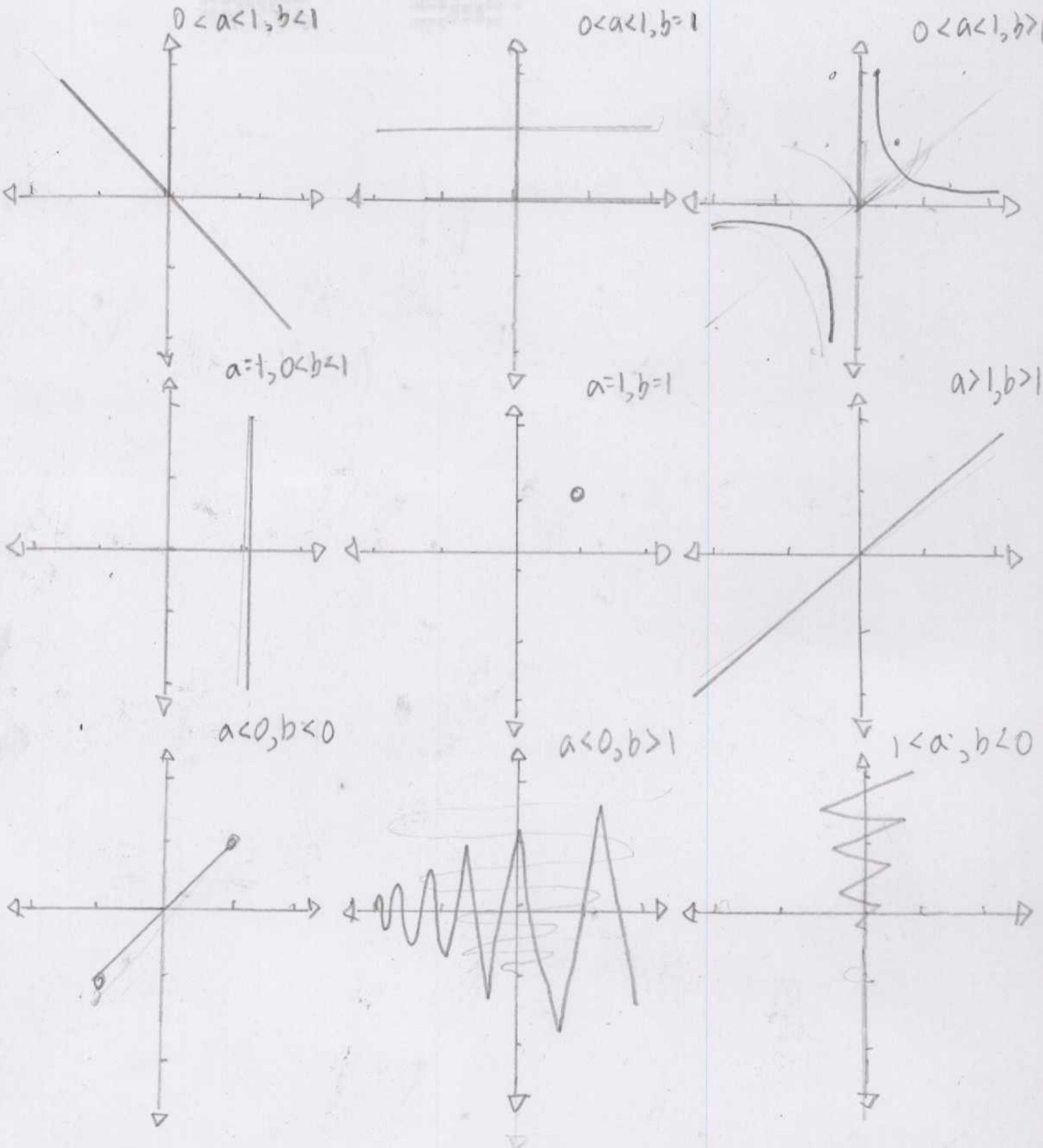
Chapter 12: Strange Attractors:

$x_{n+1} = a x_n$

$y_{n+1} = b y_n$

12.1.1. Fixed points: $x(1-a)=0$; $x^*=0$, $a=1,0$

$y(1-b)=0$; $y^*=0$, $b=1,0$



$x_{n+1} = a x_n + b y_n$

$y_{n+1} = c x_n + d y_n$

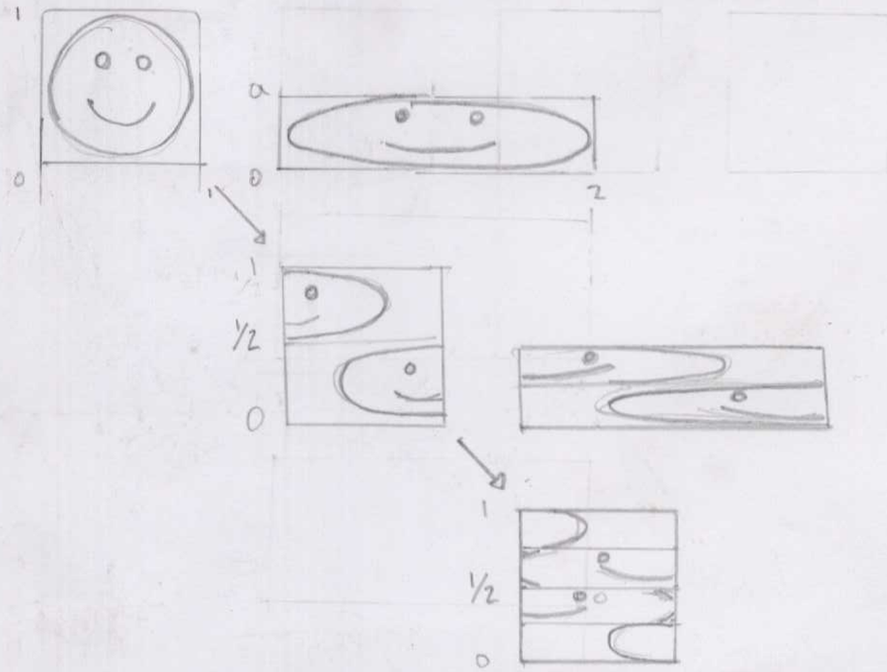
12.1.2. Fixed Points: $\begin{vmatrix} x_{n+1} \\ y_{n+1} \end{vmatrix} = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \begin{vmatrix} x_n \\ y_n \end{vmatrix}$

$$(a-\lambda)(d-\lambda) - bc < 0$$

$$\lambda = \left| \frac{(a+d) \pm \sqrt{a^2 - 2ad + 4bc + d^2}}{2} \right| < 1$$

$$ad - bc < 0.$$

$(x_{n+1}, y_{n+1}) = \begin{cases} (2x_n, ay_n) & 0 \le x \le 1/2 \\ (2x_n - 1, ay_n + 1/2) & 1/2 \le x \le 1 \end{cases}$ 12.1.3 Figure 12.1.4:



$(x_{n+1}, y_{n+1}) = \begin{cases} (2x_n, ay_n) & 0 \le x \le 1/2 \\ (2x_n - 1, ay_n + 1/2) & 1/2 \le x \le 1 \end{cases}$ 12.1.4

a) $B^2(x_0, y_0)$; Height $= a^n = a^2$; $a = \sqrt{\frac{1}{4}} = 1/2$

Box Height = 1

Number of strips = 4.

b) $B^3(x_0, y_0)$; Height $= a^n = a^3$; $a = \sqrt[3]{\frac{1}{8}} = 1/2$

Box Height = 1

Number of strips = 8

c) $B^n(x_0, y_0)$; Height $= a^n = a$; $a = \sqrt[n]{\frac{1}{2^n}} = 1/2$

Box Height = 1

Number of Strips $= 2^n$

12.1.5.

a) $(x, y) = (.a_1 a_2 a_3 \ldots \; .b_1 b_2 b_3 \ldots)$

$B(x, y) = (0.a_1 a_2 a_3 \ldots, 0.b_1 b_2 b_3 \ldots)$

b) $B^1(x, y) = (0.101, 0.010)$

$B^2(x, y) = (0.010, 0.101)$

c) The binary set is countable because the natural $(\mathbb{N})$ amount of digits.

d) An irrational value between zero and one is uncountable and aperiodic.

