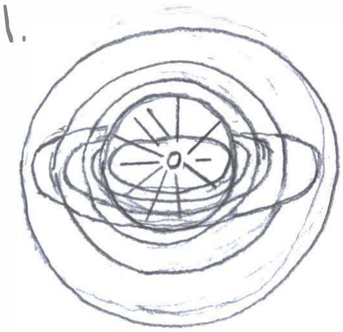
Chapter 6: Maxwell Equations, Macroscopic Electromagnetism, Conservation Laws

6.1.



"A point source flashing as a spherical shell disturbance"

(6.32) Wave Equation

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -4\pi f(x,t)$$

(6.44) $$G^*(R,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{\pm ikR}}{R}e^{-i\omega\tau}\,d\omega$$

(6.47) $$\psi(x,t) = \int\frac{[f(x',t')]_{ret}}{|x-x'|}d^3x$$

a) $f(x',t') = \delta(x'-x_0)\delta(y'-y_0)\delta(t'-t_0)$

pg 246, $[f(x',t')]_{ret}$ has $t' = t - \frac{|x-x'|}{c}$

So, $f(x',t') = \delta(x'-x_0)\delta(y'-y_0)\delta(t - \frac{|x-x'|}{c} - t_0)$

$$\psi(x,t) = \int\frac{\delta(x'-x_0)\delta(y'-y_0)\delta(t-\frac{|x-x'|}{c}-t_0)}{|x-x'|}d^3x$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dx\,dy\,dz\,\frac{\delta(x'-x_0)\delta(y'-y_0)\delta(t-\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}/c-t_0)}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$= \int_{-\infty}^{\infty}\frac{\delta(t-\sqrt{\rho^2+(z-z')^2}/c)}{\sqrt{\rho^2+(z-z')^2}/c}dz'$$

$= 0$ ◁— a solution by root

... when $z-z' = \pm\sqrt{c^2t^2-\rho^2}$

and $$\frac{\sum_i\delta(z-z')}{\left|\left(\frac{\partial F}{\partial z}\right)_{z=z_i}\right|} = 0$$

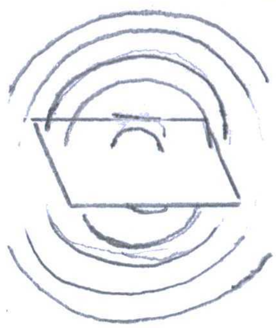
$$\Psi(x,t) = \int_{-\infty}^{\infty} \frac{\dfrac{\delta(z'-z-\sqrt{c^2t^2-\rho^2})}{\frac{1}{c}\frac{\sqrt{c^2t^2-\rho^2}}{\sqrt{\rho^2+(z-z')^2}}} + \dfrac{\delta(z'-z+\sqrt{c^2t^2-\rho^2})}{\frac{1}{c}\frac{\sqrt{c^2t^2-\rho^2}}{\sqrt{\rho^2+(z-z')^2}}}}{\sqrt{\rho^2+(z-z')^2}}\, dz'$$

$$= \frac{2c}{\sqrt{c^2t^2-\rho^2}} \quad \therefore \text{ when } c^2t^2-\rho^2>0$$

$$= \frac{2c\,\Theta(ct-\rho)}{\sqrt{c^2t^2-\rho^2}} \quad \therefore \text{ if } \Theta(ct-\rho) = \begin{cases} 0 & ct-\rho<0 \\ 1 & ct-\rho>0 \end{cases}$$

The flashing light has a temporal and spatial delay from a distant reference. 'A delay retards' is the concept in the book.

b)



"A sheet source equivalent to a point pulsed source"

$$f(x',t') = \delta(x')\,\delta(t)$$
$$= \delta(x')\,\delta(t-|x-x'|/c)$$

$$\Psi(x,t) = \int \frac{[f(x',t')]_{ret}}{|x-x'|}\, d^3x$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\delta(x')\,\delta(t-\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}/c)}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\, dx\,dy\,d$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\delta(t-\sqrt{(x)^2+(y)^2+(z)^2}/c)}{\sqrt{(x)^2+(y)^2+(z)^2}}\, dy\,dz \quad @\ x'=0,\ y'=0,\ z'=0$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\delta(t-\sqrt{(x)^2+\rho^2}/c)}{\sqrt{(x)^2+\rho^2}}\, dy\,dz$$

ooo when $y^2+z^2=\rho^2$

$$y=\rho\cos\theta$$

$$z=\rho\sin\theta$$

$$=\int_0^{\infty}\frac{\delta(t-\sqrt{(x)^2+\rho^2}/c}{\sqrt{(x)^2+\rho^2}}\rho d\rho\int_0^{2\pi}d\phi$$

$$=0$$

oooo if $\dfrac{\sum_i \delta(x-x')}{\left|\left(\frac{\partial F}{\partial z}\right)_{z=z_i}\right|}=0$

and $x=\pm\sqrt{c^2t^2-x^2}$

$$=2\pi\int_0^{\infty}\frac{c^2t\,\dfrac{\delta(\rho'-\sqrt{c^2t^2-x^2})}{\sqrt{c^2t^2-x^2}}+c^2t\,\dfrac{\delta(\rho'+\sqrt{c^2t^2-x^2})}{\sqrt{c^2t^2-x^2}}}{\sqrt{(x)^2+\rho^2}}d\rho$$

$$=2\pi\int_0^{\infty}\frac{c^2t\left(\dfrac{\delta(\rho'-\sqrt{c^2t^2-x^2}}{\sqrt{c^2t^2-x^2}}\right)}{\sqrt{(x)^2+\rho^2}}d\rho$$ ooo when $c^2t^2-x^2>0$

$=2\pi c$ ooo again when $|ct|>|x|$

$=2\pi c\,\Theta(ct-|x|)$ ooo $\Theta(ct-|x|)=\begin{cases}0 & |ct|<0\\ 1 & |ct|>0\end{cases}$

The sheet source projects in the positive and negative directions. Rays direct in x-only

6.2. $\rho(x,t) = q\delta[x' - r(t')]$ $\quad J(x',t') = qv(t')\delta[x' - r(t')]$

a) $r(t) = (z_0 + v_z t)\hat{k}$

$t_{ret}(t,z) = t - z/c$ ∴ so $r(t) = (z_0 + v_z t_{ret}(t,z))\hat{k}$

$$\delta(x - r(t_{ret}(x,t))) = \delta(x)\delta(y)\delta\{z - [z_0 + v_z t_{ret}(t,z)]\}$$

$$= \delta(x)\delta(y)\delta\{z - [z_0 + v_z(t - z/c)]\}$$

$$= \delta(x)\delta(y)\delta\{z - z_0 - v_z t + \frac{v_z}{c} z\}$$

$$= \delta(x)\delta(y)\delta\{(1 + \frac{v_z}{c})z - |z_0 + v_z t|\}$$

$$= \left(\frac{1}{1 + v_z/c}\right)\delta(x)\delta(y)\delta\left\{z - \frac{z_0 + v_z t}{1 + v_z/c}\right\}$$

$$\int \delta^3(x - r(t_{ret}(x,t)))\, d^3x = \int \delta^3(x - x_0)\left|\frac{\partial}{\partial x}\left[x - r\left(t - \frac{|x - x'|}{c}\right)\right]\right|^{-1}_{x_i = r_i} \cdot d^3x$$

$$= \int \delta^3(x - x_0)\left[\delta_{ij} - \left(\frac{\partial r}{\partial t}\right)\left(\frac{1}{c}\frac{(x - x')}{|x - x'|}\right)\right]^{-1}_{x_i = r_i} \cdot d^3x$$

$$= \int \delta^3(x - x_0)\left[\delta_{ij} - v_j\left(\frac{1}{c}\frac{x_i - r_i(t)}{|x - r(t)|}\right)\right]^{-1}_{x_i = r_i} \cdot d^3x$$

$$= \int \delta^3(x - x_0)\left[\delta_{ij} - \frac{v_j(t)}{c} \circ R(x,t)\right]^{-1} \cdot d^3x$$

∴ where $R(x,t) = \dfrac{x_i - r_i(t)}{|x - r(t)|}$

$$= \int \delta^3(x - x') \left[\delta_{ij} - \frac{V_i(t)}{c} \circ R(x,t)\right]^{-1} \cdot d^3x$$

$$= \frac{1}{K} \quad \ldots \text{when } K = 1 - \frac{V_i(t) \circ R}{c}$$

b) (6.55) "Jiffimenko generalization of Coulombs Law"

$$E(x,t) = \frac{1}{4\pi\epsilon_0} \int d^3x \left\{ \frac{\hat{R}}{R^2} [\rho(x',t')]_{ret} + \frac{\hat{R}}{cR} \left[\frac{\partial \rho(x',t')}{\partial t'}\right]_{ret} - \frac{1}{c^2 R} \left[\frac{\partial J(x',t')}{\partial t'}\right]_{ret} \right\}$$

(6.56) "Jiffimenko generalization of Biosavarts law"

$$B(x,t) = \frac{\mu_0}{4\pi} \cdot \int d^3x \left\{ [J(x',t')]_{ret} \times \frac{\hat{R}}{R^2} + \left[\frac{\partial J(x',t')}{\partial t}\right]_{ret} \times \frac{\hat{R}}{cR} \right\}$$

$$\rho(x,t) = q\delta^3(x' - r(t)) \qquad J(x',t') = qv(t')\delta^3(x' - r(t'))$$

$$E(x,t) = \frac{1}{4\pi\epsilon_0} \int d^3x \left\{ \frac{\hat{R}}{R^2} q [\delta^3(x - r(t))]_{ret} + \frac{\hat{R}}{cR} \left[\frac{\partial}{\partial t'} q\delta^3(x - r(t))\right]_{ret} - \frac{1}{c^2 R} \left[\frac{\partial}{\partial t'} qv(t')\delta^3(x' - r(t'))\right]_{ret} \right\}$$

$$= \frac{q}{4\pi\epsilon_0} \int d^3x \left\{ \left[\frac{\hat{R}}{R^2} \delta^3(x' - r(t'))\right]_{ret} + \frac{\partial}{\partial t} \frac{\hat{R}}{cR} [\delta^3(x - r(t'))]_{ret} - \frac{\partial}{\partial t} \frac{1}{c^2 R} v(t')[\delta(x' - r(t'))]_{ret} \right\}$$

$$= \frac{q}{4\pi\epsilon_0} \left\{ \left[\frac{\hat{R}}{KR^2}\right]_{ret} + \frac{1}{c} \frac{\partial}{\partial t} \left[\frac{\hat{R}}{KR}\right]_{ret} - \frac{v}{c^2} \frac{\partial}{\partial t} \left[\frac{v(t')}{KR}\right]_{ret} \right\}$$

$$B(x,t) = \frac{1}{4\pi\epsilon_0} \int d^3x \left\{ q v(t')\delta^3(x'-r(t')) \times \frac{R}{R^2} + \frac{\partial}{\partial t} q v(t')\delta^3(x'-r(t)) \times \frac{\hat{R}}{cR} \right\}$$

$$= \frac{\mu_0 R}{4\pi}\left[\frac{v \times \hat{R}}{kR^2} + \frac{1}{c}\frac{\partial}{\partial t}\frac{v \times \hat{R}}{kR}\right]$$

c) $$E(x,t) = \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\hat{R}}{R^2}\right]_{ret} + \frac{[R]_{ret}}{c}\frac{\partial}{\partial t}\left[\frac{\hat{R}}{R^2}\right] + \frac{\partial^2}{c^2\partial t^2}[R]_{ret}\right\}$$

$$B(x,t) = \frac{\mu_0 q}{4\pi}\left\{\frac{\bar{v} \times R}{k^2 R^2} + \frac{1}{c[R]_{ret}}\frac{\partial}{\partial t}\frac{v \times \hat{R}}{k}\right\}$$

