$$\boxed{\text{Chapter 9: Lorenz Equations}}$$

$$M = \int_0^{2\pi} m(\theta, t)\, d\theta$$

9.1.1:

a) $$I = I_{Wheel} + I_{Water} = m R_{Wheel}^2 + M R_{Water}^2$$

$$= m R_{Wheel}^2 + \int_0^{2\pi} m(\theta, t)\, d\theta \; R_{Water}^2$$

b) $$\dot{M} = \frac{dM}{dt} = \int_0^{2\pi} \frac{dm(\theta, t)}{dt}\, d\theta$$

$$= \int_0^{2\pi} \left[(\text{Mass pumped in}) - (\text{Mass pumped out})\right] d\theta$$

$$= \int_0^{2\pi} (Q - Km)\, d\theta$$

c) If $\dot{M} = Q - Km$, then $\dot{I} = \dot{M} R^2$

$$= Q R_{Wheel}^2 - Km R_{Water}^2$$

$$= Q R_{Wheel}^2 - K I$$

$$t = \int \frac{dI}{QR^2 - KI} = -\frac{1}{K} \int \frac{du}{u}$$

$$= -\frac{1}{K} \ln QR^2 - KI + C$$

$$I = (C) e^{-Kt} + QR^2$$

$$\lim_{t \to \infty} I(t) = QR^2$$

$$= \text{constant.}$$

$Q(\theta) = q_1 \cos\theta$ 9.1.2.

a) If $n \neq 1$, then a lagrange multiplier about the coefficients, $a(t) + b(t) = 1$.

$$Q(\theta) = q_1 \cos\theta + \lambda(a(t) + b(t))$$

$$= q_1 \cos\theta + \lambda(\mathring{a}(t) + \mathring{b}(t))$$

where $\frac{da}{dt} = \lambda a$; $a = C_1 e^{\lambda t}$

and

$\frac{db}{dt} = \lambda b$; $b = C_2 e^{\lambda t}$

Thus, $\lim_{t \to \infty} C(t) e^{\lambda t} = 0$; $\lim_{t \to \infty} C(t) = a(t) = 0$

b) If $Q(\theta) = \sum_{n=0}^{\infty} q_n \cos n\theta$, then the coefficients $a(t)$ and $b(t)$ become $a_n(t)$ and $b_n(t)$, respectively.

$\mathring{a} + a = n x b(t)$ and $\mathring{b} + b = -n x a(t) + C$

where $C = q_n / K$

The automous system arrives at a solution:

$$a(t) = e^{-t}\left(a(0) + n\int_0^{\tau} x(t) b(t) e^{t} dt\right)$$

$$b(t) = C + e^{-t}\left[b(0) - C - n\int_0^{t} x(t) a(t) e^{t} dt\right]$$

As $t \to \infty$, then $a(t) = 0$ and $b(t) = C = \frac{q}{K}$.

(Kolar and Gumbs, 1992)

$\mathring{a}_1 = \omega b_1 - K a_1$

$\mathring{b}_1 = -\omega a_1 + q_1 - K b_1$

$\mathring{\omega} = -\frac{v}{I}\omega + \frac{\pi g r}{I} a_1$

$\mathring{x} = \sigma(y - x)$

$\mathring{y} = rx - xz - y$

$\mathring{z} = xy - bz$

9.1.3. The hint states x is like ω, so σ relates the coefficients in $\mathring{\omega}$. $\sigma \propto \frac{v}{I}$ and $\sigma \propto \frac{\pi g r}{I}$

$x \propto \omega$; $\omega = \alpha x$

The hint also states y is like a_1, so

$y \propto a_1$; $a_1 = \beta y$

Lastly, z is similar to b_1.

$z \propto b_1$; $b_1 = \varepsilon z$

A dimensional problem frequently shifts time:

$t = \xi \cdot \tau$

$$\mathring{\omega} = -\frac{v}{I}\omega + \frac{\pi g r_w}{I} a_1 = \alpha \frac{1}{\xi}\frac{dx}{d\tau} = -\frac{v}{I}\alpha x + \frac{\pi g r_w}{I}\beta y$$

$$\mathring{a}_1 = \omega b_1 - K a_1 = \beta \frac{1}{\xi}\frac{dy}{d\tau} = \alpha x \cdot \varepsilon z - K \cdot \beta \cdot y$$

$$\mathring{b}_1 = -\omega a_1 + q_1 - K b_1 = \varepsilon \frac{1}{\xi}\frac{dz}{d\tau} = -\alpha x \beta y + q_1 - K \varepsilon z$$

$$x' = -\frac{v\xi}{I} x + \frac{\pi g r \beta \xi}{\alpha I} y = \sigma(y - x)$$

$$y' = \frac{\alpha \xi \varepsilon}{\beta} xz - \xi K y = rx - xz - y$$

$$z' = -\frac{\alpha \beta \xi}{\varepsilon} xy + \frac{\xi}{\varepsilon}(q_1 - K\varepsilon z) = xy - bz$$

Where $\sigma = \frac{v\varepsilon}{I} = \frac{\pi g r \beta \varepsilon}{\alpha I}$; $b = 1$

$\varepsilon = (X + \frac{4}{Z})$; $1 = \frac{\alpha \xi X}{\beta}$

$\xi = \frac{1}{K}$; $0 = \frac{\xi}{\varepsilon}(q_1 - KX)$

$r = \frac{\alpha}{\beta}\frac{4}{Z}$

$\mathring{E} = K(P-E)$

$\mathring{P} = \gamma_1(ED-P)$

$\mathring{D} = \gamma_2(\lambda+1-D-\lambda EP)$

9.1.4.

a) $\mathring{D} = 0 = \gamma_2(\lambda+1-D-\lambda EP)$ at $E^* = 0$

$\lambda = D-1$

$$\begin{bmatrix} \mathring{E} \\ \mathring{P} \\ \mathring{D} \end{bmatrix} = \begin{bmatrix} K & K & 0 \\ \gamma_1 D & 0 & \gamma_1 E \\ -\gamma_2\lambda P & -\gamma_2\lambda E & -\gamma_2 \end{bmatrix} \begin{bmatrix} E \\ P \\ P \end{bmatrix}$$

$$A_{E^*=0} = \begin{bmatrix} K-\lambda & K & 0 \\ \gamma_1 D & -\lambda & 0 \\ -\gamma_2\lambda P & 0 & -\gamma_2-\lambda \end{bmatrix}$$

$$\lambda_1 = -\gamma_2 \; ; \; \lambda_{2,3} = \frac{K \pm \sqrt{K^2+4D\cdot\gamma_1\cdot K}}{2}$$

$$\Delta = -\gamma_2(2D\gamma_1 K) \; ; \; \tau = -\gamma_2 + K$$

$$\tau^2 - 4\Delta = \gamma_2 - 2K\gamma_2 + K^2 - 8\gamma_2 D\gamma_1 K$$

If $\gamma_1, \gamma_2 >> K$, then $\Delta > 0 ; \tau < 0$

$\tau^2 - 4\Delta < 0$

"Stable Node"

b) $\mathring{E}$ is proportional to $\mathring{x}$, by the look.

$E = x$; $P = y$; $D = (\alpha - \beta) z$

$$\beta = \frac{\xi}{\gamma_1} \; ; \; r = \frac{\gamma_1}{\xi \alpha z} \; ; \; b = \frac{\gamma_2}{\xi}$$

$$\lambda = \left(\frac{\xi}{\gamma} + 1\right) xy - 1 \; ; \; \frac{\gamma_1 P}{\xi} = y$$

Lorenz's equations fit the jitter within a laser.

$Q(\theta) = \sum_{n=0}^{\infty} q_n \cos n\theta$ 9.1.5

$\mathring{a}_1 = \omega b_1 - K a_1$

$\mathring{b}_1 = -\omega a_1 - K b_1 + q_1$

$\mathring{\omega} = (-v\omega + \pi g r a_1)/I$

$$\frac{\partial m}{\partial t} = Q - Km - \omega \frac{d\theta}{d\theta} \quad (9.1.2)$$

$$m(\theta, t) = \sum [a_n(t) \sin n\theta + b_n(t) \cos n\theta] \quad (9.1.4)$$

$$Q(\theta) = \sum_{n=0}^{\infty} p_n \sin(n\theta) + q_n \cos(n\theta)$$

The equation relating change of mass per time and ~~ce~~ change of mass per angle.

$$\frac{\partial}{\partial t}\left[\sum_{n=0}^{\infty}[a_n(t)\sin(n\theta) + b_n(t)\cos(n\theta)]\right]$$

$$= \sum_{n=0}^{\infty}[p_n \sin(n\theta) + q_n \cos(n\theta)] - K \sum_{n=0}^{\infty}[a_n(t)\sin(n\theta) + b_n(t)\cos(n\theta)]$$

$$- \omega \frac{\partial}{\partial \theta} \sum_{n=0}^{\infty}[a_n(t)\sin(n\theta) + b_n(t)\cos(n\theta)]$$

$$\sum_{n=0}^{\infty} \mathring{a}_n(t)\sin(n\theta) + \mathring{b}_n(t)\cos(n\theta)$$

$$= \sum_{n=0}^{\infty}[p_n \sin(n\theta) + q_n \cos(n\theta)] - K \sum_{n=0}^{\infty}[a_n(t)\sin(n\theta) + b_n(t)\cos(n\theta)]$$

$$- \omega \sum_{n=0}^{\infty} n [a_n(t)\cos(n\theta) - b_n(t)\sin(n\theta)]$$

The similar terms on the left and right are grouped:

$\mathring{a}_n = n\omega b_n(t) - K a_n(t) + p_n$

$\mathring{b}_n = -n\omega a_n(t) - K b_n(t) + q_n$

$$\mathring{\omega} = \frac{-v\omega + gr\int_0^{2\pi} m(\theta, t)\sin\theta \, d\theta}{I} = \frac{-v\omega + \pi g r a}{I}$$

Fixed Points: $\dot{a} = 0 = \omega b_1 - Ka_1 + p_1$

$\dot{b} = 0 = -\omega a_1 - Kb_1 + q_1$

$\dot{\omega} = 0 = \dfrac{-\nu\omega + \pi g r a_1}{I}$

$$\omega^* = 0, \pm\sqrt{\frac{\pi g r q_1}{\nu} - K^2}$$

A square root is a pitchfork bifurcation, but imperfect when $p_1 \neq 0$.

