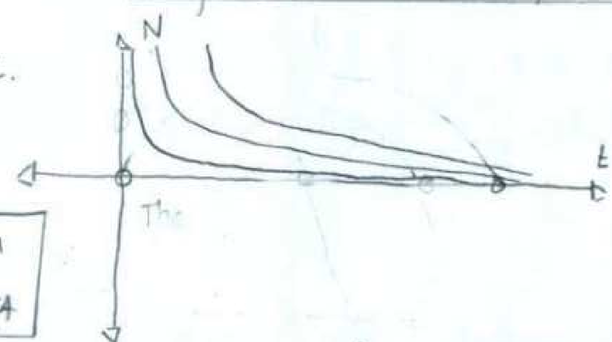
$\lim_{N\to\infty} \frac{\dot{N}}{N} = \lim_{N\to 0} r - a(N-b)^2 = r - \infty = 0$; $r = \infty$:

Each case of competition model at infinite population or extremely small populations that food amount or rate of consumption are insignificant to the competitors.

b) Fixed points of stability:

$N = 0$	source (unstable)
$N = \sqrt{\frac{r}{a}} + b$	sink (stable)

C.



d) The solutions of the logistic equation $y = Ce^{-rt} + \frac{1}{k}$ are similar, if not exact to the Allele Effect.

$t = \int \frac{du}{(b+u)(r-au^2)} = \int \frac{A}{b+u}du + \int \frac{Bu+C}{r-au^2}du$

$A(r-au^2) + (Bu+C)(b+u) = 1$

@ $u = \sqrt{\frac{r}{a}}$; $(B\sqrt{\frac{r}{a}} + C)(b + \sqrt{\frac{r}{a}}) = 1$

$B = \sqrt{\frac{a}{a}}$; $C = \sqrt{\frac{r}{a}}$

@ $u = -b$ $A = \frac{1}{r-(ab)^2}$

$= \frac{1}{r-(ab)^2}\int \frac{du}{b+u} + \frac{1}{\sqrt{ra}}\int \frac{ab\cdot u + r}{r - au^2}du$ Partial Fractions (x2)

$= \frac{\ln N}{r-(ab)^2} + \frac{b}{4}\sqrt{\frac{a}{r}}\tanh^{-1}([N-b]^2) + \frac{1}{\sqrt{a}}\tanh^{-1}\left(\sqrt{\frac{a}{r}}(N-b)\right) + C$

$\dot{X} = aX$
$\dot{Y} = bY$ 2.3.5 a) $x(t) = \frac{X(t)}{[X(t)+Y(t)]} = \frac{e^{at}}{e^{at}+e^{bt}}$; $\lim_{t\to\infty} x(t) \boxed{\approx 1}$

b) $\dot{x}(t) = \frac{ae^{at}(e^{at}+e^{bt}) - e^{at}(ae^{at}+be^{bt})}{(e^{at}+e^{bt})^2}$

$= \frac{e^{at}[a-b]e^{bt}}{(e^{at}+e^{bt})^2} = \frac{[a-b]e^{at}}{(e^{at}+e^{bt})}\left(\frac{e^{bt}+e^{at}-e^{at}}{e^{at}+e^{bt}}\right)$

$\boxed{= x[a-b](1-x)}$

$\dot{x}=(1-x)P_{YX}-xP_{XY}$ 2.3.6 a. $x=0$

$P_{YX}=sx^a$; $P_{XY}=(1-s)(1-x)^a$ $\quad x=1$

$$\boxed{x=\frac{\sqrt[a-1]{\frac{(1-s)}{s}}}{1+\sqrt[a-1]{\frac{(1-s)}{s}}}}$$

b. A plot of $s(1-x)x^a$ and $-(1-s)x(1-x)^a$ demonstrate $-(1-s)x(1-x)^a > s(1-x)x^a$ for $x=0$ and $x=0$, indicating each fixed point is stable.

c. For $x=\frac{\sqrt[a-1]{\frac{(1-s)}{s}}}{1+\sqrt[a-1]{\frac{(1-s)}{s}}}$ the plot of $s(1-x)x^a > (1-s)x(1-x)^a$, suggesting a source.

$\dot{x}=x(1-x)$ 2.4.1 $\dot{x}=f(x)=f(x^*+x)=f(x^*)+xf'(x^*)+O(x^2)$

$=xf'(x^*)+O(x^2)$

$\boxed{=x(1-2x)}$

$x=0$; $f'(x^*)=1$: Unstable (source)

$x=1$; $f'(x^*)=-1$: Stable (sink)

$\dot{x}=x(1-x)(2-x)$ 2.4.2 $\dot{x}=f(x)=f(x^*+x)\boxed{=xf'(x^*)}$

$\boxed{=x\cdot 2x(1-x)}$

$x=0$ $f'(x^*)=0$ Half-stable

$x=1$ $f'(x^*)=0$ Half-stable

$x=2$ $f'(x^*)=-4$ sink (stable)

$\dot{x}=\tan x$ 2.4.3 $\dot{x}=f(x)=f(x^*+x)=\boxed{x\sec^2(x)}$

$x=n\pi$ $f(x)=(+)$ source (unstable)

$\dot{x}=x^2(6-x)$ 2.4.4 $\dot{x}=f(x)=f(x^*+x)=\boxed{x[12x-3x^2]}$

$x=0$ $f'(x)=0$ Half-stable

$x=6$ $f'(x)=-36$ sink (stable)

$\dot{x}=1-e^{-x^2}$ 2.4.5 $\dot{x}=f(x)=f(x^*+x)=\boxed{x[2e^{-x^2}]}$

$x=0$ $f'(x^*)=0$ Half-stable

$\dot{x}=\ln x$ 2.4.6 $\dot{x}=f(x)=f(x^*+x)=\boxed{1}$

$x=1$ $f'(x^*)=1$ source (unstable)

$\dot{x}=ax-x^3$ 2.4.7 $\dot{x}=f(x)=f(x^*+x)=\boxed{x[a-3x^2]}$

a	(+)	(−)	(0)
$x=0$	source	sink	Half-stable
$x=+\sqrt{a}$	sink	source	source
$x=-\sqrt{a}$	source	sink	Half-stable

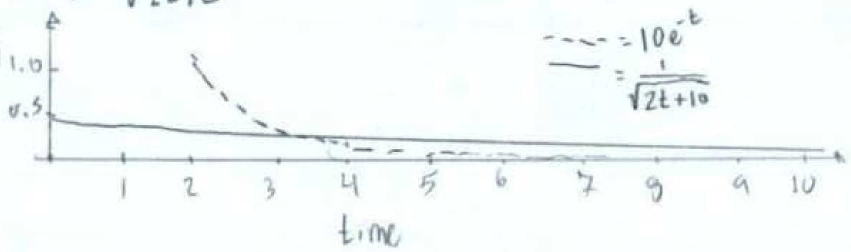
$\dot{N}=-aN\ln(bN)$ 2.4.8 $\dot{N}=f(N)=f(N+N^*)=\boxed{-\frac{aN}{b}[1+b\ln(bN)]}$

$N=0$: source (unstable)

$N=\frac{1}{b}$: sink (stable)

$\dot{x}=-x^3$ 2.4.9 a. $t=-\int\frac{dx}{x^3}=\frac{1}{2x^2}+c$; $x(t)=\sqrt{\frac{1}{2t+c}}$

$\lim_{x\to 0} t=\frac{1}{0}+c=\boxed{\infty}$



b. if $x_0=10$

$t=-\int\frac{1}{x}=-\ln x$;

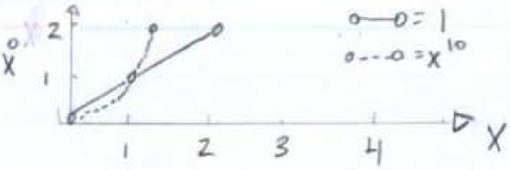
$\boxed{x(t)=x_0e^{-t}=10e^{-t}}$

$\dot{x}=-x^c$ 2.5.1 a. $c=0$

b. $dx=-dt$; $x(t)=-t$; if $t=0$ is considered finite time, then yes.

$t=-\int\frac{dx}{x^c}=\frac{-x^{1-c}}{1+c}$; $t(x=1)-t(x=0)=-\frac{1^{1-c}}{1-c}+\frac{0^{1-c}}{1-c}=\boxed{\frac{1}{c+1}}$

$\dot{x}=1+x^{10}$; 2.5.2.



$\boxed{\lim_{\dot{x}\to\infty} \dot{x} = \infty}$

$\dot{x}=rx+x^3$ 2.5.3

$t=\int \frac{dx}{x(r+x^2)} = \int \frac{A}{x}dx + \int\frac{Bx+C}{r+x^2}dx$; $A(r+x^2)+(Bx+C)(x)=1$

$x=0$; $A=\frac{1}{r}$; $1+\frac{x^2}{r}+Bx^2+Cx=1$

$(B+\frac{1}{r})x^2+Cx=0$

$(B+\frac{1}{r})x=-C$

$\boxed{x=0; C=0}$

$\boxed{x=1; B=-\frac{1}{r}}$

$=\frac{1}{r}\ln x - \frac{1}{2r}\ln r+x^2 = \frac{1}{r} \ln\frac{x}{\sqrt{r+x^2}}$

$x^2Ce^{-2rt}=(r+x^2)$; $x=\sqrt{\frac{r}{1+Ce^{-2rt}}}$

if $x_0\neq 0$; $\lim_{t\to\infty}x(t)=\infty$

$\dot{x}=x^{1/3}$ 2.5.4 $x(0)=0$; $t=\int\frac{dx}{x^{1/3}}=\frac{3}{2}x^{2/3}$; $\boxed{x(t)=\sqrt{(\frac{2}{3}t-\frac{2}{3}c)^3}}$

$\dot{x}=|x|^{p/q}$ 2.5.5. $x(0)=0$; a) $t=\frac{q}{p+q}(x)^{\frac{p+q}{q}}$; $x(t)=\left(\frac{p+q}{q}(t+C)\right)^{q/p+q}$; c = many solutions at zero because of root.

$\boxed{x(t)=\left(\frac{p+q}{q}(t+C)\right)^{q/p+q}}$.;

b) $x(t)=\left(\frac{p+q}{q}(t+c)\right)^{q/p+q}$; if $p>q$; $x(0)=\left(\frac{p+q}{q}(0+c)\right)^{q/p+q}=0$; $\boxed{c=0}$

h(t)=height

t=time

A=cross-sectional Area

v(t)=velocity

$av(t)=A\dot{h}(t)$

2.5.6 a) Newton's first law that for every force there exist an equal and opposite counter force.

b) $\frac{1}{2}mv^2=mgh$; $\boxed{v^2=2gh}$;

c) $\boxed{\dot{h}(t)=-\sqrt{\frac{a}{A}2gh}}$

d) $\boxed{h(0)=0;\ t=-\sqrt{\frac{Ah}{2a}}\,2\sqrt{h};\ h(t)=-\sqrt{\frac{a}{2A}}\,t}$

$m\ddot{x}=-kx$ 2.6.1 The text states, "there are no periodic solutions to $\dot{x}=f(x)$ because undamped systems do not oscillate, and damped oscillations do not occur for first order systems." Strogatz' statement does not fit the equation of 2.6.1.

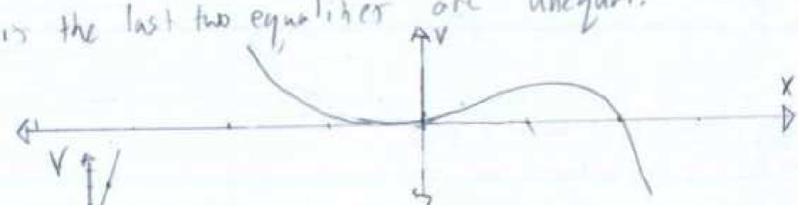
$\dot{x}=f(x)$

$x(t)=x(t+T)$

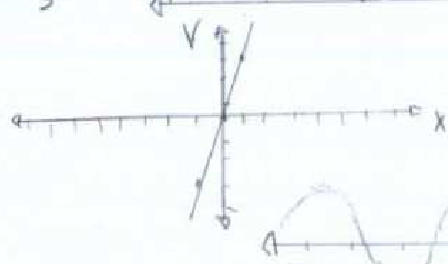
2.6.2 $\int_t^{t+T} f(x)\frac{dx}{dt}dt=\int_t^{t+T} f(x)\dot{x}(t)dt=\int_t^{t+T}f(x)\dot{x}(t+T)d(t+T)$

The contradiction of the argument is the last two equalities are unequal.

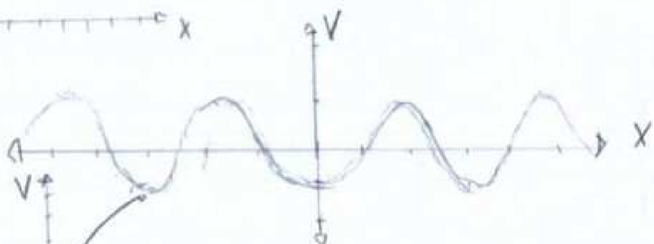
$\dot{x}=x(1-x)$ 2.7.1 $\frac{dV}{dx}=\dot{x}=x(1-x)$; $V=(1-\frac{x}{3})x^2$



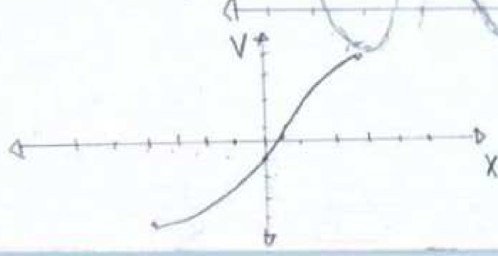
$\dot{x}=3$ 2.7.2 $\frac{dV}{dx}=\dot{x}=3$; $V=3X$



$\dot{x}=\sin x$ 2.7.3 $\frac{dV}{dx}=\dot{x}=\sin x$; $V=-\cos(x)$

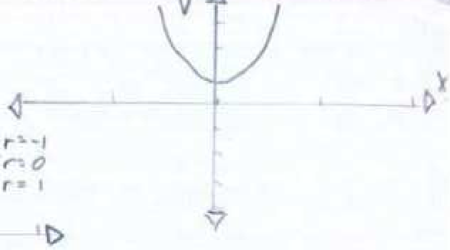


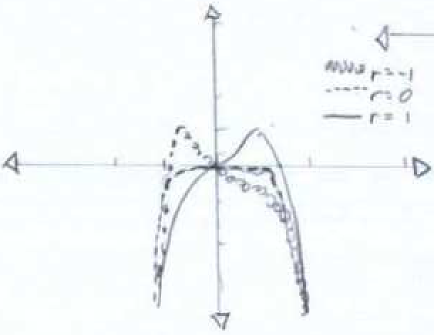
$\dot{x}=2+\sin x$ 2.7.4 $\frac{dV}{dx}=\dot{x}=2+\sin x$; $V=2X-\cos(x)$



$\dot{x}=-\sinh x$ 2.7.5. $\frac{dV}{dx}=-\sinh x$; $V=-\cosh(x)$

$\dot{x}=r+x-x^3$ 2.7.6 $\frac{dV}{dx}=r+x-x^3$; $V=rx+\frac{x^2}{2}-\frac{x^4}{4}$





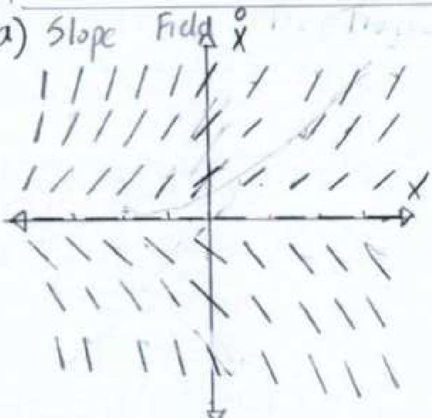
$\dot{x}=f(x)$ 2.7.7. $\frac{dV}{dx}=\dot{x}=f(x)$ ∴ $V=\frac{df(x)}{dx}dx+c$

$f(x)=\frac{d(V-c)}{dx}dx$

The solution $x(t)$ cannot oscillate because of the existence and uniqueness of $f(x)$, and the solutions for $f(x)=0$, that $V=C$ or $C=0$; withstanding, $\frac{d(V-C)}{dx}=\frac{dx}{dt}$, then the solution $x(t)$ also corresponds to a nonperiodic function.

$\dot{x}=x(1-x)$ 2.8.1 The horizontal lines are to be expected in Figure 2.8.2 because of the slope being zero at $x=1$.

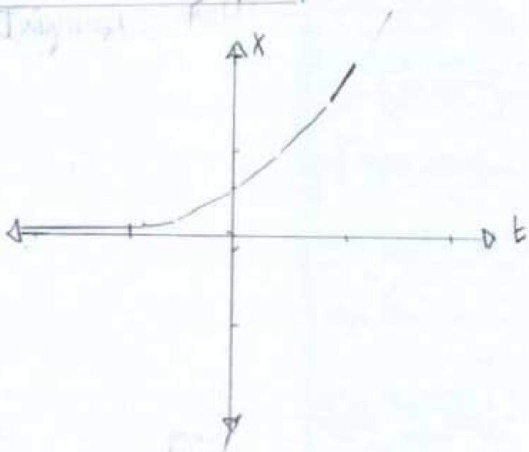
$\dot{x}=x$ 2.8.2.a) Slope Field



$\frac{dx}{dt}=x$

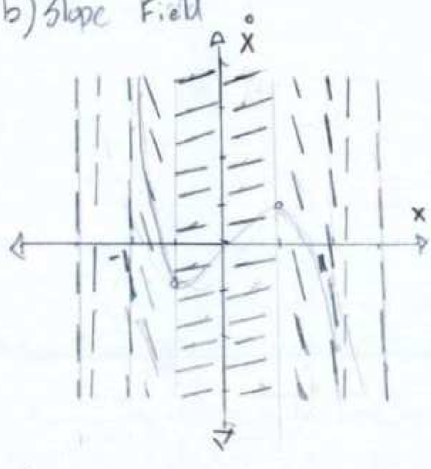
$\ln x=t$

$x=e^t$



$\dot{x}=1-x^2$

b) Slope Field



$\frac{dx}{dt}=1-x^2$

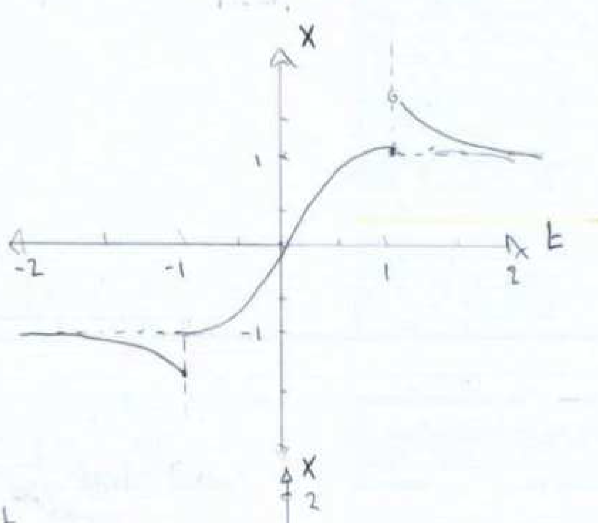
$\int\frac{dx}{1-x^2}=t$

$\frac{1}{2}\ln\frac{1+x}{x-1}=t$

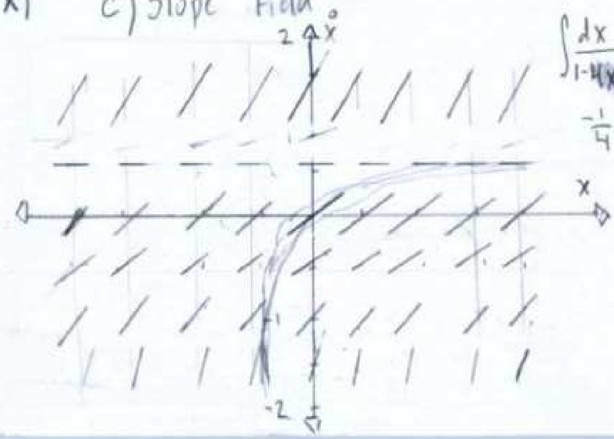
$x<-1$

$-1<x<1$

$x>1$



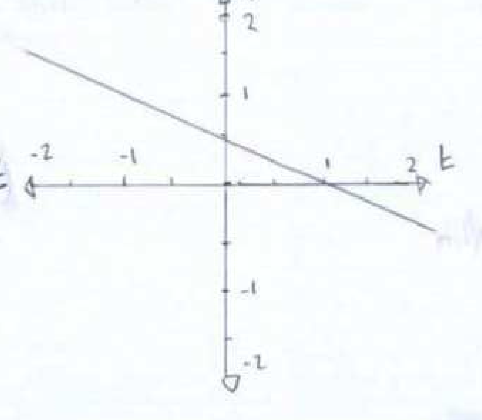
$\dot{x}=1-4x(1-x)$ c) Slope Field



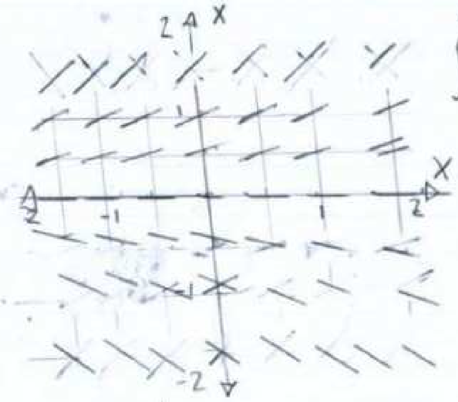
$\int\frac{dx}{1-4x(1-x)}=t$

$-\frac{1}{4}\left(\frac{1}{x-0.5}\right)=t$

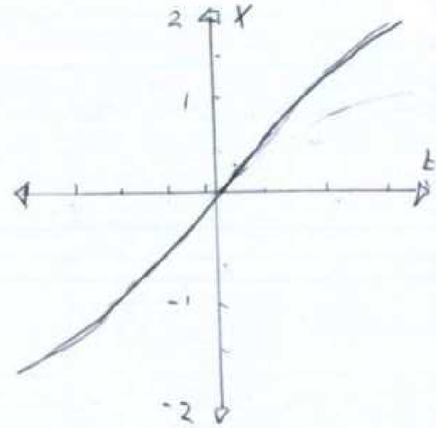
$x=0.5-\frac{1}{4}t$



$\dot{x} = \sin x$ 2.3.2 d) Slope Field



$$\frac{dx}{dt} = \sin x$$
$$\int \cosh x = t$$
$$x = \sinh^{-1}(t)$$



Euler's Method

$\dot{x} = -x ; x(0) = 1$ 2.8.3 a) $x(t) = Ce^{-t} ; C = 1 ; x(t) = e^{-t}$

b) $\Delta t = 1 ; x(t_0 + \Delta t) \approx x_0 + f(x_0)\Delta t ; x(t_0 + \Delta t) = 0 + e^{-1} \cdot 10^{0} = 0.3679$

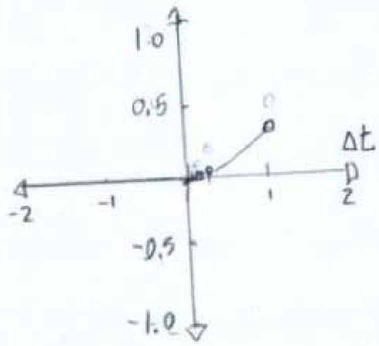
$\Delta t = 10^{-n}$ $n = 1$ $x_n + e^{-x_n} 10^{-1} = 0.36341$

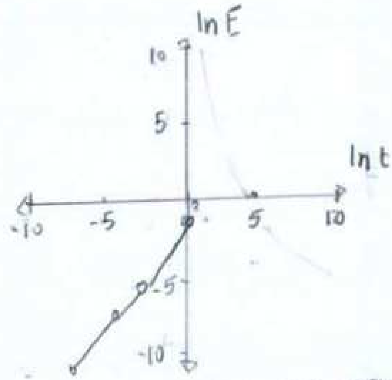
$n = 2$ $x_n + e^{-x_n} 10^{-2} = 0.36697$

$n = 3$ $x_n + e^{-x_n} 10^{-3} = 0.36785$

$n = 4$ $x_n + e^{-x_n} 10^{-4} = 0.36787$

c) $E = |\hat{x}(1) - x(1)|$

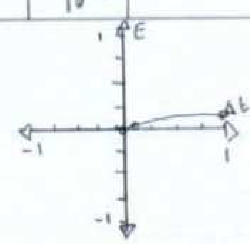


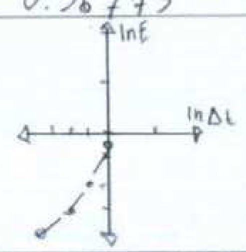


The results of $E = |\hat{x}(1) - x(1)|$ vs Δt represent error of Euler's method. The results of $E = |\hat{x}(1) - x(1)|$ vs Δt characterizes nothing informative, while the plot of $\ln E$ vs $\ln t$ characterizes nothing informative.

$\dot{x} = -x ; x(0) = 1$ 2.8.4 $x(t) = e^{-t}$;

n	Δt	$f(x)$	$x_n = x_{n-1} + f(x_{n-1})\Delta t$	$E = \lvert\hat{x}(1) - x(1)\rvert$	$\ln E$
0	10^{0}		0.36788	0.00	-1.00
1	10^{-1}	$\exp(x_{n-1})$	0.33527	0.03261	-3.42
2	10^{-2}		0.36577	0.0211	-6.16
3	10^{-3}		0.36773	0.0002	-8.33
4	10^{-4}		0.36773	0.0000	-10.78

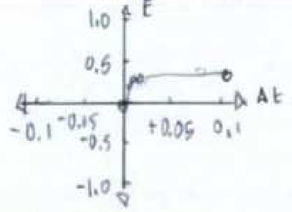


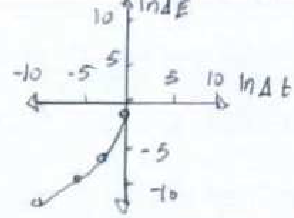


Improved Eulers Method

$\dot{x} = -x ; x(0) = 1$ 2.8.5. $x(t) = e^{-t}$

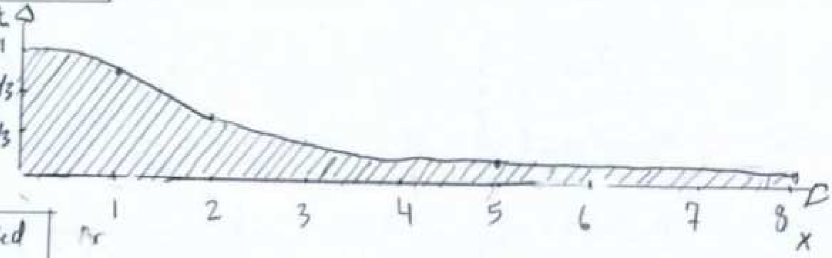
n	Δt	$f(x)$	$x_n = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$	$E = \lvert\hat{x}(1) - x(1)\rvert$	$\ln E$
0	10^{0}				
1	10^{-1}				
2	10^{-2}	$\exp(x_{n-1})$			
3	10^{-3}				
4	10^{-4}				



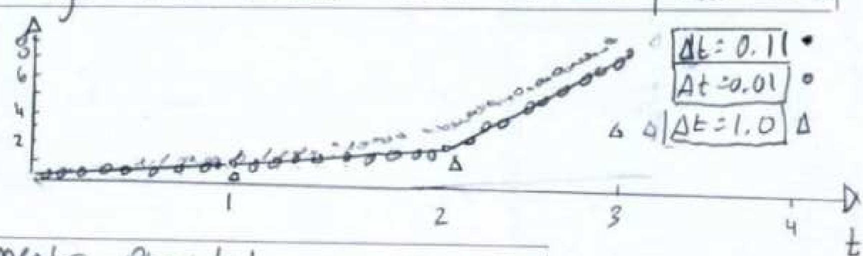


The Euler method aided the analysis of numerical methods; including, Precision. Euler's Improved Method approached the solution of $f(x)=e^{-x}$ with less round-off error. Runge-Kutta's Routine provided the least round-off errors with 10^{-20} across the spreadsheet, and necessitated high-precision.

$\dot{x}=x+e^{-x}$ 2.8.6. a: $t=\int \frac{1}{x+e^{-x}}\,dx = \int \frac{e^{x}}{e^{x}x+1}\,dx$;

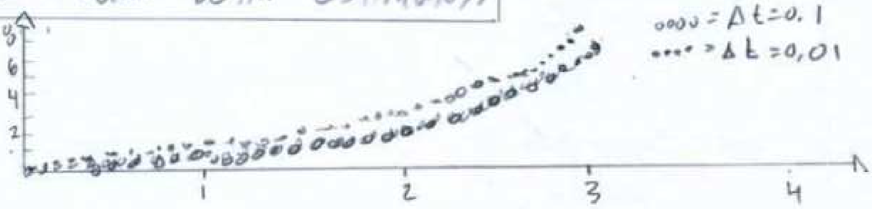


I noticed the book asked for x(t) (and not t(x)).
This led me to investigate a Numerical method of integration; Withstanding, Runge-Kutta Routine aided with the plot of x(t).



b. At $t=0$, analytical arguments provided an $x=1.011$.

c. Stepsizes of $\Delta t=0.1$ and 0.01 had different results, including inaccuracies above and below better estimates.



d) See part a.

$x_1=x_0+f(x_0)\Delta t$ 2.8.7 a) $x(t_1)=x(t_0+\Delta t)$

Taylor Series:

$$x(t+\Delta t)=\sum_{n=0}^{\infty}\frac{x^{(n)}(t)}{n!}(\Delta t)^{n} = x(t)+x'(t)\cdot\Delta t+O(\Delta t^{2})$$

$$f(t+\Delta t)=f(t)+f'(t)\cdot\Delta t+O(\Delta t^{2})=\boxed{x_0+f'(t)\cdot\Delta t}$$

b) $|x(t_1)-x_1| = |x(t_1)-x(t_1)-x'(t)\cdot\Delta t-O(\Delta t^{2})| = |O(\Delta t^{2})| = \frac{x''(t)\Delta t^{2}}{2!} = C(\Delta t^{2})$

$$\boxed{C=\frac{x''(t)}{2!}}$$

Taylor Series: $f(x+h) = \sum \frac{f^{n}(x)h^{n}}{n!}$

2.8.8. $\dot{x} = x + e^{-x}$; $|x(t_0) - t_0| = |x(t_0) - x(t_0) - x(t_0)'\Delta t - \frac{x''(t_0)\Delta t^2}{2!}| = \frac{x''(t_0)\Delta t^2}{2}$

$$\boxed{= O(\Delta t^2)}$$

$\dot{x} = x + e^{-x}$

2.8.9 Runge-Kutta: $x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ Where $k_1 = f(x_n)\Delta t$

$k_2 = f(x_n + \frac{1}{2}k_1)\Delta t$

$k_3 = f(x_n + \frac{1}{2}k_2)\Delta t$

$k_4 = f(x_n + k_3)\Delta t$

$$x(t+\Delta t) = x(t_0) + x'(t_0)\Delta t + \frac{x''(t_0)\Delta t^2}{2!} + O(\Delta t^3)$$

$$k_1 = f(x_n)\Delta t = x'(t_0)\Delta t$$

$$k_2 = f(x_n + \tfrac{1}{2}k_1)\Delta t = f(x_n) + f'(x_n)\tfrac{1}{2}k_1 + O[(\tfrac{1}{2}k_1)^2]$$

$\therefore$

$$k_3 = f(x_n + \tfrac{1}{2}k_2)\Delta t = f(x_n) + f'(x_n)\tfrac{1}{2}k_2 + O[(\tfrac{1}{2}k_2)^2]$$

$$= f(x_n) + f'(x_n)\tfrac{1}{2}\left[f(x_n) + f'(x_n)\tfrac{1}{2}k_1 + O[(\tfrac{1}{2}k_1)^2]\right] + O[(\tfrac{1}{2}k_2)^2]$$

$$k_4 = f(x_n + k_3)\Delta t = f(x_n) + f'(x_n)\cdot k_3 + O[k_3^2]$$

$$= f(x_n) + f'(x_n)\left[f(x_n) + f'(x_n)\cdot\tfrac{1}{2}\left[f(x_n) + f'(x_n)\tfrac{1}{2}k_1 + O[(\tfrac{1}{2}k_1)^2]\right] + O[(\tfrac{1}{2}k_2)]\right] + O[k_3]$$

$$x_{n+1} = x_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = x_n + \tfrac{1}{6}\Big(x'(t_0)\Delta t + 2x'(t_0) + x''(t_0)k_1$$
$$+ 2x'(t_0) + x''(t_0)[x'(t_0) + x''(t_0)x'(t)\Delta t]$$
$$+ 2x'(t_0) + x''(t_0)\Big[x(t) + \frac{x'(t)}{2}\Big[x'(t_0) + x''(t_0)\frac{x'(t)}{2}\Big]\Big]\Big)$$

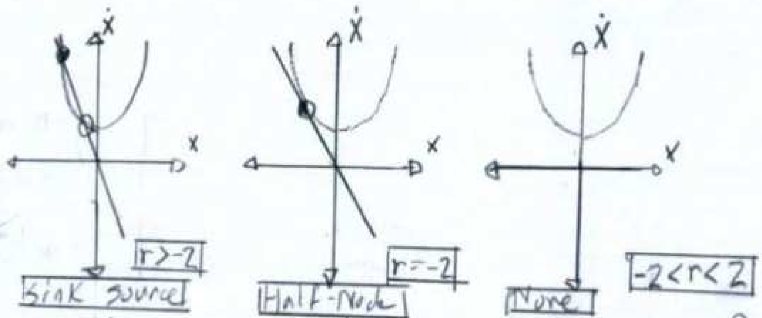
$$\boxed{|x(t_1) - x_1| = |x(t_0 + \Delta t) - x_{n+1}| = O(\Delta t^5)}$$

Chapter 3

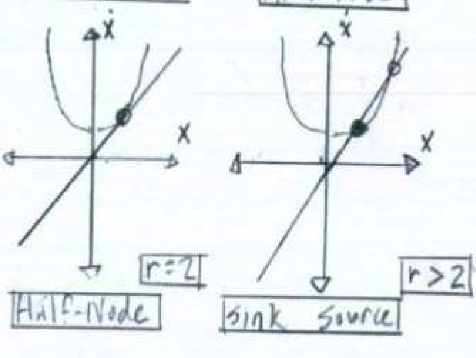
$\dot{x} = 1 + rx + x^2$ 3.1.1. Vector Field:

$$x = \frac{r \pm \sqrt{r^2 - 4}}{2}$$

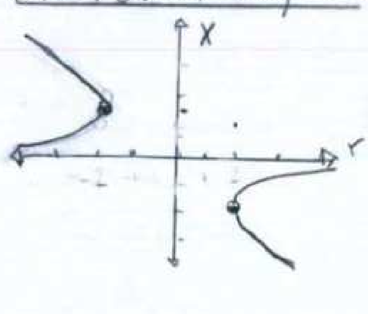
$$= \frac{r \pm \sqrt{(r-2)(r+2)}}{2}$$



r	Bifurcations
>-2	Two
-2	One
0	zero
2	one
>2	Two



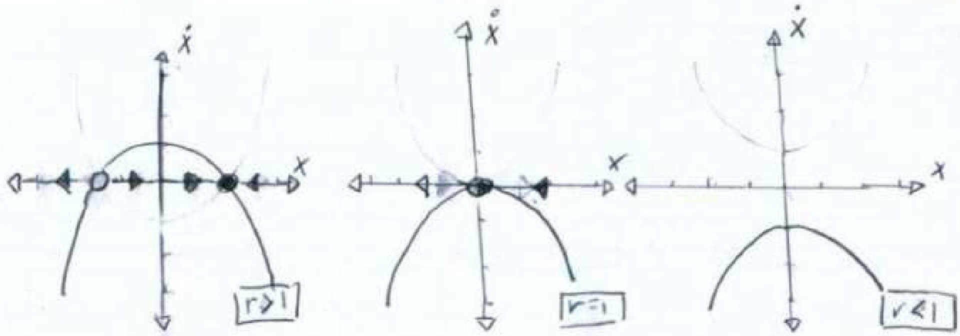
Bifurcation Diagram:



$\dot{x} = r - \cosh x$ 3.1.2 Vector Field

$r = \cosh(x)$

r	Bifurcations
<1	Zero
=1	One
>1	Two



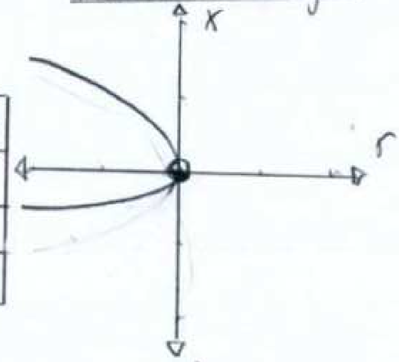
$\dot{x} = r + x - \ln(1+x)$ 3.1.3

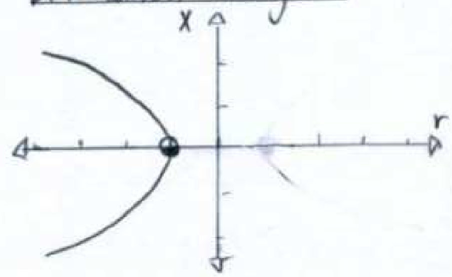
Vector Field

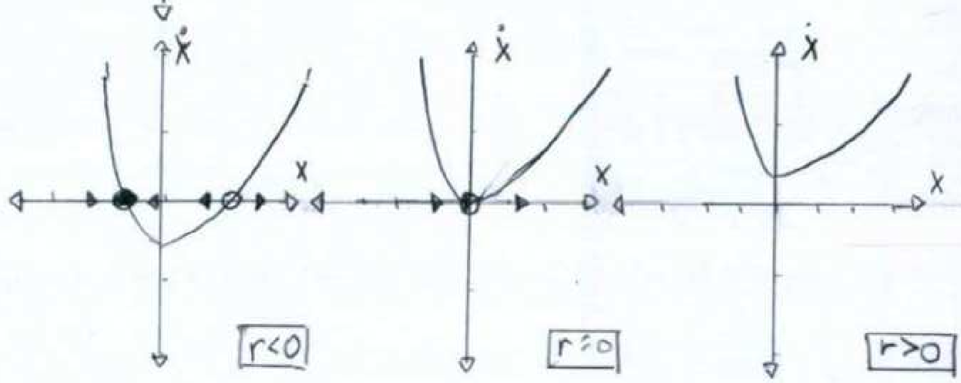
Bifurcation Diagram:

Bifurcation Diagram

r	Bifurcation
>0	Zero
=0	One
<0	Two

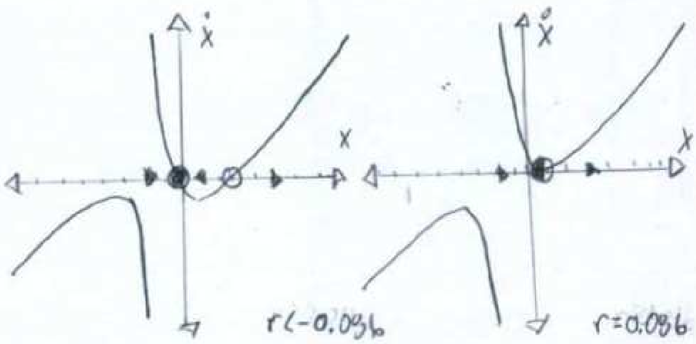


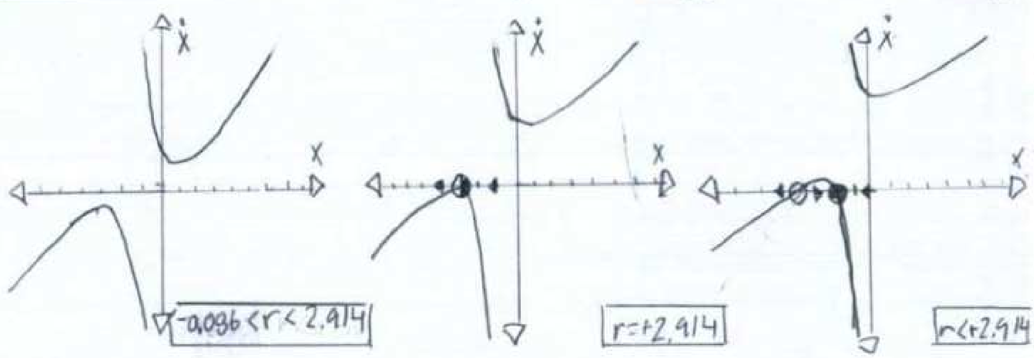




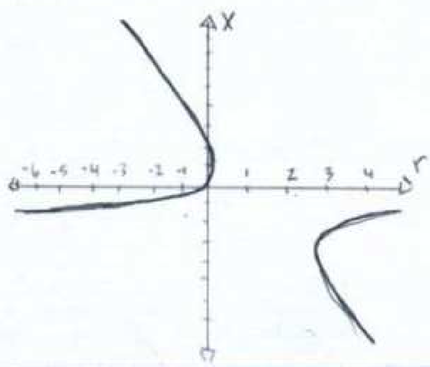
$\dot{x} = r + \frac{1}{2}x - x/(1+x)$ 3.1.4. Vector Field:

r	Bifurcations
< -0.096	Two
= -0.096	one
-0.096 < r < 2.914	zero
= +2.914	one
< +2.914	Two.



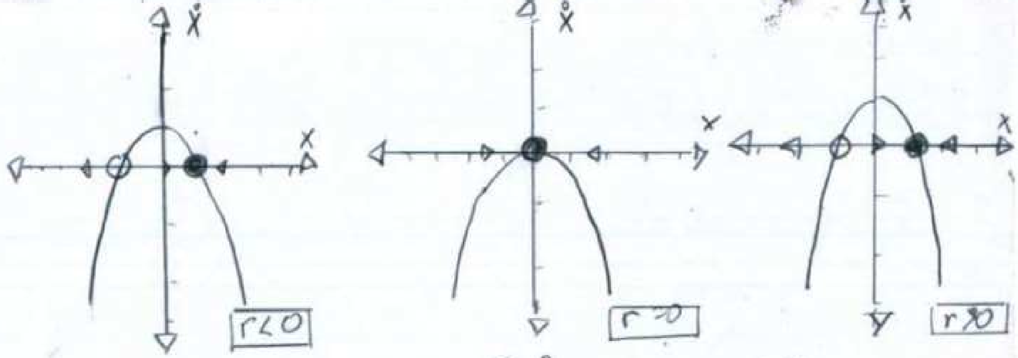


Bifurcation Diagram:



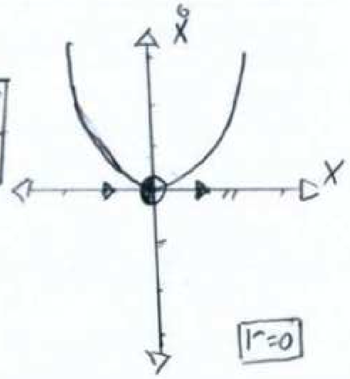
$\dot{x}=r^2-x^2$ 3.1.5. a) Vector Field:

r	Bifurcations
>0	Two
=0	one
<0	Two

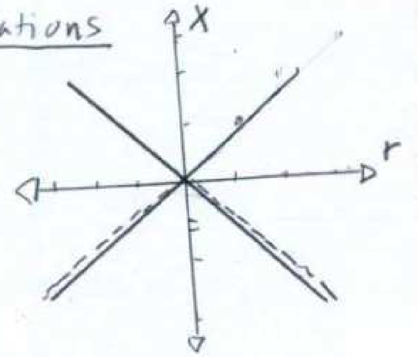


$\dot{x}=r^2+x^2$ b) Vector Field:

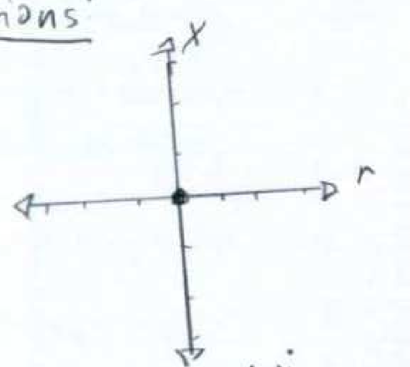
r	Bifurcations
0	one



Bifurcations



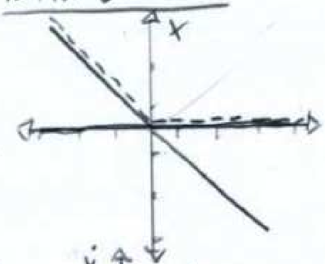
Bifurcations:

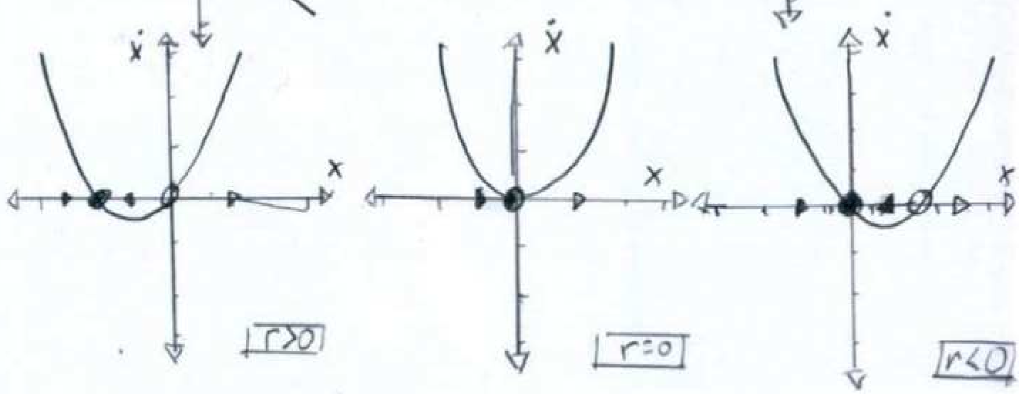


$\dot{x}=rx+x^2$ 3.2.1 Vector Field:

r	Bifurcations
>0	Two
=0	one
<0	Two

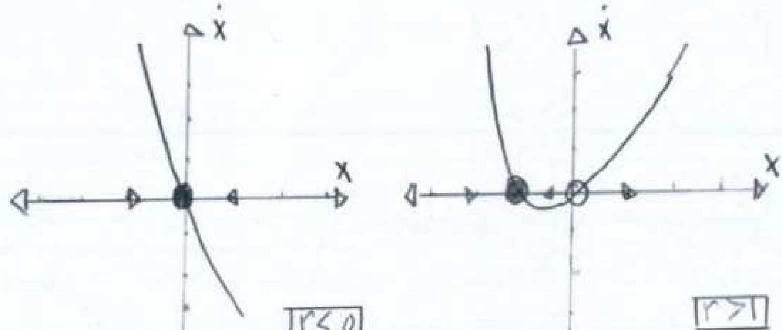
Bifurcations:

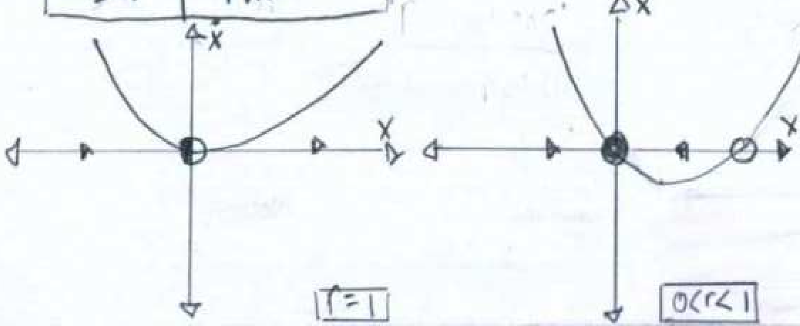




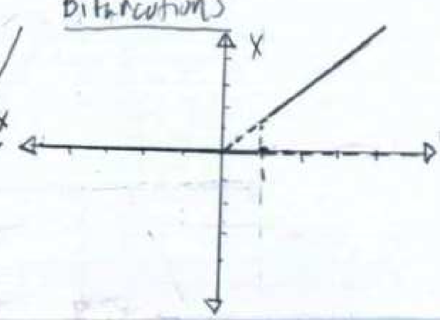
$\dot{x}=rx-\ln(1+x)$ 3.2.2. Vector Field:

r	Bifurcations
<0	one
0<r<1	Two
r=1	One
r>1	Two



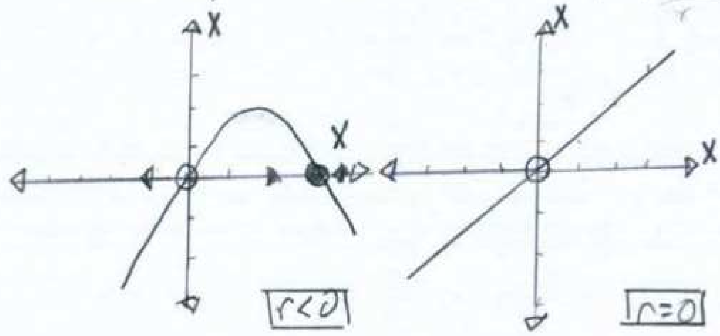


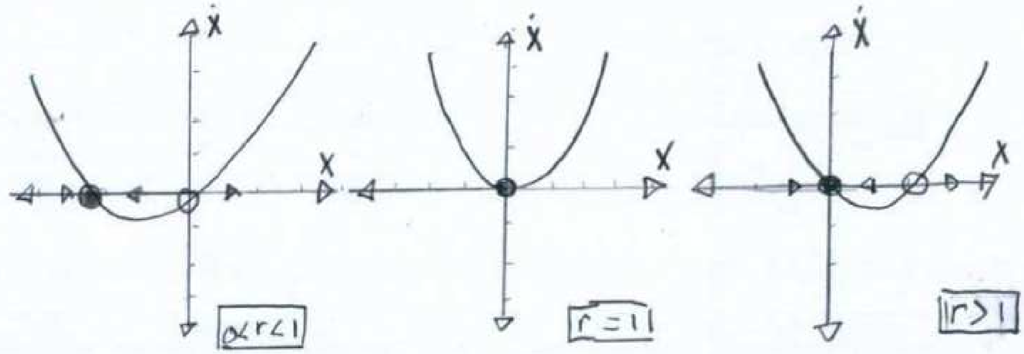
Bifurcations



$\dot{x} = x - rx(1-x)$ 3.2.3. Vector Field:

r	Bifurcations
≤0	Two
=0	One
0<r<1	Two
r=1	One
>1	Two

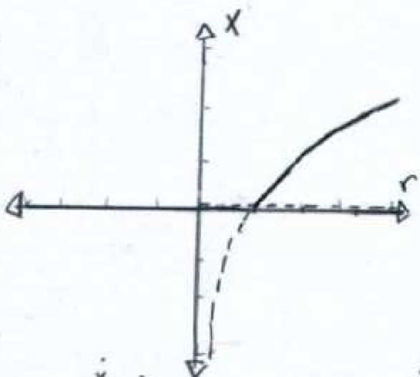




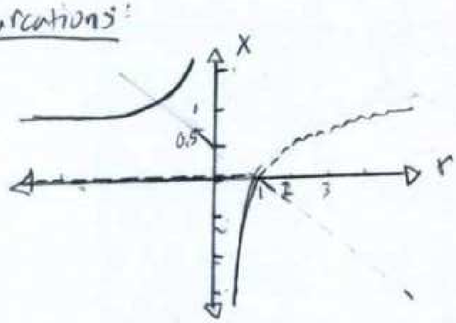
$\dot{x} = x(r - e^x)$ 3.2.4. Vector Field:

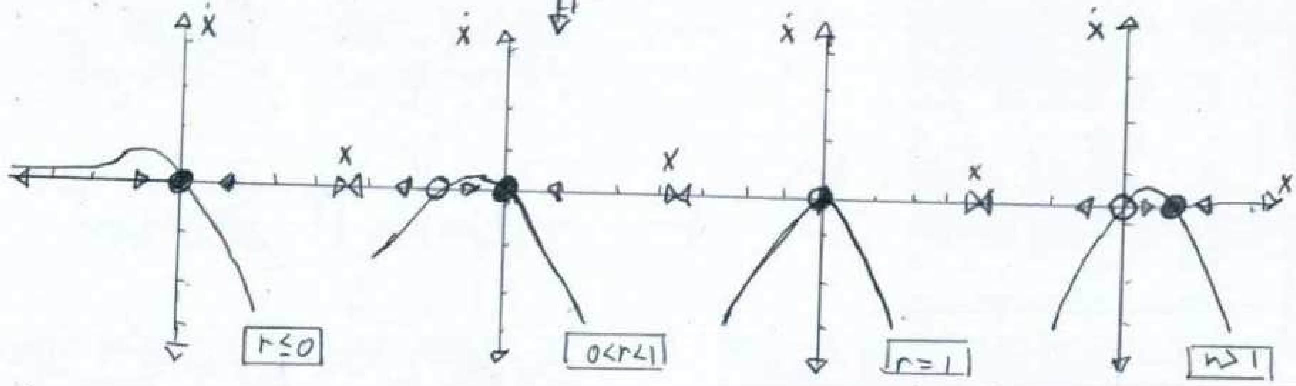
r	Bifurcations
≤0	one
0<r<1	Two
r=1	one
r>1	Two

Bifurcations:



Bifurcations:





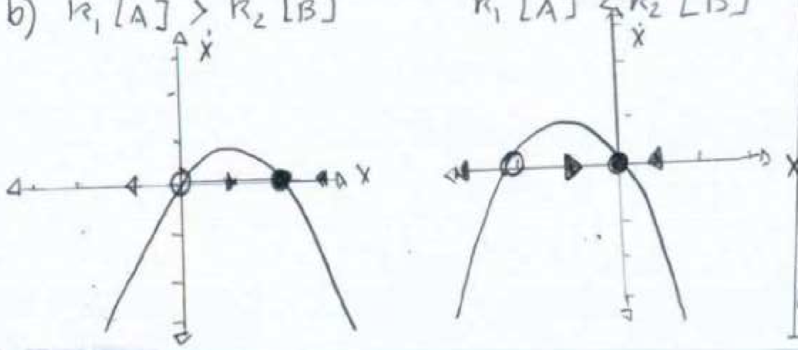
$\dot{x} = c_1 x - c_2 x^2$ 3.2.5 a) $\frac{d[A]}{dt} = -k_1[A][X] + k_2[X]^2$; $\boxed{\frac{d[X]}{dt} = (k_1[A] - k_2[B])[X] - k_{-1}[X]^2}$

$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X$

$\frac{d[B]}{dt} = -k_2[B][X]$; $\frac{d[C]}{dt} = k_2[B][X]$

$X + B \xrightarrow{k_2} C$

b) $k_1[A] > k_2[B]$ $\qquad$ $k_1[A] < k_2[B]$



Chemically, a rate of change $\frac{d[x]}{dt}$ that approaches zero, then remains zero is of greater stability than a rate of change which increases from zero.

$\dot{n} = GnN - kn$

$\dot{N} = -GnN - fN + p$

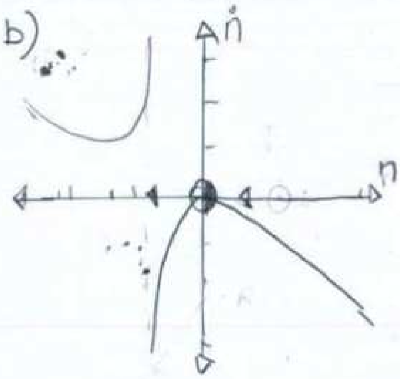
3.3.1 a) Suppose $\dot{N} > \dot{n}$, then $\dot{N} \approx 0$. "Adiabatic Elimination"

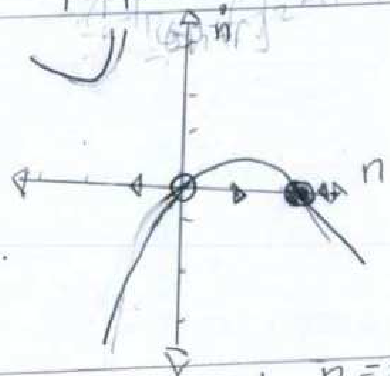
$GnN + fN = p$; $\dot{n} = -fN + p - kn$;

$N = \dfrac{p}{Gn + f}$; $\boxed{\dot{n} = -f\left[\dfrac{p}{Gn+f}\right] + p - kn}$

$\boxed{p > \dfrac{kn\,[Gn+f]}{1-pf} = p_c}$

b)





c) A transcritical bifurcation occurs at $n = 0$ because of the stability change for the fixed point.

d) $G, n, p, f > 0$, $N = 0$, a constant amount of excited photons

$\dot{E} = K(P - E)$

$\dot{P} = \gamma_1(ED - P)$

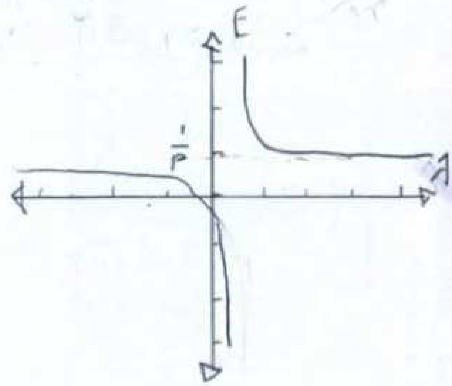
$\dot{D} = \gamma_2(\lambda + 1 - D - \lambda EP)$

3.3.2 a) Assume $\dot{P} \approx 0$, $\dot{D} \approx 0$; $P = ED$; $D = \lambda + 1 - \lambda EP$

$\dot{E} = K(ED - E) = K(E(\lambda + 1 - \lambda EP) - E)$

b) Fixed Points : $\boxed{E = 0, \frac{1}{P}}$

c) Bifurcation Diagram:

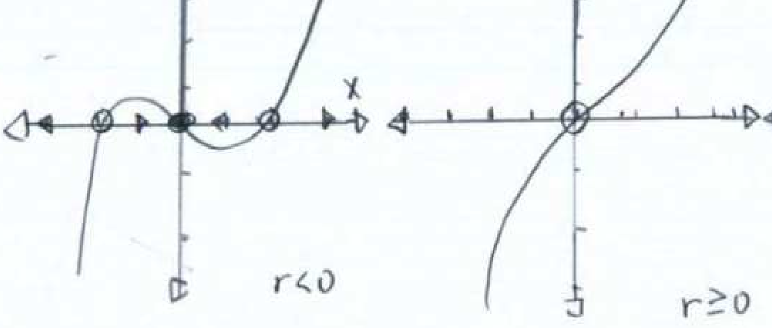


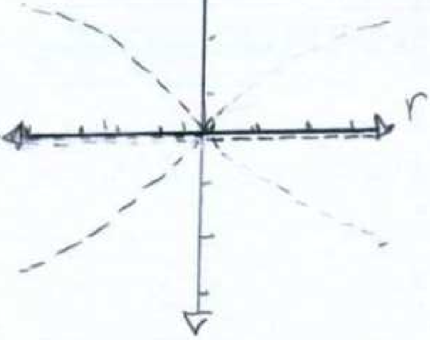
$\dot{x} = rx + 4x^3$

3.4.1 Vector Field:

r	Bifurcations
< 0	Three
≥ 0	One

Bifurcations: Subcritical

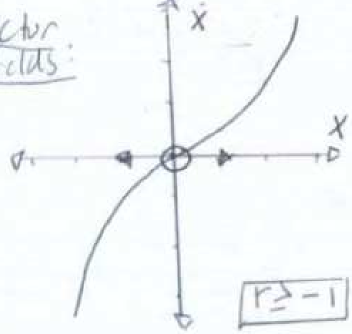




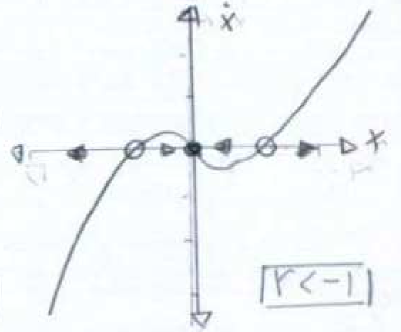
$\dot{x} = rx - \sinh x$ 3.4.2.

r	Bifurcations
≥ -1	One
< -1	Three

Vector Fields:



$\boxed{r \geq -1}$

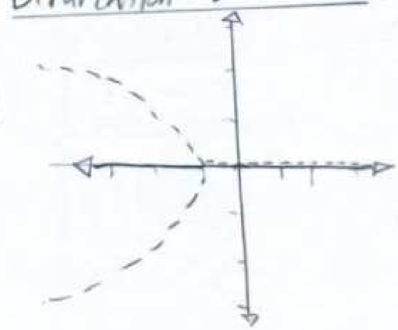


$\boxed{r < -1}$

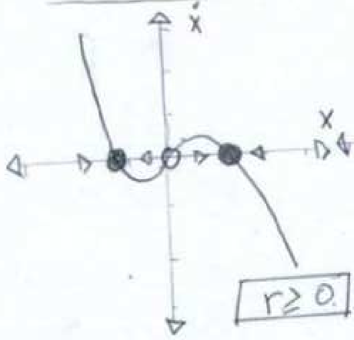
$\dot{x} = rx - 4x^3$ 3.4.3

r	Bifurcations
≥ 0	Three
≤ 0	one

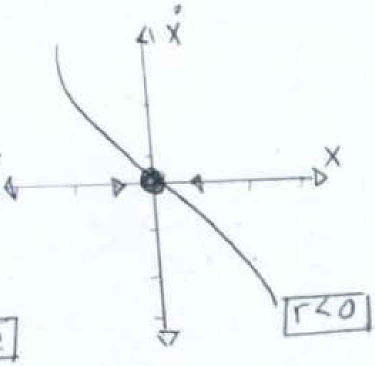
Bifurcation: subcritical



Vector Field:



$\boxed{r \geq 0.}$

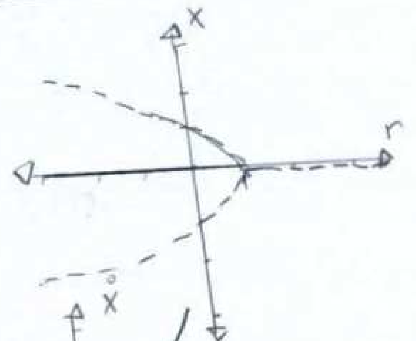


$\boxed{r < 0}$

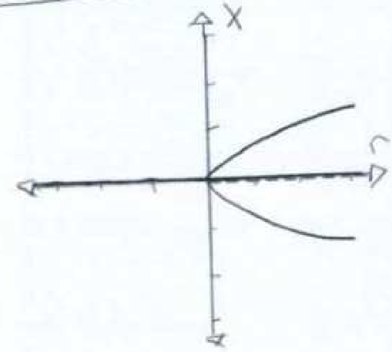
$\dot{x} = x + \frac{rx}{1+x^2}$ 3.4.4

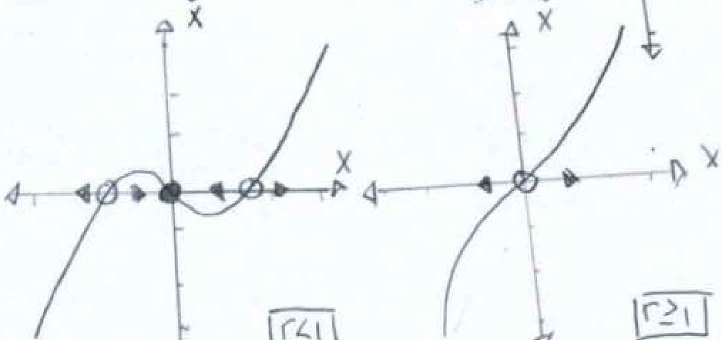
r	Bifurcations
< 1	Three
≥ 1	one

Bifurcations: Supercritical



Bifurcation: Supercritical





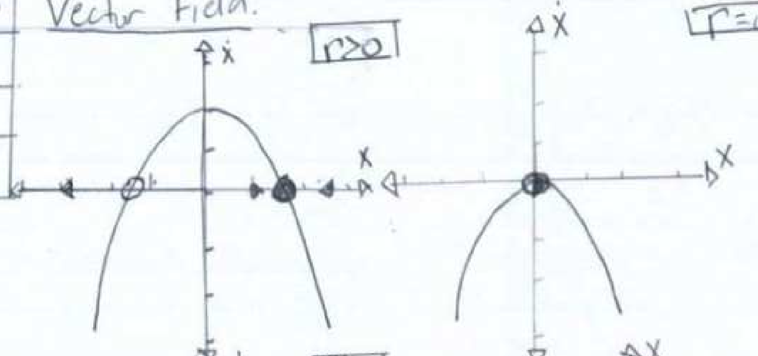
$\boxed{r < 1}$ $\boxed{r \geq 1}$

$\dot{x} = r - 3x^2$ 3.4.5

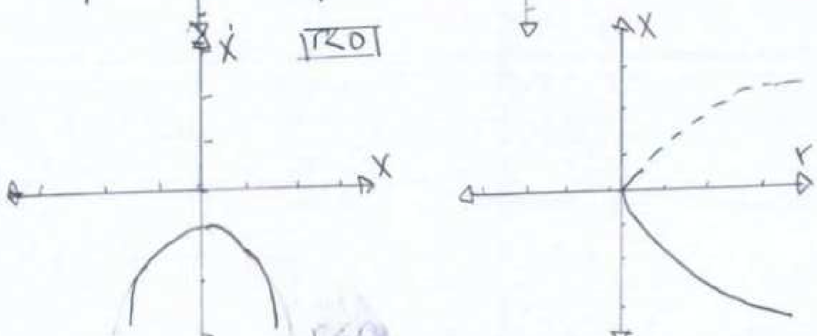
r	Bifurcations
> 0	Two
$= 0$	one
< 0	zero

Vector Field:

$\boxed{r > 0}$ $\boxed{r = 0}$



$\boxed{r < 0}$

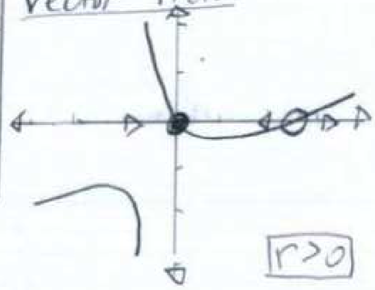


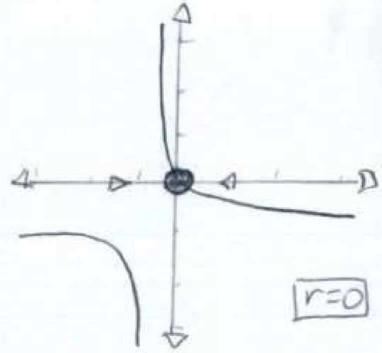
Bifurcation: Saddlenode

$\dot{x} = rx - \frac{x}{1+x}$ 3.4.6.

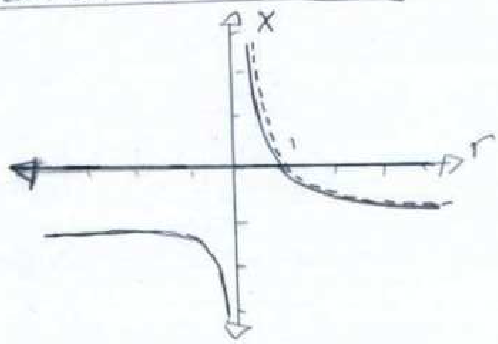
r	Bifurcations
>0	Two
=0	One
<0	Two

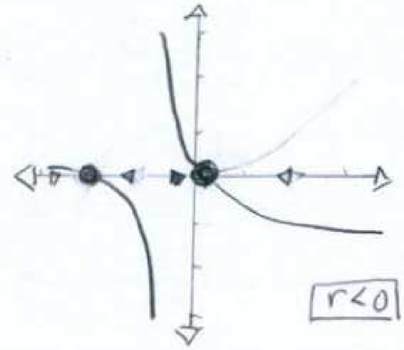
Vector Field:





Bifurcation: Transcritical

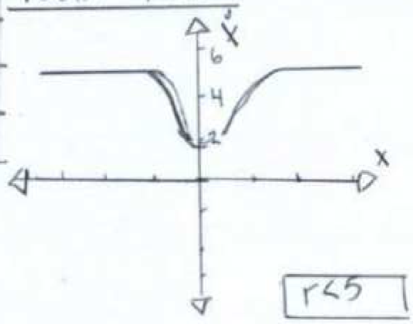


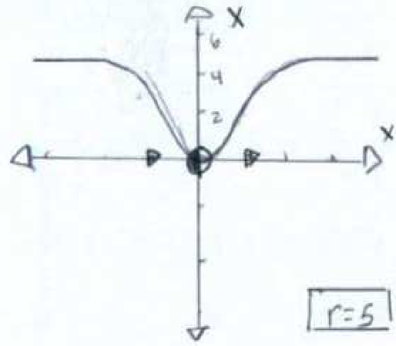


$\dot{x} = 5 - re^{-x^2}$ 3.4.7

r	Bifurcations
<5	zero
=5	one
>5	TWO

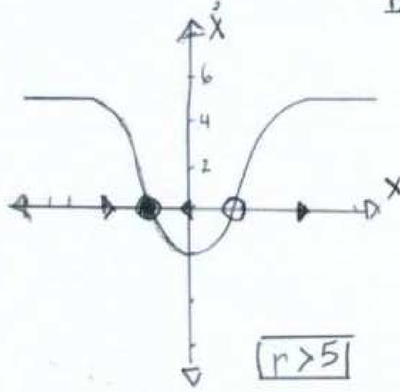
Vector Field:



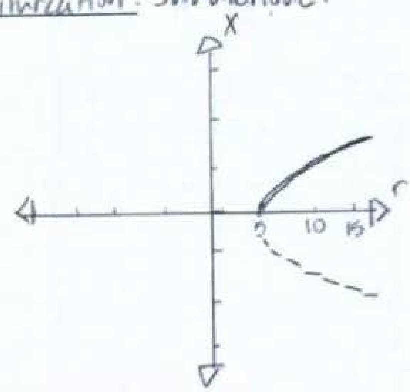


$\dot{x} = rx - \frac{x}{1+x^2}$ 3.4.8.

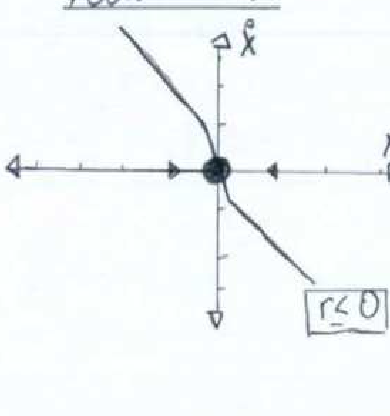
r	Bifurcations
≤0	one
0<r<1	three
≥1	one

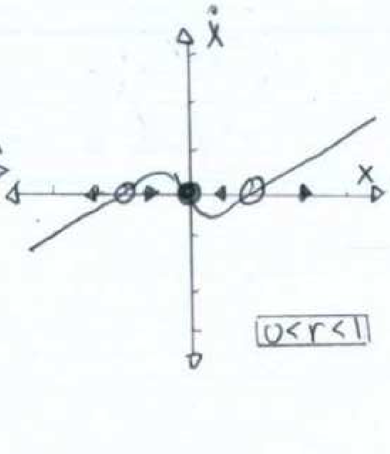


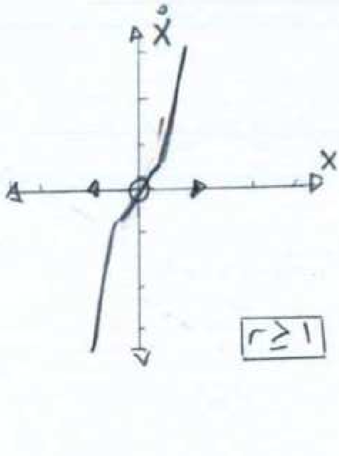
Bifurcation: Saddlenode.



Vector Field:



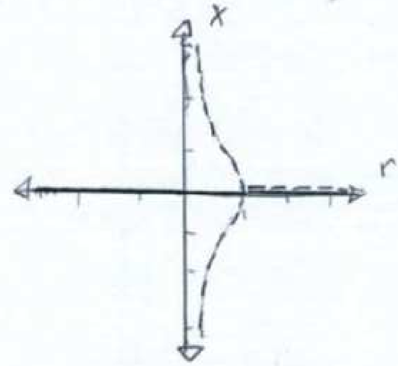




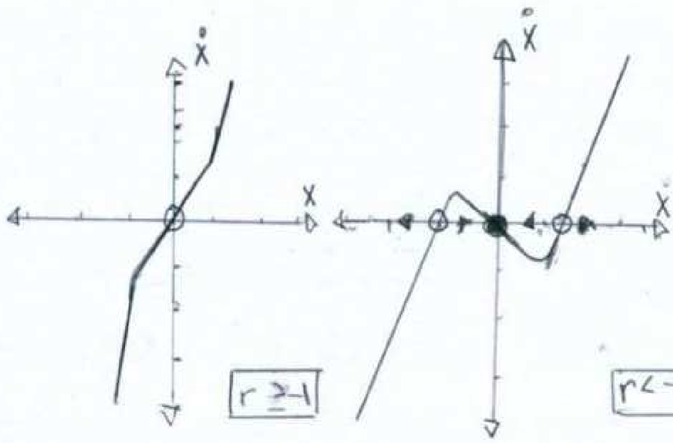
$\dot{x} = x + \tanh(rx)$ 3.4.9

r	Bifurcations
≤ -1	one
> -1	three

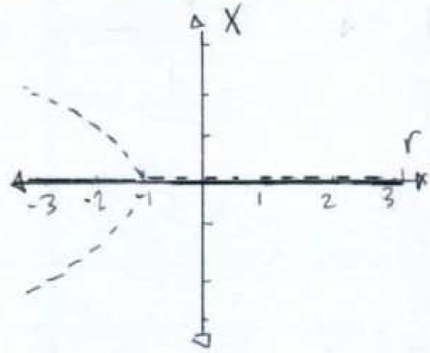
Bifurcation: Transcritical



Vector Field:



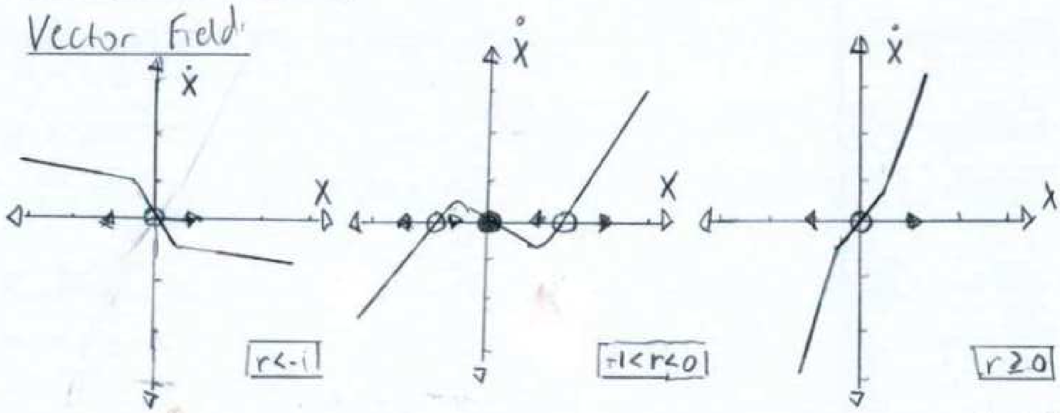
Bifurcation: subcritical



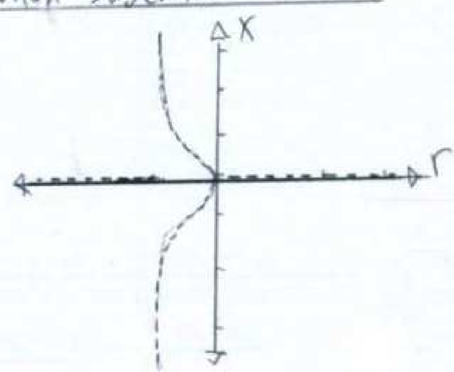
$\dot{x} = rx + \dfrac{x^3}{1+x^2}$ 3.4.10.

r	Bifurcations
<-1	one
$-1<r<0$	three
≤ 0	one

Vector Field:



Bifurcation: Subcritical Pitchfork



$\dot{x} = rx - \sin x$ 3.4.11 a) If $r=0$, then $\dot{x} = -\sin x$

Fixed points: stable $= (2k+1)\pi$

unstable $= 2k\pi$

where $k \in \mathbb{Z}$

b) If $r>1$, $\boxed{\dot{x}=0 \text{ is unstable}}$

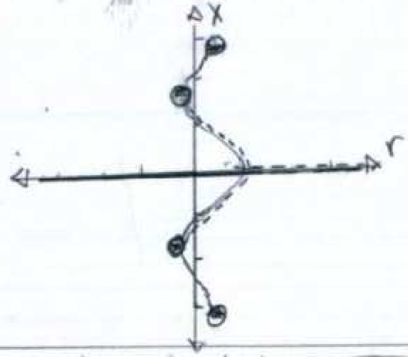
c) As $r = \infty \to 0$, then a subcritical pitchfork best describes the bifurcation.

d) $\dot{x} = rx - \sin(x)$; $r = \dfrac{\sin(x)}{x} = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + O(x^4)$

$$\boxed{x = \pm\left(6[1-r]\right)^{1/2}}$$

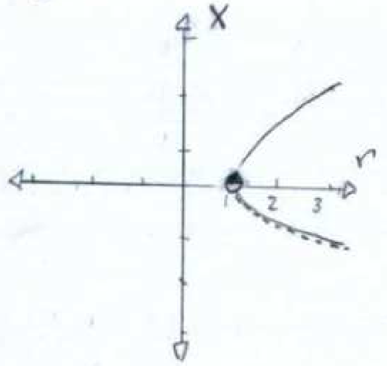
e) As $r = -\infty \to 0$, then a supercritical pitchfork occurs across the function $\dot{x} = rx - \sin(x)$

f) Bifurcations:

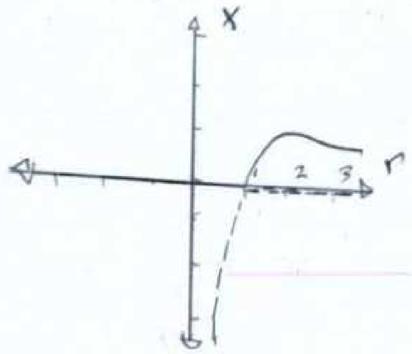


$\dot{x} = f(x,r)$ 3.4.12 A "quadrification" function is $x = \frac{1}{2}\left(3 \pm \sqrt{1 \pm 4\sqrt{r}}\right)$ where $\dot{x} = f(x,r) = (x-2)^2(x-1)^2 - r$. This function has even polynomial multiplicities to describe zero bifurcations $r<0$ and four when $r>0$.

$\dot{x} = r - x - e^{-x}$ 3.4.13 a) Best guess of roots: $r=1$, $x=0$

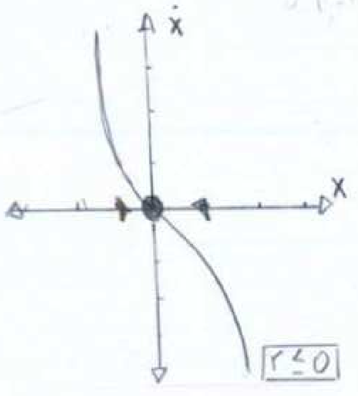


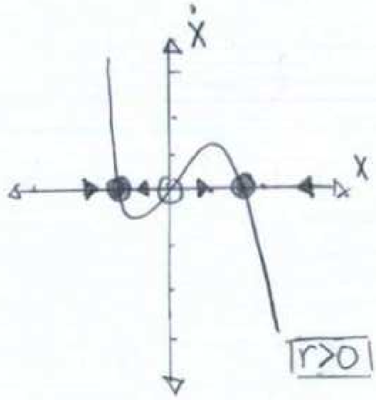
$\dot{x} = 1 - x - e^{-rx}$ b) Best Guess of roots: $r=1$, $x=0$



$\dot{x} = rx + x^3 - x^5$ 3.4.14 a) $\boxed{\ddot{x} = 0 = r + 3x^2 - 5x^4}$ -or- $\boxed{r = x^2(x^2-1)}$

b) Vector Field:





c) $\boxed{r_c \leq 0}$

$\dot{x} = rx + x^3 - x^5$ 3.4.15. $-\frac{dV(x)}{dx} = \dot{x} = 0 = -r - x^2 + x^4$

where $a = x^2$

$$a_1, a_2 = \frac{+1 \pm \sqrt{(+1)^2 - 4(1)(-r)}}{2(1)}$$

$$= \left(\frac{1 \pm \sqrt{1 + 4r}}{2}\right)$$

$$x_1 = +\sqrt{\frac{1+\sqrt{1+4r}}{2}} \; ; \; x_2 = -\sqrt{\frac{1+\sqrt{1+4r}}{2}}$$

$$x_3 = +\sqrt{\frac{1-\sqrt{1+4r}}{2}} \; ; \; x_4 = -\sqrt{\frac{1-\sqrt{1-4r}}{2}}$$

$$x_5 = 0$$

$$V(x) = -r\frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6}.$$

$$V(x_1) = V(x_2) = V(x_3) = V(x_4) = V(x_5) = 0$$

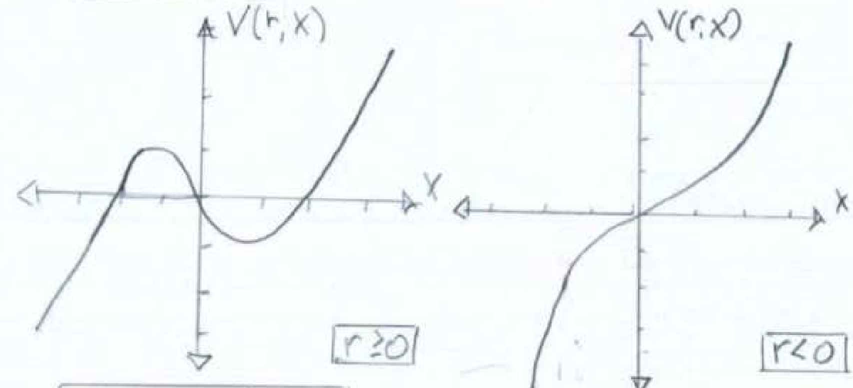
$$@\; V(x_1) = \frac{-r}{2}\left(\frac{1+\sqrt{1+4r}}{2}\right) + \frac{1}{4}\left(\frac{1+\sqrt{1+4r}}{2}\right)^2 - \frac{1}{6}\left(\frac{1+\sqrt{1+4r}}{2}\right)^3 \boxed{=0}$$

$r = -3/16$

$\dot{x} = r - x^2$ 3.4.16 a) $-\frac{dV}{dx} = \dot{x} = r - x^2$; $\boxed{V(r,x) = \frac{x^3}{3} - rx}$

r	Bifurcations
≥ 0	Three
< 0	One

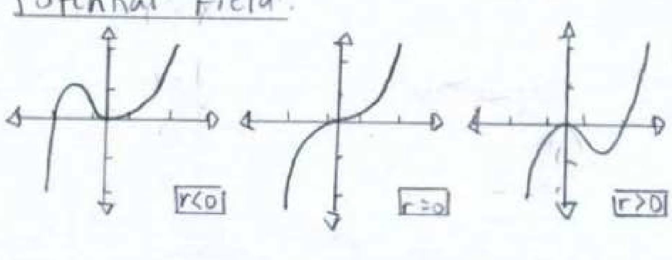
Potential Field



$\dot{x} = rx - x^2$

b) $-\frac{dV}{dx} = \dot{x} = rx - x^2$; $\boxed{V(r,x) = \frac{x^3}{3} - \frac{rx^2}{2}}$

Potential Field

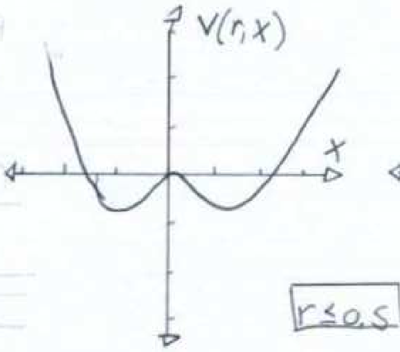


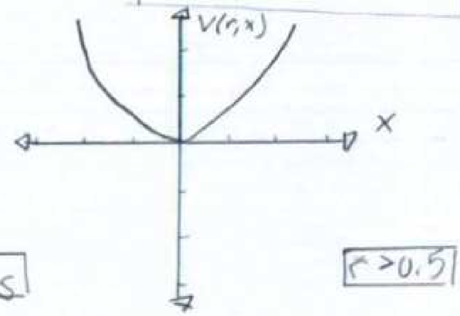
r	Bifurcations
< 0	Two
$= 0$	One
> 0	Two

$\dot{x} = rx + x^3 - x^5$ c) $-\frac{dV}{dx} = rx + x^3 - x^5$; $\boxed{V(r,x) = \frac{x^6}{6} - \frac{x^4}{4} - rx^2}$

r	Bifurcations
≤0.5	Three
>0.5	One.

Potentials:

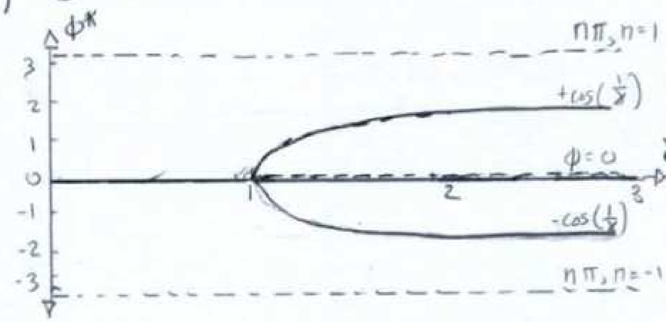




$b\dot{\phi} = -mg\sin\phi + mr\omega^2 \sin\phi\cos\phi$ 3.5.1. A better representation of $b\dot{\phi} = mg\sin\phi + mr\omega^2\sin\phi\cos\phi$ is $b\dot{\phi} = mg\sin\phi\left(\frac{r\omega^2}{g}\cos\phi - 1\right)$, which best represents the maximum angle of $\phi = \pi/2$. If the bead approaches a fixed point during rotation, then $b\dot{\phi} = 0$; $\frac{r\omega^2}{g}\cos\phi = 1$; $\cos\phi = \frac{g}{r\omega^2}$; and, $\frac{g}{r\omega^2}$ requires a positive value above zero.

$\frac{d\phi}{d\tau} = F(\phi)$
$= -\sin\phi + \gamma\sin\phi\cos\phi$
$= \sin\phi(\gamma\cos\phi - 1)$

3.5.2 $F(\phi) = \sin\phi(\gamma\cos\phi - 1)$
$f'(\phi) = \gamma[\cos^2\phi - \sin^2\phi - 1] = \gamma[\cos 2\theta - 1]$
$f''(\phi) = -2\gamma[\sin 2\theta]$
$\phi^* = n\pi$; $f'(\phi') = 0$; Half-Node
$\phi^* = \cos^{-1}\left(\frac{1}{\gamma}\right)$



$\frac{d\phi}{d\tau} = f(\phi)$
$= -\sin\phi + \gamma\sin\phi\cos\phi$
$= \sin\phi(\gamma\cos\phi - 1)$

3.5.3. If $\phi \approx 0$, then $\sin\phi \approx \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!}$
$\approx \phi$
$\cos\phi \approx 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!}$ and $\frac{d\phi}{d\tau} = \phi\left(\gamma\left[1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!}\right] - 1\right)$
$= \gamma\phi - \frac{\gamma\phi^3}{2!} + \frac{\gamma\phi^5}{4!}$

Where $\boxed{\frac{d\phi}{d\tau} = A\phi - B\phi^3 + O(\phi^3) \text{ ; } A = \gamma,\ B = \frac{\gamma}{2},\ O(\phi^5) = \frac{\gamma\phi^5}{4!}}$

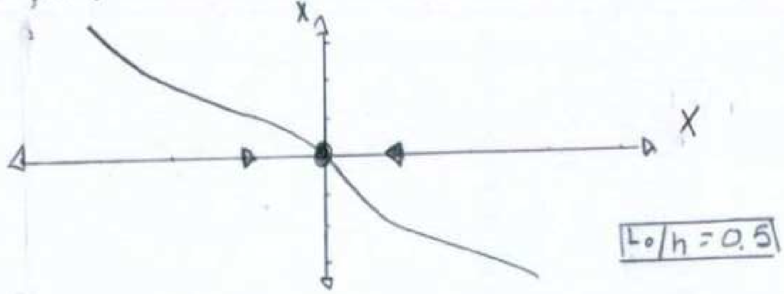
$m\ddot{x} = -F_{spring} - F_{fric}$ 3.5.4. $m\ddot{x} = -k \cdot l\cos\phi - kL_0\cos\phi - b\dot{\phi}$
$= -k(l - L_0)\cos\phi - b\dot{\phi} = -k\left(\sqrt{x^2+h^2} - L_0\right)\frac{x}{l} - b\dot{\phi}$
$= -k\left(\sqrt{h^2+h^2} - L_0\right)\frac{x}{\sqrt{h^2+x^2}} - b\dot{x}$
$\boxed{= -k\left(1 - \frac{L_0}{\sqrt{h^2+x^2}}\right)x - b\dot{x}}$

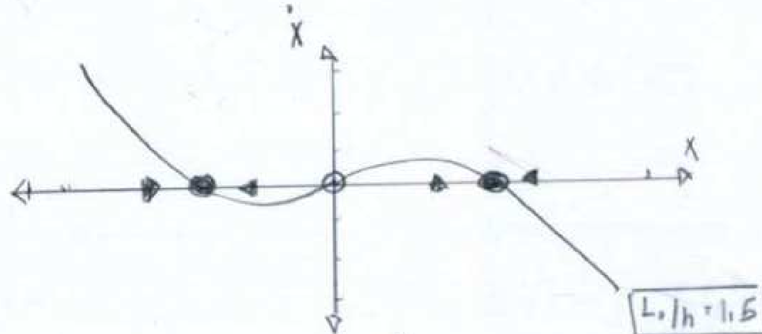
b. $m\ddot{x} + b\dot{x} + k\left(1 - \frac{L_0}{\sqrt{x^2+h^2}}\right)x = 0$

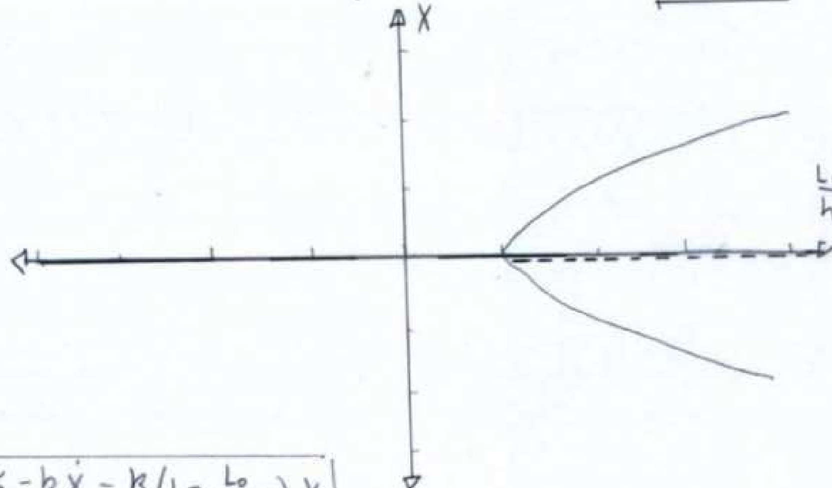
if $\dot{x} = 0$, $\boxed{x^* = \sqrt{L_0^2 - h^2},\ 0}$

c. If $m = 0$, $b\dot{x} + k\left(1 - \frac{L}{\sqrt{x^2+h^2}}\right)x = 0$, then $\boxed{x^* = \sqrt{L_0^2 - h^2},\ 0}$

Bifurcation Diagram







d. $\boxed{\text{If } m \neq 0, \text{ then } m\ddot{x} \ll -b\dot{x} - k\left(1 - \frac{L_0}{\sqrt{x^2+h^2}}\right)x}$

$\varepsilon \frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau} = f(\phi)$ 3.5.5. a) $\frac{d\phi}{d\tau} = f(\phi)$; T_{fast} is estimated to be

$$\varepsilon^{1-2k}\frac{d^2\phi}{d\tau^2} + \varepsilon^{-k}\frac{d\phi}{d\tau} = f(\phi);$$

Where $k = 1$, $\varepsilon^{1-2k} = \varepsilon^{-k} \gg 1$

$k = \frac{1}{2}$, $\varepsilon^{1-2k} = 1 \gg e^{-k}$

$k = 0$, $\varepsilon^{-k} = 1 \gg e^{1-2k}$

$$T = \varepsilon\frac{b}{mg} = \frac{m^2gr}{b^2}\,\frac{b}{mg} = \boxed{\frac{mr}{g}}$$

b) If $\tau = \varepsilon\zeta$, then $\varepsilon\frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau} = \varepsilon\frac{d^2\phi}{d(\varepsilon\zeta)^2} + \frac{1}{\varepsilon}\frac{d\phi}{d\zeta} = f(\phi)$

$\boxed{\frac{d^2\phi}{d\zeta^2} + \frac{d\phi}{d\zeta} = \varepsilon f(\phi)}$ "Rescaled"

c. $T_{fast} = \varepsilon T_{slow}$

$\varepsilon \ddot{x} + \dot{x} + x = 0$ 3.5.6 $x(0) = 1; \dot{x}(0) = 0$

a) General solution: $x(t) = C_1 e^{\lambda t}$ $\quad \varepsilon\lambda^2 + \lambda + 1 = 0$

$\dot{x}(t) = \lambda C_1 e^{\lambda t}$; $\lambda = \dfrac{-1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon}$

$\ddot{x}(t) = \lambda^2 C_1 e^{\lambda t}$; $x(t) = C_1 e^{(-1+\sqrt{1-4\varepsilon})x/2\varepsilon} + C_2 e^{(-1-\sqrt{1-4\varepsilon})x/2\varepsilon}$

$x(0) = C_1 + C_2 = 1$

$\dot{x}(t) = C_1\left(\dfrac{-1+\sqrt{1-4\varepsilon}}{2\varepsilon}\right) e^{(-1+\sqrt{1-4\varepsilon})x/2\varepsilon}$

$+ C_2\left(-\dfrac{(1-\sqrt{1-4\varepsilon})}{2\varepsilon}\right) e^{(-1-\sqrt{1-4\varepsilon})x/2\varepsilon}$

$\dot{x}(0) = C_1\left(-\dfrac{(1+\sqrt{1-4\varepsilon})}{2\varepsilon}\right) + C_2\left(\dfrac{-1-\sqrt{1-4\varepsilon}}{2\varepsilon}\right)$

$= \dfrac{-C_1\sqrt{1-4\varepsilon}}{2\varepsilon} + \dfrac{C_1}{2\varepsilon} + (1-C_1)\left(\dfrac{-1-\sqrt{1-4\varepsilon}}{2\varepsilon}\right)$

$= -\dfrac{C_1\sqrt{1-4\varepsilon}}{2\varepsilon} + \dfrac{C_1}{2\varepsilon} + \dfrac{(1+\sqrt{1-4\varepsilon})}{2\varepsilon}$

$- \dfrac{C_1(-1-\sqrt{1-4\varepsilon})}{2\varepsilon}$

$= -\dfrac{C_1\sqrt{1-4\varepsilon}}{2\varepsilon} + \dfrac{C_1}{2\varepsilon} + \dfrac{(1+\sqrt{1-4\varepsilon})}{2\varepsilon}$

$+ \dfrac{C_1}{2\varepsilon} + \dfrac{C_1\sqrt{1-4\varepsilon}}{2\varepsilon} = 0$

$= C_1 ; = \dfrac{(1+\sqrt{1-4\varepsilon})}{2}$

$$\boxed{\text{Therefore, } x(t) = \left(\frac{1+\sqrt{1-4\varepsilon}}{2}\right)\left(e^{(-1+\sqrt{1-4\varepsilon})x/2\varepsilon}\right) + \left(1 - \frac{(1+\sqrt{1-4\varepsilon})}{2}\right)\left(e^{(-1-\sqrt{1-4\varepsilon})x/2\varepsilon}\right)}$$

b. $\varepsilon \ddot{x} + \dot{x} + x : \varepsilon \dfrac{dx^2}{dt^2} + \dfrac{dx}{dt} + x = 0$

$\dfrac{\varepsilon}{T^2}\dfrac{d^2x}{d\tau^2} + \dfrac{1}{T}\dfrac{dx}{d\tau} = -x$ $\quad$ where $T = \dfrac{t}{\tau} = \varepsilon$

$\dfrac{1}{T}\dfrac{d^2x}{d\tau^2} + \dfrac{1}{T}\dfrac{dx}{d\tau} = x$; $\ddot{x} + \dot{x} = Tx = \varepsilon x$; $\boxed{\ddot{x} + \dot{x} - Tx = 0}$

$\dot{N} = rN(1-N/K)$ 3.5.7. a) $N(0) = N_0$;

Parameter	Dimensions
r	Per time (rate
K	Same as N (amount)
N_0	Same as N (amount)

b) $\frac{dN}{dt} = rN(1-N/K)$; If $\frac{N}{K} = x$, then $dN = K\,dx$

$\frac{dx}{dt} = rx(1-x)$; If $t = \frac{\tau}{r}$, then $dt = \frac{d\tau}{r}$

$$\boxed{\frac{dx}{d\tau} = x(1-x)}$$

c) $u = x$; $\frac{du}{d\tau} = u(1-u)$; $u(0) = u_0$

$\int \frac{du}{u(1-u)} = d\tau$; $\int \frac{A}{u}du + \int \frac{B}{(1-u)}du = \int \frac{du}{u} + \int \frac{du}{(1-u)} = \ln \frac{u}{1-u} = \tau + C$

$A(1-u) + Bu = 1$

$\frac{1-u}{u} = Ce^{-\tau}$

$1 = u(1+Ce^{-\tau})$

$u = \frac{1}{1+Ce^{-\tau}}$

$u(0) = u_0 = \frac{1}{1+C}$

$C = \frac{1-u_0}{u_0}$

$$\boxed{u(\tau) = \frac{1}{1+\left(\frac{1-u_0}{u_0}\right)e^{-\tau}}}$$

d) An advantage of the dimensionless functions are lower degrees of freedom during analysis. The graphical representations do not have further axis to plot, and the functions are closer to the basic functions of precalculus.

$\dot{u} = au + bu^3 - cu^5$ 3.5.8. Prove $\frac{dx}{d\tau} = rx + x^3 - x^5$, where $x = \frac{u}{U}$

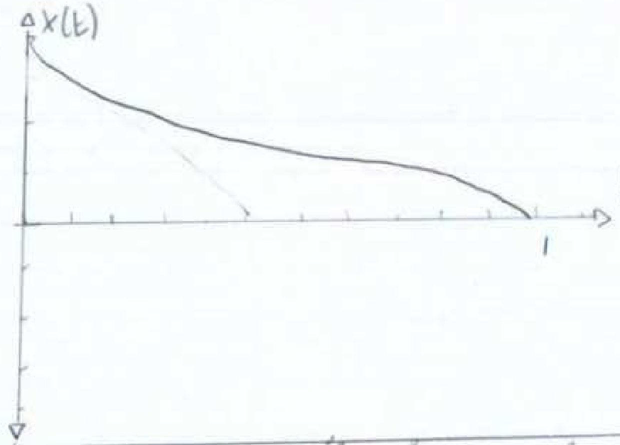
$\tau = \frac{t}{T}$

$\frac{\cancel{U}}{T}\frac{dx}{d\tau} = a\cancel{U}x + bU^{\cancel{3}\,2}x^3 - cU^{\cancel{5}\,4}x^5$

$\frac{dx}{d\tau} = Tax + TbU^2x^3 - TcU^4x^5$; $a = \frac{r}{T}$, $b = \frac{1}{TU^2}$, $c = \frac{1}{TU^4}$

$$\boxed{\frac{dx}{d\tau} = rx + x^3 - x^5}$$

3.6.1 Figure 3.6.3b corresponds to Figure 3.61b; specifically, the relationship between $y = h$, and $y = rx - x^3$. The dotted lines support a single bifurcation to two bifurcations at h_c, then three when $h > h_c$. To answer the question, Figure 3.6.3b has information of $h < 0$ and $h > 0$.

c) 

d) If $\varepsilon \ll 1$, then $\varepsilon\ddot{x} + \dot{x} + x \approx \dot{x} + x$ and is a similar model to the boundary conditions

e) Mechanical System: An extremely visious solution for an oscillating Newtonian device.

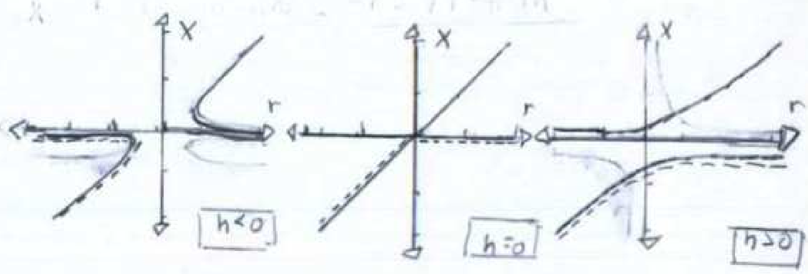
Electrical System: An electrical system of the form $v = Ri + L\frac{di}{dt} + \frac{1}{c}\int i\,dt$

where $\varepsilon = \frac{1}{c} \ll 1$.

$\dot{x} = h + rx - x^2$ 3.6.2. a)

$\frac{-r \pm \sqrt{r^2 - 4(-1)(h)}}{2(-1)}$

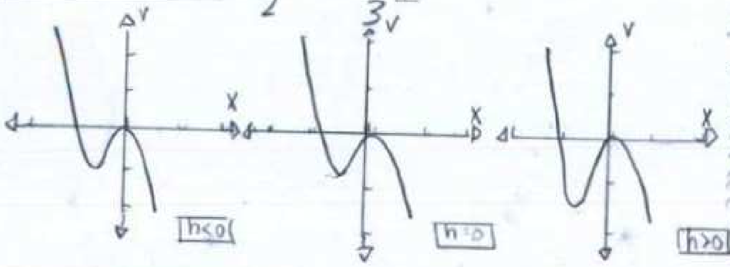
h	Bifurcations
<0	zero/one/two
=0	one/Two
>0	Two

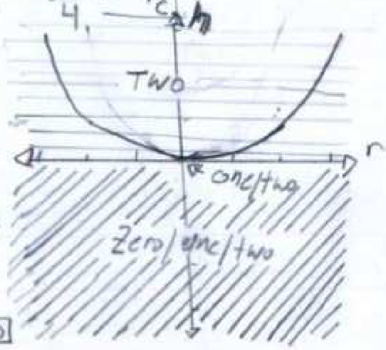


b) (r,h) Plane

$$\frac{d}{dx}(rx - x^2) = r - 2x \; ; \; x_{max} = \frac{r}{2} \; ; \; \frac{r^2}{2} - \frac{r^2}{4} = \boxed{\frac{r^2}{4} = h_c}$$

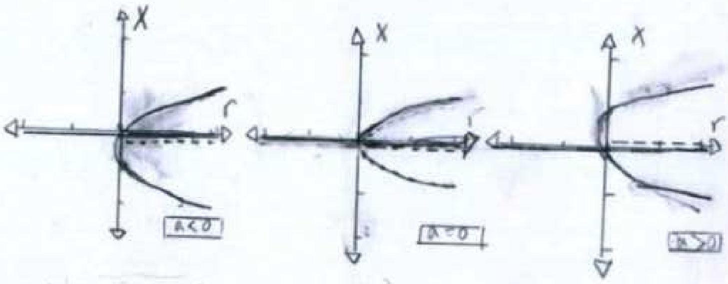
c) $\boxed{V(x,h,r) = hx + \frac{rx^2}{2} - \frac{x^3}{3}}$





$\dot{x} = rx + ax^2 - x^3$ 3.6.3 a)

a	Bifurcations
<0	one/two/three
=0	one/three
>0	one/two/three



b) (r,a) plane

$$\frac{d}{dx}(rx + ax^2 - x^3) = r + 2ax - 3x^2 = 0 \; ;$$

$$r + ax - x^2 \; ; \; \boxed{a = \frac{x^2 - r}{x}}$$

3.6.4 A small imperfection to a saddlenode bifurcation shifts the cusp either left or right.

$mg\sin\theta = kx\left(1 - \frac{L}{\sqrt{x^2+a^2}}\right)$ 3.6.5 a) $F = -F_{spring} = F_{g,x} = F_g \sin\theta = mg\sin\theta = k(x - x\sin\theta)$

$$= k\left(x - x \cdot \frac{L_0}{\sqrt{x^2+a^2}}\right)$$

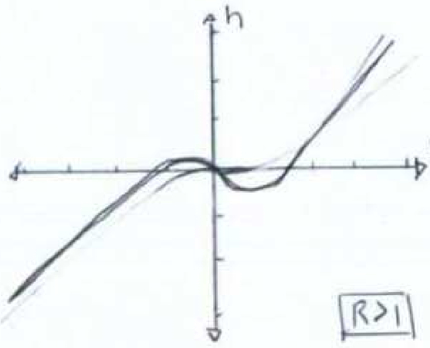
$$\boxed{= kx\left(1 - \frac{L_0}{\sqrt{x^2+a^2}}\right)}$$

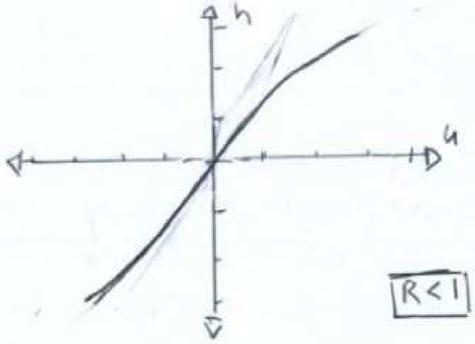
b) Prove $1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}}$

If $1 - \frac{mg\sin\theta}{kx} = \frac{L_0}{a\sqrt{\left(\frac{x}{a}\right)^2 + 1}}$, then $u = \frac{x}{a}$, $R = \frac{L_0}{a}$, $h = \frac{mg\sin(\theta)}{ka}$.

and $\boxed{1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}}}$

c)





The variable h, as a function of u, has a single equilibrium point for both R>1 and R<1.

d) If $r=R-1$, $1-\frac{h}{u}=\frac{r+1}{\sqrt{1+u^2}}$; $u-h=\frac{(r+1)u}{\sqrt{1+u^2}}$; $u\sqrt{1+u^2}-h\sqrt{1+u^2}=(r+1)u$

$$u\left(1+\tfrac{1}{2}u^2+O(u^4)\right)-h\left(1+\tfrac{1}{2}u^2+O(u^4)\right)=(r+1)u$$

$$\cancel{u}+\frac{u^3}{2}-h-\frac{h}{2}u^2=ru+\cancel{u}$$

$$\boxed{h+ru+\frac{h}{2}u^2-\frac{1}{2}u^3\approx 0}$$

e) $h\left(1+\frac{u^2}{2}\right)=\frac{1}{2}u^3-ru$

$\frac{d}{du}h\left(1+\frac{u^2}{2}\right)=\frac{d}{du}\left(\frac{1}{2}u^3-ru\right)$; $hu=\frac{3}{2}u^2-r$; r_{max} $\frac{3}{2}u^2-hu$

$h\left(1+\frac{u^2}{2}\right)=\frac{1}{2}u^3-\left(\frac{3}{2}u^2-hu\right)u$; $h+\frac{hu^2}{2}=\frac{1}{2}u^3-\frac{3}{2}u^3+hu^2$

$h\left(1-\frac{1}{2}u^2\right)=-u^3$; $h=\frac{2u^3}{u^2-2}$

$$r_{max}=\frac{3}{2}u^2-hu=\frac{3}{2}u^2-\left(\frac{2u^3}{u^2-2}\right)u$$

$$=\frac{3}{2}u^2-\frac{2u^4}{u^2-2}$$

$$=\frac{u^4+3u^2}{2(1-u^2)}\ \boxed{=R-1}$$

f) $1-\frac{h}{u}=\frac{R}{\sqrt{1+u^2}}$; $\frac{d}{du}\left(1-\frac{h}{u}\right)=\frac{d}{du}\left(\frac{R}{\sqrt{1+u^2}}\right)$; $\frac{h}{u^2}=-\frac{1}{2}\frac{R(2u)}{(1+u^2)^{3/2}}$

$h(1+u^2)^{3/2}=-R\cdot u^3$; $R=-\frac{h(1+u^2)^{3/2}}{u^3}$

$1-\frac{h}{u}=\frac{-h(1+u^2)^{3/2}}{u^3\sqrt{1+u^2}}=-\frac{h(1+u^2)}{u^3}$; $u-h=-\frac{h(1+u^2)}{u^2}$

$u^3 - hu^2 = -h(1+u^2)$; $u^3 = -h - hu^2 + hu^2$; $h = -u^3$

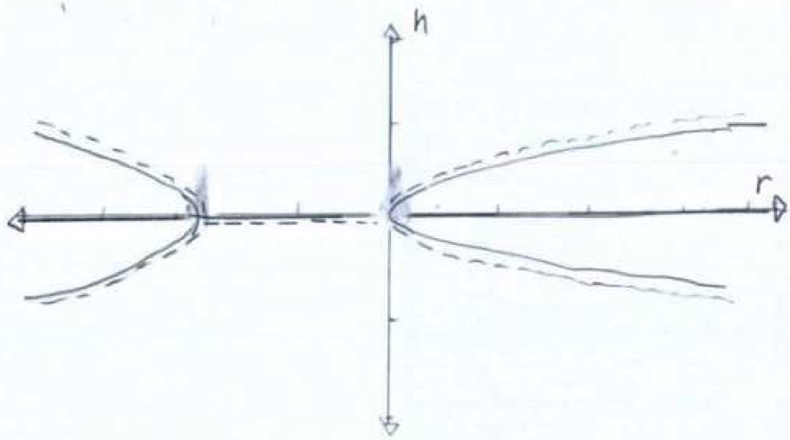
$R = \frac{-h(1+u^2)^{3/2}}{u^3} = (1+u^2)^{3/2}$

$\boxed{\lim_{u \to 0} h = -u^3 \approx \frac{2u^3}{u^2-2} \approx u^3}$

$\boxed{\lim_{u \to 0} R = (1+u^2)^{3/2} \approx \frac{u^4+3u^2}{2(1-u^2)} + 1 = r+1}$

g) $R = (1+u^2)^{3/2} = r+1$; $r(u) = (1+u^2)^{3/2} - 1$; $u = \sqrt{(r+1)^{2/3} - 1}$

$\boxed{h = -u^3 = \pm\left(\sqrt{(r+1)^{2/3}-1}\right)^3}$



h) $h = -u^3 = -\left(\frac{x}{a}\right)^3 = \frac{mg\sin\theta}{ka}$

$R = \left(1 + \left(\frac{x}{a}\right)^2\right)^{3/2} = \frac{L_0}{a}$

The bifurcation plot represents the points of stability for the oscillating system.

$\tau \mathring{A} = \varepsilon A - gA^3$

$\tau \mathring{A} = \varepsilon A - gA^3 - kA^3$

3.6.6. $A(t)$ = Amplitude; τ = typical timescale; ε = dimensionless parameter

3.6.6. supercritical : $g > 0$, subcritical: $g < 0$, $K > 0$

"Landaus Equation"

a) Landau's Equation describes the change of amplitude for a fluid system.

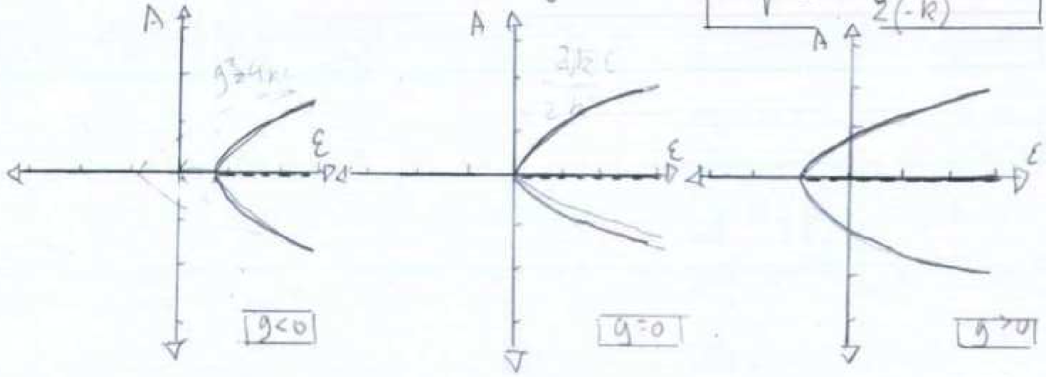
b). $\tau \mathring{A} = \varepsilon A - gA^3 - kA^3$; if $g = 0$, then $\tau \mathring{A} = \varepsilon A - kA^3$; $A = \sqrt{\frac{\varepsilon}{k}}$

The function $A(\varepsilon)$ is a tricritical bifurcation because $A = 0$ is a solution; in addition to, $A = +\sqrt{\frac{\varepsilon}{k}}$, and $A = -\sqrt{\frac{\varepsilon}{k}}$

c) $\tau \mathring{A} = h + \varepsilon A - gA^3 - kA^5$; An approximation $h \approx 0$, $0 = \varepsilon A - gA^3 - kA^5$

$= \varepsilon - gA^2 - kA^4$

Where $A^2 = b$: $0 = \varepsilon - gb - kb^2$; $\boxed{A = \sqrt{\frac{g \pm \sqrt{g^2 - 4(-k)(\varepsilon)}}{2(-k)}}}$

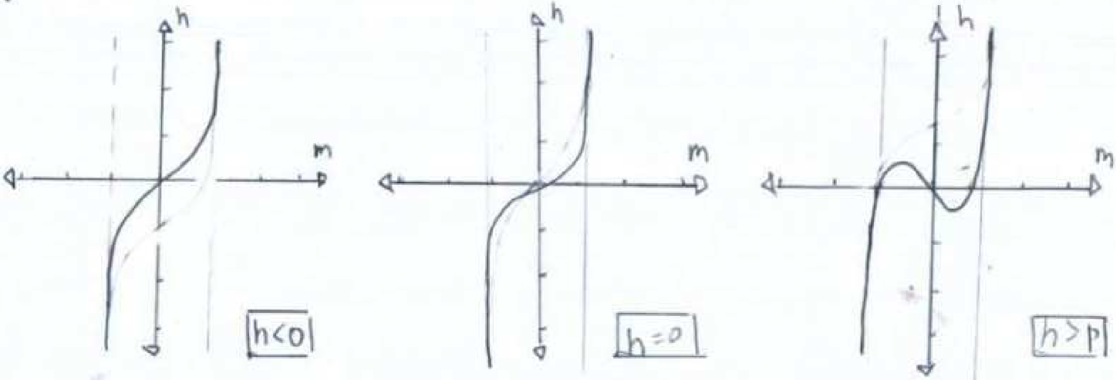


d) The graphs appearance represent the relationship of amplitude vs. time and if ϵ is large, then the first order term approaches the steadystate condition more rapidly.

$m = \left|\frac{1}{N}\sum_{i=1}^{N} S_i\right|$

$h = T\tanh^{-1}(m) - Jnm$

3.6.7 a)



b) $h = T\tanh^{-1}(m) - Jnm$; If $h=0$, then $\boxed{T_c = \frac{Jnm}{\tanh^{-1}(m)}}$

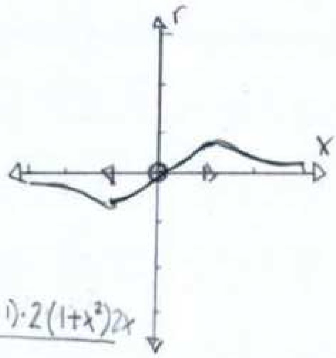
$\frac{dx}{d\tau} = rx\left(1-\frac{x}{K}\right) - \frac{x^2}{1+x^2}$ 3.7.1. @ $x^*=0$; $0 < rx - \left(\frac{1}{K}+\frac{1}{1+x^2}\right)x^2$; $\left(\frac{1}{K}+\frac{1}{1+x^2}\right)x < r$; $\boxed{0<r \text{ is positive and unstable.}}$

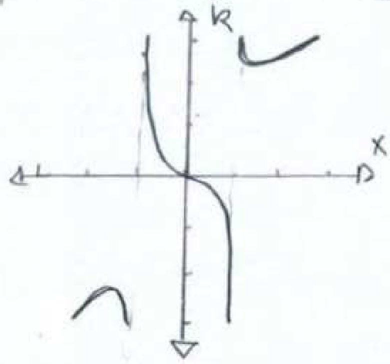
$r = \frac{2x^3}{(1+x^2)^2}$

$K = \frac{2x^3}{x^2-1}$

3.7.2. a) $\boxed{\lim_{x\to 1} r = \frac{2}{4}}$; $\boxed{\lim_{x\to\infty} r = 0}$

$\boxed{\lim_{x\to 1} K = -\infty}$; $\boxed{\lim_{x\to\infty} K = \infty}$





b) $r = \frac{(x^2-1)K}{(1+x^2)^2}$; $\frac{dr}{dx} = \frac{2x(1+x^2)^2 - (x^2-1)\cdot 2(1+x^2)2x}{(1+x^2)^4}$

$= \frac{2x(1+x^2)^2 - 4x^3(1+x^2) + 4x(1+x^2)}{(1+x^2)^4} = 0$

$= 2x(1+x^2) - 4x^3 + 4x = 2(1+x^2) - 4x^2 + 4 = (1+x^2) - 2x^2 + 2 = 0$

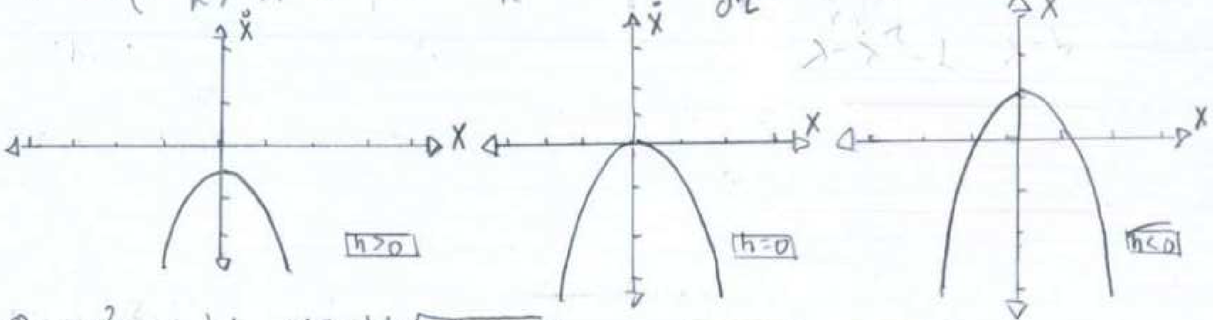
$3 = x^3$; $\boxed{x = \sqrt{3}}$

$r_{max} = \frac{(3-1)K}{(1+3)^2} = \frac{1}{9}K_{max}$; $\boxed{r_{max} = \frac{2\cdot 3^{3/2}}{(1+3)^2} = 0.6495}$; $\boxed{K_{max} = 5.1961}$

$\frac{dx}{d\tau} = x(1-x) - h$ 3.7.3. $\dot{N} = rN\left(1-\frac{N}{K}\right) - H$; $h = HKr$; $x = \frac{N}{K}$; $\tau = tr$; $\frac{dx}{d\tau} = x(1-x) - h$

a)

b)



c) $0 = -x^2 + x - h$; $x = \frac{-1 \pm \sqrt{1-4(-1)(-h)}}{2(-1)} = \frac{1 \pm \sqrt{1-4h}}{2}$; $\boxed{h_c = 0}$

d) The long-term behavior of the fish population is to reduce the total population as population rises.

$\dot{N} = rN\left(1-\frac{N}{K}\right) - H\frac{N}{A+N}$ 3.7.4. a) The variable A could represent the amount of fish in a school, and if A is large, then less fish are harvested.

b) $\boxed{x = \frac{N}{K};\ \tau = tr;\ h = HRK;\ a = A}$

c) $\frac{dx}{d\tau} = x(1-x) - h\frac{x}{a+x} = 0;\ x(1-x)(a+x) = (x-x^2)(a+x) = ax + x^2 - x^2a - x^3$

$$0 = (a-h)x + (1-a)x^2 - x^3$$

$$0 = (a-h) + (1-a)x - x^2$$

$$x_1 = 0,\ x_{2,3} = \frac{-(1-a) \pm \sqrt{(1-a)^2 - 4(-1)(a-h)}}{2(-1)}$$

$$= \frac{(1-a) \pm \sqrt{(1-a)^2 + 4(a-h)}}{2}$$

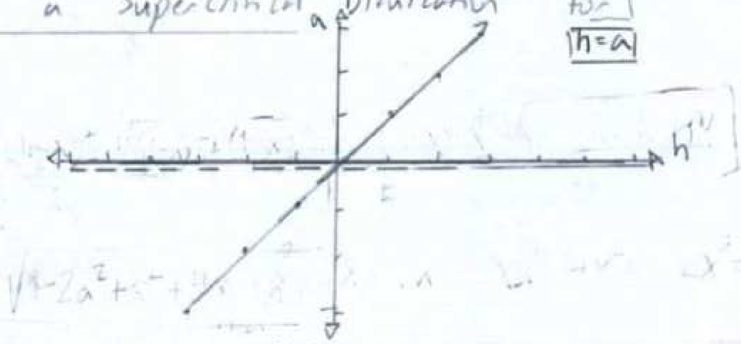
Fixed Point	$a > h$	$a < h$
0	unstable	stable
$\frac{(1-a)+\sqrt{(1-a)^2+4(a-h)}}{2}$	stable	/
$\frac{(1-a)-\sqrt{(1-a)^2+4(a-h)}}{2}$	stable	/

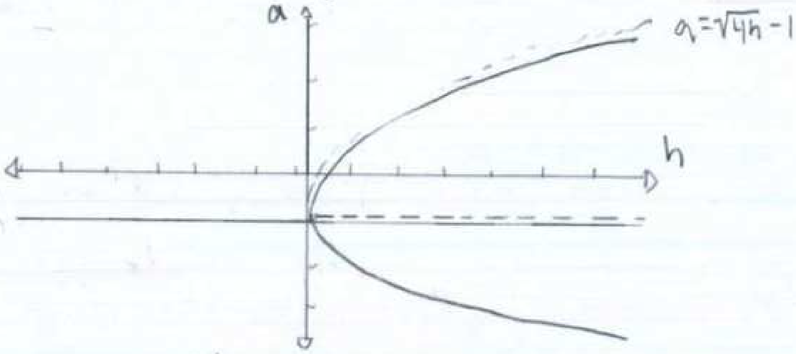
d) At $x=0$, when $h=a$, the half-node indicates a transcritical bifurcation is about to occur when h becomes less than a.

e) The graph shows a supercritical bifurcation for $\boxed{h=a}$

$h = \frac{(a+1)^2}{4}$.

f) $\boxed{a = \frac{h}{x-1} - x}$





$\dot{g} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2}$ 3.7.5.

a) $\frac{k_4}{k_3}\frac{dg}{dt} = \frac{k_4 \cdot k_1}{k_3}s_0 - \frac{k_4 k_2}{k_3}g + \frac{\left(\frac{g}{k_4}\right)^2}{1+\left(\frac{g}{k_4}\right)^2};\ x = \frac{g}{k_4};\ r = \frac{k_4 k_2}{k_3};\ s = \frac{k_1}{k_3}s_0$

$$\boxed{\frac{dx}{d\tau} = s - rx + \frac{x^2}{1+x^2}}$$

$$\boxed{\tau = \left(\frac{k_3}{k_4}\right)t}$$

b) $0 = -rx + \frac{x^2}{1+x^2};\ rx = \frac{x^2}{1+x^2};\ r(1+x^2) = x;\ rx^2 - x + r = 0$

$$\boxed{x_{1,2} = \frac{1 \pm \sqrt{1-4r^2}}{2r}}$$

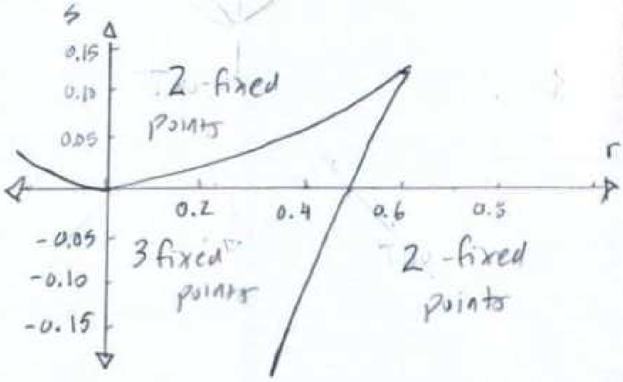
c) $g(0)=0$; $\frac{dg}{dt} = k_1 s_0 - k_2(0) + \frac{k_3(0)}{k_4^2+(0)^2} = k_1 s_0$; $g = k_1 s_0 t$; $g(t)$ increases with additional s_0.