6.3. (5.160) $\nabla^2 A = \mu\sigma \frac{\partial A}{\partial t}$

General solution: $A(x,t) = \int d^3x' \underbrace{G(x-x', t)}_{\text{"Kernel"}} \cdot \underbrace{A(x',0)}_{\text{"Initial field configuration"}}$

a) (6.80) "Fourier Transform"

$$A(x,t) = \frac{1}{(2\pi)^3}\int \vec{A}(k,t) e^{ikx} d^3k$$

... when $A(x,t) = g(x,t)$
$A(k,t) = g(k,t)$

$$\nabla^2 A = \frac{\partial^2}{\partial x^2} A(x,t)$$

$$= \frac{\partial^2}{\partial x^2}\frac{1}{(2\pi)^3}\int \vec{A}(k,t) e^{ikx} d^3x$$

$$= |ik|^2 A(x,t)$$

$$= \mu\sigma\frac{\partial}{\partial t}A(x,t)$$

$$\frac{\partial}{\partial t}A(x,t) = \frac{-k^2}{\mu\sigma}A(x,t)$$

General Solution: $A(k,t) = A(k,0)\, e^{-k^2 t/\mu\sigma}$

Fourier Transform: $A(k,t) = \frac{1}{(2\pi)^3}\int A(k,0)\, e^{-k^2 t/\mu\sigma} \cdot e^{ikx}\, d^3x$

where $A(k,0) = \int A(x,0)\, e^{-ikx}\, d^3x$

$$A(x,0) = \frac{1}{(2\pi)^3}\int\int A(x',0)\, e^{-k^2 t/\mu\sigma} \cdot e^{ik(x-x_0)}\, d^3k\, d^3x$$

$$= \int G(x-x',t) \cdot A(x',0)\, d^3x$$

where $G(x-x',t) = \frac{1}{(2\pi)^3}\int e^{-k^2 t/\mu\sigma} \cdot e^{ik(x-x)}\, d^3x$

b) $\nabla^2 G = \mu\sigma \frac{\partial G}{\partial t}$ $\qquad \frac{\partial G}{\partial t} - \frac{1}{\mu\sigma}\nabla^2 G = \sigma^{(3)}(x-x')\sigma(t)$

... becomes

$$G(x,t) = \frac{1}{(2\pi)^4}\int G(k,w)\, e^{i(kx-wt)}\, d^3k\, dw$$

$$[(-iw) - |ik|^2/\mu\sigma]\, G(x,t) = e^{-ikx}$$

$$G(x,t) = \frac{e^{-ikx}}{[k^2/\mu\sigma - iw]}$$

$$G(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\vec{k},w)\, e^{-iwt}\, dw$$

$$= \frac{i e^{-ikx}}{2\pi} \int \frac{e^{-i\omega t}}{\omega + ik^2/\mu\sigma} \, d\omega$$

If $t>0$, then $G(k,t) = (-2\pi i)\dfrac{i e^{-ikx}}{2\pi} e^{-k^2 t/\mu\sigma}$

$$= e^{-k^2 t/\mu\sigma} \cdot e^{-ikx} \quad \text{...when } \omega = -ik^2/\mu\sigma$$

If $t<0$, then $G(x-x,t) = \dfrac{\Theta(t)}{(2\pi)^3} \displaystyle\int e^{-k^2 t/\mu\sigma} \cdot e^{ik(x-x')} \, d^3k$

$$= 0$$

c) $G(x-x';t) = \dfrac{\Theta(t)}{(2\pi)^3} \displaystyle\int e^{-k^2 t/\mu\sigma} \cdot e^{ik(x-x')} \, d^3k$

$\boxed{\text{Complete the square (with } k)}$

$$-\frac{k^2 t}{\mu\sigma} + ik(x-x') =$$

$$= \frac{-t}{\mu\sigma}\left(k - \frac{i\mu\sigma(x-x')}{2t}\right)^2 - \frac{\mu\sigma(x-x')^2}{4t}$$

$$= \frac{\Theta(t)}{(2\pi)^3} e^{-\mu\sigma|x-x'|^2/4t} \int e^{-t/\mu\sigma\left(k - \frac{i\mu\sigma(x-x')}{2t}\right)^2} \cdot d^3k$$

$$= \frac{\Theta(t)}{(2\pi)^3} \left(\frac{\pi\mu\sigma}{t}\right)^{3/2} \cdot e^{-\mu\sigma|x-x'|^2/4t}$$

$$= \Theta(t) \left(\frac{\mu\sigma}{4\pi t}\right)^{3/2} \cdot e^{-\mu\sigma|x-x'|^2/4t}$$

d) $D = 1/\mu\sigma$ and $\dfrac{\partial \rho}{\partial t} = \vec{\nabla}\cdot(D\vec{\nabla}\rho)$

$$A(x,t) = \int G(x-x';t)\, A(x';0)\, d^3x$$

$$= \Theta(t)\left(\frac{\mu\sigma}{4\pi t}\right)^{3/2} \cdot \int A(x,0) \cdot e^{-\mu\sigma(x-x')^2/4t} d^3x$$

If $|\vec{x}| = r >> d$, $|\vec{x} - x'|^2 \approx d^2$

$$A(r,t) = \Theta(t)\left(\frac{\mu\sigma d}{4\pi t}\right)^{3/2} e^{-\mu\sigma r^2/4t} \cdot \langle A \rangle$$

If $t << \frac{\mu\sigma}{4\pi}$, $A(r,t) \approx 0$

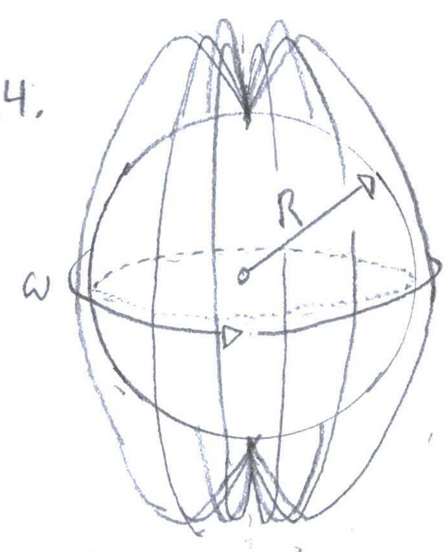
If $\frac{\mu\sigma r^2}{4} \leq t \leq 2\cdot\left(\frac{\mu\sigma r^2}{4}\right)$, $A(r,t) \propto \Theta(t)$

If $2\left(\frac{\mu\sigma r^2}{4}\right) << t$, $A(r,t) = \frac{\Theta(t)}{t^{3/2}}$

Note: The book defines $T_2 = 2\left(\frac{\mu\sigma r^2}{4}\right)$

$= 2 \cdot T_1$

6.4.



"A uniformly magnetized and conducting sphere"

Total magnetic moment

$m = 4\pi M R^3/3$

Steady state = No current

$\vec{J} = 0$

a) $\vec{m} = \vec{M} v$ when $v = \frac{4}{3}\pi R^3$ and $M = M z$

(5.105) $B_{in} = \frac{2\mu_0 M}{3}$

$= \frac{2}{3}\mu_0 \left(\frac{m}{v}\right)$

$= \frac{2}{3}\mu_0 \left(\frac{3}{4\pi R^3} m \hat{z}\right)$

$= \frac{\mu_0 m}{2\pi R^3} z$

(5.159) $J = \sigma E$ "Stationary Frame"

$= \sigma(E + v \times B)$ "Moving Frame"

$= 0$ "No current"

$E = -v \times B$

$= -(\vec{w} \times r) \times B$

$= -\frac{\mu_0 m}{2\pi R^3} (w \times r) \times z$

$= \frac{-\mu_0 m}{2\pi R^3} w \rho$ ... when $r = \rho$

$\boxed{\begin{array}{c}\underline{\text{Physics relation}} \\ v = w \times r\end{array}}$

$\boxed{\vec{B} = \frac{\mu_0 m}{2\pi R^3} \cdot \vec{z}}$

$\boxed{\begin{array}{c}\underline{\text{Gauss Law}} \\ \nabla \cdot E = \rho/\epsilon_0\end{array}}$

$\rho = \epsilon_0 \nabla \cdot E$

$= \epsilon_0 \frac{\partial E}{\partial \rho}$

$= -\frac{\mu_0 \epsilon_0 m w}{2\pi R^3} = -\frac{m w}{2\pi c^2 R^3}$ ... if $\frac{1}{c} = \sqrt{\mu_0 \epsilon_0}$

b) $E = -\nabla\phi$

$$\phi = -\int E \circ dl$$

$$= -\int E_\rho \cdot d\rho$$

$$= \frac{-\mu_0 m \omega}{2\pi R^3}\int_0^\rho \rho\, d\rho$$

$$= \frac{-\mu_0 m \omega \rho^2}{4\pi R^3}$$

$$= -\frac{\mu_0 m \omega r^2 \sin^2\theta}{4\pi R^3}$$

$$= \frac{-\mu_0 m \omega^2 r^2}{4\pi R^3}\left[\frac{2}{3}\left(1 - P_2(\cos\theta)\right)\right]$$

"Legendre's Polynomial 2nd order"

$$= -\frac{\mu_0 m \omega^2 r^2}{6\pi R^3}\left(1 - P_2(\cos\theta)\right)$$

$$= -\frac{\mu_0 m \omega^2 r^2}{6\pi R^3} + \frac{\mu_0 m \omega^2 r^2 P_2(\cos\theta)}{6\pi R^3}$$

(4.1) $$\phi(x) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\, q_{lm}\,\frac{Y_l^m(\theta,\phi)}{r^{l+1}}$$

$$\phi(r,\theta) = -\sqrt{\frac{4\pi}{5}}\,\frac{\mu_0 m \omega r^2}{6\pi R^3}\, Y_{2,0}(\theta,\phi)$$