$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r-1) = 0$

9.2.1. $\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Eigenvalues:

$$(A - \lambda) = \begin{bmatrix} -\sigma-\lambda & \sigma & 0 \\ r-z & -1-\lambda & -x \\ y & x & -b-\lambda \end{bmatrix} = 0$$

Fixed Points: $\dot{x} = 0 = \sigma(z - x)$

$\dot{y} = 0 = rx - y - xz$

$\dot{z} = 0 = xy - bz$

$(x^*, y^*, z^*) = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$

Jacobian Adjustment:

$$(A - \lambda) = \begin{bmatrix} -\sigma-\lambda & \sigma & 0 \\ -1 & -1-\lambda & \pm\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b-\lambda \end{bmatrix}$$

$= \lambda^3 + (\sigma + 1 + b)\lambda^2 + b(r+\sigma)\lambda + 2b\sigma(r-1)$

b) If $r = r_H = \sigma\left(\frac{\sigma + b + 3}{\sigma - b - 1}\right)$, then eigenvalues become cubic roots. The proposition $\sigma > b+1$ comes from $r_H = \sigma\left(\frac{\sigma+b+3}{\sigma-(b+1)}\right)$ and the a cubic cubic solution's necessity for positive values.

c) The third eigenvalue is $\lambda_3 = -(\sigma + 1 + b)$

$rx^2 + \sigma y^2 + \sigma(z-2r)^2 \leq C$

9.2.2. Equation of a Ellipse: $F(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$

When $V(x,y,z) = rx^2 + \sigma y^2 + \sigma(z-2r)^2$

$$\dot{V}(x,y,z) = 2rx\dot{x} + 2\sigma y\dot{y} + 2\sigma(z-2r)\dot{z}$$

$$= 2rx(\sigma(z-x)) + 2\sigma y(rx - y - xz) + 2\sigma(z-2r)(xy - bz)$$

$$= 2rx\sigma z - 2rx^2\sigma + 2\sigma yrx - 2\sigma y^2 - 2\sigma yxz + (2\sigma z - 4\sigma r)(xy - bz)$$

$$\frac{\dot{V}(x,y,z)}{2} = rx\sigma z - rx^2\sigma + \sigma yrx - \sigma y^2 - \sigma yxz + \sigma xyz - \sigma bz^2 - 2\sigma rxy + 2\sigma rbz$$

$$= -r\sigma x^2 - \sigma y^2 + \sigma(rxz + rxy - bz^2 - 2rxy + 2rbz)$$

$$= -r\sigma x^2 - \sigma y^2 + \sigma(-bz^2 + (rx + 2rb)z) - \sigma rxy$$

$$= -r\sigma x^2 - \sigma y^2 - \sigma b\left(z - \frac{rx + 2rb}{2b}\right)^2 + \frac{\sigma(rx + 2rb)^2}{4} - \sigma rxy$$

$$= -r\sigma\left(x+\frac{rb+y}{2\sigma(1+4r)}\right)^2+\left(\frac{1}{4\sigma(1+4r)}-1\right)y^2-\sigma b\left(z-\frac{rx+2rb}{2b}\right)^2+b^2\left(\frac{1}{4\sigma(1+4r)}+r^2\right)$$

$$-r\sigma\left(x+\frac{rb+y}{2\sigma(1+4r)}\right)^2+\left(\frac{1}{4\sigma(1+4r)}-1\right)y^2-\sigma b\left(z-\frac{rx+2rb}{2b}\right)^2+b^2\left(\frac{1}{4\sigma(1+4r)}+r^2\right)<0$$

$$1<\underbrace{\frac{+r\sigma}{b^2\left(\frac{1}{4\sigma(1+4r)}+r^2\right)}}_{1/a^2}\left(x+\frac{rb+y}{2\sigma(1+4r)}\right)^2+\underbrace{\frac{\left(\frac{1}{4\sigma(1+4r)}-1\right)}{b^2\left(\frac{1}{4\sigma(1+4r)}+r^2\right)}}_{1/b^2}y^2-\underbrace{\frac{\sigma b}{\left(\frac{1}{4\sigma(1+4r)}+r^2\right)}}_{1/c^2}\left(z-\frac{rx+2rb}{2b}\right)^2$$

The equation of the ellipse above is co-dependent. With ~~z~~ z about x-values and x about y-values. When modeled without co-dependent axis, then a coefficient becomes co-dependent.

In all cases, an ellipse centered at $\left(\frac{-rb+y}{2\sigma(1+4r)}, 0, \frac{rx+2rb}{2b}\right)$ with a maximum distance from the center:

$$\left(\sqrt{\frac{r\sigma}{b^2\left(\frac{1}{4\sigma(1+4r)}+r^2\right)}}, \sqrt{\frac{1-\frac{1}{4\sigma(1+4r)}}{b^2\left(\frac{1}{4\sigma(1+4r)}+r^2\right)}}, \sqrt{\frac{ab}{\left(\frac{1}{4\sigma(1+4r)}+r^2\right)}}\right),$$

Another sphere fits in to the ellipsoid centered ~~at~~ at the same coordinates with a minimal ellipsoid radius from the center.

$x^2+y^2+(z-r-\sigma)^2=c$

9.2.3. Equation for a sphere: $x^2+y^2+z^2=f(x,y,z)$

$$V(x,y,z)=x^2+y^2+(z-r-\sigma)^2$$

$$\dot V(x,y,z)=2x\dot x+2y\dot y+2(z-r-\sigma)\dot z$$

$$\frac{\dot V(x,y,z)}{2}=x[\sigma(z-x)]+y(rx-y-xz)+(z-r-\sigma)(xy-bz)$$

$$=-\sigma x^2-y^2-b\left(z-\frac{r+\sigma}{2}\right)^2+b\frac{(r+\sigma)^2}{4}$$

$$-\sigma x^2-y^2-b\left(z-\frac{r+\sigma}{2}\right)^2+b\frac{(r+\sigma)^2}{4}<0$$

$$1<\underbrace{\frac{4\sigma}{b(r+\sigma)^2}}_{a}x^2+\underbrace{\frac{4}{b(r+\sigma)^2}}_{b}y^2+\underbrace{\frac{4}{(r+\sigma)^2}}_{c}\left(z-\frac{r+\sigma}{2}\right)^2$$

A sphere centred at $\left(0,0,\frac{r+\sigma}{2}\right)$

with a maximum radius $\left(\sqrt{\frac{b(r+\sigma)^2}{4\sigma}},\sqrt{\frac{b(r+\sigma)^2}{4\sigma}},\sqrt{\frac{(r+\sigma)^2}{4}}\right)$

9.2.4 $\dot x=\sigma(y-x)$ The z-axis is an invariant

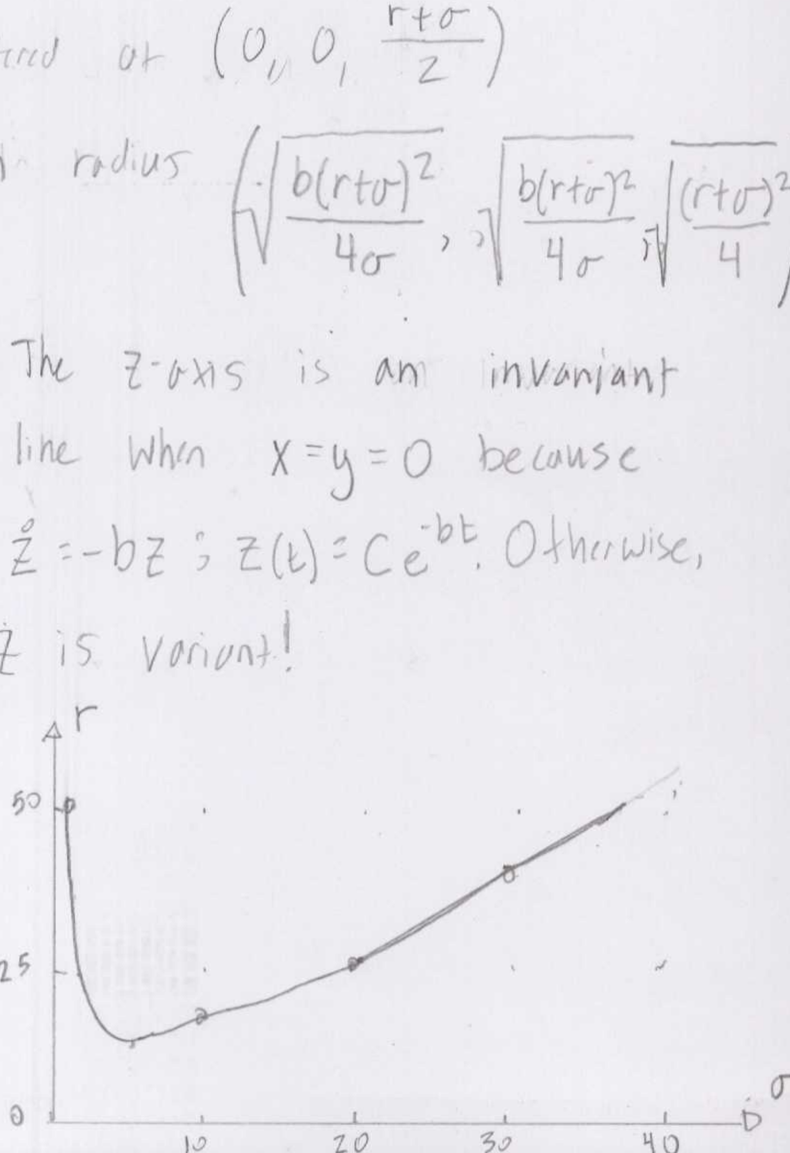
$\dot y=rx-xz-y$ line when $x=y=0$ because

$\dot z=xy-bz$ $\dot z=-bz$; $z(t)=Ce^{-bt}$. Otherwise,

z is variant!

9.2.5. The relationship between r and σ is in Problem 9.2.1.

$$r=\sigma\frac{(\sigma+3+b)}{(\sigma-1-b)}$$



$\dot{x} = -\nu x + zy$

$\dot{y} = -\nu y + (z-a)x$

$\dot{z} = 1 - xy$

9.2.6.

a) A dissapative system's volume contracts under flow.

$v(x,y,z) = x^2 + y^2 + z^2$

If dissapative, then $\dot{v}(x,y,z) < 0$ .. or $\nabla \cdot f < 0$.

$$\nabla \cdot f = \frac{\partial}{\partial x}\left[-\nu x + zy\right] + \frac{\partial}{\partial y}\left[-\nu y + (z-a)x\right] + \frac{\partial}{\partial z}\left[1 - xy\right]$$

$$= -\nu - \nu = -2\nu < 0$$

$$\dot{V} = \int_V \nabla \cdot f \, dV = -2\int_V \nu \cdot dV = -2\nu V$$

$$V(t) = V(0)e^{-2\nu t}$$

The volume shrinks with time!

b) <u>Fixed Points:</u> $\dot{x} = 0 = -\nu x + zy$

$\dot{y} = 0 = -\nu y + (z-a)x$

$\dot{z} = 0 = 1 - xy$

$(x^*, y^*, z^*) = (\pm a, \pm a, \nu)$

where $a = \nu\left(x^2 - 1/x^2\right)$

c) <u>Bifurcations:</u>

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\nu & z & y \\ z-a & -\nu & x \\ -x & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A = \begin{bmatrix} -v & z & y \\ z-a & -v & x \\ -y & -x & 0 \end{bmatrix}$$

$$(A - \lambda I) = \begin{bmatrix} -v-\lambda & z & y \\ z-a & -v-\lambda & x \\ -y & -x & -\lambda \end{bmatrix} = 0$$

If $x = \pm 1$, $y = \pm 1$, and $z = v$, then

$\lambda_1 \approx 1.41i$; $\lambda_2 \approx -1.41i$; $\lambda_3 = -2v$

Hopf Bifurcation = Spiral Node

$\dot{\theta}_1 = w_1$

$\dot{\theta}_2 = w_2$

9.3.1.
a) The solution (or time-dependent motion) is periodic as $t \to \infty$, so not a chaotic system.

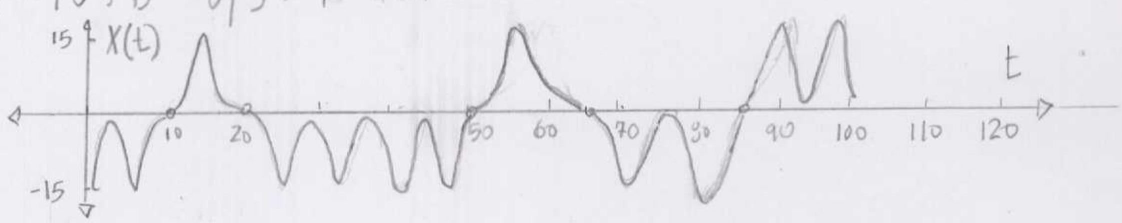
b) A large Liapunov exponent is zero.