$\boxed{\text{If } s_0 \text{ is large, then gene production has higher likelihood of rising.}}$

d) $\frac{d}{dx}\left(s - rx + \frac{x^2}{1+x^2}\right) = -r + \frac{2x}{(1+x^2)^2} = 0$; $r = \frac{2x}{(1+x^2)^2}$; $s - \left(\frac{2x}{(1+x^2)^2}\right) + \frac{x^2}{1+x^2} = 0$

$$\boxed{s = \frac{x^2(1-x^2)}{(x^2+1)^2}}$$

e) Parametric plot of (r,s)



$\dot x = -kxy$
$\dot y = kxy - ly$
$\dot z = ly$

3.7.6. $x(t)$ = number of healthy people
$y(t)$ = number of sick people
$z(t)$ = number of dead people.

a) $\dot N = \dot x + \dot y + \dot z = -kxy + kxy - ly + ly = 0$; therefore $N = x + y + z$.

b) $\dot x = -kxy$; $\dot z = ly$; $\dot x = -kx\frac{dz}{dt}\left(\frac{1}{l}\right)$; $\ln x = -\frac{kz}{l} + C$; $x(t) = Ce^{-kz/l} = \boxed{x_0 e^{-kz/l}}$

c) $\dot z = ly = l[N - x - z] = \boxed{l\left[N - z - x_0 e^{-kz/l}\right]}$

d) $\boxed{u = \frac{kz}{l} \ ; \ b = \frac{l}{kx_0} \ ; \ a = \frac{lN}{kx_0} \ ; \ \tau = \frac{l}{kx_0}t}$

e) $\boxed{\text{If } k, l, N, \text{ and } x_0 \text{ are positive, then both } a \text{ and } b \text{ are positive.}}$

f)

b \ a		>1
=0	@u=0, unstable	@u<0, unstable
>0	@u=0, unstable @u>0, stable	@u<0, unstable @u>0, stable

g) $\ddot u = -b\dot u + \dot u e^{-u} = \boxed{0}$; $u = -\ln b$; $\dot u = a - b\ln b + b$

$\ddot z = l\dot y = l(kxy - ly) = l(kx - l)y = \boxed{0}$; $y = kx$; $y = Ce^{\mp(kx-l)t}$

h) $b<1$; $\ddot u = -b + ue^{-u}$; $\boxed{\text{Through plotting of } b \text{ and } ue^{-u}\text{, at time zero, } b > ue^{-u} \text{; thus, } \dot u \text{ is increasing.}}$

t_{peak} ; $\ddot u = -b + ue^{-u} = 0$; $\dddot u = e^{-u} - u^2 e^{-u} \stackrel{?}{=} 0$; $u \stackrel{?}{=} 1$

$\dot u = a - bu - e^{-u}$ @ $u=1$; $\dot u = a - b - \frac{1}{e}$; $u = (a - b - \frac{1}{e})\tau = 1$; $\boxed{t = \frac{1}{a - b - \frac{1}{e}}\left(\frac{l}{kx_0}\right)}$

$\lim_{u\to\infty} \dot u = \lim_{u\to\infty}\left[a - bu - e^{-u}\right] = -b\cdot\infty - \frac{1}{e^{\infty}} = \boxed{-\infty}$

i) If $b>1$, $\dot{u}=a-bu-e^{-u}$; $\ddot{u}=-b\dot{u}+ue^{-u}$ & $\dddot{u}\cdots\ddddot{u}$ does not contain a logical maximum/minimum/inflection for an epidemic with t_{peak} at zero.

j) The variable b is assigned as $\frac{l}{kx_0}$. If $b=1$, then $\frac{l}{kx_0}=1$.

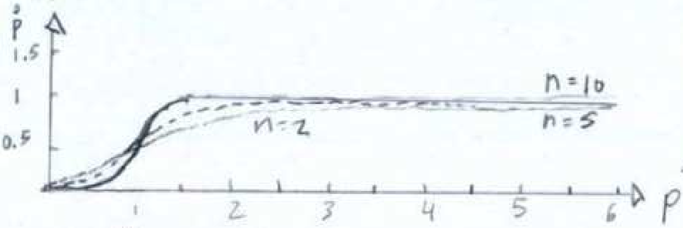
A threshold condition is when the rate of dying persons is greater than the rate of infection.

k) Autoimmunodeficiency is a disease following human immunodeficiency virus. The delayed onset from infection is time-dependent, showing that a model likely requires a time-dependent term or relationship.

$\dot{p}=\alpha+\frac{\beta p^n}{K^n+p^n}-\delta p$ 3.7.7 α = Basal Transcription Rate; β = Maximal Transcription Rate

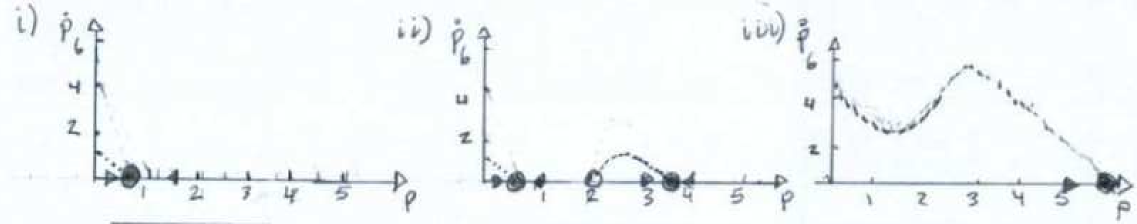
K = Activation Coefficient; δ = Decay Rate of Protein.

a)



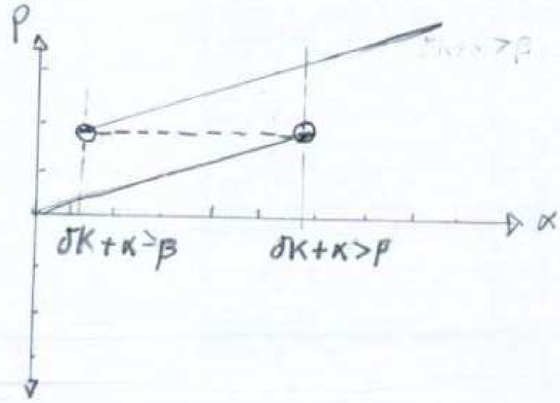
The shape of the function is a sigmoid about the point (1, 0.5) for $K=1$, $b=1$.

b)



$\boxed{\delta K-\alpha>\beta}$ $\boxed{\delta K-\alpha=\beta/2}$ $\boxed{\delta K-\alpha<0}$

c) Assume $\delta K>\beta$, $\alpha=-\frac{\beta^n}{K^n+p^n}+\delta p$ at $\alpha\geq 0$



d) When protein levels are dependent upon α, then up till $\alpha>\delta K$, protein production rate decreases until zero. While , above a threshold $\delta K-\beta>\alpha$, there is active production of further protein, proving concentration regions of protein production.

$\dot{A}_p = K_p s A + \beta \frac{A_p^n}{K^n + A_p^n} - K_d A_p$; A = unphosphorylated concentration ; A_p = phosphorylated concentration ; $A_T = A + A_p$

K_p = phosphorylation rate ; K_d = dephosphorylation rate.

Assume $K = A_T/2$; $\beta = K_d A_T$

3.7.9a) $x = A_p/K$; $\tau = K_d t$; $s = K_p S/K_d$; $b = \beta/(K_d K)$

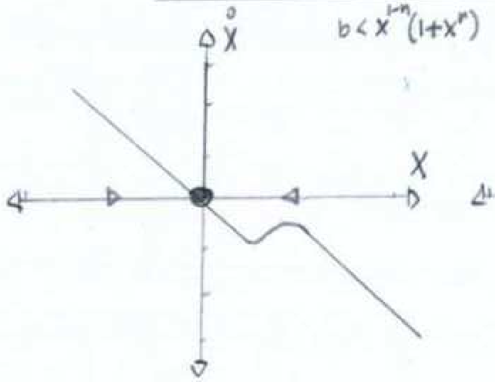
$$K\cancel{K_d}\frac{dx}{d\tau} = \cancel{K_d} s A + \cancel{K_d} K \cdot b \cdot \frac{K^n x^n}{K^n + K^n x^n} - \cancel{K_d} K x$$

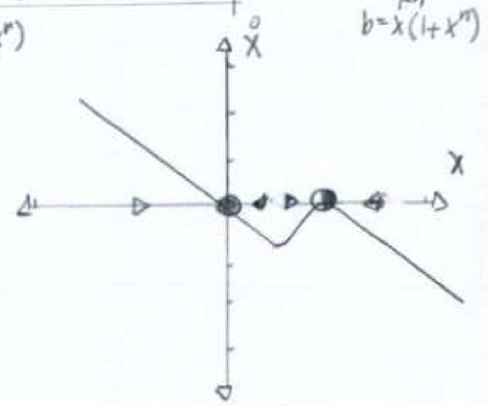
$$\frac{dx}{d\tau} = \frac{sA}{K} + b\frac{x^n}{1+x^n} - x = \frac{s(A_T - A_p)}{K} + b\frac{x^n}{1+x^n} - x$$

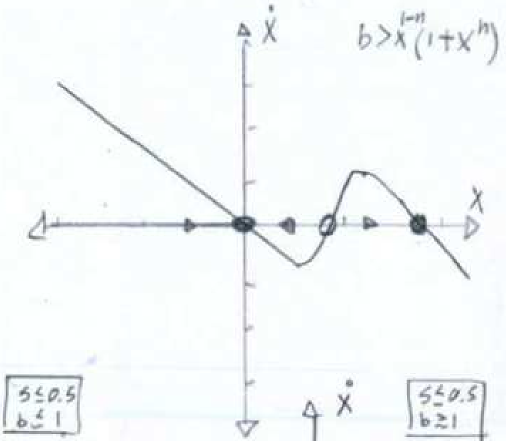
$$= \frac{s(2K - Kx)}{K} + b\frac{x^n}{1+x^n} - x$$

$$\boxed{= s(2-x) + b\frac{x^n}{1+x^n} - x}$$

b) If $s = 0$, then

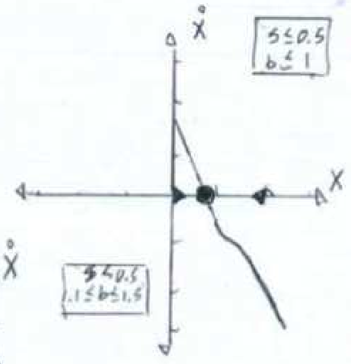


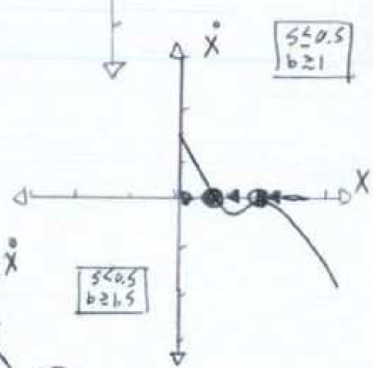


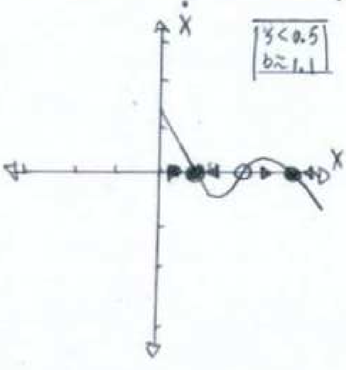


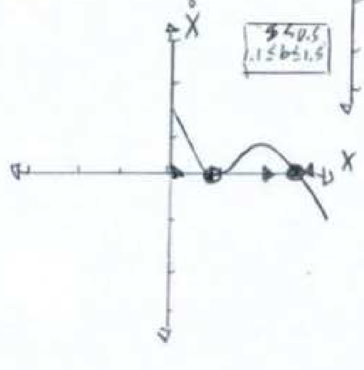
c) If $s > 0$, then a variety of bifurcations are produced.

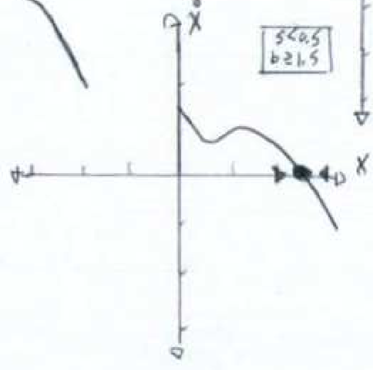
b \ s	$\lesssim 0.5$	$\gtrsim 0.5$
$b \lesssim 1$	1, Stable	
$b \gtrsim 1$	2, Stable Half node	
$b \cong 1.1$	3, Stable unstable Stable	1, stable
$1.1 \le b \le 1.5$	2, Half node Stable	
$b \ge 1.5$	1, Stable	

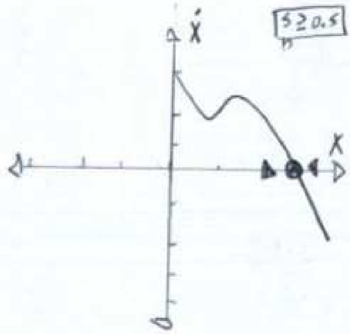








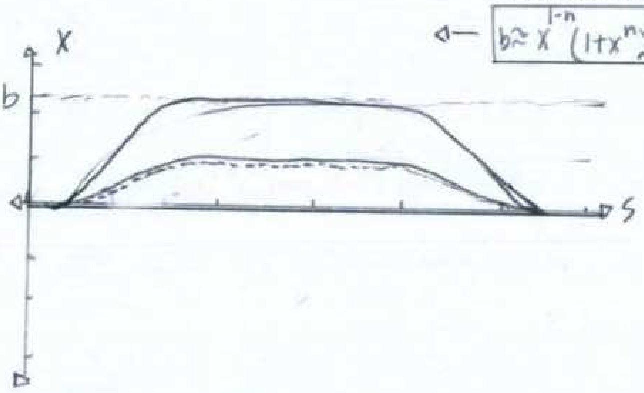




d)

Translation of bifurcation plot occurs near the left box.

$\boxed{b \approx x^{1-n}(1+x^n)}$



Incorrect

Chapter 4: Flows on the circle

$\dot{\theta} = \sin(a\theta)$ 4.1.1. The real values of a, which give a well-defined vector field, on a circle for the function, $\dot{\theta} = \sin(a\theta)$, are fixed to $n\pi$ where $n \in \mathbb{Z}$.

$\dot{\theta} = 1 + 2\cos\theta$ 4.1.2. $\boxed{\text{Fixed points}}$ $\theta = \cos^{-1}\left(-\frac{1}{2}\right)$

$= \frac{2}{3}\pi, \frac{5}{3}\pi \cdots \left(n + \frac{2}{3}\right)\pi$ "stable"

$= \frac{4}{3}\pi, \frac{7}{3}\pi \cdots \left(n + \frac{4}{3}\right)\pi$ "unstable"

where $n \in \mathbb{Z}$

$\boxed{\text{Phase Portrait}}$

$\dot{\theta} = \sin 2\theta$ 4.1.3. $\boxed{\text{Fixed Point}}$ $\theta = \frac{\sin^{-1}(0)}{2}$

$= 0\pi, 1\pi, 2\pi \cdots (n\pi)$ "unstable"

$= \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi \cdots \left(n + \frac{1}{2}\right)\pi$ "stable"

where $n \in \mathbb{Z}$

$\boxed{\text{Phase Portrait}}$

$\dot{\theta} = \sin^3\theta$ 4.1.4 $\boxed{\text{Fixed points}}$ $\theta = \sin^{-1}(0)$

$= 0\pi, 2\pi \ldots (2n)\pi$ "unstable"

$= 1\pi, 3\pi \ldots (2n+1)\pi$ "stable"

where $n \in \mathbb{Z}$

$\boxed{\text{Phase Portrait:}}$

$\dot{\theta} = \sin\theta + \cos\theta$

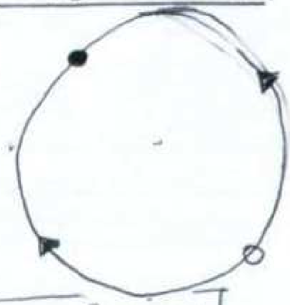
4.1.5. $\sin\theta = -\cos\theta$

$\theta = \frac{3}{4}\pi, \frac{11}{4}\pi, \frac{19}{4}\pi \dots (n+\frac{3}{4})\pi$ "stable"

$= \frac{7}{8}\pi, \frac{15}{8}\pi, \frac{23}{8}\pi \dots (n+\frac{7}{8})\pi$ "unstable"

where $n \in \mathbb{Z}$

Phase Portrait

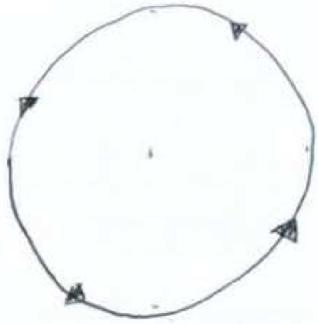


$\dot{\theta} = 3 + \cos 2\theta$

4.1.6. $\theta = \dfrac{\cos^{-1}(3)}{2}$

θ = undefined

Phase Portrait



$\dot{\theta} = \sin k\theta$ where $k \in \mathbb{N}$

4.1.7. $\dot{\theta} = \sin k\theta$

$\theta = \dfrac{\sin^{-1} 0}{k}$

where $k \in \mathbb{N}$

Phase Portrait

Phase Portrait

$\dot{\theta} = \cos\theta$

4.1.8. a) $\frac{-dV}{d\theta} = \frac{d\theta}{dt} = \dot{\theta} = \cos\theta$; $V(\theta) = -\sin\theta$

$\theta = \sin^{-1}(0)$

$= 0, 2\pi, 4\pi \dots (2n)\pi$ "stable"

$= \pi, 3\pi, 5\pi \dots (2n+1)\pi$ "unstable"

b) $\dot{\theta} = 1$; $V(\theta) = -\theta$ The non-uniqueness of $V(\theta)$ does not imply regularity for a vector field on a circle.

c) $\dot{\theta} = f(\theta)$ has a single-valued potential for periodic functions with periodic solutions of 2π intervals.

4.1.9. Exercise 2.6.2 provided a contradiction that

$$\int_t^{t+T} f(x)\dot{x}(t)\,dt \neq \int_t^{t+T} f(x)\dot{x}(t+T)\,d(t+T)$$

Exercise 2.7.7 described a potential which could not oscillate because of the existence and uniqueness of $f(x) = \frac{d(V-c)}{dx} dx$.

Each of these arguments do not carry over to periodic solutions because another solution could be similar within $2n\pi$ $(n \in \mathbb{Z})$ intervals

$T_{lap} = \frac{2\pi}{\omega_1 - \omega_2} = \left[\frac{1}{T_1} - \frac{1}{T_2}\right]^{-1}$ 4.2.1. $T_1 = 3\,sec$, $T_2 = 4\,sec.$

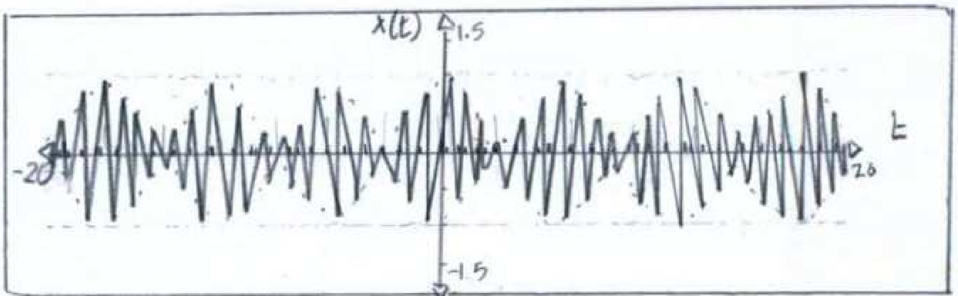
Common sense method:

# Rings	0	1	2	3	4	5
Bell #1	0	3	6	9	12	15
Bell #2	0	4	8	12	16	20

$\boxed{\text{Bell \#1 would ring four times while Bell \#2 three before ringing together again.}}$

Example 4.2.1 method: $T_{lap} = \left[\frac{1}{3sec} - \frac{1}{4sec}\right]^{-1} = \boxed{12\ sec}$.

$x(t) = \sin 8t + \sin 9t$ 4.2.2.



a) $T_{lap} = \left[\frac{1}{8} - \frac{1}{9}\right]^{-1} = \boxed{72}$

b) $x(t) = \sin 8t + \sin 9t = \boxed{2\sin\left(\frac{17}{2}t\right)\cos\left(\frac{1}{2}t\right)}$

4.2.3. 12:00pm is when long-hand angle is equal to short-hand.

$\boxed{\text{Common sense method}}$: short-hand period $[T_1]$ = 12 hour

long-hand period $[T_2]$ = 1 hour

$T_{lap} = \left[\frac{1}{1} - \frac{1}{12}\right]^{-1} = \frac{12}{11}$ hour

$\boxed{\text{Alternative method}}$: $x(t) = \sin(12t) + \sin(t) = 2\sin\left(\frac{11}{2}t\right)\cos\left(\frac{13}{2}t\right)$

Sum frequency   Difference frequency

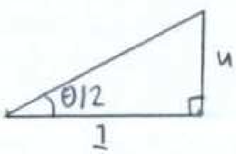
$T_{bottleneck} = \int_{-\infty}^{\infty} \frac{dx}{r + x^2}$ 4.3.1. $x = \sqrt{r}\tan\theta$; $1 + \tan^2\theta = \sec^2\theta$

$$T = \int_{-\infty}^{\infty} \frac{dx}{r + r\tan^2\theta} = \frac{1}{r}\int_{-\infty}^{\infty} \frac{dx}{1 + \tan^2\theta} = \frac{1}{r}\int_{-\infty}^{\infty} \frac{dx}{\sec^2\theta}$$

$$= \frac{1}{r}\int_{-\infty}^{\infty} \frac{\sqrt{r}\sec^2\theta\, d\theta}{\sec^2\theta} = \frac{1}{\sqrt{r}}\,\theta\Big|_{-\infty}^{\infty} = \frac{1}{\sqrt{r}}\arctan\left(\frac{x}{\sqrt{r}}\right)\Big|_{-\infty}^{\infty} = \boxed{\frac{\pi}{\sqrt{r}}}$$

$T = \int_{-\pi}^{\pi} \frac{d\theta}{\omega - a\sin\theta}$ 4.3.2.

a) $u = \tan\frac{\theta}{2}$; $du = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta$; $d\theta = \frac{2\,du}{\sec^2\frac{\theta}{2}} \boxed{= \frac{2\cdot du}{\sec^2\left[\frac{\arctan^{-1}(2u)}{2}\right]}}$

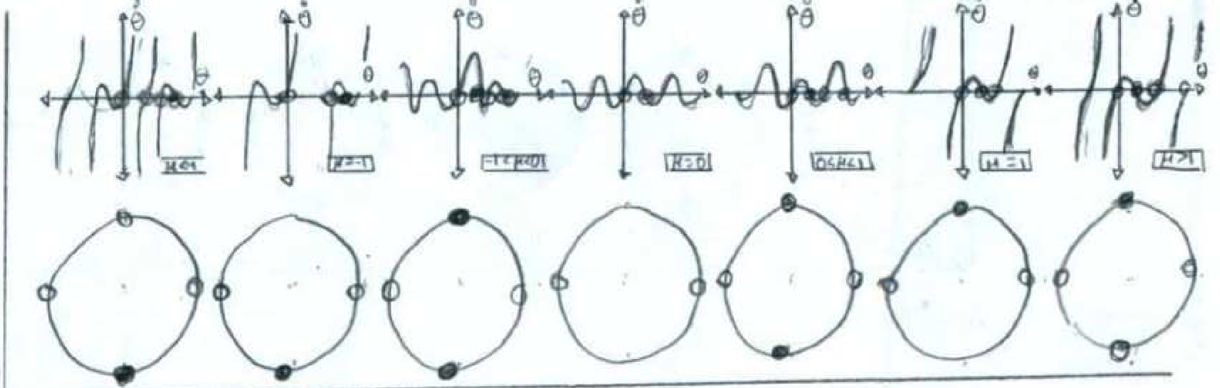
b) 

$\sin\theta \cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}} \boxed{\frac{2u}{1+u^2}}$

c) $\lim_{\theta\to\pm\pi} \sin\theta = \boxed{\lim_{u\to ?} \frac{2u}{1+u^2}}$, for $u \neq 0$, $u \to \pm\infty$

d) $T = \frac{1}{\omega}\int_{-\infty}^{\infty} \frac{1}{1 - \frac{a}{\omega}\left[\frac{2u}{1+u^2}\right]} \cdot \frac{2\cdot du}{\sec^2\left[\frac{\arctan^{-1}(2u)}{2}\right]} \boxed{= \frac{1}{\omega}\int_{-\infty}^{\infty} \frac{1}{1 - \frac{a}{\omega}\left[\frac{2u}{1+u^2}\right]} \cdot \frac{2\,du}{\left[\frac{1}{1+u^2} - \frac{u^2}{1+u^2}\right]^2}}$

e)



$r^{a-b}\frac{du}{d\tau}=r+r^{2a}u^2$ 4.3.9. $T_{bottleneck}\sim O(r^{-1/2})$

a) $O(r^a)$; $x=r^a u$, where $u\sim O(1)$. $t=r^b\tau$, with $\tau\sim O(1)$

$$\dot{x}=r+x^2=r+(r^a u)^2=r+r^{2a}u^2 \; ; \; \boxed{r^{a-b}\frac{du}{d\tau}=r+r^{2a}u^2}$$

b) $\boxed{r^{a-b}=r=r^{2a} \; ; \; a=\frac{1}{2} \; ; \; b=-\frac{1}{2}}$

$\dot{x}=r+x^{2n}$ 4.3.10. $x=r^a u$; $t=r^b\tau$; $r^{a-b}\frac{du}{d\tau}=r+r^{2an}u^{2an}$; $a=\frac{1}{2}$; $b=-\frac{1}{2}$;

$$\frac{du}{d\tau}=1+r^n u^n \; ; \; r^b\tau=$$

$mL^2\ddot{\theta}+b\dot{\theta}+mgL\sin\theta=\Gamma$ 4.4.1. $\ddot{\theta}=0$ -or- $\ddot{\theta}\ll 1$; $t=T\tau$; $\frac{mL^2}{T^2}\frac{d^2\theta}{d^2\tau}+\frac{b}{T}\frac{d\theta}{d\tau}+mgL\sin\theta=\Gamma$

$$\frac{L}{gT^2}\frac{d^2\theta}{d^2\tau}+\frac{b}{mgL}\frac{d\theta}{d\tau}+\sin\theta=\frac{\Gamma}{mgL}$$

$$\frac{b}{mgL\Gamma}=1 \; ; \; T=\frac{b}{mgL}$$

$$\frac{m^2gL^3}{b^2}\frac{d^2\theta}{d\tau^2}+\frac{d\theta}{d\tau}+\sin(\theta)=\frac{\Gamma}{mgL}$$

$$\boxed{m^2gL^3\ll b^2}$$

$\theta'=\gamma-\sin\theta$ 4.4.2 $\int\frac{d\theta}{w-\sin\theta}=dt$; $t=-\int\frac{d\theta}{\frac{2\tan(\frac{\theta}{2})}{\tan^2(\frac{\theta}{2})+1}-a}=-2\int\frac{du}{au^2-2u+a}$; where $u=\tan(\frac{\theta}{2})$

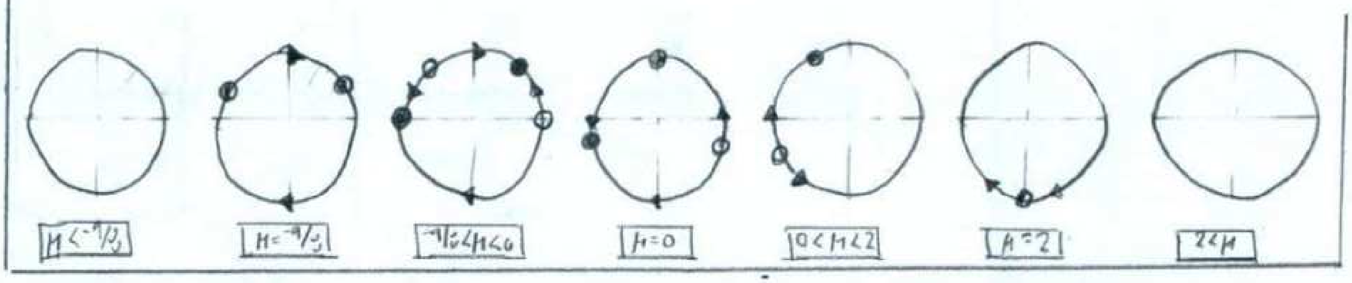
$$du=\frac{\sec^2(\frac{\theta}{2})d\theta}{2}$$

$$d\theta=\frac{1}{u^2+1}$$

$$=-2\int\frac{du}{(\sqrt{a}u-\frac{1}{\sqrt{a}})^2+a-\frac{1}{a}} \; ; \text{ where } v=\frac{au-1}{\sqrt{a}\sqrt{a-1/a}}$$

$$du=\frac{\sqrt{a-1/a}}{\sqrt{a}}dv$$

$$\boxed{=\frac{-2}{\sqrt{a}\sqrt{a-\frac{1}{a}}}\int_{-\pi}^{\pi}\frac{dv}{v^2+1}=\frac{2\arctan\left(\frac{a\tan(\frac{\theta}{2})-1}{\sqrt{a}\sqrt{a-1/a}}\right)}{\sqrt{a}\sqrt{a-1/a}}+C}$$



$\dot{\theta}=\frac{\sin\theta}{\mu+\sin\theta}$

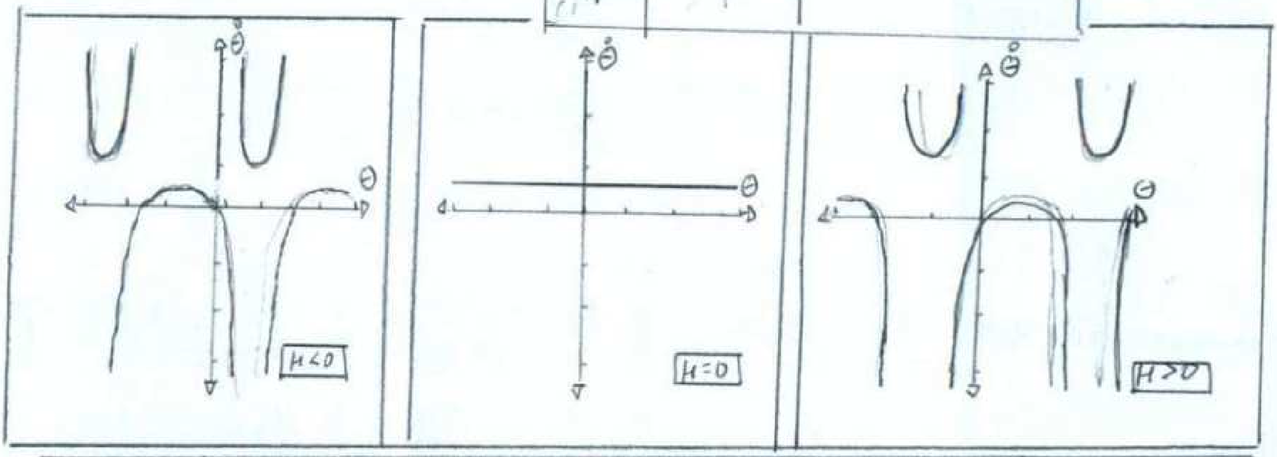
4.3.7 $0=\frac{\sin\theta}{\mu+\sin\theta}$

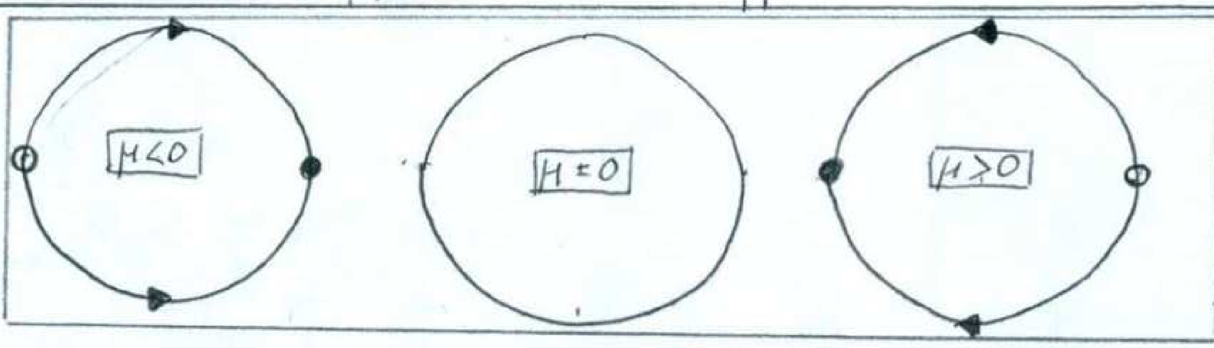
$0=\mu+\sin\theta$

$\mu=-\sin\theta$

$\theta=\sin^{-1}(-\mu)$

Θ	μ	Bifurcations
0, π	<0	Two
N/A	$=0$	Zero
0, π	>0	Two





$\dot{\theta}=\frac{\sin 2\theta}{1+\mu\sin\theta}$

4.3.8. $0=\frac{\sin 2\theta}{1+\mu\sin\theta}$

$0=1+\mu\sin\theta$

$\boxed{\theta=\sin^{-1}\left(\frac{-1}{\mu}\right)}$

θ	μ	Bifurcations
0, $\pi/2$, π, $3\pi/2$	<-1	Four
0, π, $3\pi/2$	$=-1$	Three
0, $\pi/2$, π, $3\pi/2$	$-1<\mu<0$	Four
0, $\pi/2$, π	$=0$	Three
0, $\pi/2$, π, $3\pi/2$	$0<\mu<1$	Four
0, $\pi/2$, π	$=1$	Three
0, $\pi/2$, π, $3\pi/2$	>1	Four

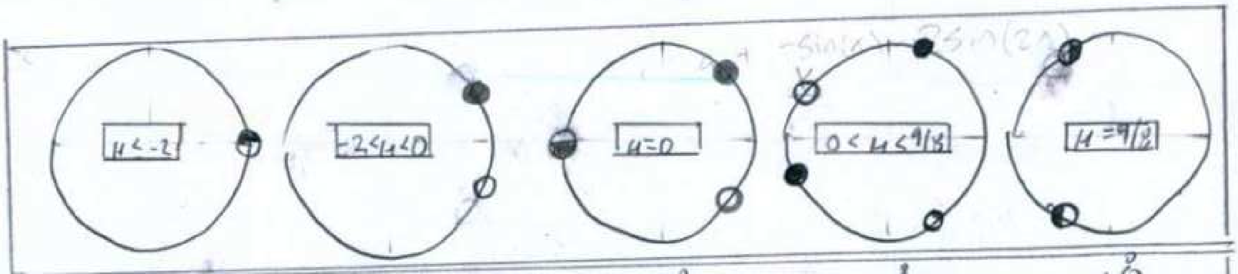
$\dot{\theta} = \mu + \cos\theta + \cos 2\theta$ 4.3.5. $0 = (\mu - 1) + \cos\theta + 2\cos^2\theta$

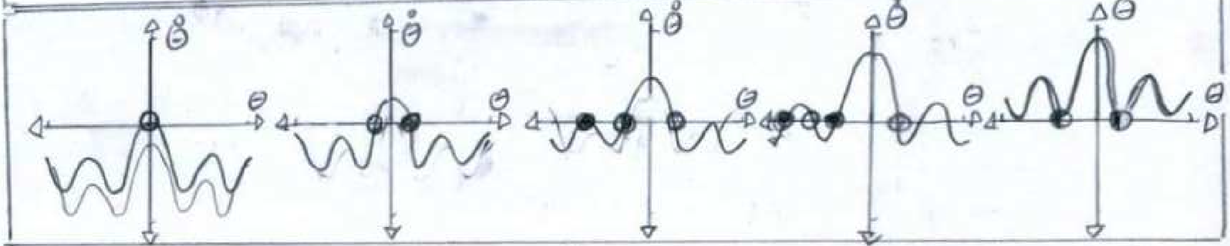
$$\cos\theta = \frac{-1 \pm \sqrt{1 - 4(2)(\mu - 1)}}{2(2)}$$

$$= \frac{-1 \pm \sqrt{9 - 8\mu}}{4}$$

$$\theta = \cos^{-1}\left(\frac{-1 \pm \sqrt{9 - 8\mu}}{4}\right)$$

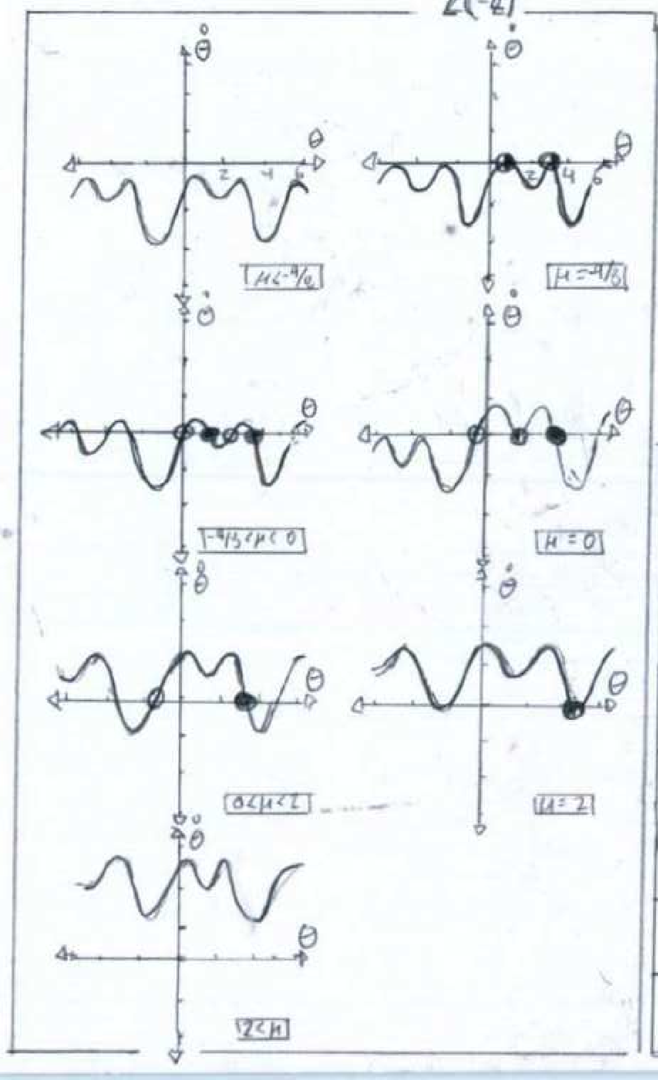
μ	Bifurcations
≤ -2	One
$-2 < \mu < 0$	Two
$= 0$	Three
$0 < \mu < 9/8$	Four
$9/8$	Two





$\dot{\theta} = \mu + \sin\theta + \cos 2\theta$ 4.3.6. $\dot{\theta} = (\mu + 1) + \sin\theta - 2\sin^2\theta$

$$\sin\theta = \frac{-1 \pm \sqrt{1 - 4(-2)(\mu + 1)}}{2(-2)}$$

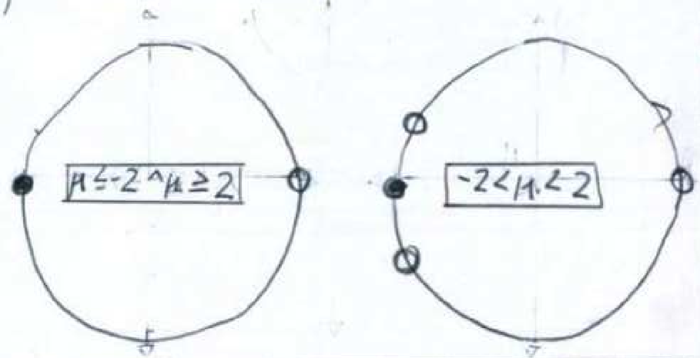


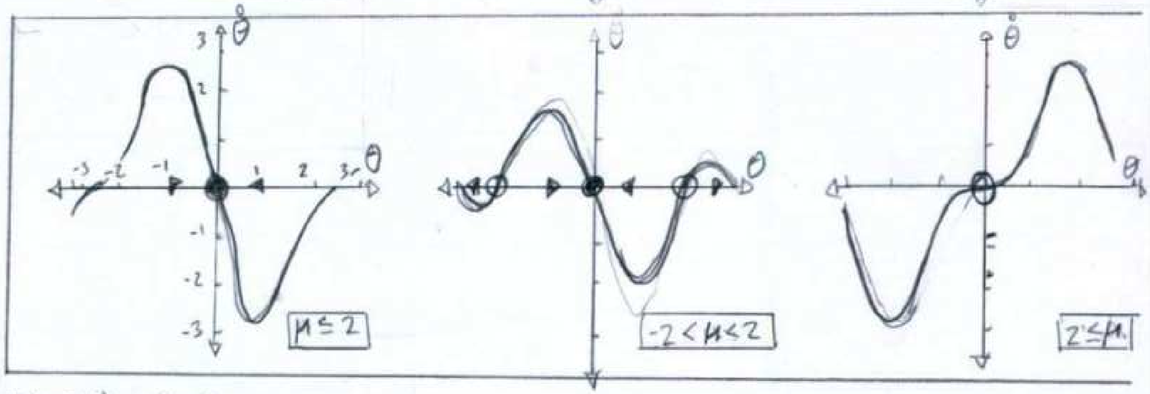
θ	μ	Bifurcations
NA	$< -9/8$	Zero
$\arcsin(1/4)$, $\arcsin(\pi - 1/4)$	$= -9/8$	Two
$\arcsin(1/4)$ to 2π	$-\frac{9}{8} < \mu < 0$	Four
$\pi/2$, $\frac{7}{6}\pi$, $\frac{11}{6}\pi$	$= 0$	Three
$\frac{7}{6}\pi < \theta < \frac{11}{6}\pi$	$0 < \mu < 2$	Two
$3\pi/2$	$= 2$	One
NA	$2 < \mu$	Zero

$\dot{\theta} = \mu\sin\theta - \sin 2\theta$ 4.3.3. $\mu = \dfrac{\sin 2\theta}{\sin\theta} = 2\cos x$; $x = \cos^{-1}\left(\dfrac{\mu}{2}\right)$

μ	Bifurcations
≤ -2	Two
$-2 \leq \mu \leq 2$	Four
≥ 2	Two

Phase Portrait: Saddle-node Bifurcation





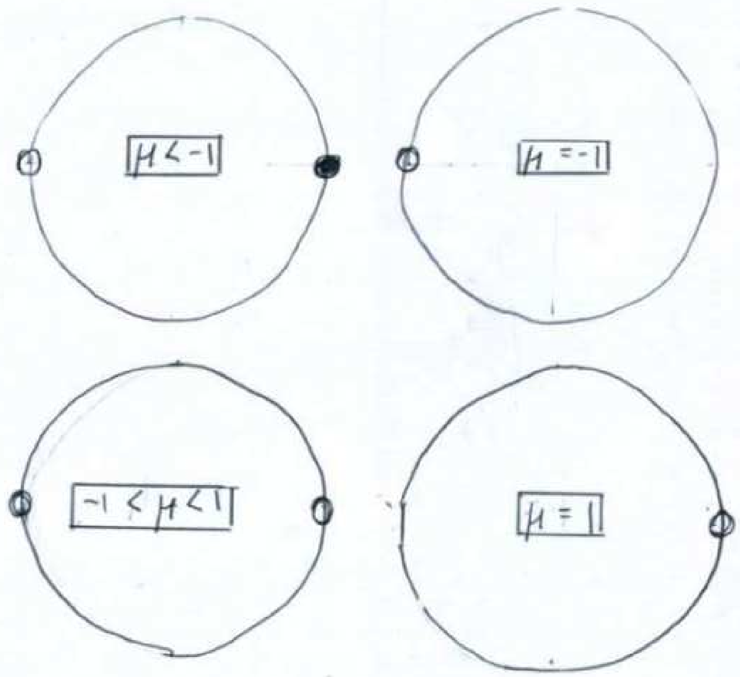
$\dot{\theta} = \dfrac{\sin\theta}{\mu + \cos\theta}$ 4.3.4. $0(\mu + \cos\theta) = \sin\theta$

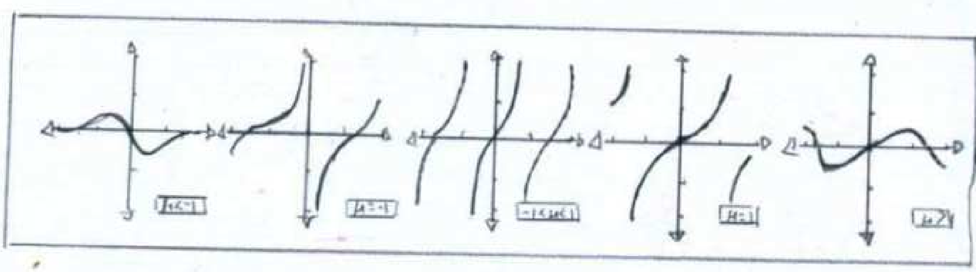
$\mu = -\cos\theta$

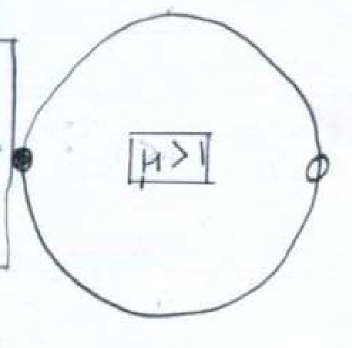
$\theta = \cos^{-1}(-\mu)$

μ	Bifurcations
<-1	Two
$=-1$	one
$-1<\mu<1$	Two
$=1$	one
>1	Two

Phase Portrait Transcritical Bifurcation







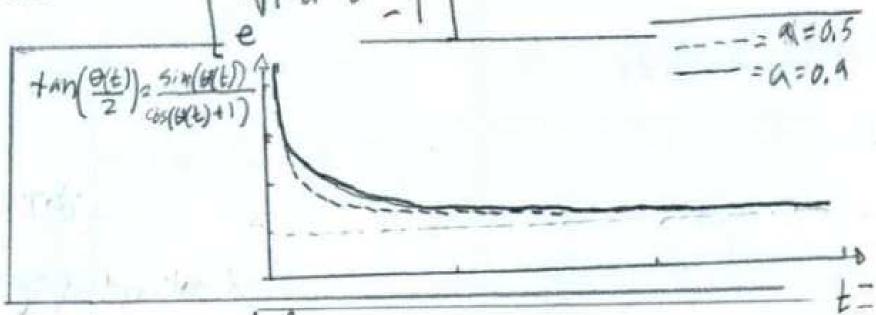
$$t = \ln\left(\frac{\left|\frac{a\sin(\theta)}{\cos\theta+1} + \frac{-2\sqrt{1-a^2}-2}{2}\right|}{\left|\frac{a\sin(\theta)}{\cos(\theta)+1} + \frac{2\sqrt{1-a^2}-2}{2}\right|}\right) \Big/ \sqrt{1-a^2}$$

$$e^{\sqrt{1-a^2}\cdot t} \circ \left|\frac{a\sin(\theta)}{\cos(\theta)+1} + \frac{2\sqrt{1-a^2}-2}{2}\right| = \left|\frac{a\sin(\theta)}{\cos(\theta)+1} - \frac{2\sqrt{1-a^2}-2}{2}\right|$$

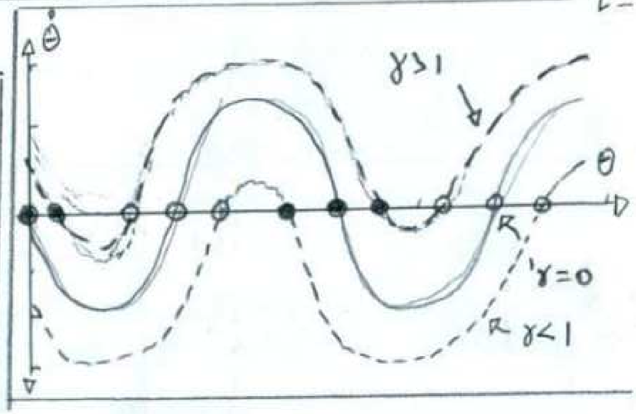
$$\frac{a\sin(\theta)}{\cos(\theta)+1}\left[e^{\sqrt{1-a^2}\,t} - 1\right] = \frac{-2\sqrt{1-a^2}-2}{2}\left[e^{\sqrt{1-a^2}\cdot t} + 1\right]$$

$$\frac{\sin(\theta)}{\cos(\theta)+1} = \frac{(-2\sqrt{1-a^2}-2)}{2a} \circ \frac{\left[e^{\sqrt{1-a^2}\,t} + 1\right]}{\left[e^{\sqrt{1-a^2}\,t} - 1\right]}$$

A graph of $\sin(\theta(t))$ vs t represented as $\tan\left(\frac{\theta(t)}{2}\right)$.



4.4.3. $\dot{\theta} = \gamma - \sin(\theta(t))$



$\boxed{\text{Similar to Question 4.4.3, } \gamma \text{ is the relationship of } T/mgL.}$ If torque (T) is greater than mass×gravity×length, then motion is directed; moreover, if torque (T) is zero, then there is no direction.

$b\dot{\theta} + mgL\sin\theta = T - k\theta$

4.4.4.

a. $\boxed{\theta = \{0, \frac{\pi k}{T}, \frac{2\pi k}{T}, \ldots, \frac{n\pi k}{T}\}}$

b. $b\dot{\theta} + mgL\sin\Theta = T - k\theta$

$\frac{b}{mgL}\dot{\theta} + \sin\theta = \frac{T - k\theta}{mgL}$; if $\tau = \frac{mgL}{b}t$; $\gamma = \frac{T}{mgL}$ $\boxed{;\ \mu = +\frac{k}{mgL}}$

$\theta' + \sin\theta = \gamma + \mu\theta$; $\theta' = \gamma - \mu\theta - \sin\theta$

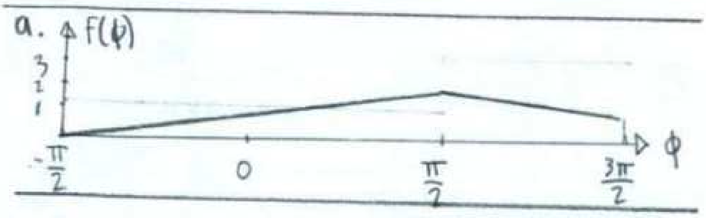
c. As the pendulum angle increases, the dampening lowers rate of angle change (θ').

d. As k is varied from 0 to ∞, then θ' is equal to zero at $\frac{\gamma - \sin\theta}{\mu}$. The bifurcation type is subcritical.

$\dot{\theta} = \omega + A f(\Theta - \theta)$ 4.5.1. $f(\phi) = \begin{cases} \phi & -\frac{\pi}{2} \le \phi \le \frac{\pi}{2} \\ \pi - \phi & \frac{\pi}{2} \le \phi \le \frac{3\pi}{2} \end{cases}$

$\dot{\Theta} = \Omega$

a.



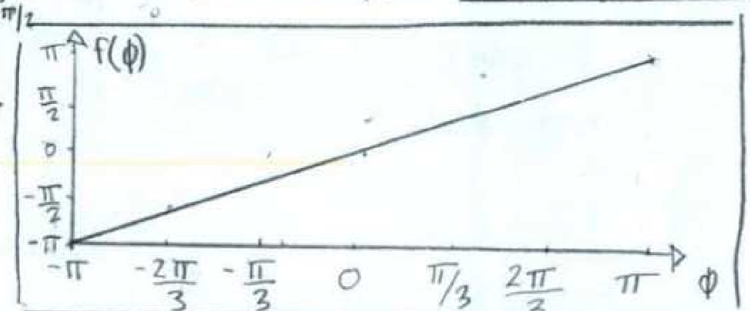
b. Range of Entrainment: $\boxed{-\frac{\pi}{2} \le f(\phi) \le \frac{\pi}{2}}$

c. $\phi^* = \dot{\Theta} - \dot{\theta} = \Omega - \omega - A\left[\frac{\pi}{2}\right]$; $\boxed{|\Omega - \omega| \le \pi/2}$

d. $T_{drift} = \int_{0}^{2\pi} \frac{d\phi}{\Omega - \omega - A[\pi - \phi]} = \frac{1}{A}\int_{-\pi/2}^{3\pi/2} \frac{du}{u} = \frac{2}{A}\ln\left(\frac{\Omega - \omega - A\pi/2}{\Omega - \omega + A\pi/2}\right) = \boxed{\frac{2}{A}\ln\left(\frac{\Omega - \omega + A\pi/2}{\Omega - \omega + A\pi/2}\right)}$

$\dot{\theta} = \omega + A f(\Theta - \theta)$ 4.5.2

$f(\phi) = \phi \quad -\pi < \phi < \pi$



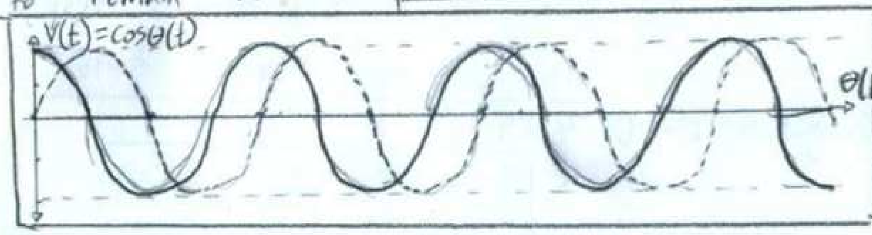
Range of Entrainment: $-\pi < f(\phi) < \pi$

$\phi^* = \dot{\Theta} - \dot{\theta} = \Omega - \omega - A[\pi]$; $|\Omega - \omega| \le \pi$

$T_{drift} = \int_{0}^{2\pi} \frac{d\phi}{\Omega - \omega - A[\phi]} = \frac{1}{A}\int_{\pi}^{\pi} \frac{du}{u} = \boxed{\frac{1}{A}\ln\left(\frac{\Omega - \omega + A[\pi]}{\Omega - \omega - A[\pi]}\right)}$

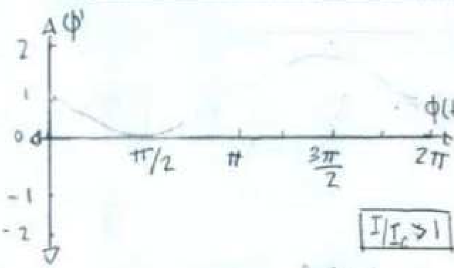
$\dot{\theta} = \mu + \sin\theta$ 4.5.3. a) The 'rest state' and 'threshold' are described by the neurons ability to remain at rest, or fire, respectively.

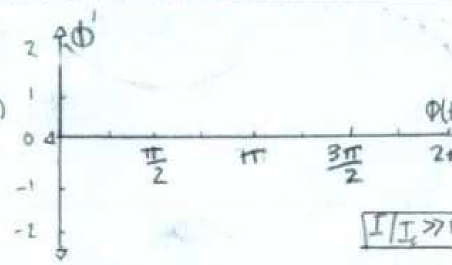
b) $V(t) = \cos\theta(t)$



4.6.1 $\beta = 0$

a. $\phi' = \frac{I}{I_c} - \sin\phi(t)$





$t = \int \frac{d\phi}{I/I_c - \sin\phi}$

$= -\int \frac{d\phi}{\sin\phi - I/I_c}$

$= -\frac{I_c}{I}\int \frac{du}{\left[\frac{2u}{1+u^2}\left(\frac{I}{I_c}\right) - 1\right]} \frac{2}{\sec^2\left(\frac{\phi}{2}\right)} = -\frac{I_c}{I}\int \frac{du}{\left[\frac{2u}{1+u^2}\left(\frac{I}{I_c}\right) - 1\right]} \frac{2}{[u^2+1]}$

Method #1: $u = \tan\left(\frac{\phi}{2}\right)$; $du = \frac{\sec^2\left(\frac{\phi}{2}\right) d\phi}{2}$

Method #2:

$\sin\phi = 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}$

$= 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}} = \frac{2u}{1+u^2}$

$= -\frac{2I_c}{I}\int \frac{du}{2u\left(\frac{I}{I_c}\right) - u^2 - 1} = \boxed{+\frac{2I_c}{I}\int \frac{du}{\left(u - \frac{I}{I_c}\right)^2 - \left(\frac{I}{I_c}\right)^2 + 1}}$

Where $v = \frac{(u - I/I_c)^2}{\sqrt{1 - I/I_c}}$

$$= 2\frac{I}{I_c}\int \frac{\sqrt{1-(I/I_c)^2}}{(1-(I/I_c)^2)V^2-(I/I_c)^2+1}\,dV = \frac{2I/I_c}{\sqrt{1-(I/I_c)^2}}\int\frac{dV}{V^2+1} = \frac{1}{\sqrt{1-(I/I_c)^2}}\arctan(V)$$

$$= 2\left(\frac{I}{I_c}\right)\frac{\arctan\left(\frac{u-I/I_c}{\sqrt{1-(I/I_c)^2}}\right)}{\sqrt{1-(I/I_c)^2}} = 2\,\frac{I}{I_c}\,\frac{\arctan\left(\frac{\tan\frac{\phi}{2}-I/I_c}{\sqrt{1-(I/I_c)^2}}\right)}{\sqrt{1-(I/I_c)^2}} + C \text{ ; where } C=0$$

$$t = \frac{I}{I_c}\ln\left(\frac{1-\frac{\tan\phi/2+I/I_c}{\sqrt{1-(I/I_c)^2}}}{1+\frac{\tan\phi/2-I/I_c}{\sqrt{1-(I/I_c)^2}}}\right) \Big/ \sqrt{(I/I_c)^2-1}$$

$$e^{(I_c/I)\sqrt{(I/I_c)^2-1}\cdot t}\cdot\left(1+\frac{\tan\phi/2-(I/I_c)}{\sqrt{1-(I/I_c)^2}}\right) = 1-\frac{\tan\phi/2+I/I_c}{\sqrt{1-(I/I_c)^2}}$$

$$\frac{\tan\phi/2-(I/I_c)}{\sqrt{1-(I/I_c)^2}}\left[e^{\frac{I_c}{I}\sqrt{(I/I_c)^2-1}\cdot t}+1\right] = 1-e^{\frac{I_c}{I}\sqrt{(I/I_c)^2-1}\cdot t}$$

$$\sin\left(\frac{\phi}{2}\right) = \cos\left(\frac{\phi}{2}\right)\left[\frac{\left(1-e^{I_c/I\sqrt{(I/I_c)^2-1}\cdot t}\right)}{\left(1+e^{I_c/I\sqrt{(I/I_c)^2-1}\cdot t}\right)}\sqrt{1-(I/I_c)^2}+I/I_c\right]$$

$$\sin\phi = 2\cos^2\left(\frac{\phi}{2}\right)\left[\sqrt{1-(I/I_c)^2}\,\frac{\left(1-e^{I_c/I\sqrt{(I/I_c)^2-1}\cdot t}\right)}{\left(1+e^{I_c/I\sqrt{(I/I_c)^2-1}\cdot t}\right)}+I/I_c\right]$$

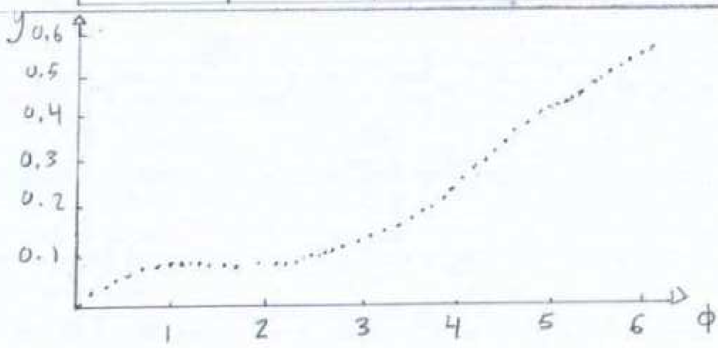
$\phi' = \frac{I}{I_c} - \sin\phi$

4.6.2. Numerical Integration: Runge-Kutta 4th Order

ϕ	k_1	k_2	k_3	k_4
0.0	$\Delta h \cdot f(\phi)$	$\Delta h \cdot f(\phi + \frac{\Delta h}{2})$	$\Delta h \cdot f(\phi + \frac{\Delta h}{2})$	$\Delta h \cdot f(\phi + \Delta h)$
...	...	...	...	...
6.0	0.1379	0.1331	0.1331	0.1282

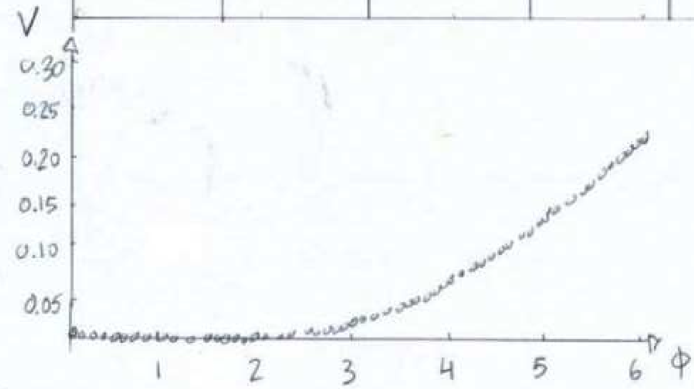
$y_{n+1} = y_n + \frac{\Delta h}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

where $\Delta h = 0.1$



ϕ	k_1	k_2	k_3	k_4
0.0	$\Delta t \cdot \Delta h f(\phi)$	$\Delta t \Delta h \cdot f(\phi + \frac{\Delta h}{2})$	$\Delta t \cdot \Delta h \cdot f(\phi + \frac{\Delta h}{2})$	$\Delta t \cdot \Delta h f(\phi + \Delta h)$
...	...	...	...	...
6.0	0.63	-0.0103	-0.0103	-0.0098

$V_{n+1} = V_n + \frac{\Delta h}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

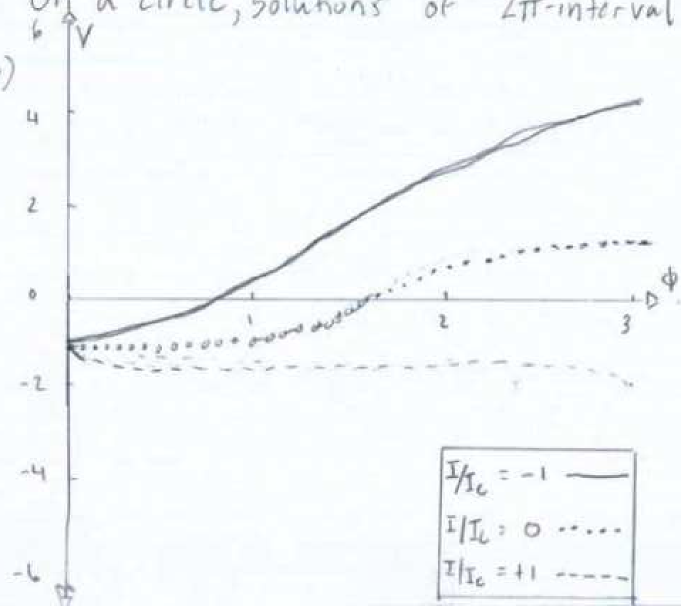


4.6.3.

a) $V = -\dot{x}\,dx$; $V = -\dot{\phi}\,d\phi = -\left[I/I_c - \sin\phi\right]d\phi = -\left[\cos\phi + \frac{I}{I_c}\phi\right]$

On a circle, solutions of 2π-interval exist: $\phi = \arcsin(I/I_c)$

b)

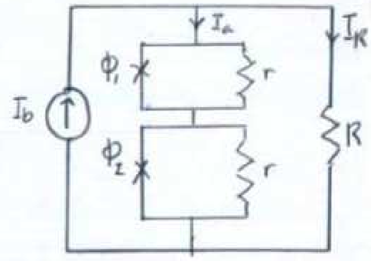


c) The increase of current (I) lowers the potential (V) per 2π oscillation.

$I_a = I_c \sin\phi_1 + V_1/r$

$I_b = I_c \sin\phi_K + \frac{\hbar}{2er}\dot{\phi}_K + \frac{\hbar}{2eR}(\dot{\phi}_1 + \dot{\phi}_2)$

4.6.4. a) $I_b = I_a + I_R$



b) Kirchoff's Law: Parallel Circuit

$$I_a = I_{a1} + I_{ar} \qquad ; \qquad I_a = I_{a2} + I_{ar}$$
$$= I_a \sin\phi_1 + \frac{V_1}{r} \qquad \boxed{= I_a \sin\phi_2 + \frac{V_2}{r}}$$

c) If $k = 1,2$, then $\boxed{V_k = \begin{cases} \frac{\hbar}{2eR}\dot{\phi}_1 \\ \frac{\hbar}{2eR}\dot{\phi}_2 \end{cases}}$

d) $I_b = I_{a1} + I_{1ar} + I_{2ar} + I_R = I_c \sin\phi_k + \frac{\hbar}{2eR}\dot{\phi}_1 + \frac{\hbar}{2eR}\dot{\phi}_2 + \frac{V_k}{R}$

$$\boxed{= I_c \sin\phi_k + \frac{\hbar}{2eR}\left[\dot{\phi}_1 + \dot{\phi}_2\right] + \frac{\hbar}{2eR}\dot{\phi}_k}$$

where $k = 1,2$.

e) $I_b = I_c \sin(\phi_K) + \frac{\hbar}{2er}\dot{\phi}_K + \frac{\hbar}{2eR}\sum_{i=1}^{N}\dot{\phi}_i$

$= I_c \sin(\phi_K) + \frac{h}{2er}\dot{\phi}_K + \frac{Nr}{R}\left(I_c \sin(\phi_K) + \frac{\hbar}{2er}\dot{\phi}_K - I_c\sum_{i=1}^{N}\sin(\phi_i)\right)$

$= \left(1 + \frac{Nr}{R}\right) I_c \sin(\phi_K) + \left(\frac{h}{2er} + \frac{N\hbar}{2eR}\right)\dot{\phi}_K - \frac{r}{R} I_c \sum_{i=1}^{2}\sin(\phi_j)$

$\hbar\left(\frac{1}{2er} + \frac{N}{2eR}\right)\dot{\phi}_K = I_b - \left(1 + \frac{Nr}{R}\right) I_c \sin(\phi_K) + \frac{r}{R} I_c \sum_{i=1}^{2}\sin(\phi_j)$

$\frac{\hbar(R+Nr)}{2erR}\dot{\phi}_K = I_b - \frac{R+Nr}{R} I_c \sin(\phi_K) + \frac{r}{R} I_c \sum_{i=1}^{2}\sin(\phi_j)$

$\dot{\phi}_K = \frac{2erR I_b}{\hbar(R+Nr)} - \frac{2er}{\hbar} I_c \sin(\phi_K) + \frac{2er^2}{\hbar(R+Nr)} I_c \sum_{i=1}^{2}\sin(\phi_j)$

$$\boxed{\Omega = \frac{2erR I_b}{\hbar(R+Nr)} \; ; \; a = \frac{-2er}{\hbar} I_c \; ; \; K = \frac{2er^2 I_c}{\hbar(R+Nr)}}$$

4.6.5 $I_b = I_c \sin(\phi_K) + \frac{\hbar}{2er}\dot{\phi}_K + \frac{\hbar}{2eR}\sum\dot{\phi}_i$

$= I_c \sin(\phi_K) + \frac{\hbar}{2er}\dot{\phi}_K + \frac{Nr}{R}\left(I_c \sin(\phi_K) + \frac{\hbar}{2er}\dot{\phi}_K - I_c\sum_{i=1}^{N}\sin(\phi_i)\right)$

$= \left(1 + \frac{Nr}{R}\right) I_c \sin\phi_K + \left(\frac{h}{2er} + \frac{N\hbar}{2eR}\right)\dot{\phi}_K - \frac{r}{R} I_c \sum_{i=1}^{2}\sin(\phi_j)$

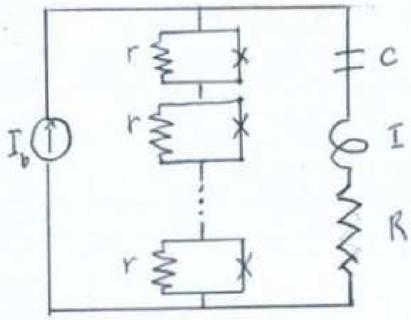
$\hbar\left(\frac{1}{2er} + \frac{N}{2eR}\right)\dot{\phi}_K = I_b - \frac{R+Nr}{R} I_c \sin\phi_K + \frac{r}{R} I_c \sum_{i=1}^{2}\sin(\phi_j)$

$\frac{\hbar(R+Nr)}{2erR}\dot{\phi}_K = I_b - \frac{R+Nr}{R} I_c \sin\phi_K + \frac{r}{R} I_c \sum_{i=1}^{2}\sin(\phi_j)$

$\frac{\hbar(R+Nr)}{2Ner^2 I_c}\dot{\phi}_K = \frac{R I_b}{Nr I_c} - \frac{R+Nr}{Nr}\sin\phi_K + \frac{1}{N}\sum_{i=1}^{2}\sin\phi_j$

$\frac{d\phi_K}{d\tau} = \Omega + a\sin\phi_K + \frac{1}{N}\sum_{j=1}^{2}\sin\phi_j \; ; \; \Omega = \frac{R I_b}{Nr I_c} \; ; \; \boxed{a = \frac{-(R+Nr)}{Nr} \; ; \; \tau = \frac{2Ner^2 I_c}{\hbar(R+Nr)} t}$

$\dot{\phi} = \Omega + a\sin\phi_k + K\sum_{j=1}^{2}\sin\phi_j$ 4.6.6.



$$\frac{\hbar}{2er}\cdot\frac{d\phi_K}{dt} + I_c\sin\phi_K + \frac{dQ}{dt} = I_b$$

$$L\frac{d^2Q}{d^2t} + R\frac{dQ}{dt} + \frac{Q}{C} = \boxed{\frac{\hbar}{2e}\sum_{j=1}^{N}\frac{d\phi_j}{dt}}$$

Chapter 5: Linear Systems

$\dot{x} = v$
$\dot{v} = -\omega^2 x$

5.1.1. a. $\frac{\dot{x}}{\dot{v}} = \frac{dx}{dv} = \frac{v}{-\omega^2 x}$; $-\omega^2 x + C = v$; $\boxed{\omega^2 x^2 + v^2 = C}$

b. Conservation of Energy: $\sum\frac{1}{2}mv^2 = E$; $\frac{1}{2}m\omega^2 x^2 + \frac{1}{2}mv^2 = C$

$$\boxed{KE_{rot} + KE_{Lin} = KE_{TOT.}}$$

$\dot{x} = ax$
$\dot{y} = -y$

5.1.2 $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{-y}{ax} = \frac{-e^{-t}}{ae^{at}} = \frac{-1}{ae^{(a+1)t}}$; $\lim_{t\to\infty}\frac{dy}{dx} = \lim_{t\to\infty}\frac{-1}{ae^{(a+1)t}} = \boxed{-\infty}$ $\parallel$ y-axis

$\lim_{t\to-\infty}\frac{dy}{dx} = \lim_{t\to-\infty}\frac{-1}{ae^{(a+1)t}}$ $\boxed{= 0}$ $\parallel$ x-axis.

$\dot{x} = -y$
$\dot{y} = -x$

5.1.3. $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \boxed{\begin{bmatrix}-1\\ -1\end{bmatrix}\begin{bmatrix}x & y\end{bmatrix}}$

$\dot{x} = 3x - 2y$
$\dot{y} = 2y - x$

5.1.4 $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \boxed{\begin{bmatrix}3 & -2\\ -1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}}$

$\dot{x} = 0$
$\dot{y} = x + y$

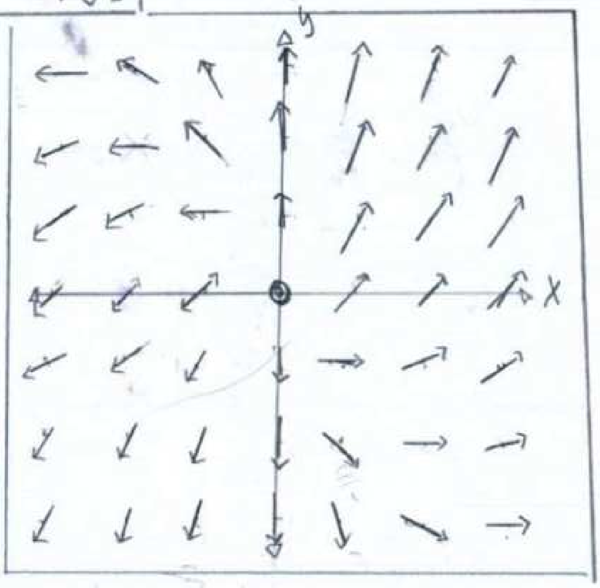
5.1.5. $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \boxed{\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}}$

$\dot{x} = x$
$\dot{y} = 5x + y$

5.1.6. $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \boxed{\begin{bmatrix}1 & 0\\ 5 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}}$

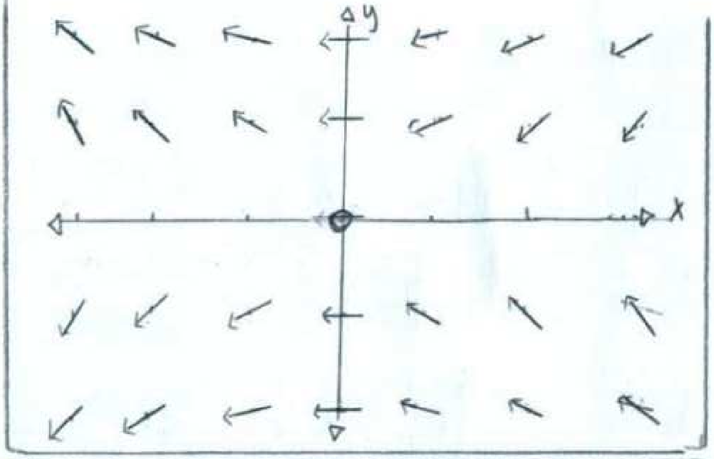
$\dot{x} = x$
$\dot{y} = x + y$

5.1.7. $\frac{dy}{dx} = \frac{x+y}{x}$



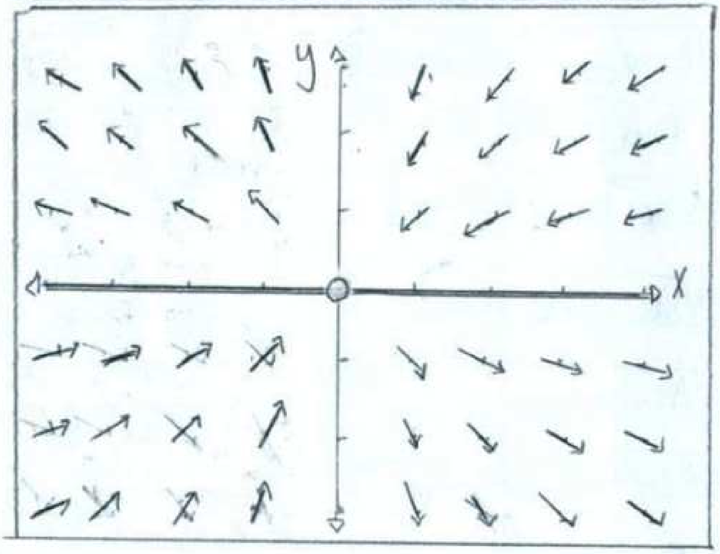
$\dot{x} = -2y$ 5.1.8. $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{-x}{-2y} = \frac{x}{2y}.$

$\dot{y} = -x$



$\dot{x} = -y$ 5.1.9 a) $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{y}{x}$

$\dot{y} = -x$



b) $x\dot{x} = -xy$; $y\dot{y} = -xy$; therefore, $x\dot{x} = y\dot{y}$; $x\dot{x} - y\dot{y} = 0$

and $\boxed{x\,dx - y\,dy = x^2 - y^2 = C}$

c) $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$; $\begin{pmatrix}-\lambda & -1\\ -1 & -\lambda\end{pmatrix} = \lambda^2 + 1 = (\lambda+1)(\lambda-1) = 0$; $\lambda_1 = 1$, $\lambda_2 = -1$

$\lambda_1 = 1$; $\begin{pmatrix}+1 & -1\\ -1 & +1\end{pmatrix}\begin{pmatrix}V_{11}\\ V_{12}\end{pmatrix} = 0$; Guess $V_{11} = 1$, $V_{12} = -1$; $x = -C_1 e^{t}$; $y = C_2 e^{t}$

$\lambda_2 = -1$; $\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}V_{11}\\ V_{12}\end{pmatrix} = 0$; Guess $V_{11} = 1$, $V_{12} = 1$; $x = C_1 e^{-t}$; $y = C_2 e^{-t}$

General Solution: $x(t) = C_1 e^{-t} - C_1 e^{t}$; $y(t) = C_2 e^{-t} + C_2 e^{t}$

$\lim_{t\to\infty} x(t) = -\infty$; $\lim_{t\to-\infty} x(t) = \infty$: $\boxed{\text{Unstable Manifold}}$

$\lim_{t\to\infty} y(t) = \infty$; $\lim_{t\to-\infty} y(t) = \infty$: $\boxed{\text{Stable Manifold}}$

d) $u = x + y$; $\dot{u} = \dot{x} + \dot{y} = -y - x = -u$; $\boxed{u(t) = u_0 e^{-t}}$

$v = x - y$; $\dot{v} = \dot{x} - \dot{y} = x - y = v$; $\boxed{v(t) = v_0 e^{t}}$

e) $\lim_{t\to\infty} u(t) = 0$; $\lim_{t\to-\infty} u(t) = \infty$; $\lim_{t\to\infty} v(t) = \infty$; $\lim_{t\to-\infty} v(t) = 0$;

$\boxed{\text{Stable}}$ $\boxed{\text{Arbitrary}}$ $\boxed{\text{Arbitrary}}$ $\boxed{\text{unstable}}$

f) See part C.

5.1.10

$\dot{x}=y; \dot{y}=-4x$ a) $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}=\begin{bmatrix}0 & 1\\ -4 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$; $\begin{bmatrix}-\lambda & 1\\ -4 & -\lambda\end{bmatrix}=\lambda^2+14=0$; $\lambda_1=\pm 2i$; $\lambda_2=\pm 2i$

$\lambda_1=2i$; $\begin{bmatrix}-2i & 1\\ -4 & -2i\end{bmatrix}\begin{bmatrix}v_{11}\\ v_{12}\end{bmatrix}=0$; $-2iv_{11}+v_{12}=0$; $v_{11}=1$; $v_{22}=2i$

$-4v_{11}-2iv_{12}=0$

$\lambda_2=-2i$; $\begin{bmatrix}2i & 1\\ -4 & 2i\end{bmatrix}\begin{bmatrix}v_{11}\\ v_{12}\end{bmatrix}=0$; $2v_{11}+v_{12}=0$; $v_{11}=1$; $v_{22}=2i$

$-4v_{11}+2v_{12}=0$;

$\boxed{\text{Liapunov Stable}}$

Identity: $e^{\lambda t}=\cos(t)+i\sin(t)$

$e^{\lambda t}V=e^{\lambda t}\begin{bmatrix}v_{11}\\ v_{12}\end{bmatrix}=\begin{bmatrix}\cos(2t)+i\sin(2t)\\ 2i(\cos(2t)+i\sin(2t))\end{bmatrix}$

$\dot{x}=2y; \dot{y}=x$ b) $\boxed{\text{None of the Above}}$

$\dot{x}=0, \dot{y}=x$ c) $\boxed{\text{None of the Above}}$

$X_1=\begin{bmatrix}x=C_2\cos(2t)+C_1\sin(2t)\\ y=2C_2\cos(2t)-2C_1\sin(2t)\end{bmatrix}$

$\dot{x}=0, \dot{y}=-y$ d) $\boxed{\text{None of the Above}}$

$\dot{x}=-x; \dot{y}=-5y$ e) $\boxed{\text{Asymptotically stable}}$

$X_i=C_1 Re(e^{\lambda t}V_1)+C_2 Im(e^{\lambda t}V_1)$

$\dot{x}=x; \dot{y}=y$ f) $\boxed{\text{Asymptotically stable}}$

5.1.11 a) $\|x(t)=C\cos(2t)+C\sin(2t)-x^*\|<C^2=\epsilon$

$\|x(0)=C-x^*\|<C+\delta$

b) $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}=\begin{bmatrix}0 & 2\\ 1 & 0\end{bmatrix}\begin{bmatrix}v_{11}\\ v_{12}\end{bmatrix}$; $\begin{bmatrix}-\lambda & 2\\ 1 & -\lambda\end{bmatrix}=\lambda^2-2=0$; $\lambda_1=+\sqrt{2}$; $\lambda_2=-\sqrt{2}$

$\lambda_1=+\sqrt{2}$; $\begin{bmatrix}-\sqrt{2} & 2\\ 1 & -\sqrt{2}\end{bmatrix}\begin{bmatrix}v_{11}\\ v_{12}\end{bmatrix}=0$; $-\sqrt{2}v_{11}+2v_{12}=0$; $v_{11}=1$; $v_{12}=\frac{1}{\sqrt{2}}$

$v_{11}-\sqrt{2}v_{12}=0$;

$\lambda_2=-\sqrt{2}$; $\begin{bmatrix}\sqrt{2} & 2\\ 1 & \sqrt{2}\end{bmatrix}\begin{bmatrix}v_{11}\\ v_{12}\end{bmatrix}=0$; $\sqrt{2}v_{11}+2v_{12}=0$; $v_{11}=1$; $v_{12}=-\frac{1}{\sqrt{2}}$

$x(t)=C_1\cosh(t)+C_2\sinh(t/\sqrt{2})$; $y(t)=x_0\cosh(t)+C_4\sinh(-t/\sqrt{2})$

$=x_0\cosh(t)+\frac{y_0}{\sqrt{2}}\sinh(t/\sqrt{2})$; $y(t)=x_0\cosh(t)-\frac{y_0}{\sqrt{2}}\sinh(-t/\sqrt{2})$

$\|x(t)-x^*\|=\|x_0\cosh(t)+\frac{y_0}{\sqrt{2}}\sinh(t/\sqrt{2})-0\|\,!=\epsilon$

$\|x(0)-x^*\|=\|x_0\|<\delta$ $\boxed{\text{None of the above}}$

c $\mathring{x}=0$; $x=1+C$; $\mathring{y}=x$; $\boxed{\text{None of the Above}}$

$=C$ $\quad y=x_0t+C$

$=x_0$ $\quad =x_0t+y_0$

d. $\mathring{x}=0$; $x=1+C$; $\mathring{y}=x$; $\boxed{\text{None of the above}}$

$=C$ $\quad =x_0t+C$

$=x_0$ $\quad =x_0t+y_0$

e. $\mathring{x}=-x$; $\mathring{y}=-5y$; $\boxed{\text{Asymptotically Stable}}$

$x=x_0e^{-t}$ $\quad \mathring{y}=y_0e^{-5t}$

f. $\mathring{x}=x$; $\mathring{y}=y$; $\boxed{\text{Asymptotically Stable}}$.

$x=e^{t}$; $y=e^{+t}$

$\mathring{x}=v$; $\mathring{v}=-x$ 5.1.12 v-axis @ $(0,-v_0)$; x-axis @ $(x,0)$; $\boxed{\mathring{v}(0)=-v=v_0 \; ; \; \mathring{x}(x)=0}$

5.1.13 The "saddle point" is a category of bifurcation that is parabolic beyond a coordinate. A connection to real saddles is the "curved" shape where the rider sits

$\mathring{x}=4x-y$
$\mathring{y}=2x+y$

5.2.1 a. $\mathring{x}=A\vec{x}$; $\begin{bmatrix}\mathring{x}\\ \mathring{y}\end{bmatrix}=\begin{bmatrix}4 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$; $\begin{bmatrix}4-\lambda & -1\\ 2 & 1-\lambda\end{bmatrix}=(4-\lambda)(1-\lambda)+2=\lambda^2-5\lambda+6=0$

$\lambda_1=2$; $\lambda_2=3$

$\lambda_1=2$; $\begin{bmatrix}2 & -1\\ 2 & -1\end{bmatrix}\begin{bmatrix}v_{11}\\ v_{12}\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$; $2v_{11}-v_{12}=0$, $v_{11}=1$; $v_{12}=2$; $\vec{v}_1=\begin{bmatrix}1\\ 2\end{bmatrix}$

$\lambda_2=3$; $\begin{bmatrix}1 & -1\\ 2 & -2\end{bmatrix}\begin{bmatrix}v_{21}\\ v_{22}\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$; $v_{21}-v_{22}=0$, $v_{21}=1$; $v_{22}=1$; $\vec{v}_2=\begin{bmatrix}1\\ 1\end{bmatrix}$

b) General Solution: $X(t)=\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}=C_1e^{\lambda_1 t}V_1+C_2e^{\lambda_2 t}V_2=C_1e^{2t}\begin{bmatrix}1\\ 2\end{bmatrix}+C_2e^{3t}\begin{bmatrix}1\\ 1\end{bmatrix}$

$\boxed{x(t)=C_1e^{2t}+C_2e^{3t}}$

$\boxed{y(t)=2C_1e^{2t}+C_2e^{3t}}$

c) $\boxed{\text{Stable}}$

d) $(x_0,y_0)=(3,4)$; $3=C_1+C_2$; $4=2C_1+C_2$

$C_1=3-C_2$; $4=2(3-C_2)+C_2=6-2C_2+C_2$

$=6-C_2$; $C_2=2$; $C_1=1$

$\boxed{x(t)=e^{2t}+2e^{3t}}$

$\boxed{y(t)=2e^{2t}+2e^{3t}}$

$\dot{x} = x - y$
$\dot{y} = x + y$

5.2.2. a) $X = Ax$; $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$; $\begin{bmatrix}1-\lambda & -1\\ 1 & 1-\lambda\end{bmatrix} = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2$

$\lambda_1 = 1 - i$; $\lambda_2 = 1 + i$

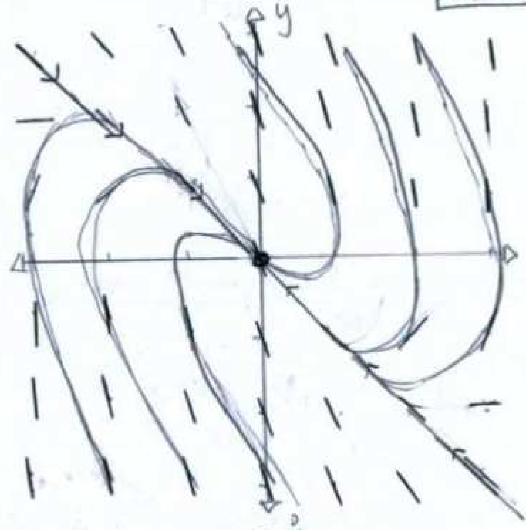
$\lambda_1 = 1 - i$; $\begin{bmatrix}1-(1-i) & -1\\ 1 & 1-(1-i)\end{bmatrix}\begin{bmatrix}V_{11}\\ V_{12}\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$; $iV_{11} - V_{12} = 0$; $\boxed{\vec{V}_1 = \begin{bmatrix}1\\ i\end{bmatrix}}$

$\lambda_2 = 1 + i$; $\begin{bmatrix}1-(1+i) & -1\\ 1 & 1-(1+i)\end{bmatrix}\begin{bmatrix}V_{21}\\ V_{22}\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$; $-iV_{21} - V_{22} = 0$; $\boxed{\vec{V}_2 = \begin{bmatrix}1\\ -i\end{bmatrix}}$

b. General Solution: $\vec{X}(t) = V_1 e^{\lambda_1 t} + V_2 e^{\lambda_2 t i} = \boxed{\begin{bmatrix}x(t) = e^t \cdot 2\cos(t)\\ y(t) = e^t \cdot 2i\sin(t)\end{bmatrix}}$

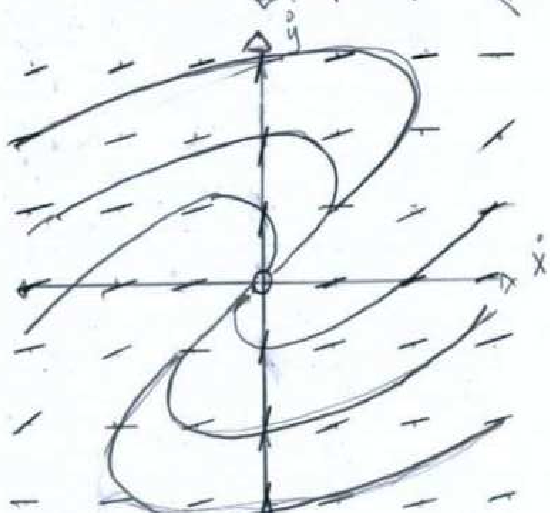
$\dot{x} = y$
$\dot{y} = -2x - 3y$

5.2.3. $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = -2 \cdot \frac{x}{y} - 3$



$\dot{x} = 5x + 10y$
$\dot{y} = -x - y$

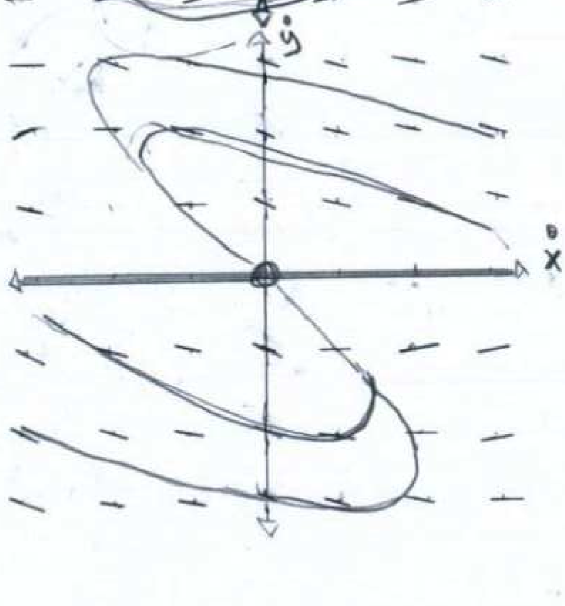
5.2.4 $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{-x-y}{5x+10y}$



Switched

$\dot{x} = 3x - 4y$
$\dot{y} = x - y$

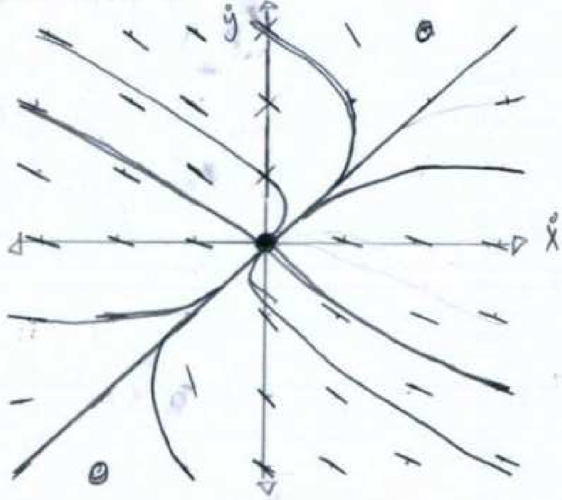
5.2.5. $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{x-y}{3x-4y}$



$\dot{x} = -3x + 2y$

$\dot{y} = x - 2y$

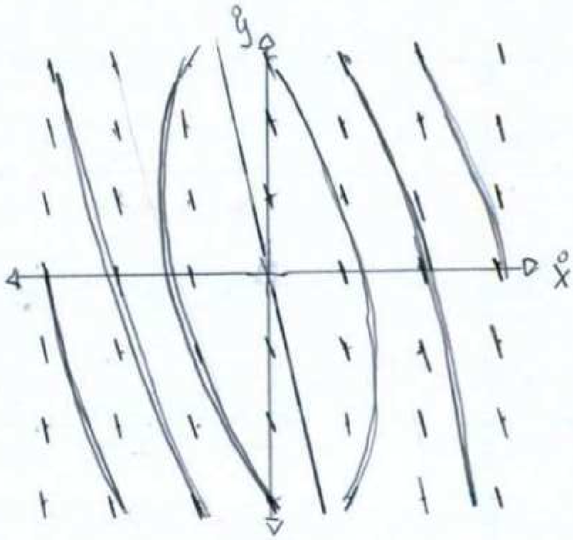
5.2.6 $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{x - 2y}{-3x + 2y}$



$\dot{x} = 5x + 2y$

$\dot{y} = -17x - 5y$

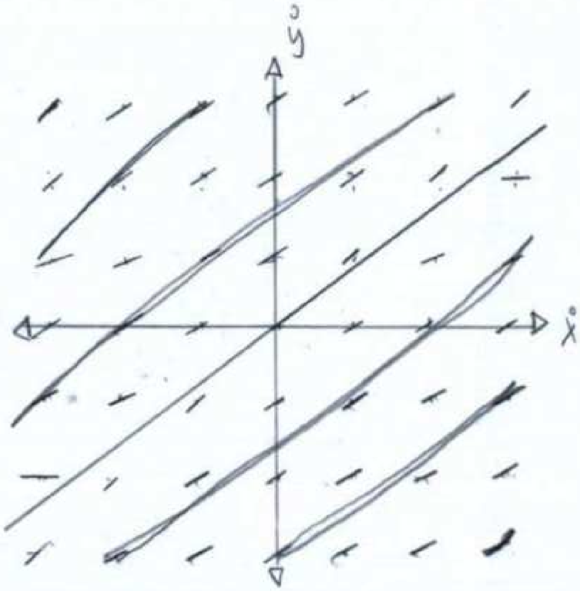
5.2.7. $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{-17x - 5y}{5x + 2y}$



$\dot{x} = -3x + 4y$

$\dot{y} = -2x + 3y$

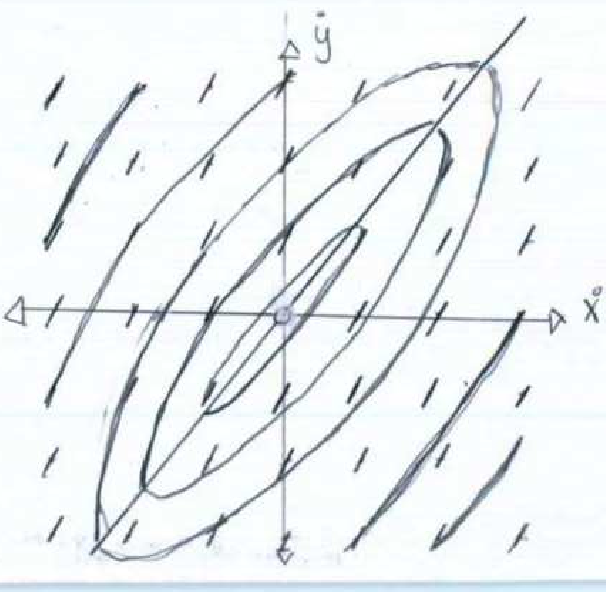
5.2.8. $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{-2x + 3y}{-3x + 4y}$



$\dot{x} = 4x - 3y$

$\dot{y} = 8x - 6y$

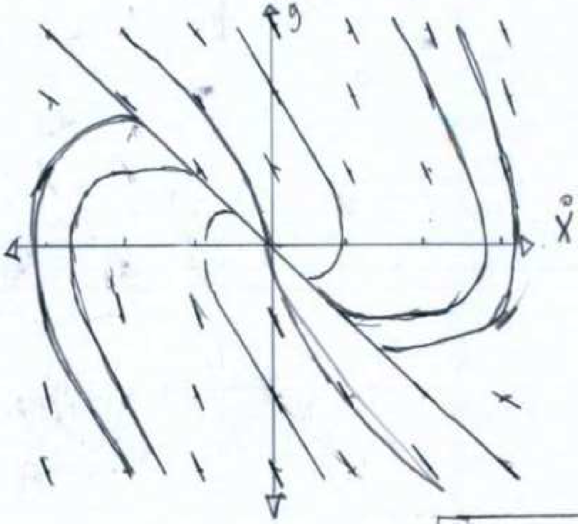
5.2.9. $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{8x - 6y}{4x - 3y}$



$\dot{x} = y$

$\dot{y} = -x-2y$

5.2.10. $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{-x-2y}{y}$

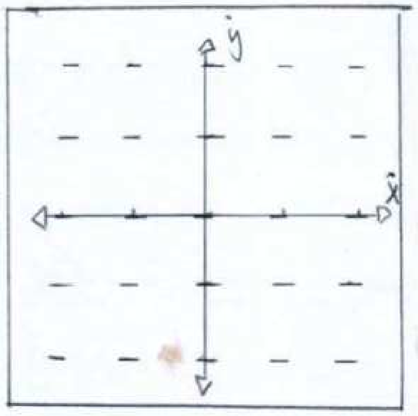


$A = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}$

5.2.11 $A - \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix} = \lambda^2 = 0 ; \lambda = 0$

$\dot{x} = Ax ; \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \lambda & b \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} ; \dot{x} = by ; \dot{y} = 0 ; \frac{\dot{y}}{\dot{x}} = 0$

The book shows a typeset to the correct solution.



$L\ddot{I} + R\dot{I} + \frac{I}{C} = 0$

5.2.12

a) $x = I \quad y = \dot{I}$

$\dot{x} = \dot{I} \quad \dot{y} = \ddot{I} = -R\dot{I} - \frac{I}{C} = -Ry - \frac{x}{C} ; \boxed{\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{C} & -R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}$

b) $R = 0 ; \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/C & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} ; \begin{bmatrix} -\lambda & 1 \\ -1/C & -\lambda \end{bmatrix} = \lambda^2 + \frac{1}{C} = 0 ; \lambda_{1,2} = \frac{i}{\sqrt{C}}$

$\lambda_{1,2} = \frac{i}{\sqrt{C}} ; \begin{bmatrix} -i/C & 1 \\ -1/C & -i/\sqrt{C} \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = 0 ; \frac{V_{11}}{C} + \frac{iV_{12}}{\sqrt{C}} = 0 ; V_{11} = -i\sqrt{C} ; V_{12} = 1$

General Solution: $e^{\lambda_{1,2} t} = \cos\left(\frac{t}{\sqrt{C}}\right) + i\sin\left(\frac{t}{\sqrt{C}}\right)$

$$e^{\lambda_{1,2} t} \cdot \vec{V} = \begin{bmatrix} \sqrt{C}\sin(t/\sqrt{C}) - i\sqrt{C}\cos(t/\sqrt{C}) \\ \cos(t/\sqrt{C}) + i\sin(t/\sqrt{C}) \end{bmatrix}$$

$$X = \begin{bmatrix} x = C_1\sin(t/\sqrt{C}) - C_2\cos(t/\sqrt{C}) \\ y = C_1\cos(t/\sqrt{C}) + C_2\sin(t/\sqrt{C}) \end{bmatrix}$$

$\boxed{\text{Neutrally Stable}}$

$R > 0 ; \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{C} & -R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} ; \begin{bmatrix} -\lambda & 1 \\ -1/C & -R-\lambda \end{bmatrix} = \lambda(R+\lambda) + \frac{1}{C} = 0 ; \lambda_{1,2} = \frac{-R \pm \sqrt{R^2 - 4(1)(1/C)}}{2(1)}$

$\lambda_1 = \frac{-R \pm \sqrt{R^2 - 4(1)(1/C)}}{2} ; \begin{bmatrix} \frac{R - \sqrt{R^2 - 4/C}}{2} & 1 \\ -1/C & \frac{-R - \sqrt{R^2 - 4/C}}{2} \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = 0$

$$\frac{R-\sqrt{R^2-4/C}}{2}\cdot v_{11} + v_{12} = 0 \; ; \quad v_{11} = 1 \; ; \; v_{12} = -\frac{R+\sqrt{R^2-4/C}}{2}$$

$$\lambda_2 = \frac{-R-\sqrt{R^2-4/C}}{2} \; ; \quad \begin{bmatrix} \frac{R+\sqrt{R^2-4/C}}{2} & 1 \\ -1/C & \frac{-R+\sqrt{R^2-4/C}}{2} \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$\frac{R+\sqrt{R^2-4/C}}{2}\cdot v_{11} + v_{12} = 0 \; ; \quad v_{11} = 1 \; ; \; v_{12} = \frac{+R+\sqrt{R^2-4/C}}{2}$$

General Solution:

$$X_i = C_i e^{\lambda_i t} V_i$$

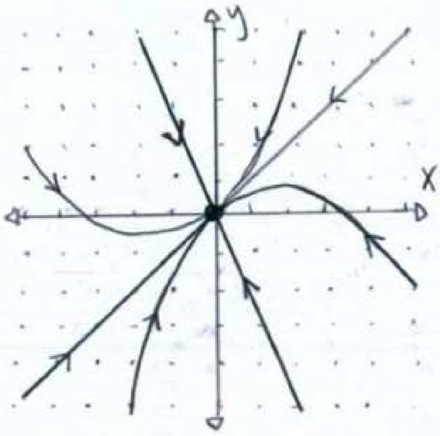
$$X_1 = C_1 e^{\frac{-R+\sqrt{R^2-4/C}}{2}t} \begin{bmatrix} 1 \\ -\frac{R+\sqrt{R^2-4/C}}{2} \end{bmatrix}$$

$$X_2 = C_2 \cdot e^{\frac{-R-\sqrt{R^2-4/C}}{2}t} \begin{bmatrix} 1 \\ \frac{R+\sqrt{R^2-4/C}}{2} \end{bmatrix}$$

$$\bar{X} = X_1 + X_2 = \begin{bmatrix} x(t) = C_1 e^{\frac{-R+\sqrt{R^2-4/C}}{2}t} + C_2 e^{\frac{-R-\sqrt{R^2-4/C}}{2}t} \\ y(t) = C_1 e^{\frac{-R+\sqrt{R^2-4/C}}{2}t}\cdot\left(-\frac{R+\sqrt{R^2-4/C}}{2}\right) + C_2 e^{\frac{-R-\sqrt{R^2-4/C}}{2}t}\cdot\left(\frac{R+\sqrt{R^2-4/C}}{2}\right) \end{bmatrix}$$

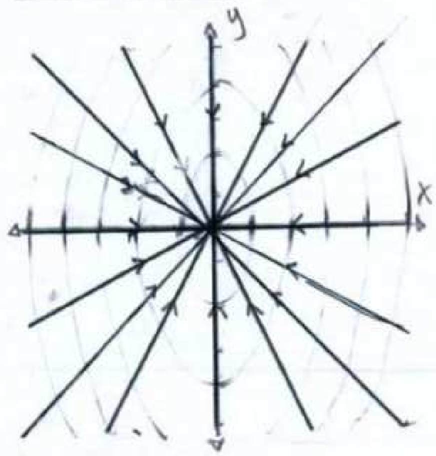
$\boxed{\text{Asymptotically Stable}}$

C. $R^2C-4L>0$



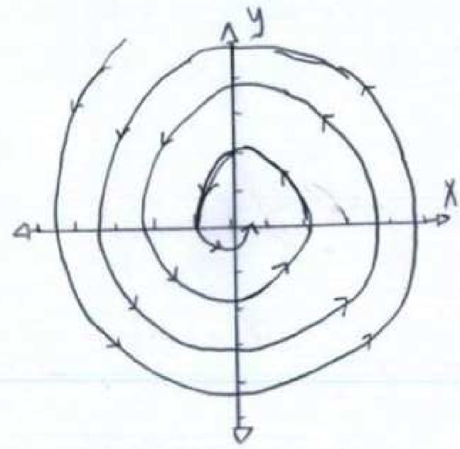
Unstable Node

$R^2C-4L=0$



Star, Degenerate Node

$R^2C-4L<0$



Unstable Spiral

$R^2C > 4L$

$m\ddot{x} + b\dot{x} + kx = 0$ 5.2.13:

a. $i = x \quad j = \dot{x}$

$\dot{i} = \dot{x} \quad \dot{j} = \ddot{x} = -\frac{b\dot{x}}{m} - \frac{kx}{m} = -\frac{b}{m}j - \frac{k}{m}i$; $\boxed{\begin{bmatrix} \dot{i} \\ \dot{j} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{b}{m} & -\frac{k}{m} \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix}}$

b. $\dot{\vec{I}} = A\vec{i}$; $\begin{bmatrix} -\lambda & 1 \\ -\frac{b}{m} & -\frac{k}{m} - \lambda \end{bmatrix} = \lambda^2 + \frac{k}{m}\lambda + \frac{b}{m} = 0$; $\lambda_{1,2} = \dfrac{-\frac{k}{m} \pm \sqrt{\left(\frac{k}{m}\right)^2 - 4(1)(b/m)}}{2(1)}$

$$\lambda_1 = \frac{-\frac{k}{m} + \sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)}}{2} ; \begin{bmatrix} \dfrac{\frac{k}{m} - \sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)}}{2} & 1 \\ -\dfrac{b}{m} & \dfrac{-\frac{k}{m} - \sqrt{(k/m)^2 - 4(b/m)}}{2} \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = 0$$

$$\left(\frac{k}{m} - \sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)}\right) V_{11} + V_{12} = 0$$

$$V_{11} = 1 ; V_{12} = -\frac{k}{m} + \sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)}$$

$$\lambda_2 = \frac{-\frac{k}{m} - \sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)}}{2} ; \begin{bmatrix} \dfrac{\frac{k}{m} + \sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)}}{2} & 1 \\ -\dfrac{b}{m} & \dfrac{-\frac{k}{m} + \sqrt{(k/m)^2 - 4(b/m)}}{2} \end{bmatrix} \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = 0$$

$$\frac{\left(\frac{k}{m}\right) + \sqrt{(k/m)^2 - 4(b/m)}}{2} V_{21} + V_{22} = 0$$

$$V_{21} = 1 ; V_{22} = \frac{-\left(\frac{k}{m}\right) - \sqrt{(k/m)^2 - 4(b/m)}}{2}$$

$$x(t) = C_1 e^{\frac{-\left(\frac{k}{m}\right) + \sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)}}{2} t} + C_2 e^{\frac{-\left(\frac{k}{m}\right) - \sqrt{(k/m)^2 - 4(b/m)}}{2} t}$$

$$y(t) = C_1 \cdot \frac{\left(\frac{-k}{m}\right) + \sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)}}{2} \cdot e^{\frac{-\left(\frac{k}{m}\right) + \sqrt{(k/m)^2 - 4(b/m)}}{2} t} + C_2 \frac{\left(\frac{-k}{m}\right) - \sqrt{(k/m)^2 - 4(b/m)}}{2} \cdot e^{\frac{-\left(\frac{k}{m}\right) - \sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)}}{2} t}$$

Unstable Spiral : $\sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)} < 0$

Star, Degenerate Node : $\sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)} = 0$

Unstable Node ; $\sqrt{\left(\frac{k}{m}\right)^2 - 4(b/m)} > 0$

c. Star, Degenerate Node is critically damped. An unstable spiral is underdamped. While an unstable node is an unstable node.

$\dot{x} = Ax$ 5.2.14 $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$;

$\lambda_2 < \lambda_1 < 0$: Stable Node

$\lambda_{1,2} = \kappa \pm i\omega < 0$: Stable Spiral

$\lambda_1 = \lambda_2 = \lambda$: Star Node, Degenerate Node

$\lambda_{1,2} = \kappa + i\omega > 0$: Unstable Spiral

$\lambda_2 > \lambda_1 > 0$: Unstable Node

$$A = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

$$\lambda_{1,2} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

Pseudo-code

If $(a+d)^2 - 4(ad-bc) < 0$, then λ_1 & λ_2 are imaginary.

If $(a+d) > 0$, then λ_1, λ_2 are an unstable spiral.

else, λ_1, λ_2 are a stable spiral.

Else

Doub $\lambda_1 = \dfrac{(a+d) + \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

Doub $\lambda_2 = \dfrac{(a+d) - \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

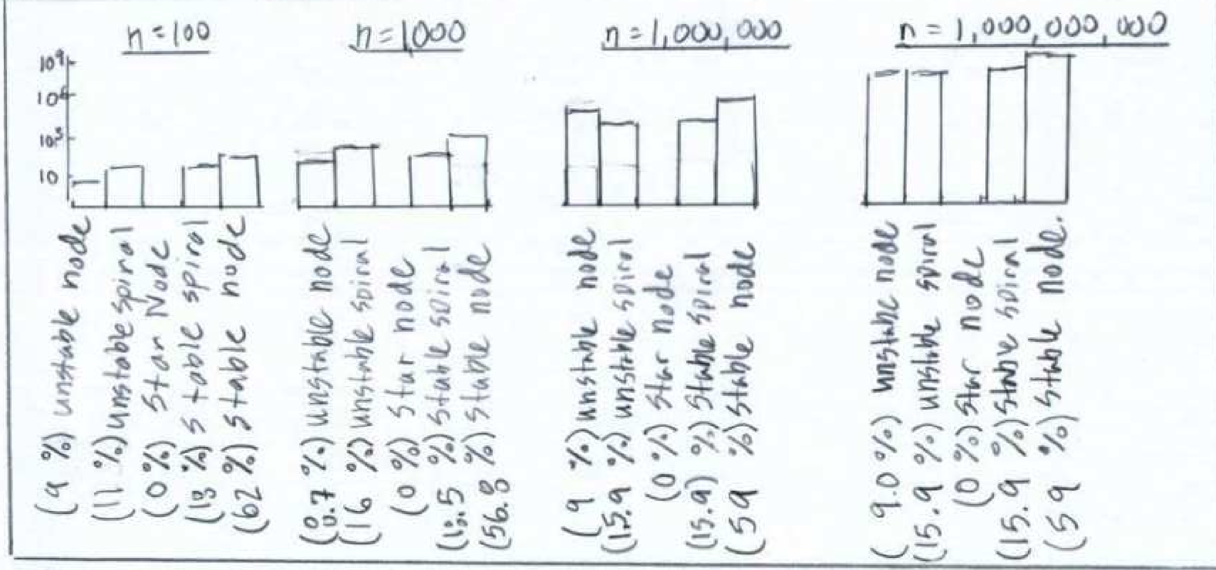
If $(\lambda_1 = \lambda_2)$, then Star Node, Degenerate Node

Else if $(\lambda_1$ & $\lambda_2 > 0)$, then Unstable Node

else, Stable node.

Real-code

```
#include <iostream>
#include <random>
using namespace std;
int main() {
  int i, s_node = 0, u_node = 0, star = 0, s_spiral = 0, u_spiral = 0, trials = 100;
  double l1, l2, a, b, c, d;
  std::default_random_engine_generator;
  std::uniform_real_distribution(-1,1);
  if(((a+d)-4(ad-bc))<0){
     if((a+d)>0) u_spiral+=1;
     else s_spiral+=1;
  }else{
    l1=((a+d)+sqrt(pow((a+d),2)-4(ad-bc)))/2;
    l2=((a+d)-sqrt(pow((a+d),2)-4(ad-bc)))/2;
    if(l1=l2) star+=1
    else if(l1&&l2>0) u_node+=1
    else s_node+=1;
  }
} }
```



An unstable spiral approaches the limit of 9%; while, stable spirals the most common, at 59%.
A normal distribution produced greater proportions of the stable phase plots (stable node [83%], stable spiral [14.9%], ...) than the uniform distribution modelled.

$\dot{R} = aR + bJ$ 5.3.1. R = Romeo's love/hate; J = Juliet's love/hate; a = b = romantic style.

$b \backslash a$	(+)	(−)
(+)	Star crossed romance	Cautious romance
(−)	Indifferent romance	Sensitive romance

$\dot{R} = J$

$\dot{J} = -R + J$

5.3.2.
a. $\begin{bmatrix} \dot{R} \\ \dot{J} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} R \\ J \end{bmatrix}$ "cautious romance"

b. $(R, J) = (0,0)$ = Neverending love/hate = Stable Node

c. $R(0) = 1$; $J(0) = 0$;

$$\begin{bmatrix} 0-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix} = -\lambda(1-\lambda) + 1 = \lambda^2 - \lambda + 1 = 0$$

$$\lambda_{1,2} = \frac{+1 \pm \sqrt{1 - 4(1)(1)}}{2} = \frac{1 \pm \sqrt{3}i}{2}$$

$$\lambda_{1,2} = \frac{1+\sqrt{3}i}{2}; \quad \begin{bmatrix} \frac{-1-\sqrt{3}i}{2} & 1 \\ -1 & -\frac{1-\sqrt{3}i}{2} \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad \frac{-1+\sqrt{3}i}{2} v_{11} + v_{12} = 0; \quad \vec{V}_{1,2} = \begin{bmatrix} 1 \\ +\frac{1+\sqrt{3}i}{2} \end{bmatrix}$$

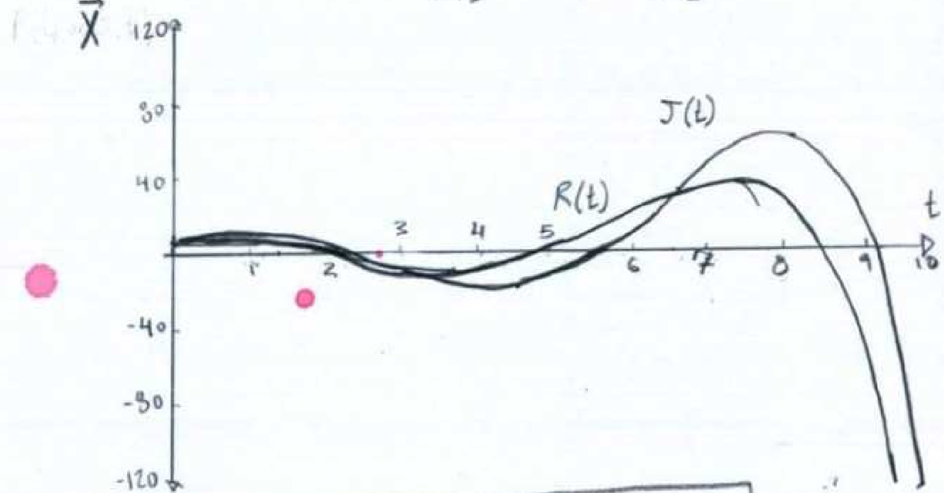
General Solution:
$$\vec{X} = \begin{bmatrix} R(t) = e^{\frac{t}{2}} \cdot 2\left[C_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right] \\ J(t) = e^{\frac{t}{2}} \cdot (1+\sqrt{3}i) \cdot \left[C_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right] \end{bmatrix}$$

$$\vec{X} = \begin{bmatrix} R(t) = 2e^{\frac{t}{2}} \cdot \left[C_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}t\right)\right] \\ J(t) = (1+\sqrt{3}) e^{\frac{t}{2}} \cdot C_1 \cos\left(\frac{\sqrt{3}}{2}t\right) + (1-\sqrt{3}) e^{\frac{t}{2}} C_2 \sin\left(\frac{\sqrt{3}}{2}t\right) \end{bmatrix}$$

$R(0) = 1$; $C_1 = \frac{1}{2}$

$J(0) = 0$; $0 = C_1 + \sqrt{3} C_2$; $C_2 = \frac{-1}{2\sqrt{3}}$

Final Solution: $\vec{X} = \begin{bmatrix} R(t) = e^{t/2}\left[\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right] \\ J(t) = e^{t/2}\left[\frac{(1+\sqrt{3})}{2}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{(1-\sqrt{3})}{2\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right] \end{bmatrix}$



$\dot{R} = aJ$ 5.3.3.
$\dot{J} = bR$

b \ a	(+)	(−)
(+)	$a^2>b^2$: Stable $a^2<b^2$: Unstable $a=b$: Mutual	Stable center of neverending love & hate
(−)	Stable center of neverending love & hate	$a^2>b^2$: Stable $a^2<b^2$: Unstable $a=b$: Mutual

Romeo and Juliet live happily when Juliet's love is of greater amounts.

$\dot{R} = aR+bJ$ 5.3.4.
$\dot{J} = -bR-aJ$

b \ a	(+)	(−)
(+)	$a^2>b^2$: Unstable $a^2<b^2$: Stable $a=b$: Star, degen node	$a^2>b^2$: Unstable $a^2<b^2$: Stable $a=b$: Star, degen node
(−)	$a^2>b^2$: Unstable $a^2<b^2$: Stable $a=b$: Star, degen node	$a^2>b^2$: Unstable $a^2<b^2$: Stable $a=b$: Star, degen node

Yes, opposites attract when the proportion of Romeo's love is larger.

$\dot{R} = aR+bJ$ 5.3.5.
$\dot{J} = bR+aJ$

b \ a	(+)	(−)
(+)	$a^2>b^2$: Unstable Node $a^2\le b^2$: Unstable Saddle	$a^2>b^2$: Unstable Node $a^2\le b^2$: Unstable Saddle
(−)	$a^2>b^2$: Unstable Node $a^2\le b^2$: Unstable Saddle	$a^2>b^2$: Unstable Node $a^2\le b^2$: Unstable Saddle

The marriage of Romeo and Juliet of exact clone demonstrate an unstable relationship for all time.

$\mathring{R} = 0$ 5.3.6.