So, $$q_{2,0} = -4\pi\epsilon_0\sqrt{\frac{5}{4\pi}}\,\frac{\mu_0 m \omega R^2}{6\pi}$$

$$= -\sqrt{\frac{5}{4\pi}}\,\frac{2 m \omega R^2}{3c^2}$$

$$Q_{33} = 2\sqrt{\frac{4\pi}{5}}\, q_{2,0}$$

$$= -\frac{4 m \omega R^2}{3c^2}$$

$$Q_{11} = Q_{22} = -\frac{1}{2} Q_{33}$$

c) $$\sigma = -\epsilon_0 \left[\frac{\partial \Phi_{out}}{\partial r} - \frac{\partial \Phi_{in}}{\partial r} \right]_{r=0}$$ "Boundary Condition"

$$= -\epsilon_0 \left[\frac{\mu_0 m \omega}{2\pi} \frac{R^2}{r^4} P_2(\cos\theta) - \frac{\mu_0 m \omega}{2\pi R^3} r \sin^2\theta \right]$$

$$= \frac{1}{4\pi R^2} \left(\frac{m\omega}{3c^2} \right) \left[1 - \frac{5}{2} P_2(\cos\theta) \right]$$

$$E = -\int_0^{\pi/2} E \cdot dl$$

$$= -R \int_0^{\pi/2} E_\theta \cdot d\theta$$

$$= \frac{\mu_0 m \omega}{2\pi R} \int_0^{\pi/2} \cos\theta \cdot \sin\theta \, d\theta$$

$$= \frac{\mu_0 m \omega}{4\pi R}$$

6.5 a) (6.117) $$P_{field} = \epsilon_0 \int_V E \times B \, d^3x$$

$$= \mu_0 \epsilon_0 \int_V E \times H \, d^3x$$

$$= -\frac{1}{c^2} \int_V \nabla\Phi \times H \, d^3x$$

$$= -\frac{1}{c^2} \int_V \left[\frac{\partial(\Phi H)}{\partial x} - \Phi \frac{\partial H}{\partial x} \right] d^3x$$

$$= \frac{1}{c^2} \int \Phi \cdot J \, d^3x$$

b) $$\Phi = \Phi(0) - r \cdot E(0) + \cdots$$
$$= -r \cdot E(0)$$

So, $P_k = \frac{-1}{c^2} E(0) \int r \circ J \, d^3x$

$$= \frac{E(0)}{c^2} \cdot J \cdot V \quad \ldots \text{ when } \quad m = \frac{1}{\mu_0} BV \quad \text{and} \quad B = \mu_0 H$$

$$= \frac{E(0)}{c^2} m$$

$$= \frac{1}{c^2} [E(0) \times m]$$

c) $P = \epsilon_0 \int E \times B \, d^3x$

$$= \epsilon_0 \int E \times \left(\frac{\mu_0}{4\pi} \left[\frac{3n(n \circ m) - m}{|x|^3} \right] \right) d^3x \quad \text{by "Magnetic induction of a loop"}$$

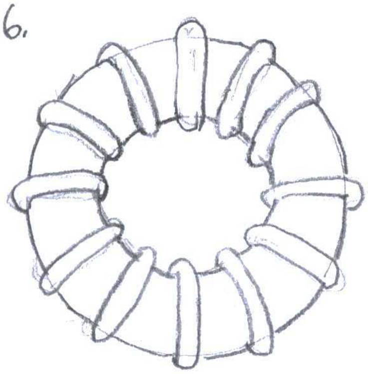
$$= \frac{E}{4\pi c^2} \left(\frac{4}{3} \pi x^3 \right) \circ \frac{2}{|x|}$$

$$= \frac{2 E \times m}{3c^2}$$

(6.117) $P_{field} = \epsilon_0 \int E \times B \, d^3x$

$= \mu_0 \epsilon_0 \int E \times H \, d^3x$ .. is an exact result to part c.

6.6.



"Circular toroidal coil"

a) (1.1b) Gauss' Law:

$$\nabla \circ E = \frac{\rho(x)}{\epsilon_0}$$

$$E \int dS = \frac{q}{\epsilon_0}$$

$$E \int_0^{2\pi} \int_0^{\pi} \int_0^{2a} r \sin\theta \, dr \, d\theta \, d\phi = \frac{q}{\epsilon_0}$$

$$E = \frac{Q}{4\pi \epsilon_0 a^2}$$

$$\hat{r} = (\cos\phi \cdot \sin\theta, \sin\phi \cdot \sin\theta, \cos\theta)$$

$$\hat{\phi} = (-\sin\phi, \cos\phi, 0)$$

$$E \times H = \pm \frac{IN}{2\pi a} \frac{Q}{4\pi\epsilon_0 a^2} (-\cos\phi\cos\theta, \sin\phi\cos\theta, \sin\theta)$$

$$\cong \pm \frac{IN}{2\pi a} \frac{Q}{4\pi\epsilon_0 a^2} (\theta, \theta, 1)$$

$$P = \mu_0\epsilon_0 \int E \times H \, d^3x$$

$$= \mu_0\epsilon_0 \int \pm \frac{IN}{2\pi a} \frac{Q}{4\pi\epsilon_0 a^2} (-\cos\phi\cos\theta, \sin\phi\cos\theta, \sin\theta) d^3x$$

$$= \frac{\mu_0 IN}{2\pi a} \frac{Q}{4\pi a^2} \int_0^{2\pi} \int_0^{a} \int_0^{A} d\phi \, d\theta \, dz$$

$$= \frac{\mu_0 IN}{2\pi a} \frac{Q}{4\pi a^2} (2\pi a A)$$

$$= \pm \frac{INQA\mu_0}{4\pi a^2}$$

A comparison to Problem 6.5b: $P = \frac{1}{c^2} E(0) \times m$

$$= \pm \frac{N}{c^2} \frac{QIA}{4\pi\epsilon_0 a^2}$$

$$= \pm \frac{INQA\mu_0}{4\pi a^2}$$

b) $Q = 10^{-6}C$; $N = 2000$; $a = 0.1m$

$I = 1.0A$; $A = 10^{-4}m^2$;

$$E = \frac{10^{-6}C}{4\pi(8.854 \times 10^{-12} m^{-3} Kg^{-1} s^4 A^2)(0.01 m^2)}$$

$$= 898774V$$

$$B = \frac{1A \times 2000 \times 1.256 \times 10^{-6}}{2\pi \cdot 0.1m}$$

$$= 4 \times 10^{-3} T$$

$$P = \frac{200\times10^{-6}C \times 10^{-4}m^2 \times 1.25\times10^{-6}}{4\pi(0.1m)^2}$$

$$= 2\times10^{-12}\ N\cdot s$$

$$P_{insect} = m\cdot V$$

$$= 10\mu g \times 0.1 m/s$$

$$= 10^{-7}\ N\cdot s$$

$$> P$$

6.7.

"Point charge with a velocity"

a) (6.92) $D_\alpha = \epsilon_0 E_\alpha + P_\alpha - \sum_\beta \frac{\partial Q'_{\alpha\beta}}{\partial x_\beta} + \cdots$

(6.97) $J(x,t) = \left\langle \sum_j q_j v_j \delta(x-x_j) + \sum q_n v_n \delta(x-x_n) \right\rangle$

(6.98) $M(x,t) = \left\langle \sum_n m_n \delta(x-x_n) \right\rangle$

(6.92)

(6.96) $\langle j_\alpha(x,t)\rangle = J_\alpha(x,t) + \frac{\partial}{\partial t}\left[D_\alpha(x,t) - \epsilon_0 E_\alpha(x,t)\right]$

$+ \sum_{\beta\gamma} \epsilon_{\alpha\beta\gamma} \cdot \frac{\partial}{\partial x_\beta} M_\gamma(x,t)$ (6.98)

$+ \sum_\beta \frac{\partial}{\partial x_\beta} \left\langle \sum_{n \atop \text{(molecules)}} \left[(p_n)_\alpha (v_n)_\beta - (p_n)_\beta (v_n)_\alpha\right]\right.$

$- \frac{1}{6} \sum_{\beta\gamma} \frac{\partial^2}{\partial x_\beta \partial x_\gamma} \left\langle \sum_n \left[(Q_n)_{\alpha\beta}(v_n)_\gamma \right.\right.$

$\left.\left. - (Q_n)_{\gamma\beta}(v_n)_\alpha\right]\delta(x-x_n)\right\rangle + \cdots$

(6.97)

<u>Taylor Expansion:</u>

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$\langle j_n(x,t)\rangle = J(x,t) + \sum q_i \left[x_{jn} \cdot \nabla f(x-x_n) + -\sum (x_{jn})_\alpha (x_{jn})_\beta \frac{\partial^2}{\partial x_\alpha \partial x_\beta} f(x-x_n)\right]$$

$$= J(x,t) + \Big[-q_i \cdot x_{jn} \circ \nabla f(x-x_n) + \frac{q_i}{2}(x_{jn})_\alpha (x_{jn})_\beta \frac{\partial^2}{\partial x_\alpha \partial x_\beta} f(x-x$$

$$- \frac{q_i}{6}\sum (Q_n)_{\alpha\beta} \frac{\partial^2 f(x-x_n)}{\partial x_\alpha \partial x_\beta} \Big]$$

$$= J(x,t) - q_i \cdot x_{jn} \circ \nabla f(x-x_n) + \frac{q_i}{2}(x_{jn} \times v_{jn})$$

$$- \frac{1}{6}\sum \frac{\partial^2}{\partial x_\beta \partial x_\gamma} \langle \sum (Q_n')\delta(x-x_n) \rangle$$

b) $\frac{1}{\mu_0}B - H = M + (D - \epsilon_0 E) \times v$

(6.63) $D = \epsilon_0 E + P \qquad H = \frac{1}{\mu_0}B - M$

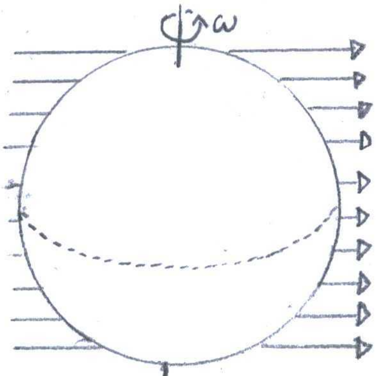
(6.69) $$\left(\frac{1}{\mu_0}B - H\right)_\alpha = M_\alpha + \langle \sum{}' (p \times v_n)_\alpha \delta(x-x_n) \rangle$$

$$- \frac{1}{6}\sum \epsilon_{\alpha\beta\gamma} \frac{\partial}{\partial x_\beta} \langle \sum{}' (Q_n')_{\gamma\beta} (v_n)_\gamma \delta(x-x_n$$

(6.89) $$P(x,t) = \langle \sum_n{}' p_n \delta(x-x_n) \rangle$$

$$\frac{1}{\mu_0}B - H = M + P \times v$$

$$= M + (D - \epsilon_0 E) \times v$$

6.8. 

