Chapter 12: Dynamics of Relativistic Particles and Electromagnetic fields:

12.1

"relativistic charge ...interacting with an external field"

a) (12.3) "Covariant form"

$$\frac{dU^\alpha}{d\tau} = \frac{e}{mc} F^{\alpha\beta} \cdot U_{\alpha\beta} \qquad \text{...when } U^\alpha = (\gamma c, \gamma u)$$

(12.5) "Lagrangian Equation of Motion"

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$$\frac{\partial}{\partial \tau}\left(\frac{\partial L}{\partial U^\gamma}\right) = \frac{\partial L}{\partial x^\gamma}$$

$$= \partial_x L$$

$$L = -\frac{m}{2} U_\alpha U^\alpha - \frac{q}{c} U_\alpha A^\alpha$$

$$= -\frac{m}{2} g_{\alpha\beta} U_\alpha U^\alpha - \frac{q}{c} g_{\alpha\beta} \cdot U^\beta \cdot A^\alpha \qquad \text{...(11.72) "contraction"}$$

$$\frac{\partial L}{\partial U^\gamma} = -\frac{m}{2} g_{\alpha\beta}\left[\delta^\beta{}_\gamma \cdot U^\alpha + U^\beta \cdot \delta^\beta{}_\gamma\right]$$

$$- \frac{q}{c} g_{\alpha\beta} \cdot \delta^\beta{}_\gamma \cdot A^\alpha \qquad \text{...(11.71) "Kronecker delta"}$$

$$= -\frac{m}{2}\left[g_{\alpha\gamma} \cdot U^\alpha + g_{\gamma\beta} \cdot U^\beta\right] - \frac{q}{c} g_{\alpha\gamma} A^\alpha$$

...(11.71) "inverted Kronecker delta"

$$= -\frac{m}{2}\left[U_\gamma + U_\gamma\right] - \frac{q}{c} A_\gamma$$

$$= -m U_\gamma - \frac{q}{c} A_\gamma$$

$$\frac{\partial L}{\partial x^\gamma} = -\frac{q}{c} g_{\alpha\beta} \cdot U^\beta \cdot \partial_\gamma \cdot A^\alpha \qquad \text{...(11.73) "inverse contraction"}$$

$$= -\frac{q}{c} U_\alpha \partial_\gamma A^\alpha$$

$$= -\frac{q}{c} U^\alpha \partial_\gamma A_\alpha \qquad \text{...(11.70) "flat space time"}$$

$$\frac{d}{d\tau}\frac{\partial L}{\partial U^\gamma} = \frac{\partial L}{\partial x^\gamma} - \frac{d}{d\tau}\left[m U_\gamma + \frac{q}{c}A_\gamma\right]$$

$$= -\frac{q}{c}U^\alpha \partial_\gamma A_\alpha$$

$$= -m\partial_\tau U_\gamma + \frac{q}{c}\partial_\tau \cdot A_\gamma =$$

$$m\frac{d}{d\tau}U_\gamma = \frac{q}{c}\left[U^\alpha \partial_\gamma A_\alpha - \frac{d}{d\tau}A_\gamma\right]$$

$$= \frac{q}{c}\left[U^\alpha \partial_\gamma A_\alpha - \frac{dx^\alpha}{d\tau}\frac{\partial}{\partial x^\alpha}A_\gamma\right]$$

$$= \frac{q}{c}\left[U^\alpha \partial_\gamma A_\alpha - U^\alpha \partial_\alpha A_\gamma\right]$$

$$= \frac{q}{c}F_{\gamma\alpha} \cdot U^\alpha$$

$$= \frac{q}{c}F^{\gamma\alpha}U_\alpha$$

... (11.70) "Flat space time"

The covariant Lorentz is a notation about force and charge in relativistic systems.

b) (12.33) "Conjugate Momentum 4-vector"

$$p^\alpha = \frac{\partial L}{\partial\left(\frac{\partial x_n}{\partial s}\right)} = mU^\alpha + \frac{e}{c}A^\alpha$$

(1234) "Effective Hamiltonian"

$$\tilde{H} = P_\alpha U^\alpha + \tilde{L}$$

$$= \left(mU^\alpha + \frac{q}{c}A^\alpha\right)U_\alpha - \frac{mU_\alpha U^\alpha}{2} - \frac{q}{c}U_\alpha A^\alpha$$

$$= \frac{mU^\alpha U_\alpha}{2}$$

$$= \frac{1}{2m}\left(P^\alpha - \frac{q}{c}A^\alpha\right)\left(P_\alpha - \frac{q}{c}A_\alpha\right)$$

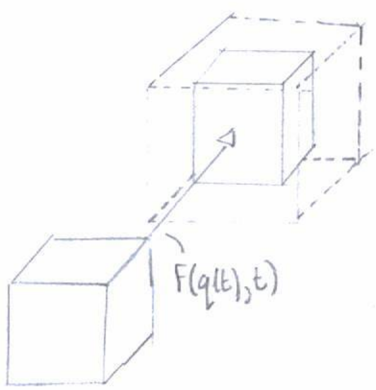
"Space time form"

$$p^\alpha = \frac{q}{c}A^\alpha = \begin{pmatrix} p^0 - \frac{q}{c}\Phi(x,t) \\ p - \frac{q}{c}A(x,t) \end{pmatrix}$$

$$H = \frac{1}{2m}\Big(\underbrace{(p^0)^2}_{\text{"Initial Kinetic energy"}} + \underbrace{p^2}_{\text{"Kinetic" energy}} + \underbrace{\frac{q^2}{c^2}\left[\Phi^2 - A^2\right]}_{\text{"Potential" energy}} - \underbrace{\frac{2q}{c}\left[P\cdot A + P^0\Phi\right]}_{\text{"relativistic energy"}}\Big)$$

"Initial Kinetic energy" + "Kinetic" energy + "Potential" energy − "relativistic energy"

12.2



"Hamiltons Principle ∘∘ Euler-Lagrange Principle"

a) $\Delta L = L_2 - L_1$

$$= \frac{d}{dt} f(q(t), t)$$

$$L_2 = L_1 + \frac{d}{dt} f(q(t), t)$$

$$= L_1 + \frac{\partial f}{\partial q}\dot{q}(t) + \frac{\partial f}{\partial t}$$

$$\frac{\partial L_2}{\partial q} = \frac{\partial L_1}{\partial q} + \frac{\partial^2 f}{\partial^2 q}\dot{q}(t) + \frac{\partial f}{\partial q dt}$$

$$= \frac{\partial L_1}{\partial \dot{q}} + \frac{\partial f}{\partial q}$$

"Hamiltons Principle"

$$\frac{\partial L_2}{\partial q_i} - \frac{\partial}{\partial t}\frac{\partial L_2}{\partial \dot{q}} = \frac{\partial L_1}{\partial q_i} - \frac{d}{dt}\frac{\partial L_1}{\partial \dot{q}} + \frac{\partial^2 F}{\partial q \partial q}\dot{q} - \frac{\partial^2 f}{\partial q^3} - \frac{d}{dt}\frac{\partial f}{\partial q}$$

$$= \frac{\partial L_1}{\partial q} - \frac{d}{dt}\frac{\partial L_1}{\partial \dot{q}}$$ ∘∘ when high-order terms equal zero.

b) <u>Gauge Transform</u>: $A^\alpha \longrightarrow A^\alpha + \partial^\alpha A$

(12.12) "charged Particle Lagrangian"

$$L = -mc^2\sqrt{1 - v^2/c^2} + \frac{e}{c}u\cdot A - e\phi$$

$$\cong L + \delta L$$

Gauge Transform: $\phi \rightarrow \phi + \frac{1}{c}\frac{\partial}{\partial t}A$

$A \rightarrow A + \nabla A$

$$\delta L = -\frac{\epsilon}{c}\left(\frac{\partial}{\partial t} + u\cdot\nabla\right)A$$

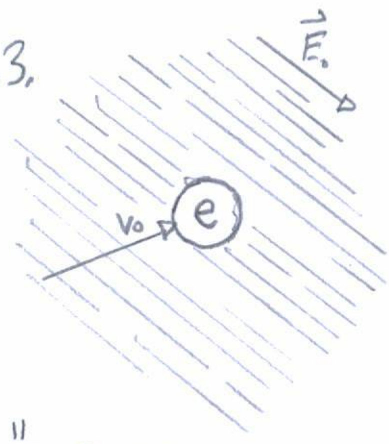
$$= -\frac{\epsilon}{c}\frac{\partial A}{\partial t}$$

Thus, $L = L + \delta L$

$$= L - \frac{\epsilon}{c}\frac{\partial A}{\partial t}$$

$$= L + \frac{d}{dt}f(q(t), t)$$

12.3.



