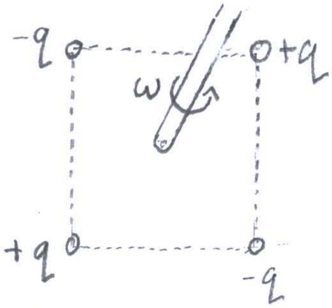
Chapter 9: Radiating Systems, Multipole Fields, and Radiation:

9.1



"... radiating quadropole"

a) Real-time dependent moments:

(9.1) "charge density ... in time"

$$\rho(x,t) = \rho(x)\, e^{-i\omega t}$$

(4.3) "Multipole moments"

$$q_{\ell m} = \int Y^*_{\ell m}(\theta', \phi')\, r'^{\ell} \rho(x')\, d^3x$$

$$= \int Y^*_{\ell m}(\theta, \phi)\, r'^{\ell}\, \mathrm{Re}\left[\rho(x) e^{-i\omega t}\right] d^3x$$

$$q^*_{\ell m}(t) = \int (-1)^m Y^*_{\ell,-m}(\theta,\phi)\, r^{\ell} \cdot \mathrm{Re}\left[\rho(x) e^{-i\omega t}\right] d^3x$$

$$= Q_{\ell,-m}(t)$$

$$\mathrm{Re}\left[q_{\ell m}(t) + i q^*(t)\right] = \mathrm{Re}\left[q_{\ell m}(t) - (-1)\, q_{\ell m}(t)\right]$$

$$= 2 q_{\ell m}(t)$$

$$q_{\ell m}(t) = \frac{1}{2}\left[Q_{\ell m}(t) + (-1)^m Q^*_{\ell,-m}(t)\right]$$

$$= \frac{1}{2}\left[Q_{\ell m} e^{-i\omega t} + (-1)^m Q^*_{\ell,-m}\, e^{-i\omega t}\right]$$

Multipole Moments at harmonic frequency by Fourier Decomposition:

(4.3) "Multipole moments"

$$q_{\ell m}(t) = \int Y^*_{\ell m}(\theta,\phi)\, r^{\ell} \rho_R(r,\theta,\phi - \omega_0 t)\, d^3x$$

$$= \int Y^*_{\ell m}(\theta, \phi' + \omega_0 t)\, r^\ell \rho_R(r,\theta,\phi')\, r^2 dr\, d\cos\theta\, d\phi'$$

$$= \int Y^*_{\ell m}(\theta, \phi')\, e^{-im\omega_0 t} \cdot r^\ell \rho_R(r,\theta,\phi')\, r^2 dr\, d\cos\theta\, d\phi'$$

$$= q_{\ell m}' e^{-im\omega_0 t}$$

$$= \frac{1}{2}\left[Q_{\ell m} e^{-i\omega t} + (-1)^m Q^*_{\ell,-m} e^{-i\omega t}\right]$$

If $m<0$,

$$q_{\ell m} = \frac{1}{2} Q_{\ell m} \quad \text{and} \quad \omega = -m\omega_0$$

If $m>0$,

$$q_{\ell m} = \frac{(-1)^m}{2} Q_{\ell,-m} \quad \text{and} \quad \omega = m\omega_0$$

If $m=0$

$$q_{\ell m} = \mathrm{Re}(Q_m) \quad \text{and} \quad \omega = 0$$

b) $$\rho(x,t) = \rho_0(x) + \sum_{n=1}^{\infty} \mathrm{Re}\left[2\rho_n(x) e^{-in\omega_0 t}\right]$$

$\therefore$ where $$\rho_n(x) = \frac{1}{T}\int_0^T \rho(x,t) e^{-in\omega_0 t} dt$$

$$\rho(x,t) = \rho_0(x) + \sum_{n=1}^{\infty}\left[\rho_n(x) e^{-im\omega_0 t} + \rho_n^*(x) e^{im\omega_0 t}\right]$$

$$q_{\ell m}(t) = \int_0^\infty Y^*_{\ell m}(\theta,\phi)\, r^\ell \rho_R(x,t)\, d^3x$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} Y^*_{\ell m}(\theta,\phi)\, r^\ell \left[\rho_n(x) e^{-im\omega_0 t} + \rho_n^*(x) e^{-im\omega_0 t}\right]$$

If $m \neq 0$,

$$= \frac{1}{2}\left[Q_{\ell m} e^{-i\omega t} + (-1)^m Q_{\ell,-m} e^{i\omega t}\right] \quad \therefore \text{when } \omega_0 = \omega,\ m \neq 0$$

If $m=0$,

$$= Q_{\ell,0} \quad \text{and} \quad w_0 = 0$$

c) $q_{\ell m}(t) = \int Y^*_{\ell m}(\theta,\phi)\, r^\ell \cdot r\, dr\, d\phi\, dz$

$$= \int Y^*_{\ell m}(\theta,\phi)\, r^\ell \cdot \frac{q}{r}\, r\, dr\, d\phi\, dz$$

$$= \int_0^\infty \int_0^\pi \int_0^R Y^*_{\ell m}(\theta,\phi) \cdot r^\ell \cdot q \cdot dr\, d\phi \cdot dz$$

$$= q \cdot R^\ell \cdot Y^*_{\ell m}(\theta,\phi)$$

Spherical Harmonics

"Jackson (3.53)"

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\phi}$$

If $\theta = \frac{\pi}{2}$ and $\phi = w_0 t$,

when $\ell = 0, m = 0$.

$$q_{00}(t) = q \cdot R^0 \cdot Y^*_{00}\left(\tfrac{1}{2}\pi, w_0 t\right)$$

$$= q \cdot \sqrt{\frac{2\cdot 0+1}{4\pi}\frac{(0-0)!}{(0+0)!}}\, P_0^0\left(\cos\frac{\pi}{2}\right) e^{-i\cdot 0\cdot w_0 t}$$

$$= \frac{q}{\sqrt{4\pi}}$$

"Jackson 3.50"

Legendre Polynomial

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2} \times \frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell$$

If $\theta = \frac{\pi}{2}$ and $\phi = w_0 t$

when $\ell = 1, m = -1, 0, 1$

$$q_{10}(t) = q \cdot R^1 \cdot Y^*_{10}\left(\tfrac{1}{2}\pi, w_0 t\right)$$

$$= q \cdot R \sqrt{\frac{2\cdot \ell+1}{4\pi}\frac{(1-0)!}{(1+0)!}}\, P_1^0\left(\cos\frac{\pi}{2}\right) e^{i\cdot 0\cdot w_0 t}$$

$$= 0$$

$$q_{1,\pm 1}(t) = q \cdot R \cdot Y^*_{1,\pm 1}\left(\tfrac{1}{2}\pi, w_0 t\right)$$

$$= q \cdot R \sqrt{\frac{2\cdot 1+1}{4\pi}\frac{(1\pm 1)!}{(1\pm 1)!}}\, P_1^{\pm 1}\left(\cos\frac{\pi}{2}\right) e^{\mp i\cdot 0\cdot w_0 t}$$

$$= \mp\sqrt{\frac{3}{8\pi}}\, q\, R e^{\mp i\omega_0 t}$$

So, $q_{em} = -m q R \sqrt{\frac{3}{8\pi}}$ .. when $m=-1,0,1$

... by Fourier decomposition:

$$\rho_n(x) = \frac{1}{T}\int_0^T \rho(x) e^{im\omega_0 t}\, dt$$

$\downarrow m=0$ $\qquad\qquad$ $\searrow m=-1,1$

$$\frac{q}{R\omega_0 T}\int_0^T [\rho_0(x)] e^{im\phi}\, dT \qquad\qquad \frac{q}{R\omega_0 T}\int_0^T 2\cdot \rho_n(x) e^{im\phi}\, dt$$

$$= \frac{q}{2\pi R} \text{ .. when } \omega_0 T = 2\pi \qquad\qquad = \frac{q}{\pi R} e^{im\phi} \text{ ... when } \omega_0 T = 2\pi$$

