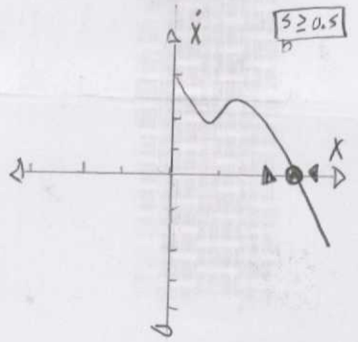
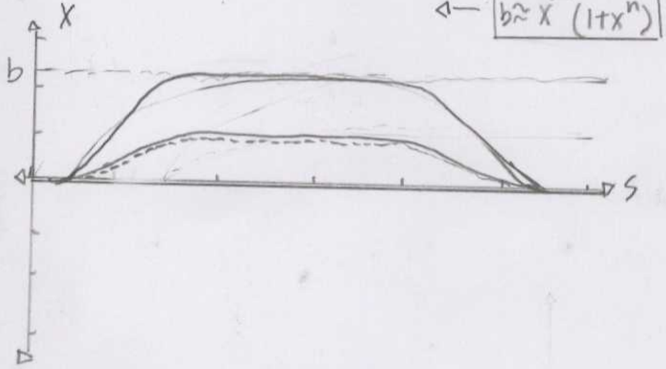


d)



Translation of bifurcation plot occurs $\leftarrow \boxed{b \approx x^{1-n}(1+x^n)} \rightarrow$ near the left box.

Incorrect

Chapter 4: Flows on the circle.

$\dot{\theta} = \sin(a\theta)$ 4.1.1. The real values of a, which give a well-defined vector field, on a circle for the function, $\dot{\theta} = \sin(a\theta)$, are fixed to $n\pi$ where $n \in \mathbb{Z}$.

$\dot{\theta} = 1 + 2\cos\theta$ 4.1.2. Fixed points: $\theta = \cos^{-1}\left(\frac{-1}{2}\right)$

$= \frac{2}{3}\pi, \frac{5}{3}\pi \cdots \left(n + \frac{2}{3}\right)\pi$ "stable"

$= \frac{4}{3}\pi, \frac{7}{3}\pi \cdots \left(n + \frac{4}{3}\right)\pi$ "unstable"

where $n \in \mathbb{Z}$

Phase Portrait

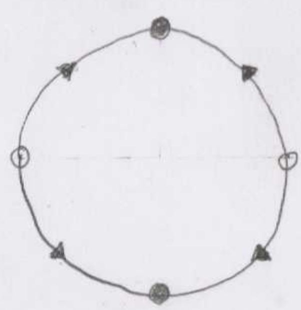
$\dot{\theta} = \sin 2\theta$ 4.1.3. Fixed Points: $\theta = \frac{\sin^{-1}(0)}{2}$

$= 0\pi, 1\pi, 2\pi \cdots (n\pi)$ "unstable"

$= \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi \cdots \left(n + \frac{1}{2}\right)\pi$ "stable"

where $n \in \mathbb{Z}$

Phase Portrait



$\dot{\theta} = \sin^3\theta$ 4.1.4 Fixed points: $\theta = \sin^{-1}(0)$

$= 0\pi, 2\pi \ldots (2n)\pi$ "unstable"

$= 1\pi, 3\pi \ldots (2n+1)\pi$ "stable"

where $n \in \mathbb{Z}$

Phase Portrait:

$\dot{\theta} = \sin\theta + \cos\theta$

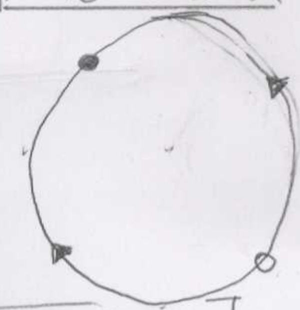
4.1.5. $\sin\theta = -\cos\theta$

$\theta = \frac{3}{4}\pi, \frac{11}{4}\pi, \frac{19}{4}\pi \ldots (n+\frac{3}{4})\pi$ "stable"

$= \frac{7}{8}\pi, \frac{15}{8}\pi, \frac{23}{8}\pi \ldots (n+\frac{7}{8})\pi$ "unstable"

where $n \in \mathbb{Z}$

$\boxed{\text{Phase Portrait}}$

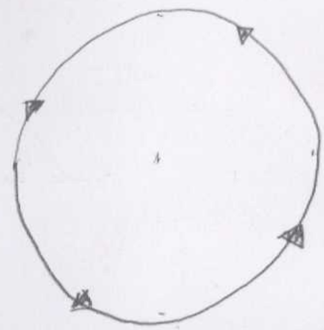


$\dot{\theta} = 3 + \cos 2\theta$

4.1.6. $\theta = \frac{\cos^{-1}(3)}{2}$

θ = undefined

$\boxed{\text{Phase Portrait}}$



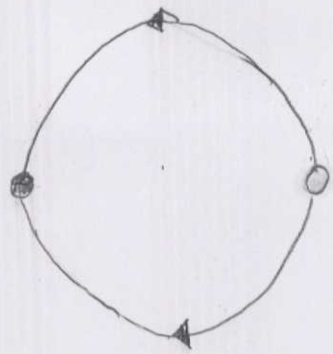
$\dot{\theta} = \sin k\theta$ where $k \in \mathbb{N}$

4.1.7. $\dot{\theta} = \sin k\theta$

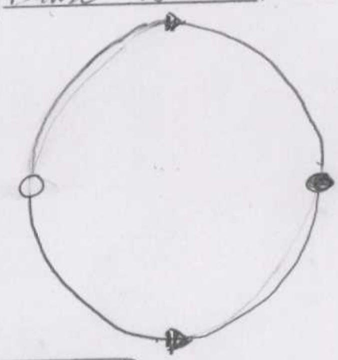
$\theta = \frac{\sin^{-1} 0}{k}$

where $k \in \mathbb{N}$

$\boxed{\text{Phase Portrait}}$



$\boxed{\text{Phase Portrait}}$



$\dot{\theta} = \cos\theta$

4.1.8. a) $\frac{-dV}{d\theta} = \frac{d\theta}{dt} = \dot{\theta} = \cos\theta \;;\; V(\theta) = -\sin\theta$

$\theta = \sin^{-1}(0)$

$= 0, 2\pi, 4\pi \ldots (2n)\pi$ "stable"

$= \pi, 3\pi, 5\pi \ldots (2n+1)\pi$ "unstable"

b) $\dot{\theta} = 1 \;;\; V(\theta) = -\theta$. The non-uniqueness of $V(\theta)$ does not imply regularity for a vector field on a circle.

c) $\dot{\theta} = f(\theta)$ has a single-valued potential for periodic functions with periodic solutions of 2π intervals.

4.1.9. Exercise 2.6.2 provided a contradiction that

$$\int_t^{t+T} f(x)\dot{x}(t)\,dt \neq \int_t^{t+T} f(x)\dot{x}(t+T)\,d(t+T)$$

Exercise 2.7.7 described a potential which could not oscillate because of the existance and uniqueness of $f(x) = \frac{d(V-c)}{dx}\,dx$.

Each of these arguments do not carry over to periodic solutions because another solution could be similar within $2n\pi$ $(n \in \mathbb{Z})$ intervals.

$T_{lap} = \frac{2\pi}{\omega_1 - \omega_2} = \left[\frac{1}{T_1} - \frac{1}{T_2}\right]^{-1}$ 4.2.1. $T_1 = 3sec$, $T_2 = 4 sec.$

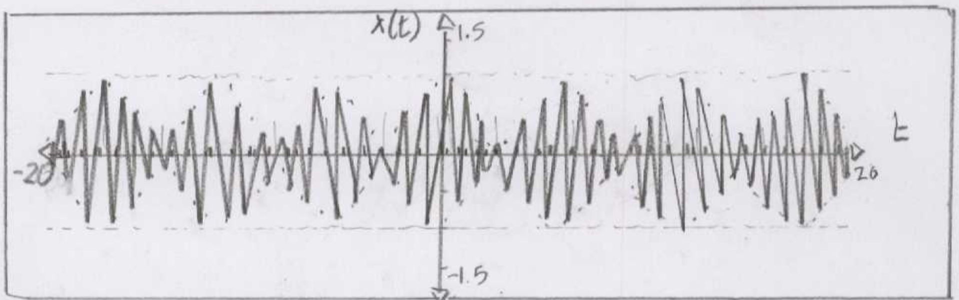
Common sense method:

# Rings	0	1	2	3	4	5
Bell #1	0	3	6	9	12	15
Bell #2	0	4	8	12	16	20

Bell #1 would ring four times while Bell #2 three before ringing together again.

