Chapter 2:

i $\dot{x} = \sin x$ 2.1.1. $\dot{x} = 0 = \sin x$; $\boxed{x = n\pi}$ 2.1.2 $\boxed{(n+\frac{1}{2})\pi \text{ where } n \text{ is even.}}$

2.1.3. a) $\ddot{x} = \cos x \sin x$ b) $\frac{1}{2}\sin(2x) = \cos(x)\sin(x)$; $\boxed{\ddot{x} = \frac{1}{2}\sin(2x);\ x = (n+\frac{1}{4})\pi;\ n \in \mathbb{Z}}$

2.1.4.a) $x_0 = \pi/4$; $t = \ln|(\csc x_0 + \cot x_0)|(\csc x + \cot x)|$

$$e^t = \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} = \frac{\csc \pi/4 + \cot \pi/4}{\csc x + \cot x} = \frac{\frac{2}{\sqrt{2}}+1}{\csc x + \cot x} = \frac{\sqrt{2}+1}{\csc x + \cot x}$$

$$\frac{1}{\frac{1}{\sin x} + \frac{\cos x}{\sin x}} = \frac{\sin x}{1+\cos x} = \frac{\sin(2\frac{x}{2})}{1+\cos(2\frac{x}{2})} = \frac{2\cos(\frac{x}{2})\sin(\frac{x}{2})}{1+2\cos^2(\frac{x}{2})-1} = \tan\left(\frac{x}{2}\right) = \frac{e^t}{\sqrt{2}+1}$$

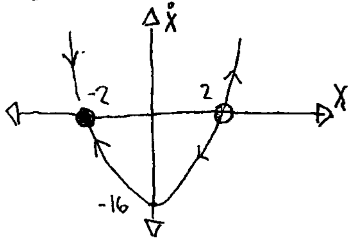
$$x(t) = 2\tan^{-1}\left(\frac{e^t}{\sqrt{2}+1}\right);\ \lim_{t\to\infty} x(t) = 2\tan^{-1}(\infty) = \boxed{2\cdot\frac{\pi}{2} = \pi}$$

b) $\boxed{x(t) = 2\tan^{-1}\left(\frac{e^t}{\csc x_0 + \cot x_0}\right)}$

2.1.5a) A mechanical analog of $\dot{x} = \sin x$ is the undamped pendulum having an $x_0$ of the maximal point

b) Unstable points are described by a positive slope (source) and stable points (sink), a negative slope. The function $\dot{x} = \sin x$ at $x^* = 0$ is unstable, while $x^* = \pi$, is stable.

$\dot{x} = 4x^2 - 16$ 2.2.1 Vector Field:




Fixed Points: | Stability:
$x = 2$ | source (unstable)
$x = -2$ | sink (stable)

Solving for x(t):

$$\frac{dx}{(x^2-4)} = 4t$$

$$\int \frac{A}{(x-2)}dx + \int \frac{B}{(x+2)}dx = 4t$$

$A(x+2) + B(x-2) = 1$

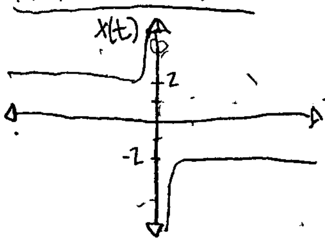
$A = 1/4$ @ $x = 2$

$B = -1/4$ @ $x = -2$

$$\ln\frac{|x-2|}{|x+2|} = 16t + C$$

$$\boxed{x(t) = 2(e^{16t+C} + 1)}$$

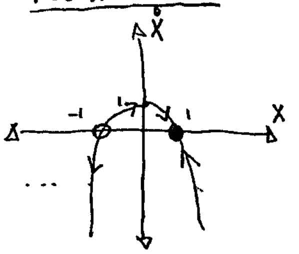
Plot of x(t):




Analytical solution of x(t):

$$x(t) = \frac{2\left(e^{16t}\cdot\left(\frac{x_0-2}{2+x_0}\right)+1\right)}{1 - e^{16t}\left(\frac{x_0-2}{2+x_0}\right)}$$

$\dot{x} = 1 - x^{14}$ 2.2.2 Vector Field:



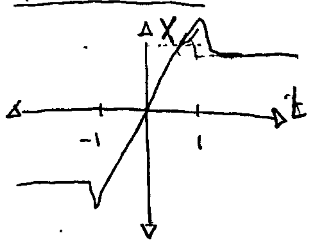

Fixed Points: | Stability:
$\boxed{x = 1}$ | $\boxed{\text{source (unstable)}}$
$\boxed{x = -1}$ | $\boxed{\text{sink (stable)}}$

Solving for x(t)

$$t = \int \frac{dx}{1-x^{14}}$$

Unsolvable

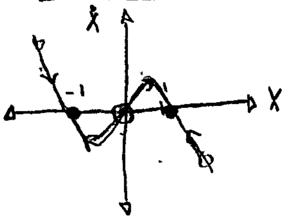
Plot of X(t):




Analytical Solution of x(t):

$\dot{x} = x - x^3$ 2.2.3

Vector Field:



Fixed Points: Stability:

$x = -1$ stable (sink)

$x = 0$ unstable (source)

$x = 1$ stable (sink)

Solving for $x(t)$:

$$t = \int \frac{dx}{x - x^3} = \int \frac{dx}{x(1-x^2)}$$

$$= -\frac{1}{2}\int \frac{A\,du}{(1-u)u} = -\frac{1}{2}\int \frac{A}{1-u}du - \frac{1}{2}\int \frac{B}{u}du$$

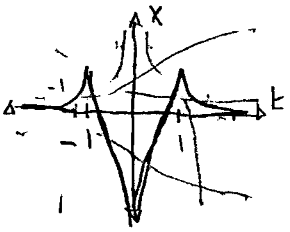
$$= -\frac{1}{2}\ln\left|\frac{1-u}{u}\right| = -\frac{1}{2}\ln\left|\frac{x^2}{1-x^2}\right|$$

$$= \frac{1}{2}\ln\left|\frac{1-x^2}{x^2}\right| + C$$

$$(e^{2t}+1)x^2 = 1$$

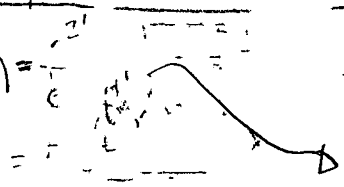
$$\boxed{x = \sqrt{\frac{1}{e^{2t+C}+1}}}$$

Plot of $x(t)$:



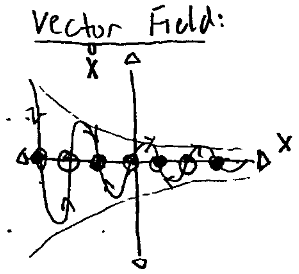
Analytical Solution of $x(t)$:

$x(t) =$



$\dot{x} = e^{-x}\sin x$ 2.2.4

Vector Field:



Fixed Points: Stability:

$x = 2n\pi$ source (unstable)

$x = (2n+1)\pi$ sink (stable)

$n = 2k+1$

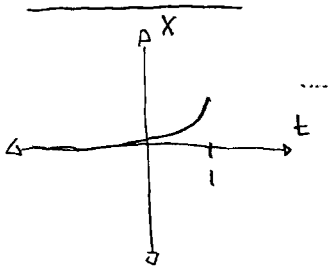
Solving for $x(t)$:

$$t = \int \frac{e^x}{\sin x}dx = \int dx + \int \cot(x)\,dx$$

$$= 1 + \ln(\sin x) + C$$

$$\boxed{x(t) = \arcsin^{-1}(C e^{t} \cdot e^{-1})}$$

Plot of $x(t)$:

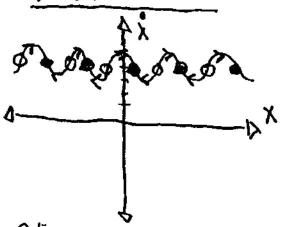


Analytical solution of $x(t)$

$$\boxed{x(t) = \arcsin^{-1}(Ce^{t}-1) \text{ where } C = -(1+\ln(\sin x_0))}$$

$\dot{x} = 1 + \frac{1}{2}\cos x$ 2.2.5.

Vector Field:



Fixed Points Stability:

$x = (4n+1)\frac{\pi}{2}$ sink (stable)

$x = (4n-1)\frac{\pi}{2}$ source (unstable)

Solving for $x(t)$:

$$t = \int \frac{1}{1+\frac{1}{2}\cos x}dx$$

$$= \int \frac{dx}{\frac{1}{2}+\cos^2(\frac{x}{2})} = \int \frac{\sec^2(\frac{x}{2})dx}{\frac{\sec^2(\frac{x}{2})}{2}+1}$$

$$\boxed{= \int \frac{\sec^2(\frac{x}{2})dx}{3+\tan^2(\frac{x}{2})}}$$

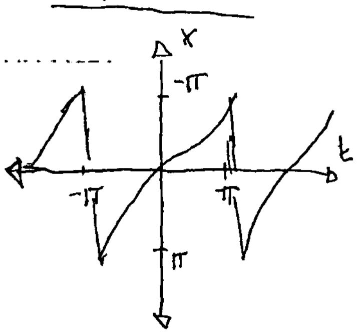
$$u = \frac{\tan(\frac{x}{2})}{\sqrt{3}};\quad \frac{du}{dx} = \frac{\sec^2(\frac{x}{2})}{2\sqrt{3}}$$

$$= \int \frac{2\sqrt{3}}{3u^2+3}du = \frac{2}{\sqrt{3}}\int \frac{du}{u^2+1}$$

$$= \frac{2}{\sqrt{3}}\arctan(u) + C$$

$$\boxed{\frac{\frac{2}{\sqrt{3}}\arctan\left(\frac{\tan(\frac{x}{2})}{\sqrt{3}}\right)}{\sqrt{3}} + C}$$

Plot of $x(t)$:

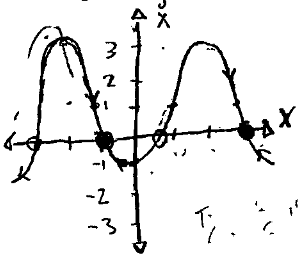


Analytical solution of $x(t)$:

$$\frac{\frac{2}{\sqrt{3}}\arctan\left(\frac{\tan(\frac{x}{2})}{\sqrt{3}}\right)}{\sqrt{3}} + C.$$

$\dot{x} = 1 - 2\cos x$ 2.2.6.

Vector Field

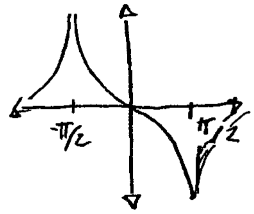



Fixed Points | Stability

$x = (n+\frac{1}{2})\pi$ ; n=even sink (stable)

$x = (n \mp \frac{1}{2})\pi$ ; n=odd source (unstable)

Plot of x(t)




Analytical Solution of x(t)

$$x(t) = \frac{\ln}{\sqrt{3}}\left|\frac{3\tan(\frac{x}{2}) - \sqrt{3}}{3\tan(\frac{x}{2}) + \sqrt{3}}\right|$$

Solving for x(t)

$$dt = \int \frac{dx}{1 - 2\cos x} ; \frac{1 - \tan^2(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})} = \cos x$$

$$= -\int \frac{dx}{2\left[\frac{1-\tan^2(\frac{x}{2})}{1+\tan^2(\frac{x}{2})}\right] - 1}$$

$$= -\int \frac{dx}{2\left[\frac{1-u^2}{1+u^2}\right] - 1} ; \quad u = \tan(\frac{x}{2}) ; \frac{du}{dx} = \frac{\sec^2(\frac{x}{2})}{2}$$

$$= -\int \frac{2\,du}{\sec^2(\frac{x}{2})\left[\frac{1-u^2}{1+u^2}\right] - 1}$$

$$= -\int \frac{2\,du}{[u^2+1](2\left[\frac{1-u^2}{1+u^2}\right] - 1)}$$

$$= -\int \frac{2\,du}{2 - 2u^2 - u^2 - 1}$$

$$= -\int \frac{2\,du}{-3u^2 - 1}$$

$$= -2\left(\frac{3}{1}\right)\int \frac{du}{(3u - \sqrt{3})(3u + \sqrt{3})}$$

$$= -6\left[\int \frac{A\,du}{(3u - \sqrt{3})} + \int \frac{B\,du}{(3u + \sqrt{3})}\right]$$

$$= -6\left[\frac{\sqrt{3}}{-2\sqrt{3}}\int \frac{du}{(3u - \sqrt{3})} - \frac{\sqrt{3}}{2\sqrt{3}}\int \frac{du}{(3u + \sqrt{3})}\right]$$

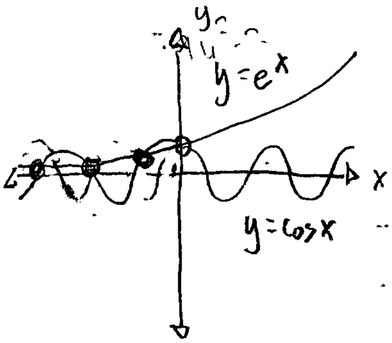
$$= \frac{\ln(3u + \sqrt{3})}{\sqrt{3}} - \frac{\ln(3u + \sqrt{3})}{\sqrt{3}}$$

$$= \frac{\ln(3\tan(\frac{x}{2}) - \sqrt{3})}{\sqrt{3}} - \frac{\ln(3\tan(\frac{x}{2}) + \sqrt{3})}{\sqrt{3}}$$

$$\boxed{= \frac{\ln}{\sqrt{3}}\left|\frac{3\tan(\frac{x}{2}) - \sqrt{3}}{3\tan(\frac{x}{2}) + \sqrt{3}}\right| + C}$$

$x_0$

$\dot{x} = e^x - \cos x$ 2.2.7.




Points of Stability

$e^x = \cos x$

$\boxed{\begin{array}{l} x_1 = 0 \text{ Source (unstable)} \\ x_2 = -1.29 \text{ Sink (Stable)} \\ x_3 = -4.72 \text{ Source (unstable)} \end{array}}$

$\dot{x} = f(x)$ 2.2.8


$f(x) = -(x+1)(1-x)^3$

Slope zero Negative Positive

2.2.9. $\boxed{\begin{array}{l} x_0 = 2 \sim \\ x_0 = 1 \sim \\ x_0 = 0.5 \sim \\ x_0 = -1 \sim \end{array}}$ $f(x) = x(1-x)$

Fixed points
@ x=1
@ x=0

$\dot{x} = f(x)$ 2.2.10 a, A periodic function having solutions $n\pi$

b, A periodic function with $n\pi$ solutions

c, $F(x) = x^3$ 1

d, $F(x) = x^2 + 1$

e, $F(x) = x^{101}$

$\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}$ 2.2.11. $Q(0) = 0$ ; $t = RC\int \frac{dQ}{V_0 C - Q} = -RC\ln V_0 C - Q + C$ ; $C = RC\ln V_0 C$ ; $\boxed{t = RC\ln\frac{V_0 C}{V_0 C - Q}}$

$Q = V_0 C(1 - e^{-t/RC})$

$\dot{Q} = g(v) - \frac{Q}{RC}$ 2.2.12 $V = g(v) - V_{cap.} = V_0 - \frac{Q}{C}$ ; $-g(v) + RI + \frac{Q}{C} = 0$ ; $-g(v) + R\dot{I} + \frac{Q}{C} = -g(v) + R\dot{Q} + \frac{Q}{C} = 0$

$\dot{Q} = g(v) - \frac{Q}{RC}$ ;

Fixed Points: $g(v) = Q/RC$

The nonlinearity of the resistor has a relationship to resistance

Stability: Source (unstable)

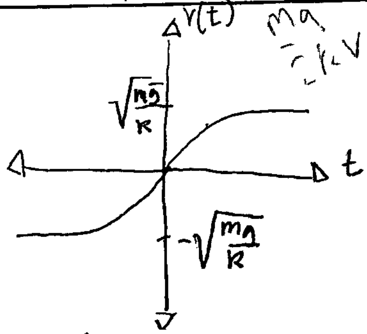
$m\dot{v} = mg - kv^2$ 2.3.13 ; where $m$ = mass, $g$ = acceleration, $k>0$ = air resistance.

a) $\int \frac{dv}{g - \frac{k}{m}v^2} = \frac{1}{g}\int \frac{dv}{1-\frac{k}{mg}v^2} = \frac{1}{g}\left[\int \frac{dv}{1-\sqrt{\frac{k}{mg}}v} + \int \frac{dv}{1+\sqrt{\frac{k}{mg}}v}\right] = \sqrt{\frac{mg}{k}}\frac{1}{2g}\left[\ln\left|1+\sqrt{\frac{k}{mg}}v\right| - \ln\left|1-\sqrt{\frac{k}{mg}}v\right|\right]$

$t = \frac{1}{2}\sqrt{\frac{m}{kg}}\ln\left|\frac{1+\sqrt{\frac{k}{mg}}v}{1-\sqrt{\frac{k}{mg}}v}\right|$ ; $\sqrt{\frac{kg}{m}}t = \frac{\ln}{2}\left|\frac{1+\sqrt{\frac{k}{mg}}v}{1-\sqrt{\frac{k}{mg}}v}\right| = \boxed{\tanh^{-1}\left(\sqrt{\frac{k}{mg}}v\right)}$

b) $\boxed{\lim_{t\to\infty} v(t) = \sqrt{\frac{mg}{k}} = \text{"terminal velocity"}}$ $\qquad v(t) = \sqrt{\frac{mg}{k}}\tanh\sqrt{\frac{kg}{m}}t$

c)




d. $V_{avg} = \frac{(31{,}400 - 2100)\,ft}{116\,sec} = 252\,\frac{ft}{sec}$.

e. $\dot{s} = \frac{ds}{dt} = v = V\tanh\sqrt{\frac{kg}{m}}t$ ; $s(t) = V\int \tanh\sqrt{\frac{kg}{m}}t\,dt$

$= V\int \frac{\sinh\sqrt{\frac{kg}{m}}t}{\cosh\sqrt{\frac{kg}{m}}t}dt$

$= V\int \frac{1}{u}dt$

$= \frac{m}{k}\ln\cosh\sqrt{\frac{gk}{m}}t$

$= \frac{V^2}{g}\ln\cosh\frac{g}{V}t$

$29{,}300 = \frac{V^2}{32.2\,ft/sec^2}\ln\cosh\frac{32.2\,ft/sec^2 \cdot 116\,sec}{V}$

$e^{9.43\times10^5/V^2} = \frac{e^{\frac{3735\,ft/sec}{V}} + e^{-\frac{3735\,ft/sec}{V}}}{2}$

$\boxed{V = 266\,ft/sec.}$

$V \approx V_{avg} = 252\,ft/sec$

$\frac{gt}{V} = \frac{32.2\,ft/sec^2 \cdot 116\,sec}{252\,ft/sec} = 14.8$,

$\frac{V^2}{g}\ln\cosh\frac{gt}{V} \approx \frac{V^2}{g}\left[\frac{gt}{V} - \ln 2\right] = 265\,ft/sec$.

$\dot{N} = rN(1-N/K)$ 2.3.1 a. $t = \frac{1}{r}\int \frac{dN}{N(1-N/K)}$

$= \frac{1}{r}\left[\int \frac{A\,dN}{N} + \int \frac{B\,dN}{(1-N/K)}\right]$

$= \frac{1}{r}\left[\int \frac{1\,dN}{N} + \frac{1}{K}\int \frac{dN}{(1-N/K)}\right]$

$= \frac{1}{r}\ln N - \ln|1-N/K| + C$

$= \frac{1}{r}\ln\frac{N}{1-N/K} + C$

$Ce^{-rt} = \frac{1-N/K}{N}$ ; $N\left(1+\frac{Ce^{-rt}}{K}\right) = 1$

$N = \frac{1}{\left(1+\frac{Ce^{-rt}}{K}\right)} = \boxed{\frac{N_0}{1\,e^{-rt}}}$

General Solution: $x = Ce^{-rt}$

$x = Ce^{-rt} + \frac{1}{K}$

b. $x = 1/N$ ; $dN = -\frac{dx}{x^2}$ ; $\dot{x} = -rx\left(1 - \frac{1}{Kx}\right) = \frac{r}{K} - rx$ ; $\dot{x} + rx - \frac{r}{K} = \dot{x} + r\left(x - \frac{1}{K}\right) = 0$

$\boxed{\dot{x} = r\left(\frac{1}{K} - x\right)}$

$N(t) = \frac{K}{KCe^{-rt}(+C)}$

$N = \frac{N_0}{e^{-rt}}$

$\dot{x} = k_1ax - k_{-1}x^2$ 2.3.2

$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X$

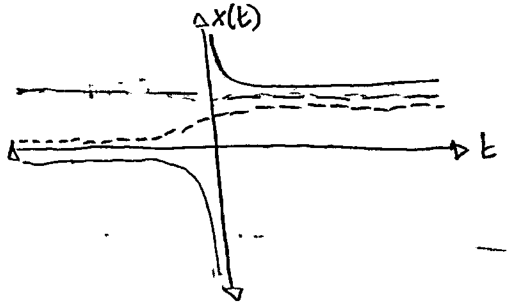
a. $t = \int \frac{dx}{k_1ax - k_{-1}x^2} = \frac{1}{k_1a}\int \frac{dx}{x - \frac{k_{-1}}{k_1a}x^2} = \frac{1}{k_1a}\int \frac{dx}{x\left(1-\frac{k_{-1}}{k_1a}x\right)}$

$= \frac{1}{k_1a}\left[\int \frac{A\,dx}{x} + \int \frac{B\,dx}{\left(1-\frac{k_{-1}}{k_1a}x\right)}\right] = \frac{1}{k_1a}\left[\int \frac{1\,dx}{x} + \frac{k_{-1}}{k_1a}\int \frac{dx}{1-\frac{k_{-1}}{k_1a}x}\right]$

$= \frac{1}{k_1a}\left[\ln|x| - \ln\left|1-\frac{k_{-1}}{k_1a}x\right|\right] = \frac{1}{k_1a}\ln\left|\frac{x}{1-\frac{k_{-1}}{k_1a}x}\right| + C$ ; $\left(Ce^{-t/k_1a} + \frac{k_{-1}}{k_1a}\right)x = 1$

$x(t) = \frac{1}{\frac{k_{-1}}{k_1a} + Ce^{-t/k_1a}}$ ; $C = \frac{1}{x_0} - \frac{k_{-1}}{k_1a}$ ; $\boxed{\text{Fixed points of stability: } x = \frac{k_1a}{k_{-1}} \text{ source unstable}}$

b.