$l=0$ $\quad Q_{0,0} = \int Y^*_{00}(\theta,\phi) r^0 \cdot \rho(x)\, r\, dr\, d\phi\, dz$

$$= \int \sqrt{\frac{(2\cdot 0+1)(0-0)!}{4\pi\,(0+0)!}}\, P_0^0(\cos\tfrac{\pi}{2})\, e^{i\cdot 0\cdot \omega t} \cdot r^0 \rho(x)\, dr\, d\phi\, dz$$

$$= \frac{1}{\sqrt{4\pi}} \int \rho(x) \cdot r\, dr\, d\phi\, dz$$

$$= \frac{1}{\sqrt{4\pi}} \int_0^\infty \int_0^{2\pi} \int_0^R \frac{q}{2\pi R}\, dr\, d\phi\, dz$$

$$= \frac{q}{\sqrt{4\pi}}$$

$l=1$ $\quad Q_{1,0} = \frac{q}{\pi R}\int Y^*_{em}(\theta,\phi)\, e^{im\phi}\, r\, dr\, d\phi\, dz$

$$= \frac{q}{\pi R}\int_0^{2\pi}\int_0^1\int_0^R \sqrt{\frac{2\cdot 1+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_1^m(\cos\tfrac{\pi}{2})\, e^{im\omega_0 t}\, e^{im\phi}\, dR\, dz\, d\phi$$

$$Q_{1,0} = \frac{qR}{2\pi}\sqrt{\frac{3}{4\pi}}\cos\frac{\pi}{2}\, e^{\mp i\cdot 0\cdot t}$$

$$= 0$$

$$Q_{1,-1} = \frac{qR}{2\pi}\sqrt{\frac{3}{8\pi}}\int_0^{2\pi} e^{i(-1-1)\phi}\, d\phi$$

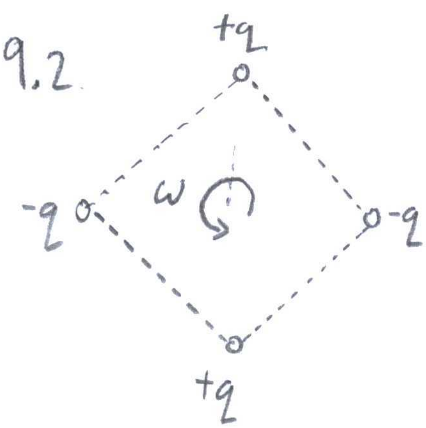
$$= 0$$

$$Q_{1,+1} = \frac{-qR}{2\pi}\sqrt{\frac{3}{8\pi}}\int_0^{2\pi} e^{i(-1+1)\phi}\, d\phi$$

$$= -qR\sqrt{\frac{3}{8\pi}}$$

Higher Harmonic frequencies correspond to $\omega = m\omega_0$ for $m = 0, 1, 2, \ldots$

9.2



"radiating quadropole consists of a square."

Charge	x	y	z
$-q$	$\pm\frac{a}{\sqrt{2}}\cos\omega t$	$\sin\omega t$	0
$+q$	$\pm\frac{a}{\sqrt{2}}\sin\omega t$	$\mp\cos\omega t$	0

(4.9) "Traceless quadropole moment tensor"

$$Q_{ij} = \int (3x_i' x_j' - r'^2\delta_{ij})\rho(x')\, d^3x$$

.. where $(r^{(k)})^2 = (x_1^{(k)})^2 + (y_1^{(k)})^2$

$$= \frac{a^2}{2}$$

$$\sum_k q_k (r^{(k)})^2 = \frac{1}{2}a^2 \sum q_k$$

$$= 0$$

$$Q_{11} = \sum_3 q\left[3x_i^{(k)}x_j^{(k)} - r^{(k)2}\right]$$

$$= q\cdot 3\cdot\frac{a^2}{2}\left[\cos^2\omega t + \cos^2\omega t - \sin^2\omega t + \sin^2\omega t\right]$$

$$= 3 \cdot a^2 \cdot q \left[\cos^2 wt - \sin^2 wt\right]$$

$$= 3 \cdot a \cdot q \cos 2wt$$

$$Q_{12} = \sum_{k=3} q \left[3 x_i^{(k)} x_j^{(k)}\right]$$

$$= 6^2 q \sin 2wt$$

$$Q_{13} = \frac{3a^2 q}{2}\left[-\cos wt \sin wt + \cos wt \sin wt + \cos wt \sin wt - \cos wt \sin wt\right]$$

$$= 0$$

$$Q_{21} = Q_{12}$$

$$Q_{22} = \frac{3a^2 q}{2}\left[-\cos 2wt\right]$$

$$Q_{23} = 0$$

$$Q_{31} = 0$$

$$Q_{32} = 0$$

$$Q_{33} = Q_{13} = Q_{31} = Q_{23} = Q_{32} = 0$$

$$Q_{ij} = 3a^2 q \begin{pmatrix} \cos 2wt & \sin 2wt & 0 \\ \sin 2wt & -\cos 2wt & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= 3a^2 q \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot e^{-2wt}$$

$$= Q_{ij} e^{-2wt}$$

(9.1) "charge density in time"

$$\rho(x,t) = \rho(x) e^{-2wt}$$

$$Q_{ij} = \int (3x_i x_j - r^2\delta_{ij})\rho(x)\,d^3x$$

$$= 3a^2 q \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Q_1 = 3a^2 q \sin\theta e^{i\phi}$$

$$Q_2 = 3a^2 q i \sin\theta e^{i\phi}$$

$$Q_3 = 0$$

(9.44) "Magnetic Field"

$$H = \frac{-ick^3}{24\pi}\frac{e^{ikr}}{r} n \times Q(n)$$

$$n \times Q(n) = 3a^2 q \sin\theta e^{i\theta} \det \begin{pmatrix} x & y & z \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 1 & 0 & 0 \end{pmatrix}$$

$$= -3a^2 q i \sin\theta e^{i\phi}\left[\cos\theta(\hat{x} + i\hat{y}) - \sin\theta e^{i\phi}\hat{z}\right]$$

$$H = \frac{-ck^3}{8\pi}\frac{e^{ikr}}{r} a^2 q \sin\theta e^{i\phi}\left[\cos\theta(x + iy) - \sin\theta e^{i\phi}\hat{z}\right]$$

$$He^{-2wt} = -\frac{ck^3a^2q}{8\pi r}\sin\theta\{\hat{x}\cos\theta\cos(kr - 2wt + \phi) - \hat{y}\cos\theta\sin(kr - 2wt + \phi) - \hat{z}\sin\theta\cos(kr - 2wt + 2\phi)\}$$