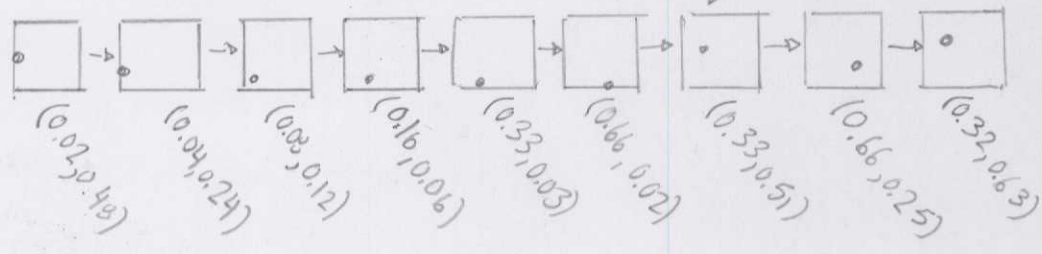
e) Dense map - given a point p, an initial condition q, and $\varepsilon > 0$, the trajectory passes through at q.

$$B^n(x,y) = (0.a_1a_2a_3\ldots, 0.b_1b_2b_3\ldots)$$
$$= (0.p_1p_2p_3\ldots, 0.q_1q_2q_3\ldots)$$

where $p_1 = q_1$, $p_2 = q_2$, $p_3 = q_3 \ldots$

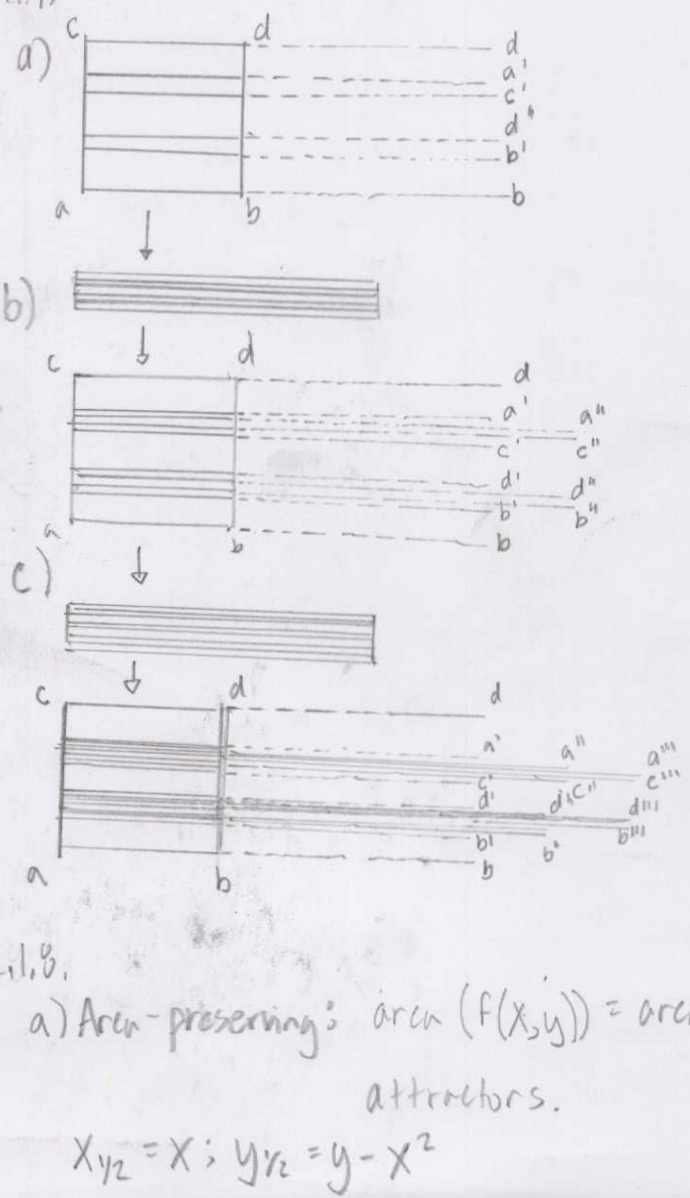
with initial condition $(0.p, 0.q)$.

12.1.6.



Smale's Horshoe

12.1.7. #1



$x_{n+1} = x_n \cos\alpha - (y_n - x_n^2)\sin\alpha$

$y_{n+1} = x_n \sin\alpha + (y_n - x_n^2)\cos\alpha$

12.1.8.

a) Area-preserving: $\text{area}(f(x,y)) = \text{area}(x,y)$, also no attractors.

$$x_{1/2} = x;\ y_{1/2} = y - x^2$$

A measure with a Jacobian determines area-change:

$$\begin{pmatrix} X_{n+1} \\ y_{n+1} \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} \frac{\partial X_{n+1}}{\partial x} & \frac{\partial X_{n+1}}{\partial y} \\ \frac{\partial y_{n+1}}{\partial x} & \frac{\partial y_{n+1}}{\partial y} \end{pmatrix} \begin{pmatrix} X_{1/2} \\ y_{1/2} \end{pmatrix}$$

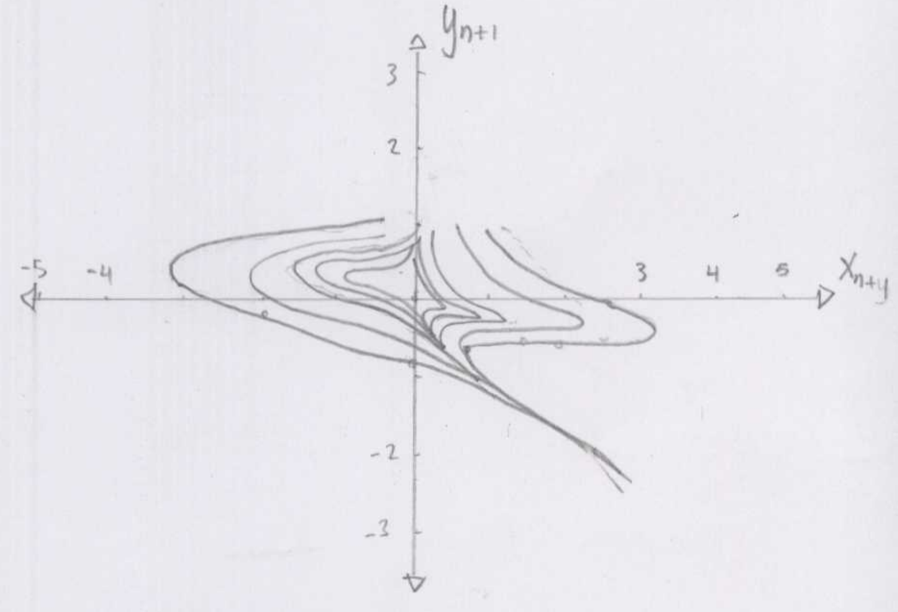
$$|J| = \left| \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \right| = 1 \quad \text{"Area-preserving"}$$

b) $\underline{T^{-1} \text{ from } T}$:

$$\left. \begin{aligned} X_{n+1} &= X_n \cos\alpha - (y_n - X_n^2)\sin\alpha \\ y_{n+1} &= X_n \sin\alpha + (y_n - X_i^2)\cos\alpha \end{aligned} \right\} T$$

$$\left. \begin{aligned} y_n &= (X_{n-1} - X_n \cos\alpha)/\sin\alpha \\ X_n^2 \sin\alpha + 2X_n\cos\alpha - X_{n-1} - X_{n+1} &= 0 \end{aligned} \right\} T^{-1}$$

c) $\cos\alpha = 0.24$, $X_0 \approx 0.57$, $y_0 \approx 0.16$.



12.1.9.

$X_{n+1} = X_n + y_{n+1}$

$y_{n+1} = y_n + k\sin X_n$

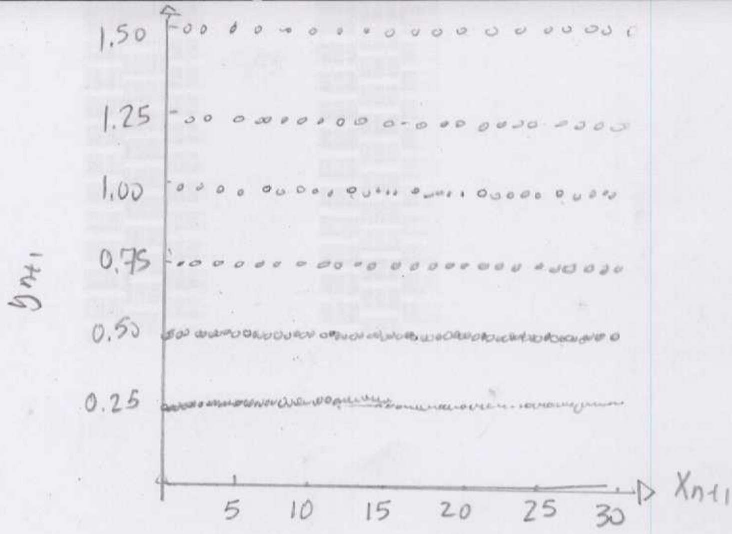
a) $\begin{bmatrix} X_{n+1} \\ y_{n+1} \end{bmatrix} = J \begin{bmatrix} X_n \\ y_n \end{bmatrix}$; $J = \begin{bmatrix} 1+k\cos X_n & 1 \\ k\cos X_n & 1 \end{bmatrix}$