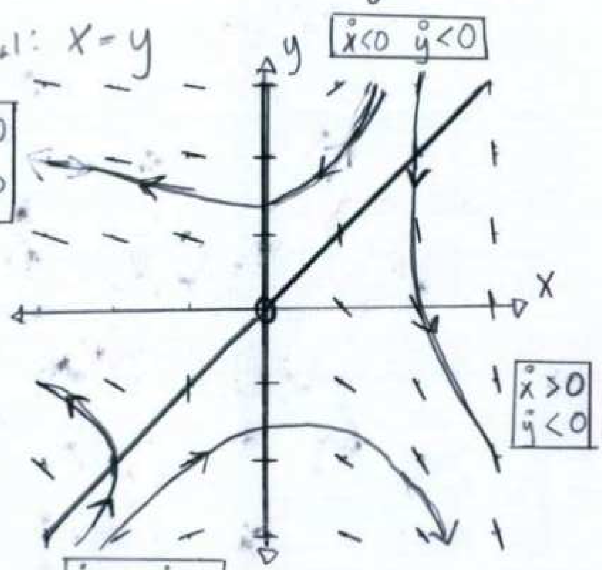
$\mathring{J} = aR + bJ$

b \ a	(+)	(−)
(+)	Unstable and Fixed Relationship	$a^2 > b^2$: Stable $a^2 < b^2$: Unstable $a = b$: Isolated
(−)	$a^2 > b^2$: Unstable $a^2 < b^2$: Stable $a = b$: Isolated.	Stable and Fixed Relationship

Chapter 6: Phase Plane:

$\mathring{x} = x - y$ 6.1.1: $\boxed{\text{Fixed Points}}$: $\mathring{x} = 0 = x - y$; $x = y$; $\mathring{y} = 0 = 1 - e^x$; $x = 0$; $(x^*, y^*) = (0,0)$

$\mathring{y} = 1 - e^x$ $\boxed{\text{Nullclines}}$: Horizontal: $x = 0$; Vertical: $x = y$



$\mathring{x} = x - x^3$ 6.1.2: $\boxed{\text{Fixed Points}}$: $\mathring{x} = 0 = x - x^3$

$\mathring{y} = -y$

$x^* = 1, 0, -1$

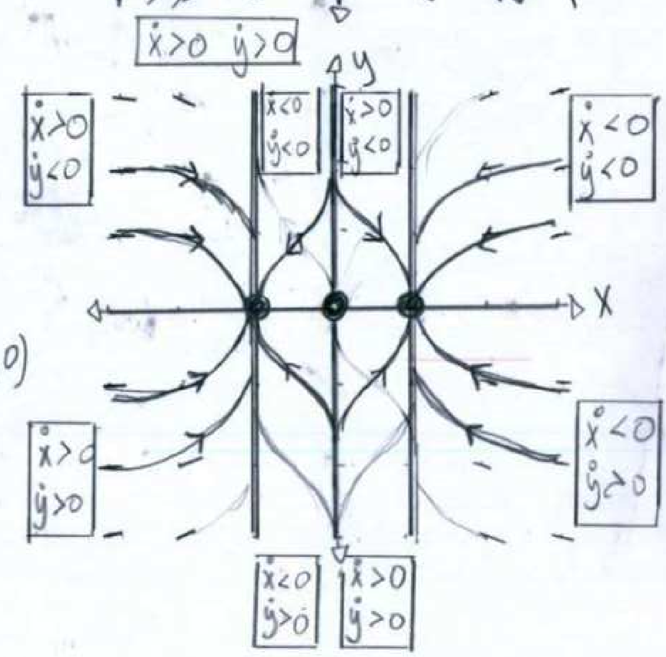
$\mathring{y} = 0 = -y$

$y^* = 0$

$(x^*, y^*) = (1,0), (0,0), (-1,0)$

$\boxed{\text{Nullclines}}$: Horizontal: $y = 0$

Vertical: $x = 0, 1, -1$



$\dot{x} = x(x-y)$

$\dot{y} = y(2x-y)$

6.1.3. $\boxed{\text{Fixed Points}}$: $\dot{x} = 0 = x(x-y)$

$\dot{y} = 0 = y(2x-y)$

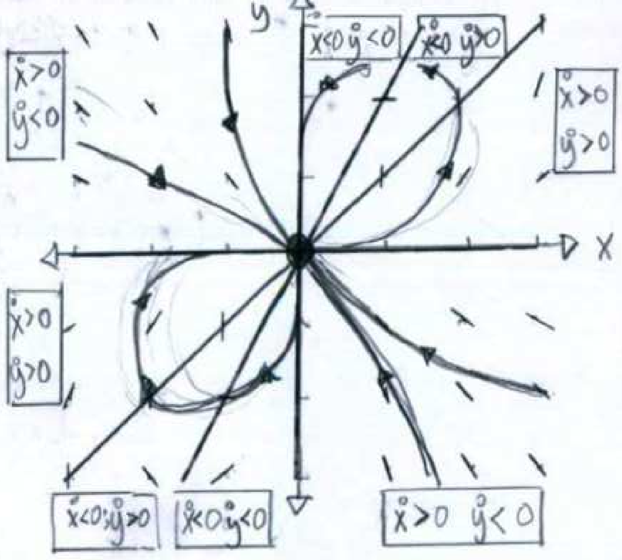
$(x^*, y^*) = (0,0)$

$\boxed{\text{Nullclines}}$: Horizontal: $y = 2x$

$y = 0$

Vertical: $y = x$

$x = 0$



$\dot{x} = y$

$\dot{y} = x(1+y) - 1$

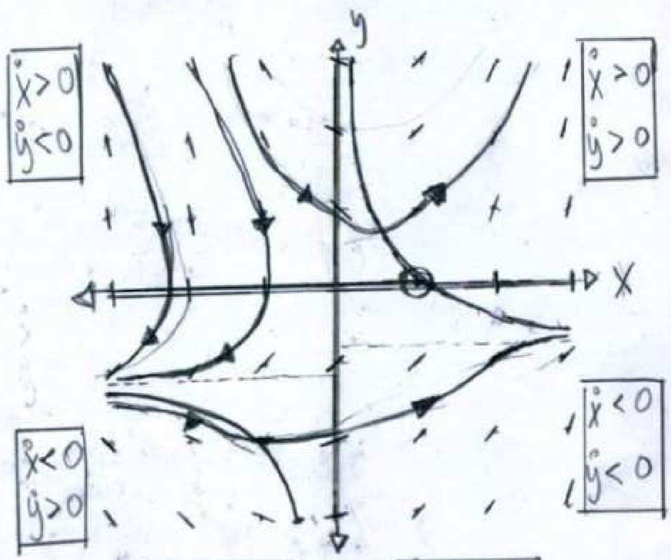
6.1.4: $\boxed{\text{Fixed Points}}$: $\dot{x} = 0 = y$

$\dot{y} = 0 = x(1+y) - 1$

$(x^*, y^*) = (1, 0)$

$\boxed{\text{Nullclines}}$: Horizontal: $y = \frac{1}{x} + 1$

Vertical: $y = 0$



$\dot{x} = x(2-x-y)$

$\dot{y} = x - y$

6.1.5. $\boxed{\text{Fixed Points}}$: $\dot{x} = 0 = x(2-x-y)$

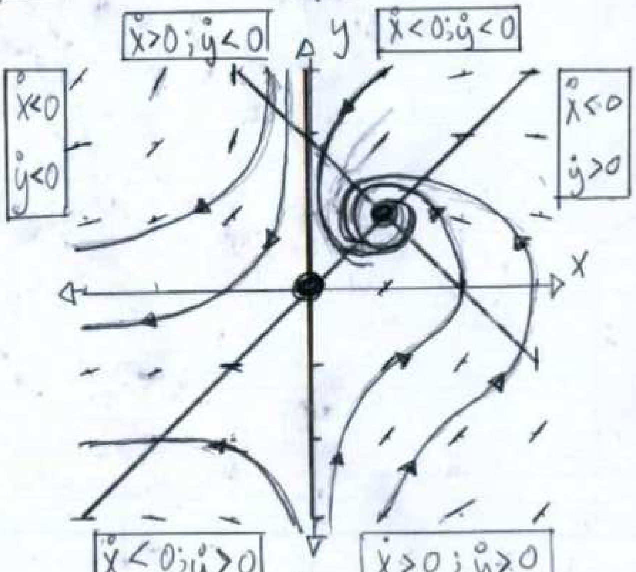
$\dot{y} = 0 = x - y$

$(x^*, y^*) = (0,0), (1,1)$

$\boxed{\text{Nullcline}}$: Horizontal: $y = x$

Vertical: $x = 0$

$y = 2 - x$



$\dot{x} = x^2 - y$

$\dot{y} = x - y$

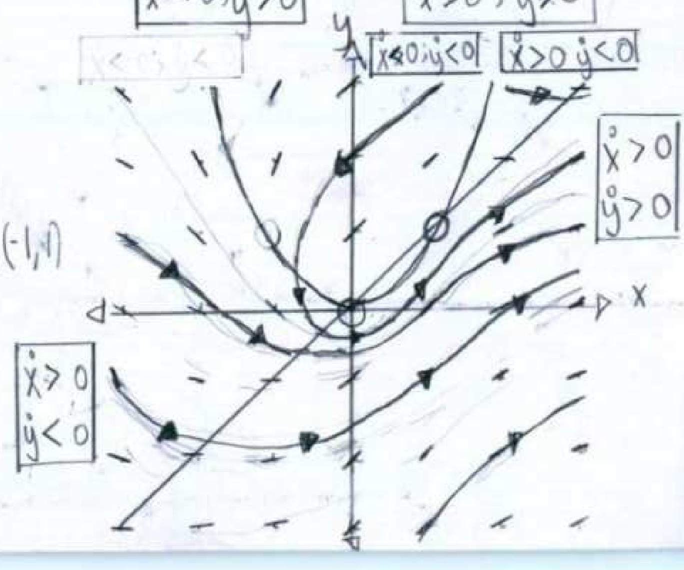
6.1.6. $\boxed{\text{Fixed Points}}$: $\dot{x} = 0 = x^2 - y$

$\dot{y} = 0 = x - y$

$(x^*, y^*) = (0,0), (1,1), (-1,1)$

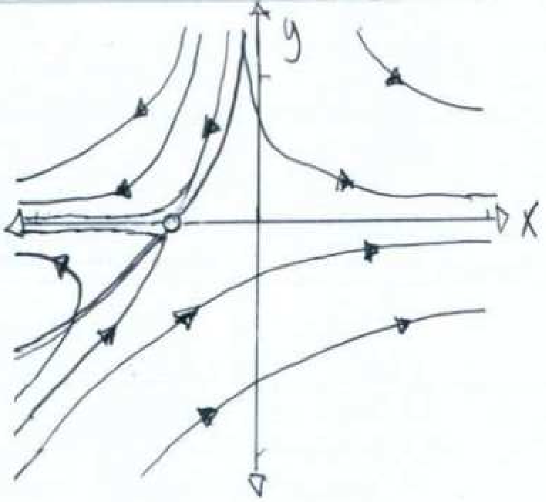
$\boxed{\text{Nullcline}}$: Horizontal: $y = x$

Vertical: $y = x^2$



$\dot{x} = x + e^{-y}$ 6.1.7

$\dot{y} = -y$



$\dot{x} = y$

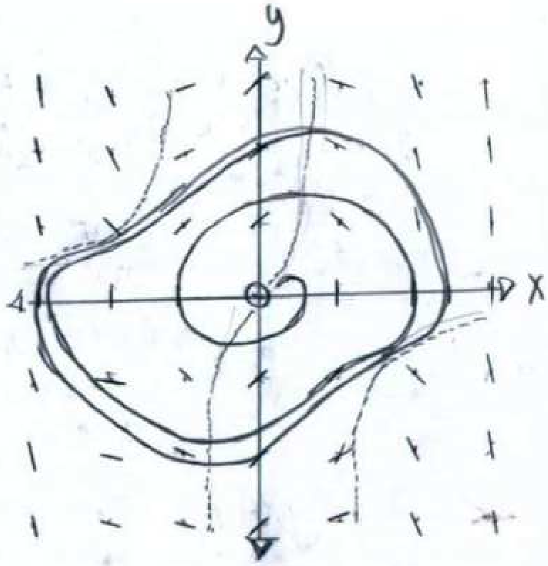
$\dot{y} = -x + y(1-x^2)$ 6.1.8 (Van der Poll Oscillator)

$\boxed{\text{Fixed points}}$ $\dot{x} = 0 = y$

$\dot{y} = 0 = -x + y(1-x^2)$

$(x^*, y^*) = (0,0)$

$\boxed{\text{Nullclines}}$ $y = \dfrac{x}{1-x^2}$



$\dot{x} = 2xy$ 6.1.9. (Dipole Fixed Point)

$\dot{y} = y^2 - x^2$

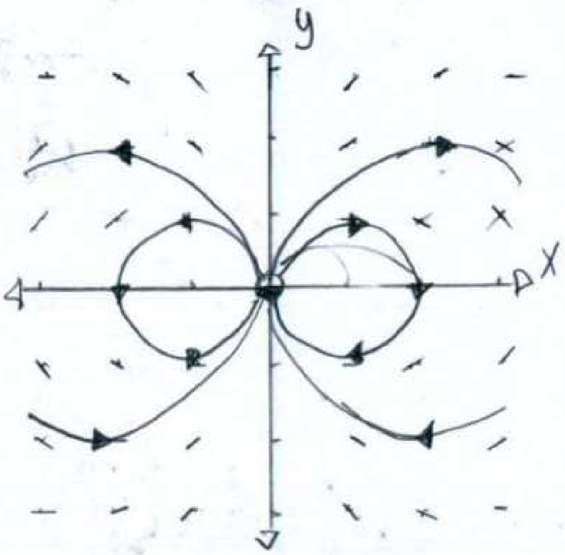
$\boxed{\text{Fixed Point}}$ $\dot{x} = 0 = 2xy$

$\dot{y} = 0 = y^2 - x^2$

$(x^*, y^*) = (0,0)$

$\boxed{\text{Nullcline}}$ $y = x$; $y = 0$; $x = 0$.

$y = -x$



$\dot{x} = y + y^2$ 6.1.10

$\dot{y} = -\frac{1}{2}x + \frac{1}{5}y - xy + \frac{6}{5}y^2$ (Two-eyed Monster)

$\boxed{\text{Fixed Points}}$

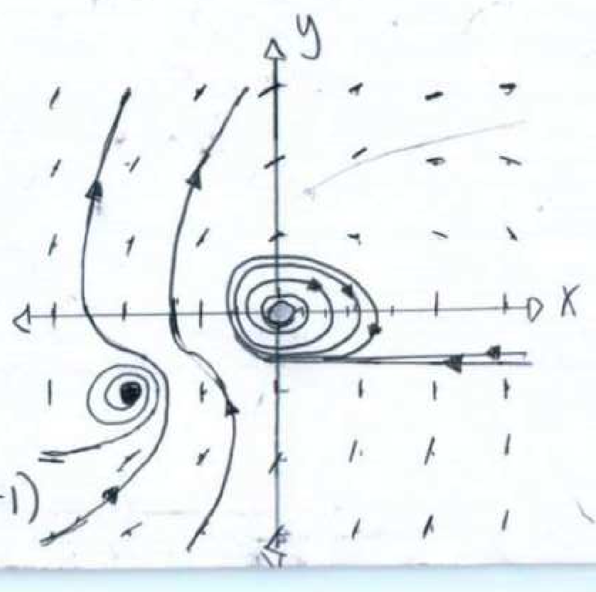
$\dot{x} = 0 = y + y^2$

$\dot{y} = 0 = -\frac{1}{2}x + \frac{1}{5}y - xy + \frac{6}{5}y^2$

$(x^*, y^*) = (0,0), (-2,-1)$

$\boxed{\text{Nullclines}}$ $y = 0$; $x = 0$

$y = \frac{1}{12}\left(\sqrt{25x^2 + 50x + 1} + 5x - 1\right)$



$\dot{x} = y + y^2$

$\dot{y} = -x + \frac{1}{5}y - xy + \frac{6}{5}y^2$

6.1.11 $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = y + y^2$

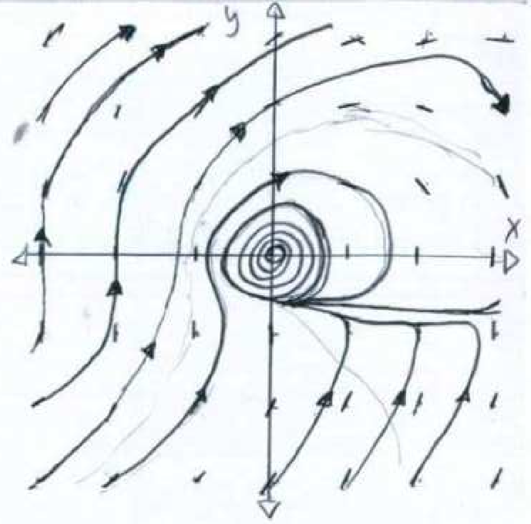
$\dot{y} = 0$

$= -x + \frac{1}{5}y - xy + \frac{6}{5}y^2$

$(x^*, y^*) = (0,0), (4,4)$

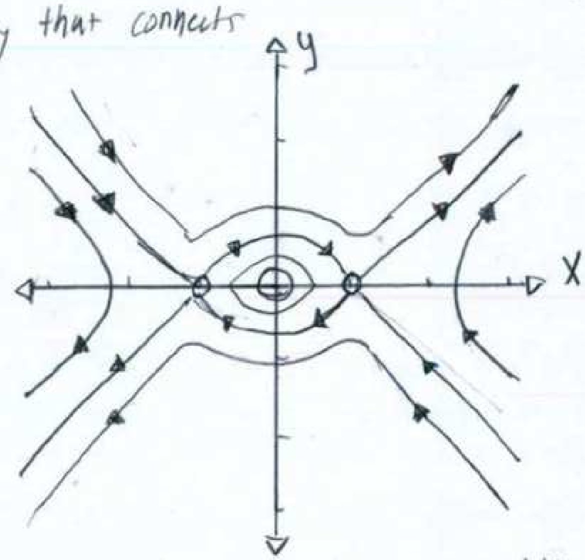
$\boxed{\text{Nullcline}}$

$y = \frac{1}{12}\left(\pm\sqrt{25x^2 + 110x + 1} + 5x - 1\right)$

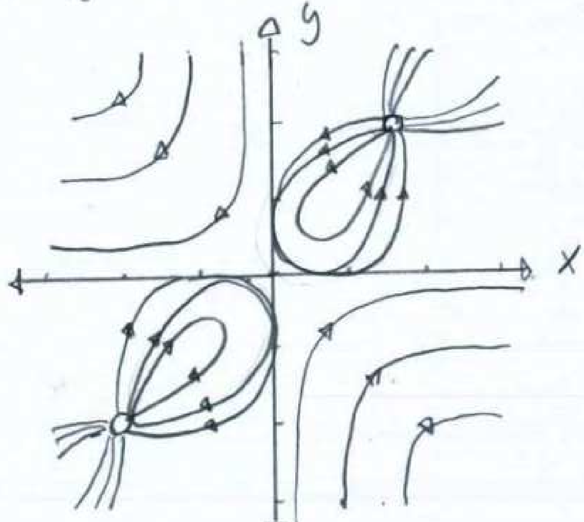


6.1.12.

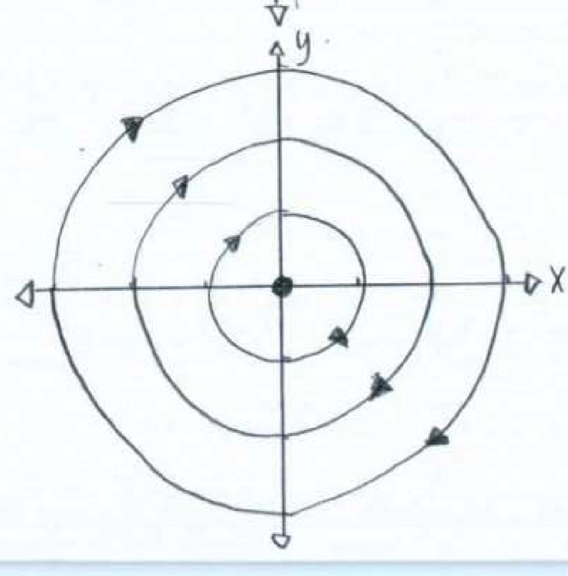
a. a single trajectory that connects the saddles.



b. there is no trajectory that connects the saddles



6.1.13: A phase portrait with three closed orbits and one fixed point.



$\dot{x} = x + e^{-y}$ 6.1.14.

$\dot{y} = -y$

a. $u = x+1$; $\frac{dy}{du} = \frac{dy}{dt}\frac{du}{dt} = -y$; $\frac{du}{dt} = \frac{dx}{dt} = u - 1 + (1 - y + \frac{y^2}{2} - \cdots)$

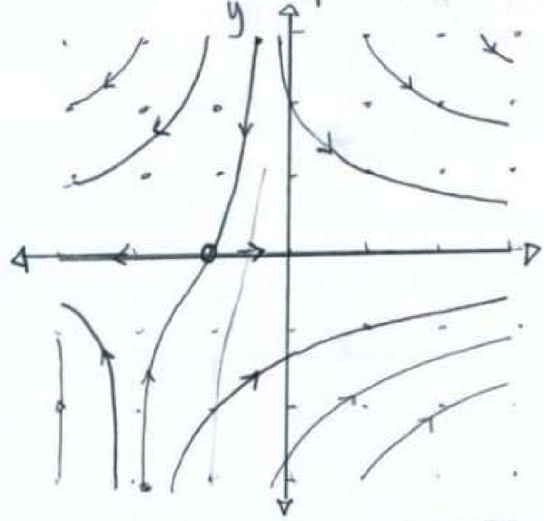
$$\frac{dy}{du} = -\frac{y}{u - y + y^2/2 - y^3/6 + \cdots}$$

$$x = \frac{a_1}{a_1 - 1} + \frac{a_1^3 - 2a_2}{2(a_1-1)^2}u + \frac{2a_1^4 + a_1^5 - 13a_1^2 a_2 + 12(a_2^2 + a_3 - a_1 a_3)}{12(a_1-1)^3}u^3$$

$$\boxed{= a_1 + 2a_2 u + 3a_3 u^3 + \cdots}$$

$a_1 = 2$; $a_2 = 4/3$; $a_3 = 10/9$

b.



$y = \frac{4}{3}(x+1) + \frac{10}{9}(x+1)^3$

Note: The function plotted is not exact.

6.2.1: Yes, trajectories do not intersect, however, may seem so for low resolution plots.

$\dot{x} = y$

$\dot{y} = -x + (1 - x^2 - y^2)y$

6.2.2: a. $D: x^2 + y^2 < 4$

Poincare - Bendixon Theorem : no fixed points and a bounded region, then the trajectory is a closed orbit, and approaches the closed orbit.

Bounded Region - $D: x^2 + y^2 < 4$

Fixed points: Zero, outside of the center

Existness and Uniqueness is satisfied for a Closed orbit.

b. If $y(t) = \cos(t)$, then $\dot{y} = 0 = -x + (1 - (x^2 + \cos(t)^2))y$ @ $t = 0$

Identity: $x^2 + y^2 = 1$

then, $x = 0$ @ $t = 0$

$\boxed{x(t) = \sin(t)}$

c. $x(0)=\frac{1}{2}$; $y(0)=0$; $x(t)^2+y(t)^2$ must be less than one because a larger value forces $\dot{y}$ to become negative and not a closed orbit.

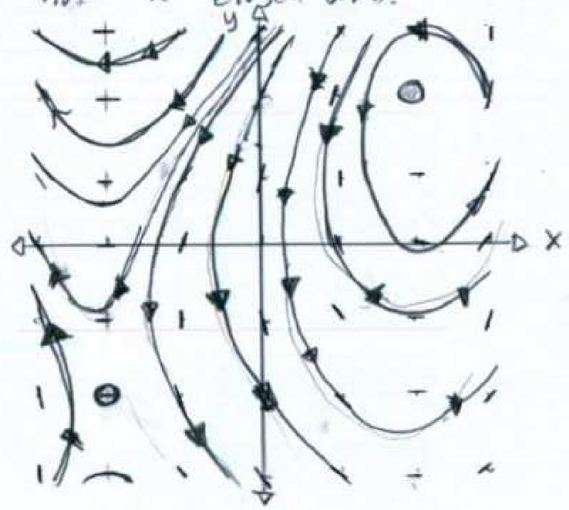
$\dot{x}=x-y$
$\dot{y}=x^2-4$

6.3.1 $\boxed{\text{Fixed points}}$ $\dot{x}=0=x-y$

$\dot{y}=0=x^2-4$

$(x^*,y^*)=(2,2),(-2,-2)$

"unstable" "unstable"



$\dot{x}=\sin y$
$\dot{y}=x-x^3$

6.3.2 $\boxed{\text{Fixed points}}$ $\dot{x}=0=\sin y$

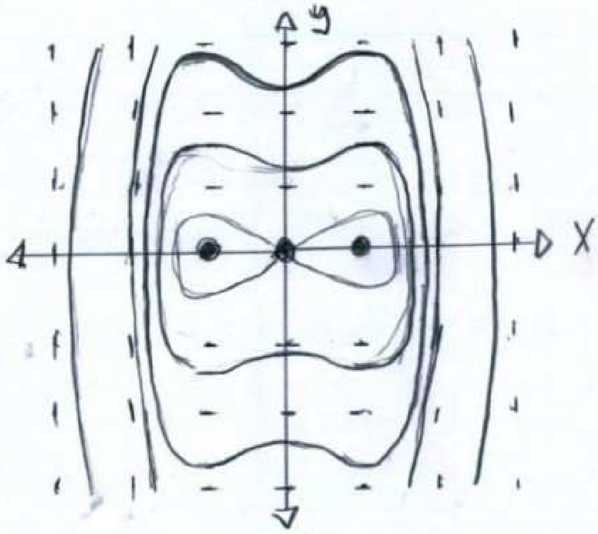
$\dot{y}=0=x-x^3$

$(x^*,y^*)=(1,n\pi)$

$(-1,n\pi)$

$(0,n\pi)$

"stable"



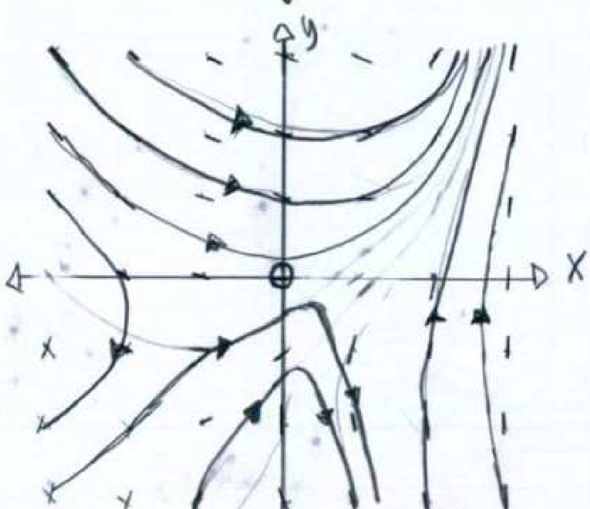
$\dot{x}=1+y+e^{-x}$
$\dot{y}=x^3-y$

6.3.3 $\boxed{\text{Fixed Points}}$ $\dot{x}=1+y-e^{-x}=0$

$\dot{y}=x^3-y=0$

$(x^*,y^*)=(0,0)$

"unstable"



$\dot{x}=y+x-x^3$
$\dot{y}=-y$

6.3.4: $\boxed{\text{Fixed Points}}$ $\dot{x}=0=y+x-x^3$

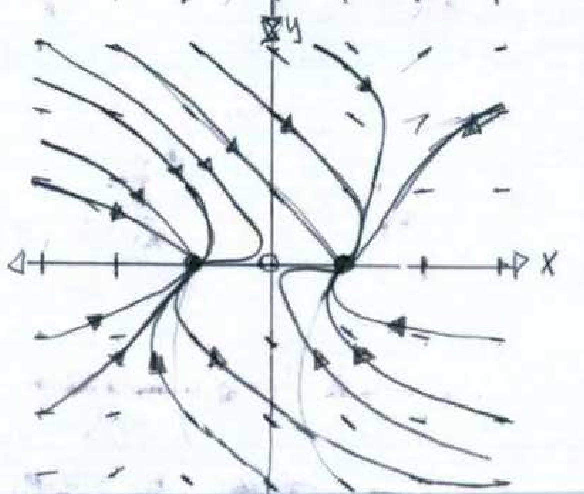
$\dot{y}=0=-y$

$(x^*,y^*)=(1,0),(-1,0)$

"stable"

$(0,0)$

"unstable"



$\dot{x} = \sin y$
$\dot{y} = \cos x$

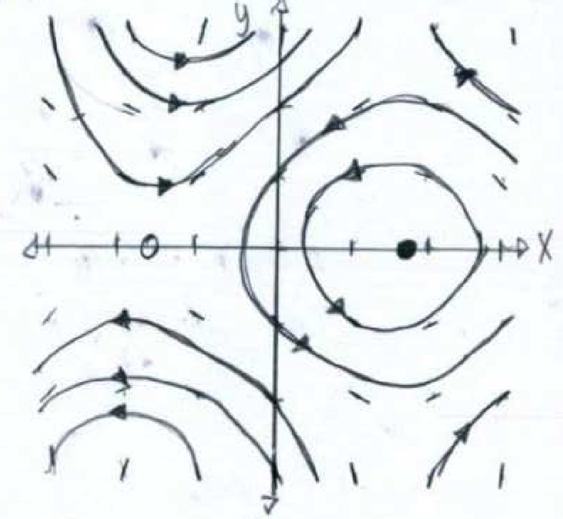
6.3.5. $\boxed{\text{Fixed Points}}$: $\dot{x} = 0 = \sin y$

$\dot{y} = 0 = \cos x$

$(x^*, y^*) = ((n+\tfrac{1}{2})\pi,\ n\pi)$

n is odd "stable"

n is even "unstable"



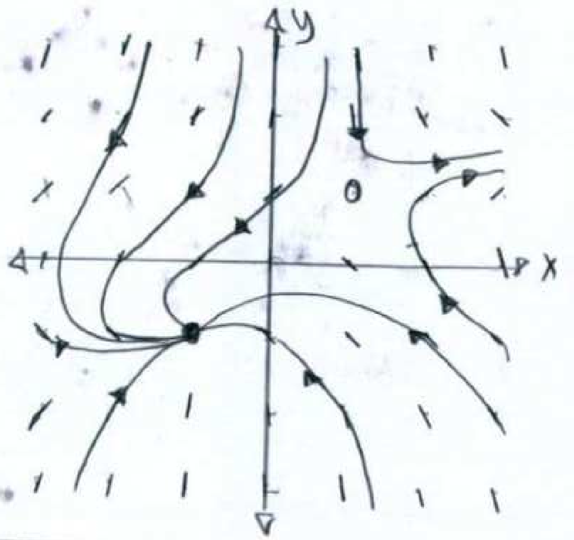
$\dot{x} = xy - 1$
$\dot{y} = x - y^3$

6.3.6. $\boxed{\text{Fixed Points}}$. $\dot{x} = 0 = xy - 1$

$\dot{y} = 0 = x - y^3$

$(x^*, y^*) = (1,1), (-1,-1)$

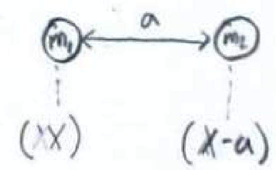
"ustable" "stable"



6.3.7. The phase portraits of problems 6.3.1-6.3.6 are computer generated.

$\ddot{x} = \dfrac{Gm_2}{(x-a)^2} - \dfrac{Gm_1}{x^2}$

6.3.8. a. $g = \ddot{x} = \dfrac{Gm}{r^2}$;



$\ddot{x}_1 = \dfrac{Gm_1}{x^2}$ $\ddot{x}_2 = \dfrac{Gm_2}{(x-a)^2}$; $\ddot{x} = \ddot{x}_2 - \ddot{x}_1 \overset{!}{=} \boxed{\dfrac{Gm_2}{(x-a)^2} - \dfrac{Gm_1}{x^2}}$

b. Equilibrium Position: $\ddot{x} = 0 = \dfrac{Gm_2}{(x-a)^2} - \dfrac{Gm_1}{x^2}$; $m_1(x-a)^2 = m_2 \cdot x^2$

$m_1(x^2 - 2xa + a^2) = m_2 x^2$

$(m_1 - m_2)x^2 - 2xa + a^2 = 0$

$x = \dfrac{2a \pm \sqrt{4a^2 - 4a^2(m_1 - m_2)}}{2(m_1 - m_2)}$

when $m_1 \neq m_2$
"stable"

$\dot{x}=y^3-4x$
$\dot{y}=y^3-y-3x$

6.3.9. $\boxed{\text{Fixed Points:}}$ $\dot{x}=0=y^3-4x$; $(x^*,y^*)=(0,0),(2,2),(-2,-2)$

a. $\dot{y}=0=y^3-y-3x$ "stable" "unstable" "unstable"

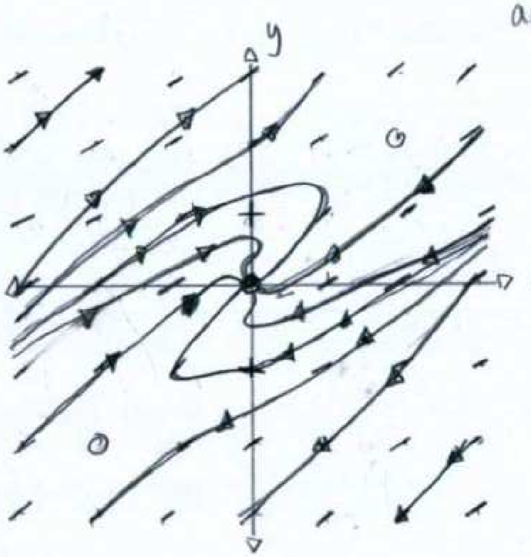
b. If $x=y$, then $\dot{x}=x^3-4x$ and $\dot{y}=x^3-4x$, so $\boxed{\frac{dy}{dx}=1}$

c. $\lim\limits_{t\to\infty}|\dot{x}-\dot{y}| = \lim\limits_{t\to\infty}|y^3-4x-y^3+y+3x| \overset{!}{=} \boxed{\lim\limits_{t\to\infty}|y-x|}$

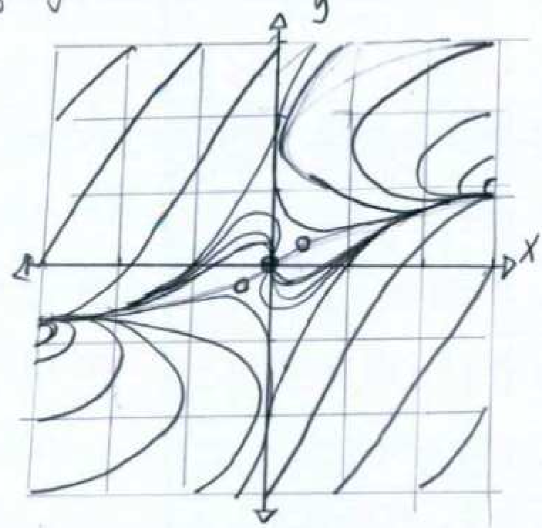
If $\dot{u}=y-x$, then $y-x=Ce^{-t}$, then $\lim\limits_{t\to\infty}|Ce^{-t}|=\boxed{0}$

and $\lim\limits_{t\to\infty}|y-x|=\boxed{0}$

d.



e.



$\dot{x}=xy$
$\dot{y}=x^2-y$

6.3.10

a. $u=x-x^*$; $v=y-y^*$; $\dot{u}=\dot{x}=f(x^*+u,y^*+v)\doteq f(x^*,y^*)+u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}+\cdots$

$\dot{v}=\dot{y}=g(x^*+u,y^*+v)=g(x^*,y^*)+u\frac{\partial g}{\partial x}+v\frac{dg}{dy}+\cdots$

$$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix}=\begin{bmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{bmatrix}\begin{bmatrix}u\\ v\end{bmatrix} ; A=\begin{bmatrix}y & x\\ 2x & -1\end{bmatrix}$$

$\boxed{\text{Fixed Point}}$ $(0,0)$; $A=\begin{bmatrix}0 & 0\\ 0 & -1\end{bmatrix}$, so the origin is a non-isolated fixed point because $\Delta=0$.

b. $\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix}=0$: $(A-\lambda)U=\dot{U}=0$; $(A-\lambda)=\begin{bmatrix}y-\lambda & x\\ 2x & -1-\lambda\end{bmatrix}=(y-\lambda)(-1-\lambda)-2x^2=0$

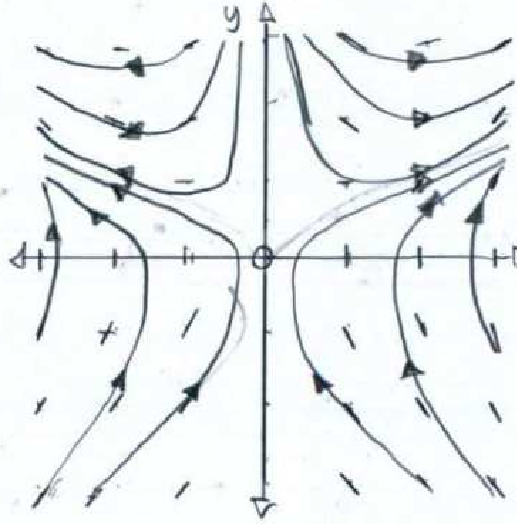
$$\lambda=\frac{(y-1)\pm\sqrt{8x^2+(y-1)^2}}{2}$$

$\boxed{\text{Thus, } \Delta=\lambda_1\lambda_2\neq 0 \text{ and the center is an isolated fixed point.}}$

c. $\boxed{\text{Nullclines}}$ $y = \pm\sqrt{x}$

$y = 0;\ x = 0$

"Saddle Point"



$\boxed{\text{d. See Part C}}$

$\dot{r} = -r$

$\dot{\theta} = \frac{1}{\ln r}$

6.3.11. a. $r(t) = Ce^{-t};\ \theta(t) = \ln\frac{\ln C}{\ln C - t} + D$ Given (r_0, θ_0); then $r(t) = r_0 e^{-t}$

$$\boxed{\theta(t) = \ln\frac{\ln r_0}{|\ln r_0 - t|} + \theta_0}$$

b. $\lim_{t\to\infty} |\theta(t)| = \lim_{t\to\infty} \left| \ln\frac{\ln r_0}{|\ln r_0 - t|} + \theta_0 \right| \ \boxed{\neq \infty}$

$\lim_{t\to\infty} r(t) = \lim_{t\to\infty} r_0 e^{-b} \stackrel{!}{=} \boxed{0}$

c. $\dot{r} = -\sqrt{x^2+y^2}$; $\dot{\theta} = \boxed{\frac{1}{\ln\sqrt{x^2+y^2}}}$

d. $\dot{r} = \frac{d}{dt}\sqrt{x^2+y^2} = \frac{x\dot{x} + y\dot{y}}{\sqrt{x^2+y^2}} = -r = -\sqrt{x^2+y^2}$

$$x\dot{x} + y\dot{y} = -x^2 - y^2$$

$\dot{\theta} = \frac{d}{dt}\arctan\left(\frac{y}{x}\right) = \frac{x\dot{y} - \dot{x}y}{x^2+y^2} = \frac{1}{\ln(\sqrt{x^2+y^2})}$;

$$x\dot{y} - \dot{x}y = \frac{2(x^2+y^2)}{\ln(x^2+y^2)}$$

$$x(x\dot{x} - y\dot{y}) - y(x\dot{y} - y\dot{x}) = (x^2+y^2)\dot{x} = -x(x^2+y^2) - y\frac{2(x^2+y^2)}{\ln(x^2+y^2)}$$

$$\boxed{\dot{x} = -x - \frac{2y}{\ln(x^2+y^2)}}$$

$$x(x\dot{y} - y\dot{x}) + y(x\dot{x} + y\dot{y}) = (x^2+y^2)\dot{y} = -y(x^2+y^2) + x\frac{2(x^2+y^2)}{\ln(x^2+y^2)}$$

$$\boxed{\dot{y} = -y + \frac{2x}{\ln(x^2+y^2)}}$$

d. $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = A\begin{pmatrix}x\\ y\end{pmatrix}$; $A = \begin{pmatrix} -1 + \frac{4xy}{(x^2+y^2)\ln^2(x^2+y^2)} & \frac{4y^2}{(x^2+y^2)\ln^2(x^2+y^2)} - \frac{2}{\ln(x^2+y^2)} \\ \frac{2}{\ln(x^2+y^2)} - \frac{4x^2}{(x^2+y^2)\ln^2(x^2+y^2)} & \frac{-4xy}{(x^2+y^2)\ln^2(x^2+y^2)} - 1 \end{pmatrix}$

$$A_{(0,0)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix};\ \boxed{\dot{x} = -x,\ \dot{y} = -y}$$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

6.3.12. $\dot{\theta} = \frac{d}{dt}\tan^{-1}\left(\frac{y}{x}\right) = \frac{\frac{d}{dt}\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = \frac{\dot{y}x - \dot{x}y}{x^2+y^2} \overset{!}{=} \boxed{\frac{x\dot{y} - y\dot{x}}{r^2}}$

$\dot{x} = -y - x^3$

$\dot{y} = x$

6.3.13. Linearization: $u = x - x^*$; $v = y - y^*$

$$\dot{x} = \dot{u} = f(x,y) = f(u+x^*, v+y^*) = f(x^*,y^*) + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + \cdots$$

$$\dot{y} = \dot{v} = g(x,y) = g(u+x^*, v+y^*) = g(x^*,y^*) + u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y} + \cdots$$

$$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}\begin{pmatrix}u\\ v\end{pmatrix};\ A = \begin{bmatrix} -3x^2 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A_{(0,0)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\Delta = 0;\ \tau = 0;\ \boxed{\text{center}}$$

Eigenvalues: $\dot{\vec{U}} = A\vec{U}$; $\lambda U = AU$; $(A-\lambda)U = 0$;

$$(A-\lambda) = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

$$\lambda_{1,2} = \pm i$$

$\boxed{\text{Thus, } \Delta = -1,\ \tau = 0, \text{ so the center is a spiral, also supported by } \tau^2 - 4\Delta > 0}$

$\dot{x} = -y + ax^2$

$\dot{y} = x + ay^3$

6.3.14. $a > 0$; Fixed points: $\dot{x} = 0 = -y + ax^2$;

$$\dot{y} = 0 = x + ay^3;$$

$$(x^*, y^*) = (0,0)$$

Linearization: $u = x - x^*$; $v = y - y^*$

$$\dot{x} = \dot{u} = f(x,y) = f(u+x^*, v+y^*) = f(x^*,y^*) + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + \cdots$$

$$\dot{y} = \dot{v} = g(x,y) = g(u+x^*, v+y^*) = g(x^*,y^*) + u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y} + \cdots$$

$$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}\begin{pmatrix}u\\ v\end{pmatrix};\ A = \begin{bmatrix} 2ax & -1 \\ 1 & 3ay^2 \end{bmatrix};$$

$$A_{(0,0)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} ; \ \Delta = 0, \ \tau = 0 ; \text{ center}$$

Eigenvalues: $\dot{\mathbb{U}} = A\mathbb{U} ; \ \lambda \mathbb{U} = A\mathbb{U} \therefore (A-\lambda)\mathbb{U} = 0$

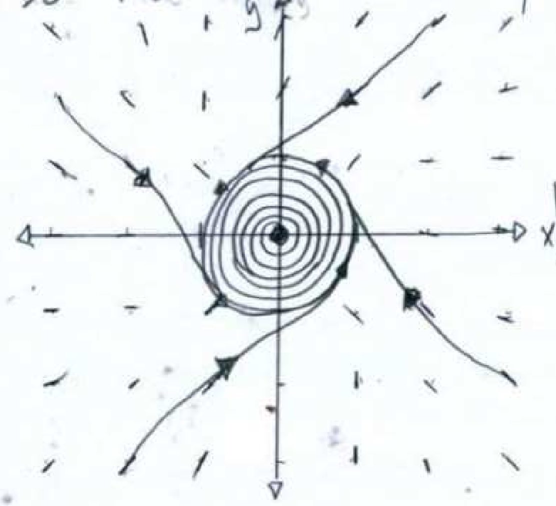
$$(A-\lambda) = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

$$\lambda_{1,2} = \pm i$$

Thus, eigenvalues demonstrate $\Delta = -1, \ \tau = 0, \ \tau^2 - 4\Delta > 0$, so the center is a spiral.

$\dot{r} = r(1-r^2)$ 6.3.15

$\dot{\theta} = 1 - \cos\theta$

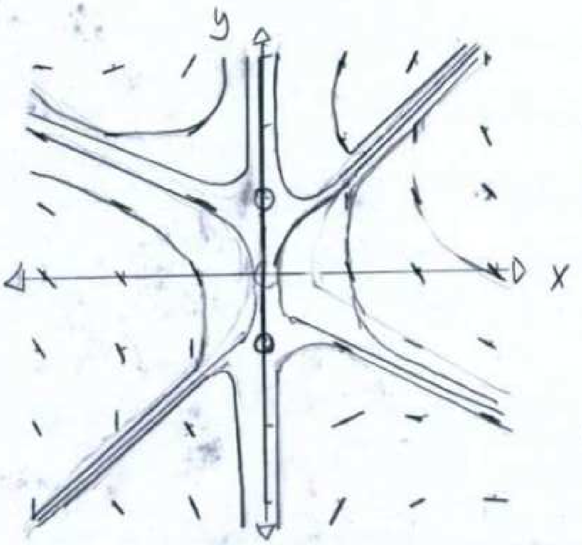


"Liapunov Stable."

$\dot{x} = a + x^2 - xy$ 6.3.16

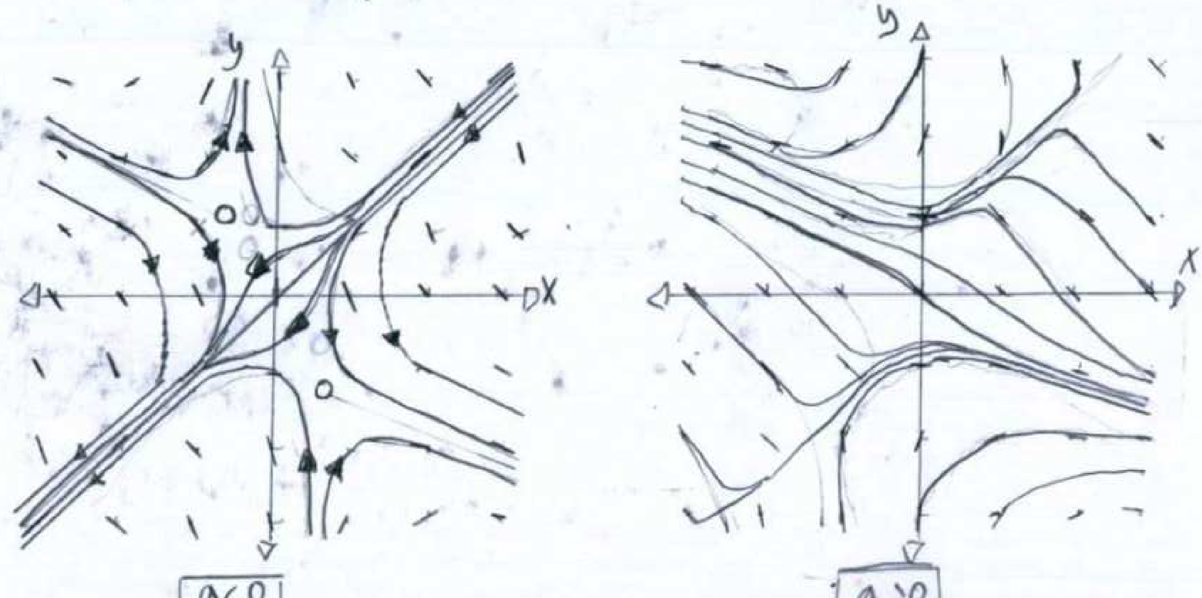
$\dot{y} = y^2 - x^2 - 1$

a.



"Saddle connection"

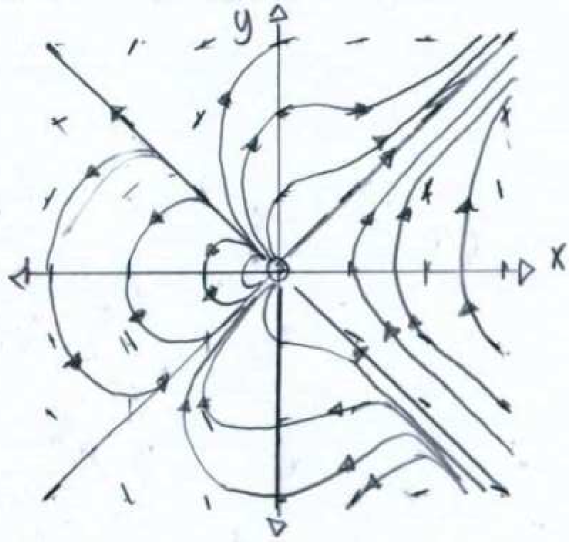
b.



$a < 0$ $a > 0$

$\dot{x} = xy - x^2y + y^3$ 6.3.17.

$\dot{y} = y^2 + x^3 - xy^2$



$\dot{x} = x(3-x-y)$ 6.4.1 $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = x(3-x-y)$

$\dot{y} = y(2-x-y)$

$\dot{y} = 0 = y(2-x-y)$

$(x^*, y^*) = (0,0)$ "unstable"

$(3,0)$ "stable"

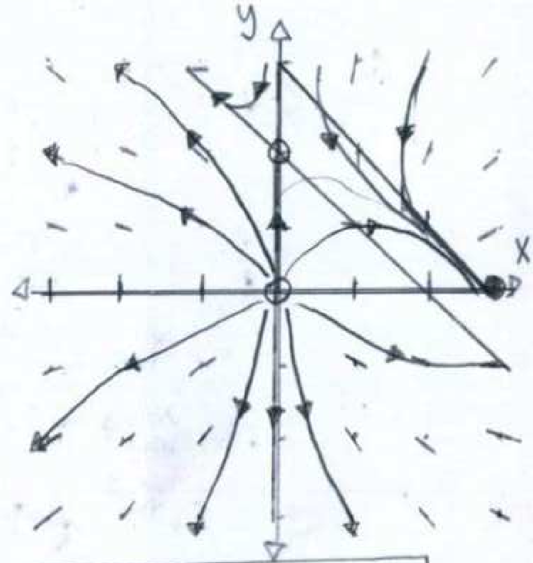
$(0,2)$ "unstable"

$\boxed{\text{Nullclines}}$ $x = 0$

$y = 0$

$y = 2-x$; $y = 3-x$

$\boxed{\text{Basin of Attraction}}$ $x \geq 0 \wedge y \geq 0$



$\boxed{\text{I forgot } (x \text{ and } y) \geq 0}$

$\dot{x} = x(3-2x-y)$ 6.4.2 $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = x(3-2x-y)$

$\dot{y} = y(2-x-y)$

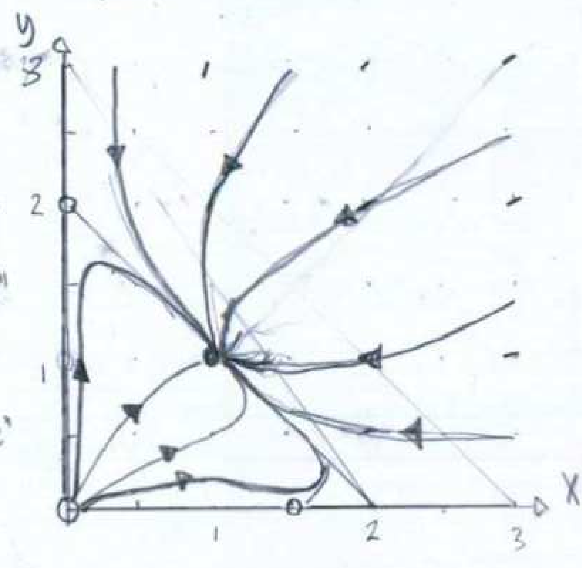
$\dot{y} = 0 = y(2-x-y)$

$(x^*, y^*) = (0,0)$ "unstable"

$(0,2)$ "unstable"

$(1,1)$ "stable"

$(3/2, 0)$ "unstable"



$\boxed{\text{Nullclines}}$ $x = 0$; $y = 0$

$y = 3-2x$

$y = 2-x$

$\boxed{\text{Basin of Attraction}}$ $(x \geq 0) \wedge (y \geq 0)$

$\dot{x} = x(3-2x-2y)$
$\dot{y} = y(2-x-y)$

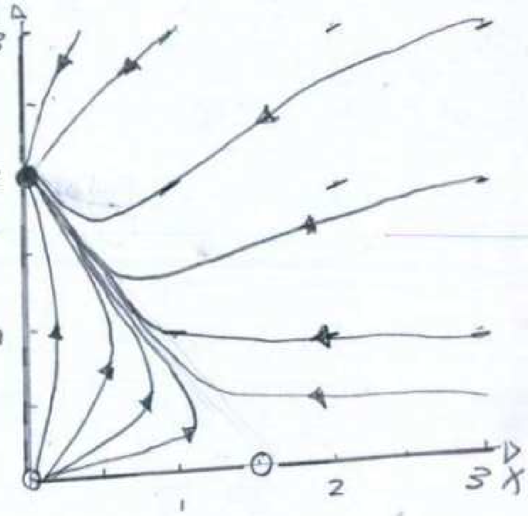
6.4.3. $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = x(3-2x-2y)$

$\dot{y} = 0 = y(2-x-y)$

$(x^*, y^*) = (0,0)$ "unstable"

$(0,2)$ "stable"

$(3/2, 0)$ "unstable"



$\boxed{\text{Nullclines}}$ $x = 0$; $y = 0$

$y = 2 - x$; $y = \frac{3}{2} - x$

$\boxed{\text{Basin of Attraction}}$ $(x>0) \wedge (y>0)$

$\dot{N}_1 = r_1 N_1 - b N_1 N_2$
$\dot{N}_2 = r_2 N_2 - b_2 N_1 N_2$

6.4.4.

a. The N_1 and N_2 model is less realistic because population for rabbits and sheep decreases from an interaction.

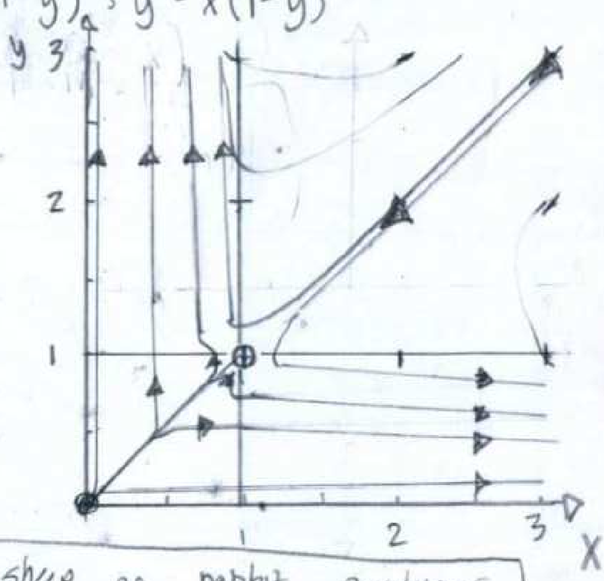
b. Unable to complete problem without $r_1 = r_2 = b = b_2 = 1$

$x = N_1$; $y = N_2$; $t = \tau$; $x' = x(1-y)$; $y' = x(1-y)$

c. $\boxed{\text{Fixed Points}}$: $x' = 0 = x(1-y)$

$y' = 0 = y(1-x)$

$(x^*, y^*) = (0,0), (1,1)$



$\boxed{\text{Nullclines}}$ $y = 1$; $x = 1$

$y = 0$; $x = 0$

d. See part c in order to denote sheep or rabbit populations approach infinity when rabbit per sheep is less than 1 or sheep per rabbit is less than 1.

e. $\frac{dx}{dy} = \frac{x}{y}\frac{(1-y)}{(1-x)}$; $\int \frac{(1-x)}{x} dx = \int \frac{(1-y)}{y} dy$; $\ln x - x = \boxed{\ln y - y + C}$

when $\rho = 1$

$\dot{N}_1 = r_1 N_1 (1 - N_1/K_1) - b_1 N_1 N_2$
$\dot{N}_2 = r_2 N_2 - b_2 N_1 N_2$

6.4.5. $\frac{dN_1}{dt}\left(\frac{1}{K_1}\right) = r_1 N_1 \left(\frac{1}{K_1}\right)(1 - N_1/K_1) - b_1 N_1 N_2 \left(\frac{1}{K_1}\right)$; $x = N_1/K_1$

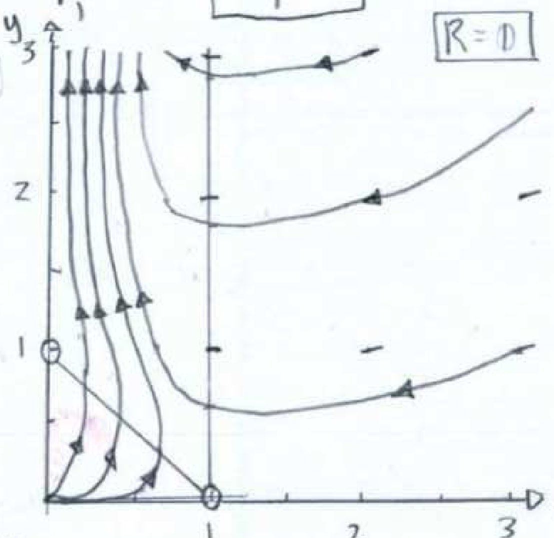
$\frac{dx}{dt} = r_1 x(1-x) - b x N_2$; $\frac{dx}{dt}\left(\frac{1}{r_1}\right) = x(1-x) - \frac{b}{r_1} x N_2$

$\rho = \frac{b_1}{r_1}$; $\tau = tr$; $N_2 = y$

$\boxed{x' = x(1-x) - \rho_1 x \cdot y}$

$y' = \frac{r_2}{r_1}N_2 - \frac{b_2}{r_1}N_1N_2 = y' = Ry - \rho_2 x y$

where $R = \frac{r_2}{r_1}$; $\rho_2 = \boxed{\frac{b_2}{r_1}K_1}$



Fixed Points $x' = 0 = x(1 - x - \rho_1 y)$

$y' = 0 = y(R - \rho_2 x)$

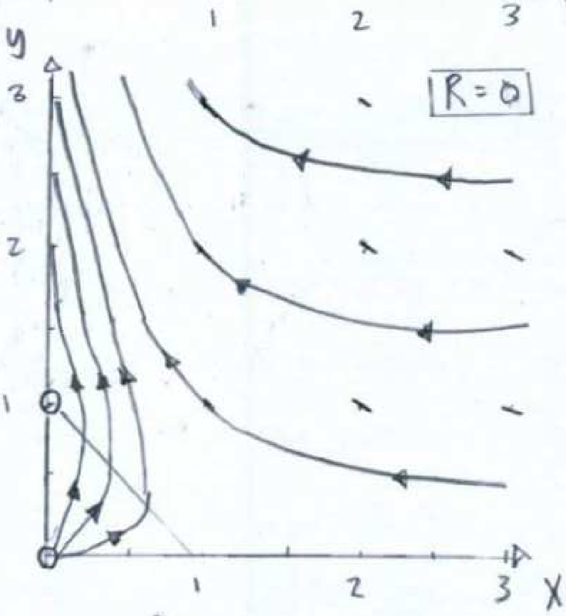
$(x^*, y^*) = (0,0), (1,0)$

If $R = 0$, $(0, y)$

and $\rho_1 = \rho_2 = 1$

$\dot{N}_1 = r_1 N_1 (1 - N_1/K_1) - b_1 N_1 N_2$

$\dot{N}_2 = r_2 N_2 (1 - N_2/K_2) - b_2 N_1 N_2$



6.4.6.

a. $\dot{N}_1\left(\frac{1}{K_1}\right) = r_1 \frac{N_1}{K_1}(1 - N_1/K_1) - b_1 \frac{N_1}{K_1} N_2$

$x = \frac{N_1}{K_1}$

$\frac{dx}{dt} = r_1 x (1 - x) - b_1 x N_2$

$\dot{N}_2\left(\frac{1}{K_2}\right) = r_2 \frac{N_2}{K_2}\left(1 - \frac{N_2}{K_2}\right) - b_2 N_1 \frac{N_2}{K_2}$

$y = N_2/K_2$

$\frac{dy}{dt} = r_2 y (1 - y) - b_2 N_1 y$

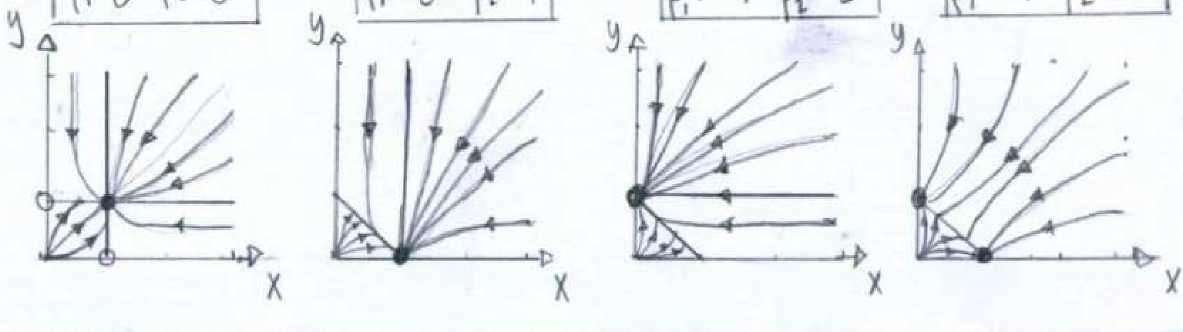
$\tau = t \cdot r_1$; $R = r_2/r_1$; $\rho_1 = \left(\frac{b_1}{r_1}\right)K_2$; $\rho_2 = \left(\frac{b_2}{r_1}\right)K_1$

$\dot{x} = x(1 - x - \rho_1 y)$; $\dot{y} = y(1 - y - \rho_2 x)$ when $R = 1$

A total of six dimensionless groups suffice.

b.

$\rho_1 = 0 \quad \rho_2 = 0$ | $\rho_1 = 0 \quad \rho_2 = 1$ | $\rho_1 = 1 \quad \rho_2 = 0$ | $\rho_1 = 1 \quad \rho_2 = 1$



c. $\boxed{\text{The species coexist when } \rho_1 = \rho_2 = 0\text{. This parameter describes the interaction between the rabbits and sheep as noncompetitive.}}$

$\dot{n}_1 = G_1 N n_1 - K_1 n_1$

$\dot{n}_2 = G_2 N n_2 - K_2 n_2$

6.4.7. $N(t) = N_0 - \alpha_1 n_1 - \alpha_2 n_2$

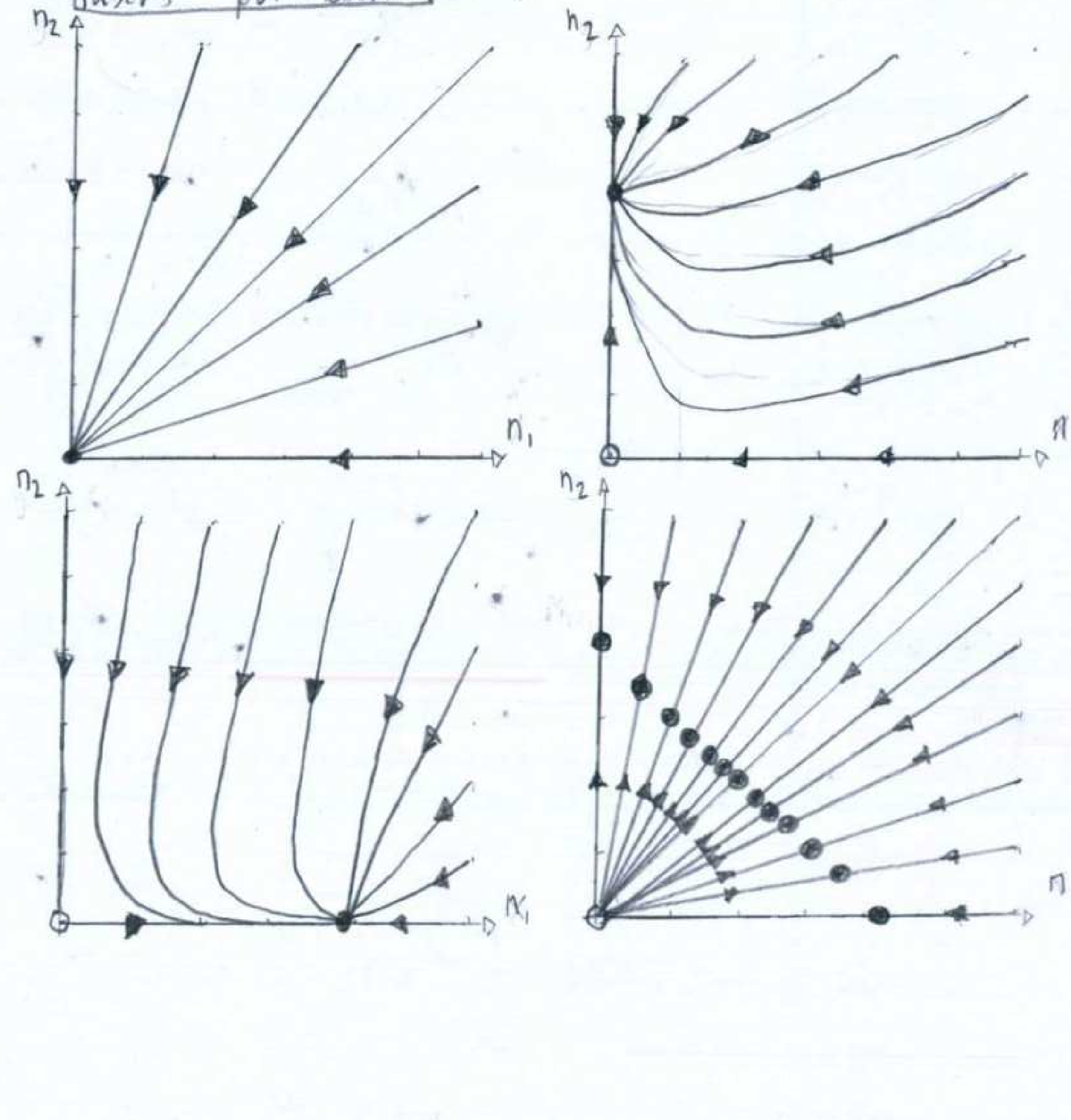
a. $$A = \begin{pmatrix} \frac{d\dot{n}_1}{dn_1} & \frac{d\dot{n}_1}{dn_2} \\ \frac{d\dot{n}_2}{dn_1} & \frac{d\dot{n}_2}{dn_2} \end{pmatrix} = \begin{pmatrix} G_1 N - K_1 & 0 \\ 0 & G_2 N - K_2 \end{pmatrix}$$

$\Delta = (G_1 N - K_1)(G_2 N - K_2)$; $\tau = (G_1 + G_2)N - (K_1 + K_2)$

$\boxed{\tau^2 - 4\Delta > 0 \text{ ; Unstable Node}}$

b. $\boxed{\text{The other fixed points are } G_1 N = K_1 \text{ and } G_2 N = K_2}$

c. $\boxed{\text{Four phase portraits appear by varying the lasers parameters.}}$



$\dot{x} = ax^c - \phi x$

$\dot{y} = by^c - \phi y$

$\phi \equiv ax^c + by^c$

6.4.8. a. If $x_0 + y_0 = 1$, $\dot{x}+\dot{y} = ax^c - (ax^c + by^c)x + by^c - (ax^c+by^c)y$

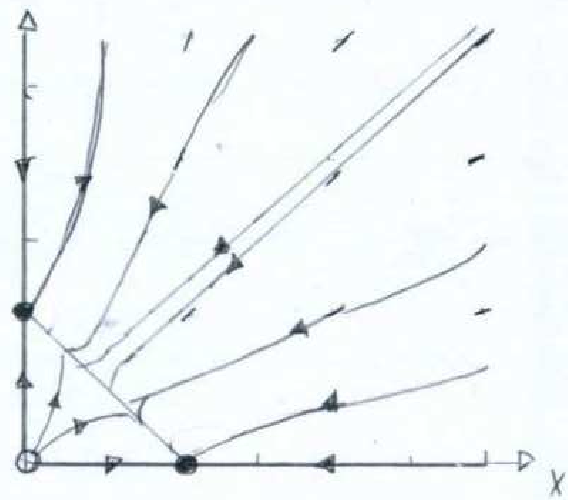
$$= a(1-x-y)x^c + b(1-x-y)y^c$$

then $\boxed{\dot{x}+\dot{y} = 0 \quad \text{and} \quad x(t)+y(t) = 1}$

b. $$\lim_{x\to\infty} \dot{y}/\dot{x} = \frac{by^c - \phi y}{ax^c - \phi x} = \frac{by^c - (ax^c+by^c)y}{ax^c - (ax^c+by^c)x} \simeq \frac{-ax^c}{-ax^{c+1}} = \frac{1}{x} \doteq \boxed{0}$$

$$\lim_{y\to\infty} \dot{y}/\dot{x} = \frac{by^c - \phi y}{ax^c - \phi x} = \frac{by^c - (ax^c+by^c)y}{ax^c - (ax^c+by^c)x} \simeq \frac{-by^{c+1}}{-by^c x} \doteq \boxed{\infty}$$

c. If $c=1$,



d. $\boxed{\text{If } c>1, \text{ then radial nullclines become generated.}}$

e. $\boxed{\text{If } c<1, \text{ then monotonically decreasing nullclines become generated.}}$

$\dot{I} = I - \alpha C$

$\dot{C} = \beta(I - C - G)$

6.4.9. $I \geq 0$: National Income ; $C \geq 0$: Rate of Consumer Spending.

$G \geq 0$: Rate of Government Spending.

$1 < \alpha < \infty$ and $1 \leq \beta < \infty$

a. $\boxed{\text{Fixed Points}}$ $\dot{I} = 0 = I - \alpha C$; $\dot{C} = 0 = \beta(I - C - G)$

$$(I^*, C^*) = \left(\frac{\alpha G}{\alpha - 1}, \frac{G}{\alpha - 1}\right)$$

$$\dot{\mathbb{I}} = A\mathbb{I} \ ; \ A = \begin{pmatrix} \frac{\partial \dot{I}}{\partial I} & \frac{\partial \dot{I}}{\partial C} \\ \frac{\partial \dot{C}}{\partial I} & \frac{\partial \dot{C}}{\partial C} \end{pmatrix} = \begin{pmatrix} 1 & -\alpha \\ \beta & -\beta \end{pmatrix}$$

If $\beta = 1$, $A = \begin{pmatrix} 1 & -\alpha \\ 1 & -1 \end{pmatrix}$, $\Delta = -(1-a)$; $\tau = 0$; $\tau^2 - 4\Delta = 4(1-a)$

$\boxed{\text{A center node}}$

b. $G = G_0 + \kappa I$; $\kappa > 0$; $I \geq 0$, $C \geq 0$

$\boxed{\text{Fixed Point}}$ $(I^*, C^*) = \left(\frac{\alpha G_0}{\alpha(1-\kappa) - 1}, \frac{G_0}{\alpha(1-\kappa)-1} \right)$

If $\kappa < \kappa_c = 1 - \frac{1}{\alpha}$, then I & $C > 0$

$\mathring{I} = AI$; $A = \begin{pmatrix} 1 & -\alpha \\ \beta(1-\kappa) & -\beta \end{pmatrix}$; $(A - \lambda I) = \begin{pmatrix} 1-\lambda & -\alpha \\ \beta(1-\kappa) & -\beta-\lambda \end{pmatrix}$

$$(1-\lambda)(-\beta-\lambda) + \alpha \cdot \beta(1-\kappa) = 0$$

$$\lambda_{1,2} = \frac{-(\beta-1) \pm \sqrt{(\beta-1)^2 - 4(\alpha(1-\kappa)-1)\beta}}{2}$$

$$A\vec{v}_1 = \begin{pmatrix} (1+\beta) + \sqrt{(\beta-1)^2 - 4(\alpha(1-\kappa)-1)\beta} & -\alpha \\ \beta(1-\kappa) & -\frac{(1+\beta)+\sqrt{(\beta-1)^2-4(\alpha(1-\kappa)-1)\beta}}{2} \end{pmatrix} \begin{pmatrix} V_{11} \\ V_{12} \end{pmatrix}$$

$$\left[(1+\beta) + \sqrt{(\beta-1)^2 - 4(\alpha(1-\kappa)-1)\beta}\right] V_1 - \alpha_1 V_2 = 0$$

$$V_{11} = 1 \; ; \; V_{12} = \frac{(1+\beta) + \sqrt{\beta^2 + (2-4\alpha(1-\kappa))\beta}}{\alpha}$$

$$\boxed{\vec{V}_1 = \begin{pmatrix} 1 \\ \frac{(1+\beta) + \sqrt{\beta^2 + (2-4\alpha(1-\kappa))\beta}}{\alpha} \end{pmatrix}}$$

$$A\vec{V}_2 = \begin{pmatrix} (1+\beta) - \sqrt{(\beta-1)^2 - 4(\alpha(1-\beta)-1)\beta} & -\alpha \\ \beta(1-\kappa) & -\frac{(1+\beta)-\sqrt{(\beta-1)^2-4(\alpha(1-\kappa)-1)\beta}}{2} \end{pmatrix} \begin{pmatrix} V_{21} \\ V_{22} \end{pmatrix}$$

$$\left[(1+\beta) - \sqrt{(\beta-1)^2 - 4(\alpha(1-\kappa)-1)\beta}\right] V_1 - \alpha_1 V_2 = 0$$

$$V_{21} = 1 \; ; \; V_{22} = \frac{(1+\beta) - \sqrt{\beta^2 + (2-4\alpha(1-\kappa))\beta}}{\alpha}$$

$$\boxed{\vec{V}_2 = \begin{pmatrix} 1 \\ \frac{(1+\beta) - \sqrt{\beta^2 + (2-4\alpha(1-\kappa))\beta}}{\alpha} \end{pmatrix}}$$

When $k > k_c$, the economy gravitates to the positive eigendirection

c) $G = G_0 + kI^2$; $\dot{I} = I - \alpha C = 0$; $\dot{C} = \beta(I - C - G_0 - k_0 I^2) = 0$

Fixed Points: $0 = \beta(I(1 - 1/\alpha) - G_0 - k_0 I^2) = -k_0 I^2 + I(1 - 1/\alpha) - G_0$

$$(I^*, C^*) = \left(\frac{+(1 - 1/\alpha) + \sqrt{(1-1/\alpha)^2 - 4k_0 G_0}}{2k_0}, \frac{(1-1/\alpha) + \sqrt{(1-1/\alpha)^2 - 4k_0 G_0}}{2k_0 \alpha} \right)$$

$$, \left(\frac{(1-1/\alpha) - \sqrt{(1-1/\alpha)^2 - 4k_0 G_0}}{2k_0}, \frac{(1-1/\alpha) - \sqrt{(1-1/\alpha)^2 - 4k_0 G_0}}{2k_0 \alpha} \right)$$

If $G_0 < \frac{(\alpha-1)^2}{4\alpha^2 k_0}$, then two positive fixed points exist

If $G_0 = \frac{(\alpha-1)^2}{4\alpha^2 k_0}$, then one fixed point exists in quadrant #1

If $G_0 > \frac{(\alpha-1)^2}{4\alpha^2 k_0}$, then zero fixed points exist because of the imaginary radical.

$\dot{x}_i = x_i \left(x_{i-1} - \sum_{j=1}^{n} x_j x_{j-1} \right)$

6.4.10.

a. If $n = 2$, $\dot{x}_1 = x_1 (x_0 - \sum_{j=1}^{n} x_1 x_0) = x_1 (x_2 - \sum_{j=1}^{2} x_1 x_2) = x_1 (x_2 - 2x_1 x_2)$

$\dot{x}_2 = x_2 (x_1 - \sum_{j=1}^{2} x_2 x_1) = \boxed{x_2 (x_1 - 2x_2 x_1)}$

b. $\dot{x}_1 = 0 = x_1 (x_2 - 2x_1 x_2)$; $\dot{x}_2 = 0 = x_2 (x_1 - 2x_2 x_1)$

$(x_1^*, x_2^*) = (1/2, 1/2)$; $A = \begin{pmatrix} x_2 - 4x_1 x_2 & x_1(1 - 2x_1) \\ x_2(1 - 2x_2) & x_1 - 4x_1 x_2 \end{pmatrix}$

$A_{(1/2, 1/2)} = \begin{pmatrix} -1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$; $\Delta = \frac{1}{4}$, $\tau = -1$

$\tau^2 - 4\Delta = 0$

Degenerate and Stable Node

c. $u = x_1 + x_2$; $\dot{u} = \dot{x}_1 + \dot{x}_2 = x_1 (x_2 - 2x_1 x_2) + x_2 (x_1 - 2x_1 x_2)$

$= x_1 x_2 - 2x_1^2 x_2 + x_1 x_2 - 2x_1 x_2^2$

$= 2x_1 x_2 (1 - x_1 - x_2) = 2x_1 x_2 (1 - u)$

$u(t) = 1 - e^{-2x_1 x_2 t}$

$\boxed{\lim_{t \to \infty} u(t) = 1}$

d. $v = x_1 - x_2$; $\dot{v} = \dot{x}_1 - \dot{x}_2$; $\dot{v} = x_1(x_2 - 2x_1x_2) - x_2(x_1 - 2x_1x_2)$

$$= -2x_1x_2(x_1 - x_2) = -2x_1x_2 \cdot v$$

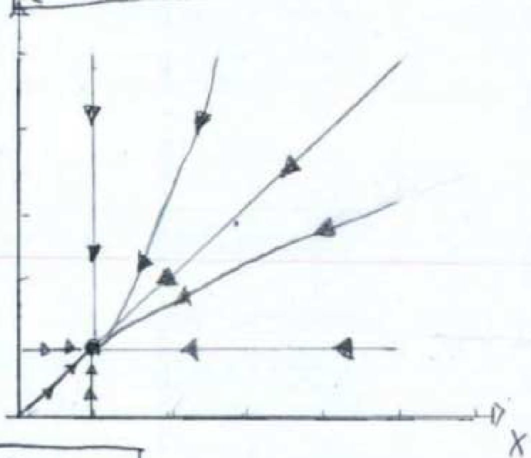
$$v(t) = e^{-2x_1x_2 t}$$

$$\boxed{\lim_{t\to\infty} v(t) = 0}$$

e. $\lim_{t\to\infty} [u(t) + v(t)] = 1 = 2x_1$; $x_1 = 1/2$

$\lim_{t\to\infty} [u(t) - v(t)] = 1 = 2x_2$; $x_2 = 1/2$; $\boxed{(x_1, x_2) \to (1/2, 1/2)}$

f. A large n value generates a plot which seems to converge to zero, but actually, converges to a positive value close to zero. This argument implies RNA remain at low concentrations indefinitely.



$\dot{x} = rxz$
$\dot{y} = ryz$
$\dot{z} = -rxz - ryz$

6.4.11
a. $\dot{z} = -\dot{x} - \dot{y}$; $0 = \dot{x} + \dot{y} + \dot{z}$; $\boxed{1 = x + y + z}$

b. The limit of the function is bounded by the invariance equation in part a.

Fixed Points $\dot{x} = 0 = rxz$; $\dot{y} = ryz$; $\dot{z} = -rxz - ryz$

$$(x^*, y^*, z^*) = (x, y, 0), (0, 0, z)$$

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial z} \\ \frac{\partial \dot{z}}{\partial x} & \frac{\partial \dot{z}}{\partial y} & \frac{\partial \dot{z}}{\partial z} \end{pmatrix} = \begin{pmatrix} rz & 0 & rx \\ 0 & rz & ry \\ -rz & -rz & -rx-ry \end{pmatrix}$$

$$A_{(x,y,0)} = \begin{pmatrix} 0 & 0 & rx \\ 0 & 0 & ry \\ 0 & 0 & -rx-ry \end{pmatrix} ; A_{(0,0,z)} = \begin{pmatrix} rz & 0 & 0 \\ 0 & rz & 0 \\ -rz & -rz & 0 \end{pmatrix}$$

$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = -rx - ry$ $\lambda_1 = \lambda_2 = 0, \lambda_3 = rz$

The eigenvectors point in the direction of λ_3 for each fixed point

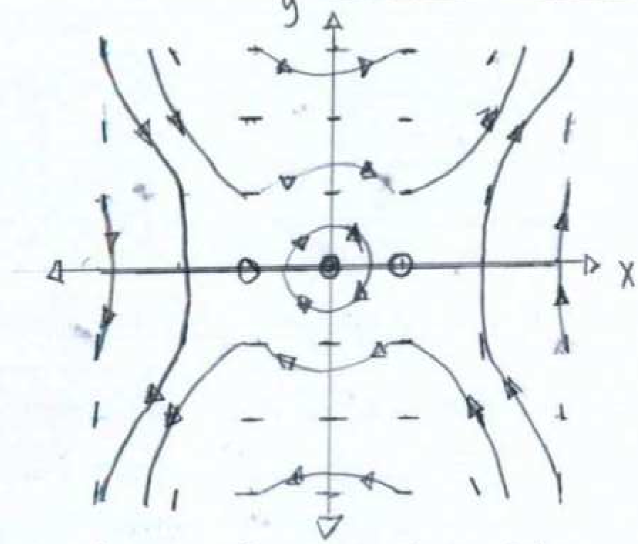
c. An interpretation from the political terms is $r<0$, the centrist pull the extremists to the centrist, while $r>0$, the extremist separate the centrists.

$\ddot{x} = x^3 - x$ 6.5.1. a. $\dot{x} = y$; $\dot{y} = x^3 - x$; $A = \begin{pmatrix} 0 & 1 \\ 3x-1 & 0 \end{pmatrix}$;

$\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = y$; $\dot{y} = 0 = x^3 - x$; $(x^*, y^*) = (-1, 0)$ "center"
$(0, 0)$ "saddle"
$(1, 0)$ "center"

b. $E = \frac{1}{2}\dot{x}^2 - \int x^3 - x\,dx = \boxed{\frac{1}{2}\dot{y}^2 - \frac{x^4}{4} + \frac{x^2}{2} + C}$

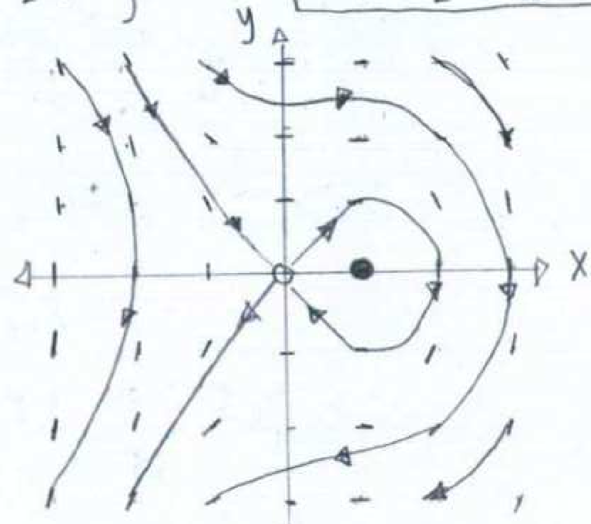
c.



$\ddot{x} = x - x^2$ 6.5.2. a. $\dot{x} = y$; $\dot{y} = x - x^2$; $A = \begin{pmatrix} 0 & 1 \\ 1-2x & 0 \end{pmatrix}$

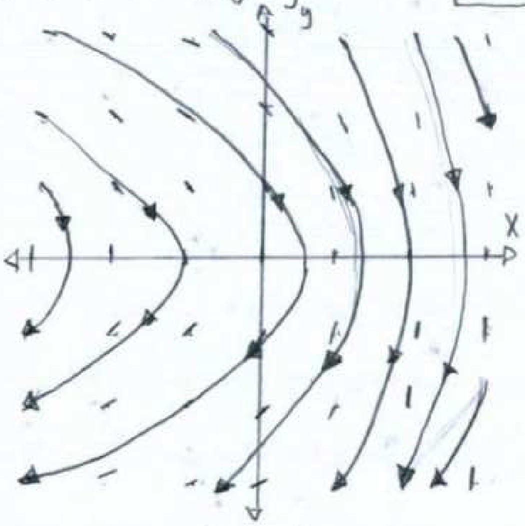
$\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = y$; $\dot{y} = 0 = x - x^2$; $(x^*, y^*) = (+1, 0)$ "center"
$(0, 0)$ "saddle"

b. $E = \frac{1}{2}\dot{x}^2 - \int (x - x^2)\,dx = \boxed{\frac{1}{2}y^2 - \frac{x^2}{2} + \frac{x^3}{3} + C}$

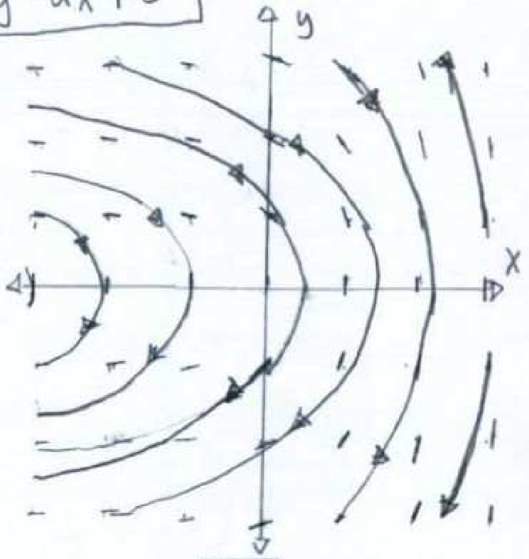


c. $\boxed{F = \frac{1}{2}y^2 - \frac{x^2}{2} + \frac{x^3}{3} + c.}$

$\ddot{x} = a - e^x$ 6.5.3. $\dot{x} = y$; $\dot{y} = a - e^x$; $E = \frac{1}{2}y^2 - \int_y a - e^x \, dx = \boxed{\frac{1}{2}y^2 - ax + e^x}$



a<0



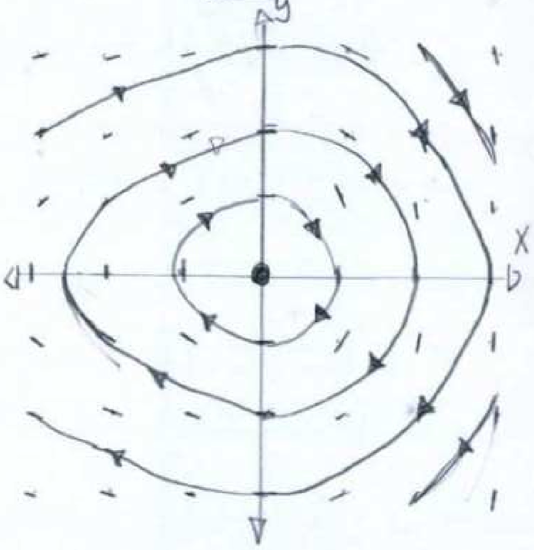
a=0

$\ddot{x} = ax - x^2$ 6.5.4. $\dot{x} = y$; $\dot{y} = ax - x^2$

Conserved Quantity:

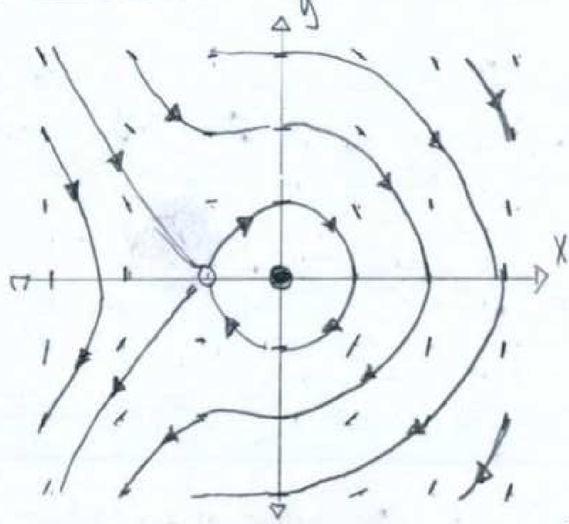
$E = \frac{1}{2}y^2 - \int ax - x^2 \, dx$

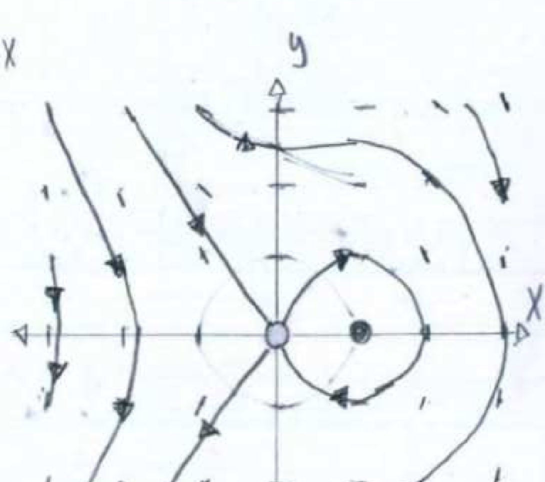
$\boxed{= \frac{1}{2}y^2 - \frac{ax^2}{2} + \frac{x^3}{3} + C}$



a>0

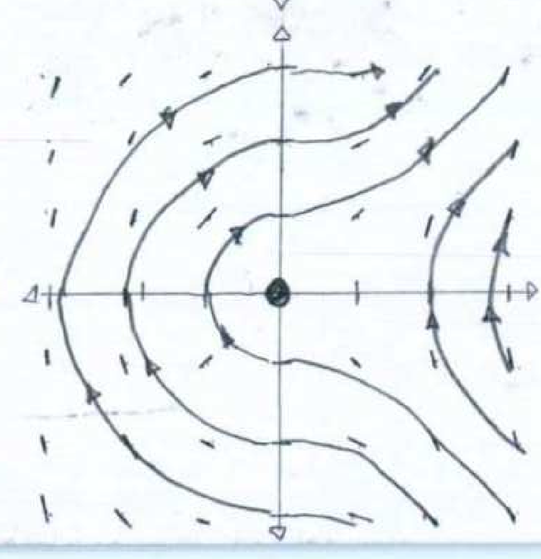
a<0





a>0

a=0



$\ddot{x} = (x-a)(x^2-a)$ 6.5.5 $\dot{x} = y$; $\dot{y} = (x-a)(x^2-a)$

$\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = y$

$\dot{y} = 0 = (x-a)(x^2-a)$

$(x^*, y^*) = (a, 0)$

$(\sqrt{a}, 0)$

$(-\sqrt{a}, 0)$

If $a=1$, then one fixed point exists in quadrant #1 and #4.

If $0<a<1$ or $a>1$, then two fixed points exist in quadrants #1 and #4

$\dot{x} = -kxy$
$\dot{y} = kxy - ly$

6.5.6. a. $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = -kxy$; $\dot{y} = 0 = kxy - ly$

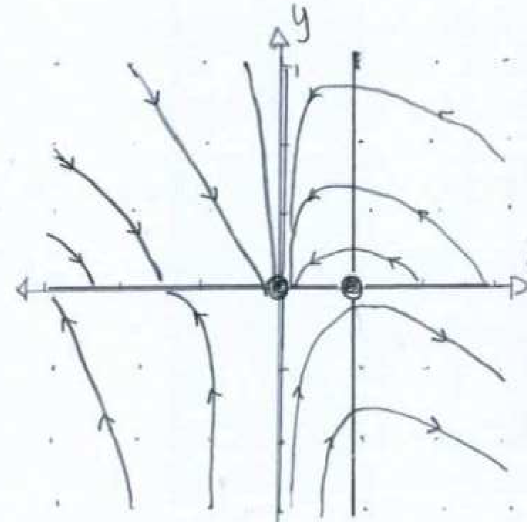
$(x^*, y^*) = (0,0)$ "center" ; $A = \begin{pmatrix} -ky & -kx \\ ky & kx-l \end{pmatrix}$

$(l/k, 0)$ "center"

b. $\boxed{\text{Nullclines}}$ $x = 0$

$y = 0$

$x = \frac{l}{k}$



c. $\boxed{\frac{dy}{dx} = -1 + l/kx \text{ ; } y = -x + \frac{l}{k}\ln x + c}$

d. $\boxed{\text{See part c}}$

e. $\boxed{\text{A population is sick from infection when } y_0 \geq 0.}$

$\frac{d^2u}{d\theta^2} + u = \alpha + \varepsilon u^2$ 6.5.7 $u = 1/r$;

a. $\boxed{v^2 + u = \alpha + \varepsilon u^2 \text{ where } v = du/d\theta.}$

b. $\boxed{\text{Fixed Points}}$: $\dot{u} = 0 = v$

$\dot{v} = 0 = \kappa + \varepsilon u^2 - u$

$(u^*, v^*) = \left(\frac{1+\sqrt{1-4\kappa\varepsilon}}{2\varepsilon}, 0 \right), \left(\frac{1-\sqrt{1-4\kappa\varepsilon}}{2\varepsilon}, 0 \right)$

c. $A = \begin{pmatrix} 0 & 1 \\ 2\varepsilon u - 1 & 0 \end{pmatrix}$; $\lambda_{1A,B} = \pm i\sqrt{1-4\alpha\varepsilon}$; $\lambda_{2A,B} = \pm\sqrt{1-4\cdot\kappa\varepsilon}$

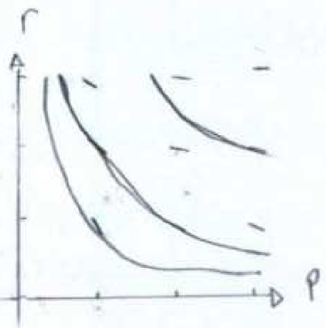
"Saddle point" $\boxed{\text{"linear "center"}}$

d. $\frac{1}{r} = u = \frac{1-\sqrt{1-4\kappa\varepsilon}}{2\varepsilon}$; $\boxed{r = \frac{2\varepsilon}{1-\sqrt{1-4\alpha\varepsilon}}}$

$H = \frac{p^2}{2m} + \frac{kx^2}{2}$ 6.5.8 $\underbrace{\dot{q} = \frac{p}{m}}_{\text{"Momentum"}}$; $\underbrace{\dot{p} = -kx}_{\text{"Force"}}$; $H = \underbrace{\frac{p^2}{2m}}_{\text{"Kinetic Energy"}} + \underbrace{\frac{kx^2}{2}}_{\text{"Potential Energy"}}$

$H(x,p)$ 6.5.9 $\dot{H} = \frac{p}{m}\dot{p} + kx\dot{x} = \frac{p}{m}(-kx) + kx\left(\frac{p}{m}\right) = \boxed{0}$

$H(p,r) = \frac{p^2}{2} + \frac{h^2}{2r^2} - \frac{k}{r}$ 6.5.10 a. The Hamiltonian plot is similar to the potential plot of $1/r^2$ where $k = 1$ and $h = 6.626 \times 10^{-34}$ J·s.



b.

$-k^2/2h^2 < E < 0$	$E = 0$	$E > 0$
• Slope is negative	Slope is zero	slope is positive
• Momentum is decreasing	• Momentum is constant	• Momentum is increasing
• radius is increasing	• Radius is increasing	• radius is increasing.

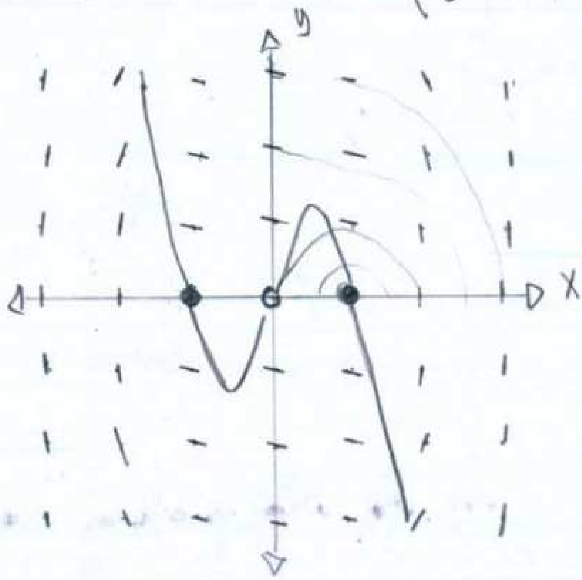
c. If $\kappa < 0$, then $A = \begin{pmatrix} v & 0 \\ 0 & h^2u + u \end{pmatrix}$; $\Delta = v(h^2u + u)$

$\tau = v + h^2u + u$

$\tau^2 - 4\Delta > 0.$

$\boxed{\text{No Periodic orbits}}$

$\dot{x} = y$
$\dot{y} = -by + x - x^3$
6.5.11



$\dot{x} = xy$

$\dot{y} = -x^2$

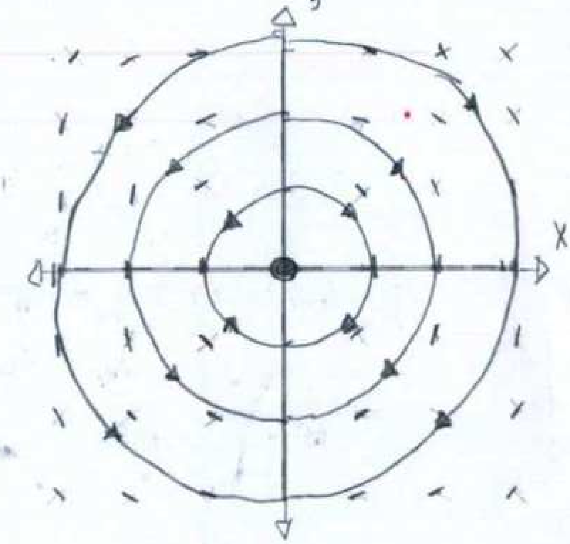
6.5.12.

a. $E = x^2 + y^2$; $E' = 2x \cdot \dot{x} + 2y \cdot \dot{y} = 2x^2y - 2y^2x \boxed{=0}$

b. $(x^*, y^*) = (0,0)$; $A = \begin{pmatrix} y & x \\ -2x & 0 \end{pmatrix}$; $A_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \boxed{=0}$

$(0, y)$; $A_{(0,y)} = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} \boxed{=0}$

c. See part B; Non-isolated Fixed Point.



$\ddot{x} + x + \varepsilon x^3 = 0$

6.5.13.

a. $E = \frac{1}{2}\dot{x}^2 - \int(-x - \varepsilon x^3)\,dx$

$= \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + \frac{\varepsilon}{4}x^4$

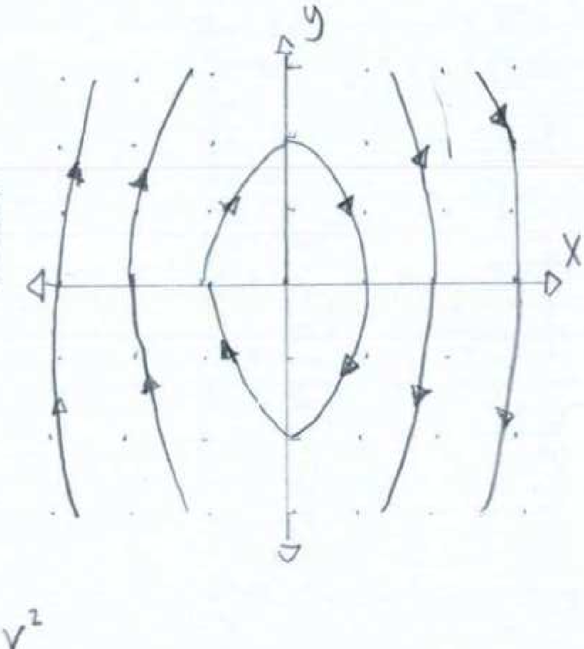
$\begin{vmatrix} E_{xx} & E_{x\dot{x}} \\ E_{\dot{x}x} & E_{\dot{x}\dot{x}} \end{vmatrix} = \begin{vmatrix} 1 + 3\varepsilon x^2 & \dot{x} + x + \varepsilon x^3 \\ \dot{x} + x + \varepsilon x^3 & 1 \end{vmatrix}_{(0,0)} \boxed{=1}$, a continuous derivative exists. at the center i.e. nonlinear center.

b. If $\varepsilon < 0$, a hyperbola trajectory is the closed orbit about $(0,0)$.

$\dot{x} = y$

$\dot{y} = -x - \varepsilon x^3$

For from the origin when $\varepsilon > 0$, this phase plot appears.



$\dot{v} = -\sin\theta - Dv^2$

$v\dot{\theta} = -\cos\theta + v^2$

6.5.14

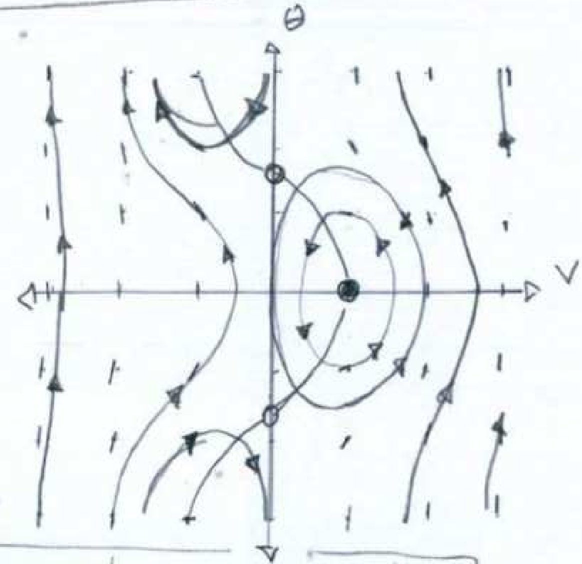
a. If $D = 0$, then $\dot{v} = -\sin\theta$

$v\dot{\theta} = -\cos\theta + v^2$

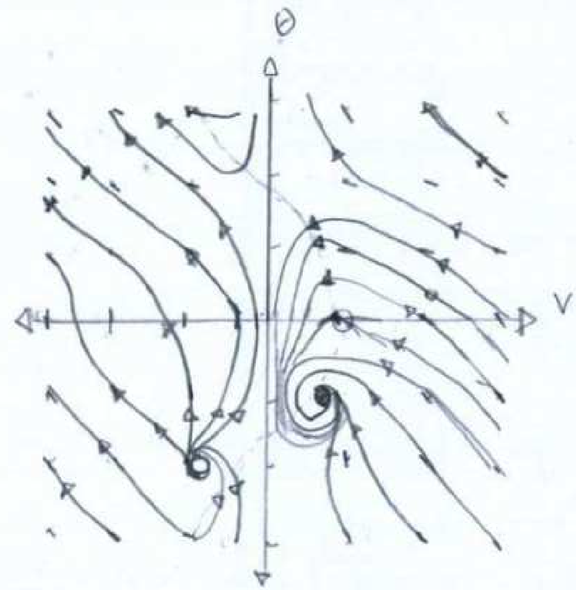
$E = \frac{1}{2}v^2 - \int v\dot{\theta}\,dv = \frac{1}{2}mv^2 + v\cos\theta - \frac{v^3}{3} \boxed{=0}$

$\frac{1}{2}v^2 - 3v\cos\theta + v^3 = 0$; $\frac{dE}{dv} = v - 3\cos\theta + 3v^2$; $\boxed{\text{Fixed Points:}}$
$(v^*, \theta^*) = (1,0), (0,0)$

$\boxed{\text{The potential energy } V(v,\theta) = -3\cos\theta + 3v^2 \text{ has a fixed point at } (0, 2n\pi).}$



b. If $D>0$, then as the glider approaches $v \to \infty$, then the angle becomes more positive and the effect of lift propels the glider upward.



$mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi$

6.5.15

a. $b = 0$

$mr\ddot{\phi} = -mg\sin\phi + mr\omega^2\sin\phi\cos\phi$

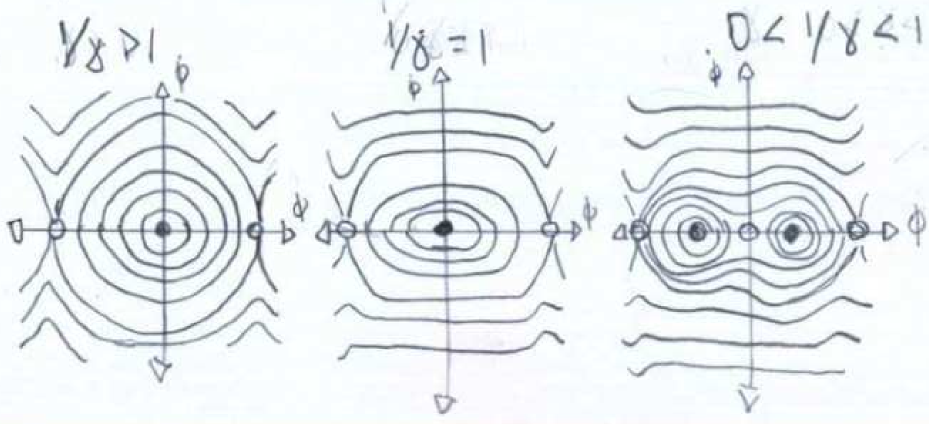
If $\gamma = r\omega^2/g$, then

$\ddot{\phi}\left(\frac{1}{g}\right) = -\sin\phi + \gamma\sin\phi\cos\phi$

$\ddot{\phi}\left(\frac{1}{\gamma g}\right) = \sin\phi(\cos\phi - \gamma^{-1})$

If $\tau = \omega t$, then $\boxed{\phi'' = \sin\phi(\cos\phi - \gamma^{-1})}$

b. $1/\gamma > 1$ $\quad$ $1/\gamma = 1$ $\quad$ $0 < 1/\gamma < 1$



c. The graphs $1/\gamma > 1$ and $1/\gamma = 1$ suggest a periodic stable point when the hoop spins, while $1/\gamma = 1$, a bead that doesn't stay in one place, and spins around the hoop.

6.5.16. $mr\ddot{\phi} = -mg\sin\phi + mr\omega^2 \sin\phi\cos\phi$

$$\ddot{\phi} = -\frac{g}{r}\sin\phi + \omega^2\sin\phi\cos\phi = \sin\phi\left(\omega^2\cos\phi - g/r\right)$$

$$0 = \sin\phi\left(\omega^2\cos\phi - g/r\right) ; \boxed{\phi = \pm\frac{\pi}{2} ; \arccos\frac{g}{r\omega^2}}$$

6.5.17: $mr\ddot{\phi} = -mg\sin\phi + mr\omega^2\sin\phi\cos\phi$

$$E = KE + PE = \frac{1}{2}\dot{\phi}^2 - \int \sin\phi\left(\omega^2\cos\phi - g/r\right)d\phi$$

$$= \frac{1}{2}\dot{\phi}^2 + \cos(\phi)\left(\omega^2\cos\phi - \frac{2g}{r\omega^2}\right)$$

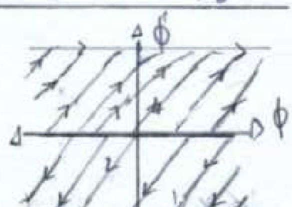
$$\ldots = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\omega^2\cos^2\phi - \frac{g}{r}\cos\phi = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\omega^2(1-\sin^2\phi) - mgr(1-\cos\phi) - mgr$$

$$\boxed{= (KE_{Trans} - KE_{RoT}) + PE}$$

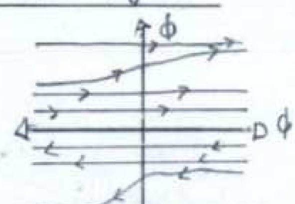
In terms of seperation of motion, the bead hoop problem has translational and rotational energy.

6.5.18. $mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi$

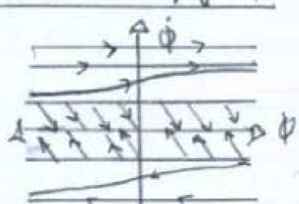
$0 < b < 1 \quad 1/\gamma > 1$



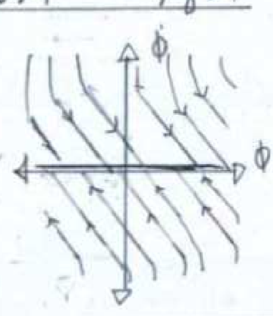
$0 < b < 1 \quad 1/\gamma = 1$



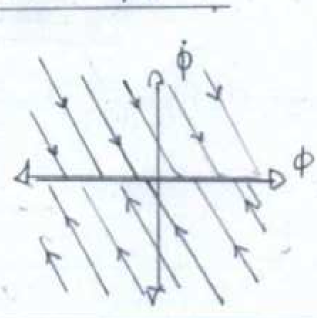
$0 < b < 1 \quad 0 < 1/\gamma < 1$



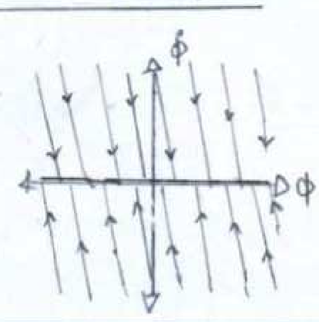
$b > 1 \quad 1/\gamma > 1$



$b > 1 \quad 1/\gamma = 1$



$b > 1 \quad 0 < 1/\gamma < 1$



$\mathring{R} = aR - bRF$
$\mathring{F} = -cF + dRF$

6.5.19. Lotka-Volterra Predator-Prey Model

a. Term

aR : Growth of the rabbit population

$-bRF$: Decrease of the rabbit population by interacting foxes

$-cF$: Decrease of the fox population

dRF: Growth of the fox population by eating rabbits.

An unrealistic assumption is foxes do not decrease when rabbits are not present.

b. $\mathring{R} = R(a-bF)$; $\mathring{R}\left(\frac{1}{a}\right) = \frac{R}{a}\left(1-\frac{b}{a}F\right)$; $x = \frac{d}{c}R$; $y = \frac{b}{a}F$; $\tau = at$

$\mathring{F} = F(dR-c)$; $\mathring{y} = \frac{cy}{a}(x-1)$; $\boxed{\mathring{y} = \mu y(x-1) \; ; \; \mathring{x} = x(1-y)}$

c. $\mathring{x} = 0 = x(1-y)$; $\mathring{y} = 0 = \mu y(x-1)$; $(x^*, y^*) = \boxed{\begin{matrix}(0,0)\\(1,1)\end{matrix}}$

d. $A = \begin{pmatrix} 1-y & -xy \\ \mu y & \mu(x-1) \end{pmatrix}$

$A_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}$; $\Delta = \mu$; $\tau = 1+\mu$; $\tau^2 - 4\Delta > 0$

$\boxed{\text{"Unstable node"}}$

$A_{(1,1)} = \begin{pmatrix} 0 & -1 \\ \mu & 0 \end{pmatrix}$; $\Delta = -\mu$; $\tau = 0$; $\tau^2 - 4\Delta > 0$

$\boxed{\text{"Center" cycle.}}$

$\mathring{P} = P(R-S)$
$\mathring{R} = R(S-P)$
$\mathring{S} = S(P-R)$

6.5.20.

a. The terms found in $\mathring{P}, \mathring{R}$, and $\mathring{S}$ relate the existence of paper, rock, and scissors, but also, a relationship when each type of species is present at any given time.

b. $P + R + S = PR - PS + RS - RP + SP - SR = \boxed{0}$

c. $E(P,R,S) = P + R + S$; $E_2(R,R,S) = PRS$

"Plane" "Multiplane"

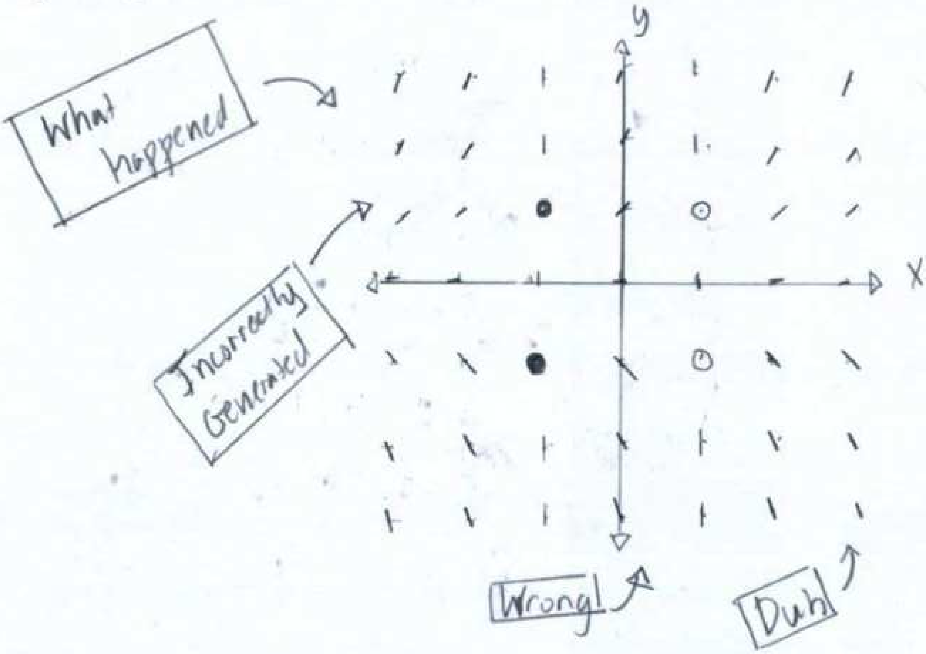
As $t \to \infty$, then a discrete solution exists of integer values between planes or amounts of P, R, S

$\dot{x} = y(1-x^2)$
$\dot{y} = 1-y^2$

6.6.1. Reversible if $t \to -t$, $x \to -x$, $y \to -y$

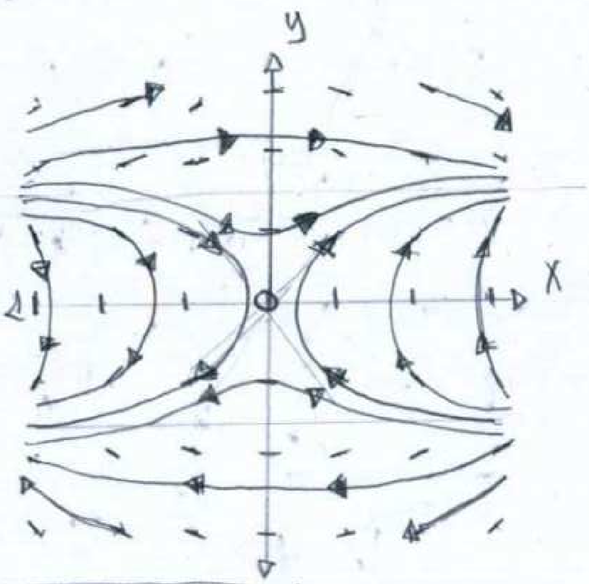
$\boxed{\text{Fixed Points:}}$ $\dot{x} = y(1-x^2) = 0$

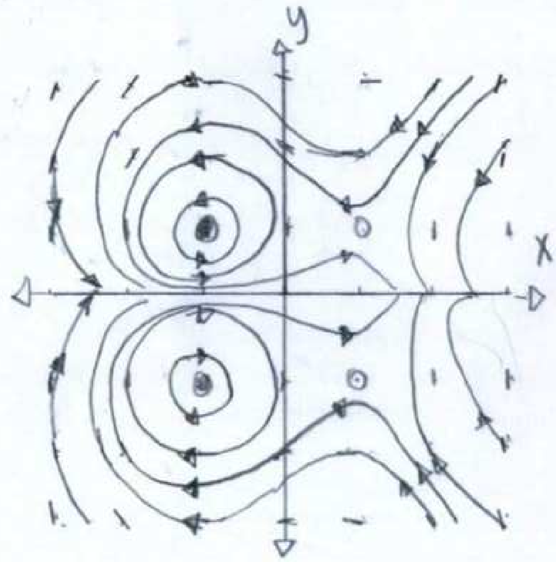
$\dot{y} = 1-y^2 = 0$; $(x^*, y^*) = (\pm 1, \pm 1)$



$\dot{x} = y$
$\dot{y} = x\cos y$

6.6.2.





$\boxed{\text{Fixed Points:}}$ $\dot{x} = 0 = y$;

$\dot{y} = 0 = x\cos y = x\cos(-y) = -x\cos(-y)$

$(x^*, y^*) = (0, 0)$

$\dot{x} = \sin y$
$\dot{y} = \sin x$

6.6.3.

a. $\frac{dy}{dx} = \left(\frac{-1}{-1}\right)\frac{dy}{dx} = \frac{-\sin x}{-\sin y} \overset{!}{=} \boxed{\frac{\sin x}{\sin y}}$

b. $\boxed{\text{Fixed Points:}}$ $\dot{x} = 0 = \sin y$; $(x^*, y^*) = (n\pi, n\pi)$

$\dot{y} = 0 = \sin x$

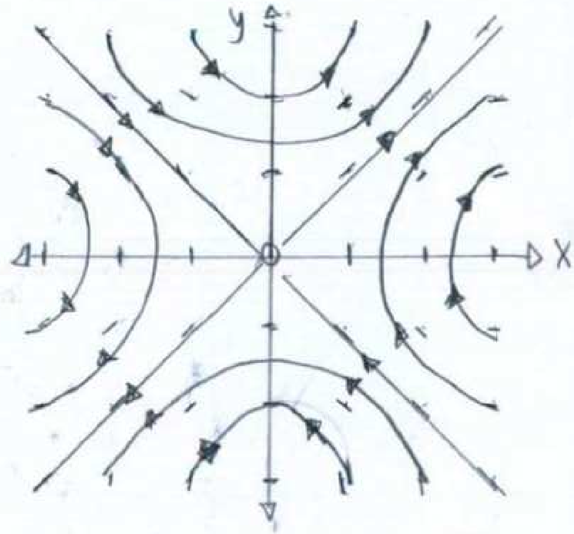
Where $n \in \mathbb{R}$

If n is even, stable node, else unstable node.

c. $\dot{x} = \sin y$; $\sin y = \sin x$; $\boxed{y = \pm x}$

$\dot{y} = \sin x$

d. $\rightarrow$



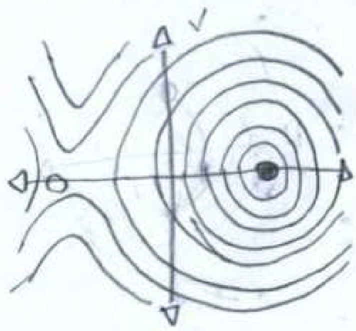
$\ddot{x} + (\dot{x})^2 + x = 3$

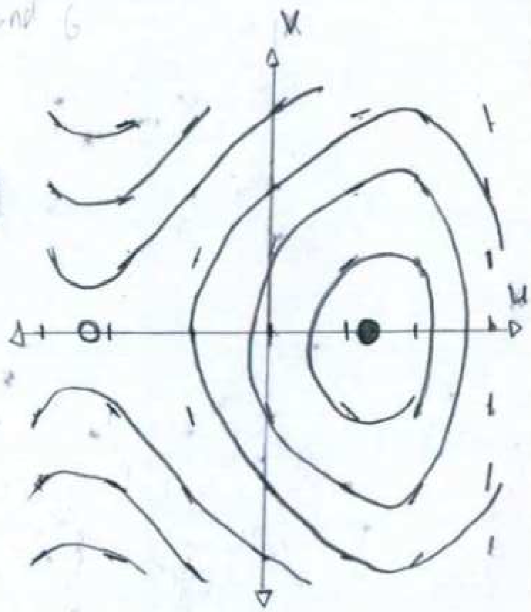
6.6.4. $\dot{u} = \dot{x} = v$

a. $\dot{v} = \ddot{x} = 3 - \dot{x}^2 + x = 3 - u^2 + u$

$\boxed{\text{Fixed Points}}$ $(u^*, v^*) = \left(0, \frac{-1}{2} \pm \frac{\sqrt{13}}{2}\right)$

$\boxed{\text{Guess}}$





$\dot{x} = y - y^3$

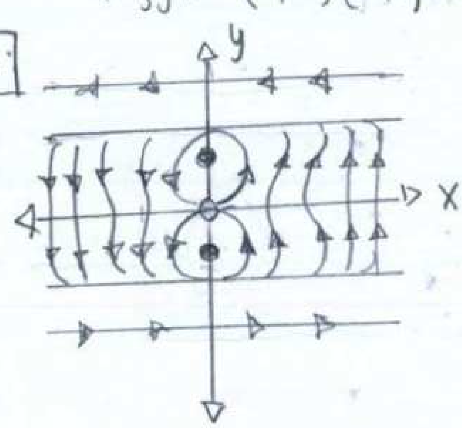
$\dot{y} = x\cos y$

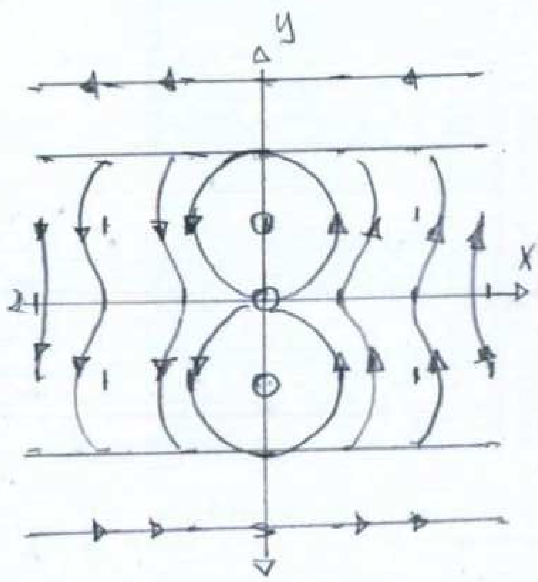
b. $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = y - y^3$

$\dot{y} = 0 = x\cos y$

$(x^*, y^*) = (0,0), (\mp x, 1)$

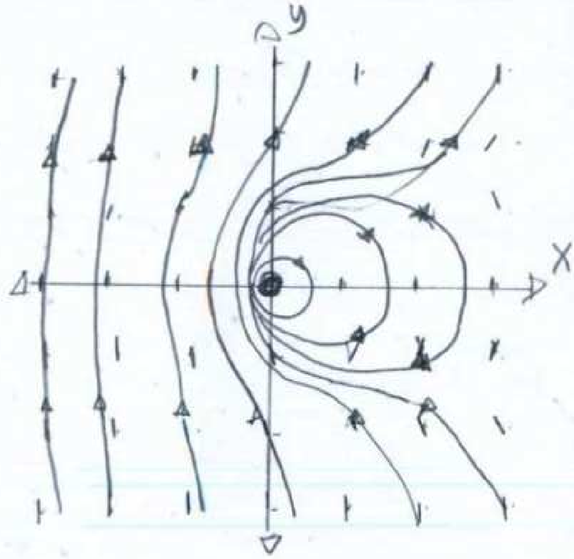
$\boxed{\text{Guess}}$



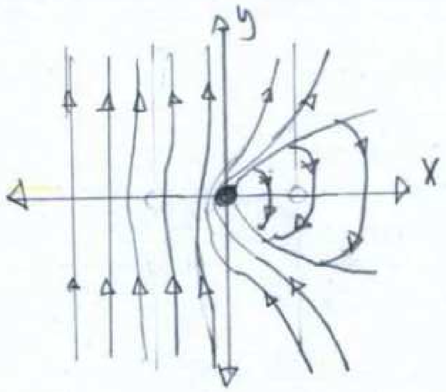


$\dot{x} = \sin y$
$\dot{y} = y^2 - x$

c. **Fixed Points** $\dot{x} = 0 = \sin y$
$\dot{y} = 0 = y^2 - x$
$(x^*, y^*) = (0,0), (1, \pm 1)$



Guess:



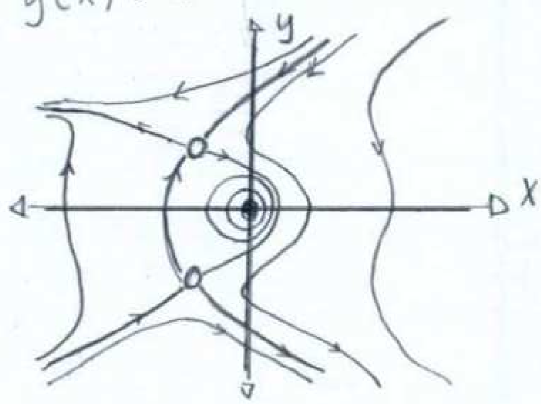
$\ddot{x} + f(\dot{x}) + g(x) = 0$ 6.6.5. f is an even function; g is an odd function
f & g are smooth

a. $-\ddot{x} + f(-\dot{x}) + g(-x) = -\ddot{x} - f(\dot{x}) - g(x) = \boxed{\ddot{x} + f(\dot{x}) + g(x)}$

b. $\dot{u} = \dot{x} = v$ Definition of a reversible system
$\dot{v} = \ddot{x} = -f(\dot{x}) - g(x)$; is no stable nodes or spirals.

$\dot{x} = y - y^3$
$\dot{y} = -x - y^2$

6.6.6.
a. $\boxed{\dot{x} = 0; \dot{y} = 0}$



b. Quadrant #1: $\dot{x} < 0$; $\dot{y} < 0$ Quadrant #3: Mixed
Quadrant #2: Mixed Quadrant #4: $\dot{x} > 0$; $\dot{y} < 0$

c. $A = \begin{pmatrix} 0 & 1-2y^2 \\ -1 & -2y \end{pmatrix}$; $A_{(-1,\pm 1)} = \begin{pmatrix} 0 & -1 \\ -1 & \pm 2 \end{pmatrix}$; $\Delta = (1-\sqrt{2})(1+\sqrt{2})$
$\tau = 2$
$\tau^2 - 4\Delta > 0$

$\lambda_1 = (1-\sqrt{2})$; $\boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

$\lambda_2 = (1+\sqrt{2})$; $\boxed{v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

d. $(-1,-1)$; If Quadrant #2 and #3 are mixed sign then a possible trajectory through $x<0$ may exist. A heteroclinic trajectory that does cross from $(-1,-1)$ to $(-1,1)$ is present because of the reversible function.

e. Other examples of a heteroclinic trajectory relate to the third fixed point. See part b.