$-$ = $X_0 > \frac{k_1 a}{k_{-1}}$

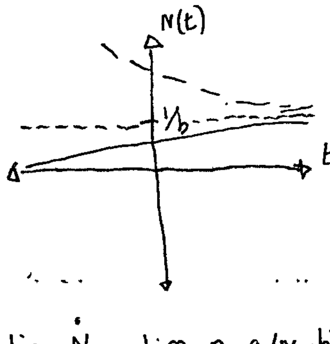
$--$ = $X_0 = \frac{k_1 a}{k_{-1}}$

$----$ = $X_0 < \frac{k_1 a}{k_{-1}}$

"Gompertz Law"

$\dot{N} = -aN\ln(bN)$ 2.3.3a) $t = -\frac{1}{a}\int \frac{dN}{N\ln(bN)} = -\frac{b}{a}\int \frac{du}{u} = -\frac{b}{a}\ln[\ln bN]$ ; $N(t) = \frac{Ce^{e^{-\frac{ab}{b}}}}{b} = \frac{Ce^{-\frac{ate}{b}}}{b}$ ; $a$ = rate constant, $b$ = Max amount of cells.

b)




$-$ = $C < b$

$---$ = $C = 1$

$--$ = $C > b$

$\frac{\dot{N}}{N} = r - a(N-b)^2$ 2.3.4. a). $\lim_{N\to 0} \frac{\dot{N}}{N} = \lim_{N\to b} r - a(N-b)^2 = r - ab^2 = \infty$ ; $r = \infty$

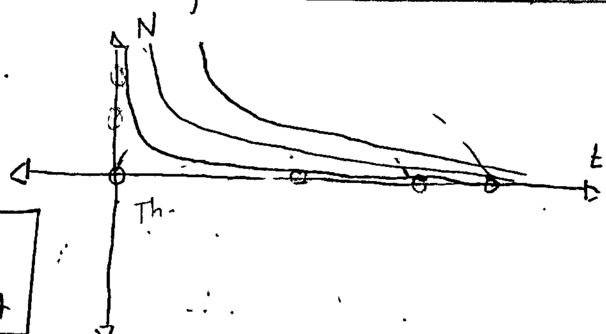
$\lim_{N\to\infty} \frac{\dot{N}}{N} = \lim_{N\to 0} r - a(N-b)^2 = r - \infty = 0$ ; $r = \infty$

Each case of competition model at infinite population or extremely small populations that food amount or rate of consumption are insignificant to the competitors.

b) Fixed points of stability:

| | |
|---|---|
| $N = 0$ | Source (unstable) |
| $N = \sqrt{\frac{r}{a}} + b$ | Sink (stable) |

c.




d) The solutions of the logistic equation $y = Ce^{-rt} + \frac{1}{k}$ are similar, if not exact to the Allee Effect.

$t = \int \frac{du}{(b+u)(r-au^2)} = \int \frac{A}{b+u}du + \int \frac{Bu+C}{r-au^2}du$

$A(r-au^2) + (Bu+C)(b+u) = 1$

@ $u = \sqrt{\frac{r}{a}}$ ; $(B\sqrt{\frac{r}{a}} + C)(b + \sqrt{\frac{r}{a}}) = 1$

$B = \sqrt{\frac{a}{a}}$ ; $C = \sqrt{\frac{r}{a}}$

@ $u = -b$ $A = \frac{1}{r-(ab)^2}$

$= \frac{1}{r-(ab)^2}\int \frac{du}{b+u} + \frac{1}{\sqrt{ra}}\int \frac{ab\cdot u + r}{r-au^2}du$ Partial Fractions (x2)

$= \frac{\ln N}{r-(ab)^2} + \frac{b}{4}\sqrt{\frac{a}{r}}\tanh^{-1}([N+b]^2) + \frac{1}{\sqrt{a}}\tanh^{-1}\left(\sqrt{\frac{a}{r}}(N+b)\right) + C$

$\dot{X} = aX$
$\dot{Y} = bY$
2.3.5 a) $x(t) = \frac{X(t)}{[X(t)+Y(t)]} = \frac{e^{at}}{e^{at}+e^{bt}}$ ; $\lim_{t\to\infty} x(t) \approx 1$

b) $\dot{x}(t) = \frac{ae^{at}(e^{at}+e^{bt}) + e^{at}(ae^{at}+be^{bt})}{(e^{at}+e^{bt})^2}$

$= \frac{e^{at}[a-b]e^{bt}}{(e^{at}+e^{bt})^2} = \frac{[a-b]e^{at}}{(e^{at}+e^{bt})}\left(\frac{e^{bt}+e^{ab}-e^{ab}}{e^{at}+e^{bt}}\right)$

$\boxed{= x[a-b](1-x)}$

$\dot{x} = (1-x)P_{yx} - xP_{xy}$ 2.3.6. a. $x = 0$

$P_{yx} = sx^a$; $P_{xy} = (1-s)(1-x)^a$ $\quad x = 1$

$$\boxed{x = \frac{\sqrt[a-1]{\frac{(1-s)}{s}}}{1+\sqrt[a-1]{\frac{(1-s)}{s}}}}$$

b. A plot of $s(1-x)x^a$ and $-(1-s)x(1-x)^a$ demonstrate $-(1-s)x(1-x)^a > s(1-x)x^a$ For $x=0$ and $x=0$, indicating, each fixed point is stable.

c. For $x = \frac{\sqrt[a-1]{\frac{(1-s)}{s}}}{1+\sqrt[a-1]{\frac{(1-s)}{s}}}$ the plot of $s(1-x)x^a > (1-s)x(1-x)^a$, suggesting a source.

$\dot{x} = x(1-x)$ 2.4.1 $\dot{x} = f(x) = f(x^*+x) = f(x^*) + x f'(x^*) + O(x^2)$
$= x f'(x^*) + O(x^2)$
$\boxed{= x(1-2x)}$

$x = 0$; $f'(x^*) = 1$ : Unstable (source)
$x = 1$; $f'(x^*) = -1$ : Stable (sink)

$\dot{x} = x(1-x)(2-x)$ 2.4.2 $\dot{x} = f(x) = f(x^*+x) = \boxed{x f'(x^*)}$
$\boxed{= x\,2x(1-x)}$

$x = 0$ $f'(x^*) = 0$ Half-stable
$x = 1$ $f'(x^*) = 0$ Half-stable
$x = 2$ $f'(x^*) = -4$ sink (stable)

$\dot{x} = \tan x$ 2.4.3 $\dot{x} = f(x) = f(x^*+x) = \boxed{x\sec^2(x)}$

$x = 0, \pi$ $f(x) = (+)$ source (unstable)

$\dot{x} = x^2(6-x)$ 2.4.4. $\dot{x} = f(x) = f(x^*+x) = \boxed{x[12x - 3x^2]}$

$x = 0$ $f'(x) = 0$ Halfstable
$x = 6$ $f'(x) = -36$ sink (stable)

$\dot{x} = 1 - e^{-x^2}$ 2.4.5. $\dot{x} = f(x) = f(x^*+x) = \boxed{x[2e^{-x^2}]}$

$x = 0$ $f'(x^*) = 0$ Half-stable

$\dot{x} = \ln x$ 2.4.6 $\dot{x} = f(x) = f(x^*+x) = \boxed{1}$

$x = 1$ $f'(x^*) = 1$ source (unstable)

$\dot{x} = ax - x^3$ 2.4.7. $\dot{x} = f(x) = f(x^*+x) = \boxed{x[a - 3x^2]}$

| $a$ | $(+)$ | $(-)$ | $(0)$ |
|---|---|---|---|
| $x=0$ | source | sink | Half-stable |
| $x=+\sqrt{a}$ | sink | source | source |
| $x=-\sqrt{a}$ | source | sink | Half-stable |

$\dot{N} = -aN\ln(bN)$ 2.4.8 $\dot{N} = f(N) = f(N+N^*) = \boxed{\frac{aN}{b}[1 + b\ln(bN)]}$

$N = 0$ : source (unstable)
$N = \frac{1}{b}$ : sink (stable)

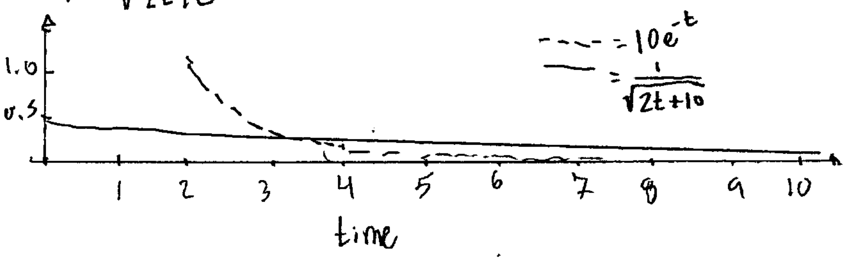
$\dot{x} = -x^3$ 2.4.9 a. $t = \int \frac{dx}{x^3} = \frac{1}{2x^2} + c$ ; $x(t) = \sqrt{\frac{1}{2t+c}}$

$\lim_{x \to 0} t = \frac{1}{0} + c = \boxed{\infty}$

b. if $x_0 = 10$

$t = -\int \frac{1}{x} = -\ln x$;

$\boxed{x(t) = x_0 e^{-t} = 10e^{-t}}$

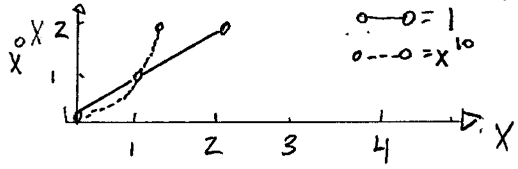



$\dot{x} = -x^c$ 2.5.1 a. $c = 0$

b. $dx = -dt$ ; $x(t) = -t$ ; if $t = 0$ is considered finite time, then yes.

$t = -\int \frac{dx}{x^c} = -\frac{x^{1-c}}{1-c}$ ; $t(x=1) - t(x=0) = -\frac{1^{1-c}}{1-c} + \frac{0^{1-c}}{1-c} = \boxed{\frac{1}{c+1}}$

$\dot{x} = 1 + x^{10}$; 2.5.2.




$\boxed{\lim_{\dot{x}\to\infty} \dot{x} = \infty}$

$\dot{x} = rx + x^3$ 2.5.3

$$t = \int \frac{dx}{x(r+x^2)} = \int \frac{A}{x}dx + \int \frac{Bx+C}{r+x^2}dx \; ; \; A(r+x^2) + (Bx+C)(x) = 1$$

$x = 0$; $A = \frac{1}{r}$; $1 + \frac{x^2}{r} + Bx^2 + Cx = 1$

$(B + \frac{1}{r})x^2 + Cx = 0$

$(B + \frac{1}{r})x = -C$

$\boxed{x = 0 ; C = 0}$

$\boxed{x = 1 ; B = -\frac{1}{r}}$

$$= \frac{1}{r}\ln x - \frac{1}{2r}\ln r + x^2 = \frac{1}{r}\ln\frac{x}{\sqrt{r+x^2}}$$

$$x^2 C e^{-2rt} = (r + x^2) \; ; \; x = \sqrt{\frac{r}{1 + Ce^{-2rt}}}$$

If $x_0 \neq 0$; $\lim_{t\to\infty} x(t) = \infty$.

$\dot{x} = x^{1/3}$ 2.5.4. $x(0) = 0$; $t = \int \frac{dx}{x^{1/3}} = \frac{3}{2}x^{2/3}$; $\boxed{x(t) = \sqrt{\left(\frac{2}{3}t - \frac{2}{3}c\right)^3}}$

$\dot{x} = |x|^{p/q}$ 2.5.5. $x(0) = 0$; a) $t = \frac{q}{p+q}(x)^{\frac{p+q}{q}}$; $x(t) = \left(\frac{p+q}{q}(t+C)\right)^{\frac{q}{p+q}}$; c = many solutions at zero because of root.

$\boxed{x(t) = \left(\frac{p+q}{q}(t+C)\right)^{q/p+q}}$;

b) $x(t) = \left(\frac{p+q}{q}(t+c)\right)^{q/p+q}$; if $p > q$; $x(0) = \left(\frac{p+q}{q}(0+c)\right)^{q/p+q} = 0$; $\boxed{c = 0}$

h(t) = height
t = time
A = cross-sectional Area
v(t) = velocity
$av(t) = A\dot{h}(t)$

2.5.6 a) Newton's first law that for every force there exists an equal and opposite counter force.

b) $\frac{1}{2}mv^2 = mgh$; $\boxed{v^2 = 2gh}$;

c) $\boxed{\dot{h}(t) = -\sqrt{\frac{a}{A}2gh}}$

d) $\boxed{h(0) = 0; \; t = -\sqrt{\frac{Ah}{2a}}\,2\sqrt{h} \; ; \; h(t) = -\sqrt{\frac{a}{2A}}\,t}$

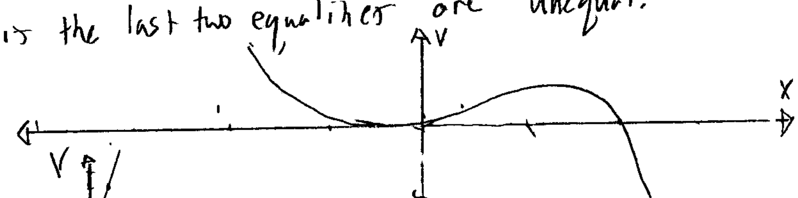
$m\ddot{x} = -kx$ 2.6.1 The text states, there are no periodic solutions to $\dot{x} = f(x)$ because undamped systems do not oscillate, and, "damped oscillations do not occur for first order systems." the equation of ~~Strogatz~~ Strogatz statement does not fit 2.61.

$\dot{x} = f(x)$ 2.6.2 $\int_t^{t+T} f(x)\frac{dx}{dt}dt = \int_t^{t+T} f(x)\dot{x}(t)dt = \int_t^{t+T} f(x)\dot{x}(t+T)\,d(t+T)$.

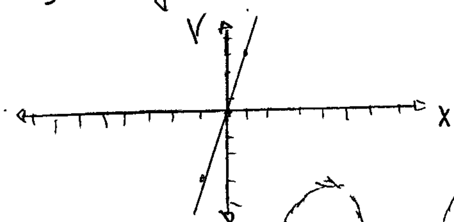
$x(t) = x(t+T)$

The contradiction of the argument is the last two equalities are unequal.

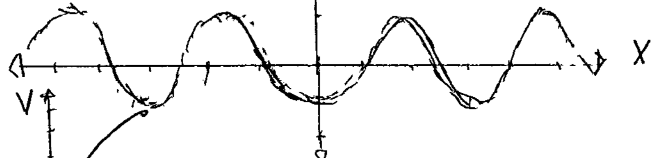
$\dot{x} = x(1-x)$ 2.7.1 $\frac{dV}{dx} = \dot{x} = x(1-x)$; $V = (1 - \frac{x}{3})x^2$



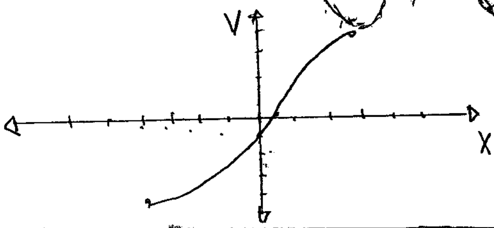

$\dot{x} = 3$ 2.7.2 $\frac{dV}{dx} = \dot{x} = 3$; $V = 3x$



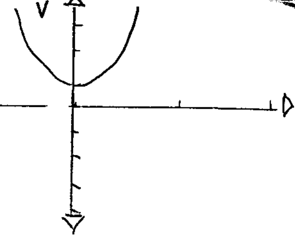

$\dot{x} = \sin x$ 2.7.3 $\frac{dV}{dx} = \dot{x} = \sin x$; $V = -\cos(x)$



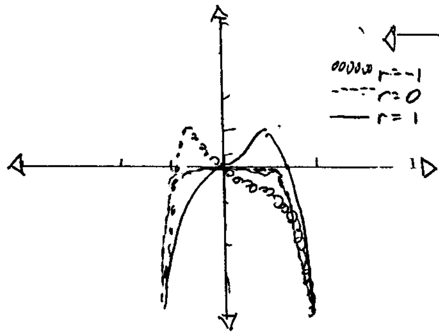

$\dot{x} = 2 + \sin x$ 2.7.4 $\frac{dV}{dx} = \dot{x} = 2 + \sin x$; $V = 2x - \cos(x)$



$\dot{x} = -\sinh x$ 2.7.5. $\frac{dV}{dx} = -\sinh x$ ; $V = -\cosh(x)$




$\dot{x} = r + x - x^3$ 2.7.6 $\frac{dV}{dx} = r + x - x^3$ ; $V = rx + \frac{x^2}{2} - \frac{x^4}{4}$



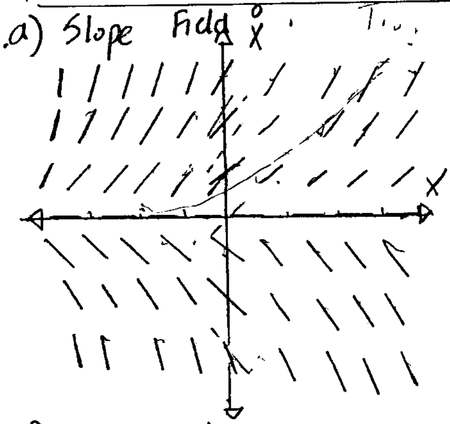

$\dot{x} = f(x)$ 2.7.7. $\frac{dV}{dx} = \dot{x} = f(x)$ ; $V = \frac{df(x)}{dx} dx + c$

$$f(x) = \frac{d(V-c)\,dx}{dx}$$

The solution $x(t)$ cannot oscillate because of the existence and uniqueness of $f(x)$, and the solutions for $f(x)=0$; that $V=C$ or $C=0$; withstanding, $\frac{c\,d(V-c)}{dx} = \frac{dx}{dt}$, then the solution $x(t)$ also corresponds to a nonperiodic function.

$\dot{x} = x(1-x)$ 2.8.1 The horizontal lines are to be expected in Figure 2.8.2 because of the slope being zero at $x=1$.