Example 4.2.1 method: $T_{lap} = \left[\frac{1}{3sec} - \frac{1}{4sec}\right]^{-1} = \boxed{12 sec}$.

$x(t) = \sin 8t + \sin 9t$ 4.2.2.



a) $T_{lap} = \left[\frac{1}{8} - \frac{1}{9}\right]^{-1} = \boxed{72}$

b) $x(t) = \sin 8t + \sin 9t = \boxed{2\sin\left(\frac{17}{2}t\right)\cos\left(\frac{1}{2}t\right)}$

4.2.3. 12:00pm is when long-hand angle is equal to short-hand.

Common sense method: short-hand period $[T_1] = 12$ hour

long-hand period $[T_2] = 1$ hour

$T_{lap} = \left[\frac{1}{1} - \frac{1}{12}\right]^{-1} = \frac{12}{11}$ hour

Alternative method: $x(t) = \sin(12t) + \sin(t) = 2\sin\left(\frac{11}{2}t\right)\cos\left(\frac{13}{2}t\right)$

sum frequency

Difference frequency

$T_{bottleneck} = \int_{-\infty}^{\infty} \frac{dx}{r + x^2}$ 4.3.1. $x = \sqrt{r}\tan\theta$; $1 + \tan^2\theta = \sec^2\theta$

$$T = \int_{-\infty}^{\infty} \frac{dx}{r + r^2\tan^2\theta} = \frac{1}{r}\int_{-\infty}^{\infty} \frac{dx}{1 + \tan^2\theta} = \frac{1}{r}\int_{-\infty}^{\infty} \frac{dx}{\sec^2\theta}$$

$$= \frac{1}{r}\int_{-\infty}^{\infty} \frac{\sqrt{r}\sec^2\theta \, d\theta}{\sec^2\theta} = \frac{1}{\sqrt{r}}\,\theta\Big|_{-\infty}^{\infty} = \frac{1}{\sqrt{r}}\arctan\left(\frac{x}{\sqrt{r}}\right)\Big|_{-\infty}^{\infty} = \boxed{\frac{\pi}{\sqrt{r}}}$$

$T = \int_{-\pi}^{\pi} \frac{d\theta}{\omega - a\sin\theta}$ 4.3.2.

a) $u = \tan\frac{\theta}{2}$; $du = \frac{1}{2}\sec^2\frac{\theta}{2}d\theta$; $d\theta = \frac{2du}{\sec^2\frac{\theta}{2}}$ $\boxed{= \frac{2 \cdot du}{\sec^2\left[\frac{\arctan^{-1}(2u)}{2}\right]}}$

b)

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}} = \boxed{\frac{2u}{1+u^2}}$

c) $\lim_{\theta \to \pm\pi} \sin\theta = \boxed{\lim_{u \to ?} \frac{2u}{1+u^2}}$, for $u \neq 0$, $u \to \pm\infty$

d) $T = \frac{1}{\omega}\int_{-\infty}^{\infty} \frac{1}{1 - \frac{a}{\omega}\left[\frac{2u}{1+u^2}\right]} \cdot \frac{2 \cdot du}{\sec^2\left[\frac{\arctan^{-1}(2u)}{2}\right]}$ $\boxed{= \frac{1}{\omega}\int_{-\infty}^{\infty} \frac{1}{1 - \frac{a}{\omega}\left[\frac{2u}{1+u^2}\right]} \cdot \frac{2du}{\left[\frac{1}{1+u^2} - \frac{u^2}{1+u^2}\right]^2}}$

e) $T = \frac{1}{\omega}\int_{-\infty}^{\infty} \frac{1}{1 - \frac{a}{\omega}[}$

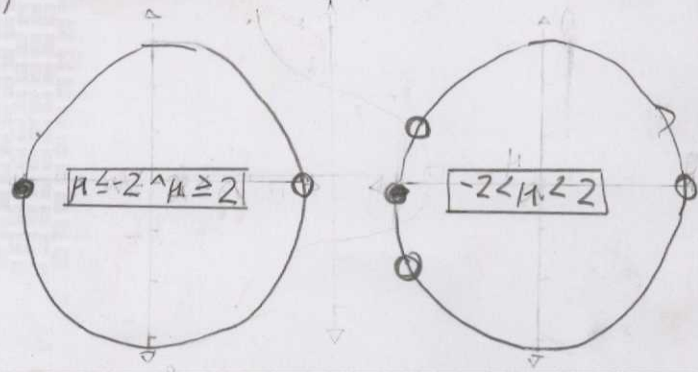
$1 - u^2$

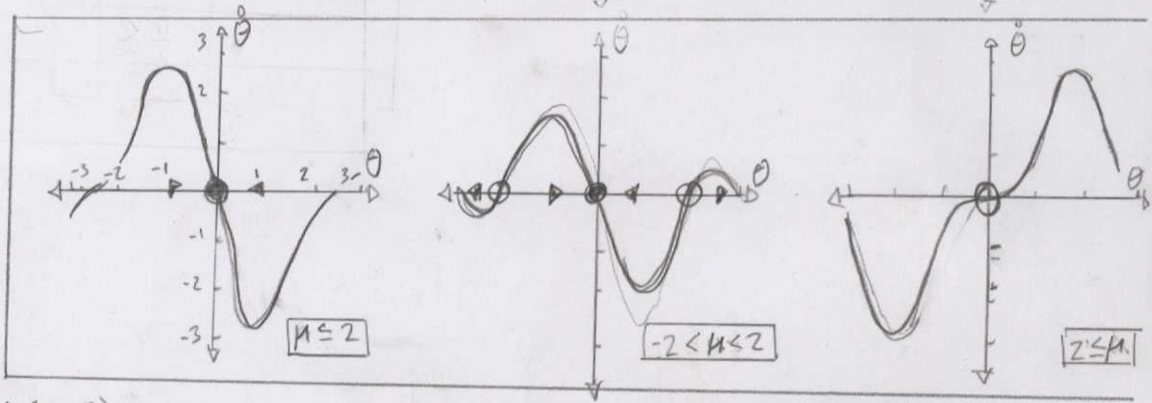
$\frac{1}{(1+u^2)^2} - \frac{2u^2}{(1+u^2)^2} + \frac{u^4}{(1+u^2)^2}$

$\dot{\theta} = \mu \sin\theta - \sin 2\theta$ 4.3.3. $\mu = \frac{\sin 2\theta}{\sin\theta} = 2\cos x$; $x = \cos^{-1}\left(\frac{\mu}{2}\right)$