"A dielectric sphere in a uniform electric field"

<u>From Section 4.4:</u>

$$\sigma_{pol} = 3\left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\right)\epsilon_0 \cdot E_0 \hat{r} \cdot \hat{x}$$

$$= 3\left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\right)\epsilon_0 E_0 \sin\theta \cos\phi \quad \text{"spherical coordinate}$$

$$= 3\left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\right)\epsilon_0 E_0 \left(\frac{x}{a}\right) \quad \text{"rectangular coordinates"}$$

$$K_M = \sigma v$$

$$= \sigma \omega \times r$$

$$= \sigma \omega \hat{z} \times a \hat{r}$$

$$= \sigma \omega a \sin\theta\, \hat{\phi}$$

$$= 3\left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\right)\epsilon_0 E_0 \omega\, x \sin\theta\, \hat{\phi}$$

$$K = M \times \hat{r}$$

$$= M \sin\theta\, \hat{\phi}$$

$$M \sin\theta\, \hat{\phi} = 3\left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\right)\epsilon_0 E_0 \omega\, x \sin\theta\, \hat{\phi}$$

$$M = 3\left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\right)\epsilon_0 E_0 \omega\, x\, \hat{z}$$

$$\sigma = 3\left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\right)\epsilon_0 E_0 \frac{\omega}{a}\, x\, \hat{z}$$

(3.70) $$\frac{1}{|x - x'|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta', \phi') Y_{lm}(\theta, \phi)$$

(4.1.5) $$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(x')}{|x - x'|} d^3x$$

$$= \frac{1}{4\pi}\int \frac{\sigma_m}{|x - x'|} da'$$

$$= 3\left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\right)\epsilon_0 E_0 \frac{\omega}{a}\int_0^{\pi}\int_0^{2\pi} a^2 \sin\theta' \cos\theta' \cos\phi'$$

$$\times 4\pi \sum\sum \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta', \phi') Y_{lm}(\theta, \phi) a^2 \sin\theta\, d\theta$$

... if $x = z = a$

$$= 3\left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\right)\epsilon_0 E_0 \omega a^3 \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta, \phi)$$

$$\times \int_0^{\pi}\int_0^{2\pi} \sin\theta' \cos\theta' \cos\phi'\, Y_{lm}(\theta', \phi') \sin\theta'\, d\theta' d\phi'$$

... when $\sin\theta\cos\theta\cos\phi = \sqrt{\frac{2\pi}{15}}\left(Y_{2,-1} - Y_{2,1}\right)$

$$= 3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0 \omega a^3 \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\phi)$$

$$\times \int_0^{\pi}\int_0^{2\pi} \sqrt{\frac{2\pi}{15}}\left(Y_{2,-1} - Y_{2,1}\right) Y_{lm}^*(\theta',\phi')\sin\theta' d\theta' d\phi'$$

$$= 3\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0 \omega a^3 \sum\sum \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\phi) Y_{lm}^*(\theta',\phi')$$

$$\times \sqrt{\frac{2\pi}{15}}\left[\delta_{l,2}\delta_{m,-1} - \delta_{l,2}\delta_{m,1}\right]$$

@ $l=2$ $$= \frac{3}{5}\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0 \omega a^3 \frac{r_<^2}{r_>^3}\frac{x\cdot z}{r^2}$$

If $r_< = r$ and $r_> = a$, then

$$\phi = \frac{3}{5}\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0 \omega x z$$

If $r_< = a$ and $r_> = r$, then

$$\phi = \frac{3}{5}\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0 \omega \frac{a^5}{r^5} x\cdot z$$

In total,

$$\phi_m = \frac{3}{5}\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)\epsilon_0 E_0 \omega \left(\frac{a}{r_>}\right)^2 x z$$

6.9. Poynting vector:

(6.103/6.104) $$\int_V J\cdot E\, d^3x = \int\left[E\cdot(\nabla\times H) - E\cdot\frac{\partial D}{\partial t}\right]d^3x$$ "Ampere's Law relation"

(6.105) $$= -\int_V\left[\nabla\cdot(E\times H) + E\cdot\frac{\partial D}{\partial t} + H\cdot\frac{\partial B}{\partial t}\right]d^3x$$

"Faraday's Law relation"

(6.106) $u = \frac{1}{2}(E \cdot D + B \cdot H)$

(6.107) $-\int_V J \cdot E\, d^3x = \int_V \left[\frac{\partial u}{\partial t} + \nabla \cdot (E \times H)\right] d^3x$

(6.108) $-J \cdot E = \frac{\partial u}{\partial t} + \nabla \cdot S$

(6.109) $S = E \times H$

Energy Density:

(5.84) $B = \mu H$

(4.37) $D = \epsilon E$

(6.106) $u = \frac{1}{2}(E \cdot D + B \cdot H)$

$= \frac{1}{2}(\epsilon E^2 + \mu H^2)$

Field-momentum Density:

(6.118) $g = \frac{1}{c^2}(E \times H)$

$= \mu \epsilon (E \times H)$

$\boxed{c = \frac{1}{\sqrt{\mu \epsilon}}}$

Maxwell Stress Tensor:

(6.120) $T_{\alpha\beta} = \epsilon_0\left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}(E \cdot E + c^2 B \cdot B)\delta_{\alpha\beta}\right]$

$= \left[\epsilon E_i E_j + \mu H_i H_j - \frac{1}{2}\delta_{ij}(\epsilon E^2 + \mu H^2)\right]$

∴ when $\alpha = i$ and $\beta = j$

Position effects the Maxwell stress tensor by ϵ and μ. Energy per meter^3 or Magnetic field per meter^3 describes force per area, a common definition about stress.

$$(6.106) \quad u = \frac{1}{2}(E \cdot D + B \cdot H)$$

$$(6.107) \quad -\int_V J \cdot E \, d^3x = \int_V \left[\frac{\partial u}{\partial t} + \nabla \cdot (E \times H)\right] d^3x$$

$$(6.108) \quad -J \cdot E = \frac{\partial u}{\partial t} + \nabla \cdot S$$

$$(6.109) \quad S = E \times H$$

Energy Density:

$$(5.34) \quad B = \mu H$$

$$(4.37) \quad D = \epsilon E$$

$$(6.106) \quad u = \frac{1}{2}(E \cdot D + B \cdot H)$$

$$= \frac{1}{2}(\epsilon E^2 + \mu H^2)$$

Field-momentum Density:

$$(6.118) \quad g = \frac{1}{c^2}(E \times H)$$

$$S = \mu\epsilon(E \times H)$$

$$\boxed{c = \frac{1}{\sqrt{\mu\epsilon}}}$$

Maxwell Stress Tensor:

$$(6.120) \quad T_{\alpha\beta} = \epsilon_0\left[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}(E \cdot E + c^2 B \cdot B)\delta_{\alpha\beta}\right]$$

$$= \left[\epsilon E_i E_j + \mu H_i H_j - \frac{1}{2}\delta_{ij}(\epsilon E^2 + \mu H^2)\right]$$

$\therefore$ when $\alpha = i$ and $\beta = j$

Position effects the Maxwell stress tensor by ϵ and μ. Energy per meter^3 or Magnetic field per meter^3 describes force per area, a common definition about stress.

6.10. "Field Angular Momentum" $\mathcal{L}_{field} = X \times g$

$$= \mu \epsilon \cdot X \times (E \times H)$$

"Flux of Angular Momentum" $\ddot{M} = \ddot{T} \times X$

"Mechanical Momentum"

(6.114) $$\frac{dP_{mech}}{dt} = \int_V (\rho E + J \times B) d^3x$$

"Torque"

$$\frac{dL_{mech}}{dt} = \int_V X \times (\rho E + J \times B) d^3X$$

$$= \int_V X \times \epsilon_0 \left[E(\nabla \cdot E) + B \times \frac{\partial E}{\partial t} - c^2 B \times (\nabla \times B) \right] d^3X$$

$$= \int_V X \times \epsilon_0 \left[E(\nabla \cdot E) - \frac{\partial}{\partial t}(E \times B) + E \times \frac{\partial B}{\partial t} - c^2 B \times (\nabla \times B) \right] d^3X$$

$\therefore$ when $c^2 B(\nabla \cdot B) = 0$

and $\nabla \times E = -\frac{\partial B}{\partial t}$ "Faradays Law"

$$= \int_V X \times \epsilon_0 \left[E(\nabla \cdot E) + c^2 B(\nabla \cdot B) - E \times (\nabla \times E) - c^2 B \times (\nabla \times B) - \frac{\partial}{\partial t}(E \times B) \right] d^3X$$

$$= \int_V X \times \left[\epsilon E(\nabla \cdot E) - \epsilon E \times (\nabla \times E) + \mu H(\nabla \cdot H) - \mu H \times (\nabla \times H) - \mu \epsilon \times \frac{\partial}{\partial t}(E \times H) \right] d^3x$$

"Conservation Law"

(6.121) $$\frac{\partial}{\partial t}(\mathcal{L}_{mech} + \mathcal{L}_{field}) = \frac{\partial}{\partial t}\left(\int_V (\mathcal{L}_{mech} + \mathcal{L}_{field}) d^3x \right)$$

$$= \int_V X \times \left[\epsilon E(\nabla \cdot E) - \epsilon E \times (\nabla \times E) + \mu H(\nabla \cdot H) - \mu H \times (\nabla \times H) \right] d^3X$$

$$(6.120)\quad T_{11} = \frac{\epsilon_0}{2}\left[E^2 + c^2B^2\right]$$

$$T_{22} = \frac{\epsilon_0}{2}\left[E^2 - c^2B^2\right]$$

$$T_{33} = \frac{\epsilon_0}{2}\left[-E^2 + c^2B^2\right]$$

$$\frac{dP_{mech}}{dt} = \int \frac{d}{dx_\beta} T_{11}\, d^3x - \epsilon_0 \frac{d}{dt}\int (E \times B)\, d^3x$$

$$= -\int \left[\frac{d}{dx_\beta}\left(E^2 + c^2B^2\right) + \epsilon_0 \frac{d(E\circ B)}{dt}\right] d^3x$$

$$|\text{Pressure}| = \left| \int_{-\infty}^{0} \frac{d}{dv} \frac{d(P_{mech})}{dt} \right|$$

$$= \left| -\frac{\epsilon_0}{2}\left[E^2 + c^2B^2\right] + \epsilon_0 \int_0^\infty dx \frac{d(E\circ B)}{dt} \right|$$

∘∘ when constant $E(t)$ and $B(t)$ "time averaged"

$$= \frac{\epsilon_0}{2}\left[E^2 + c^2B^2\right]$$

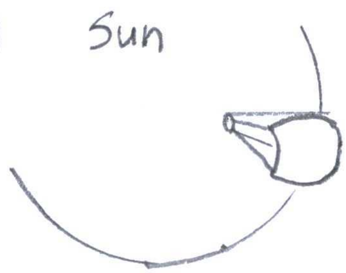
(6.106) "Total Energy Density"

$$u = \frac{1}{2}(E\circ D + B\circ H)$$

$$= \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2$$

$$= \text{Pressure}$$

b)



"Interplanetary sail plane near the sun"

$$(6.190) \quad \left[E(\nabla\cdot E) - E\times(\nabla\times E)\right] = \sum \frac{\partial}{\partial x_\beta}\left[(E_\alpha B_\beta) - \frac{1}{2}E\cdot E\,\delta_{\alpha\beta}\right.$$

$$\text{So,} \quad \left[B(\nabla\cdot B) - B\times(\nabla\times B)\right] = \sum \frac{\partial}{\partial x_\beta}\left[(B_\alpha B_\beta) - \frac{1}{2}B\cdot B\,\delta_{\alpha\beta}\right.$$

$$(6.120) \quad T_{k\ell} = \epsilon_0\left[E_k E_\ell + c^2 B_k B_\ell - \frac{1}{2}(E\cdot E + c^2 B\cdot B)\right]$$

Then
$$\frac{d}{dt}\left(\mathcal{L}_{mech} + \mathcal{L}_{field}\right) = \int_V \epsilon_{ijk}\, \rho_i\, x_j \frac{\partial}{\partial x_\ell} T_{k\ell}\, d^3x$$

$$= \int_V \epsilon_{ijk}\, \rho_i \left[\frac{\partial}{\partial x_\ell}(\epsilon_{ijk} x_j T_{k\ell}) - \epsilon_{jk} T_{kj}\right] d^3x$$

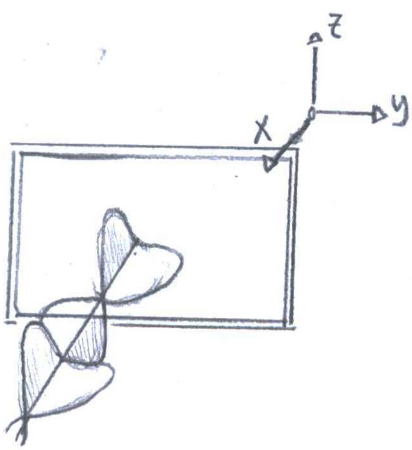
$$= \int_V \nabla\cdot(x\times T)\, d^3x$$

$$= -\int \nabla\cdot M\, d^3x$$

$$\frac{d}{dt}\left(\mathcal{L}_{mech} + \mathcal{L}_{field}\right) + \nabla\cdot M = 0$$

$$\frac{d}{dt}\int_V \left(\mathcal{L}_{mech} + \mathcal{L}_{field}\right) d^3x + \int_S n\cdot M\, da = 0$$

6.11.