"a particle ... moves in a uniform, static, electric field"

a) (12.3) "Covariant equation of motion"

$$\frac{dU^\mu}{d\tau} = \frac{e}{mc}F^{\mu\nu}U_\nu$$

$$\frac{dU^0}{d\tau} = \frac{e}{mc}E_0\cdot U$$

$$\frac{dU}{d\tau} = \frac{e}{mc}E_0\cdot U^0$$

$$\frac{d^2U^0}{d^2\tau} = \frac{e}{mc}E_0\frac{du}{d\tau}$$

$$= \left(\frac{eE_0}{mc}\right)^2 U^0$$

$$\frac{d^2U^0}{d^2\tau} - \left(\frac{eE_0}{mc}\right)^2 U^0 = 0$$

"Homogeneous second order linear differential Equation"

$$ax'' + bx' + cx = 0$$

$$x(t) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = \gamma_0 \quad b = 0 \quad c = -\left(\frac{eE_0}{mc}\right)^2$

Solution #1:

$$U_1^0 = c_1 e^{\frac{eE_0\gamma}{mc}}$$

Solution #2:

$$U_2^0 = c_1 e^{\frac{-eE_0\gamma}{mc}}$$

$$U^0 = \frac{U_1^0 + U_2^0}{2}$$

$$U^0 = \gamma_0 \cosh\left(\frac{eE_0\tau}{mc}\right)$$ ... where $c_1 = \gamma_0$

$$\begin{aligned} ct &= x^0 \\ &= \int_0^\tau U^0 d\tau \\ &= \int_0^\tau \gamma_0 \cosh\left(\frac{eE_0\tau}{mc}\right) d\tau \\ &= \frac{mc^2\gamma_0}{eE_0} \sinh\left(\frac{eE_0\tau}{mc}\right) \end{aligned}$$

$$\begin{aligned} x &= \int_0^\tau u \, d\tau \\ &= \gamma_0 u_0 c\tau + \frac{mc^2\gamma_0}{eE_0} E_0 \left[\cosh\left(\frac{eE_0\tau}{mc}\right) - 1\right] \end{aligned}$$

$$t = \frac{mc\gamma_0}{eE_0} \sinh\left(\frac{eE\tau}{mc}\right)$$

$$\tau = \frac{mc}{eE_0} \sinh^{-1}\left(\frac{eE_0 t}{mc\gamma_0}\right)$$

$$\begin{aligned} \gamma &= \frac{U}{c} \\ &= \gamma_0 \cosh\left(\frac{eE_0 t}{mc}\right) \\ &= \gamma_0 \cosh\left(\frac{eE_0}{mc} \frac{mc}{eE_0} \sinh^{-1}\left(\frac{eE_0 t}{mc\gamma_0}\right)\right) \\ &= \gamma_0 \sqrt{1 + \left(\frac{eE_0 t}{mc\gamma_0}\right)^2} \end{aligned}$$

$$u = \gamma_0 \left(v_0 + \frac{eE_0 t}{m\gamma_0}\right)$$

Velocity:

$$v(t) = \frac{u}{\gamma} = \frac{v_0 + \frac{eE_0 t}{mc\gamma_0}}{\left[1 + \left(\frac{eE_0 t}{mc\gamma_0}\right)^2\right]^{1/2}}$$

Position:

$$X(t) = \frac{mc\gamma_0}{eE_0} V_0 \sinh^{-1}\left(\frac{eE_0 t}{mc\gamma_0}\right) + \frac{m c^2 \gamma_0}{e E_0} E_0\left[\sqrt{1+\left(\frac{eE_0 t}{mc\gamma_0}\right)^2} - 1\right]$$

b) $$X_{\parallel}^{(\tau)} = \frac{mc^2\gamma_0}{eE_0}\left[\cosh\left(\frac{eE_0\tau}{mc}\right) - 1\right]$$

$$= \frac{mc^2\gamma_0}{eE_0}\left[\cosh\left(\frac{eE_0}{mc^2\gamma_0}\frac{X_\perp}{V_0}\right) - 1\right]$$ ... when $X(\tau)_\perp = \gamma_0 V_0 \tau$

If $\frac{mc\gamma_0}{eE_0} \gg 1$

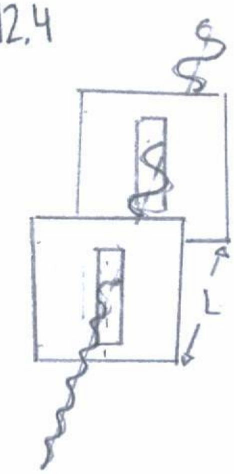
$$X_{\parallel}^{(\tau)} \cong \frac{eE_0}{2m\gamma_0 V_0^2} X_\perp^2$$

Identity:
$\cosh(x) = 1 + \frac{x^2}{2!} + \ldots$

If $\frac{mc\gamma_0}{eE_0} \ll 1$

$$X_{\parallel}^{(\tau)} \cong \frac{mc^2\gamma_0}{eE_0} e^{\left(\frac{eE_0}{mc\gamma_0}\frac{X_\perp}{V_0}\right)}$$

12.4



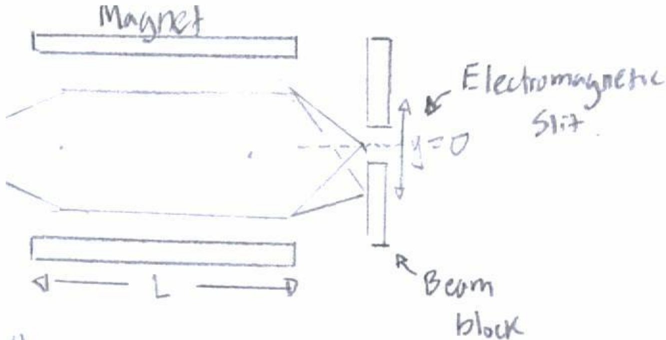
"uniform, static, crossed electric and magnetic fields ... entrance and exit slits"

Citation: Coombes et al. Phys. Rev 112, 1303 (1958)
"Antiproton-proton cross sections at 113, 197, 265, and 333 MeV"
P. Eberhard, M L Good, and H.K. Ticho Rev, Sci Instrum. 31, 1054 (1960).
"Separated 1.17 BeV/c K^- Meson beam"

Note: Mostly about spectrometer plate assembly.

Mean value: $u = cE/\beta$

In spectrometer, $y = \frac{eL^2E}{2pc}\left[\frac{1}{\beta_0} - \frac{1}{\beta}\right]$; $\Theta = \frac{eLE}{pc}\left[\frac{1}{\beta_0} - \frac{1}{\beta}\right]$



"Optical lens in a spectrometer"

E = Electric field e = charge

$\beta_0 = E/B$ p = momentum [m·v]

L = Length

If $L = 3$ meters, $E_{max} = 3\times10^{6}\,V/m$, $\Delta X = 10^{-4}m$

and $u = \frac{1}{2}(0.995c)$, $e = -1.602\times10^{-21}C$

then, $y = \dfrac{e(3m)^2 \cdot 3\times10^{6}\,V/m}{2\cdot1.672\times10^{-27}Kg\cdot(\frac{1}{2}0.995c^2)}\left[\dfrac{1}{1} - \dfrac{1}{\beta}\right]$

$= 10^{-4}m$

Velocity calculation:

$$\beta^{-1} = \frac{e(3m)^2 \cdot 3\times10^{6}\,V/m}{2\cdot1.672\times10^{-27}Kg(\frac{1}{2}\cdot0.995c^2)} + 10^{-4}m$$

$= 1.00346$

$= c/v$

$v = 0.99655\,c$

Angle calculation:

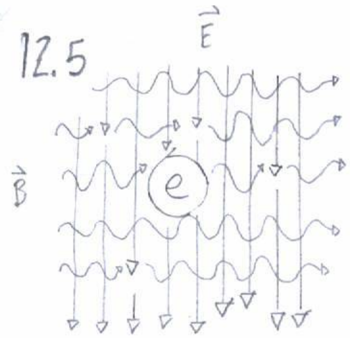
$$\Theta = \frac{eLE}{pc}\left[\frac{1}{\beta_0} - \frac{1}{\beta}\right]$$

$= 6.60\times10^{-5}\,rad$

$= 0.003827°$

Note: Papers state ±50% rejection in spectrometers.