$\boxed{\text{Phase Portrait}}$ Saddle-node Bifurcation

μ	Bifurcations
≤ -2	Two
$-2 \leq \mu \leq 2$	Four
≥ 2	Two





$\dot{\theta} = \frac{\sin\theta}{\mu + \cos\theta}$

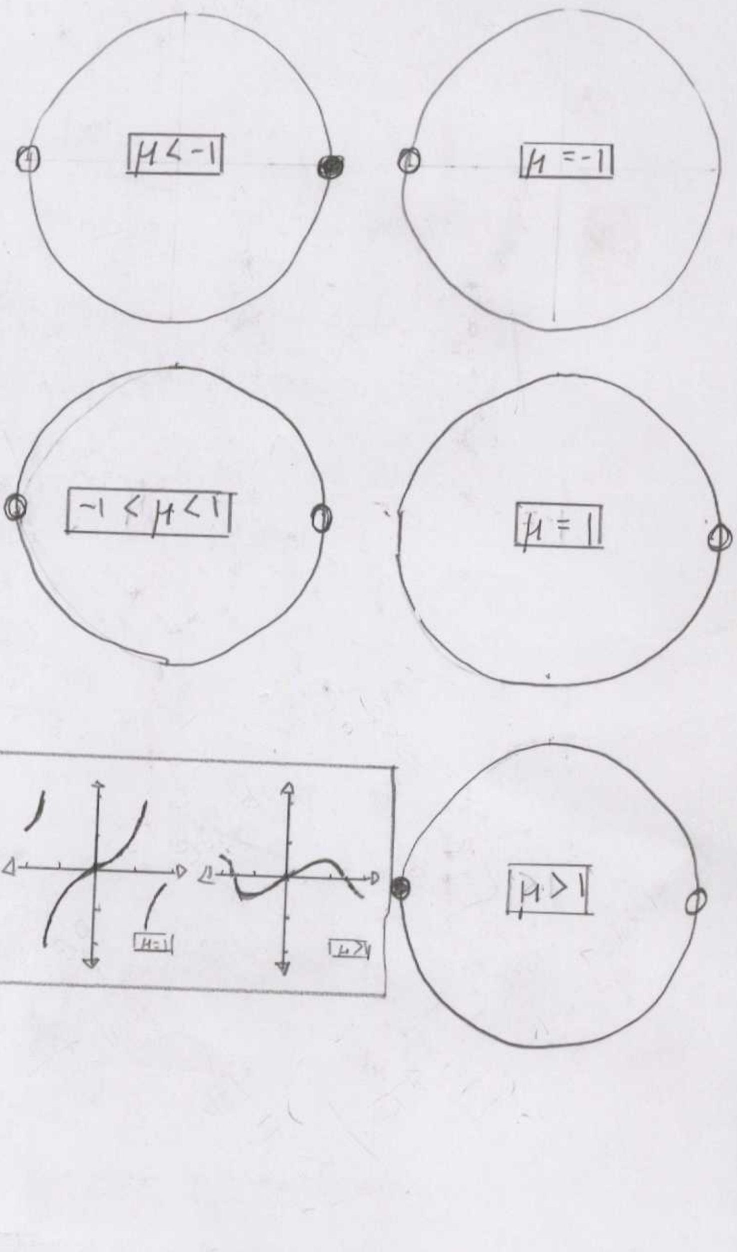
4.3.4. $0(\mu + \cos\theta) = \sin\theta$

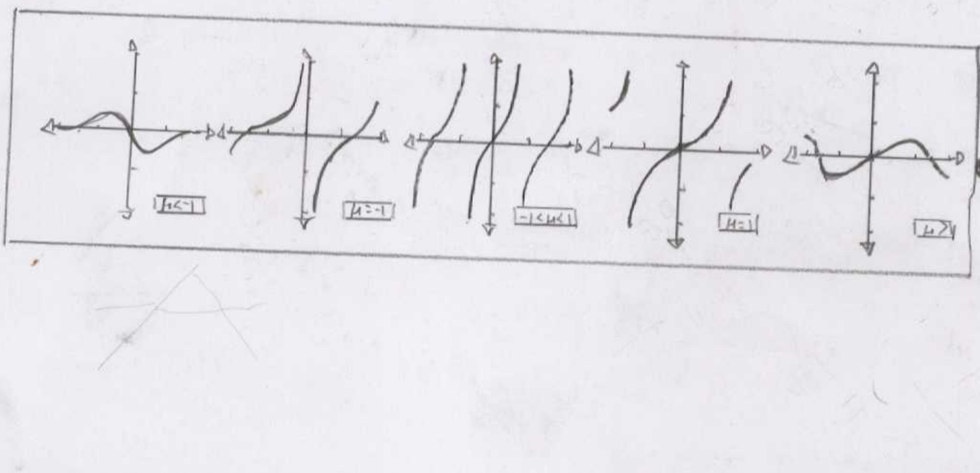
$\mu = -\cos\theta$

$\theta = \cos^{-1}(-\mu)$

$\boxed{\text{Phase Portrait}}$ Transcritical Bifurcation

μ	Bifurcations
< -1	Two
$= -1$	one
$-1 < \mu < 1$	Two
$= 1$	one
> 1	Two





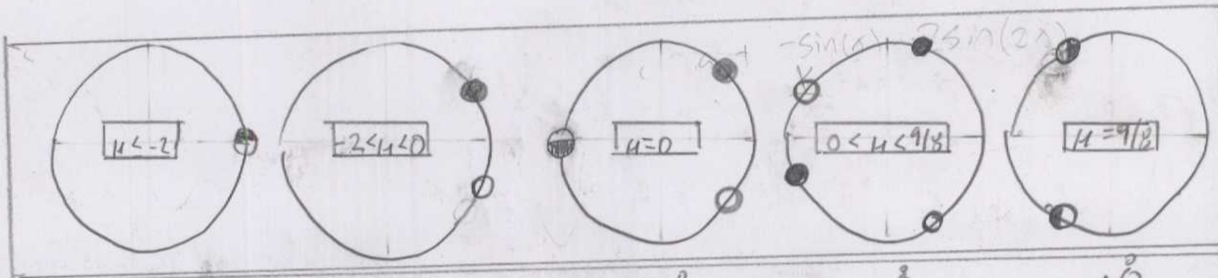
$\dot{\theta} = \mu + \cos\theta + \cos 2\theta$ 4.3.5. $0 = (\mu - 1) + \cos\theta + 2\cos^2\theta$

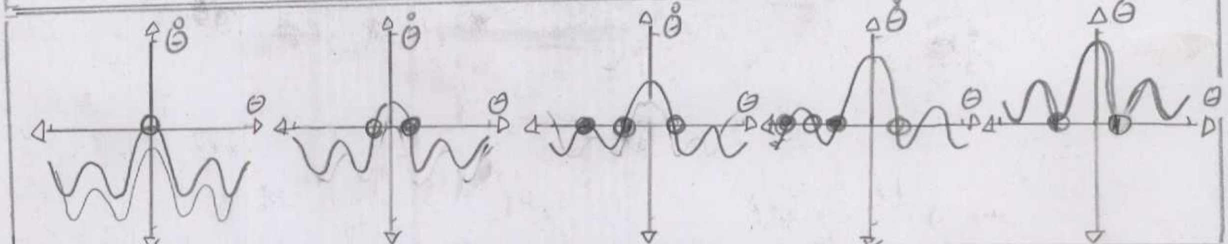
$$\cos\theta = \frac{-1 \pm \sqrt{1 - 4(2)(\mu - 1)}}{2(2)}$$

$$= \frac{-1 \pm \sqrt{9 - 8\mu}}{4}$$

$$\theta = \cos^{-1}\left(\frac{-1 \pm \sqrt{9 - 8\mu}}{4}\right)$$

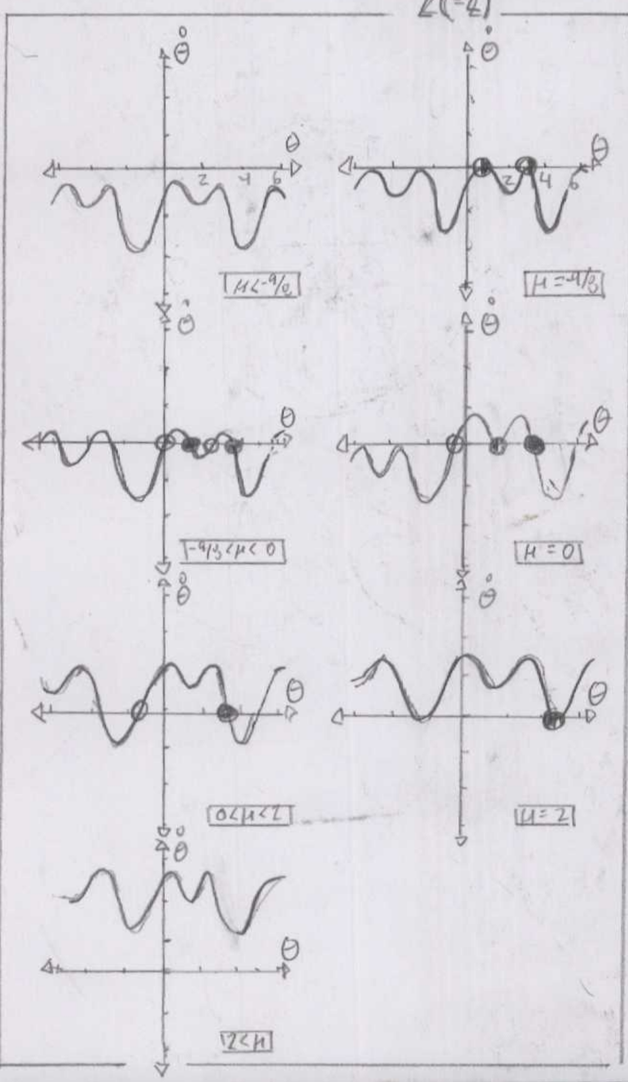
μ	Bifurcations
≤ -2	One
$-2 < \mu < 0$	Two
$= 0$	Three
$0 < \mu < 9/8$	Four
$9/8$	Two