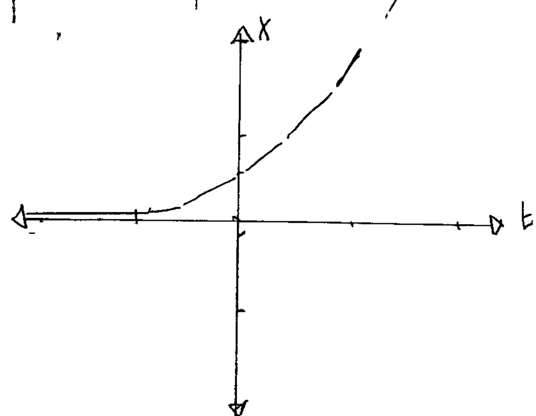
$\dot{x} = x$ 2.8.2.a) Slope Field



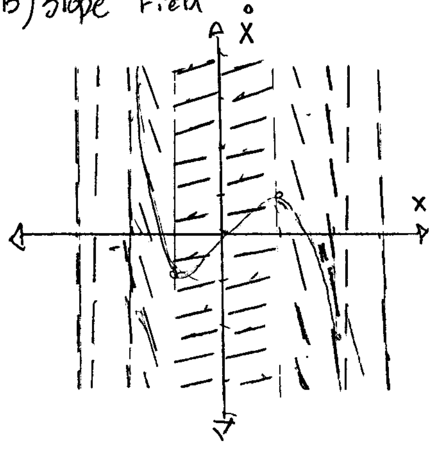

$\frac{dx}{dt} = x$

$\ln x = t$

$x = e^t$




$\dot{x} = 1 - x^2$ b) Slope Field




$\frac{dx}{dt} = 1 - x^2$

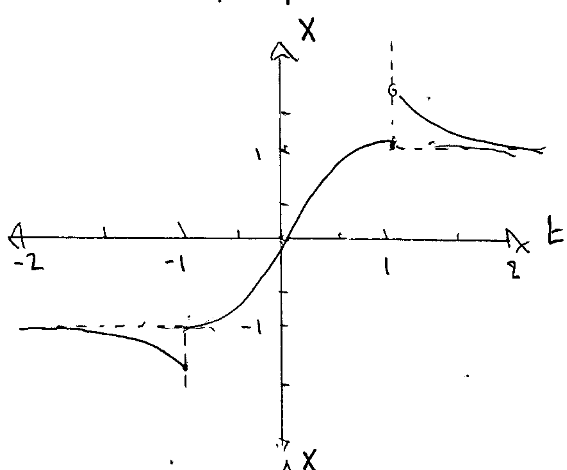
$\int \frac{dx}{1-x^2} = t$

$\frac{1}{2}\ln\frac{1+x}{x-1} = t$

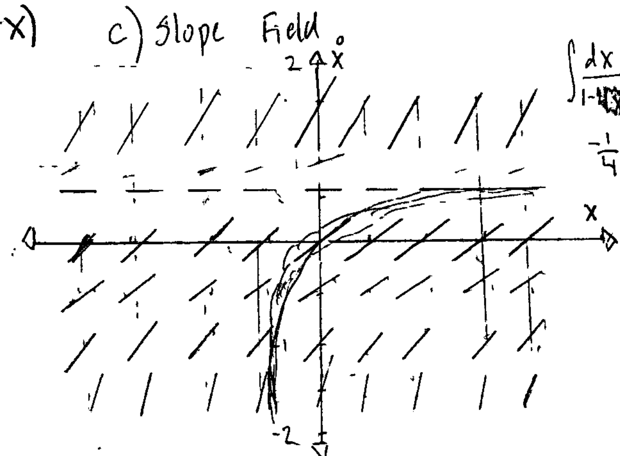
$x < -1$

$-1 < x < 1$

$x > 1$



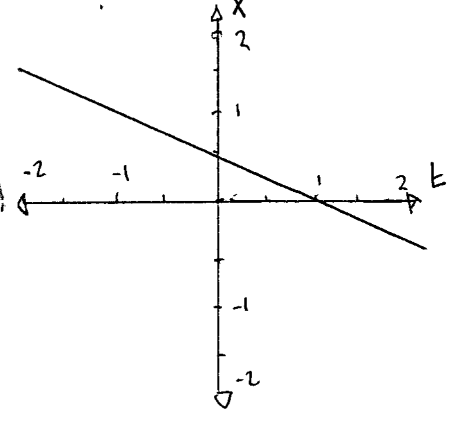

$\dot{x} = 1 - 4x(1-x)$ c) Slope Field



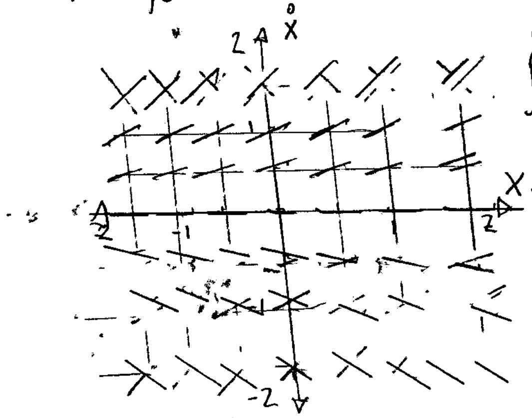

$\int \frac{dx}{1-4x(1-x)} = t$

$-\frac{1}{4}\left(\frac{1}{x-0.5}\right) = t$

$x = 0.5 - \frac{1}{4t}$



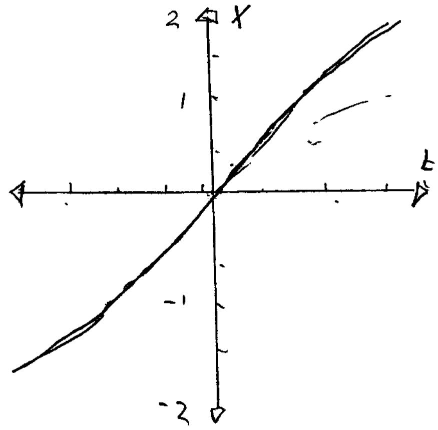
$\dot{x} = \sin x$ 2.32 d) Slope Field




$\frac{dx}{dt} = \sin x$

$\int \cosh x = t$

$x = \sinh^{-1}(t)$




**Euler's Method**

$\dot{x} = -x ; x(0) = 1$ 2.8.3. a) $x(t) = Ce^{-t}$ ; $C=1$ ; $x(t) = e^{-t}$

b) $\Delta t = 1$ ; $x(t_0 + \Delta t) \approx x_0 + f(x_0)\Delta t$ ; $x(t_0 + \Delta t) = 0 + e^{-1} \cdot 10^0 = 0.3679$

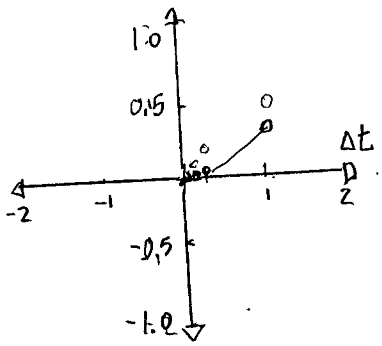
$\Delta t = 10^{-n}$ $n=1$ $x_n + e^{-x_n} 10^{-1} = 0.36391$

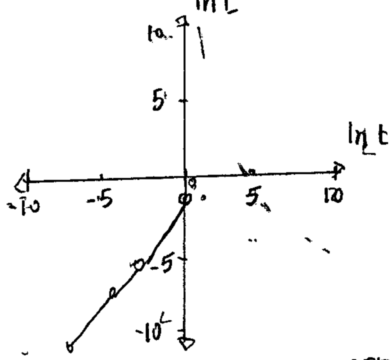
$n=2$ $x_n + e^{-x_n} 10^{-2} = 0.36697$

$n=3$ $x_n + e^{-x_n} 10^{-3} = 0.36785$

$n=4$ $x_n + e^{-x_n} 10^{-4} = 0.36787$

c) $E = |\hat{x}(1) - x(1)|$

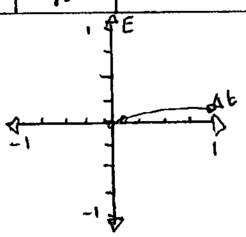


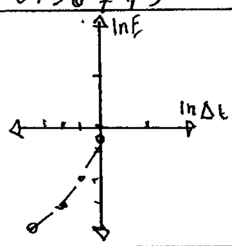




The results of $E = |\hat{x}(1) - x(1)|$ vs $\Delta t$ represent error of Euler's method. While the plot of $\ln E$ vs $\ln t$ characterizes nothing informative.

$\dot{x} = -x ; x(0) = 1$ 2.8.4. $x(t) = e^{-t}$;

| n | $\Delta t$ | f(x) | $X_n = X_{n-1} + f(X_{n-1})\Delta t$ | $E = \lvert\hat{x}(1) - x(1)\rvert$ | ln E |
|---|---|---|---|---|---|
| 0 | $10^0$ | | 0.36788 | 0.00 | -11.00 |
| 1 | $10^{-1}$ | exp(Xn-1) | 0.33527 | 0.03269 | -3.42 |
| 2 | $10^{-2}$ | | 0.31577 | 0.0211 | -6.16 |
| 3 | $10^{-3}$ | | 0.36773 | 0.0002 | -8.33 |
| 4 | $10^{-4}$ | | 0.36773 | 0.0000 | -10.78 |

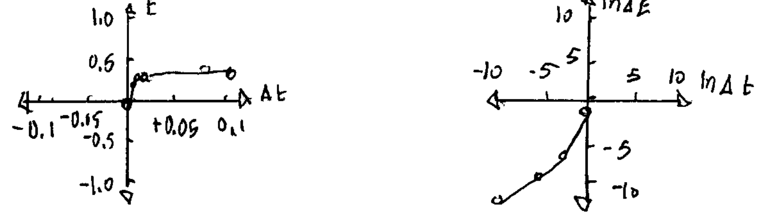



**Improved Eulers Method**

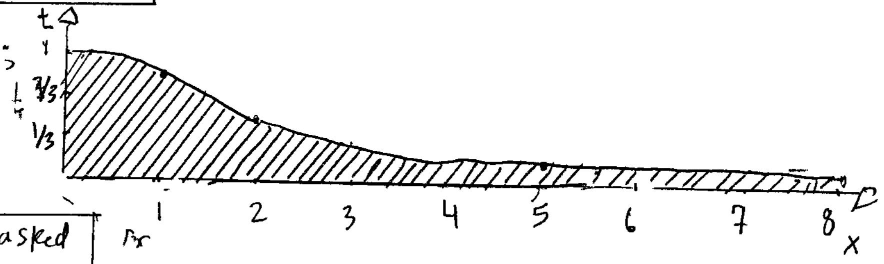
$\dot{x} = -x ; x(0) = 1$ 2.8.5. $x(t) = e^{-t}$

| n | $\Delta t$ | f(x) | $X_n = X_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ | $E = \lvert\hat{x}(1) - x(1)\rvert$ | ln E |
|---|---|---|---|---|---|
| 0 | $10^0$ | | | | |
| 1 | $10^{-1}$ | | | | |
| 2 | $10^{-2}$ | $\exp(X_{n-1})$ | | | |
| 3 | $10^{-3}$ | | | | |
| 4 | $10^{-4}$ | | | | |

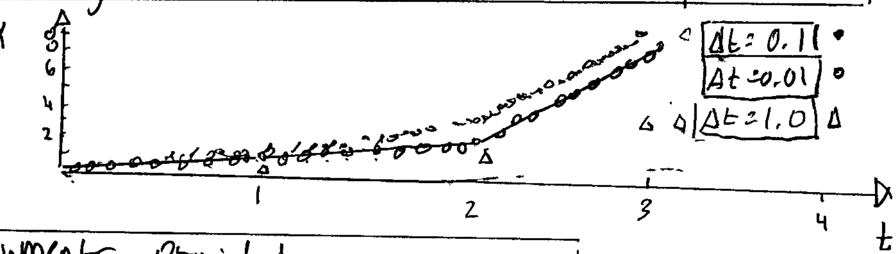



The Euler method aided the analysis of numerical methods; including, Precision. Euler's Improved Method approached the solution of $f(x)=e^{-x}$ with less round-off error, Runge-Kutta's Routine provided the least round-off errors with $10^{-20}$ across the spreadsheet, and necessitated high-precision.

$\dot{x}=x+e^{-x}$ 2.8.6. a: $t=\int \frac{1}{x+e^{-x}}\,dx=\int \frac{e^{x}}{e^{x}x+1}\,dx$ ;

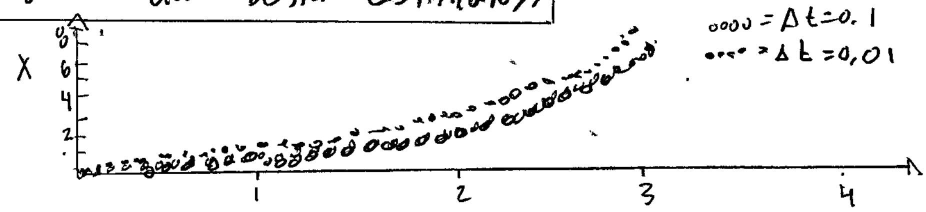



I noticed the book asked for x(t) (and not t(x)). This led me to investigate a Numerical method of integration; withstanding, Runge-Kutta Routine aided with the plot of x(t).




b. At t=0, analytical arguments provided an x=1.011.

c. Stepsizes of Δt = 0.1 and 0.01 had different results, including inaccuracies above and below better estimates.




d) See part a.

$x_1=x_0+f(x_0)\Delta t$ 2.8.7 a) $x(t_1)=x(t_0+\Delta t)$

<u>Taylor Series:</u>

$$x(t+\Delta t)=\sum_{n=0}^{\infty}\frac{x^{(n)}(t)}{n!}(\Delta t)^n=x(t)+x'(t)\cdot\Delta t+O(\Delta t^2)$$

$$f(t+\Delta t)=f(t)+f'(t)\cdot\Delta t+O(\Delta t^2)=\boxed{x_0+f'(t)\cdot\Delta t}$$

b) $|x(t_1)-x_1|=|x(t_1)-x(t_1)-x'(t)\cdot\Delta t+O(\Delta t^2)|=|O(\Delta t^2)|=\frac{x''(t)\Delta t^2}{2!}=C(\Delta t^2)$

$$\boxed{C=\frac{x''(t)}{2!}}$$

Taylor Series: $f(x+h) = \sum \frac{f^n(x)h^n}{n!}$

2.8.8. $\dot{x} = x + e^{-x}$ ; $|x(t_0) - x_0| = |x(t_0) - x(t_0) - x(t_0)'\Delta t - \frac{x''(t_0)\Delta t^2}{2!}| = \frac{x''(t_0)\Delta t^2}{2}$

$\boxed{= O(\Delta t^2)}$

$\dot{x} = x + e^{-x}$

2.8.9 Runge-Kutta: $x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ Where $k_1 = f(x_n)\Delta t$

$k_2 = f(x_n + \frac{1}{2}k_1)\Delta t$

$k_3 = f(x_n + \frac{1}{2}k_2)\Delta t$

$k_4 = f(x_n + k_3)\Delta t$

$x(t+\Delta t) = x(t_0) + x'(t_0)\Delta t + \frac{x''(t_0)\Delta t^2}{2!} + O(\Delta t^3)$

$k_1 = f(x_n)\Delta t = x'(t_0)\Delta t$

$k_2 = f(x_n + \frac{1}{2}k_1)\Delta t = f(x_n) + f'(x_n)\frac{1}{2}k_1 + O[(\frac{1}{2}k_1)^2]$

$k_3 = f(x_n + \frac{1}{2}k_2)\Delta t = f(x_0) + f'(x_n)\frac{1}{2}k_2 + O[(\frac{1}{2}k_2)^2]$

$= f(x_n) + f'(x_n)\frac{1}{2}[f(x_n) + f'(x_n)\frac{1}{2}k_1 + O[(\frac{1}{2}k_1)^2]] + O[(\frac{1}{2}k_2)^2]$

$k_4 = f(x_n + k_3)\Delta t = f(x_n) + f'(x_n)\cdot k_3 + O[k_3^2]$

$= f(x_n) + f'(x_n)[f(x_n) + f'(x_n)\cdot\frac{1}{2}[f(x_n) + f'(x_n)\frac{1}{2}k_1 + O[(\frac{1}{2}k_1)]] + O[(\frac{1}{2}k_2)]]$

$+ O[k_3]$

$x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = x_n + \frac{1}{6}(x'(t_0)\Delta t + 2\,x'(t_0) + x''(t_0)k_1$

$+ 2x'(t_0) + x''(t_0)[x'(t_0) + x''(t_0)x'(t)\Delta t]$

$+ 2x'(t_0) + x''(t_0)[x(t) + \frac{x'(t)}{2}[x'(t_0) + x''(t_0)\frac{x'(t)}{2}]])$

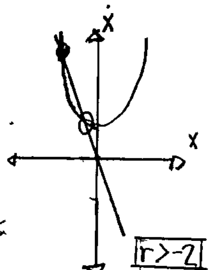
$\boxed{|x(t_1) - x_1| = |x(t_0 + \Delta t) - x_{n+1}| = O(\Delta t^5)}$

## Chapter 3:

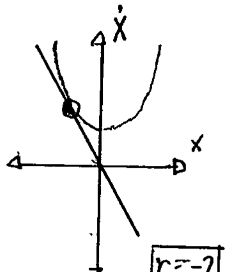
$\dot{x} = 1 + rx + x^2$ 3.1.1. Vector Field:

$x = \frac{r \pm \sqrt{r^2 - 4}}{2}$

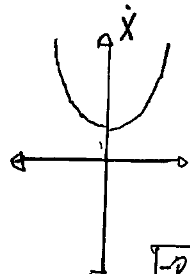
$r = \frac{r \pm \sqrt{(r-2)(r+2)}}{2}$




$\boxed{r>-2}$ $\boxed{\text{Sink source}}$

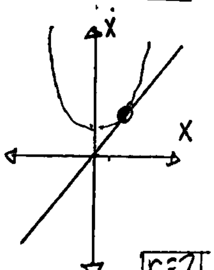



$\boxed{r=-2}$ $\boxed{\text{Half-Node}}$

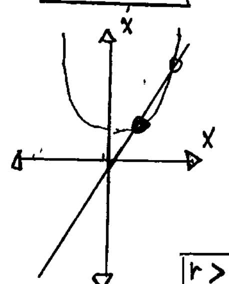



$\boxed{-2<r<2}$ $\boxed{\text{None}}$

| r | Bifurcations |
|---|---|
| >-2 | Two |
| -2 | One |
| 0 | zero |
| 2 | one |
| >2 | Two. |

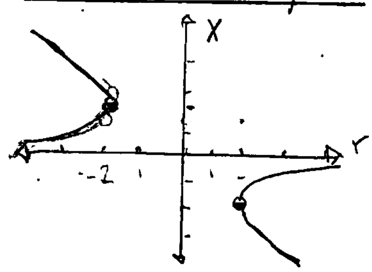



$\boxed{r=2}$ $\boxed{\text{Half-Node}}$




$\boxed{r>2}$ $\boxed{\text{Sink Source}}$

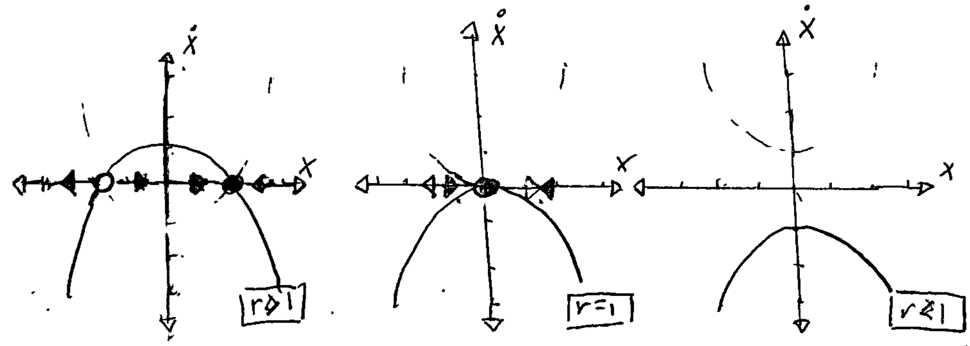
Bifurcation Diagram:



$\dot{x} = r - \cosh x$ 3.1.2. Vector Field

$r = \cosh(x)$

| r | Bifurcations |
|---|---|
| <1 | Zero |
| =1 | One |
| >1 | Two |

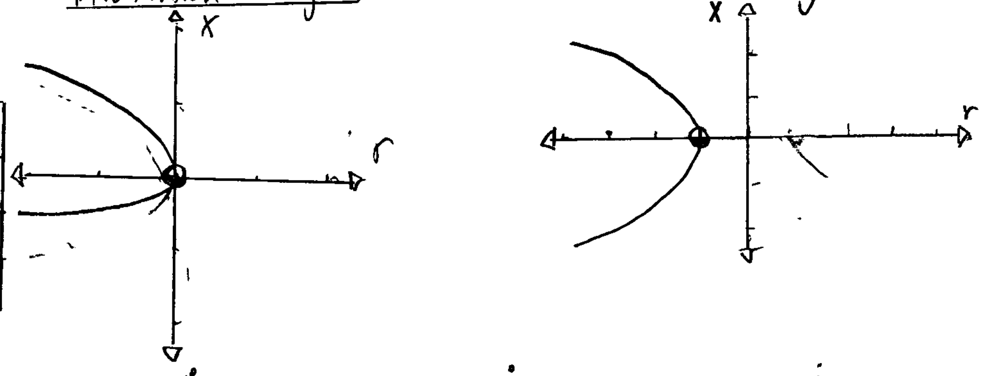


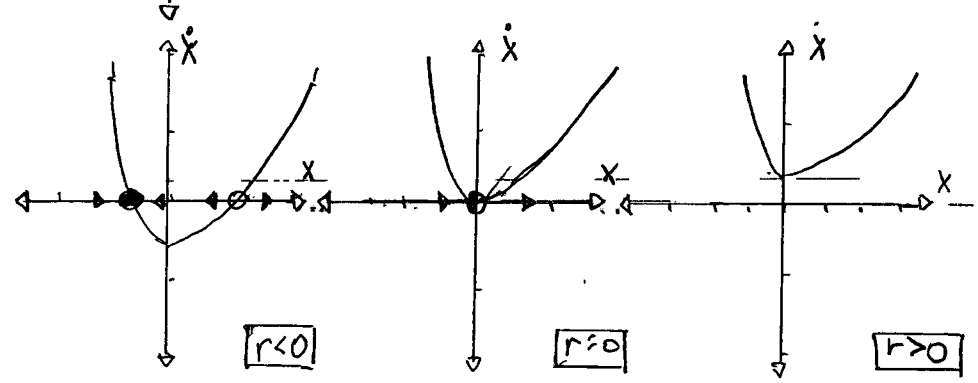

$\dot{x} = r + x - \ln(1+x)$ 3.1.3

Vector Field

Bifurcation Diagram: Bifurcation Diagram.