(9.39) "Electric Field"

$$E = ikZ_0(n \times A) \times n/\mu_0$$

$$= Z_0(\hat{H} \times \hat{n})$$

$$= -\frac{Z_0 k^3}{8\pi}\frac{e^{ikr}}{r} a^2 q \sin\theta e^{i\theta} \cdot \det \begin{pmatrix} x & y & z \\ \cos\theta & i\cos\theta & -\sin\theta e^{i\phi} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix}$$

$$= -\frac{Z_0 k^3}{8\pi}\frac{e^{ikr}}{r}a^3 q \sin\theta e^{i\phi}\left\{\hat{x}(\sin^2\sin\phi e^{i\phi} + i\cos^2\theta)\right.$$
$$-\hat{y}(\sin^2\theta\cos\phi e^{i\phi} + \cos^2\theta)$$
$$\left.-i\hat{z}(\sin\theta\cos\theta e^{i\phi})\right\}$$

$$\underline{E} e^{-2wt} = -\frac{Z_0 k^3 a q}{8\pi r}\sin\theta\left\{\hat{x}\left[\sin^2\theta\sin\phi\cos(kr-2wt+2\phi)\right.\right.$$
$$\left.-\cos^2\theta\sin(kr-2wt+\phi)\right]$$
$$-\hat{y}\left[\sin^2\theta\cos\phi\cos(kr-2wt+2\phi)\right.$$
$$\left.+\cos^2\theta\cos(kr-2wt+\phi)\right]$$
$$\left.+\hat{z}\sin\theta\cos\theta\sin(kr-2wt+2\phi)\right\}$$

(9.45) "Time-averaged Power radiated per unit solid angle"

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{1152\pi^2}k^6\left|[n \times Q(n)] \times n\right|^2$$
$$= \frac{c^2 Z_0}{1152\pi^2}\cdot k^6\left[Q^*\cdot Q - [n\cdot Q]^2\right]$$
$$= \frac{c^2 Z_0}{1152\pi^2}\cdot k^6\left[18a^4q^2\sin^2\theta - (3a^2q\sin^2\theta e^{2i\phi})^2\right]$$
$$= \frac{c^2 Z_0 a^4 q^2 k^6}{128\pi^2}\sin^2\theta(1+\cos^2\theta)$$
$$= \frac{Z_0 a^4 q^2 w^6}{2\pi^2 c^4}\sin^2\theta(1+\cos^2\theta)$$

$$P = \int_{-1}^{1} d\Omega\,\frac{Z_0 a^4 q^2 w^6}{2\pi^2 c^4}\sin^2\theta(1+\cos^2\theta)$$
$$= \frac{Z_0 a^4 q^2 w^6}{1440\pi}\cdot\frac{16\pi}{5}$$

(9.49) "Total power radiated by a quadropole"

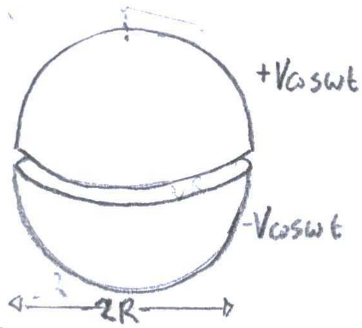
$$P = \frac{c^2 Z_0 k^6}{1440\pi}\sum_{\alpha,\beta}|Q_{\alpha\beta}|^2$$

$$=\frac{c^2 Z_0 k^6}{1440\pi}(36a^4q^2)$$

Frequency:

Experimental frequency is $\sin(\omega_0 t)$, but mathematically $\sin(2\omega t)$. So, $\omega_0 = 2\omega$ by Q_{22}, the lowest quadrupole in the two-dimensional system.

9.3.



"Two halves of a spherical metallic shell... separated by a very small insulating gap."

(Problem 2.22)

$$\phi(x,\theta,\phi) = V\frac{3}{2}\frac{R^2}{r^2}\left[\cos\theta - \frac{7a^2}{12x^2}\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) + \dots\right]$$

$$\cong V\frac{3}{2}\frac{R^2}{r^2}\cos\theta$$

$$\phi_{Elec}(x,\theta,\phi) \cong \frac{1}{4\pi\epsilon_0}\frac{p}{r^2}\cos\theta$$

$$p \cong 6\pi\epsilon_0 V R^2 \hat{z}$$

(9.18) "Magnetic Field"

$$H = \frac{ck^2}{4\cdot\pi}(n\times p)\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)$$

$$B(r,t) = \mu_0 H$$

$$\cong \frac{\mu_0 c k^2 p}{4\pi}(k\times p)\frac{e^{i(kr-\omega t)}}{r}$$

$$\cong -\frac{3}{2}\frac{Vk^2R^2}{c}\frac{e^{i(kr-\omega t)}}{r}\sin\theta\,\hat{\phi}$$

(9.18) "Electric Field"

$$E = \frac{1}{4\pi\epsilon_0}\left\{k^2(n\times p)\times n\,\frac{e^{ikr}}{r} + [3n(n\cdot p)-p]\left(\frac{1}{r^3}-\frac{ik}{r^2}\right)e^{ikr}\right\}$$

$$\simeq \frac{-1}{4\pi\epsilon_0}\left\{k^2\, n\times(n\times p)\,\frac{e^{ikr}}{r}\right\}$$

$$E(r,t) = \frac{-k^2}{4\pi\epsilon_0}\,\hat{k}\times(k\times p)\,\frac{e^{i(kr-\omega t)}}{r}$$

$$= -\frac{3}{2}Vk^2R^2\,\frac{e^{i(kr-\omega t)}}{r}\sin\theta\,\hat{\phi}$$

Time-averaged Angular Distribution:

(9.21) $$\frac{dP}{d\Omega} = \frac{1}{2}\mathrm{Re}\left[r^2\hat{r}\cdot E\times H^*\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[r^2\cdot r\cdot\left[-\frac{3}{2}Vk^2R^2\,\frac{e^{i(kr-\omega t)}}{r}\sin\theta\,\hat{\phi} \times \frac{-1}{\mu_0}\,\frac{3}{2}\,\frac{V^2k^2R^2}{c}\,\frac{e^{-i(kr-\omega t)}}{r}\sin\theta\,\hat{\phi}\right]\right]$$

$$= \frac{9}{8}\,\frac{V^2k^4R^4}{\mu_0 c}\sin^2\theta$$

<u>Total radiated Power:</u>

$$P = \int \frac{9}{8}\,\frac{V^2k^2R^4}{\mu_0 c}\sin^2\theta\, d\Omega$$

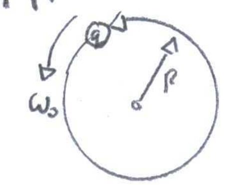
$$= \int_0^{\pi} \frac{9}{8}\,\frac{V^2k^2R^4}{\mu_0 c}\sin^3\theta\cdot d\theta$$

$$= \frac{3}{2}\,\frac{V^2k^2R^2}{\mu_0 c}$$

$$P_{TOTAL} = 2\cdot P$$

$$= \frac{3\cdot V^2k^2R^2}{\mu_0 c}$$

9.4.