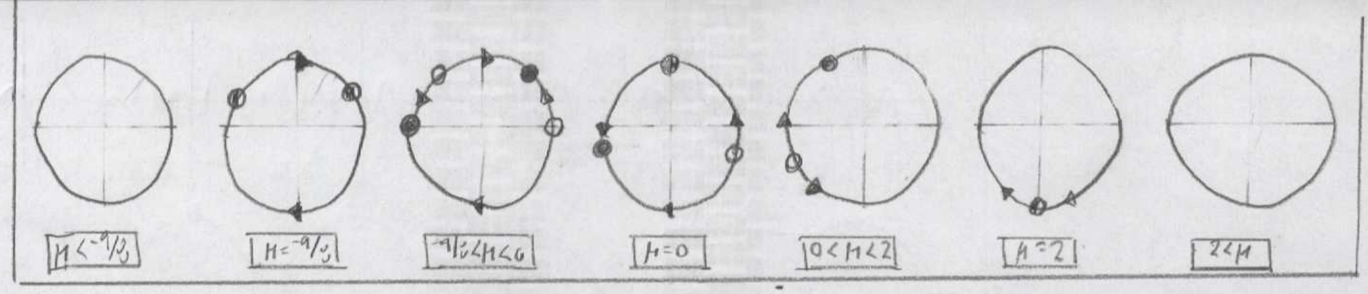


$\dot{\theta} = \mu + \sin\theta + \cos 2\theta$ 4.3.6. $\dot{\theta} = (\mu + 1) + \sin\theta - 2\sin^2\theta$

$$\sin\theta = \frac{-1 \pm \sqrt{1 - 4(-2)(\mu + 1)}}{2(-2)}$$



θ	μ	Bifurcations
NA	$< -9/8$	Zero
$\arcsin(1/4)$, $\arcsin(\pi - 1/4)$	$= -9/8$	Two
$\arcsin(1/4)$ to 2π	$-9/8 < \mu < 0$	Four
$\pi/2$, $\frac{7}{6}\pi$, $\frac{11}{6}\pi$	$= 0$	Three
$\frac{7}{6}\pi < \theta < \frac{11}{6}\pi$	$0 < \mu < 2$	Two
$3\pi/2$	$= 2$	One
NA	$2 < \mu$	Zero



$\dot{\theta} = \dfrac{\sin\theta}{\mu + \sin\theta}$

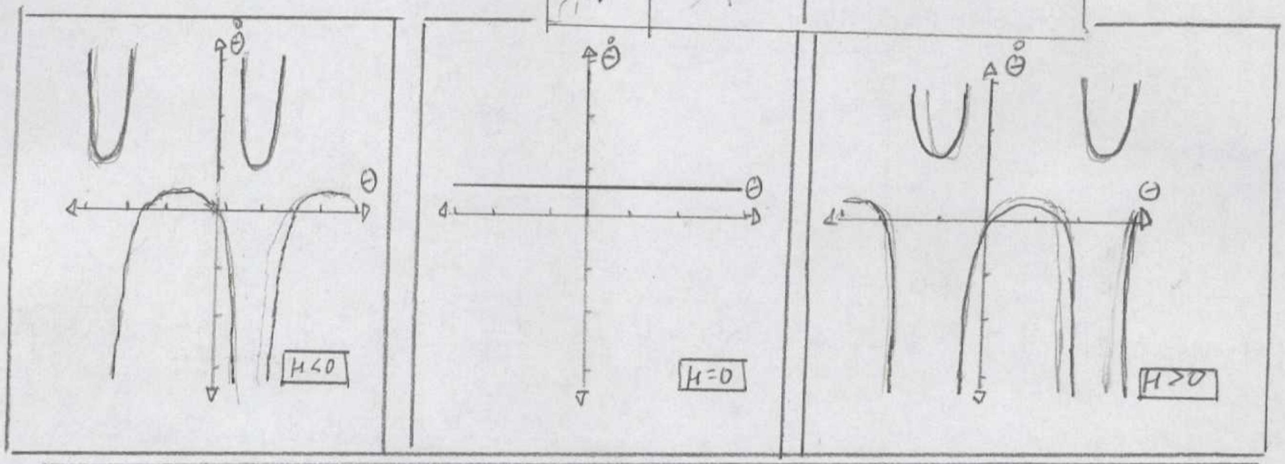
4.3.7 $0 = \dfrac{\sin\theta}{\mu + \sin\theta}$

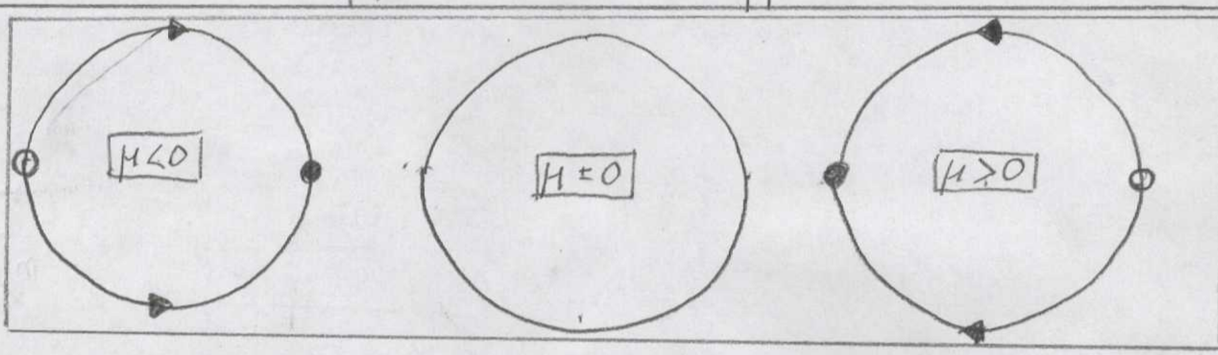
$0 = \mu + \sin\theta$

$\mu = -\sin\theta$

$\theta = \sin^{-1}(-\mu)$

Θ	μ	Bifurcations
0, π	<0	Two
N/A	$=0$	Zero
0, π	>0	Two





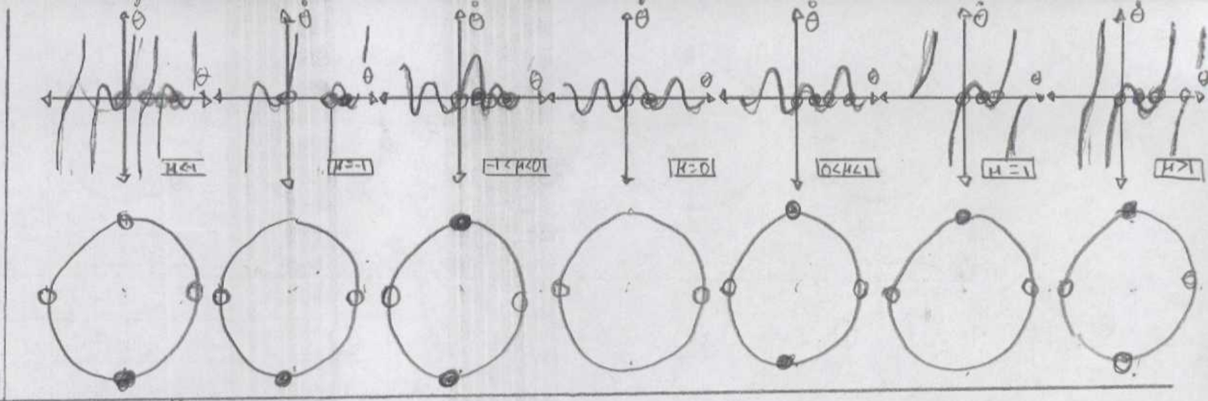
$\dot{\theta} = \dfrac{\sin 2\theta}{1 + \mu\sin\theta}$