"A transverse plane incident to an absorbing flat screen."

a) $E = E\hat{y}$ and $B = B\hat{z}$

$$\frac{d}{dt}\left(P_{mech} + P_{field}\right) = \sum_\beta \int_V \frac{\partial}{\partial x_\beta} T_{\alpha\beta}\, d^3x$$

$$\frac{dP_{mech}}{dt} = \sum \int_V \left(\frac{\partial}{\partial x_\beta} T_{\alpha\beta} - \frac{dP_{field}}{dt}\right) d^3x$$

$$= \sum \int \frac{\partial}{\partial x_\beta} T_{\alpha\beta}\, d^3x - \epsilon_0 \frac{d}{dt}\int (E\times B)\, d^3x$$

$$\text{Force} = \sum \frac{d}{dV}\frac{d(P_{mech})}{dt}$$

$$= \sum \frac{d}{\partial x_\beta} T_{\alpha\beta} - \epsilon_0 \sum \frac{\partial(E\times B)}{dt}$$

$$= \frac{1}{4}(G - iB) \cdot V^2$$

$$G - iB = \frac{2}{V^2} \cdot \left(\int_V J^* \cdot E \, d^3x + 2i\omega \int_V (w_e - w_m) \, d^3x + \oint_S S \cdot n \, da \right)$$

Conductance $= G$

$$= \frac{1}{V^2} \int J^* \cdot E \, d^3x + 2 \int S \cdot n \, da$$

Susceptance $= B$

$$= -\frac{4}{V^2} \omega \int (w_e - w_m) \, d^3x$$

b) (6.139) $\quad R = \frac{1}{|I^2|} \int_V \sigma |E|^2 d^3x$

(6.140) $\quad X = \frac{4\omega}{|I^2|} \int_V (w_m - w_e) \, d^3x$ ... at low frequencies $J = \sigma E$

$$G = \frac{1}{V^2} \int \sigma |E|^2 d^3x$$

$$B = -\frac{4\omega}{V^2} \int (w_e - w_m) \, d^3x$$

Lesson: Impedance generate in electrical terminals at specific frequencies. A large impedance raises voltage and electrical current into a magnetic field. Essentially, a resonant circuit or antenna.

$$P = u = \frac{S}{c} = \frac{1.4 \times 10^{3}\, W/m^{2}}{3 \times 10^{8}\, m/s} = 5 \times 10^{-6}\, N/m^{2}$$

$$a = \frac{P \cdot A}{m} = \frac{5 \times 10^{-6}\, N/m^{2}}{10^{-3}\, Kg/m^{2}} = 5 \times 10^{-3}\, m/s^{2}$$

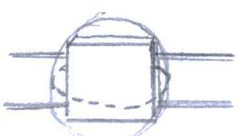
Corpuscular Radiation - a particle radiation by fast moving subatomic particles

$$a \simeq \frac{(\text{Velocity} \times \text{Density}) \circ \text{mass proton} \circ \text{velocity}}{\text{mass sailplane}}$$

$$\simeq \frac{5 \times 10^{5}\, m/s \times 3 \times 10^{7}\, 1/m^{3} \times 1.6 \times 10^{-27}\, Kg \times 7 \times 10^{5}\, m/s}{10^{-3}\, Kg/m^{2}}$$

$$\approx 2 \times 10^{-6}\, m/s^{2}$$

6.12



"Two Terminal linear passive network"

a) (6.134) "Power"

$$= \frac{1}{2}\int_V J^{*} \cdot E\, d^{3}x + 2i\omega \int_V (w_e - w_m)\, d^{3}x + \oint_S S \cdot n\, da$$

$$= \frac{1}{2} I_i^{*} \cdot V_i$$

$$= 0$$

Voltage = Impedance x Current

$$V = Z \circ I$$

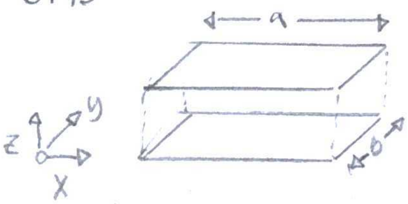
$$I = V \circ \frac{1}{Z}$$

$$= V \circ Y$$

$$= V \circ (G - iB)$$

$$\text{Power} = \frac{1}{2} I_i^{*} \circ V_i$$

6.13



"A parallel plate capacitor of two flat rectangular sheets"

a) (1.1b) "Amperes Law"

$$\nabla \times B = \mu_0 J + \frac{1}{c^2}\frac{\partial E}{\partial t} \quad \ldots \text{if } J=0$$

$$= \frac{1}{c^2}\frac{\partial E}{\partial t}$$

General Solution: $E = \left(\frac{-V}{d}\right) e^{-i\omega t} \cdot \hat{z}$

$$\nabla \times B^{(1)} = \frac{i\omega}{c^2}\frac{V}{d} e^{-i\omega t}\hat{z} \quad @\, t=0$$

$$= \frac{i\omega}{c^2}\frac{V}{d}\hat{z}$$

$$B_1 = \frac{i\omega}{c^2}\frac{V}{d}(x-x_0)\hat{y}$$

(1.1a) "Faradays Law"

$$\nabla \times E^{(2)} = -\frac{\partial B}{\partial t}$$

$$= \frac{i\omega^2}{c^2}\frac{V}{d}$$

$$E_2 = \frac{\omega^2}{c^2}\frac{V}{d}\hat{z}\int (x-a)\,dx$$

$$= \frac{\omega^2}{c^2}\frac{V}{d}\left[\frac{1}{2}(x-a)^2 - a^2\right]\hat{z}$$

"Ampere's-Maxwell Equation"

$$\nabla \times B^{(3)} = \frac{-i\omega}{c^2}E_2$$

$$= \frac{-i\omega^2}{c^4}\frac{V}{2d}\left[\frac{1}{3}(x-a)^3 - a^2(x-a)\right]$$

$$E = E^{(0)} + E^{(1)} + \ldots$$

$$= -\frac{V}{d}\left(1 - \frac{\omega^2}{2c^2}\left[(x-a)^2 - a^2\right] + \ldots\right)\hat{z}$$

$$B = B^{(1)} + B^{(3)} + \cdots$$

$$= \frac{iV\omega}{d}\frac{(x-a)}{c^2}\left(1 - \frac{\omega^2}{2c^2}\left[\frac{1}{3}(x-a)^2 - a^2\right] + \cdots\right)\hat{y}$$

$$E \approx -\frac{V}{d}\left(1 - \frac{\omega^2}{2c^2}\left[(x-a)^2 - a^2\right]\right)\cos(\omega t) \cdot z$$

$$B \approx \frac{V}{d}\frac{\omega}{c^2}(x-a)\left(1 - \frac{\omega^2}{2c^2}\left[\frac{1}{3}(x-a)^2 - a^2\right]\right)\sin(\omega t) y$$

b) (6.140)

$$X = \frac{4\omega}{|I_1|^2}\int_V (w_m - w_e)\, d^3x$$

$$E^{(0)} = -\left(\frac{V}{d}\right)e^{-i\omega t}\hat{z} \qquad B^{(0)} = \frac{i\omega}{c^2}\frac{V}{d}(x-a)\hat{y}$$

$$w_e = \frac{\epsilon_0}{4}|E|^2 \simeq \frac{\epsilon_0 |V_i|^2}{4d^2} \qquad w_m = \frac{1}{4\mu_0}|B|^2 \simeq \frac{\omega^2}{4\mu_0}\frac{|V_i|^2}{c^4 d^2}(x-a)^2$$

$$E \simeq \frac{-V}{d}\sigma = \epsilon_0 \frac{V}{d} \qquad Q = \sigma \times \text{Area} = \frac{\epsilon_0 V a b}{d}$$

$$I = \frac{dQ}{dt} = -i\omega Q = -\frac{i\omega \epsilon_0 V a b}{d}$$

So, $w_e = \dfrac{|I_1|^2}{4\epsilon_0 \omega^2 a^2 b^2}$

and

$$w_m = \frac{\mu_0 |I|^2}{4a^2b^2}(x-a)^2$$

$$\int w_e d^3x = \frac{|I|^2 d}{4\epsilon_0 \omega^2 ab} \qquad \int w_m d^3x = \frac{\mu_0 |I|^2 d}{4a^2 b}\int_0^a (x-a)dx$$

$$= \frac{\mu_0 |I|^2 ad}{12b}$$

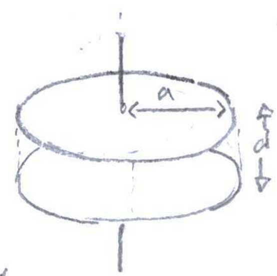
$$X = \frac{\mu_0 \omega ad}{3b} - \frac{d}{\epsilon_0 \omega ab}$$

$= \omega L$ "in inductance"

$= \frac{-1}{\omega C}$ "in capacitance"

$$C = \frac{\epsilon_0 ab}{d} \qquad L = \frac{\mu ad}{3b}$$

6.14.



"A capacitor in an AC circuit with circular plates"

$$\nabla \times E = -\frac{\partial E}{\partial \rho} = i\omega B_\phi$$

$$\nabla \times B = \frac{1}{\rho}\frac{\partial \rho B_z}{\partial \rho} = \frac{-i\omega}{c^2} E_z$$

$$= -\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{E}{\omega i}\right)$$

$$\frac{1}{\rho}\frac{\partial E}{\partial \rho} + \frac{\omega^2}{c^2}E_z = 0$$

$$\frac{\partial^2 E}{\partial^2 \rho} + \frac{1}{\rho}\frac{\partial E}{\partial \rho} + \frac{\omega^2}{c^2}E_z = 0$$

General Solution: $E_z(\rho) = AJ_0(k\rho)$

$$B_\phi(\rho) = \frac{i}{kc}\frac{\partial E}{\partial \rho}$$

$$= -\frac{i}{c}AJ_1(k\rho)$$

where $k^2 = \omega^2/c^2$

(16.1) $\sigma(\rho) = \epsilon_0 E_z(\rho)$

$= \epsilon_0 A\, J_0(k\rho)$

( ) $Q = 2\pi\epsilon_0 A \int_0^a \rho\, J_0(k\rho)\, d\rho$

$= 2\pi\epsilon_0 \left(\frac{a}{k}\right) A\, J_1(ka)$

Lesson: ① Maxwell's Equations are iterative, specifically Faraday's Law and Ampere's Law with no current.

$$\nabla \times E^{(1)} = -\frac{\partial B^{(1)}}{\partial t} \rightarrow \nabla \times B^{(1)} = \frac{1}{c^2}\frac{\partial E^{(2)}}{\partial t} \rightarrow \nabla \times E^{(2)} = \dots$$