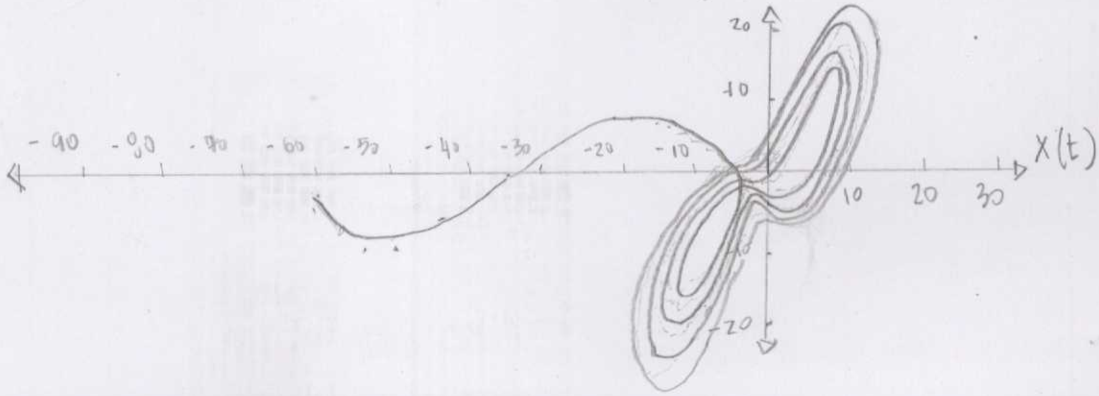
$\dot{x} = \sigma(z - x)$

$\dot{y} = rx - xz - y$

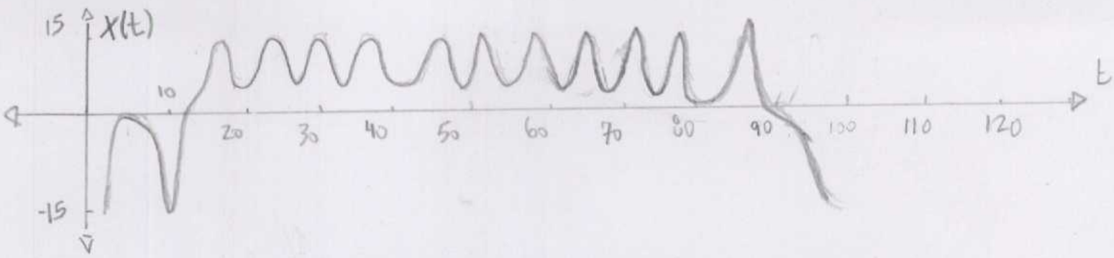
$\dot{z} = xy - bz$

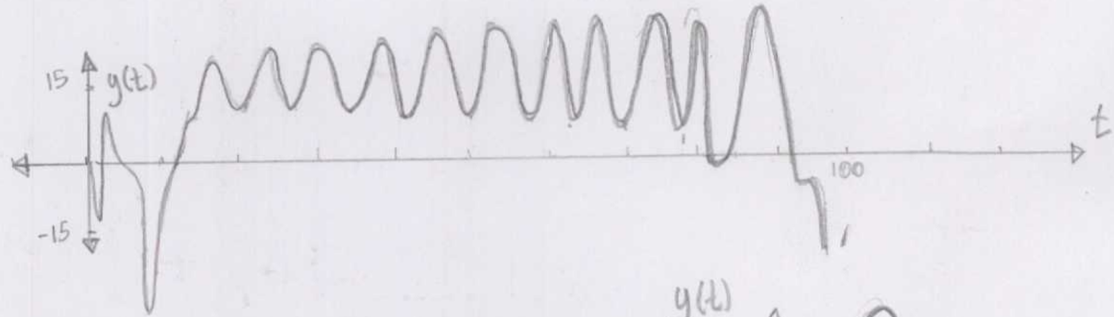
9.3.3. $\sigma = 10$; $b = 8/3$; $r = 22$

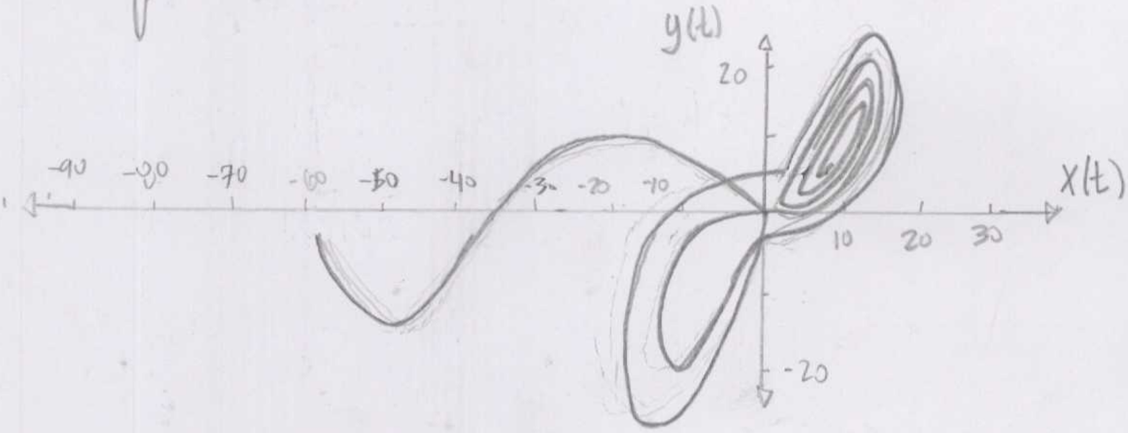




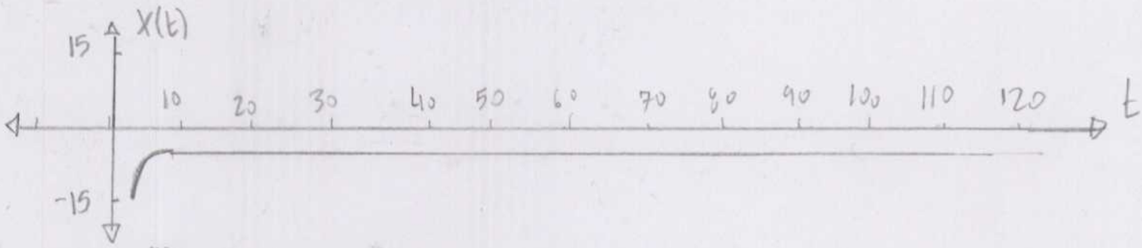
9.3.4 $\sigma = 10$; $b = 8/3$; $r = 24.5$

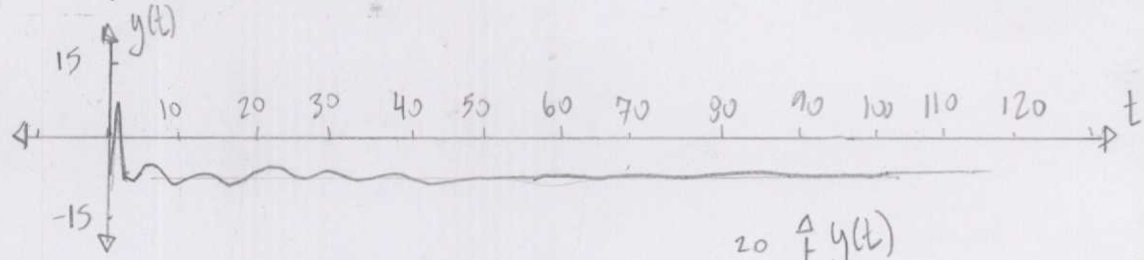


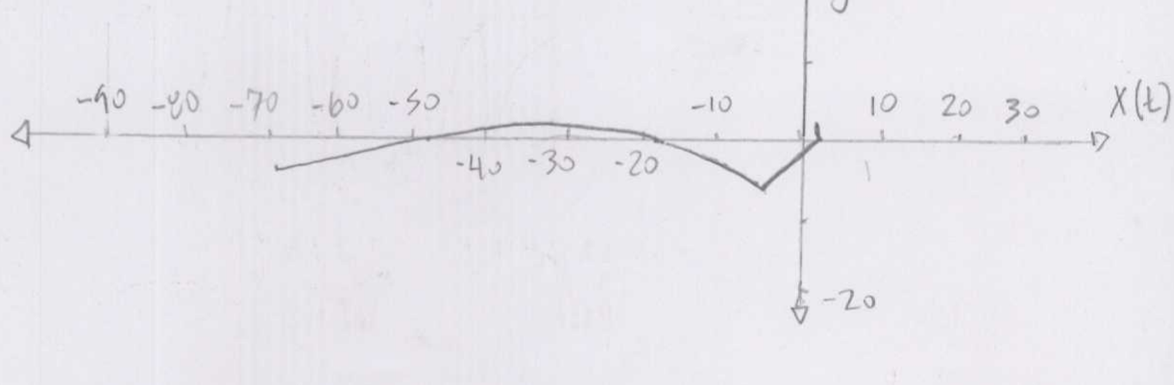




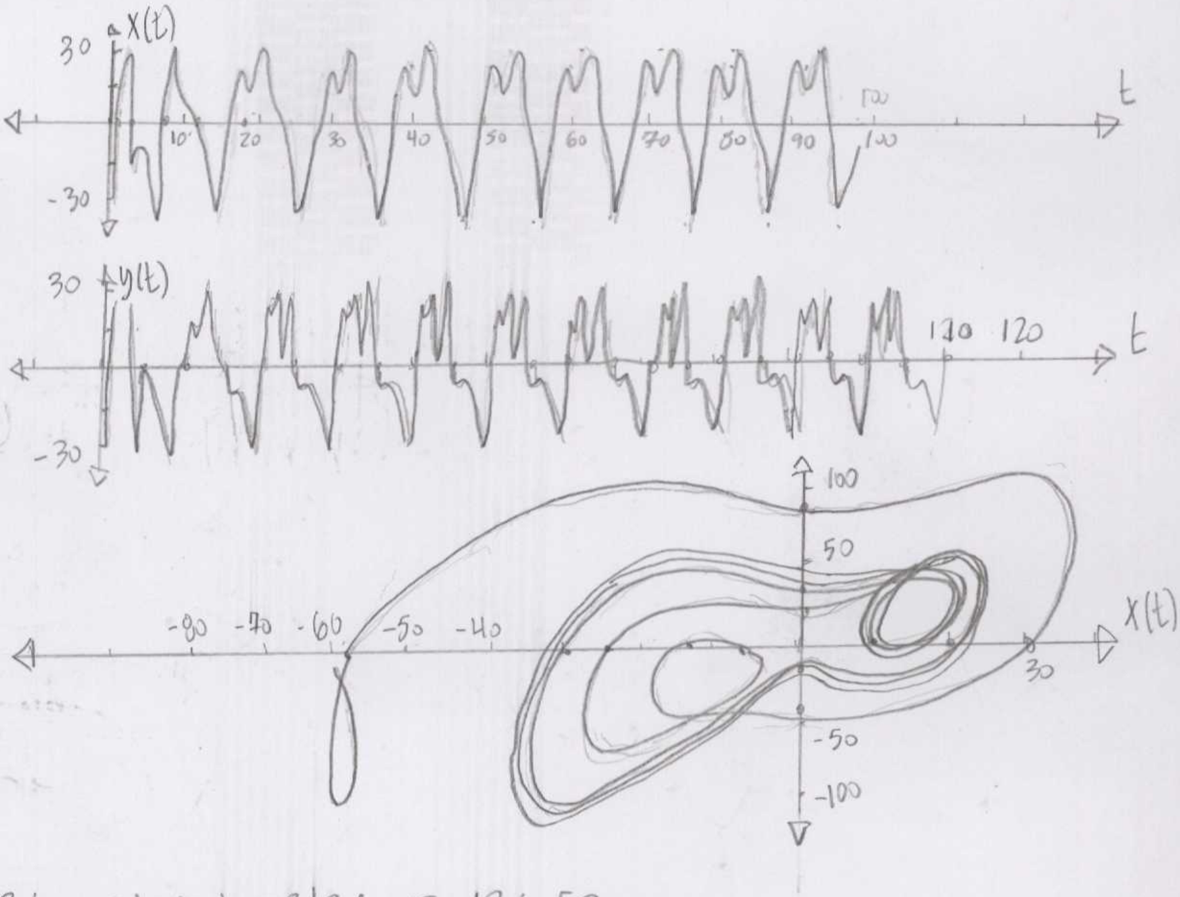
9.3.2 $\sigma = 10$; $b = 8/3$; $r = 10$