4.3.8, $0 = \dfrac{\sin 2\theta}{1 + \mu\sin\theta}$

$0 = 1 + \mu\sin\theta$

$\boxed{\theta = \sin^{-1}\left(\frac{-1}{\mu}\right)}$

θ	μ	Bifurcations
0, $\pi/2$, π, $3\pi/2$	<-1	Four
0, π, $3\pi/2$	$=-1$	Three
0, $\pi/2$, π, $3\pi/2$	$-1<\mu<0$	Four
0, $\pi/2$, π	$=0$	Three
0, $\pi/2$, π, $3\pi/2$	$0<\mu<1$	Four
0, $\pi/2$, π	$=1$	Three
0, $\pi/2$, π, $3\pi/2$	>1	Four



$r^{a-b}\frac{du}{d\tau}=r+r^{2a}u^2$ 4.3.9. $T_{bottleneck} \sim O(r^{-1/2})$

a) $O(r^a)$; $x=r^a u$, where $u \sim O(1)$. $t=r^b \tau$, with $\tau \sim O(1)$

$$\dot{x}=r+x^2=r+(r^a u)^2=r+r^{2a}u^2 ; \; r^{a-b}\frac{du}{d\tau}=\boxed{r^1+r^{2a}\cdot u^2}$$

b) $\boxed{r^{a-b}=r=r^{2a} \; ; \; a=\frac{1}{2} ; b=-1/2}$

$\dot{x}=r+x^{2n}$ 4.3.10. $x=r^a u$; $t=r^b\tau$; $r^{a-b}\frac{du}{d\tau}=r+r^{2an}u^{2an}$; $a=\frac{1}{2}$; $b=-\frac{1}{2}$;

$$\frac{du}{d\tau}=1+r^n u^n ; \; r^b\tau=$$

$mL^2\ddot{\theta}+b\dot{\theta}+mgL\sin\theta=\Gamma$ 4.4.1. $\ddot{\theta}=0$ -or- $\ddot{\theta}<<1$; $t=T\tau$; $\frac{mL^2}{T^2}\frac{d^2\theta}{d^2\tau}+\frac{b}{T}\frac{d\theta}{d\tau}+mgL\sin\theta=\Gamma$

$$\frac{L^2}{gT^2}\frac{d^2\theta}{d^2\tau}+\frac{b}{mgL}\frac{d\theta}{d\tau}+\sin\theta=\frac{\Gamma}{mgL}$$

$$\frac{b}{mgL\Gamma}=1 ; \; T=\frac{b}{mgL}$$

$$\frac{m^2gL^3}{b^2}\frac{d^2\theta}{d\tau^2}+\frac{d\theta}{d\tau}+\sin(\theta)=\frac{\Gamma}{mgL}$$

$$\boxed{m^2gL^3<<b^2}$$

$\theta'=\gamma-\sin\theta$ 4.4.2 $\int\frac{d\theta}{w-\sin\theta}=dt$; $t=-\int\frac{d\theta}{\frac{2\tan(\frac{\theta}{2})}{\tan^2(\frac{\theta}{2})+1}-a}=-2\int_{-\pi}^{\pi}\frac{du}{au^2-2u+a}$; where $u=\tan(\frac{\theta}{2})$

$du=\frac{\sec^2(\frac{\theta}{2})d\theta}{2}$

$d\theta=\frac{1}{u^2+1}$

$$=-2\int_{-\pi}^{\pi}\frac{du}{(\sqrt{a}u-\frac{1}{\sqrt{a}})^2+a-\frac{1}{a}}$$; where $v=\frac{au-1}{\sqrt{a}\sqrt{a-1/a}}$

$du=\frac{\sqrt{a-1/a}}{\sqrt{a}}dv$

$$\boxed{=\frac{-2}{\sqrt{a}\sqrt{a-\frac{1}{a}}}\int_{-\pi}^{\pi}\frac{dv}{v^2+1}=\frac{2\arctan\left(\frac{a\tan(\frac{\theta}{2})-1}{\sqrt{a}\sqrt{a-1/a}}\right)}{\sqrt{a}\sqrt{a-1/a}}+C}$$

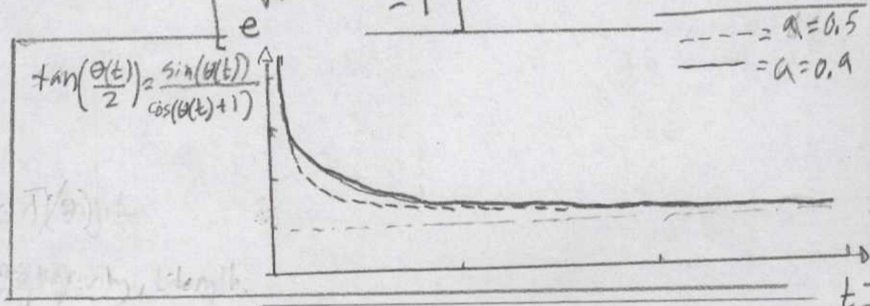
$$t = \ln\left(\frac{\left|\frac{a\sin(\theta)}{\cos\theta+1} + \frac{-2\sqrt{1-a^2}-2}{2}\right|}{\left|\frac{a\sin(\theta)}{\cos(\theta)+1} + \frac{2\sqrt{1-a^2}-2}{2}\right|}\right) \Big/ \sqrt{1-a^2}$$

$$e^{\sqrt{1-a^2}\cdot t} \circ \left|\frac{a\sin(\theta)}{\cos(\theta)+1} + \frac{2\sqrt{1-a^2}-2}{2}\right| = \left|\frac{a\sin(\theta)}{\cos(\theta)+1} - \frac{2\sqrt{1-a^2}-2}{2}\right|$$

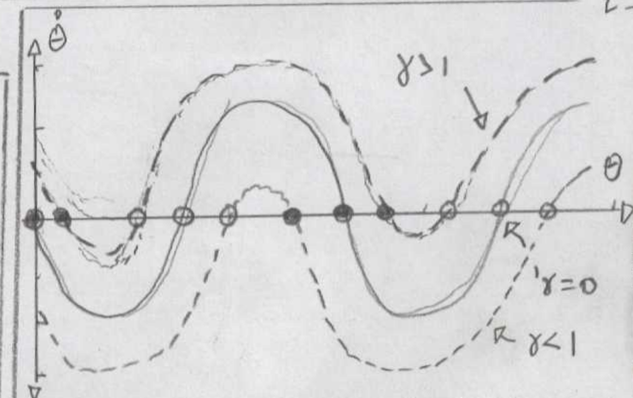
$$\frac{a\sin(\theta)}{\cos(\theta)+1}\left[e^{\sqrt{1-a^2}\,t} - 1\right] = \frac{-2\sqrt{1-a^2}-2}{2}\left[e^{\sqrt{1-a^2}\cdot t} + 1\right]$$

$$\frac{\sin(\theta)}{\cos(\theta)+1} = \frac{(-2\sqrt{1-a^2}-2)}{2a} \circ \frac{\left[e^{\sqrt{1-a^2}\,t} + 1\right]}{\left[e^{\sqrt{1-a^2}\,t} - 1\right]}$$

A graph of $\sin(\theta(t))$ vs t represented as $\tan\left(\frac{\theta(t)}{2}\right)$.