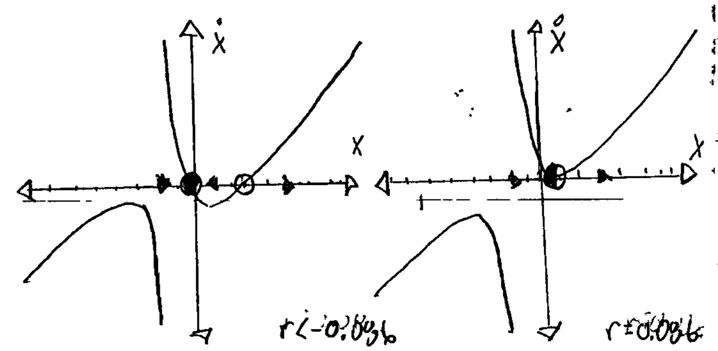
| r | Bifurcation |
|---|---|
| >0 | Zero |
| =0 | One |
| <0 | Two |

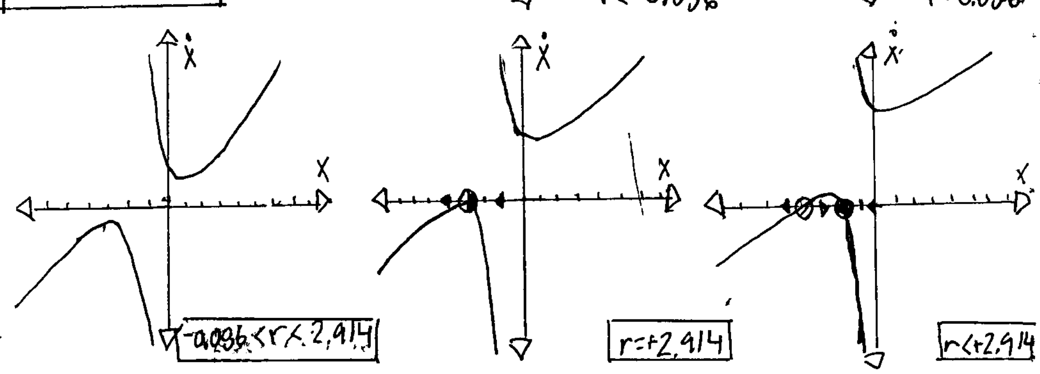


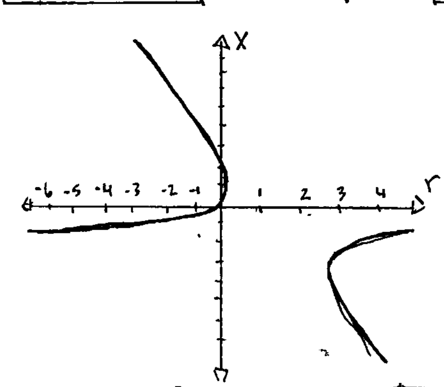

$\dot{x} = r + \frac{1}{2}x - x/(1+x)$ 3.1.4. Vector Field:

| r | Bifurcations |
|---|---|
| ≤ −0.086 | Two |
| = −0.086 | one |
| −0.086 < r < 2.914 | zero |
| = +2.914 | one |
| < +2.914 | Two. |



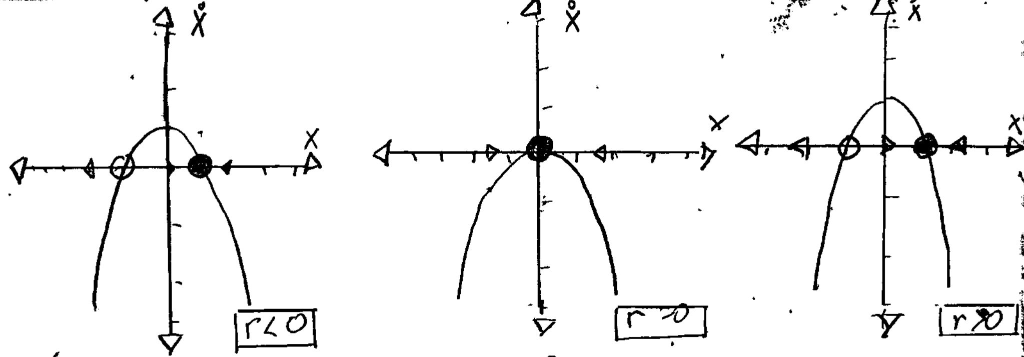




Bifurcation Diagram:



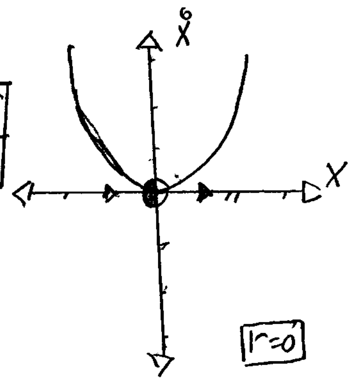
$\dot{x} = r^2 - x^2$ 3.1.5. a) Vector Field:

| r | Bifurcations |
|---|---|
| >0 | Two |
| =0 | one |
| <0 | Two |

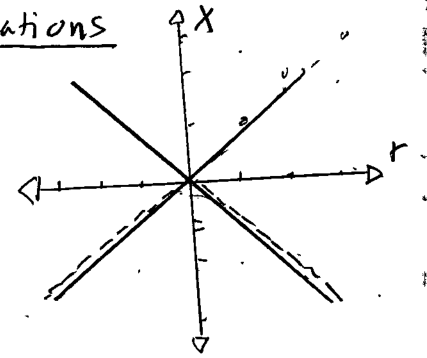



$\dot{x} = r^2 + x^2$ b) Vector Field:

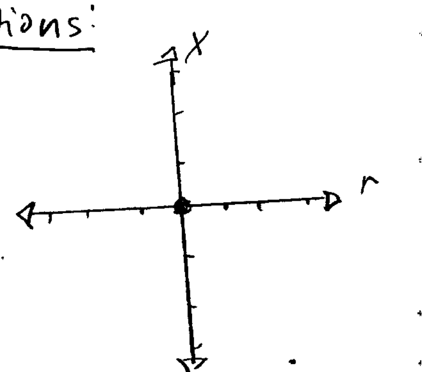
| r | Bifurcations |
|---|---|
| 0 | One |




Bifurcations



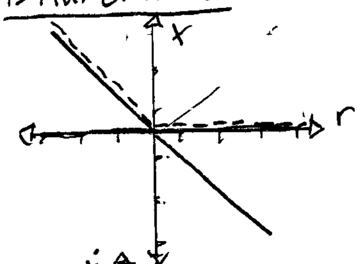

Bifurcations:

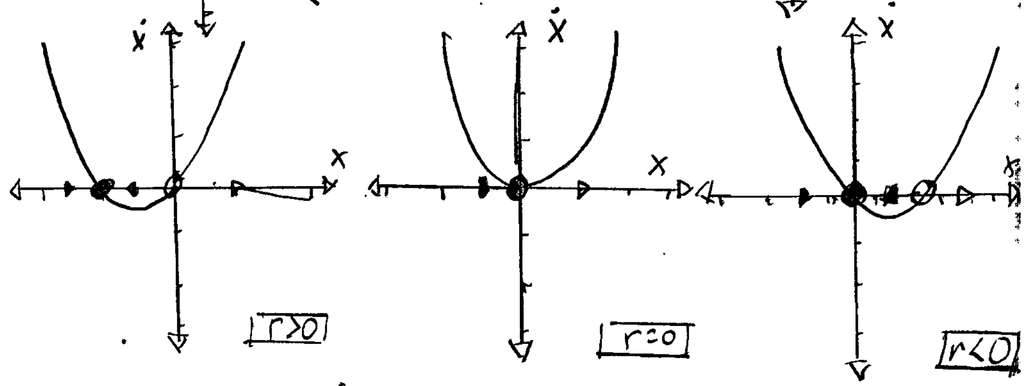



$\dot{x} = rx + x^2$ 3.2.1 Vector Field:

| r | Bifurcations |
|---|---|
| >0 | Two |
| =0 | one |
| <0 | Two |

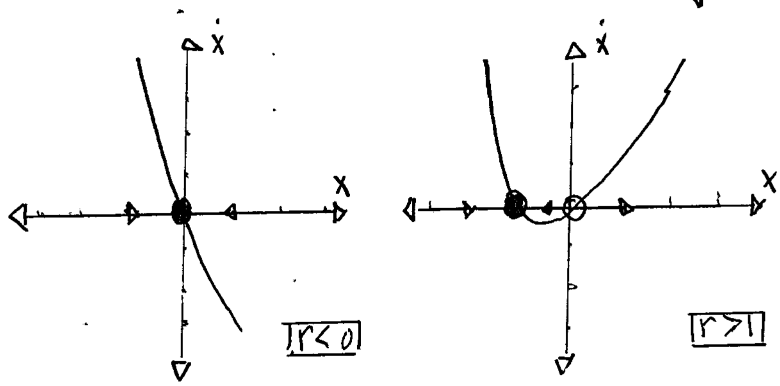
Bifurcations

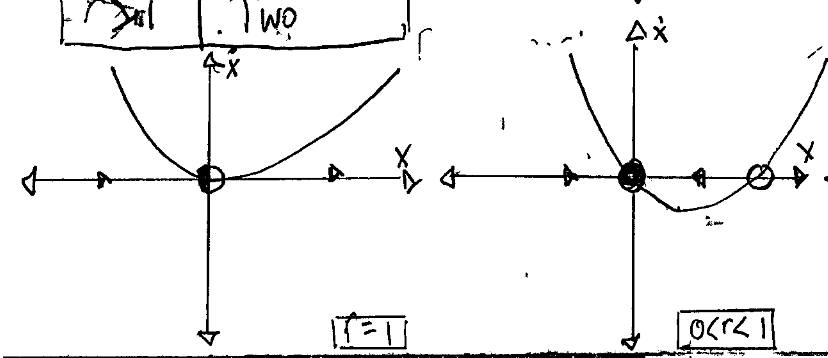


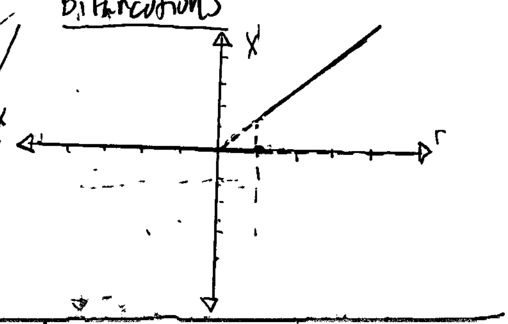

$\dot{x} = rx - \ln(1+x)$ 3.2.2. Vector Field:

| r | Bifurcations |
|---|---|
| <0 | One |
| 0<r<1 | Two |
| r=1 | One |
| r>1 | Two |



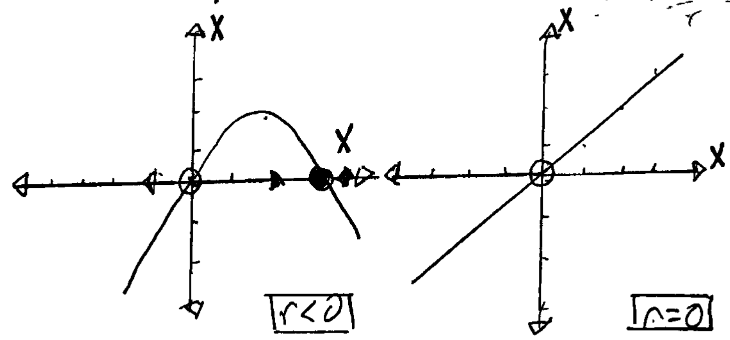



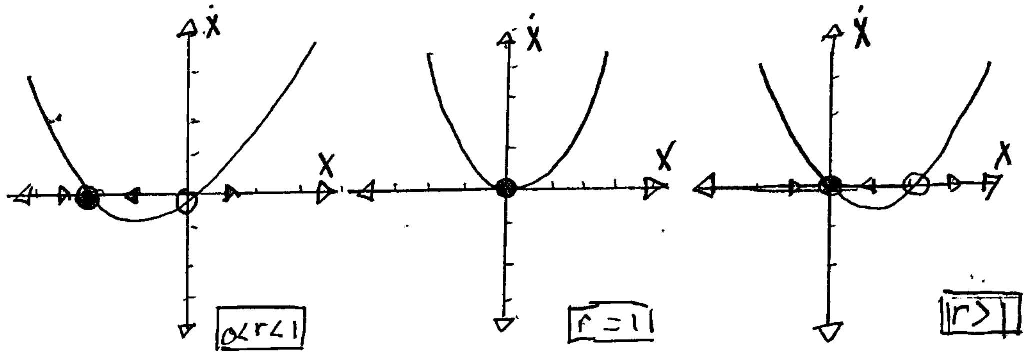

Bifurcations



$\dot{X} = X - rX(1-X)$ 3.2.3. Vector Field:

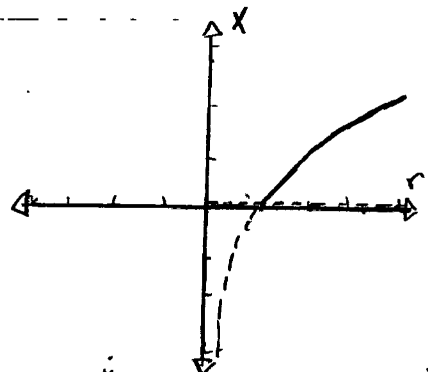
| r | Bifurcations |
|---|---|
| $<0$ | Two |
| $=0$ | (one) |
| $0<r<1$ | Two |
| $r\geq 1$ | One |
| $>1$ | Two |

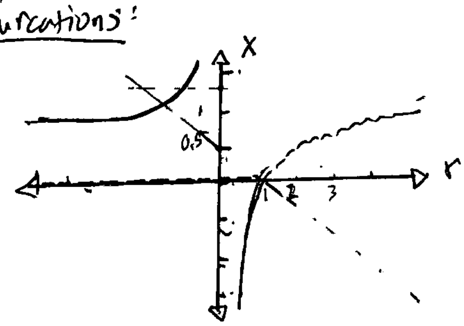


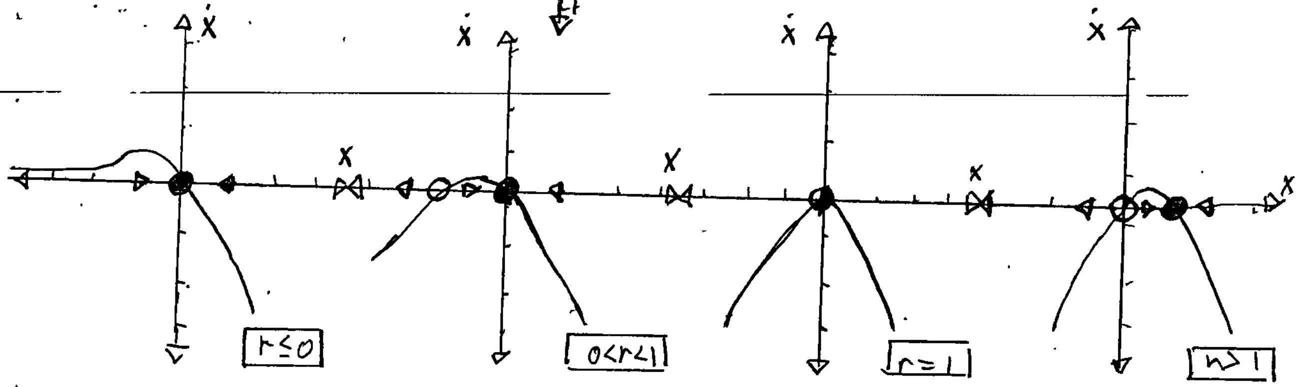




$\dot{X} = X(r - e^{X})$ 3.2.4. Vector Field: Bifurcations: Bifurcations:

| r | Bifurcations |
|---|---|
| $\leq 0$ | one |
| $0<r<1$ | Two |
| $0\leq r<1$ | One |
| $r\geq 1$ | Two |



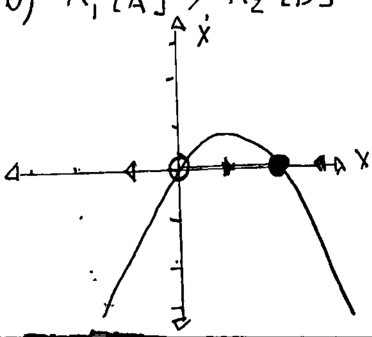

$\dot{X} = c_1 X - c_2 X^2$ 3.2.5 a) $\frac{d[A]}{dt} = -k_1[A][X] + k_{-1}[X]^2$ ; $\boxed{\frac{d[X]}{dt} = (k_1[A] - k_2[B])[X] - k_{-1}[X]^2}$

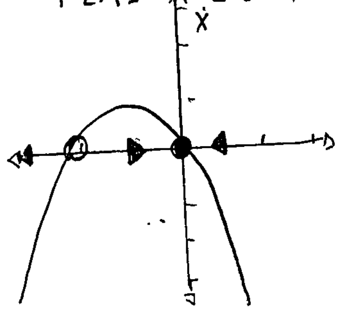
$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X$

$\frac{d[B]}{dt} = -k_2[B][X]$ ; $\frac{d[C]}{dt} = k_2[B][X]$

$X + B \xrightarrow{k_2} C$

b) $k_1[A] > k_2[B]$ $\qquad$ $k_1[A] < k_2[B]$




Chemically, a rate of change $\frac{d[X]}{dt}$ that approaches zero, then remains zero is of greater stability than a rate of change which increases from zero.

