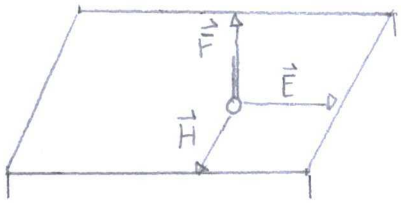
Chapter 8: Waveguides, Resonant Cavities, and Optical Fibers:

8.1



"surface region of an excellent conductor"

a) Time-averaged Force per area:

$$(5.12) \quad F = \int J(x) \times B(x)\, d^3x$$

$$= \mu_c \int J(x) \times H\, dx$$

$$= \mu_c \sigma \int E \times H\, dx$$

Electromagnetic Induction $B = \mu_c H$

Current $J = \sigma E$

$$(8.9) \quad H_c = H e^{-\xi/\delta} \cdot e^{i\xi/\delta}$$

$$(8.10) \quad E_c \simeq \sqrt{\frac{\mu_c \omega}{2\sigma}}(1-i)(n \times H_{\parallel}) e^{-\xi/\delta} \cdot e^{i\xi/\delta}$$

$$F = \mu_c \sigma \int \left(\sqrt{\frac{\mu_c \omega}{2\sigma}}(1-i)(n \times H_{\parallel}) e^{-\xi/\delta} \cdot e^{i\xi/\delta}\right) \times$$

$$\cdot \left(H e^{-\xi/\delta} \cdot e^{i\xi/\delta}\right) dx$$

$$= \mu_c \sigma \int \left(\sqrt{\frac{\mu_c \omega}{2\sigma}}(n \times H) \cos(x/\delta - \omega t) e^{-x/\delta}\right) \times$$

$$\cdot \left(H \cos(x/\delta - \omega t) e^{-x/\delta}\right) dx$$

$$= -n \mu \sigma \sqrt{\frac{\mu_c \omega}{2\sigma}} |H|^2 \int e^{-2x/\delta} \cos^2(x/\delta - \omega t)\, dx$$

$$= -n \mu \frac{|H|^2}{2\delta} \int e^{-2x/\delta}\, dx$$

$$= -n \mu_c \frac{|H|^2}{4}$$

b) (6.120) "Maxwell Stress Tensor"

$$T_{\alpha\beta} = \epsilon_0 \left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}(E \cdot E + c^2 B \cdot B)\delta_{\alpha\beta}\right]$$

Yes, additional magnetic forces reduce to Maxwell's Stress Tensor from Chapter 5, specifically μ and μ_c. Different permeabilities generate forces, also from part a:

$$F_c = -n\mu_c \frac{|H|^2}{4}$$

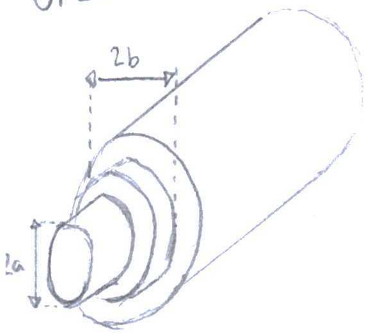
$$F = -n\mu \frac{|H|^2}{4}$$

Forces never change from Electric forces because a constant current on a single propagation path:

$$(1.1) \quad F = qE$$
$$= \int \rho E \, d^3x$$
$$= \int \frac{i}{\omega} \nabla \cdot J \cdot E \, d^3x$$
$$= 0$$

c) $|H_{\parallel}|^2$ is from $-\infty$ to ∞ and $2\langle|H_{\parallel}|\rangle^2$ from $t=0$ to infinity. So both forms about superposition equate.

8.2



"..a transmission line of two concentric cylinders"

Shape: Circle
Dimension: Plane [2d]
Charge: λ
Coordinates: cylindrical

a) Transverse Electric Mode:

(I.1b) "Gauss' Law"

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$

$$\oint E \cdot ds = \frac{1}{\epsilon_0}\int \rho(X)\, d^3X$$

$$E \oint ds = \frac{1}{\epsilon_0} \cdot \lambda$$

$$E = \frac{\lambda}{\epsilon_0} \frac{1}{\int_0^{2\pi}\int_0^{a} d\rho\, d\phi}$$

$$= \frac{\lambda}{2\pi \epsilon_0 a}$$

(8.28) $$B_{TEM} = \sqrt{\mu\epsilon}\, \hat{z} \times E_{TEM}$$

$$= \sqrt{\mu\epsilon}\, \frac{\lambda}{2\pi\epsilon_0 a}$$

H_0 = Peak Magnetic Field

$$= \frac{1}{\sqrt{\mu\epsilon}} \frac{\lambda}{2\pi a}$$

$$\lambda = 2\pi a H_0 \sqrt{\mu\epsilon}$$

Static Solutions
$$\begin{cases} E = \sqrt{\frac{\mu}{\epsilon}}\, H_0 \frac{a}{\rho}\hat{\rho} \\ B = \mu \cdot H_0 \frac{a}{\rho}\hat{\phi} \end{cases}$$

Waveguide Solutions
$$\begin{cases} E = \sqrt{\frac{\mu}{\epsilon}}\, H_0(z) \frac{a}{\rho} e^{ikz-i\omega t} \\ B = \mu \cdot H_0(z) \cdot \frac{a}{\rho} e^{ikz-i\omega t} \end{cases}$$

Energy Flux: (8.47) "Complex poynting vector"

$$S = \frac{1}{2}(E \times H^*)$$

$$\langle S \rangle = \frac{1}{2}\int E \times H^* dt$$

$$= \frac{1}{2}\int \left[\sqrt{\frac{\mu}{\epsilon}}|H_0(z)|\frac{a}{\rho}\cos(kz-wt)\right]$$

$$\times\left[\frac{1}{\mu}|H_0(z)|\frac{a}{\rho}\cos(kz-wt)\right]$$

$$= \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}|H_0(z)|^2\frac{a^2}{\rho^2}$$

Total Power: (8.49) "Total Power Flow"

$$P = \int_A S \cdot \hat{z}\, da$$

$$= \int_0^{2\pi}\int_a^b \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}|H_0(z)|^2\frac{a^2}{\rho^2}\rho\, d\rho\, d\phi$$

$$= \sqrt{\frac{\mu}{\epsilon}}\pi a^2 |H_0(z)|^2 \cdot \ln\left(\frac{b}{a}\right)$$

b) (8.12) "Time-averaged Power absorbed per unit area"

$$\frac{dP}{da} = \frac{1}{4}\mu_c w \delta |H_0|^2$$

(8.58) "Attenuation constant"

$$\beta_\lambda = -\frac{1}{2P}\frac{dP}{dz}$$

$$= +\frac{1}{2P}\frac{1}{2\sigma\delta}\oint_C |n \times H|^2 dl \quad \text{.. from (8.58)}$$

$$= \frac{1}{2\left[\sqrt{\frac{\mu}{\epsilon}}\pi a^2 |H_0|^2 \ln\left(\frac{a}{b}\right)\right]} \cdot \oint_c |n \times H|^2 dl$$

$$= \frac{1}{2\left[\sqrt{\frac{\mu}{\epsilon}}\pi a^2 |H_0|^2 \ln\left(\frac{a}{b}\right)\right]} \cdot \left(\frac{1}{\sigma\delta}\right) \cdot \oint_c \begin{Bmatrix} |n \times H(a)|^2 \\ |n \times H(b)|^2 \end{Bmatrix} dl$$

... similar to 8.59, with two boundary conditions

$$= \frac{1}{2\left[\sqrt{\frac{\mu}{\epsilon}}\pi a^2 |H_0|^2 \ln\left(\frac{a}{b}\right)\right]} \cdot \frac{1}{\sigma\delta} \cdot \int_0^{2\pi} \begin{pmatrix} \frac{1}{4}\int_0^a \sqrt{\frac{\mu}{\epsilon}} |H_{\parallel}|^2 \rho \, d\rho \\ \frac{1}{4}\int_0^b \sqrt{\frac{\mu}{\epsilon}} |H_{\parallel}|^2 \rho \, d\rho \end{pmatrix}$$

$$= \frac{1}{2\left[\sqrt{\frac{\mu}{\epsilon}}\pi a^2 |H_0|^2 \ln\left(\frac{a}{b}\right)\right]} \cdot \frac{1}{\sigma\delta} \cdot \pi \mu_c \omega \delta \left[H(a)^2 a + H(b)^2 b\right]$$

$$= \frac{1}{2\left[\sqrt{\frac{\mu}{\epsilon}}\pi a^2 |H_0|^2 \ln\left(\frac{a}{b}\right)\right]} \cdot \frac{1}{\sigma\delta} \cdot \pi \mu_c |H_0|^2 \left[\frac{1}{a} + \frac{1}{b}\right]$$

$$= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}} \frac{1}{\sigma\delta} \frac{\left(\frac{1}{a} + \frac{1}{b}\right)}{\ln\left(\frac{b}{a}\right)}$$

Skin Depth (8.8)

$$\delta = \left(\frac{2}{\mu_c \omega \sigma}\right)^{1/2}$$

(8.56) $P(z) = P_0 e^{-2\beta z}$

$$= P_0 e^{-2\gamma z}$$

$$\gamma = \beta$$

$$= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}} \frac{1}{\sigma\delta} \frac{\left(\frac{1}{a} + \frac{1}{b}\right)}{\ln\left(\frac{b}{a}\right)}$$

c) $Z = \frac{\Delta V}{I}$

$$= \frac{\int_a^b \sqrt{\frac{\mu}{\epsilon}} H_0 \frac{a}{\rho} d\rho}{I}$$

$$= \frac{\sqrt{\frac{\mu}{\epsilon}} H_0 a \ln\left(\frac{b}{a}\right)}{I}$$

(8.14) $K_{eff} = n \times H$

$$= \rho \times H_0 \frac{a}{\rho}$$

$$I = 2\pi a H_0$$

$$Z = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}} \ln\left(\frac{\rho}{a}\right)$$

d) Total Resistance per unit length:

"Normal" $\longleftrightarrow$ "Harmonic Wave"

$P = I^2 R$ $\qquad$ $P = \left(\frac{I}{\sqrt{2}}\right)^2 R$

$$R = \frac{2}{I^2}\frac{dP}{dz}$$

$$= \frac{2}{(2\pi a H_0)^2}\left[\frac{\pi}{\sigma\delta}|H_0|^2\left[\frac{1}{a} + \frac{1}{b}\right]\right]$$

$$= \frac{1}{2\pi a^2 \sigma \delta}\left(\frac{1}{a} + \frac{1}{b}\right)$$

Total Inductance per unit length:

$$W = \frac{1}{2} L I^2$$

$$L =$$

$$= \frac{4}{I^2}\left[W_{in} + W_{mid} + W_{out}\right]$$

$$= \frac{4}{I^2}\left[\frac{1}{4\mu_c}\int |B_{in}|^2 da + \frac{1}{4\mu_c}\int |B_{mid}|^2 da + \frac{1}{4\mu_c}\int |B_{out}|^2 da\right]$$

$$= \frac{4}{I^2}\left[\frac{1}{4\mu_c}\int_0^{2\pi}\int_0^{a} |B_{in}|^2 \rho \, d\rho \, d\phi + \frac{1}{4\mu_c}\int_0^{2\pi}\int_a^{b} |B_{mid}| \rho \, d\rho \, d\phi + \frac{1}{4\mu_c}\int_0^{2\pi}\int_b^{\infty} |B_{out}| \rho \, d\rho \, d\phi\right]$$

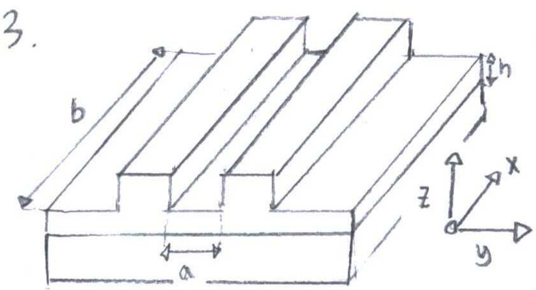
$$= \frac{2\pi}{I^2}|\mu_0|^2\left[\mu_c \int_0^{a} e^{-2(a-\rho)/\delta} \rho \, d\phi + \mu a_c^2 \int_a^{b} \frac{1}{\rho} d\rho + \mu_c \frac{a^2}{b^2}\int_b^{\infty} e^{-2(\rho-b)/\delta} \rho \, d\rho\right]$$

$$= \frac{2\pi}{I^2}|\mu_0|^2\left[\mu_c e^{-2a/\delta} \cdot \frac{\delta^2}{4} + \mu a^2 \ln\left(\frac{b}{a}\right) + \mu_c a^2 \left(\frac{1}{a} + \frac{1}{b}\right)\delta/2\right]$$

$$= \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right) + \frac{\mu}{4}\frac{\delta}{\pi}\left[\left(\frac{1}{a} + \frac{1}{b}\right) + \frac{1}{a^2} e^{-2a/\delta} \cdot \frac{\delta}{2}\right]$$

$$\cong \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right) + \frac{\mu_c}{4}\frac{\delta}{\pi}\left(\frac{1}{a} + \frac{1}{b}\right)$$

0.3.