"... particle rotating about the origin."

a) Charge density: $\rho(r) = \frac{q}{R}\delta(r-R)\cdot\delta(\phi-\omega t)\delta(z)$

Current density: $J = \rho(r)\cdot v$

$$= \frac{q\omega\cdot R}{R}\delta(r-R)\delta(\phi-\omega t)\delta(z)$$

$$= q\omega\,\delta(r-R)\,\delta(z)\,\delta(\phi-\omega t)$$

(9.32) "Magnetization"

$$M = \frac{1}{2}(x \times J)$$

$$= \frac{1}{2}(r\hat{r} + z\hat{z}) \times (q\omega\,\delta(r-R)\delta(\phi-\omega t)\delta(z))$$

$$= \frac{1}{2}q\omega r\,\delta(r-R)\,\delta(z)\,\delta(\phi-\omega t)\,(\hat{r}\times\hat{\phi})$$

$$= \frac{1}{2}q\omega R\,\delta(r-R)\delta(z)\delta(\phi-\omega t)\,\hat{z}$$

$$\nabla\cdot J_x = \nabla\cdot q\omega\cdot\delta(r-R)\,\delta(z)\,\delta(\phi-\omega t)$$

$$= x\cdot q\omega\cdot\delta(\tfrac{1}{r}-R)\,\delta(z)\delta(\phi-\omega t)$$

$$\nabla\cdot J_y = \nabla\cdot q\omega\cdot\delta(r-R)\delta(z)\delta(\phi-\omega t)$$

$$= y\,q\omega\cdot\delta(\tfrac{1}{r}-R)\delta(z)\delta(\phi-\omega t)$$

$$\nabla\cdot J_y = 0$$

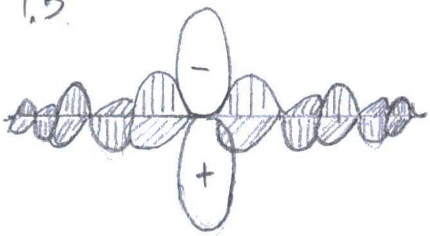
b) An antisymmetric system, so $m=0$ allowed.

A circular rotation, $l=0$ characterizes the long-wavelength magnetic multipole order.

c) If $R = a/\sqrt{2}$ from Problem 9.2, then the Q_{22} magnetic multipole becomes $Q_{22} = \frac{qa^2}{2}\sqrt{\frac{15}{\pi}}$. The quadrupole tensor evaluates to $Q = 3qa^2\begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

The power radiated is now $P = \frac{3 Z_0 \omega^6 q^2 a^4}{5\pi c}$.

9.5



"Electric dipole...
harmonic time
variation"

(6.7) "Magnetic Induction"

$$B = \nabla \times A$$

(pg 272) "Faraday's Law"

$$\nabla \times B = \mu_0 J - \frac{1}{c^2}\frac{\partial E}{\partial t}$$

$$\nabla \times (\nabla \times A) = \mu_0 J - \frac{1}{c^2}\frac{\partial E}{\partial t}$$

Vector Identity:
$\boxed{\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - (\nabla \cdot \nabla)}$

..if $E = -\nabla \phi - \frac{\partial A}{\partial t}$

$$\nabla \times (\nabla \times A) = \mu_0 J - \frac{1}{c^2}\nabla \frac{\partial}{\partial t}\phi - \frac{1}{c^2}\frac{\partial^2 A}{\partial^2 t}$$

(6.16) "Gauss' Law"

$$\nabla^2 \phi + \frac{\partial}{\partial t}(\nabla \cdot A) = -\frac{\rho}{\epsilon_0}$$

$$\downarrow$$

$$\phi(x,t) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{|x-x'|}\rho(x', t - \tfrac{1}{c}|x'-x'|)\,dx'$$

(6.16) "Lorenz gauge"

$$\nabla \cdot A = -\frac{1}{c^2}\frac{\partial \phi}{\partial t}$$

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\mu_0 J$$

$$\downarrow$$

$$A(x,t) = \frac{\mu_0}{4\pi}\int \frac{1}{|x-x'|} J(x', t - \tfrac{1}{c}|x-x'|)\,dx'$$

(9.1) "Harmonic variation"

$$A(x,t) = A(x)e^{-i\omega t}\ ; \ J(x,t) = J(x)e^{-i\omega t}\ ; \ \phi(x,t) = \phi(x)e^{-i\omega t}$$

$$A(x) = \frac{\mu_0}{4\pi}\int \frac{1}{|x-x'|} J(x') e^{ik|x-x'|}\,dx'$$

$$\phi(x) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{|x-x'|}\rho(x') e^{ik|x-x'|}\,dx'$$

Long-wavelength limit:

$$|x-x'| = r - r'\hat{x}\hat{x}'$$

Note: Later in the chapter, long wavelength limit becomes $\frac{\omega}{c} << 1$, lowest modes, and non-relativistic.

$$A(x) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left[\int (1+(r'/r)\hat{x}\cdot x')\, J(x')\, dx - ik\int (1+(r'/r)\hat{x}\cdot x')\, J(x) \cdot (r'\cdot\hat{x}\cdot\hat{x}')\, dx + \ldots\right]$$

$$\phi(x) = \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\left[\int (1+(r'/r)\hat{x}\cdot x')\, \rho(x')\, dx' - ik\int (1+(r'/r)\hat{x}\cdot x')\, J(x) \cdot (r'\cdot\hat{x}\cdot\hat{x}')\, dx' + \ldots\right]$$

With only electric dipole terms...

$\boxed{\text{Identity: } e^x = 1 + x + \frac{1}{2}x^2 + \ldots}$

$$A(x) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int J(x')\, dx'$$

$$\phi(x) = \frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r^2}\, n\cdot p(1-ikr) \quad \text{.. where } p = \int x'(\rho(x'))\, dx'$$

(9.17) "Electric Dipole Moment"

b) (9.16) "Vector Potential"

$$A(x) = -\frac{i\mu_0\omega}{4\pi}\, p \cdot \frac{e^{ikr}}{r}$$

$$H = \frac{1}{\mu_0}\nabla \times A$$
$$= \frac{1}{\mu_0}\left(-\frac{i\omega}{4\pi}\frac{e^{ikr}}{r}\cdot p\right)$$
$$= \frac{-i\omega}{4\pi}(n\times p)\frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r}\right)$$
$$= \frac{-i\omega}{4\pi}(n\times p)\left(\frac{-e^{ikr}}{r^2} + \frac{ike^{ikr}}{r}\right)$$

$\boxed{\text{Identity: } \nabla\times[p f(r)] = (n\times p)\frac{\partial f(r)}{\partial r}}$

$$= \frac{ck^2}{4\pi}(n \times p)\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)$$ "Magnetic Field from (9.19)"

$$E = -\nabla\Phi - \frac{\partial A}{\partial t}$$

$$= -\nabla\left(\frac{1}{4\pi\epsilon_0}\frac{e^{ikr}}{r^2}\, n\cdot p(1 - ikr)\right) - \frac{\partial}{\partial t}\left(\frac{-i\omega\mu_0}{4\pi}\frac{e^{ikr}}{r}\cdot p\right)$$

$$= -\frac{1}{4\pi\epsilon_0}\left(\nabla\left(\frac{e^{ikr}}{r^2}n\cdot p\right) - ik\nabla\left(\frac{e^{ikr}}{r}n\cdot p\right)\right) + \frac{\omega^2\mu_0}{4\pi}\frac{e^{ikr}}{r}\cdot p$$

$$= \frac{e^{ikr}}{4\pi\epsilon_0}\left(\frac{k^2}{r}(n \times p)\times n + (3n(n\cdot p) - p)\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)\right)$$

Identity:

$\nabla[f(r)\, n\cdot A]$:

$= \frac{f(r)}{r}A + (n\cdot A)\cdot n\cdot\left(\frac{\partial f(r)}{\partial r} - \frac{f(r)}{r}\right.$