4.4.3. $\ddot{\theta} = \gamma - \sin(\theta(t))$



| Similar to Question 4.4.3, γ is the relationship of T/mgL. If torque(T) is greater than mass x gravity x length, then motion is directed; more over, if torque(T) is zero, then there is no direction. |

$b\dot{\theta} + mgL\sin\theta = T - k\theta$ 4.4.4.

a. $\boxed{\theta = \{0, \frac{\pi k}{T}, \frac{2\pi k}{T}, \ldots, \frac{n\pi k}{T}\}}$

b. $b\dot{\theta} + mgL\sin\theta = T - k\theta$

$\frac{b}{mgL}\dot{\theta} + \sin\theta = \frac{T - k\theta}{mgL}$; if $\tau = \frac{mgL\,t}{b}$; $\boxed{\gamma = \frac{T}{mgL} ; \mu = +\frac{k}{mgL}}$

$\theta' + \sin\theta = \gamma + \mu\theta$; $\theta' = \gamma - \mu\theta - \sin\theta$

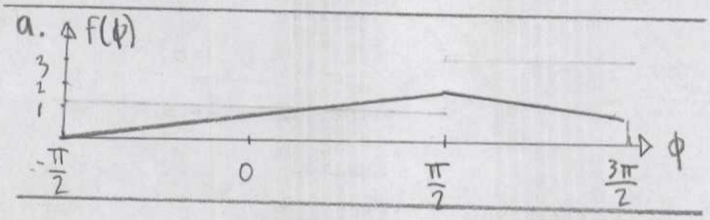
c. As the pendulum angle increases, the dampening lowers rate of angle change (θ').

d. As K is varied from 0 to ∞, then θ' is equal to zero at $\frac{\gamma - \sin\theta}{\mu}$. The bifurcation type is subcritical.

$\dot{\Theta} = \omega + A f(\Theta - \theta)$

$\dot{\Theta} = \Omega$

4.5.1. $f(\phi) = \begin{cases} \phi & -\frac{\pi}{2} \le \phi \le \frac{\pi}{2} \\ \pi - \phi & \frac{\pi}{2} \le \phi \le \frac{3\pi}{2} \end{cases}$

a. 

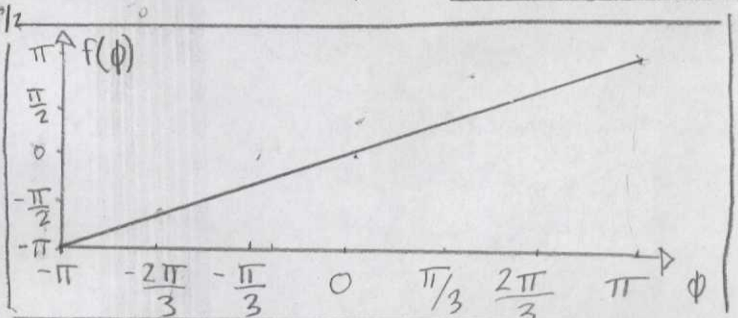
b. Range of Entrainment: $\boxed{-\frac{\pi}{2} \le f(\phi) \le \frac{\pi}{2}}$

c. $\phi^* = \dot{\Theta} - \dot{\theta} = \Omega - \omega - A[\frac{\pi}{2}$ - $\boxed{|\Omega - \omega| \le \pi/2}$

d. $T_{drift} = \int_{0/2}^{2\pi} \frac{d\phi}{\Omega - \omega - A[\pi - \phi]} = \frac{1}{A}\int_{-\pi/2}^{\frac{3\pi}{2}} \frac{du}{u} = \frac{2}{A} \ln\left(\frac{\Omega - \omega - A\pi/2\pi}{\Omega - \omega + A\pi/2}\right) = \boxed{\frac{2}{A}\ln\left(\frac{\Omega - \omega - A\pi/2}{\Omega - \omega + A\pi/2}\right)}$

$\dot{\theta} = \omega + A f(\Theta - \theta)$ 4.5.2

$f(\phi) = \phi \quad -\pi < \phi < \pi$



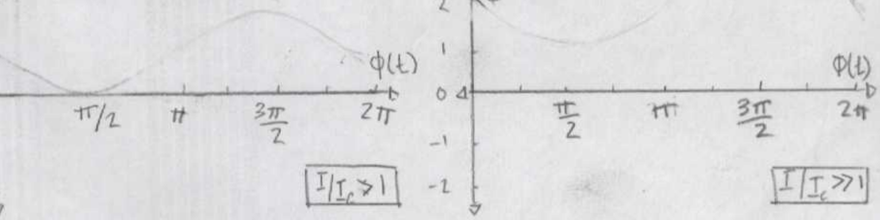
Range of Entrainment: $-\pi < f(\phi) < \pi$

$\phi^* = \dot{\Theta} - \dot{\theta} = \Omega - \omega - A[\pi]$; $|\Omega - \omega| \le \pi$

$T_{drift} = \int_{0}^{2\pi} \frac{d\phi}{\Omega - \omega - A[\phi]} = \frac{1}{A}\int_{\pi}^{\pi} \frac{du}{u} = \boxed{\frac{1}{A} \ln\left(\frac{\Omega - \omega + A[\pi]}{\Omega - \omega - A[\pi]}\right)}$

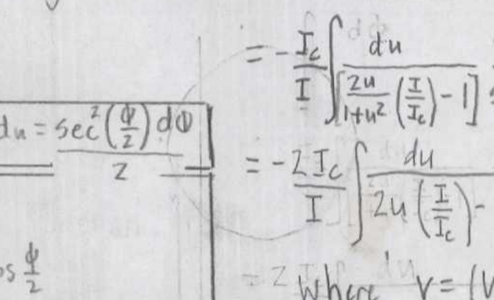
$\dot{\theta} = \mu + \sin\theta$ 4.5.3. a) The 'rest state' and 'threshold' are described by the nuerons ability to remain at rest or fire, respectively.

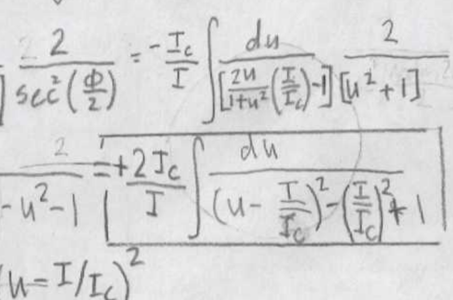
b) $V(t) = \cos\theta(t)$



4.6.1 $\beta = 0$

a. $\phi' = \frac{I}{I_c} - \sin\phi(t)$

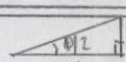




$t = \int \frac{d\phi}{I/I_c - \sin\phi}$

$= -\int \frac{d\phi}{\sin\phi - I/I_c}$

Method #1: $u = \tan(\frac{\phi}{2})$; $du = \frac{\sec^2(\frac{\phi}{2})\, d\phi}{2}$

Method #2: 

$\sin\phi = 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}$

$= 2 \frac{u}{\sqrt{1+u^2}} \frac{1}{\sqrt{1+u^2}} = \frac{2u}{1+u^2}$

$= -\frac{I_c}{I}\int \frac{du}{\frac{2u}{1+u^2}\left(\frac{I}{I_c}\right) - 1} \frac{2}{\sec^2(\frac{\phi}{2})} = -\frac{I_c}{I}\int \frac{du}{\left[\frac{2u}{1+u^2}\left(\frac{I}{I_c}\right) - 1\right]} \frac{2}{[u^2+1]}$