$\dot{n} = GnN - kn$

$\dot{N} = -GnN - fN + p$

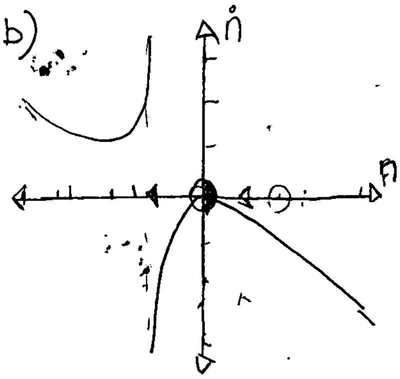
3.3.1 a) Suppose $\dot{N} \gg \dot{n}$, then $\dot{N} \approx 0$. "Adiabatic Elimination"

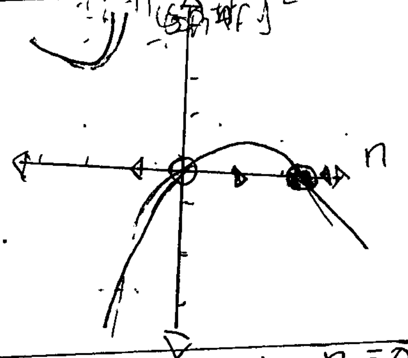
$GnN + fN = p$; $\dot{n} = -fN + p - kn$;

$N = \frac{p}{Gn + f}$; $\boxed{\dot{n} = -f\left[\frac{p}{Gn+f}\right] + p - kn}$

$\boxed{p > \frac{kn\,[Gn+f]}{1 - pf} = p_c}$

b)





c) A transcritical bifurcation occurs at $n = 0$ because of the stability change for the fixed point.

d) $G, n, p, f > 0$, $N = 0$, a constant amount of excited photons

$\dot{E} = K(P - E)$

$\dot{P} = \gamma_1(ED - P)$

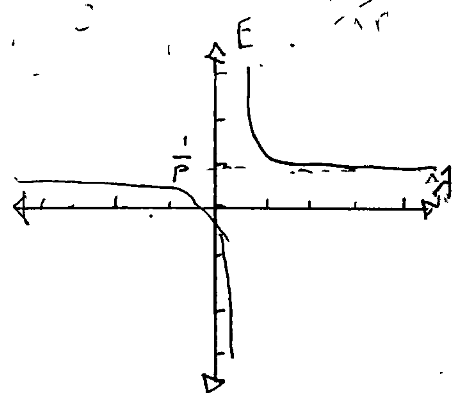
$\dot{D} = \gamma_2(\lambda + 1 - D - \lambda EP)$

3.3.2 a) Assume $\dot{P} \approx 0$, $\dot{D} \approx 0$; $P = ED$; $D = \lambda + 1 - \lambda EP$

$\dot{E} = K(ED - E) = K(E(\lambda + 1 - \lambda EP) - E)$

b) Fixed Points: $\boxed{E = 0, \frac{1}{P}}$

c) Bifurcation Diagram:

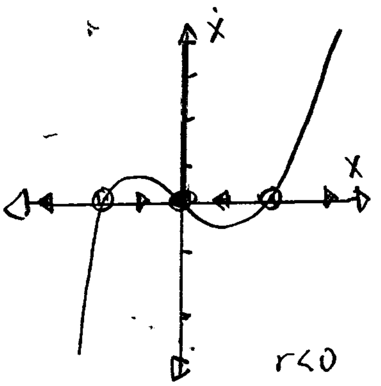


$\dot{x} = rx + 4x^3$ 3.4.1 Vector Field:

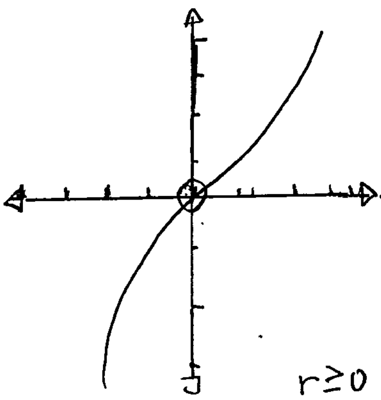
$r = -4x^2$

| $r$ | Bifurcations |
|---|---|
| $< 0$ | Three |
| $\geq 0$ | One |

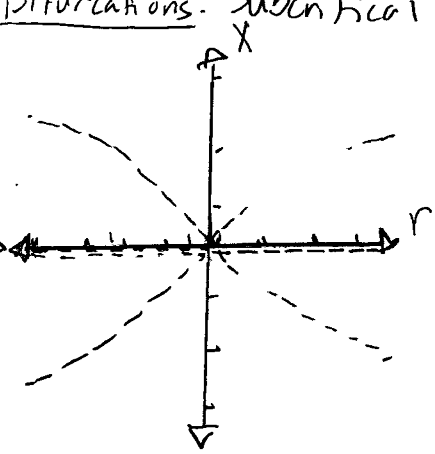
Bifurcations: Subcritical



$r < 0$



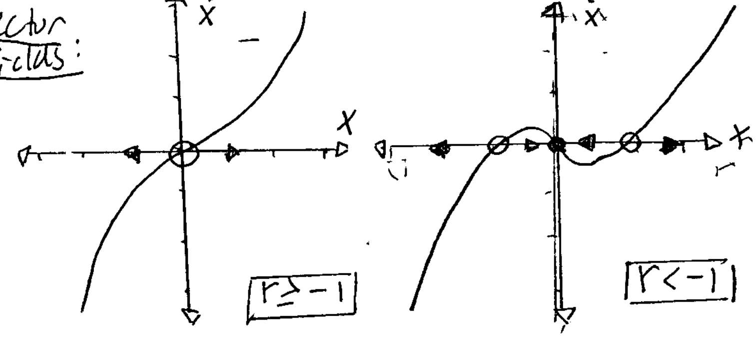
$r \geq 0$



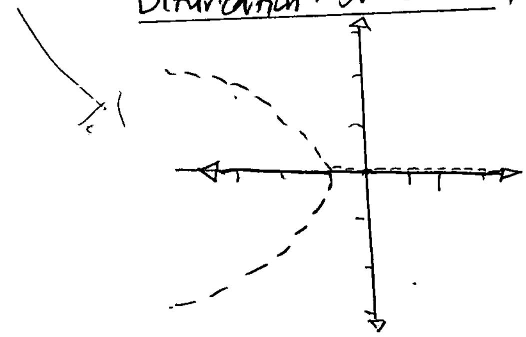
$\dot{x} = rx - \sinh x$ 3.4.2.

| r | Bifurcations |
|---|---|
| $\geq -1$ | One |
| $< -1$ | Three |

Vector Fields:



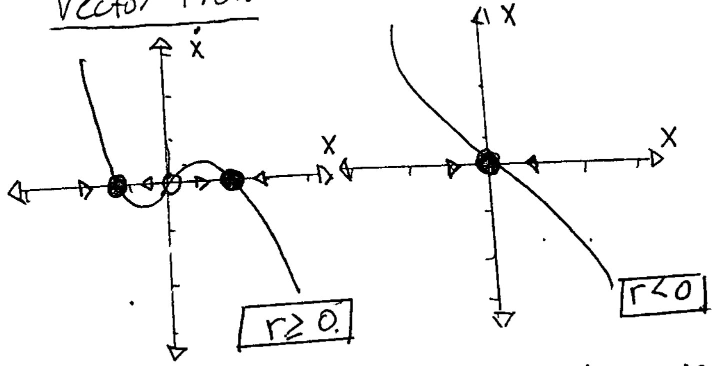

Bifurcation: subcritical



$\dot{x} = rx - 4x^3$ 3.4.3

| r | Bifurcations |
|---|---|
| $\geq 0$ | Three |
| $\leq 0$ | One |

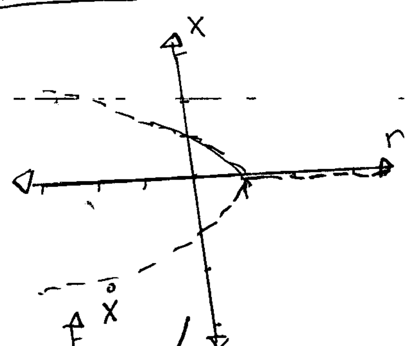
Vector Field:



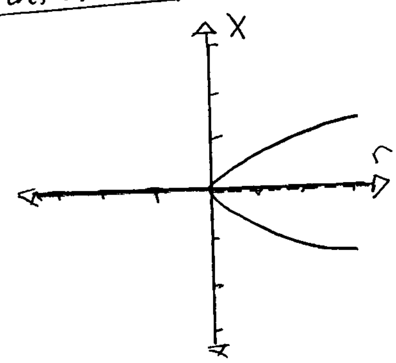

$\dot{x} = x + \frac{rx}{1+x^2}$ 3.4.4

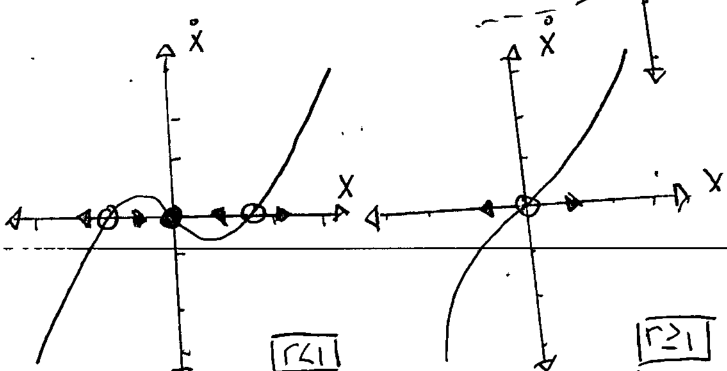
| r | Bifurcations |
|---|---|
| $< 1$ | Three |
| $\geq 1$ | One |

Bifurcations: Supercritical




Bifurcation: supercritical.

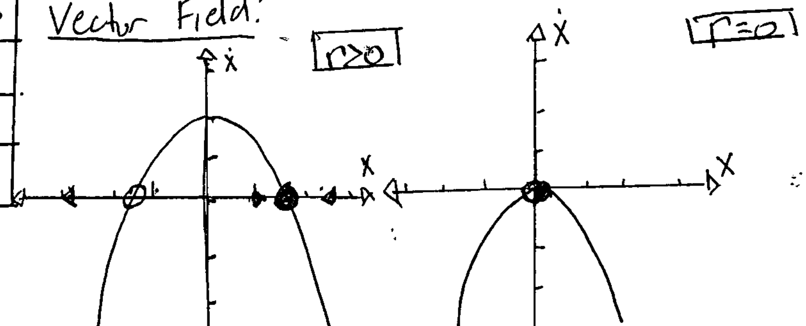


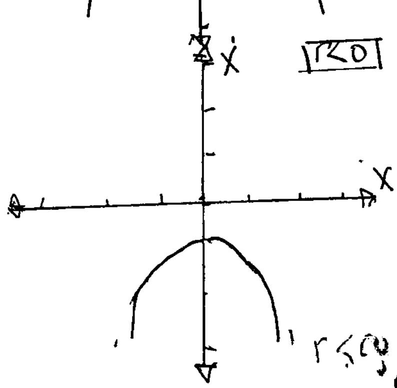



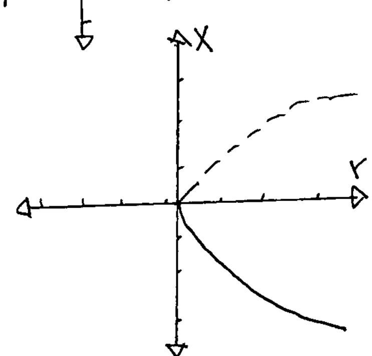

$\dot{x} = r - 3x^2$ 3.4.5

| r | Bifurcations |
|---|---|
| $\geq 0$ | Two |
| $= 0$ | One |
| $< 0$ | zero |

Vector Field:



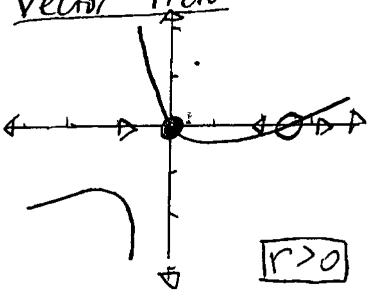



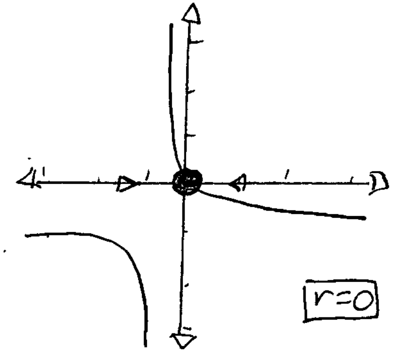

Bifurcation: Saddlenode

$\dot{x} = rx - \frac{x}{1+x}$ 3.4.6.

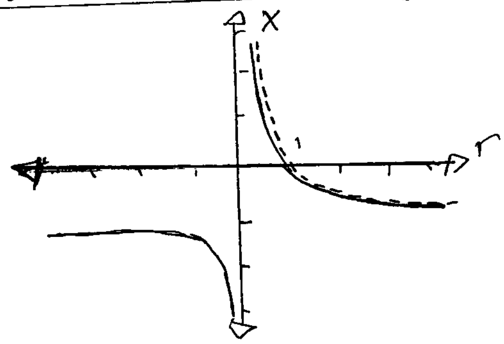
| r | Bifurcations |
|---|---|
| $\geq 0$ | Two |
| $= 0$ | one |
| $< 0$ | Two |

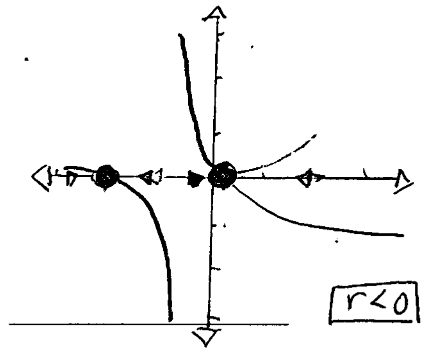
Vector Field:




Bifurcation: Transcritical

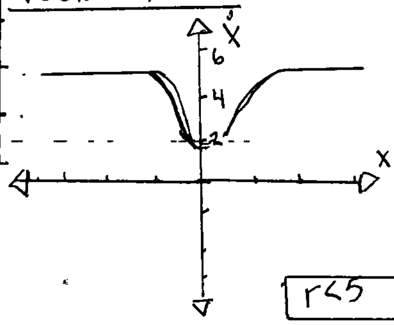


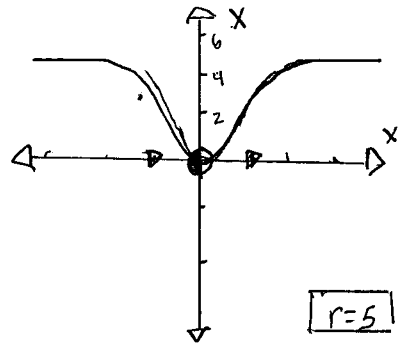



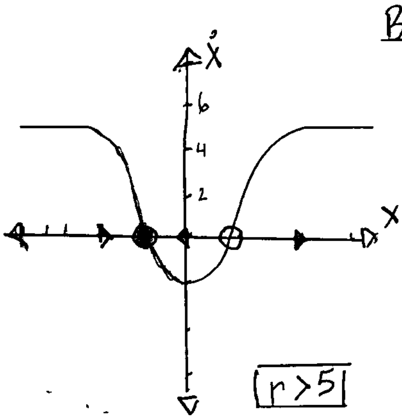

$\dot{x} = 5 - re^{-x^2}$ 3.4.7

| r | Bifurcations |
|---|---|
| $<5$ | zero |
| $=5$ | one |
| $>5$ | Two |

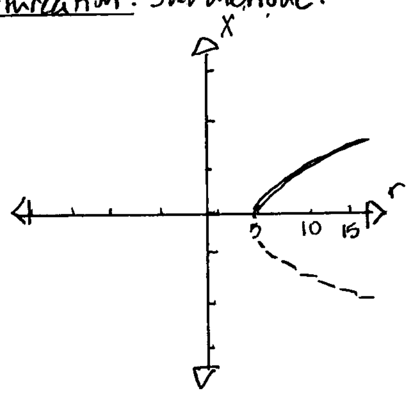
Vector Field:



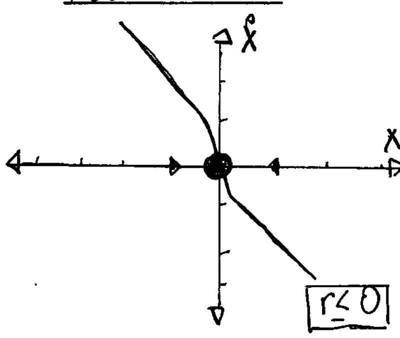

Bifurcation: Saddlenode.

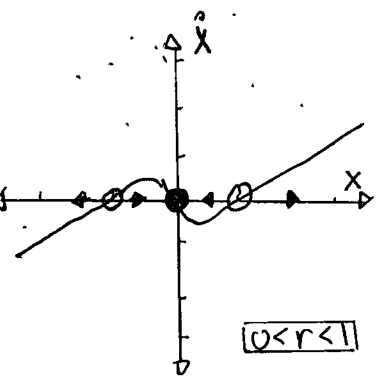


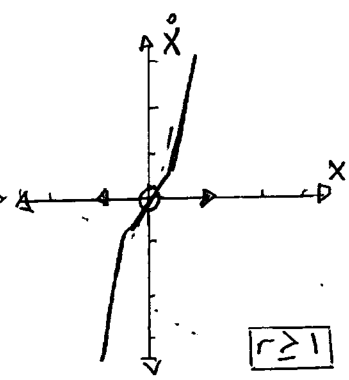

$\dot{x} = rx - \frac{x}{1+x^2}$ 3.4.8.

| r | Bifurcations |
|---|---|
| $\leq 0$ | one |
| $0<r<1$ | three |
| $\geq 1$ | one |

Vector Field:



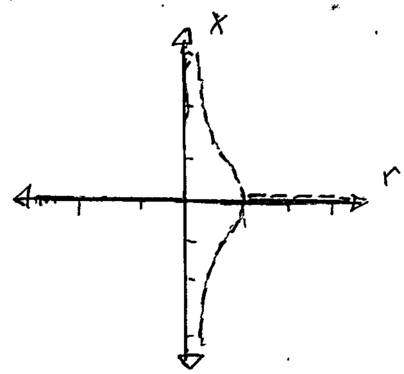



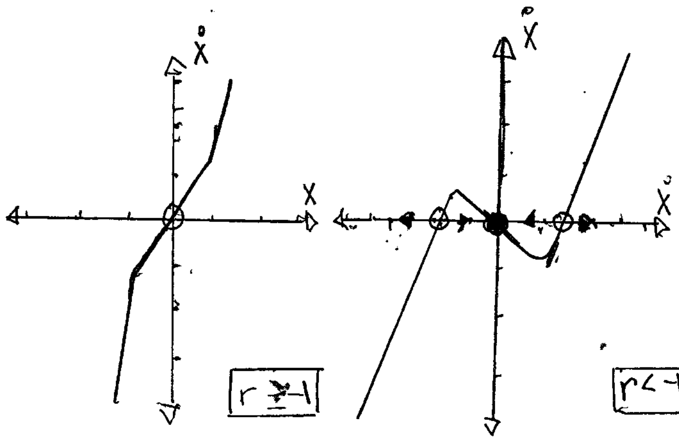
$\dot{x} = x + \tanh(rx)$ 3.4.9

| $r$ | Bifurcations |
|---|---|
| $\leq -1$ | one |
| $> -1$ | three |

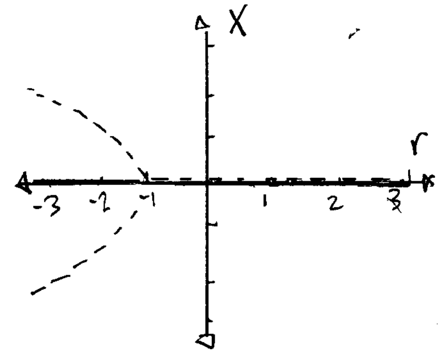
Bifurcation: Transcritical




Vector Field:



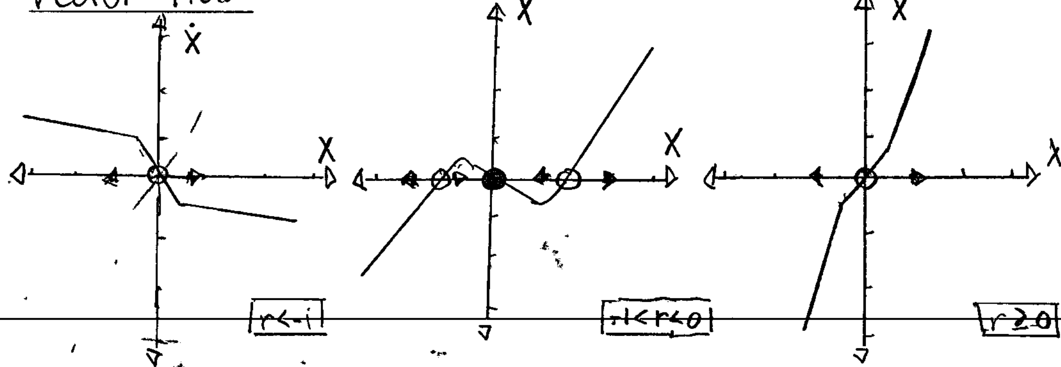

Bifurcation: subcritical




$\dot{x} = rx + \dfrac{x^3}{1+x^2}$ 3.4.10.

| $r$ | Bifurcations. |
|---|---|
| $< -1$ | one |
| $-1 \leq r < 0$ | three |
| $\geq 0$ | one |

Vector Field:



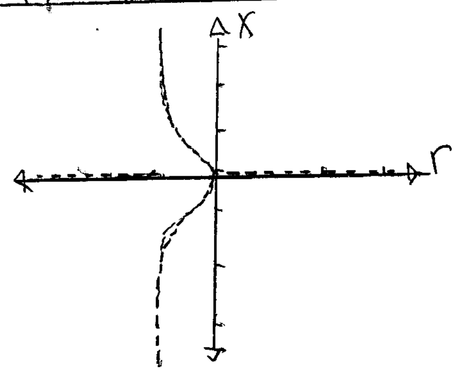

$\dot{x} = rx - \sin x$ 3.4.11 a) If $r=0$, then $\dot{x} = -\sin x$

Fixed points: Stable $= (2k+1)\pi$

unstable $= 2k\pi$

Where $k \in \mathbb{Z}$

Bifurcation: Subcritical Pitchfork




b) If $r > 1$, $\boxed{\dot{x} = 0 \text{ is unstable}}$

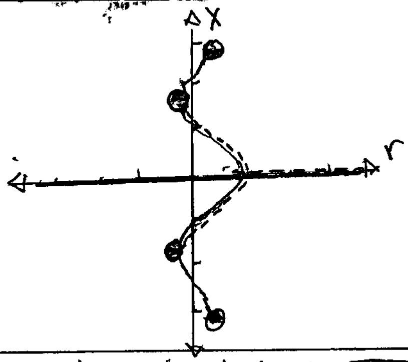
c) As $r = \infty \to 0$, then a subcritical pitchfork best describes the bifurcation.

d) $\dot{x} = rx - \sin(x)$ ; $r = \dfrac{\sin(x)}{x} = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + O(x^4)$

$$\boxed{x = \pm(6[1-r])^{1/2}}$$

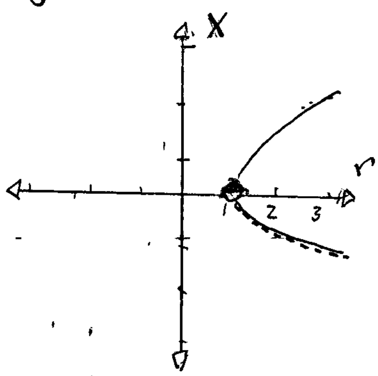
e) As $r = -\infty \to 0$, then a supercritical pitchfork occurs across the function $\dot{x} = rx - \sin(x)$

f) Bifurcations:

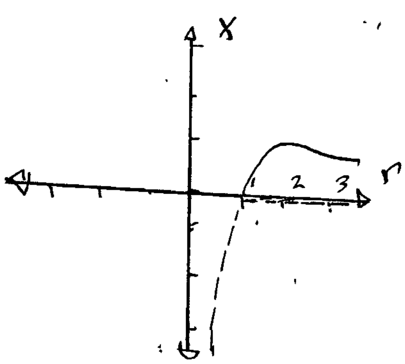



$\dot{x} = f(x,r)$ 3.4.12 A "quadrification" function is $x = \frac{1}{2}(3 \pm \sqrt{1 \pm 4\sqrt{r}})$ where $\dot{x} = f(x,r) = (x-2)^2(x-1)^2 - r$. This function has even polynomial multiplicities to describe zero bifurcations $r<0$ and four when $r>0$.