$$|J| = |1 + k\cos X_n - k\cos X_n| = 1$$

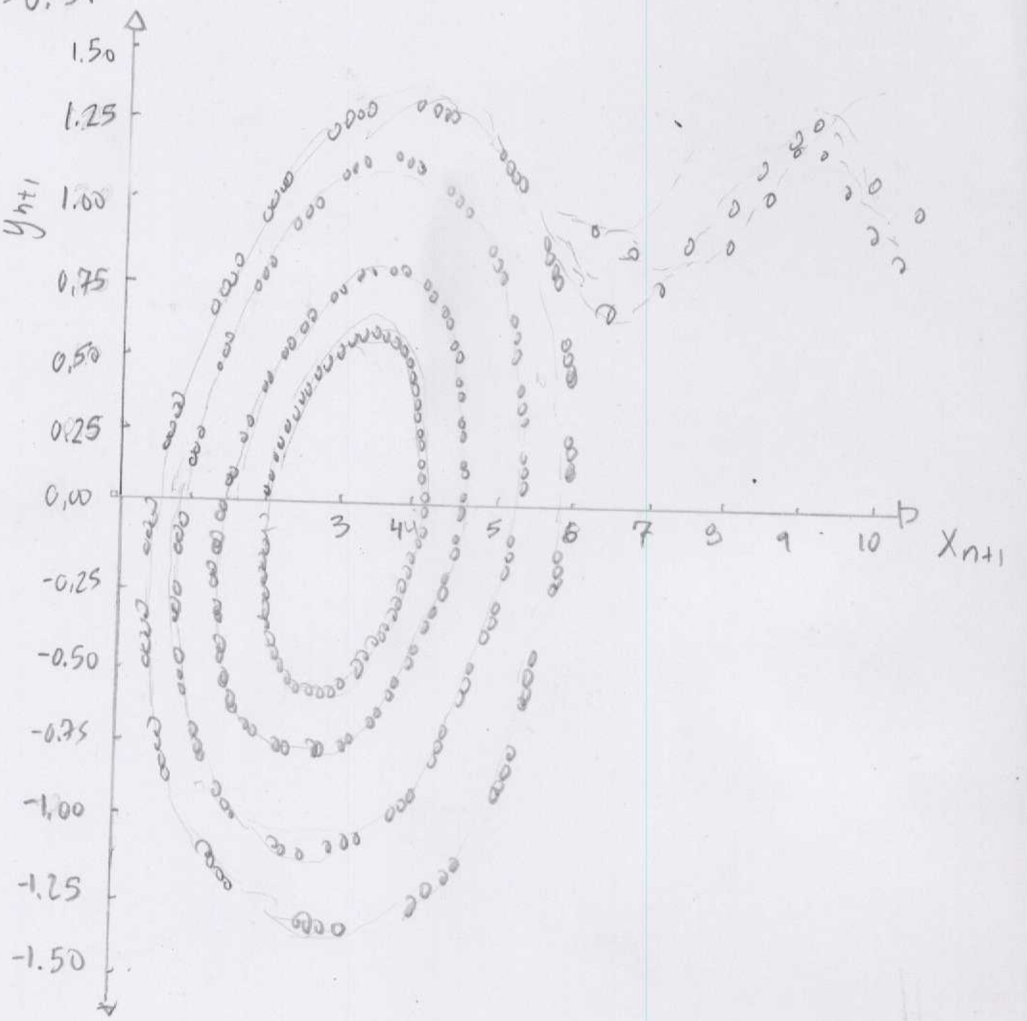
b) $k=0$: $X_{n+1} = X_n + y_n$

$y_{n+1} = y_n$

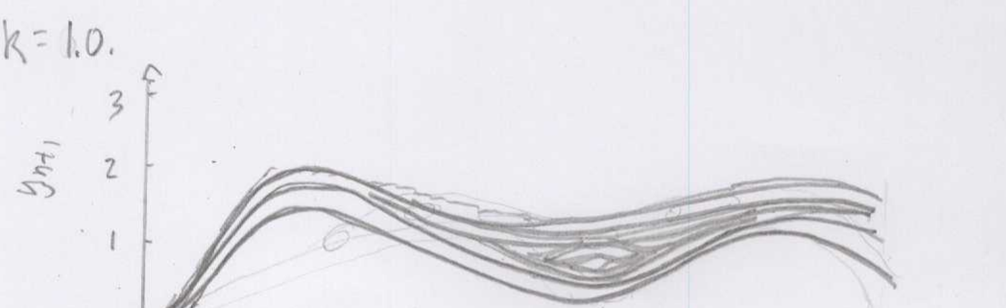
$\underline{\text{Fixed points}}$: $(x^*, y^*) = (0,0), (x,$


1.50
1.25
1.00
0.75
0.50
0.25
y_{n+1}
5 10 15 20 25 30
X_{n+1}

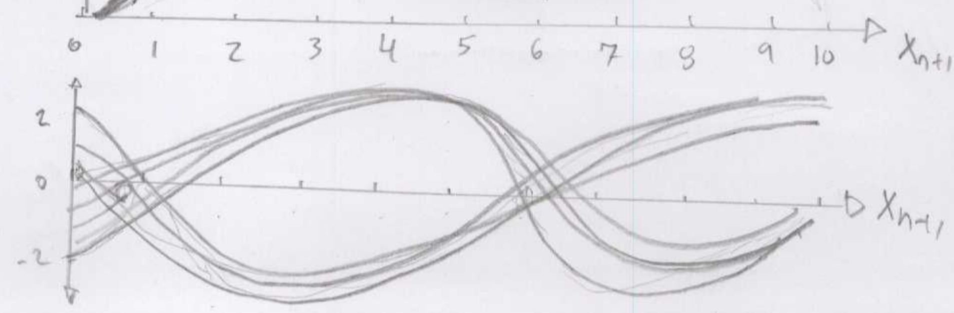
b) k=0.5.


1.50
1.25
1.00
0.75
0.50
0.25
0.00
-0.25
-0.50
-0.75
-1.00
-1.25
-1.50
y_{n+1}
3 4 5 6 7 8 9 10
X_{n+1}

d) k=1.0.


3
2
1
y_{n+1}
0 1 2 3 4 5 6 7 8 9 10
X_{n+1}

e)


2
0
-2
y_{n+1}
X_{n+1}

$T'''T''T'$

$x_{n+1} = y_n + 1 - a x_n^2$

$y_{n+1} = b x_n$

12.2.1 $T = T'''T''T'$

T: $x_{n+1} = y_n + 1 - a x_n^2$ "original"

$y_{n+1} = b x_n$

T''': $x''' = y_{n+1}$ "inverted x, y"

$y''' = x_{n+1}$

T'': $x'' = x$ "substituted the x"

$y'' = y_n + 1 - a x^2$

T': $x' = x_n$

$y' = y_n + 1 - a x_n^2$ "substituted the y"

12.2.2. T': $J = \begin{vmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -2ax_n & 1 \end{vmatrix}$

$|J| = |(1)(1) + (0)(2ax_n)| = 1$

T'': $J = \begin{vmatrix} \frac{\partial x''}{\partial x} & \frac{\partial x''}{\partial y} \\ \frac{\partial y''}{\partial x} & \frac{\partial y''}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$

$|J| = |b(1) - (0)(1)| = 1$

T''': $J = \begin{vmatrix} \frac{\partial x'''}{\partial x} & \frac{\partial x'''}{\partial y} \\ \frac{\partial y'''}{\partial x} & \frac{\partial y'''}{\partial y} \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$

$|J| = |(0)(0) - (0)(0)| = 0$

12.2.3.

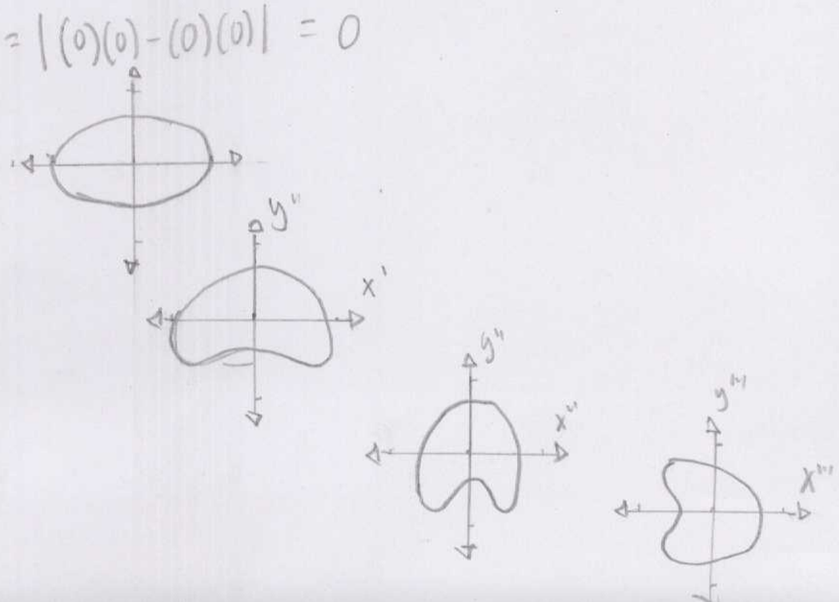
a).

T

T'

T''

T'''



The flip bifurcation ($|\lambda| = -1$) and $a = \frac{3}{4}(1-b)^2$

has a solution $x = \dfrac{2-2b}{3b^2-6B+3}$

(Henon Map) 12.2.7. $-1 < b < 1$; $x_{n+1} = y_n + 1 - a x_n^{\,2}$

$$y_{n+1} = b x_n$$

Question 12.2.7 shows the solution, $x^* = \dfrac{2-2b}{3b^2-6b+3}$

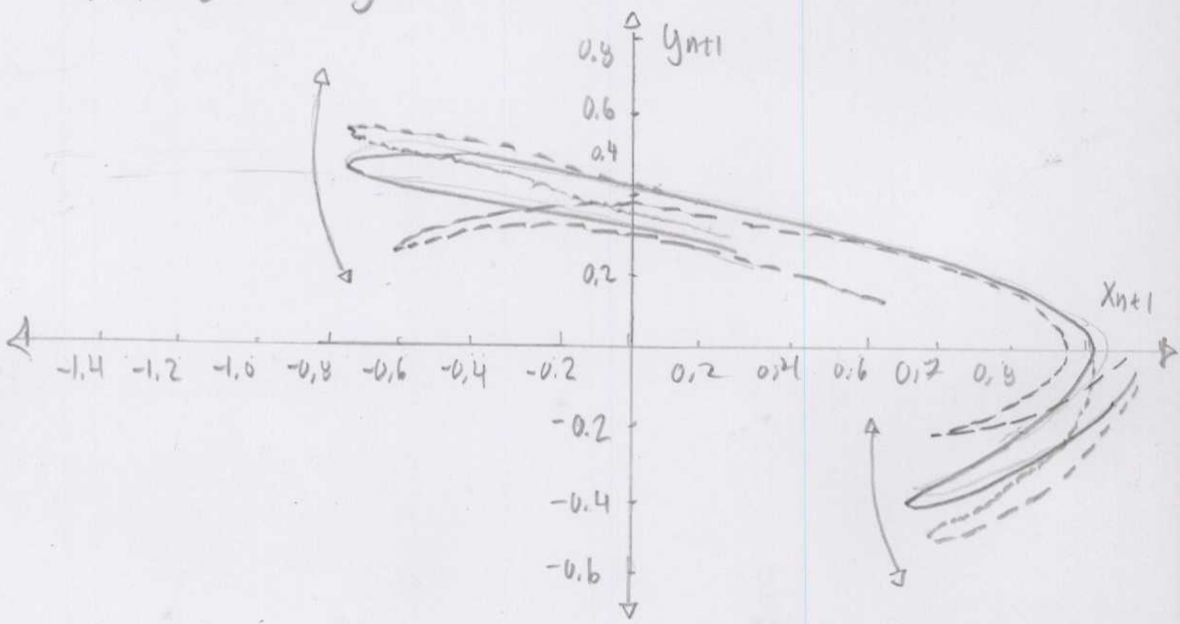
with a flip bifurcation ($\lambda = -1$) and $a = \frac{3}{4}(1-b)^2$.

where $-1 < b < 1$.

