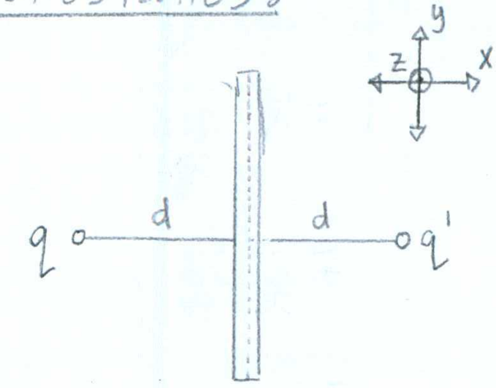
Chapter 2: Boundary-Value Problems in Electrostatics:

2.1a Surface Charge Density:



Charge q near an infinite plane conductor

Shape = Plane
Dimension = Area
Charges = q > q'

(Equation 1.48)

$$\Phi(x_i) = \frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{n-1}\frac{q_j}{|x_i - x_j|}$$

$$\cong \frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{n-1}\frac{q_j}{r_i}$$

$$\Phi(x,y,z) = \frac{q}{\sqrt{(x-d)^2+y^2+z^2}} + \frac{q'}{\sqrt{(x+d)^2+y^2+z^2}}$$

(Equation 2.5)

$$\sigma = -\varepsilon_0 \left.\frac{\partial \Phi}{\partial x}\right|_{x=0}$$

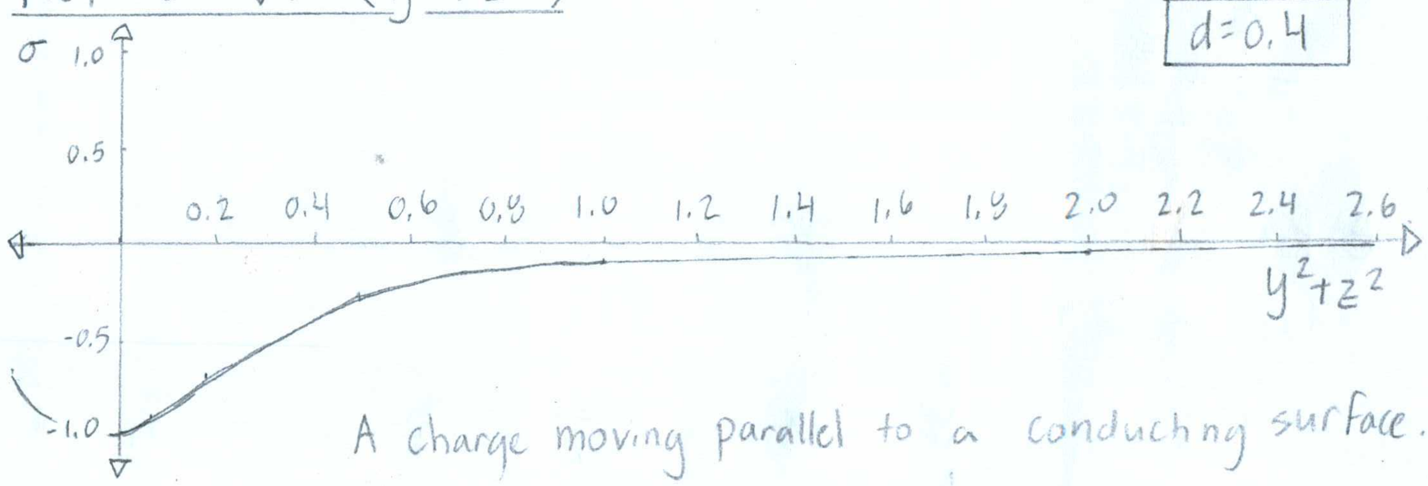
$$= -\varepsilon_0 \frac{\partial}{\partial x}\left[\frac{q}{\sqrt{(x-d)^2+y^2+z^2}} + \frac{q'}{\sqrt{(x+d)^2+y^2+z^2}}\right]_{x=0}$$

$$= -\frac{1}{2\pi}\frac{d}{(d^2+y^2+z^2)^{3/2}}$$

$$= \frac{-1}{2\pi d^2}\cdot\frac{1}{\left(1+\left(\frac{y}{d}\right)^2+\left(\frac{z}{d}\right)^2\right)^{3/2}}$$

Plot σ vs. (y^2+z^2):

$\boxed{d=0.4}$



A charge moving parallel to a conducting surface.

b) Force from Coulombs Law:

$$F = k \cdot \frac{q \cdot q'}{|x - x'|}$$

$$= \frac{1}{4\pi \cdot \varepsilon_0} \frac{q \cdot q'}{(2d)^2}$$

$$= \frac{1}{16\pi\varepsilon_0} \cdot \frac{q \cdot q'}{d^2}$$

c) Total Force acting on a plane:

$$F = \int \frac{\sigma^2}{2\varepsilon_0}$$

$$\sigma^2 = \frac{q^2}{4\pi^2} \frac{4d^2}{(d^2+y^2+z^2)^3}$$

$$= \frac{1}{2\varepsilon_0} \int_0^{\infty} \int_0^{2\pi} \frac{q^2}{4\pi^2\varepsilon_0} \frac{d^2}{(r^2+d^2)^3} \, r \, d\theta \, dr$$

$$= \frac{1}{16\pi\varepsilon_0} \frac{q \cdot q}{d^2}$$

d) Work necessary for removing a charge:

$$W = \int F \cdot r \, dr$$

$$= \int_d^{\infty} \frac{1}{16\pi\varepsilon_0} \frac{q^2}{r^2} \, dr$$

$$= \frac{q^2}{16\pi\varepsilon_0 d}$$

e) Potential energy between a charge and image:
(Equation 1.53)

$$\text{Potential} = \frac{1}{2} \int \rho(x) \phi(x) d^3x = \frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\right) \frac{q \cdot q}{d} = \frac{1}{8\pi\varepsilon_0} \frac{q \cdot q}{d}$$

f) Work in electronvolts:

$$W = \frac{1}{16\pi\varepsilon_0}\frac{q^2}{d}$$

$$= \frac{(1.602\times10^{-19}C)^2}{16\cdot3.142\cdot8.854\times10^{-12}\,F/m\cdot10^{-10}m}$$

$$= 3.73\,eV$$

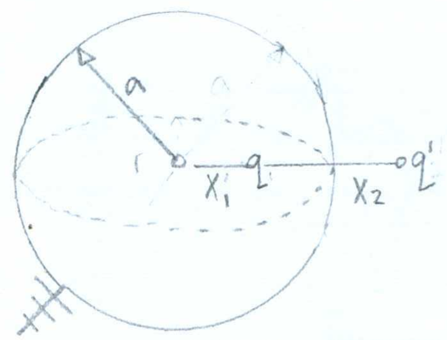
"Vacuum Permittivity" $= 8.854\times10^{-12}\,F/m$

$\pi = 3.142$

$d = 1\mathring{A} = 10^{-10}\,m$

"An electron's charge" $= 1.602\times10^{-19}C$

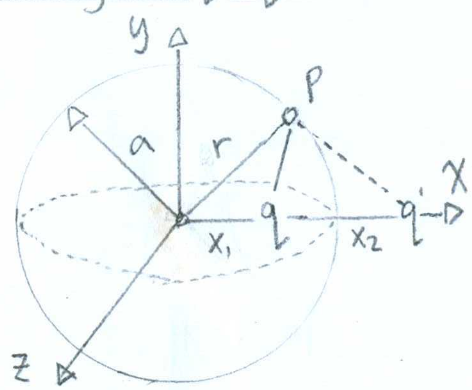
2.2



Real Image

Charge inside a hollow, grounded, conducting sphere

Shape: sphere
Dimension: Area [2D]
Charges: q, q'



Cartesian to polar coordinate system

a) Potential inside a sphere:

(Equation 1.49)

$$\Phi(x_i) = \frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{n-1}\frac{q_j}{|x_i - x_j|}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{(x-x_1)^2+y^2+z^2}} + \frac{q_2}{\sqrt{(x-x_2)^2+y^2+z^2}}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x-x_1)^2+y^2+z^2}} - \frac{\left(\frac{a}{x_1}\right)}{\sqrt{(x-\frac{a^2}{x_1})^2+y^2+z^2}}\right]$$

where $q' = -\frac{a}{x_1}q$ and $x_2 = \frac{a^2}{x_1}$

In polar coordinates,

$$\Phi(r,\theta) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2+x_1^2-2x_1r\cos\theta}} - \frac{\left(\frac{a}{x_1}\right)}{\sqrt{r^2+\left(\frac{a^2}{x_1}\right)^2-2\frac{a^2}{x_1}r\cos\theta}}\right]$$

Notes: 1) Z is always zero in the system because coordinate rotation

2) $(x-x_1)^2+y^2+z^2 = r^2+x_1^2-2x_1r\cos\theta$

and

$(x-\frac{a^2}{x_1})^2+y^2+z^2 = r^2+\left(\frac{a^2}{x_1}\right)^2-2\frac{a^2}{x_1}r\cos\theta$

b) Induced surface-charge density:

(Equation 2.5) $\sigma = -\varepsilon_0 \left.\frac{\partial \Phi}{\partial X}\right|_{x=0}$

$$= -\varepsilon_0 \left.\frac{\partial \Phi}{\partial r}\right|_{r=a}$$

$$= \frac{-q}{4\pi} \frac{\partial}{\partial r}\left[\frac{1}{\sqrt{r^2+x^2-2xr\cos\theta}} - \frac{\left(\frac{a}{X_1}\right)}{\sqrt{r^2+\left(\frac{a^2}{X}\right)^2-2\frac{a^2}{X}r\cos\theta}}\right]_{r=a}$$

$$= -\frac{q}{4\pi} \frac{X_1^2-a^2}{a(a^2+x^2-2a\cos\theta)^{3/2}}$$

$$= -\frac{q}{4\pi a^2}\left(\frac{a}{X_1}\right) \frac{(1-a^2/x^2)}{\left(1+\left(\frac{a}{X}\right)^2-2\frac{a}{X}\cos\theta\right)^{3/2}}$$

c) The magnitude and direction of Force:

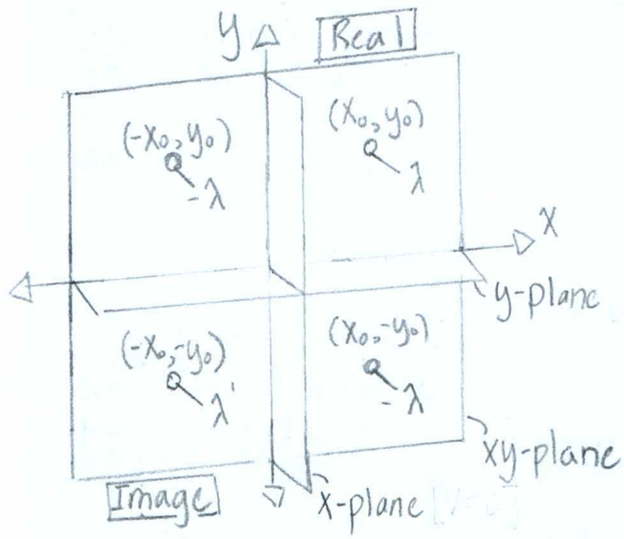
$$F = k\frac{q \cdot q'}{|X_1-X_2|} = \frac{1}{4\pi\varepsilon_0}\frac{q \cdot q'}{\left|X_1-\frac{a^2}{X_1}\right|^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{a}{y}\right)\frac{q \cdot q \cdot y^2}{(a^2-y^2)}$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q \cdot q\,(ay)}{(a^2-y^2)}$$

d) Multiple outcomes occur with a sphere at a fixed potential V. Cases include charge on the inside vs outside surface, sign, relative quantity, and location for the real charge. The derivation changes from the beginning at Equation 1.48 with a third term.

2.3.



Constant linear charge Density perpendicular to the x-y plane

Shape: Plane
Dimension: Area [2D]
Charge: $\lambda, -\lambda$

a) Single Line charge Potential:

$$\phi(x,y) = \frac{\lambda}{4\pi \cdot \varepsilon_0} \ln\left(\frac{R^2}{r^2}\right)$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{R^2}{(x-x_0)^2+(y-y_0)^2}\right)$$

Potential for real and image:

Single line charge:

$$\phi(x,y) = \frac{\lambda}{4\pi\varepsilon_0}\left[\ln\left(\frac{[(x-x_0)^2+(y+y_0)^2][(x+x_0)^2+(y-y_0)^2]}{[(x-x_0)^2+(y-y_0)^2][(x+x_0)^2+(y+y_0)^2]}\right)\right.$$

Potential at boundary conditions:

Lemma: Potential at boundary conditions is zero.

Proof by Deduction

- Hypothesis: A charge at a boundary condition has coordinates $x_0 = y_0 = 0$.
- Outcome:

$$\phi(0,0) = \frac{\lambda}{4\pi\varepsilon_0}\left[\ln\frac{(x_0^2+y_0^2)(x_0^2+y_0^2)}{(x_0^2+y_0^2)(x_0^2+y_0^2)}\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left[\ln(1)\right]$$

$$= \phi$$

Tangential Electric Field at boundry conditions:

$$E_y(x,y) = -\frac{\partial \phi(x,y)}{\partial y}\Bigg|_{x=0}$$

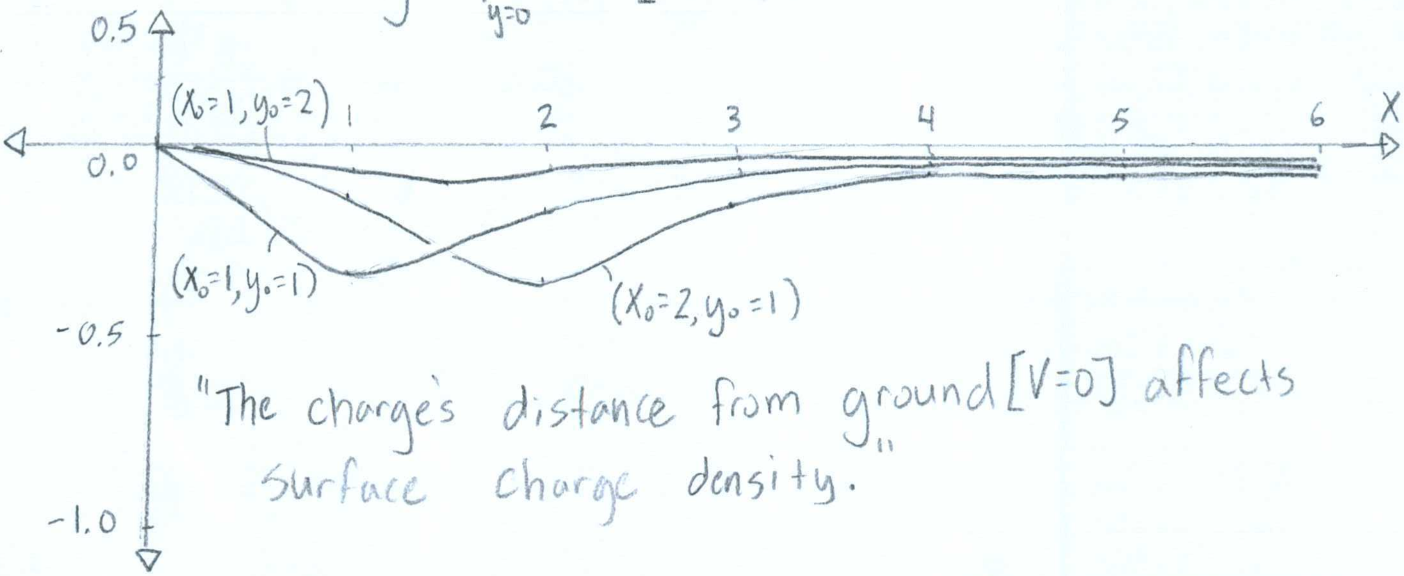
$$= -\frac{\lambda}{2\pi\varepsilon_0}\left[\frac{(y-y_0)}{(x-x_0)^2+(y-y_0)^2} + \frac{(y+y_0)}{(x+x_0)^2+(y+y_0)^2} - \frac{(y+y_0)}{(x-x_0)^2+(y+y_0)^2} - \frac{(y-y_0)}{(x+x_0)^2+(y-y_0)^2}\right]_{x=0}$$

$$E_y(0,y) = -\frac{\lambda}{2\pi\varepsilon_0}\left[\frac{(y-y_0)}{x_0^2+(y-y_0)^2} + \frac{(y+y_0)}{x_0^2+(y+y_0)^2} - \frac{(y+y_0)}{x_0^2+(y+y_0)^2} - \frac{(y-y_0)}{x_0^2+(y-y_0)^2}\right]$$

$$= \emptyset$$

b) Surface charge density [σ] on the plane $y=0, x \geq 0$:

$$\sigma(x,0) = -\varepsilon_0 \cdot \frac{\partial \phi(x,y)}{\partial y}\Bigg|_{y=0} = -\frac{\lambda}{2\pi}\left[\frac{-y_0}{(x-x_0)^2+(y-y_0)^2} + \frac{y_0}{(x+x_0)^2+(y+y_0)^2} - \frac{y_0}{(x-x_0)^2+(y+y_0)^2} - \frac{-y_0}{(x+x_0)^2+(y-y_0)^2}\right]$$



"The charge's distance from ground [V=0] affects surface charge density."

c. Total Charge:

$$Q_x = \int_0^\infty \sigma(x, y=0)\, dx$$

$$= \frac{\lambda}{2\pi} \int_0^\infty \left[\frac{-y_0}{(x-x_0)^2 + y_0^2} + \frac{y_0}{(x+x_0)^2 + y_0^2} - \frac{y_0}{(x-x_0)^2 + y_0^2} \quad \frac{y_0}{(x-x_0)^2 + y_0^2} \right] dx$$

$$= \frac{\lambda y_0}{2\pi} \int_0^\infty \left[\frac{1}{(x+x_0)^2 + y_0^2} - \frac{1}{(x-x_0)^2 + y_0^2} \right] dx$$

$$\boxed{u = \frac{x + x_0}{y_0} ; \; du = \frac{1}{y_0}}$$

$$= \frac{\lambda y_0}{2\pi} \int_0^\infty \left[\frac{y_0}{y_0^2 u^2 + y_0^2} - \frac{y_0}{y_0^2 (-u)^2 + y_0^2} \right] du$$

$$= \frac{\lambda}{2\pi} \int_0^\infty \left[\frac{1}{u^2 + 1} - \frac{1}{u^2 + 1} \right] du$$

$$\boxed{\begin{aligned} v &= \tan(u) \\ dv &= \sec^2(u)\, du \\ &= \tan^2(u) + 1 \end{aligned}}$$

$$= \frac{\lambda}{2\pi} \int_0^\infty dv$$

$$= -\frac{2\lambda}{\pi} \tan^{-1}\left(\frac{x_0}{y_0}\right)$$

$$Q_y = \frac{-2\lambda}{\pi} \tan^{-1}\left(\frac{y_0}{x_0}\right)$$

d) Far from the origin, Potential:

$$\phi(x,y) = \frac{\lambda}{4\pi\varepsilon_0} \ln\left[\frac{((x-x_0)^2 + (y+y_0)^2)((x+x_0)^2 + (y-y_0)^2)}{((x-x_0)^2 + (y-y_0)^2)((x+x_0)^2 + (y+y_0)^2)} \right]$$

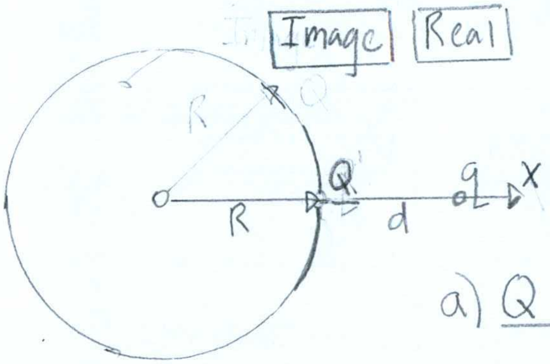
$$= \frac{\lambda}{4\pi\varepsilon_0} \ln\left[\frac{(\rho^2+\rho_0^2-2xx_0+2yy_0)(\rho^2+\rho_0^2+2xx_0-2yy_0)}{(\rho^2+\rho_0^2-2xx_0-2yy_0)(\rho^2+\rho_0^2+2xx_0+2yy_0)}\right]$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \ln\left[\frac{1+\frac{2\rho_0^2}{\rho^2}-\frac{4x^2x_0^2}{\rho^4}-\frac{4y^2y_0^2}{\rho^4}+\frac{8xyx_0y_0}{\rho^4}+\cdots}{1+\frac{2\rho_0^2}{\rho^2}-\frac{4x^2x_0^2}{\rho^4}-\frac{4y^2y_0^2}{\rho^4}-\frac{8xyx_0y_0}{\rho^4}+\cdots}\right]$$

$$\cong \frac{\lambda}{4\pi\varepsilon_0} \ln\left[1+\frac{16xyx_0y_0}{\rho^4}+\cdots\right] \text{ when } \boxed{\frac{1+x}{1-x} \cong 1+2x}$$

$$\cong \frac{4xyx_0y_0}{\pi\varepsilon_0\rho^4} \qquad \frac{\#\text{ Quadrants} \times \text{Charge area}}{\text{Square distance}}$$

2.4



Equally charged isolated and conducting sphere

$\boxed{\begin{array}{l}\text{Shape} = \text{Sphere} \\ \text{Dimension} = \text{Area}[2D] \\ \text{Charge} = q, Q\end{array}}$

a) Q vs q stability:

$\Phi(x) =$ Real charge + Image charge + Average charge

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{-q}{(R-\frac{dr}{R})} + \frac{Q}{r} + \frac{q}{r(\frac{d}{R})}\right]$$

$$F = q\cdot E$$

$$= \left[-q\cdot\nabla\Phi\right]_{r=d}$$

$$= \frac{-q}{4\pi\varepsilon_0}\frac{\partial}{\partial r}\left[\frac{-q}{(R-\frac{dr}{R})} + \frac{Q}{r} + \frac{q}{r(\frac{d}{R})}\right]$$

$$= \frac{-q}{4\pi\varepsilon_0}\left[\frac{+q\cdot d}{(R-\frac{dr}{R})^2\cdot R} - \frac{Q}{r^2} - \frac{q}{r^2(\frac{d}{R})}\right]$$

$$= \frac{-q}{4\pi\varepsilon_0}\left[\frac{q\cdot d}{(1-\frac{d^2}{R^2})^2\cdot R^2} - \frac{Q}{d^2} - \frac{Rq}{d^3}\right]$$

$$0 = \frac{-q \cdot q}{4\pi\varepsilon_0}\left[\frac{R d^3}{\left(1-\frac{d^2}{R^2}\right)^2 R^4} - \frac{Q}{q} - \frac{R}{d}\right]$$

$$= \frac{Q}{q}\left(\frac{d}{R}\right)^5 - \frac{2Q}{q}\left(\frac{d}{R}\right)^3 - 2\left(\frac{d}{R}\right)^2 + \frac{Q}{q}\left(\frac{d}{R}\right) + 1$$

When $\frac{Q}{q} = 1$, then a charge sits at $\frac{d}{R} = 1.6838$

b) $a = d - R$

$a + R = d$

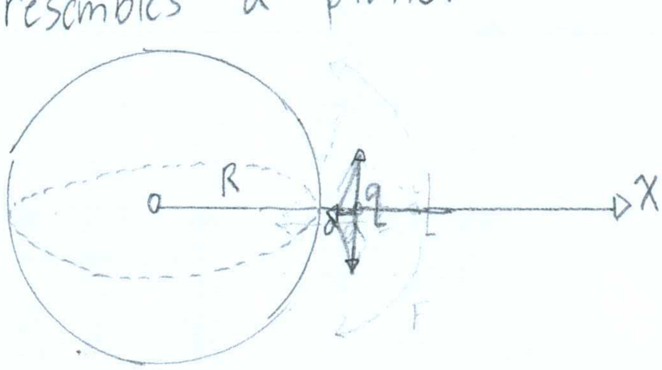
$$F = \frac{q}{4\pi\varepsilon_0}\left[\frac{d^3}{\left(1-\frac{d^2}{R^2}\right)^2 R^3} - \frac{Q}{d^2 q} - \frac{R}{d^3}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{R(a+R)^3}{\left(1-\left(\frac{a+R}{R}\right)^2\right)^2 R^3} - \frac{Q}{(a+R)^2 q} - \frac{R}{(a+R)^3}\right]$$

$$= \frac{-q}{4\pi\varepsilon_0}\left[\frac{a^6 q R + 6a^5 q R^2 + a^5 Q + 15a^4 q R^3 + a^4 q R + 5a^4 Q R + 20a^3 q R^4 + 4a^3 q R^2 + 8a^3 Q R^2 + 15a^2 q R^5 + 4a^2 q R^3 \ldots 4a^2 Q R^3 + 6a q R^6 + q R^7}{a^2 q (a+R)^3 (a+2R)^2}\right]$$

When $a \ll R$

$= \frac{-q^2}{16\pi\varepsilon_0 a^2}$ at the distance $d = a + R$, the force diagram resembles a plane.



c) When $Q = 2q$,

$$F = 0$$

$$= \frac{Q}{q}\left(\frac{d}{R}\right)^5 - \frac{2Q}{q}\left(\frac{d}{R}\right)^3 - 2\left(\frac{d}{R}\right)^2 + \frac{Q}{q}\left(\frac{d}{R}\right) + 1$$

$$= 2\left(\frac{d}{R}\right)^5 - 4\left(\frac{d}{R}\right)^3 - 2\left(\frac{d}{R}\right)^2 + 2\left(\frac{d}{R}\right) + 1$$

$$\frac{d}{R} = 1.4276$$

2.5 a) (Equation 2.6) $|F| = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a^2}\left(\frac{a}{y}\right)^3\left(1 - \frac{a^2}{y^2}\right)^{-2}$

$$W = -\int_r^\infty F \cdot dl$$

$$= -\int_r^\infty \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a^2}\left(\frac{a}{y}\right)^3\left(1 - \frac{a^2}{y^2}\right)^{-2} \cdot dy$$

$$= \frac{-q^2}{4\pi\varepsilon_0}\int_r^\infty \frac{y \cdot a}{(y^2 - a^2)^2}\, dy \qquad u = y^2 - a^2 \quad du = 2y\, dy$$

$$= \frac{-q^2}{8\pi\varepsilon_0} a \int_r^\infty \frac{1}{u^2}\, du$$

$$= \frac{-q^2 a}{8\pi\varepsilon_0 (r^2 - a^2)}$$

(Equation 2.3) $\phi(x = a) = \frac{q/4\pi\varepsilon_0}{a|n - \frac{y}{a}n'|} + \frac{-q/4\pi\varepsilon_0}{y|n - \frac{a}{y}n|}$

The equation 2.3 relates to the example because a symmetric system forms a potential with similar coefficient and summand. An example potential from Section 1.11 is $\phi(x) = \frac{1}{8\pi\varepsilon_0}\sum_i\sum_j \frac{q_i q_j}{|x - x_i|}$

b) (Equation 2.9) $F = \frac{1}{4\pi\varepsilon_0}\frac{q}{y^2}\left[Q - \frac{qa^3(2y^2 - a^2)}{y(y^2-a^2)^2}\right]$

$$W = -\int_r^\infty F \cdot dl$$

$$= \frac{-q}{4\pi\varepsilon_0}\int_r^\infty \frac{1}{y^2}\left[Q - \frac{qa^3(2y^2-a^2)}{y(y^2-a^2)^2}\right]dy$$

$$= \frac{-q}{4\pi\varepsilon_0}\left[\int_r^\infty \frac{Q}{y^2}dy - qa^3\int_r^\infty \frac{(2y^2-a^2)}{y^3(y^2-a^2)^2}dy\right]$$

$u = y^2 - a^2$

$du = 2y\,dy$

$y^2 = u + a^2$

$$= \frac{-q}{4\pi\varepsilon_0}\left[\frac{+Q}{r} - \frac{qa^3}{2}\left[\int_r^\infty \frac{2u + a^2}{u^2(u+a^2)^2}du\right]\right]$$

$v = u(u + a^2)$

$dv = (2u + a^2)du$

$$= \frac{-q}{4\pi\varepsilon_0}\left[\frac{+Q}{r} - \frac{qa^3}{2}\int \frac{1}{v^2}dv\right]$$

$$= \frac{-q}{4\pi\varepsilon_0}\left[\frac{+Q}{r} - \frac{qa^3}{2r^2(r^2-a^2)}\right]$$

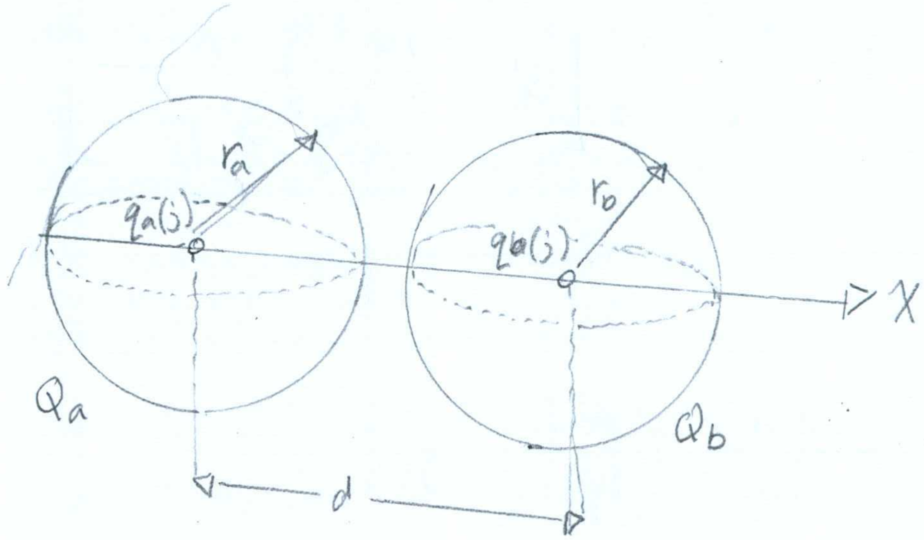
$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q^2a}{2(r^2-a^2)} - \frac{q^2a}{2r^2} - \frac{qQ}{r}\right]$$

(Equation 2.8)

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|x-y|} - \frac{aq}{y\left|x-\frac{a^2}{y^2}y\right|} + \frac{Q+\frac{a}{y}q}{|x|}\right]$$

The derived problem has similar terms to Equation 2.8, except for a sign and ambiguous ½.

2.6.

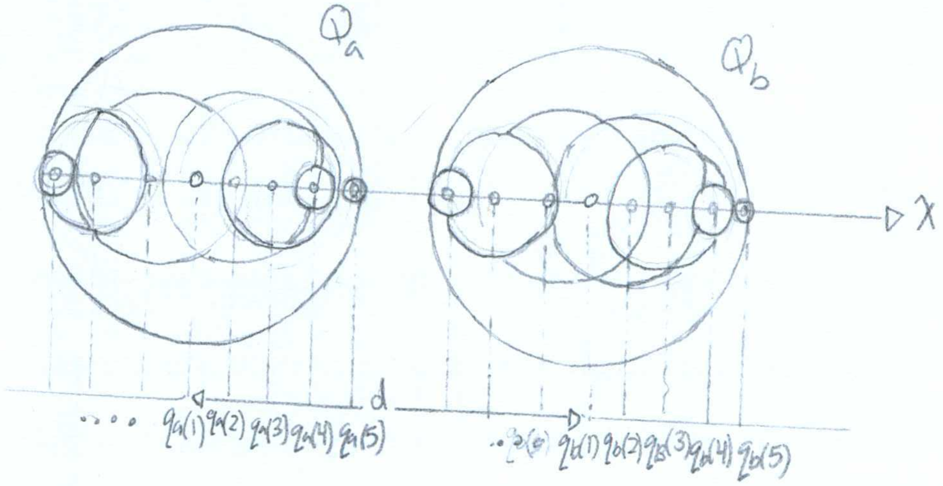


Two conducting spheres described by iteration

Shape = sphere
Dimension = Line [1D]
Charge : Q_a, Q_b

a)

j	$q_a(j)$	$q_b(j)$	$X_a(j)$	$X_b(j)$	d_a	d_b
1	q_a	q_b	r_a^2/d	r_b^2/d	d	d
2	$\frac{-r_a q_b}{d}$	$\frac{-r_b q_a}{d}$	r_a^2/d	r_b^2/d	d	d
3	$\frac{(-r_a)^2(-r_b)q_b}{d^2}$	$\frac{(-r_a)^2(-r_a)q_a}{d^2}$	r_a^2/d	r_b^2/d	d	d
⋮	⋮	⋮	⋮	⋮	⋮	⋮
n	$\frac{(-r_a)^{j-1}(-r_b)^{j-2}q_b}{d^{j-1}}$	$\frac{(-r_b)^{j-1}(-r_b)^{j-2}q_a}{d^{j-1}}$	r_a^2/d	r_b^2/d	d	d



The total sphere's charge by fractionally smaller charges (spheres) with equal distances.

b) Locations are the x-values in part (a).

Potential	$Q_a = \sum_{j=2}^{n} q_a(j)$ $Q_b = \sum_{j=2}^{n} q_b(j)$
Total Potential	$\phi(x) = \frac{1}{4\pi\varepsilon_0}\left(\sum_{j=2}^{n} \frac{q_a(j)}{\lvert x - x_a(j)\rvert} + \sum_{j=2}^{n} \frac{q_b(j)}{\lvert x - d_b(j)\rvert}\right)$
Force	$F = \frac{1}{4\pi\varepsilon_0} \sum_{j=2}^{n} \sum_{k=2}^{n} \frac{q_a(j)\, q_b(k)}{(d - x_a(j) - x_b(k))^2}$

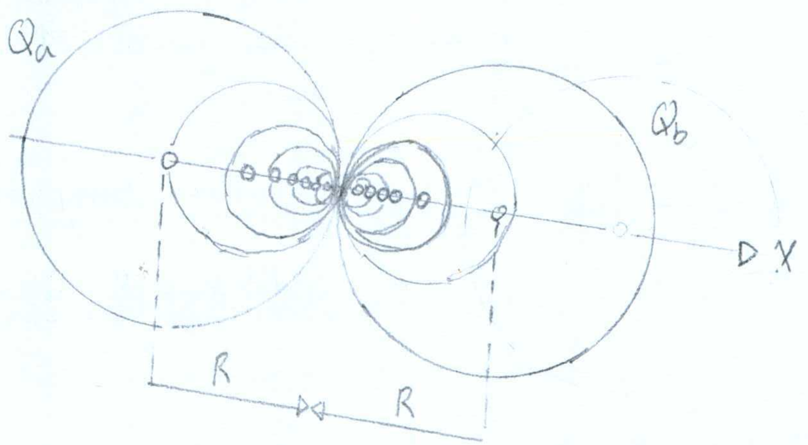
c)

j	q(j)	x(j)
1	q	0
2	$-q/2$	$R/2$
3	$q/3$	$2R/3$
4	$-q/4$	$3R/4$
5	$q/5$	$4R/5$
⋮	⋮	⋮
n	$\frac{(-1)^{j+1} \cdot q}{j}$	$\frac{(j-1)R}{j}$

$$Q = \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j} q$$

$$= q \cdot \ln 2$$

$$q = \frac{Q}{\ln 2}$$



The total sphere's charge by other fractionally smaller charges(spheres) with unequal distances.

Force: $$F = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{R^2}\sum_{j=2}^{n}\sum_{k=2}^{n}\frac{(-1)^{j+k}}{j\cdot k\left[2-\frac{(j-1)}{j}-\frac{(k-1)}{k}\right]^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{R^2}\sum_{j=2}^{n}\sum_{k=2}^{n}\frac{(-1)^{j+k}\cdot j\cdot k}{(j+k)^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{(\ln 2)^2}\frac{1}{R^2}(0.0739)$$

$$= 0.615 F_p \quad \text{when} \quad F_p = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{R^2 4}$$

Capacitance on the surface:

$$\phi(R) = \frac{1}{4\pi\varepsilon_0}\frac{q}{2R}$$

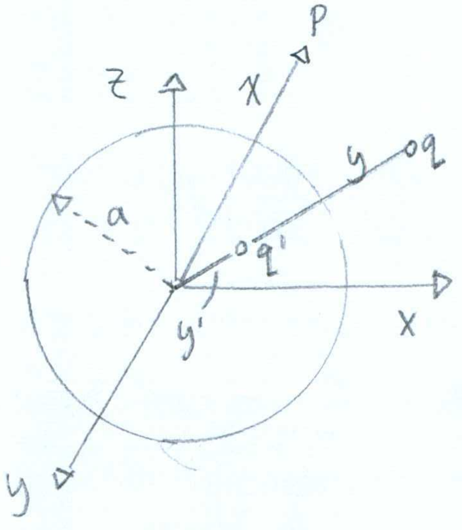
$$= \frac{1}{4\pi\varepsilon_0}\frac{Q}{2\ln 2 R}$$

$$= \frac{Q}{C}$$

$$\frac{C}{4\pi\varepsilon_0 R} = 2\ln 2$$

$$= 1.386$$

2.7.



sphere of radius a, centered at origin

Shape: sphere
Dimension: Volume/Area [3D/2D]
Charge: q, q'

(Page 38) Dirichlet Boundary Condition

$$\int_V |\nabla U|^2 d^3x = 0$$

Potential on a sphere's surface, at distance, is zero.

(Equation 2.16) Green's Theorem

$$G(x, x') = \frac{1}{|x-x'|} - \frac{a}{x\left|x - \frac{a^2}{x^2}x'\right|}$$

a) $\Phi_1 = \Phi_2$

$$\underbrace{\frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}}_{\text{Potential \#1}} = \underbrace{\frac{1}{4\pi\varepsilon_0} \cdot \frac{q'}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}}_{\text{Potential \#2}}$$

Potential #1 = Potential #2

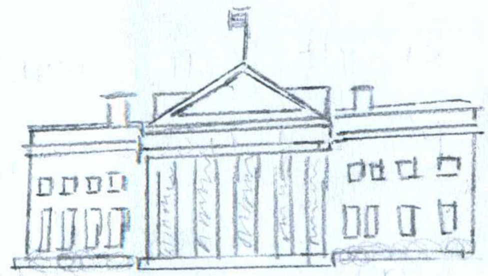
0 = Potential #1 - Potential #2

$$= \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}$$

$$= G(x, x')$$

b) (Equation 1.44)

$$\Phi(X) = \underbrace{\frac{1}{4\pi\varepsilon_0}\int_V \rho(V)\, G_D(X,X')\, d^3X}_{=0,\ \text{for Dirichlet's Boundary}} - \frac{1}{4\pi}\oint\left(\Phi\frac{dG_D}{dn}\right)dn'$$



$$= -\frac{1}{4\pi}\oint\left(\Phi\frac{dG_D}{dn}\right)da'$$

$$= -\frac{1}{4\pi}\int_0^a\int_0^{2\pi}\left(\Phi\frac{dG_D}{dn}\right)\rho'\,d\phi'\,d\rho'$$

$$= \frac{-V}{4\pi}\int_0^a\int_0^{2\pi}\frac{dG_D}{dn}\rho'\,d\phi'\,d\rho'$$

$$= \frac{V}{4\pi}\int_0^a\int_0^{2\pi}\frac{dG_D}{dz}\rho'\,d\phi'\,d\rho'$$

$$= \frac{V}{4\pi}\int_0^a\int_0^{2\pi}\frac{d}{dz}\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right]\rho'\,d\phi'\,d\rho'$$

Cylindrical coordinates: $\rho = \sqrt{x^2+y^2}$; $\phi = \tan^{-1}\left(\frac{y}{x}\right)$; $z = z'$

$$= \frac{V}{4\pi}\int_0^a\int_0^{2\pi}\frac{d}{dz}\left[\frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z-z')^2}} - \frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z+z')^2}}\right]\rho'\,d\phi'\,d\rho'$$

$$= \frac{V}{4\pi}\int_0^a\int_0^{2\pi}\left[\left(-\tfrac{1}{2}\right)\frac{-2(z-z')}{(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z-z')^2)^{3/2}} - \frac{1}{2}\,\frac{2(z+z')}{(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z+z')^2)^{3/2}}\right]\rho'\,d\phi'\,d\rho'$$

$$\Phi(x) = \frac{ZV}{2\pi}\int_0^a\int_0^{2\pi} \frac{\rho\, d\phi' d\rho'}{(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z'^2)^{3/2}} \quad @ z=0$$

c) General equation:

$$\Phi(x) = \frac{ZV}{2\pi}\int_0^a\int_0^{2\pi} \frac{\rho' d\phi' d\rho}{(\rho'^2+z^2)^{3/2}} \qquad \boxed{u=\rho'^2+z^2} \quad \boxed{du = 2\rho' d\rho'}$$

$$= \frac{ZV}{2}\int_0^a \frac{du}{u^{3/2}}$$

$$= -ZV\left[\frac{1}{\sqrt{u}}\right]_0^a$$

$$= \frac{-ZV}{\sqrt{a^2+z^2}} + \frac{ZV}{\sqrt{z^2}}$$

$$= V\left[1 - \frac{Z}{\sqrt{a^2+z^2}}\right]$$

"Potential decreases with larger sphere sizes in Dirichlet's condition"

d) $\rho^2+z^2 \gg a^2$

$$\Phi(x) = \frac{ZV}{2\pi(\rho^2+z^2)^{3/2}}\int_0^{2\pi} d\phi' \int_0^a d\rho'\rho' \left(1+\frac{\rho'^2-2\rho\rho'\cos(\phi-\phi')}{\rho^2+z^2}\right)^{-3/2}$$

$\boxed{\text{Binomial Expansion}}$

$$|1+x|^n = 1+nx+\frac{n(n-1)}{2}x^2+\dots$$

$$= \frac{ZV}{2\pi(\rho^2+z^2)^{3/2}}\int_0^{2\pi} d\phi' \int_0^a \rho' \Big(1-\frac{3}{2}(\rho^2+z^2)^{-1}(\rho'^2-2\rho\rho'\cos(\phi-\phi'))$$

$$+\frac{15}{8}(\rho^2+z^2)^{-2}(\rho'^2-2\rho\rho'\cos(\phi-\phi'))^2+\dots$$

$$= \frac{zV}{2\pi} \frac{1}{(\rho^2+z^2)^{3/2}} \left[\pi a^2 - \frac{3}{2}(\rho^2+z^2)^{-1}\left[\frac{\pi}{2}a^4\right] + \frac{15}{8}(\rho^2+z^2)^{-2} \cdot \left[2\pi\frac{a^6}{6} + 4\rho^2\frac{a^2}{4}\pi\right] + \dots \right.$$

@ $\rho = 0$

$$\phi(x) = \frac{Va^2}{2}\frac{1}{z^2}\left[1 - \frac{3}{4}\frac{a^2}{z^2} + \frac{5}{8}\frac{a^4}{z^4} + \dots\right]$$