$\dot{x} = r - x - e^{-x}$ 3.4.13 a) Best guess of roots: $r=1$, $x<0$

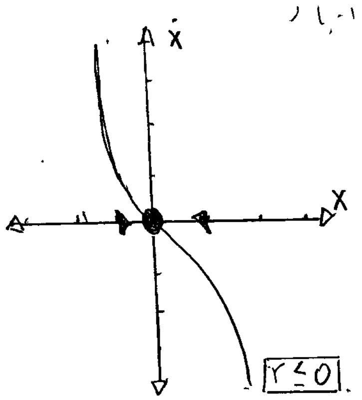


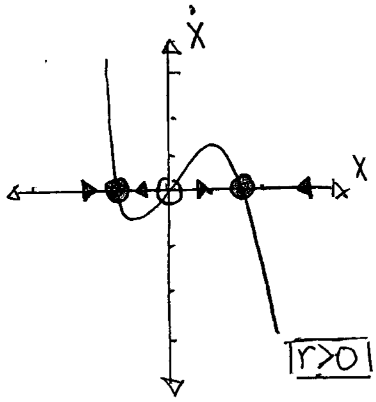

$\dot{x} = 1 - x - e^{-rx}$ b) Best Guess of roots: $r=1$, $x=0$




$\dot{x} = rx + x^3 - x^5$ 3.4.14 a) $\boxed{\ddot{x} = 0 = r + 3x^2 - 5x^4}$ -or- $\boxed{r = x^2(x^2-1)}$

b) Vector Field:




c) $\boxed{r_c \leq 0}$

$\dot{x} = rx + x^3 - x^5$ 3.4.15. $-\frac{dV(x)}{dx} = \dot{x} = 0 = r - x^2 + \frac{x^4}{4}$

Where $a = x^2$

$a_1, a_2 = \frac{+1/4 \pm \sqrt{(1/4)^2 - 4(1)(-r)}}{2(1)}$

$= \frac{1 \pm \sqrt{1+4r}}{2}$

$x_1 = +\sqrt{\frac{1+\sqrt{1+4r}}{2}}$ ; $x_2 = -\sqrt{\frac{1+\sqrt{1+4r}}{2}}$

$x_3 = +\sqrt{\frac{1-\sqrt{1+4r}}{2}}$ ; $x_4 = -\sqrt{\frac{1-\sqrt{1-4r}}{2}}$

$x_5 = 0$

$V(x) = -r\frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6}$,

$V(x_1) = V(x_2) = V(x_3) = V(x_4) = V(x_5) = 0$

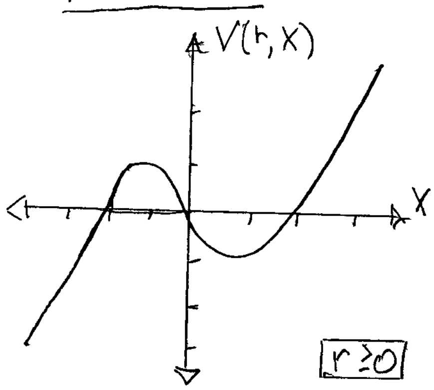
@ $V(x_1) = -\frac{r}{2}\left(\frac{1+\sqrt{1+4r}}{2}\right) + \frac{1}{4}\left(\frac{1+\sqrt{1+4r}}{2}\right)^2 - \frac{1}{6}\left(\frac{1+\sqrt{1+4r}}{2}\right)^3 = \boxed{0}$

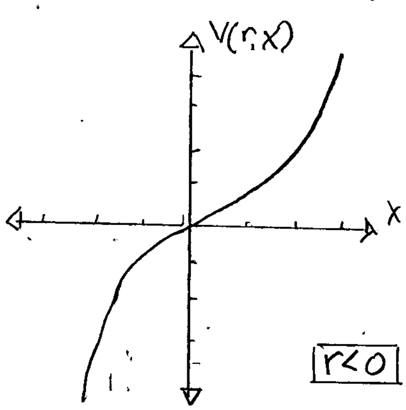
$r = -3/16$

$\dot{x} = r - x^2$ 3.4.16 a) $-\frac{dV}{dx} = \dot{x} = r - x^2$ ; $\boxed{V(r,x) = \frac{x^3}{3} - rx}$

| r | Bifurcations |
|---|---|
| ≥0 | Three |
| <0 | One |

Potential Field

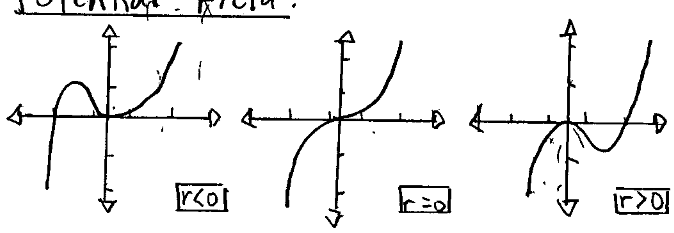



$\dot{x} = rx - x^2$

b) $-\frac{dV}{dx} = \dot{x} = rx - x^2$ ; $\boxed{V(r,x) = \frac{x^3}{3} - \frac{rx^2}{2}}$

Potential Field




| r | Bifurcations |
|---|---|
| <0 | Two |
| =0 | One |
| >0 | Two |

$\dot{x} = rx + x^3 - x^5$ c) $-\frac{dV}{dx} = rx + x^3 - x^5$ ; $\boxed{V(r,x) = \frac{x^6}{6} - \frac{x^4}{4} - rx^2}$

| r | Bifurcations |
|---|---|
| ≤0.5 | Three |
| >0.5 | One. |

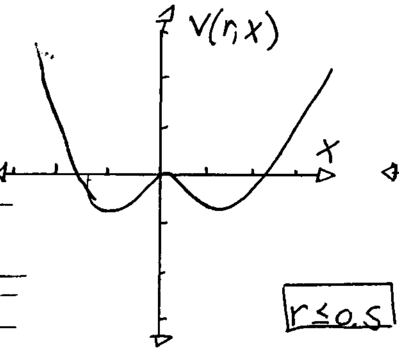
Potential $s$:

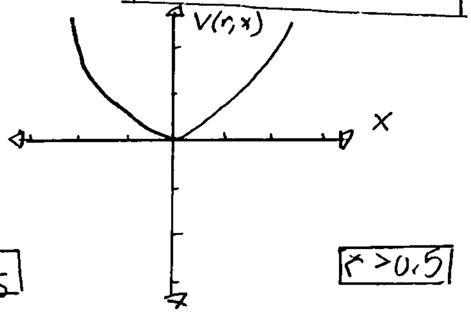
$r + x^2 - x^4$

$= r + a - a^2$

$= \frac{-1 \pm \sqrt{1 - 4(-1)r}}{2(1)}$

$x^* = \sqrt{\frac{-1 \pm \sqrt{1+4r}}{2}}$




$b\dot{\phi} = -mg\sin\phi + mr\omega^2\sin\phi\cos\phi$ 3.5.1.

$\boxed{\text{A better representation of } b\dot{\phi} = mg\sin\phi + mr\omega^2\sin\phi\cos\phi \text{ is } b\dot{\phi} = mg\sin\phi\left(\frac{r\omega^2}{g}\cos\phi - 1\right)}$, which best represents the maximum angle of $\phi = \pi/2$. If the bead approaches a fixed point during rotation, then $b\dot{\phi} = 0$ ; $\frac{r\omega^2}{g}\cos\phi = 1$ ; $\cos\phi = \frac{g}{r\omega^2}$ ; and, $\frac{g}{r\omega^2}$ requires a positive value above zero.

$\frac{d\phi}{d\tau} = F(\phi)$

$= -\sin\phi + \gamma\sin\phi\cos\phi$

$= \sin\phi(\gamma\cos\phi - 1)$

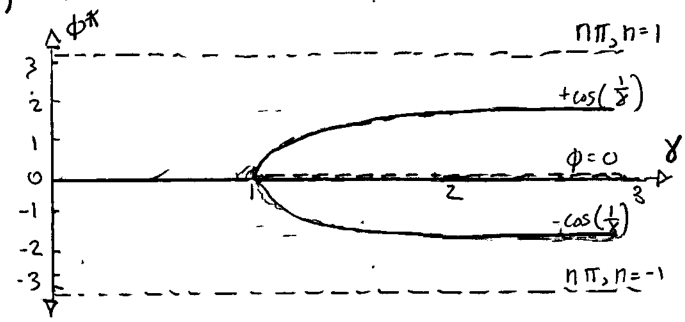
3.5.2 $F(\phi) = \sin\phi(\gamma\cos\phi - 1)$

$f'(\phi) = \gamma[\cos^2\phi - \sin^2\phi - 1] = \gamma[\cos 2\theta - 1]$

$f''(\phi) = -2\gamma[\sin 2\theta]$

$\phi^* = n\pi$ ; $f'(\phi') = 0$ ; Half-Node

$\phi^* = \cos^{-1}\left(\frac{1}{\gamma}\right)$




$\frac{d\phi}{d\tau} = f(\phi)$

$= -\sin\phi + \gamma\sin\phi\cos\phi$

$= \sin\phi(\gamma\cos\phi - 1)$

3.5.3. If $\phi \approx 0$, then $\sin\phi \approx \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!}$

$\approx \phi$

$\cos\phi \approx 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!}$ and $\frac{d\phi}{d\tau} = \phi\left(\gamma\left[1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!}\right] - 1\right)$

$= \gamma\phi - \frac{\gamma\phi^3}{2!} + \frac{\gamma\phi^5}{4!}$

Where $\boxed{\frac{d\phi}{d\tau} = A\phi - B\phi^3 + O(\phi^3) ; A = \gamma, B = \frac{\gamma}{2}, O(\phi^5) = \frac{\gamma\phi^5}{4!}}$

$m\ddot{x} = -F_{spring} - F_{Fric}$ 3.5.4. $m\ddot{x} = -k \cdot l\cos\phi - kL_0\cos\phi - b\dot{\phi}$

$= -k(l - L_0)\cos\phi - b\dot{\phi} = -k(\sqrt{x^2+h^2} - L_0)\frac{x}{l} - b\dot{\phi}$

$= -k(\sqrt{h^2+h^2} - L_0)\frac{x}{\sqrt{h^2+x^2}} - b\dot{x}$

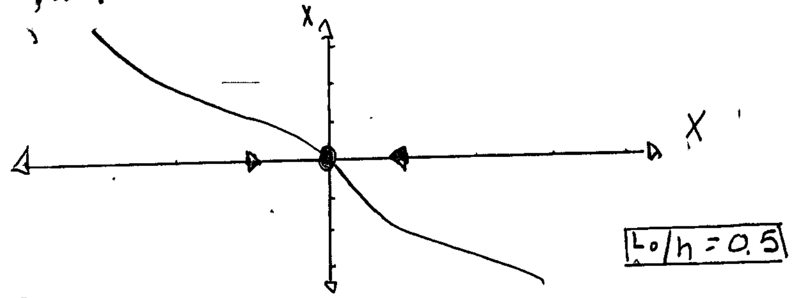
$\boxed{= -k\left(1 - \frac{L_0}{\sqrt{h^2+x^2}}\right)x - b\dot{x}}$

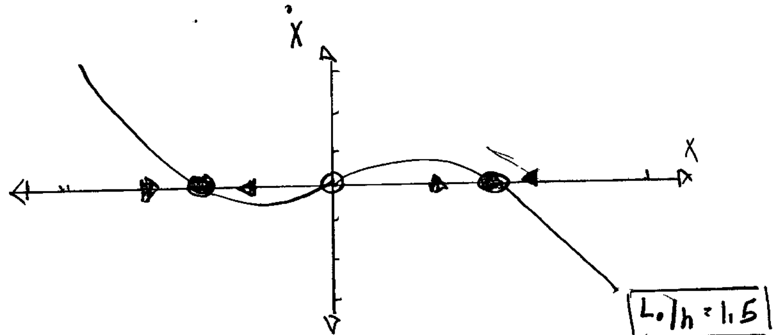
b. $m\ddot{x} + b\dot{x} + k\left(1 - \frac{L_0}{\sqrt{x^2+h^2}}\right)x = 0$

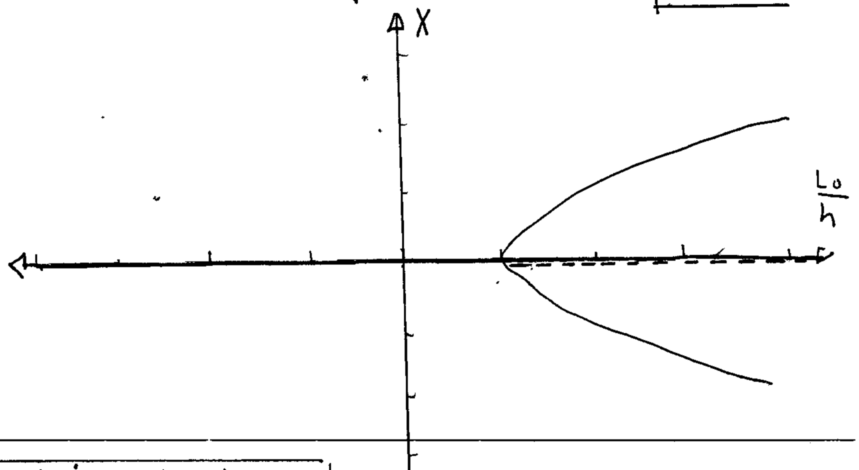
if $\dot{x}=0$, $\boxed{x^* = \sqrt{L_0^2 - h^2},\ 0}$

c. If $m=0$, $b\dot{x} + k\left(1-\frac{L}{\sqrt{x^2+h^2}}\right)x = 0$, then $\boxed{x^* = \sqrt{L_0^2-h^2},\ 0}$

Bifurcation Diagram:




d. $\boxed{\text{If } m \neq 0, \text{ then } m\ddot{x} \ll -b\dot{x} - k\left(1-\frac{L_0}{\sqrt{x^2+h^2}}\right)x}$

$\varepsilon \frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau} = f(\phi)$ 3.5.5. a) $\frac{d\phi}{d\tau} = f(\phi)$; $T_{fast}$ is estimated to be:

$\varepsilon^{1-2k}\frac{d^2\phi}{d\tau^2} + \varepsilon^{-k}\frac{d\phi}{d\tau} = f(\phi)$;

Where $k=1$, $\varepsilon^{1-2k} = \varepsilon^{-k} \gg 1$

$k=\frac{1}{2}$, $\varepsilon^{1-2k} = 1 \gg e^{-k}$

$k=0$, $\varepsilon^{-k} = 1 \gg e^{1-2k}$

$T = \varepsilon\frac{b}{mg} = \frac{m^2 g r}{b^2}\frac{b}{mg} = \boxed{\frac{mr}{g}}$

b) If $\tau = \varepsilon z$, then $\varepsilon\frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau} = \varepsilon\frac{d^2\phi}{d(\varepsilon z)^2} + \frac{1}{\varepsilon}\frac{d\phi}{dz} = f(\phi)$

$\boxed{\frac{d^2\phi}{dz^2} + \frac{d\phi}{dz} = \varepsilon f(\phi) \quad \text{"Rescaled"}}$

c. $T_{fast} = \epsilon T_{slow}$

$\epsilon\ddot{x} + \dot{x} + x = 0$ 3.5.6 $x(0) = 1$; $\dot{x}(0) = 0$

a) General Solution: $x(t) = C_1 e^{\lambda t}$ $\quad \epsilon\lambda^2 + \lambda + 1 = 0$

$\dot{x}(t) = \lambda C_1 e^{\lambda t}$; $\lambda = \dfrac{-1 \pm \sqrt{1-4\epsilon}}{2\epsilon}$

$\ddot{x}(t) = \lambda^2 C_1 e^{\lambda t}$; $x(t) = C_1 e^{(-1+\sqrt{1-4\epsilon})x/2\epsilon} + C_2 e^{(-1-\sqrt{1-4\epsilon})x/2\epsilon}$

$x(0) = C_1 + C_2 = 1$

$$\dot{x}(t) = C_1\left(\frac{-(1+\sqrt{1-4\epsilon})}{2\epsilon}\right)e^{-(1+\sqrt{1-4\epsilon})x/2\epsilon} + C_2\left(-\frac{(1-\sqrt{1-4\epsilon})}{2\epsilon}\right)e^{-(1-\sqrt{1-4\epsilon})x/2\epsilon}$$

$$\dot{x}(0) = C_1\left(\frac{-(1+\sqrt{1-4\epsilon})}{2\epsilon}\right) + C_2\left(\frac{-(1-\sqrt{1-4\epsilon})}{2\epsilon}\right)$$

$$= \frac{-C_1\sqrt{1-4\epsilon}}{2\epsilon} + \frac{C_1}{2\epsilon} + (1-C_1)\left(\frac{-(1-\sqrt{1-4\epsilon})}{2\epsilon}\right)$$

$$= \frac{-C_1\sqrt{1-4\epsilon}}{2\epsilon} + \frac{C_1}{2\epsilon} + \frac{(1+\sqrt{1-4\epsilon})}{2\epsilon}$$

$$+ \frac{C_2(1-\sqrt{1-4\epsilon})}{2\epsilon}$$

$$= \cancel{\frac{-C_1\sqrt{1-4\epsilon}}{2\epsilon}} + \cancel{\frac{C_1}{2\epsilon}} + \frac{(1+\sqrt{1-4\epsilon})}{2\epsilon}$$

$$+ \cancel{\frac{C_1}{2\epsilon}} + \cancel{\frac{C_1\sqrt{1-4\epsilon}}{2\epsilon}} = 0$$

$$= C_1 = \frac{(1+\sqrt{1-4\epsilon})}{2}$$

$$\boxed{\text{Therefore, } x(t) = \left(\frac{1+\sqrt{1-4\epsilon}}{2}\right)\left(e^{(-1+\sqrt{1-4\epsilon})x/2\epsilon}\right) + \left(1 - \frac{(1+\sqrt{1-4\epsilon})}{2}\right)\left(e^{(-1-\sqrt{1-4\epsilon})x/2\epsilon}\right)}$$

b. $\epsilon\ddot{x} + \dot{x} + x = \epsilon\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + x = 0$

$\dfrac{\epsilon}{T^2}\dfrac{d^2x}{d\tau^2} + \dfrac{1}{T}\dfrac{dx}{d\tau} = -x$ $\quad$ where $T = \dfrac{t}{\tau} = \epsilon$

$\dfrac{1}{T}\dfrac{d^2x}{d\tau^2} + \dfrac{1}{T}\dfrac{dx}{d\tau} = x$; $\ddot{x} + \dot{x} = Tx = \epsilon x$; $\boxed{\ddot{x} + \dot{x} - Tx = 0}$

$\dot{N} = rN(1 - N/k)$ 3.5.7. a) $N(0) = N_0$;

| Parameter | Dimensions |
|---|---|
| $r$ | Per time (rate |
| $K$ | Same as $N$ (amount) |
| $N_0$ | Same as $N$ (amount) |

b) $\frac{dN}{dt} = rN(1 - N/K)$; If $\frac{N}{K} = x$, then $dN = K\,dx$

$\frac{dx}{dt} = rx(1-x)$; If $t = \frac{\tau}{r}$, then $dt = \frac{d\tau}{r}$

$$\boxed{\frac{dx}{d\tau} = x(1-x)}$$

c) $u = x$; $\frac{du}{d\tau} = u(1-u)$; $u(0) = u_0$

$\int \frac{du}{u(1-u)} = d\tau$; $\int \frac{A}{u}du + \int \frac{B}{(1-u)}du = \int \frac{du}{u} + \int \frac{du}{(1-u)} = \ln \frac{u}{1-u} = \tau + c$

$A(1-u) + Bu = 1$

$\frac{1-u}{u} = Ce^{-\tau}$

$1 = u(1 + Ce^{-\tau})$

$u = \frac{1}{1 + Ce^{-\tau}}$

$u(0) = u_0 = \frac{1}{1 + C}$

$C = \frac{1 - u_0}{u_0}$

$$\boxed{u(\tau) = \frac{1}{1 + \left(\frac{1-u_0}{u_0}\right)e^{-\tau}}}$$

d) An advantage of the dimensionless functions are lower degrees of ~~freedom~~ during analysis. The ~~graphical~~ representations do not have further axis to plot, and the functions are closer to the basic functions of precalculus.