"two identical strips of metal..."

a) Power:

$$B = \mu K \hat{x}$$
$$= \mu K_0 e^{-i(kx-\omega t)}$$

$$H = \frac{B}{\mu}$$
$$= K_0 e^{i(kx-\omega t)}$$

(I.1b) "Ampere's Law"

$$\nabla \times B - \frac{\partial E}{c^2 \partial t} = \mu_0 J \quad \text{.. if } J = 0$$

$$\nabla \times B = \mu \epsilon \frac{\partial E}{\partial t}$$

$$= -i\mu\omega\epsilon E \quad \text{... when } E = e^{-i(kx-\omega t)}$$

$$E = \frac{-\nabla \times B}{i\mu\epsilon\omega}$$

$$= \frac{k \times B}{\mu \epsilon \omega}$$

$$= \frac{-k H_0}{\epsilon \omega} e^{i(kz - \omega t)}$$

(8.47) "Poynting vector"

$$S = \frac{1}{2}(E \times H^*)$$

$$= \frac{1}{2}\left\{\frac{-k H_0}{\epsilon \omega}\right\} \hat{y} \times (H_0 \hat{x})$$

$$= \frac{k|H_0|^2}{2\epsilon\omega}$$

$$= \frac{\sqrt{\mu\epsilon}\,|H_0|^2}{2\epsilon}.$$ ... when index of refraction equals one

(8.49) "Total Power flow"

$$P = \int S \cdot \hat{z}\, da$$

$$= \int_0^a \int_0^b \sqrt{\frac{\mu}{\epsilon}} \frac{|H_0|^2}{2} da$$

$$= \frac{ab}{2}\sqrt{\frac{\mu}{\epsilon}} |H_0|^2$$

(8.15) "Power dissipated in Ohmic losses per area"

$$\frac{dP_{loss}}{da} = \frac{-1}{2\sigma\delta}|K_{eff}|^2$$

$$= \frac{-1}{2\sigma\delta}|H_{\parallel}|^2$$

$$= \frac{-1}{2\sigma\delta}|H_0|^2$$

(8.58) "Power dissipated in Ohmic Losses per length"

$$\frac{dP}{dz} = \frac{-1}{2\sigma\delta}\oint_c |n \times H|^2 dl$$

$$= 2b \frac{dP}{da}$$

$$= -\frac{b}{\sigma\delta} |H_0|^2$$

$$= -2\left\{\frac{1}{a \cdot \sigma \cdot \delta} \sqrt{\frac{\epsilon}{\mu}}\right\} P$$

$$= -2 \cdot \gamma \cdot P$$

(8.56) "Power flow"

$$P(z) = P_0 e^{-2\gamma z}$$

$$V = \int E \cdot dl$$

$$= \int_0^a \frac{k H_0}{\epsilon \omega} \cdot e^{i(kz - \omega t)} dz$$

$$= \frac{k H_0 a}{\epsilon \omega} e^{i(kz - \omega t)}$$

(Fig 6.4) "Impedance"

$$Z = \frac{V}{I}$$

$$= \frac{V}{K b}$$

$$= \frac{k a}{\epsilon \omega}$$

$$= \frac{a}{b} \sqrt{\frac{\mu}{\epsilon}}$$

(6.139) "Resistance per unit length"

$$R \approx \frac{1}{|I|^2} \int_V \sigma |E|^2 d^3x = -\frac{2}{|I|^2} \frac{dP}{dz}$$

$$= -\frac{2}{|H_0|^2 b^2} \left(-\frac{b}{\sigma\delta} |H_0|^2\right)$$

$$= \frac{2}{\sigma \delta b}$$

(5.157) "Inductance per unit length"

$$L = \frac{1}{I^2}\int \frac{B \cdot B}{\mu} d^3x$$

$$= \frac{1}{I^2}\int B \cdot H \, da$$

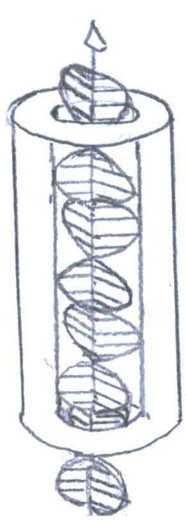
$$= \frac{1}{|H_0|^2 b^2}\left\{a \cdot b \cdot \mu |H_0|^2 + 2\int_{\text{conductor}} B \cdot H^* da\right\}$$

$$= \frac{1}{|H_0|^2 \cdot b^2}\left\{a \cdot b \cdot \mu |H_0|^2 + 2 \cdot \mu_c \int_0^\infty |H_0|^2 e^{-2\zeta/\delta} \cdot (b \cdot d\zeta)\right\}$$

$$= \frac{1}{|H_0|^2 \cdot b^2}\left\{a \cdot b \cdot \mu \cdot |H_0|^2 + 2 \cdot \frac{\mu_c}{2} \delta |H_0|^2 b\right\}$$

$$= \frac{\mu a + \mu_c \delta}{b}$$

b) A larger lower layer ($b \gg h$) has no effect on the previous calculations because the exact same dielectric and permeability constants. While ($b \ll h$) in a problem examples a plane from infinite distances, rather than two strips.

8.4. 

"Transverse electric and magnetic waves propagate along a hollow, right, circular cylinder."

Note about solving the problem:

1. Boundary Conditions
2. Laplace's Equation
3. Solution to variables in Laplaces
4. General Solution
5. Variables with the Boundary Conditions
6. Exact Solution

Cylindrical Harmonics
"Bessell Differential Equation"

∴ a harmonics matching the shape: spherical, planar, conal, elliptic.

① Boundary Conditions:

Transverse Electric Waves:

$$E_z(\rho=R,\theta,\phi)=0 \text{ everywhere} \quad \text{or} \quad \frac{\partial B(\rho=R,\theta,\phi)}{\partial\rho}=0$$

Transverse Magnetic Waves:

$$B_z(\rho=0,\theta,\phi)=0 \text{ everywhere}$$

② Laplace's Equation:

$$\nabla\Phi=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial^2\phi}+\frac{\partial^2\Phi}{\partial^2 z}=0$$

③ Solution to Variables in Laplace's Equation:

If $\Phi(\rho,\phi,z)=R(\rho)\cdot Q(\Phi)\cdot Z(z)$, then

Ⓐ Variable Separation:

$$\nabla^2\Phi=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial^2\phi}+\frac{\partial^2\Phi}{\partial^2 z}$$

$$=\frac{Q(\phi)Z(z)}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)}{\partial\rho}\right)+\frac{R(\rho)Z(z)}{\rho^2}\frac{\partial^2 Q(\phi)}{\partial^2\phi}+R(\rho)Q(\phi)\frac{\partial^2 Z(z)}{\partial^2 z}$$

$$= \frac{1}{R(\rho)\cdot\rho}\frac{\partial}{\partial\rho}\left(r\cdot\frac{\partial R(\rho)}{\partial\rho}\right) + \frac{1}{Q(\Phi)\cdot\rho^2}\frac{\partial^2 Q(\Phi)}{\partial^2\Phi} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z}$$

$$= 0$$

Ⓑ Radial Eigenvalues:

$$\frac{\rho}{R(\rho)}\frac{\partial R(\rho)}{\partial\rho} + \frac{\rho^2}{R(\rho)}\frac{\partial^2 R(\rho)}{\partial\rho^2} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z} = \lambda\rho^2$$

$$\frac{\rho}{R(\rho)}\frac{\partial R(\rho)}{\partial\rho} + \frac{\rho^2}{R(\rho)}\frac{\partial^2 R(\rho)}{\partial\rho^2} - \lambda\rho^2 = -\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z}$$

$$\frac{\rho}{R(\rho)}\frac{\partial R(\rho)}{\partial^2\rho} + \frac{\rho^2}{R(\rho)}\frac{\partial^2 R(\rho)}{\partial r^2} - \lambda\rho^2 = k^2$$

If $p = x$ and $\lambda = \frac{1}{p^2}$, then:

$$\frac{\partial^2 R(x)}{\partial^2 x} + \frac{1}{x}\frac{\partial R(x)}{\partial x} - \left(1 + \frac{k^2}{x^2}\right)R(x) = 0$$

Ⓒ Angular Eigenvalues:

$$\frac{1}{Q(\Phi)}\frac{\partial^2 Q(\Phi)}{\partial^2\Phi} = m^2 \quad ; \quad \frac{\partial^2 Q(\Phi)}{\partial^2\Phi} + m^2 Q = 0$$

Ⓓ Vertical Eigenvalues:

$$\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z} = k^2 \quad ; \quad \frac{\partial^2 Z(z)}{\partial^2 z} - k^2 Z = 0$$

Ⓔ General Solutions:

$$\Phi(\rho,\Phi,z) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} R(\rho)\cdot Q(\Phi)\cdot Z(z)$$

④ General Solution:

$$R(\rho) = A\cdot J_v(kx) + B\cdot Y_v(kx)$$

where
$$J_v(kx) = \sum_{m=0}^{\infty}\frac{(-1)^m\left(-\frac{1}{2}kx\right)^{v+2m}}{m!\,(m+v)!}$$

$$Y_v(kx) = \sum_{m=0}^{\infty}\frac{J_v(kx)\cos(v\pi) - J_{-v}(kx)}{\sin(v\pi)}$$

$$Q(\phi) = C e^{i(m\phi)}$$

$$Z(z) = D e^{i(kz-\omega t)}$$

⑤ Variables by Boundary Conditions:

$$\Phi(\rho,\phi,z) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \left[A J_\nu(kx) + B Y_\nu(kx)\right]\left[C e^{i(m\phi)}\right]\left[D e^{i(kz-\omega t)}\right]$$

$$B \quad \Phi(\rho=0,\theta,\phi) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \left[A J_\nu(k\cdot 0) + B Y_\nu(k\cdot 0)\right]\left[C e^{i(m\phi)}\right]\left[D\cdot e^{i(kz-\omega t)}\right]$$

$$= 0,\ B = 0$$

$$R \quad \frac{\partial \Phi(\rho=R,\phi,z)}{\partial\rho} = \frac{\partial}{\partial\rho}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} A\cdot J(k\cdot R)\cdot e^{i(kz-\omega t+m\phi)}$$

$$= 0,\ k = \frac{X'_{mn}}{R}$$

⑥ Exact solution:

$$B_z(\rho,\phi,z) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} A\cdot J_\nu\left(X'_{mn}\cdot\frac{\rho}{R}\right)\cdot e^{i(kz-\omega t+m\phi)}$$

$$E_z(\rho,\phi,z) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} A\cdot J_\nu\left(X_{mn}\cdot\frac{\rho}{R}\right)\cdot e^{i(kz-\omega t+m\phi)}$$

Cutoff Frequencies:

$$\frac{k}{\omega} = \frac{n}{c}$$

$$\omega = c\cdot k = c\cdot\frac{X'_{mn}}{R}$$

X_{mn}	$\nu=0$	$\nu=1$	$\nu=2$
n=1	2.40	3.83	5.14
n=2	5.52	7.02	8.42
n=3	8.65	10.17	11.62

X'_{mn}	$\nu=0$	$\nu=1$	$\nu=2$
n=1	3.83	1.84	3.05
n=2	7.02	5.33	6.71
n=3	10.17	8.54	9.97

cite: Wolfram Mathworld: Bessel Functions

The five lowest cutoff modes:

$TE_{11} = \omega_1 = 1.84 \frac{c}{R}$

$TM_{01} = \omega_2 = 2.40 \frac{c}{R}$

$TE_{21} = \omega_3 = 3.05 \frac{c}{R}$

TE_{01} and $TM_{11} = \omega_{4,5} = 3.83 \frac{c}{R}$

Primary frequencies by ratio:

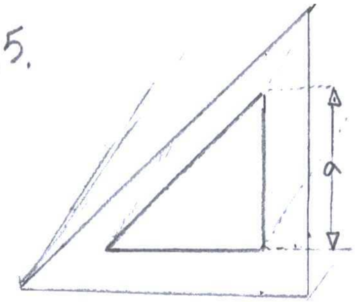
$TE_{11} = \omega_1/\omega_1 = 1.00$

$TM_{01} = \omega_2/\omega_1 = 1.30$

$TE_{21} = \omega_3/\omega_1 = 1.66$

TE_{01} and $TM_{11} = 2.08.$

0.5.



"waveguide.. constructed ... the form of a right triangle"

a) Modes of Propagation:

Obsolete in today's world?

① Boundary Conditions:

Transverse Magnetic Waves:

$B_z = 0$ everywhere, $E_z|_s = 0$

Transverse Electric Waves:

$E_z = 0$ everywhere, $\left.\frac{\partial B_z}{\partial n}\right|_n = 0$

Other denoting

$B(x, y=x, z, t) = 0$

$E(x, y=x, z, t) = 0$

$\frac{\partial B(x=0, y=x, z)}{\partial x} = 0$

$\frac{\partial B(x=0, y=x, z)}{\partial y} = 0$

$\frac{\partial E(x=a, y=x, z)}{\partial x} = 0$

$\frac{\partial E(x=a, y=x, z)}{\partial y} = 0$

$B(x=a, y, z=0, t=0) = B_0$

$B(x, y=a, z=0, t=0) = B_0$

$E(x=a, y, z=0, t=0) = E_0$

$E(x, y=a, z=0, t=0) = E_0$

② Laplace's Equation:

$$\nabla^2 \psi = \left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y} + \frac{\partial^2}{\partial^2 z} + \gamma^2 \right) \psi = 0$$

③ Solution to Laplace's Equation:

Ⓐ Variable Separation: $\frac{1}{X}\frac{\partial^2 X}{\partial^2 x} + \frac{1}{Y}\frac{\partial^2 Y}{\partial^2 Y} + \frac{1}{Z}\frac{\partial^2 Z}{\partial^2 Z} = 0$

Ⓑ Linear Eigenvalues: $\frac{\partial^2 X}{\partial^2 x} = -\lambda_1 X$; $\frac{\partial^2 X}{\partial^2 X} + \lambda_1 X = 0$

$$\frac{\partial^2 Y}{\partial^2 y} = -\lambda_2 Y; \quad \frac{\partial^2 Y}{\partial^2 Y} + \lambda_2 Y = 0$$

$$\frac{\partial^2 Z}{\partial^2 z} = -\lambda_3 Z; \quad \frac{\partial^2 Z}{\partial^2 Z} + \lambda_3 Z = 0$$

Ⓒ General Solutions:

$$X = A\cos(\lambda_1 x) + B\sin(\lambda_1 x)$$

$$Y = C\cos(\lambda_2 y) + D\sin(\lambda_2 y)$$

$$Z = E e^{i(kz - \omega t)}$$

④ General Solution to Laplace's Equation:

$$\Psi(x,y,z) = X \cdot Y \cdot Z$$
$$= (A\cos(\lambda_1 x) + B\sin(\lambda_1 x))(C\cos(\lambda_2 y) + D\sin(\lambda_2 y))E^{i(kz-\omega t)}$$

⑤ Variables by Boundary Conditions:

Transverse Electric Modes

$$B \quad \frac{\partial \Psi(x=0, y=x, z)}{\partial x} = \frac{\partial}{\partial x}(A\cos(\lambda_1 \cdot 0) + B\sin(\lambda_1 \cdot 0))$$
$$\cdot (C\cos(\lambda_2 \cdot 0) + D\sin(\lambda_2 \cdot 0)) \cdot e^{i(kz-\omega t)}$$
$$= (-A \cdot \lambda_1 \sin(0) + B \cdot \lambda_1 \cos(0))$$
$$\cdot (C\cos(0) + D\sin(0)) \cdot e^{i(kz-\omega t)}$$
$$= 0, \quad B = 0$$

$$D \quad \frac{\partial \Psi(x=0, y=x, z)}{\partial y} = \frac{\partial}{\partial y}(A\cos(\lambda 0))(C\cos(\lambda_2 \cdot 0) + D\sin(\lambda_2 \cdot 0)) \cdot e^{i(kz-\omega t)}$$
$$= A\cos(0) \cdot (-C \cdot \lambda_2 \sin(0) + D\lambda_2 \cos(0)) \cdot e^{i(kz-\omega t)}$$
$$= 0, \quad D = 0$$

$$A, \lambda_1 \quad \Psi(x=a, y, z=0, t=0) = A \cdot \cos(\lambda_1 \cdot a) \cdot \cos(\lambda_2 \cdot y) e^{i(k 0 - \omega}$$
$$= B_0, \quad A = B_0, \quad \lambda_1 = \frac{m\pi}{a}$$