9.6 (9.2) "Vector Potential"

$$A(x,t) = \frac{\mu_0}{4\pi}\int d^3x \int dt' \frac{J(x', t')}{|x - x'|}\delta\left(t' + \frac{|x - x'|}{c} - t\right)$$

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(x', t - |x - x'|/c)}{|x - x'|}d^3x$$

... if $|x - x'| \cong r - \hat{n}\cdot x$

$t' = t - \frac{|x - x'|}{c}$

$\cong t_{ret} + \frac{\hat{n}\cdot x}{c}$

where $t_{ret} = t - r/c$

$$\rho(x', t) = \rho(x', t_{ret}) + \frac{n\cdot x'}{c}\frac{\partial\rho(x', t)}{\partial t} + \dots$$

$$= \rho_{ret} + \frac{\hat{n}\cdot x}{c}\frac{\partial\rho_{ret}}{\partial t} + \dots$$

$$\Phi(x) = \frac{1}{4\pi\epsilon_0 r}\int\left[\rho_{ret} + \hat{n}\cdot x'\left(\frac{1}{r\cdot\rho_{ret}} + \frac{1}{c}\frac{\partial\rho_{ret}}{\partial t}\right) + \dots\right]d^3x$$

$$= \frac{1}{4\pi\epsilon_0 r}\left[Q + \hat{n}\cdot\left(\frac{1}{r\cdot p_{ret}} + \frac{1}{c}\frac{\partial p_{ret}}{\partial t}\right) + \dots\right]$$

... where $Q = \int\rho_{ret}\, d^3x$ "charge" $\qquad p_{ret} = \int x'\rho_{ret}\, d^3x$ "electric dipole"

$$\cong \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r^2}\hat{n}\cdot p_{ret} + \frac{1}{cr}\hat{n}\cdot\frac{\partial p_{ret}}{\partial t}\right]$$ .. after $Q = 0$

$$A(x) = \frac{\mu_0}{4\pi}\int \frac{J(x', t-|x-x'|/c)}{|x-x'|} d^3x$$

$$= \frac{\mu_0}{4\pi r}\int \left[J_{ret} + \ldots \right] d^3x$$

$$= \frac{\mu_0}{4\pi r}\int \frac{\partial x_i}{\partial x_j} J_{ret}\, d^3x$$

$$= \frac{-\mu_0}{4\pi r}\int x_i (\nabla \cdot J_{ret}) d^3x$$

$$= \frac{\mu_0}{4\pi r}\int x_i \frac{\partial \rho_{ret}}{\partial t} d^3x$$

$$\cong \frac{\mu_0}{4\pi r} \frac{\partial \rho_{ret,i}}{\partial t}$$

b) (6.7) "Magnetic Induction"

$$B = \nabla \times A$$

$$\rho_{ret} = p(t - r/c)$$

$$\frac{\partial \rho_{ret}}{\partial r} = -\frac{1}{c}\frac{\partial \rho_{ret}}{\partial t}$$

$$B = \frac{\mu_0}{4\pi}\nabla \times \left(\frac{1}{r}\frac{\partial \rho_{ret}}{\partial t}\right)$$

$$= \frac{\mu_0}{4\pi} \times \left(-\frac{1}{r^2}\frac{\partial \rho_{ret}}{\partial t} - \frac{1}{cr}\frac{\partial^2 \rho_{ret}}{\partial^2 t}\right)$$

(6.9) "Electric Field in a scalar potentia"

$$E = -\nabla \phi - \frac{\partial A}{\partial t}$$

$$= -\frac{1}{4\pi\epsilon_0}\nabla\left(\frac{x}{r^3}\rho_{ret} + \frac{x}{cr^2}\frac{\partial \rho_{ret}}{\partial t}\right) - \frac{\mu_0}{4\pi r}\frac{\partial^2 \rho_{ret}}{\partial t^2}$$

$$= -\frac{1}{4\pi\epsilon_0}\Bigg[\frac{p_{ret}}{r^3} - 3\frac{x(x\cdot p_{ret})}{r^5} - \frac{x}{cr^4}\left(x\cdot\frac{\partial p_{ret}}{\partial t}\right) + \frac{1}{cr^2}\frac{\partial p_{ret}}{\partial t}$$

$$- \frac{2x}{cr^4}\left(x\cdot\frac{\partial p_{ret}}{\partial t}\right) - \frac{x}{c^2r^3}\left(x\cdot\frac{\partial^2 p_{ret}}{\partial^2 t}\right)\Bigg] - \frac{1}{4\pi\epsilon_0}\frac{1}{c^2 r}\frac{\partial^2 p_{ret}}{\partial^2 t}$$

$$= -\frac{1}{4\pi\epsilon_0}\Bigg[\frac{p_{ret} - 3n(n\cdot p_{ret})}{r^3} + \frac{1}{cr^2}\frac{\partial}{\partial t}\left(p_{ret} - 3n(\hat{n}\cdot p_{ret})\right)$$

$$+ \frac{1}{c^2 r}\frac{\partial^2}{\partial t^2}\left(p_{ret} - \hat{n}(\hat{n}\cdot p)\right)\Bigg]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\left(1 + \frac{r}{c}\frac{\partial}{\partial t}\right)\frac{3n(n\cdot p_{ret}) - p_{ret}}{r^3} + \frac{1}{c^2 r} n\times\left(n\times\frac{\partial^2 p_{ret}}{\partial^2 t}\right)\right]$$

c) (9.18) "Magnetic Field with a dipole"

$$H = \frac{ck^2}{4\pi}(n\times p)\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)$$

$$E = \frac{1}{4\pi\epsilon_0}\left\{k^2(n\times p)\times n\frac{e^{ikr}}{r} + [3n(n\cdot p) - p]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr}\right\}$$

$$H = \frac{1}{4\pi} n\times\left(-\frac{1}{r^2}(-i\omega)p - \frac{1}{cr}(-\omega^2 p)\right)e^{ikr}$$

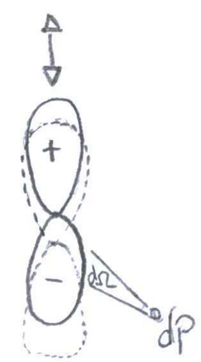
$$= \frac{\omega^2}{4\pi c r}(n\times p)\left(1 - \frac{c}{i\omega r}\right)e^{ikr}$$

$$= \frac{ck^2}{4\pi}(n\times p)\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)$$

$$E = \frac{1}{4\pi\epsilon_0}\left[\left(1 + \frac{r}{c}(-i\omega)\right)\frac{3\hat{n}(\hat{n}\cdot p) - p}{r^3} + \frac{1}{c^2 r}(-\omega^2) n\times(n\times p)\right]e^{ikr}$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{e^{ikr}}{r^3}(1 - ikr)(3n(\hat{n}\cdot p) - p) - k^2\frac{e^{ikr}}{r} n\times(n\times p)\right]$$

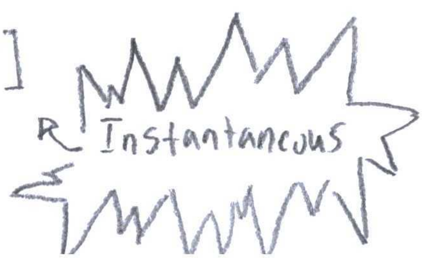
9.7



"real electric dipole"

a) (9.21) "Angular distribution radiated"

$$\frac{dP}{d\Omega} = r^2\hat{r}\cdot\mathrm{Re}[r^2\cdot\hat{r}\, E\times H^*]$$



$$= r^2 \hat{r}\left[\int_{-\infty}^{\infty} E(\omega) e^{-i\omega t} d\omega \times \int_{-\infty}^{\infty} H(\omega') \cdot e^{-i\omega' t} d\omega'\right]$$