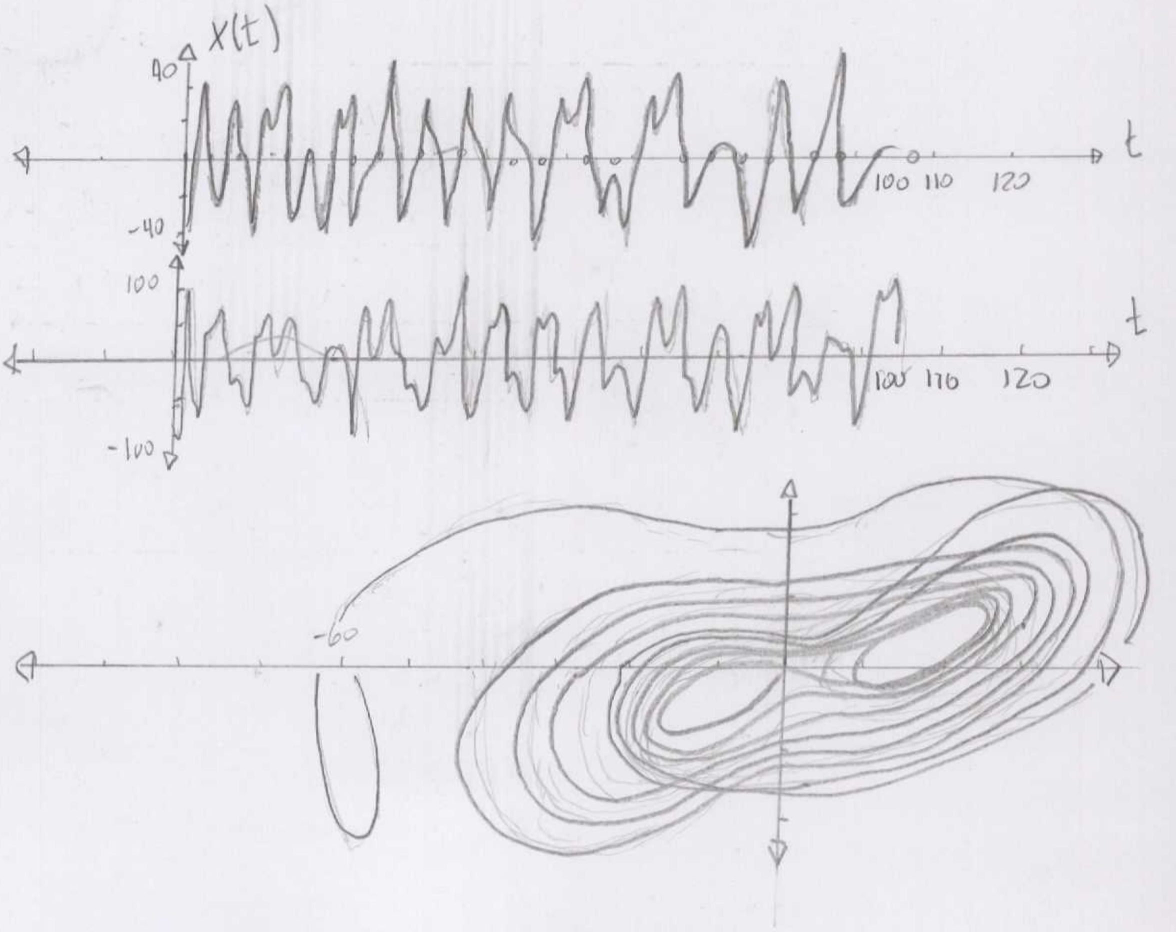




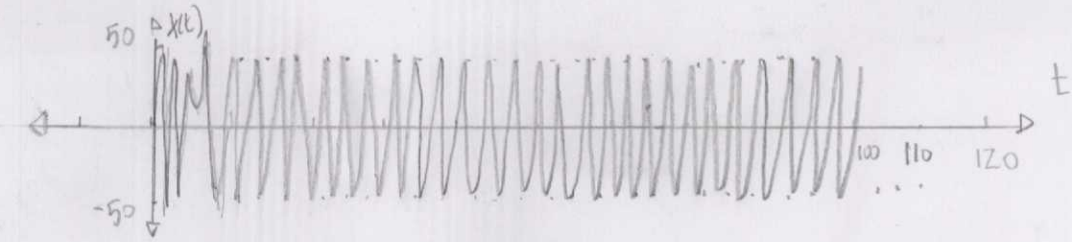
9.3.5. $\sigma = 10$; $b = 8/3$; $r = 100$

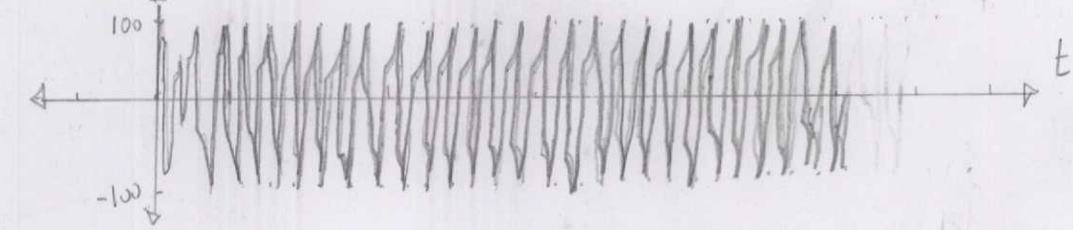


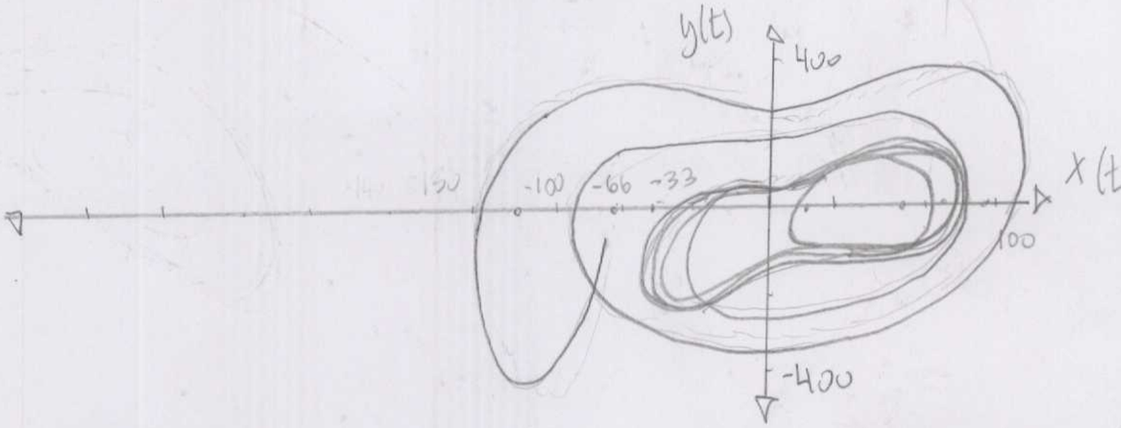
9.3.6. $\sigma = 10$; $b = 8/3$; $r = 126.52$



9.3.7. $\sigma = 10$; $b = 8/3$; $r = 400$







Note: Runge-Kutta 4th order $x_0 = -5.0$ $y_0 = -3.3$, $z_0 = 12.2$ $\Delta h = 0.1$

x_n	$x_{n-1} + \frac{\Delta h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
y_n	$y_{n-1} + \frac{\Delta h}{6}(l_1 + 2l_2 + 2l_3 + l_4)$
z_n	$z_{n-1} + \frac{\Delta h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$
k_1	$x(t, x, y)\cdot\Delta h$
l_1	$y(t, x, y, z)\cdot\Delta h$
m_1	$z(t, x, y, z)\cdot\Delta h$
k_2	$x(t + \frac{\Delta h}{2}, x + \frac{\Delta h k_1}{2}, y + \frac{\Delta h l_1}{2})\cdot\Delta h$
l_2	$y(t + \frac{\Delta h}{2}, x + \frac{\Delta h k_1}{2}, y + \Delta h\frac{l_1}{2}, z + \Delta h\frac{m_1}{2})\cdot\Delta h$
m_2	$z(t + \frac{\Delta h}{2}, x + \frac{\Delta h k_1}{2}, y + \frac{\Delta h l_1}{2}, z + \Delta h\frac{m_1}{2})\cdot\Delta h$
k_3	$x(t + \frac{\Delta h}{2}, x + \frac{\Delta h k_2}{2}, y + \Delta h\frac{l_2}{2})\cdot\Delta h$
l_3	$y(t + \frac{\Delta h}{2}, x + \frac{\Delta h k_2}{2}, y + \Delta h\frac{l_2}{2}, z + \Delta h\frac{m_2}{2})\cdot\Delta h$
m_3	$z(t + \frac{\Delta h}{2}, x + \frac{\Delta h k_2}{2}, y + \Delta h\frac{l_2}{2}, z + \Delta h\frac{m_2}{2})\cdot\Delta h$
k_4	$x(t + \Delta h, x + \Delta h k_3, y + \Delta h l_3)\cdot\Delta h$
l_4	$y(t + \Delta h, x + \Delta h k_3, y + \Delta h l_3, z + \Delta h m_3)\cdot\Delta h$
m_4	$z(t + \Delta h, x + \Delta h k_3, y + \Delta h l_3, z + \Delta h m_3)\cdot\Delta h$

$\dot{r} = r(1-r^2)$ 9.3.8.

$\dot{\theta} = 1$

a) Invariant Set: a set of points (states) in a dynamic system which are mapped into other points in the same set by the dynamic evolution operator.

Yes, the equation system is invariant when $r \leq 1$ because the constant outcome in the dynamical system

b) Open set: a union containing every point in the collection or every subset.