② Capacitors have a relationship to the initial charge (Q_0).

$$Q_0 = \frac{i I_0}{\omega} \qquad A = \frac{k A_0}{2\pi a \epsilon_0 J_1(ka)}$$

$$E^{(2)}(\rho) = \frac{Q_0}{\pi a^2 \epsilon_0}\left[1 + \left(\frac{a^2}{8} - \frac{\rho^2}{4}\right)k^2 + \dots\right]$$

$$B^{(2)}(\rho) = \frac{\mu_0 I_0 \rho}{2\pi a^2}\left[1 + \left(\frac{a^2}{8} - \frac{\rho^2}{8}\right)k^2 + \dots\right]$$

... where $I_0 = -i\omega Q_0$

b) $\omega_e^{(2)} = \frac{\epsilon_0}{4} 2\pi d \int_0^a \rho |E_z|^2 d\rho \qquad \omega_m^{(2)} = \frac{1}{4\pi\mu_0} 2\pi d \int_0^a \rho |B_\rho|^2 d\rho$

$= \frac{1}{4\pi\epsilon_0} \frac{|I_0|^2 d}{\omega^2 a^2} \qquad = \frac{\mu_0}{4\pi}\frac{|I_0|^2 d}{8}\left(1 + \frac{a^2k^2}{12}\right)$

c) (6.140) $X_L = \omega L$

$$= \frac{4\omega}{|I_0|^2} W_m = \frac{\omega \mu_0 d}{8\pi}$$

$$X_c = \frac{1}{\omega C}$$

$$= \frac{4\omega}{|I_0|^2} W_e = \frac{d}{\omega \epsilon_0 \pi a^2}$$

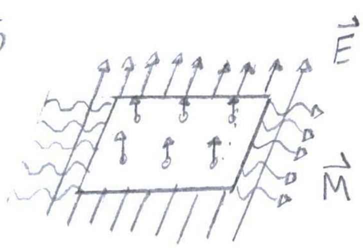
$$C = \frac{\epsilon_0 \pi a^2}{d} \qquad L = \frac{\mu_0 d}{8\pi}$$

Resonant Frequency: $\omega_0 = \frac{1}{\sqrt{LC}}$

$$= \sqrt{\frac{8}{\epsilon_0 \mu a^2}}$$

$$= \sqrt{8}\,\frac{c}{a}$$

6.15



"A semiconductor with an applied electric field and transverse magnetic field".

Pg 272, Symmetry Properties:

- Rotational and Spatial Inversion from <u>even</u> time-derivatives
- Time Reversal from <u>odd</u> time-derivatives.

B_0 [1st order]
- Rotational inversion, spatial inversion, and time reversal

B_0 [2nd order]
- Rotational inversion, spatial inversion,

(5.159) "Ohms Law"

$$J = \sigma E$$

$$E = \frac{J}{\sigma}$$

$$= \rho J$$

$$\boxed{\begin{array}{l}\underline{\text{Taylor Expansion:}} \\ f(x) = \sum \frac{f^n(x)}{n!} x^n\end{array}}$$

$$= \frac{\gamma J H^0}{0!} + \frac{\alpha (J \circ H)^1}{1!} + \frac{\beta \cdot J \circ H^2}{2!}$$

$$\gamma = \rho \qquad \alpha = R \qquad \frac{\beta}{2!} = (\beta_1 + \beta_2)$$

$$= \underbrace{\rho J}_{\text{odd}} + \underbrace{R(H \times R)}_{\text{odd}} + \underbrace{\beta_1 H^2 J}_{\text{odd}} + \underbrace{\beta_2 (H \cdot J) H}_{\text{odd}}$$

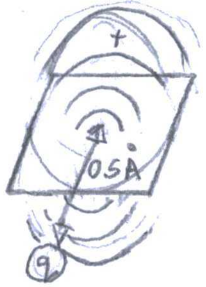
= Acceptable rotational and spatial inversion

b)

$$E = \underbrace{\rho J}_{t\text{-odd}} + \underbrace{R(H \times R)}_{t\text{-even}} + \underbrace{\beta_1 H^2 J}_{t\text{-odd}} + \underbrace{\beta_2 (H \cdot J) H}_{t\text{-odd}}$$

= $R(H \times R)$ is the acceptable term in time reversal.

6.16.



"Dirac monopole... from a median plane of a magnetic dipole"

a) (6.113)

$$F = gH$$

$$= g\frac{B}{H}$$

$$B = \frac{m\mu_0}{2\pi r^3} \quad \text{where } g = \frac{2\pi\hbar}{e} h$$

$$m = \frac{e\hbar}{2m_0}$$

(6.113) "Lorentz Force"

$$F = q(E + v \times B)$$

(6.151) $\underline{E} = E'\cos\xi + Z_0 H'\sin\xi$

$$B = -Z_0 D'\sin\xi + B'\cos\xi$$

A duality transformation by E's and B's.

$$\underline{E} = E\cos\xi + cB\sin\xi$$

$$B = -\frac{E}{c}\sin\xi + B\cos\xi$$

$$F = q\cos\xi E + q\sin\xi cB + q\cos\xi (v \times B) - \frac{q\sin\xi}{c}(v \times E$$

If $q_e = q\cos\xi$ and $q_m = qc\mu_0\sin\xi$

$$F = q_e E + q_m \frac{B}{\mu_0} + q_e(v \times B) - q_e \epsilon_0 (v \times E)$$

b) (6.151) $E = E'\cos\xi + Z_0 H'\sin\xi$ $\quad Z_0 D = Z_0 D'\cos\xi + B'\sin\xi$

$Z_0 H = -E'\sin\xi + Z_0 H'\cos\xi$ $\quad B = -Z_0 D'\sin\xi + B'\cos\xi$

(6.152) $Z_0 \rho_e = Z_0 \rho_e'\cos\xi + \rho_m'\sin\xi$ $\quad Z_0 J_e = Z_0 J_e'\cos\xi + J_m'\sin\xi$

$\rho_m = -Z_0 \rho_e'\sin\xi + \rho_m'\cos\xi$ $\quad J_m = -Z_0 J_e'\sin\xi + J_m'\cos\xi$

$$F = q_e' E' + q_m' \frac{B'}{\mu_0} + q_e'(V \times B) - q_m \epsilon_0 (V \times E)$$

$$= q_e'(E\cos\xi + cB\sin\xi) + q_m\left(\frac{B\cos\xi - \frac{E}{c}\sin\xi}{\mu_0}\right)$$

$$+ q_e' V \times \left(B\cos\xi - \frac{E}{c}\sin\xi\right) - q_m \epsilon_0 (V \times (E\cos\xi + cB\sin\xi))$$

$$= \left(q_e'\cos\xi - \frac{q_m'\sin\xi}{\mu_0 c}\right)E + \left(q_m'\cos\xi + q_e'\mu_0 c\sin\xi\right)\frac{B}{\mu_0}$$

$$+ \left(q_e'\cos\xi - \frac{q_m'\sin\xi}{\mu_0 c}\right)v \times B - \left(q_m'\cos\xi + q_e'\mu_0 c\sin\xi\right)\epsilon_0 V \times E$$

$$F_m = \left(\frac{2\pi h}{e} n\right)\left(\frac{e\hbar}{2m_p}\right)\left(\frac{1}{2\pi r^3}\right)$$

$$= \frac{\hbar}{r^3 e} n \cdot \frac{e\hbar}{2m_p} \quad \therefore \text{when } n=1$$

$$= \frac{6.626\times10^{-34}\,\mathrm{J\cdot s}/2\pi \times 6.626\times10^{-34}\,\mathrm{J\cdot s}/2\pi}{(0.5\,\mathring{A})^3\left(\frac{1\,\mathrm{m}}{10^{10}\,\mathring{A}}\right)^3 \cdot 2 \cdot 1.67\times10^{-27}\,\mathrm{kg}}$$

$$= 2.7\times10^{-11}\,\mathrm{N}$$

b)

$$F_d = \frac{ke^2}{r^2}$$

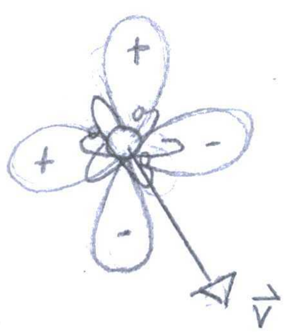
$$= \frac{(1.44\,\mathrm{eV\cdot nm})\left(\frac{1\,\mathrm{m}}{10^9\,\mathrm{nm}}\right)(1.6\times10^{-19}\,\mathrm{J/eV})}{(0.5\,\mathring{A})^2\left(\frac{1\,\mathrm{m}}{10^{10}\,\mathring{A}}\right)}$$

$$= 9\times10^{-8}\,\mathrm{N}$$

$$> F_m$$

The book references a paper, "D. Sivers, Phys. Rev. D2, 2048 (1970)." Equation 4.15 matches the force equation above and "k" by ∴

∴ $k = E_0 \cdot r_0^2$ where E_0 From Table V.

6.17. 

"A particle with electronic and magnetic charges"

$$= \frac{g}{4\pi}\int_{-\infty}^{0} dz' \, \frac{\hat{z}\times\hat{x}}{|\rho^2+(z-z')^2|^{3/2}}$$

$$= \frac{g}{4\pi}(\hat{z}\times\hat{x})\int_{-\infty}^{-z} \frac{du}{(\rho^2+u^2)^{3/2}}$$

Identity #1:

$$\int \frac{du}{(a^2+u^2)^{3/2}} = \int \frac{a\sec^2 x\,dx}{(a^2+a^2\tan^2 x)}$$

$$= \frac{g}{4\pi}(\hat{z}\times\hat{x})\int_{-\infty}^{-z} \frac{\rho\sec^2 x\,dx}{(\rho^2+\rho^2\tan^2 x)^{3/2}}$$

... when $u^2 = a^2\tan^2 x$

$du = a^2\sec^2 x\,dx$

$$= \frac{g}{4\pi}(\hat{z}\times\hat{x})\int_{-\infty}^{-z} \frac{dx}{\rho^2\sec x}$$

... as $\cos^2 x + \sin^2 x = 1$

$a^2\tan^2 x + a^2 = a^2\sec^2 x$

$$= \frac{g}{4\pi}(\hat{z}\times\hat{x})\left(\frac{1}{\rho^2}\right)\int \cos x\,dx$$

$$= \frac{g}{4\pi}(\hat{z}\times\hat{x})\frac{1}{\rho^2}\sin x$$

$$= \frac{g}{4\pi}(\hat{z}\times\hat{x})\frac{1}{\rho^2}\sin\left(\tan^{-1}\left(\frac{u}{\rho}\right)\right)$$

Identity #2:

$$\sin(\tan^{-1}(z)) = \frac{z}{\sqrt{z^2+1}}$$

$$= \frac{g}{4\pi}(\hat{z}\times\hat{x})\frac{u/\rho}{\rho^2\sqrt{(u/\rho)^2+1}}$$

$$= \frac{g}{4\pi}(\hat{z}\times\hat{x})\frac{u/\rho}{\rho^2\sqrt{u^2+\rho^2}}$$

$$= \frac{g\,(\hat{z}\times\hat{x})}{4\pi\rho^2}\left(1-\frac{z}{r}\right)$$

$$= \frac{g}{4\pi}\frac{r-z}{r^2\sin\theta}\hat{\phi}$$

... because $\hat{z}\times\hat{x} = \rho\hat{\phi}$

$= r\sin\theta$

$$= \frac{g}{4\pi}\frac{1-\cos\theta}{r\sin\theta}\hat{\phi}$$

$$= \left(\frac{g}{4\pi r}\right)\tan\left(\frac{\theta}{2}\right)$$

$$q_e = q_e' \cos\xi - q_m' \sin\xi / Z_0$$

$$q_m = q_m' \cos\xi + q_e' Z_0 \sin\xi$$

Lesson: charge is invariant in time when invariant to a transformation.