$= -\frac{2I_c}{I}\int \frac{du}{2u\left(\frac{I}{I_c}\right) - u^2 - 1} = \boxed{+\frac{2I_c}{I}\int \frac{du}{\left(u - \frac{I}{I_c}\right)^2 - \left(\frac{I}{I_c}\right)^2 + 1}}$

where $v = \frac{(u - I/I_c)^2}{\sqrt{1 - I/I_c}}$

$$= 2\frac{I}{I_c}\int \frac{\sqrt{1-(I/I_c)^2}}{(1-(I/I_c)^2)v^2-(I/I_c)^2+1}\,dv = \frac{2I/I_c}{\sqrt{1-(I/I_c)^2}}\int\frac{dv}{v^2+1} = \frac{1}{\sqrt{1-(I/I_c)^2}}\arctan(v)$$

$$= 2\left(\frac{I}{I_c}\right)\frac{\arctan\left(\frac{u-I/I_c}{\sqrt{1-(I/I_c)^2}}\right)}{\sqrt{1-(I/I_c)^2}} = 2\,\frac{I}{I_c}\,\frac{\arctan\left(\frac{\tan\frac{\phi}{2}-I/I_c}{\sqrt{1-(I/I_c)^2}}\right)}{\sqrt{1-(I/I_c)^2}} + C \text{ ; where } C=0$$

$$t = \frac{I}{I_c}\ln\left(\frac{1-\frac{\tan\phi/2+I/I_c}{\sqrt{1-(I/I_c)^2}}}{1+\frac{\tan\phi/2-I/I_c}{\sqrt{1-(I/I_c)^2}}}\right)\Big/\sqrt{(I/I_c)^2-1}$$

$$e^{(I_c/I)\sqrt{(I/I_c)^2-1}\cdot t}\circ\left(1+\frac{\tan\phi/2-(I/I_c)}{\sqrt{1-(I/I_c)^2}}\right) = 1-\frac{\tan\phi/2+I/I_c}{\sqrt{1-(I/I_c)^2}}$$

$$\frac{\tan\phi/2-(I/I_c)}{\sqrt{1-(I/I_c)^2}}\left[e^{\frac{I_c}{I}\sqrt{(I/I_c)^2-1}\cdot t}+1\right] = 1-e^{\frac{I_c}{I}\sqrt{(I/I_c)^2-1}\circ t}$$

$$\sin\left(\frac{\phi}{2}\right) = \cos\left(\frac{\phi}{2}\right)\left[\frac{\left(1-e^{I_c/I\sqrt{(I/I_c)^2-1}\cdot t}\right)}{\left(1+e^{I_c/I\sqrt{(I/I_c)^2-1}\circ t}\right)}\sqrt{1-(I/I_c)^2}+I/I_c\right]$$

$$\sin\phi = 2\cos^2\left(\frac{\phi}{2}\right)\left[\sqrt{1-(I/I_c)^2}\,\frac{\left(1-e^{I_c/I\sqrt{(I/I_c)^2-1}\cdot t}\right)}{\left(1+e^{I_c/I\sqrt{(I/I_c)^2-1}\cdot t}\right)}+I/I_c\right]$$

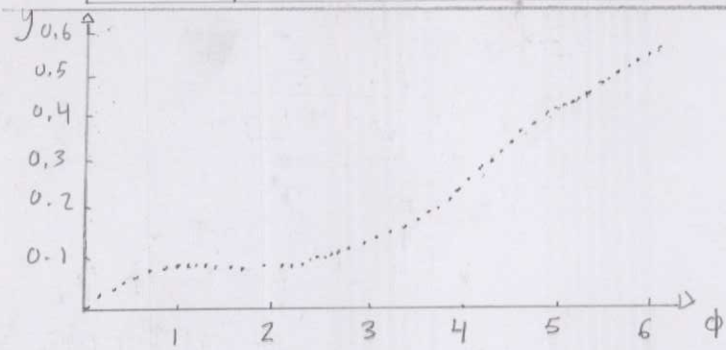
$\phi' = \frac{I}{I_c} - \sin\phi$

4.6.2. Numerical Integration : Runge-Kutta 4th Order

ϕ	k_1	k_2	k_3	k_4
0.0	$\Delta h \cdot f(\phi)$	$\Delta h \cdot f(\phi + \frac{\Delta h}{2})$	$\Delta h \cdot f(\phi + \frac{\Delta h}{2})$	$\Delta h \cdot f(\phi + \Delta h)$
...	...	...	...	...
6.0	0.1379	0.1331	0.1331	0.1282

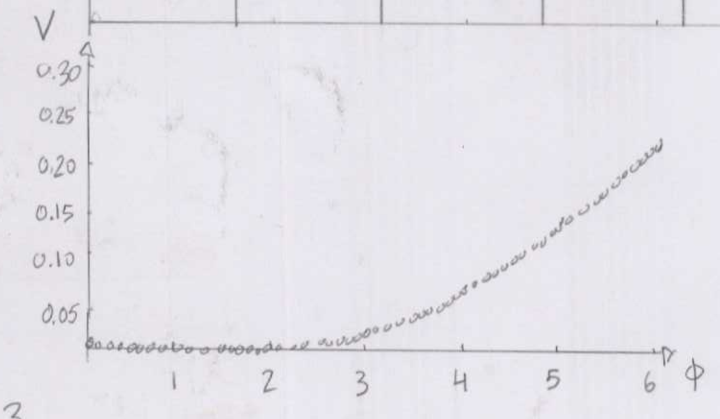
$y_{n+1} = y_n + \frac{\Delta h}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

where $\Delta h = 0.1$



ϕ	k_1	k_2	k_3	k_4
0.0	$\Delta t \cdot \Delta h f(\phi)$	$\Delta t \Delta h \cdot f(\phi + \frac{\Delta h}{2})$	$\Delta t \cdot \Delta h \cdot f(\phi + \frac{\Delta h}{2})$	$\Delta t \cdot \Delta h f(\phi + \Delta h)$
...	...	...	...	...
6.0	0.63	-0.0103	-0.0103	-0.0098

$V_{n+1} = V_n + \frac{\Delta h}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

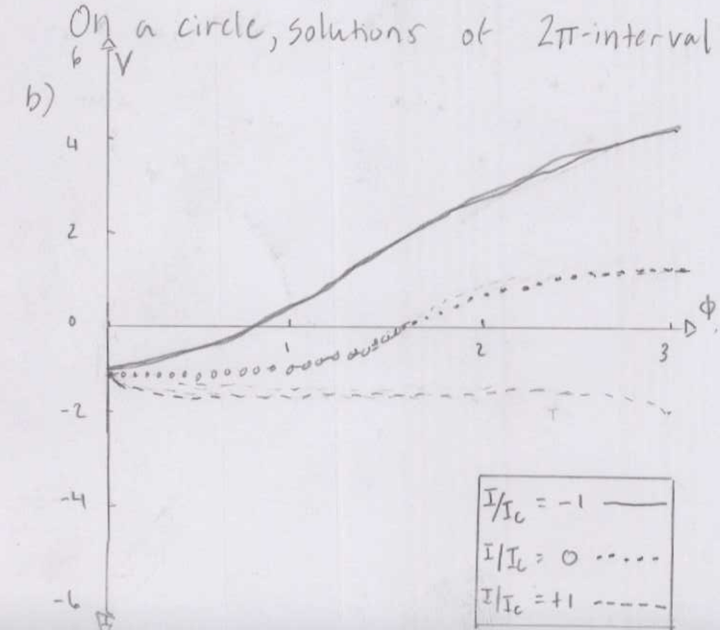


4.6.3.

a) $V = -\dot{x}\,dx$; $V = -\dot{\phi}\,d\phi = -[I/I_c - \sin\phi]\,d\phi = -[\cos\phi + \frac{I}{I_c}\phi]$

On a circle, solutions of 2π-interval exist : $\phi = \arcsin(I/I_c)$

b)

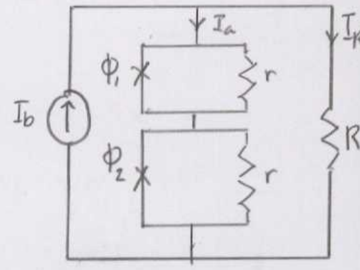


c) The increase of current (I) lowers the potential (V) per 2π oscillation.