12.5



"Particle ... in crossed, static, uniform, electric and magnetic fields"

a) $|E|<|B|$

(12.43) "Boost velocity"

$$u = c\frac{E\times B}{B^2} = \frac{cE}{B}\hat{z}$$

Center of Mass Frame:

$$E'=0,\ B'=\frac{B}{\gamma} = \frac{B_y}{\gamma} \quad ; \quad \gamma = \frac{1}{\sqrt{1-(u/c)^2}}$$

$$x'(t') = a\cos(\omega_B t')$$

$$y'(t') = v_{\parallel} t'$$

$$z'(t') = a\sin(\omega_B t') \quad \ldots \text{ where } \omega_B = \frac{qB'}{\gamma m c}$$

$$\gamma_a = \frac{1}{\sqrt{1-\frac{v_{\parallel}^2+\omega_B^2 a^2}{a^2}}}$$

$$x(t') = x'(t') = a\cos(\omega_B t')$$

$$y(t') = y'(t') = v_{\parallel} t$$

$$z(t') = \gamma(z'(t') + ut') = \gamma(a\sin(\omega_B t') + ut')$$

$$t(t') = \gamma\left(t' + \frac{u}{c^2}z'(t')\right) = \gamma\left(t' + \frac{u}{c^2}a\sin(\omega_B t')\right)$$

b) $|E|>|B|$

Lab Frame:

$$u = c\frac{E\times B}{E^2} = \frac{cB}{E}z$$

Center of Mass Frame:

$$E' = \frac{E}{\gamma} = \frac{E}{\gamma}\hat{x} \ ; \ B'=0 \ ; \ \gamma = \frac{1}{\sqrt{1-(u/c)^2}}$$

∘ Parallel Force:

$$F_{\parallel} = ma = m \frac{d}{dt} \gamma_n (t') v(t')$$

∘ Perpendicular Force:

$$F_{\perp} = ma = m \frac{d}{dt} \gamma \, v_{\perp}(t')$$

If $v_{\perp}(t'=0) = v_0$, then

$$P_{\perp,0} = m \gamma_a (t') v_{\perp}(t')$$
$$= m \gamma_0 v_0$$

$$P_{\parallel,0} = m \gamma_a (t') v_{\parallel}(t')$$
$$= qEt'$$

$$v^2(t') = \frac{c^2(\gamma_0^2 v_0^2 + q^2 E'^2 t'^2/m^2)}{\gamma_0^2 c^2 + q^2 E'^2 t'^2/m^2}$$

$$\gamma_a^2(t') = \frac{1}{(1 - v^2/c^2)}$$
$$= \gamma_0^2 + \frac{q^2 E'^2 t'^2}{m^2 c^2}$$

$$m v_{\perp}(t') = m \gamma_0 v_0$$

$$v_{\perp}(t) = \frac{\gamma_0 v_0}{\gamma_a(t')} = \frac{\gamma_0 v_0}{\sqrt{\gamma_0^2 + \frac{q^2 E'^2 t'^2}{m^2 c^2}}}$$

$$m v_{\parallel}(t) = qEt'$$

$$v_{\parallel}(t) = \frac{qEt'}{m\sqrt{\gamma_0^2 + \frac{q^2 E'^2 t'^2}{m^2 c^2}}}$$

$$x_{\perp}(t) = \int v_{\perp} \, dt'$$

$$= \int \frac{\gamma_0 v_0}{\sqrt{\gamma_0^2 + \frac{q^2 E'^2 t'^2}{m^2 c^2}}} dt'$$

$$= \frac{\gamma_0 v_0 m c}{q E'} \sinh^{-1}\left(\frac{q E' t'}{\gamma_0 m c}\right)$$

$$X_{\parallel}(t) = \int V_{\parallel}(t') dt'$$

$$= \int_0^{t'} \frac{q E' t'}{m \sqrt{\gamma_0^2 + \frac{q^2 E'^2 t'^2}{m^2 c^2}}} dt'$$

$$= \frac{\gamma_0 m c^2}{q E'} \left(\sqrt{1 + \left(\frac{q E' t'}{\gamma_0 m c}\right)^2} - 1 \right)$$

$$x'(t') = X_{\parallel}(t)$$

$$y'(t') = \cos\phi_0 \cdot X_{\perp}(t') = \cos\phi_0 \frac{\gamma_0 m c^2}{q E'} \left(\sqrt{1 + \left(\frac{q E' t'}{\gamma_0 m c}\right)^2} - 1 \right)$$

$$z'(t) = \sin\phi_0 \cdot X_{\perp}(t') = \sin\phi_0 \frac{\gamma_0 m c^2}{q E'} \left(\sqrt{1 + \left(\frac{q E' t'}{\gamma_0 m c}\right)^2} - 1 \right)$$

Rest frame:

(12.39) "Precession frequency modified"

$$\delta = \frac{q E'}{\gamma_0 m c}$$

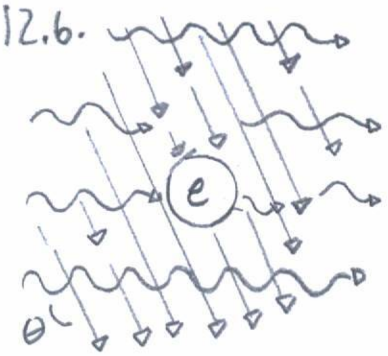
$$x(t') = \frac{c}{\delta} \left(\sqrt{1 + (\delta t')^2} - 1 \right)$$

$$y(t') = \cos\phi_0 \frac{v_0}{\delta} \sinh^{-1}(\delta t')$$

$$z(t') = \gamma\left(z'(t') + u t'\right)$$

$$= \gamma\left(\sin\phi_0 \frac{v_0}{\delta} \sinh^{-1}(\delta t') + u t'\right)$$

12.6.



"Static, uniform electric and magnetic fields... an angle θ"

a) (12.42.5) "Lorentz Force Equation"

$$\frac{dP}{dt} = e\left(E' + \frac{v' \times B}{c}\right)$$

(11.149) "Transformation Equations"

$$E' = \gamma(E + \beta \times B) - \frac{\gamma^2}{\gamma+1}\beta(\beta \cdot E)$$

$$B'_{\parallel} = \gamma(B_{\parallel} - \beta \times E_{\parallel}) - \frac{\gamma^2}{\gamma+1}\beta(\beta \cdot B)$$

$$\text{So, } E' = \frac{u}{c}(E + \beta \times B) - \frac{1}{\frac{1-\left(\frac{u}{c}\right)^2}{\left(\frac{u}{c}\right)^2}}\beta \cdot (\beta \cdot E_{\parallel})$$

$$= \frac{u}{c}\left(E + \frac{u}{c} \times B\right) \quad \text{... when } B_{\parallel} = 0$$

$$B = \frac{1}{\gamma}B'$$

$$= \left(\frac{1-\left(\frac{v}{c}\right)^2}{1}\right)^{1/2} \cdot B'$$

$$= \left(\frac{B^2 - E^2}{B^2}\right)^{1/2} \cdot B' \quad \text{... when } v = c\left(\frac{E}{B}\right)\cos\alpha$$

$$\frac{dP}{dt} = e\left(\gamma\left[E + \frac{u}{c} \times B\right] + \frac{u}{c} \times \left[\frac{B^2 - E^2\cos^2\alpha}{B^2}\right]^{1/2} \cdot B\right)$$

$$= e\left[\gamma[E + \beta \times B] + \beta \times \left[\frac{B^2 - E^2\cos^2\alpha}{B^2}\right]^{1/2} \cdot B\right]$$

b) $\frac{dP}{dt} = \left\langle \frac{e}{c}v_y B, \frac{e}{c}v \times B, eE \right\rangle$

(Problem 12.3)

$$Z(t) = \frac{\epsilon_0}{eE}\left(\sqrt{1 + \left(\frac{eEct}{\epsilon_0}\right)^2} - 1\right)$$

where $\epsilon_0 = \gamma_0 m c^2 = mc^2 / \sqrt{1-\beta^2}$

$$x = AR\sin\phi$$

$$y = -AR\cos\phi$$

$$z = \frac{R}{\rho}\sqrt{1+A^2}\cosh(\rho\phi)$$

$$ct = \frac{R}{\rho}\sqrt{1+A^2}\sinh(\rho\phi)$$

$$\frac{d}{dt}(P_x + iP_y) = -i\frac{eB}{\gamma mc}(P_x + iP_y)$$

$$= -\frac{ieB}{\gamma mc} P_\perp e^{-i\phi}$$ ∘∘∘ when $P_\perp = \sqrt{Px^2 + Py^2}$

$$\frac{d\phi}{dt} = \frac{eBc}{\sqrt{\epsilon_0^2 + (eEct)^2}}$$

$$\phi = \int \frac{eBc}{\sqrt{\epsilon_0^2 + (eEct)^2}} dt$$

$$= \frac{B}{c}\sinh^{-1}\left(\frac{eEct}{\epsilon_0}\right)$$

Inverse Hyperbolic Function

$$\boxed{\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + c}$$

$$Z(t) = \frac{\epsilon_0}{eE}\left(\cosh\left(\frac{E}{B}\phi\right) - 1\right)$$

$$= \frac{mc^2}{eE}\sqrt{1 + (p_\perp/mc)^2}\cosh\left(\frac{E}{B}\phi\right)$$

∘∘ because $\epsilon_0^2 = p_\perp^2 \circ c^2 + m^2c^4$

If $\vec{P} = \langle P_\perp \cos\phi, P_\perp \sin\phi \rangle$, then

$$X = \int_0^\phi \frac{P_\perp \cos\phi}{\gamma m} d\phi$$

$$= \frac{P_\perp \cdot e}{cB}\sin\phi$$

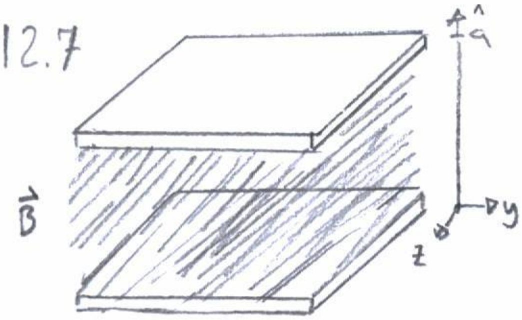
∘∘∘ when B in units "The Tesla-meter"

$$y = \int_0^\phi \frac{P_\perp \sin\phi}{\gamma m} d\phi$$

$$= -\frac{P_\perp \cdot c}{eB}\cos\phi$$

For the problem, $A = \frac{P_\perp}{mc}$, $R = \frac{mc^2}{eB}$, $\rho = E/B$

12.7



"constant uniform magnetic induction exists in a region .. by planes"

a) Nonrelativistic motion:

(1.3) "Electric Field"

$$E = q\frac{r-r_0}{|r-r_0|^3}$$

$$= q\frac{\vec{r}'}{r'^3}$$

$$= q\frac{x'x + y'y + z'z}{(x'^2+y'^2+z'^2)^{3/2}}$$

$$B = -B\hat{z}$$

(12.106) "Power"

$$P_{field} = \frac{1}{4\pi c}\int (E \times B)\, d^3x$$

$$= \frac{1}{4\pi c}\int_{x=x_0}^{a-x_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{q}{(x'^2+y'^2+z'^2)^{3/2}}\begin{vmatrix} x & y & z \\ x' & y' & z \\ 0 & 0 & -B \end{vmatrix} dx\,dy\,dz$$

$$= \frac{-qB}{4\pi c}\int_{-x_0}^{a-x_0}\int_{\rho=0}^{\infty}\int_{0}^{2\pi}\frac{y'x - x'y}{(x'^2+y'^2+z'^2)^{3/2}}\, d^3x$$

$$= \frac{qB}{4\pi c}\int_{-x_0}^{a-x_0}\int_{\rho=0}^{\infty}\int_{0}^{2\pi}\frac{x'dx'\rho\, d\rho\, d\theta}{(x'^2+\rho^2)^{3/2}}$$

$$= \frac{qB}{c}\hat{y}\begin{cases} a/2 & x_0 < 0 \\ a/2 - x_0 & 0 < x_0 < a \\ a/2 & x_0 > 0 \end{cases}$$

(12.14) "Conjugate Momentum"

$$P_{particle} = \gamma m v + \frac{q}{c}A$$

$$= \gamma m v - \frac{q x_0 B}{c} \hat{y}$$

(Prob. Advice) "Electromagnetic Momentum"

$$G = P_{field} + P_{particle}$$

$$= \gamma m v - \frac{q x_0 B}{c} \hat{y} + \frac{qB}{c} \hat{y} \begin{cases} a/2 & x_0 < 0 \\ a/2 - x_0 & 0 < x_0 < a \\ -a/2 & x_0 > a \end{cases}$$

"Total Momentum" = "Momentum in $\hat{x}$" + "momentum in $\hat{y}$" $\pm$ "momentum inside region (or outside)"

b) $P = \gamma \frac{qBa}{c} - q\frac{x_0 B}{c}\hat{y} + q\frac{B\hat{y}}{c}\left(\frac{a}{2} - x_0\right)$

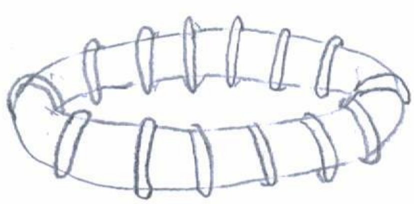
$$= \gamma \frac{qBa}{c} - q\frac{B}{c}\left(\frac{a}{2}\right)\hat{y}$$

c) If $P < qBa/2c$, then

$$P_{outer} > \frac{qBa}{2c}\hat{x} - q\frac{x_0 B}{c}\hat{y} + q\frac{B\hat{y}}{c}\left(\frac{a}{2} - x_0\right)$$

$$> \frac{qB}{2c} a\left(\gamma \hat{x} - \hat{y}\right)$$

12.8.



"current carrying toroid at rest"

Citation: Vaidman, L. "Torque and force on a magnetic dipole" University of South Carolina (1990)

$p = \gamma m v$; $J = env$

(Problem 6.5) "Momentum field"

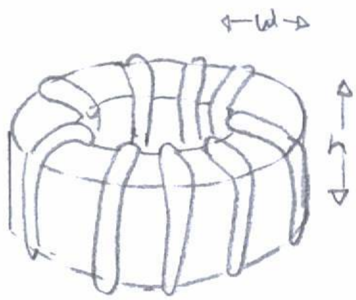
$$P_{Field} = \frac{1}{c^2}\int \Phi J d^3x$$

(Vaidman, equation 14)

$$\Delta P_{TOT} = P_{field} + P_{Mech}$$

$$= \frac{1}{c^2}\int \Phi J d^3x - \frac{1}{c^2}\int \Phi J d^3x$$

$$= 0$$



"toroid... rectangular cross section"

b) Citation: "Radio instruments and Measurements"

U. S. Department of Commerce

National Bureau of Standards

(1.1b) "Gauss' Law"

$$\nabla \cdot E = \rho / \epsilon_0$$

$$E = \frac{\lambda}{2\pi r \epsilon_0} \quad q \cdot 2\pi a$$ ... chapter 5: answer key

(6.117) "Total electromagnetic momentum modified"

$$P = -\int_a^{a+w} \frac{q}{4\pi^2 \epsilon_0 r} dr$$

$$= -\frac{q_0}{4\pi^2 \epsilon_0 a^2}\int_a^{a+w} \frac{1}{r} dr$$

$$= -\frac{q_0}{4\pi^2 \epsilon_0 a^2} \ln\left(\frac{a+w}{a}\right)$$

$$\cong -\frac{q_0}{4\pi^2 \epsilon_0 a^2}\left(\frac{w}{a}\right)$$

Maclaurin Series

$$\boxed{\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \ldots}$$

$$q = NQIA_0$$

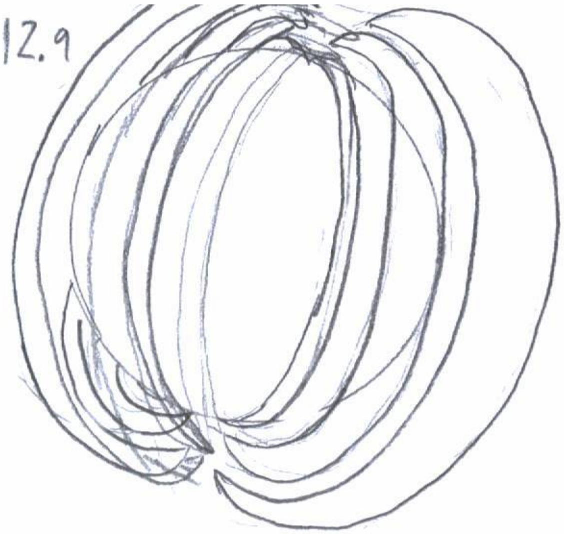
$$P_n = \frac{-Nq_0\omega}{4\pi^2 \epsilon_0 a^2}$$

$$= \frac{QINA_0}{4\pi^2 a^2 \epsilon_0}$$

$$= \frac{\mu_0 QINA}{4\pi a^2 c^2}$$

(Problem 6.6)

12.9



"Magnetic field of the earth"

-Van Allen electronbelts"

a) Magnetic Moment $(M) = 8.1 \times 10^{25}$ Gauss·cm^3

(5.107) "Magnetic Dipole Moment modified"

$$\vec{m} = -M z$$

(9.20) "Magnetic field from Electric field"

$$B = \frac{3r(r \cdot m) - m}{r^3}$$

$$= \frac{M}{r^3}(\hat{z} - 3\cos\theta\,\hat{r})$$

$$= -\frac{M}{r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) \quad \ldots \text{ when } \hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

$$\frac{\partial}{\partial\lambda} = \frac{\partial r}{\partial\lambda}\hat{r} + r\frac{\partial\theta}{\partial\lambda}\hat{\theta}$$