c) (6.153) $\dfrac{eg}{4\pi\hbar} = \dfrac{\alpha g}{Z_0 e} = \dfrac{n}{2}$ $\quad (n = 0, \pm 1, \pm 2 \ldots)$

(6.156) $\Delta L_z = b \Delta p_y = \dfrac{eg}{2\pi}$

$$\Delta p = \frac{e_2 g_1 - e_1 g_2}{2\pi b}$$

$$\Delta L = \frac{e_2 g_1 - e_1 g_2}{2\pi}$$

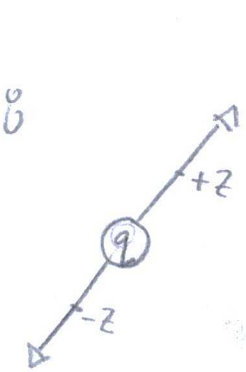
$$= \left(e_2' \cos\xi - g_2' \frac{\sin\xi}{Z_0}\right)\left(g_1' \cos\xi + e_1' Z_0 \sin\xi\right)/\hbar$$

$$- \left(e_1' \cos\xi - g_1' \frac{\sin\xi}{Z_0}\right)\left(g_2' \cos\xi + e_2' Z_0 \sin\xi\right)/\hbar$$

$$= \frac{e_2' g_1' - g_2' e_1'}{\hbar}$$

$$= 2\pi\hbar$$

6.18



a) $A(x) = \dfrac{g}{4\pi} \displaystyle\int_L \frac{dl' \times (x - x')}{|x - x|^3}$ ...if $dl' = z\,dz'$ and $x = z z'$

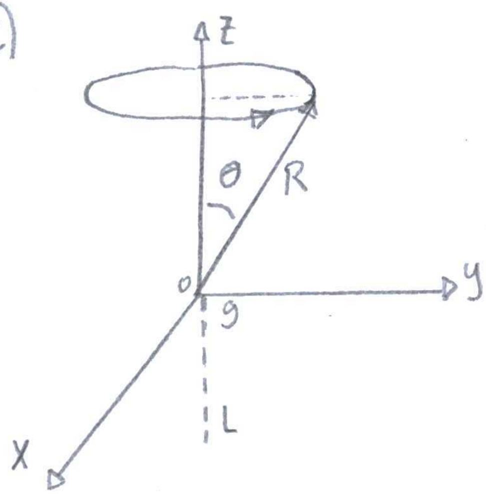
$$= \frac{g}{4\pi} \int_{-\infty}^{0} \frac{z \times (x - z z')}{|x - z z'|^3}$$

"Magnetic monopole... located at origin... on a string"

b) $$B = \nabla \times A = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{g}{4\pi}\frac{1-\cos\theta}{r\sin\theta}\right) = \frac{g}{4\pi r^2}\left(\frac{1}{\cos\theta+1}\right)$$

$$\propto \frac{1}{r^2} \quad \text{ooo except at } \theta=\pi$$

c)



"Total magnetic flux around a loop"

$$\Phi = \int B\cdot n\, da$$

$$= \int B_z\, da$$

$$= \frac{g}{4\pi}\int \frac{z}{(\rho^2+z^2)^{3/2}}\rho\, d\rho\, d\phi$$

$$= \frac{gz}{4}\int_0^{(R\sin\theta)^2} \frac{du}{(u+z^2)^{3/2}}$$

$$= -\frac{gz}{z}\frac{1}{\sqrt{u+z^2}}\Bigg|_0^{(R\sin\theta)^2}$$

$$= \frac{gR\cos\theta}{2}\left(\frac{1}{R|\cos\theta|}-\frac{1}{R}\right)$$

$$= \frac{g}{2}\left(\mathrm{sgn}(\cos\theta)-\cos\theta\right) \quad \text{ooo when } z=R\cos\theta$$

If $\theta<\pi/2$,

$$= \frac{g}{2}(1-\cos\theta)$$

If $\theta>\pi/2$,

$$= \frac{g}{2}(-1-\cos\theta)$$

d) $$\oint A\cdot dl = \int_0^{2\pi} A_\phi R\sin\theta\, d\phi$$

$$= \int_0^{2\pi}\frac{g}{4\pi}\left(\frac{1-\cos\theta}{\sin\theta\, r}\right)R\sin\theta\, d\phi$$

$$= \frac{g}{4\pi}\left(\frac{1-\cos\theta}{\sin\theta R}\right)2\pi R\sin\theta$$

$$= \frac{g}{2}(1-\cos\theta)$$

$$= \phi_{\theta<\pi/2}$$

$$= \phi_{\theta>\pi/2} + g$$

6.19.

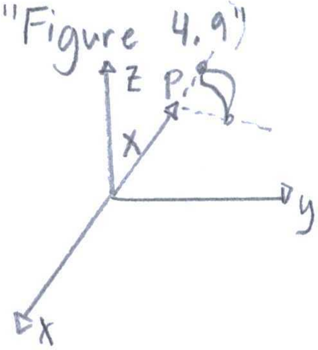
a) Exactly Problem 6.18 b

b) $\delta A = A' - A$

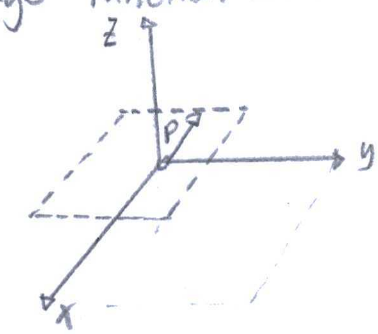
$$= -\frac{g}{4\pi r}\cot\left(\frac{\theta}{2}\right) - \frac{g}{4\pi r}\tan\left(\frac{\theta}{2}\right)$$

$$= -\frac{g}{4\pi}\nabla A$$ ooo as with (6.162) $\delta A = \frac{g}{4\pi}\nabla\Omega_c(x)$

c) "Figure 4.9"



"Gauge Function in a rectangle"



$x \to x' = x + \nabla X$

$y \to y' = y + \nabla X$

$z \to z' = z$

$A \to A' = A + \nabla X$

$\phi \to \phi' = \phi - (1/c)\left(\frac{\partial X}{\partial t}\right)$

$X = \frac{g}{4\pi}\Omega_c$

6.20. $\rho(x,t) = \delta(x)\delta(y)\delta'(z)\delta(t)$

$J_z(x,t) = -\delta(x)\delta(y)\,J(z)\,\delta'(t)$

a) (6.23) $$\phi(x,t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(x',t)}{|x-x'|}d^3x'$$

(6.48) "Greens Function Retarded"

$$\phi(x,t) = \frac{1}{4\pi\epsilon_0}\int d^3x\,\frac{1}{R}\left[\rho(x',t')\right]_{ret}$$ where $R = x - x'$

c) (6.47) $\Psi(x,t) = \int \frac{[f(x,t)]_{ret}}{|x-x'|} d^3x$

(6.40) $G_R^{(\pm)} = \frac{e^{\pm ikR}}{R}$

(6.42) $G_R^{(\pm)}(R,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm ikR}}{R}\cdot e^{-i\omega\tau}\, d\omega$ ∘∘ where $\tau = t-t'$

$= \frac{1}{R}\delta\left(\tau \mp \frac{R}{c}\right)$

(6.44, 5) $\Psi^{(\pm)}(x,t) = \iint G^{(-)}(x,t;x',t')\, f(x',t')\, d^3x\, dt'$

(6.48) $= \frac{1}{4\pi\epsilon_0}\int d^3x \frac{1}{R}[\rho(x',t')]$

(1.16) $E = -\nabla\phi$

(6.51) $E = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{1}{R}\left[-\nabla'\rho - \frac{1}{c^2}\frac{\partial J}{\partial t'}\right]$

(6.55) $= \frac{1}{4\pi\epsilon_0}\int d^3x' \left\{\frac{1}{R^2}[\rho(x,t)]_{ret} + \frac{1}{cR}\left[\frac{\partial\rho(x,t)}{\partial t'}\right] - \frac{1}{c^2R}\left[\frac{\partial J(x',t')}{\partial t'}\right]\right\}$

∘∘∘ Legendre

∘∘∘ unsure on $\delta(x,t) \to \delta(r-ct)$

$E = \frac{1}{4\pi\epsilon_0}\frac{c}{r}\left[-\delta'''(r-ct) + \frac{3}{r}\delta''(r-ct) - \frac{3}{r^2}\delta(r-ct)\right]\sin\theta\cos\theta \cdot \begin{cases}\cos\phi & E_x \\ \sin\phi & E_y\end{cases}$

∘∘∘ looks as a two-term Legendre expansion

$E_z = \frac{1}{4\pi\epsilon_0}\cdot\frac{c}{r}\left[\sin^2\theta\,\delta(r-ct) + (3\cos^2\theta - 1)\cdot\left(\frac{\delta'(r-ct)}{r} - \frac{\delta(r-ct)}{r^2}\right)\right]$

$$\phi(x,t) = \frac{1}{4\pi\epsilon_0}\int \frac{\delta(x)\delta(y)\delta'(z)\delta(t)}{|x-x'|}d^3x$$

$$= \frac{\delta(t)}{4\pi\epsilon_0}\int \frac{\delta(x)\delta(y)\delta(z)}{|x-x'|}d^3x$$

$$= \frac{z\,\delta(t)}{4\pi\epsilon_0}\cdot\int \frac{\delta(x)\delta(y)}{|x-x'|}d^3x$$

Citation: "Causality in the Coulomb Gauge"
Brill, O.L., Goodman, B. Kansas State.
University of Cincinnati, 7 March, 1967

$$= \frac{z}{4\pi\epsilon_0}\delta(t)\int\frac{1}{r^4}d^3r$$

$$= -\frac{1}{4\pi\epsilon_0}\delta(t)\int\frac{z}{r^3}$$

b) (4.20) $E(x) = \frac{1}{4\pi\epsilon_0}\left[\frac{3n(p\cdot n)-p}{|x-x_0|^3} - \frac{4\pi}{3}p\,\delta(x-x_0)\right]$

(5.22) $\nabla\times B = \mu_0 J$ .. from Faraday's law at steady-state $\left(\frac{\partial E}{\partial t}=0\right)$"

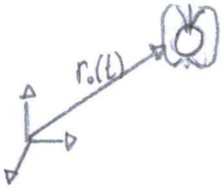
$$\Phi(x,t) = \frac{-\delta(t)}{4\pi\epsilon_0}\epsilon_0^2\left[\frac{3n(p\cdot n)-p}{|x-x_0|^3} - \frac{4\pi}{3}p\,\delta(x-x_0)\right]$$

$$= \frac{-\delta(t)}{4\pi}\epsilon_0\left[\frac{3n(p\cdot n)-p}{r^3} - \frac{4\pi}{3}\cdot p\,\delta(x-x_0)\right]$$

$$= -\delta(t)\left[\frac{2}{3}\cancel{\epsilon_0\delta(x)} - \frac{3n(\epsilon_0\cdot n)-\epsilon_0}{4\pi r^3}\right]$$

$$= -\delta(t)\left[\frac{2}{3}\epsilon_0\,\delta(x) - \frac{\epsilon_0}{4\pi r^3} + \frac{3}{4\pi r^3}\cdot n(\epsilon_3\cdot n)\right]$$