...where $m = 0, 1, 2, 3...$

$$\lambda_2 \quad \Psi(x, y=a, z=0, t=0) = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos(\lambda_2 \cdot a) e^{i(k \cdot 0 - \omega \cdot 0)}$$
$$= B_0, \quad \lambda_2 = \frac{n\pi}{a}$$

...where $n = 0, 1, 2, 3...$

$$\frac{\partial B(x,y,z,t)}{\partial x} = \frac{\partial B(x,y,z,t)}{\partial y} = \frac{\partial B(y,x,z,t)}{\partial x} = \frac{\partial B(y,x,z,t)}{\partial y}$$

$$B_z = B(x,y,z,t) + B(y,x,z,t)$$

Transverse Magnetic Modes

$$A\ \frac{\partial \psi(x=a, y=x, z, t)}{\partial x} = \frac{\partial}{\partial x}\left(A\cos\left(\frac{m\pi a}{a}\right) + B\sin\left(\frac{m\pi a}{a}\right)\right)\cdot\left(C\cdot\cos\left(\frac{n\pi a}{a}\right) + D\sin\left(\frac{n\pi a}{a}\right)\right)\cdot e^{i(kz-wt)}$$

$$= \left(-A\frac{m\pi}{a}\sin\left(\frac{m\pi a}{a}\right) + B\cdot\frac{m\pi}{a}\cdot\cos\left(\frac{m\pi a}{a}\right)\right)\cdot\left(C\cdot\cos\left(\frac{n\pi a}{a}\right) + D\sin\left(\frac{n\pi a}{a}\right)\right)\cdot e^{i(kx-wt)}$$

$$= 0\ ,\ A = 0$$

$$C\ \frac{\partial \psi(x=a, y=x, z, t)}{\partial y} = \frac{\partial}{\partial y}\left(B\sin\left(\frac{m\pi a}{a}\right)\right)\left(C\cos\left(\frac{n\pi a}{a}\right) + D\sin\left(\frac{n\pi a}{a}\right)\right)\cdot e^{i(kx-wt)}$$

$$= B\sin\left(\frac{m\pi a}{a}\right)\left(-C\cdot\frac{n\pi a}{a}\sin\left(\frac{n\pi a}{a}\right) + D\frac{n\pi a}{a}\cos\left(\frac{n\pi a}{a}\right)\right)\cdot e^{i(kx-wt)}$$

$$= 0\ ,\ C = 0$$

$$B\ \psi(x=a, y, z, t) = B\sin\left(\frac{m\pi a}{a}\right)\sin\left(\frac{n\pi a}{a}\right)e^{i(kx-wt)}$$

$$= E_0\ ,\ B = E_0$$

$$\frac{\partial E(x,y,z,t)}{\partial x} = \frac{\partial E(x,y,z,t)}{\partial y} = \frac{\partial E(y,x,z,t)}{\partial x} = \frac{\partial E(y,x,z,t)}{\partial y}$$

(5) Exact Solutions

$$B(x,y,z,t) = B_0\left[\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right) + \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right)\right]$$

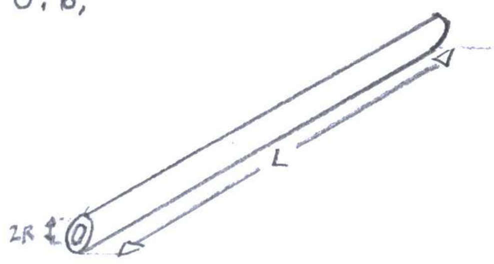
$$E(x,y,z,t) = E_0\left[\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right) - \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\right]$$

Cutoff Frequencies:

$$\frac{k}{w} = \frac{n}{c}$$

$$\omega = c\cdot k = c\,\frac{\pi}{a}\sqrt{m^2+n^2}$$

8.6.



"A resonant cavity of copper... consists of a hollow, right, circular cylinder... with flat faces."

a) ① Boundary Conditions:

Transverse Magnetic Waves:

$B_z = 0$ everywhere, $E_z|_s = 0$

Transverse Electric Waves:

$E_z = 0$ everywhere, $\left.\frac{\partial B_z}{\partial n}\right|_n = 0$

② Laplace's Equation:

$$\nabla \Phi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \cdot \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial^3 \phi} + \frac{\partial^2 \phi}{\partial^2 z}$$

$$= 0$$

③ Solution to variables in Laplace's Equation:

If $\Phi(\rho, \phi, z) = R(\rho) \cdot Q(\phi) \cdot Z(z)$, then

Ⓐ Variable Separation:

$$\nabla \Phi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \cdot \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial^2 \phi} + \frac{\partial^2 \Phi}{\partial^2 z}$$

$$= \frac{Q(\phi)Z(z)}{\rho}\frac{\partial}{\partial \rho}\left(\rho \cdot \frac{\partial R(\rho)}{\partial \rho}\right) + \frac{R(\rho)Z(z)}{\rho^2}\frac{\partial^2 Q(\phi)}{\partial^2 \phi} + R(\rho)Z(z)\frac{\partial^2 Z(z)}{\partial^2 z}$$

$$= \frac{1}{R(\rho) \cdot \rho}\frac{\partial}{\partial \rho}\left(\rho \cdot \frac{\partial R(\rho)}{\partial \rho}\right) + \frac{1}{Q(\phi)\rho^2}\frac{\partial^2 Q(\phi)}{\partial^2 \phi} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z}$$

$$= 0$$

Ⓑ Radial Eigenvalues:

$$\frac{\rho}{R}\frac{\partial R}{\partial \rho} + \frac{\rho^2}{R}\frac{\partial^2 R}{\partial^2 \rho} + \frac{1}{Z}\frac{\partial^2 Z}{\partial^2 z} = \lambda \rho^2$$

$$\frac{\rho}{R}\left(\frac{\partial R}{\partial \rho}\right)+\frac{\rho^2}{R}\frac{\partial^2 R}{\partial^2 \rho}-\lambda\rho^2=-\frac{1}{Z}\frac{\partial^2 Z}{\partial^2 Z}$$

$$\frac{\rho}{R}\left(\frac{\partial R}{\partial \rho}\right)+\frac{\rho^2}{R}\left(\frac{\partial^2 R}{\partial^2 \rho}\right)-\lambda\rho^2=k^2$$

If $\rho=x$ and $\lambda=1/\rho^2$, then.

$$\frac{\partial^2 R}{\partial^2 x}+\frac{1}{x}\frac{\partial^2 R}{\partial x}-\left(1+\frac{k^2}{x^2}\right)R=0$$

Ⓒ Angular Eigenvalues:

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial^2 \phi}=m \quad ; \quad \frac{\partial^2 Q}{\partial^2 \phi}+m^2 Q=0$$

Ⓓ Vertical Eigenvalues:

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial^2 Z}=k^2 \quad ; \quad \frac{\partial^2 Z}{\partial^2 Z}-k^2 Z=0$$

Ⓔ General Solution:

$$\Phi(\rho,\phi,z)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}R(\rho)\cdot Q(\phi)\cdot Z(z)$$

④ General Solution to Laplaces Equation:

$$R(\rho)=A J_v(kx)+B\cdot Y_v(kx)$$

where $J_v=\sum_{m=0}^{\infty}\frac{(-1)^m\left(-\frac{1}{2}kx\right)^{v+2m}}{m!(m+v)!}$

$$Y_v=\sum_{m=0}^{\infty}\frac{J_v(kx)\cos(v\pi)-J_v(kx)}{\sin(v\pi)}$$

$$Q(\phi)=Ce^{im\phi}$$

$$Z(z)=De^{kz}$$

$$\Phi(\rho,\phi,z)=(AJ_v(kx)+BY_v(kx))\cdot Ce^{im\phi}\cdot De^{kz}$$

⑤ Variables by Boundary Conditions:

Transverse Magnetic Modes

$$B \quad \psi(\rho=0,\phi,z) = \left(A J_v(k\cdot 0) + B Y_v(k\cdot 0)\right)\cdot C e^{im\phi}\cdot D e^{kz}$$

$$= 0, \quad B = 0$$

$$k \quad \frac{\partial \psi(\rho=R,\phi,z=0)}{\partial x} = \frac{\partial}{\partial x}\left(A J_v(k\cdot R)\cdot e^{im\phi}\right)$$

$$= 0, \quad k = \frac{X_{mn}}{R}$$

$$A \quad \psi(\rho=R,\phi,z=0) = A J_v\left(\frac{X_{mn}}{R}\cdot R\right) e^{im\phi}$$

$$= E_0 \text{ or } B_0 \quad \therefore A = E_0 \text{ or } B_0$$

⑥ Exact Solutions:

$$E(\rho,\phi) = E_0 J_v(k\rho)\cdot e^{im\phi}$$

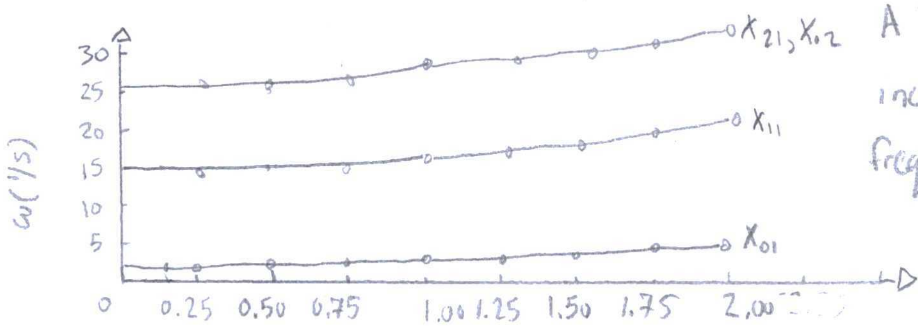
$$B(\rho,\phi) = B_0 J_v(k\rho)\cdot e^{im\phi}$$

Cutoff Frequencies:

$$\frac{k}{\omega} = \frac{n}{c}$$

$$\omega = c\cdot k$$

$$= \frac{1}{\sqrt{\mu\epsilon}\,R}\sqrt{X_{mn}^2 + \left(\frac{\rho\pi R}{L}\right)^2}$$



A larger radius increases the frequency in a tube.

b) If $R = 2\text{ cm}$ and $L = 3\text{cm}$

(8.96) "Q-factor"

$$Q = \omega_0 \frac{\text{Stored energy}}{\text{Lossed Power}} \propto \frac{1}{\text{Skindepth}} \propto \frac{1}{\text{FWHM}} \propto \frac{\text{frequency}}{\text{Change of frequency}}$$

(8.9) "Skin depth"

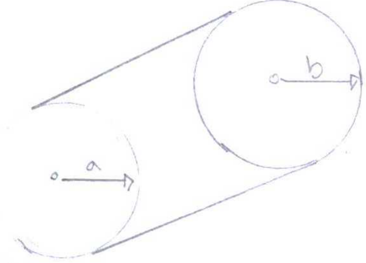
$$\delta = \left(\frac{2}{\mu_c \omega_0 \sigma}\right)^{1/2}$$

(8.96) "Q-factor with Skin depth"

$$Q = \frac{\mu}{\mu_c}\left(\frac{\text{Volume}}{\text{Surface Area} \cdot \delta}\right)$$

$$= \frac{\mu_{copper}}{\mu_{atmosphere}}\left(\frac{\pi R^2 \cdot L}{[2\pi R^2 + L \cdot \pi \cdot 2R] \cdot \left(\frac{2}{\mu_{atmosphere} \cdot \omega_{111} \cdot \sigma}\right)}\right)$$

8.7.



"A resonant cavity consists of the empty space between two perfectly conducting sphere"

a) (8.102) "Magnetic Induction"

$$B_\phi(r,\theta) = \frac{u_\ell(r)}{r} P_\ell^1(\cos\theta)$$

(8.103) "Angular Dependence"

$$\left[\frac{d^2u(r)}{dr^2} + \frac{\omega^2}{c^2} - \frac{\ell(\ell+1)}{r^2}\right]u(r) = 0$$

Solution:

$$u(r) = r\left[A_\ell j_\ell(kr) + B_\ell n_\ell(kr)\right]$$

$$\frac{du(r)}{dr} = A_\ell\left[j_\ell(kr) + kr j_\ell'(kr)\right] + B_\ell\left[n_\ell(kr) + kr n_\ell'(kr)\right]$$

Spherical Bessel's

$$j_n(x) = \sqrt{\frac{\pi}{2}} \frac{J_{n+1/2}(z)}{\sqrt{z}}$$

$$n_\ell(x) = \sqrt{\frac{\pi}{2}} \frac{Y_{n+1/2}(z)}{\sqrt{z}}$$

$$\begin{pmatrix} j_\ell(kr) + ka j_\ell'(ka) & n_\ell(ka) + ka n_\ell'(ka) \\ j_\ell(kb) + kb j_\ell'(kb) & n_\ell(kb) + kb n_\ell'(kb) \end{pmatrix}\begin{pmatrix} A_\ell \\ B_\ell \end{pmatrix} = 0$$

$$\frac{j_\ell(ka) + ka j_\ell'(ka)}{n_\ell(ka) + ka n_\ell'(ka)} = \frac{j_\ell(kb) + kb j_\ell'(kb)}{n_\ell(kb) + kb n_\ell'(kb)}$$

b) If $\ell = 1$, $j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$, $n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$

$$\frac{j_1(x) + x j_1'(x)}{n_1(x) + x n_1'(x)} = \frac{x\cos x - (1-x^2)\sin x}{x\sin x + (1-x^2)\cos x}$$

$$\frac{ka\cos ka - (1-(ka)^2)\sin(ka)}{ka\sin ka + (1-(ka)^2)\cos(ka)} = \frac{kb\cos kb - (1-(kb)^2)\sin kb}{kb\sin kb + (1-(kb)^2)\cos kb}$$

$$\left[(ka)(kb) + (1-(ka)^2)(1-(kb)^2)\right]\sin(kb-ka)$$

$$+\left[ka(1-(kb)^2) - kb(1-(ka)^2)\right]\cos(kb-ka) = 0$$

$$\tan kh = \frac{kb(1-(ka)^2) - ka(1-(kb)^2)}{(ka)(kb) + (1-(ka)^2)(1-(kb)^2)}$$ ... when $h = b-a$)

$$= \frac{kh(1+(ka)(kb))}{(ka)(kb) + (1-(ka)^2)(1-(kb)^2)}$$

$$= kh\,\frac{k^2 + 1/ab}{k^2 + ab\left(k^2 - \frac{1}{a^2}\right)\left(k^2 - \frac{1}{b^2}\right)}$$

c) When $h/a \ll 1$, $b = a+h$
$= a(1+h/a)$

$$\frac{\tan kh}{kh} = \frac{k^2 + \frac{1}{a^2} - \frac{1}{a^2}\frac{h}{a} + O((h/a)^2)}{k^4a^2 - k^2 + \frac{1}{a^2} + \left(k^4a^2 - \frac{1}{a^2}\right)\frac{h}{a} + O((h/a)^2)}$$

$$= \frac{(ka)^2+1}{(ka)^4-(ka)^2+1} - \frac{(ka)^2((ka)^4+2(ka)^2-2)}{((ka)^4-(ka)^2+1)^2}\frac{h}{a} + \cdots$$

$$= 1 + O((h/a)^2)$$

$$0 = -\frac{(ka)^2((ka)^2-2)}{(ka)^4-(ka)^2+1} - \frac{(ka)^2((ka)^4+2(ka)^2-2)}{((ka)^4-(ka)^2+1)^2}\frac{h}{a} + \cdots$$