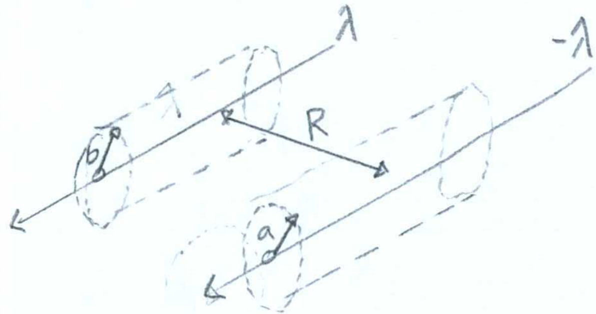
$$= V\left[1 - \left(1 - \frac{a^2}{2z^2} + \frac{3a^4}{8z^4} - \frac{5a^6}{16z^6} + \dots\right)\right]$$

Root expansion
$\hookrightarrow (1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots$

$$= V\left[1 - \left(1 + \frac{a^2}{z^2}\right)^{-1/2}\right]$$

$$= V\left[1 - \frac{z}{\sqrt{a^2+z^2}}\right]$$

2.8



Two straight, parallel line charges

Shape: Line
Dimension: Line [1D]
Charge: $\lambda, -\lambda$

a) Gauss's Law:

$$\int E \cdot ds = \frac{1}{\varepsilon_0}\int \rho(x)\, dl$$

$$E\int ds = \frac{\lambda}{\varepsilon_0}$$

$$E\int_0^{2\pi}\int_0^{r}\int_0^{1} dL\, dr\, d\phi = \frac{\lambda}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi r \varepsilon_0}$$

$$= -\nabla\phi$$

$$\phi = -\int_{x-R}^{x} \frac{\lambda}{2\pi r \varepsilon_0}\, dr$$

$$= \frac{\lambda}{2\pi \varepsilon_0} \ln \frac{x}{x-R}$$

$= V$ "Dirichlet's Boundary condition"

$$\frac{x}{x-R} = e^{\frac{V 2\pi \varepsilon_0}{\lambda}}$$

$$x e^{\frac{-V 2\pi \varepsilon_0}{\lambda}} = x - R$$

$$x^2 e^{\frac{-V \cdot 4\pi \varepsilon_0}{\lambda}} = (x-R)^2$$

$$\left(1 - e^{\frac{-4V\pi\varepsilon_0}{\lambda}}\right) x^2 - 2xR + R^2 = 0$$

$$\left(x - \frac{xR}{1 - e^{\frac{-4V\pi\varepsilon_0}{\lambda}}}\right)^2 = \frac{e^{\frac{-4V\pi\varepsilon_0}{\lambda}} R^2}{\left(1 - e^{\frac{-4V\pi\varepsilon_0}{\lambda}}\right)^2}$$

$\boxed{\text{Square complete}}$

$$= (x - x_0)^2$$

$$= \rho^2$$

$$x_0 = \frac{R}{1 - e^{-2V\pi\varepsilon_0/\lambda}}$$

$$\boxed{\sinh x = \frac{e^{x} - e^{-x}}{2}}$$

$$\rho = \frac{e^{\frac{-2V\pi\varepsilon_0}{\lambda}} R}{1 - e^{\frac{-2V\pi\varepsilon_0}{\lambda}}}$$

$$= \frac{R}{2\sinh[2\pi\varepsilon_0 V/\lambda]}$$

b) $d = a + b$

$$= \frac{R}{2\sinh[2\pi\varepsilon_0 V_A/\lambda]} - \frac{R}{2\sinh[2\pi\varepsilon_0 V_B/\lambda]}$$

Cosh relationship:

$$\cosh(a-b) = \cosh(a)\cosh(b) - \sinh(a)\sinh(b)$$

$$\cosh^2(a-b) = (\cosh(a)\cosh(b) - \sinh(a)\sinh(b))^2$$

$$2\sinh(a)\sinh(b)\cosh(a-b) = 0$$

$$= \sin^2(a-b) + \sinh^2(a) + \sinh^2(b)$$

$$= \frac{R}{2}\,\frac{\sinh[2\pi\varepsilon_0 V_A/\lambda] - \sinh[2\pi\varepsilon_0 V_B/\lambda]}{\sinh[2\pi\varepsilon_0 V_A/\lambda]\circ\sinh[2\pi\varepsilon_0 V_B/\lambda]}$$

$$= \frac{R}{2}\,\frac{\sinh[2\pi\varepsilon_0 (V_A - V_b)/\lambda]}{\sinh[2\pi\varepsilon_0 V_A/\lambda]\sinh[2\pi\varepsilon_0 V_B/\lambda]}$$

Cosh relationship:

$$\cosh(a-b) = \cosh(a)\cosh(b) - \sinh(a)\sinh(b)$$

$$\cosh^2(a-b) = (\cosh(a)\cosh(b) - \sinh(a)\sinh(b))^2$$

$$2\sinh(a)\sinh(b)\cosh(a-b) = 0$$

$$= \sinh^2(a-b) + \sinh^2(a) + \sinh^2(b)$$

$$\cosh(a-b) = \frac{1}{2}\left(\frac{\sinh(a)}{\sinh(b)} + \frac{\sinh(b)}{\sinh(a)} - \frac{\sinh^2(a-b)}{\sinh(a)\sinh(b)}\right)$$

$$\cosh\left(\frac{2\pi\varepsilon_0(V_A - V_b)}{\lambda}\right) = \frac{1}{2}\left(\frac{-a}{b} - \frac{b}{a} + \frac{d^2}{ab}\right)$$

$$= \frac{d^2 - a^2 - b^2}{2ab}$$

$$\text{Capacitance} = \frac{\lambda}{V_B - V_A} = 2\pi\varepsilon_0 \cosh\left(\frac{d^2 - a^2 - b^2}{2ab}\right)$$

$$\frac{1}{C} = \frac{1}{2\pi\varepsilon_0}\cosh^{-1}\left(\frac{d^2-a^2-b^2}{2ab}\right)$$

c) Identity : $\cosh^{-1}(z) = \log\left(z+\sqrt{z^2-1}\right)$

$$\frac{1}{C} = \frac{1}{2\pi\varepsilon_0}\log\left(z+\sqrt{z^2+1}\right) \dots \text{ when } z = \frac{d^2-a^2-b^2}{2ab}$$

Puiseux Series : A power series for negative and fractional exponents because Taylor series' fractional factorial problem.

- "Recursive, inductive, residual, or remainder process"
- $F(x) = \sum_{k \geq m} a_k t^{k/m}$

Identity : $\log\left(z+\sqrt{z^2-1}\right) = \log(2z) - \frac{1}{4z^2} + O\left(\left(\frac{1}{z}\right)^4\right)$

$$\frac{1}{C} = \frac{1}{2\pi\varepsilon_0}\log\left(z+\sqrt{z^2-1}\right)$$

$$= \frac{1}{2\pi\varepsilon_0}\left[\log(2z) - \frac{1}{4z^2} + O\left(\left(\frac{1}{z}\right)^4\right)\right]$$

$$= \frac{1}{2\pi\varepsilon_0}\left[\log\left(\frac{d^2}{ab}\right) + \log\left(1-\frac{a^2+b^2}{d^2}\right) - \frac{a^2b^2}{d^2}\left(1-\frac{a^2+b^2}{d^2}\right)^{-2} + \dots\right]$$

$$C \cong \frac{2\pi\varepsilon_0}{\log\left(\frac{d^2}{a\cdot b}\right)} \text{ if radius } a = \text{radius } b$$

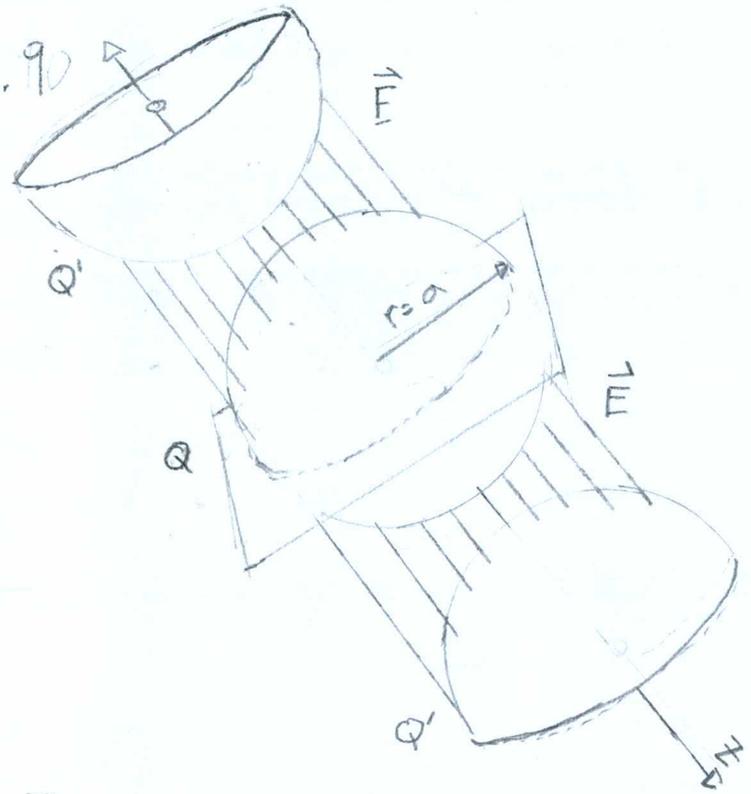
$$\cong \frac{\pi\varepsilon_0}{\log\left(\frac{d}{a}\right)}$$

d) $\frac{1}{C} = \frac{1}{2\pi\varepsilon_0}\cosh^{-1}\left(\frac{a^2+b^2-d^2}{2ab}\right)$ when $d=0$

$$= \frac{1}{2\pi\varepsilon_0}\cosh^{-1}\left(\frac{a^2+b^2-d^2}{2ab}\right)$$

$$\cong \frac{\log(|a/b|)}{2\pi\varepsilon_0}$$

2.9)



Insulated, spherical conducting shell

Shape: Sphere
Dimension: Surface area [2D]
Charge: Zero or Q

a) (Section 2.5)

(Equation 2.12) Potential

$$\phi = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{(r^2+R^2+2rR\cos\theta)^{1/2}} - \frac{1}{(r^2+R^2-2rR\cos\theta)^{1/2}} - \frac{a}{R\left(r^2+\frac{a^4}{R^2}+\frac{2a^2r}{R}\cos\theta\right)^{1/2}} - \frac{a}{R\left(r^2+\frac{a^4}{R^2}-\frac{2a^2r}{R}\cos\theta\right)^{1/2}}\right]$$

(Equation 2.13)

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{-2Q}{R^2}r\cos\theta + \frac{2Q}{R^2}\frac{a^3}{r^2}\cos\theta\right] + \cdots$$

$$E = -\nabla\phi$$

$$= -\left[\frac{\partial}{\partial r} + \frac{\partial}{\partial\theta}\right]\phi\Big|_{r=a,\ \theta=0}$$

$$= \left[E_0 - \frac{E_0 a^3}{r^3}\right]_{r=a}(2\cos\theta + \sin\theta)$$

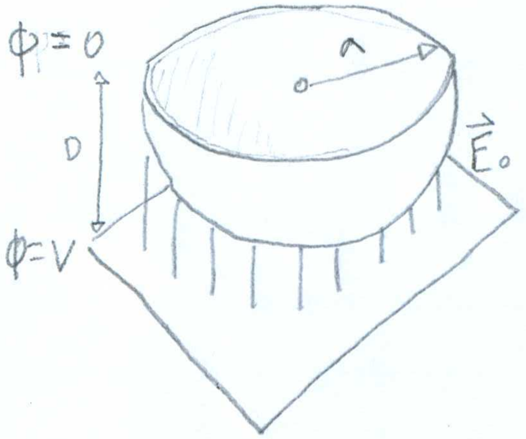
$$= -3E_0\cos\theta$$

$$\sigma = -\varepsilon_0\nabla\phi$$

$$= +3\varepsilon_0 E_0\cos\theta$$

b) $Q = \int \sigma\, da = \int_0^{2\pi}\int_0^{\pi/2} 3\varepsilon_0 E_0 \cos\theta\, r^2 \sin\theta\, d\theta\, d\phi = 3\varepsilon_0 E_0 \pi a^2$ when $r=a$

2.10



A large parallel plate capacitor with a boss

Shape: Plane, hemisphere
Dimension: Surface Area [2D]
Charge: Q

a) Surface-charge density:

(Equation 2.13)

$$Q = \frac{1}{4\pi\varepsilon_0}\left[\frac{-2Q}{R^2}\cos\theta + \frac{2Q}{R^2}\frac{a^3}{r^2}\cos\theta\right]$$

$$\phi = -E_0\left(r - \frac{a^3}{r^2}\cos\theta\right)$$

$$\sigma = -\varepsilon_0\nabla\phi$$

$$= -\varepsilon_0\left[\frac{\partial}{\partial r} + \frac{\partial}{\partial\theta}\right]\phi$$

$$= 3\varepsilon_0 E_0\cos\theta$$

b) Gauss' Law: $\nabla\cdot E = \frac{\sigma}{\varepsilon_0}$

-or

$$\int E\cdot ds = \frac{\sigma}{\varepsilon_0}$$

$$\frac{Q}{\varepsilon_0} = \int E\cdot ds$$

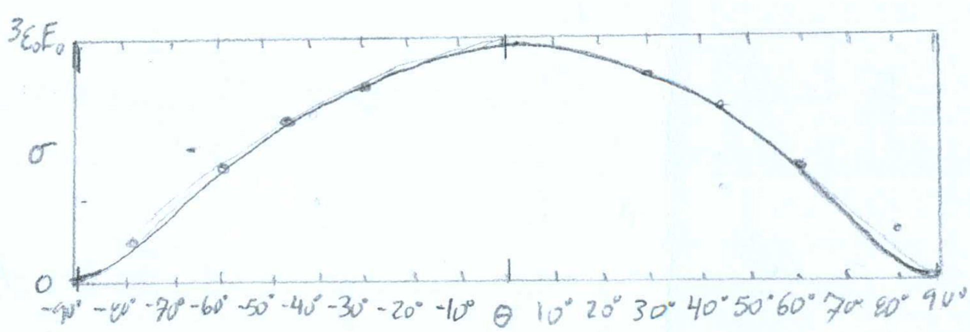
$$Q = \varepsilon_0\int E\cdot ds$$

$$= \varepsilon_0 E_0\int 3\cos\theta\, ds$$

$$= 3\varepsilon_0 E_0\int_0^{2\pi}\int_0^{\pi/2}\cos\theta\, r^2\sin\theta\, d\theta\, d\phi$$

$$= 3\varepsilon_0 E_0(2\pi)\cdot\frac{\pi}{2}a^2$$

$$= 3\varepsilon_0 E_0 a^2\pi$$



The surface charge density largest amplitude is up.

$F_z = F\cos\theta$

$= \sigma E \cos\theta$

$= \dfrac{\int \sigma^2 \cos\theta}{\varepsilon_0 \int dz}$ because Gauss' law

$= \dfrac{1}{2\pi} \int_0^{2\pi}\int_0^{\pi/2} (3\varepsilon_0 E_0 \cos^2\theta)^2 \cos\theta\, r^2 \sin\theta\, d\theta\, d\phi$

$= \dfrac{a^2}{2\varepsilon_0}(9\varepsilon_0 \pi E_0^2) \cdot 2\pi \int_0^{\pi/2} \cos^3\theta \sin\theta\, d\theta$

$= \dfrac{9}{4}\pi \varepsilon_0 E_0^2$

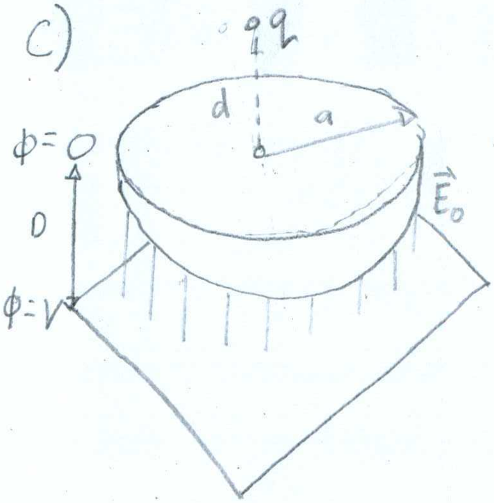
c) $\sigma = 3\varepsilon_0 E_0 \cos\theta + \dfrac{Q}{4\pi a^2}$

$F = \dfrac{a^2}{2\varepsilon_0} \int_0^{2\pi}\int_0^{\pi/2} \left(3\varepsilon_0 E_0 \cos\theta + \dfrac{Q}{4\pi a^2}\right)^2 \cos\theta \sin\theta\, d\theta\, d\phi$

$= \dfrac{a^2}{2\varepsilon_0} \int_0^{2\pi}\int_0^{\pi/2} \left(9\varepsilon_0 E_0^2 \cos^2\theta + 6\varepsilon_0 E_0 \cos\theta \dfrac{Q}{4\pi a^2} + \dfrac{Q}{16\pi^2 a^4}\right) \cos\theta \sin\theta\, d\theta\, d\phi$

$= \dfrac{9}{4}\pi a^2 \varepsilon_0 E_0^2 + \dfrac{Q^2}{32\pi \varepsilon_0 a^2} + \dfrac{E_0 Q}{2}$

"Charges" "Force between spheres" "Base electric field" c)



A large parallel plate capacitor, with a boss and a charge

Shape: Plane, hemisphere
Dimension: Surface Area [2D]
Charges: Q, q

(Equation 2.5)

$$\sigma = -\frac{q}{4\pi a^2}\,\frac{a}{d}\,\frac{1-\frac{a^2}{d^2}}{\left(1+\frac{a^2}{d^2}-2\frac{a}{d}\cos\theta\right)^{3/2}}$$

Total charge:

$$Q = \int \sigma\, da$$

$$= \left(\frac{-q}{4\pi a^2}\right)\left(\frac{a}{d}\right)\left(1-\frac{a^2}{d^2}\right)\int \frac{dA}{\left(1+\frac{a^2}{d^2}-2\left(\frac{a}{d}\right)\cos\theta\right)^{3/2}}$$

$$= \left(\frac{-q}{4\pi a^2}\right)\left(\frac{a}{d}\right)\left(1-\frac{a^2}{d^2}\right)\int_0^{2\pi} d\phi \int_0^{\pi/2} \frac{r^2 \sin\theta\, d\theta}{\left(1+\frac{a^2}{d^2}-2\left(\frac{a}{d}\right)\cos\theta\right)^{3/2}}$$

When $r = a$

$$= \left(\frac{-q}{2}\right)\left(\frac{a}{d}\right)\left(1-\frac{a^2}{d^2}\right)\int_0^{\pi/2} \frac{1}{\left(\left(\frac{a}{d}\right)^2+1\right)^{3/2}}\left(1+\frac{-2\left(\frac{a}{d}\right)\cos\theta}{\left(\frac{a}{d}\right)^2+1}\right)^{-3/2}\sin\theta\, d\theta$$

$$= \left(\frac{-q}{2}\right)\left(\frac{a}{d}\right)\left(1-\frac{a^2}{d^2}\right)\cdot\frac{1}{\left(\left(\frac{a}{d}\right)^2+1\right)^{3/2}}\int_0^{\pi/2}\Big[1-\frac{3}{2}\left(\left(\frac{a}{d}\right)^2+1\right)^{-1}\left(-2\left(\frac{a}{x}\right)\cos\theta\right)\cdots$$
$$+\frac{15}{8}\left(\left(\frac{a}{d}\right)^2+1\right)^{-2}\left(-2\left(\frac{a}{x}\right)\cos\theta\right)^2+\cdots\Big]\sin\theta\, d\theta$$

Binomial expansion
$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots$$

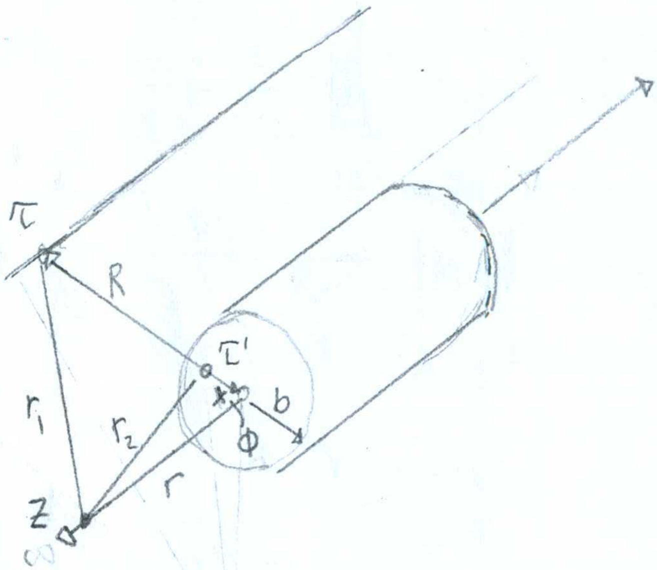
$$= \left(\frac{-q}{2}\right)\left(\frac{a}{d}\right)\left(1-\frac{a^2}{d^2}\right)\cdot\frac{1}{\left(\left(\frac{a}{d}\right)^2+1\right)^{3/2}}\left[1-\frac{3}{2}\left(\left(\frac{a}{d}\right)^2+1\right)^{-1}\cdot\left(-\left(\frac{a}{d}\right)\right)^4+\frac{5}{8}\left(\left(\frac{a}{d}\right)^2+1\right)^{-2}\left(\frac{a}{d}\right)^2+\cdots\right]$$

$$= -q\left[1 + \left[\frac{d^2-a^2}{d(a^2+d^2)^{1/2}}\right]\left(1 - \frac{1}{2}\left(\frac{a}{d}\right)^2\left(1+\frac{a}{d}\right)^2 + \frac{3}{4}\left(\frac{a}{d}\right)^4\left(1+\frac{a}{d}\right)^4 - \frac{5}{8}(\ldots)\right)\right]$$

$$\cong -q\left(1 - \frac{d^2-a^2}{d\sqrt{a^2+a^2}}\right)$$

$$\cong -q\left(1 - \frac{d^2-a^2}{d\sqrt{a^2+d^2}}\right)$$

2.11.



