Different Probabilities: $\rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1| = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}$

Incoherence: $\rho = \sum p_i |i\rangle\langle i|$ if Hamiltonian $H = \frac{\Omega}{2}\sigma_z$ ‖ $\frac{d\rho}{dt} = -i[H,\rho] + \underbrace{\frac{\gamma}{2}(\sigma_z \rho \sigma_z - \rho)}_{\text{action of environment}}$

γ = coupling strength

Amplitude Damping: $\frac{d\rho}{dt} = -i[H,\rho] + \gamma[\sigma_- \rho \sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}]$

Bloch Sphere: $\rho = \frac{1}{2}(1 + s \cdot \sigma) = \frac{1}{2}\begin{pmatrix} 1+s_z & s_x - is_y \\ s_x + is_y & 1 - s_z \end{pmatrix}$

where $s = (s_x, s_y, s_z)$; $s_i = \langle \sigma_i \rangle = tr(\sigma_i \rho)$.

The relation between these parameters:

$\langle \sigma_x \rangle = tr(\sigma_x \rho) = tr\left(\sigma_x \begin{pmatrix} \rho_+ & q \\ q^* & \rho_- \end{pmatrix}\right) = q + q^*$

$\langle \sigma_y \rangle = tr(\sigma_y \rho) = tr\left(\sigma_y \begin{pmatrix} \rho_+ & q \\ q^* & \rho_- \end{pmatrix}\right) = i(q - q^*)$

$\langle \sigma_z \rangle = tr(\sigma_z \rho) = tr\left(\sigma_z \begin{pmatrix} \rho_+ & q \\ q^* & \rho_- \end{pmatrix}\right) = \rho_+ - \rho_-$

$tr(\rho^2) = \frac{1}{2}(1 + s^2)$

$s^2 = s_x^2 + s_y^2 + s_z^2 \leq 1$

$\frac{d\rho}{dt} = \begin{pmatrix} \frac{d\rho_+}{dt} & \frac{dq}{dt} \\ \frac{dq^*}{dt} & \frac{d\rho_-}{dt} \end{pmatrix} = -i[H,\rho] + \frac{\gamma}{2}(\sigma_z \rho \sigma_z - \rho)$

$\frac{d\rho_+}{dt} = 0$; $= -iH\rho + i\rho H + \frac{\gamma}{2}(\sigma_z \rho \sigma_z - \rho)$

$= -i\left(\frac{\Omega}{2}\sigma_z \rho + i\rho \frac{\Omega}{2}\sigma_z\right) + \frac{\gamma}{2}(\sigma_z \rho \sigma_z - \rho)$

$= -i(\Omega \sigma_z \rho) + \frac{\gamma}{2}(\sigma_z \rho \sigma_z - \rho)$

$= -i\Omega \sigma_z \rho + \frac{\gamma}{2}\sigma_z \rho \sigma_z - \frac{\gamma}{2}\rho$

$\frac{dq}{dt} = -(i\Omega + \gamma)q$

$\ln q = -(i\Omega + \gamma)t$; $q = e^{-(i\Omega+\gamma)t} q(0)$

$\rho(t) = \begin{pmatrix} |a|^2 & ab^* e^{-(i\Omega+\gamma)t} \\ a^* b e^{-(i\Omega+\gamma)t} & |b|^2 \end{pmatrix}$

After $\lim_{t \to 0} \rho(t) = \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix}$

$Tr[\hat{\rho}] = 1$, $Tr[\hat{\rho}^2] \leq 1$; $\langle O \rangle = Tr[\hat{O}\rho]$

Evolution Operator: $U(t) = e^{-iHt/\hbar}$, $|\psi\rangle = U(t)|\psi(0)\rangle$

Density Matrix Evolution: $i\hbar \partial_t \hat{\rho} = [\hat{H}, \hat{\rho}]$

or equivalently $\hat{\rho}(t) = U(t)\rho(0)U^\dagger(t)$, von Neumann Entropy $S_{VN} = -Tr[\hat{\rho}\ln\rho] = -\langle \ln \rho \rangle = -\sum \lambda_n \ln \lambda_n$

Shannon Entropy $S_S = -\sum P_q \ln P_q = -\langle \ln P_q \rangle$

Equilibrium Density Matrix: Microcanonical: $\rho_{mc} = \frac{1}{\Omega}\delta(E - H)$

Canonical: $\rho_c = \frac{1}{Z} e^{-H/k_B T}$

Grand Canonical: $\rho_{gc} = \frac{1}{Z} e^{-(H - \mu N)/k_B T}$

An unnormalized density matrix: $\rho_c^u(\beta) = e^{-H/k_B T}$ satisfies the differential

$\partial_\beta \rho_c^u = -H \rho_c^u(\beta)$ where $\tau = it$ $t \in [0, \beta\hbar]$

Pure States in Quantum Mechanics: $\langle O \rangle = \langle \psi | O | \psi \rangle$

Dirac Basis Representation: $\rho = |\psi\rangle\langle\psi|$ in terms $\langle O \rangle = \langle \psi | O | \psi \rangle = Tr[O\rho]$

$Tr[\hat{\rho}] = 1$; $Tr[\hat{\rho}^2] = 1$

Basis Representation: $|\psi\rangle = \sum c_n |n\rangle$; $\hat{\rho} = \sum \rho_{nm} |n\rangle\langle m|$ with $\rho_{nm} = \langle n|\rho|m\rangle = c_n c_m^*$

$\langle O \rangle = Tr[O\rho] = \sum_{nm} O_{mn} \rho_{nm}$

$$|\psi\rangle\langle\psi| = \begin{bmatrix}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_N\end{bmatrix}\begin{bmatrix}\alpha_0^* & \alpha_1^* & \cdots & \alpha_N^*\end{bmatrix} = \begin{bmatrix}|\alpha_0|^2 & \alpha_0\alpha_1^* & \cdots & \alpha_0\alpha_N^*\\ \alpha_1\alpha_0^* & |\alpha_1|^2 & & \vdots\\ & & \ddots & \\ \alpha_N\alpha_0^* & \alpha_N\alpha_1^* & \cdots & |\alpha_N|^2\end{bmatrix}$$