If $r = r_0 \sin^2\theta$, then

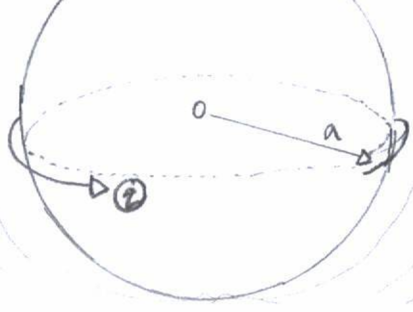
$$\frac{1}{r}\frac{dr}{d\theta} = \frac{2\sin\theta\cos\theta \cdot r_0}{r_0 \sin^2\theta}$$

$$= \frac{2\cos\theta}{\sin\theta}$$

$$= 2\cot\theta$$

$$\frac{dr}{dr} = 2\cot\theta\, d\theta$$

"Particle circles around a line of force in an equatorial plane... mean radius"



$$B = \frac{M}{r^3}\sqrt{1 + 3\cos^2\theta}$$

$$= \frac{M\sqrt{1 + 3\cos^2\theta}}{r_0^3 \sin^6\theta}$$

b) (12.61) "Drift Velocity"

$$V_D = \frac{1}{\omega_B R_c}\left(V_\parallel^2 + \frac{1}{2}V_\perp^2\right)(R_c \times B)$$

$$\approx \frac{1}{2\omega_B R_c} v_\perp^2 \left(R_c \times (-\theta)\right)$$

$$\approx -\frac{\omega_B}{2R_c} a^2 \hat{\phi}$$

(12.60) "

$$\frac{\nabla_\perp B}{B} = -\frac{\hat{R}_c}{R_c} \quad \ldots \text{when } B = M/cr^3$$

$$= -\frac{\frac{3M}{cr^4}}{\frac{M}{cr^3}} \quad \ldots \text{as } R_c = R/3$$

$$V_D \approx -\frac{\omega_B}{2R_c} a^2 \hat{\phi}$$

$$\approx -\frac{3a^2}{2R} \omega_B \hat{\phi}$$

$$\approx -\frac{3}{2}\left(\frac{a}{R}\right)^2 \omega_B (t - t_0)$$

"circular motion ... with small oscillation"

c) $\Omega = (3/\sqrt{2})(a/R)\omega_B$

If $v_\parallel < v_\perp$,

(12.72) "Parallel velocity"

$$V_0^2 = V_\parallel^2 + V_{\perp,0}^2 \frac{B(z)}{B_0}$$

$$\simeq V_\parallel^2 + \omega_B^2 a^2 \frac{B(z)}{B_0}$$

$$B\left(\theta = \frac{\pi}{2} - \frac{z}{R}\right) = \frac{M\sqrt{1+3\cos^2\theta}}{r_0^3 \sin^6\theta}$$

$$= \frac{M\sqrt{1+3\sin^2(z/R)}}{R^3\cos^6(z/R)}$$

$$\approx \frac{M}{R^3}\left(1+3\left(\frac{z}{R}\right)^{2}-\cdots\right)^{+1/2} \circ \left(1-\frac{1}{2}\left(\frac{z}{R}\right)^2+\cdots\right)^{-6}$$

$$\approx \frac{M}{R^3}\left(1+\frac{9}{2}\left(\frac{z}{R}\right)^2\right)$$

Taylor Series

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$$

Small Angle Identity

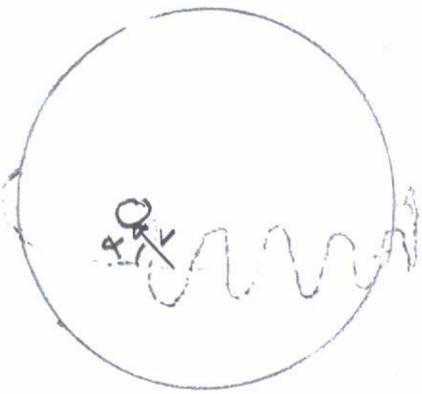
$$\sin x \approx x$$

$$V_0^2 \approx V_{\parallel}^2 + \omega_B^2 a^2 \frac{B(z)}{B_0}$$

$$\approx V_{\parallel}^2 + \omega_B(a^2)\left(1+\frac{9}{2}\left(\frac{z}{R}\right)^2\right)$$

$$\approx V_{\parallel}^2 + (\omega_B a)^2 + \frac{9}{2}\left(\frac{\omega_B a}{R}\right)^2 z^2$$

12.10.



"a charged particle in the equatorial plane ... makes an angle ... with the equatorial plane"

Angle: $\frac{V}{V_0} = \tan\alpha$

Gyration radius: $a \ll R$

Maximum magnetic latitude: λ

(12.72) "Motion Magnetic Latitude"

$$V_{\parallel}^2 = V_0^2 - V_{\perp,0}^2 \frac{B}{B_0}$$

$$= (V_{\parallel,0}^2 + V_{\perp,0}^2) - V_{\perp,0}^2 \frac{M\sqrt{3\cos^2\theta+1}}{B_0 r_0^3 \sin^6\theta}$$

@ λ, $V_{\parallel} = 0$

$$\frac{V_{\parallel,0}}{V_{\perp,0}^2} = \frac{M\sqrt{3\cos^2\theta+1}}{B_0 r_0^3 \cos^6\lambda} - 1$$

$$= \frac{M\sqrt{3\sin^2\lambda+1}}{B_0 r_0^3 \cos^6\lambda} - 1 \quad \text{.. when } \theta_{max} = \frac{\pi}{2}-\lambda$$

$$= \tan^2\alpha$$

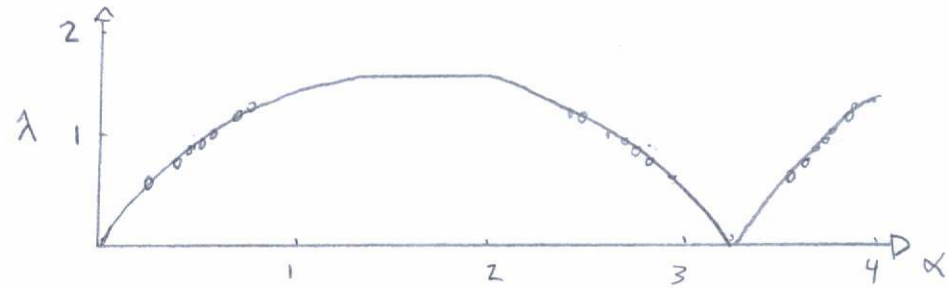
$$= \tan^2\lambda_{max}$$

$$\lambda = \arctan\left[\left(\frac{\sqrt{3\sin^2\alpha + 1}}{\cos^6\alpha} - 1\right)^{1/2}\right]$$

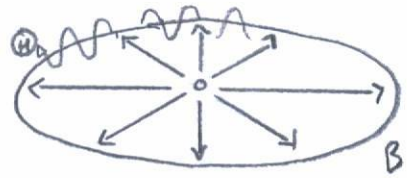
(Problem 12.9)

$r = r_0 \sin^2\theta$ and $\alpha =$

$\alpha = \arccos\left(\sqrt{\frac{R}{R}}\right)$



12.11



"precession of the spin of a muon ... longitudinal polarized ... moving in a circular orbit ... perpendicular to a uniform magnetic field"

(Sec 11.8) "Thomas Precession Formula"

$$\left(\frac{ds}{dt}\right)_{lab} = \frac{1}{\gamma}\left(\frac{ds}{d\tau}\right)_{rest} - \omega_T \times s$$

... where (11.119)

$$\omega_T = \frac{\gamma^2}{1+\gamma}\,\frac{a \times v}{c^2}$$

(1.3) "Lorentz Equation"

$$E_{Tot} = q(E + v \times B)$$

$$\frac{dP_{Tot}}{dt} = \frac{e}{c}(v \times B) \qquad \text{... when } E = 0$$

$$\gamma \quad = \gamma \frac{dp}{dt}$$

$$\frac{dV}{dt} = \frac{e}{\gamma mc}(V \times B)$$

$$= \omega_B \times V \qquad \text{... when } \omega_B = \frac{-(eB)}{\gamma mc}$$

$$= a$$

$$\omega_T = \frac{\gamma^2}{1+\gamma}\frac{1}{c^2}\frac{e}{\gamma mc}(V \times B) \times V$$

$$= \frac{\gamma - 1}{\gamma}\frac{eB}{mc}$$

(11.101) "Thomas precession"

$$\left(\frac{ds}{dt}\right)_{rest} = \mu \times B'$$

$$= \frac{ge}{2mc} \cdot s \times B$$

where $B' = \gamma\left(B - \frac{v}{c} \times E\right) - \frac{\gamma^2}{1+\gamma}\frac{v}{c}\left(\frac{v \cdot B}{c}\right)$

$$= \gamma B$$

$$\left(\frac{ds}{dt}\right)_{lab} = \frac{1}{\gamma}\left(\frac{ge}{2mc}s\right) \times (\gamma B) + \left(\frac{\gamma - 1}{\gamma}\frac{eB}{mc}\right) \times s$$

$$= \frac{e}{mc}\left\{\frac{\gamma - 1}{\gamma} - \frac{g}{2}\right\} B \times s$$

$$= W \times s$$

(Sec 11.8) "Spin precession Frequency"

$$W = \left(1 - \frac{g}{2} - \frac{1}{\gamma}\right)\frac{e}{mc}B$$

Difference between spin precession and gyromagnetic Frequency:

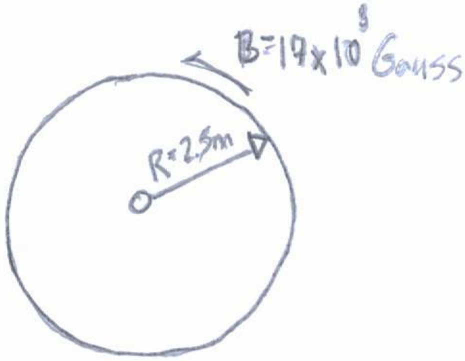
$$\Omega = W - \omega_B$$

$$= \left(1 - \frac{g}{2} - \frac{1}{\gamma}\right)\frac{eB}{mc} + \frac{eB}{\gamma mc}$$

$$= \frac{eB}{mc}\frac{(2-g)}{2},$$ ∘∘∘ close when $v \approx c$

$$\cong \frac{-eB}{m_\mu c}\frac{\hat{B}}{B}$$ ∘∘ when $B = \left(\frac{g-2}{2}\right)$

$$\cong \frac{e\hat{B}}{m_\mu c}$$ ∘∘ and $m = m_\mu$



"C.E.R.N. muon Storage ring"

b) $m_e = 105.66\,\text{MeV}$

$\tau_0 = 2.2 \times 10^{-6}\,s$

$a \approx \alpha/2\pi$

Muon Momentum:

$$F = \frac{dP}{dt} = \frac{e}{c} v \times B$$

$$= \frac{\gamma m v^2}{R}$$

$$P = \gamma m v$$

$$= \frac{eRB}{c}$$

$$= 1.28 \times 10^3\,\text{MeV}/c$$

Time Dialation:

(11.51) "Lorentz Boost Factor"

$$\gamma = \frac{E}{mc^2}$$

$$= \frac{\sqrt{p^2c^2 - m^2c^4}}{mc^2}$$

$$= 12.1$$

Periods of Precession per observed Laboratory lifeti[me]

$$\frac{\gamma \tau_0}{\tau} = \frac{\gamma \tau_0 \Omega}{2\pi}$$

$$= \frac{eBa\,\gamma \tau_0}{2\pi mc}$$

$$= \frac{eB\alpha\,\gamma \tau_0}{(2\pi)^2 mc}$$

$$= 7.12$$

c) <u>Particles precessing per rotation:</u>

300 MeV Muon:

$$\gamma = E/m_\mu c^2 = 2.83$$

$$\Omega = \left(\frac{\alpha\gamma}{2\pi}\right)\omega_B = 0.0033\,\omega_B$$

300 MeV Electron:

$$\gamma = E/m_e c^2 = 597$$

$$\Omega = \left(\frac{\alpha\gamma}{2\pi}\right)\omega_B = 0.697\,\omega_B$$

5 GeV Electron:

$$\gamma = E/m_e c^2 = 9.78\times10^3$$

$$\Omega = \left(\frac{\alpha\gamma}{2\pi}\right)\omega_B = 11.4\,\omega_B$$

12.14

a) $\mathcal{L} = -\frac{1}{8\pi}\partial_\alpha A_\beta \partial^\alpha A^\beta - \frac{1}{c} J_\alpha A^\alpha$

$= -\frac{1}{8\pi} g_{\alpha\mu} \circ g_{\beta\nu} \circ \partial_\alpha^\alpha A_\beta^\beta \partial^\mu A^\gamma - \frac{1}{c} J_\alpha A^\alpha$

... (11.70) "flat space-time"

(11.73) "inverse contraction"

$= -\frac{1}{8\pi} g_{\alpha\mu} \cdot g_{\beta\nu} \left(\partial^\mu_\circ A^\gamma + \partial^\alpha A^\beta\right) - \frac{1}{c} J_\alpha A^\alpha$

$= -\frac{1}{4\pi} \partial^\alpha A^\beta - \frac{1}{c} J_\alpha A^\alpha$

... (11.72) "contracted space-time"

$= 0$

$\partial^\alpha A^\beta = -\frac{4\pi}{c} J_\alpha \cdot A^\alpha$

$\partial^\alpha \partial_\alpha A_\beta = -\frac{4\pi}{c} J_\alpha$ ... (11.70) "flat space-time"

(11.136) "Antisymmetric field-strength tensor"

$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$

(11.141) "Covariant 4-current"

$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta$

$\partial^\alpha\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) = -\frac{4\pi}{c} J_\alpha$

$\partial^\alpha \partial_\alpha A_\beta - \partial_\beta \partial^\alpha A_\alpha = -\frac{4\pi}{c} J_\alpha$

$= \partial^\alpha \partial_\alpha A_\beta$ ... because $\partial_\beta \partial^\alpha A_\alpha = 0$

The Euler Lagrange equation reduced because a Maxwell equation $\nabla \cdot B = \nabla \cdot \nabla A = 0$

b) (12.84) "Action Integral"

$$A = \iint \mathcal{L}\, d^3x\, dt = \int \mathcal{L}\, d^4x$$

(12.85) "Electromagnetic Lagrangian Density"

$$\mathcal{L} = -\frac{1}{16\pi} F_{\alpha\beta} \circ F^{\alpha\beta} - \frac{1}{c} J_\alpha A^\alpha$$

(12.89) "Equation of Motion"

$$\frac{1}{4\pi} \partial^\beta \circ F_{\beta\alpha} = \frac{1}{c} J_\alpha$$

$$\Delta\mathcal{L} = \frac{1}{16\pi}\left(F_{\alpha\beta} \circ F^{\alpha\beta} - \partial_\alpha A_\beta \partial^\alpha A^\beta\right)$$

$$= \frac{1}{16\pi}\left[(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta \partial^\alpha) - \partial_\alpha A_\beta \partial^\alpha A^\beta\right]$$

$$= \frac{1}{16\pi}\left[\partial_\alpha A_\beta \partial^\alpha A^\beta - \partial_\beta A_\alpha \partial^\alpha A_\beta - \partial_\alpha A_\beta \partial^\beta A^\alpha + \partial_\beta A_\alpha \partial^\beta A^\alpha - \partial_\alpha A_\beta \partial^\alpha A^\beta\right]$$

$$= -\frac{1}{16\pi} \partial_\beta A_\alpha \partial^\alpha A_\beta$$

$$= -\frac{1}{16\pi} \partial_\beta \underbrace{(A_\alpha \partial^\alpha A_\beta)}_{\text{"4-divergent"}}$$

The affect to the action:

$$A = \int L\, d^4x$$

$$\Delta A = \int -\frac{1}{16\pi} \partial_\beta (A_\alpha \partial^\alpha A_\beta)\, d^4x$$

$$= 0$$

The trajectory never changes.

12.15

"Steady-state distribution of current in Earth's magnetic field"

a) "Current Density"

$$J = c(\nabla \times M)$$

If $M = m f(x)$, then

(   ) "Vector Potential"

$$A_\mu(x) = \frac{1}{c}\int J_\mu(x') G(x, x') d^3x$$

(12.92) "Proca Equations"

$$[\partial^\lambda \partial_\lambda + \mu^2] A_\mu = \frac{4\pi}{c} J_\mu$$

$$[\nabla^2 - \mu^2] A_\mu = -\frac{4\pi}{c} J_\mu$$

(Ch. 1.) "Time independent Greens Function"

$$G(x, x') = \\ = \frac{e^{-\mu|x-x'|}}{|x-x'|}$$

$$A_\mu(x) = \frac{1}{c}\int J_\mu(x') G(x, x') d^3x$$

$$= -m \nabla \times \int f(x') \frac{e^{-\mu|x-x'|}}{|x-x'|} d^3x$$

... if $M = mf(x)$

... and since $G(x, x') = -G(x, x')$

b) If $f(x')$ were $\delta(x)$,

$$A = -m \nabla \times \int f(x) \frac{e^{-\mu|x-x'|}}{|x-x'|} d^3x$$

$$= -m \nabla \times \int \delta(x) \frac{e^{-\mu r}}{r} d^3x$$

$$= -m\nabla \times \frac{e^{-\mu|r|}}{|r|}$$

$$= (1+\mu r)\,\frac{e^{-\mu|r|}}{r^3}\, m \times r$$

(5.27) "Magnetic field"

$$B = \nabla \times A$$

$$= \nabla\left((1+\mu r)\frac{e^{-\mu r}}{r^3}\right) \times (m \times r) + (1+\mu r)\frac{e^{-\mu r}}{r^3}\nabla \times (m \times r)$$

$$= -(3+3\mu r+\mu^2 r^2)\frac{e^{-\mu r}}{r^3}\,\hat{r} \times (\hat{m} \times \hat{r})$$

$$+ (1+\mu r)\frac{e^{-\mu r}}{r^3}\left(m(\nabla \cdot r) - (m \cdot \nabla) r\right)$$

$$= -(3+3\mu r+\mu^2 r^2)\frac{e^{-\mu r}}{r^3}\left(m - r(r \cdot m)\right) + (2+2\mu r)\frac{e^{-\mu r}}{r^3} m$$

$$= \left(3r(r \cdot m) - m\right)\left(1+\mu r+\frac{\mu^2 r^2}{3}\right)\frac{e^{-\mu r}}{r^3} - \frac{2}{3}\mu^2 m \frac{e^{-\mu r}}{r}$$