Line charge next to a conducting cylinder

Shape : Line
Dimension : Line
Charge : τ

a) <u>Magnitude:</u> $\tau = |\tau'|$

<u>Radial Distance for an image charge:</u>

$$\tau^2(b^2+R^2-2Rb\cos\phi) = \tau'^2(b^2+x^2-2Rx\cos\phi)$$

$$b^2+R^2 = b^2+x^2$$

$$R = x$$

Quadratic: $R^2 - x^2\left[1+\left(\frac{b}{x}\right)^2\right] - b^2 = 0$

$$R^2 - xR\left[1+\left(\frac{b}{x}\right)^2\right] - b^2 = 0$$

$$R = \frac{x}{2}\left[1+\left(\frac{b}{x}\right)^2\right] \pm \sqrt{\left\{\frac{x}{2}\left[1+\left(\frac{b}{x}\right)^2\right]\right\}^2 - b^2}$$

$$= \frac{x}{2}\left[1+\left(\frac{b}{x}\right)^2\right] \pm \left[1-\left(\frac{b}{x}\right)^2\right]$$

$$R = \frac{b^2}{x}$$

Distance: $x = \frac{b^2}{R}$ From cylinder center to balance charges

b) $$\phi(r,\phi) = \frac{\tau}{4\pi\varepsilon_0}\ln\left(r^2+\left(\frac{b^2}{R}\right)^2 - 2r\left(\frac{b^2}{R}\cos\phi\right)\right) + \frac{\tau'}{4\pi\varepsilon_0}\ln\left(r^2+R^2-2Rr\cos\phi\right)$$

$$= \frac{\tau}{4\pi\varepsilon_0} \ln\left(\frac{R^2r^2+b^4-2rRb^2\cos\phi}{R^2(r^2+R^2-2Rr\cos\phi)}\right)$$

at large distances $r >> R$

$$\cong \frac{\tau}{4\pi\varepsilon_0} \ln\left(\frac{(Rr)^2}{(Rr)^2} \frac{\left(1-\frac{2b^2}{Rr}\cos\phi\right)}{\left(1-\frac{2R}{r}\cos\phi\right)}\right)$$

$$\cong \frac{\tau}{4\pi\varepsilon_0} \ln\left(\left(1-\frac{2b^2}{Rr}\cos\theta\right)\left(1+\frac{2R}{r}\cos\phi\right)\right)$$

$$\cong \frac{\tau}{4\pi\varepsilon_0} \ln\left(\left(1+\left(\frac{2R}{r}\cos\phi - \frac{2b^2}{Rr}\cos\phi\right)\right)\right) \quad \text{when } \frac{4b^2}{r^2}\cos^2\phi = 0 \text{ at } r >> R$$

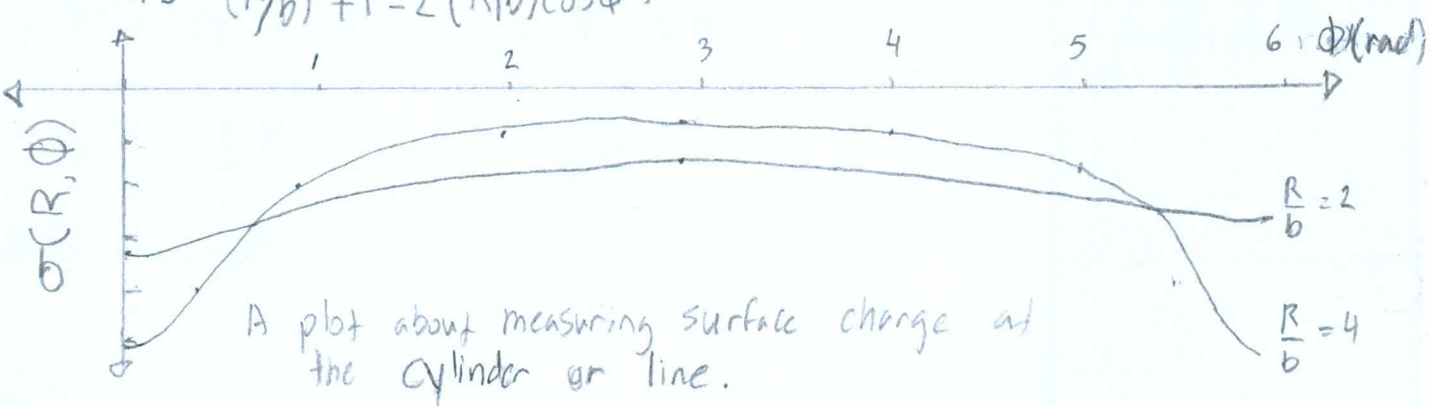
$$\cong \frac{\tau}{2\pi\varepsilon_0} \frac{(R^2-b^2)}{Rr}\cos\phi \quad \boxed{\ln(1+x) \approx x}$$

c) $\sigma = -\varepsilon_0 \frac{\partial\Phi}{\partial r}\Big|_{r=b}$

$$= -\varepsilon_0 \frac{\partial}{\partial r}\left[\frac{\tau}{4\pi\varepsilon_0} \ln\left(\frac{R^2r^2+b^4-2rRb^2\cos\phi}{R^2(r^2+R^2-2Rr\cos\phi)}\right)\right]$$

$$= \frac{-\tau}{4\pi}\left(\frac{R^2(r^2+R^2-2Rr\cos\phi)}{R^2r^2+b^4-2rRb^2\cos\phi}\right)\left(\frac{(2Rr-2Rb^2\cos\phi)(R^2(r^2+R^2-2Rr\cos\phi)) - (R^2-r^2+b^4-2rRb^4\cos\phi)(R^2(2r-2R\cos\phi))}{[R^2(r^2+R^2-2Rr\cos\phi)]^2}\right)$$

$$= -\frac{\tau}{2\pi b}\left(\frac{(R/b)^2-1}{(R/b)^2+1-2(R/b)\cos\phi}\right)$$



A plot about measuring surface charge at the cylinder or line.

d) Gauss' Law: $\int E \cdot dl = \frac{\tau}{\varepsilon_0}$

$$E = \frac{\tau}{\int_0^{2\pi}\int_0^{d} \varepsilon_0}$$

$$= \frac{\tau}{2\pi \varepsilon_0 d} \qquad d = R - \frac{b^2}{R}$$

Force by Lorentz's Law in the absence of a magnetic field:

$$F = \tau E$$
$$= \frac{\tau^2 R}{2\pi \varepsilon_0 (R^2 - b^2)}$$

2.12. (Equation 2.71)

$$\Phi(\rho, \phi) = a_0 + b_0 \ln \rho + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n) + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\phi + \beta)$$

Lesson: Normally, four variables $\rightarrow$ four equations

General solutions, four variables $\rightarrow$ four boundary conditions

Boundary conditions: min $\Phi(\rho, 0)$ max $\Phi(\rho, 2\pi)$

min $\Phi(0, \phi)$ max $\Phi(b, \phi)$

in the $\mathbb{R}$eal domain

When $\rho = 0$, $\Phi(\rho=0, \phi) = a_0 + b_0 \ln(0)$
$= \infty$

So, $b_0 = 0$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\Phi(\rho, \phi) = a_0 + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha)$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \rho^n \frac{e^{i(n\phi+\alpha)} - e^{-i(n\phi+\alpha)}}{2i}$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2i} \rho^n \left[c e^{in\phi} + c e^{-in\phi} \right]$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{1}{2i} \left(\frac{\rho}{b}\right)^n \left[c e^{in\phi} + c e^{-in\phi} \right]$$

"The unknown coefficient"

$$V(\phi) = \sum_{n=-\infty}^{\infty} c e^{in\phi}$$

Boundary condition: $\rho = b$

$$V(\phi) e^{-in'\phi} = \sum_{n=-\infty}^{\infty} c e^{in\phi} e^{-in'\phi}$$

$$\int_0^{2\pi} V(\phi) e^{-in'\phi} d\phi = \int_0^{2\pi} \sum_{n=-\infty}^{\infty} c e^{-in\phi} e^{-in'\phi} d\phi$$

$$= \sum_{n=-\infty}^{\infty} c_n (2\pi \delta)$$

Dirac's Delta Function

$$\delta(p) = \frac{1}{2\pi} \int_0^{2\pi} e^{ipx} dp$$

$$C_n = \int_0^{2\pi} \frac{V(\phi) e^{-in'\phi} d\phi}{2\pi}$$

∞ ... when 100% certain experiment shows a paper answer, then $\delta = 1$.

$$\Phi(\rho, \phi) = \frac{1}{2\pi} \int_0^{2\pi} V(\phi) d\phi \sum_{n=\infty}^{\infty} \left(\frac{\rho}{b}\right)^n e^{-in(\phi-\phi')} d\phi$$

$$= \frac{1}{2\pi} \int_0^{2\pi} V(\phi) d\phi \left[-1 + \sum_{n=0}^{\infty} \left(\frac{\rho}{b}\right)^n e^{in(\phi-\phi')} + \sum_{n=0}^{\infty} \left(\frac{\rho}{b}\right) e^{-in(\phi-\phi')} \right]$$

Geometric Identity : $\sum_0^{\infty} r^n = \frac{1}{1-r}$

$$= \frac{1}{2\pi} \int_0^{2\pi} V(\phi) d\phi \left[-1 + \frac{1}{1-\left(\frac{\rho}{b}\right)^n e^{i(\phi-\phi')}} + \frac{1}{1-\left(\frac{\rho}{n}\right)^n e^{-i(\phi-\phi')}} \right]$$

$$= \frac{1}{2\pi} \int_0^{2\pi} V(\phi) d\phi \left[\frac{1-\left(\frac{\rho}{b}\right) e^{i(\phi-\phi')} + 1 - \left(\frac{\rho}{b}\right) e^{-i(\phi-\phi')} - \left(1-\left(\frac{\rho}{b}\right) e^{i(\phi-\phi')}\right)\left(1-\left(\frac{\rho}{b}\right) e^{-i(\phi-\phi')}\right)}{\left(1-\left(\frac{\rho}{b}\right) e^{i(\phi-\phi')}\right)\left(1-\left(\frac{\rho}{b}\right) e^{-i(\phi-\phi')}\right)} \right]$$

$\boxed{\Phi(b, \phi) = V(\phi)}$ $\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$

$$= \frac{1}{2\pi} \int_0^{2\pi} \Phi(b, \phi) \frac{b^2 - \rho^2}{b^2 + \rho^2 - 2b\rho\cos(\phi-\phi')} d\phi$$ "Inside" cylinder

$$= \frac{1}{2\pi} \int_0^{2\pi} \Phi(b, \phi) \frac{\rho^2 - b^2}{b^2 + \rho^2 - 2b\rho\cos(\phi-\phi')} d\phi$$ "Outside" cylinder

④ General Solution to Laplace's Equation 2^{nd} order O. D. E.:

$$\Phi(\rho,\phi)=\underbrace{(a_0+b_0\ln\rho)}_{R_0}\times\underbrace{(A_0+B_0\phi)}_{\Theta_0}+\sum\underbrace{(a_\nu\rho^{\nu}+b_\nu\rho^{-\nu})}_{R_n}\times\underbrace{(A_\nu e^{i\nu\phi}+B_\nu e^{-i\nu\phi})}_{\Theta_n}$$

⑤ Variables by boundary conditions and shift:

b_0, b_ν $\quad \Phi(\rho=0,\phi)=(a_0+b_\nu\ln 0)(A_0+B_0\phi)+\sum(a_\nu 0^{\nu}+b_\nu 0^{-\nu})(A_\nu e^{0}+B_\nu e^{0})$

$$=\pm\infty$$

So, $b_0=0$ and $b_\nu=0$ for a real solution.

B_0 $\quad \Phi(\rho,\phi)=\Phi(\rho,\phi+2\pi)$

$$=a_0(A_0+B_0\phi)+\sum a_\nu\rho^{\nu}(A_\nu e^{i\nu\phi}+B_\nu e^{-i\nu\phi})$$

$$=a_0(A_0+B_0(\phi+2\pi))+\sum a_\nu\rho^{\nu}(A_\nu e^{i\nu(\phi+2\pi)}+B_\nu e^{-i\nu(\phi+2\pi)})$$

$$B_0\phi\neq B_0(2\pi+\phi)$$

So, $B_0=0$

A_0 $\quad \Phi(\rho=b,\phi)=V(\phi)$

$$V(\phi)=A_0+\sum_{n=1}^{\infty}b^{n}(A_n e^{in\phi}+B_n e^{-in\phi})$$

$$\int_0^{2\pi}V(\phi)=\int_0^{2\pi}A_0+\sum_{n=1}^{\infty}b^{n}(A_n e^{in\phi}+B_n e^{-in\phi})\,d\phi$$

Eulers Identity

$e^{i\pi}+1=0$

$e^{in\pi}+1=0$

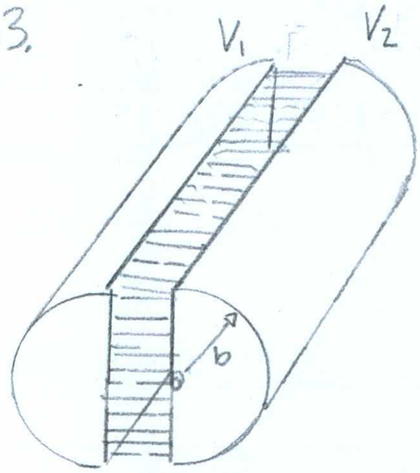
$$\int_{\pi/2}^{\pi/2}V_1\,d\phi+\int_{\pi/2}^{3\pi/2}V_2\,d\phi=A_0 2\pi$$

$$V_1\pi+V_2\pi=A_0 2\pi$$

$$A_0=\frac{V_1+V_2}{2}$$

Note: The exact Solution above describes each, and together: spheres, bosses, cylinders, corners ($90°$) and curved cylinder corners.

2.13.



Two hollow half conducting cylinders

Shape:	Half-cylinder
Dimension:	Area [2D]
Charge:	q

a) Potential Derivation:

① Laplace Equation

$$\nabla^2\phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\phi^2}$$

$$= 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) = \frac{-1}{\rho^2}\frac{\partial^2\phi}{\partial\phi^2}$$

② Boundary Conditions:

Angular Boundaries:

$$\phi(\rho,\phi) = R(\rho)\Psi\left(\frac{3\pi}{2} < \phi < \frac{\pi}{2}\right) = V_1$$

$$\phi(\rho,\phi) = R(\rho)\Psi\left(\frac{\pi}{2} < \phi < \frac{3\pi}{2}\right) = V_2$$

Radial Boundaries:

$$\phi(\rho=0,\phi) = \text{Real, finite}; \ \phi(\rho=b,\phi) = V(\phi)$$

③ Laplace Equation Solutions:

Ⓐ Variable Separation: $f = R''\Theta + \frac{1}{\rho}R'\Theta + \frac{1}{R^2}R\Theta'' = 0.$

Ⓑ Angular Eigenvalues: $\Theta'' + \lambda\Theta = 0$

Ⓒ Radial Eigenvalues: $\rho R' + \lambda R = 0$

Ⓓ General Solution: $\phi(\rho,\theta) = R_0\Theta_0 + R_n\Theta_n$

$$B_n = \frac{1}{2\pi b^n}\int_0^{2\pi} V(\phi) e^{i\pi\phi}\, d\phi$$

$$= \frac{1}{2\pi b^n}\left[V_1 \int_{-\pi/2}^{\pi/2} e^{in\phi}\, d\phi + V_2 \int_{\pi/2}^{3\pi/2} e^{in\phi}\, d\phi\right]$$

$$= \frac{(-1)^{(n+1)/2}}{-n\pi b^n}[V_1 - V_2]$$

$$\Phi(\rho,\phi) = \frac{V_1+V_2}{2} + \sum_{n=1}^{\infty} \rho^n \frac{(-1)^{(n+1)/2}}{-n\pi b}[V_1-V_2]\left[e^{in\phi} + e^{-in\phi}\right]$$

$$= \frac{V_1+V_2}{2} + \frac{V_1-V_2}{2}\frac{4}{2}\sum_{n=1}^{\infty} \frac{(-1)(-1)^{(n+1)/2}}{n}\left(\frac{\rho e^{i\phi}}{b}\right)^n$$

$$\boxed{\text{Taylor's Expansion for: } \arctan^{-1}(x) = \sum_{i=1}^{n} \frac{(-1)(-1)^{(n+1)/2}}{n} x^n}$$

$$= \frac{V_1+V_2}{2} + \frac{V_1-V_2}{2}\frac{4}{\pi}\tan^{-1}\left(\frac{\rho e^{i\phi}}{b}\right)$$

$$= \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)$$

b) $\sigma = -\varepsilon_0 \frac{\partial \Phi}{\partial \rho}$

$$= \varepsilon_0 \frac{\partial}{\partial \rho}\left(\frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)\right) \qquad \boxed{\frac{d}{dx}\tan(x) = \frac{x}{x^2+1}}$$

$$= \varepsilon_0 \frac{V_2-V_1}{\pi} \frac{1}{\frac{2b\rho}{b^2-\rho^2}\cos\phi + 1} \frac{2b\rho}{b^2-\rho^2}\cos\phi$$

A_n $\quad \Phi(\rho=b,\phi)=V(\phi)$

$$V(\phi)=A_0+\sum_{n=1}^{\infty}b^n(A_n e^{in\phi}+B_n e^{-in\phi})$$

$$\int_0^{2\pi}V(\phi)e^{-in\phi}d\phi=A_0\int_0^{2\pi}e^{-in'\phi}+\sum_{n=1}^{\infty}b^n\left(A_n\int_0^{2\pi}e^{i(n-n')\phi}d\phi+B_0\int_0^{2\pi}e^{-i(n-n')\phi}d\phi\right)$$

$$=2\pi b^n A_n$$

$$A_n=\frac{1}{2\pi b^n}\int_0^{2\pi}V(\phi)e^{-i\pi\phi}d\phi$$

$$=\frac{1}{2\pi b^n}\left[V_1\int_{-\pi/2}^{\pi/2}e^{-in\phi}d\phi+V_2\int_{-\pi/2}^{3\pi/2}e^{-in\phi}d\phi\right]$$

$$=\frac{1}{2\pi b^n}\left[V_1\left[\frac{e^{-in\phi}}{-in}\right]_{-\pi/2}^{\pi/2}+V_2\left[\frac{e^{-in\phi}}{-in}\right]_{\pi/2}^{3\pi/2}\right]$$

$$=\frac{(-1)^{(n+1)/2}}{-n\pi b^n}(V_1-V_2)$$

B_n $\quad \Phi(\rho=b,\Phi)=V(\phi)$

$$V(\phi)=A_0+\sum_{n=1}^{\infty}b^n(A_n e^{in\phi}+B_n e^{-in\phi})$$

$$\int_0^{2\pi}V(\phi)e^{in\phi}d\phi=A_0\int_0^{2\pi}e^{in\phi}+\sum_{n=1}^{\infty}b^n\left(A_n\int_0^{2\pi}e^{i(n+n')\phi}d\phi+B_n\int_0^{2\pi}e^{i(n'-n)\phi}d\phi\right.$$

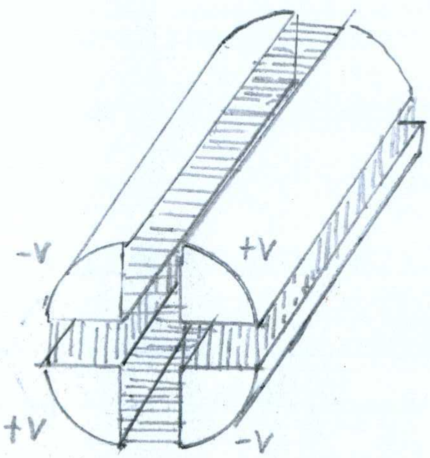
$$\int_0^{2\pi}V(\phi)e^{-in\phi}d\phi=2\pi b^n B_n$$

$$= \varepsilon_0 \frac{V_1 - V_2}{\pi} \left(\frac{2b\cos\phi\,(b^2+\rho^2)}{(b^2-\rho^2)^2 + (2b\rho\cos\theta)^2} \right)$$

2.14

a) (Equation 2.71)

$$\Phi(\rho,\phi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha) + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\phi + \beta)$$



Four hollow quarter Conducting cylinders

Shape: Quarter Cylinder

Dimension: Area [2D]

Charge: q

Potential Derivation:

① Laplace Equation

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) = \frac{-1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}$$

② Boundary Conditions:

Angular Boundaries:

$$\Phi(\rho,\phi) = \Phi(\rho, 0<\phi<\tfrac{\pi}{2}) = \Phi(\rho, \pi<\phi<\tfrac{3\pi}{2}) = +V$$

$$\Phi(\rho,\phi) = \Phi(\rho, \tfrac{\pi}{2}<\phi<\pi) = \Phi(\rho, \tfrac{3\pi}{2}<\phi<2\pi) = -V$$

$$\Phi(\rho,\phi) = \Phi(\rho,0) = \Phi(\rho,2\pi)$$

Radial Boundaries:

$$\Phi(\rho=0,\phi) = \text{finite}$$

$$\Phi(\rho=b,\phi) = V(\phi)$$

③ Laplace Equation Solutions:

① Variable Separation: $f = R''\Theta + \frac{1}{\rho}R'\Theta + \frac{1}{R^2}R\Theta'' = 0$

② Angular Eigenvalues: $\Theta'' + \lambda\Theta = 0$

③ Radial Eigenvalues: $\rho R' + \lambda R = 0$

④ General Solution: $\Phi(\rho, \phi) = R_0\Theta_0 + R_n\Theta_n$

④ General Solution to Laplaces Equation:

$$\Phi(\rho, \phi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n) + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\phi + \beta)$$

Note: Another general solution is prior to Equation 2.71. Each function works with either: two summations for twice the amount; or an unknown coefficient as twice the total amount.