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} ; \rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}| = \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & 0 & 1\end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{bmatrix}$$

$\langle A\rangle_\psi = \langle\psi|A|\psi\rangle$ "pure state"

Density Matrix: $\rho = |\psi\rangle\langle\psi|$; $\rho^2 = \rho$: projector

$\rho^\dagger = \rho$: hermiticity

Trace: $\mathrm{Tr}\,D = \sum \langle n|D|n\rangle$

$\mathrm{Tr}(\rho) = 1$: Normalization

$\mathrm{Tr}\,D = \sum \langle n|\psi\rangle\langle\phi|n\rangle$

$= \sum \langle\phi|n\rangle\langle n|\psi\rangle = \langle\phi|\psi\rangle$

$\rho \geq 0$: positivity

$\langle\phi|\rho|\phi\rangle = \langle\phi|\psi\rangle\langle\psi|\phi\rangle$

$= |\langle\phi|\psi\rangle|^2 \geq 0$

$\langle A\rangle_\rho = \mathrm{Tr}(\rho A)$

$|\psi\rangle = \sum c_n|n\rangle$, where $A|n\rangle = a_n|n\rangle$ || $\langle A\rangle_\psi = \sum |c_n|^2 a_n = \sum \frac{N_n}{N} a_n$

Mixed state: $p_i = \frac{N_i}{N}$ where $\sum_i p_i = 1$ || $\rho_{mix} = \sum p_i \rho_i^{pure} = \sum p_i |\psi_i\rangle\langle\psi_i|$; $\langle A\rangle = \mathrm{Tr}(\rho A)$

$\langle A\rangle_{\rho_{mix}} = \sum p_i \langle\psi_i|A|\psi_i\rangle$; $\mathrm{Tr}(\rho A) = \mathrm{Tr}\left(\sum p_i|\psi_i\rangle\langle\psi_i|A\right) = \sum\sum p_i\langle n|\psi_i\rangle\langle\psi_i|A|n\rangle$

$= \sum p_i\langle\psi_i|A\sum|n\rangle\langle n|\psi_i\rangle = \sum p_i\langle\psi_i|A|\psi_i\rangle$

$\rho_{mix}^2 = \sum\sum p_i p_j|\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j| = \sum p_i^2|\psi_i\rangle\langle\psi_i| \neq \rho_{mix}$

$\mathrm{Tr}\,\rho_{mix}^2 = \sum\langle n|\sum\sum p_i p_j|\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j|n\rangle$

$= \sum\sum p_i p_j\langle\psi_i|\psi_j\rangle\langle\psi_j|\sum|n\rangle\langle n|\psi_i\rangle = \sum\sum p_i p_j|\langle\psi_i|\psi_j\rangle|^2 \leq \sum p_i^2 < \sum p_i = 1$

Time evolution ; Density Matrices

$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle \xrightarrow{\dagger} -i\hbar\langle\dot\psi| = \langle\psi|H$

$i\hbar\frac{\partial}{\partial t}\rho = i\hbar\sum p_i\left(|\dot\psi_i\rangle\langle\psi_i| + |\psi_i\rangle\langle\dot\psi_i|\right) = i\hbar\sum p_i\left(\frac{-i}{\hbar}H|\psi_i\rangle\langle\psi_i| + |\psi_i\rangle\left(\frac{i}{\hbar}\langle\psi|H\right)\right)$

$= \sum p_i(H\rho_i^{pure} - \rho^{pure}H) = [H,\rho]$ || $i\hbar\frac{\partial}{\partial t}\rho = [H,\rho]$

Time shift operator: $U(t,t_0) = e^{-\frac{i}{\hbar}H(t-t_0)}$; $\rho(t) = U(t,t_0)\rho(t_0)U^\dagger(t,t_0)$

$\mathrm{Tr}\,\rho^2(t) = \mathrm{Tr}(U\rho(t_0)U^\dagger U\rho(t_0)U^\dagger) = \mathrm{Tr}(\rho(t_0)\rho(t_0)U^\dagger U) = \mathrm{Tr}\,\rho^2(t_0)$

if $|n\rangle$ be $\hat{H}$, such that $\hat{H}|n\rangle = E|n\rangle$, then $\rho_{nm}(t) = \rho_{nm}(0)e^{-i(E_n-E_m)t/\hbar}$

General: $\hat{\rho} = \sum p_i |\psi_i\rangle\langle\psi_i|$ with normalization $\sum p_i = 1$ in terms $\langle O\rangle = \sum p_i \langle\psi_i|\hat{O}|\psi_i\rangle = Tr[O\hat{\rho}]$

$Tr[\rho] = 1$, $Tr[\rho^2] \le 1$, $p_i^2 \le 1$

Thermal Equilibrium: Hamiltonian eigenstates representation

$\rho = \frac{1}{Z}\sum e^{-E_n/k_BT}|n\rangle\langle n|$ Taking the diagonal form $p_n = \frac{1}{Z}e^{-E_n/k_BT}$

Position Representation: $\hat{x}|x\rangle = x|x\rangle$

$\rho_{nm} = \frac{1}{Z}e^{-E_n/k_BT}\delta_{nm} = p_n\delta_{nm}$

The canonical representation: $\rho^u(x, \hat{x}, \beta)$

$= \langle x|e^{-\beta H}|x'\rangle = \sum \langle x|n\rangle\langle n|e^{-\beta H}|m\rangle\langle m|\hat{x}\rangle$

$= \sum \psi_n(x)\psi_n^* e^{-\beta E_n}$

Note $\rho^u(x, \hat{x}; \beta) \sim U(x, \hat{x}; t)$ when $t = -i\tau = -i\beta\hbar$

$U(t) = e^{-iHt/\hbar}$ to $\rho^u(\beta) = e^{-H\tau/\hbar}$; $\rho(\beta) = U(-i\beta\hbar)$