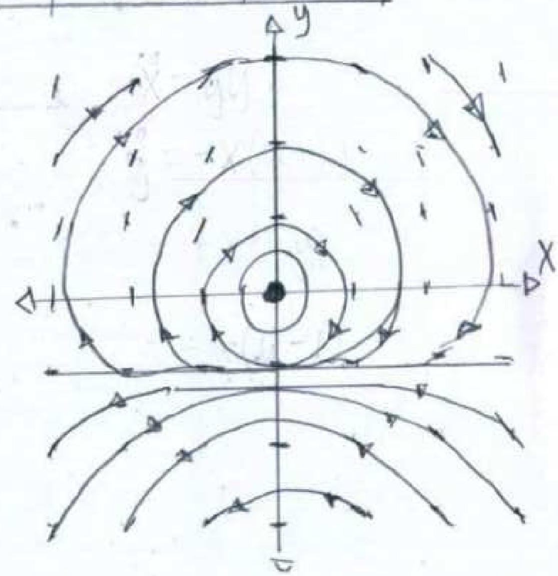
$\ddot{x}+x\dot{x}+x=0$ 6.6.7. $\dot{x}=y$

$$\dot{y}=-x(\dot{x}+1)=-x(y+1)$$

$\boxed{\text{Reversibility}}$ $-\ddot{x}-x\cdot(-\dot{x})-x$

$$=\ddot{x}+x\dot{x}+x$$

$$=0$$



$\dot{x}=\frac{\sqrt{2}}{4}x(x-1)\sin\phi$

$\dot{\phi}=\frac{1}{2}\left[\beta-\frac{1}{\sqrt{2}}\cos\phi-\frac{1}{8\sqrt{2}}x\cos\phi\right]$

6.6.8. a. $\boxed{\text{Reversibility}}$ $x\to -x$; $\phi\to -\phi$

$$\dot{x}=\frac{\sqrt{2}}{4}-x(-x-1)\sin(-\phi)$$

$$=-\frac{\sqrt{2}}{4}x(1-x)\sin(\phi)$$

$$=\frac{\sqrt{2}}{4}x(x-1)\sin\phi$$

$$\dot{\phi}=\frac{1}{2}\left[\beta-\frac{1}{\sqrt{2}}\cos(-\phi)-\frac{1}{8\sqrt{2}}(-x)\cos(-\phi)\right]$$

$$=\frac{1}{2}\left[\beta-\frac{1}{\sqrt{2}}\cos(\phi)+\frac{1}{8\sqrt{2}}x\cos\phi\right]$$

b. $\dot{x}=0=\frac{\sqrt{2}}{4}x(x-1)\sin\phi$

$$\dot{\phi}=\frac{1}{2}\left[\beta-\frac{1}{\sqrt{2}}\cos\phi-\frac{1}{8\sqrt{2}}x\cos\phi\right]=0$$

$(x^*,\phi^*)=$ $(0, 2\pi n-\cos^{-1}(\sqrt{2}\cdot\beta))$, $(0, 2\pi n+\cos^{-1}(\sqrt{2}\beta))$

$\left(1, 2\pi n-\cos^{-1}\left(\frac{8\sqrt{2}\beta}{x+8}\right)\right)$, $(0,0)$

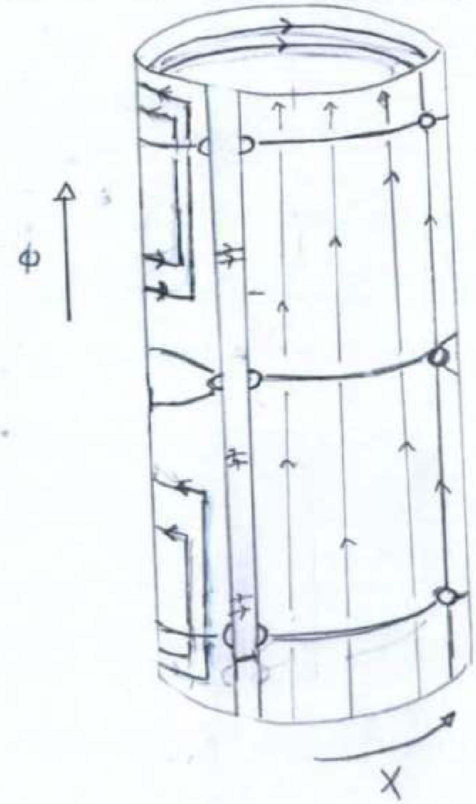
$\left(1, 2\pi n+\cos^{-1}\left(\frac{8\sqrt{2}\beta}{x+8}\right)\right)$

A homoclinic orbit is a nullcline.

$$\mathring{x} = 0 = \frac{\sqrt{2}}{4} x(x-1)\sin\phi \; ; \; \mathring{\phi} = 0 = \frac{1}{2}\left[\beta - \frac{1}{\sqrt{2}}\cos\phi - \frac{1}{8\sqrt{2}} x\cos\phi\right]$$

$$\boxed{x = -4\sqrt{2}\sec(\phi)(\sqrt{2}\cos(\phi) - 2\beta)}$$

c. $\lim_{\beta \to \frac{1}{\sqrt{2}}} 2\pi n - \cos^{-1}(2\sqrt{\beta}) = 2\pi n$; , then $(x^*, \phi^*) = (0, 2\pi n)$ and the node on the line $\phi = 0$ is closer to $(0,0)$, and the cylinder becomes a smaller diameter shape. With less closed orbits.



d. See Part c: cylinder

$$\frac{d\phi_k}{d\tau} = \Omega + a\sin\phi_k + \frac{1}{N}\sum_j^N \sin\phi_j$$

6.6.9.

a. $\theta_K = \phi_K - \frac{\pi}{2}$; $\frac{d\theta}{d\tau} = \Omega + a\cos\theta_k + \frac{1}{N}\sum_i \cos\theta_K$

$$= \Omega + a\cos(-\theta_K) + \frac{1}{N}\sum_i \cos(-\theta_K)$$

b. Fixed Points: $\mathring{\theta} = 0 = \Omega + a\cos\theta_K + \frac{1}{N}\sum_i \cos\theta_K$

$$\Omega = -a\cos\theta_K - \frac{1}{N}\sum_i \cos\theta_K$$

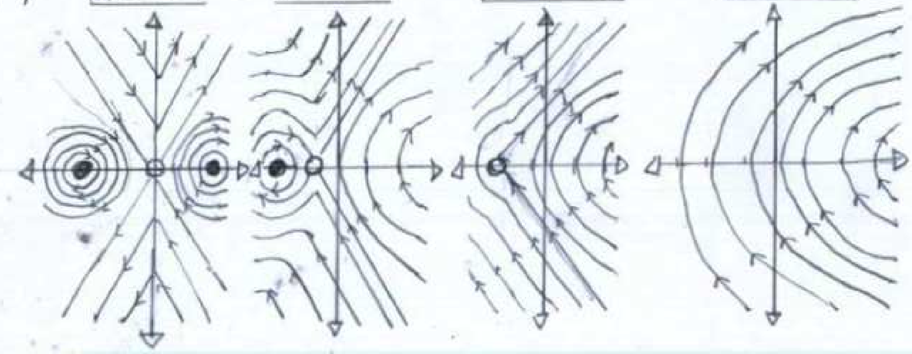
$$= -\cos\theta_K(a+1)$$

$$\cos\theta_K = \left|\frac{\Omega}{a+1}\right|$$

If $\left|\frac{\Omega}{a+1}\right| < 1$, then $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

If $\left|\frac{\Omega}{a+1}\right| > 1$, then no fixed point is generated because $\cos\theta$ is never greater than 1.

c. $a = 1$; $\boxed{\Omega = 0}$ $\boxed{\Omega = 1}$ $\boxed{\Omega = 2}$ $\boxed{\Omega = 3}$



$\dot{x} = -y - x^2$ 6.6.10 $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = -y - x^2$; $(x^*, y^*) = (0, 0)$

$\dot{y} = x$ $\dot{y} = 0 = x$

$$A = \begin{pmatrix} -2x & -1 \\ 1 & 0 \end{pmatrix} ; \quad A_{(0,0)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\Delta = -1$; $\tau = 0$; $\tau^2 - 4\Delta > 0$

"Saddle Point"

No, a nonlinear center is an isolated fixed point with closed orbits

$\dot{\theta} = \cot\phi \cos\theta$ 6.6.11. a. $\boxed{\text{Reversibility}}$ $t \to -t$; $\theta \to -\theta$

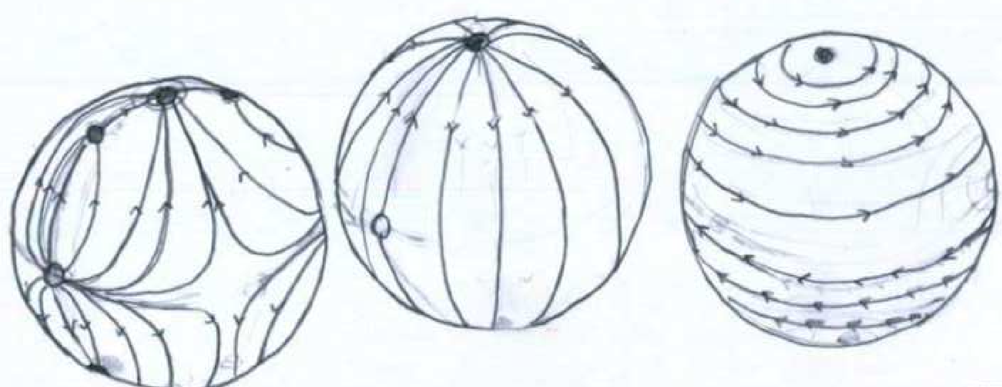
$\dot{\phi} = (\cos^2\phi + A\sin^2\phi)\sin\theta$

$$\dot{\theta} = \cot(+\phi)\cos(-\theta)$$
$$= \cot(+\phi) \cdot \cos(\theta)$$

$t \to -t$; $\phi \to -\phi$

$$\dot{\phi} = [\cos^2(-\phi) + A\sin(-\phi)^2] \cdot \sin(\theta)$$
$$= [\cos^2(\phi) + A\sin^2(\phi)] \cdot \sin(\theta)$$

b. $\boxed{A = -1}$ $\boxed{A = 0}$ $\boxed{A = 1}$

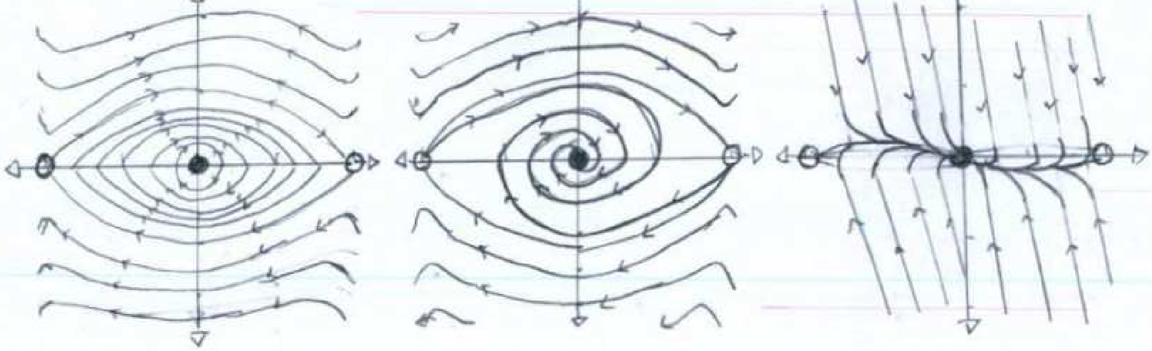


c. As $t \to \infty$, each case of shear flow trajects to a stable node. This implies rotation of a body does not freely rotate in medium.

$\ddot{\theta} + b\dot{\theta} + \sin\theta = 0$ 6.7.1. $\boxed{\text{Fixed Points}}$ $\dot{x} = \dot{\theta} = y$

$\dot{y} = \ddot{\theta} = -(by + \sin x)$

$\boxed{b=0}$ $\boxed{0 < b \leq 1}$ $\boxed{1 < b}$



$\ddot{\theta} + \sin\theta = \gamma$ 6.7.2 a. $\boxed{\text{Fixed Points}}$ $\dot{x} = \dot{\theta} = y = 0$

$\dot{y} = \ddot{\theta} = \gamma - \sin\theta = \gamma - \sin x$

$(x^*, y^*) = (\arcsin\gamma, 0)$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}; \quad A = \begin{pmatrix} 0 & 1 \\ \gamma - \cos x & 0 \end{pmatrix}$$

$\Delta = \cos x - \gamma$; $\tau = 0$; $\tau^2 - 4\Delta > 0$

If $\gamma = 0$, (x^*, y^*) is a center

If $0 < \gamma < 1$, (x^*, y^*) is a center

If $1 < \gamma$, (x^*, y^*) is a saddle point.

b. $\boxed{\text{Nullclines}}$ $y = \gamma - \sin x$

$E = \frac{1}{2}\dot{x}^2 - \int \gamma - \sin x \, dx$

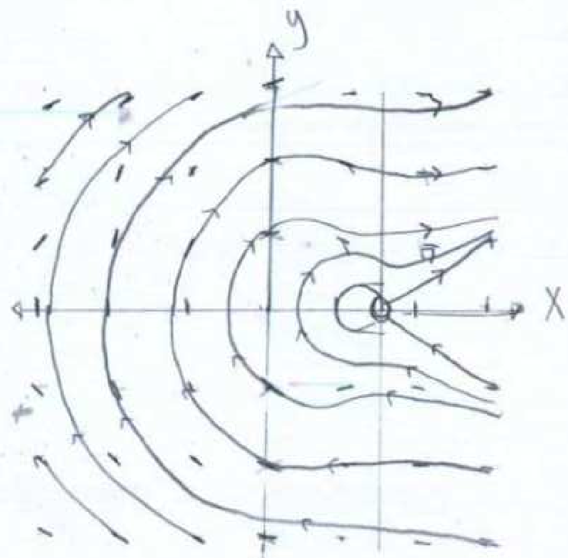
$= \frac{1}{2}y^2 - \gamma x - \cos x$

The system is not conservative because of no closed loops

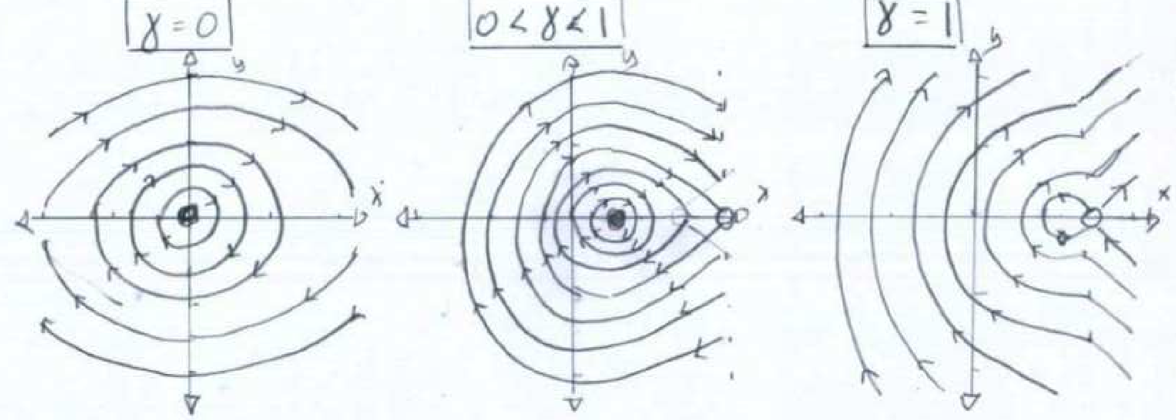
$\boxed{\text{Reversibility}}$

$\dot{x} = -y \neq y$

The system is not reversible.



d



e. $\gamma = 0$; $\dot{x} = 0 = y$; $\dot{y} = 0 = -\sin x$; $y = \dot{\theta} = -\sin\theta$

$0 < \gamma < 1$; $\dot{x} = 0 = y$; $\dot{y} = 0 = \gamma - \sin x$; $y = \dot{\theta} = \gamma - \sin\gamma$

$\gamma = 1$; ... $\dot{\theta} = 1 - \sin\theta$

$\ddot{\theta} + (1 + a\cos\theta)\dot{\theta} + \sin\theta = 0$

6.7.3 $\dot{x} = \dot{\theta} = y$

$\dot{y} = -(1 + a\cos x)\,y - \sin x$

Fixed Points: $(x^*, y^*) = (0, 0)$

	Yes	No
Reversible	☐	☑
Conservative	☐	☑

$\ddot{\theta} + \sin\theta = 0$ 6.7.4

a. $PE = mgh = mgL(1 - \cos(\theta))$; $KE = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{\theta})^2$

$E = PE + KE = mgL(1 - \cos(\theta)) + \frac{1}{2}m\dot{\theta}^2 = 0$

$\dot{\theta}^2 = -2gL(1 - \cos(\theta))$; If $\theta = \alpha$ = max height; $\dot{\theta}^2 = 0$

$\boxed{= 2(\cos(\theta) - \cos(\alpha))}$

$$T = 4\int_0^\alpha dt = 4\int_0^\alpha \frac{d\theta}{\dot{\theta}} = 4\int_0^\alpha \frac{d\theta}{\sqrt{2(\cos\theta - \cos\alpha)}}$$

b. Half-Angle Formula: $\cos(2A) = 1 - 2\sin^2 A$ where $A = \frac{\theta}{2}$ or $\frac{\alpha}{2}$

$$\boxed{T = 4\int_0^\alpha \frac{d\theta}{[4(\sin^2\frac{1}{2}\alpha - \sin^2\frac{1}{2}\theta)]^{1/2}}}$$

c. Half-Angle Formula: $(\sin\frac{1}{2}\alpha)\sin\phi = \sin\frac{1}{2}\theta$

$$\frac{1}{2}\sin\frac{1}{2}\alpha\cos\phi\frac{d\phi}{d\theta} = \frac{1}{2}\cos\frac{\theta}{2}$$

$$d\theta = \frac{\cos\theta/2}{\sin\alpha/2\,\cos\phi}\,d\phi$$

$$T = 4\int_0^{\pi/2} \frac{d\phi}{\cos\theta/2} = 4 \cdot x \boxed{\underbrace{\left[\int_0^{\pi/2} \frac{d\phi}{(1-m\sin^2\phi)^{1/2}}\right]}_{\text{"Elliptic Integral"}}}$$

d. Binomial Series $\frac{1}{(1-x)^{1/2}} = 1 + \frac{1}{2}x + \cdots$

$$T = 4 \times \int_0^{\pi/2} \left(1 + \tfrac{1}{2} m \sin^2\phi + \cdots\right) d\phi \; ; \; m = \sin^2 \frac{x}{2}$$

$$\boxed{= 2\pi\left[1 + \tfrac{1}{16}x^2 + \cdots\right]}$$

6.7.5.

Numerical Integration of $T = 4 \times \sum_{i=0}^{18}\sum_{j=0}^{9}\left(1 + \frac{1}{2}\left[\sin^2\frac{10i}{2}\right]\sin^2\frac{10j}{2} + \cdots\right)$

i \ j	0	1	2	3	4	5	6	7	8	9
0	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00
1	4.00	5.69	4.54	4.78	5.53	4.05	5.80	4.34	5.02	5.33
2	4.00	4.54	4.18	4.25	4.49	4.01	4.59	4.11	4.33	4.43
3	4.00	4.78	4.25	4.36	4.70	4.01	4.83	4.16	4.48	4.61
4	4.00	5.53	4.49	4.70	5.39	4.03	5.63	4.31	4.93	5.21
5	4.00	4.03	4.01	4.01	4.03	4.00	4.03	4.01	4.02	4.03
6	4.00	5.80	4.59	4.83	5.63	4.03	5.91	4.36	5.09	5.41
7	4.00	4.34	4.11	4.16	4.31	4.01	4.36	4.07	4.20	4.27
8	4.00	5.02	4.33	4.47	4.93	4.02	5.09	4.20	4.62	4.80
9	4.00	5.33	4.43	4.61	5.21	4.03	5.41	4.27	4.80	5.05
10	4.00	4.13	4.04	4.06	4.11	4.00	4.13	4.03	4.08	4.10
11	4.00	5.94	4.59	4.85	5.67	4.04	5.95	4.25	5.11	5.45
12	4.00	4.17	4.05	4.08	4.15	4.00	4.18	4.22	4.10	4.13
13	4.00	5.26	4.40	4.58	5.14	4.02	5.33	4.25	4.76	4.99
14	4.00	5.10	4.35	4.51	5.00	4.02	5.17	4.22	4.67	4.87
15	4.00	4.28	4.09	4.13	4.23	4.01	4.29	4.06	4.17	4.22
16	4.00	5.92	4.59	4.84	5.65	4.03	5.93	4.36	5.10	5.43
17	4.00	4.06	4.02	4.03	4.05	4.00	4.06	4.01	4.03	4.04
18	4.00	5.47	4.47	4.68	5.33	4.03	5.56	4.29	4.89	5.16

$$\boxed{T = 852.06}$$

6.8.1

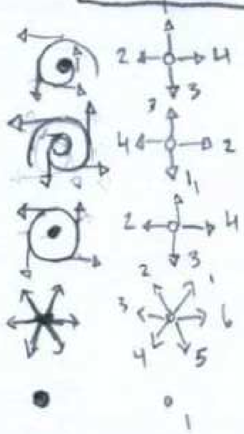
a. Stable Spiral $I_c = +1$

b. Unstable spiral $I_c = +1$

c. center $I_c = +1$

d. Star $I_c = +1$

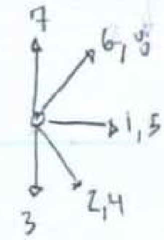
e. Degenerate Node. $I_c = +1$

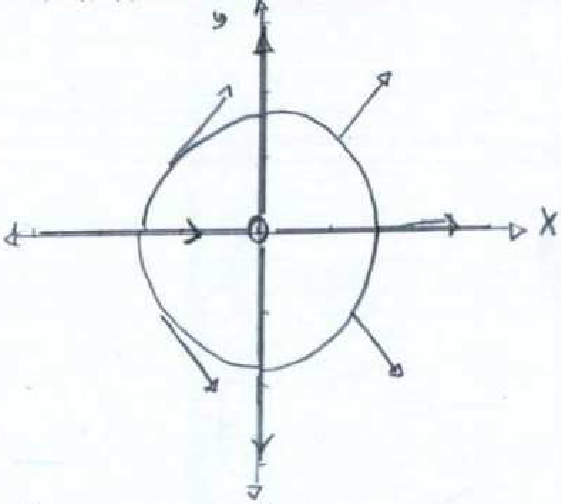


$\dot{x} = x^2$ 6.8.2 $\boxed{\text{Fixed Points}}$: $\dot{x} = 0 = x^2$; $A = \begin{pmatrix} 2x & 0 \\ 0 & 1 \end{pmatrix}$: $\Delta = 0$; $\tau = 1$; $\tau^2 - 4\Delta > 0$
$\dot{y} = y$ $\dot{y} = 0 = y$

"Non-isolated Fixed Points"

$(x^*, y^*) = (0,0)$; $\boxed{\text{Index}}$



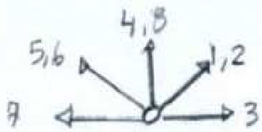


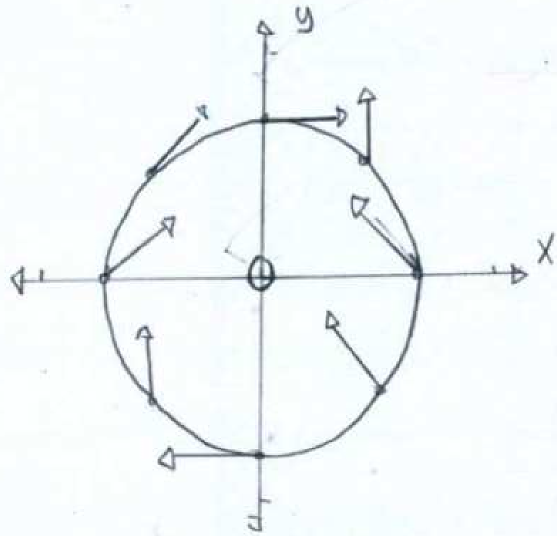
$I_c = 0$

$\dot{x} = y - x$ 6.8.3. $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = y - x$; $A = \begin{pmatrix} -1 & 1 \\ 2x & 0 \end{pmatrix}$; $\Delta = 0$; $\tau = -1$; $\tau^2 - 4\Delta > 0$
$\dot{y} = x^2$ $\dot{y} = 0 = x^2$

"Spiral sink"

$(x^*, y^*) = (0,0)$; $\boxed{\text{Index}}$



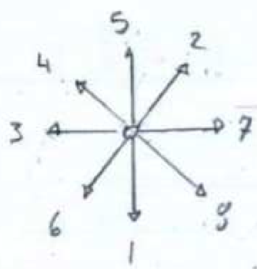


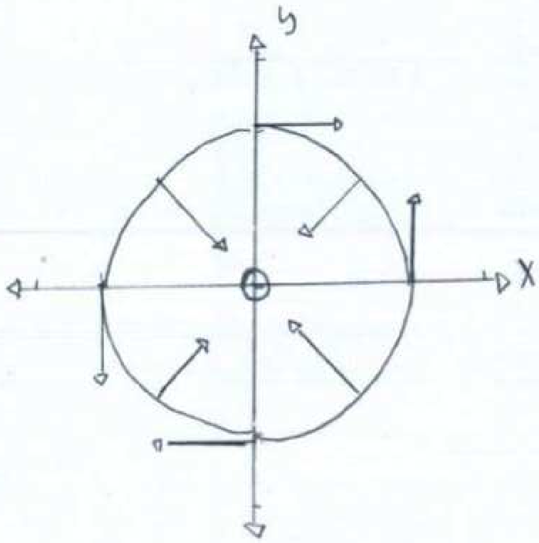
$I_c = 0$

$\dot{x} = y^3$ 6.8.4 $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = y^3$ $A = \begin{pmatrix} 0 & 3y^2 \\ 1 & 0 \end{pmatrix}$; $\Delta = 0$; $\tau = 0$; $\tau^2 - 4\Delta = 0$
$\dot{y} = x$ $\dot{y} = 0 = x$

"Saddle"

$(x^*, y^*) = (0,0)$ $\boxed{\text{Index}}$





$I_c = 0$

$\dot{x} = xy$
$\dot{y} = x + y$

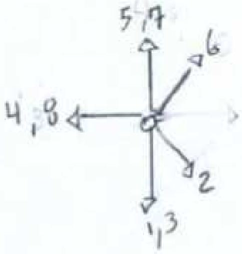
6.8.5 $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = xy$

$\dot{x} =$

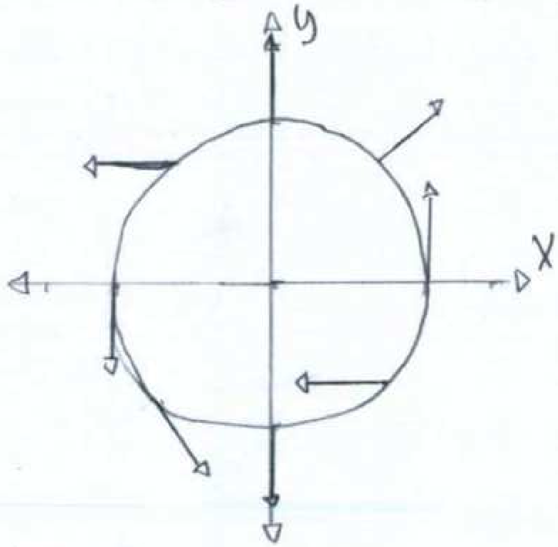
$\dot{y} = 0 = x + y$

$(x^*, y^*) = (0,0)$ $\boxed{\text{Index}}$

$A = \begin{pmatrix} y & x \\ 1 & 1 \end{pmatrix}$; $\Delta = 0$; $\tau = 0$;
$\tau^2 - 4\Delta$ "unstable saddle"



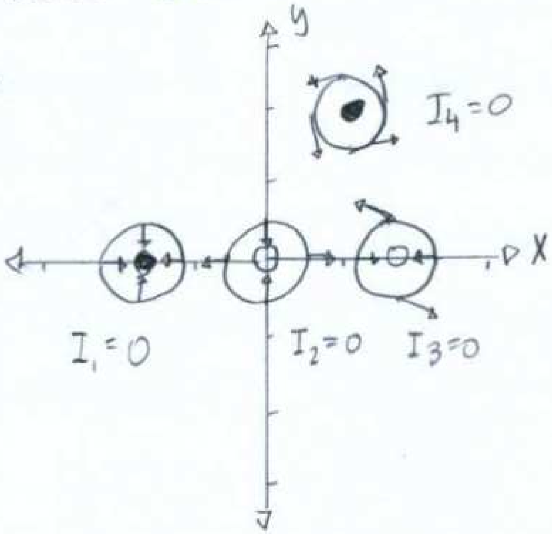
$I_c = 0$



6.8.6. Node [N] : $I_c = +1$
Spiral [F] : $I_c = +1$
Center [C] : $I_c = -1$
Saddle [S] : $I_c = 0$

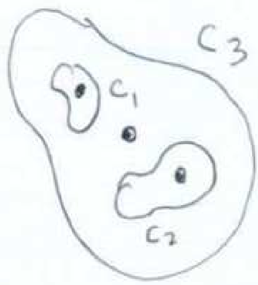
$N + S + C = +1 = 1 + S = +1$

$\dot{x} = x(4 - y - x^2)$ 6.8.7.
$\dot{y} = y(x - 1)$



The indices of each fixed point are zero ($I_c = 0$), thus, no closed orbits exist.

6.8.8. a.



b. $I_c = I_1 + I_2 + I_3 > 0$; A fixed point exists in the closed orbit.

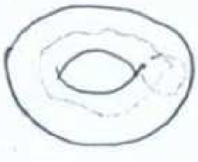
6.8.9. C_1, C_2 = Closed Trajectories

If C_1 is clockwise, $I_c < 0$

If C_2 is counter-clockwise, $I_c > 0$

$\boxed{\text{A fixed point in } C_2 \text{ is true because } I_c > 0}$

6.8.10 Torus   Cylinder   Sphere



$I_c > 0$ $I_c = 0$ $I_c > 0$

$\boxed{\text{Theorem 6.8.2 is reasonable for closed orbit shapes.}}$

$\dot{z} = z^k$

$\dot{z} = (\bar{z})^k$

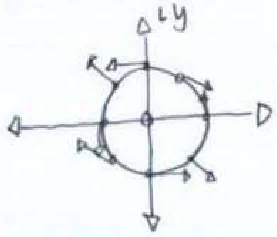
6.8.11

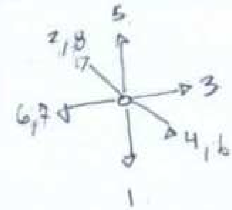
a. $k=1$; $\dot{z} = z = x+iy$; $\langle x, y\rangle$

$k=2$; $\dot{z} = z^2 = (x+iy)^2 = x^2-y^2-2ixy$; $\boxed{\langle x^2-y^2, -2xy\rangle}$

$k=3$; $\dot{z} = z^3 = (x+iy)^3 = x^3-3xy^2+i(3x^2y-y^3)$; $\boxed{\langle x^3-3xy^2, 3x^2y-y^3\rangle}$

b. $z^* = (0,0)$;





$I_c = 0$

c. The expansion is similar to a Binomial.

$$\left\langle \sum_{k=1}^{2k\le n} \binom{n}{2k} x^{n-2k} \cdot (-1)^k \cdot y^{2k}, \sum_{k=1}^{2k+1\le n} \binom{n}{2k+1} x^{n-2k} (-1)^k y^{2k} \right\rangle$$

$\dot{x} = a + x^2$

$\dot{y} = -y$

6.8.12.

a. $\boxed{\text{Fixed Points}}$ $\dot{x} = 0 = a + x^2$

$\dot{y} = 0 = -y$

$(x^*, y^*) = (\pm i\sqrt{a}, 0)$

$A = \begin{pmatrix} 2x & 0 \\ 0 & -1 \end{pmatrix}$; $\Delta_1 = -2i\sqrt{a}$; $\tau_1 = 2i\sqrt{a} - 1$

$\Delta_2 = 2i\sqrt{a}$; $\tau_2 = -2i\sqrt{a} - 1$

$\boxed{\text{Fixed points in } \mathbb{R}^3 \text{ are non-existent.}}$

b. $I_c = I_1 + I_2 = \boxed{0}$ because the imaginary fixed points $(\pm i\sqrt{a}, 0)$ are symmetric.

c. $\dot{x} = f(x, a)$ where $x \in \mathbb{R}^2$ a conserved index is independent of a, and is the sum of two indices.

$\dot{x} = f(x,y)$ 6.8.13 $\phi = \tan^{-1}(\dot{y}/\dot{x})$

$\dot{y} = g(x,y)$

a. $\frac{d}{dj}\tan^{-1} j = \frac{1}{j^2+1}$; $d\phi = \frac{1}{\left(\frac{\dot{y}}{\dot{x}}\right)^2+1} \cdot \left(\frac{\dot{y}}{\dot{x}}\right)' = \frac{\dot{x}^2}{\dot{y}^2 + \dot{x}^2} \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2}$

$$\boxed{= \frac{f\,dg - g\,df}{f^2 + g^2}}$$

b. $I_c = \frac{d}{d\phi}\tan^{-1}(\phi) = \boxed{\frac{1}{2\pi}\oint \frac{f\,dg - g\,df}{f^2+g^2}}$ where $\phi = \frac{\dot{y}}{\dot{x}}$

$\dot{x} = x\cos\alpha - y\sin\alpha$ 6.8.14.

$\dot{y} = x\sin\alpha + y\cos\alpha$

a. $\boxed{\text{Fixed Points}}$ $(x^*, y^*) = (0,0)$

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}; \quad \Delta = \cos\alpha\sin\alpha, \quad \tau = \cos\alpha + \sin\alpha$$

$\tau^2 - 4\Delta < 0$

$\boxed{\text{"Unstable Spiral"}}$

b. $$I_c = \frac{1}{2\pi}\oint \frac{(x\cos\alpha - y\sin\alpha)(x\sin\alpha + y\cos\alpha)' - (x\sin\alpha + y\cos\alpha)(x\cos\alpha - y\sin\alpha)'}{(x\cos\alpha - y\sin\alpha)^2 + (x\sin\alpha + y\cos\alpha)^2}\,dc$$

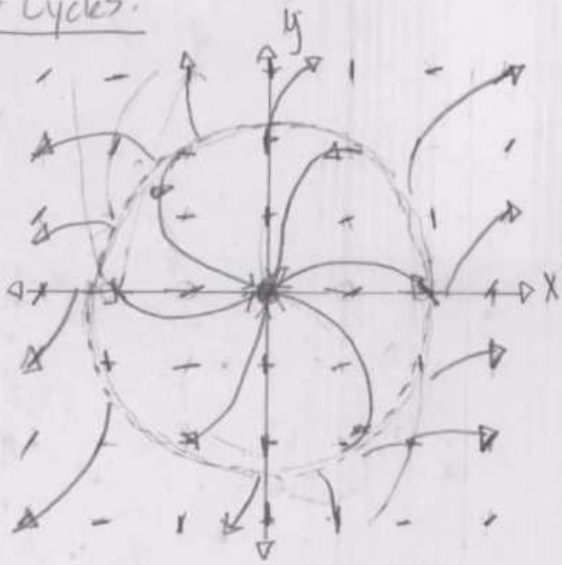
$$= \frac{1}{2\pi}\oint 1 = \boxed{1}$$

c. $\boxed{I_c = 1}$

Chapter 7: Limit Cycles:

$\dot{r} = r^3 - 4r$ 7.1.1

$\dot{\theta} = 1$



$$\frac{dy}{dx} = \frac{dy}{dx}\frac{d\theta}{d\theta} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r\cos\theta + \sin\theta\frac{dr}{d\theta}}{-r\sin\theta + \cos\theta\frac{dr}{d\theta}}$$

Where $r = \sqrt{x^2+y^2}$; $\theta = \arctan\frac{y}{x}$

and $\frac{dr}{d\theta} = (\sqrt{x^2+y^2})^3 - 4\sqrt{x^2+y^2}$

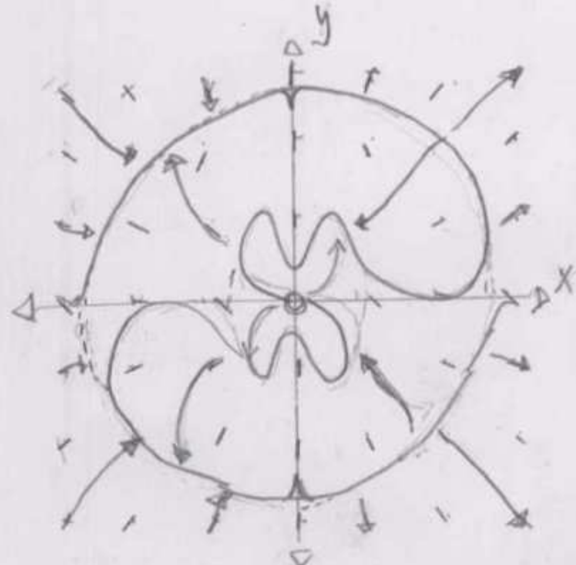
$\dot{r} = r(1-r^2)(9-r^2)$ 7.1.2

$\dot{\theta} = 1$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + \sin\theta\frac{dr}{d\theta}}{r\cos\theta + \cos\theta\frac{dr}{d\theta}}$$

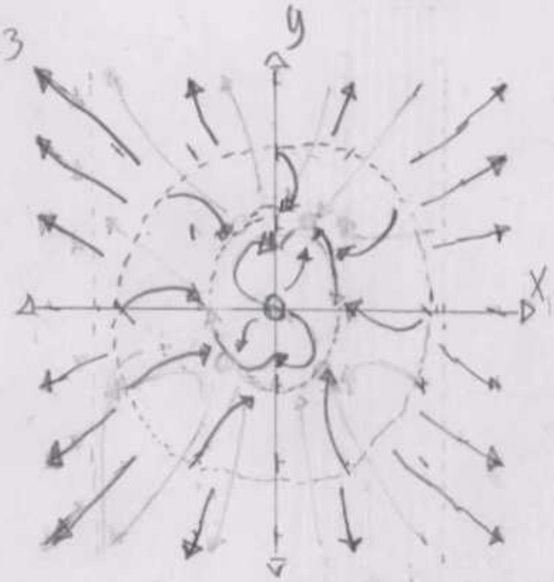
Where $r = \sqrt{x^2+y^2}$; $\theta = \arctan\left(\frac{y}{x}\right)$

and $\frac{dr}{d\theta} = r(1-r^2)(9-r^2)$



$\dot{r} = r(1-r^2)(4-r^2)$ 7.1.3

$\dot{\theta} = 2 - r^2$



$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + \sin\frac{dr}{d\theta}}{r\sin\theta + \cos\theta\frac{dr}{d\theta}}$$

Where $r = \sqrt{x^2+y^2}$; $\theta = \arctan\left(\frac{y}{x}\right)$

and $\frac{dr}{d\theta} = \frac{r(1-r^2)(4-r^2)}{2-r^2}$

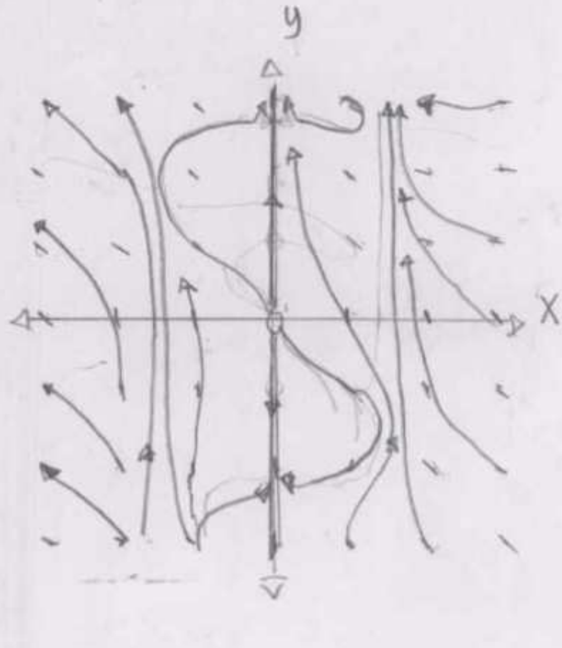
$\dot{r} = r\sin r$ 7.1.4

$\dot{\theta} = 1$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + \sin\theta\frac{dr}{d\theta}}{-r\cos\theta + \cos\theta\frac{dr}{d\theta}}$$

Where $r = \sqrt{x^2+y^2}$; $\theta = \arctan\left(\frac{y}{x}\right)$

and $\frac{dr}{d\theta} = r\sin r$



$\dot{r} = r(1-r^2)$ 7.1.5. $x = r\cos\theta$; $y = r\sin\theta$;

$\dot{\theta} = 1$

$$\dot{x} = \frac{d}{dt} r\cos\theta = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$$

$$= r(1-r^2)\cos\theta - r\sin\theta = x(1-x^2-y^2) - y$$

$$= x - x^3 - xy^2 - y = x - y - x(x^2+y^2)$$

$$\dot{y} = \frac{d}{dt} r\sin\theta = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$$

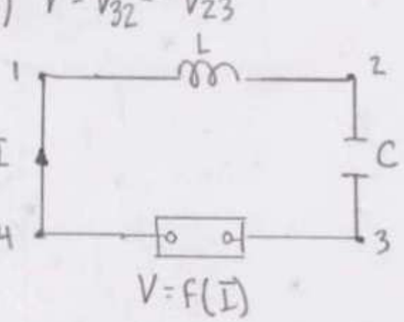
$$= r(1-r^2)\sin\theta - x = y(1-x^2-y^2) + x$$

$$= x + y - y(x^2+y^2)$$

$\dot{V} = -I/C$ 7.1.6.

$V = L\dot{I} + f(I)$

a) $V = V_{32} = -V_{23}$



$$V_{41} - V_{12} - V_{23} - V_{34} = 0$$

$$V_I - V_L - V_C - V_F = 0$$

$$V_I = V_L + V_C + V_F$$

$$= L\frac{dI}{dt} + \frac{I}{C} + f(I)$$

$$= L\frac{dI}{dt} + f(I) - \frac{I}{C}$$

$$= V + \dot{V}$$

b. If $x = \sqrt{L}\, I$; $w = \sqrt{C}\, V$; $\tau = t/\sqrt{LC}$, and $F(x) = f(x/\sqrt{L})$ and $x = I$

then $\dot{V} = -I/C = \dfrac{\frac{dw}{\sqrt{C}}}{d\tau\sqrt{LC}} = -\dfrac{x}{\sqrt{L}}\left(\dfrac{1}{C}\right)$; $\dfrac{dw}{d\tau} = -x$

and $V = L\dot{I} + f(I)$, $\dfrac{w}{\sqrt{C}} = L\dfrac{dx}{dt}\left(\dfrac{1}{\sqrt{L}}\right) + f(x/\sqrt{L})$

$$w = \sqrt{LC}\frac{dx}{d\tau\sqrt{LC}} + \sqrt{C}\, f(x/\sqrt{L})$$

$$\frac{dx}{d\tau} = w - \mu F(x) \text{ ; where } \mu = \sqrt{C}$$

$\dot{r} = r(4-r^2)$ 7.1.7.

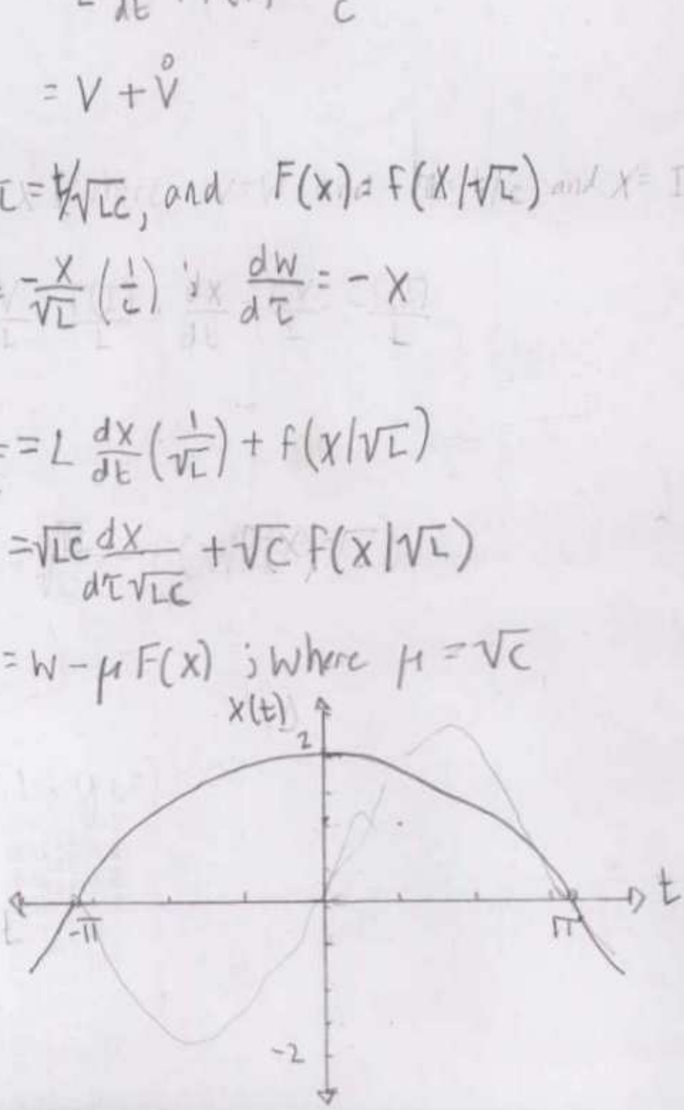
$\dot{\theta} = 1$

$x(t) = r(t)\cos\theta(t)$;

$x(0) = 0.1$; $y(0) = 0$

$t = \int \frac{dr}{r(4-r^2)} = -2\ln(4-r^2) + C$

$\theta(t) = t$; $r(t) = \sqrt{4 - Ce^{-t/2}}$



$\ddot{x} + a\dot{x}(x^2+\dot{x}^2-1) + x = 0$

7.1.8:

a. $\dot{u} = \dot{x} = v$

$\dot{v} = \ddot{x} = -a\dot{x}(x^2+\dot{x}^2-1) - x = -av(u^2+v^2-1) - u$

Fixed Points: $\dot{u} = \dot{x} = 0$

$\dot{v} = -av(u^2+v^2-1) - u = 0$

$(u^*, v^*) = (0,0)$

$\dot{\vec{U}} = A\cdot\vec{U}$; $A = \begin{pmatrix} 0 & 0 \\ -2vu-1 & -a(u^2+3v^2-1) \end{pmatrix}$

$\lambda_1 = 0$; $\lambda_2 = -a(u^2+3v^2-1)$

$\Delta = 0$; $\tau = -a(u^2+3v^2-1)$

$\tau^2 - 4\Delta > 0$ "Non-isolated Fixed Point"

b. $\dot{r} = \frac{(u\dot{u} + v\dot{v})}{r} = \frac{(r\cos\theta \cdot r\sin\theta + r\sin\theta \cdot (-av(u^2+v^2-1)-u))}{r}$

$= \frac{(r^2\cos\theta\sin\theta - ar^2\sin\theta\sin\theta(r^2\cos^2\theta + r^2\sin^2\theta - 1) - r^2\cos\theta\sin\theta}{r}$

$= -ar\sin^2\theta(r^2-1)$

$\dot{\theta} = \frac{(\dot{v}u - \dot{u}v)}{r^2} = \frac{(-a\cdot r\sin\theta(r^2-1) - r\cos\theta)r\cos\theta - r^2\sin^2\theta}{r^2}$

$= \frac{-ar^2\sin\theta\cos\theta(r^2-1) - r^2\cos^2\theta - r^2\sin^2\theta}{r^2}$

$= -a\sin\theta\cos\theta(r^2-1) - 1$

Amplitude: $-ar = -a\sqrt{u^2+v^2} = \mp a\sqrt{x^2+\dot{x}^2}$

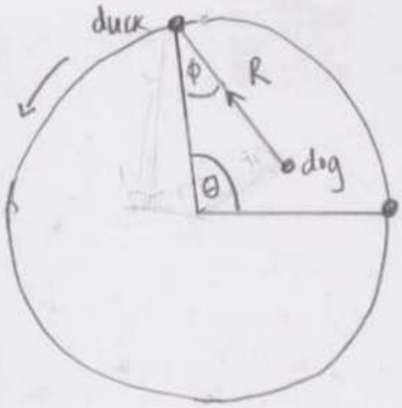
Period: $\frac{2\pi}{|T|} = 2\pi$

c. Stable limit cycle because larger a values generate a standard and periodic trajectory

d. The limit cycle is unique because solutions containing a values and c initial conditions have many solutions, for similar initial conditions.

$\frac{dR}{d\theta}, \frac{d\phi}{d\theta}$

7.1.9. a.

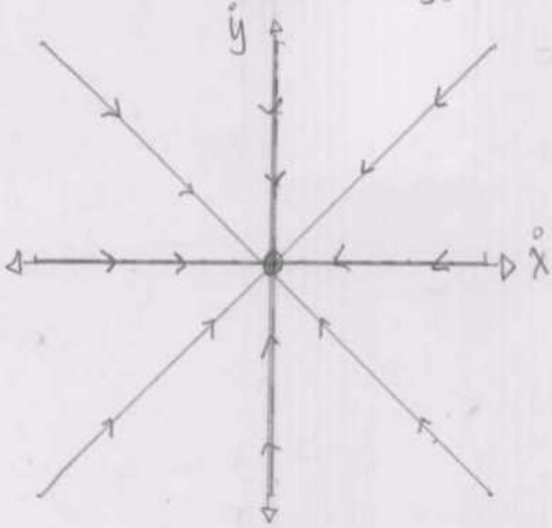


Duck: $\langle x, y \rangle = \langle r\cos k\theta, r\sin k\theta \rangle$

Dog: $\langle x, y \rangle =$

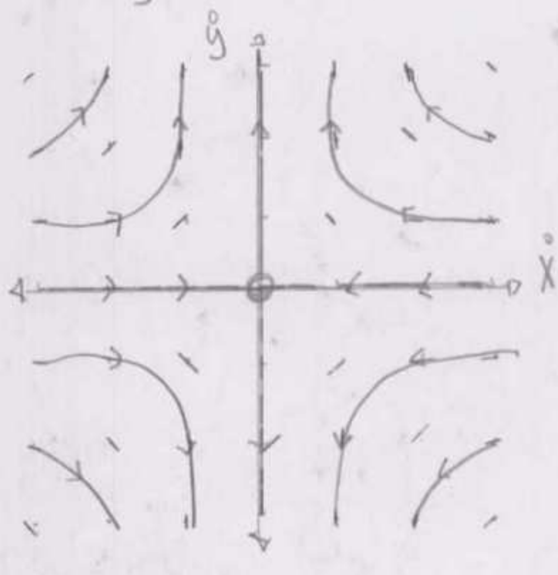
$V=x^2+y^2$ 7.2.1: $\dot{x}=-\nabla V$; $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = -\begin{bmatrix}dx\\ dy\end{bmatrix} V$; $\dot{x} = -\int x^2 = -\frac{x^3}{3}$

$\dot{y} = -\int y^2 = -\frac{y^3}{3}.$



$V=x^2-y^2$ 7.2.2. $\dot{x}=-\nabla V$, $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = -\begin{bmatrix}dx\\ dy\end{bmatrix} V$; $\dot{x} = -\int x^2 dx = -\frac{x^3}{3} + c$

$\dot{y} = +\int y^2 dx = +\frac{y^3}{3} + c$



$V=e^x \sin y$ 7.2.3. $\dot{x}=-\nabla V$

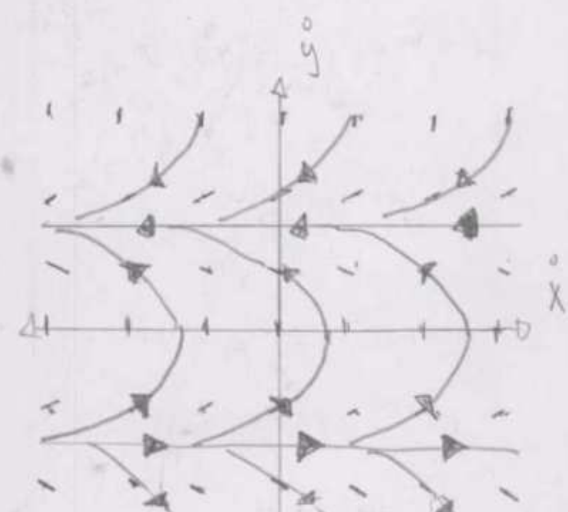
$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = -\begin{bmatrix}dx\\ dy\end{bmatrix} V$$

$$\dot{x} = -\int e^x \sin y \, dx$$

$$= -e^x \sin y + c$$

$$\dot{y} = -\int e^x \sin y \, dy$$

$$= e^x \cos y + c$$



7.2.4. Gradient System: When a system can be written as $\dot{x} = -\nabla V$, for a continuously differentiable, singled-valued scalar function

Line: A continuous function without curvature

Circle: A continuous and bounded function by an equidistant center.

Line: x; $\dot{x} = -\nabla V = -1$; A gradient system.

Circle: $\sqrt{x^2+y^2}$; $\dot{x} = -\nabla V = -\frac{x}{\sqrt{x^2+y^2}}$; A gradient system

$\dot{x} = f(x,y)$
$\dot{y} = g(x,y)$

7.2.5. a. $-\nabla V = \left(-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}\right)$; $-\frac{\partial V}{\partial x} = \dot{x} = f(x,y)$; $-\frac{\partial V}{\partial y} = \dot{y} = g(x,y)$

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

b. A sufficient condition is $p \to q$ and $\neg p \to \neg q$,

so $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ is sufficient by $V = -\int f(x,y)\,dx = -\int g(x,y)\,dy$.

$\dot{x} = y^2 + y\cos x$
$\dot{y} = 2xy + \sin x$

7.2.6
a. $V = \int \dot{x}\,dx + \int \dot{y}\,dy = xy^2 + y\sin x + xy^2 + y\sin x$
$= 2(xy^2 + y\sin x)$

$\dot{x} = 3x^2 - 1 - e^{2y}$
$\dot{y} = -2xe^{2y}$

b. $V = \int \dot{x}\,dx + \int \dot{y}\,dy = x^3 - x - xe^{2y} - xe^{2y}$
$= x(x^2 - 1 - 2e^{2y})$

$\dot{x} = y + 2xy$
$\dot{y} = x + x^2 - y^2$

7.2.7
a. If $\dot{x} = f(x,y)$ and $\dot{y} = g(x,y)$,

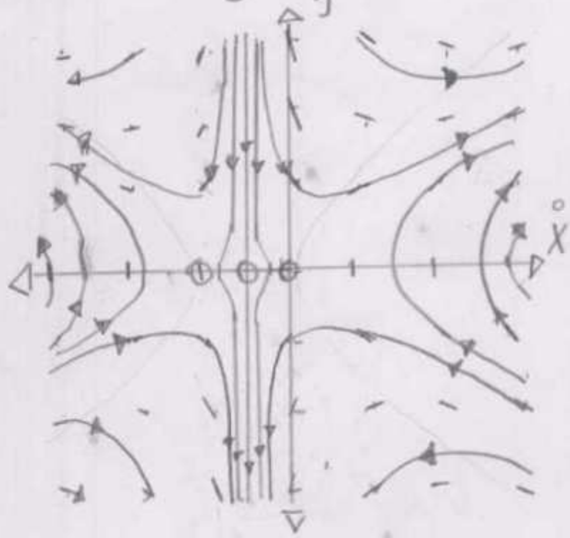
then $\frac{\partial f}{\partial y} = 1 + 2x$ and $\frac{\partial g}{\partial x} = 1 + 2x$

and $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$

b $V=\int \dot{x}\,dx+\int \dot{y}\,dy = xy + xy^2 + xy + x^2y - \frac{y^3}{3}$

$= 2xy + xy^2 + x^2y - \frac{y^3}{3}$

c.



7.2.8. If $\frac{df}{\partial y} = \frac{dg}{dx}$ at an equipotential, then $\frac{dy}{dx} = \frac{dg}{df}$;

The solution $\frac{dy}{dx} = \frac{dg}{df}$ is zero when $dg \neq df$ and one when $dy = df$, very similar to orthoganol slopes (dy/dx).

$\dot{x} = y + x^2y$
$\dot{y} = -x + 2xy$

7.2.9.

a. $V = -\int \dot{x}\,dx - \int \dot{y}\,dy = -\int y + x^2y\,dx + \int -x + 2xy\,dy$

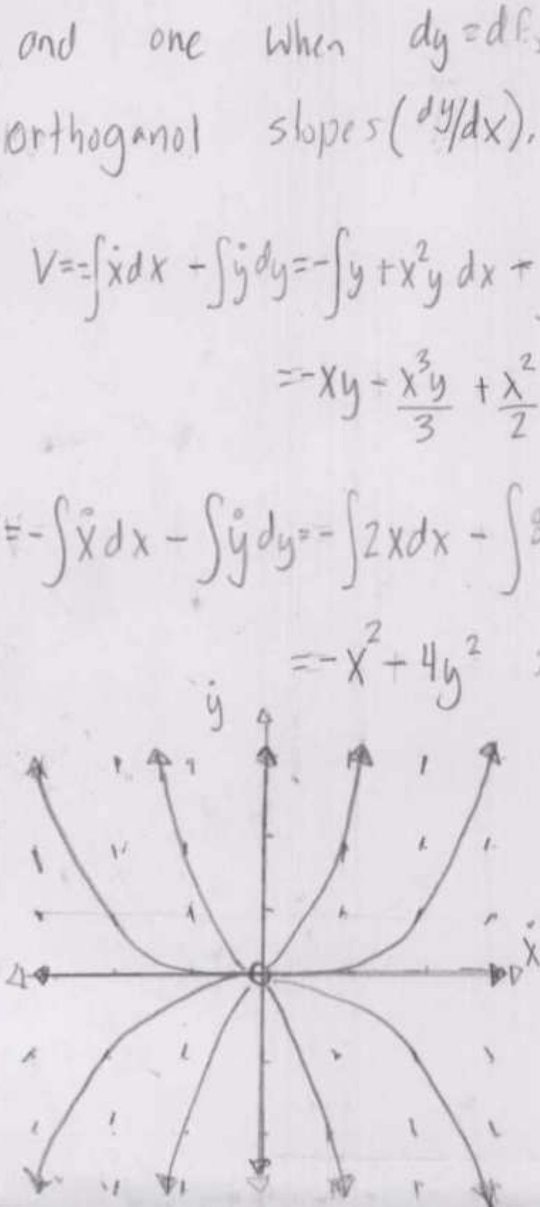
$= -xy - \frac{x^3y}{3} + \frac{x^2}{2} + x^2y$; $\frac{d\dot{x}}{dy} = \frac{d\dot{y}}{dx}$

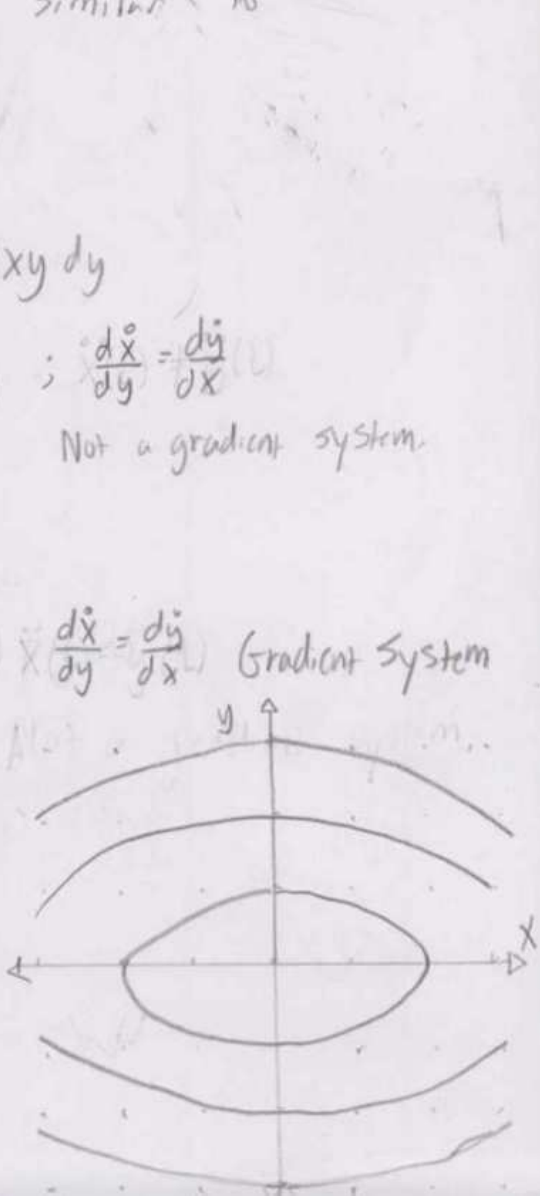
Not a gradient system.

$\dot{x} = 2x$
$\dot{y} = 8y$

b $V = -\int \dot{x}\,dx - \int \dot{y}\,dy = -\int 2x\,dx - \int 8y\,dy$

$= -x^2 + 4y^2$; $\frac{d\dot{x}}{dy} = \frac{d\dot{y}}{dx}$ Gradient System



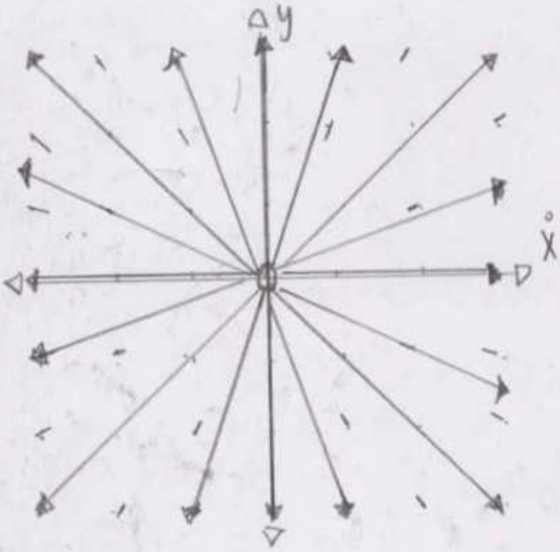


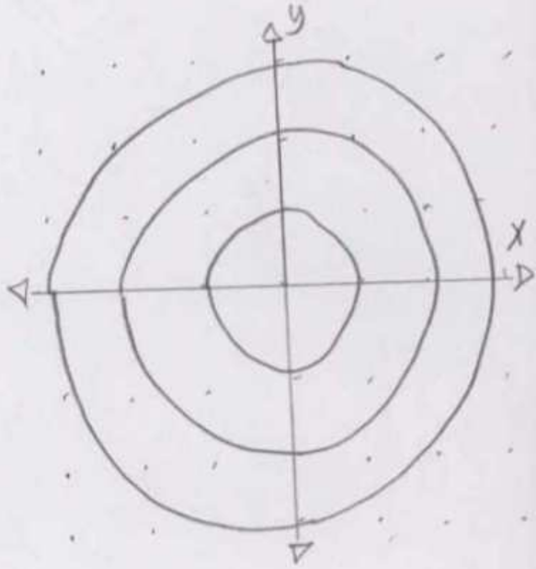
$\dot{x} = -2x e^{x^2+y^2}$ c. $V(x,y) = -\int \dot{x}\,dx - \int \dot{y}\,dy = +2\int x e^{x^2+y^2} dx + 2\int y e^{x^2+y^2} dy$

$\dot{y} = -2y e^{x^2+y^2}$

$$= e^{x^2+y^2} + e^{x^2+y^2} = 2e^{x^2+y^2}$$

$\frac{d\dot{x}}{dy} = \frac{d\dot{y}}{dx}$: Gradient system.





$\dot{x} = y - x^3$ 7.2.10. $V = -\int \dot{x}\,dx - \int \dot{y}\,dy = -\int y - x^3 dx + \int x + y^3 dy$

$\dot{y} = -x - y^3$

$$= -xy + \frac{x^4}{4} + xy + \frac{y^4}{4} = \frac{x^4}{4} + \frac{y^4}{4}$$

$a = \frac{1}{4}, b = \frac{1}{4}$

The potential function has a suitable a, and b, so this function is Liapunov stable with no closed orbits.

$V = ax^2 + 2bxy + cy^2$ 7.2.11. $\frac{\partial^2 V}{\partial x^2}\frac{\partial^2 V}{\partial y^2} - \left(\frac{\partial^2 V}{\partial x \partial y}\right)^2 = (2a)(2c) - (2b)^2 = 4(ac - b^2)$ @ (0,0)

Positive definite is a strictly positive, meaning strictly positive when $(ac - b^2) > 0$.

$\dot{x} = -x + 2y^3 - 2y^4$ 7.2.12 $V = -\int \dot{x}\,dx - \int \dot{y}\,dy = \frac{x^2}{2} - 2y^3 x + 2y^4 x + xy + \frac{y^2}{2} - \frac{xy^2}{2}$

$\dot{y} = -x - y + xy$

Fixed Points, $(x^*, y^*) = (0,0), (-2,0), (-2,1), (-1,0)$.

No periodic solutions

$\dot{N}_1 = r_1 N_1 (1 - N_1/K_1) - b_1 N_1 N_2$

$\dot{N}_2 = r_2 N_2 (1 - N_2/K_2) - b_2 N_1 N_2$

Dulac's Criterion: If $\dot{x} = f(x)$ is continuous, a real-valued function $g(x)$ such that $\nabla \cdot (g\dot{x})$ has one sign throughout R in a closed orbit.

7.2.13. $g = (N_1 N_2)^{-1}$

$$\nabla \cdot (g\dot{x}) = \frac{\partial}{\partial N_1}(g\dot{N}_1) + \frac{\partial}{\partial N_2}(g\dot{N}_2)$$

$$= \frac{\partial}{\partial N_1}\left[\frac{r_1}{N_2}(1 - N_1/K_1) - b_1\right] + \frac{\partial}{\partial N_2}\left[\frac{r_2}{N_1}(1 - N_2/K_2) - b_2\right]$$

$$= \frac{r_2}{N_1 K_2} - \frac{r_1}{N_2 K_1}$$

$$\leq 0$$

$\dot{x} = x^2 - y - 1$

$\dot{y} = y(x-2)$

7.2.14. a. Fixed Points: $\dot{x} = 0 = x^2 - y - 1$;

$\dot{y} = 0 = y(x-2)$

$2x \quad -1$

$y \quad -2$

$(x^*, y^*) = (-1, 0)$; $A_{(-1,0)} = \begin{pmatrix} -2 & -1 \\ 0 & -2 \end{pmatrix}$; Stable Node

$(1, 0)$ $A_{(1,0)} = \begin{pmatrix} 2 & -1 \\ 0 & -2 \end{pmatrix}$; Saddle Point

$(2, 3)$ $A_{(2,3)} = \begin{pmatrix} 4 & -1 \\ 3 & -2 \end{pmatrix}$; Saddle Point

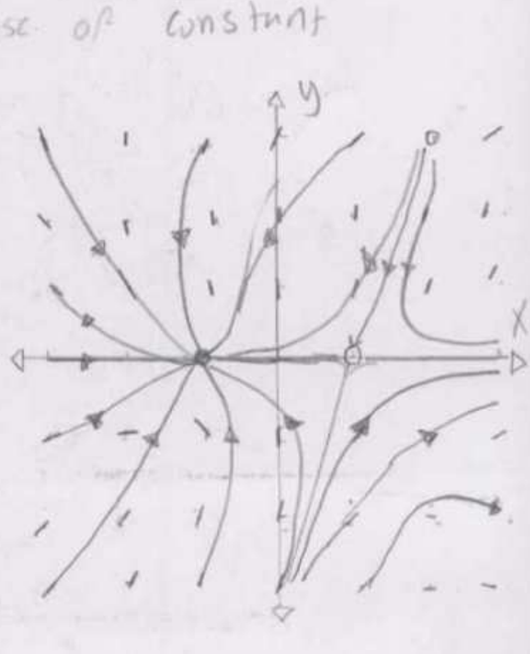
b. $(-1,0) \to (1,0)$: $\frac{dy}{dx} = 0$

$(1,0) \to (2,3)$: $\frac{dy}{dx} = x^2 - 1$

$(2,3) \to (-1,0)$: $\frac{dy}{dx} \cong -1$

Closed orbits are nonexistent because of constant trajectory between fixed points.

c.



$\dot{x} = x(2-x-y)$
$\dot{y} = y(4x-x^2-3)$

7.2.15. a. Fixed Points: $\dot{x} = 0 = x(2-x-y)$; $A = \begin{pmatrix} 2-2x-y & -x \\ 4y-2yx & 4x-x^2-3 \end{pmatrix}$

$\dot{y} = 0 = y(4x-x^2-3)$

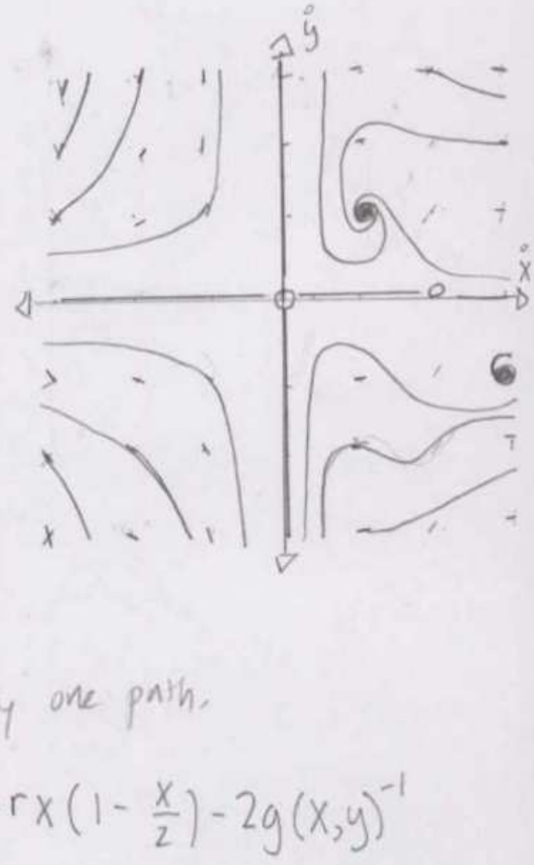
(x^*, y^*): $(0,0)$: $A_{(0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$; Saddle Point

$(1,1)$; $A_{(1,1)} = \begin{pmatrix} -1 & \cancel{2}-1 \\ 2 & 0 \end{pmatrix}$; Stable spiral

$(2,0)$: $A_{(2,0)} = \begin{pmatrix} -2 & -2 \\ 0 & 1 \end{pmatrix}$; Saddle Point

$(3,-1)$: $A_{(3,-1)} = \begin{pmatrix} 9 & -3 \\ 2 & 0 \end{pmatrix}$: Stable spiral

b. Phase Portrait



7.2.16. If R is a set of closed orbits, then $\iint_A \nabla \cdot (g\dot{x})\, dA = \oint_C g\dot{x} n\, dl$

7.2.17. If A is an annular, then Greens theorem holds true. Dulac's Criterion fails for multiple holes because the closed orbit (line integral) has only one path.

$\dot{x} = rx(1-\frac{x}{2}) - \frac{2x}{1+x}y$ 7.2.18 If $g(x,y) = \frac{1+x}{x} y^{\alpha-1}$; then $\dot{x} = rx(1-\frac{x}{2}) - 2g(x,y)^{-1}$

$\dot{y} = -y + \frac{2x}{1+x}y$ $\qquad \dot{y} = -y + 2g(x,y)^{-1}$

where $\alpha = 0$

$\nabla \cdot (g\dot{x}) = \frac{\partial}{\partial x}(g\dot{x}) + \frac{\partial}{\partial y}(g\dot{y})$

$= \frac{r(1-2x)}{2y} > 0$; No closed orbits in the positive quadrants

$\dot{R} = -R + A_s + kSe^{-S}$ 7.2.19.

$\dot{S} = -S + A_r + kRe^{-R}$

a.

Term:	Meaning:
$-R$	Rhett's decreasing love for Scarlett.
$+A_s$	Scarlett's love
$+kSe^{-S}$	Scarelett's decaying love for Rhett
$-S$	Scarlett's decreasing love for Rhett
$+A_r$	Rhett's love.
$+kRe^{-R}$	Rhett's decaying love for Scarlett.

b. $\dot{R} = 0 = -R + A_s + kSe^{-S}$; $(R^*, S^*) = (A_s + kSe^{-S}, A_r + kRe^{-R})$

$\dot{S} = 0 = -S + A_r + kRe^{-R}$; Which are greater than zero.

c. $\nabla \cdot (g\dot{x}) = \frac{\partial}{\partial R}(g\dot{R}) + \frac{\partial}{\partial S}(g\dot{S})$

$= -1 - 1 = -2 < 0$; Where $g = 1$.

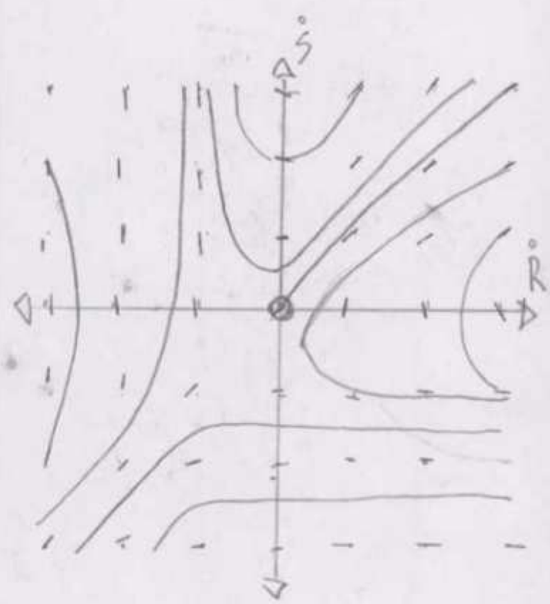
Since $x, y > 0$, there are no periodic solutions in the first quadrant.

d. Phase Portrait ; $A_s = 1.2$

$A_r = 1$

$k = 15$

$R(0) = S(0) = 0$



$\dot{x} = x - y - x(x^2 + 5y^2)$ 7.3.1 a. $A = \begin{pmatrix} 1 - x^2 - 5y^2 - 2x^2 & -1 - 10xy \\ 1 - 2xy & 1 - x^2 - y^2 - 2y^2 \end{pmatrix}$

$\dot{y} = x + y - y(x^2 + y^2)$

$= \begin{pmatrix} 1 - 3x^2 - 5y^2 & -1 - 10xy \\ 1 - 2xy & 1 - x^2 - 3y^2 \end{pmatrix}$

$A_{(0,0)} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$; $\Delta = 2$; $\tau = 2$; $\tau^2 - 4\Delta < 0$; Unstable Spiral

b. $\dot{r} = \dfrac{x\dot{x} + y\dot{y}}{r} = \dfrac{r\cos\theta\left[x - y - x(x^2+5y^2)\right] + r\sin\theta\left[x + y - y(x^2+y^2)\right]}{r}$

$$= r\cos\theta\left[\cos\theta - \sin\theta - \cos\theta(r^2\cos^2\theta + 5r^2\sin^2\theta)\right]$$
$$+ r\sin\theta\left[\cos\theta + \sin\theta - \sin\theta(r^2)\right]$$
$$= r\left[\cos\theta(\cos\theta - \sin\theta - r^2\cos\theta(\cos^2\theta + 5\sin^2\theta)\right.$$
$$\left. + \sin\theta(\cos\theta + \sin\theta - r^2\sin^2\theta)\right]$$

$$\dot{\theta} = \frac{(x\dot{y} - y\dot{x})}{r^2} = \frac{\cos\theta(x + y - y(r^2)) - \sin\theta(x - y - x(x^2+5y^2))}{r}$$

$$= \cos\theta(\cos\theta + \sin\theta - r^2\sin\theta) - \sin\theta(\cos\theta - \sin\theta - r^2\cos\theta(\cos^2\theta + 5\sin^2\theta))$$

$$= 4r^2\sin^3(\theta)\cos(\theta) + 1$$

c. $r_{min,\,outward}$ $\quad \dot{r} > 0$; $r\left[\cos\theta(\cos\theta - \sin\theta - r^2\cos\theta(\cos^2\theta + 5\sin^2\theta))\right.$
$$\left. + \sin\theta(\cos\theta + \sin\theta - r^2\sin^2\theta)\right] > 0$$

$$r_{min,\,outward} \cong -\sqrt{\frac{\sin^2\theta - \sin\theta + \cos\theta}{\sin^3\theta + 3\cos\theta - 2\cos\theta\cos(2\theta)}}$$

d. $r_{max,\,inward}$ $\quad \dot{r} < 0$; $r\left[\cos\theta(\cos\theta - \sin\theta - r^2\cos\theta(\cos^2\theta + 5\sin^2\theta))\right.$
$$\left. + \sin\theta(\cos\theta + \sin\theta - r^2\sin^2\theta\right] < 0$$

$$r_{max,\,inward} \cong \sqrt{\frac{(\sin\theta - 1)\sin\theta + \cos\theta}{\sin^3\theta + \cos\theta(3 - 2\cos 2\theta)}}$$

e. $r_{min,\,outward} < r < r_{max,\,inward}$

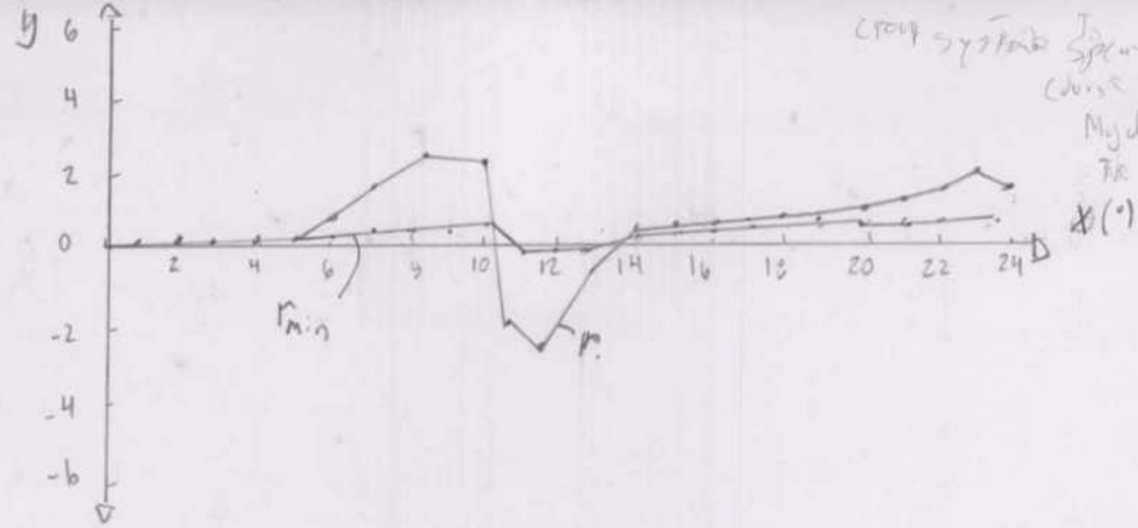
$\theta \cong 2(n\pi - 4\pi/10)$; This solution is close to the books $\theta = \frac{3\pi}{2}$

7.3.2 Runge Kutta Method [4th Order]

r_0	θ_0	$k_1(r_0, \theta_0 + \theta\Delta h/2)$	$k_1(r_0 + r\Delta h/2, \theta_0)$	$k_2(r_0 + k_1\Delta h/2, \theta_0)$	...
0.1					

...	$k_2(r_0, \theta_0 + k_1\Delta h/2)$	$k_3(r_0 + k_2\Delta h, \theta_0)$	$k_3(r_0, \theta_0 + k_2\Delta h)$	

$$r_n = r_{n-1} + \frac{\Delta h}{6}(r_0 + 2k_1 + 2k_2 + k_3) \;;\; \theta_n = \theta_{n-1} + \frac{\Delta h}{6}(\theta_0 + 2k_1 + 2k_2 + k_3)$$



$\dot{x} = x - y - x^3$
$\dot{y} = x + y - y^3$

7.3.3. $\dot{r} = \dfrac{x\dot{x} + y\dot{y}}{r} = \dfrac{r\cos\theta(x - y - x^3) + r\sin\theta(x + y - y^3)}{r}$

$= \dfrac{\cancel{r}\cos\theta\,(r\cos\theta - r\sin\theta - (r\cos\theta)^3) + \cancel{r}\sin\theta(r\cos\theta + r\sin\theta - (r\sin\theta)^3)}{\cancel{r}}$

$= r - r^3(\cos^4(\theta) + \sin^4(\theta))$

$r - r^3 < \dot{r} < r - r^3/2$ Poincare-Bendixon Theorem states at least one periodic solution.

$\dot{x} = x(1 - 4x^2 - y^2) - \frac{1}{2}y(1 + x)$
$\dot{y} = y(1 - 4x^2 - y^2) + 2x(1 + x)$

7.3.4. a. $A = \begin{pmatrix} -12x^2 - y^2 - 1 & (-2x - 1)y - 1/2 \\ x(4 - 8y) + 2 & -4x^2 - 3y^2 + 1 \end{pmatrix}$

$A_{(0,0)} = \begin{pmatrix} -1 & -1/2 \\ 2 & 1 \end{pmatrix}$; $\lambda_1 = 0$; $\lambda_2 = 0$; $\Delta = 0$; $\tau = 0$;
$\tau^2 - 4\Delta = 0$ "center"

Although graph indicates an unstable spiral.

b. $V = (1 - 4x^2 - y^2)^2$; $\dot{V} = 2(1 - 4x^2 - y^2)(1 - 4x^2 - y^2)'$

$\lim_{t \to \infty} \dot{V} = 0$; $1 - 4x^2 - y^2 = 0$; $4x^2 + y^2 = 1$

$\dot{x} = -x - y + x(x^2 + 2y^2)$
$\dot{y} = x - y + y(x^2 + 2y^2)$

7.3.5 $\dot{r} = \dfrac{x\dot{x} + y\dot{y}}{r} = \dfrac{r\cos\theta(-x - y + x(x^2 + 2y^2)) + r\sin\theta(x - y + y(x^2 + 2y^2))}{r}$

$= \dfrac{r\cos\theta(-r\cos\theta - r\sin\theta + r\cos\theta((r\cos\theta)^2 + 2(r\sin\theta)^2) + r\sin\theta(r\cos\theta - r\sin\theta + r\sin\theta((r\cos\theta)^2 + 2(r\sin\theta)^2)}{r}$

$= r^3(\sin^2(x)+1)\left(\frac{1}{2}\sin(2x)+\cos^2(x)\right)-r$

$\theta:$ $\frac{38}{100}r^3 - r < r < \frac{152}{100}r^3 - r$

A periodic solution exists by Poincare-Benedixon Theorem.

$\ddot{x}+F(x,\dot{x})\dot{x}+x=0$

7.3.6. $F(x,\dot{x})<0$, $r\le a$, else $F(x,\dot{x})>0$ if $r\ge b$ where $r^2=x^2+\dot{x}^2$

a. $\dot{u}=\dot{x}=v$

$\dot{v}=\ddot{x}=-F(x,\dot{x})\dot{x}-x$

A physical interpretation of $F(x,\dot{x})$ is an additive force to acceleration, which increases or decreases.

b. $r=\sqrt{x^2+\dot{x}^2}=\sqrt{(r\cos\theta)^2+(\dot{r}\cos\theta-r\sin\theta\,\dot{\theta})^2}$

$a<\sqrt{(r\cos\theta)^2+(\dot{r}\cos\theta-r\sin\theta\dot{\theta})^2}<b$

$\dot{x}=y+ax(1-2b-r^2)$
$\dot{y}=-x+ay(1-r^2)$

7.3.7.

a. $(0<a\le 1, 0\le b<1/2)$ and $r^2=x^2+y^2$

$$\dot{r}=\frac{x\dot{x}+y\dot{y}}{r}=\frac{r\cos\theta\,(y+ax(1-2b-r^2))+r\sin\theta\,(-x+ay(1-r^2))}{r}$$

$$=\cos\theta(r\sin\theta+ar\cos\theta(1-2b-r^2))+\sin\theta(-r\cos\theta+ar\sin\theta(1-r^2))$$

$$\dot{\theta}=\frac{x\dot{y}-\dot{x}y}{r^2}=\frac{r\cos\theta(-x+ay(1-r^2))+r\sin\theta(y+ax(1-2b-r^2))}{r^2}$$

$$=\cos\theta(-\cos\theta+a\sin\theta(1-r^2))-\sin\theta(\sin\theta+a\cos\theta(1-2b-r^2))$$

$$=ab\sin(2\theta)-1$$

b. A region of trapping $ar(1-r^2)\le\dot{r}\le ar(1-2b-r^2)$ exists as an annular cycle, $T(a,b)$.

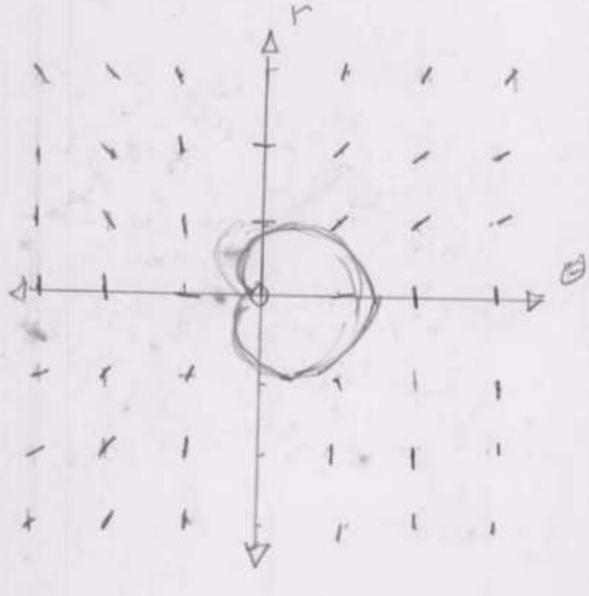
c) If $b=0$, then $ar(1-r^2) \le \dot{r} \le ar(1-r^2)$, so $\dot{r}$ must be $ar(1-r^2)$.

$\dot{r} = r(1-r^2) + \mu r\cos\theta$

$\dot{\theta} = 1$

7.3.8. $\dot{r} = 0 = r(1-r^2) + \mu r\cos\theta$

$$\mu = \frac{r(r^2-1)}{r\cos\theta}$$



If $r=0, 1,$ or -1, then the closed orbit known as the cardioid becomes absent, but a circular orbit remains.

7.3.9.

$\dot{r} = r(1-r^2) + \mu r\cos\theta$

$\dot{\theta} = 1$

a. $r(\theta) = 1 + \mu r_1(\theta) + O(\mu^2)$

$$\frac{dr}{d\theta} = r(1-r^2) + \mu r\cos\theta = \mu r_1'(\theta)$$

$$\mu r_1'(\theta) = (1+\mu r_1(\theta))(1-(1+\mu r_1(\theta))^2) + \mu(1+\mu r_1(\theta))\cos\theta$$

$r_1'(\theta) = -2r_1(\theta) + \cos(\theta)$ First-order linear differential equation.

$\cos(\theta) = r_1'(\theta) + 2r_1(\theta)$

$\cos(\theta)\,d\theta = r_1'(\theta) + 2r_1(\theta)\,d\theta$

b. $(\cos(\theta) - 2r')\,d\theta - dr(\theta) = 0$ Exact Differential Equation.

$N(r,\theta)\,d\theta + M(r,\theta)\,dr = 0$

$N'(r,\theta) \neq M'(r,\theta)$, so a multiplicative is necessary.

$N'(r,\theta) = \frac{dN}{dr} = 2$; $M'(r,\theta) = \frac{dM}{d\theta} = 0$

The assumption; $\frac{dN}{dr} = \frac{dM}{d\theta}$; $\frac{dN}{dr} - \frac{dM}{d\theta} = 0 = \mu\left(\frac{dN}{dr} - \frac{dM}{d\theta}\right)$

Also, $\mu(r,\theta) = \mu(r)$; $\frac{d\mu}{d\theta} = 0$; so $\frac{1}{\mu}\frac{d\mu}{dr} = \frac{1}{M}\left(\frac{dN}{dr} - \frac{dM}{d\theta}\right)$

$$\int \frac{d\mu}{\mu} = \int \frac{1}{M}\left(\frac{dN}{dr} - \frac{dM}{d\theta}\right) d\theta;\ \ln(\mu) = 2\theta;\ \mu = e^{2\theta}$$

Multiplying the equation by $e^{2\theta}$

$$(\cos(\theta)-2r)e^{2\theta}d\theta - e^{2\theta}dr = 0$$

$$N(r,\theta)\,d\theta - M(r,\theta)dr = 0$$

$$N'(r,\theta) = M'(r,\theta) = -2e^{2\theta}$$ Exact equation.

$$dF(r,\theta) = N(r,\theta)\,d\theta + M(r,\theta)\,dr$$

$$F(r,\theta) = \int N(r,\theta)\,dr = \int e^{2\theta}\cos(\theta) - 2e^{2\theta}\cdot r\,d\theta$$

$$= \frac{e^{2\theta}\sin(\theta)}{5} + \frac{2e^{2\theta}\cos(\theta)}{5} - e^{2\theta}r + C$$

$$r = \frac{\sin(\theta)}{5} + \frac{2\cos(\theta)}{5} + \frac{C}{e^{2\theta}}$$

$$r(\theta) = 1 + \mu\left(\frac{\sin(\theta)}{5} + \frac{2\cos(\theta)}{5}\right)$$

b. $\frac{dr}{d\theta} = 0 = \frac{\mu\cos\theta}{5} - \frac{2\sin\theta}{5}$ @ $\theta = \arctan\left(\frac{1}{2}\right) + n\pi$

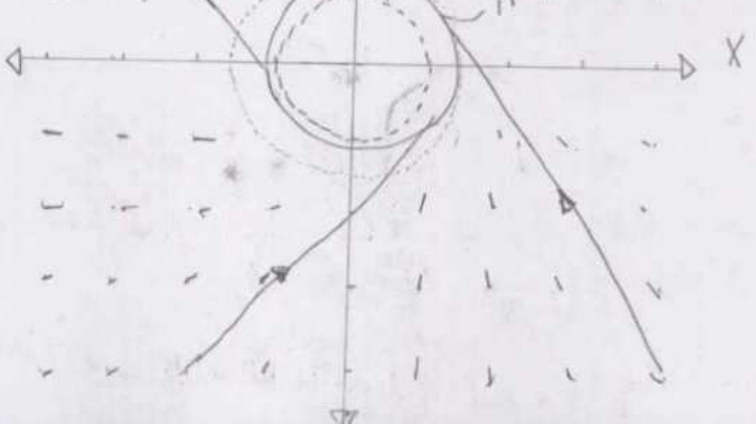
$$r\left(\arctan\left(\tfrac{1}{2}\right)\right) = 1 + \mu\left(\frac{1}{5}\frac{1}{\sqrt{5}} + \frac{2}{5}\frac{2}{\sqrt{5}}\right) = 1 + \frac{\mu}{\sqrt{5}}$$

or

$$r\left(\arctan\left(\tfrac{1}{2}\right)+\pi\right) = 1 + \mu\left(\frac{1}{5}\frac{-1}{\sqrt{5}} + \frac{2}{5}\frac{-2}{\sqrt{5}}\right) = 1 - \frac{\mu}{\sqrt{5}}$$

$$\sqrt{1-\mu} < 1 - \frac{\mu}{\sqrt{5}} < r < 1 + \frac{\mu}{\sqrt{5}} < \sqrt{1+\mu}$$

c.



$\dot{x} = Ax - r^2 x$ 7.3.10. $r = \|x\|$; $A \in \mathbb{R}$; $\lambda_{1,2} = \alpha \pm i\omega$

$\Delta = (\alpha^2 + \omega^2)$; $\tau = 2\alpha$; $\tau^2 - 4\Delta < 0$

Fixed Points: $\dot{x} = 0 = Ax - r^2 x$; $x = 0$ if $\alpha < 0$ "stable fixed point"

if $\alpha > 0$ "Unstable spiral"

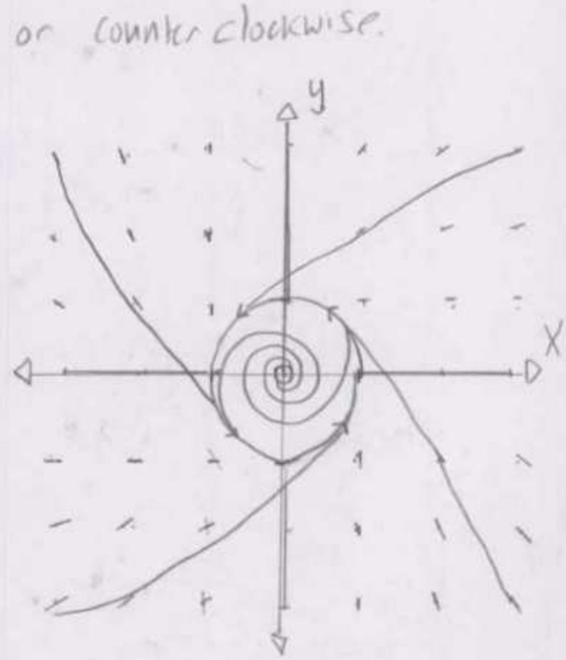
$\dot{r} = r(1-r)\left[r^2 \sin^2\theta + (r^2\cos^2\theta - 1)^2\right]$

$\dot{\theta} = r^2 \sin^2\theta + (r^2\cos^2\theta - 1)^2$

7.3.11 $\dot{x} = f(x)$ is a vector field on $\mathbb{R}^2$

Cycle graph: an invariant set containing a finite number of fixed points connected by a finite number of trajectories, all oriented clockwise or counter clockwise.

a. Phase Portrait



b. $\dot{x} = \frac{\dot{r}x}{r} - y\dot{\theta}$

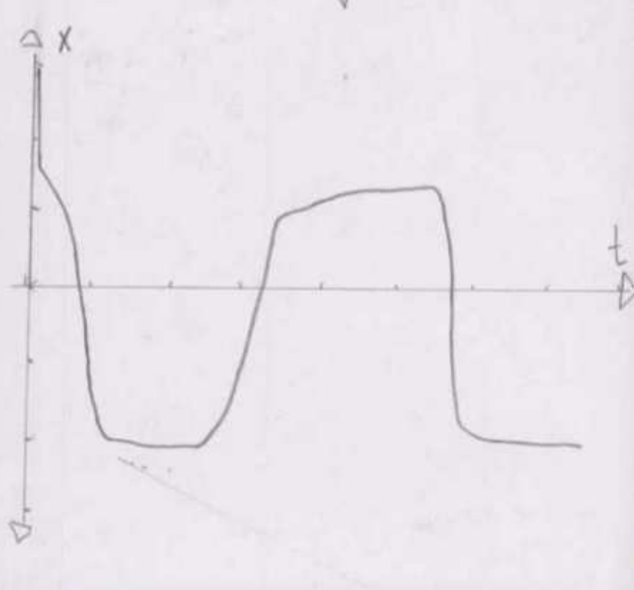
$= \frac{r(1-r)\left[r^2\sin^2\theta + (r^2\cos^2\theta - 1)^2\right]x}{r}$

$- y\left[r^2\sin^2\theta + (r^2\cos^2\theta - 1)^2\right]$

where $r = \sqrt{x^2 + y^2}$

$\theta = \operatorname{atan}(y/x)$

Runge-Kutta performance became absent because of problem sets statement "Sketch".



$\dot{P} = P[(aR-S)-(a-1)(PR+RS+PS)]$

$\dot{R} = R[(aS-P)-(a-1)(PR+RS+PS)]$

$\dot{S} = S[(aP-R)-(a-1)(PR+RS+PS)]$

$E_1(P,R,S) = P+R+S$

$E_2(P,R,S) = PRS.$

7.3.12.

a. $\dot{E}_1 = (1-E_1)(a-1)(PR+RS+PS)$; $P+R+S=1$; $E_1 = 1$

If $P=-1$ $\quad \dot{E}_1 = (1-E_1)(a-1)(RS-PR-PS) = 0$

$= (1-E_1)(a-1)(PR+RS+PS)$

If $R=-1$ $\quad \dot{E}_1 = (1-E_1)(a-1)(PS-PR-RS) = 0$

$= (1-E_1)(a-1)(PR+RS+PS)$

If $S=-1$ $\quad \dot{E}_1 = (1-E_1)(a-1)(PS-PR-RS)$

$= (1-E_1)(a-1)(PR+RS+PS)$ "Ellipsoid"

b. $P,R,S \geq 0$ & $P+R+S=1$

If $(P=1/2 \,\|\, R=1/2 \,\|\, S=1/2)$, then $\dot{E}_1 = 0$ "sphere"

c. $\dot{E}_1 = 0 = (1-E_1)(a-1)(PS+RS+PR)$

$(P^*, R^*, S^*) : (P, -(P+S), -(P+R))$

$= (-(S+R), R, -(P+R))$

$= (-(S+P), -(S+R), S)$

d. $\frac{dE_2}{dt} = \frac{d}{dt}(PRS) = R(S\cdot\dot{P} + P\cdot\dot{S}) + P\cdot S\cdot\dot{R}$

$= \frac{PRS\,(a-1)}{2}\left[(P-R)^2+(R-S)^2+(S-P)^2\right]$

e. $(P^*, R^*, S^*) = (1/3, 1/3, 1/3)$

$\dot{E}_2 = \frac{(1/3)^3(a-1)}{2}\left[0^2+0^2+0^2\right] = 0$

$(P^*, R^*, S^*) = (3, 0, 0)$; $\dot{E}_2 = 0$

f. If $a<1$, then the model F_2 trajectory is decreasing else $a=1$, then F_2 model does not change.

g. If $a<1$, then $\dot{F}_2$ is less than zero and trajects to fixed center.

$\ddot{x}+\mu(x^2-1)\dot{x}+\tanh x=0$

7.4.1. $\dot{u}=\dot{x}=v$

$\dot{v}=\mu(1-u^2)v-\tanh u$

Fixed Points: $\dot{u}=0=\dot{x}=v$

$\dot{v}=0=\mu(1-u^2)v-\tanh u$

$(u^*, v^*): (0,0)$

① u and v are continuously diffentiable

$\ddot{u}=1;\ \ddot{v}=\mu(1-2u\dot{u})v+\mu(1-u^2)\dot{v}-(1-\tanh^2 u)$

② $\dot{v}(-u,-v)=-\dot{v}(+u,+v)$ is an odd function.

$\dot{v}(-u,-v)=-\mu(1-(-u)^2)v-\tanh(-u)$

$=-[\mu(1-u^2)v-\tanh(u)]$

③ $\dot{v}(u,v)>0$ for $u,v>0$

$\dot{v}(u,v)=\underbrace{\mu(1-u^2)v}_{(+)\text{ till } u^2=1}-\underbrace{\tanh u}_{(+)}$; $\mu(1-u^2)v>\tanh u$ (+)

④ $\dot{u}(-v)=\dot{u}(v)$ is an even function

$\dot{u}(-v)=-v=0$; $0=\dot{u}(v)$ at fixed point

⑤ The odd function $F(v)=\int_0^v \dot{u}(R)\,dR$ is positive at $v=a$, and negative for $0<v<a$, is positive and nondecreasing for $v>0$, and $F(v)\to\infty$ as $v\to\infty$

A stable limit cycle.

$\ddot{x}+\mu(x^4-1)\dot{x}+x=0$ 7.4.2. $f(x)=\ddot{x}+\mu(x^4-1)x$; $g(x)=x$

a. ① $f(x)$ and $g(x)$ are continuously differentiable

$\dot{f}(x)=\ddot{x}+\mu(4x^4)+\mu(x^4-1)$; $\dot{g}(x)=1$

② $g(-x)=-g(x)$ is an odd function

$g(-x)=-g(x)$ at $x=0$

③ $f(-x)=f(x)$ is an even function

$f(-x)=-\ddot{x}-\mu[(-x)^4-1]\cdot x=-f(x)$ at $x=0$

④ $g(x)>0$ for $x>0$

⑤ $F(x)=\int_0^x f(u)\,du=\int_0^x \ddot{u}+\mu(u^2-1)\,du=\cancel{\dot{x}}+\mu(\frac{x^3}{3}-x)$

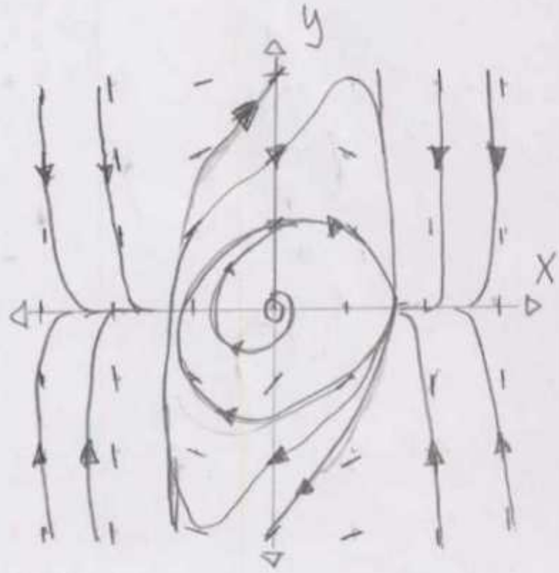
$F(x)$ has a positive root at $x=\sqrt{3}$

$F(x)$ is negative from $0<x<\sqrt{3}$

$F(x)$ is positive and nondecreasing for $\sqrt{3}<x$

and $F(x)\rightarrow\infty$ as $x\rightarrow\infty$

b. Phase Portrait



c. If $\mu<1$, then the function has an unstable periodic cycle in the opposite direction.

$x_A=2$
$x_B=1$

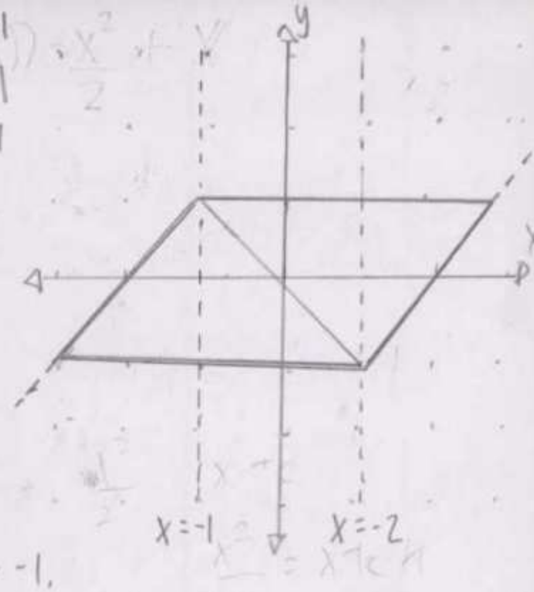
7.5.1. Van der Pol Oscillator: $\ddot{x}+\mu(x^2-1)\dot{x}+x=0$

$\mu\gg1$; $\ddot{x}+\mu(x^2-1)\dot{x}+x=\frac{d}{dt}\left(\dot{x}+\mu(\frac{x^3}{3}-x)+\frac{x^2}{2}\right)$

$F(x)=\frac{x^3}{3}-x$; $w=\dot{x}+\mu F(x)+\frac{x^2}{2}$

$\dot{x}=w-\mu F(x)-\frac{x^2}{2}$; $\dot{w}=0$

$$F(x) = \int f(x)\,dx = \begin{cases} x+2 & x \le -1 \\ -x & |x| \le 1 \\ x-2 & x \ge 1 \end{cases}$$



b. Nullclines in graph.

c. If $\mu \gg 1$, then nullcline minimum is $y=-1$ and maximum $(y=1)$.

See graph about the limit cycle about $F'(x)=-1$, $F(-1)=1$ or -1.

d. $|\dot{x}| \sim O(\mu) \gg 1$ and $|\dot{y}| \sim O(\mu^{-1}) \ll 1$

The period of the nullcline is $T \approx \mu \int_{-1}^{2} \frac{y-F(x)}{-\mu}\,dx$

$$\approx \mu \int_{-1}^{2} \frac{-x^2}{2\mu} + x\,dx$$

$$\approx \frac{-x^3}{6} + \frac{x^2}{2}\mu \Big|_{-1}^{2}$$

$$\approx \frac{-(2^3-1)}{6} + \left[\frac{4}{2} + \frac{1}{2}\right]\mu$$

$$\approx \frac{-7}{6} + \frac{5}{2}\mu$$

$\ddot{x} + \mu(x^2-1)\dot{x} + x = a$ 7.5.6.

a. Fixed Points: $\ddot{x} + \mu(x^2-1)\dot{x} + (x-a) = \frac{d}{dt}\left[\dot{x} + \mu F(x) + \frac{x^2}{2} - ax\right]$

$$F(x) = \frac{x^3}{3} - x \;;\; w = \dot{x} + \mu F(x) + \frac{x^2}{2} - ax$$

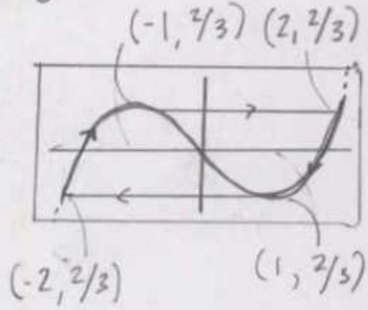
$$\dot{x} = w - \mu F(x) - \frac{x^2}{2} + ax \;;\; \dot{w} = 0$$

If $y = \dfrac{-\frac{x^2}{2} + ax + w}{\mu}$, $\dot{x} = \mu[y - F(x)]$

$$\dot{y} = \frac{a-x}{\mu}$$

$$(x^*, y^*) = (a, F(x)) = \left(a, \frac{x^3}{3} - x\right)$$

If $y = \frac{w}{\mu}$, then $\mathring{x} = \mu[y - F(x)]$ and $\mathring{y} = 0$



Nullcline minimum: $F'(x) = x^2 - 1$; $x = \pm 1$

Nullcline intersection: $F(-1) = 2/3$, $x = -1, 2$

$\mathring{x} = y$
$\mathring{y} = -x - \mu(x^2-1)$

7.5.2. Nullclines: $\mathring{x} = 0 = y$

$$\mathring{y} = 0 = -x - \mu(x^2-1)$$

$$(x^*, y^*) = \left(\frac{+1 \pm \sqrt{1+4\mu^2}}{-2\mu}, 0\right)$$

A Liénard plane provides advantages, such as separation of time scales ($\propto \mu$) from the fixed point.

$\ddot{x} + k(x^2-4)\mathring{x} + x = 1$ 7.5.2

7.5.3. $\ddot{x} + k(x^2-4)\mathring{x} + x - 1 = \frac{d}{dt}\left[\mathring{x} + k\left(\frac{x^3}{3} - 4x\right) + \frac{x^2}{2} - x\right]$

$$F(x) = \frac{x^3}{3} - 4x \text{ ; } \omega = \mathring{x} + kF(x) + \frac{x^2}{2} - x$$

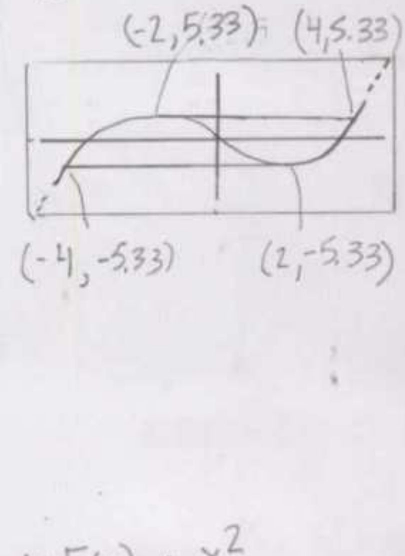
$$\mathring{x} = \omega - kF(x) - \frac{x^2}{2} + x \text{ ; } \mathring{\omega} = 0$$

If $y = \frac{\omega}{k}$, then $\mathring{x} = k(y - F(x)) - \frac{x^2}{2} + x$

$$\mathring{y} = 0$$

Nullcline minimum: $F'(x) = x^2 - 4$; $x = \pm 2$

Nullcline Intersection: $F(-2) = \frac{16}{3}$; $x = 30$



$\ddot{x} + \mu f(x)\mathring{x} + x = 0$ 7.5.4 $f(x) = -1$ for $|x| < 1$; $f(x) = 1$ for $|x| \geq 1$

a. $\ddot{x} + \mu f(x)\mathring{x} + x = \frac{d}{dt}\left[\mathring{x} + \mu\int f(x)dx + \frac{x^2}{2}\right]$

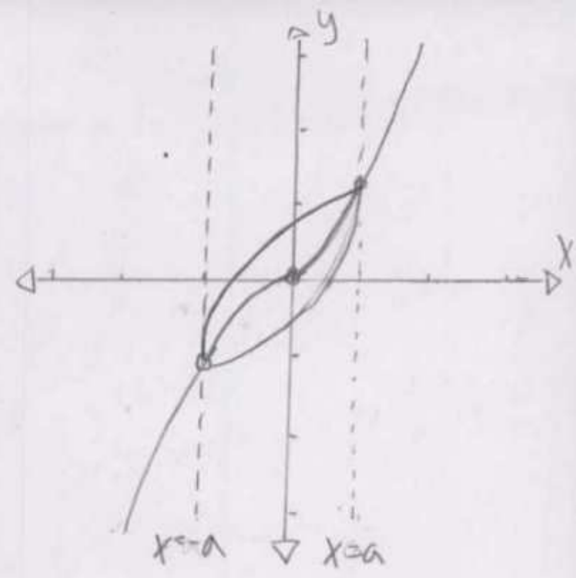
$$F(x) = \int f(x)dx \text{ ; } \omega = \mathring{x} + \mu F(x) + \frac{x^2}{2}$$

$$\mathring{x} = \omega - \mu F(x) - \frac{x^2}{2} \text{ ; } \mathring{\omega} = 0$$

If $y = \frac{-x^2}{2\mu}$, then $\mathring{x} = \mu(y - F(x))$; $\mathring{y} = \frac{-x}{\mu}$

b. Nullclines in the Lienard Plane.

The center seems stable not corresponding to the book.



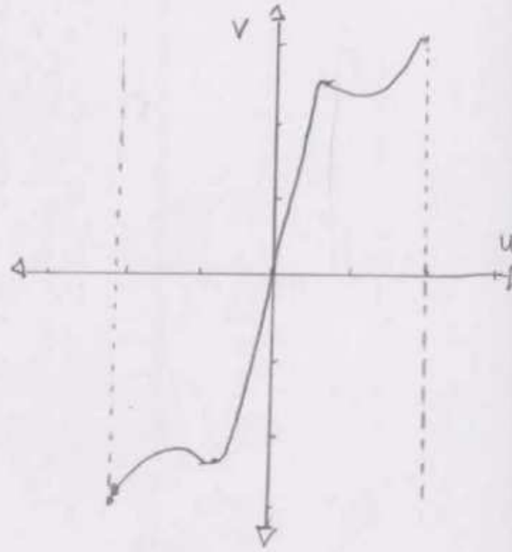
c. Nullcline Minimum: $F'(x) = x^2 - 1$

$x = \pm 1$

Nullcline Intersection: $F(-1) = \frac{2}{3}$

$|a| < 0$ because $dy/dx = 0$

d. see plot.



$\dot{u} = b(v-u)(\alpha + u^2) - u$

$\dot{v} = c - u$

7.5.7. a. Nullclines on graph.

b. $|C_1| = |C_2|$

c. A fixed point exists when c is beyond the inflection points.

$x(t,\varepsilon) = (1-\varepsilon^2)^{-1/2} e^{-\varepsilon t} \sin|(1-\varepsilon^2)^{1/2} t|$

$x(t,\varepsilon) = \sin t - \varepsilon t \sin t + O(\varepsilon^2)$

7.6.1 $x(t,\varepsilon) = (1-\varepsilon^2)^{-1/2} \circ e^{-\varepsilon t} \circ \sin|(1-\varepsilon^2)^{1/2} t|$

Identities: $(1+x)^a = 1 + ax + \frac{1}{2}(a-1)ax^2 + \cdots$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$x(t,\varepsilon) = \left[1 + \tfrac{1}{2}\varepsilon^2 + \cdots\right]\left[1 - \varepsilon t + \cdots\right] \sin\left[\left(1 - \tfrac{1}{2}\varepsilon^2 + \cdots\right)t\right]$$

$$= \left[1 - \varepsilon t + \tfrac{1}{2}\varepsilon^2 - \frac{\varepsilon^3 t}{2} + \cdots\right] \sin\left[\left(1 - \tfrac{1}{2}\varepsilon^2 + \cdots\right)t\right]$$

$$= \sin\left[t - O(\varepsilon^2)\right] - \varepsilon t \sin\left[t - O(\varepsilon^2)\right] + O(\varepsilon^2)$$

$$\cong \sin[t] - \varepsilon t \sin[t] + O(\varepsilon^2)$$

$\ddot{x} + x + \varepsilon x = 0$ 7.6.2. $x(0) = 1$; $\dot{x}(0) = 0$

$\dot{u} = \dot{x} = v$

$\dot{v} = -(1+\varepsilon)x = -(1+\varepsilon)u$

$$\vec{X} = Ax = \begin{pmatrix} 0 & 1 \\ -(1+\varepsilon) & 0 \end{pmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \; ; \; 0 = Ax = \lambda \; ; \; Ax - \lambda = 0$$

$$\begin{pmatrix} -\lambda & 1 \\ -(1+\varepsilon) & -\lambda \end{pmatrix} = 0 \; ; \; \lambda_1 = +\sqrt{1+\varepsilon} \; ; \; \lambda_2 = -\sqrt{1-\varepsilon}$$

$$\lambda_1 = +i\sqrt{1+\varepsilon} \; ; \; \begin{pmatrix} -i(\sqrt{1+\varepsilon}) & 1 \\ -(1+\varepsilon) & -i(\sqrt{1+\varepsilon}) \end{pmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$-i(\sqrt{1+\varepsilon})v_{11} + v_{12} = 0 \; ; \; \vec{V}_1 = \begin{bmatrix} 1 \\ i\sqrt{1+\varepsilon} \end{bmatrix}$$

$$\lambda_2 = -i\sqrt{1+\varepsilon} \; ; \; \begin{pmatrix} i\sqrt{1+\varepsilon} & 1 \\ -(1+\varepsilon) & i\sqrt{1+\varepsilon} \end{pmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0$$

$$i\sqrt{1+\varepsilon}\, v_{21} + v_{22} = 0 \; ; \; \vec{V}_2 = \begin{bmatrix} 1 \\ -i\sqrt{1+\varepsilon} \end{bmatrix}$$

$$\vec{X} = \vec{V}_1 C_1 e^{-\lambda_1 t} + \vec{V}_2 C_2 e^{-\lambda t}$$

$$= \begin{bmatrix} 1 \\ \sqrt{1+\varepsilon} \end{bmatrix} C_1 e^{-\sqrt{1+\varepsilon}\, t} + \begin{bmatrix} 1 \\ -\sqrt{1+\varepsilon} \end{bmatrix} C_2 e^{\sqrt{1+\varepsilon}\, t}$$

$$u(t) = C_1 e^{-\sqrt{1+\varepsilon}\, t} + C_2 e^{\sqrt{1+\varepsilon}\, t}$$

$$v(t) = \sqrt{1+\varepsilon}\, C_1 e^{-\sqrt{1+\varepsilon}\, t} - \sqrt{1+\varepsilon}\, C_2 e^{\sqrt{1+\varepsilon}\, t}$$

$$x(0) = 0 = C_1 - C_2 \; ; \; \dot{x}(0) = 0 = -C_1\sqrt{1+\varepsilon} + C_2\sqrt{1+\varepsilon}$$

$$C_1 = C_2 ;$$

$$u(t) = e^{-\sqrt{1+\varepsilon}\, t} + e^{\sqrt{1+\varepsilon}\, t}$$

$$v(t) = \sqrt{1+\varepsilon}\, e^{-\sqrt{1+\varepsilon}\, t} - \sqrt{1+\varepsilon}\, e^{-\sqrt{1+\varepsilon}\, t}$$

b. $x(t,\varepsilon) = X_0 + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + O(\varepsilon^2)$

$$0 = (x_0 + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + O(\varepsilon^2)) + \varepsilon \frac{d}{dt}[X_0 + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + O(\varepsilon^2)]$$

$$+ \varepsilon^2 \frac{d^2}{d^2t}[X_0 + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + O(\varepsilon^2)]$$

$$0=(x_0+\varepsilon x_1(t)+\varepsilon^2 x_2(t))+\varepsilon[\dot{x}_0+\varepsilon\dot{x}_1(t)+\varepsilon^2\dot{x}_2(t))$$
$$+\varepsilon^2[\ddot{x}_0+\varepsilon\ddot{x}_1(t)+\varepsilon^2\ddot{x}_2(t)]$$
$$=x_0+\varepsilon(x_1(t)+\dot{x}_0)+\varepsilon^2(x_2(t)+\dot{x}_1(t)+\ddot{x}_0)+\cdots$$

$O(1)=x_0$

$O(\varepsilon)=x_1(t)+\dot{x}_0$

$O(\varepsilon^2)=x_2(t)+\dot{x}_1(t)+\ddot{x}_0$

c. Secular terms consist of functions which go to infinity as t approaches infinity, so the function has many.

$\ddot{x}+x=\varepsilon$ 9.6.3. $x(0)=1$; $\dot{x}(0)=0$

a. $\dot{u}=\dot{x}=v$
$\dot{v}=\varepsilon-x=\varepsilon-u$

$\vec{\dot{x}}=A\vec{x}+f(t)$; $Ax+f(t)=\lambda x$; $(A-\lambda)x+f(t)=0$

$$\begin{bmatrix}-\lambda & 1\\ -1 & -\lambda\end{bmatrix};\ \lambda_1=+i\ ;\ \lambda_2=-i$$

$\lambda_1=+i$

$$\begin{bmatrix}-i & 1\\ -1 & -i\end{bmatrix}\begin{bmatrix}v_{11}\\ v_{12}\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix};\ -iv_{11}+v_{12}=0 \quad \vec{v}_1=\begin{bmatrix}1\\+i\end{bmatrix}$$

$\lambda_2=-i$

$$\begin{bmatrix}+i & 1\\ -1 & +i\end{bmatrix}\begin{bmatrix}v_{21}\\ v_{22}\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix};\ iv_{21}+v_{22}=0 \quad \vec{v}_2=\begin{bmatrix}1\\-i\end{bmatrix}$$

$$\vec{x}=\begin{bmatrix}u\\v\end{bmatrix}=c_1\begin{bmatrix}1\\+i\end{bmatrix}e^{-it}+c_2\begin{bmatrix}1\\-i\end{bmatrix}e^{it}$$

$u(t)=c_1e^{-it}+c_2e^{it}=x(t)$

$v(t)=c_1ie^{-it}+c_2ie^{it}$

$$u(0) = 1 = C_1 \sin(0) + C_2 \cos(0) + \varepsilon$$

$$\dot{u}(0) = 0 = C_1 \cos(0) - C_2 \sin(0)$$

$$1 = C_2 + \varepsilon$$

$$0 = C_1$$

$$u(t) = (1 - \varepsilon) e^{it} + \varepsilon$$

$$v(t) = (1-\varepsilon) i e^{it}$$

b. $x(t, \varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + O(\varepsilon^3)$

$$0 = x_0 + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \frac{d}{dt}\varepsilon\left[x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t)\right] + \frac{d^2}{dt^2}\varepsilon^2\left[x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t)\right]$$

$$= x_0 + \varepsilon\left[x_1(t) + \dot{x}_0(t)\right] + \varepsilon^2\left[x_2(t) + \dot{x}_1(t) + \ddot{x}_0(t)\right] + \cdots$$

$O(1) = x_0$

$O(\varepsilon) = x_1(t) + \dot{x}_0(t)$

$O(\varepsilon^2) = x_2(t) + \dot{x}_1(t) + \ddot{x}_0(t)$

c. Secular terms not present because this function of t is periodic in the real-space.