⑤ Variables by boundary conditions and shift:

b_0, b_n $\quad \Phi(\rho = 0, \phi) = a_0 + b_0 \ln 0 + \sum_{n=1}^{\infty} a_n 0^n \sin(n\phi + \alpha_n) + \sum_{n=1}^{\infty} b_n 0^{-n} \sin(n\phi + \beta)$

$= -\infty$ and not finite

So, $b_0 = 0$ and $b_n = 0$

α $\quad \Phi(\rho, 0) = \Phi(\rho, 2\pi)$

$$= a_0 + \sum_{n=1}^{\infty} a_n \rho^n \sin(0 + \alpha)$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \rho^n \left[e^{i(\alpha)} + e^{-i(\alpha)}\right]$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \rho^n \left[e^{i(2\pi + \alpha)} + e^{-i(2\pi + \alpha)}\right]$$

$$X = 0$$

$a_n \quad \Phi(\rho, \phi) = a_0 + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi)$

$$= \sum_{n=1}^{\infty} a_n \rho^n \left[e^{in\phi} - e^{-in\phi} \right]$$

$$= \sum_{m=-\infty}^{\infty} a_m \rho^m \left[e^{im\phi} \right]$$

$$\Phi(\rho = b, \phi) = V(\phi)$$

$$V(\phi) = \sum_{m=-\infty}^{\infty} a_m \rho^m e^{im\phi}$$

$$\int_0^{2\pi} V(\phi) e^{-im\phi} d\phi = \sum_{m=-\infty}^{\infty} a_m b^m \int_0^{2\pi} e^{i(m-m)\phi} d\phi$$

$$= \sum_{m=-\infty}^{\infty} a_m b^m \cdot 2\pi \delta$$

$$a_n = \frac{1}{2\pi b^m} \int_0^{2\pi} V(\phi) e^{-im\phi} d\phi$$

$$= \frac{1}{2\pi b^m} \left[V \int_0^{\pi/2} e^{-im\phi} d\phi - V \int_{\pi/2}^{\pi} e^{-im\phi} d\phi + V \int_{\pi}^{3\pi/2} e^{-im\phi} d\phi - V \int_{3\pi/2}^{2\pi} e^{-im\phi} d\phi \right]$$

$$= \frac{V}{(-im) 2\pi b^m} \left[e^{-im\pi/2} - 1 - e^{-im\pi} + e^{-im 3\pi/2} \right]$$

$$= \frac{2V}{(-im) \pi b^m} \left[(-1)^{m/2} - 1 \right]$$

$$\Phi(\rho,\phi) = \sum_{m=0}^{\infty} \frac{2V}{im\pi}\left(\frac{\rho}{b}\right)^m \left[1-(-1)^{m/2}\right]\left[e^{im\phi} - e^{-im\phi}\right]$$

$$= \sum_{m=0}^{\infty} \frac{4V}{m\pi}\left(\frac{\rho}{b}\right)^m \left[1-(-1)^{m/2}\right]\cdot \sin(m\phi) \qquad m=4n+2$$

$$= \frac{4V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2} \frac{\sin[(4n+2)\phi]}{2n+2}$$

b) $$\Phi(\rho,\phi) = \frac{4V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2} \frac{\sin[(4n+2)\phi]}{2n+2}$$

$$= \text{Imag}\left[\frac{8V}{\pi}\sum_{n=0}^{\infty}\frac{\left(\frac{\rho}{b}e^{i\phi}\right)^n}{m}\right] \qquad \text{when } m=4n+2$$

Log identity: $$\ln(1+x) = \sum_{n=0}^{\infty}\frac{x^n}{n}(-1)^{n-1}$$

$$\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}$$

$$\ln(1+x) + \ln(1-x) = -2\sum_{n=2,4,6}^{\infty}\frac{x^n}{n}$$

$$\ln[(1+x)(1-x)] = -2\sum_{n=2,4,6}^{\infty}\frac{x^n}{n}$$

$$-\frac{1}{2}\ln[(1+x^2)(1-x^2)] = -2\sum_{n=2,4,6}^{\infty}\frac{x^n}{n}$$

$$\ln[(1+z)(1-z)] - \frac{1}{2}\ln[(1+z^2)(1-z^2)] = -2\sum_{n=2,4,6}^{\infty}\frac{z^n}{n}$$

$$\Phi(\rho,\phi) = \frac{2V}{\pi}\,\text{Imag}\left[\ln\left(\frac{1+z^2}{1-z^2}\right)\right]$$

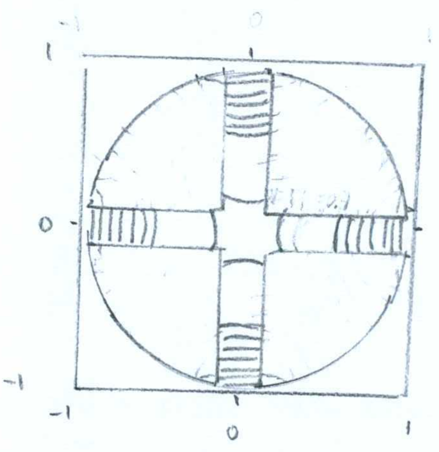
$$\boxed{\ln\left[\frac{1+X^2}{1-X^2}\right] = \tan^{-1}\left((-i)\,\frac{X-X^*}{X+X^*}\right)}$$

$$= \frac{2V}{\pi}\tan^{-1}\left((-i)\,\frac{\left(\frac{1+\left(\frac{\rho}{b}e^{i\phi}\right)^2}{1-\left(\frac{\rho}{b}e^{i\phi}\right)^2}\right) - \left(\frac{1+\left(\frac{\rho}{b}e^{i\phi}\right)^2}{1-\left(\frac{\rho}{b}e^{i\phi}\right)^2}\right)^*}{\left(\frac{1+\left(\frac{\rho}{b}e^{i\phi}\right)^2}{1-\left(\frac{\rho}{b}e^{i\phi}\right)^2}\right) + \left(\frac{1+\left(\frac{\rho}{b}e^{i\phi}\right)^2}{1-\left(\frac{\rho}{b}e^{i\phi}\right)^2}\right)^*}\right)$$

$$= \frac{2V}{\pi}\tan^{-1}\left((-i)\left(\frac{\left(\frac{\rho}{b}e^{i\phi}\right)^2 - \left(\frac{\rho}{b}e^{i\phi}\right)^{2*}}{1-\left(\frac{\rho}{b}e^{i\phi}\right)^2\left(\frac{\rho}{b}e^{i\phi}\right)^{2*}}\right)\right)$$

$$= \frac{2V}{\pi}\tan^{-1}\left(\frac{2\rho^2 b^2 \sin(2\phi)}{b^4-\rho^4}\right)$$

c)



$b=1,\ V=$ 1 0.7 ⋯ 0.1

2.15. (Equation 1.39) $\nabla_x^2 G(x,x') = -4\pi\delta(x-x')$ $\boxed{\text{one-dimensional}}$

$$\nabla_{xy}^2 G(x,y;x'y') = -4\pi\delta(x-x')\delta(y-y') \quad \boxed{\text{two-dimensional}}$$

Boundary Conditions: $G(x=0) = G(x=1) = G(y=0) = G(y=1) = 0$

$$G(x,y;x',y') = \sum_{n=1}^{\infty} \sqrt{2}\, f(x,y,y') \sin(n\pi y')$$

$$\nabla_{xy}^2 G(x,y;x';y') = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) G(x,y;x';y')$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \sqrt{2}\, f(x,y,y') \cdot \sin(n\pi x)$$

$$= \left(\frac{\partial^2}{\partial y^2} - n^2\pi^2\right) \sqrt{2}\, f(x,y,y') \sin(n\pi x)$$

$$= -4\pi\, \delta(x-x')\, \delta(y-y')$$

$$\delta(x-x') = \sqrt{2} \sin(n\pi x)$$

$$= 2 \sin(n\pi x) \sin(n\pi x')$$ ∘∘ by boundary conditions

$$\left(\frac{\partial^2}{\partial y^2} - n^2\pi^2\right) f(x,y,y') \sin(n\pi x') = -4\pi\, \delta(y-y') \sin(n\pi x) \sin(n\pi x')$$

$$f(x,y,y') = g_n(y,y') \sqrt{2} \sin(n\pi x)$$

$$\left(\frac{\partial^2}{\partial y^2} - n^2\pi^2\right) = -4\pi\, \delta(y-y')$$

b) $$g_n(y,y') = \begin{cases} g_{y'<y} & A \sinh(n\pi y') + B \cosh(n\pi y') \quad \text{if } y' < y \\ g_{y'>y} & C \sinh(n\pi y') + D \cosh(n\pi y') \quad \text{if } y' > y \end{cases}$$

By the boundary conditions, $g(y, y'=0) = 0$

$$= A \sinh(n\pi y') + B \cosh(n\pi y')$$

$$g(y, y'=1) = 0$$

$$= C \sinh(n\pi y') + D \cosh(n\pi y')$$

So, $B = 0$ and $D = 0$

$$g(y,y') = \begin{cases} g_{y'<y} & A\sinh(n\pi y') \quad \text{if } y'<y \\ g_{y'>y} & B\sinh(n\pi y') \quad \text{if } y'>y \end{cases}$$

By the boundary conditions, $g(y, y'=y) = g(y, y'^{-})$

$$A\sinh(n\pi y) = C\sinh(n\pi(1-y))$$

$$\frac{\partial}{\partial x} A\sinh(n\pi y) = 4\pi + \frac{\partial}{\partial x} C\sinh(n\pi(1-y))$$

$$A = \frac{4\sinh(n\pi(1-y))}{n\sinh(n\pi)}$$

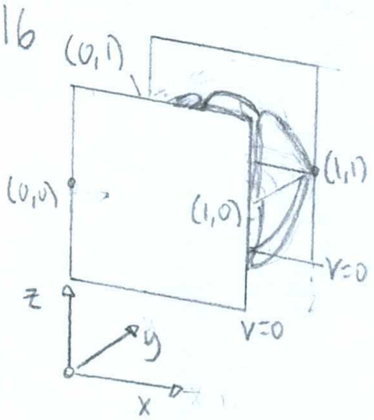
$$B = \frac{4\sinh(n\pi y)}{n\sinh(n\pi)}$$

$$g(y,y') = \frac{4}{n\sinh(n\pi)} \begin{cases} \sinh(n\pi(1-y))\sinh(n\pi y) & y'<y \\ \sinh(n\pi y)\sinh(n\pi(1-y)) & y'>y \end{cases}$$

$$g(x,y;x',y') = g(x;x')g(y;y')$$

$$= 8\sum_{n=1}^{\infty} \frac{1}{n\sinh(n\pi)} \left\{ \sinh(n\pi x)\sinh(n\pi x')\sinh(n\pi y)\sinh(1-y) \right.$$

2.16



Boundary Conditions: $\Phi(x=0) = \Phi(x=1) = 0$

$\Phi(y=0) = \Phi(y=1) = 0$

(Equation 1.44)

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(x')G_D(x,x')d^3x' - \frac{1}{4\pi\varepsilon_0}\oint \Phi(x')\frac{\partial G_D}{\partial n'}da'$$

From problem 2.15,

$$G(x,y;x',y') = 8\sum_{n=1}^{\infty}\frac{1}{n\sinh(n\pi)}\sin(n\pi x)\sin(n\pi x')\sinh(n\pi y)\sinh(n\pi(1-y))$$

$$\Phi(x,y) = \frac{8}{4\pi\varepsilon_0}\sum_{n=1}^{\infty}\frac{1}{n\sinh(n\pi)}\int_0^1\int_0^1 \sin(n\pi x)\sin(n\pi x')\sinh(n\pi y)\sinh(n\pi(1-y))\,dx'dy'$$

$$= \frac{2}{\pi\varepsilon_0}\sum_{n=1}^{\infty}\frac{1}{n\sinh(n\pi)}\left[\frac{\sin(n\pi x)(\cos(n\pi)-1)}{n\pi}\right]\left[\frac{\sinh(n\pi)-\sinh(n\pi(1-y))-\sinh(n\pi y)}{n\pi}\right]$$

$$= \frac{4}{\pi^3\varepsilon_0}\sum_{n=1}^{\infty}\frac{\sin(n\pi x)}{n^3}\left[1-\frac{\sinh[n\pi(1-y)]+\sinh(n\pi y)}{\sinh[n\pi]}\right]$$

$$= \frac{4}{\pi^3\varepsilon_0}\sum_{n=1}^{\infty}\frac{\sin(n\pi x)}{n^3}\left[1-\frac{\cosh(n\pi(y-1/2))}{\cosh(n\pi/2)}\right]$$

$$= \frac{4}{\pi^3\varepsilon_0}\sum_{n=1}^{\infty}\frac{\sin[(2m+1)\pi x]}{(2m+1)^3}\left[1-\frac{\cosh[(2m+1)\pi(y-1/2)]}{\cosh[(2m+1)\pi/2]}\right]$$

2.17. (Equation 1.40)

$$G(x,x') = \frac{1}{|\vec{x}-\vec{x}'|} \qquad |\vec{x}-\vec{x}'| = R$$
$$= \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$$

$$G(x,y;x'y') = \int_{-Z}^{Z} G(x,y,z;x',y',z')\,dz$$

$$= \int_{-Z}^{Z}\frac{dz}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \qquad @\ z'=0$$

$$= \ln\left[\left((x-x')^2+(y-y')^2+z^2\right)^{1/2}+z\right]_{-Z}^{Z}$$

$$= \ln \frac{[(x-x')^2+(y-y')^2+z^2]+z}{[(x-x')^2+(y-y')^2+z^2]-z}$$

Taylor Expansion

$$(1+a)^{1/2}+1 \cong 2+\frac{a}{2}-\frac{a^2}{8}$$

$$(1+a)^{1/2}-1 \cong \frac{a}{2}-\frac{a^2}{8}+\frac{a^3}{16}$$

$$= \ln \frac{\left[1+\frac{(x-x')^2+(y-y')^2}{z^2}\right]^{1/2}+1}{\left[1+\frac{(x-x')^2+(y-y')^2}{z^2}\right]^{1/2}-1}$$

$$\cong \ln \frac{2+\frac{(x-x')^2+(y-y')^2}{2z^2}}{\frac{(x-x')^2+(y-y')^2}{2z^2}}$$

$$\cong \ln \frac{4z^2+(x-x')^2+(y-y')^2}{(x-x')^2+(y-y')^2}$$

$$\cong -\ln[(x-x')^2+(y-y')^2] + \text{constant}$$ ... because "z is taken to be large"

$$\cong -\ln[(x-x')^2+(y-y')^2]$$

$$\cong -\ln[\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')]$$

b) $\nabla^2_{\rho\phi} G(\rho,\phi;\rho',\phi') = \frac{\delta(\rho-\rho')}{\rho'}\delta(\phi-\phi')$

$$= \frac{1}{2\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{\rho'}\frac{\partial}{\partial\rho'}\left(\rho'\frac{\partial}{\partial\rho'}\right)-\frac{1}{\rho'^2}\frac{\partial}{\partial\phi^2}\right\}e^{im(\phi-\phi')}$$

$$\delta(\rho-\rho') = \frac{1}{2\pi}\sum_{n=1}^{\infty}\left\{\frac{\partial}{\partial\rho'}\rho'\frac{\partial}{\partial\rho'}-\frac{1}{\rho'^2}\frac{\partial}{\partial\phi^2}\right\}$$

$$\delta(\phi-\phi') = e^{im(\phi-\phi')}$$

c) $\nabla^2_{\rho\phi} G(\rho,\phi;\rho',\phi') = \left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}\right\}g_m(\rho,\rho')$

$$= 0$$

$$g_m(\rho,\rho') = \begin{cases} A_m \rho'^m + B_m \rho'^{-m} & \rho'<\rho \\ A_m' \rho'^m + B_m' \rho'^{-m} & \rho'>\rho \end{cases}$$

$$g(\rho,0) = \begin{cases} A_m 0^m + B_m 0^{-m} & \rho'<\rho \\ A_m' 0^m + B_m' 0^{-m} & \rho'>\rho \end{cases}$$

$$= 0 \quad , \text{so} \quad B_m = A_m' = 0$$

$$g_m(\rho,\rho') = \begin{cases} A_m \rho'^m & \rho'<\rho \\ B_m' \rho'^m & \rho'>\rho \end{cases}$$

$$= \begin{cases} \gamma_m \\ \gamma_m \end{cases}$$

$A_m \rho^m = \gamma_m \quad ; A_m = \gamma_m \cdot \rho^{-m}$

$B_m' \rho^{-m} = \gamma_m \quad ; B_m' = \gamma_m \rho^m$

$$g_m(\rho,\rho') = \begin{cases} \gamma_m \left(\frac{\rho'}{\rho}\right)^m & \rho'<\rho \\ \gamma_m \left(\frac{\rho}{\rho'}\right)^m & \rho'>\rho \end{cases}$$

$$\frac{dG}{\partial n} = \left.\frac{\partial g}{\partial n}\right|_{\rho'=\rho} - \left.\frac{\partial g}{\partial n}\right|_{\rho'=\rho}$$

$$= -m\gamma\left(\frac{1}{\rho} + \frac{1}{\rho}\right)$$

$$= \frac{1}{\rho}$$

$$\gamma = -\frac{1}{2m}$$

$$g_m = \begin{cases} -\frac{1}{2m}\left(\frac{\rho'}{\rho}\right)^m & \rho'<\rho \\ -\frac{1}{2m}\left(\frac{\rho'}{\rho}\right)^m & \rho'>\rho \end{cases}$$

$$\nabla^2 G = -4\pi\delta(\rho-\rho')\delta(\phi-\phi')g_m(\rho,\rho')$$

$$= -4\pi \sum_{n=1}^{\infty} \frac{\nabla^2}{2\pi}\left[e^{im(\phi-\phi')}\right]\left[\frac{-1}{2m}\left(\frac{\rho_<}{\rho_>}\right)^m\right]$$

$$= \frac{1}{m}\sum_{n=1}^{\infty} \nabla^2\left[e^{im(\phi-\phi')}\right]\left[\left(\frac{\rho_<}{\rho_>}\right)^m\right]$$

$$G = \frac{1}{m}\sum_{n=1}^{\infty}\left(\frac{\rho_<}{\rho_>}\right)^m e^{im(\phi-\phi')}$$

$$= \frac{2}{m}\sum_{n=1}^{\infty}\left(\frac{\rho_<}{\rho_>}\right)^m \cos(m(\phi-\phi'))$$

"A series for changing area, in cylindrical coordinates."

2.18.

a) ① Boundary Conditions:

$g_m(\rho,\rho'=0) = 0 \quad \rho'<\rho$

$g_m(\rho,\rho'=b) = 0 \quad \rho'>\rho$

Again, Greens theorem describes areas change with a sum or series.