When $r \leq 1$, the disk is an open set, since every point space, any union, or subset frequents similar properties

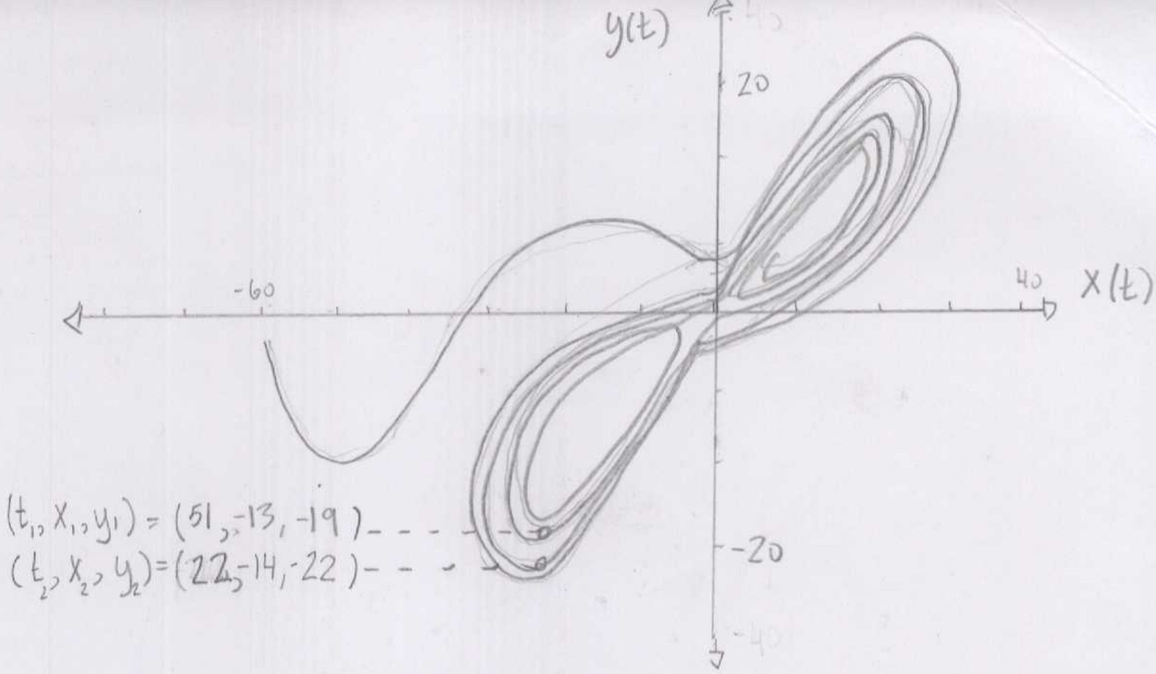
c) Attractor: a set to which all neighboring trajectories converge.

The function set shows an unstable node, with exact trajectories, so an attractor at $x^2 + y^2 = 1$.

d) $x^2 + y^2 = 1$ is an attractor.

9.3.9 $\sigma = 10$; $b = 8/3$; $r = 28$.

The time horizon determined from the graph: $t_{horizon} \sim O\left(\frac{1}{\lambda} \ln \frac{a}{\|\delta_0\|}\right) =$

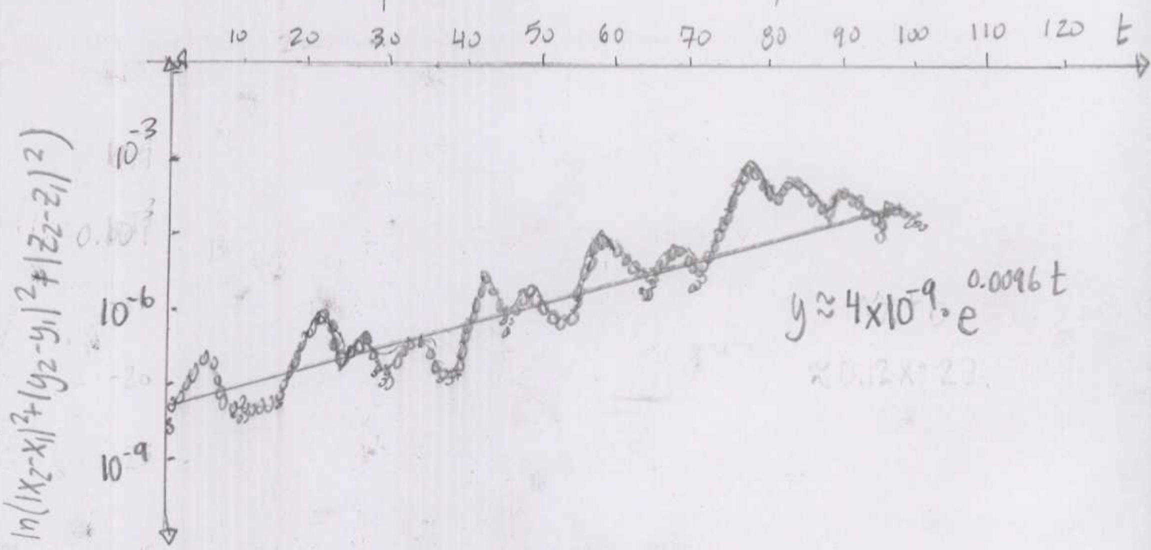


Steps for a Lyapunov Exponent:

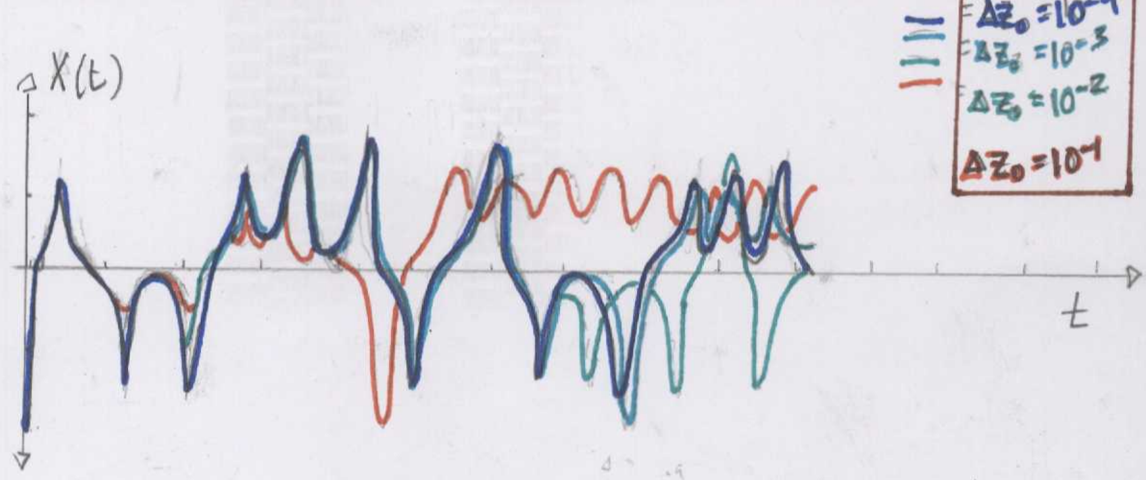
① A Runge-Kutta 4th order calculation for an equation system

② The z-variable's initial condition changes by $\delta = 1 \times 10^{-9}$ through new coordinates.

③ A plot of $\ln(|x_2-x_1|^2+|y_2-y_1|^2+|z_2-z_1|^2)$ vs time shows a linear plot with a slope.



The book shifted z_0 by 0.001 with a $\lambda = 0.8623$ result. A 1×10^{-9} δ_0-shift generated smaller Lyapunov constants at 0.01.



The new initial condition shifts the time horizon. A small shift in Δz misaligned later in time while a large value, much earlier.

9.4.1 See problem 9.3.9.

$$x_{n+1} = \begin{cases} 2x_n & 0 \le x_n \le 1/2 \\ 2 - 2x_n & 1/2 \le x_n \le 1 \end{cases}$$

9.4.2.

a) The "tent map" function peaks at $x_n = 2$.

b) Fixed Points: $0 \le x_n \le 1/2$; $x_n + 1 = 0 = 2x_n$

$x_n^* = 0$ "stable"

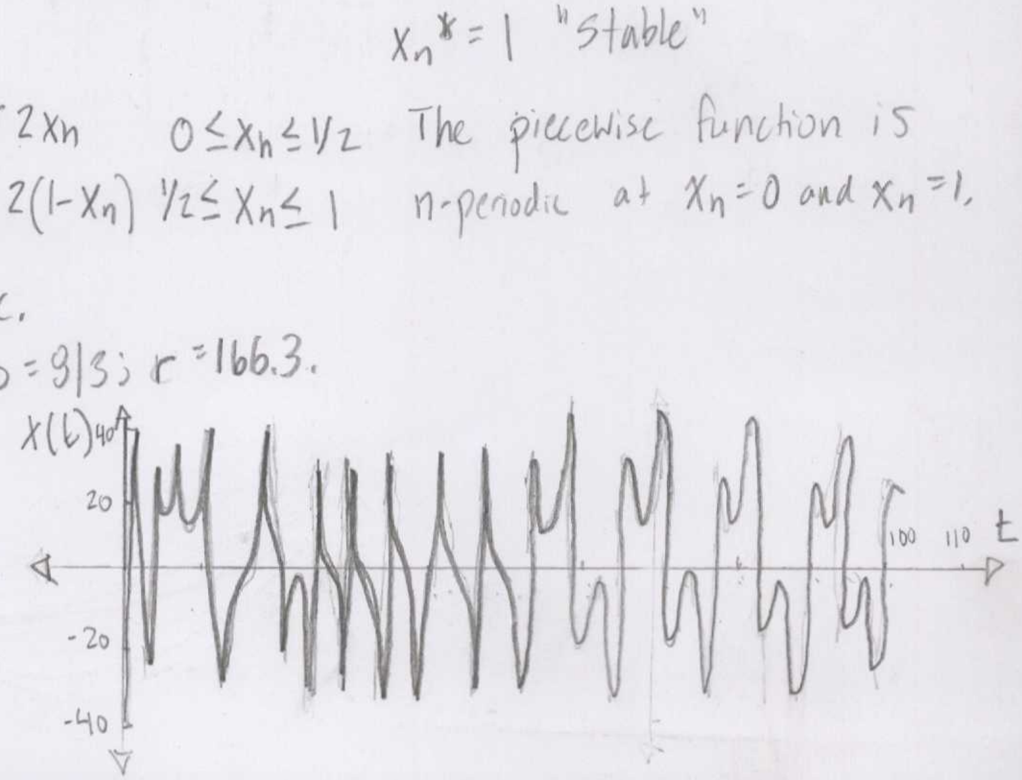
$1/2 \le x_n \le 1$; $x_{n+1} = 0 = 2 - 2x_n$

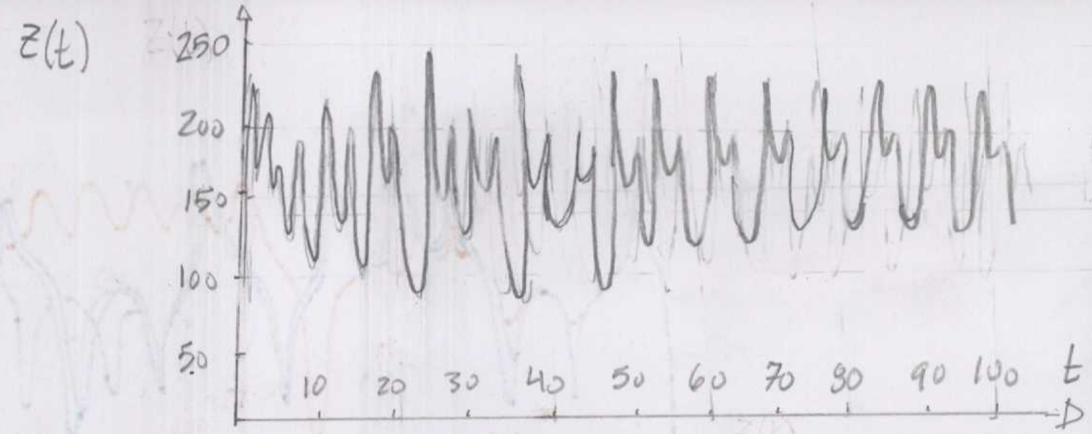
$x_n^* = 1$ "stable"