If $ka = \sqrt{2}$, then $\omega_1 = \sqrt{2}\frac{c}{a}$

If $ka = \sqrt{2} + \delta(ka)$, then

$$0 = -2\sqrt{2}\,\delta(ka)\frac{2}{3} - \frac{4}{3}\frac{h}{a}$$

$$\delta(ka) = -\frac{1}{\sqrt{2}}\frac{h}{a}$$

$$ka = \sqrt{2} - \frac{1}{\sqrt{2}}\frac{h}{a}$$

$$= \sqrt{2}(1 - h/2a + \cdots)$$

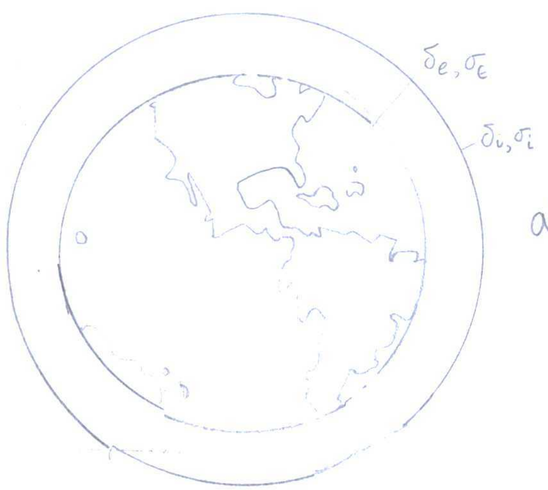
$$\omega_1 = \sqrt{2}\frac{c}{a}(1 - h/2a + \cdots)$$

$$= \sqrt{2}\frac{c}{a + h/2 + \cdots}$$

$$a + h/2 = \frac{a+b}{2}$$

8.8

"Earth .. and the ionosphere"



a) (8.105) $\omega_\ell \cong \sqrt{\ell(\ell+1)}\,\frac{c}{a}$

(8.8) $\delta = \left(\frac{2}{\mu_c \omega \sigma}\right)^{1/2}$

$$\equiv P_{earth} + P_{ionosphere}$$

$$= \mu_0 \omega (\delta_e + \delta_i) \pi a^2 |H_0|^2 \frac{l(l+1)}{2l+1}$$

(8.86) "Energy loss per cycle"

$$Q = \omega \frac{U}{P_{lost}}$$

$$\cong \frac{2h}{\delta_e + \delta_i}$$

$$= \frac{Nh}{\delta_e + \delta_i} \quad \text{∴ when} \quad N=2$$

b) If $\sigma_e = 0.1 (\Omega m)^{-1}$, $\sigma_i = 10^{-5} (\Omega m)^{-1}$, $h = 10^2 km$

(pg 376) "Resonant Frequencies at low noises on Earth"
"Schumann Resonances"

$$\frac{\omega}{2\pi} = 10.6Hz, 18.3Hz, 25.8Hz, 33.4Hz, 40.9Hz$$

$$\delta_e = \sqrt{\frac{2}{\mu_0 \sigma_e \omega_1}}$$

$$\cong 500m$$

$$\delta_i \cong \sqrt{\frac{2}{\mu_0 \sigma_i \omega_1}}$$

$$\cong 5 \times 10^4 m$$

$$Q \approx \frac{2 \times 10^5 m}{5 \times 10^2 m + 5 \times 10^4 m}$$

$$\approx 4$$

c) The schumann resonances had a specific time c,' from the past. Natural resonances have diurnal variations on Earth.

(8.103.5) "Radial Electric Field"

$$E_r = \frac{-ic^2}{\omega r}\,\ell(\ell+1)\,\frac{u_\ell(r)}{r}\,P_\ell(\cos\theta)$$

$$\cong \frac{-i}{\epsilon_0 \omega_\ell a}\,\ell(\ell+1) H_0 P_\ell(\cos\theta)$$

(8.102) "Azimuthal Magnetic Induction"

$$B_\phi = \frac{u_\ell(r)}{r} P_\ell^1(\cos\theta)$$

$$H_\phi \cong H_0 P_\ell^1(\cos\theta) \qquad \ldots \text{when} \quad \mu = \mu_0$$

(6.106) "Total energy density"

$$u = \frac{1}{2}(E\cdot D + B\cdot H)$$

$$U = ha^2 \int d\Omega \left[\frac{\epsilon_0}{4}|E_r|^2 + \frac{\mu_0}{4}|H_\phi|^2\right]$$

$$= \frac{\mu_0 h a^2}{4}|H_0|^2 \int d\Omega \left[\frac{c^2}{\omega_\ell^2 a^2}\,\ell^2(\ell+1)^2 P_\ell(\cos\theta)^2 + P_\ell^1(\cos\theta)^2\right]$$

$$= \frac{\mu_0 h a^2}{4}|H_0|^2 2\pi \int_{-1}^{1} d\cos\theta \left[\ell(\ell+1) P_\ell(\cos\theta)^2 + P_\ell^1(\cos\theta)^2\right]$$

$$= 2\mu_0 h \pi a^2 |H_0|^2 \frac{\ell(\ell+1)}{2\ell+1}$$

Legendre Polynomial

$$\int_{-1}^{1} P_\ell^m(x) P_\ell^m(x)\,dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}$$

(8.15) "Power loss modified"

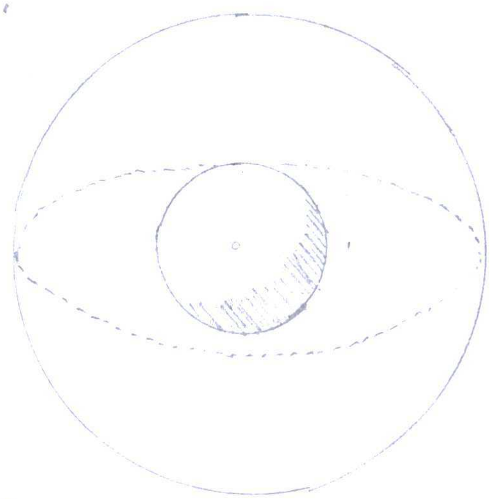
$$P = \frac{1}{2\sigma\delta}\int_S |n\times H|^2 da$$

$$\cong \frac{1}{2\sigma\delta}|H_0|^2 R^2 \int d\Omega\, P_\ell^1(\cos\theta)^2$$

$$= \frac{\pi R^2 |H_0|^2}{\sigma\delta}\int_{-1}^{1} d\cos\theta\, P_\ell^1(\cos\theta)^2$$

$$= \frac{2\pi R^2 |H_0|^2}{\sigma\delta}\frac{\ell(\ell+1)}{2\ell+1} \qquad \ldots \text{When } R = a+h \cong a$$

8.9.



"Hollow volume V containing an isotropic linear medium, bounded by a perfect, conducting closed sphere"

a) $$\frac{\int_V E^* \cdot [\nabla \times (\nabla \times E)]\, d^3x}{\int_V E^* \cdot E\, d^3x}$$

$$= \frac{\int_V E^* \cdot k^2 \cdot E\, d^3x}{\int_V E^* \cdot E\, d^3x}$$

$$= k^2$$

b) If $E = E_0 \cos(\pi\rho/2R)$, then

$$\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E$$

$$= -\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho \frac{\partial E}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 E}{\partial \phi^2} + \frac{\partial^2 E}{\partial^2 z}\right)$$

$$= E_0\left[\frac{\pi}{2R\rho}\sin\left(\frac{\pi\rho}{2R}\right) + \frac{\pi^2}{4R^2}\cos\left(\frac{\pi\rho}{2R}\right)\right]$$

$$k^2 = \frac{\int_0^R \left[\frac{\pi}{2R\rho}\cos\left(\frac{\pi\rho}{2R}\right)\sin\left(\frac{\pi\rho}{2R}\right) + \frac{\pi^2}{4R^2}\cos^2\left(\frac{\pi\rho}{2R}\right)\right]\rho\, d\rho}{\int_0^R \cos^2\left(\frac{\pi\rho}{2R}\right)\rho\, d\rho}$$

$$= \frac{\frac{1}{16}(4+\pi^2)}{\frac{1}{4\pi^2}R^2(-4+\pi^2)}$$

$$= \frac{\pi^2}{4R^2}\,\frac{\pi^2+4}{\pi^2-4}$$

$$Rk = \frac{\pi}{2}\sqrt{\frac{\pi^2+4}{\pi^2-4}}$$

$$= 2.4146$$

c) If $E = E_0\left[1 + \alpha(\rho/R)^2 - (1+\alpha)(\rho/R)^4\right]$

$$\nabla \times (\nabla \times E) = -\nabla^2 E$$

$$= -\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial E}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 E}{\partial^2 \Phi} + \frac{\partial^2 E}{\partial^2 z}\right)$$

$$= E_0 \frac{4}{R^2}(\alpha R^2 - 4r^2 - 4r^2\alpha)$$

$$k^2 = \frac{\int_0^R (1+\alpha(\rho|R)^2 - (1+\alpha)(\rho|R)^4)\frac{4}{R^4}(\alpha R^2 - 4r^2 - 4r^2\alpha)\rho\, d\rho}{\int_0^R (1+\alpha(\rho|R)^2 - (1+\alpha)(\rho|R)^4)^2 \rho\, d\rho}$$

$$= \frac{2 + \frac{4}{3}\alpha + \frac{1}{3}\alpha^2}{\frac{R^2}{60}(16 + 7\alpha + \alpha^2)}$$

$$= \frac{20}{R^2}\frac{6 + 4\alpha + \alpha^2}{16 + 7\alpha + \alpha^2}$$

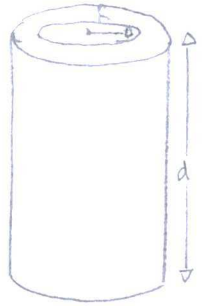
$$\frac{dk^2}{d\alpha} = \frac{20}{R^2}\frac{(16+7\alpha+\alpha^2)(4+2\alpha) - (6+4\alpha+\alpha^2)(7+2\alpha)}{(16+7\alpha+\alpha^2)^2}$$

$$\alpha = -\frac{1}{3}(10 \pm \sqrt{34})$$

$$kR = \sqrt{80\frac{17 - 2\sqrt{34}}{68 + \sqrt{34}}}$$

$$= 2.4050 \ (0.399\%)$$

8.10.



"TE_{111} mode in a right circular cylinder"

(Variational Principle)

$$k^2 = \frac{\int_V E^* \circ [\nabla \times (\nabla \times E)] d^3x}{\int_V E^* \circ E d^3x}$$

a) $$\frac{\int_V E^* \circ [\nabla \times (\nabla \times E)] d^3x}{\int_V E^* \circ E d^3x}$$

Identities
$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - (\nabla \cdot \nabla) A$
$\nabla \circ (A \times B) = B \circ \nabla \times A - A \circ \nabla \times B$

$$= \frac{\int_V \nabla \cdot (\nabla \circ E) - (\nabla \circ \nabla) E \, d^3x}{\int_V E^* \circ E d^3x}$$

$$= \frac{\int_V \nabla(\nabla \times E \times E) + \nabla \times E \circ \nabla \times E \, d^3x}{\int E^* \circ E d^3x}$$

$$= \frac{\int_V (\nabla \times E^*) \circ (\nabla \times E) d^3x}{\int E^* \circ E d^3x}$$

$$= k^2$$

b) Transverse Components in Electric Field:

$$B_z = B_0 (\rho/R)(1 - \rho/2R) \cos\phi \sin(\pi z/d)$$

$$E_\rho = \frac{\partial B_z}{\partial \phi}$$

$$= E_0 \circ \frac{\rho}{R} (1 - \rho/2R) \cos\phi \sin(\pi z/d)$$

$$E_\phi = \frac{\partial B_z}{\partial \rho}$$

$$= \frac{E_0}{R} (1 - \rho/R) \sin\phi \sin(\pi z/d) \quad \therefore \text{ when } E_0 = B_0 \ @ \ \rho = R$$

Curl in Cylindrical

$$\nabla \times E = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ E_\rho & E_\phi & E_z \end{vmatrix}$$

c) Curl of E:

$$\nabla \times E = \left(\frac{\partial}{\partial \phi} E_z - \frac{\partial}{\partial z} E_\phi\right) \frac{1}{\rho} - \left(\frac{\partial}{\partial \rho} E_z + \frac{\partial}{\partial z} E_\rho\right) \frac{\rho}{\rho} + \left(\frac{\partial}{\partial \rho} E_\phi - \frac{\partial}{\partial \phi} E_\rho\right) \frac{1}{\rho}$$

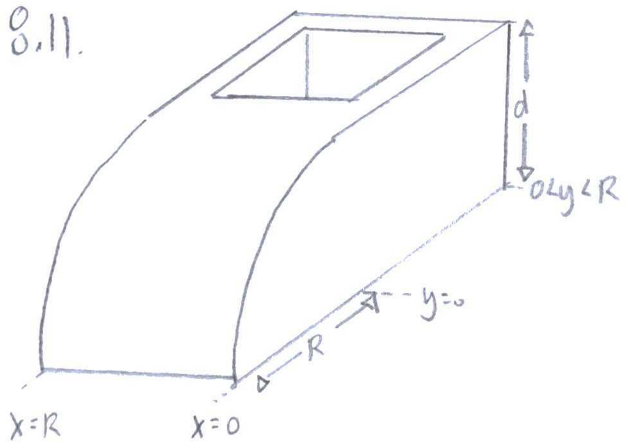
$$= \frac{1}{\rho} \left(\frac{\partial}{\partial \phi} E_0 (1 - \rho/R) \cos\phi \cos(\pi z/d) \frac{\pi}{d} - \frac{\partial}{\partial z} \rho E_0 (1 - \rho/R) \cos\phi \sin(\pi z/d) \right.$$

d) The original expression in Problem 8.9 is an equivalent integrand.

$$E^{*}\circ[\nabla\times\nabla\times E] = E^{*}[\nabla(\nabla\circ E)-\nabla^{2}E] = (\nabla\times E)\circ(\nabla\times E).$$

The merits are less computation and derivatives.

8.11.