"..at magnetic equator"

$$r \cdot m = 0$$

So, $B = B_{Dipole} + B_{external}$

$$B_{Dipole} = -m\left(1+\mu R+\frac{\mu^2 R^2}{3}\right)\left(\frac{e^{-\mu r}}{R^3}\right)$$

$$B_{external} = -m\left(\frac{2}{3}\mu^2 R^2\right)\frac{e^{-\mu R}}{R^3}$$

Lower limit:

$$\frac{B_{dipole}}{B_{external}} = \frac{-m\left(1+\mu R+\frac{\mu^2 R^2}{3}\right)\frac{e^{-\mu r}}{R^3}}{-m\left(\frac{2}{3}\mu^2 R^2\right)\frac{e^{-\mu r}}{R^3}}$$

$$= \frac{1+\mu R+\mu^2R^2/3}{\frac{2}{3}\mu^2R^2}$$

$$< 4\times10^3$$

$$\mu R < 0.08$$

$$\mu < \frac{1}{12.5 R}$$

$$< \frac{1}{8.0\times10^9 cm}$$

$\circ\circ\circ$ when $R = 6.38\times10^8 cm$

Quadratic Formula

$ax^2+bx+c=0$

$x_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Upper limit:

$$m = \frac{\mu\hbar}{c}$$

$$= \frac{1.05\times10^{-27}\ erg\cdot s}{(8.0\times10^9 cm)(3\times10^{10} cm/s)}$$

$$= 4.4\times10^{-48}\ gm$$

12.16

(12.91) "Proca Lagrangian"

$$\mathcal{L} = -\frac{1}{16\pi}F_{\mu\nu}\cdot F^{\mu\nu} + \frac{1}{8\pi}\mu^2 A_\mu A^\mu$$

(12.103) "Covariant generalization of the Hamiltonian"

$$T^{\mu\nu} = \frac{\partial L}{\partial\partial_\mu A_\lambda}\partial^\nu A_\lambda - \eta^{\mu\nu}\mathcal{L}$$

$$= -\frac{1}{4\pi}F^{\mu\sigma}\cdot\partial^\nu\lambda_\lambda + \frac{1}{16\pi}\eta^{\mu\nu}F_{\mu\nu}\cdot F^{\mu\nu} - \frac{1}{8\pi}\mu^2\eta^{\mu\nu}A_\mu A^\mu$$

$$= -\frac{1}{4\pi}\left[F^{\mu\lambda}F_\lambda^{\ \mu\nu} - \frac{1}{4}\eta^{\mu\nu}F_{\mu\nu}\cdot F^{\mu\nu} + \frac{1}{2}\mu^2\eta^{\mu\nu}A_\mu A^\mu\right]$$

$$-\frac{1}{4\pi}F^{\mu\lambda}\partial_\lambda A^\nu$$

$$= -\frac{1}{4\pi}\left[F^{\mu\lambda}\cdot F^{\nu}_{\ \lambda} - \frac{1}{4}\eta^{\mu\nu}\circ F_{\mu\nu}\circ F^{\mu\nu} + \frac{1}{2}\mu^2\eta^{\mu\nu}\cdot A_\mu A^\mu - (\partial_\lambda F^{\mu\lambda})A^\nu\right] - \frac{1}{4\pi}\partial_\lambda(F^{\mu\lambda}\cdot A^\nu)$$

(12.91) "Proca Equation of motion"

$$\partial_\lambda F^{\lambda\mu} + \mu^2 A^\mu = 0$$

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda S^{\lambda\mu\nu}$$

... where $\Theta^{\mu\nu}$ = "Stress tensor"

$$= -\frac{1}{4\pi}\left[F^{\mu\lambda}\cdot F^{\nu}_{\ \lambda} - \frac{1}{4}\eta^{\mu\nu}\cdot F_{\mu\nu}\cdot F^{\mu\nu} + \frac{1}{2}\mu^2\eta^{\mu\nu}\cdot A_\mu\cdot A^\mu - (\partial_\lambda F^{\mu\lambda})A^\nu\right]$$

$S^{\lambda\mu\nu}$ = "Antisymmetric Stress tensor"

$$= \frac{1}{4\pi}F^{\lambda\mu}\cdot A^\nu$$

b) (12.91) "Proca Equation"

$$\partial_\mu\Theta^{\mu\nu} = \partial_\mu\left[-\frac{1}{4\pi}\left[F^{\mu\lambda}\cdot F^{\nu}_{\ \lambda} - \frac{1}{4}\eta^{\mu\nu}\cdot F_{\mu\nu}\cdot F^{\mu\nu} + \frac{1}{2}\mu^2\eta^{\mu\nu}\cdot A_\mu\cdot A^\mu - (\partial_\lambda F^{\mu\lambda})A^\nu\right]\right]$$

$$= -\frac{1}{4\pi}\left[\partial_\mu F^{\mu\nu}\circ F^{\nu}_{\ \lambda} + F^{\mu\lambda}\cdot\partial_\mu\cdot F^{\nu}_{\ \lambda} - \frac{1}{2}F_{\rho\lambda}\partial^\nu F^{\rho\lambda} - \mu^2(\partial_\mu\cdot A^\mu\circ A^\nu + A^\mu\partial_\mu\cdot A^\nu - A^\lambda\partial^\nu A_\lambda)\right]$$

$$= -\frac{1}{4\pi}\left[\partial_\mu F^{\mu\lambda}F^{\nu}_{\ \lambda} + \frac{1}{2}F_{\rho\lambda}(2\partial^\rho F^{\nu\lambda} - \partial^\nu F^{\rho\lambda}) + \mu^2 A^\lambda\underbrace{(\partial^\nu A_\lambda - \partial_\lambda A^\nu)}_{=F^{\nu}_{\ \lambda}}\right]$$

$$= -\frac{1}{4\pi}\left[(\partial_\mu F^{\mu\lambda} + \mu^2 A^\lambda)F^{\nu}_{\ \lambda} \ldots\right.$$

$$+\frac{1}{2}F_{\mu\lambda}\left(\partial^{\rho}F^{\nu\lambda}+\partial^{\lambda}F^{\rho\nu}+\partial^{\nu}F^{\lambda\rho}\right)\Big]$$

$$=-\frac{1}{c}J^{\lambda}F^{\nu}_{\ \lambda}$$

$$=\frac{1}{c}J_{\lambda}F^{\lambda\nu}$$

c) (   ) "Maxwell Tensor"

Identities

(Bianchi Identity)

$$3\partial^{\rho}\cdot F^{\mu\lambda}=0$$

(Proca)

$$\partial_{\mu}A^{\mu}=0$$

If $F_{\mu\nu}\cdot F^{\mu\nu}=-2(E^2-B^2)$

$$A_{\mu}\cdot A^{\mu}=(A^0)^2-A^2$$

$$\Theta_{\mu\nu}=-\frac{1}{4\pi}\Big[F^{\mu\nu}\cdot F^{\nu}_{\ \lambda}+\frac{1}{2}\eta^{\mu\nu}(E^2-B^2)-\mu^2(A^{\mu}A^{\nu}$$

$$-\frac{1}{2}\eta^{\mu\nu}((A^0)^2-A^2)\Big]$$

Time-time component:

$$\Theta^{00}=-\frac{1}{4\pi}\Big[-\frac{1}{2}(E^2+B^2)-\frac{1}{2}\mu^2((A^0)^2+A^2)\Big]$$

$$=\frac{1}{8\pi}\Big[E^2+B^2+\mu^2((A^0)^2+A^2\Big]$$

Time-space component:

$$\Theta^{00}=-\frac{1}{4\pi}\Big[F^{0}_{j}F^{ij}-\mu^2A^0A^i\Big]$$

$$=-\frac{1}{4\pi}\Big[E^i(-\epsilon_{ijk}\cdot B^k)-\mu^2A^0\cdot A^i\Big]$$

$$=-\frac{1}{4\pi}\Big[-\epsilon_{ijk}E^iB^k-\mu^2A^0A^i\Big]$$

$$=-\frac{1}{4\pi}\Big[(E\times B)^i+\mu^2A^iA^i\Big]$$

12.17

a) (12.114)

$$\Theta^{00} = \frac{1}{8\pi}(E^2 + B^2)$$

$$\Theta^{0i} = \frac{1}{4\pi}(E \times B)$$

$$\Theta^{ij} = -\frac{1}{4\pi}\left[E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}(E^2 + B^2)\right]$$