$\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0$

7.6.4. $h(x, \dot{x}) = x$; $\ddot{x} + x + \varepsilon x = 0$

Averaged equation: $r' = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\sin(\theta)\, d\theta \equiv \langle h \sin\theta \rangle$

$$r\phi' = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\cos(\theta)\, d\theta \equiv \langle h\cos\theta\rangle$$

$$r' = \langle h\sin\theta\rangle = \langle r\cos\theta\sin\theta\rangle = \frac{r}{2\pi}\int_0^{2\pi}\cos\theta\sin\theta\, d\theta = r\frac{\sin^2\theta}{2}\Big|_0^{2\pi} = 0$$

$$r\phi' = \langle h\cos\theta\rangle = \langle r\cos^2\theta\rangle = \frac{r}{2\pi}\int_0^{2\pi}\cos^2\theta\, d\theta = \frac{r}{2\pi}\int_0^{2\pi}\frac{\cos(2\theta)+1}{2}\, d\theta$$

$$= \frac{r}{2\pi}\left[\frac{\sin(2\theta)}{4} + \frac{1}{2}\theta\right]_0^{2\pi} = \frac{r}{2}$$

Initial conditions: $r(t)=\sqrt{x(t)^2+\dot{x}(0)^2}$; $\phi(t)\approx \arctan\left(\dot{x}(t)/x(t)\right)$

$r(0)=\sqrt{a^2+0^2}=a$; $\phi(0)=\arctan(0)=0$

Amplitude/Frequency: Amplitude: $r(T)=\text{constant}$

Frequency: $\omega=\frac{d\theta}{dt}=1+\frac{d\phi}{dT}\frac{dT}{dt}=1+\varepsilon\phi'$

$=1+\frac{\varepsilon r(T)}{2}$

$=1+\frac{\varepsilon}{2}$

Solution: $x_0=r(T)\cos(\tau+\phi(T))$

$=a\cos(\tau)$

$\approx x(t,\varepsilon)$

$h(x,\dot{x})=x\dot{x}^2$ 7.6.5. $\ddot{x}+x+\varepsilon x\dot{x}^2=0$

Averaged Equation: $r'=\frac{1}{2\pi}\int_0^{2\pi}h(\theta)\sin(\theta)\,d\theta=\langle h\sin\theta\rangle$

$=\frac{1}{2\pi}\int_0^{2\pi}r\cos\theta(-r\sin\theta)^2\sin\theta\,d\theta$

$=\frac{-r^2}{2\pi}\int_0^{2\pi}\cos\theta\sin^3\theta\,d\theta=\frac{-r^2}{2\pi}\int_0 u^3\,du$

$=\frac{+r^3}{2\pi}\frac{\sin\theta}{4}\Big|_0^{2\pi}=0$

$r\phi'=\frac{1}{2\pi}\int_0^{2\pi}h(\theta)\cos(\theta)\,d\theta=\langle h\cos\theta\rangle$

$=\frac{1}{2\pi}\int_0^{2\pi}r\cos\theta(-r\sin\theta)^2\cos\theta\,d\theta$

$=\frac{r^3}{2\pi}\int_0^{2\pi}\cos^2\theta\sin^2\theta\,d\theta=\frac{r^3}{2\pi}\int_0^{2\pi}(\cos\theta\sin\theta)^2\,d\theta$

$=\frac{r^3}{2\pi}\int_0^{2\pi}\frac{\sin^2(2\theta)}{4}\,d\theta=\frac{r^3}{2\pi}\int_0^{2\pi}\frac{1-\cos(4\theta)}{8}\,d\theta$

$$= \frac{r^3}{16\pi}\left[\theta - \frac{\sin(4\theta)}{4}\right]_0^{2\pi} = \frac{r^3}{16\pi}[2\pi] = \frac{r^3}{8}$$

Initial conditions: $r(t) = \sqrt{x(t)^2 + \dot{x}(t)^2}$; $\phi(t) = \arctan(\dot{x}(t)/x(t))$

$= a$ $\qquad = 0$

Amplitude/Frequency: Amplitude: $r(T) = \text{constant}$

Frequency: $w = 1 + \frac{d\phi}{dT}\frac{dT}{dt} = 1 + \varepsilon\phi'$

$= 1 + \frac{\varepsilon a^2}{8}$; $\phi(T) = \frac{\varepsilon a^2}{8}T$

Solution: $x_0 = r(T)\cos(\tau + \phi(T))$

$= a\cos(\tau + \frac{\varepsilon a^2}{8}T)$

$x(t,\varepsilon) = a\cos\left(\left(\frac{\varepsilon a^2}{8}+1\right)t\right)$

$h(x,\dot{x}) = x\dot{x}$ 7.6.6. $\ddot{x} + x + x^2\dot{x} = 0$

Averaged Equations: $r' = \frac{1}{2\pi}\int_0^{2\pi} h\cos\theta\, d\theta$

$= \frac{-r^2}{2\pi}\int_0^{2\pi}\cos^2\theta\sin\theta\, d\theta$

$= -\frac{r^2}{2\pi}\int_0^{2\pi}\frac{u^2}{2}du = \frac{-r^2}{2\pi}\frac{\cos^2(\theta)}{2}\Big|_0^{2\pi} = \frac{-r^2}{2\pi}\left[\frac{1}{2}-\frac{1}{2}\right]$

$= 0$

$r\phi' = \frac{1}{2\pi}\int_0^{2\pi} h\sin\theta\, d\theta$

$= \frac{-r^2}{2\pi}\int_0^{2\pi}\cos\theta\sin^2\theta\, d\theta = \frac{-r^2}{2\pi}\int_0^{2\pi}\frac{u^2}{2}du$

$= -\frac{r^2}{2\pi}\frac{\sin^3\theta}{6}\Big|_0^{2\pi} = 0$

Initial conditions: $r(t) = \sqrt{x(t)^2 + \dot{x}(t)^2}$; $\phi(t) = \arctan(\dot{x}(t)/x(t))$

$= a$ $\qquad = 0$

Amplitude/Frequency: Amplitude: $r(T) = \text{constant}$

Frequency: $\omega = 1 + \phi'\frac{dT}{dt} = 1 + \varepsilon\phi' = 1$; $\phi(T) = 0$

$$= \frac{-r}{2\pi}\left[r^4\left(\frac{3\theta}{8} + \frac{3\cos\theta\sin(\theta)}{8} + \frac{\cos^3(\theta)\sin(\theta)}{4}\right.\right.$$

$$\left. - \frac{5}{6}\left[\frac{3\theta}{8} + \frac{3\cos\theta\sin\theta}{8} + \frac{\cos^3\theta\sin\theta}{4}\right]\right)$$

$$\left. - \frac{\theta}{2} - \frac{\cos\theta\sin\theta}{2}\right]_0^{2\pi}$$

$$= \frac{-r}{2\pi}\left[r^4\left(\frac{6\pi}{8} - \frac{5}{6}\left(\frac{6\pi}{8}\right)\right) - \pi\right] = \frac{-r}{2\pi}\left(r^4\left(\frac{\pi}{8}\right) - \pi\right)$$

$$= \frac{r}{2} - \frac{r^5}{16} = \frac{r}{16}(8 - r^4)$$

$$r\phi' = \frac{1}{2\pi}\int h\cos\theta\, d\theta = \frac{1}{2\pi}\int_0^{2\pi}(x^4 - 1)\dot{x}\cos\theta\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(r^4\cos^4\theta - 1)(-r\sin\theta)\cos\theta\, d\theta$$

$$= \frac{-r}{2\pi}\left[r^4\int_0^{2\pi}\cos^5\theta\sin\theta\, d\theta + \int_0^{2\pi}\sin\theta\cos\theta\, d\theta\right]$$

$$= \frac{-r}{2\pi}\left[r^4\int_0^{2\pi} -u^5\, du + \int_0^{2\pi} v\, dv\right] = \frac{-r}{2\pi}\left[r^4\frac{\cos^6\theta}{6} + \frac{\sin^2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{-r}{2\pi}[0] = 0$$

Initial Conditions:

$$r(t) = \underset{= a}{\sqrt{x(t)^2 + \dot{x}(t)^2}} \; ; \; \phi(t) = \underset{= 0}{\arctan\left(\dot{x}(t)/x(t)\right)}$$

Amplitude/Frequency: Amplitude: $16\int \frac{dr}{r(8 - r^4)} = T + c$

$$16\int \frac{dr}{r^5(8/r^4 - 1)} = 16\int \frac{r^5}{32\, r^5}\frac{1}{u}\, du$$

$$= \frac{1}{2}\ln\left(8/r^4 - 1\right) = T + C$$

$$r(T) = \sqrt[4]{\frac{8}{e^{T/2} + 1 + C}}$$

Frequency: $\omega = 1 + \varepsilon\phi' = 1 + O(\varepsilon^2)$

$$\phi(T) = O(\varepsilon^2)$$

Solution: $X_0 = r(\tau)\cos(\tau + \phi(\tau))$

$= 0$

$X(t,\varepsilon) = 0$

$h(x,\dot{x}) = (x^4-1)\dot{x}$

7.6.7. $\ddot{x} + x + (x^4-1)\dot{x} = 0$

Averaged Equations: $r' = \frac{1}{2\pi}\int_0^{2\pi} h\sin\theta\, d\theta = \langle h\sin\theta\rangle$

$$= \frac{1}{2\pi}\int_0^{2\pi}(x^4-1)\dot{x}\sin\theta\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}([r\cos\theta]^4-1)(-r\sin\theta)\sin\theta\, d\theta$$

$$= \frac{-r}{2\pi}\left[r^4\int_0^{2\pi}\cos^4\theta\sin^2\theta\, d\theta - \int_0^{2\pi}\sin^2\theta\, d\theta\right]$$

$$= \frac{-r}{2\pi}\left[r^4\int\cos^4(\theta)(1-\cos^2(\theta))\,d\theta - \int_0\sin^2\theta\, d\theta\right]$$

Reduction Formula:

$$\int\cos^n(x)\,dx = \frac{n-1}{n}\int\cos^{n-2}(x)\,dx + \frac{\cos^{n-1}(x)\sin(x)}{n}$$

$$\int\sin^n(x)\,dx = \frac{n-1}{n}\int\sin^{n-2}(x)\,dx - \frac{\cos(x)\sin^{n-1}(x)}{n}$$

$$= \frac{-r}{2\pi}\left[r^4\left(\int_0^{2\pi}\cos^4(\theta)\,d\theta - \int_0^{2\pi}\cos^6(\theta)\,d\theta\right) - \int_0^{2\pi}\sin^2(\theta)\,d\theta\right]$$

$$\int_0^{2\pi}\cos^4(\theta) = \frac{3}{4}\int_0^{2\pi}\cos^2(\theta)\,d\theta + \frac{\cos^3(\theta)\sin\theta}{4}$$

$$= \frac{3}{4}\left(\frac{\theta}{2} + \frac{\cos(\theta)\sin(\theta)}{2}\right) + \frac{\cos^3(\theta)\sin(\theta)}{4}$$

$$\int_0^{2\pi}\cos^6(\theta)\,d\theta = \frac{5}{6}\int_0^{2\pi}\cos^4(\theta)\,d\theta + \frac{\cos^5(\theta)\sin(\theta)}{6}$$

$$= \frac{5}{6}\left[\frac{3\theta}{8} + \frac{3\cos(\theta)\sin(\theta)}{8} + \frac{\cos^3(\theta)\sin(\theta)}{4}\right]$$

$$\int_0^{2\pi}\sin^2(\theta)\,d\theta = \frac{\theta}{2} + \frac{\cos(\theta)\sin(\theta)}{2}$$

Solution: $x_0 = r(\tau)\cos(\tau + \phi(\tau))$

$$= \sqrt[4]{\frac{8}{e^{\tau/2} + 1 + C}}\cos(\tau + O(\varepsilon^2))$$

$$x(t,\varepsilon) = \sqrt[4]{\frac{8}{e^{\tau/2} - 1 + \frac{8}{a^4}}}\cos(\tau + O(\varepsilon^2))$$

$h(x,\dot{x}) = (|x|-1)\dot{x}$ 7.6.8 Averaged Equations:

$$r' = \frac{1}{2\pi}\int_0^{2\pi} h\sin\theta\, d\theta = \langle h\sin\theta\rangle$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(|x|-1)\dot{x}\sin\theta\, d\theta = \frac{1}{2\pi}\int_0^{2\pi}(|r\cos\theta|-1)(-r\sin\theta)\sin\theta\, d\theta$$

$$= \frac{-r}{2\pi}\left(r\int_0^{2\pi}|\cos\theta|\sin^2\theta\, d\theta - \int_0^{2\pi}\sin^2\theta\, d\theta\right)$$

$$= \frac{-r}{2\pi}\left(r\frac{\cos\theta}{|\cos\theta|}\int_0^{2\pi}\cos\theta\sin^2\theta\, d\theta - \left(\frac{\theta}{2} - \frac{\cos(\theta)\sin(\theta)}{2}\right)\Big|_0^{2\pi}\right)$$

$$= \frac{-r}{2\pi}\left(r\frac{\cos\theta}{|\cos\theta|}\frac{\sin^3(\theta)}{3} - \frac{\theta}{2} - \frac{\cos(\theta)\sin(\theta)}{2}\right)\Big|_0^{2\pi}$$

$$= \frac{-r}{2\pi}\left[-\pi\right] = \frac{r}{2}$$

$$r\phi' = \frac{1}{2\pi}\int_0^{2\pi} h\cos\theta\, d\theta = \langle h\cos\theta\rangle$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(|x|-1)\dot{x}\cos\theta\, d\theta = \langle h\cos\theta\rangle$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(|r\cos\theta|-1)(-r\sin\theta)\cos\theta\, d\theta$$

$$= \frac{-r}{2\pi}\left(r\int_0^{2\pi}|\cos\theta|\cos\theta\sin\theta\, d\theta - \int_0^{2\pi}\sin\theta\cos\theta\, d\theta\right)$$

$$= \frac{-r}{2\pi}\left(r\frac{\cos\theta}{|\cos\theta|}\int_0^{2\pi}\cos^2\theta\sin\theta\, d\theta - \frac{\sin^2\theta}{2}\Big|_0^{2\pi}\right)$$

$$= \frac{-r}{2\pi}\left(-r\frac{\cos\theta}{|\cos\theta|}\frac{\cos^3\theta}{3}\Big|_0^{2\pi}\right) = 0$$

Initial Conditions: $r(t) = \sqrt{x(t)^2 + \mathring{x}(t)^2} = a$

$\phi(t) = \arctan\left(\mathring{x}(t)/x(t)\right) = 0$

Amplitude/Frequency: Amplitude $r(T) = a$

Frequency $w = 1 + \varepsilon\dot{\phi}$; $\phi(T) = 0$

Solution: $x_0 = r(T)\cos(\tau + T) = a\cos(t)$

$x(t, \varepsilon) = a\cos(t)$

$h(x, \dot{x}) = (x^2 - 1)\dot{x}^3$ 7.6.9. $\ddot{x} + x + (x^2 - 1)\dot{x}^3 = 0$

Average Equations: $r' = \frac{1}{2\pi}\int_0^{2\pi} h(x, \dot{x})\sin(\theta)\,d\theta = \langle h\sin\theta \rangle$

$= \frac{1}{2\pi}\int_0^{2\pi} (x^2 - 1)\dot{x}^3 \sin(\theta)\,d\theta$

$= \frac{1}{2\pi}\int_0^{2\pi} (r^2\cos^2\theta - 1)(-r^3)(\sin^3\theta)\sin(\theta)\,d\theta$

$= \frac{-r^3}{2\pi}\left[r^2\int_0^{2\pi}\cos^2\theta\sin^4\theta\,d\theta - \int_0^{2\pi}\sin^4\theta\,d\theta\right]$

Reduction Formula:

$\int_0 \cos^n(x)\,dx = \frac{n-1}{n}\int \cos^{n-2}(x)\,dx + \frac{\cos^{n-1}(x)\sin(x)}{n}$

$\int \sin^n(x)\,dx = \frac{n-1}{n}\int \sin^{n-2}(x)\,dx + \frac{\cos(x)\sin^{n-1}(x)}{n}$

$\int_0^{2\pi} \sin^2(\theta)\,d\theta = \frac{\theta}{2} + \frac{\cos(\theta)\sin(\theta)}{2}$

$-\int_0^{2\pi} \sin^6(\theta)\,d\theta = \frac{-5}{6}\int_0^{2\pi} \sin^4(\theta)\,d\theta + \frac{\cos(\theta)\sin^5(\theta)}{6}$

$\int_0^{2\pi} \sin^4(\theta)\,d\theta = \frac{3}{4}\int_0^{2\pi} \sin^2\theta\,d\theta + \frac{\cos(\theta)\sin^3(\theta)}{4}$

$$= \frac{-r^3}{2\pi}\left[r^2\left[\left[\frac{\theta}{2}+\frac{\cos(\theta)\sin(\theta)}{2}\right]-\frac{5}{6}\left[\frac{3}{4}\left[\frac{\theta}{2}+\frac{\cos(\theta)\sin(\theta)}{2}\right]+\frac{\cos(\theta)\sin^3(\theta)}{4}\right]\right.\right.$$

$$\left.\left.+\frac{\cos(\theta)\sin^5(\theta)}{6}\right]-\frac{3}{4}\left[\left[\frac{\theta}{2}+\frac{\cos\theta\sin\theta}{2}\right]+\frac{\cos\theta\sin^3\theta}{6}\right]\right]_0^{2\pi}$$

$$= \frac{-r^3}{2\pi}\left[r^2\left[\pi-\frac{15}{24}\pi\right]-\frac{3}{4}\pi\right]$$

$$= \frac{-r^3}{2\pi}\left[\frac{3}{8}\pi r^2-\frac{3}{4}\pi\right] = \frac{r^3}{16}\left(6-r^2\right)$$

$$r\phi' = \frac{1}{2\pi}\int_0^{2\pi} h\cos(\theta)\,d\theta = \langle h\cos\theta\rangle$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(x^2-1)\,\dot{x}^3\cos(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}(r^2\cos^2\theta-1)(-r^3)\sin^3\theta\cos(\theta)\,d\theta$$

$$= \frac{-r^3}{2\pi}\left[r^2\int_0^{2\pi}\cos^3\theta\sin^3\theta\,d\theta-\int_0^{2\pi}\sin^3\theta\cos\theta\,d\theta\right]$$

$$= -\frac{r^3}{2\pi}\left[r^2\int_0^{2\pi}\cos(\theta)(1-\sin^2\theta)\sin^3\theta\,d\theta-\frac{\sin^4\theta}{4}\right]_0^{2\pi}$$

$$= -\frac{r^3}{2\pi}\left[r^2\left[\int_0^{2\pi}\cos\theta\sin^3\theta\,d\theta-\int_0^{2\pi}\cos\theta\sin^5\theta\,d\theta\right]\right]$$

$$= \frac{-r^3}{2\pi}\left[r^2\left[\frac{\sin^4\theta}{4}-\frac{\sin^6\theta}{6}\right]\right]_0^{2\pi} = 0$$

Initial Conditions: $r(t)=\sqrt{x(t)^2+\dot{x}(t)^2}=a$

$$\phi(t)=\arctan\left(\dot{x}(t)/x(t)\right)=0$$

Amplitude/Frequency: Amplitude: $T=\int\frac{dr}{r^3(\;)}$

$$= -16\int\frac{dr}{r^3(r^2-6)}$$

$$= -16\int\left(\frac{A}{r}+\frac{B}{r^2}+\frac{C}{r^3}+\frac{Dr+E}{(r^2-6)}\right)dr$$

$$A=\frac{-1}{36};\ B=0;\ C=\frac{-1}{6};\ D=\frac{1}{36};\ E=0$$

$$= -16\int\frac{r}{36(r^2-6)}-\frac{1}{36r}-\frac{1}{6r}\,dr$$

$$= \frac{2}{9}\ln(|r^2-6|) + \frac{4}{9}\ln(|r|) - \frac{12}{9}\frac{1}{r^2} = T + C$$

Frequency: $\omega = 1 + \varepsilon\phi'$; $\phi(T) = \phi_0$

Solution: $x_0 = r(T)\cos(\tau + \phi(T))$

$$= \sqrt{6}\cos(\tau) + \phi_0)$$

$$x(t,\varepsilon) = \sqrt{6}\cos(\tau + \phi_0)$$

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2}$

$\sin\theta\cos^2\theta = \frac{\sin\theta + \sin 3\theta}{4}$

7.6.10. $\sin\theta\cos^2\theta = \left[\frac{e^{i\theta}-e^{-i\theta}}{2}\right]\left[\frac{e^{i\theta}+e^{-i\theta}}{2}\right]^2$

$$= \frac{1}{8}\left[e^{i\theta} - e^{-i\theta} + e^{3i\theta} - e^{-3i\theta}\right]$$

$$= \frac{\sin\theta + \sin 3\theta}{4}.$$

$$\partial_{\tau\tau}x_1 + x_1 = \left[-2r' + r - \frac{1}{4}r^3\right]\sin(\tau+\phi)$$
$$+ \left[-2r\phi'\right]\cos(\tau+\phi) - \frac{1}{4}r^3\sin 3(\tau+\phi)$$

7.6.11 $x(0) = 2$; $\dot{x}(0) = 0$

$$r(T) = \sqrt{x(t)^2 + \dot{x}(t)^2} = 2; \quad \phi(T) = \arctan\left(\frac{\dot{x}(t)}{x(t)}\right) = 0$$

$$\partial_{\tau\tau}x_1 + x_1 = [2-2]\sin(\tau+\phi) + [0]\cos(\tau+\phi) - \frac{1}{2}\sin 3(\tau+\phi)$$

$$\partial_{\tau\tau}x_1 + x_1 = -\frac{1}{2}\sin 3(t+\phi)$$

$$\underbrace{2\partial_{\tau\tau}x_1 + 2x_1 = -\sin 3(t+\phi)}$$

Linear Equation with constant coefficients.

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = f(y)$$

$$a_0\lambda^n + a_1\lambda^{(n-1)} + \cdots + a_{n-1}\lambda + a_n = 0$$

$$2\lambda^2 + 2 = 0$$

$$2(\lambda^2+1) = 0$$

$$\lambda_{1,2} = \pm i\ ;\ k=1\ ;\ \tau = C_1 \sin(y) + C_2 \cos(y)$$

Solution to a Homogenous Differential Equation:

$$y = \sum_i P_{k-1}(y)e^{\alpha y}\sin\beta y + Q_{k-1}(y)e^{\alpha y}\cos\beta y$$

Where $\lambda = \alpha \pm \beta i$

$$P_{k-1}(y)e^{\alpha y} = C_1$$

$$Q_{k-1}(y)e^{\alpha y} = C_2$$

Particular Solution: $2\partial_{\tau\tau}X_1 + 2X_1 = -\sin 3(t+T)$

Assume: $X_1 = A$, then $\partial_{\tau\tau}X_1 = 0$

$$2A = -\sin 3(t+T)$$

$$A = -\frac{1}{2}\sin 3(t+T)$$

General Solution: X = Homogenous + Particular

$$= C_1\sin\tau + C\cos\tau - \frac{\sin 3(t+T)}{2}$$

$$\langle f(\theta)\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\,d\theta$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\cos k\theta\,d\theta$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\sin k\theta\,d\theta$$

7.6.51

a. $\langle \cos k\theta \sin m\theta\rangle$ Method #1: even x odd = odd ; $\int_{-\infty}^{\infty} \text{odd} = \int_0^{2\pi}\text{odd} = 0$

Method #2: Product to Sum of Two angles

$$\cos\phi\sin\theta = \frac{\sin(\theta+\phi) - \sin(\theta-\phi)}{2}$$

$$\int \cos k\theta \sin m\theta\,d\theta = \int \frac{\sin((k+m)\theta) - \sin((k-m)\theta)}{2}\,d\theta$$

$\langle \cos k\theta \cos m\theta\rangle$ Method #1: even x even = even ; odd x odd = even ; $\int$ even =

Method #2: Product to sum of Two Angles

$$\cos\theta\cos\phi = \frac{\cos(\theta-\phi) + \cos(\theta+\phi)}{2}$$

$$= \int \frac{\cos((k-m)\theta) + \cos((k+m)\theta)}{2} \, d\theta$$

$\neq 0$ for $k \neq m$

$\langle \cos^2 k\theta \rangle = \langle \sin^2 k\theta \rangle = \frac{1}{2}$: Method #1 : even x even = even

$\int$ even = constant.

Method #2 : Produce to sum of Two Angles

$$\cos\theta\cos\phi = \frac{\cos(\theta-\phi)+\cos(\theta+\phi)}{2}$$

$$\int \cos^2 k\theta = \int \frac{1}{2} + \int \frac{\cos 2k\theta}{2} \, d\theta$$

$$= \frac{1}{2}$$

b. $h(\theta) = \sum_{k=0}^{\infty} a_k \cos k\theta + \sum_{k=1}^{\infty} b_k \sin k\theta$

$$\cos m\theta \, h(\theta) = \sum_{k=0}^{\infty} a_k \cos k\theta \cos m\theta + \sum_{k=1}^{\infty} b_k \sin k\theta \cos m\theta$$

$$\cos m\theta \, h(\theta) = \sum_{k=0}^{2\pi} a_k \cos k\theta \cos m\theta + \sum_{k=0}^{2\pi} b_k \sin k\theta \cos m\theta$$

$$\langle h(\theta) \cos m\theta \rangle = \frac{1}{2} a_k$$

c. $h(\theta) = \sum_{k=0}^{\infty} a_k \cos k\theta + \sum_{k=0}^{\infty} b_k \sin k\theta$

$$\langle h(\theta) \sin k\theta \rangle = \sum_{k=0}^{2\pi} a_k \cos k\theta \sin + \sum_{k=0}^{2\pi} b_k \sin k\theta \sin k\theta$$

$$= \frac{1}{2} b_k$$

$$\langle h(\theta) \rangle = \sum_{k=0}^{2\pi} a_k \cos k\theta + \sum_{k=0}^{2\pi} b_k \sin k\theta$$

$$= a_0$$

$\ddot{x}+x+\varepsilon x^3$ 7.6.13 $x(0)=a$; $\dot{x}(0)=0$

a. $h=x^3$; $r' = \frac{1}{2\pi}\int_0^{2\pi} h\sin\theta\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} x^3 \sin\theta\, d\theta$

$$= \frac{1}{2\pi}\int_0^{2\pi} (r\cos\theta)^3 \sin\theta\, d\theta = \frac{r^3}{2\pi}\int_0^{2\pi} \cos^3\theta \sin\theta\, d\theta$$

$$= \frac{r^3}{2\pi}\frac{\cos^3\theta}{3}\Big|_0^{2\pi} = 0$$

$T = r(T) = $ constant.

b. $$r\phi' = \frac{1}{2\pi}\int_0^{2\pi} h\cos\theta\, d\theta - \int_0 (r\cos\theta)^3 \cos\theta\, d\theta$$

$$= \frac{r^3}{2\pi}\int_0^{2\pi} \cos^3\theta \cdot \cos\theta\, d\theta = r^3 \int_0^{2\pi} \cos^4\theta\, d\theta$$

Reduction Formula:

$$\int \cos^n\theta\, d\theta = \frac{n-1}{n}\int \cos^{n-2}\theta\, d\theta + \frac{\cos^{n-1}\theta \sin\theta}{n}$$

$$\int \cos^4\theta\, d\theta = \frac{3}{4}\int \cos^2\theta\, d\theta + \frac{\cos^3\theta \sin\theta}{4}$$

$$\int \cos^2\theta\, d\theta = \frac{\theta}{2} + \frac{\cos\theta\sin\theta}{2}$$

$$r\phi' = \frac{r^3}{2\pi}\left[\frac{3}{4}\left[\frac{\theta}{2} + \frac{\cos\theta\sin\theta}{2}\right]_0 + \frac{\cos^3\theta\sin\theta}{4}\right]_0^{2\pi}$$

$$= \frac{r^3}{2\pi}\left[\frac{3}{4}\pi\right] = \frac{3}{8}r^3$$

$\phi' = \frac{3}{8}r^2$; $\omega = 1+\varepsilon\phi' = 1+\frac{3}{8}r^2\varepsilon = 1+\frac{3}{8}a^2\varepsilon$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{1+\frac{3}{8}\varepsilon a^2} = 2\pi\left(1 - \frac{3}{8}\varepsilon a^2 + \left(\frac{3}{8}\varepsilon a^2\right) + \cdots\right)$$

b. Note: The book answer is a power series about kinetic energy, while the Fourier series arrived to the same answer.

$\ddot{x} + \sin x = 0$ 7.6.15

a. $\ddot{x} = -\sin(x)$; $F = ma = m\ddot{x} = -kx = -\sin(x)$

$$\cong -\left(x - \frac{1}{6}x^3\right)$$
$$= -\left(1 - \frac{1}{6}x^2\right)x$$
$$= -kx$$

$$r\phi' = \frac{1}{2\pi}\int_0^{2\pi} h\cos\theta\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} r^3\cos^3\theta\cos\theta\, d\theta = \frac{3}{8}r^3$$

$$\omega = 1 + \varepsilon\phi' = 1 + \varepsilon\frac{3}{8}r^2 \approx 1 - \frac{1}{16}a^2$$

b. $T = \frac{2\pi}{\omega} = \frac{2\pi}{1-\frac{1}{16}a^2} = 2\pi\left(1 + \frac{1}{16}a^2 + \cdots\right)$ "Agreement"

$\ddot{x} + \varepsilon\dot{x}(x^2-1) + x = 0$ 7.6.16. Green's Theorem : $\oint_C v\cdot n\, dl = \iint_A \nabla\cdot v\, dA$

$$v = \dot{x} = (\dot{x}, \dot{y})$$
$$\dot{v} = -x - \varepsilon\dot{x}(x^2-1) = -x - \varepsilon v(x^2-1)$$
$$\iint_A \nabla\cdot v\, dA = \iint_A \frac{\partial}{\partial v}\left(-x - \varepsilon v(x^2-1)\right)dA = \iint_A -\varepsilon(x^2-1)\, dA$$
$$= \int_0^{2\pi}\int_0^a -\varepsilon(r^2\cos^2\theta - 1)\, r\, dr\, d\theta = \int_0^{2\pi} \left.\frac{-\varepsilon}{4}r^4\cos^2\theta + \frac{\varepsilon r^2}{2}\right|_0^a d\theta$$
$$= \int_0^{2\pi} \frac{-\varepsilon}{4}a^4\cos^4\theta + \frac{\varepsilon a^2}{2}\, d\theta = \frac{-\varepsilon a^4\pi}{4} + \varepsilon a^2\pi$$

$$r' = \frac{1}{2\pi}\int_0^{2\pi} r(\gamma + \cos(2t))\cos\theta\sin\theta\, d\theta$$

$$= \frac{r}{2\pi}\left[\int_0^{2\pi} \gamma\sin\theta\cos\theta\, d\theta + \int_0 \cos(2\theta - 2\phi)\cos\theta\sin\theta\, d\theta\right]$$

$$= \frac{r}{2\pi}\left[\int_0^{2\pi} \cos 2\theta\cos 2\phi\cos\theta\sin\theta\, d\theta + \int_0^{2\pi} \sin 2\theta\sin 2\phi\cos\theta\sin\theta\, d\theta\right]$$

$$= \frac{r}{2\pi}\left[\int_0^{2\pi} (2\cos^2\theta - 1)\cos 2\phi\cos\theta\sin\theta\, d\theta + \int_0^{2\pi} 2\cos\theta\sin\theta\cos\theta\sin\theta\, d\theta\right]$$

$$= \frac{r}{2\pi}\left[\frac{2\pi}{4}\sin 2\phi\right] = \frac{r}{4}\sin 2\phi$$

b. If $r=0$, $\phi' = 0 = \frac{r}{2}\left(\gamma + \frac{\cos 2\phi}{2}\right)$; $\cos 2\phi = 2\gamma$

$$r' = 0 = \frac{r}{2}\sin(\arccos(-2\gamma))$$

$$= \frac{r}{2}\sqrt{1-(-2\gamma)^2} = \frac{r}{2}\sqrt{1-4\gamma^2}$$

When $\gamma < 1/2$, then $r' > 0$

$\gamma = 1/2$, then $r' = 0$

$\gamma > 1/2$, then $r' \propto i$

Therefore $\gamma < 1/2$ is a critical value.

c. $r' = \frac{dr}{dT} = \frac{r}{4}\sqrt{1-4\gamma^2}$ @ $r=0$, then $T = \int \frac{4\, dr}{r\sqrt{1-4\gamma^2}}$

$$= \frac{4}{\sqrt{1-4\gamma^2}}\ln r$$

and $r(T) = e^{\sqrt{1-4\gamma^2}\,T/4}$

Where $k = \frac{\sqrt{1-4\gamma^2}}{4}$

$$\oint_V v\cdot n\, dl = \int_0^{2\pi} \langle a\sin(t), -a\cos(t) - \varepsilon a\sin(t)(a^2\cos^2(t)-1)\rangle \cdot \langle -a\sin(t), a\cos(t)\rangle\, dt$$

$$= \int_0^{2\pi} -\varepsilon a^4 \sin(t)\cos^3(t) + \varepsilon a^3\sin(t)\cos(t)$$

$$= \frac{a^4}{4}\varepsilon\cos^4(t) - \frac{a^2}{2}\varepsilon\cos^2(t)\Big|_0^{2\pi} = 0$$

$$\varepsilon a^2\pi\left(1-\frac{1}{4}a^2\right)=0 \;;\; a=2$$

$$\ddot{x} + (1+\varepsilon\gamma + \varepsilon\cos 2t)\sin x = 0$$

7.6.17. x = Angle Between swing and Downward Vertical

- $1+\varepsilon\gamma+\varepsilon\cos 2t$: Effect of gravity and periodic pumping

$x=0$; $\dot{x}=0$

a. $\ddot{x} + (1+\varepsilon\gamma+\varepsilon\cos 2t)x=0$

$$r\phi' = \frac{r}{2\pi}\int_0^{2\pi}((\gamma + \cos(2t))\cos^2\theta\, d\theta$$

$$= \frac{r}{2\pi}\int_0^{2\pi}(\gamma\cos^2\theta)\,d\theta + \frac{r}{2\pi}\int_0^{2\pi}\cos(2\theta-2\phi)\cos^2\theta\, d\theta$$

$$= \frac{r}{2\pi}\left[\gamma\pi + \int_0^{2\pi}(\cos2\theta\cos2\phi + \sin2\theta\sin2\phi)\cos^2\theta\, d\theta\right]$$

$$= \frac{r}{2\pi}\left[\gamma\pi + \int_0^{2\pi}\cos2\theta\cos^2\theta\cos2\phi\, d\theta + \int_0^{2\pi}\sin2\theta\sin2\phi\cos^2\theta\, d\theta\right]$$

$$= \frac{r}{2\pi}\left[\gamma\pi + \int_0^{2\pi}(2\cos^2\theta-1)\cos^2\theta\cos2\phi\, d\theta + \int_0^{2\pi}2\cos\theta\sin\theta\sin2\phi\cos^2\theta\, d\theta\right]$$

$$= \frac{r}{2\pi}\left[\gamma\pi + \cos2\phi\left[\frac{6\pi}{4}-\pi\right]\right] = \frac{r}{2}\left[\gamma + \frac{\cos2\phi}{2}\right]$$

d. $\frac{dr}{d\phi} = \frac{r'}{\phi'} = \frac{\left(\frac{r}{4}\sin(2\phi)\right)}{\left(\frac{r}{2}\left[\gamma + \frac{\cos 2\phi}{2}\right]\right)} = \frac{r\sin(2\phi)}{2\gamma + \cos(2\phi)}$

$$\int \frac{dr}{r} = \int \frac{\sin(2\phi)}{2\gamma + \cos(2\phi)}\, d\phi$$

$$\ln(r) = -\frac{1}{2}\ln(2\gamma + \cos(2\phi))$$

$r(\phi) = \frac{c}{\sqrt{2\gamma + \cos(2\phi)}}$ is a periodic function and closed orbit.

e. A physical interpretation of the form $\ddot{x} + kx = 0$ is synonymous with Hooke's law, a pendulum, or swing. In this problem, $k = \frac{\sqrt{1-4\gamma^2}}{4} = r(\gamma + \cos 2t)\cos\theta$

$\ddot{x} + (a + \varepsilon\cos t)x = 0$

7.6.19 Mathieu Equation ; $a \simeq 1$; $\tau = \varepsilon^2 t$

$$\ddot{x} + (a + \varepsilon\cos t)x = (\ddot{x}_0(\tau) + \varepsilon\ddot{x}_1(\tau) + \varepsilon^2\ddot{x}_2(\tau)) + (a + \varepsilon\cos t)(x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau))$$

$$= [\ddot{x}_0(\tau) + a x_0(\tau)]$$

$$+ \varepsilon[\ddot{x}_1(\tau) + a x_1(\tau) + \cos t\, x_0(\tau)]$$

$$+ \varepsilon^2[\ddot{x}_2(\tau) + a x_2(\tau) + \cos t\, x_1(\tau)]$$

$$+ O(\varepsilon^3)$$

$O(1)$: $\ddot{x}_0(\tau) + a x_0(\tau) = 0$

$O(\varepsilon)$: $\ddot{x}_1(\tau) + a x_1(\tau) + \cos(t)x_0(\tau) = 0$

$O(\varepsilon^2)$: $\ddot{x}_2(\tau) + a x_2(\tau) + \cos(t)x_1(\tau) = 0$

$\ddot{x} + x + \varepsilon x^3 = 0$ 7.6.19. $x(0) = a$; $\dot{x}(0) = 0$; $\boxed{\text{Poincare-Lindstedt Method}}$

a. $\tau = \omega t$; $\frac{d^2x}{d^2t} + x + \varepsilon x^3 = \omega \frac{d^2x}{d^2\tau} + x + \varepsilon x^3$

$= \omega x'' + x + \varepsilon x^3$

$= 0$

b. $x(\tau, \varepsilon) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + O(\varepsilon^3)$

$\omega = 1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + O(\varepsilon^3)$

$(1 + \varepsilon\omega_1 + \varepsilon^2\omega_2)^2 (x_0''(\tau) + \varepsilon x_1''(\tau) + \varepsilon^2 x_2''(\tau))$

$+ (x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau)) + (x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(t))^3 = 0$

$(1 + 2\varepsilon\omega_1 + 2\varepsilon^2\omega_2 + \cdots)(x_0''(\tau) + \varepsilon x_1''(\tau) + \varepsilon^2 x_2''(\tau) + \cdots)$

$+ (x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \cdots) + \varepsilon(x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon x_2(t))^3$
$= 0$

$O(1) = x_0''(\tau) + x_0(\tau) = 0$

$O(\varepsilon) = x_1''(\tau) + 2\omega_1 x_0''(\tau) + x_1(\tau) + x_0^3(\tau) = 0$

c. $x_0(0) = a$; $\dot{x}_0(0) = 0$; $x_k(0) = \dot{x}_k(0) = 0$

From the blurb, $x(0) = a$; $\dot{x}(0) = 0$; $\dot{x}_0 = (a)$; $x_0'(0) = 0$

d. $x_0''(\tau) + x_0(\tau) = 0$: $x_0 = a\cos(\tau)$

e. $x_1''(\tau) + x_1(\tau) = -2\omega_1 x_0''(\tau) - x_0^3(\tau)$

$= +2a\omega_1 \cos(\tau) - a^3\cos^3(\tau)$

$= 2a\omega_1\cos(\tau) - a^3\left[\frac{1}{4}(3\cos(\tau) + \cos(3\tau))\right]$

$= (2aw_1 - \frac{3}{4}a^3)\cos(\tau) - \frac{a^3}{4}\cos(3\tau)$; $w_1 = a^2$

F. $X_1''(\tau) + X_1(\tau) = (2aw_1 - \frac{3}{4}a^3)\cos(\tau) - \frac{a^3}{4}\cos(3\tau)$

$X_1''(\tau) + X_1 = -\frac{1}{4}a^3\cos(3\tau)$

$4X_1''(\tau) + 4X_1 = -a^3\cos(3\tau)$

$\boxed{\text{Linear Equation of Constant Coefficients.}}$ "Homogenous"

$a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$

$4(\lambda^2+1) = 0$; $\lambda^2 + 1 = 0$; $\lambda_{1,2} = \pm i$

$\boxed{\text{Solution of Homogenous Equations}}$

$t = \sum P_{K-1}(t)e^{\alpha t}\sin\beta t + Q_{K-1}(t)e^{\alpha t}\cos\beta t$

$\lambda = \alpha \pm \beta i$; $X = C_1\sin(t) + C_2\cos(t)$

$\boxed{\text{Method of Undetermined Coefficients.}}$

$X_i = t^s e^{\alpha s}(R_m(t)\cos\beta t + T_m(t)\sin\beta t) = -a^3\cos(3t)$

$S = 0$; $\alpha = 0$; $\beta = 3$

$X = B\sin(3t) + A\sin(3t)$

$X'' = -9B\sin(3t) - 9A\sin(3t)$

$-32B\sin(3t) - 32A\cos(3t) = -\frac{a^3}{4}\cos(3t)$

$A = \frac{a^3}{32}$; $B = 0$; $X_i = \frac{a^3}{32}\cos(3t)$

$\boxed{\text{General Solution}}$

$X = \frac{a^3}{32}\cos(3t) + C_1\sin(t) + C_2\cos(t)$

$X(0) = \frac{a^3}{32} + C_2 = 0$;

$X = \frac{a^3}{32}\cos(3t) - \frac{a^3}{32}\cos(t)$

$x(t,\varepsilon) = a\cos t + \varepsilon a^3\left[-\frac{3}{8}t\sin t + \frac{1}{32}(\cos 3t - \cos t)\right] + O(\varepsilon^2)$

7.6.20. $\ddot{x} + x + \varepsilon x^3 = (\ddot{x}_0 + \varepsilon\ddot{x}_1 + \varepsilon^2\ddot{x}_2) + (x_0 + \varepsilon x_1 + \varepsilon^2 x_2)$

$$+ \varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2)^3$$

$$= (\ddot{x}_0 + x_0) + \varepsilon(\ddot{x}_1 + x_1 + x_0^3)$$

$$+ \varepsilon^2(\ddot{x}_2 + x_2 + 3x_0^2 x_1) + O(\varepsilon^3)$$

$O(1)$: $\ddot{x}_0 + x_0 = 0$

$O(\varepsilon)$: $\ddot{x}_1 + x_1 + x_0^3 = 0$

$O(\varepsilon^2)$: $\ddot{x}_2 + x_2 + 3x_0^2 x_1 = 0$

Solving for x_0; $\ddot{x}_0 + x_0 = 0$; $x_0 = a\cos(t)$

Solving for x_1; $\ddot{x}_1 + x_1 = -a^3\cos^3(t)$

$\boxed{\text{Linear Equation of constant coefficients}}$

$a_0 t^{(n)} + a_1 t^{(n-1)} + \cdots + a_{n-1} t' + a_n t = f(t)$

$\boxed{\text{Solving the Homogenous Equation}}$

$a_0\lambda^n + a_1\lambda^{(n-1)} + \cdots + a_{n-1}\lambda' + a_n\lambda = 0$

$(\lambda^2 + 1) = 0$; $\lambda_{1,2} = \pm i$

$\boxed{\text{Solution of a Homogenous Equation; Summand}}$

$t = \sum P_{k-1}(t)e^{\alpha t}\sin\beta t + Q_{k-1}(t)e^{\alpha t}\cos\beta t$

where $\lambda = \alpha + \beta i$; $\alpha = 0$; $\beta = 1$; $k = 1$

so $x = A\sin(t) + B\cos(t)$

$\boxed{\text{Method for Undetermined Coefficients: Particular Solution}}$

$x_i = t^s e^{\alpha t}(R_m(t)\cos\beta t + T_m(t)\sin\beta t)$

If $s = k = 1$, $x = t(B\sin t + A\cos t)$

$$\ddot{x} = (-Bt - 2A)\sin t + (2B - At)\cos t$$

$$\ddot{x}_1 + x_1 = 2B\cos(t) - 2A\sin(t) = -a^3\cos t$$

$$2B = -a^3 \; ; \; A = 0$$

$$-2A = 0 \; ; \; B = -\frac{a^3}{2}$$

$$x_1 = -\frac{\varepsilon a^3 t \sin t}{2}$$

$\boxed{\text{General Solution = Particular + Homogenous Equation}}$

$$x_1(t) = -\frac{\varepsilon a^3 t \sin t}{2} + C_1 \sin t + C_2 \cos t$$

with initial conditions: $x(0) = a$; $\dot{x}(0) = 0$

$$x_1(t) = a\cos t$$

Solving for x_2: $\ddot{x}_2 + x_2 = -3x_0^2 x_1$

$-\frac{3a^3}{4} + C_2$

$\boxed{\text{Linear Equation with constant coefficients}}$

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \cdots + a_{n-1} t' + a_n t = f(t)$$

$\boxed{\text{Solving the Homogeneous Equation}}$

$$a_0 \lambda^n + a_1 \lambda^{(n-1)} + \cdots + a_{n-1}\lambda' + a_n \lambda = 0$$

$$(\lambda^2 + 1) = 0 \; ; \; \lambda_{1,2} = \pm i \; ; \; k = 1$$

$\boxed{\text{Solution of a Homogeneous Equation: Summand}}$

$$t = \sum P_{k-1}(t) e^{\alpha t} \sin\beta t + Q_{k-1}(t) e^{\alpha t} \cos\beta t$$

where $\lambda = \alpha + \beta i$; $\alpha = 0$; $\beta = 1$; $k = 1$

so $x = A\sin(t) + B\cos(t)$

$\boxed{\text{Method for Undetermined Coefficients: Particular Solution}}$

$$x_i = t^s e^{\alpha t}(R_m(t)\cos\beta t + T_m(t)\sin\beta t)$$

$$= t(B\sin t + A\cos t)$$

$$\ddot{X} = (-Bt-2A)\sin t + (2B-At)\cos t$$

$$\ddot{X}_2 + X_2 = 2B\cos(t) - 2A\sin(t) = -3a^3\cos^3(t)$$

$$2B = -3a^3\cos^2(t) \;;\; A=0$$

$$B = -\frac{3a^3\cos^2(t)}{2}$$

$$X_2 = -\frac{3\varepsilon a^3 t\cos^3(t)}{2}$$

$$\boxed{\text{General Solution} = \text{Particular} + \text{Homogeneous}}$$

$$X_2(t,\varepsilon) = -\frac{3\varepsilon^2 a^3 t\cos^3(t)}{2} + A\sin(t) + B\cos(t)$$

with initial conditions ; $B = a + \frac{3\varepsilon a^3 t}{2}$

$$\dot{X}_2(t,\varepsilon) = +\frac{9\varepsilon a^3 t\cos^2(t)\sin(t)}{2} + A\cos(t) - B\sin(t)$$

$$A = 0$$

$$\ddot{X}_2(t,\varepsilon) = -\frac{3\varepsilon^2 a^3 t\cos^3(t)}{2} + \left(a + \frac{3\varepsilon a^3 t}{2}\right)\cos(t)$$

$$= \frac{-3\varepsilon a^3 t}{2}\left[\frac{1}{4}\left[3\cos t + \cos 3t\right]\right] + \left(a + \frac{3\varepsilon a^2 t}{2}\right)\cos(t)$$

$$= a\cos t + \varepsilon a^3\left[\frac{3}{2}t\cos(t) + \frac{3}{8}t\left[\cos 3t - 3\cos t\right]\right]$$

This solution is not an exact answer to why the Duffing oscillator has a frequency dependent on amplitude.

$\ddot{X} + \varepsilon(X^2-1)\dot{X} + X = 0$ 7.6.21. $\boxed{\text{Poincare-Lindstedt Method.}}$

① Define a new time $\tau = \omega t$

$$\omega^2 x'' + \varepsilon\omega(x^2-1) + x = 0$$

$x_1(0)=0$; $\dot{x}_1(0)=0$;

$\ddot{x}_1+x_1=0-2w_1 a$; $w_1=0$

$\ddot{x}_1+x_1=\frac{+a^3}{4}\left[3\sin(t)-\sin(3t)\right]$

Linear Equation with constant coefficients

$a_0 t^{(n)}+a_1 t^{(n-1)}+\cdots+a_{n-1}t'+a_n t=f(t)$

Solving the Homogeneous Equation

$a_0\lambda^{(n)}+a_1\lambda^{(n-1)}+\cdots+a_{n-1}\lambda'+a_n\lambda=0$

$(\lambda^2+1)=0$; $\lambda_{1,2}=\pm i$; $K=1$

Solution to a Homogeneous Equation: Summand

$t_i=\sum P_{k-1}(t)e^{\alpha t}\sin\beta t+Q_{k-1}(t)e^{\alpha t}\cos\beta t$

where $\lambda=\alpha+\beta i$; $\alpha=0$; $\beta=1$; $K=1$

so $x=A_1\sin(t)+B_1\cos(t)$

Solving the Particular Equation: Method for Undetermined Coefficients.

$x_i=t^s e^{\alpha t}(R_m(t)\cos\beta t+T_m(t)\sin\beta t)$

$x_1=t\left[A_2\cos t+B_2\sin t\right]$

$\ddot{x}_1=(-B_2 t-2A_1)\sin(t)+(2B_2-A_1 t)\cos(t)$

$\ddot{x}_1+x_1=2B_2\cos(t)-2A_2\sin(t)=\frac{-a^3}{4}\left[3\sin(t)-\sin(3t)\right]$

$2B_2=0$; $-2A_2=\frac{a^3}{8}\left[1-\frac{2\sin(t)\sin(3t)}{\cos(2t)-1}\right]$

General Solution = Particular + Homogeneous

$x_1(t)=A_1\sin(t)+B_1\cos(t)-\frac{a^3}{4}\sin(3t)$

With initial conditions : $x_1(0)=0$; $\dot{x}_1(0)=0$

$A_1=\frac{-a^3}{4}$; $B_1=0$

$x_1(t)=-\frac{a^3}{4}\left[3\sin(t)+\sin(3t)\right]$

② Assign a new $X(\tau,\varepsilon) = X_0(\tau) + \varepsilon X_1(\tau) + \varepsilon^2 X_2(\tau) + \cdots$

$$\omega = 1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \cdots$$

③ Solve the perturbation equation

$$\omega^2 x'' + \varepsilon\omega(x^2-1)x' + x$$

$$= (1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots)^2 (\ddot{X}_0 + \varepsilon\ddot{X}_1 + \varepsilon^2\ddot{X}_2 + \cdots)$$

$$+ \varepsilon\cdot(1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots)([X_0 + \varepsilon X_1 + \varepsilon^2 X_2]^2 - 1)\,\circ$$

$$(\dot{X}_0 + \varepsilon\dot{X}_1 + \varepsilon^2\dot{X}_2) + (X_0 + \varepsilon X_1 + \varepsilon^2 X_2)$$

$$O(1):\ \ddot{X}_0 + X_0 = 0$$

$$O(\varepsilon):\ 2\omega_1\ddot{X}_0 + \ddot{X}_1 + X_0^2\dot{X}_0 - \dot{X}_0 + X_1 = 0$$

$$O(\varepsilon^2):\ \underline{2\ddot{X}_0\omega_2} + \underline{2\ddot{X}_0\omega_1^2} + 2\ddot{X}_1\omega_1 + X_0^2\dot{X}_0\omega_1 + \underline{\ddot{X}_0\omega_1^2}$$

$$+ \underline{X_0^2\dot{X}_1} + \underline{2X_0\dot{X}_0X_1} - \dot{X}_1\omega_1 - \underline{\dot{X}_1} + \ddot{X}_2 + X_2 = 0$$

$$\ddot{X}_2 + X_2 = -(\omega_1^2 + 2\omega_2)X_0'' - 2\ddot{X}_1\omega_1^2 - 2X_0\dot{X}_0X_1$$

$$+ (1 - X_0^2)(X_1' + \omega_1 X_0')$$

Solving for X_0:

$$\ddot{X}_0 + X_0 = 0\ ;\ X(0) = a\ ;\ \dot{X}(0) = 0$$

$$X_0(t) = a\cos(t)$$

Solving for X_1:

$$\ddot{X}_1 + X_1 = -2\omega_1\ddot{X}_0 + (1 - X_0^2)\dot{X}_0$$

$$= +2\omega_1 a\cos(t) - (1 - a^2\cos^2(t))\,a\sin(t)$$

$$= a\left[2\omega_1\cos(t) - \frac{a^2}{4}[3\sin(t) - \sin(3t)]\right]$$

Solving for X_2:

$$\ddot{X}_2 + X_2 = -(\omega_1^2 + 2\omega_2)\ddot{X}_0 - 2\ddot{X}_1\omega_1^2 - 2X_0\dot{X}_0 X_1 + (1 - X_0^2)(\dot{X}_1 + \omega_1\dot{X}_0)$$

$$= +2\omega_2 a\cos(t) + 2a^2\cos(t)\sin(t)\left[\frac{-1}{4}a^3\left[3\sin(t) + \sin(3t)\right]\right]$$

$$-\frac{3}{4}\left(3\sin(t)\cos(3t) + \sin(3t)\cos(t)\right)\left(1 - a^2\cos^2(t)\right)$$

$$= \frac{1}{4}\left[\cos(t)\left(8a\omega_2 - 2a^5\sin(t)\left(3\sin(t) + \sin(3t)\right)\right)\right]$$

$$+ (3a^2\cos^2(t) - 3)(3\sin(t)\cos(3t)) + \sin(3t)\cos(t)$$

Linear Equation of Constant Coefficients

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \cdots + a_{n-1} t' + a_n t = f(t)$$

Solving the Homogeneous Equation

$$a_0 \lambda^{(n)} + a_1 \lambda^{(n-1)} + \cdots + a_{n-1}\lambda^1 + a_n \lambda^0 = 0$$

$$4(\lambda^2 + 1) = 0 \;;\; \lambda_{1,2} = \pm i \;;\; k = 1$$

Solution to a Homogeneous Equation: Summand

$$t = \sum_{n-1} R(t)e^{\alpha t}\sin\beta t + Q_{k-1}(t)e^{\alpha t}\cos\beta t$$

$$X = A_1\cos t + B_1\sin t$$

Lagrange's Method of Variation of Parameters

System of Equations: $C'(t)X_1 + C_1'(t)X_2 = 0$

$$C'(t)X_1 + C_1'(t)X_2^2 = \frac{F(t)}{a_0}$$

where $X_1 = \cos(t)$; $X_2 = \sin(t)$

$$X_1' = -\sin(t) \;;\; X_2' = \cos(t)$$

$$a_0 X'' = 2$$

$$F(t) = \frac{1}{2}\left[\cos(t)\left(8a\omega_2 - 2a^5\sin(t)(3\sin(t) + \sin(3t))\right)\right]$$

$$+ 2(3a^2\cos^2(t) - 3)(3\sin(t)\cos(3t)) + \sin(3t)\cos(t)$$

Finding $C'(t), C_1'(t)$ by Cramer's Rule:

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin(t) \\ \dfrac{\cos(t)\left[8aw_2 - 2a^5\sin(t)\left(\sin(3t)+3\sin(t)\right)\right]}{2} + 2\cos(t)\sin(3t) + 6\left[3a^2\cos^2(t)-3\right]\sin(t)\cos(3t) & \cos(t) \end{vmatrix} = -\sin(t)\left[\frac{\cos(t)\left[8aw_2 - 2a^5\sin(t)\left(\sin(3t)+3\sin(t)\right)\right]}{2} + 2\cos(t)\sin(3t) + 6\left[3a^2\cos^2(t)-3\right]\sin(t)\cos(3t)\right]$$

$$W_2 = \begin{vmatrix} \cos(t) & 0 \\ -\sin(t) & \dfrac{\cos(t)\left[8aw_2 - 2a^5\sin(t)\left(\sin(3t)+3\sin(t)\right)\right]}{2} + 2\cos(t)\sin(3t) + 6\left[3a^2\cos^2(t)-3\right]\sin(t)\cos(3t) \end{vmatrix}$$

$$= \cos(t)\left[\frac{\cos(t)\left[8aw_2 - 2a^5\sin(t)\left(\sin(3t)+3a\sin(t)\right)\right]}{2} + 2\cos(t)\sin(3t) + 6\left[3a^2\cos(t)-3\right]\sin(t)\cos(3t)\right]$$

$$C'(t) = \frac{W_1}{W} = W_1 \ ; \ C_1'(t) = \frac{W_2}{W} = W_2$$

$$C(t) = \int C'(t)\,dt = \cdots \ ; \ C_1(t) = \int C_1'(t)\,dt = \cdots$$

Solving for ω_2: $\ddot{X}_2 + X_2 = \left(4\omega_2 + \frac{1}{4}\right)\cos(\tau) + \cdots$

$$\omega_2 = -\frac{1}{16}$$

Solving for ω: $\omega = (1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots)$

$$= \left(1 - \frac{1}{16}\varepsilon^2 + \cdots\right)$$

$\ddot{X} + X + \varepsilon X^2 = 0$ 7.6.22. $x(0) = a$; $\dot{x}(0) = 0$

① Define a new time $\tau = \omega t$

$$\omega^2 X'' + \omega X + \varepsilon X^2 = 0$$

$\lambda^2+1=0$; $\lambda_{1,2}=\pm i$; $K=s=1$; $\lambda=\alpha+\beta i$

$\boxed{\text{Solution to a Homogeneous Equations: Summand}}$

$$t=\sum R_{n-1}(t)e^{\alpha t}\sin\beta t+Q_{k-1}(t)e^{\alpha t}\cos\beta t$$

$$X(t)=A_1\sin(t)+B_1\cos(t)$$

$\boxed{\text{Solving the Particular Equation: Method for Undetermined Coefficients}}$

$$X_i=t^s e^{\alpha t}(R_m(t)\cos\beta t+T_m(t)\sin\beta t)$$

$$X_1=t[A_2\cos t+B_2\sin t]$$

$$\ddot{X}_1=(-B_2 t-2A_2)\sin(t)+(2B_2-A_2 t)\cos(t)$$

$$\ddot{X}_1+X_1=2B_2\cos(t)-2A_2\sin(t)=a\cos(t)(W_1+W_2)+a^2\cos^2(t)$$

Initial conditions: $X(0)=a$; $\dot{X}(0)=0$

$$2B_2=a(W_1+W_2)+a^2 \quad ; -2A_2=0$$

$$B_2=\frac{a}{2}(W_1+W_2)+\frac{a^2}{2} \; ; \quad A_2=0$$

$\boxed{\text{General Solution = Particular + Homogeneous}}$

$$X_1(t)=\left[a(W_1+W_2)+a^2\right]\cos(t)t+A_1\cos t+B_2\sin t$$

Initial conditions: $X(0)=a$; $\dot{X}(0)=0$

$$X_1(0)=a=A_1 \; ; \; A_1=a$$

$$\dot{X}_1(0)=0=B_2 \; ; \quad B_2=0$$

$$X_1(t)=\left[a(W_1+W_2)+a^2+a\right]\cos(t)$$

⑥ Solving for X_2:

$$\ddot{X}_2+X_2=-2[X_0X_1+X_0W_1W_2+\ddot{X}_1(W_1+W_2)]-\ddot{X}_0(W_1^2+W_2^2)-X_1(W_1+W_2)$$

$$=-2\left[a^2\cos^2(t)[(W_1+W_2)+a+1]+aW_1W_2\cos(t)\right.$$
$$\left.-a[(W_1+W_2)+a+1]\cos(t)(W_1+W_2)\right]$$
$$+a\cos(t)(W_1^2+W_2^2)-a[(W_1+W_2)+a+1]\cos(t)(W_1+W_2)$$

$$=a\cos(t)\left[2a\cos(t)(a+W_1+W_2+1)-(W_1+W_2)(3a+2W_1+2W_2+3)\right]$$

② Assign a new perturbation:

$$X(\tau,\varepsilon) = X_0(\tau) + \varepsilon X_1(\tau) + \varepsilon^2 X_2(\tau) + \cdots$$

$$\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots$$

③ Solve the perturbation equation:

$$\omega^2 X'' + \omega X + \varepsilon X^2 = 0$$

$$= (1+\varepsilon w_1 + \varepsilon w_2)^2(\ddot{X}_0 + \varepsilon\ddot{X}_1 + \varepsilon^2\ddot{X}_2) + (1+\varepsilon w_1 + \varepsilon w_2)(X_0 + \varepsilon X_1 + \varepsilon^2 X_2)$$

$$+ \varepsilon(X_0 + \varepsilon X_1 + \varepsilon^2 X_2)^2$$

$$O(1): \ddot{X}_0 + X_0 = 0$$

$$O(\varepsilon): X_0^2 + X_0(w_1 + w_2) + 2\ddot{X}_0(w_1 + w_2) + X_1 + \ddot{X}_1 = 0$$

$$\ddot{X}_1 + X_1 = -2\ddot{X}_0(w_1 + w_2) - X_0(w_1 + w_2) + X_0^2$$

$$O(\varepsilon^2): \ddot{X}_2 + X_2 = -2X_0X_1 - \ddot{X}_0 w_1^2 - 2\ddot{X}_0 w_1 w_2$$

$$- \ddot{X}_0 w_2^2 - X_1(w_1 + w_2) - 2\ddot{X}_1(w_1 + w_2)$$

$$= -2\left[X_0X_1 + X_0 w_1 w_2 + \ddot{X}_1(w_1 + w_2)\right]$$

$$- \ddot{X}_0(w_1^2 + w_2^2) - X_1(w_1 + w_2)$$

④ Solving for X_0:

$$\ddot{X}_0 + X_0 = 0 \;;\; X(0) = a \;;\; \dot{X}(0) = 0$$

$$X_0(t) = a\cos(t)$$

⑤ Solving for X_1:

$$\ddot{X}_1 + X_1 = 2a\cos(t)(w_1 + w_2) - a\cos(t)(w_1 + w_2) + a^2\cos^2(t)$$

$$= a\cos(t)(w_1 + w_2) + a^2\cos^2(t)$$

Linear Equation of Constant coefficients

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \cdots + a_{n-1} t^1 + a_n t^0 = f(t)$$

Solving the Homogeneous Equation

$$a_0 \lambda^{(n)} + a_1 \lambda^{(n-1)} + \cdots + a_{n-1}\lambda^1 + a_n\lambda^0 = 0$$

Linear Equation of Constant Coefficients

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \dots + a_{n-1} t^1 + a_n t^0 = f(t)$$

Solving the Homogeneous Equation

$$a_0 \lambda^{(n)} + a_1 \lambda^{(n-1)} + \dots a_{n-1} \lambda^1 + a_n \lambda^0 = 0$$

$$(\lambda^2 + 1) = 0 ; \lambda_{1,2} = \pm i ; K = s = 1$$

Solution to the Homogeneous Equation : Summand

$$t = \sum R_{m+1}(t) e^{\alpha t} \sin \beta t + Q_{K_1}(t) e^{\alpha t} \cos \beta t$$

$$X(t) = A_1 \sin(t) + B_2 \cos(t)$$

Solving the Particular Equation : Method for Undetermined Coefficients.

$$X_i = t^s \cdot e^{\alpha t} \cdot [R_m(t) \cos \beta t + T_m(t) \sin \beta t]$$

$$X_2 = t [A_2 \cos t + B_2 \sin t]$$

$$\ddot{X}_2 = (-B_2 t - 2A_2) \sin t + (2B_2 - A_2 t) \cos t$$

$$\ddot{X}_2 + X_2 = 2B_2 \cos(t) - 2A_2 \sin(t) =$$

$$= a \cos t [2a \cos(t)(a + w_1 + w_2 + 1) - (w_1 + w_2)(3a + 2w_1 + 2w_2 + 3)]$$

Initial conditions: $x(0) = a$; $\dot{x}(0) = 0$

$$B_2 = \frac{a}{2}[2a(a + w_1 + w_2 + 1) - (w_1 + w_2)(3a + 2w_1 + 2w_2 + 3)]$$

$$A_2 = 0$$

General Solution = Particular + Homogeneous

$$X_2(t) = A_1 \sin t + B_1 \cos t + \frac{at}{2}[2a(a + w_1 + w_2 + 1) - (w_1 + w_2)(3a + 2w_1 + 2w_2 + 3)] \sin(t)$$

$$X_2(0) = a = B_1 ; \dot{X}_2(0) = 0 = A_1$$

$$X_2(t) = a \cos t + \frac{at}{2}[2a(a + w_1 + w_2 + 1) - (w_1 + w_2)(3a + 2w_1 + 2w_2 + 3)] \sin(t)$$

$$KE = \frac{1}{2} \varepsilon X^2 = \frac{1}{2} \varepsilon (a - \frac{t^2}{2} a - O(t^4))^2 \approx \frac{1}{2} \varepsilon a^2 + O(t^2)^2$$

$\ddot{x} - \varepsilon x \dot{x} + x = 0$ 7.6.23. ① Assign a new time $\tau = \omega t$

$$\omega^2 x'' - \varepsilon x \omega x' + x = 0$$

② Assign a new perturbation

$$x(\tau, \varepsilon) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \cdots$$

$$\omega = 1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \cdots$$

③ Solve the perturbation equation:

$$(1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \cdots)^2 (\ddot{x}_0 + \varepsilon \ddot{x}_1 + \varepsilon^2 \ddot{x}_2 + \cdots)$$

$$- \varepsilon (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)(\dot{x}_0 + \varepsilon \dot{x}_1 + \varepsilon^2 \dot{x}_2 + \cdots)$$

$$* (1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \cdots) + (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) = 0$$

$O(1)$: $\ddot{x}_0 + x_0 = 0$

$O(\varepsilon)$: $\ddot{x}_1 + x_1 = -2\omega_1 \ddot{x}_0 + \dot{x}_0 x_0$

$O(\varepsilon^2)$: $\ddot{x}_2 + x_2 = -\omega_1^2 \ddot{x}_0 + \omega_1 \dot{x}_0 x_0 - 2\omega_1 \ddot{x}_1$

$$- 2\omega_2 \ddot{x}_0 + x_1 \dot{x}_0 + x_0 \dot{x}_1$$

④ Solving for x_0:

$\ddot{x}_0 + x_0 = 0$; $x(0) = a$; $\dot{x}(0) = 0$

$$x_0(t) = a \cos t$$

⑤ Solving for x_1:

$$\ddot{x}_1 + x_1 = -2\omega_1 x_0 + \dot{x}_0 x_0$$

$$= -2\omega_1 a \cos t - a^2 \cos t \sin t$$

$$= [-2\omega_1 a - a^2 \sin(t)] \cos t$$

$\boxed{\text{Linear equation with constant coefficients}}$

$$a_0 t^{(n)} + a_1 t^{(n-1)} + \cdots + a_{(n-1)} t^1 + a_n t^0 = f(t)$$

$\boxed{\text{Solving the Homogenous Equation}}$

$$a_0 \lambda^{(n)} + a_1 \lambda^{(n-1)} + \cdots + a_{n-1} \lambda^1 + a_n \lambda^0 = 0$$

$(\lambda^2+1)=0$; $\lambda_{1,2}=\pm i$; $K=S=1$

Solution to a Homogeneous Equation: Summand

$$t=\sum_i R_{M-1}(t)e^{\alpha t}\cos\beta t \cdot Q_{M-1}(t)e^{\alpha t}\sin\beta t$$

$\lambda=\alpha+\beta i$; $\alpha=0$; $\beta=1$

$$X=A_1\cos t+B_1\sin(t)$$

Lagrange's Method of Variation of parameters.

① Build a system

$$A_1'X_1+B_1'X_2=0$$

$$A_1'X_1'+B_1'X_2'=\frac{P(t)}{a_0}$$

Where $X_1=\cos(t)$ $X_2=\sin(t)$

$X_1'=-\sin(t)$ $X_2'=\cos(t)$

$a_0X''=1$; $P(t)=\cos(t)(-a^2\sin(t)-2aw_1)$

② Solve the System using Cramer's Rule.

$$W=\begin{vmatrix}\cos(t) & \sin(t)\\ -\sin(t) & \cos(t)\end{vmatrix}$$

$$W_1=\begin{vmatrix}0 & \sin(t)\\ \cos(t)(-a^2\sin(t)-2aw) & \cos(t)\end{vmatrix}=\cos(t)\sin(t)(-a^2\sin(t)-2aw_1)$$

$$W_2=\begin{vmatrix}\cos(t) & 0\\ -\sin(t) & \cos(t)(-a^2\sin(t)-2aw)\end{vmatrix}=\cos^2(t)(-a^2\sin(t)-2aw_1)$$

$$A_1'(t)=\frac{W_1}{W}=\cos(t)\sin(t)(-a^2\sin(t)-2aw_1)$$

$$B_1'(t)=\frac{W_2}{W}=\cos^2(t)(-a^2\sin(t)-2aw_1)$$

$$A_1=\int A_1'(t)dt=\frac{a^2\sin^3(t)}{3}+aw_1\sin^2(t)+c_1$$

$$B_1=\int B_1'(t)dt=-\frac{aw_1\sin(2t)}{2}+\frac{a^2\cos^3(t)}{3}-aw_1t+C_2$$

$$X = -\frac{aW\sin(t)\sin(2t)}{2} + \frac{a^2\cos(t)\sin^3(t)}{3} + aW\cos(t)\sin^2(t)$$

$$+ \frac{a^2\cos^3(t)\sin(t)}{3} - aWt\sin(t) + C_1\sin(t) + C_2\cos(t)$$

Initial conditions: $X(0) = 0$; $\mathring{X}(0) = 0$

$$X(0) = 0 = C_2 \ ; \ \mathring{X}(0) = 0 = 2a^2 + 6C_1 \ ; \ C_1 = -\frac{1a^2}{3}$$

$$X_1(t) = \left[-\frac{aW_1\sin(2t)}{2} + \frac{a^2\cos(t)\sin^2(t)}{3} + aW_1\cos(t)\sin^2(t)\right.$$

$$\left. + \frac{a^2\cos^3(t)}{3} - aW_1 t - \frac{a^2}{3}\right]\sin(t)$$

$W_1 = 0$ because no secular terms

$$X_1(t) = \left[\frac{a^2\cos(t)\sin^2(t)}{3} + \frac{a^2\cos^3(t)}{3} - \frac{a^2}{3}\right]\sin(t)$$

Identities: $\sin(2t) = 2\cos(t)\sin(t)$

$$\sin^2(t) = \tfrac{1}{2}(1-\cos(2t))$$

$$\cos^2(t) = 1 - \sin^2(t)$$

$$X_1(t) = \frac{1}{6}\left(-2a^2\sin(t) + a^2\sin(2t)\right)$$

③ Solving for X_2

$$\mathring{\mathring{X}}_2 + X_2 = -W_1^2\cancel{\mathring{\mathring{X}}_0} + W_1\cancel{\mathring{X}_0}X_0 - 2W_1\mathring{\mathring{X}}_1 - 2W_2\mathring{\mathring{X}}_0 + X_1\mathring{X}_0 + X_0\mathring{X}_1$$

$$= -2W_2\mathring{\mathring{X}}_0 + X_1\mathring{X}_0 + X_0\mathring{X}_1$$

$$= 2W_2 a\cos(t) - \frac{a\sin(t)}{6}\left(-2a^2\sin(t) + a^2\sin(2t)\right)$$

$$+ \frac{a\cos(t)}{3}\left(-2a^2\cos(t) + a^2\cos(2t)\right)$$

$\boxed{\text{Linear Equation of Constant Coefficients}}$

$$a_0 t^n + a_1 t^{(n-1)} + \cdots + a_{(n-1)} t^1 + a_{(n)} t^0 = f(t)$$

$\boxed{\text{Solving the Homogeneous Equation}}$

$$a_0 \lambda^n + a_1 \lambda^{(n-1)} + \cdots + a_{(n-1)} \lambda^1 + a_{(n)} \lambda^0 = 0$$

$(\lambda^2+1)=0$; $\lambda_{1,2}=\pm i$; $k=s=1$

Solution to a Homogeneous Equation: Summand

$$t=\sum e^{\alpha t}\left[R_{m-1}(t)\cos\beta t + Q_{m-1}(t)\sin\beta t\right]$$

$$X_2(t)=A_1\cos t + B_1\sin t$$

$\boxed{\text{Lagrange's Method of Variation of Parameters}}$

$$A_1' X_1 + B_1' X_2 = 0$$

$$A_1' X_1' + B_1' X_2' = \frac{f(t)}{a_0} \quad \text{where } X_1 = \cos(y) \quad X_2=\sin(y)$$

$$X_1' = -\sin(y) \quad X_2' = \cos(y)$$

$$a_0 X'' = 6$$

$$f(t) = 12w_2\cos(t) - a^3\sin t(-2\sin t + \sin 2t) + 2a^3\cos t(\cos t + \cos 2t)$$

$$W = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & \sin t \\ +2w_2\cos t - \frac{a^3\sin t}{6}(-2\sin t + \sin 2t) + \frac{a^3\cos t}{3}(\cos t + \cos 2t) & \cos t \end{vmatrix}$$

$$W_2 = \begin{vmatrix} \cos t & 0 \\ -\sin t & 2w_2\cos t - \frac{a^3\sin t}{6}(-2\sin t + \sin 2t) + \frac{a^3\cos t}{3}(\cos t + \cos 2t) \end{vmatrix}$$

$$A_1' = \frac{W_1}{W} = \sin t\left[\frac{a^3\sin t}{6}(-2\sin t + \sin 2t) - \frac{a^3\cos t}{3}(\cos t + \cos 2t) - 2w_2\cos t\right]$$

$$B_1' = \frac{W_2}{W} = \cos t\left[2w_2\cos t - \frac{a^3\sin t}{6}(-2\sin t + \sin 2t) + \frac{a^3\cos t}{3}(\cos t + \cos 2t)\right]$$

$$A_1 = \int A_1' dt = \sin^2 t\left[\frac{a^3}{4}\sin^2 t - \frac{(6aw + a^3)}{6}\right] + \cos t\left[\frac{a^3}{3} - \frac{2a^3\cos^2(t)}{9}\right] + C_1$$

$$B_1 = \int B_1' dt = \frac{a^3\sin 4t}{32} + \frac{(48aw + 8a^3)\sin 2t}{96} + \frac{2a^3\sin^3(t)}{9} - \frac{a^3\sin(t)}{3} + awt + \frac{a^3 t}{24} + C_2$$

$$X_2(t) = \left[\frac{a^3\sin 4t}{32} + \frac{(48aw+8a^3)\sin 2t}{96} + \frac{2a^3\sin^3(t)}{9} - \frac{a^3\sin t}{3} + awt + \frac{a^3t}{24} + C_1\right]\sin t + \left[\sin^2 t\left(\frac{a^3\sin^2 t}{4} - \frac{(6aw+a^3)}{6}\right) + \cos t\left(\frac{a^3}{3} - \frac{2a^3\cos^2 t}{9}\right) + C_2\right]\cos t$$

Initial conditions: $X_2(0)=0$; $\dot{X}_2(0)=0$

$$X_2(0) = 0 = \frac{a^3}{3} - \frac{2a^3}{9} + C_2 \;;\; C_2 = -\frac{a^3}{3} + \frac{2a^3}{9}$$

$$\dot{X}_2(0) = 0 = awt + \frac{a^3t}{24} + C_1 \;;\; C_1 = -awt - \frac{a^3t}{24}$$

$$X_2(t) = \left[\frac{a^3\sin 4t}{32} + \frac{(48aw_2+8a^3)\sin 2t}{96} + \frac{2a^3\sin^3(t)}{9} - \frac{a^3\sin t}{3}\right]\sin t + \left[\sin^2 t\left(\frac{a^3\sin^2 t}{4} - \frac{(6aw_2+a^3)}{6}\right) + \cos t\left(\frac{a^3}{3} - \frac{2a^3\cos^2 t}{9}\right) - aw_2t - \frac{a^3t}{24}\right]\cos t$$

$$w_2 = -\frac{1}{8}a^2$$

$$w(a) = 1 + \varepsilon w_1 + \varepsilon^2 w_2 + O(\varepsilon^3)$$
$$= 1 - \frac{1}{8}\varepsilon^2 a^2$$

$\ddot{X} + X - \epsilon X^3 = 0$ 7.6.24. $X(0)=a$; $\dot{X}(0)=0$

① Assign a new time ; $\tau = wt$

$$w^2\ddot{X} + X - \varepsilon X^3 = 0$$

② Apply perturbation equations

$$x(\tau,\varepsilon) = X_0(\tau) + \varepsilon X_1(\tau) + \varepsilon^2 X_2(\tau) + O(\varepsilon^3)$$

$$w = 1 + \varepsilon w_1 + \varepsilon^2 w_2 + O(\varepsilon^3)$$

$$(1+\varepsilon w_1 + \varepsilon^2 w_2 + \cdots)^2(\ddot{X}_0 + \varepsilon\ddot{X}_1 + \varepsilon^2\ddot{X}_2) + (X_0 + \varepsilon X_1 + \varepsilon^2 X_2) - \varepsilon(X_0 + \varepsilon X_1 + \varepsilon^2 X_2)^3 = 0$$

$O(1)$: $\ddot{X}_0 + X_0 = 0$

$O(\varepsilon)$: $\ddot{X}_1 + X_1 = -\omega_1 \ddot{X}_0 + X_0^3 - X_1$

$O(\varepsilon^2)$: $\ddot{X}_2 + X_2 = -\omega_1 \ddot{X}_1 - \omega_2 \ddot{X}_0 + 3 X_1 X_0^2$

(3) Solving for X_0:

$$\ddot{X}_0 + X_0 = 0 ;\quad X_0(0) = a ;\quad \dot{X}_0(0) = 0 ;\quad X_0(t) = a\cos t$$

(4) Solving for X_1:

$$\ddot{X}_1 + X_1 = -\omega_1 \ddot{X}_0 + X_0^3 = +\omega_1 a \cos t + a^3 \cos^3 t$$
$$= (\omega_1 a - \tfrac{3}{4}a^3)\cos t - \frac{a^3}{4}\cos 3t$$

Linear Equation with constant coefficients | $\omega_1 = a^2$

$$a_0 t^n + a_1 t^{(n-1)} + \cdots + a_{n-1} t^1 + a_n t^0 = f(t)$$

Solving the Homogeneous Equation.

$$a_0 \lambda^n + a_1 \lambda^{(n-1)} + \cdots + a_{n-1} t^1 + a_n t^0 = 0$$

$$(\lambda^2 + 1) = 0 ;\quad \lambda_{1,2} = \pm i ;\quad K = S = 1$$

Solution to Homogeneous Equation: Summand.

$$t = \sum R_{M-1}(t) e^{\alpha t} \cos\beta t + Q_{M-1}(t) e^{\alpha t} \sin\beta t$$

$$X_1(t) = A_1 \cos t + B_1 \sin t$$

Solving the particular Equation Method for Undetermined Coefficients.

$$X_i = t^S \cdot e^{\alpha t} \cdot [R_M(t) \cos\beta t + Q_M(t) \sin\beta t]$$

$$X_1 = t[A_1 \cos 3t + B_2 \sin 3t]$$

$$\ddot{X}_1 = \cos 3t(6B_2 - 9A_2 t) - 3\sin 3t(2A_2 + 3B_2 t)$$

$$\ddot{X}_1 + X_1 = \cos(3t)(6B_2 - 8A_2 t) - 2\sin(3t)(3A_2 + 4B_2 t) = \frac{-a^3}{4}\cos 3t$$

$$B_2 = -\frac{a^3}{24}$$

$$X_1(t) = \frac{-a^3}{24} t \sin 3t$$

General Solution: Homogeneous + Particular

$X_1(t) = A_1 \cos t + B_1 \sin t + \frac{a^3}{24} t \cos 3t$

Initial Conditions: $X_1(0) = 0$; $\dot{X}_1(0) = 0$

$X_1(0) = 0 = A_1$; $\dot{X}_1(0) = 0 = B_1 + \frac{a^3}{24}$; $B_1 = \frac{a^3}{24}$

$X_1(t) = \frac{a^3}{24} \sin t - \frac{a^3}{24} t \cos 3t$

(5) Solving for X_2:

$\ddot{X}_2 + X_2 = -\omega_1 \ddot{X}_1 - \omega_2 \ddot{X}_0 + 3 X_1 X_0^2$

$= \frac{a^5}{24}(-\sin t + 6 \sin 3t + 9t \cos 3t) + \omega_2 a \cos t$

$+ \frac{3a^5}{8}\left[\sin t - t \cos 3t\right] \cos^2 t$

$\omega_2 = 0$

(6) I forgot $O(\epsilon^3)$, which a modification necessitates a time-shift $(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \epsilon^3\omega_3 + \cdots)$ and a perturbation of $1 + \epsilon X_1 + \epsilon^2 X_2 + \epsilon^3 X_3 + \cdots$.

Next time, for this example, $\omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2$

$= 1 + \epsilon a^2$.

$\ddot{X} + X + \epsilon h(x, \dot{x}, t) = 0$

7.6.25. $X(t) = r(t) \cos(t + \phi(t))$

$\dot{X}(t) = -r(t) \sin(t + \phi(t))$

a. $\dot{r} = \langle \epsilon h(x, \dot{x}, t) \rangle_{2\pi} = \epsilon h \sin(t + \phi(t))$

$r\dot{\phi} = \langle \epsilon h(x, \dot{x}, t) \rangle_{2\pi} = \epsilon h \cos(t + \phi(t))$

b. $\langle r\rangle(t) = \bar{r}(t) = \frac{1}{2\pi}\int_{t-\pi}^{t+\pi} r(\tau)\,d\tau$

$$\frac{d\langle r\rangle}{dt} = \frac{d\bar{r}(t)}{dt} = \frac{1}{2\pi}\frac{d}{dt}\left(r(t+\pi) - r(t-\pi)\right) = \left\langle \frac{dr}{dt}\right\rangle$$

c. $\frac{d\langle r\rangle}{dt} = \langle \epsilon h \sin(t+\phi)\rangle = \epsilon\langle h(x,\dot{x},t)\sin t+\phi\rangle$

$$= \epsilon\langle h(r\cos(t+\phi), -r\sin(t+\phi), t)\sin t+\phi\rangle$$

d. $\frac{d\bar{r}}{dt} = \epsilon\langle h(r\cos(t+\phi), -r\sin(t+\phi), t)\sin(t+\phi)\rangle + O(\epsilon^2)$

$\bar{r}\frac{d\bar{\phi}}{dt} = \epsilon\langle h(r\cos(t+\phi), -r\sin(t+\phi), t)\sin(t+\phi)\rangle + O(\epsilon^2)$

$-(r+1)$

$\ddot{x} = -\epsilon x\sin^2 t$ 7.6.26 $0 \le \epsilon \ll 1$; $x = x_0$ @ $t=0$

a. $\ddot{x} = \frac{d^2x}{d^2t} = -\epsilon x\sin^2 t$; $x\ln x - x = -\epsilon\int\left[\int\frac{(1-\cos 2t)}{2}dt\right]dt$

$$= -\epsilon\int\left[\frac{t}{2} - \frac{\sin 2t}{4} + c\right]dt$$

$$= -\epsilon\left[\frac{t^2}{4} + \frac{\cos 2t}{8} + ct\right] \pm c$$

An alternative solution is a homogeneous plus particular, which has continuous style.

$$x = e^{-\epsilon\prod_{1}^{\infty}\log\left[\frac{t^2}{4}+\frac{\cos 2t}{8}\right]}$$

b. $\bar{x}(t) = \frac{1}{2\pi}\int_{t-\pi}^{t+\pi} x(\tau)\,d\tau$

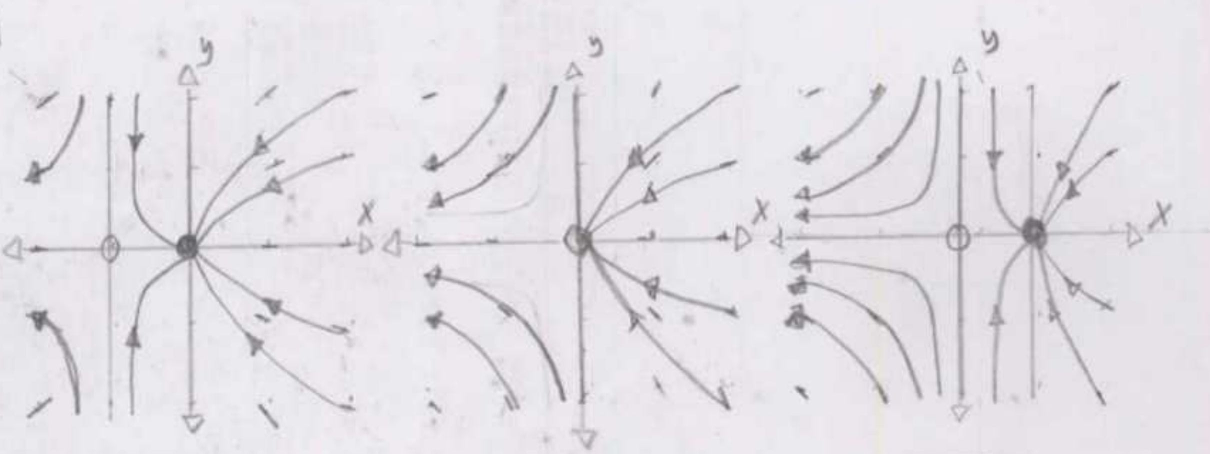
$$x(t) = \bar{x}(t) + O(\epsilon) = \frac{e^{-\epsilon\prod\log\left[\frac{1}{4}(t-\pi)^2 + \frac{\cos 2t}{8}\right]}}{\pi} + O(\epsilon)$$

c. The error depends on the exactness and amount of terms.
A product-log function isn't the common method, either.

Chapter 8: Bifurcations Revisited!

$\dot{x} = \mu x - x^2$ 8.1.1
$\dot{y} = -y$

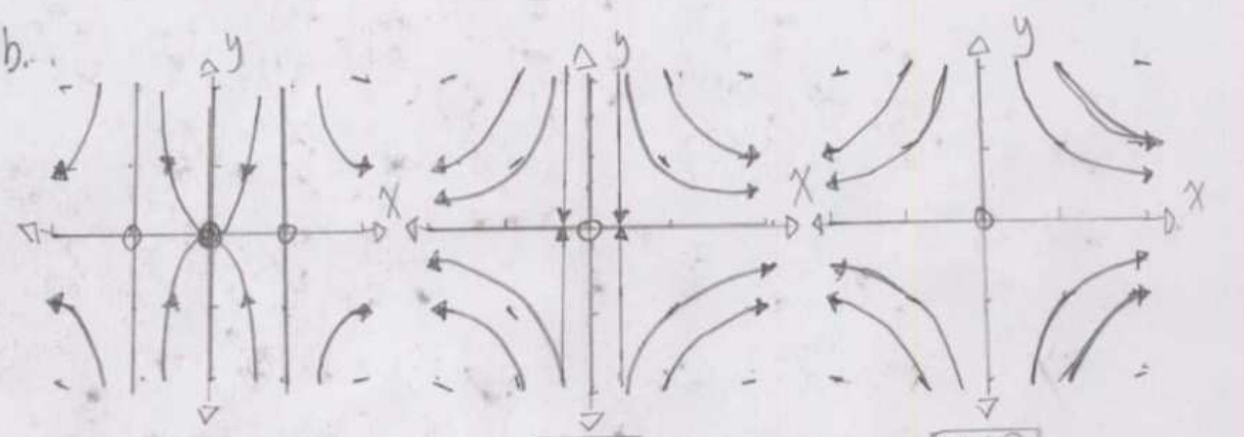
a.



$\boxed{\mu < 0}$ $\boxed{\mu = 0}$ $\boxed{\mu > 0}$

$\dot{x} = \mu x + x^3$
$\dot{y} = -y$

b.



$\boxed{\mu < 0}$ $\boxed{\mu = 0}$ $\boxed{\mu > 0}$

$\dot{x} = \mu - x^2$
$\dot{y} = -y$

8.1.2. Eigenvalues: $\dot{\vec{x}} = A\vec{x} = 0$; $A\vec{x} = 0 = \lambda\vec{x}$; $(A - \lambda)\vec{x} = 0$

$$= \begin{pmatrix} -2x - \lambda & 0 \\ 0 & -1-\lambda \end{pmatrix} = (-2x - \lambda)(-1-\lambda) = 0$$

$$\lambda_1 = -2x \,;\, \lambda_2 = -1$$

Fixed Points: $\dot{x} = 0 = \mu - x^2$
$\dot{y} = 0 = -y$; $(x^*, y^*) = (\sqrt{\mu}, 0)$

Eigenvalues + Fixed Points: $\lambda_1 = -2\sqrt{\mu}$; $\lim_{\mu \to 0} \lambda_1 = 0$

$\dot{x} = \mu x - x^2$
$\dot{y} = -y$

8.1.3. Eigenvalues: $\dot{\vec{x}} = A\vec{x} = 0$; $(A - \lambda)\vec{x} = \begin{pmatrix} \mu - 2x - \lambda & 0 \\ 0 & -1 - \lambda_2 \end{pmatrix}\vec{x} = 0$

$$\lambda_1 = \mu - 2x$$
$$\lambda_2 = -1$$

Fixed Points: $\dot{x} = 0 = \mu x - x^2$; $(x^*, y^*) = (\mu, 0)$

$\dot{y} = 0 = -y$

Eigenvalues + Fixed Points: $\lambda_1 = \mu - 2x$;

$$\lim_{\mu \to 0} \lambda_1 = \lim_{\mu \to 0} \mu - 2\mu = 0$$

$\dot{x} = \mu x + x^3$ 8.1.4: Eigenvalues: $\dot{\vec{x}} = A\vec{x} = 0$; $(A - \lambda)\vec{x} = \begin{pmatrix} \mu + 3x^2 - \lambda_1 & 0 \\ 0 & -1 - \lambda_2 \end{pmatrix} \vec{x}$

$\dot{y} = -y$

$$\lambda_1 = \mu + 3x^2$$
$$\lambda_2 = -1$$

Fixed Points: $\dot{x} = 0 = \mu x + x^3$; $(x^*, y^*) = (\sqrt{\mu}\, i, 0)$

$\dot{y} = 0 = -y$

Eigenvalues + Fixed Points: $\lambda_1 = \mu + 3(\sqrt{\mu}\, i)^2$

$$\lim_{\mu \to 0} \lambda_1 = \lim_{\mu \to 0} \mu + 3(\sqrt{\mu}\, i)^2$$
$$= \phi$$

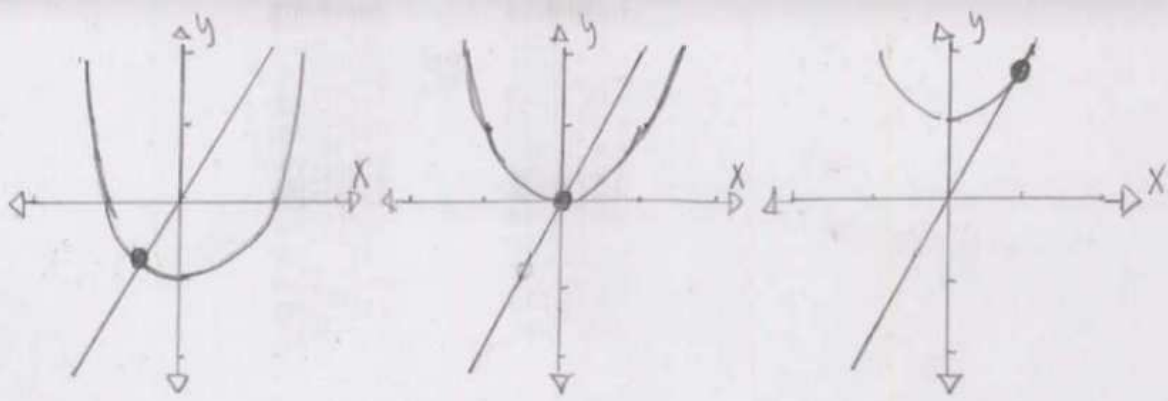
8.1.5: True, since the zero-eigenvalue bifurcation is exampled by saddle-nodes, transcritical, and pitchfork bifurcations that each have tangential intersections about their nullclines.

$\dot{x} = y - 2x$ 8.1.6:

$\dot{y} = \mu + x^2 - y$ a. Nullclines: $\dot{x} = 0 = y - 2x$

$\dot{y} = 0 = \mu + x^2 - y$

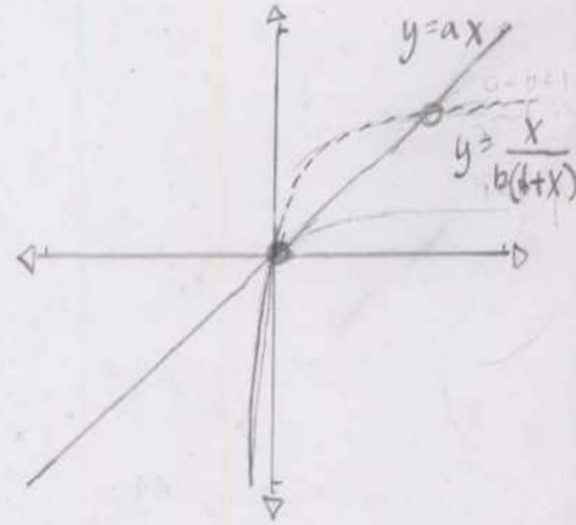
$$(x^*, y^*) = (1 - \sqrt{1-\mu}, -2(\sqrt{1-\mu} - 1))$$
$$= (1 + \sqrt{1-\mu}, 2(\sqrt{1-\mu} + 1))$$



$\boxed{\mu<0}$ $\boxed{\mu=0}$ $\boxed{0<\mu\leq 1}$

b. Saddle-node

c. See part A.



$\dot{x}=y-ax$

$\dot{y}=-by+\frac{x}{1+x}$

8.1.7. $y=ax$; $y=\frac{x}{b(1+x)}$;

$$ax=\frac{x}{b(1+x)}$$

$$x=0,\ \frac{1}{ab}-1$$

The book shows a Jacobian method. This is an equation, table or equation graph. Maybe an equation, table, and graph.

Bifurcation Amount	Conditions
2	$ab<1$
1	$ab=1$
2	$ab>1$

Transcritical Bifurcation.

$$\varepsilon\frac{d^2\phi}{d^2\tau}=-\frac{d\phi}{d\tau}-\sin\phi+\gamma\sin\phi\cos\phi$$

8.1.8, $\varepsilon>0$; $\gamma>0$;

a.) $\dot{x}=\frac{d\phi}{d\tau}=y$

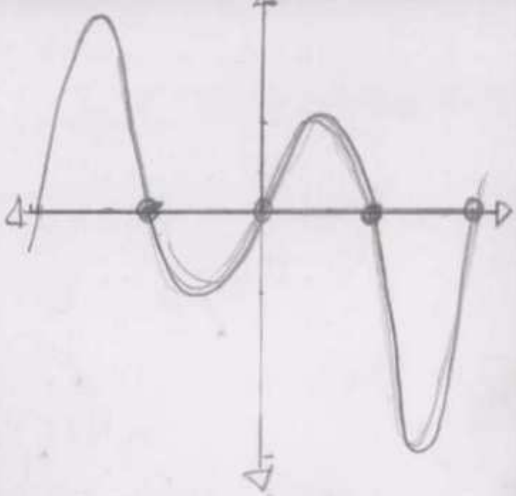
$$\dot{y}=\frac{d^2\phi}{d^2\tau}=\frac{-y+\sin x(\gamma\cos x-1)}{\varepsilon}$$

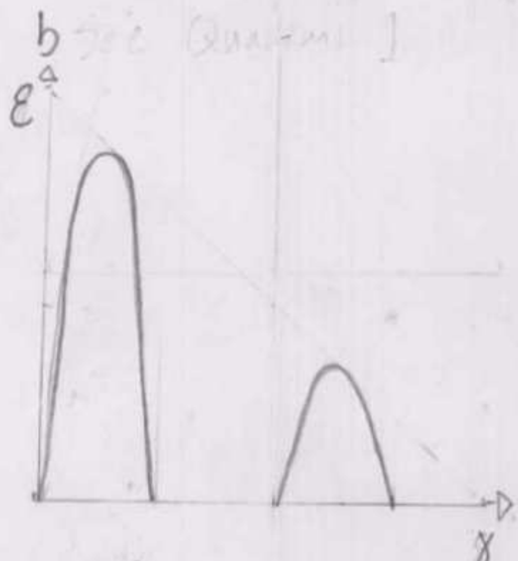
Bifurcations: $\dot{x}=0=y$

$$\dot{y}=0=\frac{-y+\sin x(\gamma\cos x-1)}{\varepsilon}$$

$y = \sin x(\gamma \cos x - 1)$

$x = 0, \arccos\left(\frac{1}{\gamma}\right)$





Bifurcation Amount	Conditions
2	$0 \leq \gamma \leq 1$
4	$1 < \gamma$

"Pitch Fork Bifurcation: Supercritical"

$\ddot{x} + b\dot{x} - kx + x^3 = 0$ 8.1.9. $\dot{x} = y$

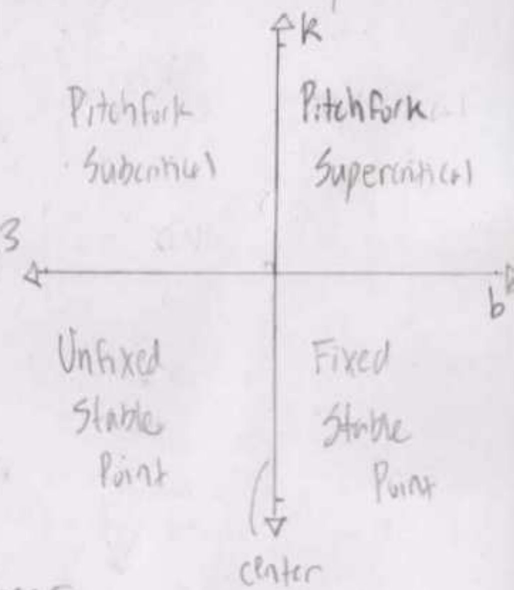
$\dot{y} = -b\dot{x} + kx - x^3 = -by + kx - x^3$

<u>Bifurcations:</u> $\dot{x} = 0 = y$

$\dot{y} = 0 = -b\dot{y} + kx - x^3$

$(x^*, y^*) = (0, 0)$

$(\pm\sqrt{k}, 0)$



$\dot{S} = r_S S\left(1 - \frac{S}{K_S}\frac{K_E}{E}\right)$

$\dot{E} = r_E E\left(1 - \frac{E}{K_S}\right) - P\frac{B}{S}$

8.1.10 $S(t)$ = Average Size of Trees

$E(t)$ = "Energy Reserve"

B = Budworm Population

$r_S, r_E, K_S, K_E, P > 0$

a. First term of $\dot{S}$ is rate of increase of average tree size
Second term of $\dot{S}$ is rate of decrease of average tree size
First term of $\dot{E}$ is rate of increase of energy reserve
Second term of $\dot{E}$ is rate of decrease of energy reserve
Third term of $\dot{E}$ is rate of decrease of energy reserve from budworms

b. Scaled budworm density $\mu = \frac{P\beta}{S}$; $\alpha = \frac{E}{K_E}$; $t = r_E \tau$

$$\dot{S} = r_E S' = r_S S\left(1 - \frac{S}{K_S}\frac{1}{\alpha}\right) ; \quad S' = \frac{r_S}{r_E} S\left(1 - \frac{S}{K_S \alpha}\right) ;$$

$$\dot{E} = r_E E' = r_E E(1 - \alpha) - \beta\mu \quad E' = E(1-\alpha) - \beta\mu$$

If $R = \frac{r_S}{r_E}$; $C = \frac{S}{K_S}$; $C' = RC\left(1 - \frac{C}{\alpha}\right)$

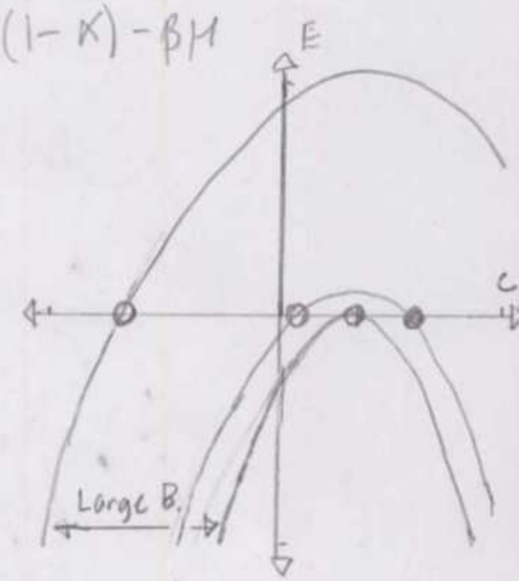
$$E' = E(1-\alpha) - \beta\mu$$

c. Nullclines: $C' = 0 = RC\left(1 - \frac{C}{\alpha}\right)$

$$E' = 0 = E(1-\alpha) - B\mu$$

$$(C^*, E^*) = \left(0, \frac{B\mu}{(1-\alpha)}\right)$$

$$\left(\alpha, \frac{B\mu}{(1-\alpha)}\right)$$



d. See part c.

Bifurcation Amount	Conditions
0	$B > \frac{E(1-\alpha)}{\mu}$
1	$B = \frac{E(1-\alpha)}{\mu}$
2	$B < \frac{E(1-\alpha)}{\mu}$

$\dot{u} = a(1-u) - uv^2$
$\dot{v} = uv^2 - (a+k)v$

8.1.11. Bifurcations:

$$\dot{u} = 0 = a(1-u) - uv^2$$

$$\dot{v} = 0 = uv^2 - (a+k)v$$

$$(u^*, v^*) = \left(\frac{a \pm \sqrt{a^2 - 4(a+k)^2}}{2a}, \frac{a \mp \sqrt{a^2-4(a+k)^2}}{2(a+k)}\right)$$

$$0 = a^2 - 4(a+k)^2 ; \ (a+k)^2 = \frac{a}{4} ; \ k = -a \pm \frac{a^{1/2}}{v}$$

$\dot{\theta}_1 = K\sin(\theta_1 - \theta_2) - \sin\theta_1$
$\dot{\theta}_2 = K\sin(\theta_2 - \theta_1) - \sin\theta_2$

8.1.12 a. Fixed Points: $\dot{\theta}_1 = 0 = K\sin(\theta_1 - \theta_2) - \sin\theta_1$

$$\dot{\theta}_2 = 0 = K\sin(\theta_2 - \theta_1) - \sin\theta_2$$

$$(\theta_1^*, \theta_2^*) = (n_1\pi, n_2\pi) \quad n_1, n_2 \in \mathbb{Z}$$

b. Identity: $\sin(a-b) = \cos b \sin a - \sin b \cos a$

$$\dot{\theta}_1 = 0 = K[\sin\theta_1 \cos\theta_2 - \sin\theta_2 \cos\theta_1] - \sin\theta_1$$

$$\dot{\theta}_2 = 0 = K[\sin\theta_2 \cos\theta_1 - \sin\theta_1 \cos\theta_2] - \sin\theta_2$$

$$0 = 2K[\sin\theta_1 \cos\theta_2 - \sin\theta_2 \cos\theta_1] + \sin\theta_2 - \sin\theta_1$$

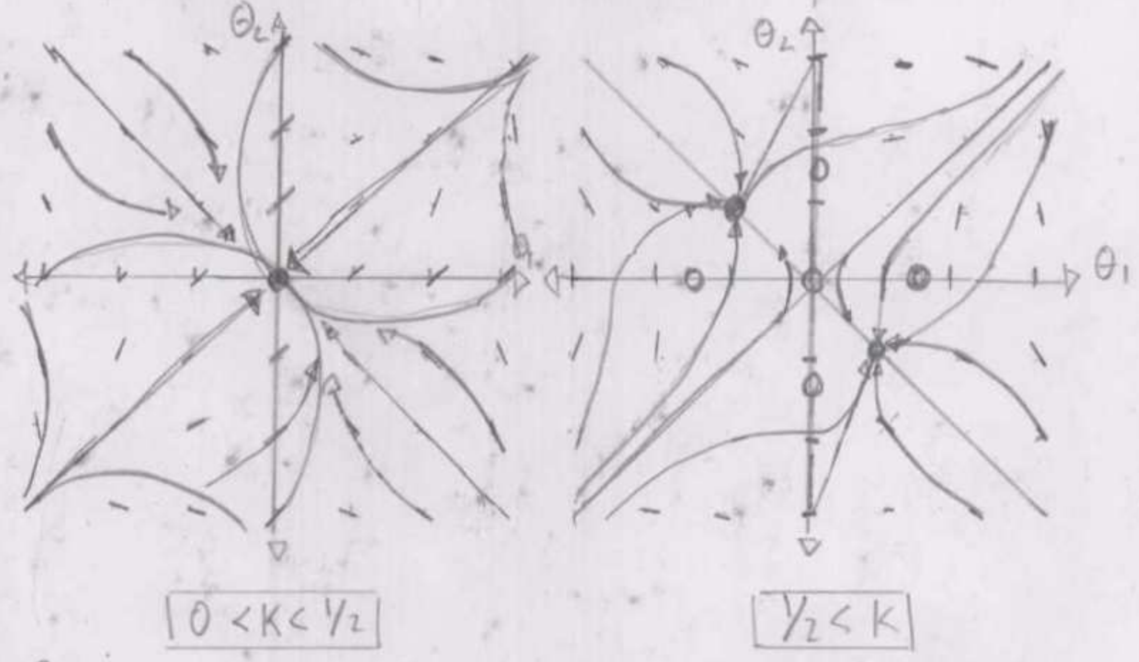
$$K = 1/2 \ @ \ (\theta_1, \theta_2) = (n_1\pi, n_2\pi)$$

c. $\dot{\theta} = -\frac{\partial V}{\partial \theta_1}$; $V(\theta_1, \theta_2) = K\cos(\theta_1 - \theta_2) - \cos\theta_1$

$= K\cos(\theta_2 - \theta_1) - \cos\theta_1$

d. $\dot{\theta}_1$ and $\dot{\theta}_2$ written as $V(\theta_1, \theta_2)$ imply a Lyapunov function. Gradient flows have no periodic orbits.

e.



$\dot{n} = GnN - kn$

$\dot{N} = -GnN - fN + p$

8.1.13

a. $N(t)$ = # of excited atoms

$n(t)$ = # of photons in laser field

G = Gain coefficient for Stimulated Emission

k = Decay rate due to loss of photons by mirror transmission

f = Decay rate for spontaneous emission

p = pump strength

If $\dot{n} = GnN - kn$, then $\frac{G}{k^2}\dot{n} = \frac{G^2 nN}{k^2} - \frac{Gn}{k}$

$\dot{N} = -GnN - fN + p$, then $\frac{G}{k^2}\dot{N} = -\frac{G^2 nN}{k^2} - \frac{G}{k^2}(fN + p)$

If $\tau = k^2 t$, $x = \frac{Gn}{k}$, $y = \frac{GN}{k}$, $a = \frac{f}{k}$, $b = \frac{pG}{k^2}$

$\dot{x} = x(y-1)$, $\dot{y} = -xy - ay + b$

b. <u>Fixed Points:</u> $\dot{x} = 0 = x(y-1)$

$$\dot{y} = 0 = -xy - ay + b$$

$$(x^*, y^*) = (0, b/a)$$
$$= (b-a, 1)$$

<u>Classification:</u> $\dot{\vec{x}} = A\vec{x}$; $A = \begin{pmatrix} y-1 & x \\ -y & -(x+a) \end{pmatrix}$

$$(A_{(0,a/b)} - \lambda) = \begin{pmatrix} \frac{b}{a} - 1 - \lambda_1 & 0 \\ -b/a & -a - \lambda_2 \end{pmatrix} = 0$$

$$(A_{(b-a,1)}, -\lambda) = \begin{pmatrix} -\lambda_1 & b-a \\ -1 & -b-\lambda_2 \end{pmatrix}$$

$\lambda_1 = \frac{b}{a} - 1$; $\lambda_2 = -a$

$\Delta = a - b$

$\tau = \frac{b}{a} - 1 - a$

$\tau^2 - 4\Delta < 0$ @ $a > b$

"Stable node"

$\tau^2 - 4\Delta > 0$ @ $a < b$

"Unstable node"

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4(b-a)}}{2}$$

$$\Delta = \frac{(b^2 - (b-4)b - 4a^2)}{4}$$

$\tau = -b$

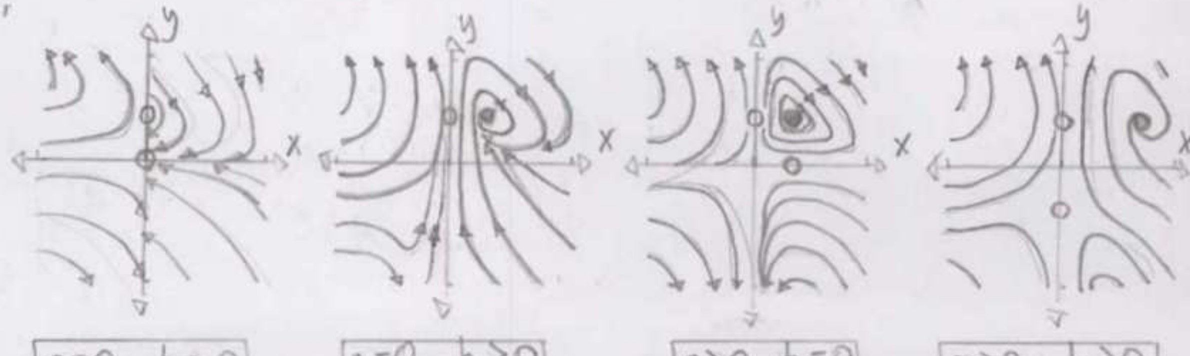
$\tau^2 - 4\Delta > 0$ @ $b < \frac{4}{3}a$

"Stable spiral when $b > 0$
Unstable node when $b < 0$"

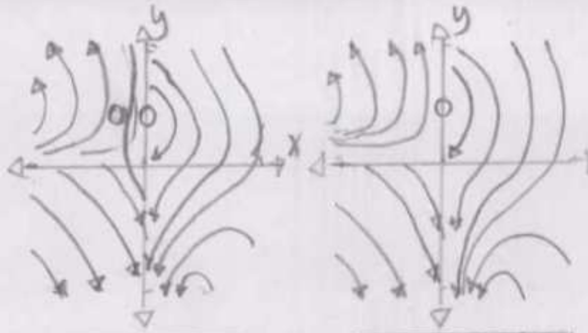
$\tau^2 - 4\Delta < 0$ @ $b > \frac{4}{3}a$

"Unstable spiral when $b < 0$
Stable node when $b > 0$"

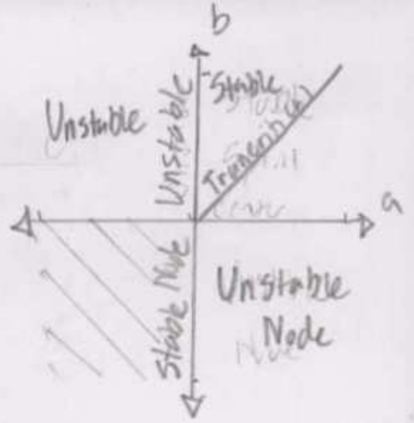
c.



$\boxed{a=0 \quad b=0}$ $\boxed{a=0 \quad b>0}$ $\boxed{a>0 \quad b=0}$ $\boxed{a>0 \quad b>0}$



$\boxed{a=0 \quad b<0}$ $\boxed{a>0 \quad b<0}$



d. Upper - Right page.

$\dot{x}_1 = -x_1 + F(I - bx_2)$

$\dot{x}_2 = -x_2 + F(I - bx_1)$

8.1.14. If $F(x) = 1/(1+e^{-x})$: Gain Function

I : Strength of the Stimulus

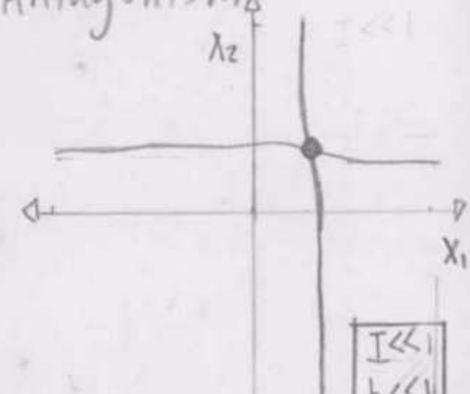
b : Strength of the Mutual Antagonism

a. <u>Nullclines</u>: $\dot{x}_1 = 0 = -x_1 + F(I - bx_2)$

$$= -x_1 + \frac{1}{1+e^{-I+bx_2}}$$

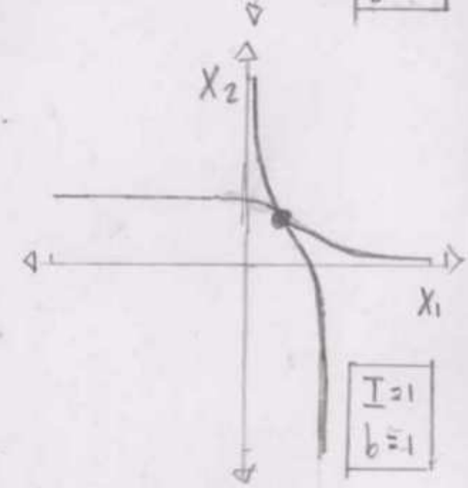
$$\dot{x}_2 = 0 = -x_2 + F(I - bx_1)$$

$$= -x_2 + \frac{1}{1+e^{-I+bx_1}}$$



$$x_2 = \frac{I + \log\left(\frac{1-x_1}{x_1}\right)}{b}$$

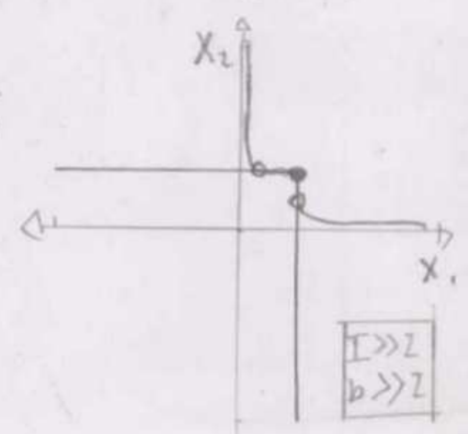
$$x_2 = \frac{1}{1+e^{-I+b\cdot x_1}}$$



b. If $x_1^* = x_2^* = x^*$, then $\dot{x}_1 = -x_1 + \frac{1}{1+e^{-I+bx_1}}$

$$\dot{x}_2 = -x_1 + \frac{1}{1+e^{-I+bx_1}}$$

$$\dot{x}_1 = \dot{x}_2 = x^*$$



$$\text{c. } \lim_{b\to\infty} \dot{x}_1 = \lim_{b\to\infty} -x_2 + \frac{1}{1+e^{-\Sigma+bx}} = -x_2$$

d. See part a; supercritical pitchfork because of the unstable center.

8.1.15.

$\dot{n}_A = (p+n_A)n_{AB} - n_A n_B$ where $n_{AB} = 1-(p+n_A)-n_B$

$\dot{n}_B = n_B n_{AB} - (p+n_A)n_{AB}$

a. The first term of $\dot{n}_A$ fits a constant (p) and changing (n_A) population of A-B.
The second term of $\dot{n}_A$ are the decreasing populations from A-B interaction.
The first term of $\dot{n}_B$ fits an increasing population from A-B interaction.
The second term of $\dot{n}_B$ fits a constant (p) and changing (n_A) population of A-B.

b. $n_B(0) = 1-p$; $n_A(0) = n_{AB}(0) = 0$

<u>Numerical Integration:</u> $\Delta h = 0.001$ $p = 0.15$

$k_{x1} = f(x_n)\Delta t$

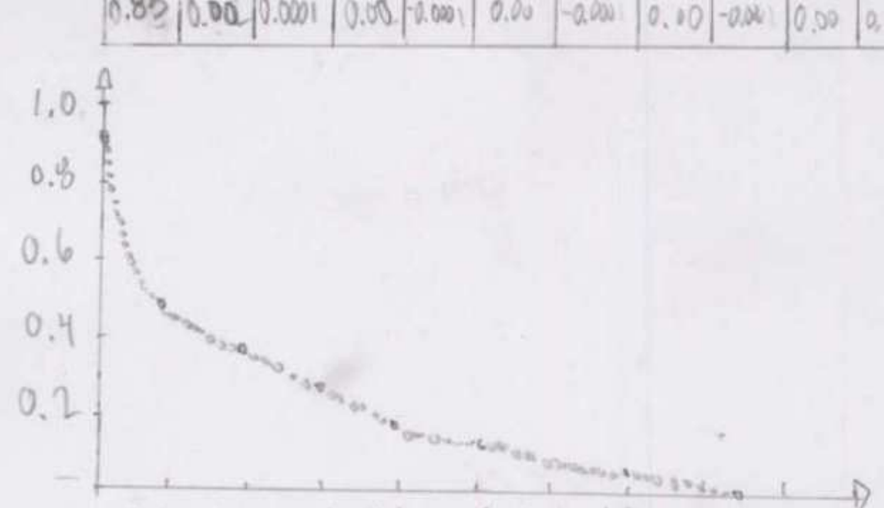
$k_{x2} = f(x_n + \frac{1}{2}k_{x1})\Delta t$

$k_{x3} = f(x_n + \frac{1}{2}k_{x2})\Delta t$

$k_{x4} = f(x_n + k_3)\Delta t$

$$n_{x+1} = n_x + \frac{(k_1 + 2k_2 + 2k_3 + k_4)}{6}$$

n_A	n_B	n_{AB}	k_{A1}	k_{B1}	k_{A2}	k_{B2}	k_{A3}	k_{B3}	k_{A4}	k_{B4}
0	0.85	0	-	-	-	-	-	-	-	-
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.85	0.00	0.0001	0.00	-0.001	0.00	-0.000	0.00	-0.00	0.00	0.00



$\ddot{x} + \mu(x^2-1)\dot{x} + x = a$ 8.2.1

<u>Van der Pol Oscillator:</u>

<u>Fixed Points:</u> $\dot{x} = y$

$\dot{y} = -\mu(x^2-1)y + (a-x)$

$(x^*, y^*) = (a, 0)$

Eigenvalues: $(A-\lambda)\vec{x}=0$; $A-\lambda=\begin{pmatrix} -\lambda & 1 \\ -2\mu xy-1 & -\mu(x^2-1)-\lambda_2 \end{pmatrix}$

$$=(\lambda)(\mu(x^2-1)+\lambda)+2\mu xy-1$$

$$A_{(a,0)}=\lambda(\mu(a^2-1)+\lambda)=0.$$

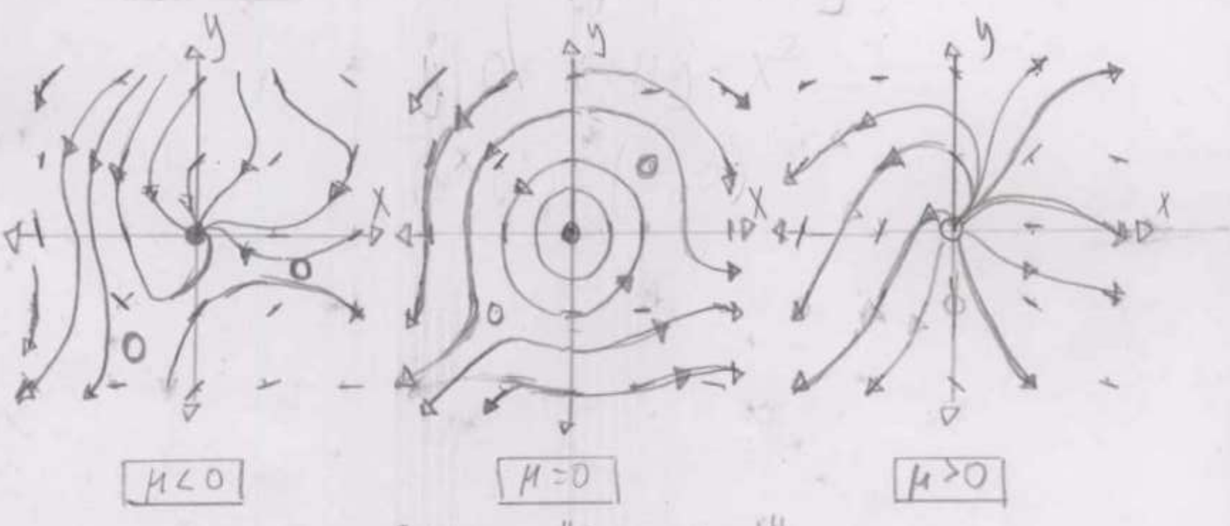
$$\lambda_1=0;\ \lambda_2=-\mu(a^2-1)$$

$$\Delta=0;\ \tau=-\mu(a^2-1)$$

$$\tau^2-4\Delta>0$$

Hopf Bifurcations: The phase plot changes when the sign of μ becomes positive, zero, or negative, in addition to, $a=\pm 1$

$\dot{x}=-y+\mu x+xy^2$ 8.2.3.
$\dot{y}=x+\mu y-x^2$



Pitchfork: "Super critical"

8.2.4.

a. $\dot{r}=\frac{d}{dt}\sqrt{x^2+y^2}=\frac{\dot{x}y+x\dot{y}}{\sqrt{x^2+y^2}}=\frac{(-y+\mu x+xy^2)y+x(x+\mu y-x^2)}{\sqrt{x^2+y^2}}$

$\dot{\theta}=\frac{d}{dt}\arctan\frac{y}{x}=\frac{x\dot{y}-y\dot{x}}{x^2+y^2}=\frac{x(x+\mu y-x^2)-y(-y+\mu x+xy^2)}{x^2+y^2}$

b. If $r<<1$, then $\dot{\theta}\approx\frac{x^2+y^2}{x^2+y^2}=1$

and $\dot{r} = \dfrac{x^2 - y^2 + 2\mu xy + xy^3 - x^3}{\sqrt{x^2+y^2}}$

c. $\dot{r} = 0 \approx \mu r + \frac{1}{8} r^3$: $r = \sqrt{-8\mu}$

If $\mu < 0$, then $r \in \mathbb{R}$, and if $\mu > 0$ $r \in \mathbb{I}$

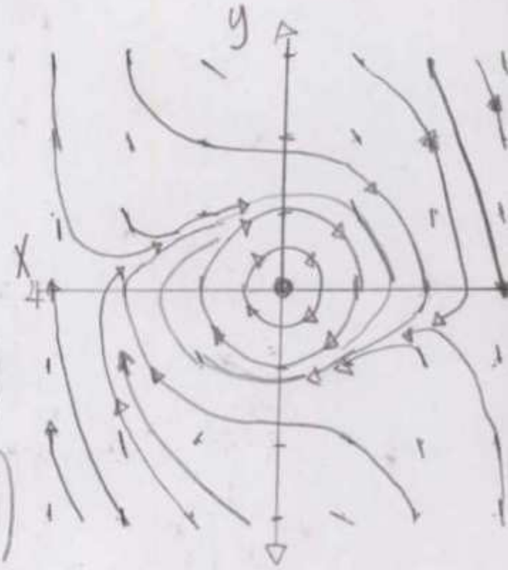
$\dot{x} = y + \mu x$ 8.2.5. Fixed Points: $\dot{x} = 0 = y + \mu x$

$\dot{y} = -x + \mu y - x^2 y$ $\quad \dot{y} = 0 = -x + \mu y - x^2 y$

$(x^*, y^*) = (0,0)$

$= \left(-\sqrt{\frac{\mu+1}{\mu}}, \sqrt{\mu(\mu^2+1)}\right)$

$= \left(\sqrt{\frac{\mu+1}{\mu}}, -\sqrt{\mu(\mu^2+1)}\right)$

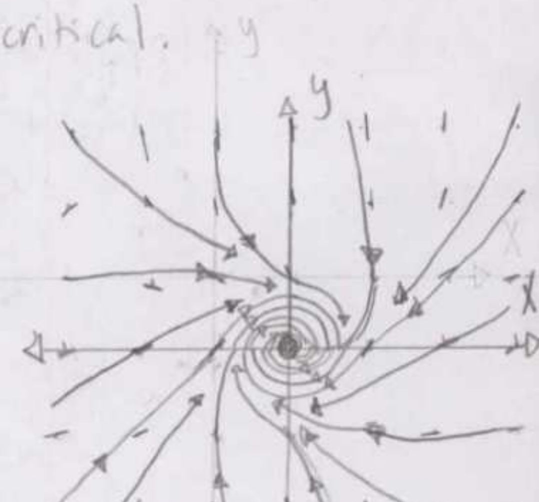


Pitchfork: Subcritical.

$\dot{x} = \mu x + y - x^3$ 8.2.6. Fixed Points: $\dot{x} = 0 = \mu x + y - x^3$

$\dot{y} = -x + \mu y - 2y^3$ $\quad \dot{y} = 0 = -x + \mu y - 2y^3$

$(x^*, y^*) = (0,0)$



Pitchfork Supercritical

$\dot{x} = \mu x + y - x^2$ 8.2.7. Fixed Points: $\dot{x} = 0 = \mu x + y - x^2$

$\dot{y} = -x + \mu y - 2x^2$ $\quad \dot{y} = 0 = -x + \mu y - 2x^2$

$(x^*, y^*) = (0,0)$

$= \left(\frac{\mu^2+1}{m-2}, \frac{(2m+1)(m^2+1)}{(m-2)^2}\right)$

$\dot{x} = x[x(1-x)-y]$ "Pitchfork : Supercritical"

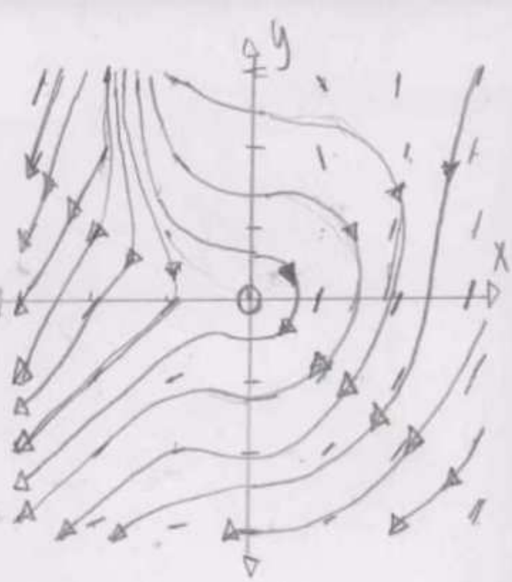
$\dot{y} = y(x-a)$

8.2.8 a. Nullclines: $\dot{x} = 0 = x[x(1-x)-y]$

$\dot{y} = 0 = y(x-a)$

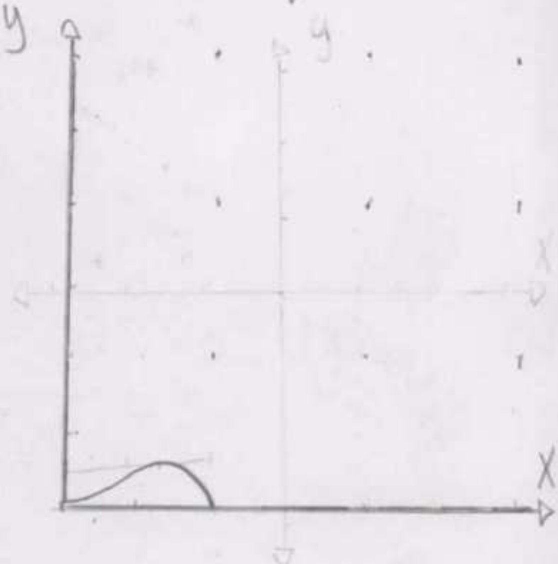
$x = 0 ; y = 0$

$x = a ; y = x^2(1-x)$



b. Fixed Points: $\dot{\vec{x}} = A\vec{x} = 0$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2x-3x^2 & -x \\ y & x-a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



$$A_{(0,0)} - \lambda = \begin{bmatrix} -\lambda & 0 \\ 0 & -(\lambda+a) \end{bmatrix}$$

$\lambda_1 = 0 ; \lambda_2 = -a$

$\Delta = 0 ; \tau = -a ; \tau^2 - 4\Delta > 0$ "stable spiral"

$$A_{(1,0)} - \lambda = \begin{bmatrix} -(\lambda+1) & -1 \\ 0 & 1-(a+\lambda) \end{bmatrix}$$

$\lambda_1 = -1 , \lambda_2 = a-1$

$\Delta = 1-a ; \tau = a-2 ; \tau^2 - 4\Delta = a^2 - 6a$

If $a > 6$, line of unstable fixed points

If $a < 6$, saddle node.