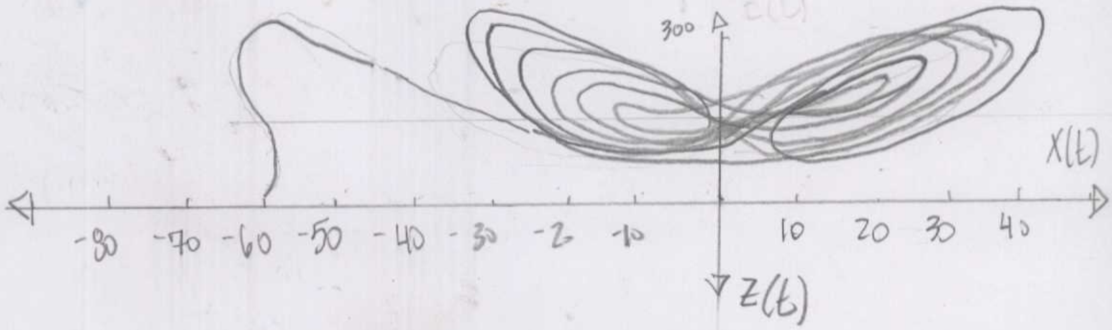
c) $x_{n+1} = \begin{cases} 2x_n & 0 \le x_n \le 1/2 \\ 2(1 - x_n) & 1/2 \le x_n \le 1 \end{cases}$ The piecewise function is n-periodic at $x_n = 0$ and $x_n = 1$.

d) See part c.

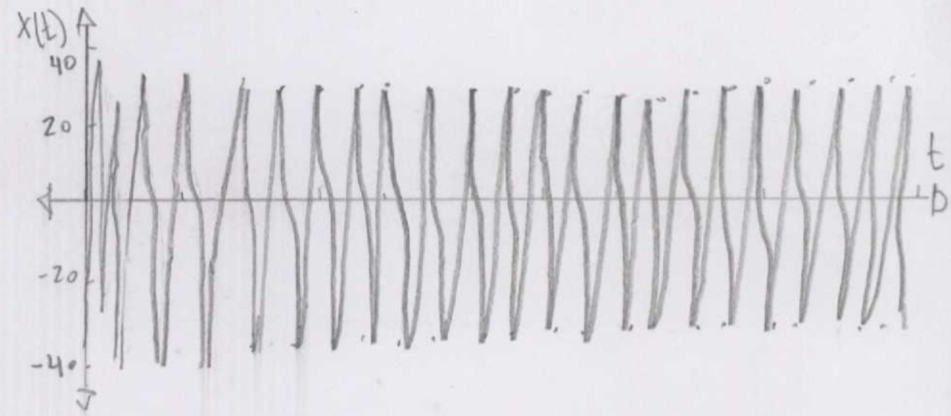
9.5.1 $\sigma = 10$; $b = 8/3$; $r = 166.3$.

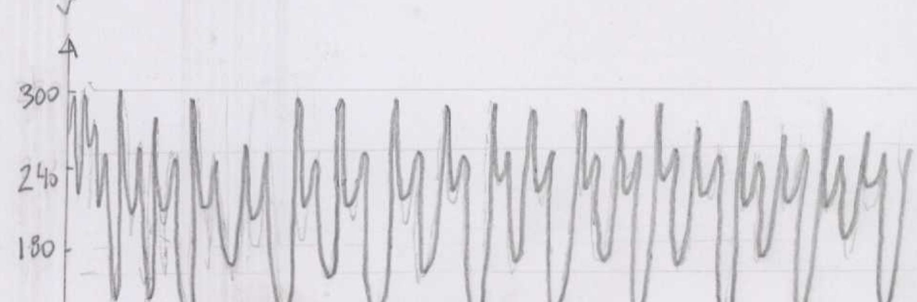


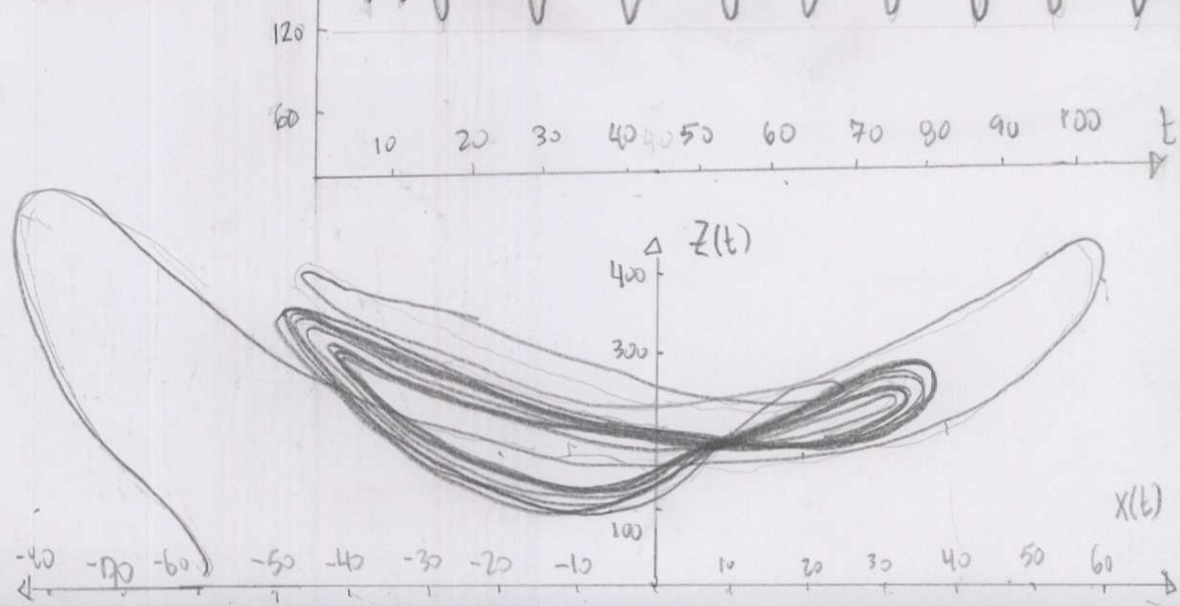




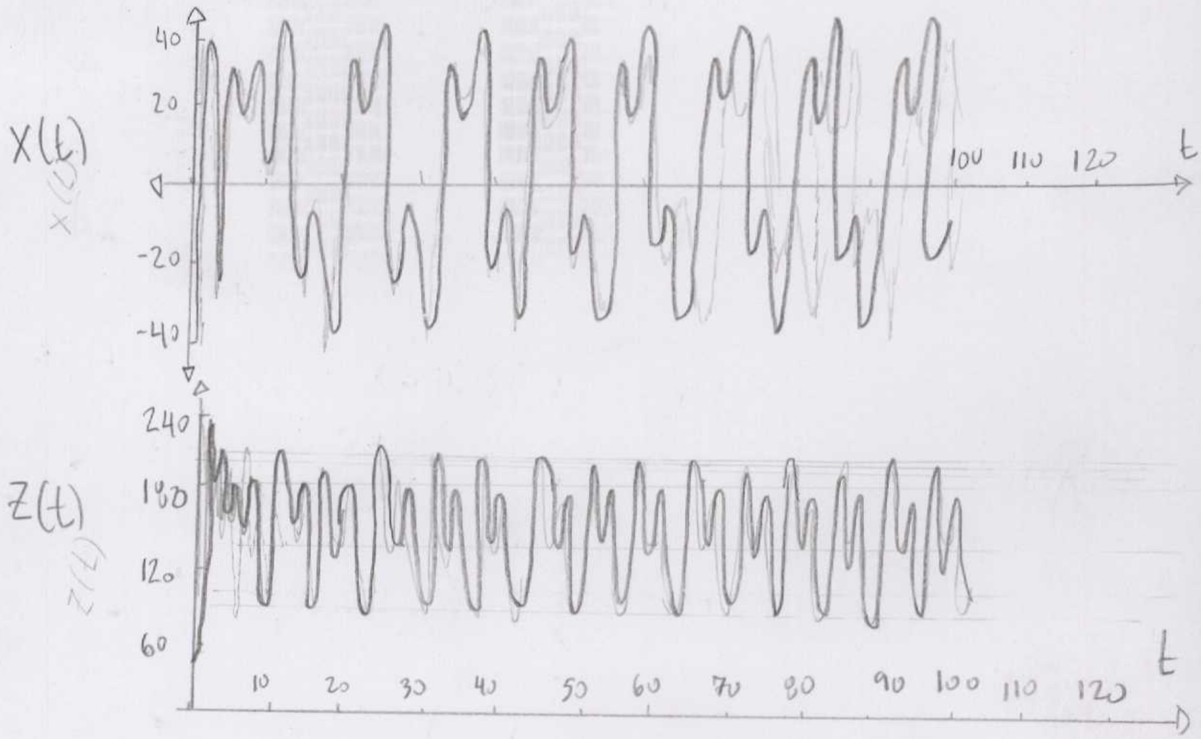
9.5.2 $\sigma = 10$; $b = 8/3$; $r = 212$

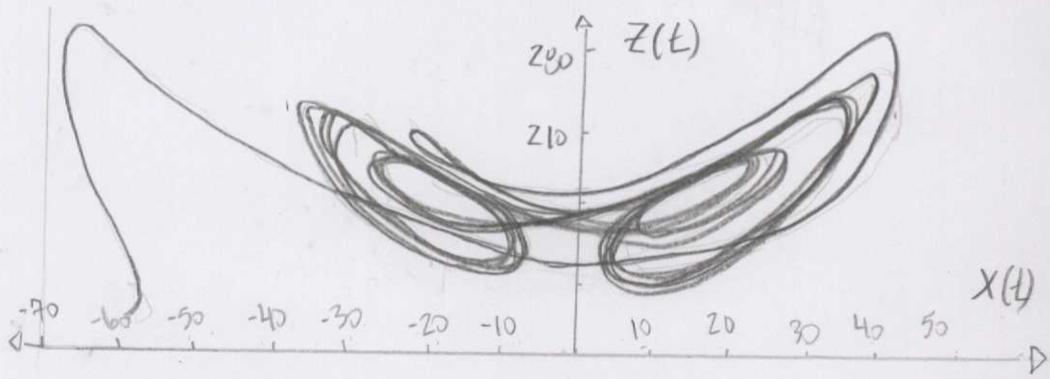




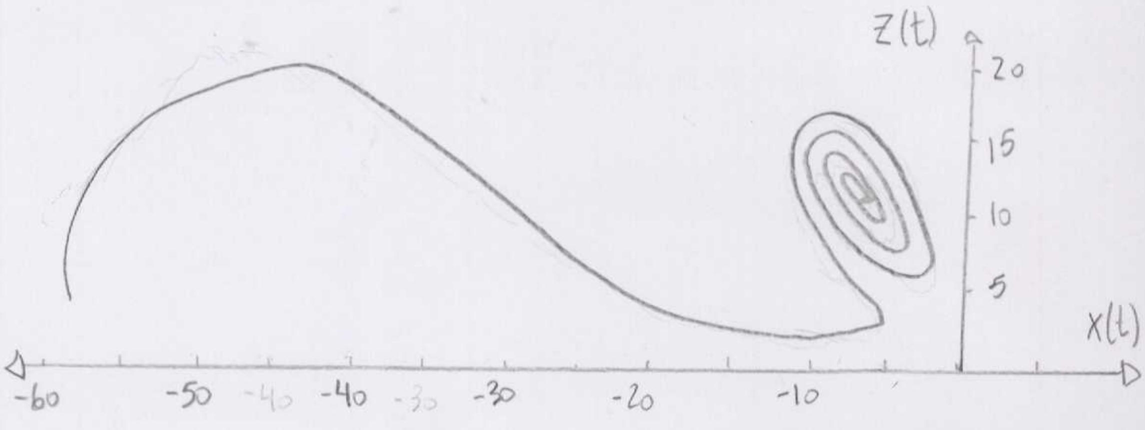


9.5.3 $\sigma = 10$; $b = 8/3$; $r = 150$





9.5.4 $\sigma = 10$; $b = 8/3$; $r = 24.4 + \sin \omega t$; $\omega = 10$



X(t) is negative when $\omega \approx 10$ and rare from 5 or more plots about Lorenz equation.