$A = \epsilon_0 \cos(kz - \omega t)$; $F_t = \omega k / 8\pi$; $F_e = (\mu/\omega)^2 F_t$

$$\frac{\langle\Theta^{30}\rangle}{\langle\Theta^{00}\rangle} = \frac{-\frac{1}{4\pi}\left[E_3 E_0 + B_3 B_0 - \frac{1}{2}\delta_{i0}(E^2 + B^2)\right]}{\frac{1}{8\pi}\left[E^2 + B^2\right]}$$

$$= \frac{k}{\omega}\mu^2$$ ... when $E = B$

(12.95) Time and Space constant

$$\omega^2 = c^2 k^2 + \mu^2 c^2$$

$$\frac{k}{\omega} = \sqrt{\frac{1}{c^2} - \frac{\mu^2}{\omega^2}}$$

$$\frac{\langle\Theta^{30}\rangle}{\langle\Theta^{00}\rangle} = \frac{k}{\omega}\mu^2$$

$$= \mu^2\sqrt{\frac{1}{c^2} - \frac{\mu^2}{\omega^2}}$$

b) (10.3) "Differential Scattering Cross Section"

$$\frac{d\sigma}{d\Omega}(n, \epsilon; n_0, \epsilon_0) = \frac{r^2 \frac{1}{2Z_0}|\epsilon^* \cdot E_{sc}|^2}{\frac{1}{2Z_0}|\epsilon^* \cdot E_{inc}|^2}$$

$$= r^2 |\epsilon^* \cdot \epsilon_0|^2 \frac{\delta \epsilon(x)}{\epsilon_0}$$ ... (pg 468) Born Approximation

$$= r^2 E_0 |\epsilon^* \cdot \epsilon_0|^2 \cdot \frac{F_{out}}{F_{in}}$$ ... $\underbrace{\frac{\delta \epsilon(x)}{\epsilon_0}}_{\text{Proca Equation}} = \underbrace{E_0}_{\text{Initial}} \cdot \underbrace{\frac{F_{out}}{F_{in}}}_{\text{Effective Factor}}$

$$E_0 = \frac{\delta \epsilon(x)}{\epsilon_0} \cdot \frac{F_{in}}{F_{out}}$$

c) (pg 695) "Dielectric constant"

$$\hat{\epsilon}_1 = \cos\theta(\epsilon_x \cos\phi + \epsilon_y \sin\phi) - e_z \sin\theta$$

$$\hat{\epsilon}_2 = -e_x \sin\phi + e_y \cos\phi$$

$$\hat{\epsilon}_1^2 = \cos^2\theta \cos^2\phi + \sin^2\phi$$ ... when $\hat{\epsilon}_x^2 = 1$

$$\hat{\epsilon}_2^2 = \cos^2\theta \sin^2\phi + \cos^2\phi$$ and $\hat{\epsilon}_y^2 = 1$

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma_{TT}}{d\Omega} + \frac{d\sigma_{TL}}{d\Omega}$$

$$= \left(\frac{r^2}{2}\right)\left[\frac{\epsilon_1^2 + \epsilon_2^2}{2}\right]_{\phi=0} + \left(\frac{r^2}{2c^2}\right)\left[\frac{\hat{\epsilon}_1^2 - \hat{\epsilon}_2^2}{2}\right]_{\theta=\pi/2}$$

$$= \frac{r^2}{2}\left[\cos^2\theta + 1 + \left(\frac{M^2}{\omega^2}\right)\sin^2\theta\right]$$

d) $$\frac{d\sigma}{d\Omega} = \frac{d\sigma_{LT}}{d\Omega} + \frac{d\sigma_{LL}}{d\Omega}$$

$$= \left(\frac{r^2}{2c^2}\right)\left[\frac{\epsilon_1^2 + \epsilon_2^2}{2}\right]_{\theta=\pi/2} + \left(\frac{r^2}{2c^2}\right)\left[\hat{\epsilon}_1^2\right]_{\phi=0}$$

$$= \frac{r^2}{2}\left(\frac{M^2}{\omega^2}\right)\left[\frac{\sin^2\theta}{2} + \cos^2\theta\right]$$

Note: $1/c^2$ is (12.95) funny business

12.18

(12.106) "Total Momentum and Electromagnetic field"

$$\int T^{00} d^3x = \frac{1}{8\pi}\int (E^2+B^2)\, d^3x$$
$$= cP_{field}$$

$$\int T^{0i} d^3x = \frac{1}{4\pi}\int (E\times B)\, d^3x$$
$$= cP_{field}$$

(12.105) "Canonical Stress Tensor"

$$T^{00} = \frac{1}{8\pi}(E^2+B^2) + \frac{1}{4\pi}\nabla\cdot(\Phi E)$$

$$T^{0i} = \frac{1}{4\pi}(E\times B) + \frac{1}{4\pi}\nabla\cdot(A\cdot E)$$

(6.112) "Total field Energy"

$$E_{field} = \frac{\epsilon_0}{2}\int E^2 + cB^2\, d^3x$$

(6.117) "Total Electomagnetic Momentum"

$$P_{field} = \epsilon_0 \int E\times B\, d^3x$$

12.19

(12.117)

$$M^{\alpha\beta\gamma} = \Theta^{\alpha\beta} x^\gamma - \Theta^{\alpha\gamma} x^\beta$$

"Source-free electromagnetic fields in space"

$$M^{0ij} = \Theta^{0i} x^j - \Theta^{0j} x^i$$

$$= c(g^i x^j - g^j x^i)$$

$$= c\,\epsilon^{ijk} (g \times x)^k$$

$$= -c\,\epsilon^{ijk} (x \times g)^k$$

$$M^{ij} = \int M^{ij} d^3x$$

$$= -c\,\epsilon^{ijk} \int (x \times g)\, d^3x$$

$$= -c\,\epsilon^{ijk} L^k$$

b) If $\beta = 0$, then

$$M^{0i} = \int M^{00i} d^3x$$

$$= \int (\Theta^{00} x^i - \Theta^{0i} x^0)\, d^3x$$

$$= \int (u x^i - c g^i x^0)\, d^3x$$

$$= \int (u x^i - c^2 t g^i)\, d^3x$$

From definition, $\int u x^i d^3x = E X^i$

$E = \int u\, d^3x$

$$M^{0i} = E X^i - c^2 t P^i$$

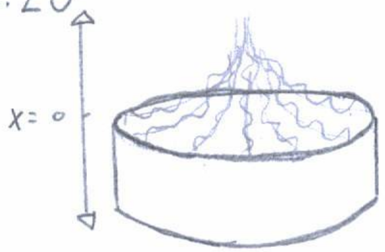
$$\frac{dM^{0i}}{dt} = \frac{d}{dt}\left[E X^i - c^2 t P^i\right]$$

$$= E \frac{dX^i}{dt} - c^2 P$$

$$= 0$$

$$\frac{dX^i}{dt} = \frac{c^2 P_{cm}}{E_{cm}}$$

12.20



"Superconductor with London penetration ... electric and magnetic fields on the surface"

$$A(x,t) = \left(a e^{ikx} + b e^{-ikx}\right) e^{-i\omega t} \begin{cases} k = k & x<0 \\ k = |k| & x>0 \end{cases}$$

... where $k = \sqrt{\mu^2 - \omega^2/c^2}$

$$E(x=0,t) = -\frac{1}{c}\frac{\partial}{\partial t} A(x=0,t)$$

$$= \frac{i\omega}{c}(a+b)\,\hat{y}\, e^{-i\omega t}$$

$$B(x=0,t) = -\nabla \times A(x=0,t)$$

$$= -\sqrt{\mu^2 - \omega^2/c^2}\,(a^2+b^2)\,\hat{z}\, e^{-i\omega t}$$

Surface Impedance:

$$Z_s = \frac{4\pi}{c}\frac{E_y}{B_z} \quad \frac{4\pi i\omega}{c^2\sqrt{\mu^2 - \omega^2/c^2}}$$

If $\mu = 1/\lambda$, $\omega = 2\pi c/\lambda$

$$Z_s = -\frac{8\pi^2 i}{c}\frac{\lambda_L}{\lambda}\left(1 - (2\pi\lambda_L/\lambda)^2\right)^{-1/2}$$

$$= -\frac{8\pi^2 i}{c}\frac{\lambda_L}{\lambda}$$ .. when $\lambda >> L$

Notes: After Chapter 12, the material accessed magnetic lenses in spectrometers, superconductors at hot or cold temperatures, gravitational lenses by relativistic expansion and contraction. Content also adventured quantum vortices, early atomic models, and cylcotrons versus synchrotrons.