② Laplaces Equation:

Ⓐ Variable separation: $F = R''\Theta + \frac{1}{\rho}R'\Theta + \frac{1}{R^2}R\Theta'' = 0$

Ⓑ Angular Eigenvalues: $\frac{1}{\Theta}\Theta'' = \lambda \; ; \quad \Theta'' + \lambda R = 0$

Ⓒ Radial Eigenvalues: $\frac{\rho}{R}\frac{d}{d\rho}\rho R' = \lambda \; ; \quad \rho R' + \lambda R = 0$

Ⓓ General Solution: $\Phi(\rho,\phi) = R_0\Phi_0 + R_n\Phi_n$

③ General Solution to Laplace's Equation:

$R_0(\rho) = a_0 + b_0 \ln\rho \qquad \Phi_0(\phi) = A_0 + B_0\phi$

$R_n(\rho) = a\rho^{\nu} + b\rho^{-\nu} \qquad \Phi(\phi) = A\cos(wt) + B\sin(wt)$

④ Variables by Boundary Conditions:

$$B_m \quad g_m(\rho,\rho'=0) = \begin{cases} A0^m + B0^{-m} & \rho'<\rho \\ C\rho^m + B\rho^{-m} & \rho<\rho' \end{cases}$$

$$= 0, \text{ so } B=0$$

$$C, D \quad g_m(\rho,\rho'=b) = \begin{cases} A\rho^m + B\rho^{-m} & \rho'<\rho \\ Cb^m + Db^{-m} & \rho'>\rho \end{cases}$$

$$= 0, \text{ so } Cb^m = \gamma \text{ and } -Db^{-m} = \gamma$$

$$C = \gamma b^{-m} \text{ and } D = -\gamma b^m$$

$$A \qquad A\rho^m = C\rho^m + D\rho^{-m}$$

$$= \gamma\left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right]$$

$$A = \frac{\gamma}{\rho^m}\left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right]$$

$$g_m(\rho,\rho'=b) = \begin{cases} \gamma\left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right]\left(\frac{\rho}{\rho'}\right)^m & \rho'<\rho \\ \gamma\left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right] & \rho'>\rho \end{cases}$$

$$\gamma \quad \frac{dG}{dn} = \left.\frac{dg}{d\rho'}\right|_{\rho'=b} - \left.\frac{dg}{d\rho'}\right|_{\rho'=b}$$

$$= m\gamma\left[\frac{\rho^{m-1}}{b^m} + \frac{b^m}{\rho^{m+1}}\right] - m\gamma\left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right]\frac{1}{\rho}$$

$$= 2m\gamma_m\left(\frac{b}{\rho}\right)^m\frac{1}{\rho} = \frac{1}{\rho}$$

$$g_m(\rho) = \begin{cases} \frac{1}{2m}\left[\left(\frac{\rho'\rho}{b^2}\right)^m - \left(\frac{\rho'}{\rho}\right)^m\right] & \rho' < \rho \\ \frac{1}{2m}\left[\left(\frac{\rho'\rho}{b^2}\right)^m - \left(\frac{\rho}{\rho'}\right)^m\right] & \rho' > \rho \end{cases}$$

$$= \frac{1}{2m}\left[\left(\frac{\rho'\rho}{b^2}\right)^m - \left(\frac{\rho_<}{\rho_>}\right)^m\right]$$

$$G(\rho,\phi;\rho',\phi') = \frac{1}{2\pi}\sum_{m=1}^{\infty}\frac{1}{m}\left[\left(\frac{\rho'\rho}{b^2}\right)^m - \left(\frac{\rho_<}{\rho_>}\right)^m\right]\cos(m(\phi-\phi'))$$

$$= \frac{1}{2\pi}\sum_{m=1}^{\infty}\frac{1}{m}\left[\int_0^{\left(\frac{\rho'\rho}{b^2}\right)} u^{m-1}\,du - \int_0^{\left(\frac{\rho_<}{\rho_>}\right)} u^{m-1}\,du\right]\cos(m(\phi-\phi'))$$

$$= \frac{1}{2\pi}\sum_{m=1}^{\infty}\frac{1}{m}\left[\int_0^{\left(\frac{\rho'\rho}{b^2}\right)} \frac{u^m\cos(m(\phi-\phi'))}{u}\,du - \int_0^{\left(\frac{\rho_<}{\rho_>}\right)} \frac{u^m\cos(m(\phi-\phi'))\,du}{u}\right]$$

$$= \frac{1}{2\pi}\sum_{m=1}\frac{1}{2}\left[\int_0^{\left(\frac{\rho'\rho}{b^2}\right)} \frac{u^m}{u}\left[e^{im(\phi-\phi')} + e^{-im(\phi-\phi')}\right]du - \int_0^{\left(\frac{\rho_<}{\rho_>}\right)} \frac{u^m}{u}\left[e^{im(\phi-\phi')} + e^{-im(\phi-\phi')}\right]du\right]$$

$$= \frac{1}{4\pi}\left[\int_0^{\left(\frac{\rho'\rho}{b^2}\right)} \frac{1}{u}\left[\frac{1}{1-ue^{i(\phi-\phi')}} + \frac{1}{1+ue^{-i(\phi-\phi')}} - 2\right]du + \int_0^{\left(\frac{\rho_<}{\rho_>}\right)} \frac{1}{u}\left[\frac{1}{1-ue^{i(\phi-\phi')}} + \frac{1}{1+ue^{-i(\phi-\phi')}} - 2\right]du\right]$$

$$= \frac{1}{4\pi}\left[\int_0^{\left(\frac{\rho'\rho}{b^2}\right)} \frac{1}{u}\left[\frac{1-ue^{-i(\phi-\phi')}-ue^{i(\phi-\phi')}-1}{1-ue^{i(\phi-\phi')}-ue^{-i(\phi-\phi')}+u^2} - 2\right]du\right.$$

$$\left. + \int_0^{\left(\frac{\rho_<}{\rho_>}\right)} \frac{1}{u}\left[\frac{1-ue^{-i(\phi-\phi')}-ue^{i(\phi-\phi')}-1}{1-ue^{i(\phi-\phi')}-ue^{-i(\phi-\phi')}+u^2} - 2\right]du\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{\left(\frac{\rho'\rho}{b^2}\right)} \frac{1}{u}\left[\frac{1-u\cos(\phi-\phi')}{1+u^2-2u\cos(\phi-\phi')} - 1\right]du + \int_0^{\left(\frac{\rho_>}{\rho_<}\right)} \frac{1}{u}\left[\frac{1-u\cos(\phi-\phi')}{1+u^2-2u\cos(\phi-\phi')} - 1\right]du\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{\left(\frac{\rho'\rho}{b^2}\right)} \frac{\cos(\phi-\phi') - u}{1+u^2-2u\cos(\phi-\phi')}du + \int_0^{\left(\frac{\rho_<}{\rho_>}\right)} \frac{\cos(\phi-\phi') - u}{1+u^2-2u\cos(\phi-\phi')}du\right]$$

$$= \frac{1}{2\pi} \left.\frac{\ln(1-2u\cos(\phi-\phi') + u^2)}{-2}\right|_0^{\left(\frac{\rho'\rho}{b^2}\right)} + \left.\frac{\ln(1-2u\cos(\phi-\phi') + u^2)}{-2}\right|_0^{\left(\frac{\rho_<}{\rho_>}\right)}$$

$$= \frac{1}{4\pi}\ln\left[\frac{\rho'^2\rho^2 + b^4 - 2\rho\rho' b^2\cos(\phi-\phi')}{b^2(\rho^2+\rho'^2-2\rho\rho\cos(\phi-\phi')}\right] \underbrace{- \frac{1}{4\pi}\ln\frac{\rho^2}{b^2}}_{\neq 0 \text{ at } g_m(\rho,\rho'=b)}$$

b) $$\Phi(\rho,\phi) = \int \Phi(\rho'=b,\phi) \left.\frac{\partial G}{\partial\rho}\right|_{\rho'=b} da$$

$$\frac{dG}{d\rho} = \frac{-1}{4\pi}\left\{\frac{2\rho^2\rho' - 2\rho b^2\cos(\phi-\phi')}{b^4+\rho^2\rho'^2-2\rho\rho' b^2\cos(\phi-\phi')} - \frac{2\rho'-2\rho\cos(\phi-\phi')}{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}\right\}$$

$$= \frac{-1}{4\pi}\ \frac{2\rho^2 b - 2\rho b^2\cos(\phi-\phi')}{b^4+\rho^2 b^2 - 2\rho b b^2\cos(\phi-\phi')} - \frac{2b-2\rho\cos(\phi-\phi')}{\rho^2+b^2-2\rho b\cos(\phi-\phi')}\Bigg\}$$

$$= \frac{-1}{2\pi}\left\{\frac{\rho^2-b^2}{b(\rho^2+b^2-2\rho b\cos(\phi-\phi'))}\right\}$$

c) <u>Green's Theorem Derivation:</u>

① <u>Boundary Conditions:</u>

$g(\rho,\rho'=b)=0$ for $\rho'<\rho$ and $g(\rho,\rho'=0)=0$ for $\rho'>\rho$

② Laplace's Equation Solution:

Ⓐ Variable Separation: $f \stackrel{L}{=} R''\Theta + \frac{1}{\rho}R'\Theta + \frac{1}{R^2}R\Theta'' = 0$

Ⓑ Angular Eigenvalues: $\frac{1}{\Theta}\Theta'' = \lambda$; $\Theta'' + \lambda\Theta = 0$

Ⓒ Radial Eigenvalues: $\frac{\rho}{R}\frac{d}{d\rho}\rho R' = \lambda$; $\rho R' + \lambda R = 0$

Ⓓ General Solutions: $\Phi(\rho, \phi) = R_0\Theta_0 + R_n\Theta_n$

③ General Solution to Laplaces Equation:

$R_0(\rho) = a_0 + b_0 \ln \rho \qquad \Theta_0(\phi) = A_0 + B_0\phi$

$R_n(\rho) = a\rho^{v} + b\rho^{-v} \qquad \Theta_n(\rho) = A\cos(v\phi) + B\sin(v\phi)$

④ Variables by Boundary Conditions:

A,B $\quad g_m(\rho, \rho' = b) = \begin{cases} A\rho^{m} + B\rho^{-m} & \rho' < \rho \\ C\rho^{m} + D\rho^{-m} & \rho' > \rho \end{cases}$

$= \begin{cases} Ab^{m} + Bb^{-m} & \rho' < \rho \\ C\rho^{m} + D\rho^{-m} & \rho' > \rho \end{cases}$

$= 0$, so $Ab^{m} = \gamma$ and $Bb^{-m} = \gamma$

$A = \gamma b^{-m}$ and $B = \gamma b^{m}$

D $\quad g_m(\rho, \rho' = 0) = \begin{cases} A\rho^{v} + B\rho^{-v} & \rho' < \rho \\ C0^{v} + D0^{-v} & \rho' > \rho \end{cases}$

$= 0$, so $D = 0$

$C \quad g(\rho, \rho'=b) = g(\rho, \rho'=0)$

$$C\rho^m = A\rho^m + B\rho^{-m}$$

$$= \gamma\left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right]$$

$$C = \gamma\rho^m\left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right]$$

$$\gamma \quad \frac{dG}{dn} = \left.\frac{dg}{dn}\right|_{\rho'=b} - \left.\frac{dg}{dn}\right|_{\rho'=b}$$

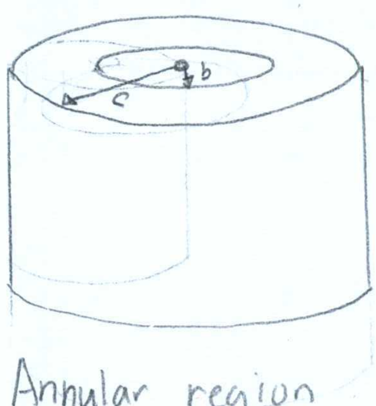
$$= -m\gamma\left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right]\frac{1}{\rho} - m\gamma\left[\left(\frac{\rho}{b}\right)^m + \left(\frac{b}{\rho}\right)^m\right]\frac{1}{\rho}$$

$$= \frac{1}{\rho}$$

$$\gamma = -\frac{1}{2m}\left(\frac{b}{\rho}\right)^m$$

$$g_m(\rho, \rho') = \frac{1}{2m}\left[\left(\frac{b^2}{\rho\rho'}\right)^m - \left(\frac{\rho_<}{\rho_>}\right)^m\right]$$

2.19.



Annular region

Shape: Cylinders Dimension: Area [2D] Charge: Q

Greens Theorem Derivation:

① Boundary conditions:

$$g(\rho, \rho'=b) = 0$$

$$g(\rho, \rho'=c) = 0$$

② Laplace's Equation Solution:

Ⓐ Variable Separation: $f' = R''\Theta + \frac{1}{\rho}R'\Theta + \frac{1}{R^2}R\Theta'' = 0$

Ⓑ Angular Eigenvalues: $\frac{1}{\Theta}\Theta'' = \lambda$; $\Theta'' + \lambda\Theta = 0$

Ⓒ Radial Eigenvalues: $\frac{\rho}{R}\frac{d}{d\rho}\rho R' = \lambda$; $\rho R' + \lambda R = 0$

Ⓓ General solution: $\Phi(\rho, \phi) = R_0\Theta_0 + R_n\Theta_n$

③ General Solution to Laplace's Equation:

$R_0(\rho) = a_0 + b_0 \ln \rho$ $\qquad \Theta_0(\phi) = A_0 + B_0\phi$

$R_n(\rho) = a\rho^{\nu} + b\rho^{-\nu}$ $\qquad \Theta_n(\rho) = A\cos(\nu\phi) + B\sin(\nu\phi)$

⑤ Variables at Boundary Conditions:

$$g_m(\rho, \rho') = \begin{cases} A\rho^m + B\rho^{-m} \\ C\rho^m + D\rho^{-m} \end{cases}$$

$$B \quad g_m(\rho, \rho' = b) = \begin{cases} Ab^m + Bb^{-m} \\ C\rho^m + D\rho^{-m} \end{cases}$$

$= 0$, so $\quad Ab^m = \gamma$ and $Bb^{-m} = \gamma$

$A = \gamma b^{-m}$ and $B = \gamma b^m$

$$D \quad g_m(\rho, \rho' = c) = \begin{cases} A\rho^m + B\rho^{-m} \\ Cc^m + Dc^{-m} \end{cases}$$

$= 0$, so $\quad Cc^m = \alpha$ and $-Bc^{-m} = \alpha$

$C = \alpha c^{-m}$ and $B = -\alpha c^m$

γ

$$\frac{dG}{dn} = \frac{dg}{d\rho}\Big|_{\rho'=b} - \frac{dg}{d\rho}\Big|_{\rho'=b}$$

$$= m\gamma\left(\left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right] - \left[\left(\frac{\rho}{c}\right)^m + \left(\frac{c}{\rho}\right)^m\right]\right)\frac{1}{\rho}$$

$$= m\gamma\left(\left[\left(\frac{b}{b}\right)^m - \left(\frac{b}{b}\right)^m\right] - \left[\left(\frac{b}{c}\right)^m + \left(\frac{c}{b}\right)^m\right]\right)\frac{1}{\rho}$$

$$= -m\gamma\left(\frac{c^{2m} + b^{2m}}{c^m \cdot b^m}\right)\frac{1}{\rho}$$

$$= \frac{1}{\rho}$$

$$\gamma = \frac{1}{m}\,\frac{-c^m \cdot b^m}{(c^{2m} + b^{2m})}$$

$$= \frac{1}{m}\,\frac{-1}{c^{2m}/b^{2m} + 1}\left(\frac{\rho}{b}\right)^m$$

α

$$\frac{dG}{dn} = \frac{dg}{d\rho}\Big|_{\rho'=c} - \frac{dg}{d\rho}\Big|_{\rho'=c}$$

$$= m\alpha\left(\left[\left(\frac{\rho}{b}\right)^m - \left(\frac{b}{\rho}\right)^m\right] - \left[\left(\frac{\rho}{c}\right)^m - \left(\frac{c}{\rho}\right)^m\right]\right)\frac{1}{\rho}$$

$$= m\alpha\left(\left[\left(\frac{c}{b}\right)^m - \left(\frac{b}{c}\right)^m\right] - \left[(1) - (1)\right]\right)\frac{1}{\rho}$$

$$= m\alpha\left(\frac{c^{2m} - b^{2m}}{b^m c^m}\right)\frac{1}{\rho}$$

$$= \frac{1}{\rho}$$

$$\alpha = \frac{1}{m}\left(\frac{1}{1 - b^{2m}/c^{2m}}\right)\left(\frac{\rho}{c}\right)^m$$

$$g_m(\rho,\rho') = \begin{cases} \dfrac{1}{m\left[c^{2m}/b^{2m}+1\right]}\left[\left(\dfrac{\rho}{\rho'}\right) - \left(\dfrac{\rho'\rho}{b^2}\right)^m\right] \\ \dfrac{1}{m\left[1-b^{2m}/c^{2m}\right]}\left[\left(\dfrac{\rho'\rho}{c^2}\right)^m - \left(\dfrac{\rho}{\rho'}\right)^m\right] \end{cases}$$

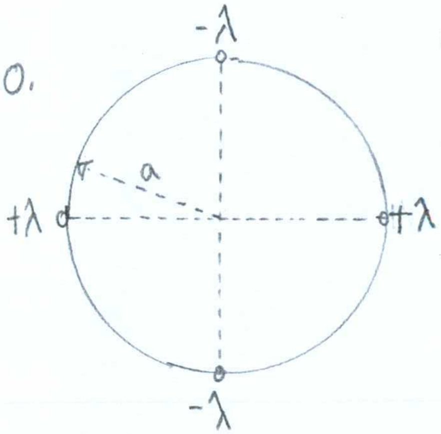
$$G(\rho,\phi;\rho',\phi') = \sum_{n=1}^{\infty} e^{i(\phi-\phi')} \cdot g(\rho, b \le \rho \le c)$$

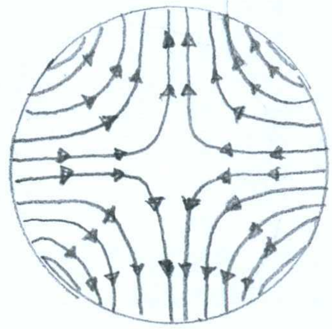
$$= \sum_{n=1}^{\infty} e^{i(\phi-\phi')} \cdot \frac{\left[\left(\frac{\rho_<}{\rho_>}\right)^m - \left(\frac{\rho_<'\rho_>}{b^2}\right)^m\right]\left[\left(\frac{\rho_>\rho_<}{c^2}\right)^m - \left(\frac{\rho_<}{\rho_>}\right)^m\right]}{m\left[c^{2m}/b^{2m}+1\right]\left[1-b^{2m}/c^{2m}\right]}$$

$$= 2\sum_{n=1}^{\infty} \frac{\cos(m(\phi-\phi'))}{m\left[1-b^{2m}/c^{2m}\right]^2}\left[\rho_<^m - \left(\frac{b^2}{\rho_<}\right)^m\right]\left[\frac{1}{\rho_>} - \left(\frac{\rho_>}{c^2}\right)^m\right]$$

Note: I left a squared denominator.

2.20.





Two-dimensional electric quadropole

Shape: Line charges
Dimension: Line [1D]
Charges: $+\lambda$, $-\lambda$

Charge Density:

$$\sigma(\rho,\phi) = \frac{\lambda}{a}\sum_{n=0}^{3}(-1)^n \delta(\rho-a)\,\delta(\phi - n\pi/2)$$

a) From part 2.17c,

$$G(\rho,\phi;\rho',\phi') = -\ln(\rho_>^2) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m \cos(m(\phi-\phi'))$$

$$\sigma = -\varepsilon_0 \frac{d\Phi}{d\rho}$$

$$\Phi(\rho,\phi) = -\frac{1}{\varepsilon_0}\int \sigma \, d\rho$$

$$= \frac{4\lambda}{4\pi\varepsilon_0 a}\int_0^a -\ln(\rho_>^2) + 2\sum_{m=1}^{\infty}\left(\frac{1}{m}\right)\left(\frac{\rho_<}{\rho_>}\right)^m \cos(m(\phi-\phi'))\, d\rho$$

$$= \frac{\lambda}{\pi\varepsilon_0}\left[2\sum_{m=1}^{\infty}\left(\frac{1}{m}\right)\left(\frac{\rho_<}{\rho_>}\right)^m \cos(m(\phi-\phi'))\right] \qquad m = 4k+2;\ \phi'=0$$

$$= \frac{\lambda}{\pi\varepsilon_0}\sum_{m=1}^{\infty}\frac{1}{2k+1}\left(\frac{\rho_<}{\rho_>}\right)^{4k+2}\cos((4k+2)\phi)$$

b) (Equation 1.59)

$$W = \frac{\sigma^2}{2\varepsilon_0}$$

$$= \frac{1}{2\varepsilon_0}\left[\frac{\lambda}{a}\sum_{n=0}^{3}(-1)^n\delta(\rho-a)\delta(\phi-n\pi/2)\right]^2$$

$$= \frac{\lambda^2}{\pi\varepsilon_0}\left[\sum_{m=0}^{3}\rho'\cos(m(\phi-\phi'))\right]^2$$

$$= \frac{2\lambda^2}{4\pi\varepsilon_0}\left[\sum_{m=0}^{3}\rho' e^{im\phi}\right]^2$$

$$= \frac{2\lambda^2}{4\pi\varepsilon_0}\left[\sum_{m=0}^{3}(z-ia)^2\right]$$

$$= \frac{2\lambda^2}{4\pi\varepsilon_0}\left[\sum_{m=0}^{3}(z-ia)(z+ia)\right]$$

$$= \frac{2\lambda^2}{4\pi\varepsilon_0}\ln\left[\frac{(z-ia)(z+ia)}{(z-a)(z+a)}\right]$$

$$= 0$$

c) (Equation 1.59)

$$W = \frac{\varepsilon_0}{2}|E|^2$$

$$w = \frac{2\lambda}{4\pi\varepsilon_0} \ln\left[\frac{(z-ia)(z+ia)}{(z-a)(z+a)}\right]$$

$$E = \frac{\lambda}{\sqrt{\pi}\varepsilon_0} \ln\left|\frac{(x+iy)^2+a^2}{(x+iy)^2-a^2}\right|^{1/2}$$

) (Equation 1.16)

$$|E| = -\nabla\phi$$

$$= -\frac{\lambda}{\pi\varepsilon_0}\frac{\partial}{\partial\rho}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{\rho_<}{\rho_>}\right)^{4k+2}\cos[(4k+2)\phi]$$

$$= -\frac{\lambda}{\pi\varepsilon_0}\sum_{k=0}^{\infty}\frac{2}{\rho_>}\left(\frac{x^2+y^2}{\rho_>}\right)^{2k}\cos\left[(4k+2)\arctan\left(\frac{y}{x}\right)\right]$$

$y=0$, $k=0$: $|E| = E_{x0} = -\dfrac{\lambda}{\pi\varepsilon_0}$

$y=0$, $k=1$: $|E| = E_{x1} = \dfrac{-\lambda}{\pi\varepsilon_0}\dfrac{2x^4}{\rho_>^3}$

$$\frac{E_{x1}}{E_{x0}} = \frac{\rho_>^3}{2x^4}$$

2.21. Cauchy's Theorem:

$$\frac{1}{2\pi i}\oint_C \frac{F(z')dz'}{z'-z} = \begin{cases} F(z) & \text{if } z \text{ is inside} \\ 0 & \text{outside} \end{cases}$$

"A closed surface C in the region R is F(z) or 0 whether in the real or complex space."

$$F(z) = \text{inside} - \text{outside}$$

$$= \frac{1}{2\pi i}\left(\oint_C \frac{F(z')dz'}{z'-z} - \oint \frac{F(z')dz'}{z'-1/z}\right)$$

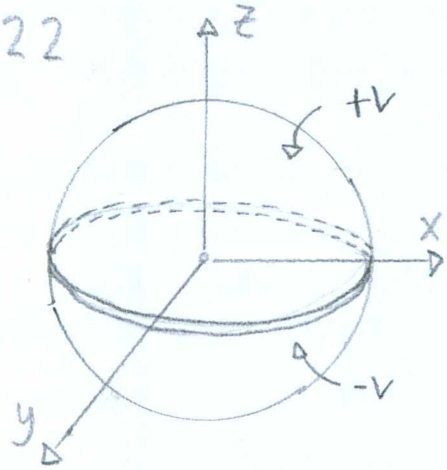
$$= \frac{1}{2\pi i}\left(\oint_C \frac{1}{z'-z} F(z+e^{in})dz - \oint_C F(\tfrac{1}{z}+e^{in}) \frac{1}{z'-\frac{1}{z}} dz\right)$$

$$= \frac{1}{2\pi}\left(\oint_C \frac{1}{z'-z}\frac{dF(z+e^{in})}{dn} - F(\tfrac{1}{z}+e^{in})\frac{d}{dn}\frac{1}{z'-1/z}\right)dn$$

Poisson's Integral (Equation 1.36)
$$= \frac{1}{4\pi}\left[\oint \frac{1}{R}\frac{d\Phi}{dn'} - \Phi \cdot \frac{d}{dn}\left(\frac{1}{R}\right)\right)da$$

Green's theorem (1824) relates to real or complex closed surfaces by Cauchy's and Poisson's integral. The fact, a surface deforms by work or vice-versa, work is a surface's deformation.