Note: Runge-Kutta 4th order employed for r-coefficient.

$X' = Y$
$Y' = -XZ$
$Z' = XY$

9.5.5.

a. $\varepsilon = r^{-1/2}$; $\dot{x} = \sigma(y-x)$; $\dot{y} = rx - xz - y$; $\dot{z} = xy - bz$
$= \sigma y - \sigma x$ $\quad = \frac{x}{\varepsilon^2} - xz - y$

$\tau = \frac{t}{\varepsilon}$; $x' = \sigma\varepsilon^2 y - \sigma\varepsilon^2 x$; $Y' = \varepsilon X - \varepsilon^3 XZ - \varepsilon^3 y$; $Z' = \varepsilon^3 xy - bz$

$X = \varepsilon x$ $\quad X' = Y - \sigma\varepsilon X$; $Y' = \sigma X - XZ - \epsilon Y$; $Z' = XY - bZ$

$Y = \varepsilon^2 \sigma y$

$Z = \sigma(\varepsilon^2 z - 1)$ $X' = Y - \sigma\varepsilon X$; $Y' = -XZ - \epsilon Y$; $Z' = XY - b\epsilon\sigma(\frac{Z}{\sigma}+1)$

$\lim_{\epsilon \to 0} X' = Y$; $\lim_{\epsilon \to 0} Y' = -XZ$; $\lim_{\epsilon \to 0} Z' = XY$

b. A constant of motion is a zero quantity, zero relationship, change of zero, or conserved system.

$(Y^2 + Z^2)' = -2YY' + 2ZZ'$
$= -2XYZ + 2XYZ$
$= 0$

$(X^2 - 2Z)' = 2XX' - 2Z'$
$= 2XY - 2XY$
$= 0$

c. A volume-preserving system follows either

$\dot{V} = \int_V \nabla \cdot f \, dV < 0$ or $\dot{V}(x, y, z) < 0$

The model provides an x, y, z vector.

$\dot{V} = \int \nabla \cdot F \, dV = \int \nabla \cdot \langle X', Y', Z' \rangle dV$

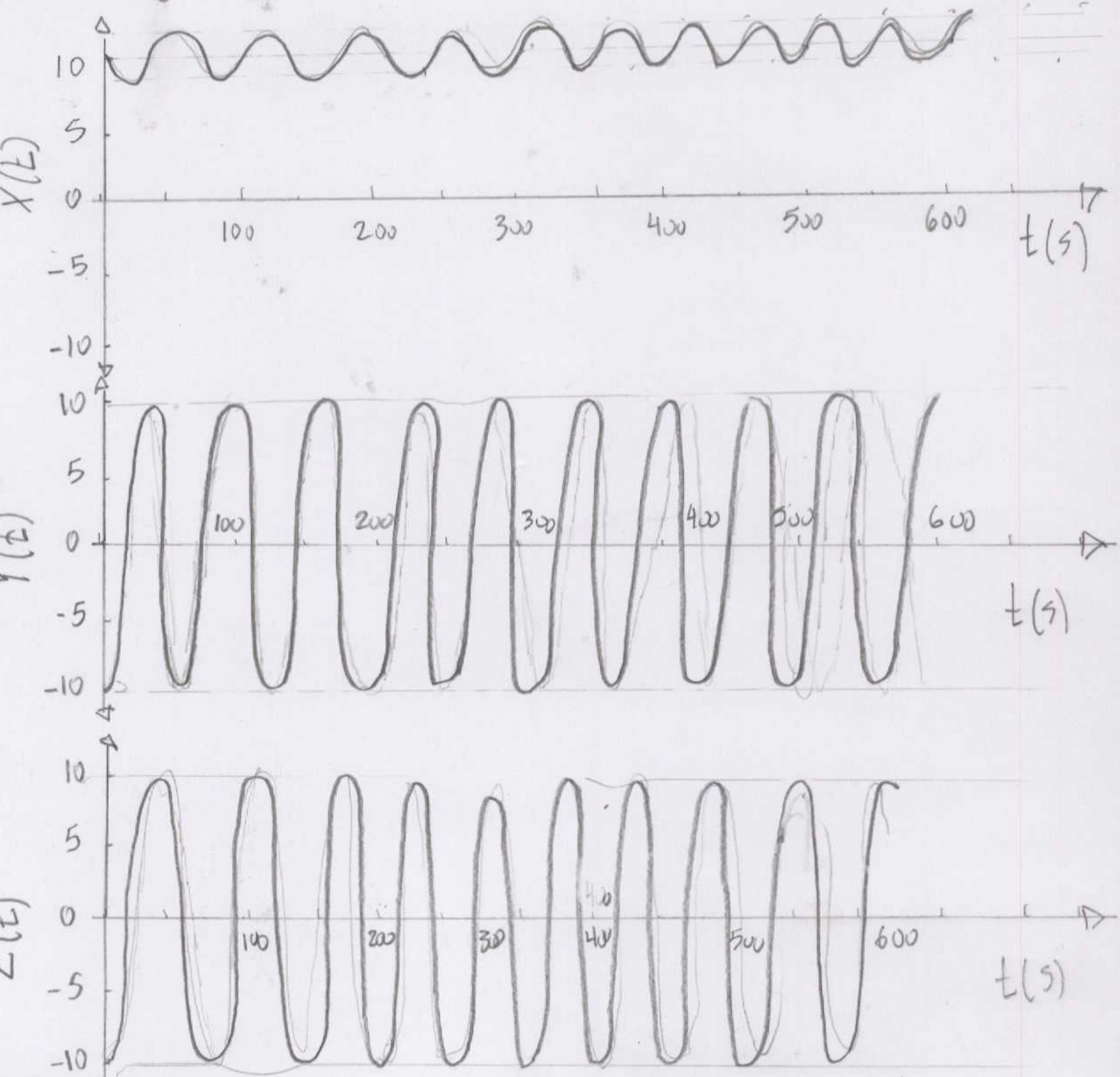
$= \int \nabla \cdot \langle Y, -XZ, XY \rangle dV = \int \langle \frac{dY}{dX} - \frac{dXZ}{dY} + \frac{dXY}{dZ} \rangle dV = 0$

$r=21$
$\sigma=10$
$b=8/3$

The systems constant ~~rate~~ of change is volume preserving.

d) r, also known as Rayleigh's number for fluids, in heat transfer or convection boundaries or diffusion limits, as a measurable criterion in mediums. The constant is an aggregate ratio at a specific temperature, ~~force~~ energy flux per dissapation. As r becomes infinity, then the ~~fluid~~ flux flows undamped.

e) Runge-Kutta 4th Order: $\Delta h = 0.1$; $(X_0, y_0, z_0) = (10, -10, 0)$



The individual plots align well with Lorentz' equations.

$V=\frac{1}{2}e_2^2+2e_3^2$ 9.6.1

$\dot{e}_1=\sigma(e_2-e_1)$

$\dot{e}_2=-e_2-20u(t)e_3$

$\dot{e}_3=5u(t)e_2-be_3$

a) $V=\frac{1}{2}e_2^2+2e_3^2$

$\dot{V}=\frac{1}{2}2e_2\cdot\dot{e}_2+4e_3\cdot\dot{e}_3$

$=-e_2^2-20u(t)\cdot e_2e_3+20u(t)e_2e_3-4be_3^2$

$=-e_2^2-4be_3^2$

$=-2(\frac{1}{2}e_2^2+2be_3^2)$

For the inequality, $K<2$ and $K<2b$,

$\dot{V}\le -kV$; $\ln V=-kt+V_0$; $0<V(t)<V_0e^{-kt}$

b) A proof about $e_2(t)$ & $e_3(t)$ quickly approaching Zero:

$$\frac{1}{2}e_2(t)^2<V(t)<V_0e^{-kt}$$

$$e_2(t)<\sqrt{2V(t)}<\sqrt{2V_0}\,e^{-kt/2}$$

and

$$2e_3(t)^2<V(t)<V_0e^{-kt}$$

$$e_3(t)<\sqrt{\frac{V(t)}{2}}<\sqrt{\frac{V_0}{2}}\,e^{-kt/2}$$

c) $e_1(t)$ becomes zero.

$\dot{e}_1(t)=\sigma(e_2-e_1)$; $\dot{e}_1(t)+\sigma e_1(t)=\sigma e_2(t)$

$<\sigma\sqrt{2V_0}\,e^{-kt/2}$

The equation isn't a Bernouilli Differential, but a Bernoulli method (of two functions)!

Function #1: $e_1(t)=uv$

Function #2: $\dot{e}_1(t)=u\dot{v}+\dot{u}v$

$$\mathring{e}_1(t) + \sigma e_1(t) = \sigma\sqrt{2V_0}\, e^{-kt/2}$$

$$u\mathring{v} + \mathring{u}v + \sigma uv = \sigma\sqrt{2V_0}\, e^{-kt/2}$$

$$\mathring{u}v + u(\mathring{v} + \sigma v) = \sigma\sqrt{2V_0}\, e^{-kt/2}$$

Solving $\mathring{v} + \sigma v = 0$:

$$\mathring{v} = -\sigma v$$

$$-\int \frac{dv}{v} = \int \sigma\, dt$$

$$v = Ce^{-\sigma t}$$

Solving $\mathring{u}v = \sigma\sqrt{2V_0}\, e^{-kt/2}$

If $v = Ce^{-\sigma t}$, then $\mathring{u} = \sigma\sqrt{2V_0}\, e^{-\frac{kt}{2} + \sigma t}$

$$\int du = \sigma\sqrt{2V_0}\int e^{-\frac{kt}{2} + \sigma t}\, dt$$

$$u = \frac{\sigma\sqrt{2V_0}}{-\frac{k}{2} + \sigma}\, e^{(\sigma - \frac{k}{2})t} + c$$

$$= \frac{-2\sigma\sqrt{2V_0}}{k - 2\sigma}\, e^{(\sigma - \frac{k}{2})t} + c$$

Substituting u and v:

$$e_2(t) = u \cdot v$$

$$= \left[\frac{-2\sigma\sqrt{2V_0}}{k - 2\sigma}\, e^{(\sigma - \frac{k}{2})t} + c\right] \cdot Ce^{-\sigma t}$$

$$= \frac{C}{e^{-\sigma t}} - \frac{2\sigma\sqrt{2V_0}}{k - 2\sigma}\, e^{-kt/2}$$

$\therefore$ The equation decay is exponential!