"breadbox cavity"

a) <u>Trial Function:</u>

$$E_z = E_0\,(\rho/R)^{\nu}(1-\rho/R)\sin(2\phi)$$

$$\nabla\times\nabla\times E = \hat{\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_\rho}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 E_\rho}{\partial^2\phi}+\frac{\partial^2 E_\rho}{\partial^2 z}-\frac{E_\rho}{\rho^2}-\frac{2}{\rho^2}\frac{\partial E_\phi}{\partial\phi}\right)$$

$$+\hat{\phi}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_\phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 E_\phi}{\partial^2\phi}+\frac{\partial^2 E_\phi}{\partial^2 z}-\frac{E_\phi}{\rho^2}-\frac{2}{\rho^2}\frac{\partial E_\rho}{\partial\phi}\right)$$

$$+\hat{z}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_z}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial^2\phi}+\frac{\partial^2 E_z}{\partial^2 z}\right)$$

$$= E_0\left(\frac{\left(\frac{\rho}{R}\right)^{\nu}(R(\nu+1)-\rho(\nu+2))}{\rho}+\frac{4\left(\frac{\rho}{R}\right)^{\nu+1}(\rho-R)}{\rho^2}\right)\sin(2\phi)$$

$$\frac{\int_V E^{*}(\nabla\times\nabla\times E)\,dV}{\int_V E^{*}\circ E\,dV} =$$

$$= \frac{\int_0^R \int_0^{2R} \int_0^d \left(E_0 \left(\frac{\rho}{R}\right)^v (1-\rho/R)\sin(2\phi)\right) \times \left(E_0 \left[\frac{\left(\frac{\rho}{R}\right)^v (R(v+1)-\rho(v+2))}{\rho} + \frac{4\left(\frac{\rho}{R}\right)^{v+1}(\rho-R)}{\rho^2}\right]\sin 2\phi\right) \cdot \sqrt{x^2+y^2}\,dx\,dy\,dz}{\int_0^R \int_0^{2R} \int_0^d \left(E_0 \left(\frac{\rho}{R}\right)^v (1-\rho/R)\sin(2\phi)\right)^2 \sqrt{x^2+y^2}\,dx\,dy\,dz}$$

Cylindrical to Cartesian
$\rho = x^2+y^2 \quad \phi = \tan^{-1}\left(\frac{y}{x}\right)$

$$= \frac{\int_0^R \int_0^{2R} \int_0^d E_0^2 \left(\frac{\sqrt{x^2+y^2}}{R}\right)^v \left(1-\sqrt{x^2+y^2}/R\right)\sin\left(2\cdot\tan^{-1}\left(\frac{y}{x}\right)\right)\left[\left(\frac{\sqrt{x^2+y^2}}{R}\right)^v\left(R(v+1)-\sqrt{x^2+y^2}(v+2)\right) + \ldots\right.}{\ldots}$$

$$\ldots \frac{\left.4\left(\frac{\sqrt{x^2+y^2}}{R}\right)^{v+1}\right]\sin\left(\tan^{-1}\left(\frac{y}{x}\right)\right)\sqrt{x^2+y^2}\,dx\,dy\,dz}{\int_0^R \int_0^{2R} \int_0^d \left(E_0 \left(\frac{\sqrt{x^2+y^2}}{R}\right)^v \left(1-\sqrt{x^2+y^2}/R\right)\sin\left(\tan^{-1}\left(\frac{y}{x}\right)\right)\right)^2 \sqrt{x^2+y^2}\,dx\,dy\,dz}$$

$$= \frac{(v+2)(2v+3)(v^2+v+4)}{v(2v+1)R^2}$$

$$= k^2$$

$$k^2R^2 = \frac{(v+2)(2v+3)(v^2+v+4)}{v(2v+1)}$$

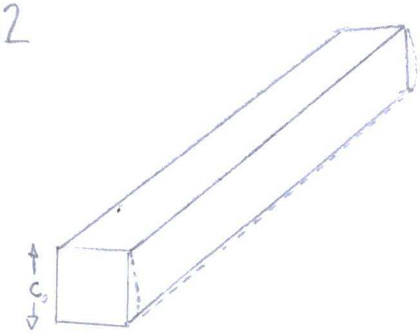
$$R^2\frac{dk^2}{dv} = \frac{8v^5+24v^4+18v^3-47v^2-96v-24}{v^2(2v+1)^2}$$

$$= 0$$

V_{max}	-1.7212	-0.2960	1.5675
kR	5.2054	Imaginary	5.2054

The variational method provided a solution with 1.36% error.

8.12



"a waveguide with lossless dielectric... departs slightly from contour"

a) Eigenvalue parameters:

$$C = (\gamma^2, \psi^2)$$

$$C_0 = (\gamma_0^2, \psi_0^2)$$

(1.35) "Green's Theorem"

$$\int_A \left[\phi \nabla_t^2 \psi - \psi \nabla_t^2 \phi\right] da$$

$$= \oint \left[\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n}\right] dl$$

(8.11) $E_{11} \cong \sqrt{\frac{\mu_c w}{2\sigma}}(1-i)(n \times H_{11})$

(8.65) "Perturbed boundary condition"

$$\psi\big|_s \cong f \frac{\partial \psi_0}{\partial n}\Big|_s$$

(8.64) "Unperturbed boundary condition"

$$(\nabla_t^2 + \gamma_0^2)\psi_0 = 0 \quad \psi_0\big|_s = 0$$

(8.67) "Perturbed longitudinal Electric field"

$$(\nabla_t^2 + \gamma^2)\psi = 0 \quad \psi\big|_s \cong f \frac{\partial \psi_0}{\partial n}\Big|_s$$

$$\int_A \left[\psi_0 \nabla^2 \psi - \psi \nabla^2 \psi_0\right] da = -\int_A (\psi_0 \gamma^2 \psi + \psi \gamma_0^2 \psi_0) da$$

$$= (\gamma_0^2 - \gamma^2) \int \psi_0^* \psi da$$

$$= f \oint \left|\frac{d\phi}{\partial n}\right|^2 dl \qquad (8.68)$$

$$\gamma_0^2 - \gamma^2 = \frac{f \int_c \left|\frac{\partial \psi_0}{\partial n}\right|^2 dl}{\int_A |\psi|^2 da}$$

$$\oint_c \left[\psi_0^* \delta(x,y) \left| \frac{\partial \psi_0}{\partial n} \right|^2 - \psi_0^* \delta(x,y) \frac{\partial^2 \psi_0}{\partial n^2} \right] dl$$

$$= -(\gamma_0^2 - \gamma^2) \int_S |\psi_0|^2 da$$

$$\gamma^2 - \gamma_0^2 = -\frac{\oint_c \delta(x,y) \left[\left| \frac{\partial \psi_0}{\partial n} \right|^2 - \psi_0^* \frac{\partial^2 \psi_0}{\partial n^2} \right] dl}{\int_{S_0} |\psi_0|^2 dx\,dy}$$

"a rectangular guide .. changed in shape."

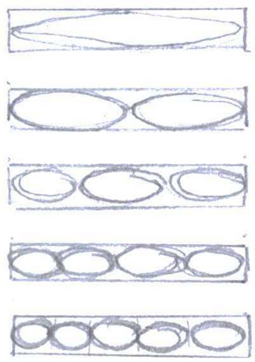
b)
$$\gamma^2 - \gamma_0^2 = -\frac{\oint_c \delta(x,y) \left| \frac{\partial \psi_0}{\partial n} \right|^2 dy}{\frac{1}{4} \int_0^a \int_0^b |\psi_0|^2 dx\,dy}$$

$$= \frac{2 \cdot \frac{\pi^2 \delta}{a^2 b} \int y \sin^2\left(\frac{\pi y}{b}\right) dy}{\frac{1}{4} \int_0^a \int_0^b dx\,dy}$$

$$= \frac{\frac{\pi^2 \delta b}{2a^2}}{\frac{1}{4} a \cdot b}$$

$$= \frac{2\pi^2 \delta}{a^3}$$

8.13



"two-dimensional (waveguide) situation .. N-fold digeneracy"

a) (Problem 8.12)

$$\gamma^2 - \gamma_0^2 = \frac{-\oint_C \sigma(x,y)\left[\left|\frac{\partial \psi_0}{\partial n}\right|^2 - \psi_0^* \frac{\partial^2 \psi_0}{\partial n^2}\right] d\rho}{\int_{S_0} |\psi_0|^2 \, dx\,dy}$$

$$= \frac{\Delta_i - \Delta_{ji}}{N_j}$$

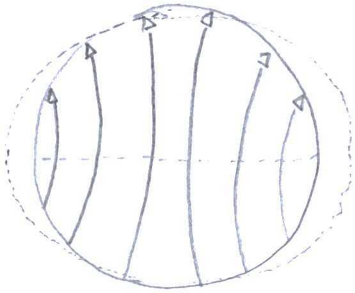
$$\left[(\gamma_0^2 - \gamma^2) N_j \delta_{ij} + \Delta_{ji}\right] a_t = 0$$

b) $$\psi^{(\pm)} = B_z = B_0 J_1(\gamma_0 \rho) \exp(\pm i\phi) \exp(ikz - i\omega t)$$

$$= 0, \text{ so } \gamma_0^2 = \gamma \pm \frac{|\Delta|}{|N|}$$

$$= \gamma^2 \pm \frac{\Delta R}{R}\left(\frac{x_{11}}{R}\right)^2$$

$$= \gamma^2\left(1 \pm \frac{\Delta R}{R}\right)$$



"the lowest mode in a circular guide of radius R.. is the twofold degenerate TE_{11} mode" "..distorted along length to an ellipse"

Another method is by eigenvalues:

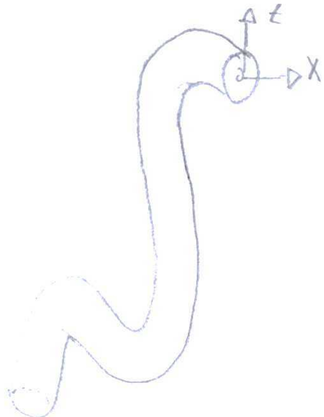
$$\begin{vmatrix} N(\gamma^2 - \gamma_0^2) & \Delta \\ \Delta & N(\gamma^2 - \gamma_0^2) \end{vmatrix} = 0$$

$\gamma_1^2 = \gamma_0^2(1 + \lambda \Delta R / R)$ and $\gamma_2^2 = \gamma_0^2(1 - \lambda \Delta R / R)$

.. where $\lambda = 1$ from prior example.

The eigenvalues turned out similarly because each zero-value in the solution. In real world, a ratio about the contour per surface area.

8.14.



"an optical fiber with graded index of refraction"

$$n(x) = n(0)\,\text{sech}(\alpha x)$$

$$\bar{n} = n(x_{max})$$

$$= n(0)\cos(\theta)$$

a) (pg 380) Eikonal Equations

$$(8.109)\ \left[\nabla^2 + \frac{\omega^2}{c^2} n^2(x)\right]\psi = 0$$

$$(8.110)\ \psi = e^{i\omega S(x)/c} \quad \ldots \text{where } S(x) = \text{"Eikonal}$$

(pg 380) Eikonal Approximation

$$\nabla S \circ \nabla S = n^2(x)$$

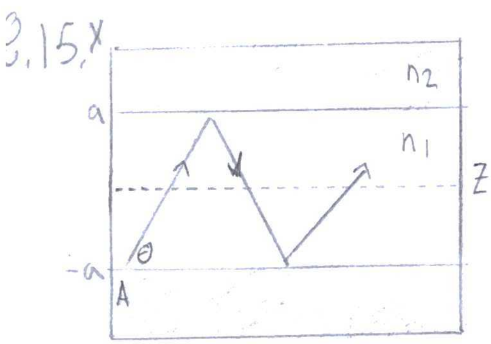
$$\sinh^2(\alpha x) = \sinh(\alpha x_{max})$$

$$n(x)\sin(\alpha z) = n_{max}(x)\sin(\alpha z)$$

$$\text{sech}(\alpha x)\sin(\alpha z) = \text{sech}(\alpha x_{max}) \cdot \sin(\alpha z)$$

$$x = \text{sech}^{-1}\left(\text{sech}(\alpha x_{max})\sin(\alpha z)\right) \quad \ldots @\, z = 0$$

8.15.



"Slab dielectric waveguide"

a) (8.121) "Internal reflection"

$$4ka\sin\theta + 2\phi = 2p\pi$$

(8.122) "Phases"

$$\phi_{TE} = -2\arctan\sqrt{\frac{2\Delta}{\sin^2\phi} - 1}$$

$$\phi_{TM} = -2\arctan\left(\frac{1}{1-2\Delta}\sqrt{\frac{2\Delta}{\sin^2\theta} - 1}\right)$$

(8.123) "Phases with variable"

$$\tan\left(V\xi - \frac{p\pi}{2}\right) = f\sqrt{\frac{1}{\xi^2} - 1} \quad \ldots \text{where } V = ka\sqrt{2\Delta}$$

$$\xi = \sin\theta / \sqrt{2\Delta}$$

$$4ka\sin\theta + 2\phi = 2p\pi$$

$$ka\sin\theta + \frac{\phi}{2} = \frac{p\pi}{2}$$

$$\left(V\xi - \frac{p\pi}{2}\right) = \frac{\phi}{2}$$

$$\tan\left(\frac{-\phi_{TE}}{2}\right) = \sqrt{\frac{2\Delta}{\sin^2\theta} - 1}$$

$$\tan\left(\frac{-\phi_{TM}}{2}\right) = \frac{1}{1-2\Delta}\sqrt{\frac{2\Delta}{\sin^2\theta} - 1}$$

$$= \tan\left(V\xi - \frac{p\pi}{2}\right)$$

The even:odd relates to the phase in tangent ; $\tan(x+\phi)$.

b) Newton's Method:

$$X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}$$

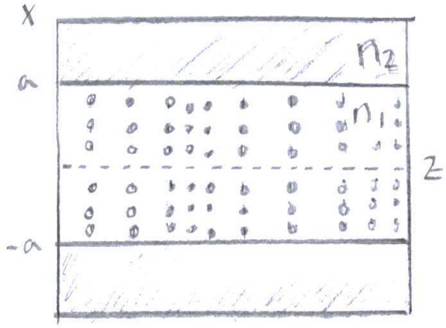
The solution in the book provides no exactness in an analysis.

Per se, $\xi = \frac{(p+1)\pi}{2(v+1)}\left[1 - \frac{(p+1)^2\pi^2}{24(V+1)^2}\right]$ is not the zero in the function $\tan\left(V\xi - \frac{p\pi}{2}\right) - \sqrt{\frac{1}{\xi} - 1} = 0$.

A graph presented no correct answer either.

c) Answer skipped.