6.21



"Electric dipole moment... at a distance"

a) (6.95) $m_n = \sum_j \frac{q_j}{2}(X_{jn} \times V_{jn})$

(6.93) $j(x,t) = \sum_j q_j v_j \delta(x - x_j(t))$

(6.76) $p_n = \sum_{j(n)} q_j X_{jn}$

(6.71) $\eta(x,t) = \sum_j q_j \delta[(x - x_j(t))]$ ... if $x_j(t) = r_j(t)$

$$= \sum_j q_j [f(x-x_n) - X_{jn} \nabla f(x-x_n) + \frac{1}{2}\sum_j (X_{jn})_\alpha (X_{jn})_\beta \circ \frac{\partial^2}{\partial x_\alpha \partial x_\beta} f(x-x_n) + \cdots$$

$$= q_n \delta(x - x(t) - p_n \circ \nabla f(x-x_n) + \frac{1}{6}\sum_{\alpha\beta}(Q_n')_{\alpha\beta} \frac{\partial^2 f(x-x_n)}{\partial x_\alpha \partial x_\beta} + \cdots$$

$$\cong q_n \delta(x-x_n) - (p \circ \nabla)\delta(x - X_n)$$

$$\cong -p_o \nabla \delta(x - X_n)$$

(6.93) $j(x,t) = \sum q_j v_j \delta(x - r_o(t))$

$$= \sum p(x,t) \circ v_j$$

$$= -v(p_o \nabla)\delta(x - r_o(t))$$

b) (6.95) $m = \sum \frac{q_n}{2}(X_{jn} \times V_{jn})$

$$= \sum \frac{p}{2} \times V_{jn}$$

$$= \frac{1}{2} p \times v$$

6.23 (6.168a) $\mu\epsilon \frac{\partial^2 \Pi_e}{\partial t^2} - \nabla^2 \Pi_e = P_{ext} - \frac{\mu_0}{\mu} \nabla \times V$

(6.168b) $\mu\epsilon \frac{\partial^2 \Pi_e}{\partial t^2} - \nabla^2 \Pi_m = M_{ext} + \frac{\partial V}{\partial t} + \nabla \frac{\partial 5}{\partial t}$

Hertz (1889) Righi (1901)

"Lorentz condition" $\nabla A + \mu\epsilon \frac{\partial \phi}{\partial t} = 0$

If $\Pi_e' = \Pi_e + \mu_0 \nabla \times G - \nabla g$

$\Pi_m' = \Pi_m - \mu \frac{\partial G}{\partial t}$

then $\left(\mu\epsilon - \nabla^2\right) \begin{Bmatrix} G \\ g \end{Bmatrix} = \begin{Bmatrix} \frac{1}{\mu}(V + \nabla 5) \\ 0 \end{Bmatrix}$

$\mu\epsilon \circ G - \nabla^2 G + \mu\epsilon g - \nabla^2 g = \frac{1}{\mu} V + \nabla \frac{5}{\mu}$

$\mu\epsilon \circ G - \nabla^2 G = \frac{1}{\mu} V$

$\mu\epsilon \cdot g - \nabla^2 g = \frac{1}{\mu} \nabla 5$

Solution: $G(X,t) = c_1(\sqrt{\mu\epsilon} \cdot X + t) + c_2(t - \sqrt{\mu\epsilon} \cdot x) - \frac{1}{\mu} V$

$g(X,t) = c_1(\sqrt{\mu\epsilon} \cdot X + t) + c_2(t - \sqrt{\mu\epsilon} \cdot X) - \frac{1}{\mu} \nabla 5$

$\Pi_e' = \Pi_e + \mu_0 \nabla \times G - \nabla g$

$= \Pi_e + \mu_0 \nabla \times G(x,t) - \nabla g(x,t)$

$\Pi_m' = \Pi_m - \mu \frac{\partial G}{\partial t}$

$= \Pi_m - \mu \frac{\partial G(x,t)}{\partial t}$

... into (6.168a)

$\mu\epsilon \frac{\partial^2 \Pi_e}{\partial^2 t} - \nabla^2 \Pi_e = P_{ext} - \frac{\mu_0}{\mu} \nabla \times V$

$$\frac{\partial^2 \Pi_e + \mu_0 \nabla \times (G(x,t) - \nabla g(x,t))}{\partial^2 t} - \nabla^2 \left(\Pi_e + \mu \nabla \times (G(x,t) - \nabla g(x,t))\right)$$

$$= \frac{\partial^2 \Pi_e}{\partial^2 t} - \nabla^2 \Pi_e + \frac{\mu_0}{\mu} \nabla \times V - \nabla^2 \Pi_e + \nabla^2 \mu \nabla \times G(x,t) - \nabla^2 \nabla g(x,t)$$

$= (6.168a)$

∘∘∘ into $(6.168b)$

$$\mu \epsilon \frac{\partial^2 \Pi_m}{\partial^2 t} - \nabla^2 \Pi_m = \mu \epsilon \frac{\partial^2}{\partial t^2}\left[\Pi_m - \mu \frac{\partial G(x,t)}{\partial t}\right] - \nabla^2 \left[\Pi_m - \mu \frac{\partial G(x,t)}{\partial t}\right]$$

$$= \mu \epsilon \frac{\partial^2}{\partial t^2} \Pi_m - \mu^2 \epsilon \frac{\partial G(x,t)}{\partial t} - \nabla^2 \Pi_m + \mu \nabla^2 \frac{\partial G(x,t)}{\partial t}$$

$$= \mu \epsilon \frac{\partial^2 \Pi_m}{\partial t^2} - \nabla^2 \Pi_m$$

$$= 0$$

$$\frac{d}{dt}\left(\mu^2 \epsilon G - \mu \nabla^2 G\right) = \frac{d}{dt}\left[\left(\frac{1}{\mu} V + \frac{\nabla \xi}{\mu}\right)\mu - \mu \epsilon g + \nabla^2 g\right]$$

$$= 0$$

$$\mu \epsilon \frac{\partial^2 \Pi_m}{\partial^2 t} - \nabla^2 \Pi_m = \frac{\partial v}{\partial t} + \frac{\partial \nabla \xi}{\partial t}$$ ∘∘ when $g(x,t) = 0$

b) (6.19) $A \longrightarrow A + \nabla \Lambda$

$\phi \longrightarrow \phi - \frac{\partial \Lambda}{\partial t}$ where $\nabla^2 \Lambda - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = 0$

If $\Lambda = \nabla A$ and $\nabla^2 \Lambda - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = \nabla A + \mu \epsilon \frac{\partial \phi}{\partial t}$

$= 0$ "Lorentz condition"

Note∘ Lorentz condition describes direct relationship between position-and-time, or in a representation field-and-phase.

6.24. (6.55) $E(x,t)=\frac{1}{4\pi\epsilon_0}\int d^3x'\left\{\frac{\hat{R}}{R^2}[\rho(x,t)]_{ret}+\frac{\hat{R}}{cR}\left[\frac{\partial\rho(x,t)}{\partial t}\right]_{ret}-\frac{1}{c^2R}\left[\frac{\partial J(x,t)}{\partial t}\right]_{ret}\right\}$

a)

(6.56) $B(x,t)=\frac{\mu_0}{4\pi}\int d^3x\left\{[J(x',t')]_{ret}\times\frac{\hat{R}}{R^2}+\left[\frac{\partial J(x',t')}{\partial t'}\right]_{ret}\times\frac{\hat{R}}{cR}\right\}$

$@(x=\hat{r}r, t)$

$$\frac{\partial B(x,t)}{\partial t}=\frac{\partial}{\partial t}\left[\frac{\mu_0}{4\pi}\int d^3x\left\{[J(x,t)]_{ret}\times\frac{\hat{R}}{R^2}+\left[\frac{\partial J(x',t')}{\partial t'}\right]\times\frac{\hat{R}}{cR}\right\}\right]$$

$$=\frac{\mu_0}{4\pi}\Big[3\cdot(m(t-r/c)\times\frac{\hat{R}}{R^3}-m(t-r/c)$$

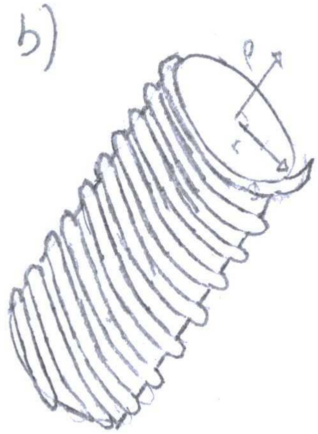
$$-\frac{1}{R^2}\frac{\partial}{\partial t}\left[3(m(t-r/c))\hat{R}-m(t-r/c)\right]\Big]$$

$$=\frac{\mu_0}{4\pi}\frac{1}{R^3}\left(1+\frac{r}{c}\frac{\partial}{\partial t}\right)\left[3(m(t-r/c)\cdot\hat{R}-m(t-r/c)\right]$$

$$\frac{\partial E(x,t)}{\partial t}=\frac{1}{4\pi\epsilon_0}\frac{\partial}{\partial t}\left[\frac{\hat{R}}{R^2}\rho(x,t)\right]$$

$$=\frac{1}{4\pi\epsilon_0}\frac{\hat{R}}{R^2}\times\frac{\partial m(t-r/c)}{\partial t}$$

b)



"right circular Solenoid"

$$\lim_{r\to\infty}B=\lim_{r\to\infty}\frac{\mu_0}{4\pi}\frac{1}{r^3}\left(1+\frac{r}{c}\frac{\partial}{\partial t}\right)\left[3(m(t-r/c)\cdot\hat{r})\hat{r}-m(t-r/c)\right]$$

$$=0$$

$$\lim_{r\to\infty}E=\lim_{r\to\infty}\frac{\mu_0}{4\pi}\frac{1}{\rho^2}\rho\times\frac{\partial m(t-\rho/c)}{\partial t}$$

$$=-\frac{\mu_0}{2}\frac{Na^2}{\rho}\frac{\partial I(t-\rho/c)}{\partial t}\hat{\Phi}$$

6.25 a) (6.114) $\frac{dP_{mech}}{dt} = \int_V (\rho E + J \times B)\, d^3x$ ... if Gauss' Law: $\nabla \cdot E = \frac{\rho}{\epsilon_0}$

and

$= \int_V \left[(\epsilon_0 \nabla \cdot E) + \left(\frac{\nabla \times B}{\mu_0} - \epsilon_0 \frac{\partial E}{\partial t}\right) \times B \right] d^3x$ Ampere's Law: $\nabla \times B = \mu_0 (J + \epsilon \frac{\partial E}{\partial t}$

$= (d \cdot \nabla) E + d \times B$

b) (6.118) $g = \frac{1}{c^2}(E \times H)$

$$\frac{dg}{dt} = \frac{d}{dt}\frac{1}{c^2}(E \times H)$$

$=$ ... no justifiable meaning

$= \frac{1}{2}(n^2 - 1)\frac{dg_{em}}{dt}$