Recalling an operator differential satisfies density matrix:

$$\partial_\beta \rho^u(x, x'; \beta) = -H(\hat{p}, x)\rho^u(x, x'; \beta)$$

where $\hat{p} = -i\hbar\frac{\partial}{\partial x}$

Propagator Property: $\rho^u(x_2, x_0; \tau_1+\tau_2) = \int dx_1 \rho^u(x_2, x_1; \tau_2)\rho^u(x_1, x_0; \tau_1)$

The corresponding expectation value

$$\langle O\rangle = \int dx\, dx'\, O(x, x')\rho(x', x)$$

Path Integral Representation: replacing $t = -i\tau$, $S_E[\{q_n(\tau)\}] = \int_0^{\beta\hbar} d\tau L[\{q_n(\tau)\}]$

$\rho^u(x, x'; \beta) = \int_{x(0)=x,\, x(\beta\hbar)=x'} [dx(\tau)] e^{-S_E[x(\tau)]/\hbar}$

$Z = \sum e^{-S_E[\{q_n(\tau)\}]/\hbar} = \int dx \rho^u[x, x; \beta]$

$= \int_{-\infty}^{\infty} dx \int_{x(0)=x,\, x(\beta\hbar)=x} dx(\tau) e^{-S_E[x(\tau)]/\hbar}$

$H_{eff}[q(\tau)] = S_E[q(\tau)]/\hbar$

$\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau L[q(\tau)] \to \beta\hbar$

Density Matrix Examples

Spin 1/2 States: $|+, n\rangle$ $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$|\uparrow\rangle, |\downarrow\rangle$ with eigenstates $S_z|+, \hat{n}\rangle$

$= \cos\frac{\theta}{2}e^{-i\phi/2}|\uparrow\rangle + \sin\frac{\theta}{2}e^{i\phi/2}|\downarrow\rangle$

Corresponding to the density Matrix:

$\hat{\rho}^{Pure,n} = |+\rangle\langle +| = \cos^2\frac{\theta}{2}|\uparrow\rangle\langle\uparrow| + \sin^2\frac{\theta}{2}|\downarrow\rangle\langle\downarrow|$

$+ \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi}|\uparrow\rangle\langle\downarrow| + \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi}|\downarrow\rangle\langle\uparrow|$

The matrix elements would be

$$\rho^{Pure}_{\sigma,\sigma'} = \begin{pmatrix} \cos^2\frac{\theta}{2} & \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi} \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi} & \sin^2\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

When photons are perfectly polarized

$Tr[\rho] = 1$

$Tr[\rho^2] = 1$ and $\hbar/2$ gives $\hat{n}\cdot\vec{S}$. For $\theta = \pi/2, \phi = 0$

We find 0 for S_z and $\hbar/2$ S_x

For later quantization $\theta = \pi/2$, $\phi = 0$

S_x is $\rho^{Pure,x} = \frac{1}{2}(|0\rangle+|1\rangle)(\langle 0|+\langle 1|) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\rho = \frac{1}{2}(\mathbb{1} + \vec{a}\cdot\vec{\sigma})$; $\vec{a} = Tr(\rho\vec{\sigma}) = \langle\vec{\sigma}\rangle$ "Bloch Vector" ; $\vec{a} = 1$

pure state $\rho^2 = \rho \Rightarrow |\vec{a}| = 1$ || $\rho_{mix} = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) = \frac{1}{2}\mathbb{1}$

mixed state $\rho^2 \neq \rho$ $\quad |\vec{a}| < 1$ || $Tr\rho_{mix} = 1$; $Tr\rho_{mix}^2 = \frac{1}{2}$

$\langle A\rangle = \sum_i q_i \langle\psi_i|A|\psi_i\rangle$

$\langle\psi_i|A|\psi_i\rangle = tr[A|\psi_i\rangle\langle\psi_i|]$

$\langle A\rangle = \sum_i q_i tr[A|\psi_i\rangle\langle\psi_i|] = tr\{A\underbrace{\sum_i q_i|\psi_i\rangle\langle\psi_i|}\}$

$\rho = \sum_i q_i|\psi_i\rangle\langle\psi_i|$; $\langle A\rangle = tr(A\rho)$

Example:

$\rho = q|0\rangle\langle 0| + (1-q)|\psi\rangle\langle\psi\rangle$

Where $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$

$\rho = q + (1-q)\left[\cos^2\frac{\theta}{2}|0\rangle\langle 0| + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}|0\rangle\langle 1| + \sin^2\frac{\theta}{2}|1\rangle\langle 1|\right]$

$$= \begin{bmatrix} q + (1-q)\cos^2\frac{\theta}{2} & (1-q)\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\ (1-q)\cos\frac{\theta}{2}\sin\frac{\theta}{2} & (1-q)\sin^2\frac{\theta}{2} \end{bmatrix}$$

Ambiguity of Mixture

$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

Properties of Density Matrix:

$q_i \in [0,1]$, $\sum_i q_i = 1$

$\rho^\dagger = \rho$ Secondly, $tr(\rho) = \sum_i q_i tr(|\psi\rangle\langle\psi|) = \sum_i q_i\langle\psi|\psi\rangle = \sum_i q_i = 1$

if: $A = 1$, since $\langle 1\rangle = 1$ we again get $tr(\rho) = 1$

$\langle\phi|\rho|\phi\rangle$ is a sum of probabilities $|\langle\phi|\psi\rangle|^2$

$\langle\phi|\rho|\phi\rangle$ = Prob. of finding the system at state $|\phi\rangle$ given ρ

Besides normalization, the main property is positive semi-definite; so $\rho \geq 0$

$\langle\phi|\rho|\phi\rangle = \sum_i q_i|\langle\phi|\psi_i\rangle|^2 \geq 0$; eigenvalues are always non-negative

$\rho = \sum p_k|k\rangle\langle k|$ if $|\phi\rangle = |k\rangle$; $p_k \in [0,1]$ $\sum p_k = 1$