12.2.8

a) $b = 0.3$, $a = 1.06$

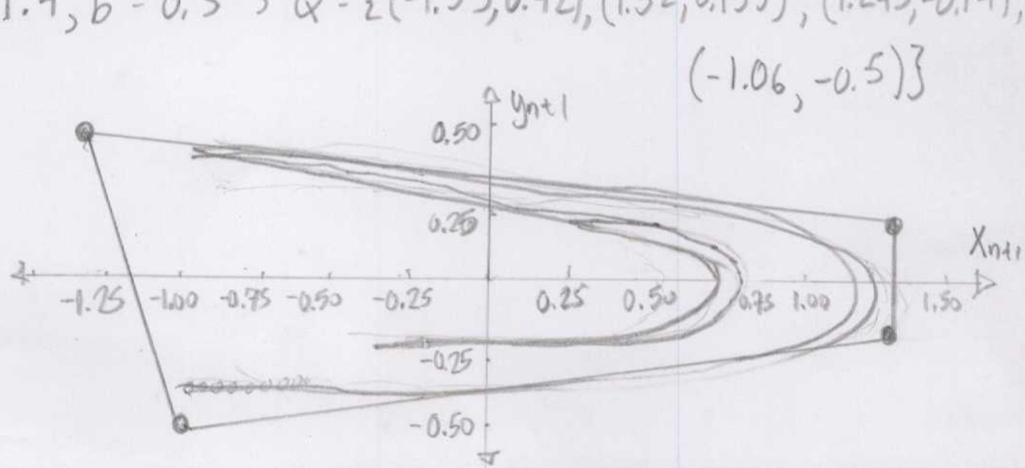
The parameters adjusted the graph, but the "b" variable a good amount.



b) The attractor at $a = 1.3$ is a Star trek insignia on a side, or about 90°-clockwise.

12.2.9. $a = 1.4$, $b = 0.3$; $Q = \{(-1.33, 0.42), (1.32, 0.133), (1.245, -0.14), (-1.06, -0.5)\}$

a)



b) See part a).

$x_{n+1} = y_n + 1 - a x_n^2$

$y_{n+1} = b x_n$

12.2.4. Fixed Points: $a x_n^2 + x - 1 = y_n = 0$

$$x^* = \frac{\pm\sqrt{4a-1} - 1}{2a} \; ; \; y^* = b\left(\frac{\pm\sqrt{4a-1} - 1}{2a}\right)$$

12.2.5. Eigenvalues:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_n}{\partial x} - \lambda & \frac{\partial x}{\partial y} \\ \frac{\partial y_n}{\partial x} & \frac{\partial y}{\partial y} - \lambda \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

$$= \begin{bmatrix} -2ax - \lambda & 1 \\ b & -\lambda \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

$$(-2ax - \lambda)(-\lambda) - b = 0$$

$$\lambda = \pm\sqrt{(ax)^2 + b} - ax$$

12.2.6. Fixed Points: From 12.2.5:

$$\lambda = \pm\sqrt{(ax)^2 + b} - ax$$

Stability: $|\lambda| < 1$: Linearly Stable

$$|\lambda_1| = \left|\sqrt{(ax)^2 + b} - ax\right| < 1$$

Stable:

$$\begin{cases} a = 0, \; 0 \le b < 1 \\ a > 0, \; b \ge 0, \; x > \frac{b-1}{2a} \\ a > 0, \; -1 < b < 0, \; x \ge \sqrt{\frac{-b}{a^2}} \\ a > 0, \; -1 < b < 0, \; \frac{b-1}{2a} < x \le -\sqrt{\frac{-b}{a^2}} \\ a > 0, \; b \le -1, \; x \ge \sqrt{\frac{-b}{a^2}} \end{cases}$$

$$|\lambda_2| = \left|-\sqrt{(ax)^2 + b} - ax\right| < 1$$

unstable:

$$\begin{cases} a < 0 \quad b \le -1 \quad x > \frac{b-1}{2a} \\ a < 0 \quad -1 < b < 0 \quad \frac{1-b}{2a} < x \le \sqrt{\frac{-b}{a^2}} \\ a < 0 \quad -1 < b < 0 \quad x \ge \sqrt{\frac{-b}{a^2}} \\ a < 0 \quad b \ge 0 \quad x > \frac{1-b}{2a} \end{cases}$$

Stable $\left[\; a = 0 \quad 0 \le b < 1 \quad x \in \mathbb{R} \right.$

b) max $T(Q) = (-1.3064, 0.396) < y = 0.42$

min $T(Q) = (-1.0565, -0.399) > y = -0.50$

12.2.10. An unstable fixed point diverges toward infinity, so $|\lambda| > 1$.

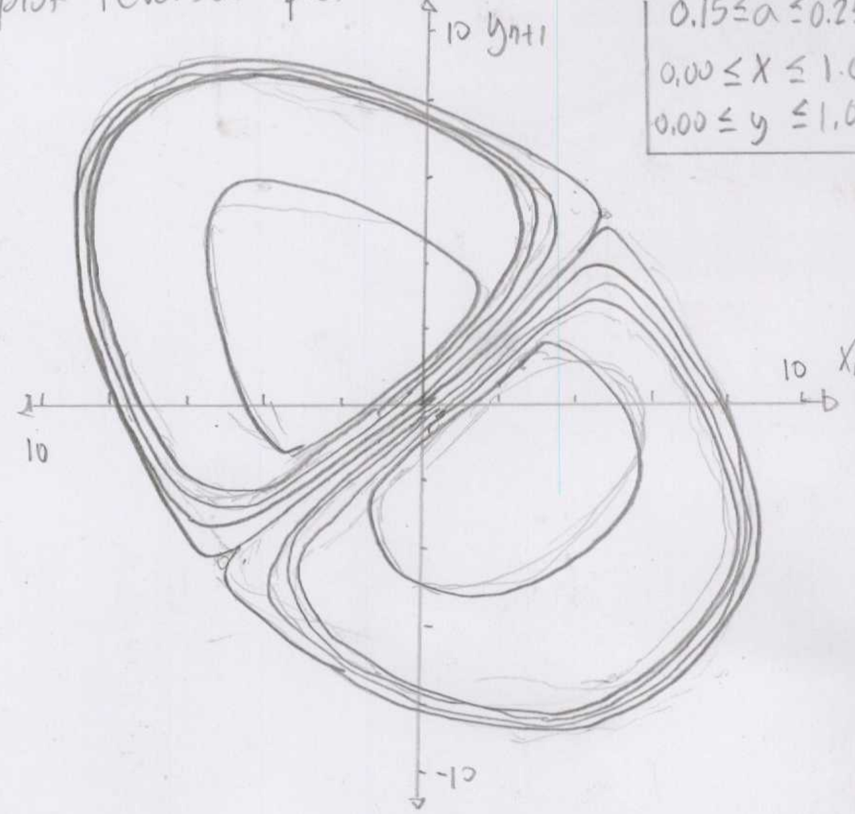
$$|\lambda| = |\pm\sqrt{(ax)^2 + b} - ax| > 1$$

If $a \geq 0$, then $a = 0$, and $b > 1$

12.2.11. $a = 0, b = 0$ produce $x_{n+1} = 1$ and $y_{n+1} = 0$.