2.22



Conducting Sphere With Hemispheres at Different Potentials

Shape: Hemisphere
Dimension: Area [2D]
Charge: q_0

a) (Equation 2.19)

$$\Phi(x) = \frac{1}{4\pi}\int \Phi(a,\theta,\phi')\frac{a(x^2-a^2)}{(x^2+a^2-2ax\cos\gamma)^{3/2}}\,d\Omega$$

$$= \frac{Va\,|x^2-a^2|}{4\pi}\int_0^{2\pi} d\phi' \int_0^1 d\cos\theta \left[\frac{1}{(x^2+a^2-2ax\cos\theta')} - \frac{1}{(x^2+a^2+2ax\cos\theta')}\right]$$

$$= \frac{Va\,|x^2-a^2|}{2x}\left[\frac{1}{(x^2+a^2-2ax\cos\theta')^{1/2}} - \frac{1}{(x^2+a^2+2ax\cos\theta')^{1/2}}\right]_0^1$$

$$= \frac{V|x^2-a^2|}{2x}\left[\frac{1}{|x-a|} - \frac{1}{|x+a|} + \frac{2}{(x^2+a^2)}\right]$$

$$= V\left(\frac{a}{x}\right)\left[\frac{|x-a||x+a|}{2a|x-a|} - \frac{|x-a||x+a|}{2a|x+a|} + \frac{a^2-x^2}{a\sqrt{a^2+x^2}}\right]$$

$$= V\left(\frac{a}{x}\right)\left[1 - \frac{a^2-z^2}{a\sqrt{a^2+x^2}}\right]$$

(Equation 2.27)

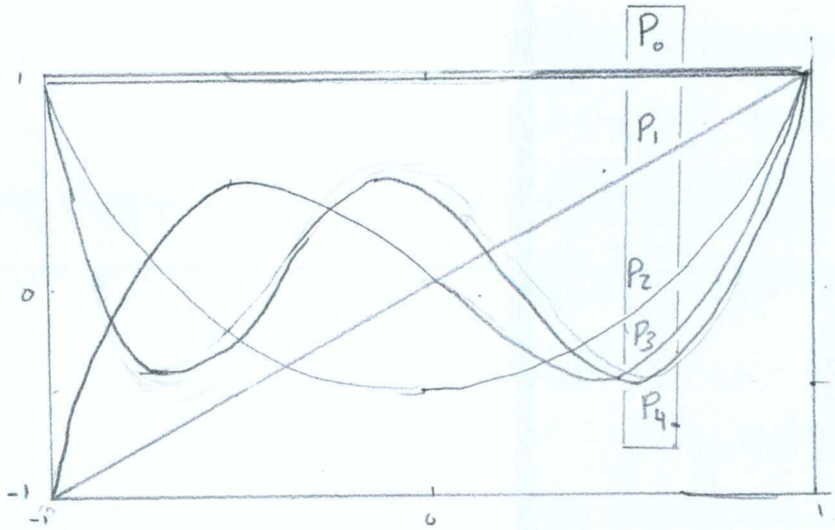
$$\Phi(x,\theta,\phi) = \frac{3Va^2}{2x^2}\Big[\underbrace{\cos\theta - \frac{7a^2}{12x^2}\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) + \cdots}_{\text{Legendre Polynomials}}\Big]$$

Legendre Polynomials

$$P_n(x) = \sum_{k=0}^{\infty} a_k X^k$$

$$a_k = \frac{-(l+k)(l+k+1)}{(k+2)(k+1)} a_k \qquad a_0 = 0 \qquad a_1 = 1$$

n	$P_n(x)$
0	$\cancel{X}$
1	X
2	$\frac{1}{2}(3X^2 - 11)$
3	$\frac{1}{2}(5X^3 - 3X)$



$$\phi(x) = \frac{Va}{z}\left[1 - \frac{a^2 - z^2}{a\sqrt{z^2 + a^2}}\right] \qquad @ \; X = Z$$

$$= V\left(\frac{a}{z}\right)\left[\frac{3z^2}{2a^2} - \frac{7z^4}{12a^2} + \frac{11}{28}\left(\frac{z}{a}\right)^6 + \dots\right]$$

$$= \frac{3Vz}{2a}\left[1 - \frac{7}{12}\frac{z^2}{a^2} + \frac{11}{28}\left(\frac{z}{a}\right)^4 + \dots\right]$$

$$= \frac{3Vz}{2a}\left[\cos(x) - \frac{7}{12}\left(\frac{z^2}{a^2}\right)\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) + \dots\right]$$

The book stated a 2-percent approximation in the second term with Legendre's Polynomials. I prefer the polynomials, themselves.

b) (Equation 2.22) $\Phi_{outer}(z) = V\left[1 - \frac{(z^2-a^2)}{z(z^2+a^2)^{1/2}}\right]$

$$E_{outer} = -\nabla\Phi_{outer}$$

$$= V\left[\frac{3a^2}{(a^2+z^2)^{3/2}} + \frac{a^4}{z^2(a^2+z^2)^{3/2}}\right]$$

$$= \frac{Va^2}{(a^2+z^2)^{3/2}}\left[3 + \left(\frac{a}{z}\right)^2\right]$$

$$\Phi_{inside}(z) = V\left(\frac{a}{z}\right)\left[1 - \frac{(a^2-z^2)}{a(a^2+z^2)^{1/2}}\right]$$

$$E_{inside}(z) = -\nabla\Phi$$

$$= V\left[\frac{a(a^3 - a^2\sqrt{a^2+z^2} - z^2\sqrt{a^2+z^2} + 3az^2)}{z^2(a^2+z^2)^{3/2}}\right]$$

$$= -\frac{V}{a}\left[\frac{3+(a/z)^2}{(1+(z/a)^2)^{3/2}} - \frac{a^2}{z^2}\right]$$

$$E_{outer}(0) = -\frac{V}{a}\left[\frac{3}{2} - \frac{21z^2}{8a^2} + \frac{55z^4}{16a^4} - \cdots\right]$$

$$\cong -\frac{3V}{2a}$$

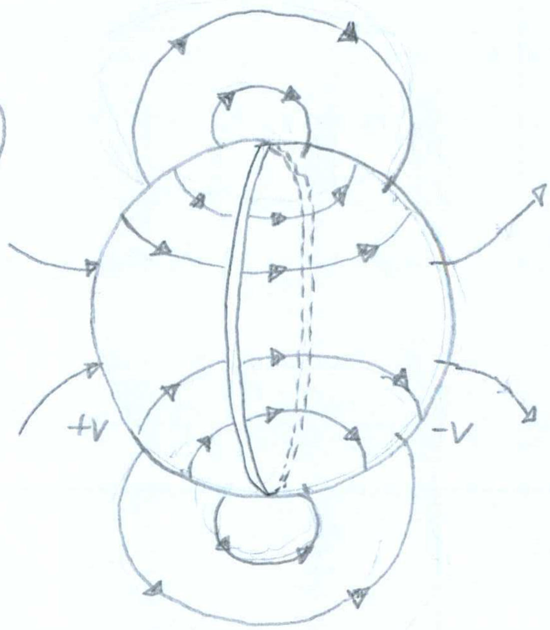
$$E_{in}(a) = -\frac{V}{a}\left[\frac{4}{2^{3/2}} - 1\right]$$

$$= -\frac{V}{a}\left[\sqrt{2} - 1\right]$$

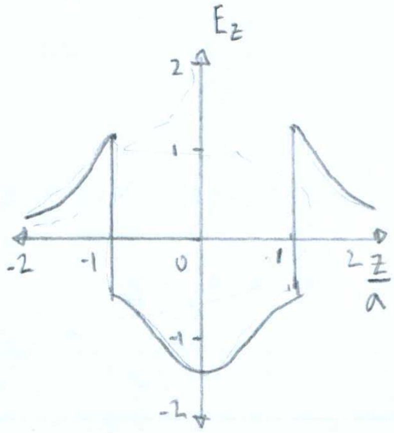
$$F_{outer}(a) = \frac{Va^2}{2^{3/2}a^3} \cdot 4$$

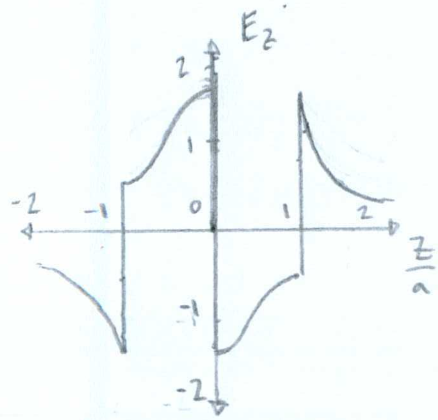
$$= \frac{\sqrt{2}V}{a}$$

c)

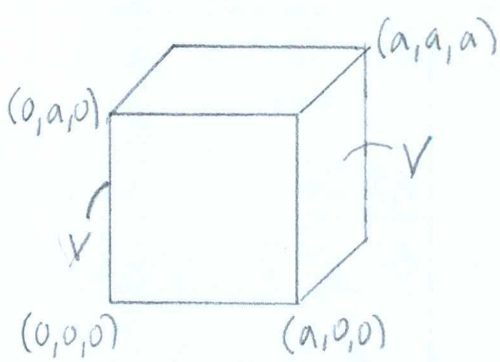


Electric Field Lines.





2.23.



Hollow-cube with conducting walls

Shape: Cube
Dimension: Volume [3D]
Charge: q

a) Potential Derivation:

① Laplace's Equation:

$$\nabla^2\Phi = \frac{1}{X}\frac{\partial^2 X}{\partial^2 x} + \frac{1}{Y}\frac{\partial^2 Y}{\partial^2 y} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z}$$

$$= 0$$

② Boundary Conditions:

$\Phi(0,y,z)=0$; $\Phi(x,0,z)=0$

$\Phi(a,y,z)=0$; $\Phi(x,a,z)=0$

$\Phi(x,y,0)=V$; $\Phi(x,y,a)=0$

③ Laplaces Equations Solutions:

Ⓐ Variable Separation: $f = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$

Ⓑ Linear Eigenvalues: $\frac{X''}{X} = -\lambda_1$; $X'' + \lambda_1 X = 0$

$\frac{Y''}{Y} = -\lambda_2$; $Y'' + \lambda_2 Y = 0$

$\frac{Z''}{Z} = -\lambda_3$; $Z'' + \lambda_3 Z = 0$

Ⓒ General Solution: $X = A\cos(\lambda_1 X) + B\sin(\lambda_1 X)$

$Y = C\cos(\lambda_2 y) + D\sin(\lambda_2 y)$

$Z = E\sinh(\lambda_3 z) + F\cosh(\lambda_3 z)$

④ General Solution to Laplaces Equation:

$$\phi(X,y,z) = \underbrace{(A\cos(\lambda_1 X) + B\sin(\lambda_1 X))}_{X} \times \underbrace{(C\cos(\lambda_2 y) + D\sin(\lambda_2 y))}_{Y} \times \underbrace{(E\sinh(\lambda_3 z) + F\cosh(\lambda_3 z))}_{Z}$$

⑤ Variables by Boundary Conditions:

A $\phi(0,y,z) = (A\cos(\lambda_1 \cdot 0) + B\sin(\lambda_1 \cdot 0))(C\cos(\lambda_2 \cdot y) + D\sin(\lambda_2 y))(E\sinh(\lambda_3 z) + F\cosh(\lambda_3 z))$

$= 0$, so $A = 0$.

C $\phi(X,0,z) = B\sin(\lambda_1 \cdot X)(C\cos(\lambda_2 \cdot 0) + D\sin(\lambda_2 \cdot 0))(E\sinh(\lambda_3 \cdot z) + F\cosh(\lambda_3 \cdot z))$

$= 0$, so $C = 0$

λ_1 $\phi(a,y,z) = B\sin(\lambda_1 a) \circ D\sin(\lambda_2 x)[E\sinh(\lambda_3 \circ z) + F\cosh(\lambda_3 \circ z)]$

$= 0$, so $\lambda_1 = \frac{n\pi}{a}$

λ_2 $\phi(X,a,z) = B\sin\left(\frac{n\pi X}{a}\right) D\sin(\lambda_2 \cdot a)[E\sinh(\lambda_3 z) + F\cosh(\lambda_3 \cdot z)]$

$= 0$, so $\lambda_2 = \frac{m\pi}{a}$

Also, $\lambda_3 = \dfrac{\pi\sqrt{n^2+m^2}}{a}$

$$\phi(x,y,z) = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left[E\sinh(\lambda_3 z) + F\cosh(\lambda_3 z)\right]$$

E $\phi(x,y,0) = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)\left[E\sinh(\lambda_3 \cdot 0) + F\cosh(\lambda_3 \cdot 0)\right]$

$= \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)F$ "Double Fourier Series"

$F = V$

$$F = \frac{4V}{a^2}\int_0^a\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx\,dy$$

$= \dfrac{16V}{\pi^2 nm}$ for odd n, m

E $\phi(x,y,a) = \phi(x,y,0)$

$= F$

$= E\sinh(\lambda_3) + F\cosh(\lambda_3)$

$$E = \frac{1-\cosh(\lambda_3)}{\sinh(\lambda_3)}$$

$$\phi(x,y,z) = \frac{16V}{\pi^2 nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)\left[\frac{1-\cosh(\lambda_3)}{\sinh(\lambda_3)}\sinh(\lambda_3 z) + \cosh(\lambda_3 z)\right]$$

b) $\phi\left(\frac{a}{2},\frac{a}{2},\frac{a}{2}\right) = \dfrac{16V}{\pi^2 nm}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{m\pi}{2}\right)\left[\dfrac{1-\cosh(\lambda_3)}{\sinh(\lambda_3)}\sinh\left(\frac{\lambda_3}{2}\right) + \cosh\left(\frac{\lambda_3}{2}\right)\right]$

@ $n = m = 1$, $\phi = 0.347546V$

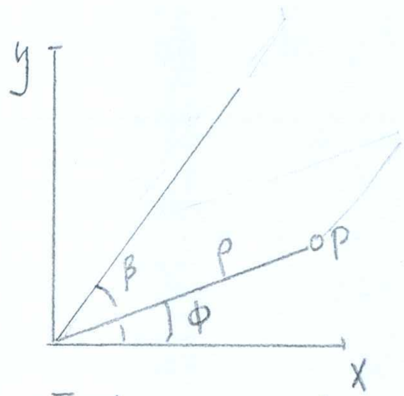
@ $n = 1, m = 3$ or $n = 3, m = 1$, $\phi = 0.332492V$

c) $\sigma = -\varepsilon_0 \frac{\partial \phi}{\partial z}\Big|_{z'=a}$

$$= -\varepsilon_0 \frac{\partial}{\partial z}\left[\frac{16V}{\pi^2 nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left[\frac{1-\cosh(\lambda_3)}{\sinh(\lambda_3)}\sinh(\lambda_3 z)+\cosh(\lambda_3 z)\right]\right]$$

$$= \frac{-16\varepsilon_0 V}{\pi^2 nma}\lambda_3 \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\left[(1-\cosh(\lambda_3))\coth(\lambda_3)+\sinh(\lambda_3)\right]$$

2.24.



Intersection of Two conducting planes

Shape: Plane
Dimension: Angle (1D)
Charge: q

Potential Derivation:

① Boundary Conditions:

$\phi(0,y)=0 \qquad \phi(x,0)=0$

$\phi(\beta,y)=\beta \qquad \phi(x,\beta)=\beta$

② Laplaces Equation:

$$\nabla^2\phi = \frac{1}{X}\frac{\partial^2 X}{\partial^2 x} + \frac{1}{Y}\frac{\partial^2 Y}{\partial^2 y}$$

③ Laplaces Equation solutions:

Ⓐ Variable Separation: $f'' = \frac{X''}{X} + \frac{Y''}{Y}$

$= 0$

Ⓑ Linear Eigenvalues: $\frac{X''}{X} = -\lambda_1 ; \; X'' + \lambda_1 X = 0$

$\frac{Y''}{Y} = -\lambda_2 ; \; Y'' + \lambda_2 Y = 0$

Ⓒ General Solution: $X = A\cos(\lambda_1 x) + B\sin(\lambda_2 x)$

$Y = C\cos(\lambda_2 y) + D\sin(\lambda_3 y)$

④ General Solution to Laplace's Equation:

$$\phi(x,y) = (A\cos(\lambda_1 x) + B\sin(\lambda_2 x))(C\cos(\lambda_2 y) + D\sin(\lambda_2 y))$$

⑤ Variables by Boundary Conditions:

A $\quad \phi(0,y) = [A\cos(\lambda_1 0) + B\sin(\lambda_1 0)][C\cos(\lambda_2 y) + D\sin(\lambda_2 y)]$

$= 0$, so $A = 0$

B $\quad \phi(x,0) = B\sin(\lambda_1 x)[C\cos(\lambda_2 \cdot 0) + D\sin(\lambda_2 \cdot 0)]$

$= 0$, so $C = 0$

$\lambda_1 \quad \phi(\beta,y) = B\sin(\lambda_1 \beta)\sin(\lambda_2 y)$

$= 0$, $\lambda_1 = \frac{m\pi}{\beta}$

$\lambda_2 \quad \phi(x,\beta) = B\sin\left(\frac{m\pi x}{\beta}\right)\sin(\lambda_2 \beta)$

$= 0$, $\lambda_2 = \frac{m\pi}{\beta}$

B is a coefficient to a first order Fourier Series, 2/Period = 2/β. The register for the integral solution failed.

$$\phi(x,y) = \frac{2}{\beta}\sin\left(\frac{m\pi x}{\beta}\right)\sin\left(\frac{m\pi y}{\beta}\right)$$

2.25. a) Green's Theorem Derivation:

① Boundary Conditions:

$\Phi(\rho, \phi=0) = 0 \quad ; \quad \Phi(\rho, \phi=\beta) = 0$

② Laplaces Equation:

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}$$

$= 0$

③ Laplace's Equation Solutions:

Ⓐ Variable Separation: $\nabla^2\Phi = \frac{\rho}{R}\frac{\partial}{\partial\rho}(\rho R') + \frac{1}{R}\Phi'' = 0$

Ⓑ Angular Eigenvalues: $\frac{\Phi''}{\Phi} = -\lambda \; ; \; \Phi'' + \Phi\lambda = 0$

Ⓒ Radial Eigenvalues: $\frac{\rho}{R}\frac{\partial}{\partial\rho}(\rho R') = -\lambda \; ; \; \rho R' + R\lambda = 0$

Ⓓ General Solution: $\Phi(\rho, \phi) = R_0\Phi_0 + R_n\Phi_n$

④ General Solution to Laplaces Equation:

$R_0 = (a_0 + b_0 \ln\rho) \qquad \Phi_0 = (A_0 + B_0\phi)$

$R_n = (a_v\rho^{v} + b\rho^{-v}) \qquad \Phi_n = A_v e^{iv\phi} + B_v e^{-iv\phi}$

⑤ Variables by Boundary conditions:

$A_0, A_v, B_v \quad \Phi(\rho, \phi=0) = (a_0 + b_0\ln\rho)(A_0 + B_0\phi) + \sum_{n=1}^{\infty}(a_v\rho^{v} + b\rho^{-v})(A_v e^{iv\phi} + B_v e^{-iv\phi})$

$= 0$, so $A_0 = 0$ and $B_v = -A_v$

$a_0, b_0,$ $\Phi(\rho, \phi = \beta) = (a_0 + b_0 \ln\rho)\beta + \sum_{\nu=1}^{\infty} (a_\nu \rho^{\nu} + b_\nu \rho^{-\nu})(Ae^{i\nu\beta} + Be^{-i\nu\beta})$

$\checkmark$

$= 0, \text{ so } a_0 = b_0 = 0 \text{ and } \nu = \frac{m\pi}{\beta}$

$$\Phi(\rho, \beta) = \sum_{\nu=1}^{\infty} (a_\nu \rho^{m\pi/\beta} + b_\nu \rho^{-m\pi/\beta}) \sin(m\pi\phi/\beta)$$

a_ν, b_ν $E_1 = -\frac{\partial\Phi}{\partial\rho}\Big|_{b=0} = -\frac{m\pi}{\beta^2} \sum_{m=1}^{\infty} a_\nu \rho^{m\pi/\beta} \sin(m\pi\phi/\beta)$

$$E_2 = -\frac{\partial\Phi}{\partial\rho}\Big|_{a=0} = +\frac{m\pi}{\beta} \sum_{m=1}^{\infty} b_\nu \rho^{-m\pi/\beta} \sin(m\pi\phi/\beta)$$

$$E = E_1 - E_2 = \frac{\lambda\delta(\phi - \phi')}{\rho\varepsilon_0}$$

$$\sum_{m=1}^{\infty} \int_0^{\beta} \sin(m\pi\phi'/\beta) \sin(n\pi\phi'/\beta)\, d\phi' \left[a_\nu\rho^{m\pi/\beta} + b_\nu\rho^{-m\pi/\beta}\right]$$

$$= \frac{\lambda \sin(n\pi\phi/\beta) \cdot \beta}{m\pi \cdot \varepsilon_0}$$

$$\left[a_\nu e^{-m\pi/\beta} + b_\nu e^{-m\pi/\beta}\right] = \frac{2\lambda \sin(n\pi\phi/\beta) \cdot \beta}{m\pi\varepsilon_0}$$

$$\frac{\partial\Phi}{\partial\rho}\Big|_{b=0} = \frac{\partial\Phi}{\partial\rho}\Big|_{a=0}$$

$$b_\nu = \rho^{m\pi/\beta} \frac{1}{\pi} \frac{\lambda}{m\varepsilon_0} \sin(m\pi\phi/\beta)$$

$$a_\nu = \rho^{-m\pi\phi/\beta} \frac{1\lambda}{\pi m\varepsilon_0} \sin(m\pi\phi/\beta)$$

$$\Phi(\rho,\phi) = \frac{\lambda}{4\varepsilon_0}\sum_{m=1}^{\infty}\frac{1}{m}\,\rho_<^{m\pi/\beta}\cdot\rho_>^{-m\pi/\beta}\sin(m\pi\phi/\beta)\sin(m\pi\phi'/\beta)$$