Defining $tr(\rho) = 1$; $\rho \geq 0$

Purity: $tr(\rho^2) = \sum p_k^2 \leq 1$; if pure state $tr(\rho^2) = 1$ and $\rho = |\psi\rangle\langle\psi|$; $p_i = 1$

Purity $= P = tr(\rho^2) \leq 1$ || $\frac{1}{d} \leq tr(\rho^2) \leq 1$ | Maximally disordered state $\rho = \frac{\mathbb{I}_d}{d}$

The Von Neumann Equation: $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$ || $\rho(t) = \sum q_i e^{-iHt}|\psi(0)\rangle\langle\psi(0)|e^{iHt} = e^{-iHt}\rho(0)e^{iHt}$

Differentiating: $\frac{d\rho}{dt} = (-iH)e^{-iHt}\rho(0)e^{iHt} + e^{-iHt}\rho(0)e^{iHt}(iH) = -iH\rho(t) + i\rho(t)H$

$\frac{d\rho}{dt} = i[H,\rho]$ where $\rho(t) = e^{-iHt}\rho(0)e^{iHt}$

Bloch's sphere and Coherence:

$\rho = \begin{pmatrix} p_+ & q \\ q^* & p_- \end{pmatrix}$ where $p_+ + p_- = 1$ and $|\psi\rangle = a|0\rangle + b|1\rangle$

$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix}$

Shannon's Entropy: $S = N^{-1}\log_2 \Omega$; $S = -p_0 \log_2(p_0) - (1-p_0)\log_2(1-p_0)$
$= -\sum_i p_i \log_2 p_i$

Appendix A: Path Integrals of Quantum Mechanics

Phase-Space Path Integral $|\psi(t)\rangle = U(t)|\psi(0)\rangle$

Given that $|\psi(t)\rangle$ satisfies the Schrödinger equation, the formal solution is given by $U(t) = e^{-\frac{i}{\hbar}Ht}$.

In a 1D coordinate representation, we have:

$$\langle x_f|\psi(t)\rangle = \int_{-\infty}^{\infty} dx_0 \langle x_f|U(t)|x_0\rangle\langle x_0|\psi(0)\rangle$$

$$\psi(x_n, t) = \int_{-\infty}^{\infty} dx_0\, U(x_n, x_0; t)\,\psi(x_0, 0)$$

Trotter Decomposition: $\hat{U}(t_n) = e^{-\frac{i}{\hbar}\hat{H}t} = \left(e^{-\frac{i}{\hbar}H\frac{t}{N}}\right)^N = U(\epsilon)U(\epsilon)\cdots U(\epsilon)$

$$= \prod_{n=1}^{N-1}\left[\int_{-\infty}^{\infty} dx_n\right] = \int_{x=0}^{x=t} Dx(t)\, e^{\frac{i}{\hbar}S[x(t)]}$$

where $S[x(t), p(t)] = \int_0^t dt\left[p\dot{x} - \frac{p^2}{2m} - V(x)\right]$

$$S[x(t)] = \int_0^t dt\left[\frac{1}{2}m\dot{x}^2 - V(x)\right]$$

$\partial_\beta \hat{\rho}(\beta) = -\hat{H}\hat{\rho}(\beta)$; $\mathrm{Tr}[T(x(\tau_1)\cdots x(\tau_n)\rho(\beta)] = \int Dx(\tau)x(\tau_1)\cdots x(\tau_n)e^{-\frac{1}{\hbar}S}$

$Z = \int_{u(r,0)=u(r,\beta\hbar)} Du(r,\tau)e^{-\frac{1}{\hbar}S_E[u(r,\tau)]}$; $S_E[u(r,\tau)] = \int_0^{\beta\hbar} d\tau\, d^d r\left[\frac{1}{2}\rho(\partial_\tau u)^2 + \frac{1}{2}\mu(\nabla u)^2\right]$

Coherent States path-integral : $|z\rangle = e^{-\frac{1}{2}|z|^2}e^{za^\dagger}|0\rangle = e^{-\frac{1}{2}|z|^2}\sum \frac{z^n}{\sqrt{n!}}|n\rangle$

$$a|z\rangle = z|z\rangle$$

Nontrivial overlap $\langle z_1|z_2\rangle = e^{-\frac{1}{2}|z_1|^2}e^{-\frac{1}{2}|z_2|^2}e^{\bar{z}_1 z_2}$

Resolution of Identity: $1 = \int \frac{d^2z}{2\pi i}|z\rangle\langle z|$; $\frac{d^2z}{2\pi i} = \frac{d\mathrm{Re}\, d\mathrm{Im}}{\pi}$

$\langle z_f|e^{-\beta H}|z_i\rangle = \langle z_n|e^{-\frac{\epsilon}{\hbar}H}|z_{n-1}\rangle\langle z_{n-1}|e^{-\frac{\epsilon}{\hbar}H}|z_{N-1}\rangle \cdots$

$\epsilon = \beta\hbar/N$; $\langle z_j|e^{-\epsilon/\hbar H}|z_{j-1}\rangle = \langle z_j|z_{j-1}\rangle\left(1 - \frac{\epsilon}{\hbar}\frac{\langle z_j|\hat{H}|z_{j-1}\rangle}{\langle z_j|z_{j-1}\rangle}\right)$

Mixed State at infinite Temperature: $\rho^{mixed} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$; Averaging in on stokes sphere || $Tr[\rho_{mixed}^2] = 1/2 < 1$

$\langle \vec{S} \rangle = Tr[\hat{\rho}\vec{S}] = 0$, where $\theta = \frac{\pi}{2}$, which $\langle S_z \rangle = 0$

Pure N-spin 1/2 states:

unEntangled Product States: Product of 2-spin are shown below:

$|\Psi_z\rangle = |\uparrow\rangle|\uparrow\rangle$

$|\Psi_+\rangle = |+\rangle|-\rangle = \frac{1}{2}(|\uparrow\rangle + |\downarrow\rangle)(|\uparrow\rangle - |\downarrow\rangle)$