12.2.12. When the sign of b is negative then the plot reverses plot order.

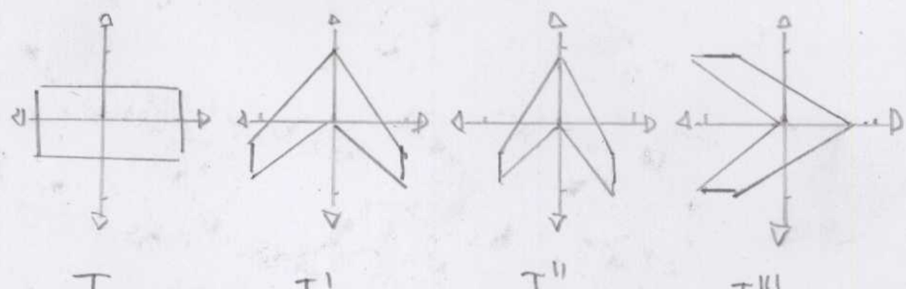
12.2.13, $b = 1$;



$0.15 \leq a \leq 0.25$
$0.00 \leq x \leq 1.00$
$0.00 \leq y \leq 1.00$

$x_{n+1} = 1 + y_n - a|x_n|$
$y_{n+1} = bx$

12.2.14 (Lozi Map)



$x_{n+1} = 1 + y_n - a|x_n|$
$y_{n+1} = bx_n$

12.2.15. $$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_{n+1}}{\partial x} - \lambda & \frac{\partial x_{n+1}}{\partial y} \\ \frac{\partial y_{n+1}}{\partial x} & \frac{\partial y_{n+1}}{\partial y} - \lambda \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

$$J = \begin{bmatrix} -a - \lambda & 1 \\ b & -0 \end{bmatrix} = -b$$

Area-contracting: $|\lambda| < 1$ or $|\det(J)| < 1$

12.2.16. Fixed Points: $x_{n+1} = 1 + y_n - a|x_n|$

$y_{n+1} = bx_n$

$(x^*, y^*) = (-1, a-2)$ for $b = 2-a$

$(1, a)$ for $b = a$

Stability: $|f'(-1, a-2)| = \begin{vmatrix} -1 \\ -b \end{vmatrix} > 1$ "unstable"

$|f'(1, a)| = \begin{vmatrix} 1 \\ 1 \end{vmatrix} > 1$ "unstable"

Stability is by ~~derivative~~ or eigenvalues with trace and determinant. The notation above fits two equation maps.

12.2.17 $x_{n+1} = 1 + y_n - a|x_n|$

$y_{n+1} = bx_n$

$x_{n+2} = 1 + bx_n - a|1 + y_n - a|x_n||$

$y_{n+2} = b(1 + y_n - a|x_n|)$

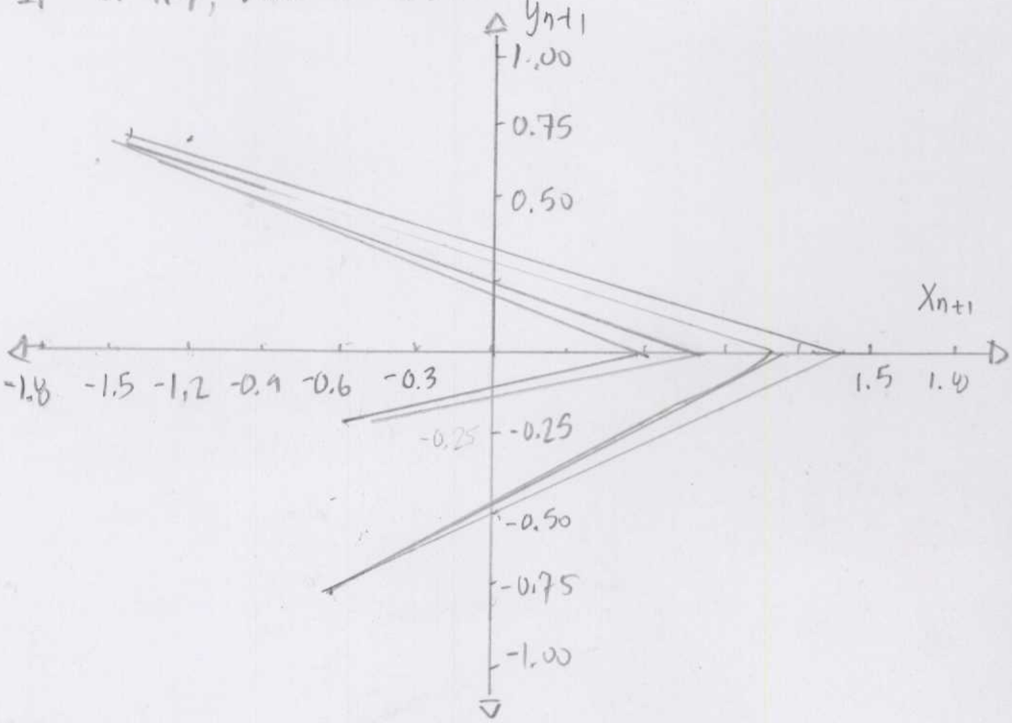
$$y = \frac{b(a|x^2| - 1)}{(b-1)}$$

$$X = bX_n + 1 - a\left|\frac{b(a|X|^2-1)}{(b-1)} + 1 - a|X|\right|$$

If $a = 0$, $b < 1-a$;

$$X = \frac{1}{a-b+1} \; ; \; y = \frac{b\left(a\left|\frac{1}{a-b+1}\right| - 1\right)}{(b-1)}$$

12.2.18. If $a = 1.7$, and $b = 0.5$:



$\dot{x} = -y - z$

$\dot{y} = x + ay$

$\dot{z} = b + z(x-c)$

12.3.1. $b = 2$, $c = 4$, $0 < a < 0.4$ (Rössler System)

a) <u>Fixed points:</u> $\dot{x} = 0 = -y - z$

$\dot{y} = 0 = x + ay$

$\dot{z} = 0 = b + z(x-c)$

$$(x^*, y^*, z^*) = \left(\frac{c \pm \sqrt{c^2-4ab}}{2}, \frac{\mp\sqrt{c^2-4ab} - c}{2a}, \frac{c \pm \sqrt{c^2-4ab}}{2a}\right)$$

<u>Coordinate change:</u> $\dot{x} = -\left(y \pm \frac{\mp\sqrt{c^2-4ab} - c}{2a}\right) - \left(z + \frac{c \pm \sqrt{c^2-4ab}}{2a}\right)$

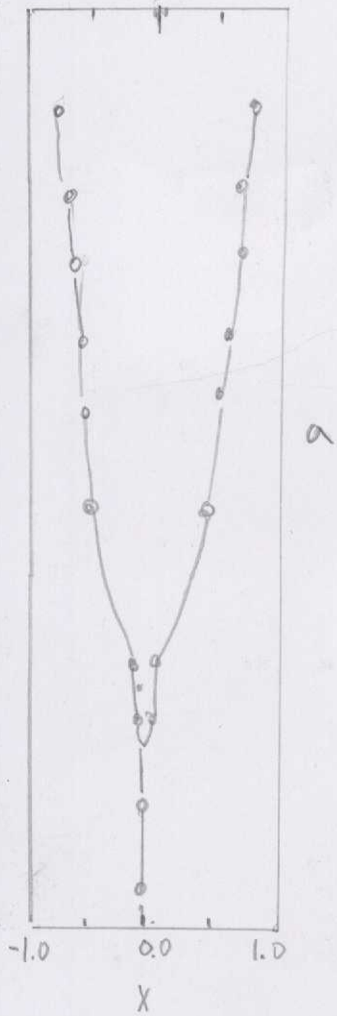
$$\dot{y} = \left(x + \frac{c \pm \sqrt{c^2-4ab}}{2}\right) + a\left(y + \frac{\mp\sqrt{c^2-4ab} - c}{2a}\right)$$

$$\dot{z} = b + \left(z + \frac{c \pm \sqrt{c^2-4ab}}{2a}\right)\left(x \pm \frac{c \pm \sqrt{c^2-4ab}}{2} - c\right)$$

<u>Eigenvalues:</u> $\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\lambda & & -1 \\ & a-\lambda & 0 \\ z + \frac{c \pm \sqrt{c^2-4ab}}{2a} & 0 & x + \frac{c \pm \sqrt{c^2-4ab}}{2} - c - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

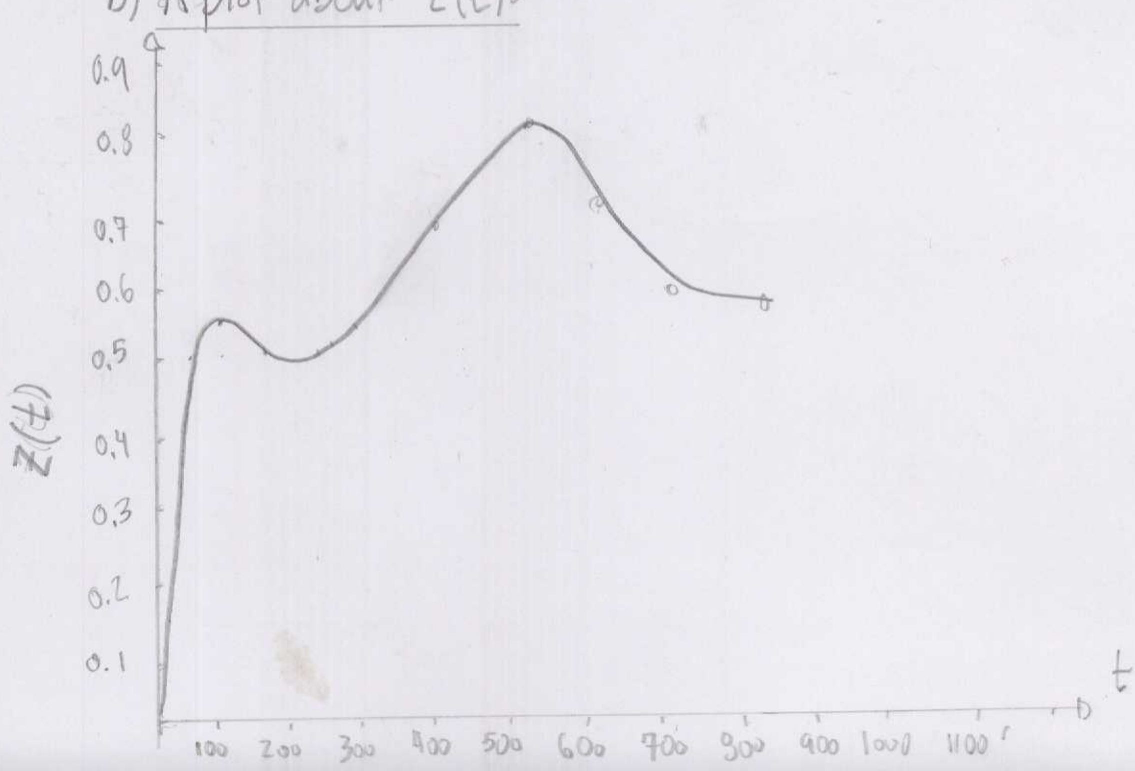
The functional method is the maximum and minimum eigenvalues as a function of "a".
They were large trinomials.
As so, a table was the next option:

a	X
0.36	0.70097, -0.76869
0.35	0.67647, -0.72824
0.34	0.636505, -0.68900
0.33	0.60562, -0.65097
	0.575621, -0.60885
	0.54649, -0.57848
⋮	
0.21	0.077587, -0.07678
0.209	0.05779, -0.05779
⋮	
<0.209	Null