If $a = 6$, unstable star / degenerate node.

$$A_{(a,a-a^2)} - \lambda = \begin{bmatrix} 2a-3a^2-\lambda & -a \\ a^2-a^2 & -\lambda \end{bmatrix}$$

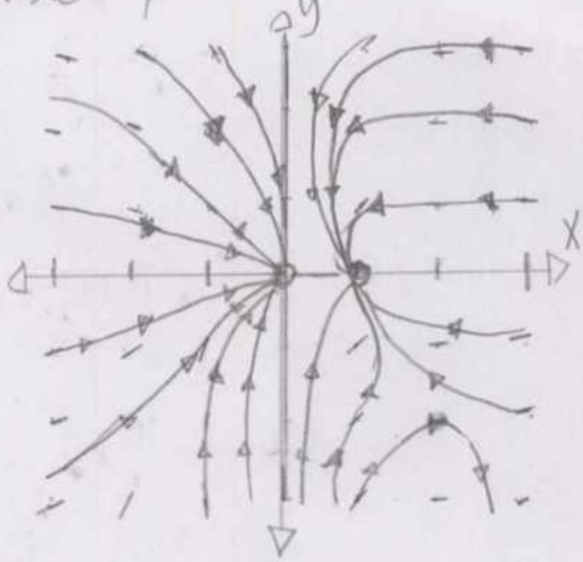
$$\lambda_1 = \frac{1}{2}\left(-\sqrt{9a-8}\,a^{3/2} - 3a^2 + 2a\right)$$

$$\lambda_2 = \frac{1}{2}\left(\sqrt{9a-8}\,a^{3/2} - 3a^2 + 2a\right)$$

$$\Delta = a^2 - a^3 ; \; \tau = 2a - 3a^2$$

$\tau^2 - 4\Delta > 0$ "Stable Spiral"

c) If $a > 1$, phase portrait.



d)

a	Bifurcations
$< 1/2$	2
$= 1/2$	3
$> 1/2$	3

Pitchfork: Supercritical Hopf

e) At Hopf Bifurcation $(\frac{1}{2}, \frac{1}{4})$,

$$\lambda_1 = \frac{1}{2}\left(-\sqrt{9/2 - 8}\,(1/2)^{3/2} - 3(1/2)^2 + 2(1/2)\right)$$

$$= \frac{1}{2}\left(-\frac{35}{100}\left(\frac{7}{2}\right)i - (3/4) + 1\right)$$

$$= \frac{1}{8} + \frac{49}{80} i$$

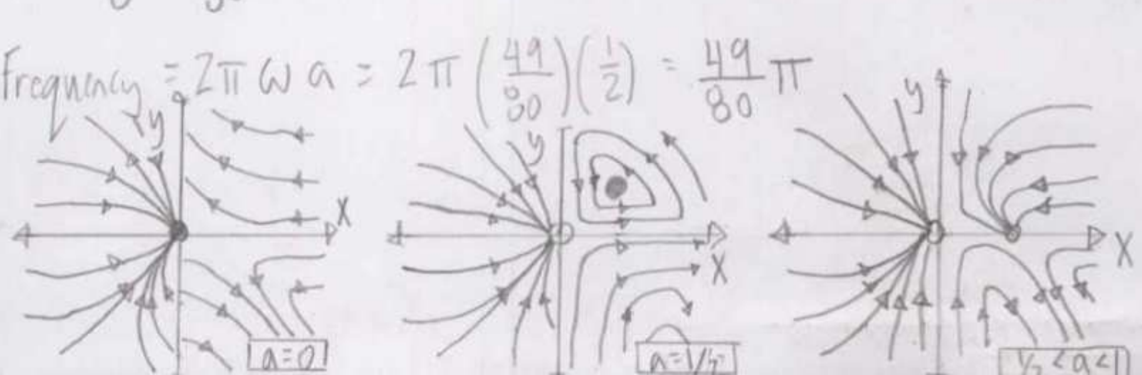
$$\lambda_2 = \frac{1}{2}\left(\sqrt{9/2 - 8}\,(1/2)^{3/2} - 3(1/2)^2 + 2(1/2)\right)$$

$$= \frac{1}{2}\left(-\frac{35}{100}\left(\frac{7}{2}\right)i - 3/4 + 1\right)$$

$$= \frac{1}{8} - \frac{49}{80} i$$

Frequency $= 2\pi\,\omega\,a = 2\pi\left(\frac{49}{80}\right)\left(\frac{1}{2}\right) = \frac{49}{80}\pi$

f)



$\dot{x} = x\left(b - x - \frac{y}{1+x}\right)$ 8.2.9. $x, y \geq 0$; $a, b > 0$

$\dot{y} = y\left(\frac{x}{1+x} - ay\right)$

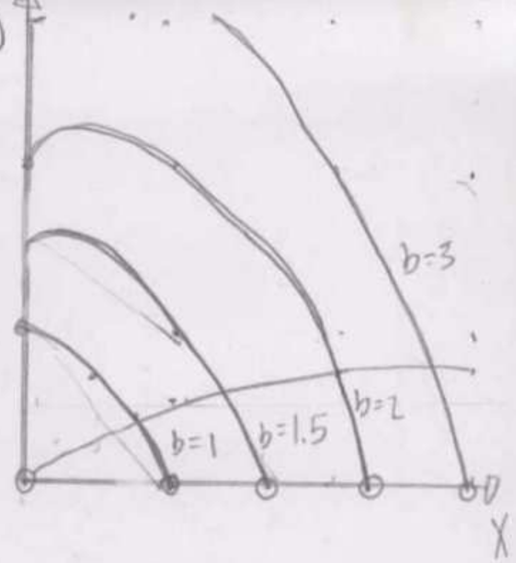
a. Nullclines: $\dot{x} = 0 = x\left(b - x - \frac{y}{1+x}\right)$

$\dot{y} = 0 = y\left(\frac{x}{1+x} - ay\right)$

$y = 0$; $x = 0$

$y = (1+x) \cdot (b - x)$

$y = \frac{x}{a(1+x)}$

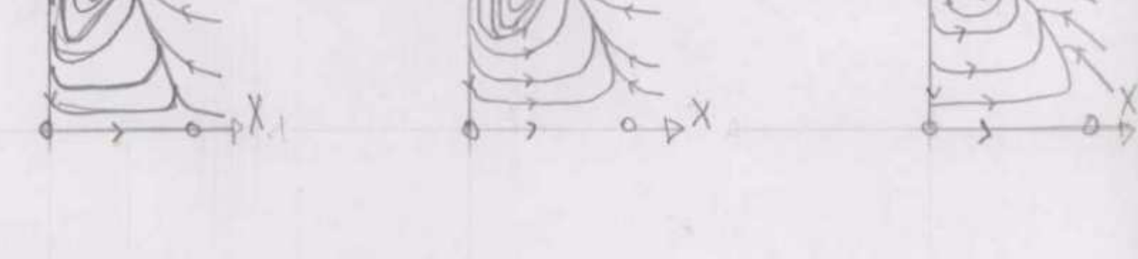


b. A graphical argument for the fixed point $x^* > 0$, $y^* > 0$ for all $a, b > 0$ displayed in part a, becomes an integer-solution.

c. If $a_c = \frac{4(b-2)}{b^2(b+2)}$, then and $b<2$, $b=2$, or $b>2$, then

$= a$	Number of Bifurcations
$< \frac{4(b-2)}{b^2(b+2)}$	3 "unstable"
$= \frac{4(b-2)}{b^2(b+2)}$	3 "stable"
$> \frac{4(b-2)}{b^2(b+2)}$	3 "stable"

d. Phase Portrait:



$\dot{x} = B - x - \frac{xy}{1+qx^2}$ 8.2.10.

$\dot{y} = A - \frac{xy}{1+qx^2}$

$\dot{x} = x\left(b - x - \frac{y}{1+x}\right)$

$\dot{y} = y\left(\frac{x}{1+x} - ay\right)$

8.2.9. $x, y \geq 0$; $a, b > 0$

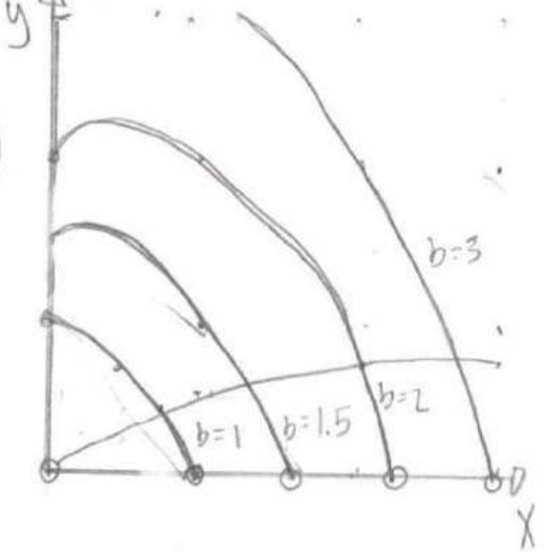
a. Nullclines: $\dot{x} = 0 = x\left(b - x - \frac{y}{1+x}\right)$

$$\dot{y} = 0 = y\left(\frac{x}{1+x} - ay\right)$$

$$y = 0 ; x = 0$$

$$y = (1+x) \cdot (b - x)$$

$$y = \frac{x}{a(1+x)}$$



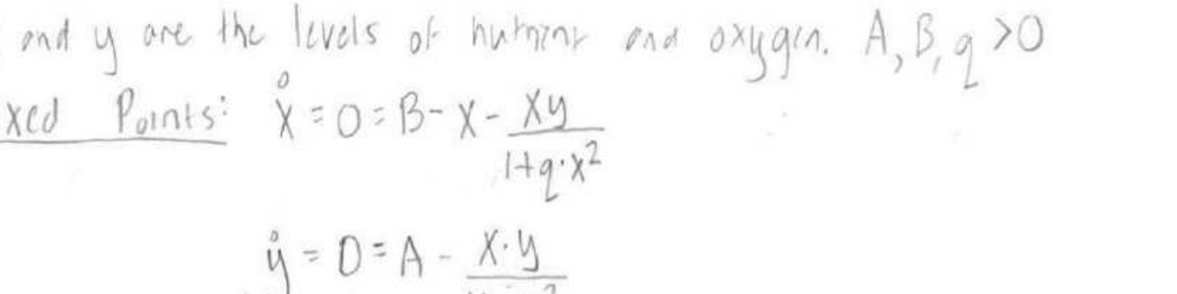
b. A graphical argument for the fixed point $x^* > 0$, $y^* > 0$ for all $a, b > 0$ displayed in part a becomes a real-solution.

c. If $a_c = \frac{4(b-2)}{b^2(b+2)}$, then

and $b < 2$, $b = 2$, or $b > 2$, then

a	Number of Bifurcations
$< \frac{4(b-2)}{b^2(b+2)}$	3 "unstable"
$= \frac{4(b-2)}{b^2(b+2)}$	3 "stable"
$> \frac{4(b-2)}{b^2(b+2)}$	3 "stable"

d. Phase Portrait:



$\dot{x} = B - x - \frac{xy}{1+qx^2}$

$\dot{y} = A - \frac{xy}{1+qx^2}$

8.2.10. x and y are the levels of nutrient and oxygen. $A, B, q > 0$

Fixed Points: $\dot{x} = 0 = B - x - \frac{xy}{1+q \cdot x^2}$

$$\dot{y} = 0 = A - \frac{x \cdot y}{1+qx^2}$$

$$(x^*, y^*) = \left(A - B, \frac{A}{B-A}\left[1 + q(A-B)^2\right]\right)$$

Nullclines: $x = A - B$

$$y = \frac{A}{B-A}\left(1 + q(A-B)^2\right)$$

Trapping Region: $\mathring{\vec{x}} = A\vec{x}$; $0 = A\vec{x} = \lambda\vec{x} \Rightarrow 0 = (A-\lambda)\vec{x}$

$$\text{where } A = \begin{pmatrix} -1 - \frac{y(1-qx^2)}{(1+qx^2)^2} - \lambda & \frac{+x}{(1+qx^2)} \\ \frac{-y(1-qx^2)}{(1+qx^2)^2} & \frac{-x}{1+qx^2} - \lambda \end{pmatrix}$$

$$0 = \left(-1 - \frac{y(1-qx^2)}{(1+qx^2)^2} - \lambda\right)\left(\frac{-x}{1+qx^2} - \lambda\right) - \left(\frac{x}{1+qx^2}\right)\left(\frac{y(1-qx^2)}{(1+qx^2)^2}\right)$$

$$\lambda_{1,2} = \frac{(q^3x^6(y-1) - q^2x^4(x+y-3) \pm \sqrt{(qx^2+1)^3(q^3x^6(y^2-2y+1) + q^2(x^5(2y-2) + x^4(3-y^2-2y+q) - 4qx^3(y+1) + x^2q(y^2+2y+3) + x^2 + 2x(y-1) + y^2 + 2y+1)} - 2qx^3 - qx^2y - 3qx^2 - x - y - 1}{2(qx^2+1)^3}$$

A stable limit cycle has three parameters, μ the stability of a fixed point at the origin, ω the frequency, and b the dependence of frequency on amplitude.

The stability of μ depends on $q^3x^6(y-1) - q^2x^4(x+y-3)/2(qx^2+1)^3$ being positive or negative.

The frequency ω about the origin is the square root in the eigenvalue being real or imaginary.

$$\ddot{x} + \mu\dot{x} + x - x^3 = 0$$

8.2.11 a. $\dot{x} = y$

$\dot{y} = -\mu y - x + x^3$

Fixed Points: $\dot{x} = 0 = y$

$\dot{y} = 0 = -\mu y - x + x^3$

$(x^*, y^*) = (0,0), (-1,0), (1,0)$

Bifurcations: $\dot{\vec{x}} = A\vec{x}$; $0 = A\vec{x} = \lambda\vec{x}$; $0 = (A-\lambda)\vec{x}$

$$A = \begin{pmatrix} -\lambda & 1 \\ -1+3x^2 & -\mu-\lambda \end{pmatrix} = \lambda(\mu+\lambda)+1-3x^2 = 0$$

$$\lambda_{1,2} = \frac{1}{2}\left(\pm\sqrt{\mu^2+12x^2-4} - m\right)$$

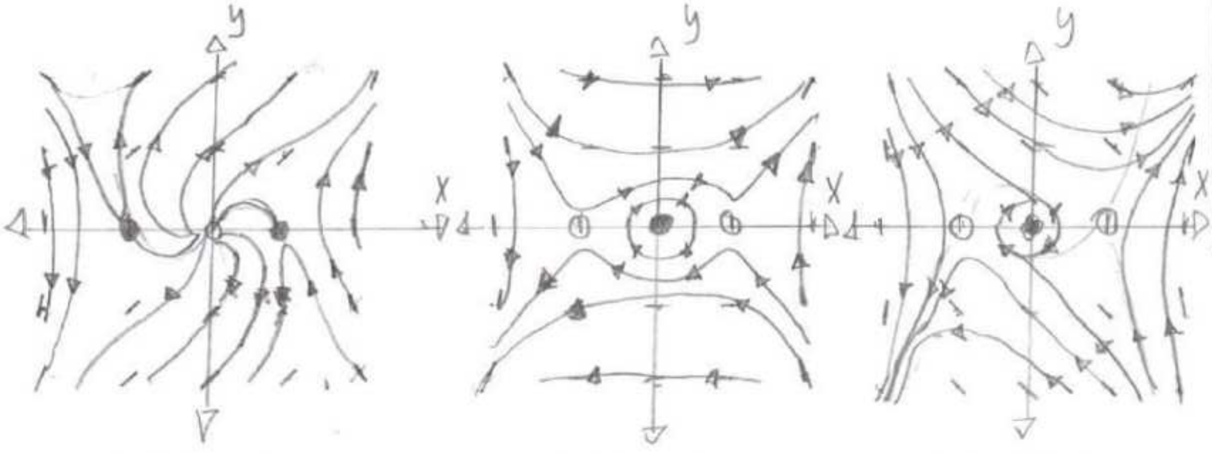
$\Delta = 1-3x^2$; $\tau = -\mu$; $\tau^2 - 4\Delta = \mu^2 - 4 - 12x^2$

If $\mu < 0$ and $x > \sqrt{\frac{1}{3}}$, unstable spiral

If $\mu = 0$ and $x > \sqrt{\frac{1}{3}}$, center

If $\mu > 0$ and $x > \sqrt{\frac{1}{3}}$, stable spiral.

b. Phase Portraits:



$\boxed{\mu < 0}$ $\boxed{\mu = 0}$ $\boxed{\mu > 0}$

$\dot{x} = -\omega y + f(x,y)$ 8.2.12. $16a = f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}$

$\dot{y} = \omega x + g(x,y)$ $+ \frac{1}{\omega}\left[f_{xy}(f_{xx}+f_{yy}) - g_{xy}(g_{xx}+g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\right]$

If $a < 0$: supercritical, $a > 0$: subcritical.

a) $\dot{x} = -y + xy^2$ $\quad \dot{y} = x - x^2$

$f = xy^2$	$g = -x^2$	$f_{xy} = 2y$	$g_{xy} = 0$
$f_x = y^2$	$g_x = -2x$	$f_{xyy} = 2$	$g_{yy} = 0$
$f_{xx} = 0$	$g_{xx} = -2$	$f_{yy} = 2x$	$g_{xxy} = 0$
$f_{xxx} = 0$	$g_{xxx} = 0$		$g_{yyy} = 0$

$$16a = 0 + 2y + 0 + 0 + \frac{1}{\omega}\left[2y(0+2x) - 0(0+0) - 0\cdot 0 + 2x\cdot 0\right]$$

$$= 2y + \frac{4yx}{\omega}$$

An evaluation at the point $(0,0)$

$16a = 2$; $a = \frac{1}{8} > 0$: subcritical

b. $\dot{x} = -y + \mu x + xy^2$; $\dot{y} = x + \mu y - x^2$

A subcritical Hopf Bifurcation occurs when $\mu = 0$.

$\dot{x} = y + \mu x$
$\dot{y} = -x + \mu y - x^2 y$

8.2.13. $f = 0$ $\quad g = -x^2 y$

$f_x = 0 \quad f_{xy} = 0 \quad f_y = 0 \quad g_x = -2xy \quad g_y = -x^2$

$f_{xx} = 0 \quad f_{xyy} = 0 \quad f_{yy} = 0 \quad g_{xx} = -2y \quad g_{yy} = 0 \quad g_{xy} = -2x$

$f_{xxx} = 0 \quad g_{xxy} = -2 \quad g_{yyy} = 0$

$$16a = 0 + 0 - 2 + 0 + \frac{1}{\omega}\left[0(0+0) + 2x(-2y+0) + 0\cdot 2y + 0\cdot 0\right]$$

$$= -2 - \frac{4xy}{\omega} = -2$$

An evaluation at the point $(0,0)$

$a = -\frac{1}{8}$; A supercritical Hopf Bifurcation

$\dot{x} = \mu x + y - x^3$
$\dot{y} = -x + \mu y + 2y^3$

8.2.14. $f = -x^3$ $\quad g = 2y^3$

$f_x = -3x^2 \quad f_{xy} = 0 \quad f_y = 0 \quad g_x = 0 \quad g_y = 6y^2$

$f_{xx} = -6x \quad f_{xyy} = 0 \quad f_{yy} = 0 \quad g_{xx} = 0 \quad g_{yy} = 12y \quad g_{xy} = 0$

$f_{xxx} = -6 \quad g_{xxy} = 0 \quad g_{yyy} = 12$

$$16a = -6 + 0 + 0 + 12 + \frac{1}{\omega}\left[0(-6x+0) - 0(0+12y) + 6x\cdot 0 + 0\cdot 12y\right]$$

$$= 6$$

An evaluation at the point $(0,0)$:

$a = \frac{3}{8}$; A subcritical Hopf Bifurcation.

$\dot{x} = \mu x + y - x^2$ 8.2.15. $f = -x^2$ $\quad g = 2x^2$

$\dot{y} = -x + \mu y + 2x^2$

$f_x = -2x \quad f_{xy} = 0 \quad f_y = 0 \qquad g_x = 4x \quad g_y = 0$

$f_{xx} = -2 \quad f_{xyy} = 0 \quad f_{yy} = 0 \qquad g_{xx} = 4 \quad g_{yy} = 0 \quad g_{xy} = 0$

$f_{xxx} = 0 \qquad g_{xxy} = 0 \quad g_{yyy} = 0$

$$16a = 0 + 0 + 0 + 0 + \frac{1}{\omega}\left[0(-2+0) - 0(4+0) + 2 \cdot 4 + 0 \cdot 0\right]$$

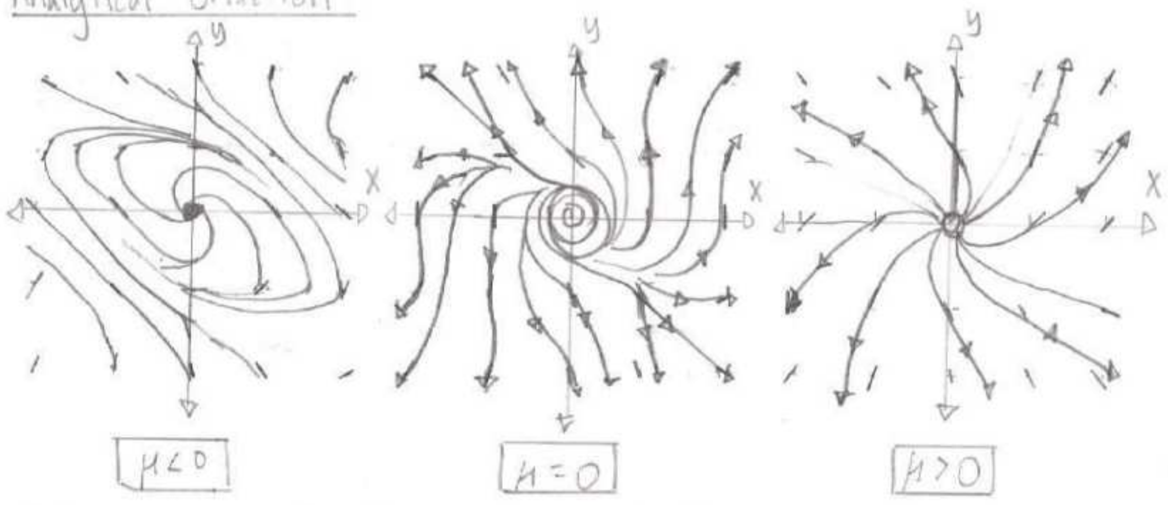
$$= \frac{8}{\omega}$$

An evaluation at point $(0,0)$:

$a = \frac{1}{2}\omega = \frac{1}{2}$; A subcritical Hopf Bifurcation.

$\dot{x} = \mu x - y + xy^2$ 8.2.16. Analytical Criterion:

$\dot{y} = x + \mu y + y^3$



Subcritical bifurcation when $\mu < 0$.

$\dot{x}_1 = -x_1 + F(I - bx_2 - gy_1)$ 8.2.17 $\quad y$ = adoption

$\dot{y}_1 = (-y_1 + x_1)/T \quad T$ = Timescale

$\dot{x}_2 = -x_2 + F(I - bx_1 - gy_2) \quad g$ = Associated neuronal population

$\dot{y}_2 = (-y_2 + x_2)/T \quad F(x) = 1/(1+e^{-x})$ = Gain Function

b = mutual strength

a) $x_1^* = y_1^* = x_2^* = y_2^* = u = 0$; $\quad u_1 = \frac{1}{1+e^{-(I - bx_2 - gy_1)}}$

b)
$$A = \begin{bmatrix} -1 & -Fg & -Fb & 0 \\ 1/T & -1/T & 0 & 0 \\ -Fb & 0 & -1 & -Fg \\ 0 & 0 & 1/T & -1/T \end{bmatrix} = \begin{bmatrix} -c_1 & -c_2 & -c_3 & 0 \\ d_1 & -d_1 & 0 & 0 \\ -c_3 & 0 & -c_1 & -c_2 \\ 0 & 0 & d_1 & -d_1 \end{bmatrix}$$

Where $c_1 = 1$, $c_2 = Fg$, $c_3 = Fb$, $d_1 = 1/T$

If $A = \begin{bmatrix} -c_1 & -c_2 \\ d_1 & -d_1 \end{bmatrix}$ and $B = \begin{bmatrix} -c_3 & 0 \\ 0 & 0 \end{bmatrix}$

then $\begin{bmatrix} A & B \\ B & A \end{bmatrix} = A^2 - B^2 = (A+B)(A-B)$

Eigenvalues of a 4x4:

$$\begin{bmatrix} -c_1 - c_3 - \lambda & -c_2 \\ d_1 & -d_1 - \lambda \end{bmatrix} \begin{bmatrix} c_3 - c_1 - \lambda & -c_2 \\ d_1 & -d_1 - \lambda \end{bmatrix} = 0$$

$$\lambda_{1,2} = \frac{\pm\sqrt{4T(Fb - Fg - 1) + (FbT - T - 1)^2} + FbT - T - 1}{2T}$$

$$\lambda_{3,4} = \frac{\pm\sqrt{(FbT + T + 1)^2 - 4T(Fb + Fg + 1)} - FbT - T - 1}{2T}$$

c. $\Delta = \lambda_1 \cdot \lambda_2 = \dfrac{F(g+b)+1}{T} > 0$

$\tau = \lambda_1 + \lambda_2 = -Fb - \frac{1}{T} - 1 < 0$

So, λ_1 and λ_2 are each negative eigenvalues.

d. $\Delta = \lambda_3 \cdot \lambda_4 = \dfrac{F(g-b)+1}{T}$; If $g > b$, then $\Delta > 0$; Hopf Bifurcation.
If $g < b$, then $\Delta < 0$; Pitchfork Bifurcation

$\tau = \lambda_3 + \lambda_4 = Fb - \frac{1}{T} - 1$; Positive or negative.

e.

$\dot{x}=1-(b+1)x+ax^2y$ 8.3.1. $a,b>0$ and $x,y\geq 0$
$\dot{y}=bx-ax^2y$

a) Fixed Points: $\dot{x}=0=1-(b+1)x+ax^2y$

$$\dot{y}=0=bx-ax^2y$$

$$(x^*,y^*)=(0,0),(1,b/a)$$

$$(A-\lambda)\vec{x}=0;\ A=\begin{pmatrix}-(b+1)+2axy & ax^2\\ b-2axy & -ax^2\end{pmatrix}$$

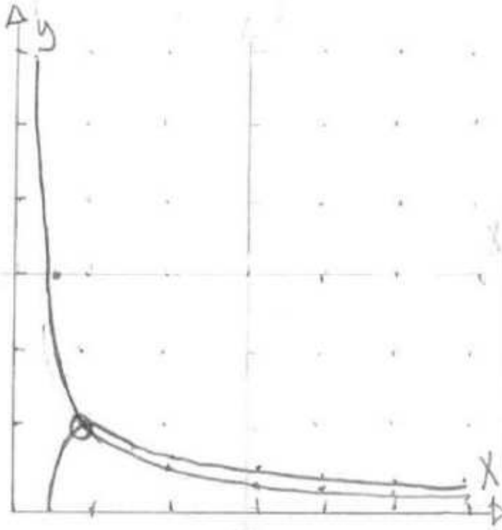
$$A_{(1,\frac{b}{a})}=\begin{pmatrix}b-1 & a\\ -b & -a\end{pmatrix}$$

$$\Delta=a;\ \tau=b-(1+a);\ \tau^2-4\Delta$$

b) Nullclines: $\dot{x}=0=1-(b+1)x+ax^2y$

$$\dot{y}=0=bx-ax^2y$$

$$y=\frac{b}{a}\left(\frac{1}{x}\right);\ y=\frac{-1+(b+1)x}{ax^2}$$



c) Bifurcations:

The bifurcation occurs at

$\tau=0=b-(1+a)$.

d) Poincaré-Bendixson theorem:

1) A single unstable node or spiral inside an invariant region

2) No critical points inside an invariant region.

If either case exists and non-periodic, then no limit cycle in the graph.

If the solution is periodic, then a limit cycle appears.

A critical point at $(1, b/a)$ fits type I and is a trapping region defined by the domain and range of the nullclines.

e) The period of the limit cycle appears from a transformation to polar coordinates or eigenvalues. An analysis about the Jacobian shows the eigenvalues, $\lambda_{1,2}$ as complex values proportional to $\sqrt{a}$, so a limit cycle has a period $\frac{2\pi}{\sqrt{a}}$.

$X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} A$

$B \xrightarrow{k_2} Y$

$2X + Y \xrightarrow{k_3} 3X$

$\dot{x} = a - x + x^2 y$

$\dot{y} = b - x^2 y$

8.3.2.

a) $\frac{\dot{y}}{\dot{x}} = \frac{b - x^2 y}{a - x + x^2 \cdot y}$; $\lim_{x \to \infty} \frac{\dot{y}}{\dot{x}} = -1$; $\lim_{x \to -\infty} \frac{\dot{y}}{\dot{x}} = -1$

$\lim_{y \to \infty} \frac{\dot{y}}{\dot{x}} = -1$; $\lim_{y \to -\infty} \frac{\dot{y}}{\dot{x}} = -1$

b) Fixed Points: $\dot{x} = 0 = a - x + x^2 \cdot y$

$\dot{y} = 0 = b - x^2 \cdot y$

$(x^*, y^*) = \left(a+b, \frac{b}{(a+b)^2}\right)$

$$A = \begin{pmatrix} -1 + 2xy & x^2 \\ -2xy & -x^2 \end{pmatrix}$$

$$A_{(x^*, y^*)} = \begin{pmatrix} -1 + \frac{2b}{(a+b)} & (a+b)^2 \\ \frac{-2b}{(a+b)} & -(a+b)^2 \end{pmatrix}$$

$\Delta = (a+b)^2 - 2b(a+b)$; $\tau = -1 + \frac{2b}{(a+b)} - (a+b)^2$

$= a^2 - b^2$

If $a > b$, "stable spiral"

If $a < b$, "saddle point"

c) Bifurcations: $\tau = -1 + \frac{2b}{(a+b)} - (a+b)^2$

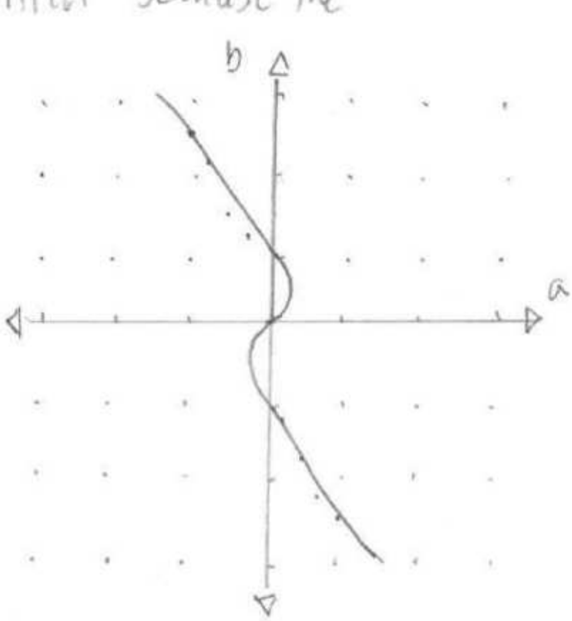
$$= \frac{-(a+b) + 2b - (a+b)^3}{(a+b)} = 0$$

$$b - a = (a+b)^3$$

d) The Hopf Bifurcation is subcritical because the center is stable.

e) Stability Diagram:

$$b = \frac{\sqrt[3]{\sqrt{3}\sqrt{27a^2-1} - 9a}}{3^{2/3}} + \frac{1}{\sqrt[3]{3}\sqrt[3]{\sqrt{3}\sqrt{27a^2-1} - 9a}} - a$$



If $x^* = (a+b)$, then

$$a = \frac{(a+b)}{2}\left(1 - (a+b)^2\right)$$

$$b = \frac{(a+b)}{2}\left(1 + (a+b)^2\right) \quad \text{and} \quad b - a = \frac{(a+b)}{2}\left[2(a+b)^2\right] = (a+b)^3$$

$$\dot{x} = a - x - \frac{4xy}{1+x^2}$$

$$\dot{y} = bx\left(1 - \frac{y}{1+x^2}\right)$$

8.3.3 Phase Plane:

$b \ll 1$;

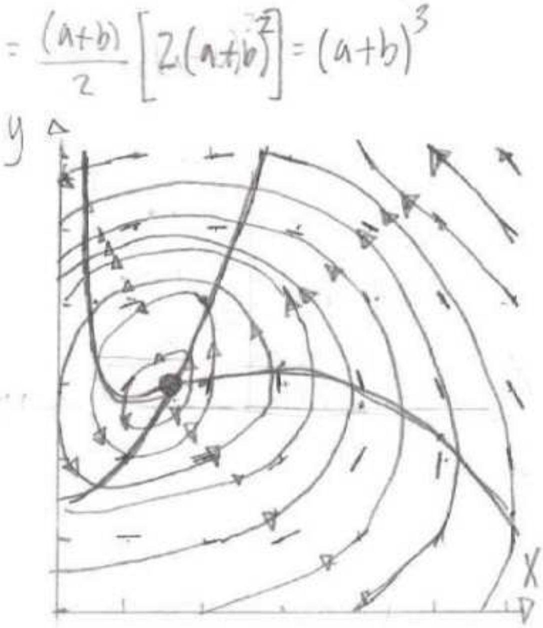
(8.3.7) $b < b_c = \frac{3a}{5} - \frac{25}{a}$

If $b = 0.5$, then $a = 6.8$

Nullclines:

$$y = 1 + x^2$$

$$y = \frac{(a-x)(1+x^2)}{4x}$$



Limit Cycle: $T = \int_{t_1}^{t_2} dt + \int_{t_2}^{t_3} dt + \int_{t_3}^{t_4} dt + \int_{t_4}^{t_1} dt$

where $\int_{t_2}^{t_3} dt = \int_{t_4}^{t_1} dt = 0$

$$T = \int_{t_1}^{t_2} dt + \int_{t_3}^{t_4} dt = \int_{t_1}^{t_2} \frac{dy}{dx}\frac{dt}{dy}\,dx + \int_{t_3}^{t_4} \frac{dy}{dx}\frac{dt}{dy}\,dx$$

$$= \int_{t_1}^{t_2} \frac{ax^2-a-2x^3}{4x^2} \frac{dx}{bx\left(1-\frac{a-x}{4x}\right)}$$

$$+ \int_{t_3}^{t_4} \frac{ax^2-a-2x^3}{4x^2} \cdot \frac{dx}{bx\left(1-\frac{a-x}{4x}\right)}$$

$$= \left.\frac{(3a^2-125)\,x\ln(5x-a) - 5a(2x^2+5) + 125x\ln(x)}{25abx}\right|_{t_1}^{t_2}$$

$$+ \left.\frac{(3a^2-125)\,x\ln(5x-a) - 5a(2x^2+5) + 125x\ln(x)}{25abx}\right|_{t_3}^{t_4}$$

$\dot{r} = r(1-r^2)$ 8.4.1 $\frac{dr}{dt} = r(1-r^2)$

$\dot{\theta} = \mu - \sin\theta$

$$t = \int \frac{dr}{r(1-r^2)} = \int \frac{A}{r}\,dr + \int \frac{B}{(1-r)}\,dr + \int \frac{C}{(1+r)}\,dr$$

$$A(1-r)(1+r) + Br(1+r) + Cr(1-r) = 1$$

$r=1;\ B = 1/2$

$r=-1;\ C = -1/2$

$r=0;\ A = 1$

$$= \int \frac{1}{r}\,dr + \int \frac{1}{(1-r)}\,dr - \int \frac{1}{(1+r)}\,dr = \ln r - \frac{\ln|1-r|}{2} - \frac{\ln|1+r|}{2}$$

$$= \ln \frac{r}{\sqrt{1-r^2}}$$

$$(1-r^2)e^{2t} = e^{2t} - r^2 e^{2t} = r^2 \;;\; -r^2 e^{2t} - r^2 + e^{t} = -r^2(e^{2t}+1) + e^{2t}$$

$$r = \frac{e^{t}}{\sqrt{e^{2t}+1}}$$

$$\frac{d\theta}{dt} = \mu - \sin\theta \;;\; t = \int \frac{d\theta}{\mu - \sin\theta} = \int \frac{d\theta}{\mu - \sin\theta} = \int \frac{d\theta}{\mu - \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}}}$$

$$= \int \frac{d\theta}{\mu - \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}}$$

$$\text{If } u = \tan\frac{\theta}{2} \;;\; \frac{du}{dx} = \frac{\sec^2\frac{\theta}{2}}{2} \;;\; dx = \frac{2\,du}{\sec^2\frac{\theta}{2}} = \frac{2}{u^2+1}\,du$$

$$= \int \frac{d}{\mu - \frac{2u}{1+u^2}} \circ \frac{2}{u^2+1}\,du = 2\int \frac{du}{\mu u^2 + \mu - 2u}$$

$$= 2\int \frac{du}{\mu\left(u - \frac{1}{\mu}\right)^2 - \frac{1}{\mu^2} + \mu}$$

$$\text{If } v = \frac{\mu u - 1}{\sqrt{\mu}\left(\sqrt{\mu - 1/\mu}\right)} \;;\; \frac{dv}{du} = \frac{\sqrt{\mu}}{\sqrt{\mu - 1/\mu}} \;;\; du = \frac{\sqrt{\mu - 1/\mu}}{\sqrt{\mu}}\,dv$$

$$= 2\int \frac{\sqrt{\mu - 1/\mu}}{\sqrt{\mu}\left(\left(\mu - \frac{1}{\mu}\right)v^2 + \mu - \frac{1}{\mu}\right)}\,dv$$

$$= \frac{2}{\sqrt{\mu}\sqrt{\mu - \frac{1}{\mu}}} \int \frac{1}{v^2+1}\,dv = \frac{\arctan(v)}{\sqrt{\mu}\sqrt{\mu - 1/\mu}}$$

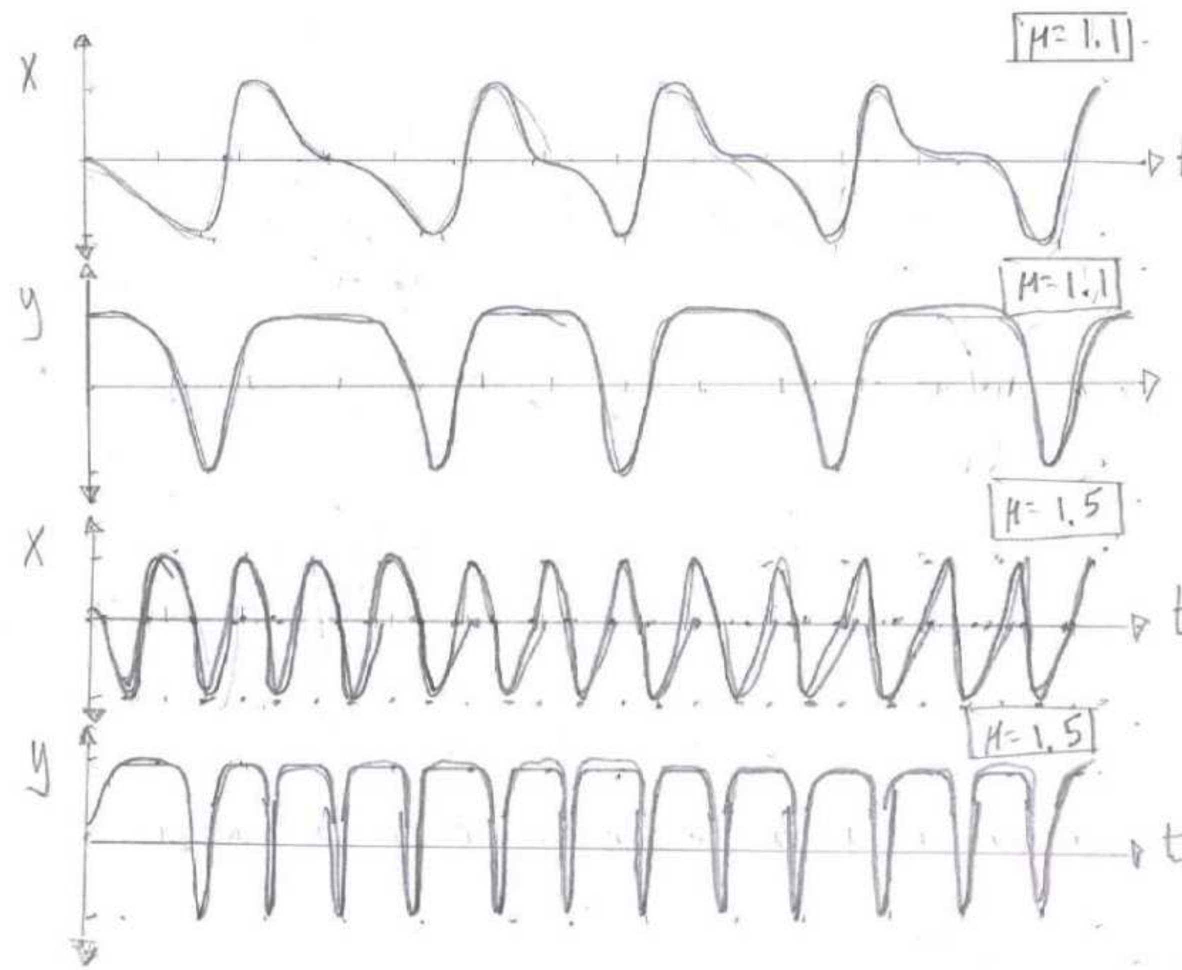
$$= \frac{2\arctan\left(\frac{\mu u - 1}{\sqrt{\mu}\sqrt{\mu - 1/\mu}}\right)}{\sqrt{\mu}\sqrt{\mu - 1/\mu}}$$

$$= \frac{2\arctan\left(\frac{\mu\tan\left(\frac{\theta}{2}\right) - 1}{\sqrt{\mu}\sqrt{\mu - 1/\mu}}\right)}{\sqrt{\mu}\sqrt{\mu - 1/\mu}}$$

$$\theta = 2\tan^{-1}\left(\frac{\sqrt{\mu}\tan\left(\frac{1}{2}\sqrt{\mu}\sqrt{\frac{\mu^2-1}{\mu}}\,t\right)}{\sqrt{\frac{\mu^2-1}{\mu}}} - \frac{\tan\left(\frac{1}{2}\sqrt{\mu}\sqrt{\frac{\mu^2-1}{\mu}}\,t\right)}{\mu^{3/2}\sqrt{\frac{\mu^2-1}{\mu}}} + \frac{1}{\mu}\right)$$

$$x(t) = r\cos\theta = \frac{e^{2t}}{\sqrt{e^{2t}+1}}\cos\left[2\tan^{-1}\left(\frac{\sqrt{\mu}\tan\left(\frac{1}{2}\sqrt{\mu}\sqrt{\frac{\mu^2-1}{\mu}}\,t\right)}{\sqrt{\frac{\mu^2-1}{\mu}}} - \frac{\tan\left(\frac{1}{2}\sqrt{\mu}\sqrt{\frac{\mu^2-1}{\mu}}\,t\right)}{\mu^{3/2}\sqrt{\frac{\mu^2-1}{\mu}}} + \frac{1}{\mu}\right)\right]$$

$$y(t) = r\sin\theta = \frac{e^{2t}}{\sqrt{e^{2t}+1}}\sin\left[2\tan^{-1}\left(\frac{\sqrt{\mu}\tan\left(\frac{1}{2}\sqrt{\mu}\sqrt{\frac{\mu^2-1}{\mu}}\,t\right)}{\sqrt{\frac{\mu^2-1}{\mu}}} - \frac{\tan\left(\frac{1}{2}\sqrt{\mu}\sqrt{\frac{\mu^2-1}{\mu}}\,t\right)}{\mu^{3/2}\sqrt{\frac{\mu^2-1}{\mu}}} + \frac{1}{\mu}\right)\right]$$



$\ddot{\theta}+(1-\mu\cos\theta)\dot{\theta}+\sin\theta=0$

8.4.4. $\dot{\phi}=\psi=$

$\dot{\psi}=-(1-\mu\cos\phi)\psi-\sin\phi$; $\dot{\psi}=0=-(1-\mu\cos\phi)\psi-\sin\psi$

$$\mu_c=\frac{\tan(\phi)}{\psi}$$

If $\mu<\mu_c$, Infinite period bifurcation.

If $\mu>\mu_c$, Infinite-period bifurcation.

If $\mu=\mu_c$, stable cycle.

$\ddot{x}+x+\epsilon(bx^3+k\dot{x}-ax-F\cos t)=0$ "Forced Duffing oscillator"

8.4.5. $$r'=\langle h\sin\theta\rangle=\frac{1}{2\pi}\int_0^{2\pi}[bx^3+k\dot{x}-ax-F\cos t]\sin\theta\,d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}(br^3\cos^3\theta-kr\sin\theta-ar\cos\theta-F\cos(\theta-\phi))\sin\theta\,d\theta$$

$$=\frac{1}{2\pi}\left[br^3\int_0^{2\pi}\cos^3\theta\sin\theta\,d\theta-kr\int_0^{2\pi}\sin^2\theta\,d\theta-ar\int_0^{2\pi}\cos\theta\sin\theta\,d\theta\right.$$
$$\left.-F\int_0^{2\pi}\cos(\theta-\phi)\sin\theta\,d\theta\right]$$

$$=\frac{1}{2\pi}\left[-br^3\int_0^{2\pi}u^3du-kr\int_0^{2\pi}\frac{1-\cos u}{4}du+ar\int_0^{}u\,du\right.$$
$$\left.-\frac{F}{2}\int_0^{2\pi}\sin(2\theta-\phi)+\sin\phi\,d\theta\right]$$

$$=\frac{1}{2\pi}\left[-\frac{br^3\cos^4\theta}{4}\Big|_0^{2\pi}-\frac{kr(2\theta)}{4}\Big|_0^{2\pi}+\frac{kr\sin2\theta}{4}\Big|_0^{2\pi}+ar\sin\theta\Big|_0^{2\pi}\right.$$
$$\left.-\frac{F}{2}\left[\theta\sin\phi\Big|_0^{2\pi}-\frac{\cos(2\theta-\phi)}{2}\Big|_0^{2\pi}\right]\right]$$

$$=\frac{1}{2\pi}\left[-kr\pi-\frac{F}{2}\left[2\pi\sin\phi-\frac{\cos(4\pi-\phi)}{2}+\frac{\cos(\phi)}{2}\right]\right]$$

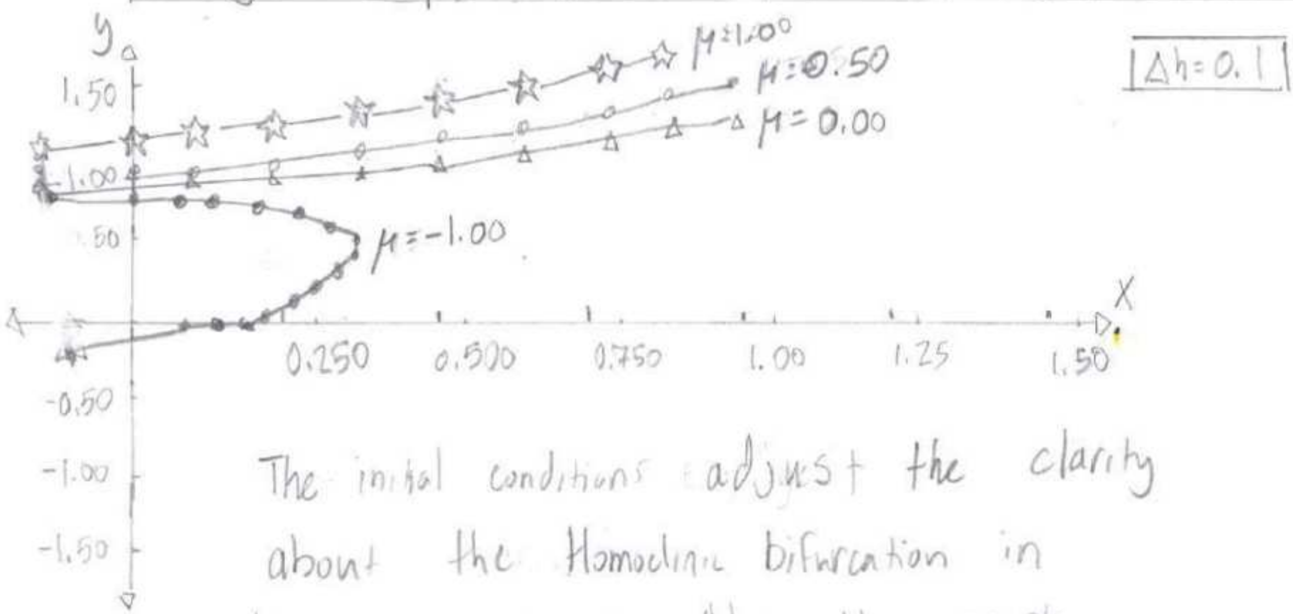
$$=\frac{-kr-F\sin\phi}{2}$$

$\dot{r} = r(\mu - \sin r)$ 8.4.2. μ describes the frequency of radial nodes, about an
$\dot{\theta} = 1$ infinite-period bifurcation. When $|\mu| > 1$, no nodes appear in the graph.

$\boxed{\Delta h = 0.1}$

$\dot{x} = \mu x + y - x^2$ 8.4.3.
$\dot{y} = -x + \mu y + 2x^2$

X_1 X_0	0
y_0	0
KX_1	$f(X, y, t)$
Ky_1	$g(X, y, t)$
KX_2	$f(X_n + \Delta h \frac{KX_1}{2}, y_n + \Delta h \frac{Ky_1}{2}, t_n)$
Ky_2	$g(X_n + \Delta h \frac{KX_1}{2}, y_n + \Delta h \frac{Ky_1}{2}, t_n)$
KX_3	$f(X_n + \Delta h \frac{KX_2}{2}, y_n + \Delta h \frac{Ky_2}{2}, t_n)$
Ky_3	$g(X_n + \Delta h \frac{KX_2}{2}, y_n + \Delta h \frac{Ky_2}{2}, t_n)$
KX_4	$f(X_n + \Delta h KX_3, y_n + \Delta h Ky_3, t_n)$
Ky_4	$g(X_n + \Delta h KX_3, y_n + \Delta h Ky_3, t_n)$
X	$X_{n+1} = X_n + \frac{\Delta h}{6}(KX_1 + 2KX_2 + 2KX_3 + KX_4)$
y	$y_{n+1} = y_n + \frac{\Delta h}{6}(Ky_1 + 2Ky_2 + 2Ky_3 + Ky_4)$



The initial conditions adjust the clarity about the Homoclinic bifurcation in Numerical Integration. Above, the graph begins at $(x = -0.1, y = -0.1)$, with an initial large step around the orbit.

$$r\phi' = \langle h\cos\theta\rangle = \frac{1}{2\pi}\int_0^{2\pi}\left(br^3\cos^3\theta - Rr\sin\theta - ar\cos\theta - F\cos(\theta-\phi)\right)\cos\theta\,d\theta$$

$$= \frac{1}{2\pi}\left[br^3\int_0^{2\pi}\cos^4\theta\,d\theta - kr\int_0^{2\pi}\sin\theta\cos\theta\,d\theta - ar\int_0^{2\pi}\cos^2\theta\,d\theta - F\int_0^{2\pi}\cos(\theta-\phi)\cos\theta\,d\theta\right]$$

$$= \int_0^{2\pi}\cos^4\theta\,d\theta = \int_0^{2\pi}\cos^2\theta - \cos^2\theta\sin^2\theta\,d\theta$$

$$= \int_0^{2\pi}\frac{1+\cos 2\theta}{2}\,d\theta - \int_0^{2\pi}\frac{\sin^2 2\theta}{4}\,d\theta$$

$$= \frac{\theta}{2} + \frac{\sin 2\theta}{4}\Big|_0^{2\pi} - \int_0^{2\pi}\frac{1-\cos 4\theta}{8}\,d\theta$$

$$= \pi - \frac{\theta}{8} + \frac{\sin 4\theta}{16}\Big|_0^{2\pi} = \pi - \frac{\pi}{4} = \frac{3}{4}\pi$$

$$\int_0^{2\pi}\cos(\theta-\phi)\cos\theta\,d\theta = \int_0^{2\pi}\frac{\cos(-\phi)+\cos(2\theta-\phi)}{2}\,d\theta$$

$$= \cos(-\phi)\cdot\pi$$

$$= \frac{1}{2\pi}\left[br^3\left(\frac{3}{4}\pi\right) - ar(\pi) - F\cos(\phi)\pi\right]$$

$$= \frac{3br^3}{8} - \frac{ar}{2} - \frac{F\cos(\phi)}{2} = \frac{3br^3 - 4ar - 4F\cos\phi}{8}$$

$$\phi' = \frac{3br^3 - 4ar - 4F\cos\phi}{8r}$$

8.4.6.

Averaged Equations: $r' = -\frac{1}{2}(kr + F\sin\phi)$

$$\phi' = -\frac{1}{8}\left(4a - 3br^2 + \frac{4F}{r}\cos\phi\right)$$

Fixed Points: $r' = 0 = -\frac{1}{2}(kr + F\sin\phi)$

$$\phi' = 0 = -\frac{1}{8}\left(4a - 3br^2 + \frac{4F}{r}\cos\phi\right)$$

$$(r^*, \phi^*) = \left(\frac{F}{\sqrt{k^2 + (\frac{3}{4}br^2 - a)^2}}, 2\pi n\right) \text{ where } n \in \mathbb{Z}$$

Phase-locked periodic solution correspondence:

The polar fixed points represent a closed orbit every 2π angles with radial ratios of $\frac{F}{\sqrt{k^2 + (\frac{3}{4}br^2 - a)^2}}$. The value is a solution to a forced duffing oscillator because the derivation was from the oscillator equation

$$\nabla \cdot x' = \frac{1}{r}\frac{\partial}{\partial r}(rr') + \frac{1}{r}\frac{\partial}{\partial \phi}(r\phi')$$

8.4.7. Dulac's criterion $g(r,\phi) \equiv 1$; $x' = (r', r\phi')$

$$\nabla \cdot x' = \frac{1}{r}\frac{\partial}{\partial r}(rr') + \frac{1}{r}\frac{\partial}{\partial \phi}(r\phi')$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{-r}{2}(kr + F\sin\phi)\right] + \frac{1}{r}\frac{\partial}{\partial \phi}\left[-\frac{1}{2}ra - \frac{3br^2}{8} + 4F\cos\phi\right]$$

$$= -k - \frac{F\sin\phi}{2r} + \frac{F\sin\phi}{2r} = -k$$

So, $\nabla \cdot (gx') < 0$ because $k > 0$, and no closed orbits exist in the averaged system

8.4.8. A sink or saddle-node are the bifurcations. With the divergence from Dulac's criterion being negative, the slope points inward.

$r^2\left[k^2+\left(\frac{3}{4}br^2-a\right)^2\right]=F^2$

8.4.9. $r'=\frac{1}{2}(kr+F\sin\phi)$ $\phi'=-\frac{1}{8}\left(4a-3br^2+\frac{4F}{r}\cos\phi\right)$

$r'=0=\frac{1}{2}(kr+F\sin\phi)$; $-kr=-F\sin\phi$

$\cos\phi=\sqrt{1-\frac{k^2r^2}{F^2}}$

$\phi'=0=-\frac{1}{8}\left(4a-3br^2+\frac{4F}{r}\cos\phi\right)$

$3br^3-4ar=4F\cos(\phi)$

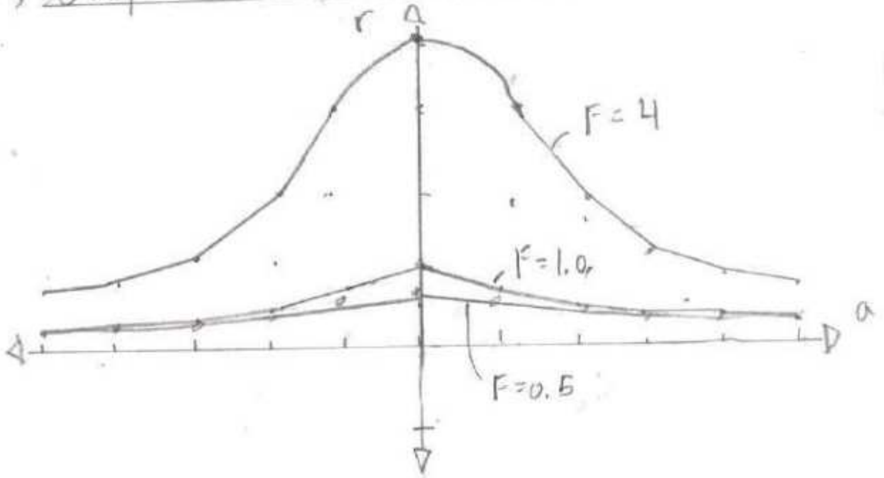
$(3br^3-4ar)^2=16F^2\cos^2(\phi)=16F^2\left(1-\frac{k^2r^2}{F^2}\right)$

$=16F^2-16k^2r^2$

$r^2\left[k^2+\left(\frac{3}{4}br^2-a\right)^2\right]=F^2$

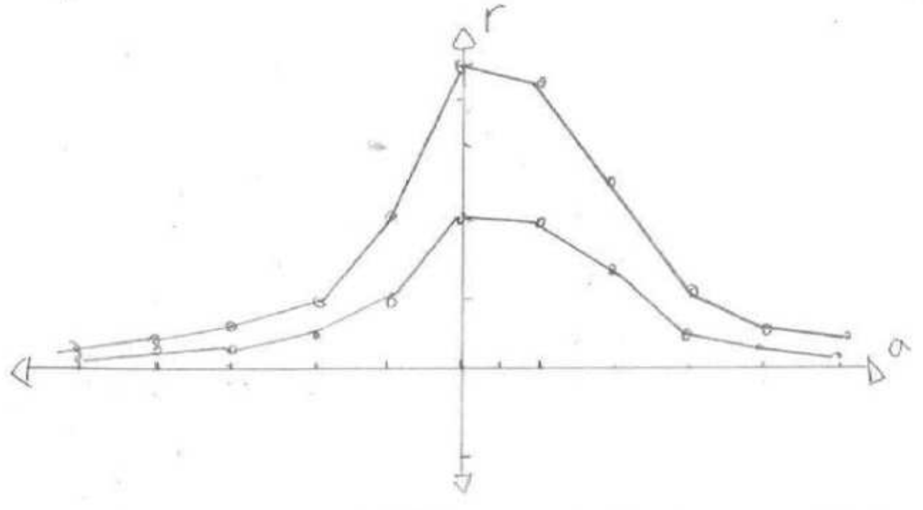
b) Graph of r vs. a at b=0: $r=\frac{F}{\sqrt{k^2+a^2}}$

$\boxed{k=1}$



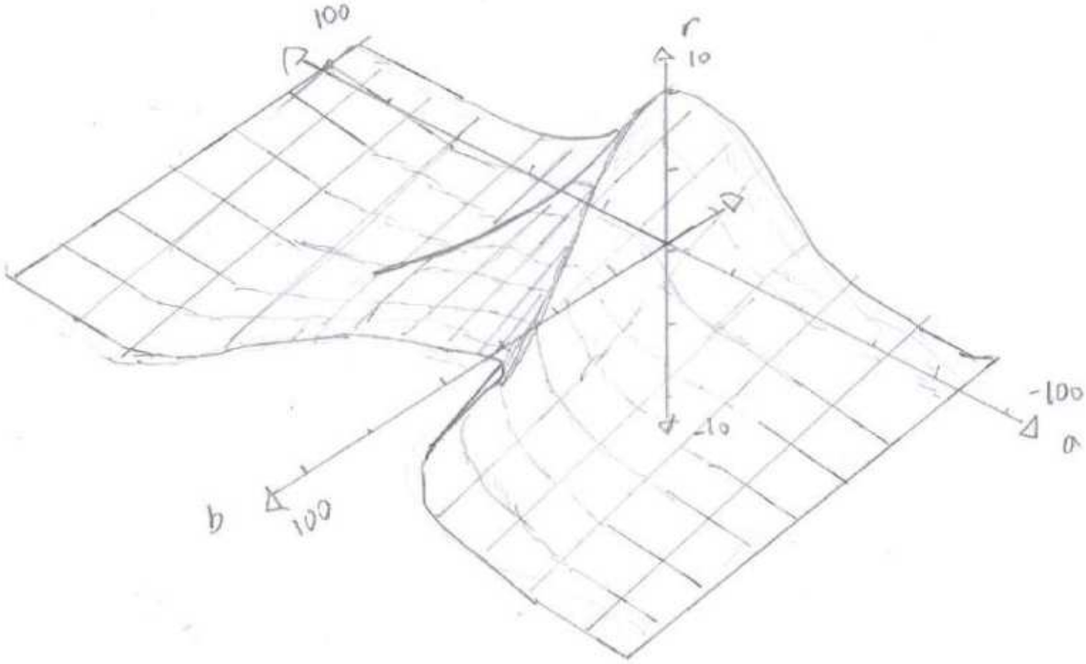
c. Graph of r vs. a at b ≠ 0: $r=\frac{F}{\sqrt{k^2+\left(\frac{3}{4}br^2-a\right)^2}}$

$\boxed{\begin{array}{l}k=1\\ b=1\end{array}}$



$$b_c = \frac{4\left(ar^4 \pm \sqrt{r^6(F-k^2r^2)}\right)}{3r^6}$$

d) Plot of (a,b) plane:

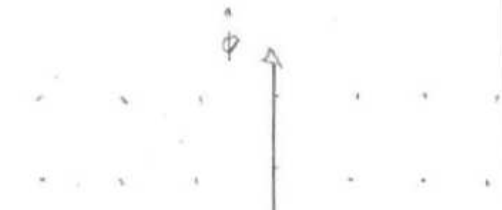


0.4.10.

a. $r' = 0 = -\frac{1}{2}(Kr + F\sin\phi)$; $\phi = \arcsin\left[\frac{rK}{F}\right]$

$\phi' = 0 = -\frac{1}{8}\left(4a - 3br^2 + \frac{4F}{r}\cos\phi\right)$; $\phi = \arccos\left[\frac{3br^3 - 4ar}{4F}\right]$

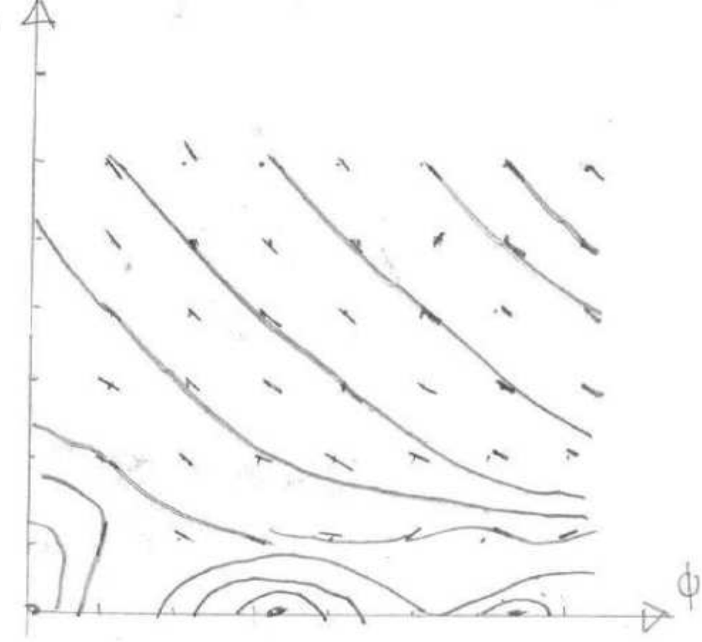
b.



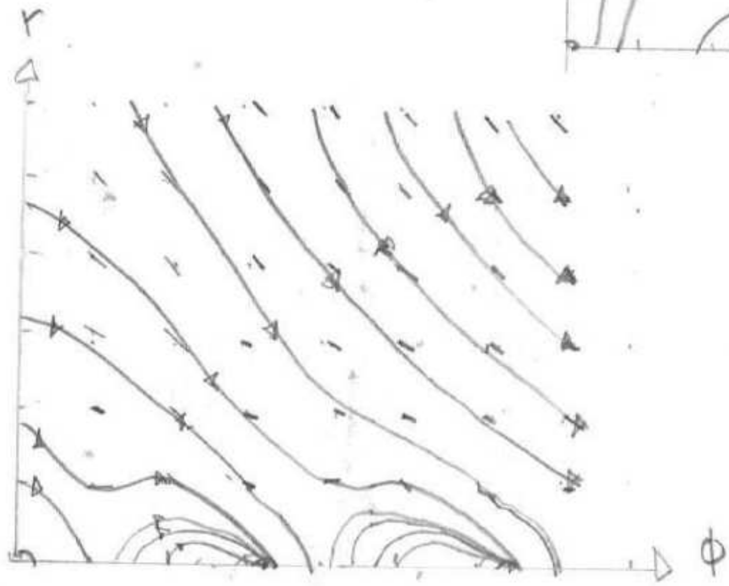
0.4.11 If $k=1$, $b=\frac{4}{3}$, $F=2$

a)

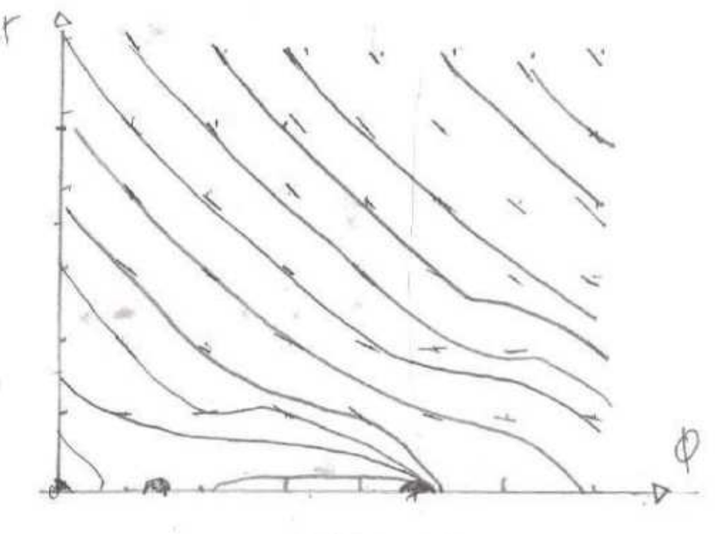
$\boxed{a=-1}$



$\boxed{a=0}$



$\boxed{a=1}$



$\boxed{a=2.5}$

b. Fixed Points:

$$r' = 0 = -\frac{1}{2}\left(kr + F\sin\phi\right)$$

$$\phi' = 0 = -\frac{1}{8}\left(4a - 3br^2 + \frac{4F}{r}\cos\phi\right)$$

$$\phi = 2\pi n \pm \cos^{-1}\left(\frac{1}{160} r\left(45r^2 - 224\right)\right)$$

C. Duffing Equation:

$$\ddot{x} + x + \varepsilon\left(bx^3 + k\dot{x} - ax - F\cos t\right) = 0$$

$$\ddot{x} = -x - \varepsilon\left(bx^3 + k\dot{x} - ax - F\cos t\right)$$

$$\dot{u} = \dot{x} = v$$

$$\dot{v} = \ddot{x} = -u - \varepsilon\left(bu^3 + k\cdot v - a\cdot u - F\cos t\right)$$

~or~

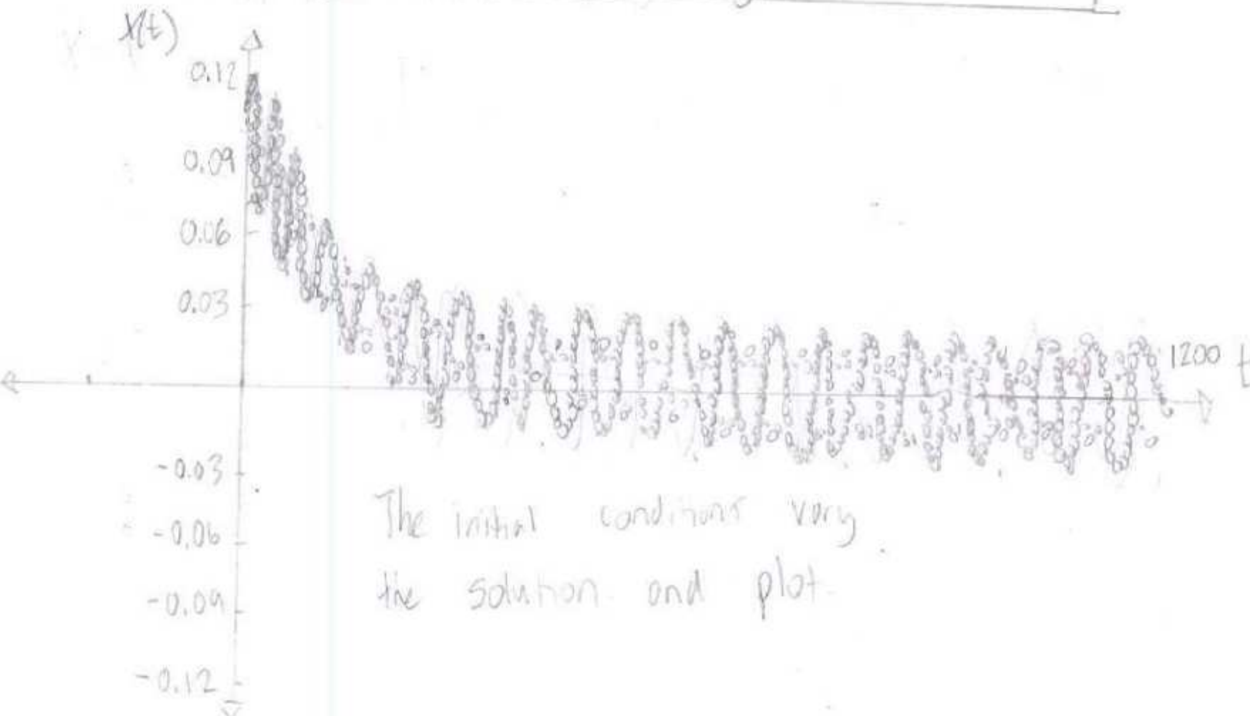
$$\mathring{u} = f(X, t)$$

$$\mathring{v} = g(u, \mathring{u}, t) = -u - \varepsilon(bu^3 + k \circ f(X,t) - a \cdot u - F\cos t)$$

$$= -u(x) - \varepsilon(b u^3(x) + k \circ f(X,t) - a \circ u(x) - F\cos t)$$

$t =$

Term	Function
u	X
$\mathring{u}$	$f(u,t) = f(X,t)$
k_{1f}	$f(u,t) = f(X,t)$
k_{2f}	$f(u + \Delta h \frac{k_{1f}}{2}, t + \frac{\Delta h}{2}) = f(X + \Delta h \frac{k_{1f}}{2}, t + \frac{\Delta h}{2})$
k_{3f}	$f(u + \Delta h \frac{k_{2f}}{2}, t + \frac{\Delta h}{2}) = f(X + \Delta h \frac{k_{2f}}{2}, t + \frac{\Delta h}{2})$
k_{4f}	$f(u + \Delta h k_{3f}, t + \Delta h) = f(X + \Delta h k_{3f}, t + \Delta h)$
$\mathring{v}$	$g(u, \mathring{u}, t) = g(X, f(X,t), t)$
k_{1g}	$g(u, \mathring{u}, t) = g(X, f(X,t), t)$
k_{2g}	$g(u + \Delta h \frac{k_{1g}}{2}, f(u,t) + \frac{\Delta h k_{1g}}{2}, t + \Delta h/2)$ $= g(X + \Delta h \frac{k_{1g}}{2}, f(X,t) + \Delta h \frac{k_{1g}}{2}, t + \Delta h/2)$
k_{3g}	$g(u + \Delta h \frac{k_{2g}}{2}, f(u,t) + \Delta h \frac{k_{2g}}{2}, t + \Delta h/2)$ $= g(X + \Delta h \frac{k_{2g}}{2}, f(X,t) + \Delta h \frac{k_{2g}}{2}, t + \Delta h/2)$
k_{4g}	$g(u + \Delta h k_{3g}, f(u,t) + \Delta h \cdot k_{3g}, t + \Delta h/2$ $= g(X + \Delta h k_{3g}, f(X,t) + \Delta h k_{3g}, t + \Delta h/2$



The initial conditions vary the solution and plot

8.4.12. $\dot{x} \approx \lambda_u x$; $\dot{y} \approx -\lambda_s y$; $(\mu, 1)$ where $\mu \ll 1$

$$t\lambda = \ln x + C_1 ; \quad t\lambda = \ln y + C_2$$

$$x(t) = C_1 e^{+\lambda t} \quad ; \quad y(t) = C_2 e^{-\lambda t}$$

$$x(0) = \mu = C_1 \qquad y(0) = 1 = C_2$$

$$t = \frac{\ln \frac{x(t)}{\mu}}{\lambda} \qquad t = \frac{\ln y(t)}{-\lambda}$$

$$t = -\frac{\ln \mu}{\lambda}$$

$\ln(I-I_c)^{-1}$

8.5.1. If $f^{(n)}(I) = \left(\frac{d}{dI}\right)^n \ln(I-I_c)^{-1}$

$$f'(I) = \frac{d}{dI} \ln(I-I_c)^{-1} = \frac{-1}{(I-I_c)(\ln(I-I_c))^2}$$

$$f''(I) = \frac{d^2}{dI^3} \ln(I-I_c)^{-1} = \frac{2}{(I-I_c)^2(\ln(I-I_c))^3} + \frac{1}{(I-I_c)^2(\ln(I-I_c))^2}$$

$$f'''(I) = \frac{d^3}{dI^3} \ln(I-I_c)^{-1} = \frac{-6}{(I-I_c)^3\ln(I-I_c)^4} + \frac{-6}{(I-I_c)^3\ln(I-I_c)^3} + \frac{-2}{(I-I_c)^3\ln(I-I_c)^2}$$

$$f^{(1)} = \frac{-f}{I-I_c} ; \quad f^{(2)} = \frac{2f}{(I-I_c)^2} + \frac{f^2}{(I-I_c)^2}$$

$$f^{(3)} = \frac{-6f^4 - 6f^3 - 2f}{(I-I_c)^3}$$

$$f^{(n)} = \frac{\sum_{k=2}^{n+1} (-1)^n n! f^k}{(I-I_c)^n}$$

$\phi'' + \alpha\phi' + \sin\phi = I$

8.5.2. $\dot{u} = \phi'$

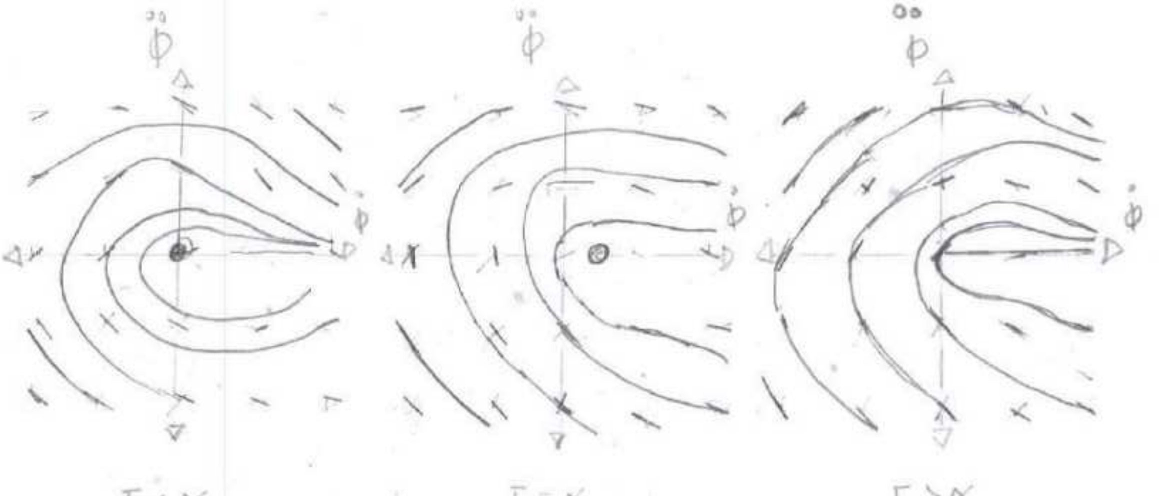
$$\dot{v} = \phi'' = -\alpha v + \sin u + I$$

Fixed points: $\dot{u} = 0$

$$\dot{v} = 0 = -\alpha v + \sin u + I$$

$$(u^*, v^*) = (0, I/v)$$

Phase Portrait:



$\dot{N}=rN(1-N/K(t))$

8.5.3.

a. Poincaré map: $-\frac{\dot{N}}{N(t)^2}+\frac{r}{N(t)^2}=\frac{r}{K(t)}$

If $X=\frac{1}{N(t)}$; $\dot{X}=\frac{-1}{N(t)^2}$

then, $\dot{X}+rX=\frac{r}{K(t)}$

Integrating factor: e^{rt}

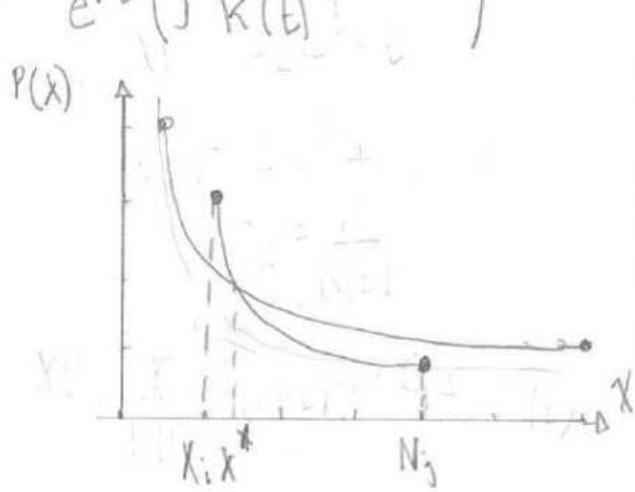
$$\dot{X}e^{rt}+r\cdot Xe^{rt}=\frac{re^{rt}}{K(t)}$$

$$\frac{d}{dt}\left(e^{rt}\cdot X\right)=\frac{re^{rt}}{K(t)}$$

$$X=\frac{1}{e^{rt}}\left(\int\frac{re^{-rt}}{K(t)}dt+C\right)$$

Similar Solutions: $t=t+T$

b. The solution is unique because $\lim_{t\to\infty}X(t)=$ constant.



$\dot{x} = rx(1-x) - h(1+\alpha \sin t)$; $r, h > 0$ and $0 < \alpha < 1$

8.5.4

a. Solution to $\dot{x} = rx(1-x) - h(1+x\sin(t))$

is periodic if $x(t) = x(t+T)$ for all t.

If t ranges from zero to one, then

$$x(1) - x(0) = \int_0^1 rx(1-x) - h(1+\alpha \sin t)\, dt$$

$$\leq \frac{r}{4} - h$$

So, if $h > \frac{r}{4}$, then $x(n+1) - x(n) < 0$ and $x(t)$ is divergent.

b. If $h < \frac{r}{4(1+\alpha)}$, then $\dot{x} > r\left[x(1-x) - \frac{h}{r}(1+\alpha \sin t)\right]$

$$> r\left[x(1-x) - \frac{1}{4}(1+\alpha \sin t)\right]$$

$$> 0$$

$x > 1/2$ when $t = n\pi$

$x < 1$ when $t = n\pi$ "stable limit"

When $0 < x < 1/2$, then $\dot{x} < 0$ and divergent, such as an unstable limit cycle.

Biological systems with a stable limit cycle survive, while unstable diverge to zero populations.

c. If $\frac{r}{4(1+\alpha)} < h < \frac{r}{4}$, then zero, one, or two periodic solutions appear in the data.

$\ddot{\theta} + \kappa\dot{\theta}|\dot{\theta}| + \sin\theta = F$

8.5.5, $\alpha > 0$ and $F > 0$

a. $\ddot{\theta} + \alpha v|v| + \sin\theta = F$ where $v = \dot{\theta}$

$$\begin{bmatrix} \dot{\theta} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} -\cos\theta & -2\alpha|v| \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ v \end{bmatrix}$$

Fixed Points: $\dot{\theta} = 0 = V$

$\dot{V} = 0 = F - \kappa V|V| - \sin\theta$

$(\ddot{\theta}, \dot{V}) = (\arcsin(F), 0); (\pi - \arcsin(F), 0)$

$$A_{(\arcsin(F),0)} = \begin{bmatrix} -\cos(\arcsin(F)) & \cancel{0} \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda = \begin{bmatrix} -\cos(\arcsin(F)) - \lambda & 0 \\ 0 & 1 - \lambda \end{bmatrix}$$

$$= [-\cos(\arcsin(F)) - \lambda][1 - \lambda] = 0$$

$$\lambda_1 = i(1-F^2)^{1/4} ; \lambda_2 = -i(1-F^2)^{1/4}$$

$$\Delta = (1-F^2)^{1/2} > 0 ; \tau = 0 ; \tau^2 - 4\Delta < 0$$

"Center"

$$A_{(\pi - \arcsin(F), 0)} = \begin{bmatrix} -\cos(\pi - \arcsin(F)) & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda = \begin{bmatrix} -\cos(\pi - \arcsin(F)) - \lambda & 0 \\ 0 & 1 - \lambda \end{bmatrix}$$

$$= (-\cos(\pi - \arcsin(F)) - \lambda)(1 - \lambda) = 0$$

$$\lambda_1 = 1 ; \lambda_2 = +\cos(\pi - \arcsin(F)) = \sqrt{1-F^2}$$

$\Delta = \sqrt{1-F^2} > 0$; $\tau = 1 + \sqrt{1-F^2}$; $\tau^2 - 4\Delta > 0$ "unstable node"

$$V(\theta, v) = KE + PE$$
$$= \int \dot{\theta}\, dv - \int \dot{v}\, d\theta$$
$$= \frac{1}{2} v^2 - F\theta - \cos(\theta) + F \arcsin(F) + \sqrt{1 - F^2}$$

$$\dot{V}(\theta, v) = v\dot{v} - F\dot{\theta} + \sin(\theta)\dot{\theta}$$
$$= v(F - \alpha v|v| - \sin(\theta)) - Fv + \sin(\theta) v$$
$$= -\alpha v^2 |v|$$

When $v = 0$, then $\dot{V} = 0$ and the center is a Liapunov function.

b) A stable limit cycle appears when $F > 1$.

Fixed Points: $\dot{v} = F - \alpha v|v| - \sin(\theta) \geq 0$

$$F - \sin(\theta) \geq \alpha v|v|$$

where $\sin(\theta)$ oscillates between -1 to 1

Limit Cycle: $F - \sin(\theta) \geq F - 1 \geq \alpha v|v|$

and

$$F - \sin(\theta) \geq F + 1 \geq \alpha v|v|$$
$$\sqrt{\frac{F-1}{\alpha}} \leq v \leq \sqrt{\frac{F+1}{\alpha}}$$

Uniqueness: A second limit cycle justification is a contradiction to uniqueness. The minimum and maximum range are singular, positive, and unique.

c. When $u = \frac{1}{2}v^2$, $\frac{du}{d\theta} = \frac{1}{2}\frac{d}{d\theta}v^2(t(\theta))$

$$= v\mathring{v}\cdot\frac{dt}{d\theta}$$

$$= \mathring{v}$$

$\frac{du}{d\theta} + 2\alpha u + \sin\theta = F$

d. $\frac{du}{d\theta} = F - 2\alpha u - \sin\theta \leq 0$

$$u \geq \frac{F - \sin\theta}{2\alpha}$$

Limit Cycle: $\frac{F-\sin\theta}{2\alpha} \geq \frac{F-1}{2\alpha} \geq u$

and

$$\frac{F-\sin\theta}{2\alpha} \geq \frac{F+1}{2\alpha} \geq u$$

$$\frac{F+1}{2\alpha} \geq u \geq \frac{F-1}{2\alpha}$$

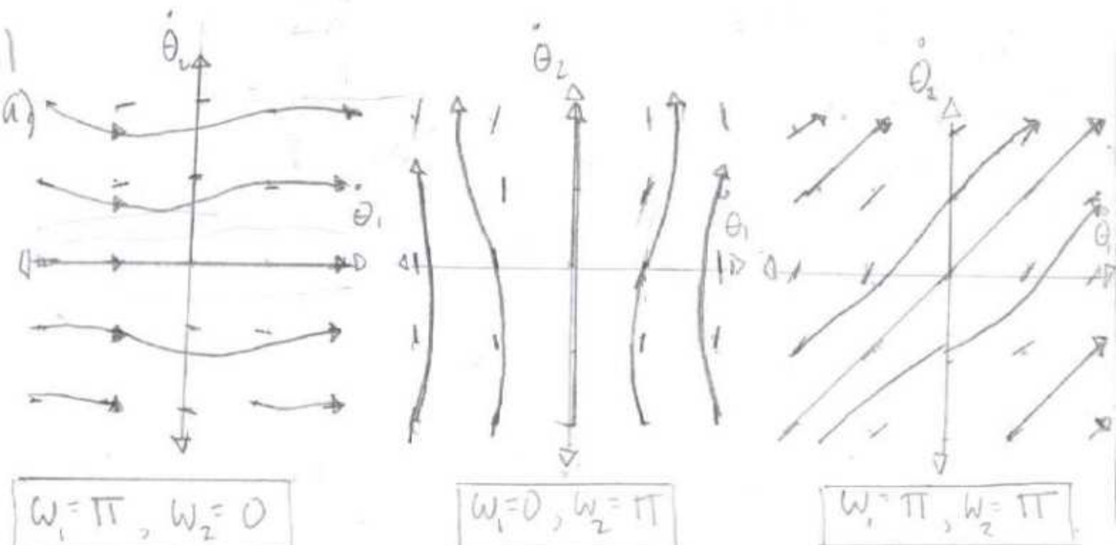
e. When $u = \frac{1}{2}v^2$, then the bifurcation occurs at $u=0$. with the limit cycle $\frac{F+1}{2\alpha} \geq 0 \geq \frac{F-1}{2\alpha}$ and bifurcation solution $F = 2\alpha$.

$\mathring{\theta}_1 = \omega_1 + \sin\theta_1\cos\theta_2$

$\dot{\theta}_2 = \omega_2 + \sin\theta_2\cos\theta_1$

8.6.1

a)



$\omega_1 = \pi$, $\omega_2 = 0$ | $\omega_1 = 0$, $\omega_2 = \pi$ | $\omega_1 = \pi$, $\omega_2 = \pi$

b) $\dot{\phi}_1 = \dot{\theta}_1 - \dot{\theta}_2 = \omega_1 - \omega_2 + \sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1$

$= \omega_1 - \omega_2 + \sin(\theta_1 - \theta_2)$

$= \omega_1 - \omega_2 + \sin\phi_1$

Fixed Points: $\dot{\phi}_1 = 0 = \omega_1 - \omega_2 + \sin\phi_1$

$\phi_1^* = 2\pi - \arcsin(\omega_1 - \omega_2)$

$\dot{\phi}_2 = 0 = \omega_1 + \omega_2 + \sin\phi_2$

$\phi_2^* = 2\pi - \arcsin(\omega_1 + \omega_2)$

Eigenvalues: $\begin{bmatrix}\dot{\phi}_1 \\ \dot{\phi}_2\end{bmatrix} = \begin{bmatrix}\cos\phi_1 & 0 \\ 0 & \cos\phi_2\end{bmatrix}\begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix}$

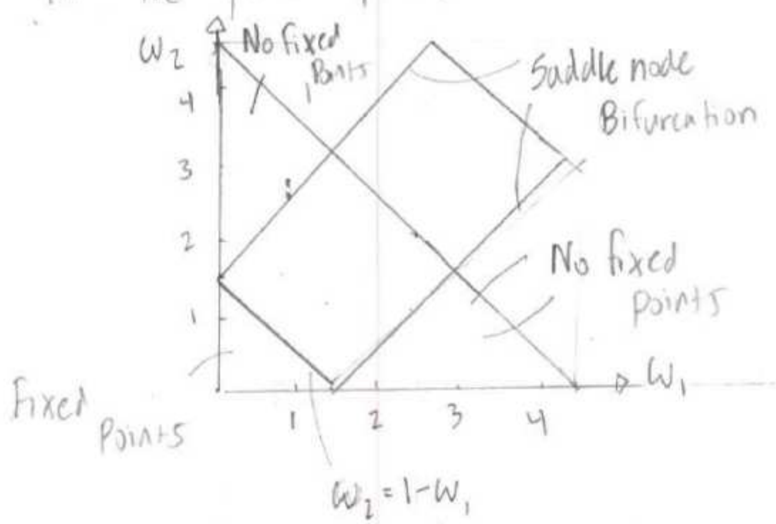
$A - \lambda = (\cos\phi_1 - \lambda)(\cos\phi_2 - \lambda) = 0$

$\lambda_1 = \cos\phi_1$; $\lambda_2 = \cos\phi_2$

$\Delta = \cos\phi_1\cos\phi_2$; $\tau = \cos\phi_1 + \cos\phi_2$; $\tau^2 - 4\Delta = (+)/(-)$

If $\omega_1 + \omega_2 = 1$, then saddle node bifurcation or $\omega_1 - \omega_2 = 1$ an infinite-period bifurcation appears in the phase plot.

c) Parameter Space:



$\dot{\theta}_1 = \omega_1 + K_1\sin(\theta_2 - \theta_1)$
$\dot{\theta}_2 = \omega_2 + K_2\sin(\theta_1 - \theta_2)$

8.6.2

a) Fixed Points: $\dot{\phi}_1 = 0 = \dot{\theta}_1 - \dot{\theta}_2$

$= \omega_1 - \omega_2 - (K_1 + K_2)\sin(\phi)$

$$\phi_1^* = \arcsin \frac{\omega_1 - \omega_2}{K_1 + K_2}$$

Although, the difference has a fixed point, the individual phases lack a positive frequency which a fixed point yields.

$$\dot{\phi}_1 = 0 = \dot{\theta}_1 - \dot{\theta}_2 ; \ \dot{\theta}_1 = \theta_2 = \omega_2 + K_2 \sin\phi^\circ$$

$$= \omega_2 + K_2 \frac{(\omega_1 - \omega_2)}{K_1 + K_2}$$

$$= \frac{K_1 \omega_2 + K_2 \omega_2}{K_1 + K_2}$$

The comprimise frequency, ω^*, is not a frequency at zero.

b. $\dot{\phi}_1 = \dot{\theta}_1 - \dot{\theta}_2 ; \ \sin(\theta_1 - \theta_2) = \dfrac{\omega_1 - \omega_2}{K_1 + K_2}$

$\dot{\phi}_2 = \dot{\theta}_1 + \dot{\theta}_2 ; \ \sin(\theta_1 + \theta_2) = \dfrac{\omega_2 - \omega_1}{K_1 + K_2}$

c) If $K_1 = K_2$, then

$\dot{\theta}_1 = \dfrac{d\theta}{dt} = \omega_1 + K \sin(\theta_2 - \theta_1)$ where $\tau = \omega_1 t$ and $a = \dfrac{K}{\omega_1}$

$$d\theta_1 / d\tau = 1 + a \sin(\theta_2 - \theta_1)$$

and

$d\theta_2 / d\tau = \dfrac{\omega_2}{\omega_1} + a \sin(\theta_1 - \theta_2)$ where $\omega = \dfrac{\omega_2}{\omega_1}$

$$= \omega + a \sin(\theta_1 - \theta_2)$$

d. Winding Number $\lim_{\tau \to \infty} \theta_1(\tau)/\theta_2(\tau)$

$$\langle d(\theta_1+\theta_2)/d\tau \rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T d(\theta_1+\theta_2)/d\tau \, d\tau$$

$$= \lim_{T\to\infty} \frac{1}{T}\sum_0^T 1 + a\sin(\theta_2-\theta_1) + \omega + a\sin(\theta_1-\theta_2)$$

$$= 0$$

$$\langle d(\theta_1-\theta_2)/d\tau \rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T d(\theta_1-\theta_2)/d\tau \, d\tau$$

$$= \lim_{T\to\infty} \frac{1}{T}\sum_0^T (1-\omega) - 2a\sin(\theta_2-\theta_1)$$

$$= 0$$

So the limit is also zero

$\dot{\theta}_1 = \omega_1$

$\dot{\theta}_2 = \omega_2$

8.6.3. $\frac{\omega_1}{\omega_2} \in \mathbb{P}$; $\frac{\omega_1}{\omega_2} = \frac{\dot{\theta}_1}{\dot{\theta}_2}$; Intersection $= \left| \frac{\dot{\theta}_1}{\dot{\theta}_2} - \varepsilon \right| = \frac{p}{q}$

$\dot{\theta}_1 = E - \sin\theta_1 + K\sin(\theta_2-\theta_1)$

$\dot{\theta}_2 = E + \sin\theta_2 + K\sin(\theta_1-\theta_2)$

8.6.4

a) <u>Fixed Points:</u> $\dot{\phi}_1 = \dot{\theta}_1 - \dot{\theta}_2 = \sin\theta_2 - \sin\theta_1 - 2K\sin(\theta_1-\theta_2)$

$= 0$

$\dot{\phi}_2 = \dot{\theta}_1 + \dot{\theta}_2 = 2E + \sin\theta_1 + \sin\theta_2 = 0$

$(\theta_1, \theta_2) = (n\pi, m\pi) \quad n, m \in \mathbb{R}$

When $E = 0$.

<u>Bifurcations:</u>
$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} -\cos\theta_1 - K\cos(\theta_2-\theta_1) & K\cos(\theta_2-\theta_1) \\ -K\cos(\theta_1-\theta_2) & \cos\theta_2 - K\cos(\theta_1-\theta_2) \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_1 \end{bmatrix}$$

$$A-\lambda = \begin{bmatrix} -\cos\theta_1 - K\cos(\theta_2-\theta_1) - \lambda & K\cos(\theta_2-\theta_1) \\ -K\cos(\theta_1-\theta_2) & \cos\theta_2 - K\cos(\theta_1-\theta_2) - \lambda \end{bmatrix}$$

$$= (-\cos\theta_1 - K\cos(\theta_2-\theta_1)-\lambda)(\cos\theta_2 - K\cos(\theta_1-\theta_2)-\lambda)$$
$$+ K^2\cos(\theta_2-\theta_1)\cos(\theta_1-\theta_2)$$

$$= 0$$

$$\lambda_1 = \frac{1}{2}\left(-\sqrt{-4K^2\cos^2(\theta_1-\theta_2) + 2\cos(\theta_1)\cos(\theta_2) + \cos^2\theta_1 + \cos^2\theta_2}\right.$$
$$\left. - 2K\cos(\theta_1-\theta_2) - \cos\theta_1 + \cos\theta_2\right)$$

$$\lambda_2 = \frac{1}{2}\left(\sqrt{-4K^2\cos^2(\theta_1-\theta_2) + 2\cos\theta_1\cos\theta_2 + \cos^2\theta_1 + \cos^2\theta_2}\right.$$
$$\left. -2K\cos(\theta_1-\theta_2) - \cos\theta_1 + \cos\theta_2\right)$$

$$\Delta = 2K^2\cos^2(\theta_1-\theta_2) + K\cos(\theta_1)\cos(\theta_1-\theta_2) - K\cos(\theta_2)\cos(\theta_1-\theta_2) - \cos(\theta_1)\cos(\theta_2)$$

$$\tau = -2K\cos(\theta_1-\theta_2) - \cos(\theta_1) + \cos\theta_2$$

$$\tau^2 - 4\Delta = -4K^2\cos^2(\theta_1-\theta_2) + 2\cos\theta_1\cos\theta_2 + \cos^2\theta_1 + \cos^2\theta_2$$

$K=0$; $E=0$: Unstable saddle

$k>0$; $E=0$: Stable sinks and Unstable sources

$K=0$; $E>0$: Unstable saddles

$k>0$; $E>0$: Stable Fixed Points

A plot become the accurate method for fixed point analysis.

b) $\dot{\theta}_1 = 0 = E - \sin\theta_1 + K\sin(\theta_2-\theta_1)$

$E = \sin\theta_1 - K\sin(\theta_2-\theta_1)$

$\dot{\theta}_2 = 0 = E - \sin\theta_2 + K\sin(\theta_1-\theta_2)$

$E = \sin\theta_2 - K\sin(\theta_1-\theta_2)$

The type of periodic solution depends on K.

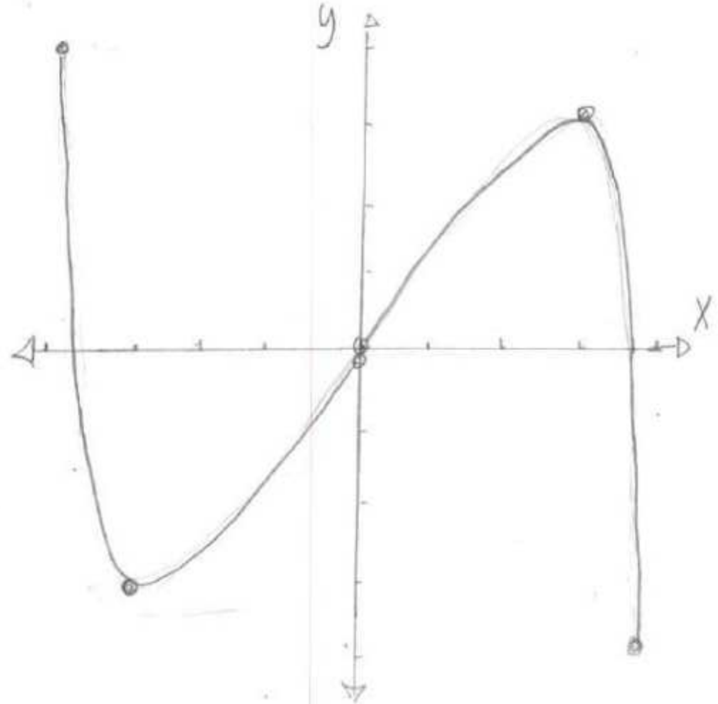
A K=0, unstable saddles become solutions, while K>0, stable fixed points.

c) See part a.

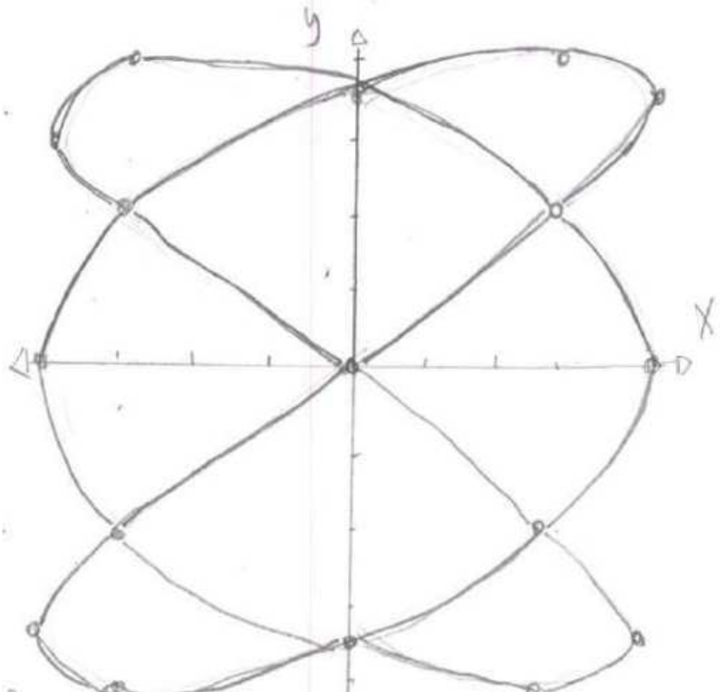
$x(t) = \sin t$
$y(t) = \sin \omega t$

8.6.5.

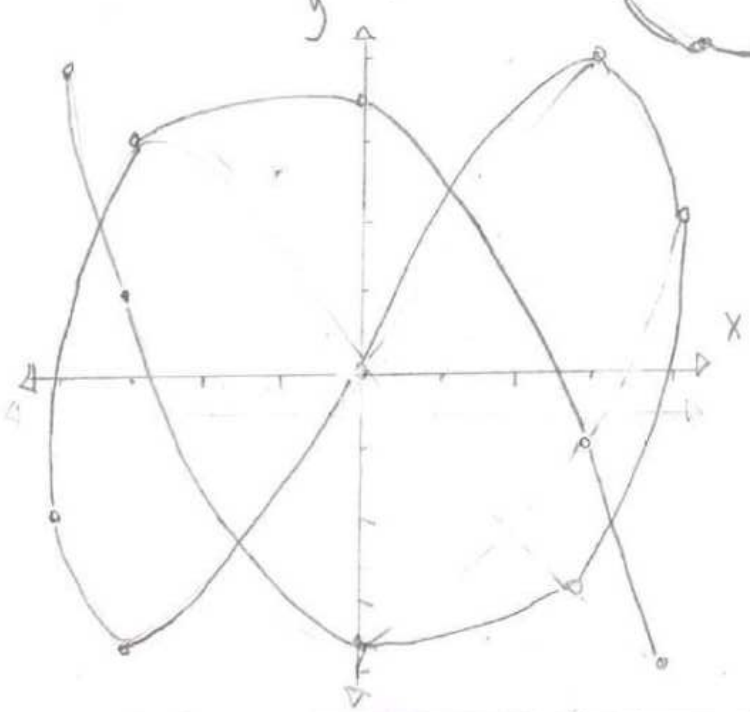
a) $\omega = 3$



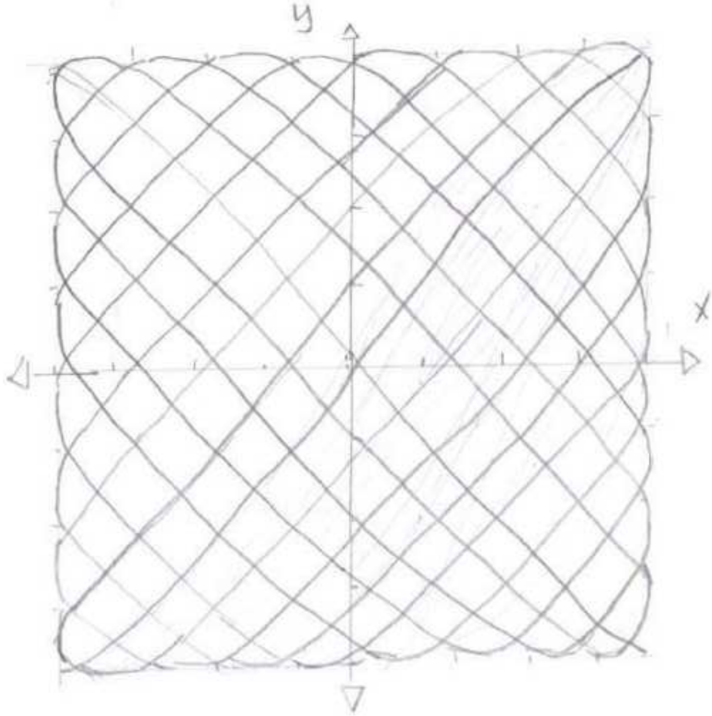
b) $\omega = \frac{2}{3}$



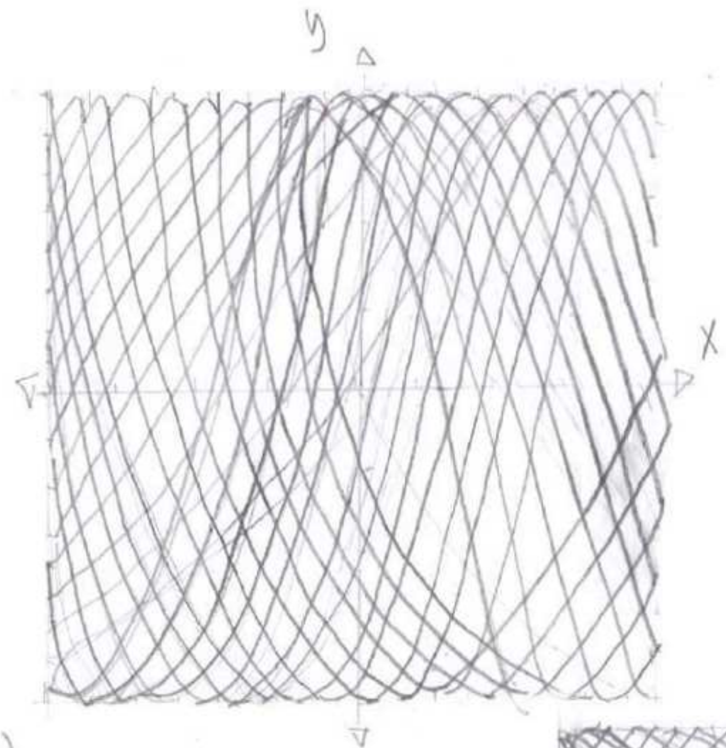
c) $\omega = 5/3$



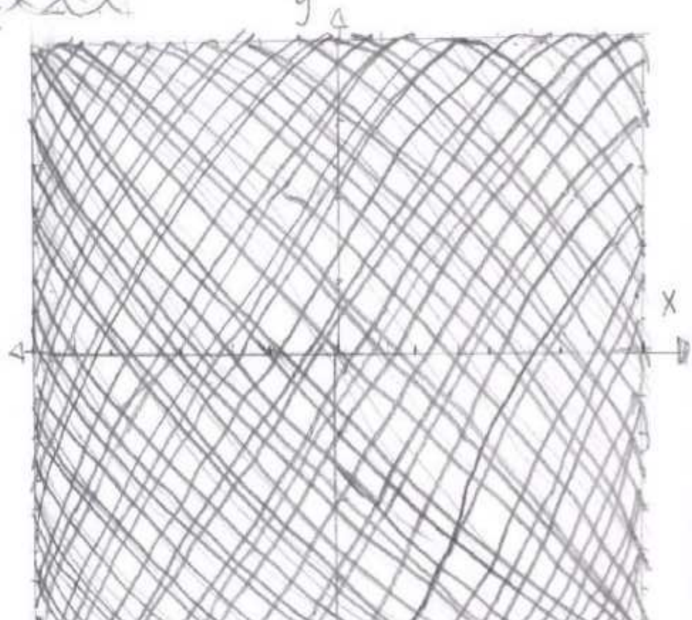
d.) $\omega = \sqrt{2}$



e) $\omega = \pi$



f) $\omega = \frac{1+\sqrt{5}}{2}$



$\ddot{x}+x=0$
$\ddot{y}+\omega^2 y=0$

8.6:6

a) If $x=A(t)\sin\theta(t)$ and $y=B(t)\sin\phi(t)$, then

$$\dot{x}=A(t)\cos\theta(t)\cdot\dot{\theta}(t)+\dot{A}(t)\sin\theta(t)$$

$$\ddot{x}=\dot{A}(t)\cos\theta(t)\cdot\dot{\theta}(t)-A(t)\sin\theta(t)\,\dot{\theta}(t)^2+A(t)\cos\theta(t)\,\ddot{\theta}(t)$$
$$+\ddot{A}(t)\sin\theta(t)+\dot{A}(t)\cos\theta(t)\,\dot{\theta}(t)$$

$$\ddot{x}+x=\dot{A}(t)\cos\theta(t)\dot{\theta}(t)-A(t)\sin\theta(t)\,\dot{\theta}(t)^2+A(t)\cos\theta(t)\,\ddot{\theta}(t)$$
$$+\ddot{A}(t)\sin\theta(t)+\dot{A}(t)\cos\theta(t)\,\dot{\theta}(t)+A(t)\sin\theta(t)$$
$$=0\text{, where }\dot{\theta}=1\text{ and }\dot{A}(t)=0$$

$$y=B(t)\sin\phi(t)$$

$$\dot{y}=B(t)\cos\phi(t)\cdot\dot{\phi}(t)+\dot{B}(t)\sin\phi(t)$$

$$\ddot{y}=\dot{B}(t)\cos\phi(t)\cdot\dot{\phi}(t)-B(t)\sin\phi(t)\,\dot{\phi}(t)^2+B(t)\cos\phi(t)\,\ddot{\phi}(t)$$
$$+\ddot{B}(t)\sin\phi(t)+\dot{B}(t)\cos\phi(t)\,\dot{\phi}(t)$$

$$\ddot{y}+\omega y=\dot{B}(t)\cos\phi(t)\dot{\phi}(t)-B(t)\sin\phi(t)\,\dot{\phi}(t)^2+B(t)\cos\phi(t)\ddot{\phi}(t)$$
$$+\ddot{B}(t)\sin\phi(t)+\dot{B}(t)\cos\phi(t)\dot{\phi}(t)+\omega B(t)\sin\phi(t)$$
$$=0\text{, where }\dot{\phi}(t)=\omega\text{ and }\dot{B}(t)=0$$

b) A two-dimensional tori appears from the four-dimensional system because constraints in the scaled equations.

c) Lissajous figures relate trajectories in the system through a constant period in the system

8.6.7

$m\ddot{r}=\dfrac{h^2}{mr^3}-K$

$\dot{\theta}=h/mr^2$

a) m = mass

K = central force of constant strength

h = constant (the angular momentum of the particle)

a) If $r = r_0$ and $\dot{\theta} = \omega_0$, then $m\ddot{r} = 0 = \dfrac{h^2}{mr_0^3} - k$

and $r_0 = \sqrt[3]{\dfrac{h^2}{mk}}$

Also, $\dot{\theta} = \dfrac{h}{mr_0^2} = \dfrac{h}{m}\left(\dfrac{mk}{h^2}\right)^{2/3} = \left(\dfrac{k^2}{mh}\right)^{1/3} = \omega_\theta$

b) $\omega_r = \sqrt{\dfrac{k}{m}}$; $\ddot{r} = \dfrac{h^2}{mr_0^3} - \dfrac{k}{m}$

$$\frac{d\ddot{r}}{dr_0} = \frac{-3h^2}{mr_0^4}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{3h^2}{mr_0^4} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\omega^2 = \dfrac{3h^2}{mr_0^4}$; $\omega_r = \sqrt{\dfrac{3h^2}{mr_0^4}} = \sqrt{3}\,\omega_0$

c) Winding Number: $\dfrac{\omega_r}{\omega_\theta} = \dfrac{\sqrt{3}\,\omega_0}{\omega_\theta} = \sqrt{3}$ and irrational

d) Eigenvalues: $(A - \lambda)I = \begin{bmatrix} -\lambda & 1 \\ -\dfrac{3h^2}{mr_0^4} & -\lambda \end{bmatrix} = \lambda^2 + \dfrac{3h^2}{mr_0^4} = 0$

$\lambda_{1,2} = \pm\sqrt{\dfrac{3h^2}{mr_0^4}}$; $\Delta = \dfrac{3h^2}{mr_0^4}$; $\tau = 0$; "center"

Also, the period : $\Delta\theta = \theta(t+T) - \theta(t)$

$= 2\pi$

Lastly, $\dot{\theta} = \omega_0$, which is a constant.

The motion is periodic for any amplitude.

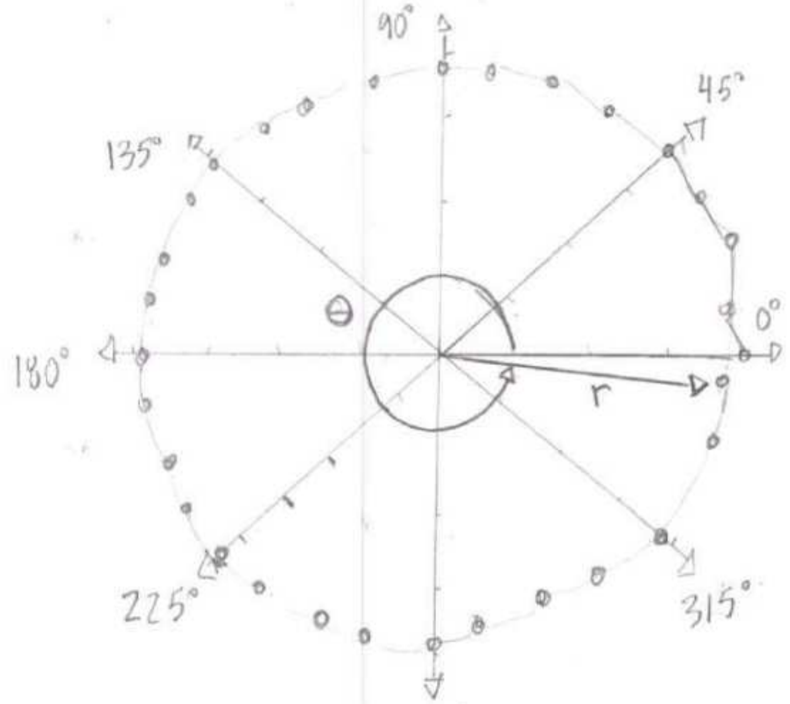
d) A mechanical realization of this system is quasiperiodic vibrations or weather patterns.

$m\ddot{r} = \frac{h^2}{mr^3} - k$ 8.6.8 <u>Runge-Kutta 4th-order:</u>

$\dot{\theta} = \frac{h}{mr^2}$

$\dot{\theta} = \omega_0$; $\theta(t) = \omega_0 t$

Parameter	Function
r_0	1
k_1	$f(r_0, t)$
k_2	$f(r + k_1/2, t)$
k_3	$f(r + k_2/2, t)$
k_4	$f(r + k_3, t)$
r_{n+1}	$r_n + (k_1 + 2k_2 + 2k_3 + k_4)/6$



$\dot{\theta}_1 = \omega + H(\theta_2 - \theta_1)$.

$\dot{\theta}_2 = \omega + H(\theta_1 - \theta_2)$

$\dot{\theta}_1 = \omega + H(\theta_2 - \theta_1) + H(\theta_3 - \theta_1)$

$\dot{\theta}_2 = \omega + H(\theta_1 - \theta_2) + H(\theta_3 - \theta_2)$

$\dot{\theta}_3 = \omega + H(\theta_1 - \theta_3) + H(\theta_2 - \theta_3)$

8.6.3. a) $\phi = \theta_1 - \theta_2$; $\dot{\phi} = \dot{\theta}_1 - \dot{\theta}_2 = H(\theta_2 - \theta_1) - H(\theta_1 - \theta_2)$

$\psi = \theta_2 - \theta_3$; $\dot{\psi} = \dot{\theta}_2 - \dot{\theta}_3 = H(\theta_1 - \theta_2) + H(\theta_3 - \theta_2) - H(\theta_1 - \theta_3) - H(\theta_2 - \theta_3)$

b) If $H(x) = a\sin x$, then $\dot{\phi} = H(\theta_1 - \theta_2) - H(\theta_2 - \theta_1)$

$$= a\sin(\theta_1 - \theta_2) - a\sin(\theta_2 - \theta_1)$$

$$= -2a\sin\phi$$

$$= 0, \text{ when } \phi = \pi.$$

$$\dot{\psi} = -2a\sin(\theta_2 - \theta_3) + a\sin(\theta_1 - \theta_2) - a\sin(\theta_1 - \theta_3)$$

$$= -2a\sin(\psi) + a\sin(\phi) - a\sin(\phi + \psi)$$

$$= 0, \text{ when } \phi = n\pi \text{ and } \psi = m\pi \quad n, m \in \mathbb{R}^3$$

$$\text{or } \phi = \frac{2n\pi}{3} \text{ and } \psi = \frac{2m\pi}{3}$$

c) $H(x) = a\sin x + b\sin 2x$

Fixed Points: $\dot{\phi} = 0 = H(-\phi) + H(-\phi - \psi) - H(\phi) - H(-\psi)$

$$= a\sin(-\phi) + b\sin(-2\phi) + a\sin(-\phi - \psi) + b\sin(-2(\phi+\psi)) - a\sin(\phi) - b\sin(2\phi) - a\sin(-\psi) - b\sin(-2\psi)$$

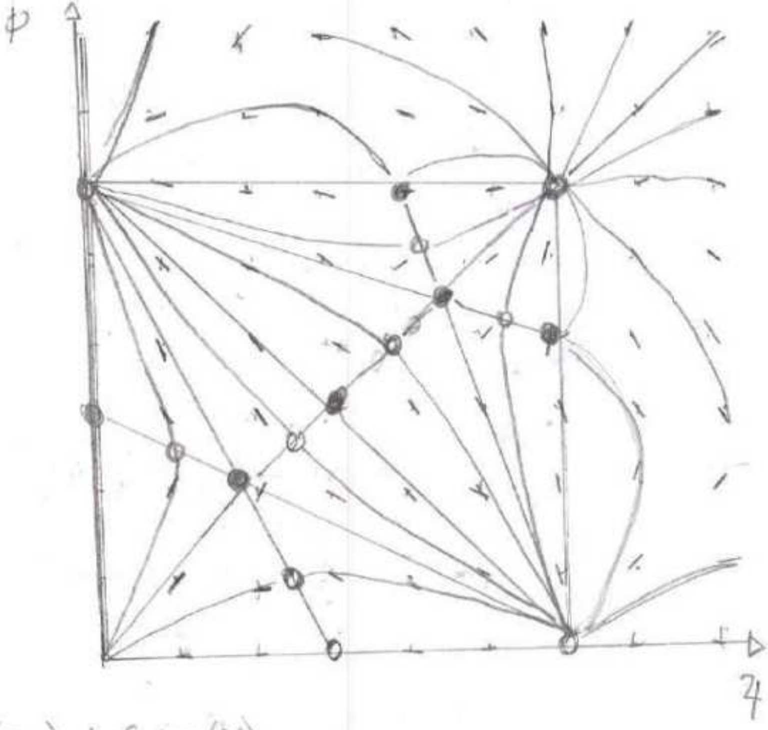
$$= -2a\sin(\phi) - 2b\sin(2\phi) - a\sin((\phi+\psi)) - b\sin(2(\phi-\psi)) + a\sin(\psi) + b\sin(2\psi)$$

$$\dot{\psi} = 0 = -2a\sin\psi + a\sin\phi - a\sin(\phi+\psi)$$

$$(a^*, b^*) = (0, 0)$$

Nullcline: $\dfrac{a}{b} = -\dfrac{2\sin(\phi)\cos(\phi - 2\psi)}{\sin(\phi+\psi) + 2\sin(\phi) - \sin(\psi)}$

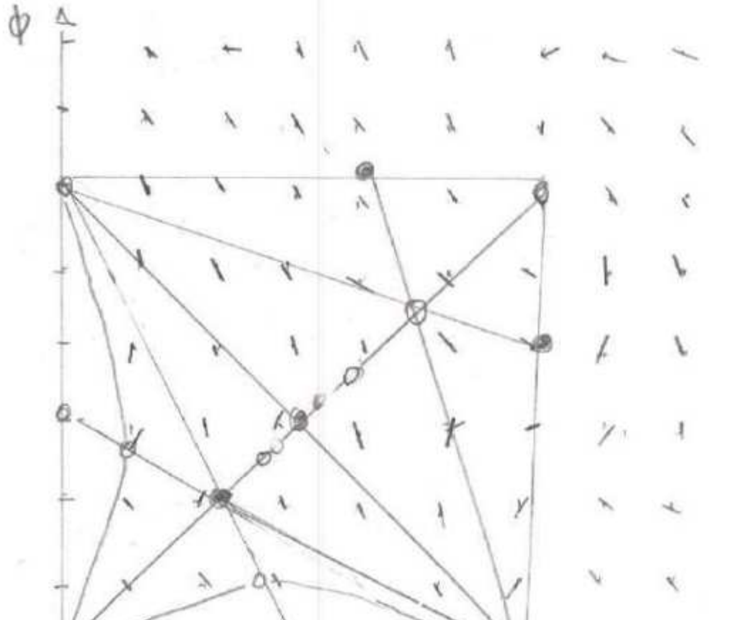
$a=-5; b=1$



a) $H(X) = a\sin(X) + b\sin(2X) + C\cos(X)$

$a=-5; b=1; C=0.1$

The zero-points were hardly comprehensible because of a 30+ term function



$r = \left[1 + e^{-4\pi}(r_0^{-2} - 1)\right]^{-1/2}$

8.7.1. $t = \int_{r_0}^{r_1} \frac{dr}{r(1-r^2)} = \int_{r_0}^{r_1} \frac{dr}{r(1+r)(1-r)} = \int_{r_0}^{r_1} \frac{A}{r} dr + \int_{r_0}^{r_1} \frac{B}{1+r} dr + \int_{r_0}^{r_1} \frac{C}{1-r} dr$

$A(1+r)(1-r) + B\cdot(1-r)r + C\cdot(1+r)r = 1$

If $r=0$, then $A=1$

If $r=1$, then $C=\frac{1}{2}$

If $r=-1$, then $B=\frac{-1}{2}$

$$= \int_{r_0}^{r_1} \frac{1}{r} dr - \frac{1}{2}\int_{r_0}^{r_1} \frac{dr}{1+r} - \frac{1}{2}\int_{r_0}^{r_1} \frac{dr}{r-1}$$

$$= \ln r_1/r_0 - \frac{1}{2} \ln \frac{1+r_1}{1+r_0} - \frac{1}{2} \ln \frac{r_1-1}{r_0-1}$$

$$= \ln \frac{r_1}{r_0} \frac{\sqrt{r_0^2-1}}{\sqrt{r_1^2-1}} = 2\pi$$

Solving for r_1:

$$\frac{r_1}{r_0} \frac{\sqrt{r_0^2-1}}{\sqrt{r_1^2-1}} = e^{2\pi}$$

$$r_1^2 = \frac{r_0^2 e^{4\pi}}{1+r_0^2 e^{4\pi} - r_0^2} = \frac{1}{-e^{-4\pi} + r_0^{-2} e^{-4\pi} + 1}$$

$$= \frac{1}{1+e^{-4\pi}(r_0^{-2}-1)}$$

$$r_1 = \frac{1}{\sqrt{1+e^{-4\pi}(r_0^{-2}-1)}}$$

where $r_{n+1} = P(r_n) = \dfrac{1}{\sqrt{1+e^{-4\pi}(r_n^{-2}-1)}}$

and $\dfrac{dP(r)}{dr} = \dfrac{e^{-4\pi} r^{-3}}{\sqrt{1+e^{-4\pi}(r^{-2}-1)}}$

$$\frac{dP(1)}{dr} = e^{-4\pi}$$

$\dot{\theta} = 1$ 8.7.2. $\theta = t$; $y = C \cdot e^{at} = C e^{a\theta}$ "Lyapunov stable = Periodic Orbit"

$\dot{y} = ay$

$\dot{x} + x = F(t)$ 8.7.3. $F(t) = \begin{cases} +A, & 0<t<T/2 \\ -A, & T/2<t<T \end{cases}$

a) $x(0) = x_0$: Bernoulli's Equation $\quad y' + P(x)y = Q(x)y^n$

$I(x) = \exp\left[\int [1-n] p(x)\, dx\right]$

$y^{1-n} = \frac{1}{I(x)}\left[\int [1-n] Q(x) I(x)\, dx\right]$

① $x' + x = F(t)$

② $I(t) = \exp\left[\int dt\right] = e^{T}$

③ $x(t) = e^{-T} \cdot \left[\int_0^T F(t) \cdot e^{t}\, dt\right] = e^{-T}\left[\int_0^{T/2} A e^{t} dt - \int_{T/2}^{T} A e^{t} dt\right]$

$= e^{-T}\left[A(e^{T/2} - 1) - A(e^{T/2} - e^{T})\right] + C \cdot e^{-T}$

④ Initial conditions: $x(0) = x_0$

$x(0) = 1 \cdot [A(0) - A(0)] + C(1) = x_0 \quad ; C = x_0$

$x(t) = x_0 e^{-T} - A(1 - e^{-T/2})^2$

b) Identity: $\tanh(x) = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

$x(t) = -x_0 e^{-T} - A(1 - e^{-T/2})^2$

$x_0 (1 - e^{-T}) = -A(1 - e^{-T/2})^2$

$x_0 = \dfrac{-A(1 - e^{-T/2})^2}{(1 - e^{-\frac{T}{2}})(1 + e^{-T/2})}$

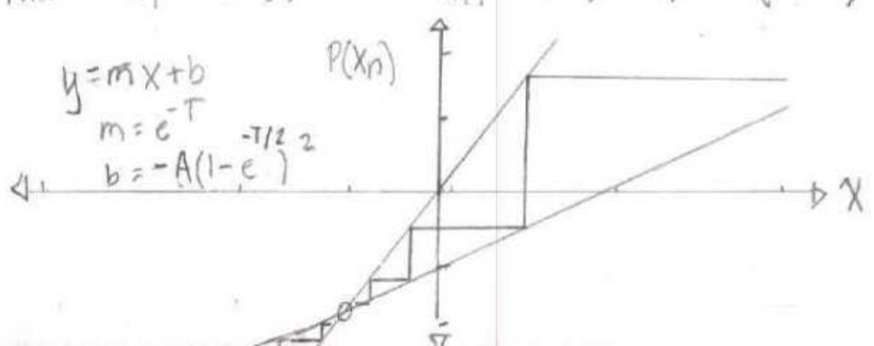
$= -A \tanh\left(\frac{T}{4}\right)$

c) $\lim_{T \to 0} x_0 = 0 \quad ; \quad \lim_{T \to \infty} x_0 = -A$

The results indicate a smaller period in an overdamped linear oscillator "strobe" little, while on always a lot for longer periods.

d) If $x_1 = x(T)$, then $x_1 = P(x_0)$ or $x_{n+1} = P(x_n) = x_n e^{-T} - A(1 - e^{-T/2})^2$

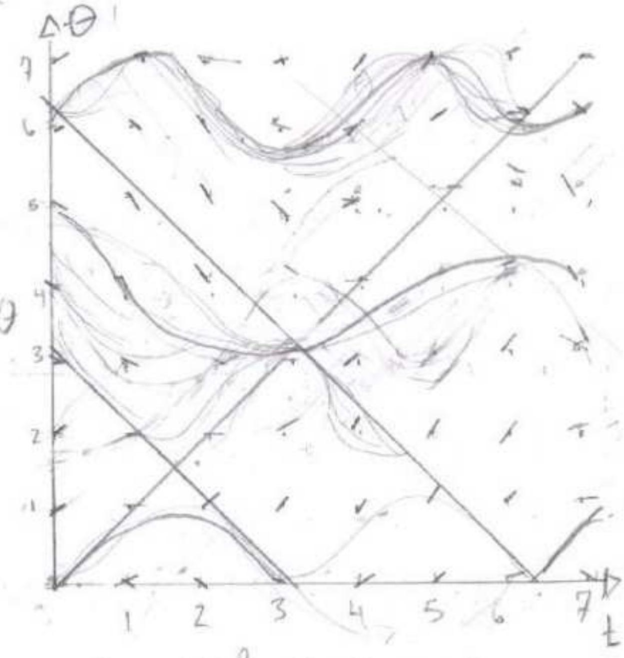
e)



$\dot{x}+x=A\sin\omega t$ 8.7.4. Solution: $P(x_0)=(x_0-c_3)e^{-2\pi/\omega}+c_3$

$$=x_0 e^{-2\pi/\omega}+c_4$$

The sign of $c_4=A$ is positive because the cobweb plot $(y=mx+b)$ has a slope $e^{-2\pi/\omega}$ and intercept $(b=c_4>0)$.