$\dot{u} = au + bu^3 - cu^5$ 3.5.8. Prove $\frac{dx}{d\tau} = rx + x^3 - x^5$, where $x = \frac{u}{U}$

$\tau = \frac{t}{T}$

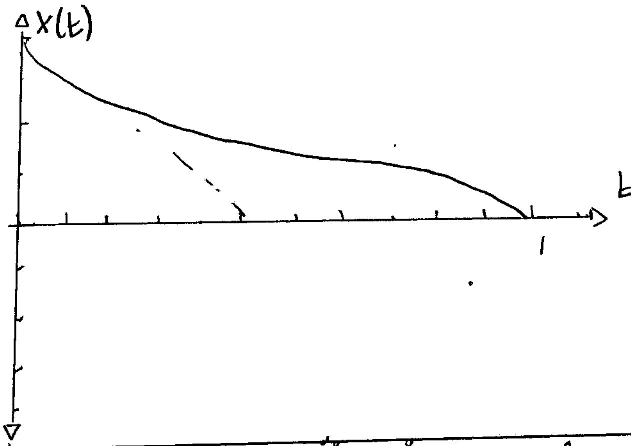
$\frac{U}{T}\frac{dx}{d\tau} = a\cancel{U}x + bU^3x^3 - cU^5x^5$

$\frac{dx}{d\tau} = Tax + TbU^2x^3 - TcU^4x^5$; $a = \frac{r}{T}$, $b = \frac{1}{TU^2}$, $c = \frac{1}{TU^4}$

$$\boxed{\frac{dx}{d\tau} = rx + x^3 - x^5}$$

3.6.1. Figure 3.6.3b corresponds to Figure 3.61b; specifically, the relationship between $y = h$, and $y = rx - x^3$. The dotted lines support a single bifurcation to two bifurcations at $h_c$, then three when $h > h_c$. To answer the question, Figure 3.6.3b has information of $h < 0$ and $h > 0$.

c)




d) If $\varepsilon \ll 1$, then $\varepsilon \ddot{x} + \dot{x} + x \approx \dot{x} + x$ and is a similar model to the boundary conditions

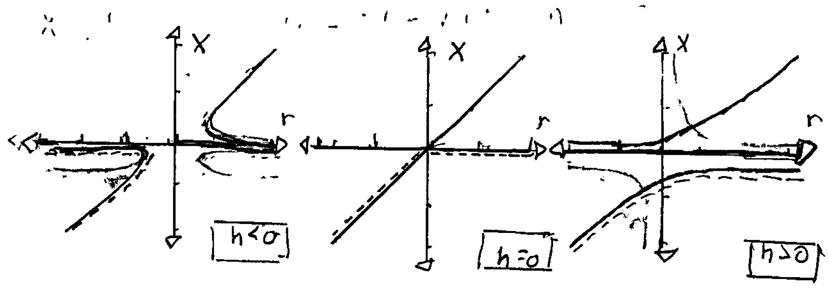
e) Mechanical System: An extremely viscous solution for an oscillating Newtonian device.

Electrical System: An electrical system of the form $V = Ri + L\frac{di}{dt} + \frac{1}{c}\int i\,dt$ where $\varepsilon = \frac{1}{c} \ll 1$.

$\dot{x} = h + rx - x^2$ 3.6.2. a)

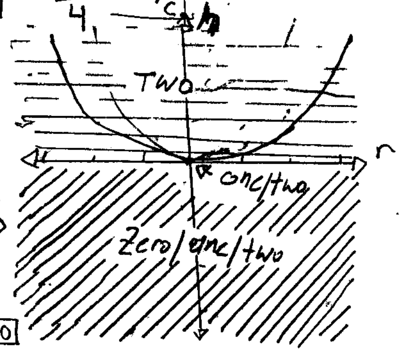
$\frac{-r \pm \sqrt{r^2 - 4(-1)(h)}}{2(-1)}$

| h | Bifurcations |
|---|---|
| <0 | ~~zero~~ one/two |
| =0 | one/Two |
| >0 | Two |

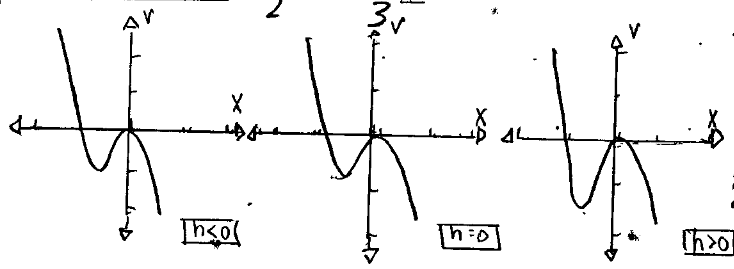



b) (r,h) Plane

$\frac{d}{dx}(rx - x^2) = r - 2x$ ; $x_{max} = \frac{r}{2}$ ; $\frac{r^2}{2} - \frac{r^2}{4} = \frac{r^2}{4} = h_c$

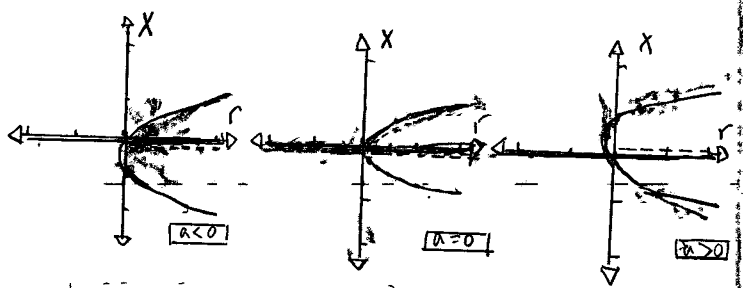



c) $\boxed{V(x,h,r) = hx + \frac{rx^2}{2} - \frac{x^3}{3}}$




$\dot{x} = rx + ax^2 - x^3$ 3.6.3 a)

| a | Bifurcations |
|---|---|
| <0 | one/two/three |
| =0 | one/three |
| >0 | one/two/three |




b) (r,a) plane

$\frac{d}{dx}(rx + ax^2 - x^3) = r + 2ax - 3x^2 = 0$ ;

$rx + ax - x^2$ ; $\boxed{a = \frac{x^2 - r}{x}}$

3.6.4 $\boxed{\text{A small imperfection to a saddlenode bifurcation shifts the cusp either left or right.}}$

$mg\sin\theta = kx\left(1 - \frac{L}{\sqrt{x^2+a^2}}\right)$ 3.6.5 a) $F = -F_{spring} = F_{g,x} = F_g \sin\theta = mg\sin\theta = k(x - x\sin\theta)$

$= k\left(x - x\frac{L_0}{\sqrt{x^2+a^2}}\right)$

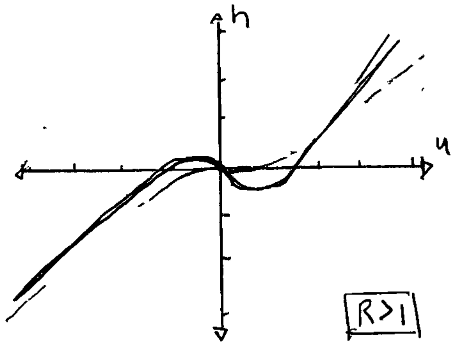
$\boxed{= kx\left(1 - \frac{L_0}{\sqrt{x^2+a^2}}\right)}$

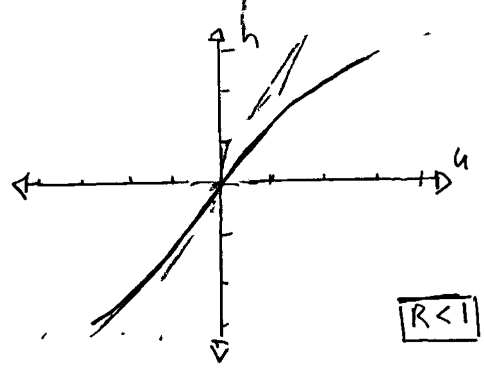
b) Prove $1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}}$

If $1 - \frac{mg\sin\theta}{kx} = \frac{L_0}{a\sqrt{\left(\frac{x}{a}\right)^2+1}}$, then $u = \frac{x}{a}$, $R = \frac{L_0}{a}$, $h = \frac{mg\sin(\theta)}{ka}$.

and $\boxed{1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}}}$

c)




$\boxed{\text{The variable } h \text{ as a function of } u \text{ has a single equilibrium point for both } R>1 \text{ and } R<1.}$

d) If $r=R-1$, $1-\frac{h}{u}=\frac{r+1}{\sqrt{1+u^2}}$; $u-h=\frac{(r+1)u}{\sqrt{1+u^2}}$; $u\sqrt{1+u^2}-h\sqrt{1+u^2}=(r+1)u$

$$u\left(1+\tfrac{1}{2}u^2+O(u^4)\right)-h\left(1+\tfrac{1}{2}u^2+O(u^4)\right)=(r+1)u$$

$$\cancel{u}+\frac{u^3}{2}-h-\frac{h}{2}u^2=ru+\cancel{u}$$

$$h+ru+\frac{h}{2}u^2-\frac{1}{2}u^3\boxed{\approx 0}$$

e) $h\left(1+\frac{u^2}{2}\right)=\frac{1}{2}u^3-ru$

$\frac{d}{du}h\left(1+\frac{u^2}{2}\right)=\frac{d}{du}\left(\frac{1}{2}u^3-ru\right)$; $hu=\frac{3}{2}u^2-r$; $r_{max}=\frac{3}{2}u^2-hu$

$h\left(1+\frac{u^2}{2}\right)=\frac{1}{2}u^3-\left(\frac{3}{2}u^2-hu\right)u$; $h+\frac{hu^2}{2}=\frac{1}{2}u^3-\frac{3}{2}u^3+hu^2$

$h\left(1-\frac{1}{2}u^2\right)=-u^3$ ; $h=\frac{2u^3}{u^2-2}$

$2-u^2$

$$r_{max}=\frac{3}{2}u^2-hu=\frac{3}{2}u^2-\left(\frac{2u^3}{u^2-2}\right)u$$

$$=\frac{3}{2}u^2-\frac{2u^4}{u^2-2}$$

$$=\frac{u^4+3u^2}{2(1-u^2)}\boxed{=R-1}$$

f) $1-\frac{h}{u}=\frac{R}{\sqrt{1+u^2}}$ ; $\frac{d}{du}\left(1-\frac{h}{u}\right)=\frac{d}{du}\left(\frac{R}{\sqrt{1+u^2}}\right)$ ; $\frac{h}{u^2}=-\frac{1}{2}\frac{R(2u)}{(1+u^2)^{3/2}}$

$\therefore h(1+u^2)^{3/2}=-R\cdot u^3$ ; $R=-\frac{h(1+u^2)^{3/2}}{u^3}$

$1-\frac{h}{u}=\frac{-h(1+u^2)^{3/2}}{u^3\sqrt{1+u^2}}=-\frac{h(1+u^2)}{u^3}$ ; $u-h=-\frac{h(1+u^2)}{u^2}$

$u^3 - hu^2 = -h(1+u^2)$ ; $u^3 = -h - hu^2 + hu^2$ ; $h = -u^3$

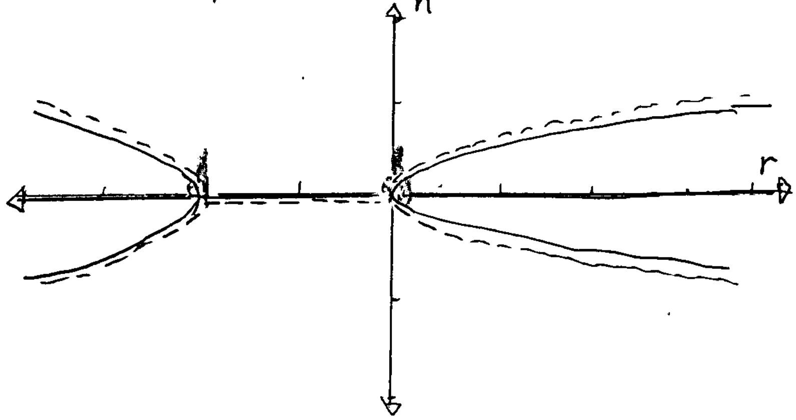
$$R = \frac{-h(1+u^2)^{3/2}}{u^3} = (1+u^2)^{3/2}$$

$$\boxed{\lim_{u \to 0} h = -u^3 \approx \frac{2u^3}{u^2-2} \approx u^3}$$

$$\boxed{\lim_{u \to 0} R = (1+u^2)^{3/2} \approx \frac{u^4+3u^2}{2(1-u^2)} + 1 = r+1}$$

g) $R = (1+u^2)^{3/2} = r+1$ ; $r(u) = (1+u^2)^{3/2} - 1$ ; $u = \sqrt{(r+1)^{2/3} - 1}$

$$\boxed{h = -u^3 = \pm\left(\sqrt{(r+1)^{2/3}-1}\right)^3}$$




h) $h = -u^3 = -\left(\frac{x}{a}\right)^3 = \frac{mg\sin\theta}{ka}$

$R = \left(1+\left(\frac{x}{a}\right)^2\right)^{3/2} = \frac{L_0}{a}$

The bifurcation plot represents the ~~points~~ of stability for the oscillating system.

$\tau\dot{A} = \varepsilon A - gA^3$ 3.6.6. $A(t)$ = Amplitude; $\tau$ = typical timescale; $\varepsilon$ = dimensionless parameter

$\tau\dot{A} = \varepsilon A - gA^3 - kA^3$ 3.6.6. Supercritical : $g > 0$, subcritical: $g < 0$, $K > 0$

"Landaus Equation"

a) Landau's Equation describes the change of amplitude for a fluid system.

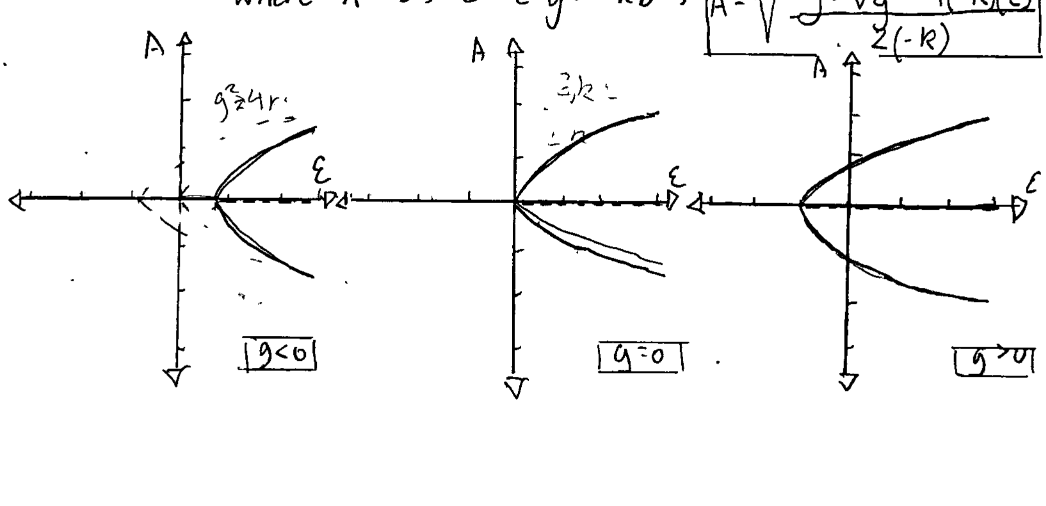
b). $\tau\dot{A} = \varepsilon A - gA^3 - kA^3$; if $g = 0$, then $\tau\dot{A} = \varepsilon A - kA^3$ ; $A = \sqrt{\frac{\varepsilon}{k}}$

The function $A(\varepsilon)$ is a tricritical bifurcation because $A = 0$ is a solution; in addition to, $A = +\sqrt{\frac{\varepsilon}{k}}$, and $A = -\sqrt{\frac{\varepsilon}{k}}$

c) $\tau\dot{A} = h + \varepsilon A - gA^3 - kA^5$; An approximation $h \approx 0$, $0 = \varepsilon A - gA^3 - kA^5$

$= \varepsilon - gA^2 - kA^4$

Where $A^2 = b$; $0 = \varepsilon - gb - kb^2$; $\boxed{A = \sqrt{\frac{-g \pm \sqrt{g^2 - 4(-k)(\varepsilon)}}{2(-k)}}}$

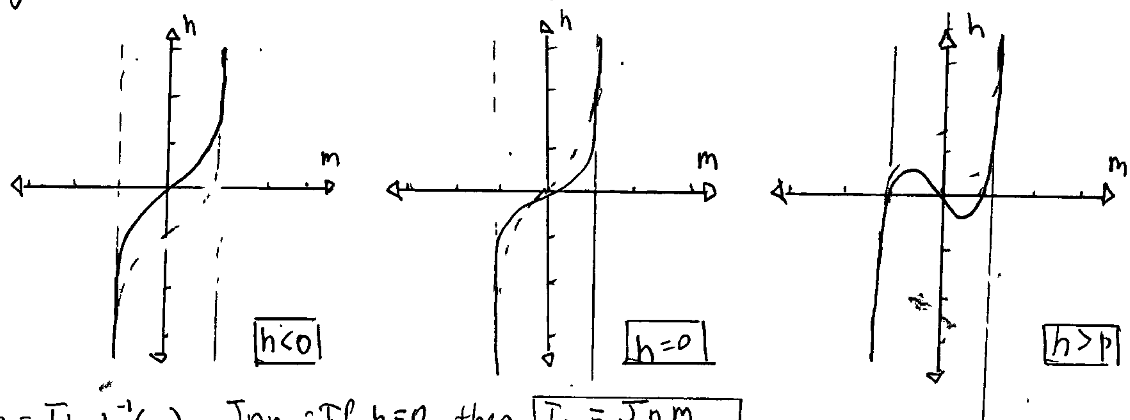


d) $\boxed{\text{The graphs appearance represent the relationship of amplitude vs. time and if } \epsilon \text{ is large, then the first order term approaches the steadystate condition more rapidly.}}$

$m = \left|\frac{1}{N}\sum_{i=1}^{N} S_i\right|$

$h = T\tanh^{-1}(m) - Jnm$

3.6.7. a)




b) $h = T\tanh^{-1}(m) - Jnm$; If $h=0$, then $\boxed{T_c = \frac{Jnm}{\tanh^{-1}(m)}}$