Initial conditions: $X_0 = 0$, $y_0 = 0$, $Z_0 = 0$.

b) A plot about $Z(t)$:

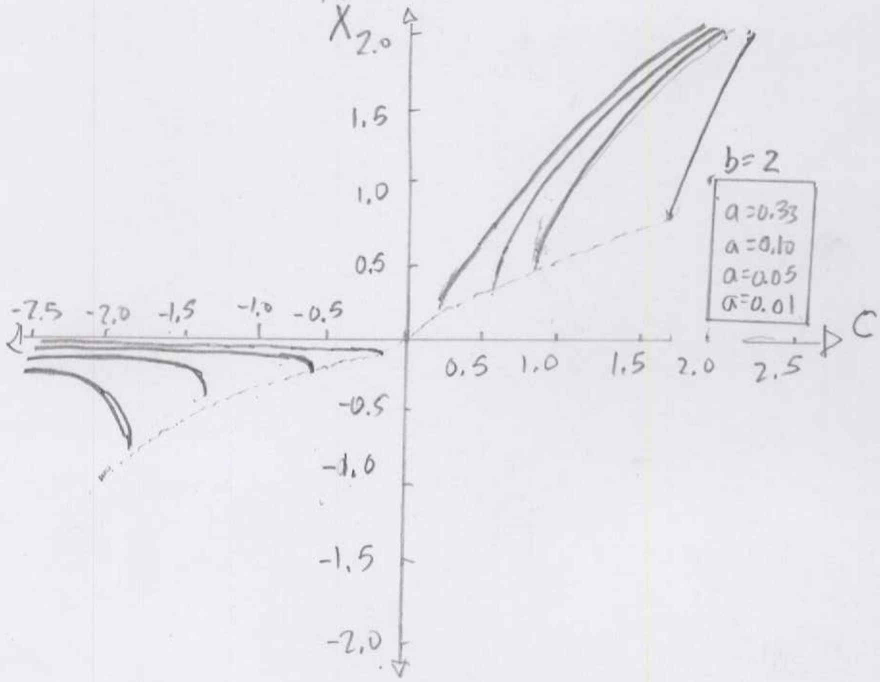


12.3.2. Fixed Points: ~~From problem 12.3.1 a)~~

From problem (12.3.1a)

$$(x^*, y^*, z^*) = \left(\frac{c \mp \sqrt{c^2-4ab}}{2}, \frac{\mp\sqrt{c^2-4ab} - c}{2a}, \frac{c \pm \sqrt{c^2-4ab}}{2a}\right)$$

"Hopf Bifurcation, supposedly."



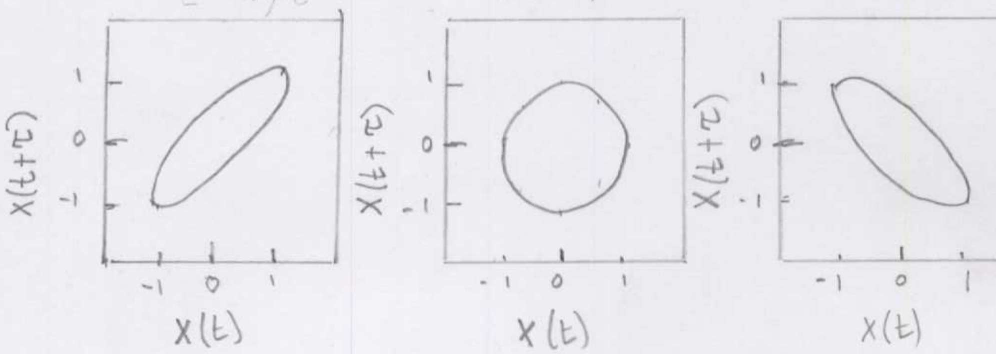
The trapping region is when $x^* = \frac{c \pm \sqrt{c^2-4ab}}{2}$.

12.3.3. The Rössler system is hard because the eigenvalues.

$x(t) = \sin(t)$

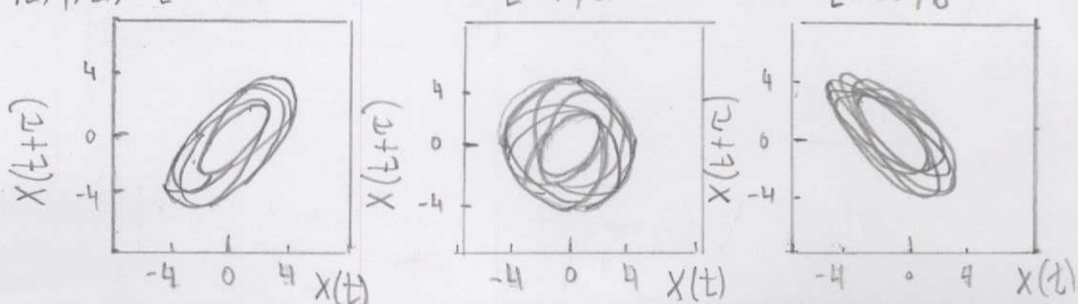
12.4.1 $x(t) = (x(t), x(t+\tau))$; $0 < \tau < \frac{\pi}{2}$

$\tau = \pi/6$ $\quad$ $\tau = \pi/2$ $\quad$ $\tau = 5\pi/6$



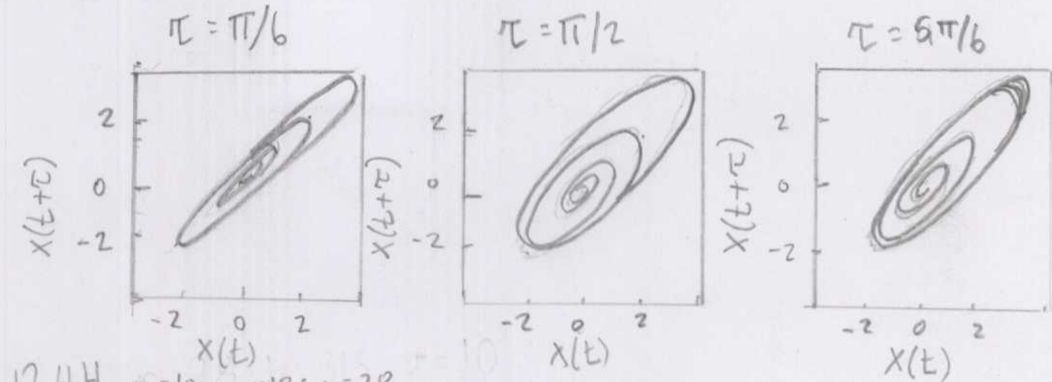
$x(t) = 3\sin t + \sin(\sqrt{2}t)$

12.4.2. $\tau = \pi/6$ $\quad$ $\tau = \pi/2$ $\quad$ $\tau = 5\pi/6$

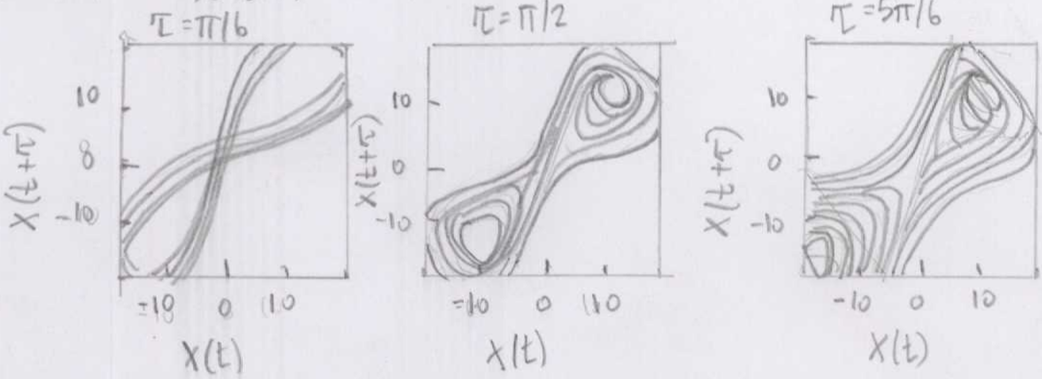


The charts look torus-like.