$\boxed{\text{Fourier decomposition}}$

(9.18) "Magnetic and Electric Field"

$$H = \frac{ck^2}{4\pi}(n \times p)\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)$$

$$E = \frac{1}{4\pi\epsilon_0}\left\{k^2(n \times p) \times n \frac{e^{ikr}}{r} + [3n(n \cdot p) - p]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right)e^{ikr}\right\}$$

When $k = \frac{\omega}{c}$, $\hat{n} = \hat{r}$, $\hat{p} = \hat{v}$ and $r \gg 1$, then

$$H = \frac{\omega^2}{4c\pi}(r \times v)\, e^{i\omega r/c}$$

$$E = \frac{\mu_0 \omega^2}{4\pi}\frac{e^{i\omega r/c}}{r}\hat{r} \times (\hat{v} \times r)$$

$$\frac{dP}{d\Omega} = r^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega\, d\omega'\, e^{-i(\omega+\omega')t} \cdot \frac{\mu_0 \omega^2}{4\pi} \cdot \frac{e^{i(\omega+\omega')}}{r^2}\frac{\omega'^2}{4c\pi} \times r$$

$$\cdot\left(\left[\hat{r} \times (V(\omega) \times r)\right] \times \left[\hat{r} \times V(\omega')\right]\right)$$

$$= \frac{\mu_0}{16c\pi^2}\left\{\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t'} \omega^2 (V(\omega))\right\}\left\{\int_{-\infty}^{\infty} d\omega' e^{-i\omega' t} \omega' V(\omega')\right\}(\delta_{jk} - n\hat{k})$$

$$= \frac{Z_0}{16c^2\pi^2}\left\{\left(\frac{d^2 p}{dt'^2}(t)\right)^2 - \left(r \cdot \frac{d^2 p}{dt'^2}\right)^2\right\} \quad \therefore \text{ when } V(\omega) = \frac{-1}{2\pi}\int_{-\infty}^{\infty} d\tau \frac{d^2 p}{d\tau^2} e^{i\omega\tau}$$

$$= \frac{Z_0}{16c^2\pi^2}\left|\left[n \times \frac{d^2 p}{dt'^2}(t')\right] \times n\right|^2 \quad \therefore \text{ when (9.46)}$$

For the magnetic dipole,

$$\frac{dP}{d\Omega}(t) = \frac{Z_0}{16\pi^2 c^4}\left|\hat{n} \times \frac{d^2 m}{dt^2}\right|^2$$

∴ by suggestion in the problem $\left(\frac{1}{c}\right)\ddot{m} \times n = (n \times \ddot{p}) \times n$

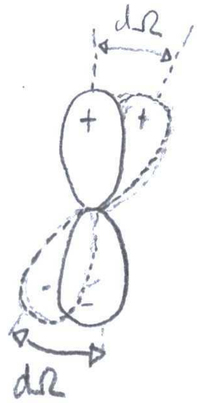
b) $$\frac{dP}{d\Omega}(t) = \frac{\mu_0}{16c\pi^2}\left\{\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t'}\, \omega\,(V(\omega))\right\}\cdot\left\{\int_{-\infty}^{\infty} d\omega' e^{-i\omega' t}\, \omega'^2 (V(\omega))\right\}(\delta_{jk} - r_j k)$$

$$= \frac{Z_0}{16\pi^2 c^2}\,\frac{1}{6^2 c^2}\left|\left(\hat{n} \times \frac{d^3}{dt'^3} Q(n,t)\right) \times \hat{n}\right|^2$$

$$= \frac{Z_0}{576\pi^2 c^4}\left[\left[n \times \frac{d^3 Q}{dt'^3}(n, t')\right] \times n\right]^2$$

$$= \frac{Z_0}{576\pi^2 c^4} \left| \left(\hat{r} \times \frac{d^3}{dt'^3} Q(\hat{r}, t') \right) \times \hat{r} \right|^2$$

9.8



"electric dipole... radiates electromagnetic angular momentum"

a) (9.18) "Magnetic Field with dipole"

$$H = \frac{ck^2}{4\pi} (n \times p) \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr} \right)$$

$$E = \frac{1}{4\pi\epsilon_0} \left\{ k^2 (n \times p) \times n \frac{e^{ikr}}{r} + [3n(n \cdot p) - p] \left(\frac{1}{r^3} - \frac{ik}{r^2} \right) e^{ikr} \right\}$$

(9.137) "Linear Momentum Density"

$$\vec{g} = \frac{1}{2c^2} E \times H^*$$

(Prob 6.10) "Angular Momentum Density"

$$m = \vec{r} \times \vec{g}$$

$$= \frac{1}{2c^2} r \times (E \times H^*)$$

$$= \frac{1}{2c^2} [E(r \cdot H) - H^*(r \cdot E)]$$

Identity:
$A \times (B \times C^*)$
$= B(A \cdot C) - C(A \cdot B)$

$$r \cdot E = r \cdot \hat{n} \cdot E$$

$$= \frac{1}{4\pi\epsilon_0} (2\hat{n} \cdot \vec{p}) \left(\frac{1}{r^2} - \frac{ik}{r} \right) e^{ikr}$$

Electric Dipole Field

$r \cdot H = 0$

$$m = -\frac{1}{2c^2} H^* (r \cdot E)$$

$$= -\frac{1}{2c^2} \frac{ck^2}{4\pi} (\hat{n} \times p^*) \frac{e^{-ikr}}{r} \left(1 + \frac{1}{ikr} \right) \frac{1}{4\pi\epsilon_0} (2n \cdot p) \cdot \left(\frac{1}{r^2} - \frac{ik}{r} \right) e^{ikr}$$

$$= \frac{ik^3}{16\pi^2 \epsilon_0 c r^2} \left(1 + \frac{1}{(kr)^2} \right) (\hat{n} \cdot \vec{p})(n \times p)$$

For the electric dipole, $V(\omega) = p(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt'' p(t'') e^{i\omega t''}$

$$\omega^2 V(\omega) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau \frac{d^2}{d^2\tau} p(\tau) e^{i\omega\tau}$$

$$\frac{-d^2}{d^2\tau} p(\tau)\Big|_{\tau=t'} = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t'} \omega^2 V(\omega)$$

$$|(r \times \vec{A}) \times r|^2 = A^2 - (r \circ A)^2$$

$$(r \times \vec{A}) = (r \times A)^2 - (r \circ (r \times A))^2$$

$$\frac{dP(t)}{d\Omega} = \frac{Z_0}{16\pi^2 c^2}\left|\left(r \times \frac{d^2}{d^2t'} p(t')\right) \times r\right|^2$$

$$= \frac{Z_0}{16\pi^2 c^4}\left|\hat{r} \times \frac{d^2}{dt'^2} m(t')\right|^2$$