8.16



"...longitudinal phase velocity in the dielectric slab waveguide"

a) $V_p = \omega / k_z$

$= c / (n_1 \cos\theta_p)$

(8.123) "Transverse Waveguide with variables"

$$\tan\left(V\xi - \frac{p\pi}{2}\right) = f\sqrt{\frac{1}{\xi^2} - 1}$$

...where $f = 1$, $V = ka\sqrt{2\Delta}$ and $\xi = \sin\theta / \sqrt{2\Delta}$

$$\tan\left(ka\sin\theta - \frac{p\pi}{2}\right) = \sqrt{\frac{2\Delta}{\sin^2\theta_p} - 1}$$

...if $k_z = k\cos\theta_p$, then $\sin\theta_p = \sqrt{1-\cos^2\theta_p}$

$$= \sqrt{1 - \frac{k_z^2}{k^2}}$$

$$= \frac{1}{k}\sqrt{k^2 - k_z^2}$$

$$\tan\left(a\sqrt{k^2-k_z^2} - \frac{p\pi}{2}\right) = \sqrt{\frac{2\Delta k^2}{k^2-k_z^2} - 1}$$

"Implicit Differentiation"

$$\frac{d}{dk}\left(\tan\left(a\sqrt{k^2-k_z^2} - \frac{p\pi}{2}\right)\right) = \frac{d}{dk}\left(\sqrt{\frac{2\Delta k^2}{k^2-k_z^2} - 1}\right)$$

$$\frac{1}{\cos^2\left\{a\sqrt{k^2-k_z^2} - \frac{p\pi}{2}\right\}} \frac{a}{2} \frac{2k - 2k_z\,(dk_z/dk)}{\sqrt{k^2-k_z^2}}$$

$$= \frac{1}{2} \frac{\frac{(4\Delta k)(k^2-k_z^2) - (2\Delta k^2)(2k - 2k_z(dk_z/dk))}{(k^2-k_z^2)^2}}{\sqrt{2\Delta k^2/(k^2-k_z^2) - 1}}$$

$$\frac{dk_z}{dk} = \frac{1}{\cos\theta_p} \cdot \frac{\cos^2\theta_p + ka\sqrt{2\Delta - \sin^2\theta_p}}{1 + ka\sqrt{2\Delta - \sin^2\theta_p}}$$

$$\boxed{\begin{aligned} k^2 &= k_z^2 + k_x^2 \\ k^2 - k_z^2 &= k_x^2 \\ &= k^2\sin\theta_p \end{aligned}}$$

$$\boxed{V_g = \frac{d\omega}{dk_z} = \frac{d\omega}{dk} \cdot \frac{dk}{dk_z}} \quad \boxed{\cos^2(x) = \frac{1}{1+\tan^2(x)}}$$

$$V_g = \frac{d\omega}{dk}\,\frac{dk}{dk_z}$$

$$= \frac{c\cdot\cos\theta_p}{n_1}\;\frac{1+k\cdot a\sqrt{2\Delta-\sin^2\theta_p}}{\cos^2\theta_p + ka\sqrt{2\Delta-\sin^2\theta_p}} \quad \ldots \text{when } \frac{d\omega}{dk}=\frac{c}{n}$$

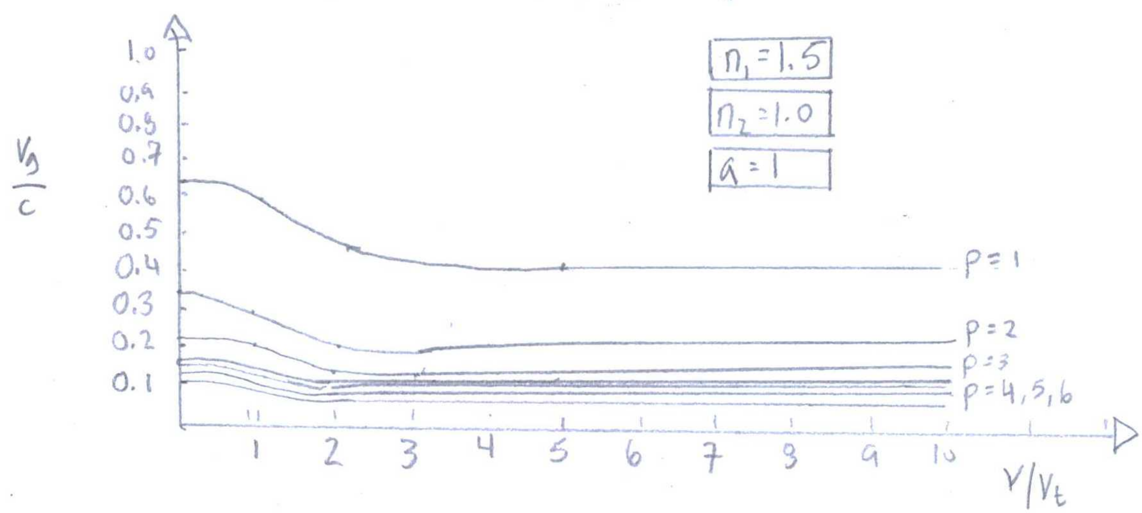
$$= \frac{c\cdot\cos\theta_p}{n_1}\;\frac{1+\beta_p a}{\cos^2\theta_p+\beta_p a} \quad \ldots \text{where } \beta_p = k\sqrt{2\Delta-\sin^2\theta_p}$$

The group velocity (V_g) is larger by a factor ($c/n_1 \cdot \cos\theta_p$) because total internal reflection. In terms, Goos-Hanchen effect measures the departure from $\frac{1+\beta_p a}{\cos^2\theta_p+\beta_p a}$ through lateral displacement in total internal reflections.

b) $$\frac{V_g}{c} = \frac{\cos\theta_p}{n_1}\left[\frac{1+\beta_p\cdot a}{\cos^2\theta_p+\beta_p\cdot a}\right]$$

$$= \frac{k_z/k}{n_1}\left[\frac{1+4\Delta^2\cdot k\sqrt{1-\sin^2\theta_p/2\Delta}\cdot a}{(k_z/k)^2+4\Delta^2\cdot k\sqrt{1-\sin^2\theta_p/2\Delta}\cdot a}\right]$$

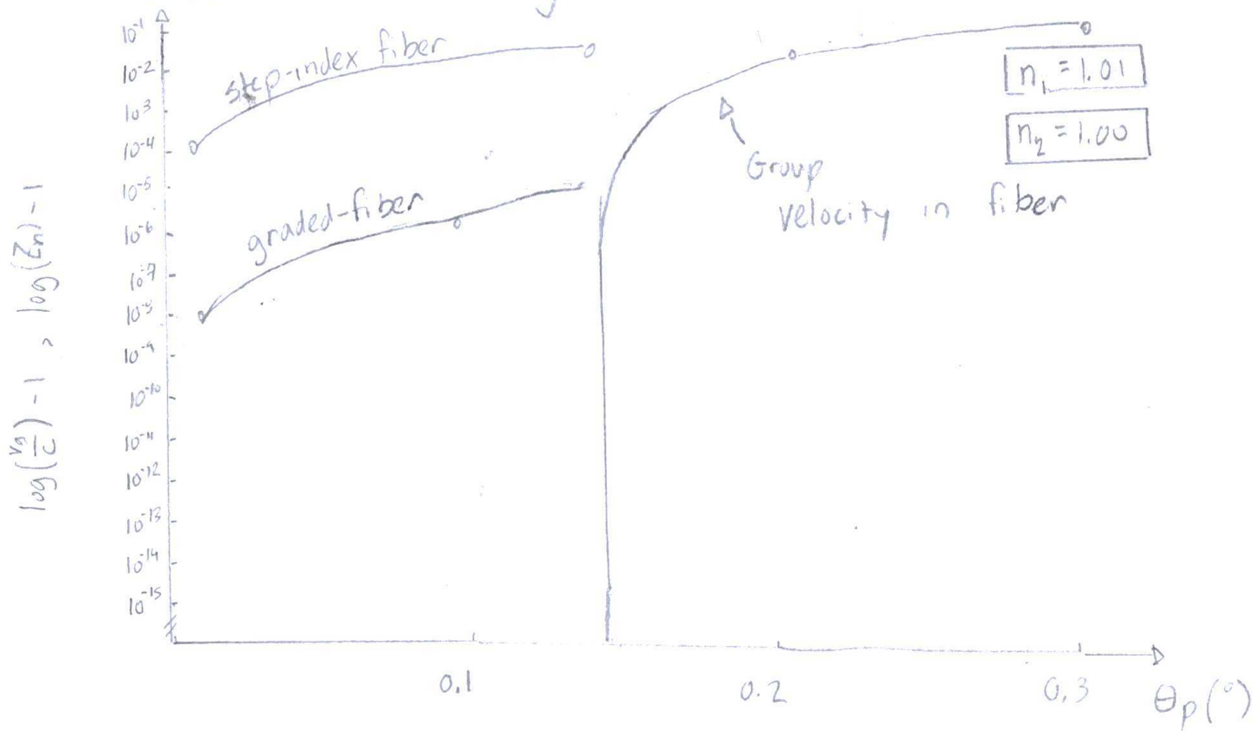
$$= \left(\frac{n_2}{n_1^2}\right)\left[\frac{1+\frac{v^4}{\sqrt{2}\,n_1^4\,a^3}}{(n_2/n_1)^2+\frac{v^4}{\sqrt{2}\cdot n_1^4\cdot a^3}}\right]$$



Note: Group velocity is a ratio between the speed of light.

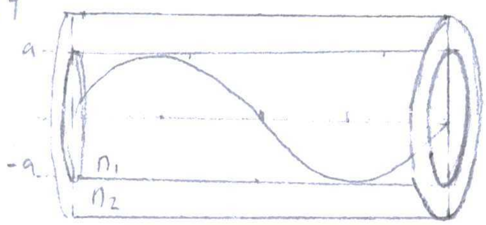
Velocity is greatest in a dielectric slab with small volumes in respect to wavelength by a transverse wave.

c) Figure 8.12b shows angle entering a fiber versus length in the fiber.



The $\log(\frac{v_g}{c})-1$ value indicates near light speed group velocity in fiber strands. Light speed in a vacuum has an index of refraction about $n_1=1.00$. While fiber optics $n_1=1.01$ provides a speed 99% the limit of light.

8.17



"... propagating modes in a cylindrical optical fiber waveguide"

a) (8.128) "Coupled/Cylindrical fields, also HF/EH modes"

$$\begin{Bmatrix} E_t \\ H_t \end{Bmatrix} = \begin{Bmatrix} A_e \\ A_n \end{Bmatrix} J_m(\gamma_1 \rho) e^{im\phi} \qquad \rho < a$$

$$\begin{Bmatrix} E_z \\ H_z \end{Bmatrix} = \begin{Bmatrix} B_e \\ B_n \end{Bmatrix} K_m(\beta \rho) e^{im\phi} \qquad \rho > a$$

...when $\gamma^2 = \frac{n_1^2 \omega^2}{c^2}$ and $\beta^2 = k_z^2 - n_2^2 \omega^2 / c^2$

(8.17) "Wave Equation"

$$\left[\nabla_t^2 + (\mu \epsilon \omega^2)\right] \begin{Bmatrix} E \\ B \end{Bmatrix} = 0$$

$$\nabla_t^2 E + \mu \epsilon \omega^2 E = 0$$

$$\nabla_t^2 E + k^2 E = 0$$

$$\nabla_t^2 B + k^2 B = 0$$

(6.1) "Faraday's Law"

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$= -\mu \frac{\partial H}{\partial t} \quad \text{... if } H = e^{-i(kx - \omega t)}$$

$$= -i\mu \omega H$$

(6.1) "Ampere's Law"

$$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \quad \text{... if } J = 0 \text{ and } E = e^{-i(kx - \omega t)}$$

$$\nabla \times H = i \epsilon_0 \omega E$$

Individual Directions:

(8.35) $\gamma^2 = \mu \epsilon \omega^2 - k^2$

$$\nabla \times H = i \epsilon_0 \omega E$$

$$= i\epsilon_0\omega(E_x - E_y + E_z)$$

$$= \frac{\partial H_z}{\partial y} + \frac{\partial H_z}{\partial x} + \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} + \gamma H_x - \gamma H_y$$

So, $$\frac{\partial H_z}{\partial y} + \gamma H_y = i\omega\epsilon E_x$$

$$\frac{\partial H_x}{\partial x} + \gamma H_x = -i\omega\epsilon E_y$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = i\omega\epsilon E_z$$

Also, $\nabla \times E = -i\omega\mu H$

$$= -i\omega\mu(-H_x + H_y - H_z)$$

$$= \frac{\partial E_z}{\partial y} + \frac{\partial E_z}{\partial x} + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \gamma E_y + \gamma E_x$$

So, $$\frac{\partial E_z}{\partial y} + \gamma E_y = -i\omega\mu H_x$$

$$\frac{\partial E_z}{\partial x} + \gamma E_x = i\omega\mu H_y$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -i\omega\mu H_z$$

In combo,

$$H_x = -\frac{\gamma}{h^2}\frac{\partial H_z}{\partial x} + \frac{i\omega\epsilon}{h^2}\frac{\partial E_z}{\partial y}$$

$$H_y = -\frac{\gamma}{h^2}\frac{\partial H_z}{\partial x} - \frac{i\omega\epsilon}{h^2}\frac{\partial E_z}{\partial x}$$

$$E_x = -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial x} - \frac{i\omega\mu}{h^2}\frac{\partial H_z}{\partial y}$$

$$E_y = -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial y} + \frac{i\omega\mu}{h^2}\frac{\partial H_z}{\partial x}$$

$$H_r = \frac{-i}{k^2-\beta^2}\left(\beta \frac{\partial H_z}{\partial r} - \frac{\omega\epsilon}{r}\frac{\partial E_z}{\partial \phi}\right)$$

$$H_\phi = \frac{-i}{k^2-\beta^2}\left(\omega\epsilon \frac{\partial E_z}{\partial r} + \frac{\rho}{r}\frac{\partial H_z}{\partial \phi}\right)$$

$$E_\phi = \frac{-i}{k^2-\beta^2}\left(\omega\epsilon \frac{\partial E_z}{\partial r} + \frac{\beta}{r}\frac{\partial H_z}{\partial \phi}\right)$$

$$E_\rho = \frac{-i}{k^2-\beta^2}\left(\frac{\omega\epsilon}{r}\frac{\partial H_z}{\partial \phi} + \beta \frac{\partial E_z}{\partial r}\right)$$