Entangled Product States: $|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle)$

$|\Psi_2\rangle = \frac{1}{\sqrt{2}}(|+\rangle|-\rangle + |-\rangle|+\rangle)$

$|\Psi_{cat}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\ldots\rangle - |\downarrow\downarrow\downarrow\ldots\rangle)$

Free Particle: $\hat{H}_0 = \frac{p^2}{2m}$; For a free particle $\rho^u(x, x'; \beta)$ with momentum eigenstates $|k\rangle$

$\rho_0^u(x, x'; \beta) = \langle x | e^{-\beta H_0} | x' \rangle = \sum_k \Psi_k(x)\Psi_k^*(x') e^{-\beta E_k} = \left(\frac{m}{2\pi\beta\hbar^2}\right)^{d/2} e^{-\frac{1}{2}m(x-x')^2/\beta\hbar}$

Where $\lambda_T^d = h/\sqrt{2\pi m K_B T}$ $= \frac{1}{\lambda_T^d} e^{-\pi(x-x')/\lambda_T^2}$

The limit as $\beta \to 0$, is the density matrix δ-function

$\rho_0^u(x, x'; 0) = \delta(x - x')$

Harmonic Oscillator: $H_{H_0} = p^2/2m + \frac{1}{2}m\omega^2\hat{x}^2$ $\frac{1}{2}kx^2$ $k\frac{1}{2}x^2$ $\frac{1}{2}kx^2 = E$ $k = m\omega^2$ $\sqrt{\frac{k}{m}} = \omega$

$\rho_{H_0}^u(x, \hat{x}; \beta)$; Hermite polynomial $H_n(x)e^{x^2}$

$\rho_{H_0}^u(x, \hat{x}; \beta) = \left(\frac{m\omega}{2\pi\hbar\sinh(\beta\hbar\omega)}\right)^{1/2} e^{-\frac{m\omega}{4\hbar}\left[(x+x')^2\tanh(\beta\hbar\omega/2) + (x+x')^2\coth(\beta\hbar\omega/2)\right]}$

Where partition function $Z = \frac{1}{2\sinh(\beta\omega\hbar/2)}$

A normalized density matrix $\rho = \rho^u/Z$

$\rho_{H_0}(x, x'; \beta) = \left(\frac{m\omega\tanh(\beta\hbar\omega/2)}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega}{2\hbar\sinh(\beta\hbar\omega)}\left[(x^2 + x'^2)\cosh(\beta\hbar\omega) - 2xx'\right]}$

Approximate Methods: Perturbation Theory: $H = H_0 + H_1$; $Z_0 = Tr[e^{-\beta H_0}]$, $Z_0 = Tr[e^{-\beta(H_0 + H_1)}]$

$H_1 \ll H_0$; $Z = Tr[e^{-\beta(H_0 + H_1)}] \approx Tr\left[e^{-\beta H_0}\left(1 - \beta H_1 + \frac{\beta^2 H_1^2}{2!} - \right)\right]$

$= Z_0\left[1 - \beta\langle H_1\rangle_0 + \frac{1}{2!}\beta^2\langle H_1^2\rangle_0 - \right]$

$\frac{Z}{Z_0}, \frac{Z'}{Z_0}, \frac{Z''}{Z_0}$

...ational Theory: F = upper bound computed using minimized variational ...

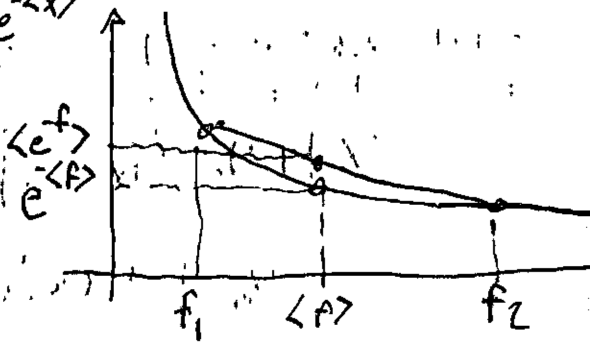
F_{var} = computed with trial Hamiltonian H_{tr}

$Z = Tr[e^{-\beta H}] = e^{-\beta F} = Tr[e^{-\beta(H - H_{tr})} e^{-\beta H_{tr}}] = \langle e^{-\beta(\ldots}$

$\geq e^{-\beta(H - \ldots}$

Where $F_{var} = F_{tr} + \langle H - H_{tr}\rangle_{tr} \geq F$

to prove $\langle e^{-x}\rangle \geq e^{-\langle x\rangle}$



Variational method is an art

...odynamics from pure closed quantum systems

...tate Thermalization Hypothesis (ETH)

$\ldots\Psi|\hat{H}|\Psi\rangle$ $\frac{1}{t}\int_0^t dt' \underbrace{\langle\Psi(t')|\hat{O}|\Psi(t')\rangle}_{t\to\infty} = \langle\hat{O}\rangle_{equilibrium}$

$\langle\Psi(t')|\hat{O}|\Psi(t')\rangle = \sum O_{nm} c_m c_n^* e^{-i(E_n - E_n)t/\ldots}$

Thus, $\frac{1}{t}\int_0^t dt' \langle\Psi(t')|\hat{O}|\Psi(t')\rangle_{t\to\infty} = \sum O_{nm}|c_n|^2 = \sum \ldots$

...ment Entropy" in closed quantum systems: $\rho = |\Psi\rangle\langle\Psi\ldots$

$\ldots[\hat{\rho}] \equiv \sum_{\Psi_B} \langle\Psi_B|\hat{\rho}|\Psi_B\rangle$; $\rho_{\sigma_A\sigma_A'} \equiv \langle\sigma_A|\hat{\rho}_A|\sigma_A'\rangle = \sum \langle\Psi_A, \Psi_B|\hat{\rho}|\Psi_{\ldots}$

$\ldots_B Tr[\rho_A \ln\rho_A]$ || $\hat{\rho}_A$ is pure for: A & B unentangled $|\Psi_A\Psi_B\rangle = |\ldots$

$\hat{\rho}_A$ is mixed for: A & B entangled $|\Psi_A\Psi_B\rangle \neq |\Psi_A\rangle|\ldots$

Thus S_E measures extent of entanglement for

Such as $|\Psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B]$ leads

ρ_A with $S_E = K_B \ln 2$ in contra...