"Electric Quadropole"

$$\omega^2 V(\omega) = \frac{i}{6c}\omega^2 \hat{r} \circ Q(\omega)$$

$$= \frac{-1}{12\pi c}\int_{-\infty}^{\infty} d\tau\, \hat{r} \circ Q(\tau) \frac{d^3}{d\tau^3} e^{i\omega\tau}$$

$$= \frac{1}{12\pi c}\int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau} \circ \hat{r} \circ \frac{d^3}{d\tau^3} Q$$

$$\frac{1}{6c}\frac{d^3}{d\tau^3} Q(\tau)\Big|_{\tau=t'} = \frac{1}{6c}\frac{d^3}{dt'^3} Q(r,t)$$

$$= \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \cdot e^{-i\omega t'} \circ \omega^2 V(\omega)$$

$$\frac{dP}{d\Omega} = \frac{Z_0}{16\pi^2 c^2}\frac{1}{6^2 c^2}\left|\left(r \times \frac{d^3}{dt'^3} Q(r,t')\right) \times \hat{r}\right|$$

$$dL = m \circ da \circ dr$$

$$= m \circ r^2 dr d\Omega$$

$$\frac{dL}{d\Omega} = m r^2 \left(\frac{dr}{dt}\right) d\Omega$$

$$= r^2 c \int m \, d\Omega$$

$$= \frac{ik^3}{16\pi^2 \epsilon_0}\left(1 + \frac{1}{(kr)^2}\right)\int (\hat{n} \circ \vec{p})(\hat{n} \times p^*) d\Omega$$

$$= \frac{ik^3}{16\pi^2 \epsilon_0}\left(1 + \frac{1}{(kr)^2}\right) \circ \frac{1}{3}\int d\Omega (p \times p)$$

$$= \frac{ik^3}{16\pi^2 \epsilon_0}\left(1 + \frac{1}{(kr)^2}\right) \circ \frac{4\pi}{3}(p \times p)$$

$$\lim_{r\to\infty} \frac{dL}{dt} = \lim_{r\to\infty}\left(\frac{ik^3}{16\pi^2 \epsilon_0}\left(1 + \frac{1}{(kr)^2}\right) \circ \frac{4\pi}{3}(p \times p)\right)$$

$$= \frac{-ik^3}{12\pi \epsilon_0} p^* \times p$$

$$= \frac{k^3}{12\pi \epsilon_0} \mathrm{Im}[p^* \times p]$$

b) Ratio of $\dfrac{\text{Angular Momentum Radiated}}{\text{Energy Radiated}}$

$$= \frac{dL/dt}{dU/dt}$$

$$= \frac{\frac{k^3}{12\pi \epsilon_0} \mathrm{Im}[p^* \times p]}{\frac{c^2 Z_0 k^4}{12\pi} |p^* \circ p|^2}$$

$$= \frac{1}{c^2 Z_0 \epsilon k} \frac{Im[p^* \times p]}{p^* \cdot p}$$

$$= \frac{1}{\omega} \frac{1}{c Z_0 \epsilon_0} \frac{Im[p^* \times p]}{p^* \cdot p}$$

$$= \frac{1}{\omega} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{Z_0} \frac{Im[p^* \times p]}{p^* \cdot p}$$

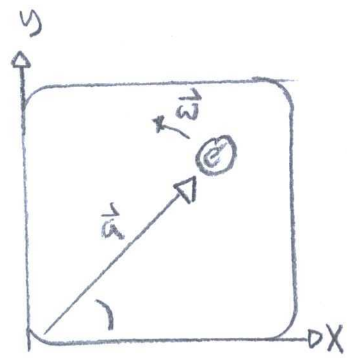
$$= \frac{1}{\omega} \frac{2 p_1 \times p_2}{p_1^2 + p_2^2}$$

$$= \frac{1}{\omega} \frac{\cancel{\theta} p_1 p_2 \sin\theta}{p_1^2 + p_2^2} \quad \therefore \theta = \text{angle}$$

(pg 408)

$$\boxed{Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \qquad \vec{p} = \vec{p}_1 + i\vec{p}_2}$$

The interpretation: radiation from a dipole when $m=1$ is per frequency, at an angle, and dependent upon orientation.



"charge rotating in x-and-y plane"

c) $p(x,t) = \int \vec{x} \rho(x) d^3x$

$$= \int \epsilon a (\hat{x}\cos\omega t + \hat{y}\sin\omega t) d^3x$$

$$= \int x c\, \delta(x - a\cos\omega t)\delta(y - a\sin\omega t)\delta(z)\, dx\, dy\, dz$$

$$= \underbrace{\epsilon a(\hat{x} - i\hat{y})}_{\text{"Real and complex dipole"}} \underbrace{e^{-i\omega t}}_{\text{"Harmonic component"}}$$

$$\frac{dL}{dt} = \frac{k^3}{12\pi\epsilon_0} Im[p^* \times p]$$

$$= \frac{k^3 \epsilon^2 a^2}{6\pi\epsilon_0} \hat{z}$$

A charge oscillating along the Z-axis has a real and imaginary component in the x-y plane.

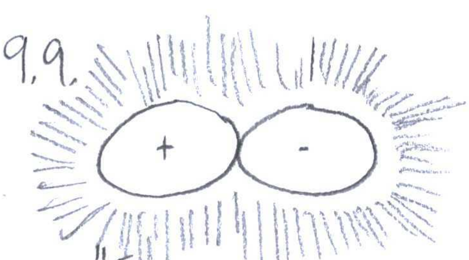
d) $E \to Z_0 H$, $p \to m/c$

$$\frac{dL}{dt} = \frac{k^3}{12\pi\epsilon_0 c^2} Im[m^* \times m]$$

$$= \frac{\mu_0 k^3}{12\pi} Im[m^* \times m]$$

$$P = \frac{Z_0 k^4}{12\pi} |\vec{m}|^2$$

9.9.



"Electric dipole fields ... rate of radiation"

a) (9.21) "Average Power per solid angle"

$$\frac{dP}{d\Omega} = \frac{1}{2} Re[r^2 n \cdot E \times H^*]$$

(9.4) "Magnetic Field"

$$H = \frac{1}{\mu_0} \nabla \times A$$

(9.13) "Vector Potential"

$$A(x) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int J(x') d^3x$$

$$H = \frac{-\mu_0}{4\pi} \frac{e^{ikr}}{r} \int J(x') d^3x$$

$$= \frac{-1}{4\pi c r} n \times \frac{\partial^2 p}{\partial^2 t}$$ ... when $p = p_{ret}(x, t)$

$= \beta e^{i(kr - wt)}$

and $\beta =$

$$(9.19)\quad E = Z_0 H \times n$$
$$= -Z_0\, n \times H$$
$$= -\frac{1}{c\epsilon}\, n \times H$$

$$\frac{dP}{d\Omega} = -\frac{r^2}{\epsilon_0 c}\, n \circ \left((\hat{n} \times \hat{H}) \times \hat{H} \right) \quad \ldots \text{when } n \circ H = 0$$

$$= -\frac{r^2 n}{\epsilon_0 c} (n \times H) \circ (n \times H)$$
$$= \frac{r^2}{\epsilon_0 c} |H|^2 \sin^2\theta \ldots \quad \theta = \text{angle}$$
$$= \frac{1}{16\pi^2 \epsilon_0 c^2} \left(\frac{\partial^2 p}{\partial t^2} \right)^2 \sin^2\theta$$

$$P = \int \frac{1}{16\pi^2 \epsilon_0 c^2} \left(\frac{\partial^2 p}{\partial t^2} \right)^2 \sin^2\theta \, d\Omega$$

$$= \frac{1}{16\pi^2 \epsilon_0 c^2} \left(\frac{\partial^2 p}{\partial^2 t} \right)^2 \circ \frac{8\pi}{3}$$
$$= \frac{1}{6\pi \epsilon_0 c^3} \left(\frac{\partial^2 p}{\partial t^2} \right)^2$$