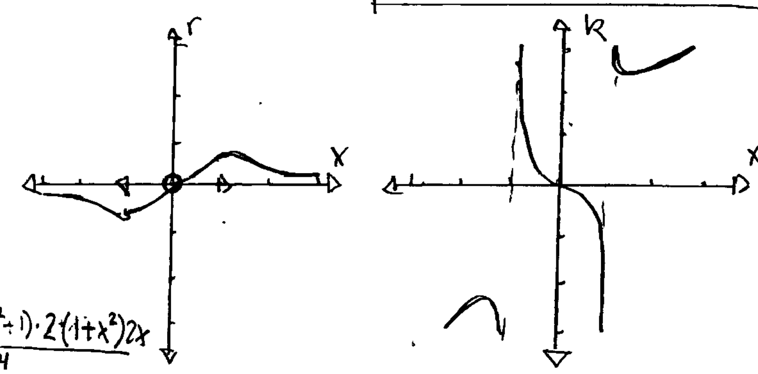
$\frac{dx}{d\tau} = rx\left(1-\frac{x}{K}\right) - \frac{x^2}{1+x^2}$ 3.7.1. @ $x^*=0$; $0 < rx - \left(\frac{1}{K}+\frac{1}{1+x^2}\right)x^2$; $\left(\frac{1}{K}+\frac{1}{1+x^2}\right)x < r$; $\boxed{0<r \text{ is positive and unstable}}$

$r = \frac{2x^3}{(1+x^2)^2}$

$K = \frac{2x^3}{x^2-1}$

3.7.2. a) $\boxed{\lim_{x\to 1} r = \frac{2}{4}}$ ; $\boxed{\lim_{x\to\infty} r = 0}$

$\boxed{\lim_{x\to 1} K = -\infty}$ ; $\boxed{\lim_{x\to\infty} K = \infty}$




b) $r = \frac{(x^2-1)K}{(1+x^2)^2}$ ; $\frac{dr}{dx} = \frac{2x(1+x^2)^2 - (x^2+1)\cdot 2(1+x^2)2x}{(1+x^2)^4}$

$= \frac{2x(1+x^2)^2 - 4x^3(1+x^2) + 4x(1+x^2)}{(1+x^2)^4} = 0$

$= 2x(1+x^2) - 4x^3 + 4x = 2(1+x^2) - 4x^2 + 4 = (1+x^2) - 2x^2 + 2 = 0$

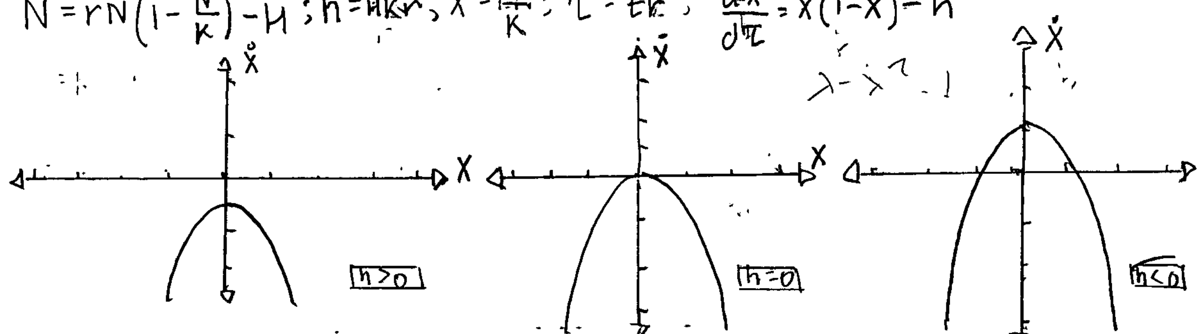
$3 = x^2$ ; $\boxed{x = \sqrt{3}}$

$r_{max} = \frac{(3-1)K}{(1+3)^2} = \frac{1}{8}K_{max}$ ; $\boxed{r_{max} = \frac{2\cdot 3^{3/2}}{(1+3)^2} = 0.6495}$ ; $\boxed{K_{max} = 5.1961}$

$\frac{dx}{d\tau} = x(1-x) - h$ 3.7.3. $\dot{N} = rN\left(1-\frac{N}{K}\right) - H$ ; $h = HKr$ ; $x = \frac{N}{K}$ ; $\tau = tr$ ; $\frac{dx}{d\tau} = x(1-x) - h$

a)

b)




c) $0 = -x^2 + x - h$; $x = \frac{-1 \pm\sqrt{1-4(-1)(-h)}}{2(-1)} = \frac{1 \pm\sqrt{1-4h}}{2}$ ; $\boxed{h_c = 0}$

d) $\boxed{\text{The long-term behavior of the fish population is to reduce the total population as population rises.}}$

$\dot{N} = rN\left(1-\frac{N}{K}\right) - H\frac{N}{A+N}$ 3.7.4. a) $\boxed{\text{The variable } A \text{ could represent the amount of fish in a school, and if } A \text{ is large, then less fish are harvested.}}$

b) $\boxed{x = \frac{N}{K};\ \tau = tr;\ h = HRK;\ a = A}$

c) $\frac{dx}{d\tau} = x(1-x) - h\frac{x}{a+x} = 0;\ x(1-x)(a+x) = (x-x^2)(a+x) = ax + x^2 - x^2a - x^3$

$0 = (a-h)x + (1-a)x^2 - x^3$

$0 = (a-h) + (1-a)x - x^3$

$x_1 = 0,\ x_{2,3} = \frac{-(1-a) \pm \sqrt{(1-a)^2 - 4(-1)(a-h)}}{2(-1)}$

$= \frac{(1-a) \pm \sqrt{(1-a)^2 + 4(a-h)}}{2}$

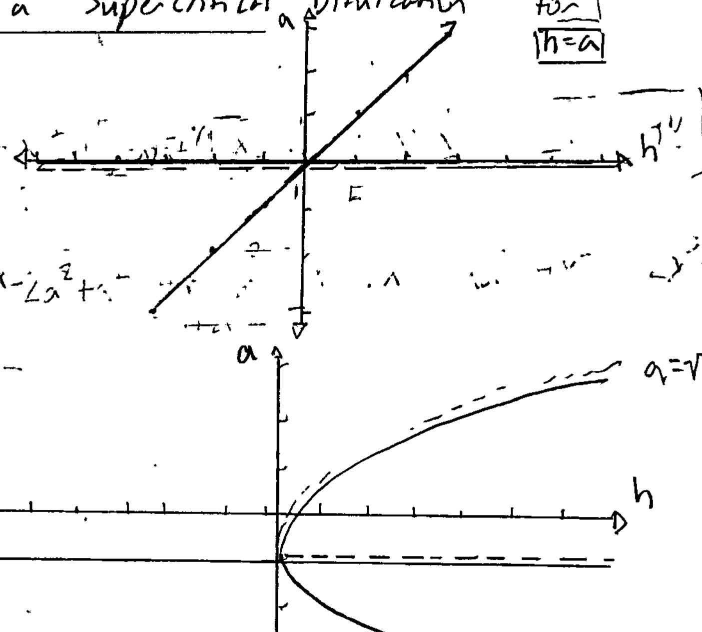
| Fixed Point | Stability | $a<h$ |
|---|---|---|
| 0 | unstable | stable |
| $\frac{(1-a)+\sqrt{(1-a)^2+4(a-h)}}{2}$ | stable | / |
| $\frac{(1-a)-\sqrt{(1-a)^2+4(a-h)}}{2}$ | stable | / |

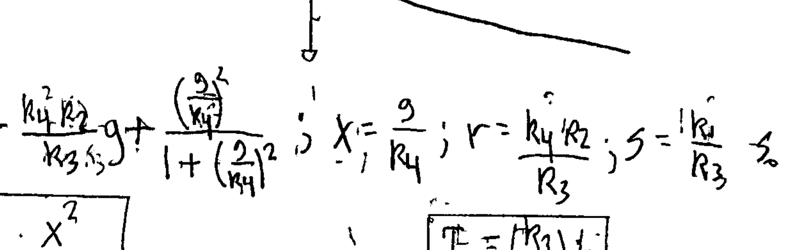
d) $\boxed{\text{At } x=0, \text{ when } h=a, \text{ the half-node indicates a transcritical bifurcation is about to occur when } h \text{ becomes less than } a.}$

e) $\boxed{\text{The graph shows a supercritical bifurcation for } h=a}$

$h = \frac{(a+1)^2}{4}$.

f) $\boxed{a = \frac{h}{x-1} - x}$




$\dot{g} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2}$ 3.7.5.

a) $\frac{k_4^2}{k_3}\frac{dg}{dt} = \frac{k_4 k_1}{k_3}s_0 - \frac{k_4^2 k_2}{k_3}g + \frac{\left(\frac{g}{k_4}\right)^2}{1+\left(\frac{g}{k_4}\right)^2};\ x = \frac{g}{k_4};\ r = \frac{k_4 k_2}{k_3};\ s = \frac{k_1}{k_3}s_0$

$\boxed{\frac{dx}{d\tau} = s - rx + \frac{x^2}{1+x^2}}$ $\boxed{\tau = \left(\frac{k_3}{k_4^2}\right)t}$

b) $0 = -rx + \frac{x^2}{1+x^2};\ rx = \frac{x^2}{1+x^2};\ r(1+x^2) = x;\ rx^2 - x + r = 0$

$\boxed{x_{1,2} = \frac{1 \pm \sqrt{1-4r^2}}{2r}}$

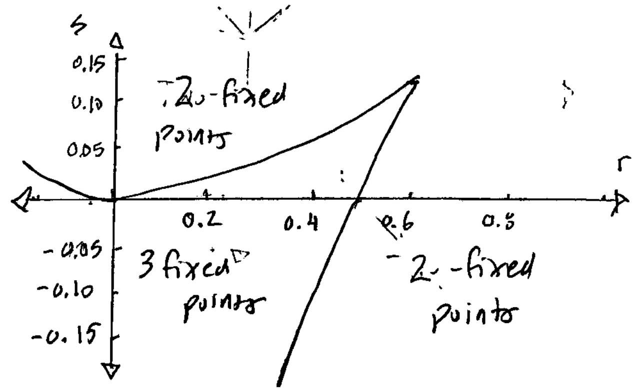
c) $g(0)=0$ ; $\frac{dg}{dt} = k_1 s_0 - k_2(0) + \frac{k_3(0)}{k_4^2+(0)^2} = k_1 s_0$ ; $g = k_1 s_0 t$ ; $g(t)$ increases with additional $s_0$.

$\boxed{\text{If } s_0 \text{ is large, then gene production has higher likelihood of rising.}}$

d) $\frac{d}{dx}\left(s - rx + \frac{x^2}{1+x^2}\right) = -r + \frac{2x}{(1+x^2)^2} = 0$ ; $r = \frac{2x}{(1+x^2)^2}$ ; $s - x\left(\frac{2x}{(1+x^2)^2}\right) + \frac{x^2}{1+x^2} = 0$

$$\boxed{s = \frac{x^2(1-x^2)}{(x^2+1)^2}}$$

e) Parametric plot of $(r,s)$




$\dot{x} = -kxy$
$\dot{y} = kxy - ly$
$\dot{z} = ly$

3.7.6. $x(t)$ = number of healthy people
$y(t)$ = number of sick people
$z(t)$ = number of dead people.

a) $\dot{N} = \dot{x} + \dot{y} + \dot{z} = -kxy + kxy - ly + ly = 0$ ; therefore $N = x + y + z$.

b) $\dot{x} = -kxy$ ; $\dot{z} = ly$ ; $\dot{x} = -kx \frac{dz}{dt}\left(\frac{1}{l}\right)$ ; $\ln x = -\frac{kz}{l} + C$ ; $x(t) = Ce^{-kz/l}$ $\boxed{= x_0 e^{-kz/l}}$

c) $\dot{z} = ly = l[N - x - z] = \boxed{l\left[N - z - x_0 e^{-kz/l}\right]}$

d) $\boxed{u = \frac{kz}{l}\ ;\ b = \frac{l}{kx_0}\ ;\ a = \frac{lN}{kx_0}\ ;\ \tau = \frac{l}{kx_0} t}$

e) $\boxed{\text{If } k, l, N, \text{ and } x_0 \text{ are positive, then both } a \text{ and } b \text{ are positive.}}$

f)

| b \ a | $\leq 1$ | $> 1$ |
|---|---|---|
| $=0$ | @$u=0$, unstable | @$u<0$, unstable |
| $>0$ | @$u=0$, unstable @$u>0$, stable | @$u<0$, unstable @$u>0$, stable |

g) $\ddot{u} = -b\dot{u} + \dot{u}e^{-u} = \boxed{0}$ ; $u = \ln b$ ; $\dot{u} = a - b\ln b + b$

$\ddot{z} = l\dot{y} = l(kxy - ly) = l(kx - l)y = \boxed{0}$ ; $y = kx$ ; $y = Ce^{(kx-l)t}$

h) $b < 1$ ; $\ddot{u} = -b + ue^{-u}$ ; $\boxed{\text{Through plotting } b \text{ and } ue^{-u} \text{ at time zero, } b > ue^{-u} \text{ ; thus } \dot{u} \text{ is increasing.}}$

$t_{peak}$ : $\ddot{u} = -b + ue^{-u} = 0$ ; $\dddot{u} = e^{-u} - ue^{-u} = 0$ ; $u = 1$

$\dot{u} = a - bu - e^{-u}$ @ $u=1$ ; $\dot{u} = a - b - \frac{1}{e}$ ; $u = (a - b - \frac{1}{e})\tau = 1$ ; $\boxed{t = \frac{1}{a - b - \frac{1}{e}}\left(\frac{l}{kx_0}\right)}$

$\lim_{u\to\infty} \dot{u} = \lim_{u\to\infty}\left[a - bu - e^{-u}\right] = -b\cdot\infty - \frac{1}{e^\infty} \boxed{= -\infty}$

i) If $b>1$; $\dot{u}=a-bu-e^{-u}$; $\ddot{u}=\ldots+u\,e^{-u}$ ... does not contain a logical maximum/minimum/inflection for an epidemic with $t_{peak}$ at zero.

j) The variable $b$ is assigned as $\frac{l}{kx_0}$. If $b=1$, then $\frac{l}{kx_0}=1$.

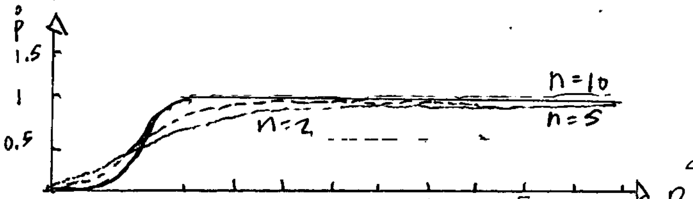
A threshold condition is when the rate of dying persons is greater than the rate of infection.

k) Autoimmunodeficiency is a disease following human immunodeficiency virus. The delayed onset from infection is time-dependent, showing that a model likely requires a time-dependent term or relationship.

$\dot{p}=\alpha+\frac{\beta p^n}{K^n+p^n}-\delta p$ 3.7.7. $\alpha$ = Basal Transcription Rate; $\beta$ = Maximal Transcription Rate

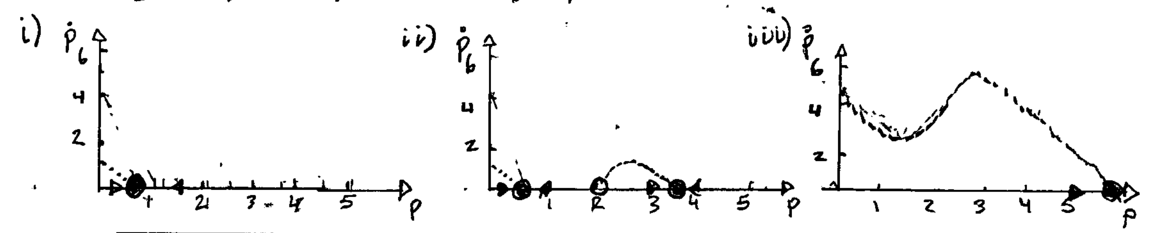
$K$ = Activation Coefficient; $\delta$ = Decay Rate of Protein.

a)



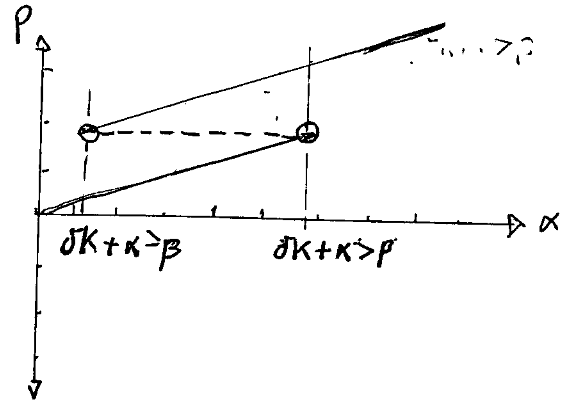

The shape of the function is a sigmoid about the point (1, 0.5) for $K=1$, $b=1$.

b)




$\boxed{\delta K-\alpha\geq\beta}$ $\boxed{\delta K-\alpha=\beta/2}$ $\boxed{\delta K-\alpha<0}$

c) Assume $\delta K>\beta$, $\alpha=-\frac{\beta p^n}{K^n+p^n}+\delta p$ at $\alpha\geq 0$




d) When protein levels are dependent upon $\alpha$, then up till $\alpha>\delta K$, protein production rate decreases until zero. While ... is above a threshold $\delta K-\beta>\alpha$, there is active production of further protein, proving concentration regions of protein production.

$\dot{A}_p = K_p S A + \beta \frac{A_p^n}{K^n + A_p^n} - K_d A_p$ ; $A$ = unphosphorylated concentration ; $A_p$ = phosphorylated concentration. ; $A_T = A + A_p$

$K_p$ = phosphorylation rate ; $K_d$ = dephosphorylation rate.

Assume $K = A_T/2$ ; $\beta = K_d A_T$

3.7.8a) $x = A_p/K$ ; $\tau = K_d t$ ; $s = K_p S/K_d$ ; $b = \beta/(K_d K)$

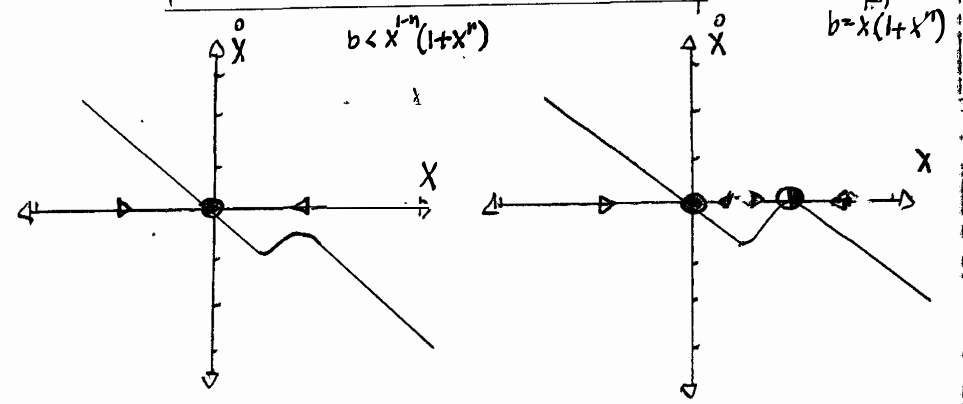
$$\cancel{K}\cancel{K_d}\frac{dx}{d\tau} = \cancel{K_d} s A + \cancel{K_d} K \cdot b \cdot \frac{\cancel{K^n} x^n}{\cancel{K^n} + \cancel{K^n} x^n} - \cancel{K_d}\cancel{K} x$$

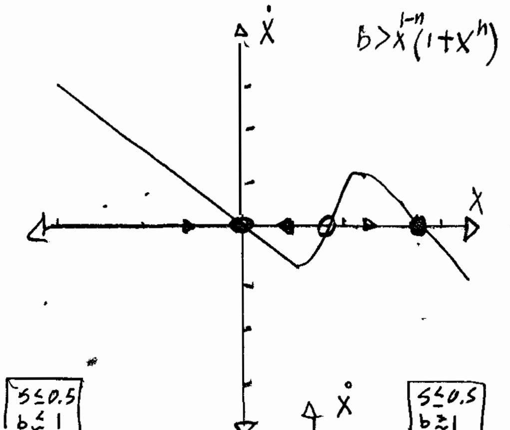
$$\frac{dx}{d\tau} = \frac{sA}{K} + b\frac{x^n}{1+x^n} - x = \frac{s(A_T - A_p)}{K} + b\frac{x^n}{1+x^n} - x$$

$$= \frac{s(2\cancel{K} - \cancel{K}x)}{\cancel{K}} + b\frac{x^n}{1+x^n} - x$$

$$\boxed{= s(2-x) + b\frac{x^n}{1+x^n} - x}$$

b) If $s=0$, then




c) If $s > 0$, then a variety of bifurcations are produced.

| b \ s | $\lesssim 0.5$ | $\gtrsim 0.5$ |
|---|---|---|
| $b \lesssim 1$ | 1, stable | 1, stable |
| $b \gtrsim 1$ | 2, stable, Half node | |
| $b \cong 1.1$ | 3, stable, unstable, stable | |
| $1.1 \le b \le 1.5$ | 2, Half node, stable | |
| $b \ge 1.5$ | 1, stable | |