$|\Psi\rangle = |\uparrow\rangle_A|\downarrow\rangle_B$ gives $S_E = 0$

...to information Theory $\Omega = \frac{N!}{M_0!M_1!} = \frac{N!}{(p_0 N)!(p_1 N)!} = \frac{N^N}{(p_0 N)^{p_0 N}(p_1 N)^{p_1 N}} \equiv$

$= p_0^{-p_0 N}(1 - p_0)^{-(1-p_0)\ldots}$

$$Tr(\rho^2) = \sum \langle \phi_m | \rho^2 | \phi_m \rangle = \sum \langle \phi_m | \rho | \phi_n \rangle \langle \phi_n | \rho | \phi_m \rangle$$

$$[Tr(\rho)]^2 \leq Tr(\rho^2) \therefore \sum p_i^2 \leq \left(\sum p_i\right)^2$$

Pure Ensemble: $\rho = |\psi\rangle\langle\psi|$; $\rho^2 = \rho$; $\rho(\rho-1) = 0$; $Tr(\rho^2) = Tr(\rho) = 1$

$$\langle X_m | \rho | X_n \rangle = \rho_m \delta_{nm} \; ; \; \langle X_m | \rho^2 | X_n \rangle = \langle X_m | \rho | X_n \rangle$$

or $\sum \langle X_m | \rho | X_k \rangle \langle X_k | \rho | X_n \rangle = \rho \delta_{nm}$

or $\sum \rho_m \delta_{mk} \rho_k \delta_{kn} = \rho_m \delta_{nm}$

i.e. $\rho_m^2 \delta_{nm} = \rho_m \delta_{nm}$

or $\rho_m (\rho_m - 1) \delta_{nm} = 0$

Remember for mixed systems: $0 < Tr(\rho^2) < [Tr(\rho)]^2$ with $Tr(\rho^2) < 1$

$$\langle \phi | \rho | \phi \rangle = \sum p_i \langle \phi | \psi \rangle \langle \psi | \phi \rangle$$
$$= \sum p_i |\langle \phi | \psi \rangle|^2 \geq 0$$

1. Mixture of two subsystems: $|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $|\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $p_1, p_2 = 1/2$

2. Mixture of three subsystems: $|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $|\psi_2\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$; $|\psi_3\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$

with probabilities $p_1, p_2, p_3 = \frac{1}{3}$

or $|\psi_1\rangle = \frac{1}{\sqrt{281}}\begin{pmatrix} 9 \\ -i10\sqrt{2} \end{pmatrix}$, $|\psi_2\rangle = \frac{1}{\sqrt{194}}\begin{pmatrix} 12 \\ -i5\sqrt{2} \end{pmatrix}$

$|\psi_3\rangle = \frac{1}{\sqrt{17}}\begin{pmatrix} -i3 \\ 2\sqrt{2} \end{pmatrix}$ $p_1 = \frac{281}{900}$, $p_2 = \frac{97}{450}$, $p_3 = \frac{17}{36}$

Reduced Density Operator and Density Matrix

ρ^{AB} ; $A \in |a_i\rangle, i = 1, 2, \ldots d_A$; $B \in |b_j\rangle, j = 1, 2, 3 \ldots d_B$

$$|\psi_{ij}\rangle \equiv |a_i\rangle \otimes |b_j\rangle \equiv |a_i b_j\rangle \equiv |ij\rangle \; ; \; \langle \psi_{ij} | \rho^{AB} | \psi_{ij} \rangle \equiv \langle ij | \rho^{AB} | ij \rangle$$

$$\langle \psi_{ij} | \Omega | \psi_{i'j'} \rangle \equiv \langle ij | \Omega_A | i'j' \rangle \equiv \langle i | \Omega_A | i' \rangle \langle j | j' \rangle \equiv \langle i | \Omega_A | i' \rangle \delta_{ij}$$

$$Tr(\rho^{AB} \Omega_A) = \sum \langle \psi_{ij} | \rho^{AB} \Omega_A | \psi_{ij} \rangle = \sum \langle \psi_{ij} | \rho^{AB} | \psi_{i'j'} \rangle \langle \psi_{i'j'} | \Omega_A | \psi_{ij} \rangle$$
$$= \sum \langle ij | \rho^{AB} | i'j' \rangle \langle i' | \Omega_A | i \rangle \delta_{jj'}$$
$$= \sum \langle ij | \rho^{AB} | i'j \rangle \langle i' | \Omega_A | i \rangle$$

$$\langle i | \rho^A | i' \rangle = \sum \langle ij | \rho^{AB} | i'j \rangle \; ; \; Tr(\rho^{AB} \Omega_A) = \sum \langle i | \rho^A | i' \rangle \langle i' | \Omega_A | i \rangle$$
$$= \sum \langle i | \rho^A \Omega_A | i \rangle = Tr(\rho^{AB} \Omega_A) = |\Omega_A|$$

$$\langle z_j|z_{j-1}\rangle e^{-\frac{\epsilon}{\hbar}\frac{\langle z_j|H|z_{j-1}\rangle}{\langle z_j|z_{j-1}\rangle}} = \langle z_j|z_{j-1}\rangle e^{-\frac{\epsilon}{\hbar}H(\bar{z}_j, z_{j-1})}$$
$$= e^{-\frac{1}{2}[\bar{z}_j(z_j - z_{j-1}) - \bar{z}_{j-1}(z_j - z_{j-1})]} e^{-\frac{\epsilon}{\hbar}H(\bar{z}_j, z_{j-1})}$$

$$\langle z_f|e^{-\beta H}|z_i\rangle = \int \prod_{j=1}^{N-1}\frac{d^2 z_j}{2\pi i} e^{-\frac{1}{\hbar}S_E[\bar{z}_j, z_j]} \cong \int_{z(\beta\hbar)}^{z(0)} D\bar{z}(\tau) D z(\tau) e^{-\frac{1}{\hbar}S_E[\bar{z}(\tau), z(\tau)]}$$

$$S_E[\bar{z}_j, z_j] = \sum_{j=1}^{N-1}\left[\frac{1}{2}\bar{z}_j(z_j - z_{j-1}) - \frac{1}{2}z_j(\bar{z}_{j+1} - \bar{z}_j) + \frac{\epsilon}{\hbar}H(\bar{z}_j, z_{j-1})\right]$$
$$+ \frac{1}{2}\bar{z}_f(z_f - z_{n-1}) - \frac{1}{2}z_i(\bar{z}_1 - \bar{z}_i)\Big]$$