Into (3.123) from above, if $\rho < a$

$$E_\phi = \frac{i}{\alpha^2}\left(\frac{-m\beta}{r} A J_m(\alpha\rho)\sin(m\phi) - \omega\mu B\cdot\alpha J_m'(\alpha\rho)\sin(m\phi)\right)$$

$$E_\rho = \frac{-i}{\alpha^2}\left(\frac{m\omega\mu}{\rho} B J_m(\alpha\rho)\cos(m\phi) + B\alpha A J_m'(\alpha\rho)\cos(m\phi)\right)$$

$$H_\phi = \frac{-i}{\alpha^2}\left(\omega\epsilon\alpha A J_m'(\alpha\rho)\cos(m\phi) + \frac{m\beta}{r} B J_m(\alpha\rho)\cos(m\phi)\right)$$

$$H_\rho = \frac{-i}{\alpha^2}\left(\beta\cdot\alpha\cdot B\cdot J_m'(\alpha\rho)\sin(m\phi) + \frac{m\omega\epsilon}{\rho} A J_m(\alpha\rho)\sin(m\phi)\right)$$

While $\rho > a$,

$$E_\phi = \frac{i}{\alpha^2}\left(-\frac{m\beta}{\rho} C\cdot K_m(\alpha\rho)\sin(m\phi) - \omega\mu D\cdot\beta\cdot K_m'(\beta\rho)\sin(m\phi)\right)$$

$$E_\rho = \frac{-i}{k^2-\beta^2}\left(\frac{\omega\mu}{\rho} D\cdot m K_m(\alpha\rho)\cos(m\phi) + \beta\alpha C K_m'(\beta\rho)\sin(m\phi)\right)$$

$$H_\phi = \frac{i}{\beta^2}\left(\omega\epsilon\beta C K_m'(\beta\rho)\cos(m\phi) + \frac{m\beta}{\rho} D\cdot K_m(\beta\rho)\cos(m\phi)\right)$$

$$H_\rho = \frac{-i}{k^2-\beta^2}\left(\beta\cdot D\cdot\alpha\cdot K_m'(\alpha\rho) + \frac{\beta}{\rho}\cdot m\cdot D\cdot K_m(\beta\rho)\cos(m\phi)\right)$$

At $\rho = a$,

$$A J_m(\alpha a) = C K_m(\beta a)$$

$$B J_m(\alpha a) = D\cdot K_m(\beta a)$$

$$\frac{1}{\alpha^2 a^2}\left(m\beta A J_m(\alpha a)+\omega\mu B\alpha a J_m'(\alpha a)\right)=\frac{-1}{\beta^2 a^2}\left(m\beta C K_m(\beta a)+\omega\mu D\beta a K_m'(\beta a)\right)$$

$$\frac{-1}{(\alpha a)^2}\left(m\beta A J_m(\alpha a)+\omega\mu B\alpha a J_m'(\alpha a)\right)=\frac{-1}{(\beta a)^2}\left(m\beta A J_m(\gamma a)+\omega\mu B\frac{J_m(\gamma a)}{K_m(\beta a)}\beta a K_m'(\beta a)\right)$$

$$\frac{-1}{(\alpha a)^2}\left(\omega\epsilon_1\alpha a J_m'(\alpha a)+m\beta B J_m(\alpha a)\right)=\frac{1}{(\beta a)^2}\left(\omega\epsilon\beta a A\frac{J_m(\alpha a)}{K_m(\beta a)}K_m'(\beta a)+m\beta B J_m(\alpha a)\right)$$

$$B\left(\frac{\mu J_m'(\alpha a)}{\alpha a J_m(\alpha a)}+\frac{\mu K_m'(\beta a)}{\beta a K_m(\beta a)}\right)=-A\left(\frac{1}{(\alpha a)^2}+\frac{1}{(\beta a)^2}\right)\frac{m\beta}{\omega\mu_0}$$

$$A\left(\frac{\epsilon J_m'(\alpha a)}{\alpha a J_m(\alpha a)}+\frac{\epsilon K_m'(\beta a)}{\beta a K_m(\beta a)}\right)=-B\left(\frac{1}{(\alpha a)^2}+\frac{1}{(\beta a)^2}\right)\frac{m\beta}{\omega\mu_0}$$

$$\left(\frac{J_m'(\alpha a)}{\alpha a J_m(\alpha a)}+\frac{K_m'(\beta a)}{\beta a K_m(\beta a)}\right)\left(\frac{n_1^2 J_m'(\alpha a)}{\alpha a J_m(\alpha a)}+\frac{n_2^2 K_m(\beta a)}{\beta a K_m(\beta a)}\right)=\frac{m^2\beta^2}{k_0^2}\left(\frac{1}{(\alpha a)^2}+\frac{1}{(\beta a)^2}\right)^2$$

$$\left(\frac{n_1^2 J_m'(\alpha a)}{\gamma J_m(\alpha a)}+\frac{n_2^2 K_m(\beta a)}{\beta K_m(\beta a)}\right)\left(\frac{1 J_m'(\alpha a)}{\gamma J_m(\alpha a)}+\frac{1 K_m'(\beta a)}{\beta K_m(\beta a)}\right)=\frac{m^2}{a^2}\left(\frac{n_1^2}{\gamma^2}+\frac{n_2^2}{\beta^2}\right)\left(\frac{1}{\gamma^2}+\frac{1}{\beta^2}\right)$$

$\therefore$ only when $n=1, k_0=1$

b) If $m=0$,

$$\left(\frac{n_1^2 J_0'(\alpha a)}{\gamma J_0(\alpha a)}+\frac{n_2^2 K_0(\beta a)}{\beta a K_0(\beta a)}\right)\left(\frac{n_1^2 J_0'(\alpha a)}{\alpha a J_0(\alpha a)}+\frac{n_2^2 K_0'(\beta a)}{\beta a K_0(\beta a)}\right)=0$$

Left or right quantity equals zero.

(pg 379) $V=ka\sqrt{2\Delta}$

(8.106) $\Delta=\dfrac{n_1^2-n_0^2}{2n_1^2}$

$$V=ka\sqrt{\frac{n_1^2-n_0^2}{n_1^2}}$$

$$= \frac{2\pi a}{\lambda}\sqrt{\frac{n_1^2 - n_0^2}{n_1^2}}$$

$$= \frac{\omega}{c} a \sqrt{\frac{n_1^2 - n_0^2}{n_1^2}}$$

$$= 2.405$$

c) When $m \geq 1$, $n_1^2 \cong n_2^2 \cong n^2$

$$\left(\frac{n^2 J_m'(\alpha a)}{\gamma J_m(\alpha a)} + \frac{n^2 K_0'(\beta a)}{\beta a K_m(\beta a)}\right)^2 \cong \frac{m^2 \beta^2}{n^2 k_0^2}\left(\frac{1}{(\alpha a)^2} + \frac{1}{(\beta a)^2}\right)$$

$$\cong m^2\left(\frac{1}{(\alpha a)^2} + \frac{1}{(\beta a)^2}\right)^2$$

$$\frac{J_m'(\alpha a)}{(\gamma) J_m(\alpha a)} \cong \frac{-K_m'(\beta a)}{\beta a K_m(\beta a)} \pm \left(\frac{1}{(\alpha a)^2} + \frac{1}{(\beta a)^2}\right)$$ [Cite: Gloge, 1971]

8.18.

(8.19) "Wave Equation"

$$\left\{\nabla_t^2+(\mu\epsilon\omega^2-k^2)\right\}\begin{bmatrix}E\\B\end{bmatrix}=0$$

... where $\nabla_t^2=\nabla^2-\frac{\partial^2}{\partial z^2}$ (8.20)

(8.34) "Scalar in Wave Equation"

$$(\nabla_t^2+\gamma^2)\psi=0$$

... where $\gamma^2=\mu\epsilon\omega^2-k^2$ (8.35)

(8.36) "Transverse Electric Modes"

$$\left.\frac{\partial\psi_n}{\partial n}\right|_s=0$$

a) (8.67) "Orthogonal Preposition"

$$\left\{\nabla_t^2+\gamma_\lambda^2\right\}\psi_\lambda=0 \quad \text{and} \quad \left\{\nabla_t^2+\gamma_\mu^2\right\}\psi_\mu=0$$

$$(\gamma_\mu^2-\gamma_\lambda^2)\psi_\mu\psi_\lambda=\psi_\mu\nabla_t^2\psi_\lambda-\psi_\lambda\nabla_t^2\psi_\mu$$

$$(\gamma_\mu^2-\gamma_\lambda^2)\int_A\psi_\mu\psi_\lambda\,da=\int_A\left[\psi_\mu\nabla_t^2\psi_\lambda-\psi_\lambda\nabla_t^2\psi_\mu\right]da$$

$$=-\oint_C\left[\psi_\mu\frac{\partial\psi_\lambda}{\partial n}-\psi_\lambda\frac{\partial\psi_\mu}{\partial n}\right]dl$$

$$=0$$

So, $\int_A\psi_\mu\psi_\lambda\,da=0$ when $(\gamma_\mu^2\neq\gamma_\lambda^2)$ and $\mu\neq\lambda$

b) (8.131) "Real Transverse Electric Fields"

$$\int E_{t,\lambda}\cdot E_{t,\mu}\,da=\delta_{\lambda,\mu}$$

Transverse Magnetic Modes | Transverse Electric Modes

(8.33) $E_t = \frac{ik}{\gamma^2} \nabla_t^2 E_z$ $\qquad$ $E_t = -\frac{i\mu\omega}{\gamma^2} \nabla_t H_z$

(8.31) $H_t = \frac{1}{Z} \hat{z} \times E_t$ $\qquad$ $H_t = \frac{1}{Z} \hat{z} \times E_t$

(8.32) $Z = \frac{k}{\epsilon\omega}$ $\qquad$ $Z = \frac{\mu\omega}{k}$

Transverse Magnetic Modes only:

(8.31)
$$\int_A E_{t,\lambda} E_{t,\mu}\, da = \frac{-k^2}{\gamma_\mu^2 \gamma_\lambda^2} \int_A \nabla_t \cdot E_{z,\lambda} \cdot \nabla_t \cdot E_{z,\mu}\, da$$

$$= \frac{-k^2}{\gamma_\mu^2 \cdot \gamma_\lambda^2} \left[-\oint_S E_{z,\lambda} \frac{\partial E_z}{\partial n}\, dl - \int_A E_{z,\lambda} \nabla_t^2 E_{z,\mu}\, da \right]$$

$$= \frac{-k^2}{\gamma_\mu^2 \gamma_\lambda^2} \int_A E_{z,\lambda} \cdot E_{\lambda,\mu}\, da$$

$= 0$ .. for $\lambda \neq \mu$

Transverse Electric Modes only:

$$\int_A E_{t,\lambda} \cdot E_{t,\mu}\, da = -\frac{\mu^2\omega^2}{\gamma_\mu^2 \gamma_\lambda^2} \int_A (\hat{z} \times \nabla_t H_{z,\lambda}) \cdot (\hat{z} \times \hat{\nabla}_t H_{z,\lambda})\, da$$

$$= -\frac{\mu^2\omega^2}{\gamma_\mu^2 \gamma_\lambda^2} \int_A [\nabla_t H_{z,\lambda} \cdot \nabla_t H_{z,\mu} - (z \cdot \nabla H_{z,\lambda}) \cdot (z \cdot \nabla H_{z,\lambda})\, da$$

$$= -\frac{\mu^2\omega^2}{\gamma_\mu^2 \gamma_\lambda^2} \int \nabla_t H_{z,\lambda} \cdot \nabla_t H_{z,\mu}\, da$$

$= 0$

Transverse Magnetic and Electric Modes

$$\int_A E_{t,\lambda} \circ E_{t,\mu}\, da = \frac{\mu \omega k}{\gamma_\mu^2 \gamma_\lambda^2} \int_A \nabla_t E_{z,\lambda} \circ (z \times \nabla_t H_{z,\mu})\, da$$

$$= \frac{-\mu \omega k}{\gamma_\mu^2 \gamma_\lambda^2} \int_A [\nabla_t E_{z,\lambda} \times \nabla H_{z,\mu}] \circ \hat{z}\, da$$

$$= \frac{-\mu \omega k}{\gamma_\mu^2 \gamma_\lambda^2} \int_A \nabla_t \times (E_{z,\lambda} \cdot \nabla_t H_{z,\mu}) \circ \hat{z}\, da$$

$$= \frac{-\mu \omega k}{\gamma_\mu^2 \gamma_\lambda^2} \oint_S E_{z,\lambda} \cdot \nabla_t H_{z,\mu} \circ dl$$

$$= 0$$

Transverse Electric Modes orthogonal to Magentic Modes:

$$(8.132) \int_A H_{t,\lambda} \circ H_{t,\mu}\, da = \frac{1}{Z_\mu \cdot Z_\lambda} \int_A (z \times E_{t,\lambda}) \circ (z \times E_{t,\mu})\, da$$

$$= \frac{1}{Z_\mu \cdot Z_\lambda} \int_A [E_{t,\lambda} \cdot E_{t,\mu} - (z \circ E_{t,\lambda})(z \cdot E_{t,\mu})]\, da$$

Vector Identity:

$$(A \times B) \times (C \times D)$$
$$= (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C)$$

$$= \frac{1}{Z_\mu \cdot Z_\lambda} \int_A E_{t,\lambda} \circ E_{t,\mu}\, da$$

$$= \frac{1}{Z_\mu Z_\lambda} \delta_{\lambda,\mu}$$

$$= \frac{1}{Z_\lambda^2} \delta_{\lambda,\mu}$$

(8.133) "Average TE and TM Mode"

$$\frac{1}{2}\int_A (E_{t,\lambda} \times H_{t,\mu}) \cdot z\, da = \frac{1}{2Z_\mu}\int_A z \cdot [E_{t,\lambda} \times (z \times E_{t,\mu})]\, da$$

$$= \frac{1}{2Z_\mu}\int_A [E_{t,\lambda} \cdot E_{t,\mu} - (z \cdot E_{t,\lambda})(z \cdot E_{t,\mu})]\, da$$

$$= \frac{1}{2Z_\mu}\int_A E_{t,\lambda} \cdot E_{t,\mu}\, da$$

$$= \frac{1}{2Z_\mu}\delta_{\lambda,\mu}$$

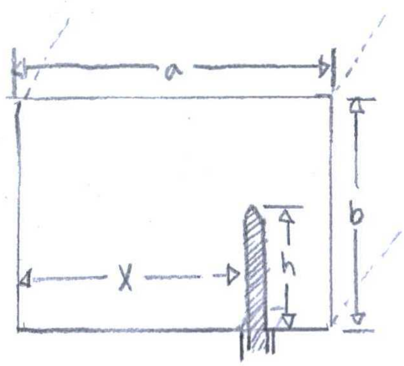
$$= \frac{1}{2Z_\mu}\delta_{\lambda,\mu}$$

(8.134) TM: $\int_A E_{z,\lambda} \cdot E_{z,\mu}\, da = -\frac{\gamma_\lambda^2}{k_\lambda^2}\delta_{\lambda,\mu}$

TE: $\int_A E_{z,\lambda} \cdot E_{z,\mu}\, da = \frac{-\gamma_\lambda^2}{\mu^2 \omega^2}\delta_{\lambda,\mu}$

$$= \frac{-\gamma_\lambda^2}{k_\lambda^2 Z_\lambda^2}$$

8.19.