(Rössler System) 12.4.3. $a=0.4, b=2; c=4$

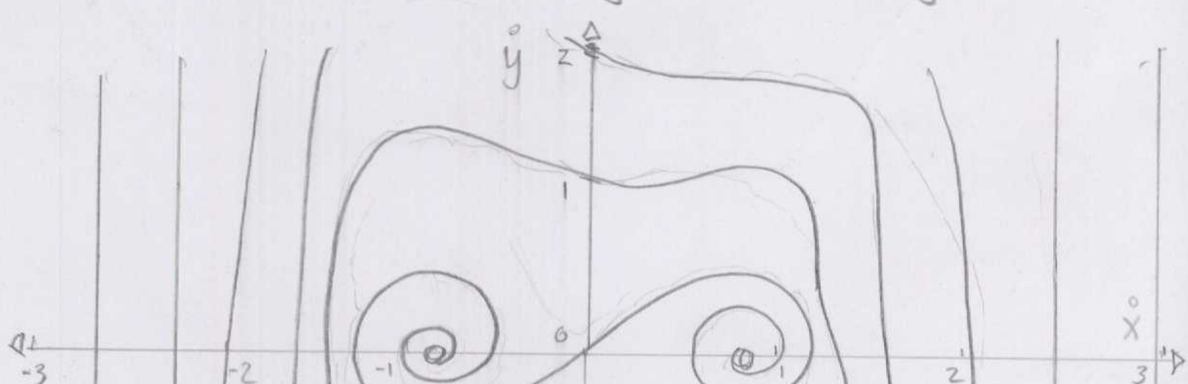


(Lorenz Equations) 12.4.4. $\sigma=10; b=8/3; r=28$



$\ddot{x}+\delta\dot{x}-x+x^3=F\cos\omega t$ 12.5.1. $\delta=0.25, F=0$; $\dot{y}=\ddot{x}=-\delta\dot{x}+x-x^3$; $\dot{x}=y$

$=-\delta y+x-x^3$ $\qquad =y$



As δ increases from 0.25, the gradient descent is steep. Also, the unforced system depends heavily on δ in an unforced system.

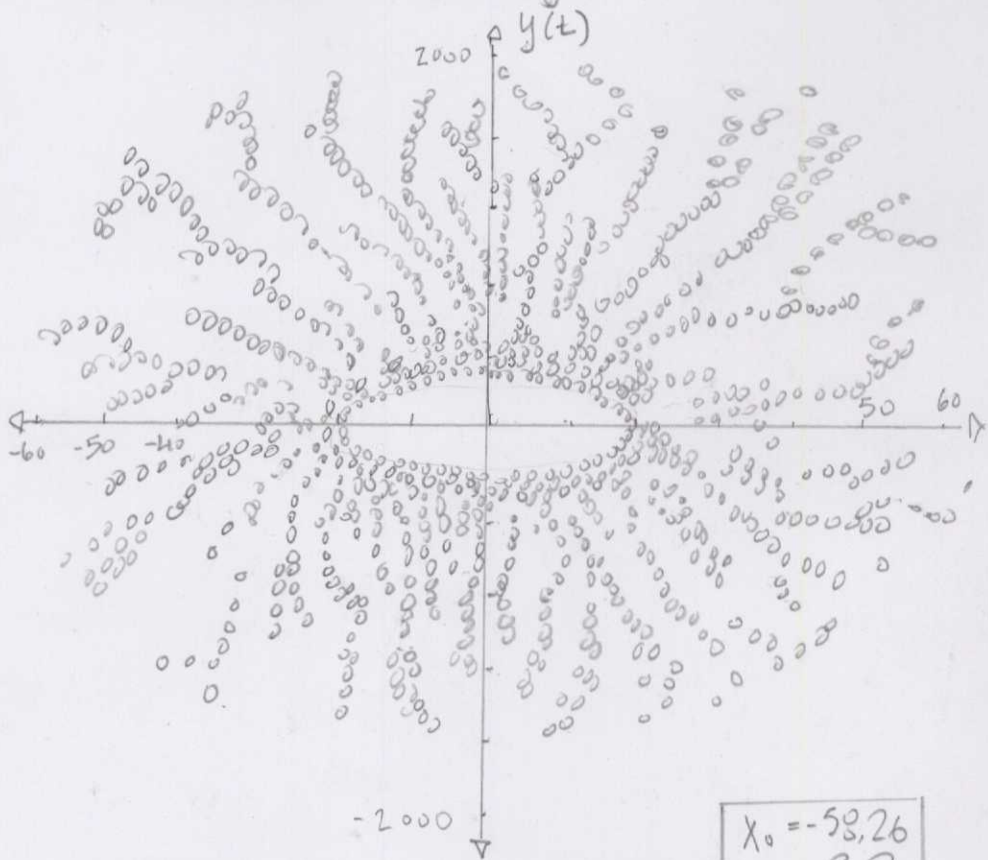
12.5.2. $$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -1+3x^2 & \delta-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\lambda_{1,2} = \pm\, 0.575724$; $x = 0.075$

$\Delta = \lambda_1 \lambda_2 = (-)$; $\tau = \lambda_1 + \lambda_2 = (0)$; $\tau^2 - 4\Delta > 0$

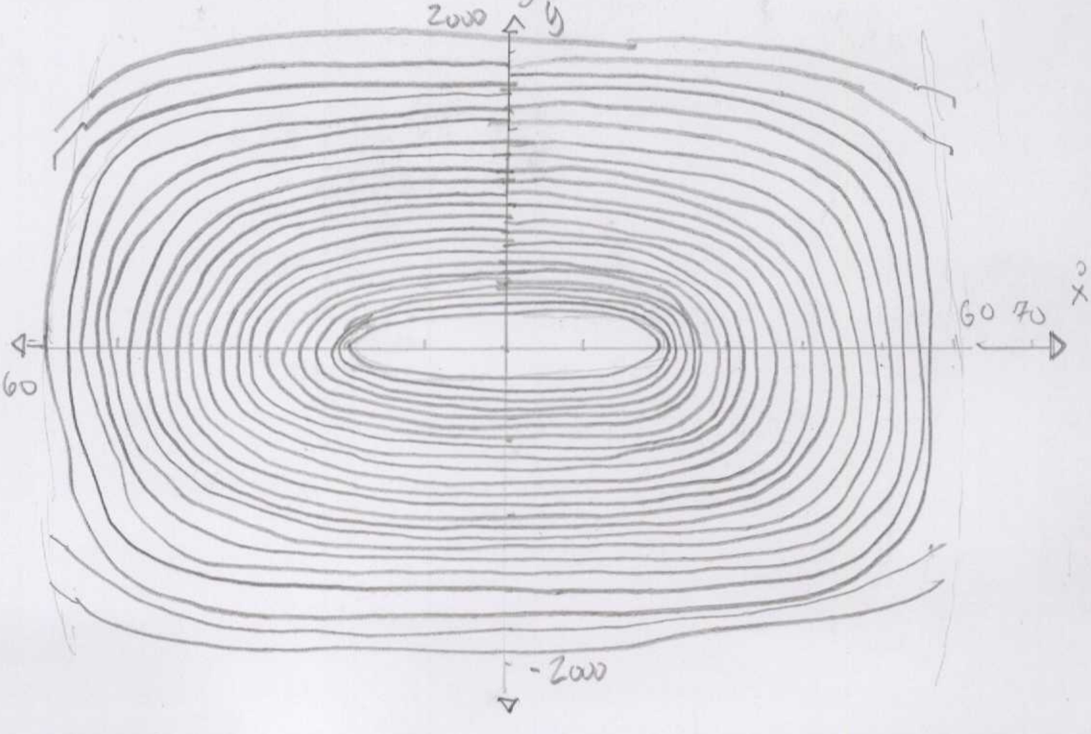
"Stable nodes, Saddle points"