Bosonic Euclidian Action: $S_E[\bar{\psi}(\tau, r), \psi(\tau, r)] = \int_0^{\beta\hbar} d\tau\, d^d r \left[\frac{1}{2}\hbar(\bar{\psi}\partial_\tau\psi - \psi\partial_\tau\bar{\psi}) - \bar{\psi}\frac{\hbar^2\nabla^2}{2m}\psi\right]$

$$= \int_0^{\beta\hbar} d\tau\, d^d r\, \bar{\psi}\left(\hbar\partial_\tau - \frac{\hbar^2\nabla^2}{2m}\right)\psi$$

Density Operator and Density Matrix

Completely Random, Pure, and Mixed States

$P_\uparrow = \frac{N_\uparrow}{N_\uparrow + N_\downarrow}$, $P_\downarrow = \frac{N_\downarrow}{N_\uparrow + N_\downarrow}$ $\quad N\uparrow = |0\rangle$, $N\downarrow = |\downarrow\rangle$

$P_\uparrow, P_\downarrow$ = Fraction population

An even mixture: $N_\uparrow = N_\downarrow$, $P_\uparrow = P_\downarrow = 0.5$

Always: $\sum_i p_i = 1 \neq \sum_i |c_i|^2$; Average Representation: $[\Omega] = \sum p_i \langle\psi_i|\Omega|\psi_i\rangle$

$$= \sum p_i \langle\psi_i|\phi_n\rangle\langle\phi_n|\Omega|\phi_m\rangle\langle\phi_m|\psi_i\rangle$$
$$= \sum p_i [\langle\phi_m|\psi_i\rangle\langle\psi_i|\phi_n\rangle]\langle\phi_n|\Omega|\phi_m\rangle$$

$$\rho \equiv \sum p_i |\psi_i\rangle\langle\psi_i|$$

$$\langle\phi_n|\rho|\phi_m\rangle = \sum p_i \langle\phi_m|\psi_i\rangle\langle\psi_i|\phi_n\rangle$$

$$[\Omega] = \sum \langle\phi_n|\rho|\phi_m\rangle\langle\phi_n|\Omega|\phi_m\rangle$$
$$= \sum\langle\phi_m|\rho\Omega|\phi_m\rangle = Tr(\rho\Omega)$$

Hermitian Conjugate:

$$\rho^\dagger = \left(\sum \rho_i|\psi_i\rangle\langle\psi_i|\right)^\dagger = \sum \rho_i|\psi_i\rangle\langle\psi_i| = \rho$$

$$Tr(\rho) = \sum\langle\phi_n|\rho|\phi_n\rangle = \sum\langle\phi_n|\left(\sum p_i|\psi_i\rangle\langle\psi_i|\right)|\phi_n\rangle$$
$$= \sum p_i\langle\phi_n|\psi_i\rangle\langle\psi_i|\phi_n\rangle$$
$$= \sum p_i\left(\sum\langle\psi_i|\phi_n\rangle\langle\phi_n|\psi_i\rangle\right)$$
$$= \sum p_i\langle\psi_i|\psi_i\rangle = \sum p_i = 1$$

Normalized: $[\Omega] = \frac{\sum p_i\langle\psi_i|\Omega|\psi_i\rangle}{\sum p_i}$

$$= \frac{Tr(\rho\Omega)}{Tr(\rho)}$$

$$\sigma_y = \frac{1}{2}\begin{cases} (\sigma_y^{(1)} \otimes 1^{(2)} + 1^{(1)} \otimes \sigma^{(2)} \\ -i(|00\rangle - |11\rangle)(\langle 01| + \langle 10|) + i(|10\rangle + |01\rangle)(\langle 00| - \langle 11|) \\ \begin{pmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{pmatrix} \end{cases}$$

Recipes
Paper
Practice
Notes

$$\sigma_z = \begin{cases} \frac{1}{2}(\sigma_z^{(1)} \otimes 1^{(2)} + 1^{(1)} \otimes \sigma_z^{(2)} \\ |00\rangle\langle 00| - |11\rangle\langle 11| \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{cases}$$

Density Matrix for Multipole Photon beams

$\langle 1m_r|\rho_r|1m_r'\rangle$; γ_r $|10\rangle, |1,+1\rangle, |1-1\rangle$; $\rho_i = \sum |0,1m_r\rangle\langle 1m_r|\rho|1m_r'\rangle\langle 0;1m_r'|$

-or-

$$\rho_i = \frac{1}{n_{\alpha 0}}\sum |0;1m_r\rangle\langle 1m_r|\rho_r|1m_r'\rangle\langle 0;1m_r'|$$

-or

$$\rho_r = \frac{1}{2}I_r\begin{pmatrix} 1+\eta_2 & 0 & -\eta_3+i\eta_1 \\ 0 & 0 & 0 \\ -\eta_3-i\eta_1 & 0 & 1-\eta_2 \end{pmatrix}$$

where $\eta_1 = \frac{1}{I_r}\left[I_r(\pi/4) - I_r(3\pi/4)\right]$

$$\eta_2 = \frac{1}{I_r}\left[I_r(+) - I_r(-)\right]$$

$$\eta_3 = \frac{1}{I_r}\left[I_r(0) - I_r(\pi/2)\right]$$

Then Linearly Polarized OX $\eta_1, \eta_2 = 0, \eta_3 = +1$
Linearly Polarized OY $\eta_1, \eta_2 = 0, \eta_3 = -1$
Right Circularly $\eta_1 = 0, \eta_2 = 1\ \eta_3 = 0$
Left Circularly $\eta_1 = 0\ \eta_2 = -1\ \eta_3 = 0$