$I_a = I_c \sin\phi_1 + V_1/r$

$I_b = I_c \sin\phi_K + \frac{\hbar}{2er}\dot{\phi}_K + \frac{\hbar}{2eR}(\dot{\phi}_1 + \dot{\phi}_2)$

4.6.4. a) $I_b = I_a + I_R$



b) Kirchoff's Law: Parallel Circuit

$$I_a = I_{a1} + I_{ar} \quad ; \quad I_a = I_{a2} + I_{ar}$$
$$= I_a \sin\phi_1 + \frac{V_1}{r} \quad ; \quad \boxed{= I_a \sin\phi_2 + \frac{V_2}{r}}$$

c) If $k = 1, 2$, then $\boxed{V_k = \begin{cases} \frac{\hbar}{2eR}\dot{\phi}_1 \\ \frac{\hbar}{2eR}\dot{\phi}_2 \end{cases}}$

d) $I_b = I_{a1} + I_{1ar} + I_{2ar} + I_R = I_c \sin\phi_k + \frac{\hbar}{2eR}\dot{\phi}_1 + \frac{\hbar}{2eR}\dot{\phi}_2 + \frac{V_k}{R}$

$$\boxed{= I_c \sin\phi_k + \frac{\hbar}{2eR}\left[\dot{\phi}_1 + \dot{\phi}_2\right] + \frac{\hbar}{2eR}\dot{\phi}_k}$$

where $k = 1, 2$.

e) $I_b = I_c \sin(\phi_K) + \frac{\hbar}{2er}\dot{\phi}_K + \frac{\hbar}{2eR}\sum_{i=1}^{N}\dot{\phi}_i$

$= I_c \sin(\phi_K) + \frac{h}{2er}\dot{\phi}_K + \frac{Nr}{R}\left(I_c \sin(\phi_K) + \frac{\hbar}{2er}\dot{\phi}_K - I_c\sum_{i=1}^{N}\sin(\phi_i)\right)$

$= \left(1 + \frac{Nr}{R}\right) I_c \sin(\phi_K) + \left(\frac{h}{2er} + \frac{N\hbar}{2eR}\right)\dot{\phi}_K - \frac{r}{R} I_c \sum_{i=1}^{2}\sin(\phi_j)$

$\hbar\left(\frac{1}{2er} + \frac{N}{2eR}\right)\dot{\phi}_K = I_b - \left(1 + \frac{Nr}{R}\right) I_c \sin(\phi_K) + \frac{r}{R} I_c \sum_{i=1}^{2}\sin(\phi_j)$

$\frac{\hbar(R + Nr)}{2erR}\dot{\phi}_K = I_b - \frac{R + Nr}{R} I_c \sin(\phi_K) + \frac{r}{R} I_c \sum_{i=1}^{2}\sin(\phi_j)$

$\dot{\phi}_K = \frac{2erR I_b}{\hbar(R + Nr)} - \frac{2er}{\hbar} I_c \sin(\phi_K) + \frac{2er^2}{\hbar(R + Nr)} I_c \sum_{i=1}^{2}\sin(\phi_j)$

$$\boxed{\Omega = \frac{2erR I_b}{\hbar(R + Nr)} \; ; \; a = \frac{-2er}{\hbar} I_c \; ; \; K = \frac{2er^2 I_c}{\hbar(R + Nr)}}$$

4.6.5 $I_b = I_c \sin(\phi_K) + \frac{\hbar}{2er}\dot{\phi}_K + \frac{\hbar}{2eR}\sum\dot{\phi}_i$

$= I_c \sin(\phi_K) + \frac{\hbar}{2er}\dot{\phi}_k + \frac{Nr}{R}\left(I_c \sin(\phi_K) + \frac{\hbar}{2er}\dot{\phi}_K - I_c\sum_{i=1}^{N}\sin(\phi_i)\right)$

$= \left(1 + \frac{Nr}{R}\right) I_c \sin\phi_K + \left(\frac{h}{2er} + \frac{N\hbar}{2eR}\right)\dot{\phi}_K - \frac{r}{R} I_c \sum_{i=1}^{2}\sin(\phi_j)$

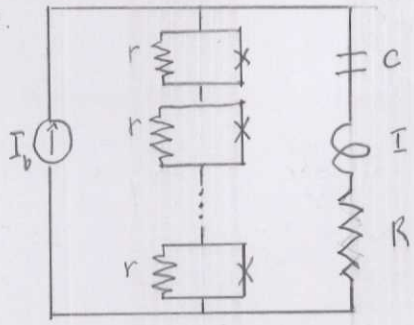
$\hbar\left(\frac{1}{2er} + \frac{N}{2eR}\right)\dot{\phi}_K = I_b - \frac{R + Nr}{R} I_c \sin\phi_K + \frac{r}{R} I_c \sum_{i=1}^{2}\sin(\phi_j)$

$\frac{\hbar(R + Nr)}{2erR}\dot{\phi}_K = I_b - \frac{R + Nr}{R} I_c \sin\phi_K + \frac{r}{R} I_c \sum_{i=1}^{2}\sin(\phi_j)$

$\frac{\hbar(R + Nr)}{2Ner^2 I_c}\dot{\phi}_K = \frac{R I_b}{Nr I_c} - \frac{R + Nr}{Nr}\sin\phi_K + \frac{1}{N}\sum_{i=1}^{2}\sin\phi_j$

$\frac{d\phi_K}{d\tau} = \Omega + a \sin\phi_K + \frac{1}{N}\sum_{j=1}^{2}\sin\phi_j \; ; \; \Omega = \frac{R I_b}{Nr I_c} \; ; \; \boxed{a = \frac{-(R + Nr)}{Nr} \; ; \; \tau = \frac{2Ner^2 I_c}{\hbar(R + Nr)} t}$

$\dot{\phi} = \Omega + a\sin\phi_k + K\sum_{j=1}^{2}\sin\phi_j$ 4.6.6.



$$\frac{\hbar}{2er}\cdot\frac{d\phi_K}{dt} + I_c\sin\phi_K + \frac{dQ}{dt} = I_b$$

$$L\frac{d^2Q}{d^2t} + R\frac{dQ}{dt} + \frac{Q}{C} = \boxed{\frac{\hbar}{2e}\sum_{j=1}^{N}\frac{d\phi_j}{dt}}$$

Chapter 5: Linear Systems

$\dot{x} = v$
$\dot{v} = -\omega^2 x$

5.1.1. a. $\frac{\dot{x}}{\dot{v}} = \frac{dx}{dv} = \frac{v}{-\omega^2 x}$; $-\omega^2 x + c = v$; $\boxed{\omega^2x^2 + v^2 = C}$

b. Conservation of Energy: $\sum\frac{1}{2}mv^2 = E$; $\frac{1}{2}m\omega^2x^2 + \frac{1}{2}mv^2 = C$

$\boxed{KE_{rot} + KE_{Lin} = KE_{TOT.}}$

$\dot{x} = ax$
$\dot{y} = -y$

5.1.2 $\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{-y}{ax} = \frac{-e^{-t}}{ae^{at}} = \frac{-1}{ae^{(a+1)t}}$; $\lim_{t\to\infty}\frac{dy}{dx} = \lim_{t\to\infty}\frac{-1}{ae^{(a+1)t}} = \boxed{-\infty}$ || y-axis

$\lim_{t\to-\infty}\frac{dy}{dx} = \lim_{t\to-\infty}\frac{-1}{ae^{(a+1)t}} \boxed{= 0}$ || x-axis.

$\dot{x} = -y$
$\dot{y} = -x$

5.1.3. $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \boxed{\begin{bmatrix}-1\\ -1\end{bmatrix}\begin{bmatrix}x & y\end{bmatrix}}$

$\dot{x} = 3x - 2y$
$\dot{y} = 2y - x$

5.1.4 $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \boxed{\begin{bmatrix}3 & -2\\ -1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}}$

$\dot{x} = 0$
$\dot{y} = x + y$

5.1.5. $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \boxed{\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}}$

$\dot{x} = x$
$\dot{y} = 5x + y$

5.1.6. $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \boxed{\begin{bmatrix}1 & 0\\ 5 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}}$

$\dot{x} = x$
$\dot{y} = x + y$

5.1.7. $\frac{dy}{dx} = \frac{x+y}{x}$