$x(t) = x(t)$

$\dot{y}_r = r x(t) - y_r - x(t) z_r$

$\dot{z}_r = x(t) y_r - b z_r$

9.6.2.

a) If $e_1 = x - x_r$, $e_2 = y - y_r$, and $e_3 = z - z_r$

$e_1 = 0$, $e_2 = y_r$, and $e_3 = z_r$

$\dot{e}_2 = -x(t) e_3 - e_2$ and $\dot{e}_3 = x(t) e_2 - b e_3$

b) $V = e_2^2 + e_3^2 =$

$\dot{V} = 2 e_2 \dot{e}_2 + 2 e_3 \dot{e}_3$

$= 2e_2 [-x(t) e_3 - e_2] + 2e_3 [x(t) e_2 - b e_3]$

$= -2e_2^2 - 2e_3^2$

$= -2V$, $V(t) = Ce^{-2t}$ ← Yes!

The Liapunov function

Whoa, nelly, jelly belly.

c) A Lorenz oscillator exhibits chaotic behavior with solely two equations, when co-dependent.

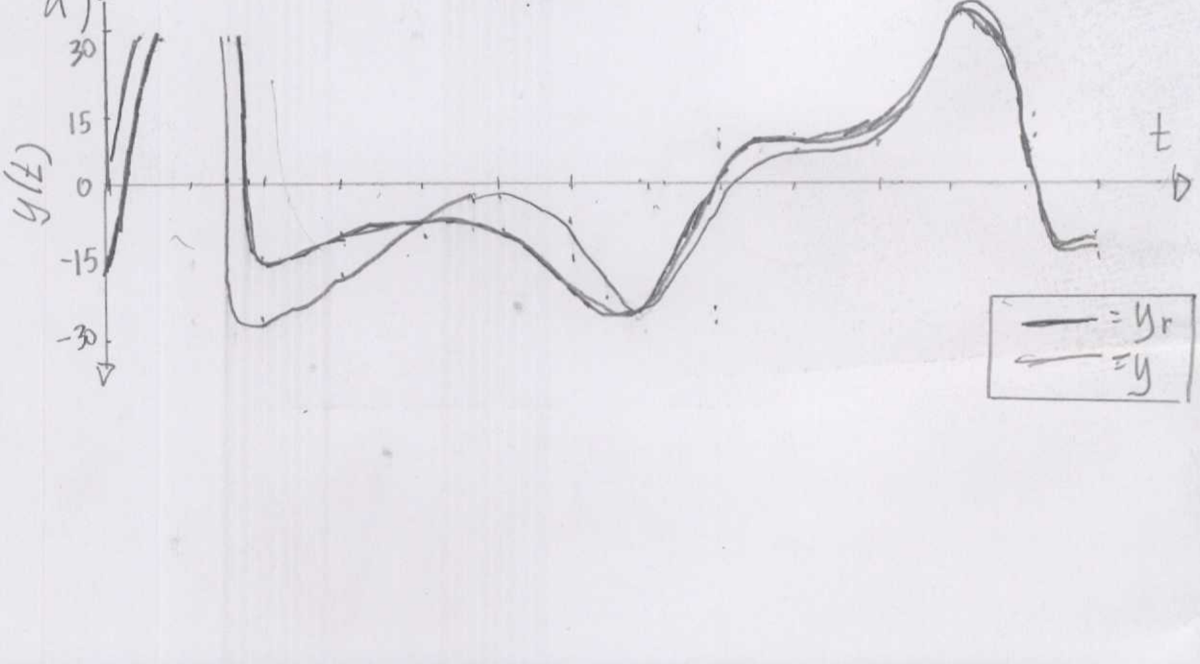
$x(t) = x(t)$

$\dot{y}_r = r x(t) - y_r - x(t) z_r$

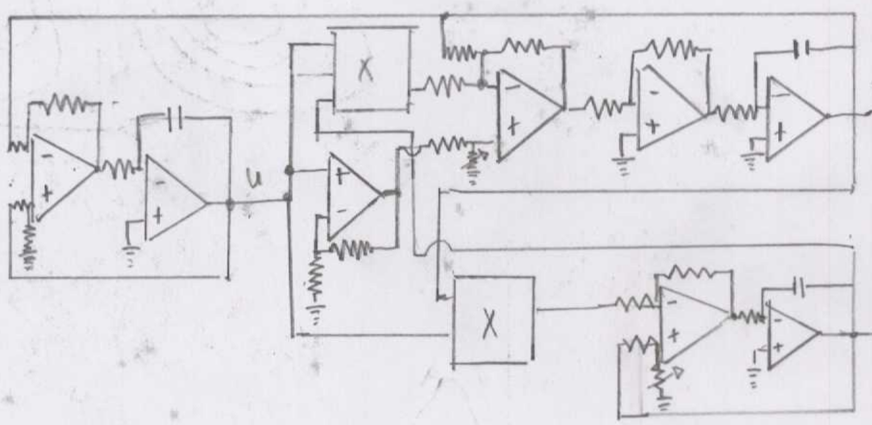
$\dot{z}_r = x(t) y_r - b z_r$

9.6.3. $r = 60$; $\sigma = 10$; $b = 8/3$.

a)



9.6.6.



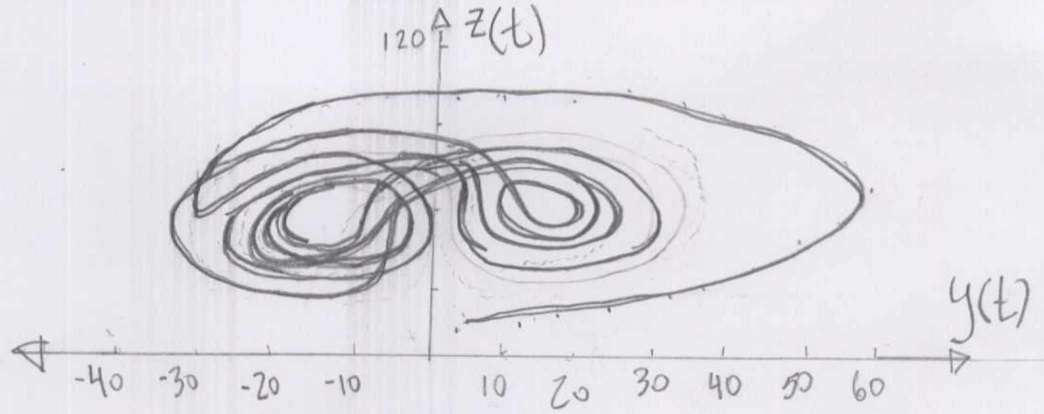
Three operational amplifier circuits are in the schematic above. One type is an integrator at the u, v, and w outputs. A second circuit is the differential amplifier before the v output - integrator, also a subtractor. Lastly, an adder following the u output near the circuit center.

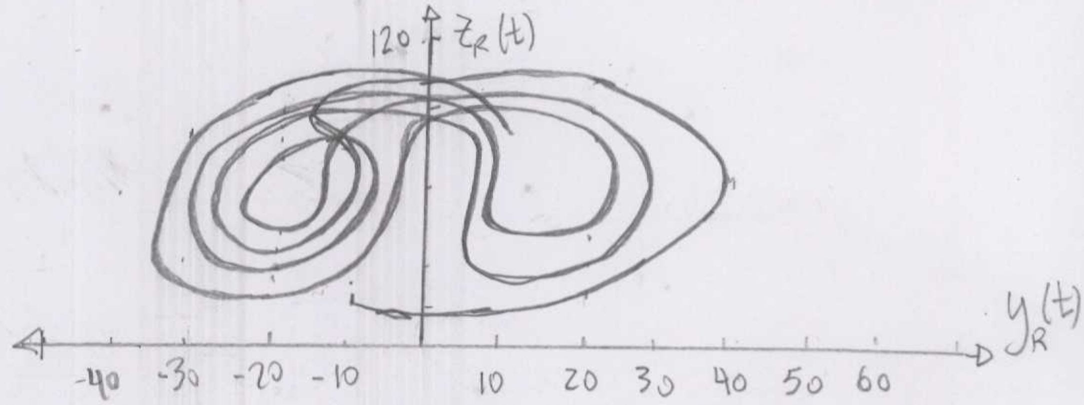
$$\mathring{u} = \sigma(v-u)$$

$$\mathring{v} = ru - v - 20uw$$

$$\mathring{w} = 5uv - bw$$

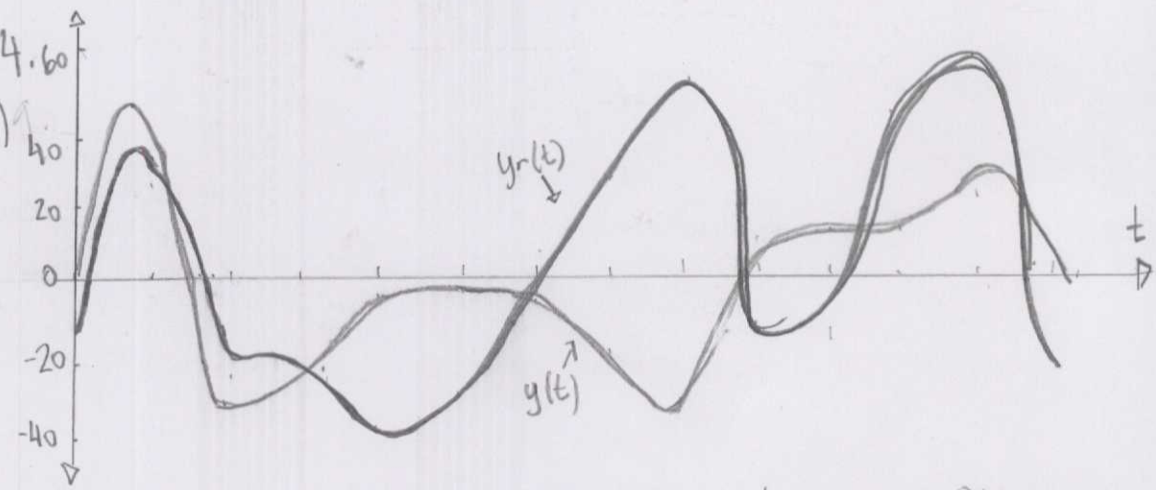
b)





9.6.4.

a)



Each plots shows an aperiodic behavior without synchronization or alignment in time.

b) $Y(t)$ as the synchronization or driving signal is perfect alignment because $y(t) = y_R(t)$

$\dot{x}_R = \sigma(y_r - x_r)$

$\dot{y}_R = rs(t) - y_R - s(t) z_R$

$\dot{z}_R = s(t) y_r - b z_R$

9.6.5

a) $s(t) = x(t) + m(t)$

where $x(t)$ = chaotic mask

$m(t)$ = low-power message

When $m(t) = \sin(t)$, then $\dot{x} = \sigma(y_r - x_r)$

$\dot{y}_R = r[x(t) + \sin(t)] - y_R - [x(t) + \sin(t)] z_R$

$\dot{z}_R = [x(t) + \sin(t)] y_r - b z_R$