If, $\lambda = 4\pi\varepsilon_0$

$$G(\rho,\phi;\rho',\phi') = 4\sum_{m=1}^{\infty}\frac{1}{m}\,\rho_<^{m\pi/\beta}\,\rho_>^{-m\pi/\beta}\,e^{im\pi\phi/\beta}\cdot e^{-im\pi\phi/\beta}$$

b) $$G(\rho,\phi;\rho',\phi') = 4\sum_{m=1}^{\infty}\frac{1}{m}\,\rho_<^{m\pi/\beta}\,\rho_>^{-m\pi/\beta}\,e^{im\pi\phi/\beta}\,e^{-im\pi\phi/\beta}$$

$$G(\rho,\phi;\rho',\phi') = \Big[-2\sum_{m=1}^{\infty}\frac{1}{m}\rho^{m\pi/\beta}\cdot\rho^{-m\pi(\phi+\phi')/\beta} + 2\sum_{m=1}^{\infty}\frac{1}{m}\rho^{m\pi/\beta}\circ\rho^{im(\phi-\phi')/\beta}\Big]$$

$$= \Big[-2\sum_{m=1}^{\infty}\frac{1}{m}z_1^m + 2\sum_{m=2}^{\infty}\frac{1}{m}z_2^m\Big]$$

$$= 2\,\mathrm{Real}\big[\ln(1-z_1) - \ln(1-z_2)\big]$$

$$= 2\,\mathrm{Real}\Big[\ln\Big(\frac{1-z_1}{1-z_2}\Big)\Big]$$

$$= \ln\left(\frac{1+|z_1|^2 - 2\,\mathrm{Real}[z_1]}{1+|z_2|^2 - 2\,\mathrm{Real}[z_2]}\right)$$

$$= \ln\left(\frac{1+\rho_<^{2\pi/\beta}\rho_>^{-2\pi/\beta} - 2\rho_<^{\pi/\beta}\rho_>^{-\pi/\beta}\cos(\pi(\phi-\phi')/\beta)}{1+\rho_<^{2\pi/\beta}\rho_>^{-2\pi/\beta} - 2\rho_<^{\pi/\beta}\rho_>^{-\pi/\beta}\cos(\pi(\phi-\phi')/\beta)}\right)$$

$$= \ln\left(\frac{\rho_>^{2\pi/\beta} + \rho_<^{2\pi/\beta} - 2(\rho_<\rho_>)^{\pi/\beta}\cos(\pi(\phi-\phi')/\beta)}{\rho_>^{2\pi/\beta} + \rho_<^{2\pi/\beta} - 2(\rho_<\rho_>)^{\pi/\beta}\cos(\pi(\phi-\phi')/\beta)}\right)$$

Imaginary Identity:

$$2\,\mathrm{Imag}(z_1)\cdot\mathrm{Imag}(z_2) = \mathrm{Real}[-z_1\circ z_2 + z_1\circ z_2]$$

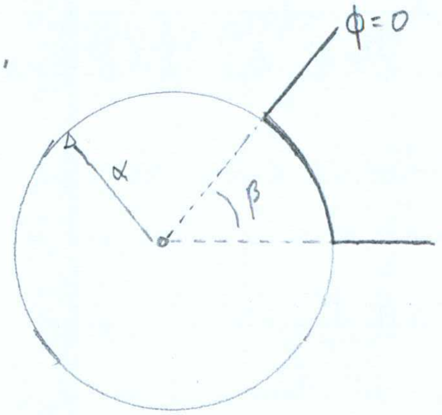
A second identity:

$$-\ln(1-z) = \sum_{m=1}^{\infty}\frac{z^m}{m}$$

Another identity:

$$2\,\mathrm{Real}[\ln z] = \ln[z\circ z^*]$$

2.26.



Rounded conducting Surface

Shape: Round Surface
Dimension: Area [2D]
Charge: q

a) Potential Derivation:

① Boundary Conditions:

$\Phi(\rho, \phi=0)=0$ $\qquad$ $\Phi(\rho,\phi=\beta)=0$

$\Phi(\rho=a,\phi)=0$

② Laplace's Equation:

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\phi^2}=0$$

③ Laplace's Equation Solutions:

Ⓐ Variable separation: $\nabla^2\Phi=\frac{\rho}{R}\frac{\partial}{\partial\rho}(\rho R')+\frac{b}{R}\phi''=0$

Ⓑ Angular Eigenvalues: $\frac{\phi''}{\phi}=-\lambda$; $\phi''+\phi\lambda=0$

Ⓒ Radial Eigenvalues: $\frac{\rho}{R}\frac{\partial}{\partial\rho}(\rho R')=-\lambda$

$\rho R'+\lambda R=0$

Ⓓ General Solution: $\Phi(\rho,\phi)=R_0\Phi_0+R_n\Phi_n$

④ General Solution to Laplace's Equation:

$R_0=a_0+\ln b_0$ $\qquad$ $\Phi_0=(A_0+B_0\phi)$

$R_n=(a_v\rho^v+b_v\rho^{-v})$ $\qquad$ $\Phi_n=A_v e^{iv\phi}+B_v e^{-iv\phi}$

⑤ Variables by Boundary Conditions:

$$\Phi(\rho,\phi)=(a_0+\ln b_0)(A_0+B_0\phi)+\sum_{v=1}^{\infty}(a_v\rho^v+b_v\rho^{-v})(A_v e^{iv\phi}+B_v e^{-iv\phi})$$

A_0 $\quad$ $\Phi(\rho,\phi=0)=(a_0+\ln b_0)(A_0)+\sum_{v=1}^{\infty}(a_v\rho^v+b_v\rho^{-v})(A_v+B_v)$

A_v, B_v $\quad$ $=0$, so $A_0=0$ and $A_v=-B_v$

$$\beta \quad \Phi(\rho; \phi=\beta) = (a_0 + b_0 \ln \rho)\beta + \sum_{m=1}^{\infty} (a_v \rho^{v} + b_v \rho^{-v}) \sin(v\beta)$$

$$= 0, \; B_0 = 0 \quad \text{and} \quad v = \frac{n\pi}{\beta}$$

$$b_n \quad \Phi(\rho = a, \phi) = (a_v a^{n\pi/\beta} + b_v a^{-n\pi/\beta}) \sin\left(\frac{n\pi\phi}{\beta}\right)$$

$$= 0, \text{ so } \quad b_n = -a_n a^{2n\pi/\beta}$$

$$\Phi(\rho, \phi) = \sum_{m-1}^{\infty} A_n \left(\left(\frac{\rho}{a}\right)^{n\pi/\beta} - \left(\frac{\rho}{a}\right)^{-n\pi/\beta} \right) \sin\left(\frac{n\pi\phi}{\beta}\right)$$

b) $E = -\nabla \Phi$

$$= -\frac{\partial \Phi}{\partial \rho} - \frac{1}{\rho}\frac{\partial \Phi}{\partial \phi}$$

$$= E_\rho + E_\phi$$

$$E_\rho = -\frac{\partial \Phi}{\partial \rho} = -\frac{\partial}{\partial \rho} \sum_{m=1}^{\infty} A_n \left(\left(\frac{\rho}{a}\right)^{n\pi/\beta} - \left(\frac{\rho}{a}\right)^{-n\pi/\beta} \right) \sin\left(\frac{n\pi\phi}{\beta}\right)$$

$$= -\sum_{m=1}^{\infty} A_n \frac{n\pi}{a\beta} \left(\left(\frac{\rho}{a}\right)^{n\pi\beta - 1} + \left(\frac{\rho}{a}\right)^{-n\pi\beta - 1} \right) \sin\left(\frac{n\pi\phi}{\beta}\right)$$

$$E_\phi = \frac{-1}{\rho}\frac{\partial \Phi}{\partial \phi} = -\frac{1}{\rho}\frac{\partial}{\partial \phi} \sum_{m=1}^{\infty} A_n \left(\left(\frac{\rho}{a}\right)^{n\pi\beta} - \left(\frac{\rho}{a}\right)^{-n\pi/\beta} \right) \sin\left(\frac{n\pi\phi}{\beta}\right)$$

$$= \frac{-A_n \pi}{a\beta} \left(\left(\frac{\rho}{a}\right)^{\pi/\beta - 1} - \left(\frac{\rho}{a}\right)^{-\pi/\beta - 1} \right) \cos\left(\frac{n\pi\phi}{\beta}\right)$$

$$\sigma(\rho,0) = +\varepsilon_0 E\Big|_{\phi=0}$$

$$= -\frac{A n\pi\varepsilon_0}{a\beta}\left(\left(\frac{\rho}{a}\right)^{\pi/\beta-1} - \left(\frac{\rho}{a}\right)^{-\pi/\beta-1}\right)$$

$$\sigma(\rho,\beta) = \varepsilon_0 E$$

$$= \frac{A n\pi\varepsilon_0}{a\beta}\left(\left(\frac{\rho}{a}\right)^{\pi/\beta-1} - \left(\frac{\rho}{a}\right)^{-\pi/\beta-1}\right)$$

$$\sigma(a,\phi) = \varepsilon_0 E$$

$$= -\frac{A_1 n\pi\varepsilon_0}{a\beta}\left(\left(\frac{a}{a}\right)^{\pi/\beta-1} + \left(\frac{a}{a}\right)^{-\pi/\beta-1}\right)\sin\left(\frac{n\pi\phi}{\beta}\right)$$

$$= -\frac{A_1 2\pi\varepsilon_0}{a\beta}\sin\left(\frac{n\pi\phi}{\beta}\right)$$

c) At Boundary $\Phi(\rho, \phi=\beta) = \Phi(\rho, \phi=\pi)$

$$E_{Tot} = E_\rho + E_\phi$$

$$= A_1\frac{n}{a}\left(\left(\left(\frac{\rho}{a}\right)^{n-1} + \left(\frac{\rho}{a}\right)^{-n-1}\right)\sin(n\phi) - \left(\left(\frac{\rho}{a}\right)^{n-1} - \left(\frac{\rho}{a}\right)^{-n-1}\right)\cos(n\phi)\right)$$

@ $n=1$

$$= -A\frac{1}{a}\left[-\left(1+\left(\frac{a}{\rho}\right)^2\right)\sin(\phi) - \left(1-\left(\frac{a}{\rho}\right)^2\right)\cos\phi\right]$$

@ $\rho \gg a$

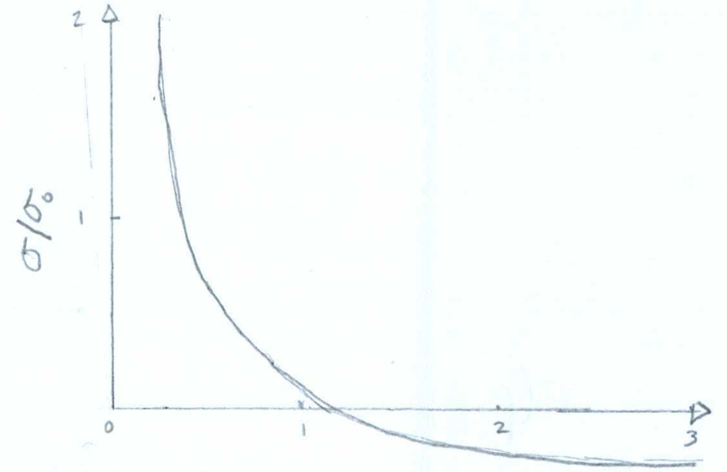
$$E_{TOT} = -\frac{A}{a}\left[\sin(\phi) + \cos(\phi)\right]$$

$$= -A/a$$

$$\sigma(a,\phi) = \frac{-A_1(2\varepsilon_0)}{a}\sin\phi$$

$$\sigma(\rho,0) = \sigma(\rho,\beta)$$

$$= \frac{\sigma_0}{2}\left(1-\left(\frac{\rho}{a}\right)^{-2}\right)$$



Small "ρ/a" implies an electromagnetic sphere around the cylinder, but larger ratios an inversion on the cylinder and into the bar magnet field lines.

Charge:

$$Q_{Half} = -A_1\frac{2\varepsilon_0}{a}\int_0^{\pi}\sin(\phi)\,a\,d\phi$$

$$= -A_1 4\varepsilon_0$$

$$\sigma(y=0) = \varepsilon_0 E$$

$$= -\varepsilon_0\frac{A_1}{a}$$

$$Q_{strip} = \sigma 2a$$

$$= -A_1 2\varepsilon_0$$

$$= \frac{1}{2}Q_{Half}$$

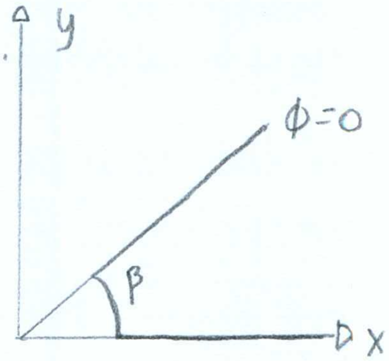
$$Q = 2\int_a^{l}(-A_1)\frac{\varepsilon_0}{a}\left(1-\left(\frac{\rho}{a}\right)^{-2}\right)d\rho + Q_{Half}$$

$$= 2A_1\varepsilon_0\left[\frac{-l}{a}-\frac{a}{l}\right] + Q_{Half}$$

@ $l \gg a$

$$Q = -\frac{2l\varepsilon_0 A_1}{a} - 4\varepsilon_0 A_1$$

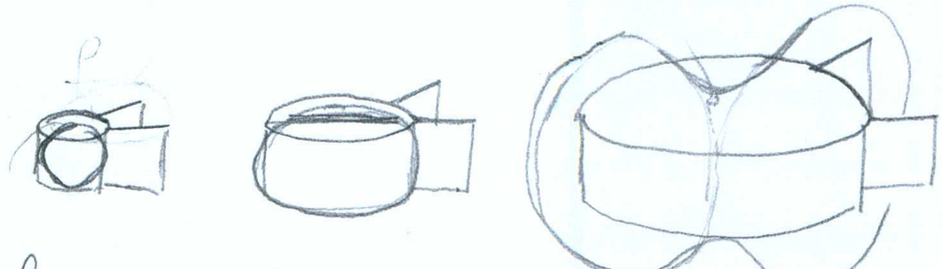
2.27.



Thin conducting sheets and a cylinder

a) From problem 2.26,

$$\sigma = \frac{\sigma_0}{2}\left(1 - \left(\frac{\rho}{a}\right)^{-2}\right)$$



$\frac{\rho}{a} < 1$ $\quad$ $\frac{\rho}{a} = 1$ $\quad$ $\frac{\rho}{a} > 1$

$\boxed{\text{Surface charge Density}}$

b) Charge:

$$Q_c = -A\left(\frac{2\varepsilon_0}{a}\right)\int_0^{2\pi} \sin(\phi)\, a\, d\phi$$

$$= 0$$

a = finite,

$$Q = 2\int_a^{\ell} (-A_1)\frac{\varepsilon_0}{a}\left(1 - \left(\frac{\rho}{a}\right)^{-2}\right) d\rho + Q_c$$

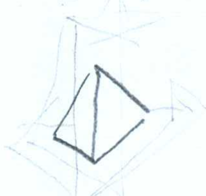
$$= -\frac{2\ell\varepsilon_0 A_1}{a}$$

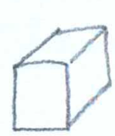
$a = 0$,

$$Q = 2\int_a^{\ell} (-A_1)\frac{\varepsilon_0}{a}\left(1 - \left(\frac{\rho}{a}\right)^{-2}\right) d\rho + Q_c$$

$= -\infty$, math falls apart

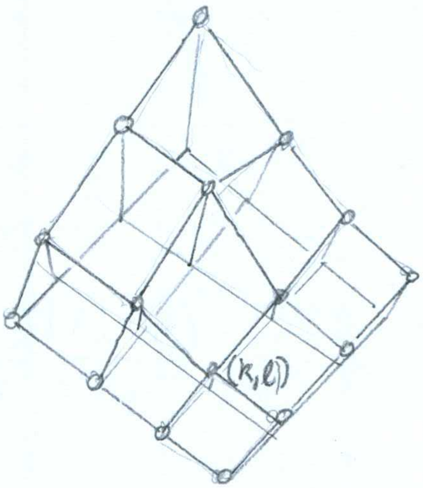
2.28 $\quad n = 4 \quad n = 6 \quad n = 8 \quad \ldots$





$$V_{TOT} = \sum_i \phi_i$$
$$= n\phi$$
$$\phi_{TOT} = \frac{\sum_i V_{TOT}}{n}$$

2.29. Galerkin's Method (1929):



Average value at a point:

$$\int_R \phi_{ij}(x,y)\,dx\,dy$$

Average value across points:

$$\int_R \nabla\phi_{ij}(x,y)\circ\nabla\phi_{kl}(x,y)\,dx\,dy$$

$$\phi_{ij}(x,y) = \left(1-\frac{|x|}{h}\right)\left(1-\frac{|y|}{h}\right)$$

$$\int_{-h}^{h}\int_{-h}^{h}\phi_{ij}(x,y)\,dxy = \int_{-h}^{h}\int_{-h}^{h}\left(1-\frac{|x|}{h}\right)\left(1-\frac{|y|}{h}\right)dx\,dy$$

$$= h^2$$

$$\int_{-h}^{h}\int_{-h}^{h}\nabla\phi_{ij}(x,y)\,\nabla\phi_{ij}(x,y)\,dx\,dy$$

$$= \int_{-h}^{h}\int_{-h}^{h}\left\langle -\frac{(h-|y|)}{h^2}, -\frac{(h-|x|)}{h^2}\right\rangle\circ\left\langle -\frac{(h-|y|)}{h^2}, -\frac{(h-|x|)}{h^2}\right\rangle dx\,dy$$

$$= \frac{4}{3}+\frac{4}{3}$$

$$= 8/3$$

$\boxed{\text{Lesson, integrals with absolutes require parcel terms about the absolute minimum and maximum.}}$

$$\int_{-h}^{h}\int_{-h}^{h} \nabla\Phi_{i+,j} \circ \nabla\Phi_{ij}\, dx\,dy$$

$$= \int_{-h}^{h}\int_{-h}^{h} \left\langle \frac{-(h-|y|)(x+h)}{h^2}, \frac{-(h-|x|)}{h^2} \right\rangle \circ \left\langle \frac{-(h-|y|)}{h^2}, \frac{-(h-|x|)}{h^2} \right\rangle dx\,dy$$

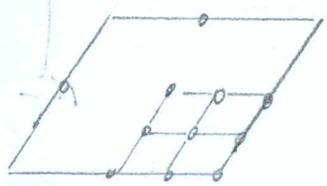
$$= -1/3$$

$$\int_{-h}^{h}\int_{-h}^{h} \nabla\Phi_{i,j+1} \circ \nabla\Phi_{i,j}\, dx\,dy$$

$$= \int_{-h}^{h}\int_{-h}^{h} \left\langle \frac{-(h-|y|)}{h^2}, \frac{-(h-|x|)(y+h)}{h^2} \right\rangle \circ \left\langle \frac{-(h-|y|)}{h^2}, \frac{-(h-|x|)}{h^2} \right\rangle dx\,dy$$

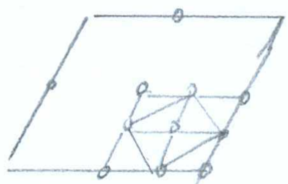
$$= -1/3$$

2.30. "Squares" "Triangles"



Unit square

Shape: Square
Dimension: Area
Charge: q



Unit Square

Shape: Square
Dimension: Area
Charge: q

(Equation 2.83) $\sum_{j=1}^{3} \psi_j \int \nabla N_i^{(e)} \cdot \nabla N_j^{(e)} \, dx\,dy = \int g N_i^{(e)} \, dx\,dy$

$$\int_V \nabla \phi_{ij} \cdot \nabla \psi \, dx\,dy = \int 4\pi\rho \, \phi_{ij} \, dx\,dy$$

$$\nabla \psi = 4\pi\rho \quad \text{where } \rho = -\varepsilon_0 \nabla \phi$$
$$= 4\pi\varepsilon_0 \phi$$

From problem 1.24,

$$4\pi\varepsilon_0 \phi(x,y) = \frac{16}{\pi^2} \sum_{m=0}^{n} \frac{\sin[(2m+1)\pi x]}{(2m+1)^3} \left\{ 1 - \frac{\cosh((2m+1)\pi(y-1/2))}{\cosh((2m+1)\pi/2)} \right\}$$

This problem,

$$4\pi\varepsilon_0 \phi(x,y) = \nabla \psi$$
$$= \psi_1 + \psi_2 x + \psi_3 y$$