"...cross-sectional view of an infinitely long rectangular waveguide ...with a coaxial line extending vertically... at $z=0$"

a) $I(y) = I_0 \sin[(\omega/c)(h-y)]$

Transverse Electric Modes:

(8.42) $H_z = H_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$

(8.33) $E_t = -\frac{i\mu\omega}{\gamma_{mn}^2}\hat{z} \times \nabla_t H_z$

So, $E_y = -\frac{i\mu_0\omega}{\gamma_{mn}^2} \cdot \frac{m\pi}{a} H_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$

...where $k_{mn}^2 = \frac{\omega^2}{c^2} - \gamma_{mn}^2$

$\gamma_{mn}^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$ (8.43)

$$H_0 = \frac{2i\gamma_{mn}}{\mu_0 \omega \sqrt{ab}} \quad \ldots \text{when } m,n \gg 1$$

Transverse Magnetic Modes:

(pg 362) $E_z = E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$

(8.31) $E_t = \frac{i k_{mn}}{\gamma_{mn}^2} \nabla_t E$

So, $E_y = \frac{-i k_{mn}}{\gamma_{mn}^2} \frac{n\pi}{b} E_0 \sin\left(\frac{m\pi X}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$

$$E_0 = \frac{2i\gamma_{mn}}{k_{mn}\sqrt{ab}} \quad \ldots \text{when } m,n \gg 1$$

Excitation Amplitudes:

(8.146) "Normalized Amplitude"

$$A_{mn}^{(\pm)} = \frac{-Z_{mn}}{2} I_0 \int_V J_0 \cdot E_{mn}^{(\mp)} d^3x$$

...where $Z_{mn} = \begin{cases} k_{mn}/\epsilon_0\omega & \text{if TM} \\ \mu_0\omega/k_{mn} & \text{if TE} \end{cases}$

$$J(x,y,z) = I(y)\cdot\delta(X)\cdot\delta(y)$$

$$= I_0 \sin\left[\frac{\omega}{c}(h-y)\right]\delta(x-X)\delta(z)\Theta(h-y)$$

$$A_{mn}^{(\pm)} = \frac{-Z_{mn}}{2} I_0 \int_0^h \sin\left[\frac{\omega}{c}(h-y)\right] E_y^{(\mp)}(X,y,0)\,dy$$

$$= \beta_{mn}^{(TM,TE)} \cdot \sin\left(\frac{m\pi X}{a}\right)\left[\left(\frac{\omega}{c}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right]^{-1}\left[\cos\left(\frac{n\pi h}{b}\right) - \cos\left(\frac{\omega h}{c}\right)\right]$$

$$\cong \frac{1}{n} \quad \text{(TM modes)}$$

$$\cong \frac{m}{n} \frac{1}{(m/a)^2 + (n/b)^2} \sim \frac{1}{n^3} \quad \text{(TE modes)} \ldots$$

$$A_{10}^{(\pm)} = \beta_{10}^{(TE)} \sin\left(\frac{\pi X}{a}\right)\left[1 - \cos\left(\frac{wh}{c}\right)\right]$$
$$= \beta_{10}^{(TE)} \sin\left(\frac{\pi X}{a}\right) \sin^2\left(\frac{wh}{2c}\right)$$

b)

$$p^{(\pm)} = \frac{1}{2Z_{mn}} \left|A_{mn}^{(\pm)}\right|^2$$
$$= \frac{\mu c^2 I^2}{wkab} \sin^2\left(\frac{\pi X}{a}\right) \sin^4\left(\frac{wh}{2c}\right)$$

c) $E^{(-)} = A_{10}^{(-)} E_t e^{-ikz}$

$= -A_{10}^{(+)} E_{t,10} e^{ik(2L-z)}$ ... at $z=L$, no electric field

$E = E^{(-)} + E^{(refL)}$ ... (8.137)

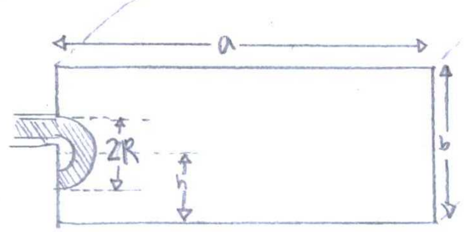
$= A_{10}(1 - e^{2ikL}) E_{10} e^{-ikz}$

$kL = (n + 1/2)\pi$ ∴ $L = (n+1/2)\pi/k$ maximizes E.

$$p^{(\pm)} = \frac{4\mu c^2 I^2}{wkab} \sin^2\left(\frac{\pi X}{a}\right) \sin^4\left(\frac{wh}{2c}\right)$$
$$= \frac{1}{2} I_0^2 R_{rad} \quad \ldots \quad R_{rad} = \frac{8\mu c^2}{wkab} \sin^2\left(\frac{\pi X}{a}\right)\sin^4\left(\frac{wh}{2c}\right)$$

$$E_{mn}^{(\pm)} = \frac{2}{\sqrt{ab}}\left[\frac{\pi}{\gamma_{mn}}\left(X \frac{m}{a} \cos\left(\frac{m\pi X}{a}\right)\sin\left(\frac{n\pi y}{b}\right) + y\frac{n}{b}\sin\left(\frac{m\pi X}{a}\right)\cdot\cos\left(\frac{n\pi y}{b}\right)\right.\right.$$
$$\left.\mp \frac{Zi\gamma_{mn}}{k_{mn}} \sin\left(\frac{m\pi X}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\right] e^{\pm i k_{mn} z}$$

8.20.



"an infinitely long rectangular waveguide has a coaxial line

a) (8.146) "Normalized Amplitude"

$$A_{mn}^{(\pm)} = -\frac{Z_{mn}}{2}\int_V J \cdot E_{mn}^{(\mp)} d^3X$$

$$= -\frac{Z_{TM}}{2}\frac{2\pi I_0 R}{\gamma_{mn}\sqrt{ab}}\int_{-\pi/2}^{\pi/2}\left\{\frac{m}{a}(-\sin\phi)\cos\left(\frac{m\pi R\cos\phi}{a}\right)\cdot\sin\left(\frac{n\pi(h+R\sin\phi)}{b}\right)\right.$$
$$\left.+\frac{n}{b}\cos\phi\sin\left(\frac{m\pi R\cos\phi}{a}\right)\cos\left(\frac{n\pi(h+R\sin\phi)}{b}\right)\right\}d\phi$$

$$A_\lambda^{(\pm)} = -\frac{Z_\lambda}{2}\int_{-\pi/2}^{\pi/2} I_0(-\sin\phi\,\hat{x}+\cos\phi\,\hat{y})\cdot E_\lambda^{(\mp)} R\,d\phi$$

∴ when $dl = R(\hat{x}\cos\phi - y\sin\phi)d\phi$

$l(\phi) = xR\sin\phi + y(h+R\cos\phi)$

$$= -\frac{1}{2}RI_0 Z_\lambda\int_{-\pi/2}^{\pi/2}\left\{-\sin\phi\left\{E_\lambda^{(\mp)}\right\}_x + \cos\phi\left\{E_\lambda^{(\mp)}\right\}_y\right\}d\phi$$

(8.135) "Normalized Electric fields"

$$\left\{E_m^{\mp}\right\}_x = \frac{2\pi m}{\gamma_{mn}a\sqrt{ab}}\cos\left\{\frac{m\pi(R\cos\phi)}{a}\right\}\sin\left\{\frac{n\pi(h+R\sin\phi)}{b}\right\}$$

$$\left\{E_m^{\mp}\right\}_y = \frac{2\pi n}{\gamma_{mn}b\sqrt{ab}}\sin\left\{\frac{m\pi(R\cos\phi)}{a}\right\}\cos\left\{\frac{n\pi(h+R\sin\theta)}{b}\right\}$$

The 'lack of excitation' is because the zero integration in the y-component (odd function) with another zero integration from the x-component because the loop rotation; $\pi/2$ to $-\pi/2$.

b) Transverse Electric Waves:

$$\left\{E_{mn}^{(\mp)}\right\}_x = -\frac{2\pi n}{\gamma_{mn}b\sqrt{ab}}\cos\left(\frac{m\pi R\cos\phi}{a}\right)\sin\left(\frac{n\pi(h+R\sin\phi)}{b}\right)$$

$$\left\{E_{mn}^{(\mp)}\right\}_y = \frac{2\pi m}{\gamma_{mn}a\sqrt{ab}}\sin\left(\frac{m\pi R\cos\phi}{a}\right)\cos\left(\frac{n\pi(h+R\sin\phi)}{b}\right)$$

$$A_{mn}^{(\pm)} = \frac{-\pi R I_0 Z_{mn}}{\gamma_{mn}\sqrt{ab}} \int_{-\pi/2}^{\pi/2} \left\{ \frac{n}{b} \sin\phi \cos\left(\frac{m\pi R\cos\phi}{a}\right) \sin\left(\frac{n\pi(h+R\sin\phi)}{b}\right) + \frac{m}{a}\cos\phi \sin\left(\frac{m\pi R\cos\phi}{a}\right)\cos\left(\frac{n\pi(h+R\sin\phi)}{b}\right)\right\} d\phi$$

When $m=1, n=0$, then

$$A_{1,0} = -\frac{\pi R I_0 Z_{1,0}}{\gamma_{1,0}\sqrt{2a^3 b}} \int_{-\pi/2}^{\pi/2} \left\{\cos\phi \sin\left(\frac{\pi R\cos\phi}{a}\right)\right\} d\phi$$

$$= -\frac{\pi R I_0 Z_{1,0}}{\gamma_{1,0}\sqrt{2a^3 b}} \pi J_1\left(\frac{\pi R}{a}\right) \quad \ldots \text{where } \gamma_{1,0} = \pi/a$$

$$\cong -\frac{\pi^3 R^2 I_0 Z_{1,0}}{\gamma_{1,0}\sqrt{8a^5 b}} \quad \ldots \text{because } J_1\left(\frac{\pi R}{a}\right) \cong \frac{\pi R}{2a}$$

c) $$P = \frac{1}{2}\int (E \times H^*) \cdot \hat{z}\, da$$

$$= \frac{1}{2}\int \left\{ \left(\sum A_\lambda E_\lambda\right) \times \left(\sum A_\lambda^* H_\lambda^*\right)\right\} z\, da$$

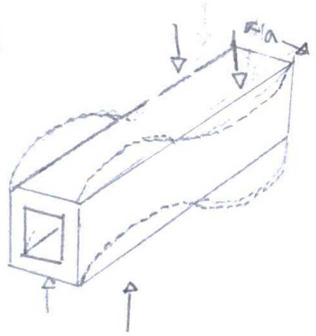
$$= \frac{1}{2}\sum A_\lambda A_\lambda^* \int (E_\lambda^* \times H_\lambda^*) \cdot z\, da$$

$$= \frac{1}{2}\sum \frac{|A_\lambda|^2}{Z_\lambda}$$

$$= \frac{1}{2}\frac{|A_{1,0}|^2}{Z_{1,0}}$$

$$= \frac{I_0^2}{16} Z_{1,0} \frac{a}{b}\left(\frac{\pi R}{a}\right)^4$$

0.21



"hollow metallic waveguide.... vicinity of distortion"

a) Lowest Propogating mode: TE_{01}

Wavefunction: $\psi = E_0 \sin\left(\frac{\pi y}{a}\right) e^{\pm ikz}$

Curvature: $K(s) = 1/R(s)$

Area: $dA = h(s,t)\,ds\,dt$ ... where s = length, t = transverse coordinate

Height: $h(s,t) = 1 - K(s)\cdot t$

Anatz: $\psi(s,t) = \frac{u(s)}{\sqrt{h(s,t)}} \cdot \sin\left[\frac{\pi t}{w(s)}\right]$

$$\nabla^2 \psi = \frac{1}{h}\frac{\partial}{\partial t}\left(h \frac{\partial \psi}{\partial t}\right) + \frac{1}{h}\frac{\partial}{\partial s}\left(\frac{1}{h}\frac{\partial \psi}{\partial s}\right)$$

$$= -\left(\frac{\pi^2}{w(s)^2} - \frac{\pi^2}{a^2}\right)\psi \cdot \sqrt{h} + \frac{1}{4}\frac{K^2 \psi \sqrt{h}}{h^2} + \frac{1}{h^2}\frac{\partial^2 \psi}{\partial^2 s} \cdot h$$

$$+ k^2 \psi \sqrt{h} - \frac{2}{h^3}\frac{\partial h}{\partial s}\frac{\partial \psi}{\partial s}\sqrt{h} - \frac{1}{2h^3}\frac{\partial^2 h}{\partial s^2}\psi\sqrt{h}$$

$$+ \frac{5}{4h^4}\left[\frac{\partial h}{\partial s}\right]^2 \psi \sqrt{h}$$

$$= 0$$

If $s \gg 1$, then higher order derivatives drop-out:

$$\nabla^2 \psi = -\left(\frac{\pi^2}{w^2} - \frac{\pi^2}{a^2}\right)u + \frac{1}{4}K^2 u + \frac{\partial^2 u}{\partial s^2} + k^2 u$$

$$= 0$$

<u>Variable separation:</u>

$$\frac{\partial^2 u}{\partial s^2} + k^2 u = V(s) \cdot u \quad ; \quad \frac{\partial^2 u}{\partial^2 s} + \left(k^2 - V(s)\right) = 0$$

$$-\left(\frac{\pi^2}{w^2} - \frac{\pi^2}{a^2}\right)u + \frac{1}{4}K^2 u = -V(s)u \;;\; V(s) = \pi^2\left(\frac{1}{w^2} - \frac{1}{a^2}\right) - \frac{1}{4}K^2$$

V(s) describes the transverse potential at zero. In terms of a Schrodinger or Laplacion, when the second derivative is zero.

b) Citation: Goldstone and Jaffe (1993)

(Equation 3.20) $\Delta_{11}(R^2) \approx -\frac{2}{8R} \frac{1}{\pi^2 - R^2 - 1/4 R^2}$

$$\leq \pi^2 \circ \left(-\frac{2}{8R} \frac{1}{\pi^2 - \pi^2 - 1/4 R^2} \right)$$

If $k^2 = \frac{\omega^2}{c^2}$, then,

$$\omega_0^2 \leq 4 \left(\frac{\pi c}{a} \right)^2$$

The solution corresponds to the book with the problems information: $R \gg a$ and $\theta a \ll 1$.

$$\omega_0^2 \leq \left(\frac{\pi c}{a} \right)^2 \left[1 - \left(\frac{\theta a}{8 \pi R} \right)^2 \right]$$

$$\approx \left(\frac{\pi c}{a} \right)^2$$

Note: The problem suggest the first and lowest frequency transverse wave in an oscillatory square tube.