Defined as a tensor operator : $\rho_r = \sum_{KQ}\langle T(1)^\dagger_{KQ}\rangle T(1)_{KQ}$

$$\langle T(1)^\dagger_{KQ}\rangle = \sqrt{2K+1}\sum_{m_r m_r'}(-1)^{1-m_r}$$

$$\begin{pmatrix} 1 & 1 & K \\ m_r & -m_r' & -Q \end{pmatrix}\langle 1m_r|\rho_r|1m_r'\rangle$$

$$\rho_r = \frac{1}{2}(|1+1\rangle\langle 1+1| + |1-1\rangle\langle 1-1|)$$

$\rho^A \equiv Tr_B(\rho^{AB})$; $\rho^B = Tr_A(\rho^{AB})$; with $\langle j|\rho^B|j'\rangle = \sum_i \langle ij|\rho^{AB}|ij'\rangle$

Miscellaneous: $\rho = \sum \lambda_i |i\rangle\langle i|$; $\rho^2 = \sum \lambda_i^2 |i\rangle\langle i|$; $Tr(\rho) = \sum \lambda_i = 1$; $Tr(\rho^2) = \sum \lambda_i^2$

Alternate Representations:

$$\rho_{\psi^+} \equiv \begin{cases} |\psi^+\rangle\langle\psi^+| = \frac{1}{2}(|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|) \\ \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{cases}$$

$$\rho_{\psi^-} \equiv \begin{cases} |\psi^-\rangle\langle\psi^-| = \frac{1}{2}(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|) \\ \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{cases}$$

$$\rho_{\phi^+} \equiv \begin{cases} |\phi^+\rangle\langle\phi^+| = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|) \\ \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \end{cases}$$

$$\rho_{\phi^-} \equiv \begin{cases} |\phi^-\rangle\langle\phi^-| = \frac{1}{2}(|00\rangle\langle 00| - |00\rangle\langle 11| - |11\rangle\langle 00| + |11\rangle\langle 11|) \\ \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \end{cases}$$

Composite systems:
$$\begin{aligned}\Omega \equiv \sum_{n=1}^{n} \Omega^{(n)} = {} & \Omega^{(1)} \otimes 1^{(2)} \otimes 1^{(3)} \otimes \cdots \otimes I^{(n-1)} \otimes I^{(n)} + \\ & 1^{(1)} \otimes \Omega^{(2)} \otimes 1^{(3)} \otimes \cdots \otimes 1^{(n-1)} \otimes I^{(n)} + \\ & 1^{(1)} \otimes 1^{(2)} \otimes 1^{(3)} \otimes \cdots \otimes \Omega^{(n-1)} \otimes I^{(n)} + \\ & 1^{(1)} \otimes 1^{(2)} \otimes 1^{(3)} \otimes \cdots \otimes 1^{(n-1)} \otimes \Omega^{(n-1)}\end{aligned}$$

$$H \equiv H_1 \otimes H_2 \otimes \cdots \otimes H_n$$

$$\sigma \equiv \frac{1}{2}(\sigma^{(1)} + \sigma^{(2)}) = \frac{1}{2}(\sigma^{(1)} \otimes 1^{(2)} + 1^{(1)} + \sigma^{(2)})$$

$$\sigma_x = \frac{1}{2}\begin{cases} (\sigma_x^{(1)} \otimes 1^{(2)} + 1^{(1)} \otimes \sigma_x^{(2)}) \\ (|1\rangle\langle 0| + |0\rangle\langle 1|)^{(1)} \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|)^{(2)} + \\ (|0\rangle\langle 0| + |1\rangle\langle 1|)^{(1)} \otimes (|1\rangle\langle 0| + |0\rangle\langle 1|)^{(2)}, \\ (|1\rangle\langle 0| + |0\rangle\langle 1|)(|0\rangle\langle 0| + |1\rangle\langle 1|) + (|0\rangle\langle 0| + |1\rangle\langle 1|)(|1\rangle\langle 0| + |0\rangle\langle 1|) \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{cases}$$

Polarization of Electromagnetic wave	Monopole Moment (K=0)	Orientation vector (K=1)	Alignment vector (K=2)
LP ($m_r=0$)	$\left\langle T(1)^{\dagger}_{00} \right\rangle = \frac{1}{\sqrt{3}}$	$\left\langle T(1)^{\dagger}_{10} \right\rangle = 0$	$\left\langle T(1)^{\dagger}_{20} \right\rangle = -\sqrt{\frac{2}{3}}$
RCP ($m_r=+1$)	$\left\langle T(1)^{\dagger}_{00} \right\rangle = \frac{1}{\sqrt{5}}$	$\left\langle T(1)^{\dagger}_{10} \right\rangle = \frac{1}{\sqrt{2}}$	$\left\langle T(1)^{\dagger}_{20} \right\rangle = \frac{1}{\sqrt{6}}$
LCP ($m_r=-1$)	$\left\langle T(1)^{\dagger}_{00} \right\rangle = \frac{1}{\sqrt{3}}$	$\left\langle T(1)^{\dagger}_{10} \right\rangle = \frac{-1}{\sqrt{2}}$	$\left\langle T(1)^{\dagger}_{20} \right\rangle = \frac{1}{\sqrt{6}}$
UP	$\left\langle T(1)^{\dagger}_{00} \right\rangle = \frac{1}{\sqrt{3}}$	$\left\langle T(1)^{\dagger}_{10} \right\rangle = 0$	$\left\langle T(1)^{\dagger}_{20} \right\rangle = \frac{1}{\sqrt{6}}$