$\dot{\theta}+\sin\theta=\sin t$ 8.7.5. $\dot{t}=1$;

$\dot{\theta}=\sin t-\sin\theta$

Nullclines: $\dot{\theta}=0=\sin t-\sin\theta$

$\theta=t$

$=\pi-t$

8.7.6 A mechanical interpretation for $\dot{\theta}=\sin t-\sin\theta$ is a pendulum in a viscous medium.

8.7.7: See 8.7.5.

$\dot{x}+x=F(t)$ 8.7.8 Solution from 8.7.3: $x(T)=x_0e^{-T}-A(1-e^{-T/2})^2$

A T-periodic system has similar solutions with new T-values. At the limit of T to infinity, $x(T)$ equals negative one, moreover, zero at some period (T). Since multiple solutions exist for $x(T)=0$ during different parameters and initial conditions, then yes, the system's T-periodic.

$\dot{r} = r - r^2$ 8.7.9.

$\dot{\theta} = 1$

a) $t = \int \frac{dr}{r(1-r)} = \int \frac{A}{r}dr + \int \frac{B}{1-r}dr = \ln r - \ln(1-r) + c$

$= \ln \frac{r}{1-r} + r_0$

$r = \frac{e^{t-r_0}}{1+e^{t-r_0}}$

$P(r_{n+1}) = \frac{e^{t-r_n}}{1+e^{t-r_n}}$ or $P(r_0) = \frac{e^{t-r_0}}{1+e^{t-r_0}}$

b) $P(r^* + v_0) = P(r^*) + DP(r^*)v_0 + O(\|v_0\|^2)$

$(n-1)\times(n-1)$ Matrix: Linearized Poincare Map.

$v_1 = [DP(r^*)]v_0$

$= [DP(r^*)] \sum_{j=1}^{n-1} v_j e_j = \sum_{j=1}^{n-1} v_j \lambda_j e_j$

$v_k = \sum_{j=1}^{n-1} v_j (\lambda_j)^k e_j$; Goal: Characteristic multipliers during a small perturbation.

Fixed Points: $\dot{r} = 0 = r - r^2$

$r^* = -1, 0, 1$

If $r = 1+\eta$, where η is infinitesimal.

$\dot{r} = \dot{\eta} = (1+\eta) - (1+\eta)^2 = 1 + \eta - 1 - 2\eta - \eta^2$

$= -\eta^2 - \eta$

$\eta(t) = \frac{-e^c}{e^c - e^t} = \frac{-1}{1-e^{t-c}} = \frac{-1}{1-Ce^t}$

The characteristic multiplier is $e^{2\pi}$, and $P(1) > 1$, unstable.

c. The characteristic multiplier is $e^{2\pi}$.

8.7.10. Floquet multipliers:

① Find the fixed points about a differential

② Perturb the system by a small η

③ Solve the differential shifted by η

④ Determine the multipliers as coefficients about η_0.

⑤ Evaluate the multipliers at 2π or $2\pi i$ intervals

$\int_{r_0}^{r_1} \frac{dr}{r(1-r^2)}$

8.7.11. $\dot{r} = r(1-r^2)$; Fixed Points: $\dot{r} = 0 = r(1-r^2)$

$r^* = 0, 1$

Perturbations: $\dot{r} = \dot{\eta} = (1+\eta)(1-(1+\eta)^2)$

$\eta(t) = \eta_0 e^{-2t}$

Poincare Map: $P'(r^*) = e^{4i\pi} > 1$, unstable

Note: A shift of -2π, rather than 2π changes the nodes stability.

$\dot{\theta}_i = f(\theta_i) + \frac{K}{N}\sum_{j=1}^{N} f(\theta_j)$

8.7.12. If $\theta(t) = \theta^*(t) + \eta(t)$, then the oscillator becomes:

$$\dot{\eta} = f(\theta_i^*)\eta_i + f(\theta_j^*)\frac{K}{N}\sum_{j=1}^{N}\eta_j$$

A substitution $\mu = \frac{K}{N}\sum_{j=1}^{N}\eta_j$ and $\epsilon = \eta_{i+1} - \eta_i$

then, $$\frac{d\epsilon}{\epsilon} = f(\theta_i^*)dt = \frac{f(\theta_i^*)\,d\theta^*}{f(\theta_i) + \frac{K}{N}\sum_{j=1}^{N} f(\theta_j)}$$

$$\oint \frac{d\epsilon}{\epsilon} = \int_0^{2\pi} \frac{f(\theta^*)\,d\theta^*}{f(\theta_i)^* + \frac{K}{N}\sum_{j=1}^{N} f(\theta_j)^*}$$

$$\ln \frac{\epsilon(T)}{\epsilon(0)} = \frac{2\pi}{\frac{K}{N} + 1}$$

If $\epsilon(T) = \epsilon(0)$ for a periodic system, then

$0 = \dfrac{2\pi}{\frac{K}{N} + 1}$ and a characteristic multiplier is $\lambda = +1$

for K approaching infinity cycles.

$\boxed{\text{Chapter 9: Lorenz Equations}}$

$M = \int_0^{2\pi} m(\theta, t)\, d\theta$

9.1.1:

a) $I = I_{Wheel} + I_{Water} = m R^2_{Wheel} + M R^2_{Water}$

$= m R^2_{Wheel} + \int_0^{2\pi} m(\theta, t)\, d\theta \; R^2_{Water}$

b) $\dot{M} = \frac{dM}{dt} = \int_0^{2\pi} \frac{dm(\theta,t)}{dt}\, d\theta$

$= \int_0^{2\pi} [(\text{Mass pumped in}) - (\text{Mass pumped out})]\, d\theta$

$= \int_0^{2\pi} (Q - Km)\, d\theta$

c) If $\dot{M} = Q - Km$, then $\dot{I} = \dot{M} R^2$

$= Q R^2_{Wheel} - K m R^2_{Water}$

$= Q R^2_{Wheel} - K I$

$t = \int \frac{dI}{QR^2 - KI} = -\frac{1}{K} \int \frac{du}{u}$

$= -\frac{1}{K} \ln QR^2 - KI + C$

$I = (C) e^{-Kt} + QR^2$

$\lim_{t \to \infty} I(t) = QR^2$

$=$ constant.

$Q(\theta) = q_1 \cos\theta$ 9.1.2.

a) If $n \neq 1$, then a lagrange multiplier about the coefficients, $a(t) + b(t) = 1$.

$$Q(\theta) = q_1 \cos\theta + \lambda(a(t) + b(t))$$
$$= q_1 \cos\theta + \lambda(\mathring{a}(t) + \mathring{b}(t))$$

where $\frac{da}{dt} = \lambda a ; a = C_1 e^{\lambda t}$

and

$$\frac{db}{dt} = \lambda b ; b = C_2 e^{\lambda t}$$

Thus, $\lim_{t \to \infty} C(t) e^{\lambda t} = 0 ; \lim_{t \to \infty} C(t) = a(t) = 0$

b) If $Q(\theta) = \sum_{n=0}^{\infty} q_n \cos n\theta$, then the coefficients $a(t)$ and $b(t)$ become $a_n(t)$ and $b_n(t)$, respectively.

$$\mathring{a} + a = n\chi b(t) \quad \text{and} \quad \mathring{b} + b = -n\chi a(t) + C$$

where $C = q_n / K$

The automous system arrives at a solution:

$$a(t) = e^{-t}\left(a(0) + n\int_0^{\tau} \chi(t) b(t) e^{t} dt\right)$$

$$b(t) = C + e^{-t}\left[b(0) - C - n\int_0^{t} \chi(t) a(t) e^{t} dt\right]$$

As $t \to \infty$, then $a(t) = 0$ and $b(t) = C = \frac{q}{K}$.

(Kolar and Gumbs, 1992)

$\dot{a}_1 = \omega b_1 - K a_1$
$\dot{b}_1 = -\omega a_1 + q_1 - K b_1$
$\dot{\omega} = -\frac{v}{I}\omega + \frac{\pi g r}{I} a_1$

$\dot{x} = \sigma(y - x)$
$\dot{y} = rx - xz - y$
$\dot{z} = xy - bz$

9.1.3. The hint states x is like w, so σ relates the coefficients in $\dot{\omega}$. $\sigma \propto \frac{v}{I}$ and $\sigma \propto \frac{\pi g r}{I}$

$x \propto \omega$; $\omega = \alpha x$

The hint also states y is like a_1, so

$y \propto a_1$; $a_1 = \beta y$

Lastly, z is similar to b_1.

$z \propto b_1$; $b_1 = \varepsilon z$

A dimensional problem frequently shifts time:

$t = \xi \cdot \tau$

$$\dot{\omega} = -\frac{v}{I}\omega + \frac{\pi g r_w}{I} a_1 = \alpha \frac{1}{\xi}\frac{dx}{d\tau} = -\frac{v}{I}\alpha x + \frac{\pi g r_w}{I}\beta y$$

$$\dot{a}_1 = \omega b_1 - K a_1 = \beta \frac{1}{\xi}\frac{dy}{d\tau} = \alpha x \cdot \varepsilon z - K \cdot \beta \cdot y$$

$$\dot{b}_1 = -\omega a_1 + q_1 - K b_1 = \varepsilon \frac{1}{\xi}\frac{dz}{d\tau} = -\alpha x \beta y + q_1 - K \varepsilon z$$

$$x' = -\frac{v\xi}{I} x + \frac{\pi g r \beta \xi}{\alpha I} y = \sigma(y - x)$$

$$y' = \frac{\alpha \xi \epsilon}{\beta} xz - \xi K y = rx - xz - y$$

$$z' = -\frac{\alpha \beta \xi}{\epsilon} xy + \frac{\xi}{\epsilon}(q_1 - K \epsilon z) = xy - bz$$

Where $\sigma = \frac{v\varepsilon}{I} = \frac{\pi g r \beta \varepsilon}{\alpha I}$; $b = 1$

$\epsilon = (\chi + \frac{4}{z})$; $1 = \frac{\alpha \xi \chi}{\beta}$

$\xi = \frac{1}{K}$; $0 = \frac{\xi}{\epsilon}(q_1 - K\chi)$

$r = \frac{\alpha}{\beta}\frac{4}{z}$

$\dot{E} = K(P-E)$
$\dot{P} = \gamma_1(ED-p)$
$\dot{D} = \gamma_2(\lambda+1-D-\lambda EP)$

9.1.4.

a) $\dot{D} = 0 = \gamma_2(\lambda+1-D-\lambda EP)$ at $E^* = 0$

$\lambda = D-1$

$$\begin{bmatrix} \dot{E} \\ \dot{P} \\ \dot{D} \end{bmatrix} = \begin{bmatrix} K & K & 0 \\ \gamma_1 D & 0 & \gamma_1 E \\ -\gamma_2 \lambda P & -\gamma_2 \lambda E & -\gamma_2 \end{bmatrix} \begin{bmatrix} E \\ P \\ P \end{bmatrix}$$

$$A_{E^*=0} = \begin{bmatrix} K-\lambda & K & 0 \\ \gamma_1 D & -\lambda & 0 \\ -\gamma_2 \lambda P & 0 & -\gamma_2-\lambda \end{bmatrix}$$

$$\lambda_1 = -\gamma_2 \; ; \; \lambda_{2,3} = \frac{K \pm \sqrt{K^2 + 4D \cdot \gamma_1 \cdot K}}{2}$$

$$\Delta = -\gamma_2 \left(2D\gamma_1 K \right) ; \; \tau = -\gamma_2 + K$$

$$\tau^2 - 4\Delta = \gamma_2 - 2K\gamma_2 + K^2 - 8\gamma_2 D \gamma_1 K$$

If $\gamma_1, \gamma_2 >> K$, then $\Delta > 0 ; \tau < 0$

$\tau^2 - 4\Delta < 0$

"Stable Node"

b) $\dot{E}$ is proportional to $\dot{x}$, by the look.

$$E = x \; ; \; P = y \; ; \; D = (\alpha - \beta) z$$

$$\beta = \frac{\xi}{\gamma_1} \; ; \; r = \frac{\gamma_1}{\xi \alpha z} \; ; \; b = \frac{\gamma_2}{\xi}$$

$$\lambda = \left(\frac{\xi}{\gamma} + 1\right) xy - 1 \; ; \; \frac{\gamma_1 p}{\xi} = y$$

Lorenz's equations fit the jitter within a laser.

$$Q(\theta) = \sum_{n=0}^{\infty} q_n \cos n\theta \quad 9.1.5$$

$$\mathring{a}_1 = \omega b_1 - K a_1$$

$$\mathring{b}_1 = -\omega a_1 - K b_1 + q_1$$

$$\mathring{\omega} = (-\nu\omega + \pi g r a_1)/I$$

$$\frac{\partial m}{\partial t} = Q - Km - \omega \frac{d\theta}{d\theta} \quad (9.1.2)$$

$$m(\theta, t) = \sum [a_n(t) \sin n\theta + b_n(t) \cos n\theta] \quad (9.1.4)$$

$$Q(\theta) = \sum_{n=0}^{\infty} p_n \sin(n\theta) + q_n \cos(n\theta)$$

The equation relating change of mass per time and change of mass per angle.

$$\frac{\partial}{\partial t}\left[\sum_{n=0}^{\infty}[a_n(t)\sin(n\theta) + b_n(t)\cos(n\theta)]\right]$$

$$= \sum_{n=0}^{\infty}[p_n \sin(n\theta) + q_n \cos(n\theta)] - K\sum_{n=0}^{\infty}[a_n(t)\sin(n\theta) + b_n(t)\cos(n\theta)]$$

$$-\omega \frac{\partial}{\partial \theta}\sum_{n=0}^{\infty}[a_n(t)\sin(n\theta) + b_n(t)\cos(n\theta)]$$

$$\sum_{n=0}^{\infty} \mathring{a}_n(t)\sin(n\theta) + \mathring{b}_n(t)\cos(n\theta)$$

$$= \sum_{n=0}^{\infty}[p_n \sin(n\theta) + q_n \cos(n\theta)] - K\sum_{n=0}^{\infty}[a_n(t)\sin(n\theta) + b_n(t)\cos(n\theta)]$$

$$-\omega \sum_{n=0}^{\infty} n[a_n(t)\cos(n\theta) - b_n(t)\sin(n\theta)]$$

The similar terms on the left and right are grouped:

$$\mathring{a}_n = n\omega b_n(t) - K a_n(t) + p_n$$

$$\mathring{b}_n = -n\omega a_n(t) - K b_n(t) + q_n$$

$$\mathring{\omega} = \frac{-\nu\omega + gr\int_0^{2\pi} m(\theta,t)\sin\theta \, d\theta}{I} = \frac{-\nu\omega + \pi g r a}{I}$$

Fixed Points: $\mathring{a} = 0 = \omega b_1 - K a_1 + p_1$

$$\mathring{b} = 0 = -\omega a_1 - K b_1 + q_1$$

$$\mathring{\omega} = 0 = \frac{-\nu\omega + \pi g r a_1}{I}$$

$$\omega^* = 0, \pm\sqrt{\frac{\pi g r q_1}{\nu} - K^2}$$

A square root is a pitchfork bifurcation, but imperfect when $p_1 \neq 0$.

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r+\sigma)b\lambda + 2b\sigma(r-1) = 0$$

9.2.1. $$\begin{bmatrix} \mathring{x} \\ \mathring{y} \\ \mathring{z} \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Eigenvalues:

$$(A-\lambda) = \begin{vmatrix} -\sigma-\lambda & \sigma & 0 \\ r-z & -1-\lambda & -x \\ y & x & -b-\lambda \end{vmatrix} = 0$$

Fixed Points: $\mathring{x} = 0 = \sigma(z - x)$

$$\mathring{y} = 0 = rx - y - xz$$

$$\mathring{z} = 0 = xy - bz$$

$$(x^*, y^*, z^*) = \left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1\right)$$

Jacobian Adjustment:

$$(A-\lambda) = \begin{bmatrix} -\sigma-\lambda & \sigma & 0 \\ -1 & -1-\lambda & \pm\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b-\lambda \end{bmatrix}$$

$= \lambda^3 + (\sigma + 1 + b)\lambda^2 + b(r+\sigma)\lambda + 2b\sigma(r-1)$

b) If $r = r_H = \sigma\left(\frac{\sigma+b+3}{\sigma-b-1}\right)$, then eigenvalues become cubic roots. The proposition $\sigma > b+1$ comes from $r_H = \sigma\left(\frac{\sigma+b+3}{\sigma-(b+1)}\right)$ and a cubic solution's necessity for positive values.

c) The third eigenvalue is $\lambda_3 = -(\sigma+1+b)$

$rx^2 + \sigma y^2 + \sigma(z-2r)^2 \leq C$

9.2.2. Equation of a Ellipse: $F(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$

When $V(x,y,z) = rx^2 + \sigma y^2 + \sigma(z-2r)^2$

$$\dot{V}(x,y,z) = 2rx\dot{x} + 2\sigma y\dot{y} + 2\sigma(z-2r)\dot{z}$$

$$= 2rx(\sigma(z-x)) + 2\sigma y(rx-y-xz) + 2\sigma(z-2r)(xy-bz)$$

$$= 2rx\sigma z - 2rx^2\sigma + 2\sigma yrx - 2\sigma y^2 - 2\sigma yxz + (2\sigma z - 4\sigma r)(xy-bz)$$

$$\frac{\dot{V}(x,y,z)}{2} = rx\sigma z - rx^2\sigma + \sigma yrx - \sigma y^2 - \sigma yxz + \sigma xyz - \sigma bz^2 - 2\sigma rxy + 2\sigma rbz$$

$$= -r\sigma x^2 - \sigma y^2 + \sigma(rxz + rxy - bz^2 - 2rxy + 2rbz)$$

$$= -r\sigma x^2 - \sigma y^2 + \sigma(-bz^2 + (rx+2rb)z) - \sigma rxy$$

$$= -r\sigma x^2 - \sigma y^2 - \sigma b\left(z - \frac{rx+2rb}{2b}\right)^2 + \frac{\sigma(rx+2rb)^2}{4} - \sigma rxy$$

$$= -r\sigma\left(x+\frac{rb+y}{2\sigma(1+4r)}\right)^2+\left(\frac{1}{4\sigma(1+4r)}-1\right)y^2-\sigma b\left(z-\frac{rx+2rb}{2b}\right)^2+b^2\left(\frac{1}{4\sigma(1+4r)}+r^2\right)$$

$$-r\sigma\left(x+\frac{rb+y}{2\sigma(1+4r)}\right)^2+\left(\frac{1}{4\sigma(1+4r)}-1\right)y^2-\sigma b\left(z-\frac{rx+2rb}{2b}\right)^2+b^2\left(\frac{1}{4\sigma(1+4r)}+r^2\right)<0$$

$$1<\underbrace{\frac{+r\sigma}{b^2\left(\frac{1}{4\sigma(1+4r)}+r^2\right)}}_{1/a^2}\left(x+\frac{rb+y}{2\sigma(1+4r)}\right)^2-\underbrace{\frac{\left(\frac{1}{4\sigma(1+4r)}-1\right)}{b^2\left(\frac{1}{4\sigma(1+4r)}+r^2\right)}}_{1/b^2}y^2-\underbrace{\frac{\sigma b}{\left(\frac{1}{4\sigma(1+4r)}+r^2\right)}}_{1/c^2}\left(z-\frac{rx+2rb}{2b}\right)^2$$

The equation of the ellipse above is co-dependent. With z about x-values and x about y-values. When modeled without co-dependent axis, then a coefficient becomes co-dependent.

In all cases, an ellipse centered at $\left(\frac{-rb+y}{2\sigma(1+4r)}, 0, \frac{rx+2rb}{2b}\right)$ with a maximum distance from the center:

$$\left(\sqrt{\frac{r\sigma}{b^2\left(\frac{1}{4\sigma(1+4r)}+r^2\right)}}, \sqrt{\frac{1-\frac{1}{4\sigma(1+4r)}}{b^2\left(\frac{1}{4\sigma(1+4r)}+r^2\right)}}, \sqrt{\frac{ab}{\left(\frac{1}{4\sigma(1+4r)}+r^2\right)}}\right),$$

Another sphere fits into the ellipsoid centered at the same coordinates with a minimal ellipsoid radius from the center.

$x^2+y^2+(z-r-\sigma)^2=C$

9.2.3. Equation for a sphere: $x^2+y^2+z^2=f(x,y,z)$

$$V(x,y,z)=x^2+y^2+(z-r-\sigma)^2$$

$$\dot{V}(x,y,z)=2x\dot{x}+2y\dot{y}+2(z-r-\sigma)\dot{z}$$

$$\frac{\dot{V}(x,y,z)}{2}=x[\sigma(z-x)]+y(rx-y-xz)+(z-r-\sigma)(xy-bz)$$

$$=-\sigma x^2-y^2-b\left(z-\frac{r+\sigma}{2}\right)^2+b\frac{(r+\sigma)^2}{4}$$

$$-\sigma x^2-y^2-b\left(z-\frac{r+\sigma}{2}\right)^2+b\frac{(r+\sigma)^2}{4}<0$$

$$1<\underbrace{\frac{4\sigma}{b(r+\sigma)^2}}_{a}x^2+\underbrace{\frac{4}{b(r+\sigma)^2}}_{b}y^2+\underbrace{\frac{4}{(r+\sigma)^2}}_{c}\left(z-\frac{r+\sigma}{2}\right)^2$$

A sphere centered at $\left(0,0,\frac{r+\sigma}{2}\right)$

with a maximum radius $\left(\sqrt{\frac{b(r+\sigma)^2}{4\sigma}},\sqrt{\frac{b(r+\sigma)^2}{4\sigma}},\sqrt{\frac{(r+\sigma)^2}{4}}\right)$

9.2.4 $\dot{x}=\sigma(y-x)$ The z-axis is an invariant

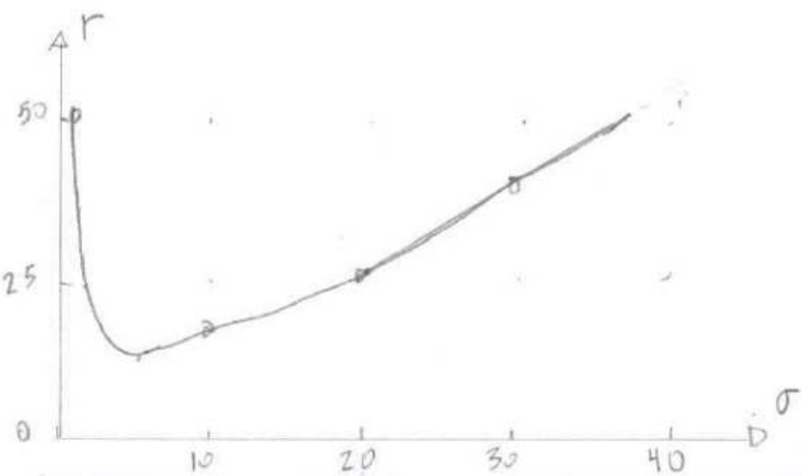
$\dot{y}=rx-xz-y$ line when $x=y=0$ because

$\dot{z}=xy-bz$ $\dot{z}=-bz$; $z(t)=Ce^{-bt}$. Otherwise,

z is variant!

9.2.5. The relationship between r and σ is in Problem 9.2.1.

$$r=\sigma\frac{(\sigma+3+b)}{(\sigma-1-b)}$$



$\dot{x} = -vx + zy$
$\dot{y} = -vy + (z-a)x$
$\dot{z} = 1 - xy$

9.2.6.

a) A dissapative systems volume contracts under flow.

$V(x,y,z) = x^2 + y^2 + z^2$

If dissapative, then $\dot{V}(x,y,z) < 0$.

$$\nabla \cdot f = \frac{\partial}{\partial x}\left[-vx + zy\right] + \frac{\partial}{\partial y}\left[-vy + (z-a)x\right] + \frac{\partial}{\partial z}\left[1 - xy\right]$$

$$= -v - v = -2v < 0$$

$$\dot{V} = \int_V \nabla \cdot f \, dV = -2\int_V v \cdot dV = -2vV$$

$$V(t) = V(0)e^{-2vt}$$

The volume shrinks with time!

b) <u>Fixed Points:</u> $\dot{x} = 0 = -vx + zy$

$\dot{y} = 0 = -vy + (z-a)x$

$\dot{z} = 0 = 1 - xy$

$(x^*, y^*, z^*) = (\pm 1, \pm 1, v)$

where $a = v(x^2 - 1/x^2)$

c) <u>Bifurcations:</u>

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -v & z & y \\ z-a & -v & x \\ -x & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A = \begin{bmatrix} -v & z & y \\ z-a & -v & x \\ -y & -x & 0 \end{bmatrix}$$

$$(A - \lambda I) = \begin{bmatrix} -v-\lambda & z & y \\ z-a & -v-\lambda & x \\ -y & -x & -\lambda \end{bmatrix} = 0$$

If $x = \pm 1$, $y = \pm 1$, and $z = v$, then

$\lambda_1 \approx 1.41i$; $\lambda_2 \approx -1.41i$; $\lambda_3 = -2v$

Hopf Bifurcation = Spiral Node

$\dot{\theta}_1 = \omega_1$

$\dot{\theta}_2 = \omega_2$

9.3.1.

a) The solution (or time-dependent motion) is periodic as $t \to \infty$, so not a chaotic system.

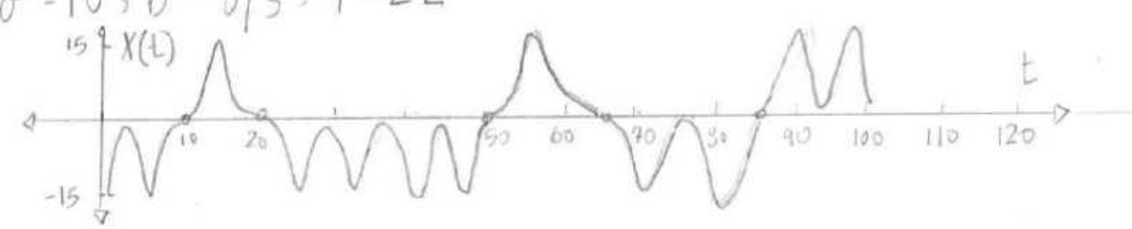
b) A large Liapunov exponent is zero.

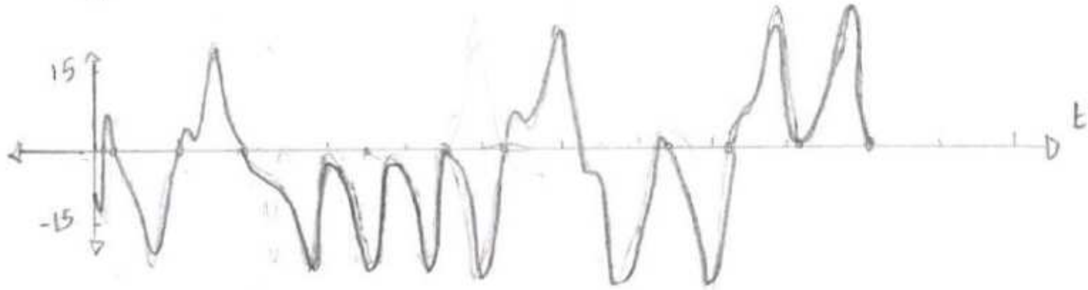
$\dot{x} = \sigma(z - x)$

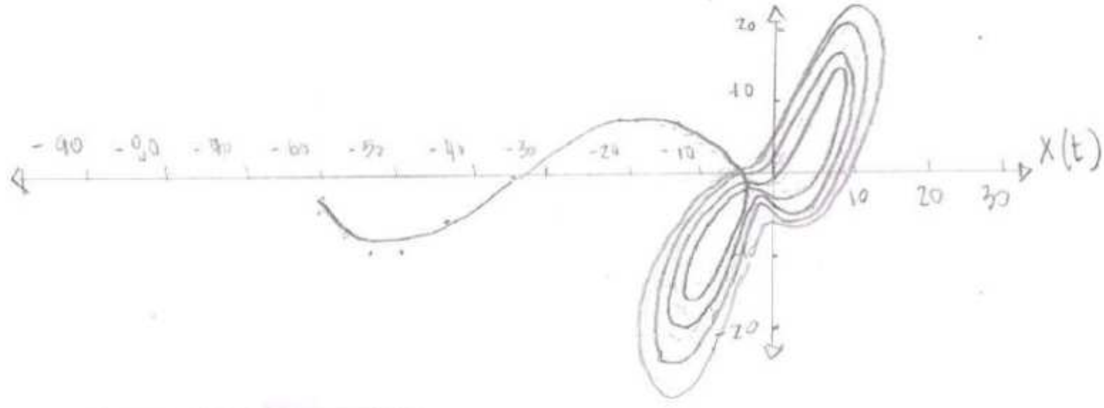
$\dot{y} = rx - xz - y$

$\dot{z} = xy - bz$

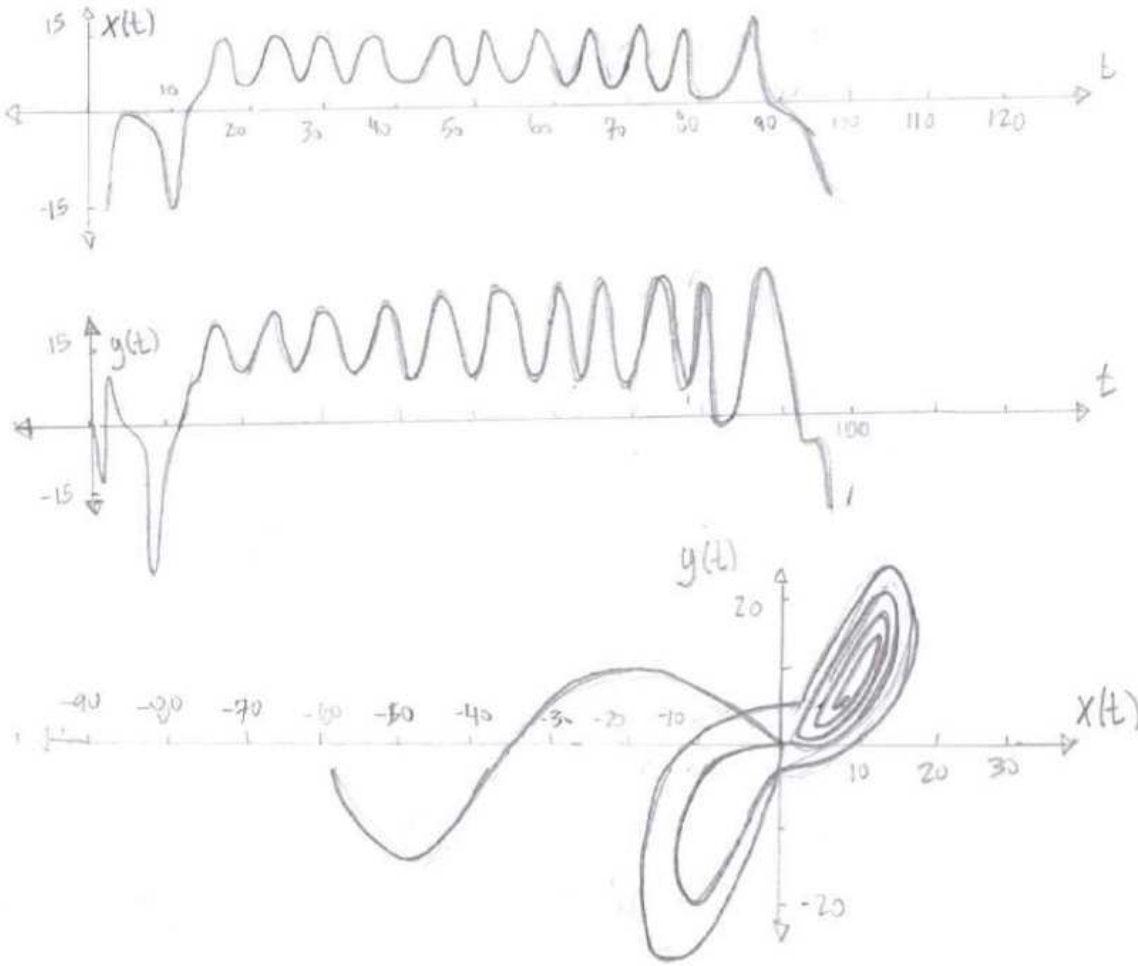
9.3.3. $\sigma = 10$; $b = 8/3$; $r = 22$



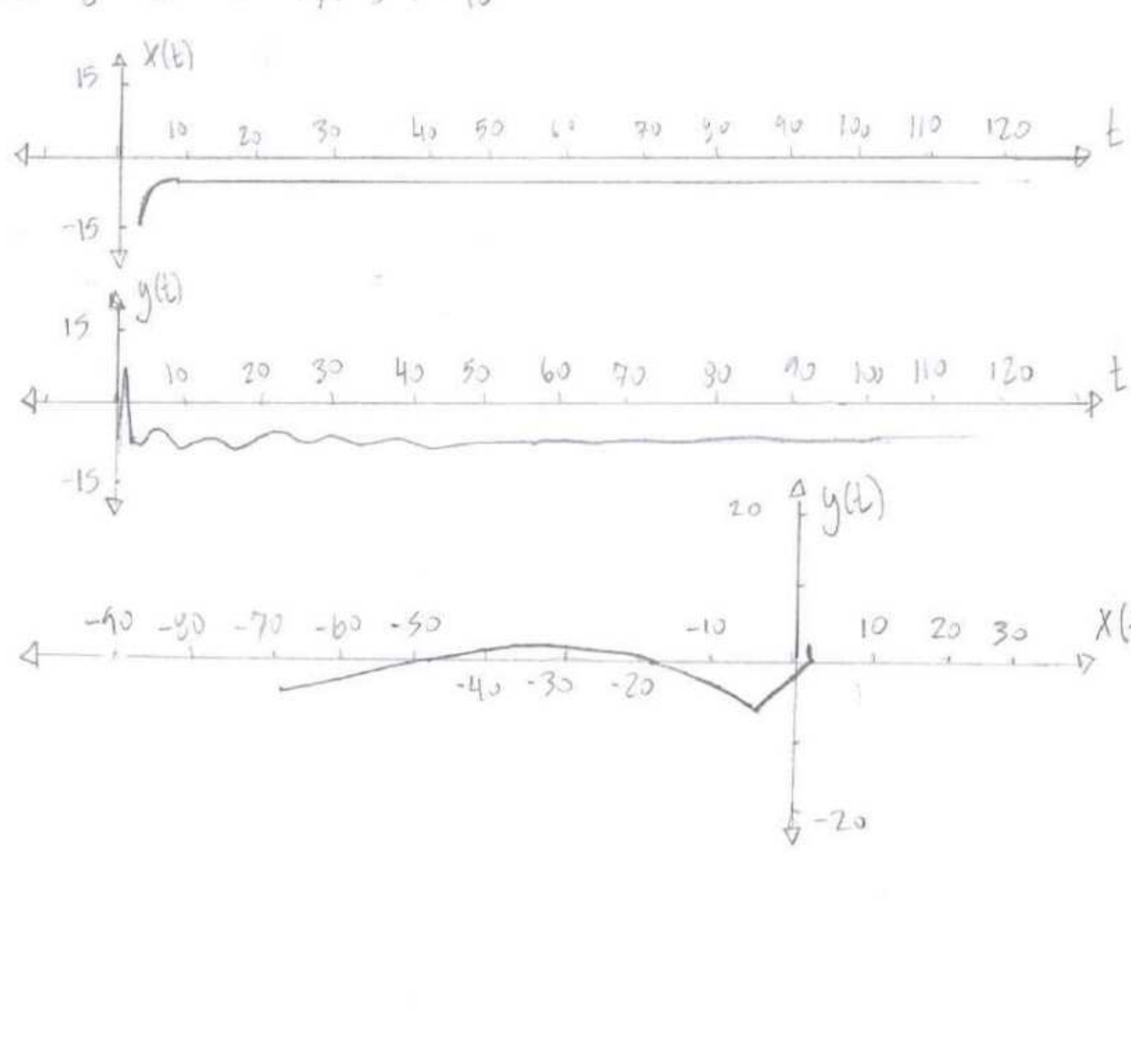




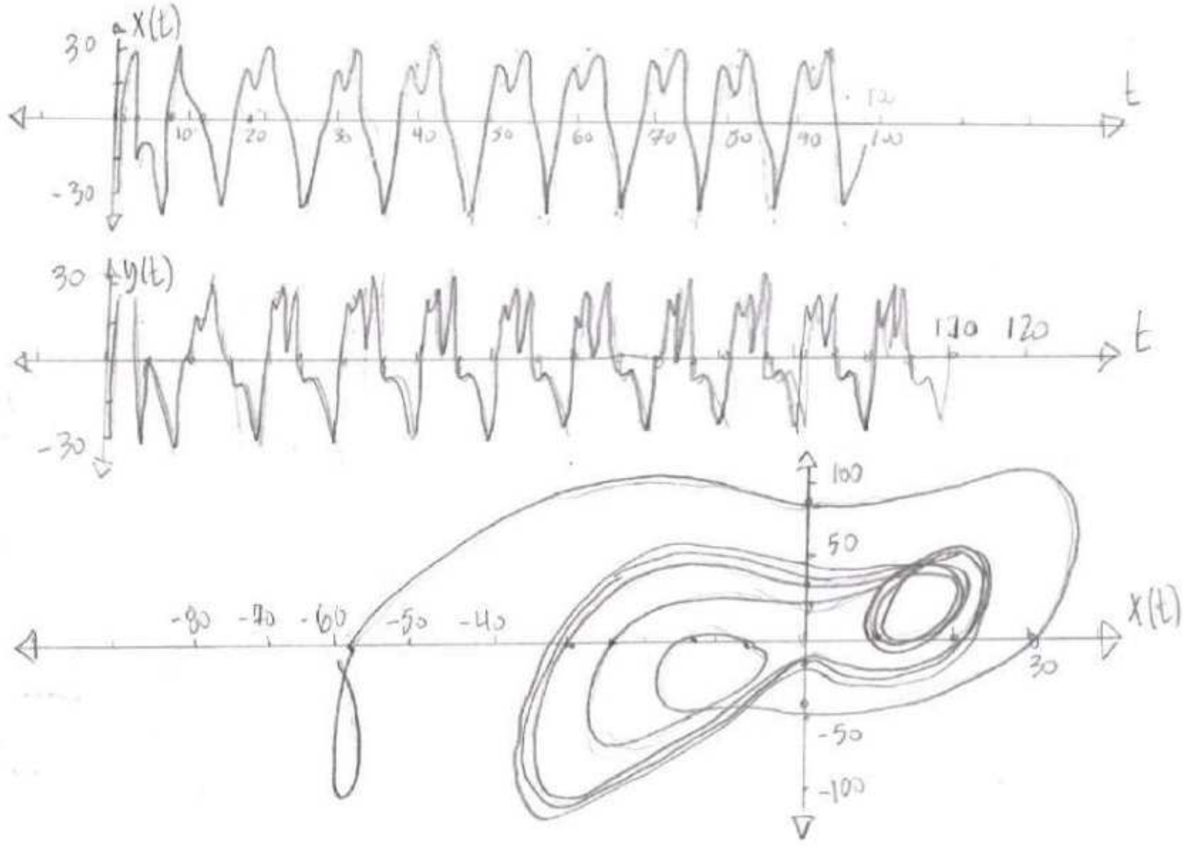
9.3.4 $\sigma = 10$; $b = 8/3$; $r = 24.5$



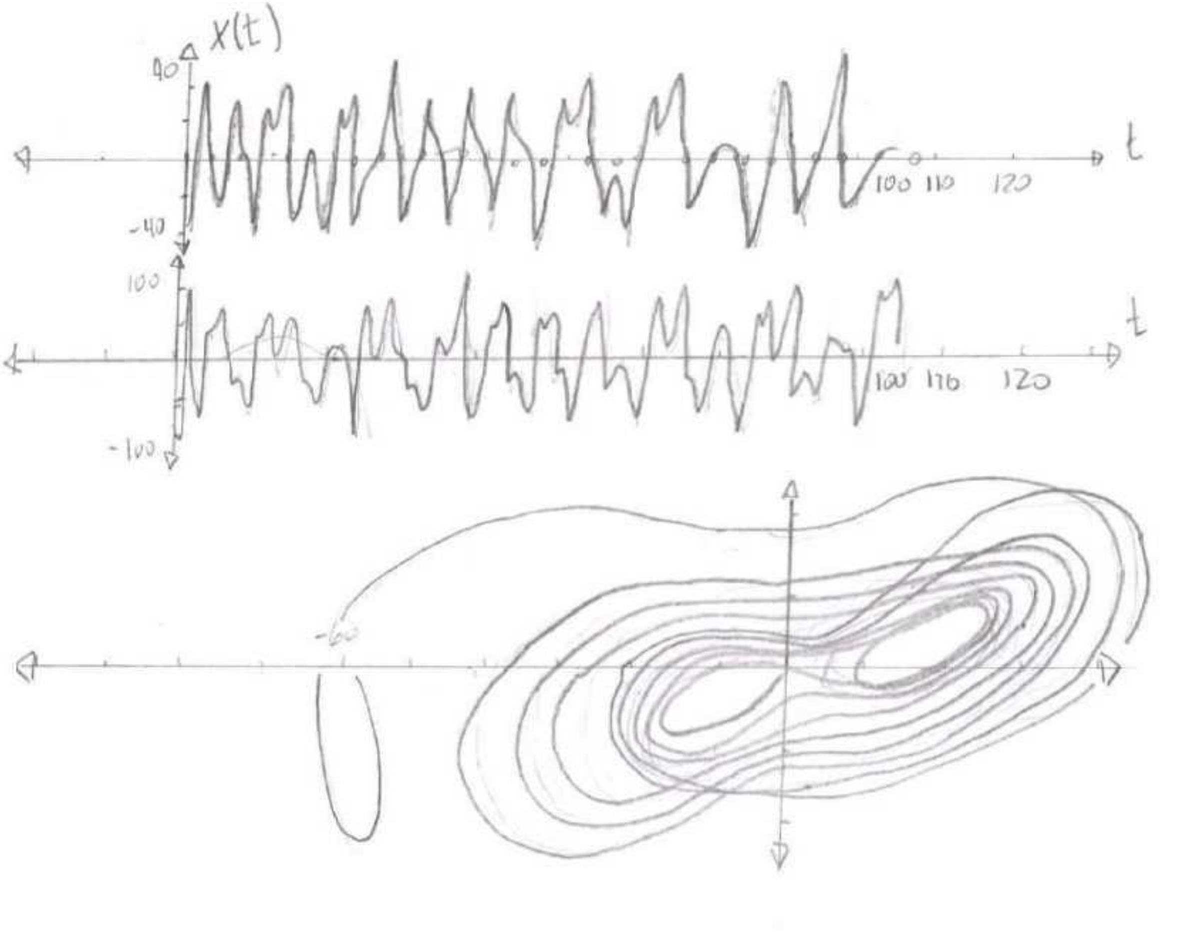
9.3.2 $\sigma = 10$; $b = 8/3$; $r = 10$



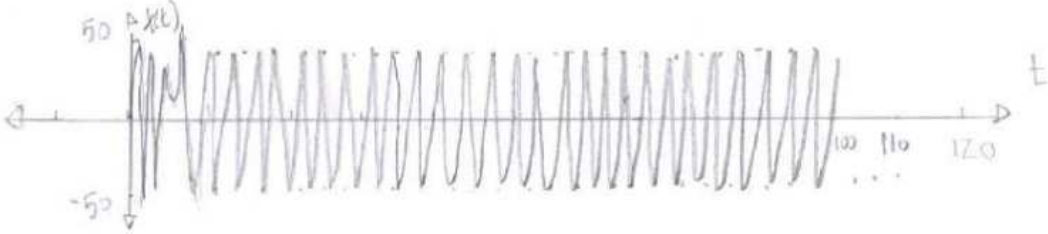
9.3.5. $\sigma = 10$; $b = 8/3$; $r = 100$

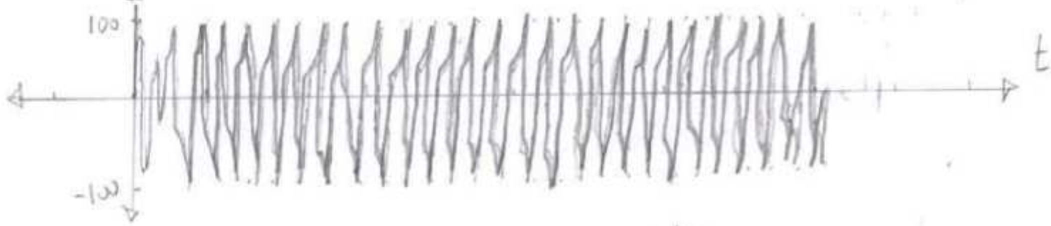


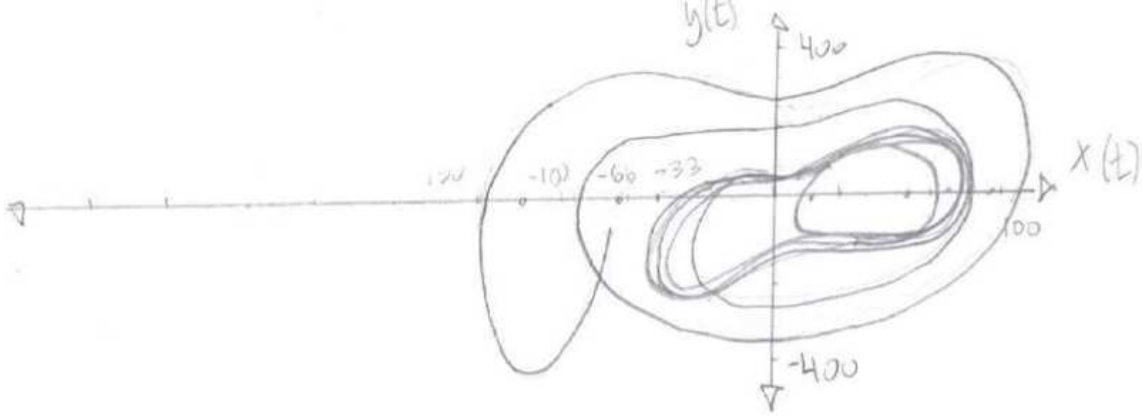
9.3.6. $\sigma = 10$; $b = 8/3$; $r = 126.52$



9.3.7. $\sigma = 10$; $b = 8/3$; $r = 400$







Note: Runge-Kutta 4th order $x_0 = -58$, $y_0 = -3.3$, $z_0 = 12.2$, $\Delta h = 0.1$

x_n	$x_{n-1} + \frac{\Delta h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
y_n	$y_{n-1} + \frac{\Delta h}{6}(l_1 + 2l_2 + 2l_3 + l_4)$
z_n	$z_{n-1} + \frac{\Delta h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$
k_1	$x(t, x, y) \cdot \Delta h$
l_1	$y(t, x, y, z) \cdot \Delta h$
m_1	$z(t, x, y, z) \cdot \Delta h$
k_2	$x(t + \frac{\Delta h}{2}, x + \frac{\Delta h k_1}{2}, y + \frac{\Delta h l_1}{2}) \cdot \Delta h$
l_2	$y(t + \frac{\Delta h}{2}, x + \frac{\Delta h k_1}{2}, y + \Delta h \frac{l_1}{2}, z + \Delta h \frac{m_1}{2}) \cdot \Delta h$
m_2	$z(t + \frac{\Delta h}{2}, x + \frac{\Delta h k_1}{2}, y + \frac{\Delta h l_1}{2}, z + \Delta h \frac{m_1}{2}) \cdot \Delta h$
k_3	$x(t + \frac{\Delta h}{2}, x + \frac{\Delta h k_2}{2}, y + \Delta h \frac{l_2}{2}) \cdot \Delta h$
l_3	$y(t + \frac{\Delta h}{2}, x + \frac{\Delta h k_2}{2}, y + \Delta h \frac{l_2}{2}, z + \Delta h \frac{m_2}{2}) \cdot \Delta h$
m_3	$z(t + \frac{\Delta h}{2}, x + \frac{\Delta h k_2}{2}, y + \Delta h \frac{l_2}{2}, z + \Delta h \frac{m_2}{2}) \cdot \Delta h$
k_4	$x(t + \Delta h, x + \Delta h k_3, y + \Delta h l_3) \cdot \Delta h$
l_4	$y(t + \Delta h, x + \Delta h k_3, y + \Delta h l_3, z + \Delta h m_3) \cdot \Delta h$
m_4	$z(t + \Delta h, x + \Delta h k_3, y + \Delta h l_3, z + \Delta h m_3) \cdot \Delta h$

$\dot{r} = r(1-r^2)$ 9.3.8.

$\dot{\theta} = 1$

a) Invariant set: a set of points (states) in a dynamic system which are mapped into other points in the same set by the dynamic evolution operator.

Yes, the equation system is invariant when $r \le 1$ because the constant outcome in the dynamical system

b) Open set: a union containing every point in the collection or every subset.

When $r \le 1$, the disk is an open set, since every point space, any union, or subset frequents similar properties

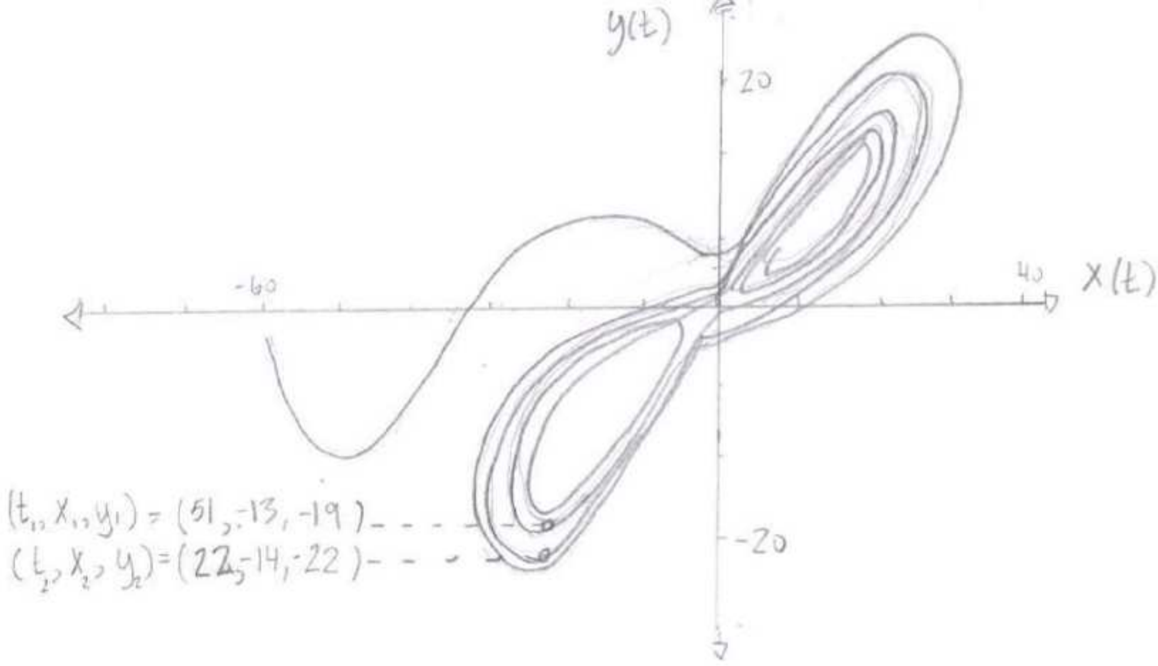
c) Attractor: a set to which all neighboring trajectories converge.

The function set shows an unstable node with exact trajectories, so an attractor at $x^2+y^2=1$.

d) $x^2+y^2=r$ is an attractor.

9.3.9 $\sigma = 10$; $b = 8/3$; $r = 28$.

The time horizon determined from the graph: $t_{horizon} \sim O\left(\frac{1}{\lambda} \ln \frac{a}{\|\delta_0\|}\right) =$



y(t)
20
-60
40
x(t)
(t1, x1, y1) = (51, -13, -19)
(t2, x2, y2) = (22, -14, -22)
-20

$\dot{r} = r(1-r^2)$ 9.3.8.

$\dot{\theta} = 1$

a) Invariant set: a set of points (states) in a dynamic system which are mapped into other points in the same set by the dynamic evolution operator.

Yes, the equation system is invariant when $r \leq 1$ because the constant outcome in the dynamical system

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When $r \leq 1$, the disk is an open set, since every point space, any union, or subset frequents similar properties

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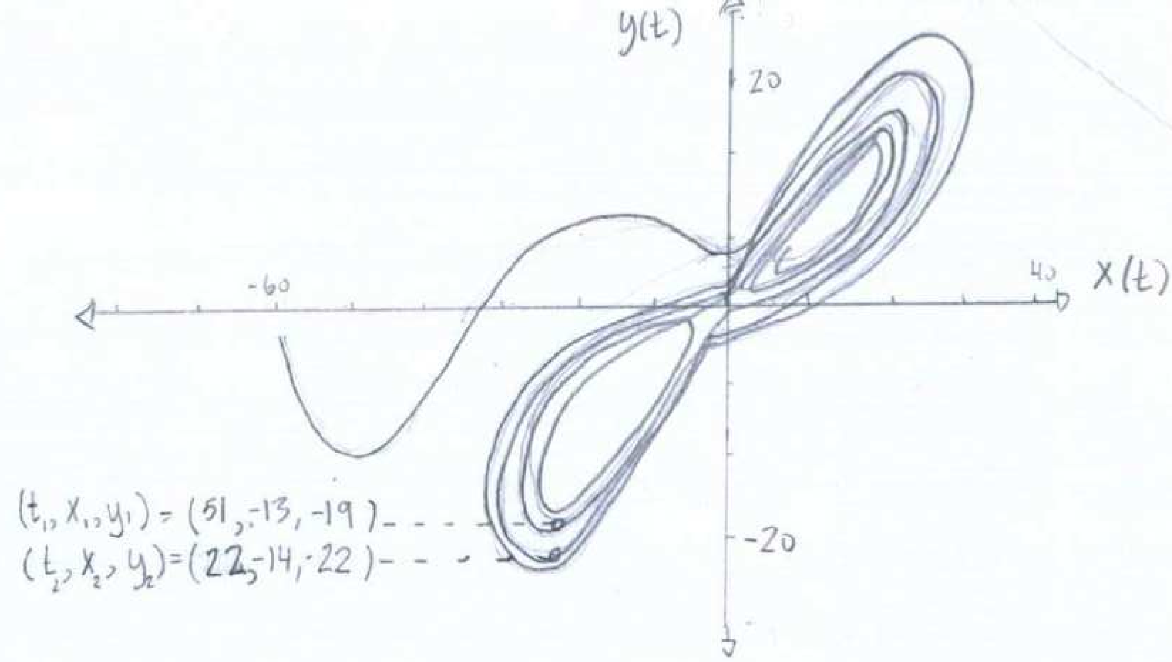
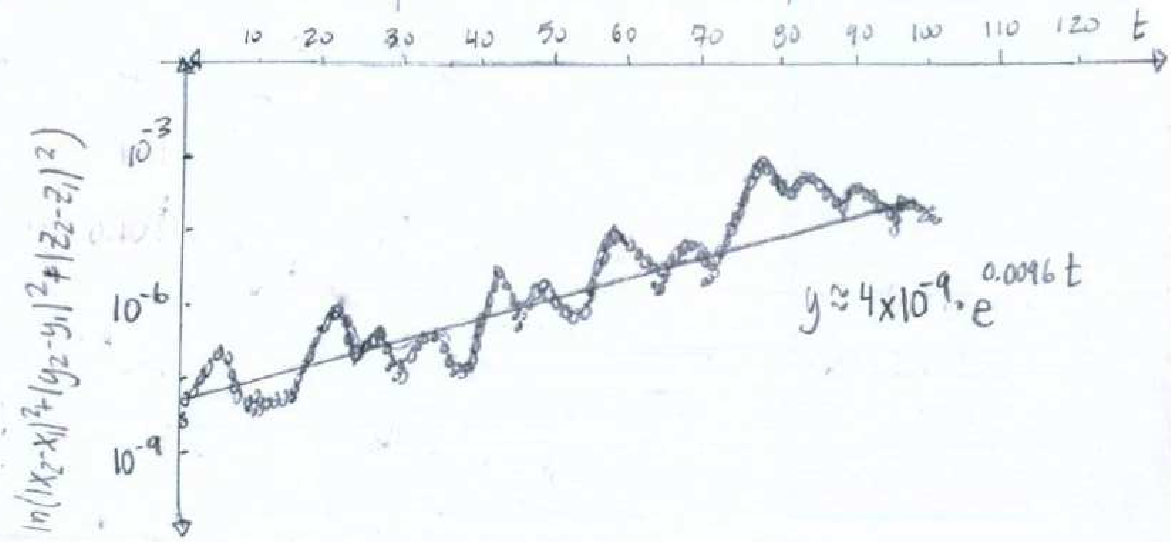


Steps for a Lyapunov Exponent λ:

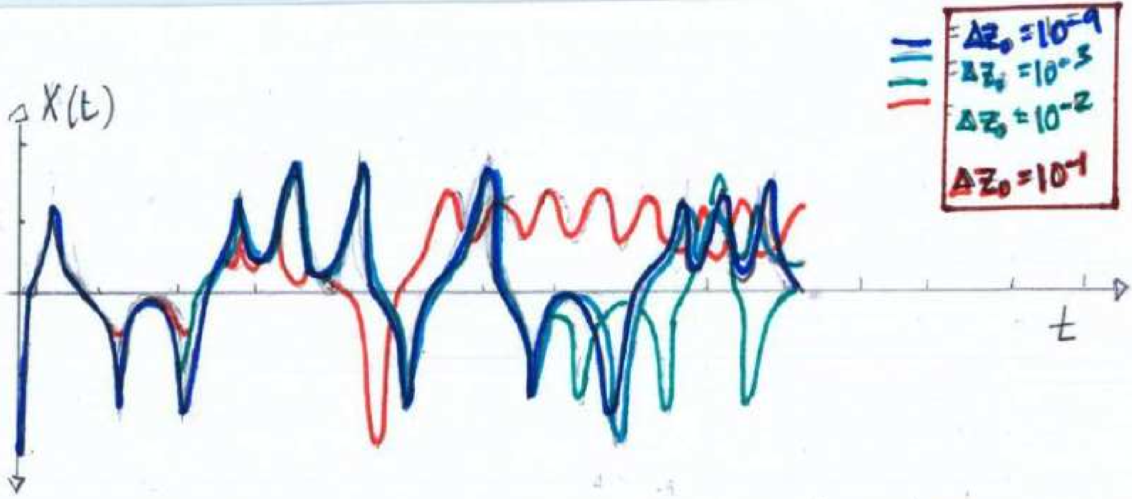
① A Runge-Kutta 4th order calculation for an equation system

② The z-variable's initial condition changes by $\delta = 1\times10^{-9}$ through new coordinates.

③ A plot of $\ln(|x_2-x_1|^2+|y_2-y_1|^2+|z_2-z_1|^2)$ vs time shows a linear plot with a slope.



The book shifted z_0 by 0.001 with a $\lambda = 0.8623$ result. A 1×10^{-9} δ_0-shift generated smaller Lyapunov constants at 0.01.



The new initial condition shifts the time horizon. A small shift in ΔZ misaligned later in time while a large value, much earlier.

9.4.1 See problem 9.3.9.

$$X_{n+1} = \begin{cases} 2X_n & 0 \le X_n \le 1/2 \\ 2-2X_n & 1/2 \le X_n \le 1 \end{cases}$$

9.4.2.

a) The "tent map" function peaks at $X_n = 2$.

b) Fixed Points: $0 \le X_n \le 1/2$; $X_n + 1 = 0 = 2X_n$

$X_n^* = 0$ "stable"

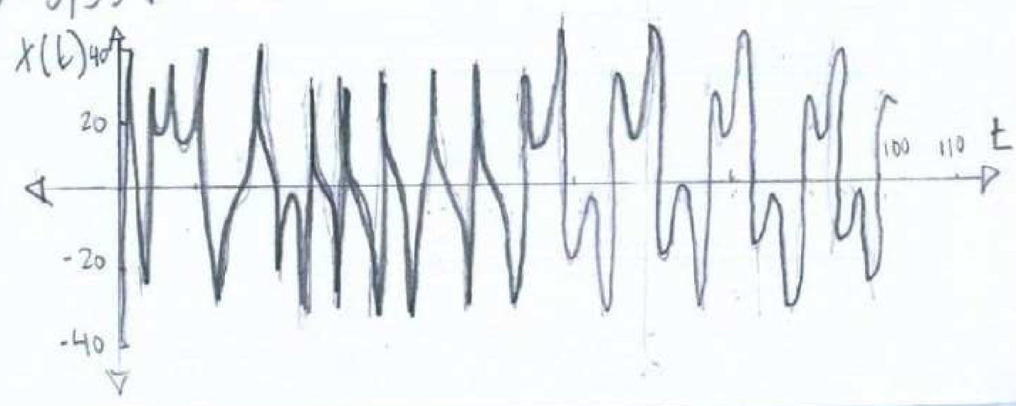
$1/2 \le X_n \le 1$; $X_{n+1} = 0 = 2 - 2X_n$

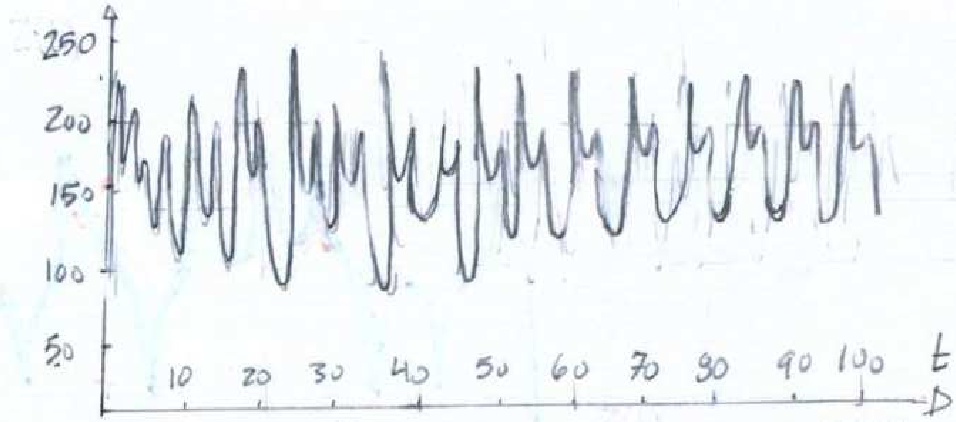
$X_n^* = 1$ "stable"

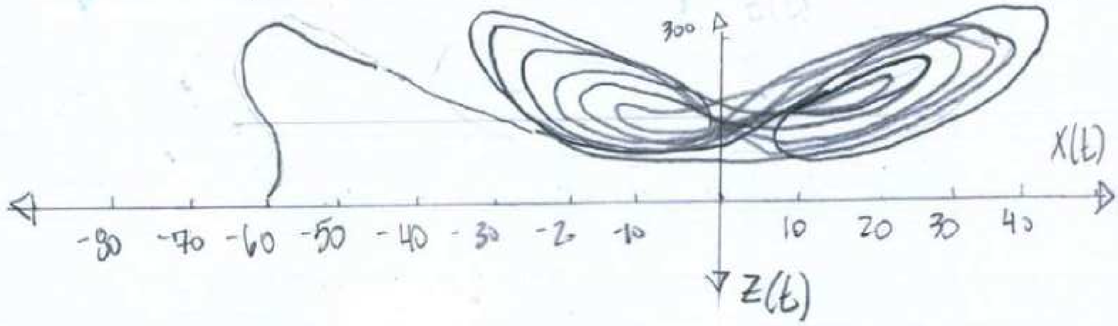
c) $X_{n+1} = \begin{cases} 2X_n & 0 \le X_n \le 1/2 \\ 2(1-X_n) & 1/2 \le X_n \le 1 \end{cases}$ The piecewise function is n-periodic at $X_n = 0$ and $X_n = 1$.

d) See part c.

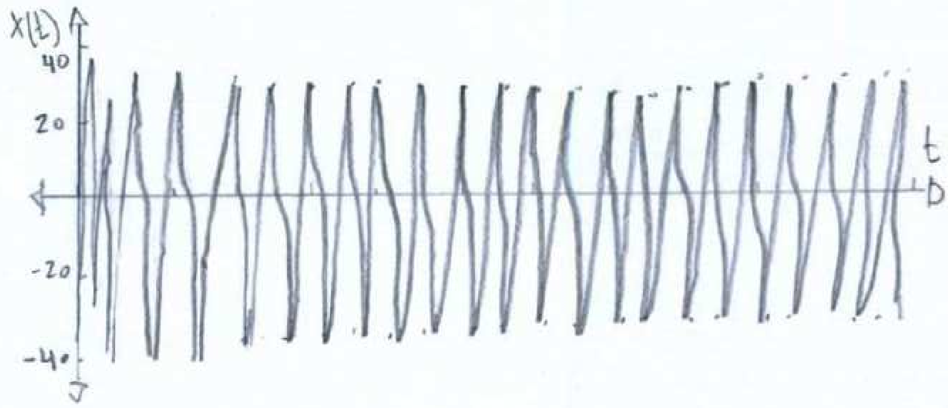
9.5.1 $\sigma = 10$; $b = 8/3$; $r = 166.3$.

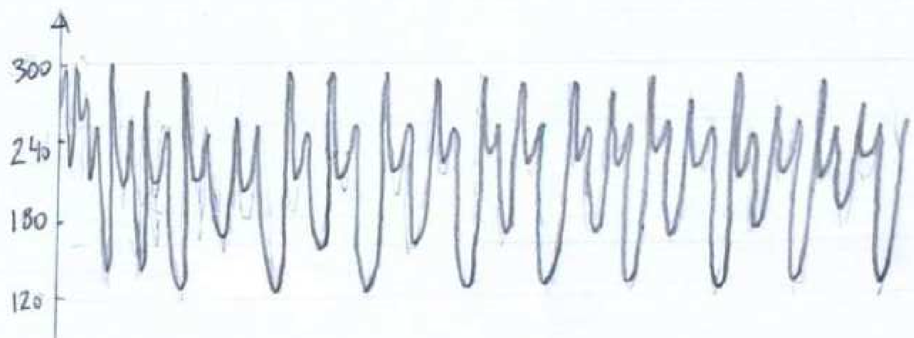


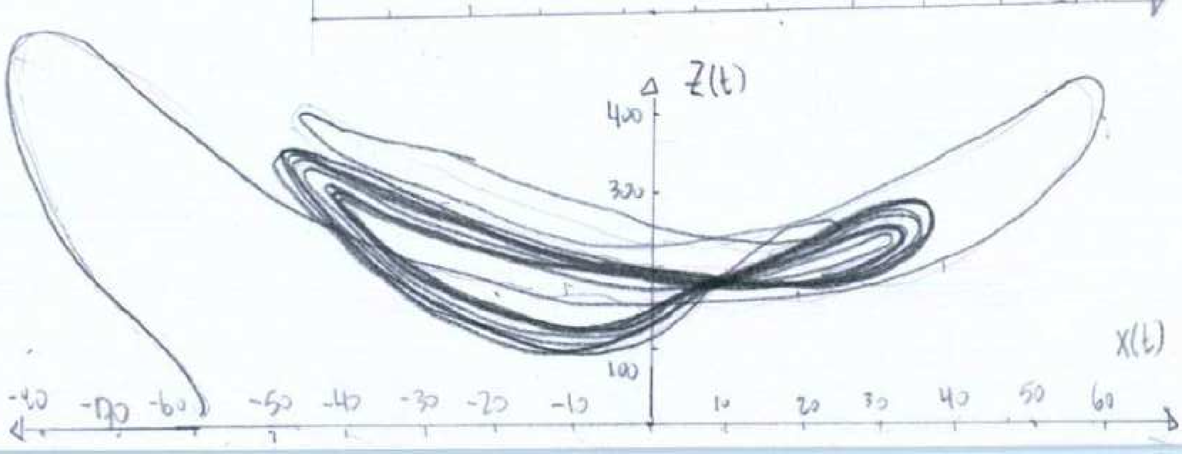




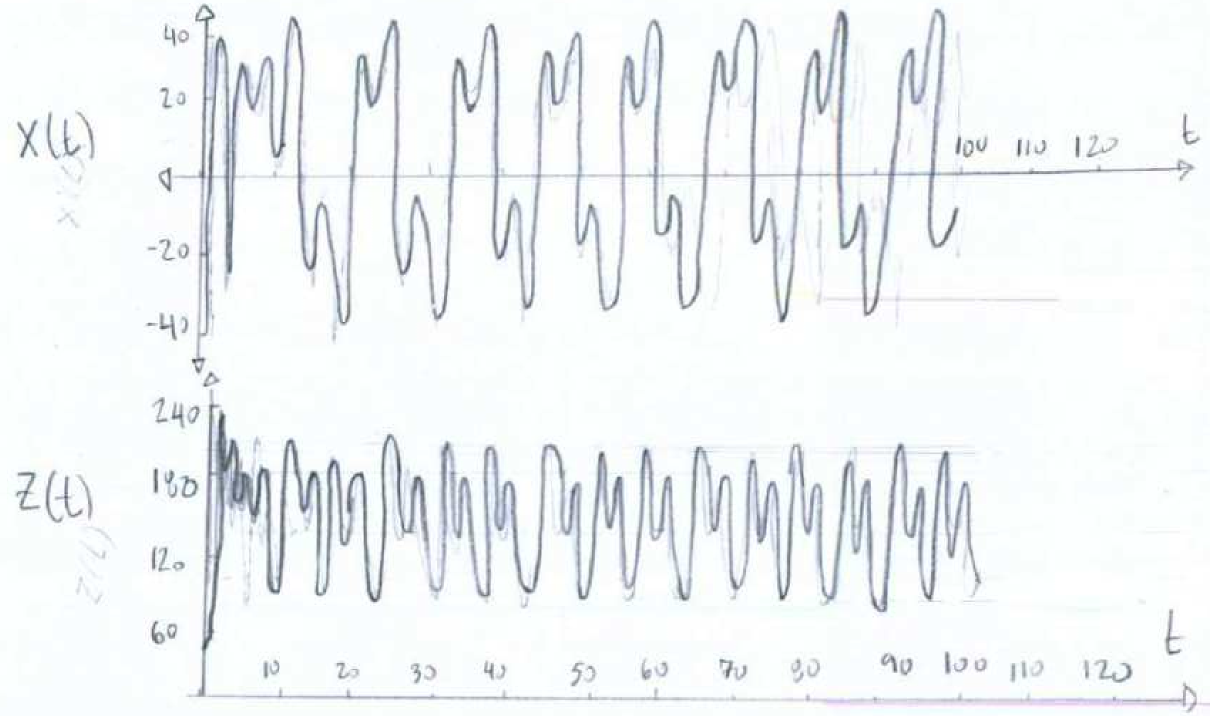
$9.5.2\ \sigma = 10;\ b = 8/3;\ r = 212$

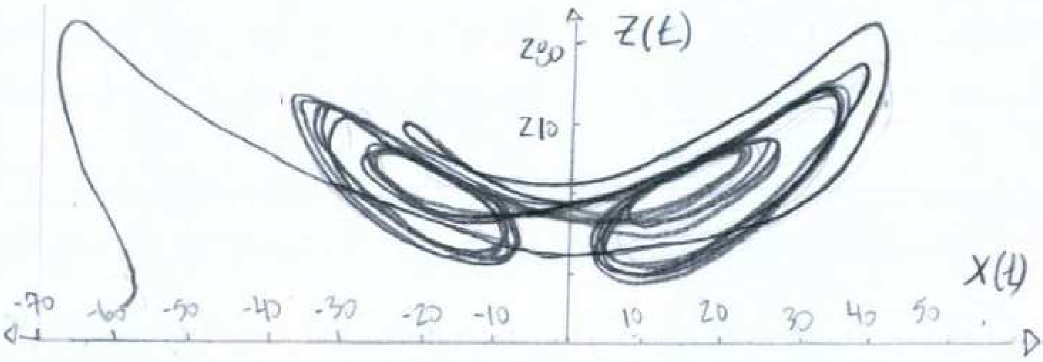




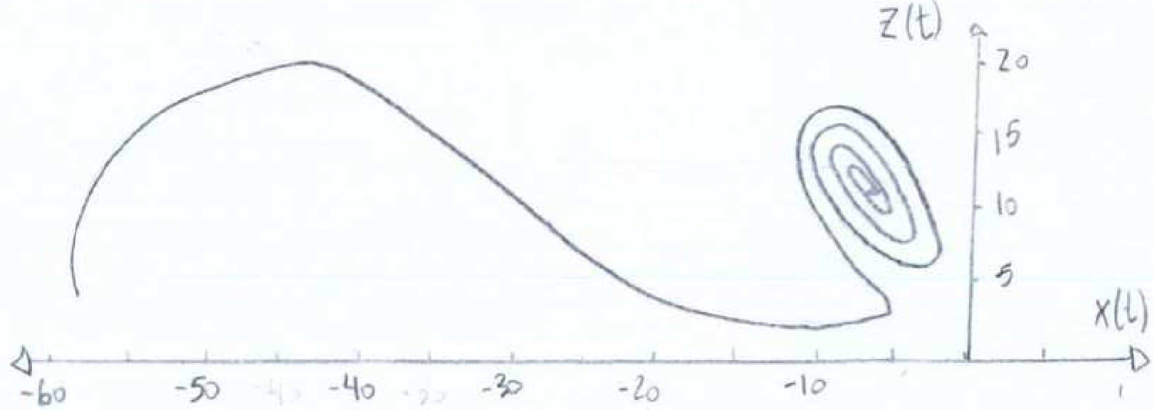


9.5.3 $\sigma=10$; $b=8/3$; $r=150$





9.5.4 $\sigma=10$; $b=8/3$; $r=24.4+\sin\omega t$; $\omega=10$



X(t) is negative when $\omega \approx 10$ and rane from 5 or more plots about Lorenz equation.

Note: Runge-Kutta 4th order employed for r-coefficient.

$X' = Y$

$Y' = -XZ$

$Z' = XY$

9.5.5.

a. $\varepsilon = r^{-1/2}$; $\dot{x} = \sigma(y-x)$; $\dot{y} = rx - xz - y$; $\dot{z} = xy - bz$

$= \sigma y - \sigma x$ $\quad = \frac{x}{\varepsilon^2} - xz - y$

$\tau = \frac{t}{\varepsilon}$; $X' = \sigma\varepsilon^2 y - \sigma\varepsilon^2 x$; $Y' = \varepsilon X - \varepsilon^3 XZ - \varepsilon^3 y$; $Z' = \varepsilon^3 xy - bZ$

$X = \varepsilon x$ $\quad X' = Y - \sigma\varepsilon X$; $Y' = \sigma X - XZ - \varepsilon Y$; $Z' = XY - bZ$

$Y = \varepsilon^2 \sigma y$

$Z = \sigma(\varepsilon^2 Z - 1)$ $X' = Y - \sigma\varepsilon X$; $Y' = -XZ - \varepsilon Y$; $Z' = XY - b\varepsilon\sigma(\frac{Z}{\sigma}+1)$

$\lim_{\varepsilon\to 0} X' = Y$; $\lim_{\varepsilon\to 0} Y' = -XZ$; $\lim_{\varepsilon\to 0} Z' = XY$

b. A constant of motion is a zero quantity, zero relationship, change of zero, or conserved system.

$(Y^2 + Z^2)' = -2YY' + 2ZZ'$

$= -2XYZ + 2XYZ$

$= 0$

$(X^2 - 2Z)' = 2XX' - 2Z'$

$= 2XY - 2XY$

$= 0$

c. A volume-preserving system follows either

$\dot{V} = \int_V \nabla \cdot f \, dV < 0$ or $\dot{V}(x, y, z) < 0$

The model provides an x, y, z vector.

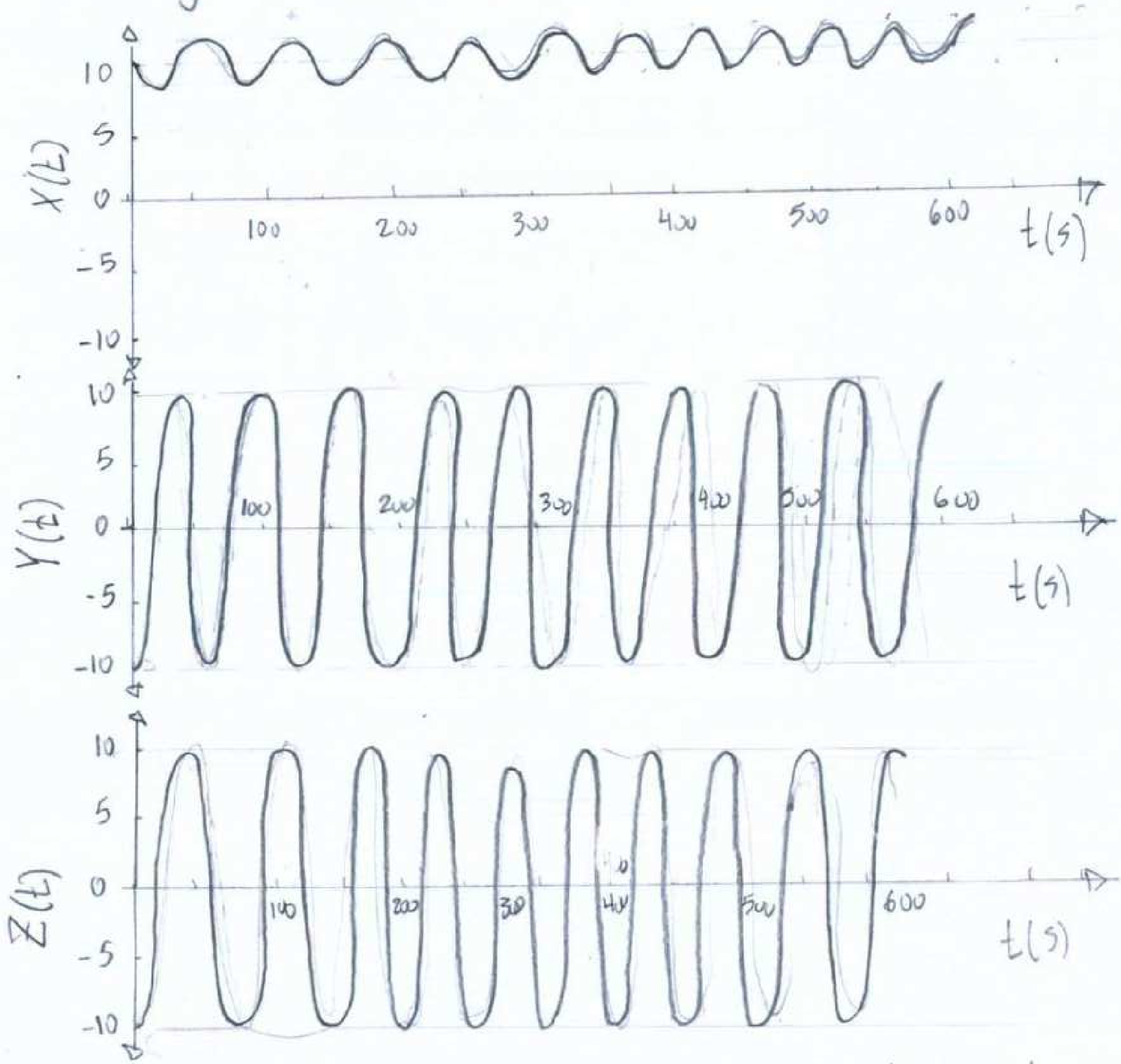
$\dot{V} = \int \nabla \cdot F \, dV = \int \nabla \cdot \langle X', Y', Z' \rangle dV$

$= \int \nabla \cdot \langle Y, -XZ, XY \rangle dV = \int \langle \frac{dY}{dX} - \frac{dXZ}{dY} + \frac{dXY}{dZ} \rangle dV = 0$

The systems constant rate of change is volume preserving.

d) r, also known as Rayleigh's number for fluids, in heat transfer or convection boundaries or diffusion limits, as a measurable criterion in mediums. The constant is an aggregate ratio at a specific temperature, for energy flux per dissapation. As r becomes infinity, then the flux flows undamped.

e) Runge-Kutta 4th Order: $\Delta h = 0.1$; $(x_0, y_0, z_0) = (10, -10, 0)$



The individual plots align well with Lorentz' equations.

$V = \frac{1}{2}e_2^2 + 2e_3^2$ 9.6.1

$\dot{e}_1 = \sigma(e_2 - e_1)$

$\dot{e}_2 = -e_2 - 20u(t)e_3$

$\dot{e}_3 = 5u(t)e_2 - be_3$

a) $V = \frac{1}{2}e_2^2 + 2e_3^2$

$$\dot{V} = \frac{1}{2}2e_2 \cdot \dot{e}_2 + 4e_3 \cdot \dot{e}_3$$

$$= -e_2^2 - 20u(t) \cdot e_2e_3 + 20u(t)e_2e_3 - 4be_3^2$$

$$= -e_2^2 - 4be_3^2$$

$$= -2\left(\frac{1}{2}e_2^2 + 2be_3^2\right)$$

For the inequality, $K < 2$ and $K < 2b$,

$\dot{V} \leq -kV$; $\ln V = -kt + V_0$; $0 < V(t) < V_0 e^{-kt}$

b) A proof about $e_2(t)$ & $e_3(t)$ quickly approaching zero:

$$\frac{1}{2}e_2(t)^2 < V(t) < V_0 e^{-kt}$$

$$e_2(t) < \sqrt{2V(t)} < \sqrt{2V_0}\, e^{-kt/2}$$

and

$$2e_3(t)^2 < V(t) < V_0 e^{-kt}$$

$$e_3(t) < \sqrt{\frac{V(t)}{2}} < \sqrt{\frac{V_0}{2}}\, e^{-kt/2}$$

c) $e_1(t)$ becomes zero.

$$\dot{e}_1(t) = \sigma(e_2 - e_1) \; ; \; \dot{e}_1(t) + \sigma e_1(t) = \sigma e_2(t)$$

$$< \sigma\sqrt{2V_0}\, e^{-kt/2}$$

The equation isn't a Bernouilli Differential, but a Bernoulli method (of two functions)!

Function #1: $e_1(t) = uv$

Function #2: $\dot{e}_1(t) = u\dot{v} + \dot{u}v$

$$\ddot{e}_1(t) + \sigma e_1(t) = \sigma\sqrt{2V_0}\, e^{-kt/2}$$

$$u\mathring{v} + \mathring{u}v + \sigma uv = \sigma\sqrt{2V_0}\, e^{-kt/2}$$

$$\mathring{u}v + u(\mathring{v} + \sigma v) = \sigma\sqrt{2V_0}\, e^{-kt/2}$$

Solving $\mathring{v} + \sigma v = 0$:

$$\mathring{v} = -\sigma v$$

$$-\int \frac{dv}{v} = \int \sigma\, dt$$

$$v = Ce^{-\sigma t}$$

Solving $\mathring{u}v = \sigma\sqrt{2V_0}\, e^{-kt/2}$

If $v = Ce^{-\sigma t}$, then

$$\mathring{u} = \sigma\sqrt{2V_0}\, e^{-\frac{kt}{2} + \sigma t}$$

$$\int du = \sigma\sqrt{2V_0} \int e^{-\frac{kt}{2} + \sigma t}\, dt$$

$$u = \frac{\sigma\sqrt{2V_0}}{-\frac{k}{2} + \sigma} e^{(\sigma - \frac{k}{2})t} + C$$

$$= -\frac{2\sigma\sqrt{2V_0}}{k - 2\sigma} e^{(\sigma - \frac{k}{2})t} + C$$

Substituting u and v:

$$e_2(t) = u \cdot v$$

$$= \left[\frac{-2\sigma\sqrt{2V_0}}{k - 2\sigma} e^{(\sigma - \frac{k}{2})t} + C\right] \cdot Ce^{-\sigma t}$$

$$= \frac{C}{e^{-\sigma t}} - \frac{2\sigma\sqrt{2V_0}}{k - 2\sigma} e^{-kt/2}$$

¡The equation decay is exponential!

$x(t) = x(t)$

$\dot{y}_r = r x(t) - y_r - x(t) z_r$

$\dot{z}_r = x(t) y_r - b z_r$

9.6.2.

a) If $e_1 = x - x_r$, $e_2 = y - y_r$, and $e_3 = z - z_r$

$e_1 = 0$, $e_2 = y_r$, and $e_3 = z_r$

$\dot{e}_2 = -x(t) e_3 - e_2$ and $\dot{e}_3 = x(t) e_2 - b e_3$

b) $V = e_2^2 + e_3^2$

$$
\begin{aligned}
\dot{V} &= 2 e_2 \dot{e}_2 + 2 e_3 \dot{e}_3 \\
&= 2e_2\left[-x(t)e_3 - e_2\right] + 2e_3\left[x(t)e_2 - be_3\right] \\
&= -2e_2^2 - 2e_3^2 \\
&= -2V, \quad V(t) = Ce^{-2t}
\end{aligned}
$$

Yes!

Th Liapunov function !

Whoa, nelly, jelly belly.

c) A Lorenz oscillator exhibits chaotic behavior with solely two equations. when co-dependent.

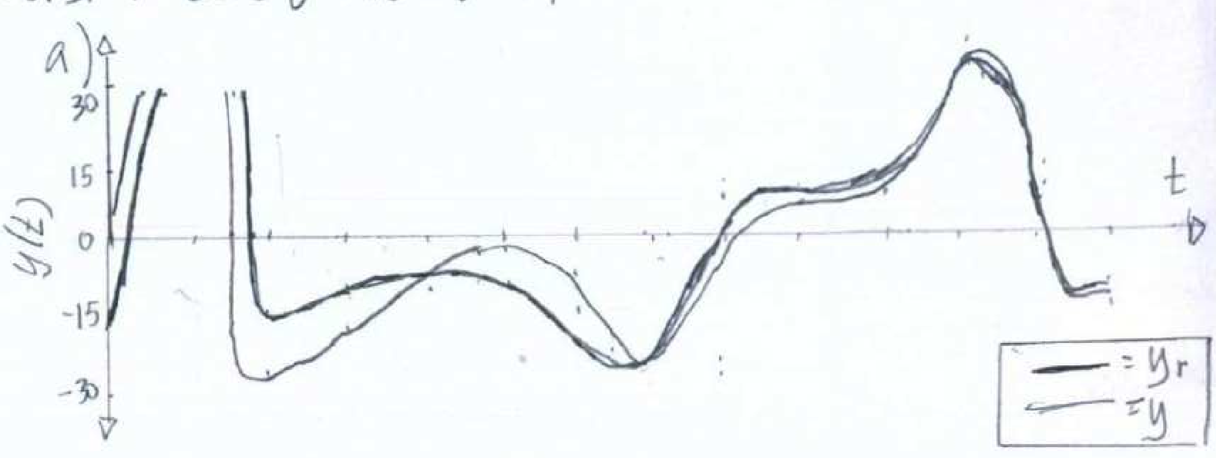
$x(t) = x(t)$

$\dot{y}_r = r x(t) - y_r - x(t) z_r$

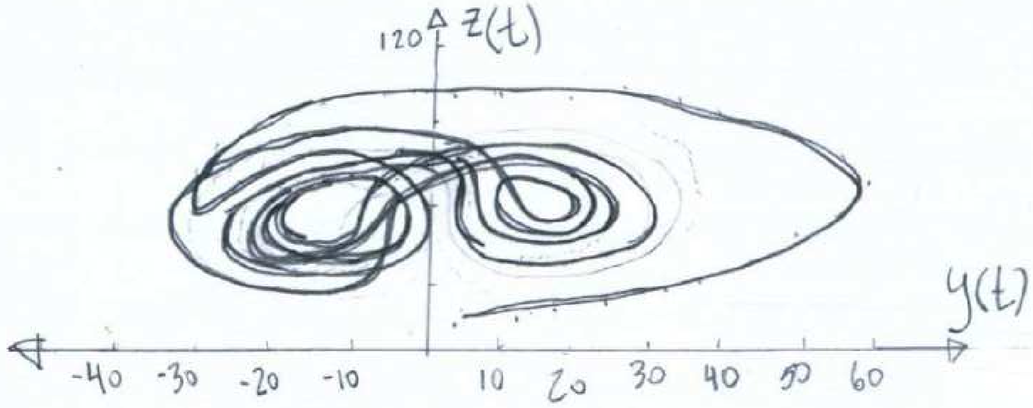
$\dot{z}_r = x(t) y_r - b z_r$

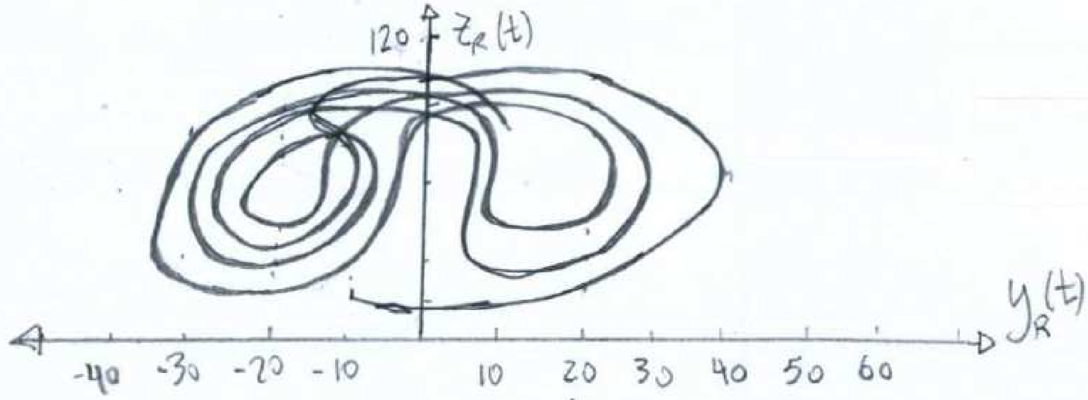
9.6.3. $r = 60$; $\sigma = 10$; $b = 8/3$.

a)



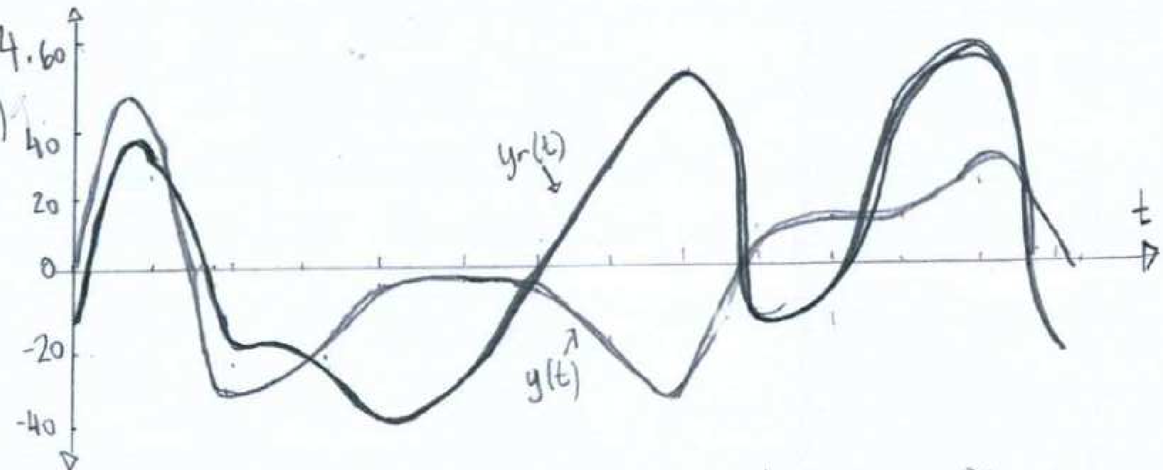
b)





9.6.4.

a)



Each plots shows an aperiodic behavior without synchronization or alignment in time.

b) Y(t) as the synchronization or driving signal is perfect alignment because $y(t) = y_R(t)$

$$\mathring{x}_R = \sigma(y_r - x_r)$$
$$\mathring{y}_R = r s(t) - y_R - s(t) z_R$$
$$\mathring{z}_R = s(t) y_r - b z_R$$

9.6.5

a) $s(t) = x(t) + m(t)$

where $x(t)$ = chaotic mask

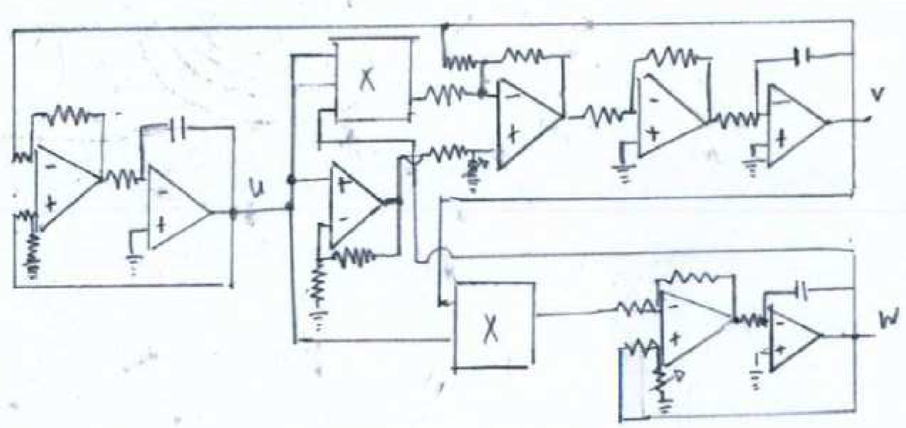
$m(t)$ = low-power message

When $m(t) = \sin(t)$, then $\mathring{x} = \sigma(y_r - x_r)$

$$\mathring{y}_R = r[x(t) + \sin(t)] - y_R - [x(t) + \sin(t)] z_R$$

$$\mathring{z}_R = [x(t) + \sin(t)] y_r - b z_R$$

9.6.6.



Three operational amplifier circuits are in the schematic above. One type is an integrator at the u, v, and w outputs. A second circuit is the differential amplifier before the v output-integrator, also a subtractor. Lastly, an adder following the u output near the circuit center.

$$\dot{u} = \sigma(v-u)$$

$$\dot{v} = ru - v - 20uw$$

$$\dot{w} = 5uv - bw$$

Chapter 10: One-Dimensional Maps:

$x_{n+1} = \sqrt{x_n}$

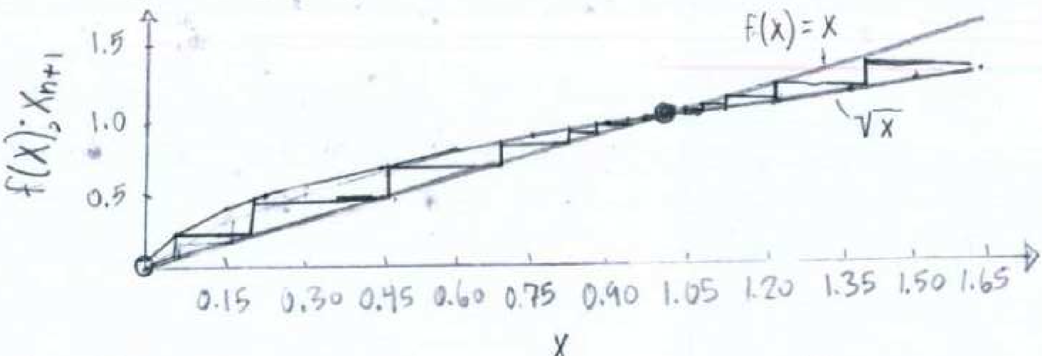
10.1.1. <u>Prediction</u>: Zero, Zero to one, one, and greater than one correspond with unique events on the calculator.

<u>Fixed Points</u>: $x_{n+1} = 0$

$x^* = 0, 1$

<u>Stability</u>: $x^* = 0$; $f'(0) = \infty$; unstable

$x^* = 1$; $f'(1) = 1/2$; stable



$x_{n+1} = x_n^3$

10.1.2. <u>Prediction</u>: Zero, zero to one, one, and greater than one are unique regions. Also, negative one to zero, negative one, and less than negative one show unique diagrams.

<u>Fixed Points</u>: $x_{n+1} = 0 = x_n^3$

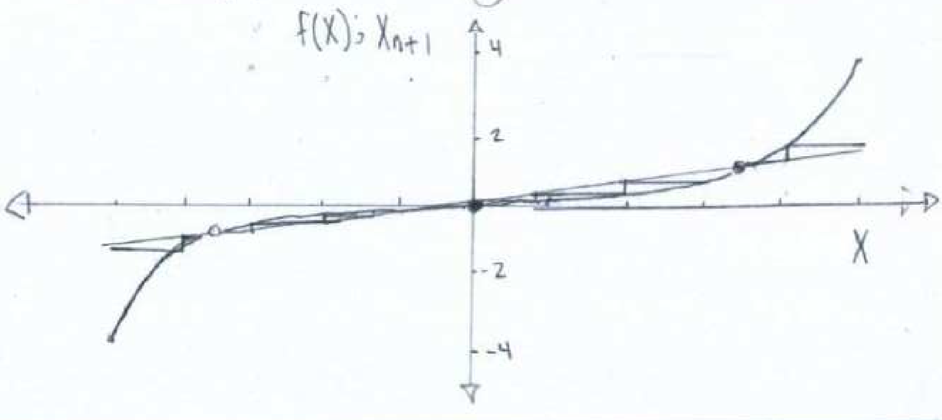
$x_n^3 - x_{n+1} = 0$

$x^* = -1, 0, 1$

<u>Stability</u>: $x^* = -1$; $f'(-1) = 2$; unstable

$x^* = 0$; $f'(0) = 0$; superstable

$x^* = 1$; $f'(1) = 1$; Marginal case



$X_{n+1} = \exp X_n$

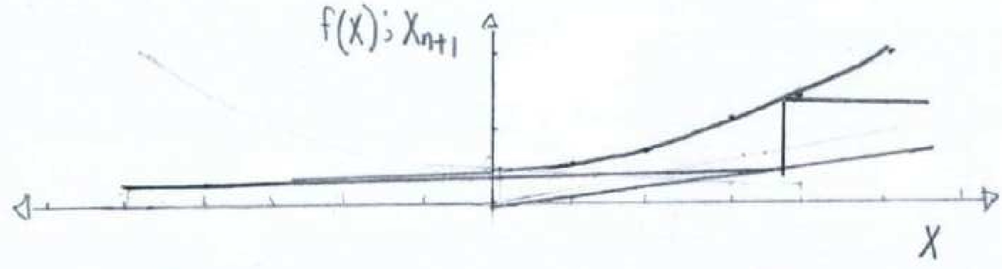
10.1.3. Prediction: Values below zero, at zero, and above zero dictate map's terminal behavior. ¡wrong!

Fixed Points: $X_{n+1} = 0 = \exp X_n$

$$\exp X_n - X_{n+1} = 0$$

$$X^* = NaN$$

Stability: Null



$X_{n+1} = \ln X_n$

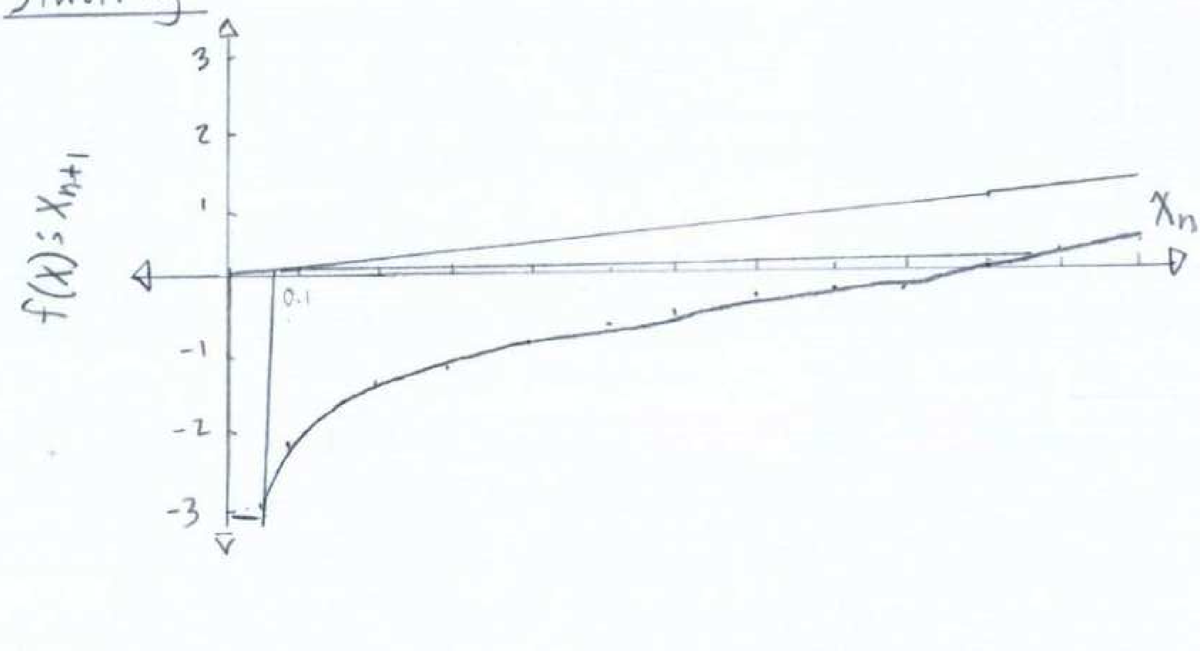
10.1.4. Prediction: The logarithm domain input values greater than zero. Successive iterations fail up till natural logarithm base, e, euler's number, 2.718..., then an exponential output.

Fixed Points: $X_{n+1} = 0 = \ln X_n$

$$\ln X_n - X_{n+1} = 0$$

$$X^* = NaN$$

Stability: Null



$X_{n+1} = \cot X_n$

10.1.5. Prediction: Similar solutions appear from periodic functions, such as cotangent because n-π relations.

Fixed Points: $X_{n+1} = 0 = \cot X_n$

$\cot X_n - X_{n+1} = 0$

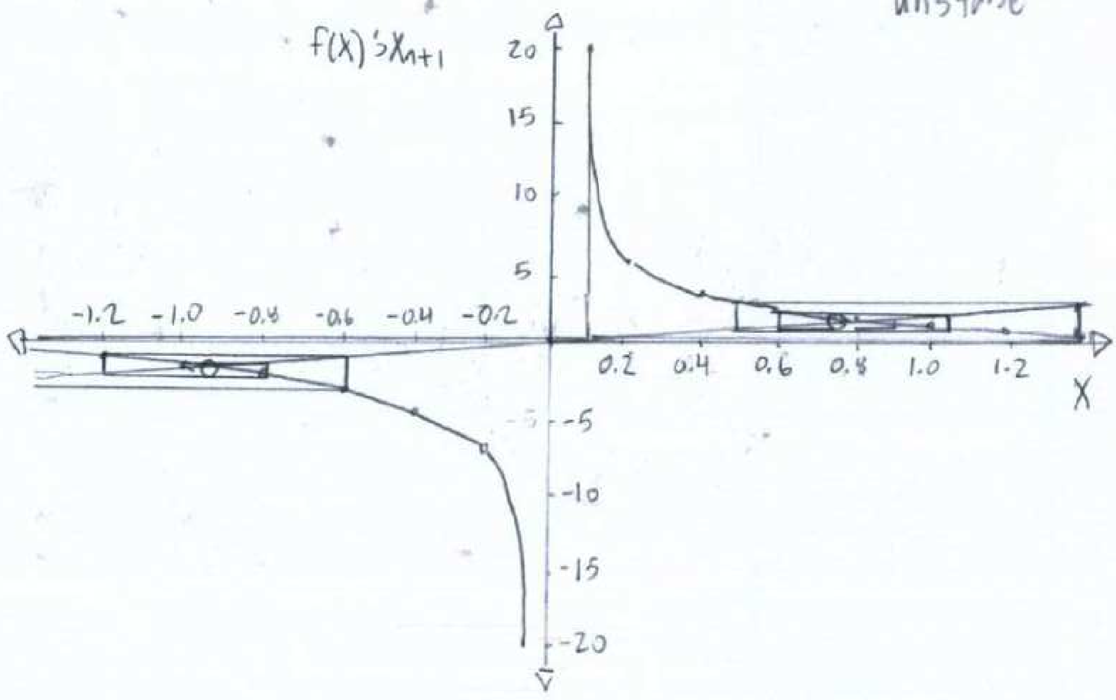
X^* when $\cot x = x$; $x^* = \pm 0.86, \pm 3.43, \ldots$

Stability: $x^* = +0.86$; $|f'(0.86)| = |-\csc^2(0.86)| = 1.74$ "unstable"

$x^* = -0.86$; $|f'(-0.86)| = |-\csc^2(-0.86)| = 1.74$ "unstable"

$x^* = 3.43$; $|f'(3.43)| = |-\csc^2(3.43)| = 12.36$ "unstable"

$x^* = -3.43$; $|f'(-3.43)| = |-\csc^2(-3.43)| = 12.36$ "unstable"



$X_{n+1} = \tan X_n$

10.1.6. Prediction: Since tan x equals zero at many points, the periodic nature has many solutions.

Fixed Points: $X_{n+1}^* = 0 = \tan X_n$

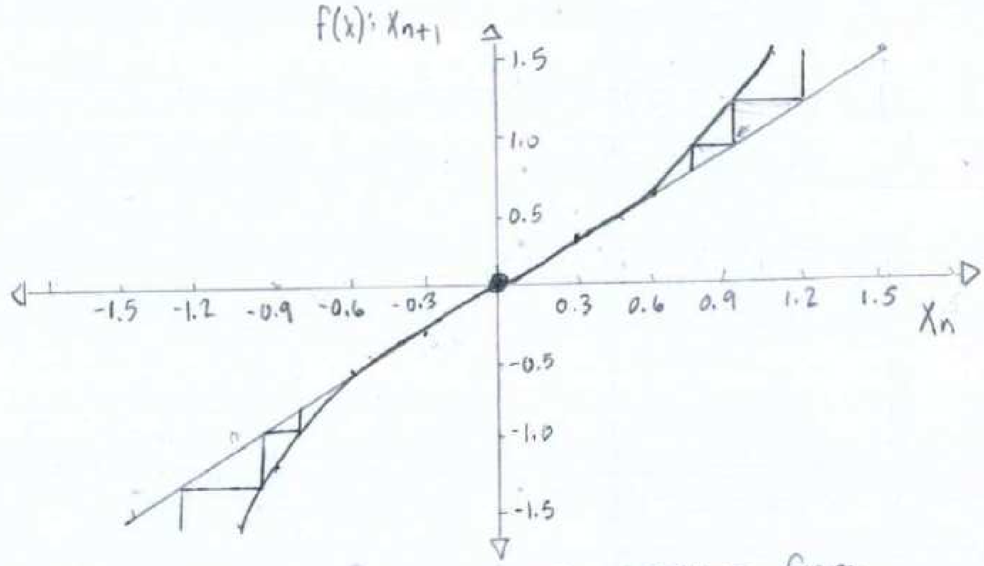
$\tan X_n - X_{n+1} = 0$

$X^* = (\pm n\pi)$ where $n \in \mathbb{Z}$

Stability: $x^* = 0$; $|f'(0)| = |1 - \tan^2(0)| = 1$ "Marginal case"

$x^* = -\pi$; $|f'(-\pi)| = |1 - \tan^2(-\pi)| = 1$ "Marginal case"

$x^* = \pi$; $|f'(\pi)| = |1 - \tan^2(\pi)| = 1$ "Marginal case"



$x_{n+1} = \sinh x_n$

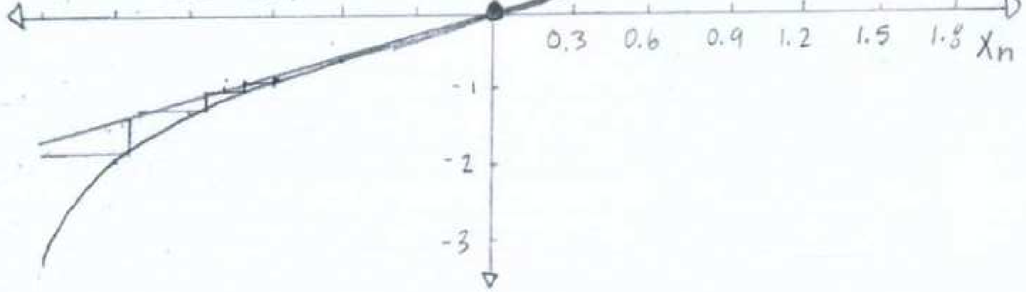
10.1.7. Prediction: Three important regions appear from $\sinh x$; less than zero, zero, and greater than zero.

Fixed points: $x_{n+1} = 0 = \sinh x_n$

$\sinh x_n - x_{n+1} = 0$

$x^* = 0$

Stability: $x^* = 0$; $|f'(0)| = |\cosh(0)| = 1$ "marginal case"



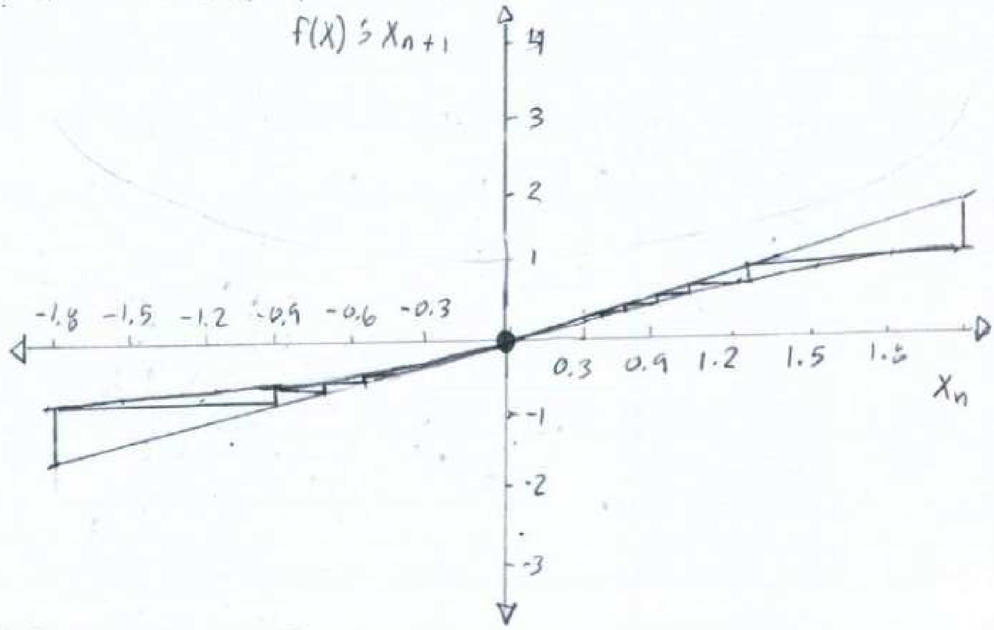
$x_{n+1} = \tanh x_n$

10.1.8 Prediction: The superstable value at zero is apparent, but x_n below zero and above generate unusual plots.

Fixed Points: $x^*_{n+1} = 0 = \tanh x_n$

$\tanh x_n - x_{n+1} = 0$; $x^* = 0$

Stability: $X^* = 0$; $|f'(0)| = |1 - \tanh(0)| = 1$ "marginal case"



$X_{n+1} = 2X_n/(1+X_n)$

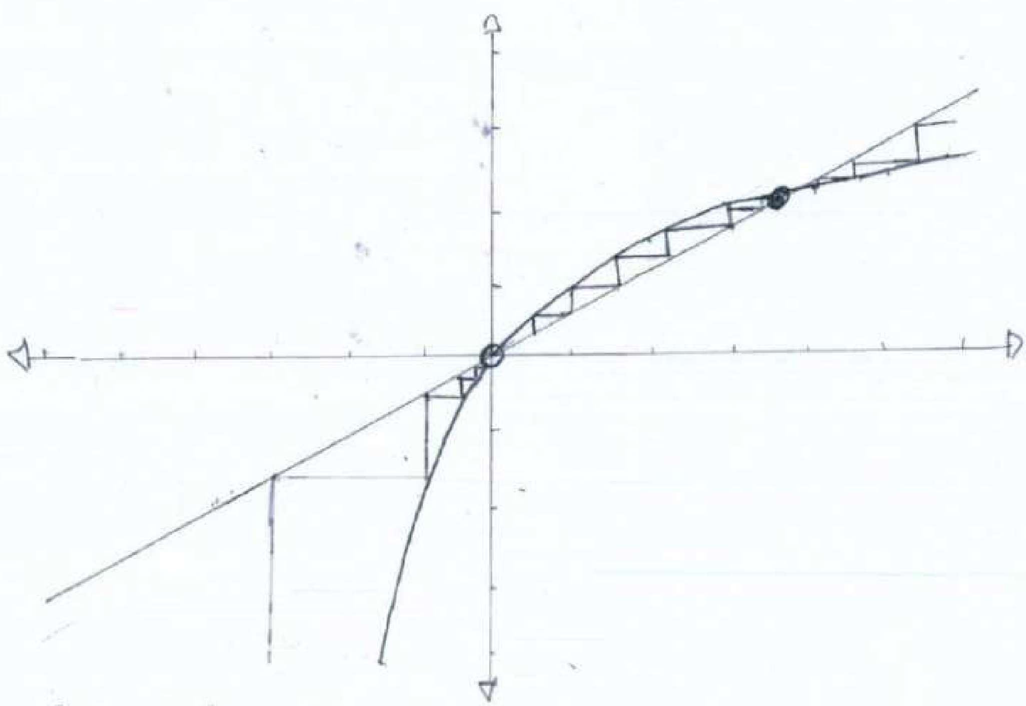
10.1.9. Fixed Points: $X_{n+1} = 0 = 2X_n/(1+X_n)$

$$\frac{2X_n}{(1+X_n)} - X_{n+1} = 0$$

$$X^* = 0, 1$$

Stability: $X^* = 0$; $|f'(0)| = \left|\frac{2-4(0)}{(1+0)^2}\right| = 2$; "unstable"

$X^* = 1$; $|f'(1)| = \left|\frac{2-4(1)}{(1+1)^2}\right| = 1$; "Marginal case"



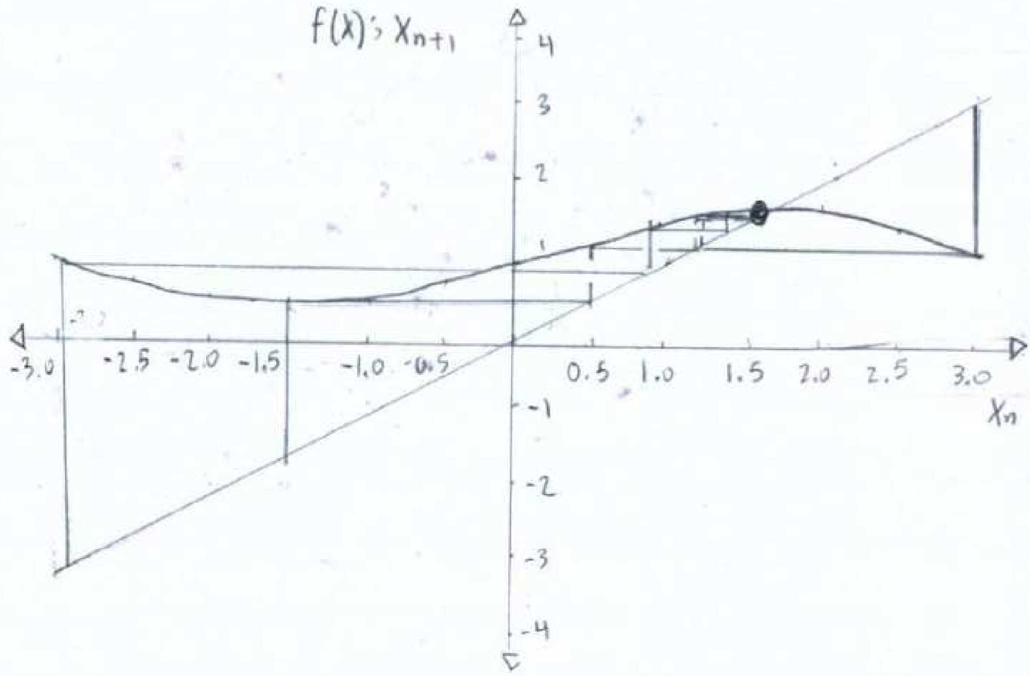
$X_{n+1} = 1 + \frac{1}{2}\sin X_n$

10.1.10 Fixed Pants: $X^*_{n+1} = 0 = 1 + \frac{1}{2}\sin X_n$

$$1 + \frac{1}{2}\sin X_n - X = 0$$

$$X^* = 1.4987...$$

Stability: $x^* = 1.4987..$; $|f'(1.4987)| = \left|\frac{\cos(1.4987)}{2}\right| = 0.03$ "stable"



$x_{n+1} = 3x_n - x_n^3$ 10.1.11

a) Fixed Points: $x_{n+1} = 0 = 3x_n - x_n^3$

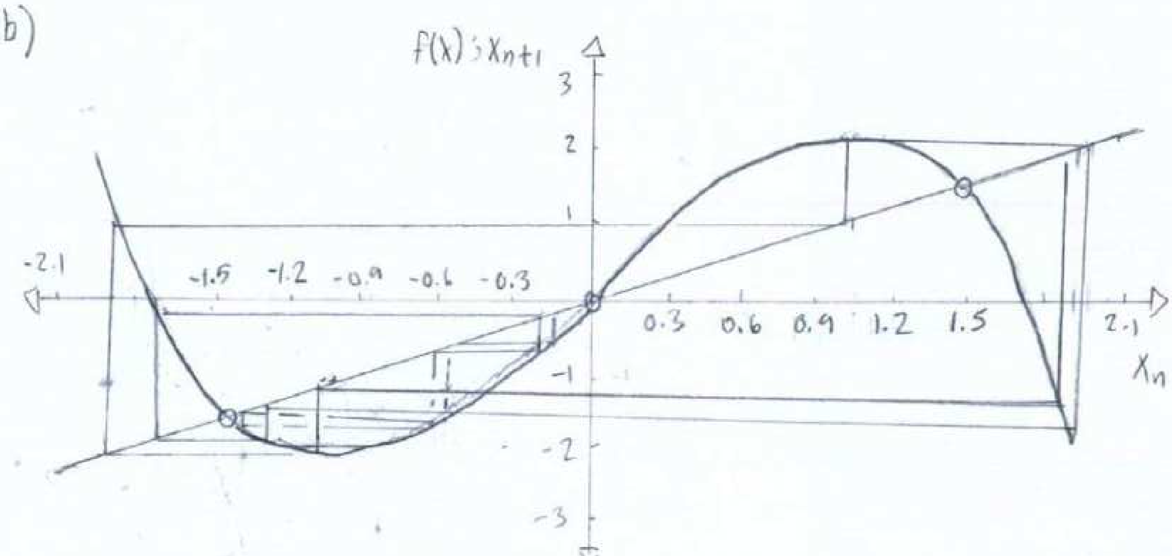
$$x^3 - 2x_n = 0$$

$$x^* = 0, \pm\sqrt{2}$$

Stability: $x^* = 0$; $|f'(0)| = |3 - 3(0)^2| = 3$ "unstable"

$x^* = \pm\sqrt{2}$; $|f'(\sqrt{2})| = |3 - 3(\sqrt{2})^2| = 3$ "unstable"

b)



c) An explicit $x_0 = 2.1$ has bounded behavior

d) The direction about x_{n+1} relates the stability in the fixed points and limit cycles.

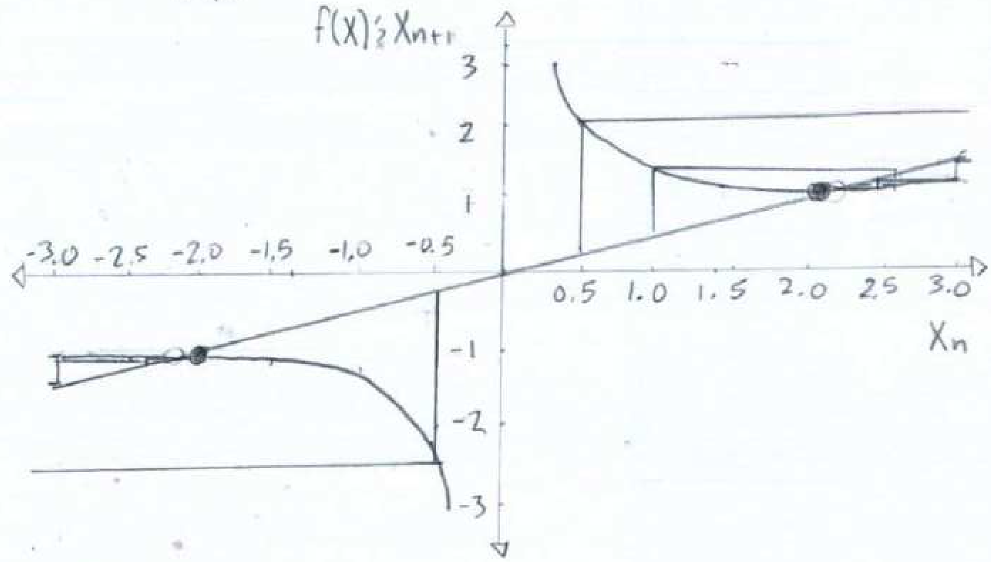
$f(x) = x_n - \frac{g(x_n)}{g'(x_n)}$

10.1.12.

a) $x_{n+1} = f(x_n)$; $g(x) = x^2 - 4$

$g'(x) = 2x$

$x_{n+1} = x_n - \frac{x_n^2 - 4}{2x_n}$



b) Fixed Points: $x_{n+1} = 0 = x_n - \frac{x_n^2 - 4}{2x_n}$

$\frac{x^2 - 4}{2x_n} = 0$

$x^* = \pm 2$

c) $x^* = 2$; $|f'(2)| = \left| 1 - \frac{4(2)^2 + 2(2^2 - 4)}{4(2)^2} \right| = 0$ "superstable"

$x^* = -2$; $|f'(-2)| = \left| 1 - \frac{4(-2)^2 + 2((-2)^2 - 4)}{4(-2)^2} \right| = 0$ "superstable"

d) Part a shows $x_0 = 1$.

10.1.13. Exercise 10.1.12, part c is an exact solution.

$x_{n+1} = -\sin x_n$

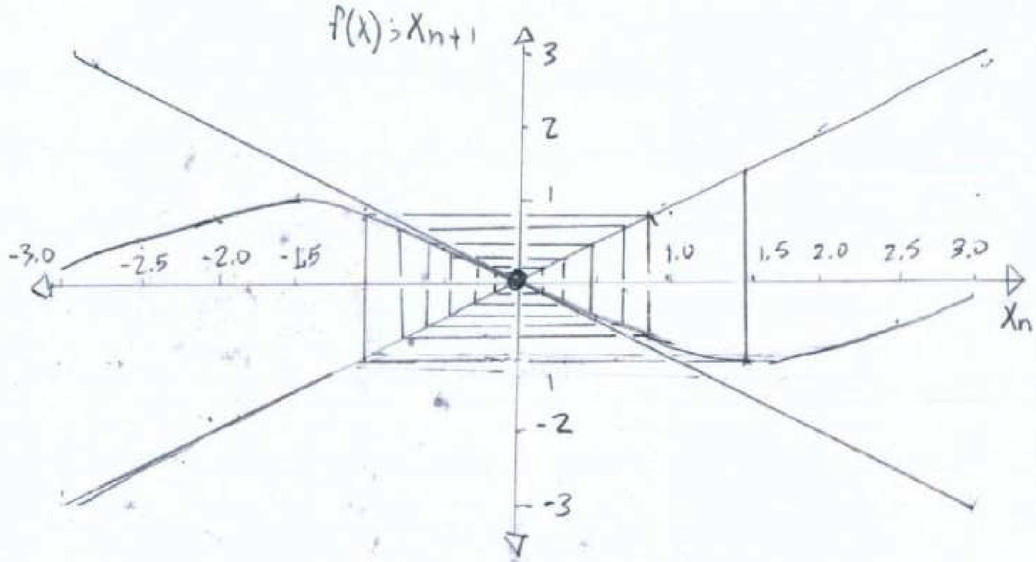
10.1.14 Fixed Points: $x_{n+1}^* = 0 = -\sin x_n$

$\sin x - x = 0$

$x^* = 0$

Stability: $x^* = 0$; $|f'(0)| = |-\cos(0)| = 1$, "Marginal case"

Note: The problem states $x^* = 0$ is stable....