Poincare Section: by Runge-Kutta 4th order



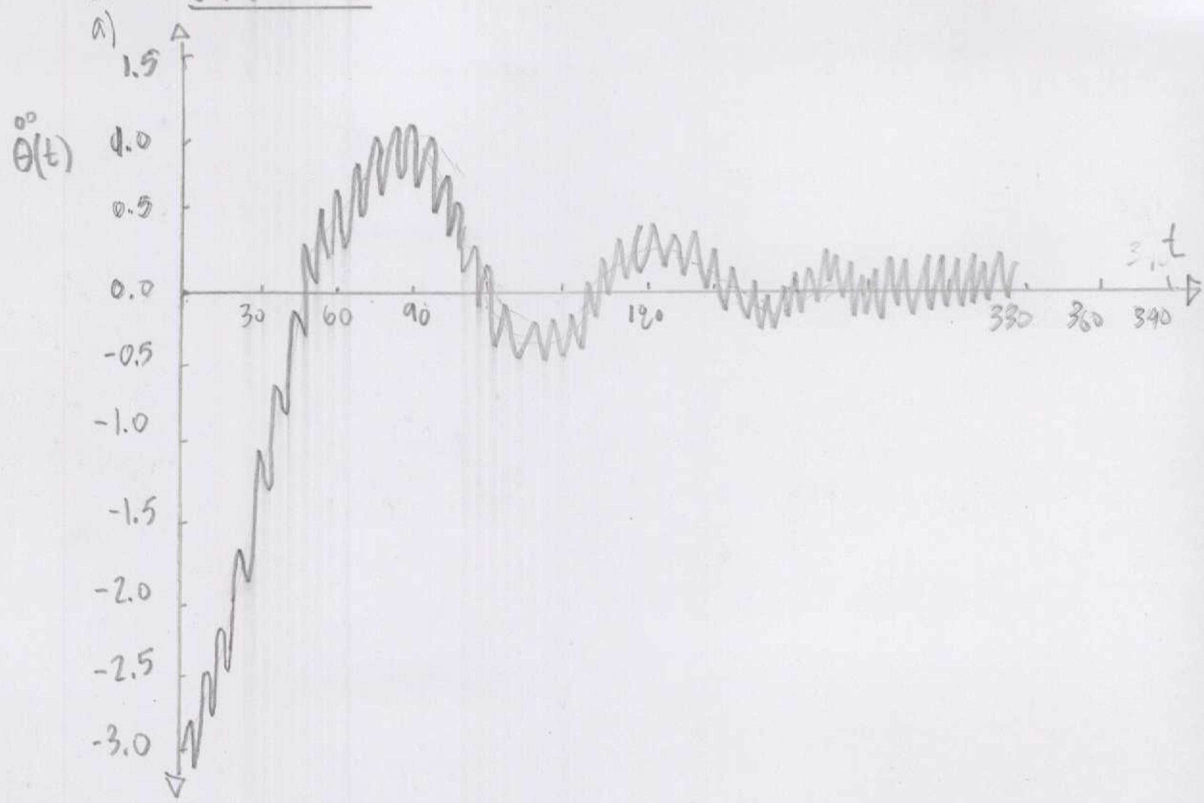
$x_0 = -58.26$
$y_0 = -3.3$

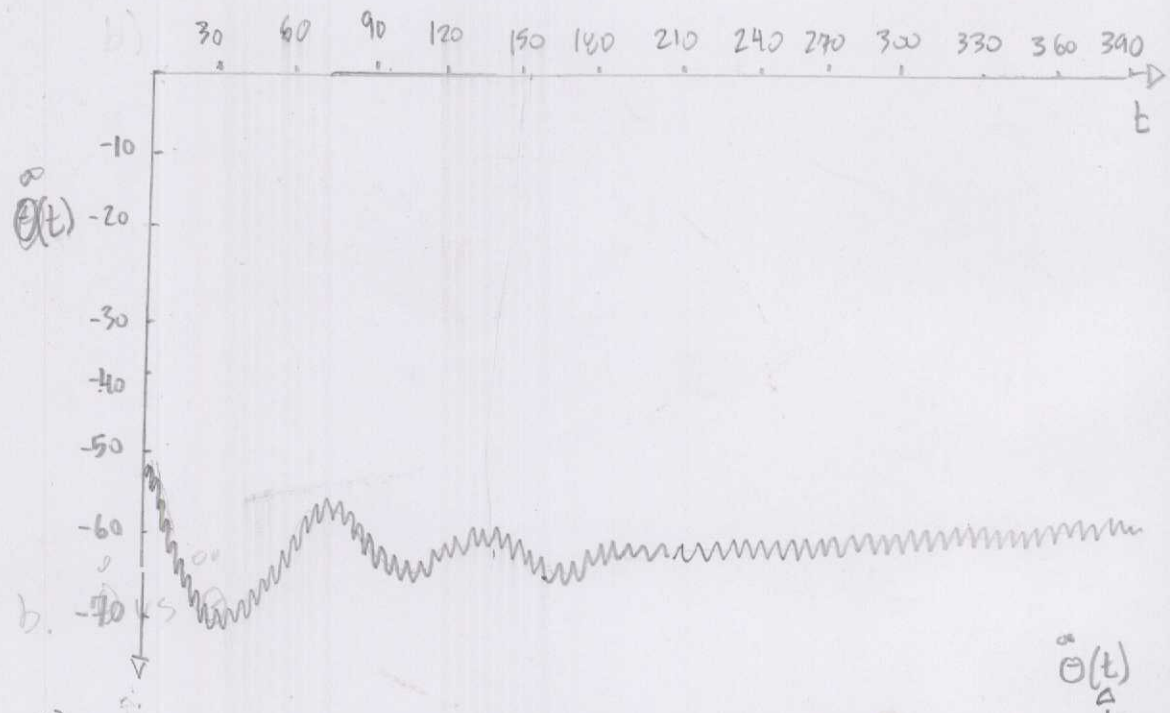
$\ddot{x} + k\dot{x} + x^3 = B\cos t$ 12.5.3, $k = 0.1$, $B = 12$, $x_0 = -58.26$, $y_0 = -3.3$,



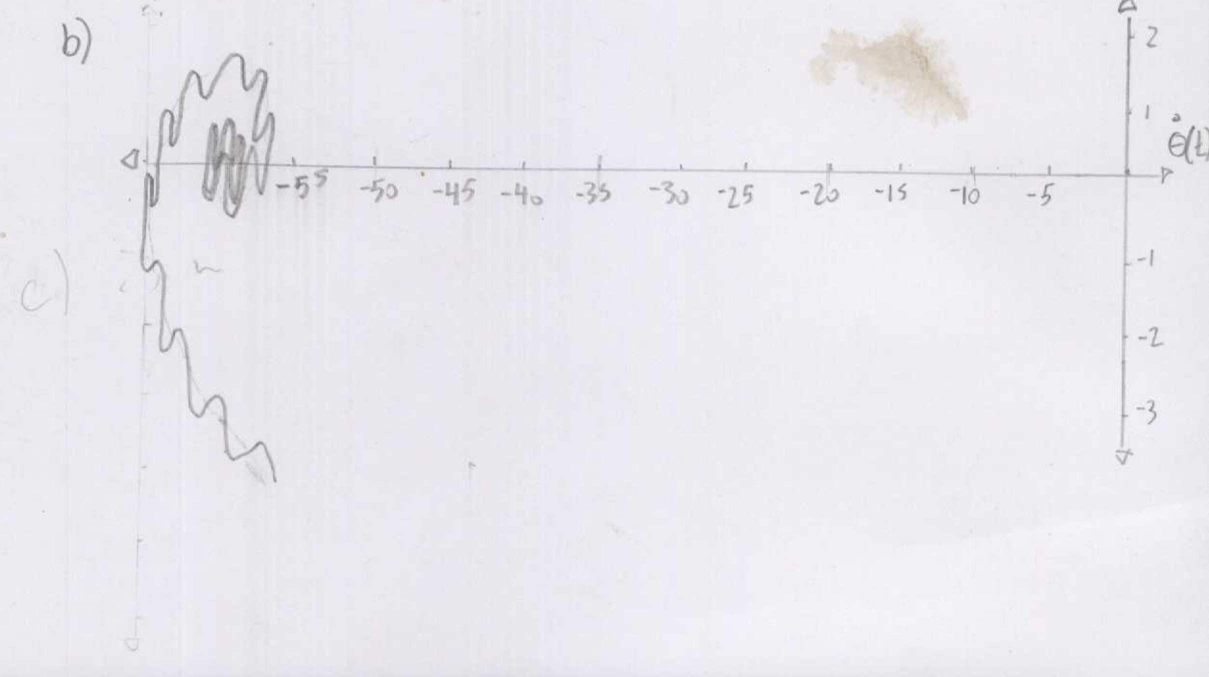
12.5.5. $\ddot{\theta}(t)$ vs t:

a)

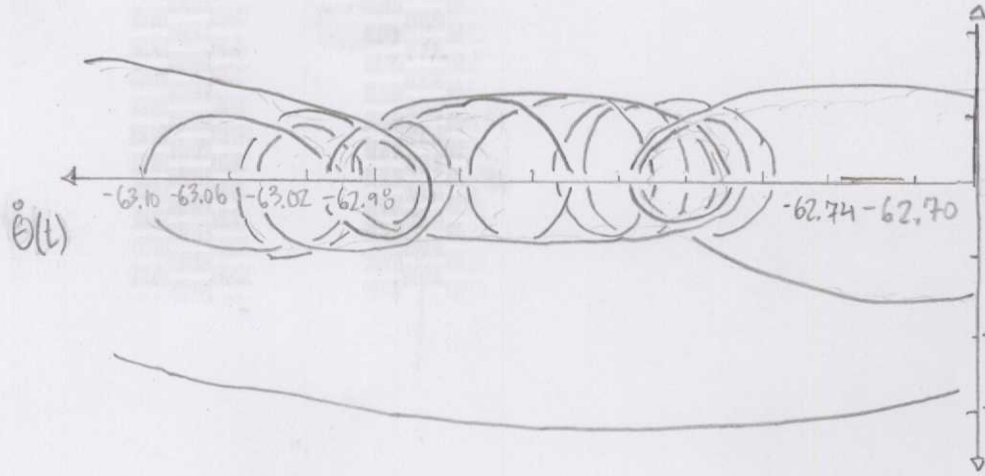




b)



c)



$\overset{\circ\circ}{\theta} + b\overset{\circ}{\theta} + \sin\theta = F\cos t$

12.5.5

a. $b = 0.2,\ F = 2$

<u>Fixed points:</u> $\overset{\circ}{u} = \overset{\circ}{\theta} = v$

$\overset{\circ}{v} = \overset{\circ\circ}{\theta} = F\cos t - b\overset{\circ}{\theta} - \sin\theta$

$= F\cos t - bv - \sin u$

$\overset{\circ}{u} = 0$

$\overset{\circ}{v} = 0 = F\cos t - bv - \sin u$

$(\overset{\circ}{u}^*, \overset{\circ}{v}^*) = (\overset{\circ}{\theta}^*, \overset{\circ\circ}{\theta}^*) = \left(0, \frac{F\cos t}{b}\right)$

$@\, t = \frac{\pi}{2}$ and $\frac{3\pi}{2}$

The pendulum oscillates both left and right in a pendular-fashion.

b). <u>Fractal</u>