$X_{n+1} = r X_n (1 - X_n)$

10.2.1.

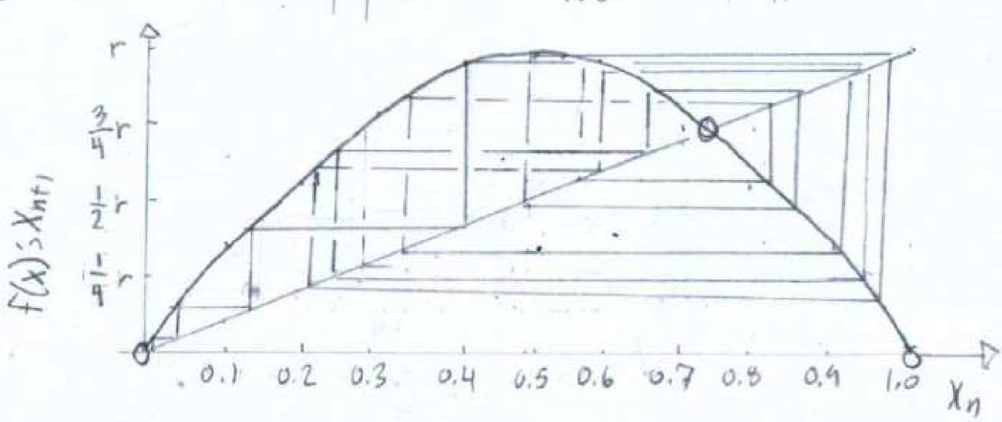
a) If $X_n > 1$, then subsequent iterations diverge toward $-\infty$

Proof by example:

n	X_n	X_{n+1}
1	-2	-2
2	-2	-6
3	-6	-42
4	-42	-1806
5	-1806	-326344
...	...	...

b) A restriction of $r \in [0,4]$ and $x \in [0,1]$ aids the biological model by max population at $r = 1$ and total population near $x = 1$.

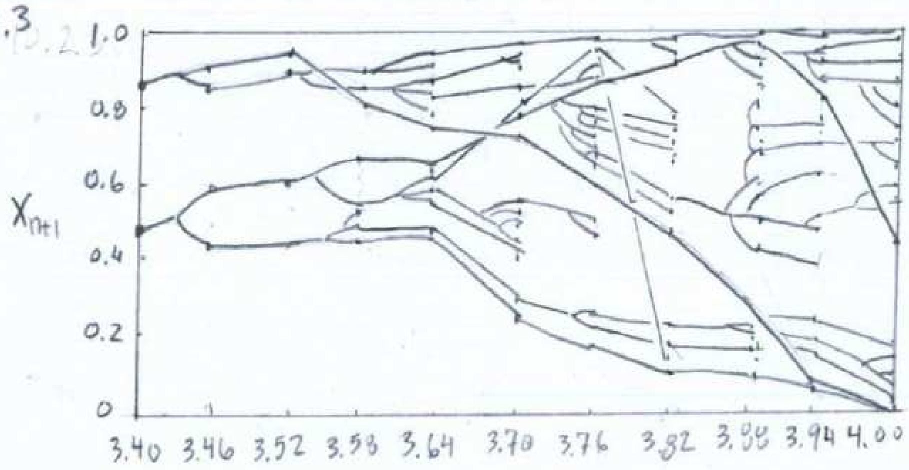
10.2.2.



A cobweb plot when $0 \le r \le 1$ approaches $x^* = 0$ because a small range less than zero

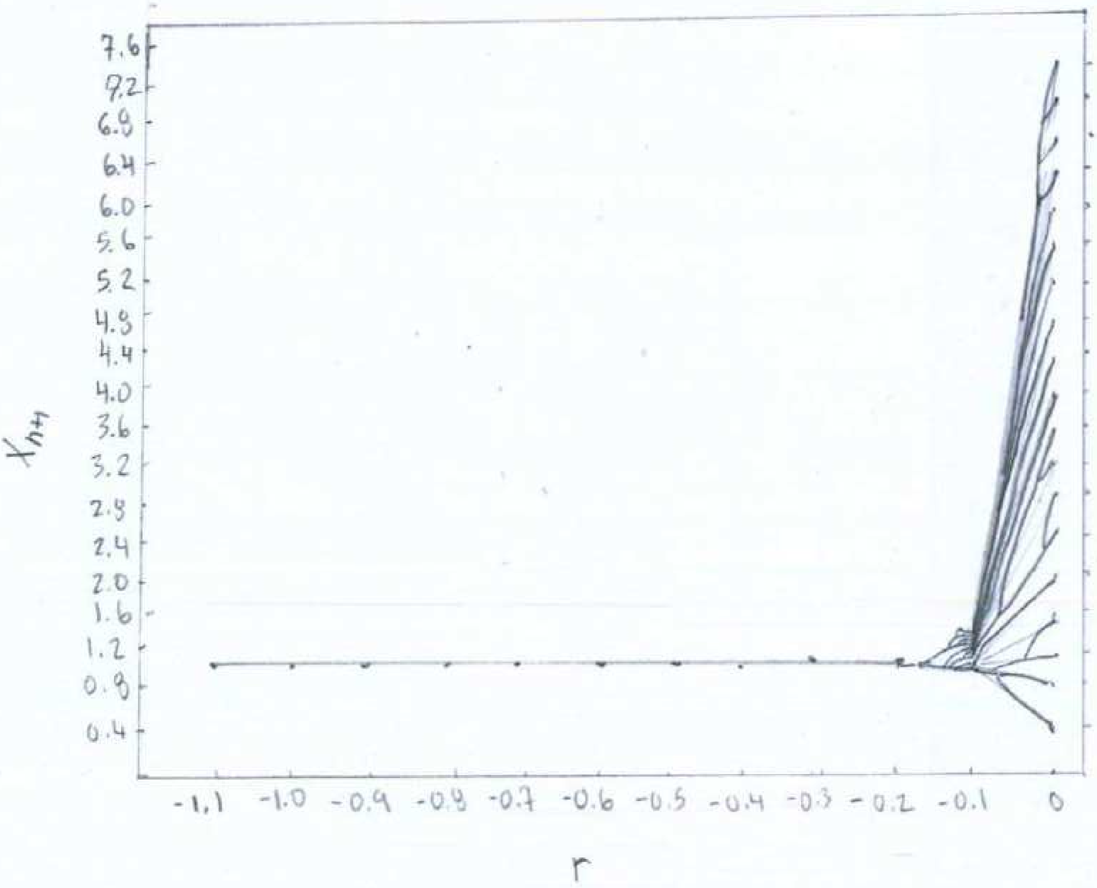
$X_{n+1} = rX_n(1-X_n)$

10.2.3



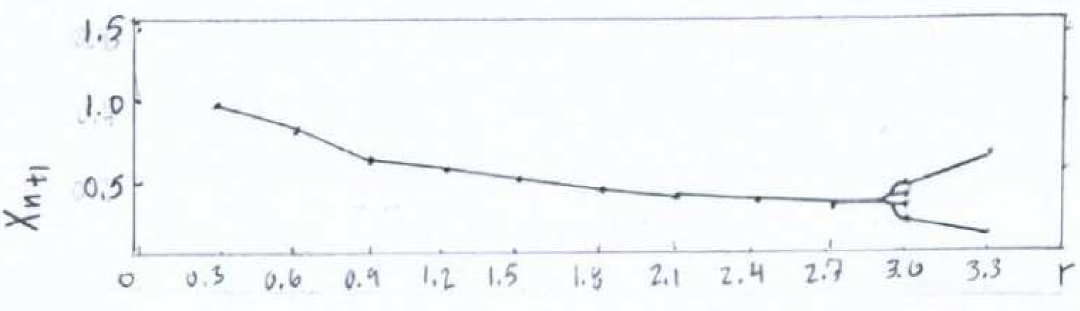
$X_{n+1} = X_n e^{-r(1-X_n)}$

10.2.4. I used $r=-1$ because the r derivatives maximum at $x=r$. With roots about r. The solution $|f'(1)| = |e^{r(1-1)}(r(1)+1)|$ is $r+1$ with stability less than zero but greater than negative one.



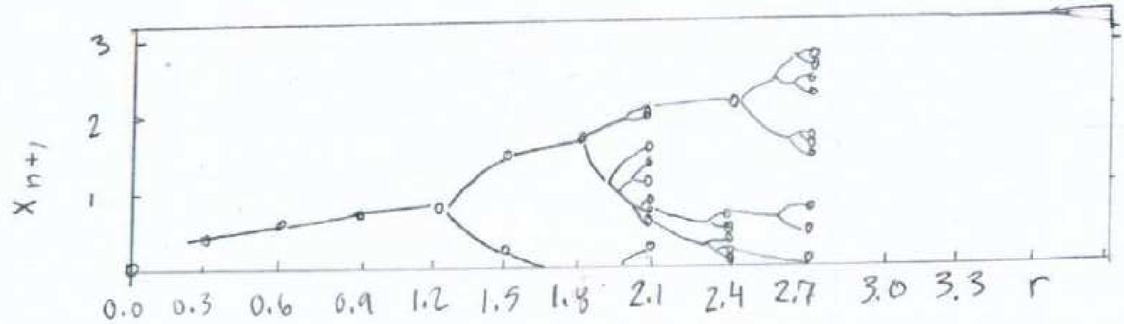
$X_{n+1} = e^{-rX_n}$

10.2.5 $\frac{dX_{n+1}}{dX_n} = -re^{-rX_n} = 0$; $X_n = \frac{\log r}{r}$; $r = e$; $X_n = \frac{1}{e}$



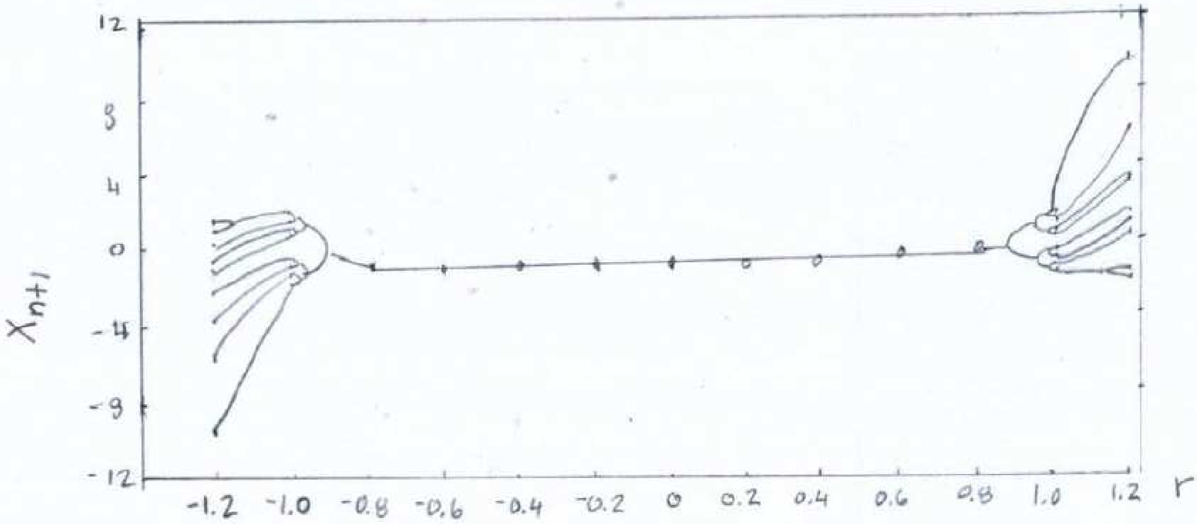
$x_{n+1} = r\cos x_n$

10.2.6. $\frac{d}{dx}x_{n+1} = -r\sin x_n = 0$; $x_n = n\pi$; $r = x$



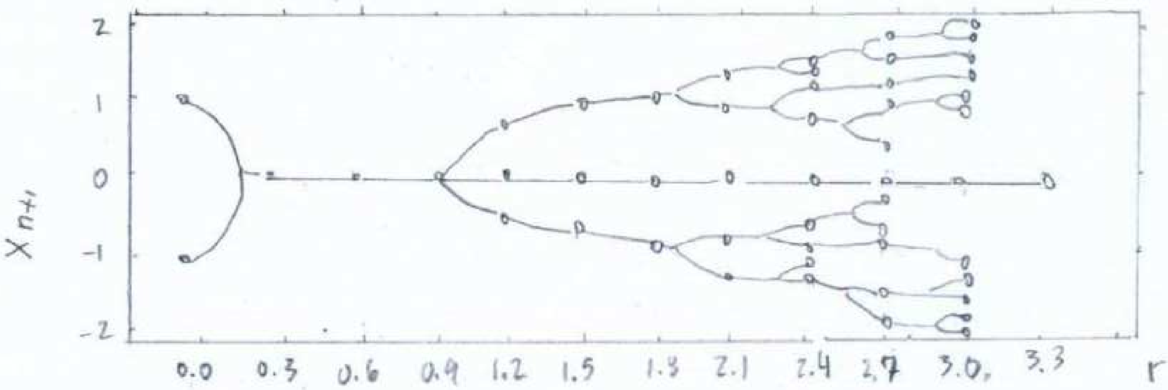
$x_{n+1} = r\tan x_n$

10.2.7. $\frac{d}{dx} r\tan x_n = r(1+\tan^2 x_n) = 0$; $x_n = \text{NaN}$; $r = 0$



$x_{n+1} = rx_n - x_n^3$

10.2.8. $\frac{d}{dx}x_{n+1} = r - 3x_n^2 = 0$; $x_n = \sqrt{\frac{r}{3}}$



$x_{n+1} = rx_n(1-x_n)$

10.3.1 $x_{n+1} = 0 = rx_n(1-x_n)$; $x_n = 0, 1$

Stability: $|f'(0)| = 0$ "superstable"

$|f'(1)| = 2$ "unstable"

$x_{n+1} = rx_n(1-x_n)$

10.3.2

a) $p = 1/2$; $q = 1/2$: $f(x)' = r - 2rx = 0$ ∴ $x = 1/2$

$f(f(x))' = -r^2(2x-1)(2r(x-1)x+1) = 0$; $x = 1/2$

b) $X_{max} = 1/2$ for $f(x)$, $f(f(x))$, and $f(f(f(x)))$ with $r = 1/2$

$X_{n+1} = rX_n/(1+X_n^2)$

10.3.3. Fixed Points: $f(X_n) = X_{n+1} = rX_n/(1+X_n^2) = 0$

$$x(1+x^2) - rx = 0$$

$$x = \pm\sqrt{r-1},\ 0$$

$$f'(X_n) = X_{n+1} = \frac{-r(x^2-1)}{(x^2+1)^2} = 0$$

Stability: $|f'(0)| = r$ "stable, marginal, or unstable"

$|f'(\pm\sqrt{r-1})| = \frac{2}{r} - 1$ "stable, marginal, or unstable"

The fixed points show direction toward the origin by an $r<1$, $r=1$, or $r>1$.

$X_{n+1} = X_n^2 + c$

10.3.4. a) Fixed Points: $0 = x^2 - x + c$

$$x^* = \frac{1 \pm \sqrt{1-4(c)}}{2}$$

Stability: $x^* = \frac{1+\sqrt{1-4c}}{2}$; $\left|f'\left(\frac{1+\sqrt{1-4c}}{2}\right)\right| = 1+\sqrt{1-4c}$

$x^* = \frac{1-\sqrt{1-4c}}{2}$; $\left|f'\left(\frac{1-\sqrt{1-4c}}{2}\right)\right| = 1-\sqrt{1-4c}$

c / Fixed Point	$< 1/4$	$= 1/4$
$1+\sqrt{1-4c}$	unstable	Marginal case
$1-\sqrt{1-4c}$	Stable	Super stable

b) Bifurcations: $x = 1+\sqrt{1-4c}$

$$c = \frac{1-(x-1)^2}{4}$$

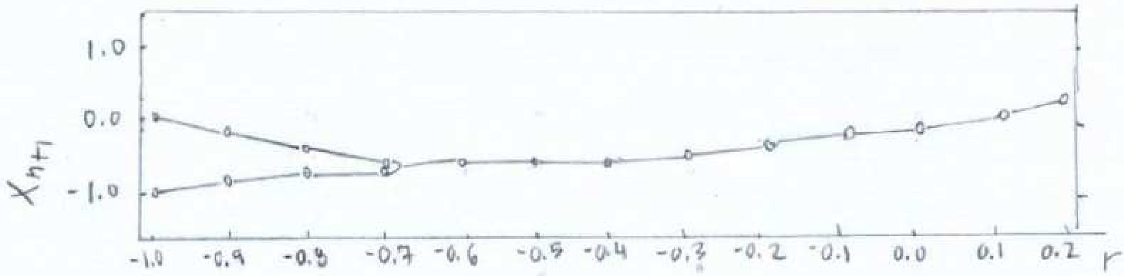
$$x = 1-\sqrt{1-4c}$$

$$c = -\frac{1+(x-1)^2}{4}$$

Saddlenode bifurcation because zero, one, then two fixed points arise from new c's.

c) A stable 2-cycle shows when $c < 1/4$ near the $x^* = 1 - \sqrt{1-4c}$ fixed point. Super stable at $c = 1/4$ with a similar maximum.

d)



$x_{n+1} = r x_n (1 - x_n)$

$y_{n+1} = y_n^2 + c$

10.3.5. $x_{n+1} = r x_n (1 - x_n) = r(y_n^2 + c)(1 - (y_n^2 + c))$

$= a y_{n+1} + b = a(y_n^2 + c) + b$

$r(ay + b)(1 - (ay + b)) = a(y_n^2 + c) + b$

$-a^2 r y^2 + ar(1-2b)y + rb(1-b) = ay^2 + ac + b$

$a = -\frac{1}{r}$; $b = \frac{1}{2}$; $c = \frac{r(2-r)}{4}$

$x_{n+1} = f(x_n)$

$f(x_n) = r x_n - x_n^3$

10.3.6. a) Fixed Points: $x_{n+1} = r x_n - x_n^3$

$x^3 + (1-r)x = 0$

$x^* = 0, \pm\sqrt{r-1}$

Stability: $x^* = 0$; $|f'(0)| = r$ "super stable"

$x^* = \sqrt{r-1}$; $|f'(\sqrt{r-1})| = -2r + 3$

"superstable, unstable"

$x^* = -\sqrt{r-1}$; $|f'(-\sqrt{r-1})| = -2r + 3$

"Superstable, unstable"

$r = 0$; $r = 3/2$; "Stable" occurs when $1 < r < \frac{3}{2}$

b) When $f(p) = q$ and $f(q) = p$, then $q = p = 0$ or

$q = p = \pm\sqrt{r-1}$

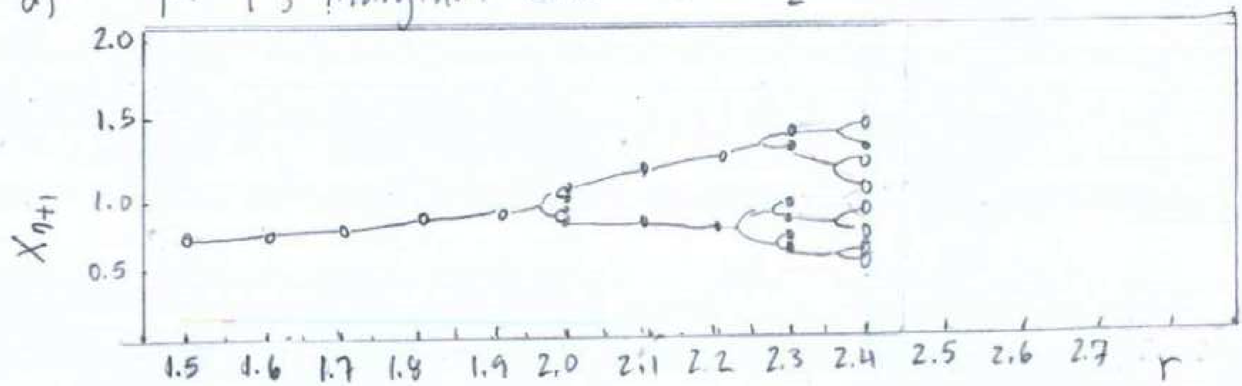
c) stability $= |f'(x)|$

$q = p = 0 : |f'(q)| = |f'(p)| = r$

$q = p = \pm\sqrt{r-1} : |f'(q)| = |f'(p)| = -2r+3$

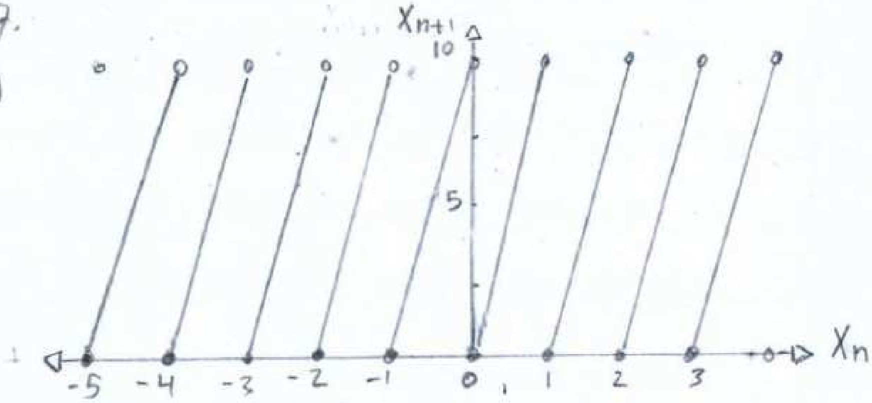
$r = 0$; "Unstable" ; $r = \frac{3}{2}$: "superstable"

d) $r = 1$; "marginal case" : $1 < r < \frac{3}{2}$ "Stable"



$X_{n+1} = 10 X_n \pmod 1$ 10.3.7.

a)



b) Fixed Points: $X_{n+1} = 10 X_n \pmod 1$

$X^* = n \cdot 1.00$ where $n \in \mathbb{R}$

c) The aperiodic orbits appear between n-values. e.g. irrational numbers. Many aperiodic solutions exist from zero to one. or one to two, etc.

d) Infinite many solutions are the irrational values from zero up till one.

e) By example:

n	1	2		1		1
X_n	0.01	0.1	...	0	...	1
X_{n+1}	0.1	1.0		0		1

$X_{n+1} = 10 X_n$

10.3.8. A dense orbit appears about initial conditions at irrational values. An irrational number shifted in base-10, $10x_n \pmod 1$, never ends as with a decimal shift map.

$X_{n+1} = 2X_n \pmod 1$

10.3.9. An aperiodic map in the function is the dense map for irrational initial conditions. A periodic orbit is when $\frac{d}{dx} f^n(x) > 1$.

$X_n = \sin^2(\pi \theta_n)$

$X_{n+1} = 4X_n(1-X_n)$

10.3.10.

a) $\theta_{n+1} = 2\theta_n \pmod 1$; $X_n = \sin^2(\pi \theta_n)$

Identity: $\sin^2(a) = [2\sin(a)\cos(a)]$

$$X_n = \sin^2(\pi\theta_n) = 2\sin(\pi\theta_n)\cos(\pi\theta_n)$$

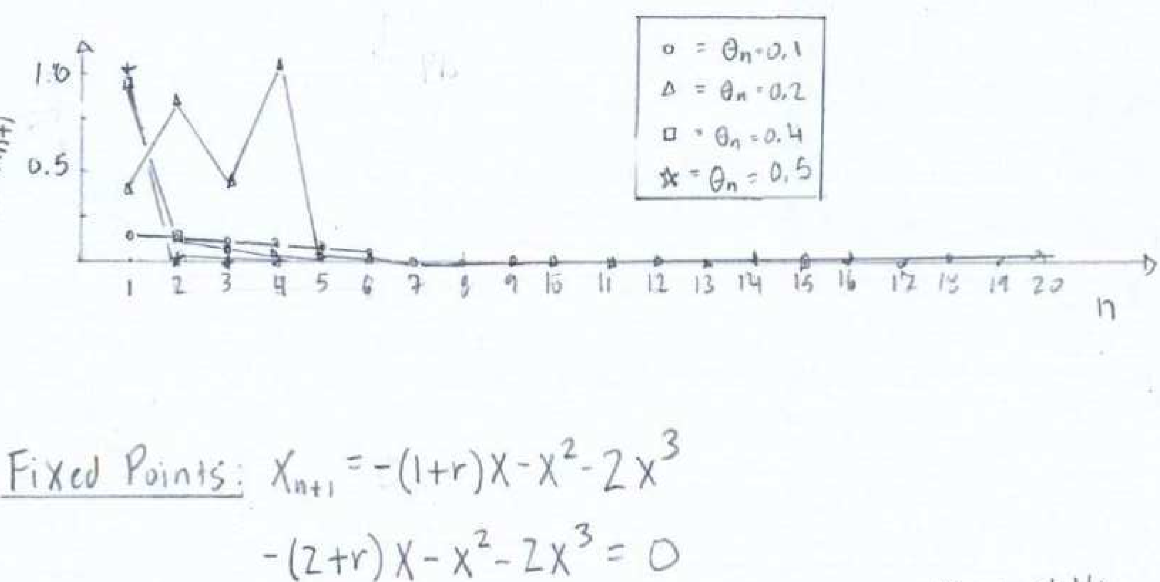
$$= \left[2\sin(2\pi\theta_n \pmod 1)\cos(2\pi\theta_n \pmod 1)\right]^2$$

If $\sqrt{X_n} = \sin(\pi\theta_n)$, then the equation becomes.

$$= \left[2\sqrt{X_n}\left(1-\sqrt{X_n}\right)\right]^2$$

$$= 4X_n(1-X_n)$$

b) Plots of X_{n+1} vs. n.



$X_{n+1} = F(X_n)$

$f(x) = -(1+r)x - x^2 - 2x^3$

10.3.11

a) Fixed Points: $X_{n+1} = -(1+r)X - X^2 - 2X^3$

$$-(2+r)X - X^2 - 2X^3 = 0$$

Stability: $|f'(0)| = |-(1+r) - 2\cdot(0) - 6(0)^2| = r-1$ "superstable, stable, marginal, unstable"

b) Flip Bifurcation - a location when period doubling shows in the map.

b) $r < 0$: $|f'(0)| = |r-1| > 1$ "Unstable"

$r = 0$: $|f'(0)| = |r-1| = 1$ "Marginal Case"

$r > 0$: $|f'(0) = |r-1| < \infty$: "Superstable, Stable, marginal case, Unstable"

A bifurcation about the fixed point occurs at $r=0$.

c) <u>Taylor Series</u>: $f(x) = f(a)\cdot\frac{(x-a)^0}{0!} + f(a)'\cdot\frac{(x-a)^1}{1!} + f(a)''\cdot\frac{(x-a)^2}{2!}$

$$= \sum_{n=0}^{\infty} f(a)^{(n)} \frac{(x-a)^n}{n!}$$

$$f(x) = -(1+r)x - x^2 - 2x^3$$

$$f^2(x) = -(1+r)[-(1+r)x - x^2 - 2x^3] - [-(1+r)x - x^2 - 2x^3]^2 - 2[-(1+r)x - x^2 - 2x^3]^3$$

$$f^2(x)' = (-r - 6x^2 - 2x - 1)(-6x^2(r + 2x^2 + x + 1)^2 + 2(r+1)x - r + 4x^3 + 2x^2 - 1)$$

$$f^2(x)'' = 12r^3x + r^2(240x^3 + 72x^2 + 36x - 2) + 2r(504x^5 + 360x^4 + 300x^3 + 48x^2 + 18 - x - 1) + 4x(288x^6 + 336x^5 + 378x^4 + 165x^3 + 70x^2 + 3x + 3)$$

$$f^2(x) = (r+1)^2x - r(r+1)x^2$$

$r < -1$ $|f^2(0)'| = |(r+1)^2| \geq 1$ "Marginal Case or Unstable"

$r = -1$ $|f^2(0)'| = |(r+1)^2| = 0$ "Superstable"

$-1 < r < 0$ $|f^2(0)'| = |(r+1)^2| < 1$ "Stable"

$r = 0$ $|f^2(0)'| = |(r+1)^2| = 1$ "Marginal case"

d) A $r < 0$ has a 2-period behavior and above zero an instability. When $r > 0$, each around the flip bifurcation.

$X_{n+1} = rx(1-x)$

10.3.12

a) $R_n = \dfrac{R_n - R_{n-1}}{R_{n+1} - R_n}$

b)

R_2	3.44949
R_3	3.54409
R_4	3.56441
R_5	3.56876
R_6	3.56969
R_7	3.56989
R_8	3.56993

c) 4.65000 "Feigenbaum's constant"

10.3.13

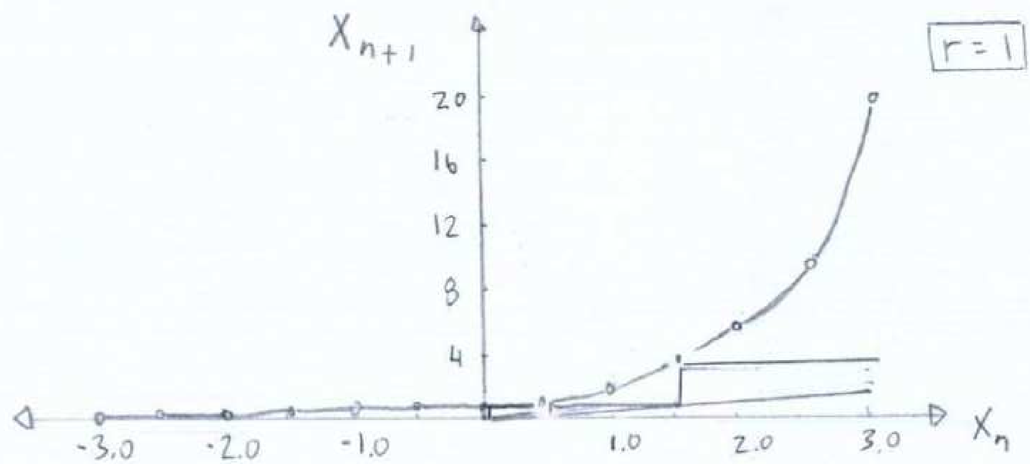
a) The dark curves represent bifurcations in the $f^p(x)$ function. A maximum value at $x_m = 1/2$ is the solution for $f^p(x)$'s maximum value.

b) r's value at the "intersecting" point shown in the orbit diagram, $R_n = \dfrac{R_n - R_{n-1}}{R_{n+1} - R_n} \approx 3.56$

$X_{n+1} = re^{X_n}$

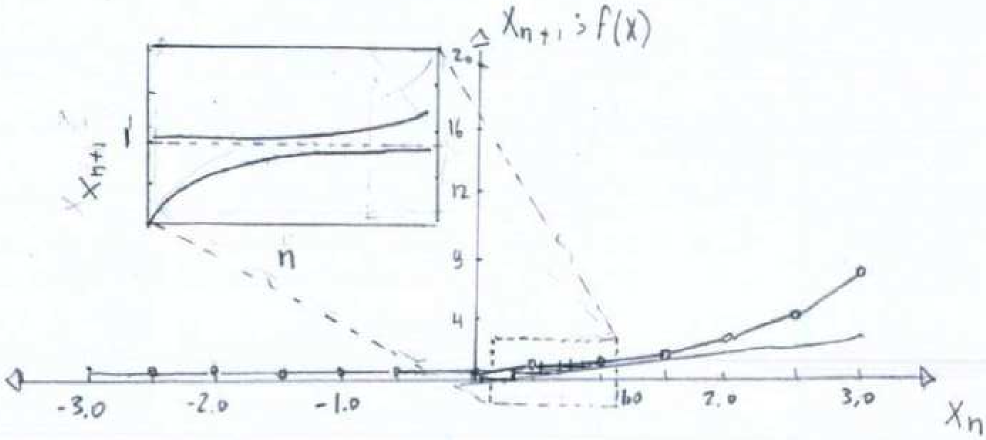
10.4.1.

a)



b) Tangent Bifurcation: A critical value where the stable and unstable coalesce.

c)



$x_{n+1} = \dfrac{r x_n^2}{(1+x_n^2)}$

10.4.2 Fixed Points: $x_{n+1} = \dfrac{r x_n^2}{(1+x_n^2)}$

$$x^* = \pm\sqrt{r-1}\ , 0$$

Stability: $x^* = \sqrt{r-1}\ ;\ |f'(\sqrt{r-1})| = \left|\dfrac{r - r(r-1)}{(r-1+1)^2}\right|$

$$= \left|\frac{2-r}{r}\right|$$

$r < 1$ "Unstable"

$r = 1$ "Marginal case"

$r > 1$ "Stable

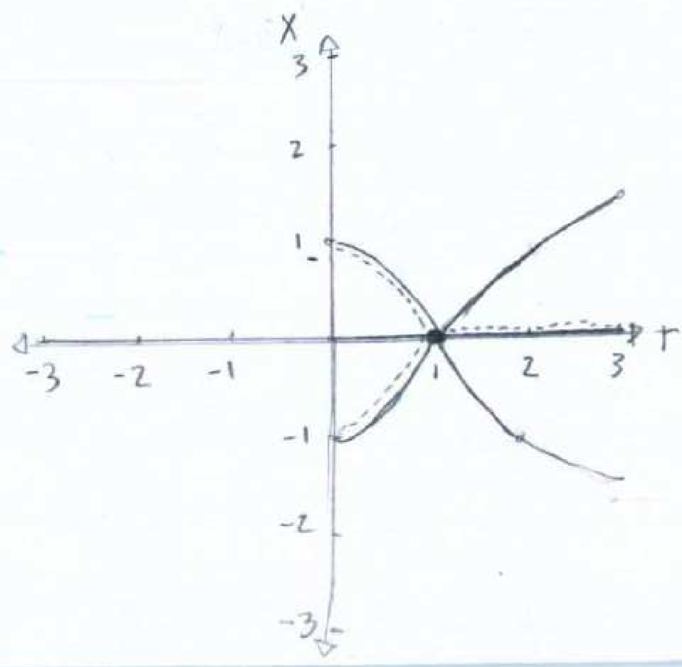
$x^* = 0\ ;\ |f'(0)| = |r|$

$r < 1$ "stable

$r = 1$ "Marginal case"

$r > 1$ "Unstable"

Bifurcations:



Intermittency - chaotic behavior below a period - 3 window
The system's 3-period behavior is fact for an investigation about intermittency.

$X_{n+1} = 1 - rX_n^2$

10.4.3. Cycles occur when $f^p(X_{n+1}) = 0$. The variable p describes p-cycles. A 3-cycle has similar outputs for each cycle of zero.

$$\frac{d}{dx} f^3(X_{n+1}) = \frac{d}{dx} f(f(f(X_{n+1})))$$
$$= f'(f(f(X_{n+1}))) \circ f'(f(X_{n+1})) \circ f'(X_{n+1})$$
$$= (-2r\, f(f(X_{n+1}))) \circ (-2r\,(f(X_{n+1}))) \circ -2r\, X_n$$
$$= -8r^3 f(f(X_{n+1})) \circ f(X_{n+1}) \circ X$$
$$= 0$$

$f(0) = 1$; Unable to be zero

$f(f(0)) = 1 - r$; Zero at $r = 1$

$f(f(f(0))) = 1 - r(1-r)^2$; Capable of zero at $r \cong 1.75$

The successive derivatives are zero with a cubic equation dependent upon r.

10.4.4 Logistic equation: $X_{n+1} = rX_n(1 - X_n)$

$f(f(f(X_n))) = 1 - r(1 - r(1 - rX^2)^2)^2 = 0$

Excel: $r = 3.82$, $0 < X < 1$

$\vdots$

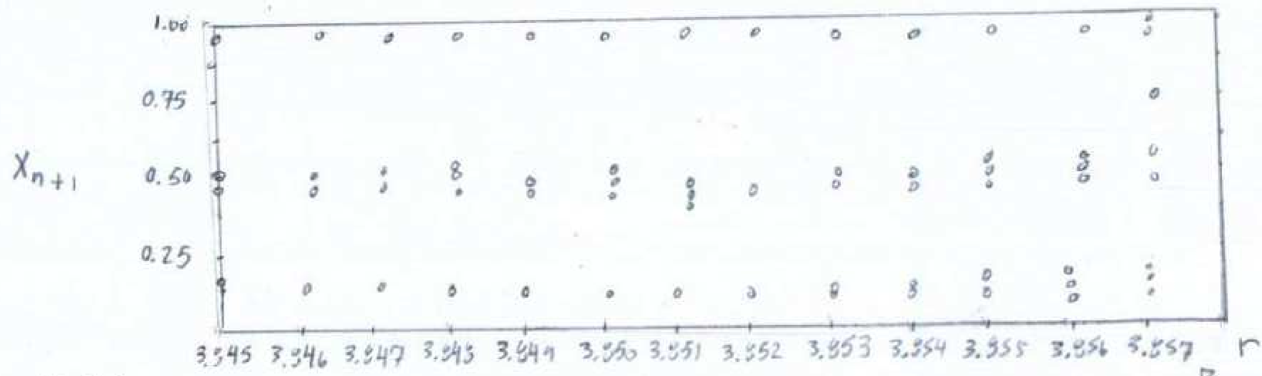
$r = 3.828$, $0 < X < 1$

$\vdots$

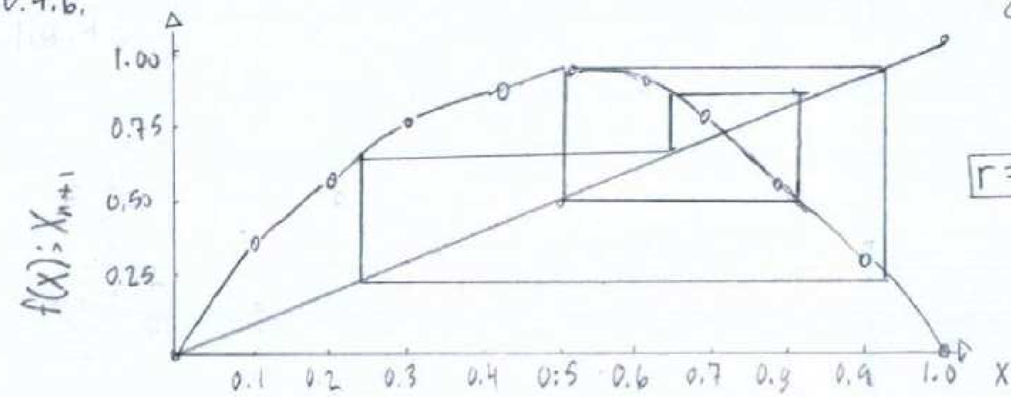
$r = 3.8284$, $0 < X < 1$

$\vdots$

10.4.5



10.4.6.



$\boxed{r = 3.738915}$

The superstable cycle's period is five because the number of iterations before $x_0 = x_5 = 1/2$.

10.4.7.

a) An $r > 1+\sqrt{5}$ generates the sequence RL in a logistic equation because p-cyles where $p > 2$ oscillates right, then left.

b. RLRR.

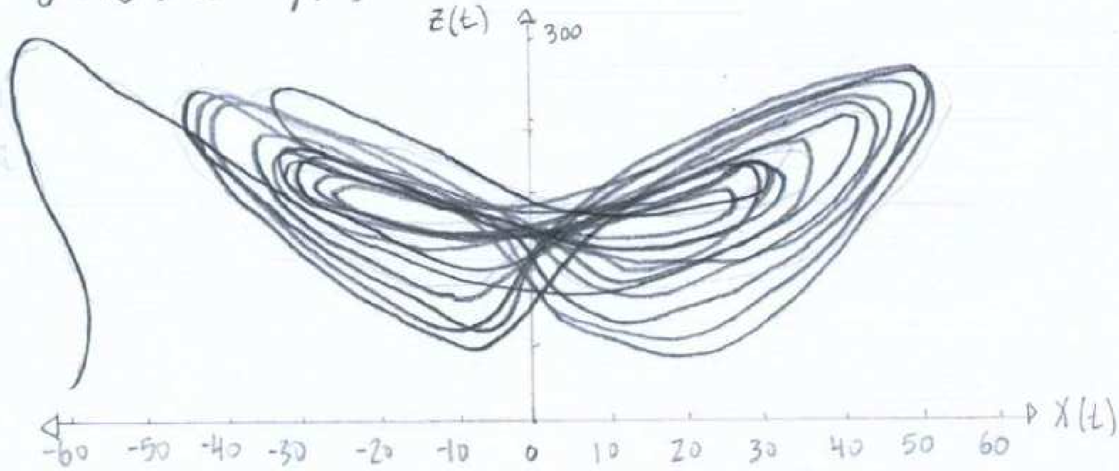
$\dot{x} = \sigma(y - x)$

$\dot{y} = rx - y - xz$

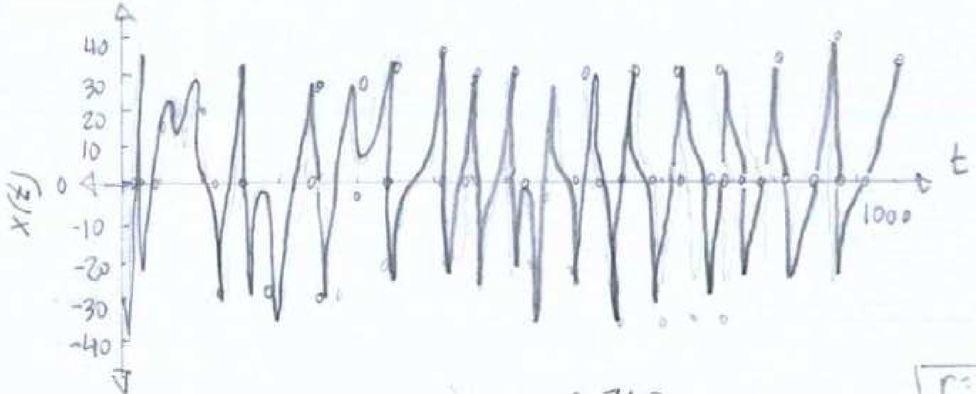
$\dot{z} = xy - bz$

10.4.8. $\sigma = 10$; $b = 8/3$; $r \approx 166$.

a)

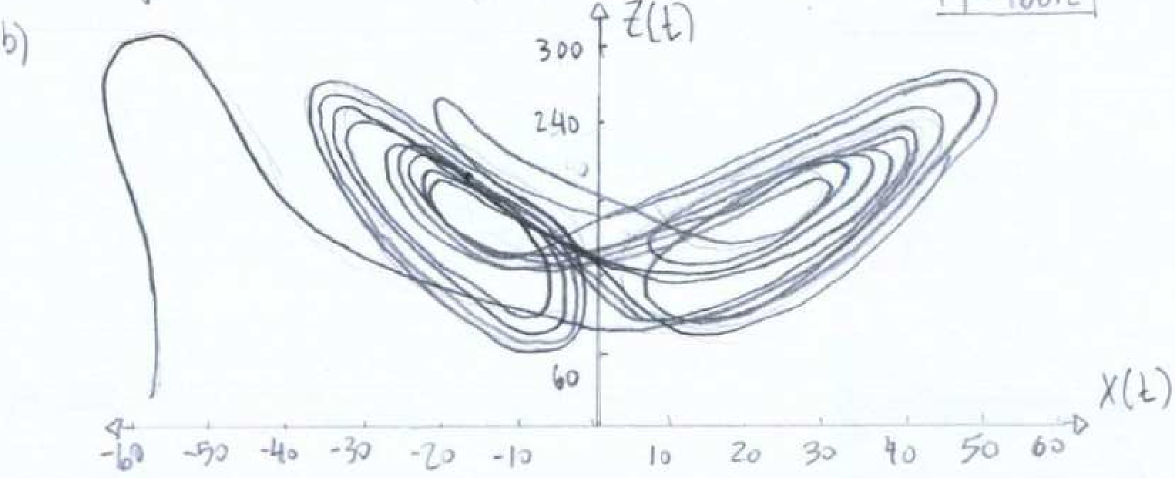


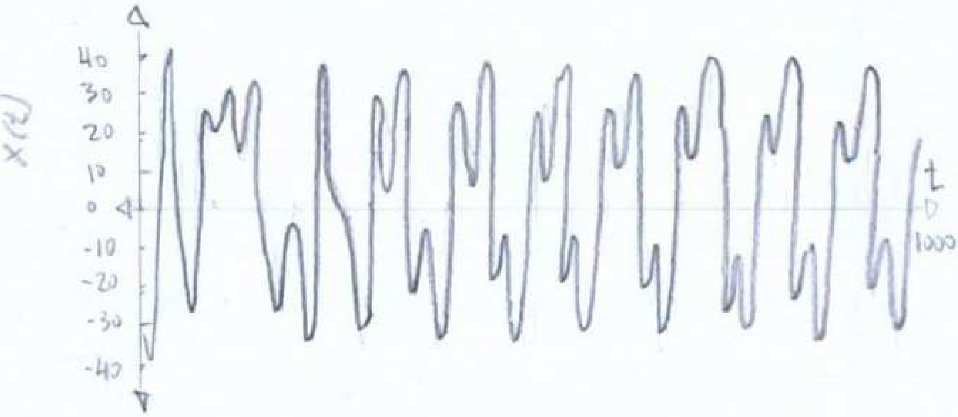
No guess work



$\boxed{r = 166.2}$

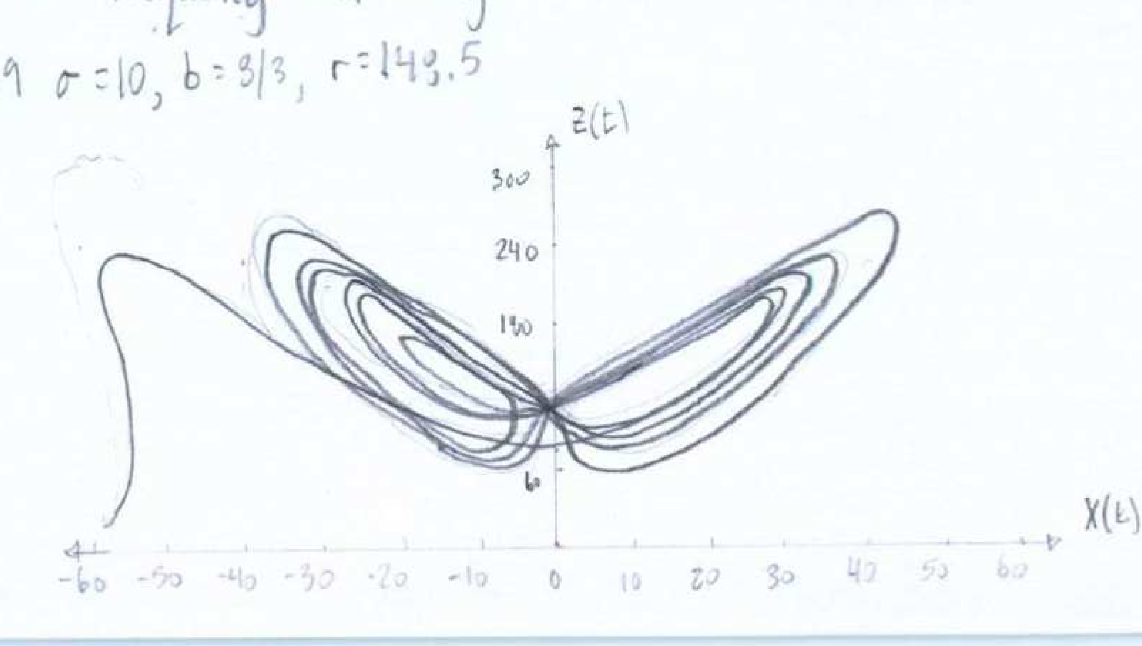
b)

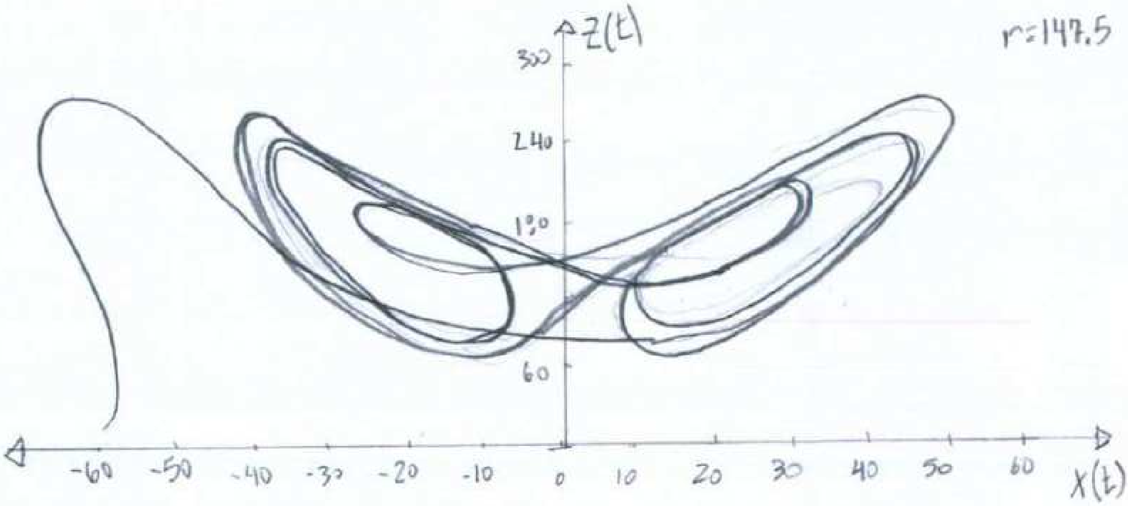




c) The regularity from $r = 166$ to $r = 166.2$ demonstrates a frequency with longer lifetimes.

10.4.9 $\sigma = 10$, $b = 8/3$, $r = 148.5$





$f(a)=b$
$f(b)=c$
$f(c)=a$

10.4.10.

$r=1+\sqrt{8}=3.8284$; a, b, c are period-3 bifurcations.

$X_{n+1}=rX_n(1-X_n)$;

Three equations representing a period-3 cycle:

$$f(a)=b=ra(1-a)$$

$$f(b)=c=rb(1-b)=f(f(a))$$

$$f(c)=a=rc(1-b)=f(f(f(a)))$$

Maximum for the period-3 cycle:

$$\frac{df(f(f(a)))}{da}=\frac{d(f^3(a))}{d(f^2(a))}\cdot\frac{d(f^2(a))}{d(f(a))}\cdot\frac{df(a)}{da}$$

$$=r^3(1-2a)(1-2b)(1-2c)$$

$$=1$$

A shifted maximum toward zero for $F(a), f(b), f(c)$:

$A=r(a-1/2)$; $B=r(b-1/2)$; $C=r(z-1/2)$

When the maximal equation becomes A, B, C:

$$r^3(1-2a)(1-2b)(1-2c)=1$$

$$8\circ A\circ B\circ C=-1$$

A second equation appears from A, B, C:

$$\frac{r^2}{4}-\frac{r}{2}=A^2+B=B^2+C=C^2+A$$

Expanding the previous equation with three relationships:

$a = A + B + C$; $b = AB + BC + CA$; $c = ABC$

$8ABC = -1 = 3A^3 = 8a^3 + 27 = (2a+3)(4a^2 - 6a + 9) = 0$

Three Identities: $A^2 + B^2 + C^2 = a^2 - 2b$

$A^3 + B^3 + C^3 = a^3 - 3ab + 3c$

$(AB)^2 + (BC)^2 + (CA)^2 = b^2 - 2ca$

A new R-equation: $R = 1/3(a^2 + a - 2b)$

$R^2 = (A^2 + B)(B^2 + C)$

A function with a and b and c:

$a^4 - 4a^3 + 14ab + a^2 + b^2 + 6ac - 4a^2b - 3b - 18c = 0$

$c = -1/8$; $b = \dfrac{16a^3 - 8a^2 - 9}{56a - 24}$

The equation with only a:

$24(2a-1)(2a+3)(4a^2 - 6a + 9)^2 = 0$; $a = 1/2$

Solving for $b = -9/4$

Solving for $R = 7/4$

Solving for $r = 1 + 2\sqrt{2}$

$x_1 = a$ 10.4.11. $x_{n+1} = a^{x_n}$

$x_2 = a^a$ The hyperpower converges when $x_1 = a = e^{-e}, e, e^e$ or e^{-1}.

$x_3 = a^{(a^a)}$ A long term behavior becomes complex with a geometric or arithmetic relationship.

$x_{n+1} = r x_n$ 10.5.1. $\lambda = \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right| = \frac{1}{n} \ln \left| \frac{r^n \delta_0}{\delta_0} \right| = \ln |r|$

$x_{n+1} = 10 x_n$ 10.5.2 $\lambda = \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right| = \frac{1}{n} \ln \left| \frac{10^{10} \delta_0}{\delta_0} \right| = \ln 10$

$f(x) = \begin{cases} rx & 0 \le x \le 1/2 \\ r - rx & 1/2 \le x \le 1 \end{cases}$

10.5.3. $\lambda = \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right| = \frac{1}{n} \ln \left| \frac{r^n \delta_0}{\delta_0} \right| = \ln r$

A Liapunov exponent with $\lambda = \ln r$ describes chaotic solutions for all $r > 1$.

10.5.4 $\lambda_{logistic} = \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right| = \frac{1}{n} \ln \left| \frac{[r^n (1-x)^n - r^n x^n] \delta_0}{\delta_0} \right|$

$$= \frac{1}{n} \ln \left| r^n (1-2x)^n \right|$$

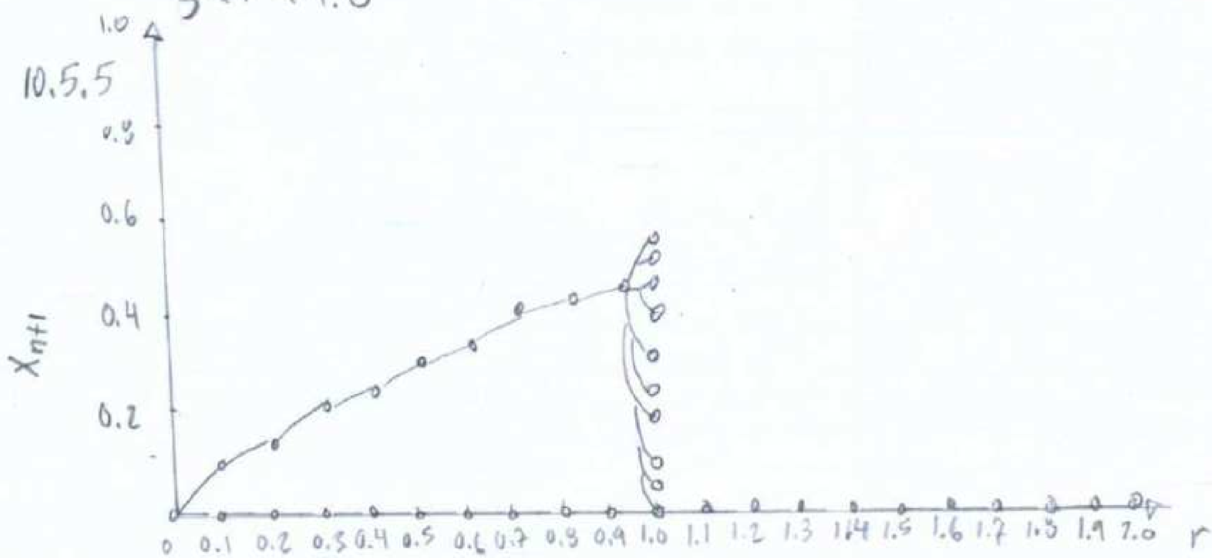
$$= \ln |r(1-2x)|$$

$\lambda_{tent} = \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right| = \frac{1}{n} \ln \left| \frac{r^n \delta_0}{\delta_0} \right|$

$$= \ln r$$

When $r > 0$, then the tent maps Liapunov constant is always greater than zero. The constant in the logistic equation alternates from positive into negative, chaotic into stable when $3 < r < 4.0$

10.5.5



$x_{n+1} = r \sin \pi x_n$

10.5.6 Fixed Points: $x_{n+1} = 0 = r \sin \pi x_n$

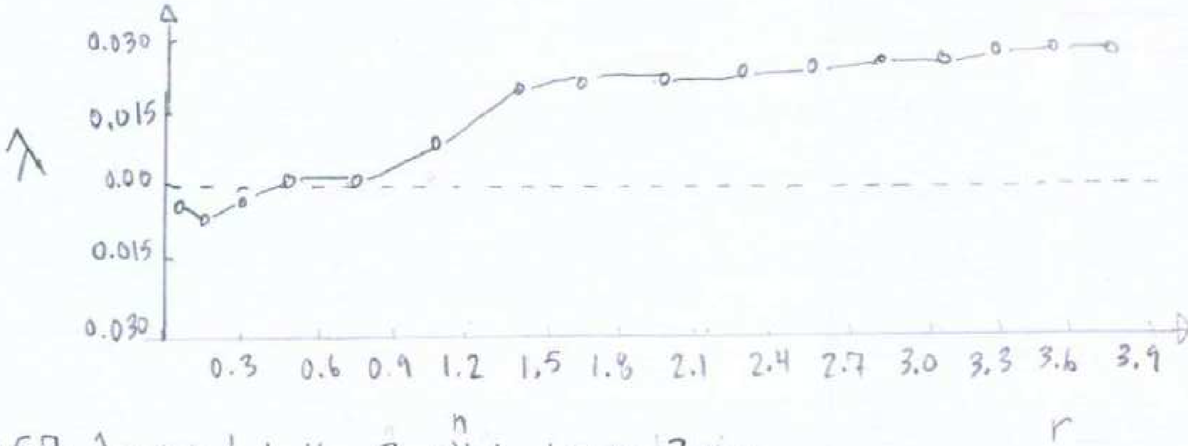
$$r = \frac{x}{\sin \pi x}$$

$$x^* = 0$$

Liapunov Exponent:

$$\lambda = \frac{1}{n}\ln\left|\frac{\delta_n}{\delta_0}\right| = \frac{1}{n}\ln\left|\frac{r^n\pi^n\cos^n\pi x_n\,\delta_0}{\delta_0}\right|$$

$$= \ln|r\pi\cos\pi x_n|$$



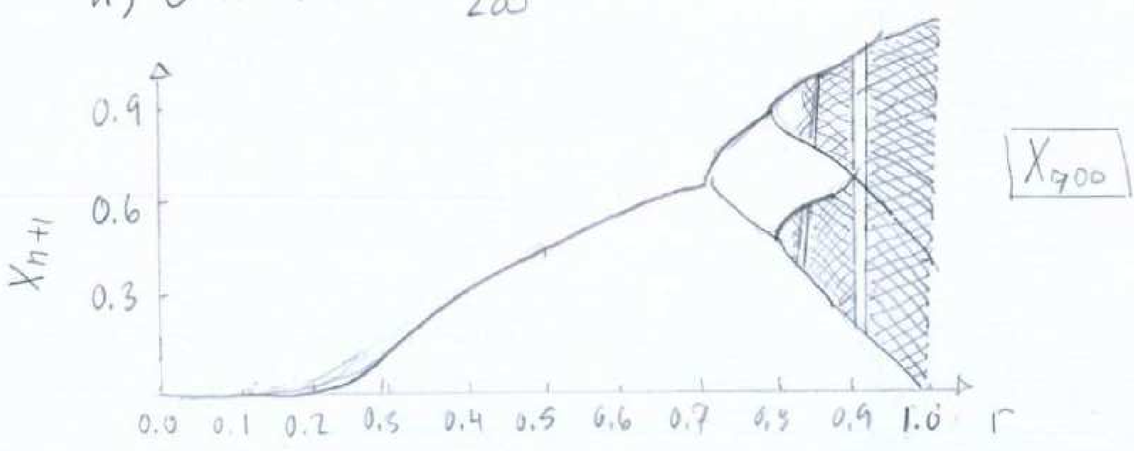
10.5.7 $\lambda = 0 = \frac{1}{n}\ln|(r-2rx)^n|$; $1 = r - 2rx$

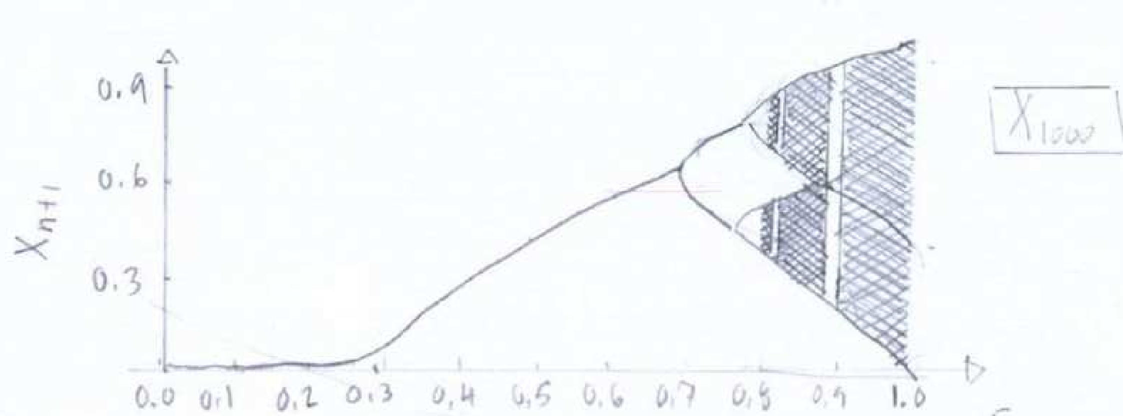
$$r = \frac{1}{1-2x}$$

$x_{n+1} = r\sin\pi x_n$

10.6.1

a) $0 < r < 1$; $\Delta r = \frac{1}{200}$;





$r_1 \to r_6$

b)

r	Value
1	0.71967
2	0.83326
3	0.85860
4	0.86405
5	0.86522
6	0.86556

c) Feigenbaum Ratio $= \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = \frac{r_4 - r_3}{r_5 - r_4} = 4.65811$

$\%$ Error $= \frac{|4.65811 - 4.66921|}{4.6692} \times 100\% = 0.08\%$

10.6.2

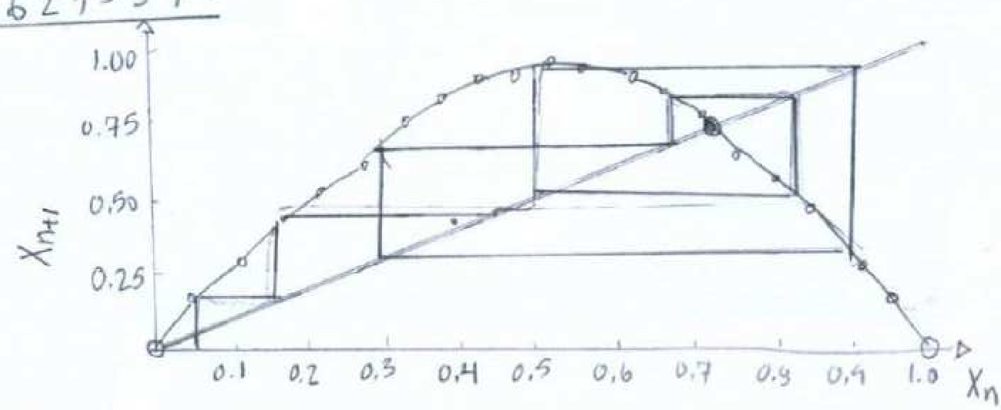
a) R_n computation is easier because R_n is found by graph, while r_n through calculating $f^n(x)$ where $n > 2$.

b) I comprehend no extreme difference. See 10.6.1.

10.6.3.

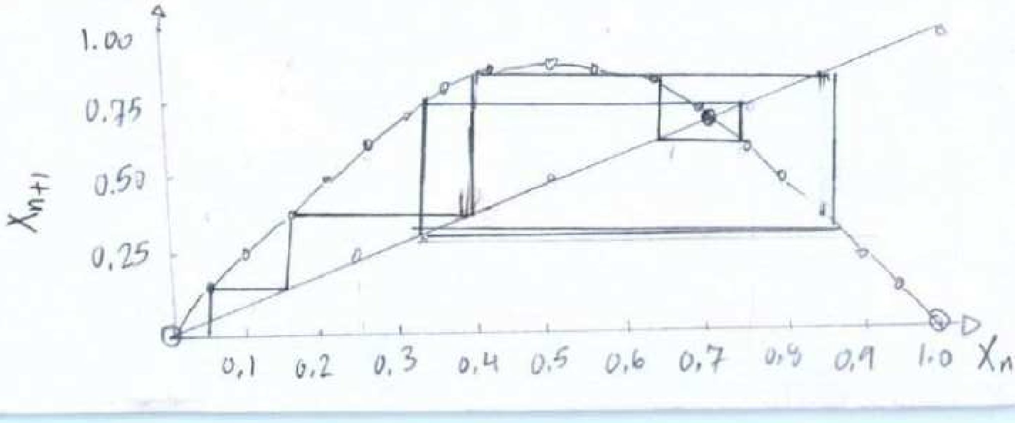
$X_{n+1} = rX_n(1 - X_n)$

a) $r = 3.6275575$



$X_{n+1} = r \sin \pi X_n$

$r = 0.8811406$



b) The 6-cycles match both graphically and functionally by:

$X_{n+1} = rX_n(1-X_n)$; $f'(X_n) = r(1-2X_n) = 0$ $X_n = 1/2$ $X_{n+1} = r\sin(\pi X_n)$; $f'(X_n) = r\pi\cos(\pi X_n) = 0$ $X_n = 1/2$.	Maximums
$X_{n+1} = 0 = rX(1-X_n)$; $X_{n+1} = 0 = r\sin(\pi X_n)$; $X^* = 0, 1$	Nodes:
$X_{n+1} = rX_n(1-X_n)$; $0 = 3.62756X_n^2 - 2.62756X_n$ $X_n = 0.724332$ $X_{n+1} = r\sin(\pi X_n)$; $0 = r\sin(\pi X_n) - X_n$ $X_n = 0.704363$	Fixed Points.

10.6.4.

a) (Metropolis et. al 1973) $r = 3.9602701$ for a period-4 cycle

RLL U-Sequence: $X_6 = 0.3$ R
$X_7 = 0.8$ L
$X_8 = 0.$ L

RLR U-Sequence: $X_0 = 0.6$ R $X_3 = 0.6$ R
$X_1 = 0.9$ L $X_4 = 0.9$ L
$X_2 = 0.2$ R $X_5 = 0.3$ R

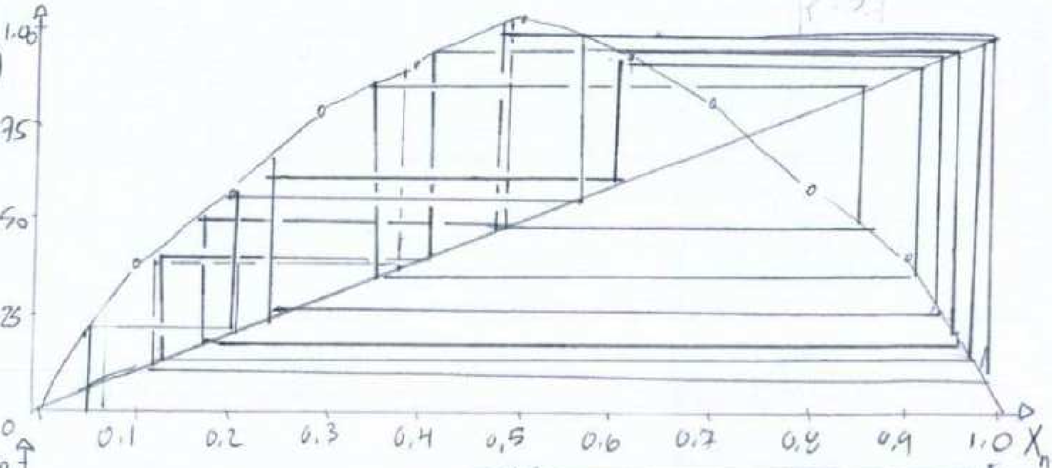
The RLL sequence seemed intermittent in the X_{n+1} analysis.

b) $X_i > X_2, X_5$ is a unique period-4 orbit with an alternating U-sequence.

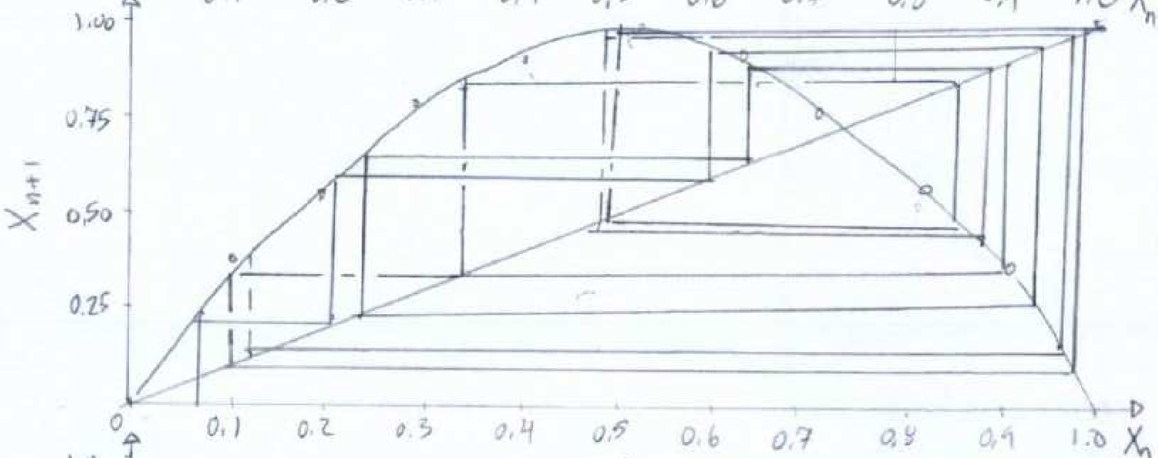
10.6.5

a)

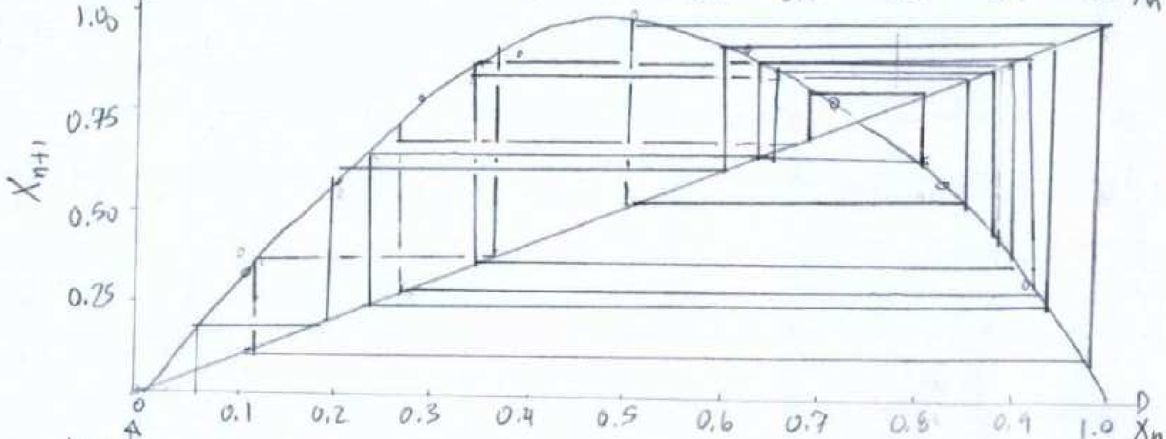
$r = 3.1057065$



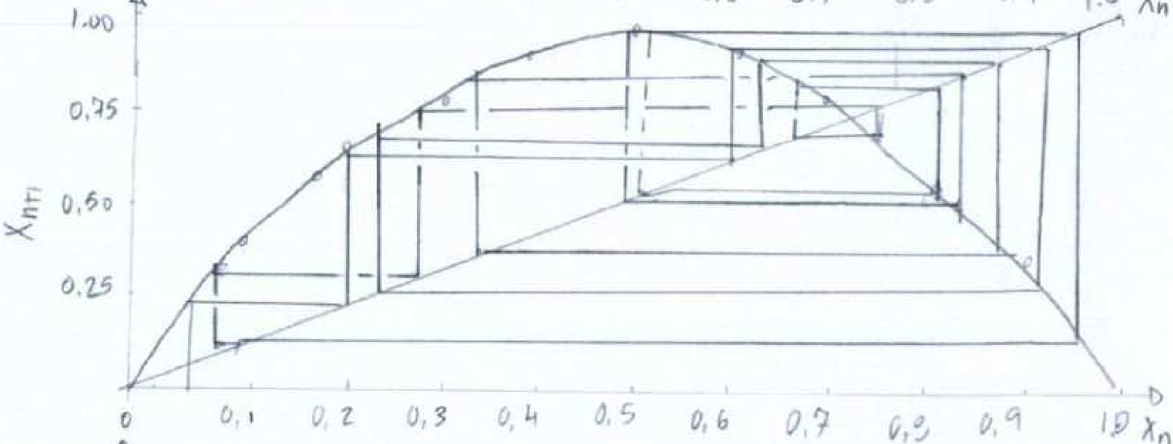
$r = 3.9375364$



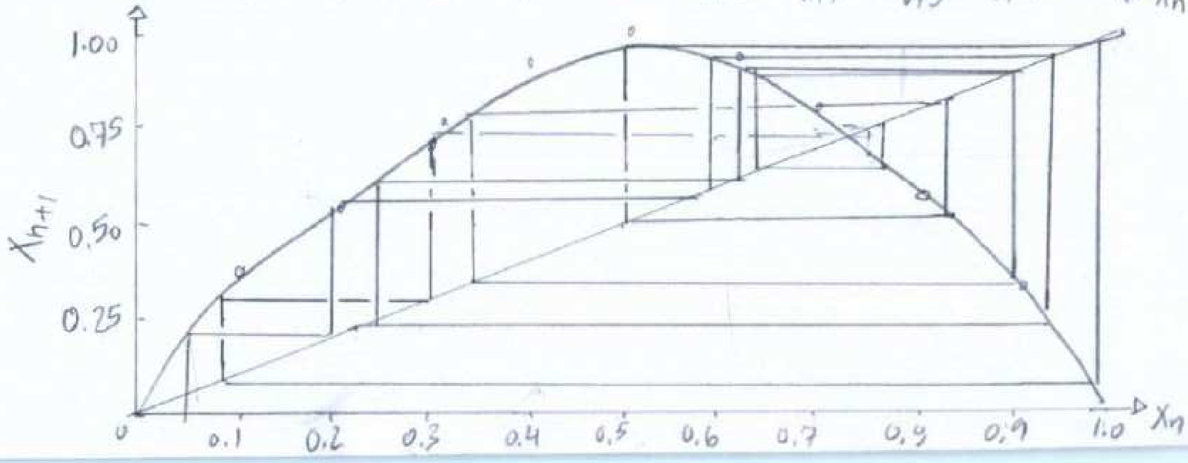
$r = 3.9602701$



$r = 3.9777664$



$r = 3.9902670$



b) Accuracy by a computer would improve results:

$x_{n+1} = r - x_n^2$

$R(y) = \sqrt{r-y}$

$L(y) = -\sqrt{r-y}$

10.6.6.

a) $RLLR(0) = R(L(L(R(0))))$

$= \sqrt{r + \sqrt{r + \sqrt{r - \sqrt{r + 0}}}}$

$= \sqrt{r + \sqrt{r + \sqrt{r - \sqrt{r}}}}$

b) $r_0 = 2$; $r_{n+1} = \sqrt{r_n + \sqrt{r_n + \sqrt{r_n - \sqrt{r_n}}}}$

$r_1 = 1.913881 \to r_2 = 1.88132 \to r_3 = 1.86877$

$\downarrow$

$r_4 = 1.86390$

$\downarrow$

$r_n \approx 1.860$

c) $R \to L \to L \to R$ around a stable node $r_{n+1} \approx 1.86$.

$g(x) = \alpha g^2(x/\alpha)$

10.7.1

a) $g(x) = 1 + c_2 x^2$; $g(x) = \alpha\left(1 + c_2\left(1 + c_2\left(\frac{x}{\alpha}\right)^2\right)^2\right)$

$= \alpha + \alpha c_2 + \frac{2c_2^2 x^2}{\alpha} + \frac{c_2^3 x^4}{\alpha^3}$

$1 + c_2 x^2 = \alpha(1 + c_2) + \frac{2c_2^2 x^2}{\alpha} + O(x^4)$

$1 = \alpha(1 + c_2)$; $1 = \frac{2c_2}{\alpha}$

$(\alpha, c) = (1, 0), \left(-1 \pm \sqrt{3}, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\right)$

b) $g(x) = 1 + c_2 x^2 + c_4 x^4$;

$= \alpha\left(1 + c_2\left(1 + c_2\left(\frac{x}{\alpha}\right)^2 + c_4\left(\frac{x}{\alpha}\right)^4\right)^2 + c_4\left(1 + c_2\left(\frac{x}{\alpha}\right)^2 + c_4\left(\frac{x}{\alpha}\right)^4\right)^4\right)$

$= \alpha(1 + c_2 + c_4) + \left(\frac{2c_2^2 + 4c_2c_4}{\alpha}\right)x^2$

$+ \left(\frac{c_2^3 + 4c_4^2 + 6c_2^2 + 2c_2c_4}{\alpha^3}\right)x^4 + O(x^6)$

$1 = \alpha(1 + c_2 + c_4)$; $1 = \frac{2c_2 + 4c_4}{\alpha}$; $c_4 = \left(\frac{c_2^3 + 4c_4^2 + 6c_2^2c_4 + 2c_2c_4}{\alpha^3}\right)$

$$(\alpha, c_2, c_4) \approx (-2.82, -1.30, -0.06)$$
$$(0.67, 0.63, -0.14)$$
$$(4.65, -3.89, 3.11)$$

$y_{n+1} = f(y_n)$

$X_n = \alpha\, y_n$

10.7.2. $X_n = \alpha\, y_n = \alpha\, f(y_n) = \alpha\, f\left(\frac{X}{\alpha}\right)$

$$X_{n+1} = \alpha\, f^2\left(\frac{X}{\alpha}\right)$$

10.7.3. Functional Equation: a function defined in terms of itself.

$$g(x) = g^2(x) = \mu g^2\left(\frac{x}{\mu}\right)$$

10.7.4. $g(x) \approx$ Parabolic. $= x^2$

$$g(x^*) = g(\alpha x^*) = \alpha^n f^{(2^n)}\left(\frac{x^*}{\alpha^n}\right)$$

$f(x, r) = r - x^2$

10.7.5.

a) $F(x, R_0) = R_0 - x^2$

$F^2(x, R_1) = R_0 - [R_1 - x^2]^2$

A solution into R_0 and R_1:

R_0: $R_0 - x^2 = R_0 - x^2$

$R_0 = R_0$

$= 0$

R_1: $R_p - x^2 = R_0 - [R_1 - x^2]^2$

$x = \pm\sqrt{R_1}, \pm\sqrt{R_1 + 1}$

$R_1 = 1$

$F(x_1, R_0) = -x^2$

$\alpha f\left(\frac{x}{\alpha}, R_1\right) = \alpha\left[1 - \left[1 - \left(\frac{x}{\alpha}\right)^2\right]^2\right]$

$= -\frac{x^4}{\alpha^3} + \frac{2x^2}{\alpha}$

b) When $\alpha = -2$, then:

$$f(x, R_0) = \alpha f^2(x/\alpha, R_1)$$

$$-x^2 = \frac{2x^2}{\alpha} - \frac{x^4}{\alpha^3} \approx \frac{2x^2}{\alpha}$$

$$-x^2 = -x^2 = \text{"Resemblance"}$$

$f(x,r) = r - x^2$

10.7.6. $\alpha f^2\left(\frac{x}{\alpha}, R_1\right) = -\frac{x^4}{\alpha^3} + \frac{2x^2}{\alpha}$

$$\alpha^2 f^4\left(\frac{x}{\alpha^2}, R_4\right) = -\alpha^2 (R_4^8 - 4R_4^7 + 6R_4^6 - 6R_4^5 + 5R_4^4 - 2R_4^3 + R_4^2 - R_4) + \frac{8R_4^5(R_4^4 - 3R_4^3 + 3R_4^2 - 2R_4 + 1)x^2}{\alpha^2} + O\left(\frac{x}{\alpha^2}\right)$$

<u>A solution for R_4:</u>

$$R_4 = -\alpha^2 (R_4^8 - 4R_4^7 + 6R_4^6 - 6R_4^5 + 5R_4^4 - 2R_4^3 + R_4^2 - R_4)$$

$$-1 = \frac{8R_4^3(R_4^4 - 3R_4^3 + 3R_4^2 - 2R_4 + 1)}{\alpha^2}$$

$$(\alpha; R_4) = (0,0), (\pm 0.747144, -0.322988), (\pm 1.00087, 1.74495)$$

$f(x,r) = r - x^4$

10.7.7 $f'(x, R_0) = R_0 - x^4$

<u>Fixed Points:</u> $f'(x, R_0) = R_0 - 4x^3 = 0$

$$x^* = \sqrt[3]{\frac{R_0}{4}}$$

When fixing to zero, $x^* = 0$, $R_0 = 0$.

<u>A solution into R_1:</u>

$$f^2(x, R_1) = R_1 - [R_1 - x^4]^4$$

$$0 = R_1 - R_1^4$$

$$R_1 = 1$$

$$f(x, R_0) = \alpha f^2\left(\frac{x}{\alpha}, R_1\right)$$

$$-x^4 = \alpha\left[1 - \left[1 - \left(\frac{x}{\alpha}\right)^4\right]^4\right]$$

$$= \frac{4x^4}{\alpha^3} - \frac{6x^3}{\alpha^7} + \frac{4x^{12}}{\alpha^{11}} - \frac{x^{16}}{\alpha^{15}}$$

$$-x^4 \approx \frac{4x^4}{\alpha^3} \quad ; \quad \alpha = \sqrt[3]{-2^2}$$

<u>Universal g-function:</u>

$$g(x) = \alpha g\left(g\left(\frac{x}{\alpha}\right)\right)$$

$$1 + c_1 x^4 = \alpha(c_1 + 1) + \frac{4c_1^2 x^4}{\alpha^3} + \frac{6c_1^3 x^8}{\alpha^7} + \frac{4c_1^4 x^{12}}{\alpha^{11}} + \frac{c_1^5 \cdot x^{16}}{\alpha^{15}}$$

$$1 = \alpha(c_1 + 1) \; ; \; 1 = \frac{4c_1}{\alpha^3}$$

$$(\alpha, c_1) = (-1.835, -1.545), (0.862, 0.160)$$

<u>δ-estimation:</u> $\delta = \dfrac{R_n - R_{n-1}}{R_{n+1} - R_n}$

<u>R_2 calculation:</u> $f^3(x, R_2) = R_2 - [R_2 - [R_2 - x^4]^4]^4$

$$= R_2 - [R_2 - [R_2]^4]^4$$

$$= R_2 - R_2^4 + 4R_2^7 - 6R_2^{10} + 4R_2^{13} - R_2^{16}$$

$$R_2 = 1.229.$$

$$\delta = \frac{1 - 0}{1.229 - 1} = 4.366$$

$$\delta_{Actual} = 4.669$$

$$\text{Error} = \frac{|\delta - \delta_{Actual}|}{\delta_{Actual}} = 6.5\%$$

$x_{n+1} = f(x_n, r)$

$f(x_n, r) = -r + x - x^2$

10.7.8

a) If $r = 0$, then.

$f(x_n, r) = x - x^2$

Tangent line: $y = mx + b$

where $m = f(x_n, r)' = 1 - 2x$

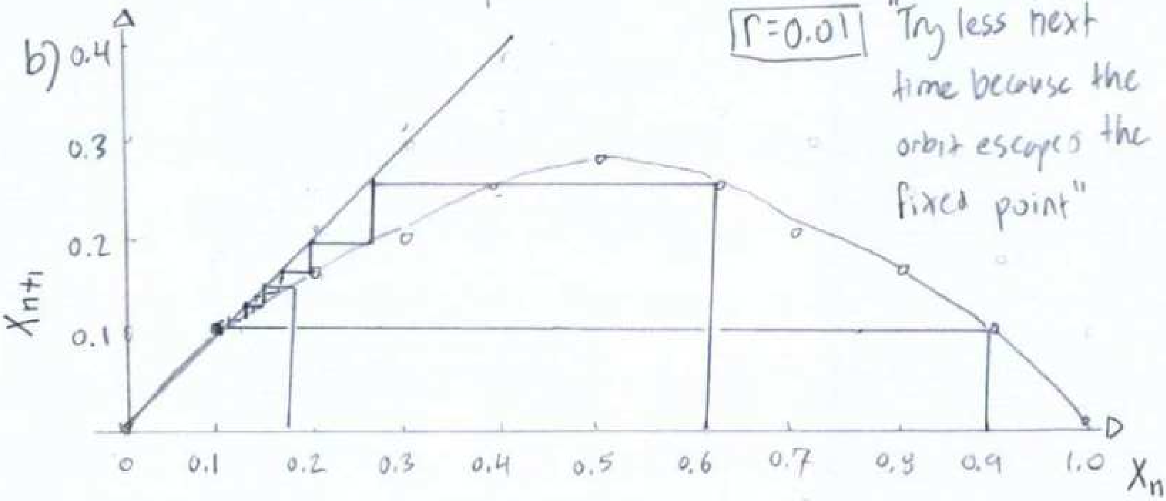
When evaluated at $x = 0, m = 1$

$y = x + b$

at the point $(0,0)$, $b = 0$

$y = x$

Essentially, a tangent line exists at $x = 0$ with n-periodic cycles at the same point.

b) 

$\boxed{r = 0.01}$ "Try less next time because the orbit escapes the fixed point"

c) $N(r)$ = typical number of iterations of f

Universal Function: $\lim_{n \to \infty} \alpha \cdot f^{(2^n)}\left(\frac{x}{\alpha^n}, R_n\right)$

$r = n$ for the problem: $\lim_{r \to 0} \alpha^r \cdot f^{(2r)}\left(\frac{x}{\alpha^r}, R_n\right)$

$N(r) = \lim_{r \to 0} 2^r \log \alpha^r \cdot f\left(\frac{x}{\alpha^r}, R_n\right)$

$N(r) = \lim_{r \to 0} 2^r \log f\left(\frac{x}{\alpha^r}, R_n\right) + r \log \alpha$

$\frac{1}{2} N(r) = 2 \log\left(f\left(\frac{x}{\alpha^2}, R_n\right)\right) + \log \alpha$

$= \log \alpha^2 f^2\left(\frac{x}{\alpha^2}, R_n\right)$

"The derivation iterations show logarithmic relation."

d) $f^2(x,R) = R+(R+x-x^2)-(R+x-x^2)^2$

$$= -R^2-2R+(2R+1)x-(2R+2)x^2+O(x^4)$$

$\frac{1}{2}N(r) = \log \alpha^2 f^2\left(\frac{x}{\alpha^2}, R_n\right)$

$N(4r) = \log \alpha^{4r} \circ f^{2^{4r}}\left(\frac{x}{\alpha^{2r}}, R_n\right) = \log\left[\alpha^2 \circ f^2\left(\frac{x}{\alpha^r}, R_n\right)\right]^{2r}$

$\frac{1}{2}N(r) \approx N(4r)$

e) $N(r) = ar^b$ is a solution where $a = 2\log \alpha f\left(\frac{x}{\alpha^2}, R_n\right)$

$b = 1$

$g(x) = \alpha g^2(x/\alpha)$

10.7.9

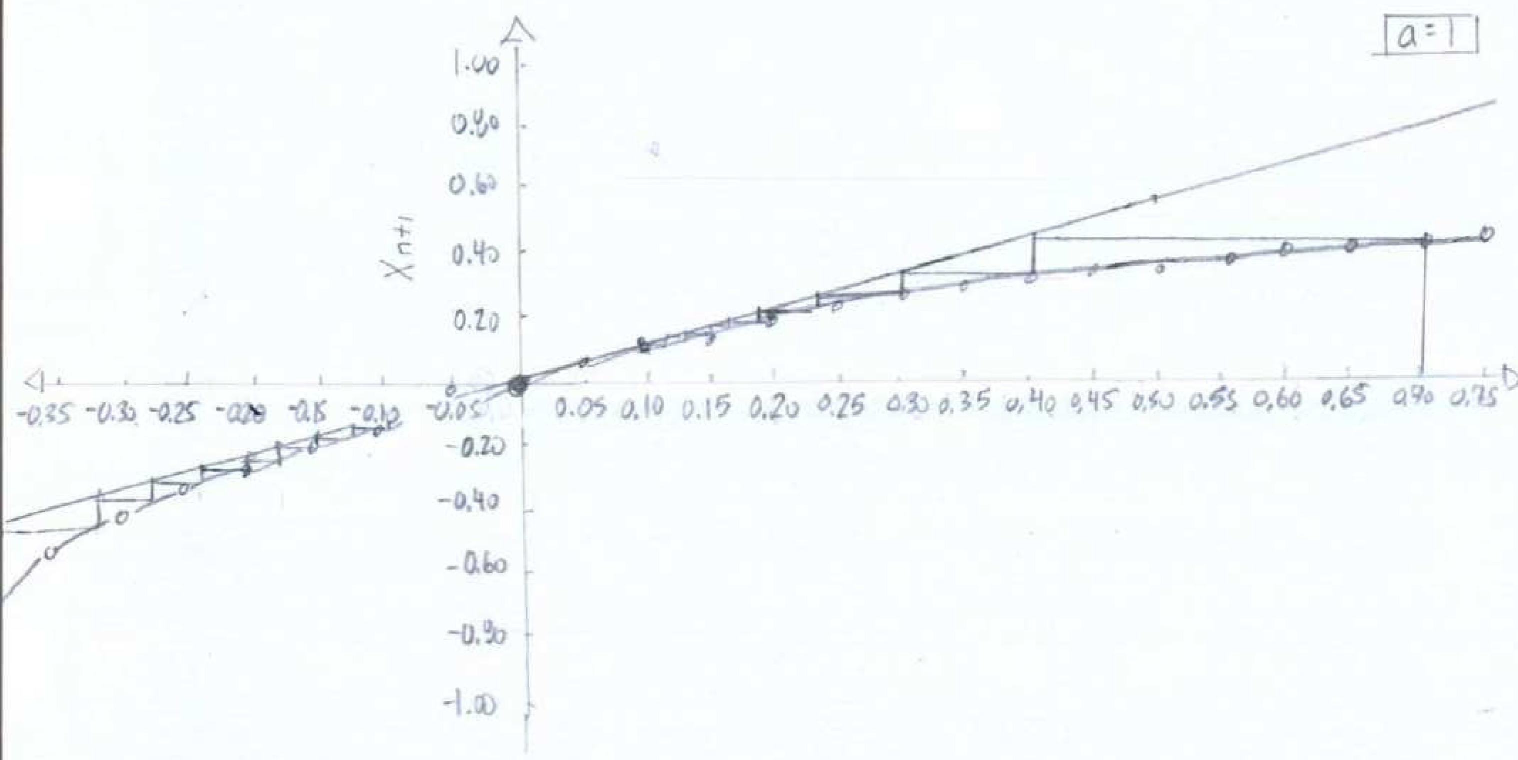
a) $\alpha = 2$, $g(x) = \frac{x}{1+ax}$

$$\alpha g^2\left(\frac{x}{\alpha}\right) = \alpha\left[\frac{\frac{(x/\alpha)}{1+a(x/\alpha)}}{1+a\left(\frac{(x/\alpha)}{1+a(x/\alpha)}\right)}\right]$$

$$= \frac{x}{1+x}$$

$$g(0) = \alpha g^2(0) = 0$$

b)



$f(x) = -(1+\mu)x + x^2$ 10.7.10

$p + \eta_{n+1} = f^2(p+\eta_n)$

$$p = \frac{\mu + \sqrt{\mu^2 + 4\mu}}{2}$$

$$p + \eta_{n+1} = f^2(p+\eta_n)$$

$$= -(1+\mu)\left[-(1+\mu)[p+\eta_n] + [p+\eta_n]^2\right]$$

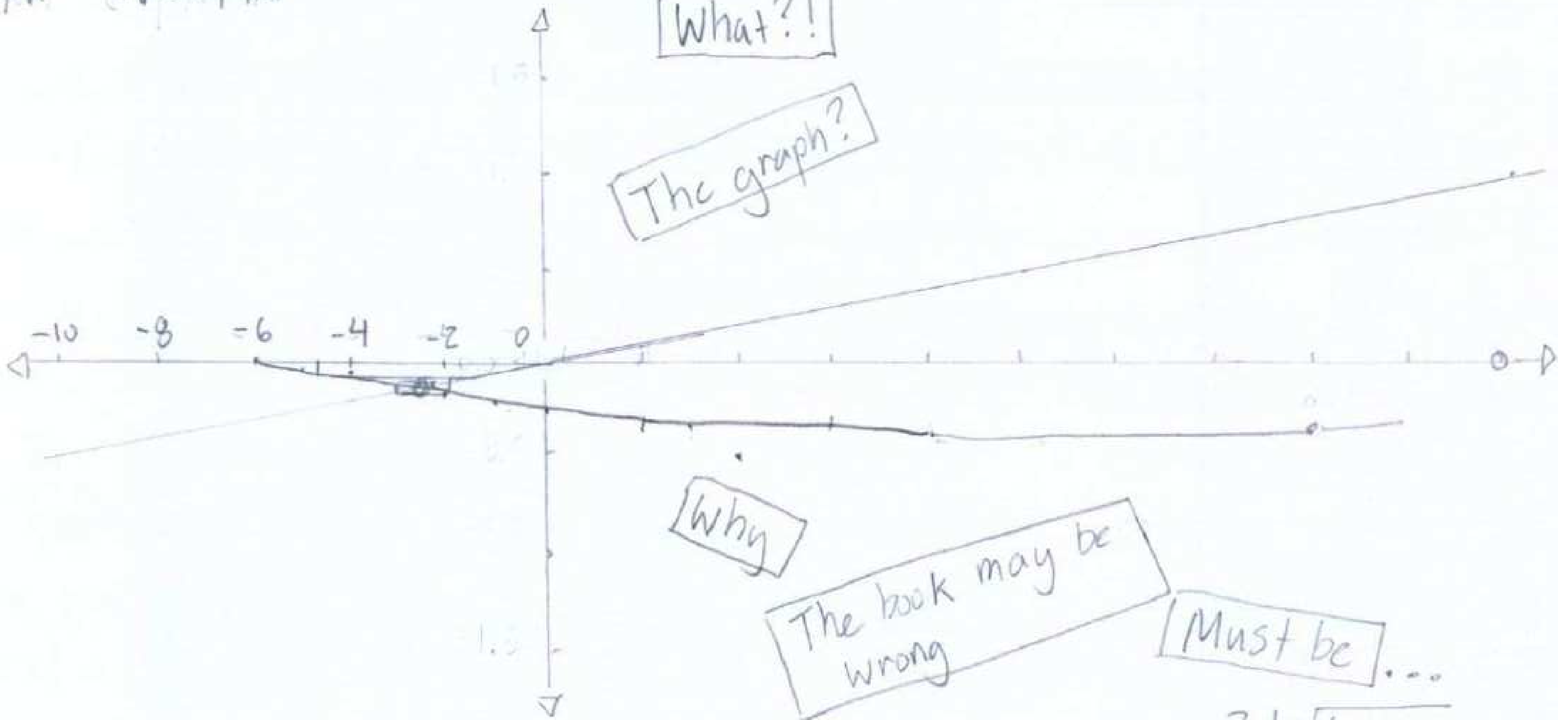
$$+ \left[-(1+\mu)[p+\eta_n] + [p+\eta_n]^2\right]^2$$

$$= \eta - 6p^2\eta + 4p^3\eta + 2\mu p + 2\mu p\eta - 6\mu p^2 + \mu^2$$

$$+ 2\mu^2 p\eta + O(\eta^2) + O(\eta^0)$$

The missing zeroth order led no solution, rather author contact.

$\mu_K = -2\sqrt{6 + \mu_{K-1}}$ 10.7.11.



$\boxed{\text{What?!}}$

$\boxed{\text{The graph?}}$

$\boxed{\text{Why}}$

$\boxed{\text{The book may be wrong}}$

$\boxed{\text{Must be}}$...

$\mu_K = -2 + \sqrt{6 + \mu_{K-1}}$

$\boxed{\text{Uhh oh}}$

$\boxed{\text{The Plus sign}}$

Chapter 11: Fractals:

$X_1 = 0.X_{11}X_{12}X_{13}X_{14}\ldots$
$X_2 = 0.X_{21}X_{22}X_{23}X_{24}\ldots$
$X_3 = 0.X_{31}X_{32}X_{33}X_{34}\ldots$
$\vdots$

11.1.1. Example 11.1.4 demonstrates the uncountable rational numbers by a "diagnol argument." A rational number is a fraction with traditional numerator and denominator. When digits never end, such as an extensive root, then the "diagnal argument" fails.

$2\mathbb{N}-1$;
$X = \{x \mid 2n-1, n \in \mathbb{N}\}$;

11.1.2 $X = \{1, 3, 5, \ldots\}$, specifically odd numbers, are countable. A one-to-one correspondence maps two sets into pairs, also counts.

$2 \longleftrightarrow 3,\ 3 \longleftrightarrow 5,\ 4 \longleftrightarrow 7,\ \ldots,\ N \longleftrightarrow 2N-1$

$\mathbb{R} - \mathbb{Q}$;
$\mathbb{R}/\mathbb{Q}$

11.1.3. The irrational numbers are uncountable because pi's infinite digits. A "diagnol argument" is endless at infinite n. Also, a one-to-one correspondence seems difficult with Euler's number and special roots because the irregular sets.

11.1.4. A repeating decimal found by:

(Magnitude #1 − Magnitude #2)α = Value at Magnitude #1 − Value at Magnitude #2.

α = Repeating decimal, such as:
$2.7272\overline{72}\ldots$

The neverending digits limit Cantor's "diagnol argument" because the finite diagnol (n) never terminates.

$X = \{X \mid (p, q, r)\}$
$Y = \{Y \mid (i, j, k, l, m)\}$

11.1.5. By induction: Base case ($n=3$) − Total Points $= n^3 - 1 = 3^3 - 1 = 26$ points

Next case ($n+2$) − Total Points $= (n+2)^3 - 1 = 98$ points

Infinite Case ($n+\infty$) − Total Points $= (\infty)^3 - 1 = \infty$ points

Each set is countable by a larger cube with a larger side length.

- Other solutions suppose exact vs. inexact counting.
- A large solution set in structure theory prove shape dialation, subdivision, enumeration, infinite rays and infinite cones for countable points.
- Alternative arguments in $\mathbb{R}^3$ are not only shapes, but polynomials and convex functions for countable points.

$X_{n+1} = 10X_n \pmod 1$

11.1.6. Fixed Points: $X_n^* \in \mathbb{Z}/10$

a) Stability: $|f(x^*)'| = |10x \pmod 1)| = 10 > 1 =$ Unstable

Proof about Countability: A one-to-one correspondence exists between $\mathbb{Z}/10$ and $\mathbb{Q}/10$, $\mathbb{Z}/10$ and $\mathbb{R}/10$, also $\mathbb{Z}/10$ and $\mathbb{C}/10$.

Each periodic orbit is unstable and countable.

b) An aperiodic behavior exhibits sensitive dependence on initial conditions, such as a value between the integers i.e. a decimal.

Proof about countability:

Contradiction: A countable set between the integers, over 10. such as 0 to 0.1 have a list $\{X_1, X_2, X_3 \ldots\}$

$X_1 = 0.X_{11}X_{12}X_{13}X_{14}\cdots$

$X_2 = 0.X_{21}X_{22}X_{23}X_{24}\ldots$

$X_3 = 0.X_{31}X_{32}X_{33}X_{34}\ldots$

$\vdots$

where X_{ij} denotes the j^{th} digit of the real number

1) The first digit isn't X_1 the second X_2, then X_{nn} digit not be X_n:

A last digit is uncountable.

c) An "eventually-fixed point" is countable because the finite nature in the function $X_{n+1} = X_n$ for all $n > N$.

$X_{n+1} = 2X_n \pmod 1$ 11.1.7. Fixed Points: $X \in \mathbb{Z}/2$

Periodic Orbits: Countable periodic orbits are true because one-to-one correspondence between $\mathbb{Z}/2$ and $\mathbb{R}/2$, $\mathbb{Q}/2$, and $\mathbb{C}/2$.

Aperiodic Orbits: The set for aperiodic orbits is not countable. Initial conditions are highly sensitive between fixed points, also involve irrational numbers that Cantors "diagnol arguments" fail to solve.

$S_0 = [0,1]$

$C = S_\infty$

11.2.1.

$S_0 = [0,1]$

$S_1 = [0, 1/3] \wedge [2/3, 1]$

$S_2 = [0, 1/9] \wedge [2/9, 3/9] \wedge [6/9, 7/9] \wedge [8/9, 1]$

$S_3 = [0, 1/27] \wedge [2/27, 3/27] \wedge [\cdots] \wedge [26/27, 1]$

$\vdots$

"Cantors Set" S_∞

Set	Length	Length Removed
S_0	$(2/3)^0$	$1-(2/3)^0$
S_1	$(2/3)^1$	$1-(2/3)^1$
S_2	$(2/3)^2$	$1-(2/3)^2$
S_3	$(2/3)^3$	$1-(2/3)^3$
S_n	$(2/3)^n$	$1-(2/3)^n$

Length Removed $= 1 - S_n$

$= 1 - \lim_{n\to\infty} (2/3)^n$

$= 1$

$\mathbb{Q}$ = Rationals

11.2.2 $\varepsilon_1 = 1/1$; $\varepsilon_2 = 1/2$; $\varepsilon_3 = 1/4$ $\cdots$ $\varepsilon_n = 1/2^n$

$\varepsilon_\infty = \lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} 1/2^n = 0$

$\mathbb{R}$ = Reals

11.2.3.

Set	Length	Length Removed
S_0	$\varepsilon(\frac{1}{2})^0$	$1-\varepsilon(1/2)^0$
S_1	$\varepsilon(\frac{1}{2})^1$	$1-\varepsilon(1/2)^1$
S_2	$\varepsilon(\frac{1}{2})^2$	$1-\varepsilon(1/2)^2$
S_n	$\varepsilon(\frac{1}{2})^3$	$1-\varepsilon(1/2)^3$

An upper limit solution, bounds and counts the lower limit set.

$0 \le \text{const} \cdot \left(\bigcup_{n=1}^{\infty} \mathbb{R}\right) = 1 + \varepsilon \sum_{1}^{\infty} \left(\frac{1}{2}\right)^n$

$= 1 + \frac{\varepsilon}{\log(4)}$

$X = \{X \mid 0 < X < 1 \wedge X \in \mathbb{R} - \mathbb{Q}\}$ 11.2.4.

a) $X = \frac{a_0}{10^0} + \frac{a_1}{10^1} + \frac{a_2}{10^2} + \cdots$ because base-10. Also, $a_0 = 0$, since not including zero or one.

$= \frac{a_1}{10^1} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \cdots$

$= 0.a_1 a_2 a_3 \ldots$ Where each constant defines a digit or magnitude in base-10 counting. They change.

The measure is $L_n = \left(\frac{1}{10}\right)^n = 0$.

b) Irrational numbers between 0 and 1 are incountable because base-10 decimals end at infinity. A proof by conjecture is the "diagnol argument."

c) Disconnected subset - When a set representation is never a union of two or more disjoint non-empty open subsets.

In the problem, $X = \{X \mid (0 < X < 1) \wedge X \in \mathbb{R} - \mathbb{Q}\}$ is diconnected from the other irrationals, $\mathbb{R} - \mathbb{Q}$.

d) Isolated Point - a subset with an element and without neighboring elements.

Irrational numbers between zero and one have an addressable representation. The elements are isolated from another element.

11.2.5

a) Base-3 expansion of 1/2:

$X = 1/2 = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^2} + \cdots = \sum_{1}^{\infty} \frac{a_i}{3^i}$

$a_1 = 1;\ X = \frac{1}{2} = \frac{1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots$

$$\frac{3}{2} - 1 = \frac{1}{2} = \frac{a_2}{3} + \frac{a_3}{3^2} + \cdots$$

$$a_2 = 1$$

$$\frac{3}{2} - 1 = \frac{1}{2} = \frac{a_3}{3} + \cdots$$

$$a_3 = 1$$

$$(0.5)_{10} = (0.\bar{1})_3$$

b) One-to-one correspondence: $c \in C$ and $x \in [0,1]$

$$\left(x = \cancel{\frac{a_1}{10^0}} + \frac{a_2}{10^1} + \frac{a_3}{10^2} + \cdots\right) = \left(c = \cancel{\frac{b_1}{3^0}} + \frac{b_2}{3^1} + \frac{b_3}{3^2} + \cdots\right)$$

$$\cancel{\frac{a_1}{10^0} = \frac{b_1}{3^0}};\ \frac{a_2}{10^1} = \frac{b_2}{3^1};\ \frac{a_3}{10^2} = \frac{b_3}{3^2};\ \ldots$$

where $x = 0.a_2 a_3 a_4$ and $c = 0.b_2 b_3 b_4$.

c) Endpoint - an extreme value or unattainable segment in a set.

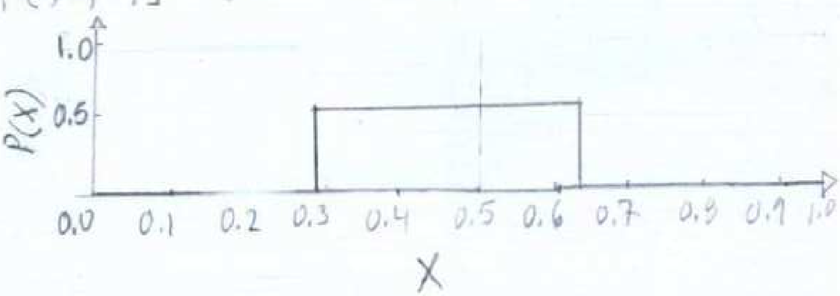
A base-3 counting system, Cantor's set has endpoints located at ternary values.

$S_0 = [0, 1]$
$S_\infty = C$

11.2.6.

a) $S_0 = [0, 1]$; $P_0 \in \text{Random}(S_0) < X$
$= 1$

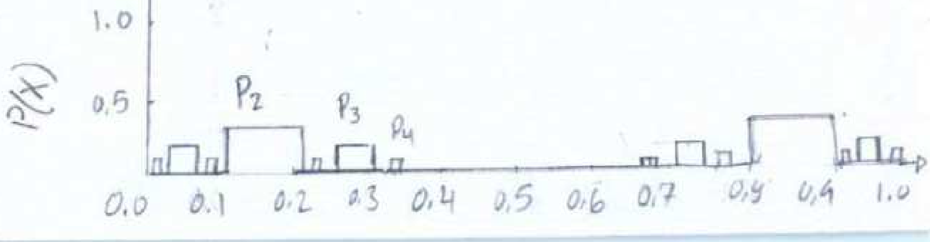
b) $S_1 = [(0, 1/3), (2/3, 1)]$; $P_0 \in \text{Random}(S_1) < X$



c) $P_2 \in \text{Random}(S_2) = 1/4 < X$

$P_3 \in \text{Random}(S_3) = 1/8 < X$

$P_4 \in \text{Random}(S_4) = 1/16 < X$



d) $P_\infty(\lambda)$ is continuous and nearly zero.
The graph is almost a line at $P(\lambda)=0$.

$S_0 = [0,1]$
$S_\infty = C_{1/2}$

11.3.1

a) $S_0 = [0,1]$

$S_1 = [(0, 1/4), (3/4, 1)]$

$S_2 = [(0, 1/16), (3/16, 4/16), (12/16, 13/16), (15/16, 16/16)]$

$\vdots$

S_∞ ;

$m = 1 \quad r = 1$
$m = 2 \quad r = 3$

$m = \infty, \; r = \infty$

Similarity dimension (d): the exponent defined by $m = r^d$
or $d = \dfrac{\ln m}{\ln r}$, where m is the number of copies and r is a scale factor.

$$d = \frac{\ln(2)}{\ln(3)} = 0.63$$

b) The sets measure is zero because the zero length at $n = \infty$.

11.3.2. $d = \dfrac{\ln(m)}{\ln(r)} = \dfrac{\ln(m)}{\ln(\alpha)}$

11.3.3.

a) $d = \dfrac{\ln(m)}{\ln(r)} = \dfrac{\ln(4)}{\ln(7)} = 0.71$

b) $d = \dfrac{\ln(m)}{\ln(r)} = \dfrac{\ln(n)}{\ln(2n+1)} = \ln(n+1)$ where $n \in \mathbb{N}$

11.3.4. $d = \dfrac{\ln(m)}{\ln(r)} = \dfrac{\ln(5)}{\ln(10)} = 0.70$

11.3.5. $d = \dfrac{\ln(m)}{\ln(r)} = \dfrac{\ln(9)}{\ln(10)} = 0.95$

11.3.6. Cantor's set is all ternary numbers without any 1's in their ternary representation.

Also, the inner intervals are not isolated because $S_\infty \in S_5 \in S_4 \in S_3 \in S_2 \in S_1 \in S_0 \in Q$

von Koch snowflake.

11.3.7.

a. 

S_0

S_1 Partial Star of David.

S_2

b. See above

c. $L_0 = \left(\frac{4}{3}\right)^0 \cdot 1$

$L_1 = \left(\frac{4}{3}\right)^1 \cdot 1$

$L_2 = (4/3)^2 \cdot 1$

$\vdots$

$L_n = \lim_{n \to \infty} \left(\frac{4}{3}\right)^n = \infty$

d) $L_0 = 0$

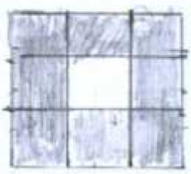
$L_1 = \left(\frac{1}{3}\right)^2 \left(\frac{\sqrt{3}}{4}\right) n^2$

$L_2 = \left(\frac{1}{3}\right)^4 \left(\frac{\sqrt{3}}{4}\right) 2^2 + \left(\frac{1}{3}\right)^2 \left(\frac{\sqrt{3}}{4}\right)$

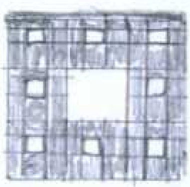
$\vdots$

$L_\infty = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n} \left(\frac{\sqrt{3}}{4}\right) n^2 = \frac{45\sqrt{3}}{1024}$

e) $d = \frac{\ln(m)}{\ln(r)} = \frac{\ln(4)}{\ln(2)} = 0.63$



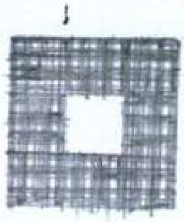
S_1



S_2

11.3.8.

a) S_3



b) $d = \frac{\ln(m)}{\ln(r)} = \frac{\ln(512)}{\ln(27)} = 1.89$ ⟵ Large value!!!

c) Area = Length x Width ; $S_0(\text{Area}) = 1$

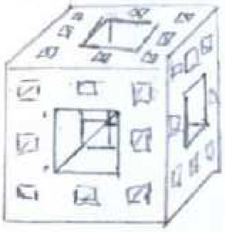
$S_1(\text{Area}) = 8/9$

$S_2(\text{Area}) = 64/81$

$S_n = (8/9)^n$

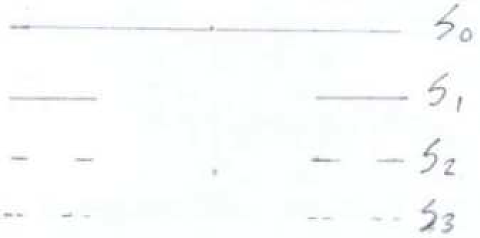
$S_\infty = 0$

11.3.9. $d = \frac{\ln(m)}{\ln(n)} = \frac{\ln(20)}{\ln(3)}$


Menger sponge

n-dimensional: $d = \frac{\ln(m)}{\ln(n)} = \frac{\ln(N2^{N-1}+2^N)}{\ln(3)}$

11.3.10.
a).



Topological Cantor Set - a closed set with

1. "Totally disconnected" elements
2. No isolated points

If $C = \bigcap_{n=0}^{\infty} C_n = C_0 \cap C_1 \cap C_2 \cap \cdots \cap C_n$

$B = \bigcap_{n=0}^{\infty} B_n = B_0 \cap B_1 \cap B_2 \cap \cdots \cap B_n$

$C_0 = B_0 = [0,1)$

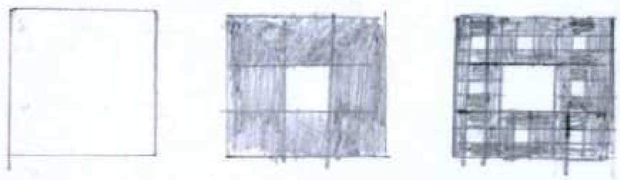
$B \in C$

b) $B_0 = 1$; $B_1 = \left(\frac{1}{2}\right)^1$; $B_2 = \left(\frac{1}{4}\right)^2$; $\lim_{n\to\infty} B_n = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$

11.4.1. Von Koch Snowflake:

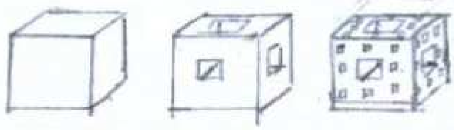
$d = \frac{\ln N}{\ln\left(\frac{1}{\varepsilon}\right)} = \frac{\ln 12}{\ln\left(1/\left(\frac{1}{3\sqrt{2}}\right)\right)} = \frac{\ln 12}{\ln(3\sqrt{2})} = 1.72$

11.4.2. Sierpinski Carpet:

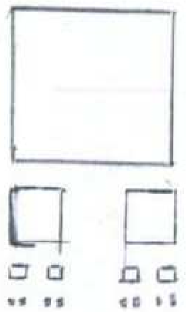


11.4.3. Menger Sponge:

$d = \frac{\ln N}{\ln(1/\varepsilon)} = \frac{\ln(8^n)}{\ln\left(1/\left(\frac{1}{3}\right)^n\right)} = \frac{\ln 8}{\ln 3} = 1.89.$



$d = \frac{\ln(N)}{\ln(1/\varepsilon)} = \frac{\ln(20)}{\ln(3)} = 2.72.$



11.4.4. The Cartesian Product of the middle-thirds Cantor set:

$$d = \frac{\ln(N)}{\ln(1/\varepsilon)} = \frac{\ln(2^2)}{\ln(1/(\frac{1}{3}))} = \frac{2\ln(2)}{\ln(3)} = 1.26.$$

11.4.5. Menger Hypersponge: $d = \frac{\ln(N)}{\ln(1/\varepsilon)} = \frac{\ln(49^N)}{\ln(3^N)} = 3.52$

$$f(x) = \begin{cases} rx & 0 \le x \le 1/2 \\ r(1-x) & \frac{1}{2} \le x \le 1 \end{cases}$$

11.4.6.

a) $x_{n+1} = f(x_n)$; $r > 2$

If $f(x_0) > 1$, then "escape."

Fixed Points: $0 \le x \le 1/2$: $x_{n+1} = rx$

$x(1-r) = 0$; $x^* = 0$

$\frac{1}{2} \le x \le 1$: $x_{n+1} = r(1-x)$

$x(1+r) - r = 0$; $x^* = \frac{r}{(1+r)}$

Stability: $x^* = 0$; $|f'(0)| = r$ "unstable"

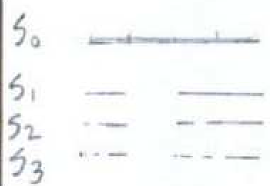
$x^* = \frac{r}{1+r}$; $|f'(\frac{r}{1+r})| = r$ "unstable"

I calculate no "escape" because $f(x_0)_{max} = 1$ and not greater than one.

b) $x_0 = \{x \mid [0,1]\}$

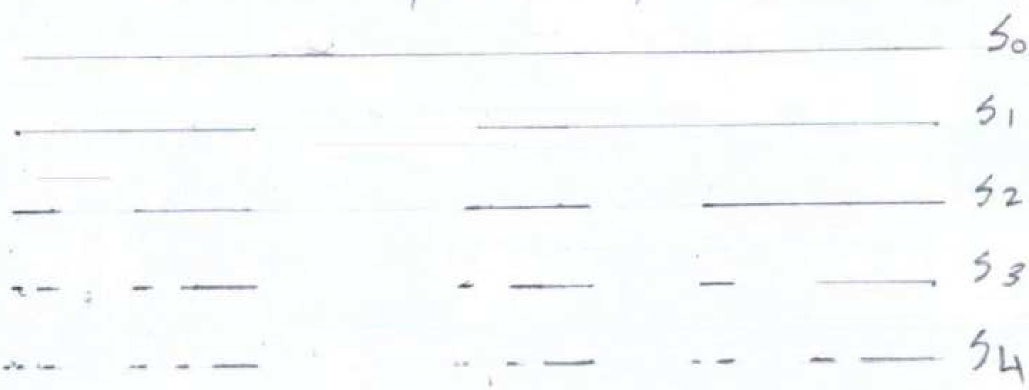
c) $d = \frac{\ln(N)}{\ln(1/\varepsilon)} = \frac{\ln(\infty)}{\ln \frac{1}{1-\frac{1}{r}}} = \infty$

d) Liapunov Exponent: An unstable fixed point has a positive exponent.



11.4.7.

a.



b) $d = \ln \frac{N}{\ln 1/\varepsilon} = \frac{\ln(3^n)}{\ln\left(1/\left(\frac{1}{4}\right)^n\right)} = 0.79.$

c) S_∞ is not self-similar because the irregular construction.

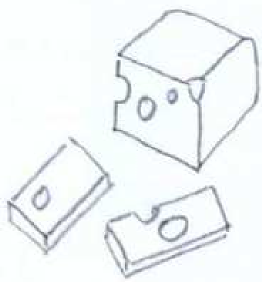
Random Fractal

11.4.8.

a) The box dimension(d) is $\frac{\ln(3^n)}{\ln\left(1/\left(\frac{1}{4}\right)^n\right)} = 0.79.$

A 50:50 coin generates no self-similar structure, but "boxable".

b) A first quarter selection makes one segment and not two, so the box dimension differs.



11.4.9. p^2 = unit square

m^2 = random square $\quad d = \frac{\ln(m)}{\ln(r)} = \frac{\ln(2m+1)}{\ln(p)}$

$p > m+1$

$1/p$ = side

11.4.10 $d = \frac{\ln(m)}{\ln(r)} = \frac{\ln(2^n)}{\ln\left(1/\left(\frac{1}{2}\right)^n\right)} = 1$

11.5.1 Correlation Dimension: density of points near an attractor.

Pseudo-code:

```
//Necessary set of attracting functions
func1 = σ(y-x).
func2 = x(ρ-x)-y
func3 = xy-bz
// Initial conditions
ρ=28; σ=10; b=8/3; dt=0.1; x=0; y=0; z=0;
// Runge-Kutta 4th-order
k1x, k2x, k3x, k4x;
k1y, k2y, k3y, k4y;
k1z, k2z, k3z, k4z;
```

```
//Iterate and fill function values
int totalIterations =1000;
int i,j; int values [totalIterations/dH][2]
for(i=0; i<totalIterations/dH ; i++){
  k1x = σ(y- x);
  k1y = x(ρ-x)-y;
  k1z = xy-bz;
  k2x = σ(y+dH·k1y/2 - x-dh·k1x/2)·dh
  k2y =[(x+k1x·dh/2)(ρ-(x+k1x·dh/2))- y-k1y·dh/2]dh
  k2z =[(x+k1x·dh/2)(y+k1y·dh/2)-b(z+k1z·dh/2)]dh
  k3x = σ(y+dh·k2y/2-x-dh·k2x/2)·dh
  k3y =[(x+k2x·dh/2)(ρ-(x+k2x·dh/2))-y-k1y·dh/2]dh
  k3z =[(x+k2x·dh/2)(y+k2y·dh/2)-b(z+k2z·dh/2)]dh
  k4x = σ(y+dh·k3y - x- dh·k3x )·dh
  k4y =[(x+k3x·dh)(ρ-(x+k3x·dh))-y-k3y·dh]dh
  k4z =[(x+k3x·dh)(y+k3y·dh)-b(z+k3z·dh)]·dh
  x = x + dh/6 (k1x +2k2x +2k3x + k4x)
  y = y + dh/6 (k1y +2k2y +2k3y + k4y)
  z = z + dh/6 (k1z +2k2z +2k3z + k4z)
  values[i][0] = x;
  values[i][1] = z;
}
//Grassberger and Procaccia (1983)
int point [2] = {rand(), rand()}
int radius = 1;
int C = 0;
```

```
for(i=0; i< totalIterations/dH; i++){
  if(sqrt([value[i][0] -point[0]]^2 + [value[i][1] -point[1]]^2) < 1){
    C+=1
  }
}
C/= totalIterations x totalIterations;
printf("%f", C);
//Real Code:
#include <iostream>
#include <cmath>
int main(){
 int i,j,exp=10,ro=100,sigma=10,radius=1,total,totalIter=10000;
 Float K1X,k2X,k3X,k4X,k1y,k2y,k3y,k4y,k1z,k2z, \
     k3z,k4z,C=0,dh=0.1,b=8/3, X=-58.26 y=-3.3, \
     z=12.2,maxX=0, maxZ=0;
 total=totalIter/dH;
 Float values[total][2];
 for(i=0;i<total;i++){
   k1x=sigma(y-x)dh;
   k1y=(x(ro-z)-y)dh;
   k1z=(xy-bz)dh
   k2x=sigma(y+k1y·dh/2-x-k1x·dh/2)dh
   k2y=(x+k1x·dh/2)(ro-z-k1z·dh/2)-y-k1y·dh/2)dh
   k2z=((x+k1x·dh/2)(y+k1y·dh/2)-b(z+k1z·dh/2))dh
   k3x=sigma(y+k2y·dh/2-x-k2x·dh/2)dh
   k3y=(x+k2x·dh/2)(ro-z-k2z·dh/2)-y-k2y·dh/2)·dh
   k3z=((x+k2x·dh/2)(y+k2y·dh/2)-b(z+k2z·dh/2))dh
   k4x=sigma(y+k3y·dh-x-k3x·dh)dh
   k4y=(x+k3x·dh)(ro-z-k3z·dh)-y-k3y·dh)·dh
   k4z=((x+k3x·dh)(y+k3y·dh)-b(z+k3z·dh))dh
   values[i][0]=x
   values[i][1]=z
```

```
        if(x>maxX){maxX=x}
        if(z>maxZ){maxZ=z}
    }
    int point[2] = {rand() % (int)maxX, rand() % (int)maxZ}
    for(i=0; i<total; i++){
     if(sqrt(pow(values[i][0]-point[0], 2)+pow(values[i][1]-point[1], 2)) <radius){
          C+=1;
     }
    }
    C/=totalIterations * totalIterations;
    std::cout << "Count: "<< C << std::endl;
    std::cout << "The exponent is the correlation dimension." << std::endl;

    return 0;
}
```

A larger ro raises the correlation dimension, in these equations, Lorenz equations, and not Lorentz' equation or Lorentz' equations.

Chapter 12: Strange Attractors:

$x_{n+1} = a x_n$ · 12.1.1. Fixed points: $x(1-a) = 0$; $x^* = 0$, $a = 1, 0$

$y_{n+1} = b y_n$ $y(1-b) = 0$; $y^* = 0$, $b = 1, 0$