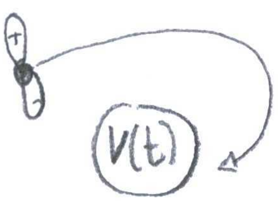
Angular Momentum radiated...

$$\frac{dL}{dt} = cr^2 \int d\Omega\, \vec{r} \times \left(\frac{1}{c^2} E \times H \right)$$
$$= \frac{r^3}{c} \int d\Omega \left[E(n \circ H) - H(n \circ E) \right]$$
$$= \frac{-r^3}{c} H(\hat{n} \circ E)$$
$$= \frac{-r^3}{c} \int d\Omega \left(\frac{-1}{4\pi r^2} \right) \left(1 + \frac{r}{c}\frac{d}{dt} \right) \hat{n} \times \frac{dp}{dt} \left(\frac{1}{4\pi\epsilon_0} \left(1 + \frac{r}{c}\frac{\partial}{\partial t} \right) \frac{2\hat{n} \circ \hat{p}}{r^3} \right)$$
$$= \frac{1}{8\pi^2 \epsilon_0 c r^2} \int d\Omega \left[\left(1 + \frac{r}{c}\frac{\partial}{\partial t} \right) \hat{n} \times \frac{dp}{dt} \right] \left[\left(1 + \frac{r}{c}\frac{\partial}{\partial t} \right) \hat{n} \circ p \right]$$

$$= \frac{1}{6\pi\epsilon_0 c r^2}\left[\left(1+\frac{r}{c}\frac{\partial}{\partial t}\right)\vec{p}\right]\times\left[\left(1+\frac{r}{c}\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial t}\right]$$

$$\lim_{r\to\infty}\frac{dL}{dt} = \lim_{r\to\infty}\left[\frac{1}{6\pi\epsilon_0 c r^2}\left[\left(1+\frac{r}{c}\frac{\partial}{\partial t}\right)\vec{p}\right]\times\left[\left(1+\frac{r}{c}\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial t}\right]\right]$$

$$= \frac{1}{6\pi\epsilon_0 c^3}\left(\frac{\partial p}{\partial t}\times\frac{\partial^2 p}{\partial t^2}\right)$$



"dipole moment... by a particle... moving nonrelativistically ... in a central potential"

b) Notes: Abraham-Lorentz Equation (Section 16.2):

Outcomes: Effective mass depends on frequency.

Challenges: The Fourier Transform in the complex plane has infinite mass .... Also, reprocussions from infinite mass are self-force, non-convergent energy, and runaway solutions.

Theory: Force from radiation has:

1) $\dot{a} = 0$ at $t = \infty$

2) No sign change from radiation

3) A characteristic time, τ.

Equation: External Force $\propto$ Radiation Force

$$m(\dot{v} - \tau\ddot{v}) = F_{ext}$$

$$F_{rad} = \frac{2}{3}\frac{e^2}{c^3}\ddot{v}$$

$$= m\tau\ddot{v}$$

( )

$$p = e \cdot X$$

$$\frac{dp}{dt} = e \cdot \mathring{X}$$

$$= \frac{e}{m}(m\mathring{X})$$

$$\frac{d^2p}{d^2t} = e\ddot{X}$$

$$= \frac{e}{m}(m\ddot{X})$$

$$= \frac{e}{m}\vec{F}$$

$$= \frac{-e}{m}\nabla V$$

$$= \frac{-e}{m}\hat{n}\frac{dV}{dr}$$

$$P = \frac{1}{6\pi\epsilon_0 c^3}\left(\frac{\partial^2 p}{\partial^2 t}\right)^2$$

$$= \frac{e^2}{6\pi\epsilon_0 m^2 c^3}\left(\frac{\partial V}{\partial r}\right)^2$$

$$= \frac{\tau}{m}\left(\frac{\partial V}{\partial r}\right)^2 \quad \ldots\ldots\ldots \text{when } \tau = \frac{e^2}{6\pi\epsilon_0 m c^3}$$

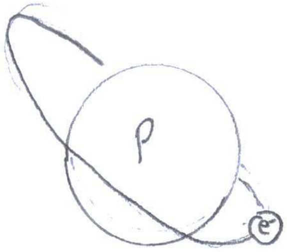
$$\frac{dL}{dt} = -\frac{e^2}{6\pi\epsilon_0 m^2 c^3}(m\mathring{x})$$

$$= \frac{1}{6\pi\epsilon_0 c^3}\left(\frac{\partial p}{\partial t} \times \frac{\partial^2 p}{\partial^2 t}\right)$$

$$= \frac{-e^2}{6\pi\epsilon_0 m^2 c^3}(m\mathring{x}) \times X\left(\frac{1}{r}\frac{\partial V}{\partial r}\right)$$

$$= \frac{\tau}{m}\left(\frac{\partial V}{r \cdot dr}\right)\vec{L} \quad \ldots \text{where } \vec{L} = \vec{X} \times (m\mathring{\vec{X}})$$

Abraham-Lorentz equation relates to Power (P) and angular momentum ($\vec{L}$) through a characteristic time. with mass



"...a charged particle in a hydrogen atom"

c) Rate of Angular Momentum Radiated

particle Angular momentum

$$= \frac{dL/dt}{L}$$

Notes: Section 9.11: Transition probability (Γ) is a reciprical to mean life for photons at energy $= \hbar\omega$.

$$\Gamma = \frac{\frac{dL}{dt}}{L}$$

$$\cong \frac{\frac{\tau}{m}\left(\frac{\partial V}{r\,dr}\right)\vec{L}}{\vec{L}}$$

$$\cong \frac{\tau}{m}\left(\frac{\partial V}{r\cdot dr}\right)$$

$$\boxed{\begin{array}{l}\text{Coulomb potential [Hydrogen]}\\ V = e^2/4\pi\epsilon_0 r\end{array}}$$

$$\cong \frac{e^2}{6\pi^2\epsilon_0 m^2 c^3}\left|\frac{1}{r}\frac{d}{dr}\frac{e^2}{4\pi\epsilon_0 r}\right|$$

$$\cong \frac{e^4}{24\pi^2\epsilon_0{}^2 m^2 c^3}\frac{1}{a_0{}^3}$$ ... when $r \approx a_0$

and $a_0 = \frac{\hbar}{\alpha mc}$

$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$

$$\cong \frac{2}{3\pi}\frac{\alpha^4 c}{a_0}$$

$$\cong 0.212 \cdot \frac{\alpha^4 c}{a_0}$$

$$\Gamma_{2p\to 1s} \cong \left(\frac{2}{3}\right)^{8}\frac{\alpha^4 c}{a_0}$$

$$= 0.039\,\frac{\alpha^4 c}{a_0}$$

d) Harmonic time dependence:

$$\frac{dL_z}{dt} = \frac{e^2 k^3 a^2}{6\pi\epsilon_0}$$

$$P = \frac{1}{6\pi\epsilon_0 c^3}\left(\frac{\partial^2 p}{\partial t^2}\right)^2$$

$$= \frac{1}{6\pi\epsilon_0 c^3}\omega^4\left(\frac{1}{2}\vec{p}\cdot\vec{p}\right) \quad \circ\circ\circ\circ \text{ when } \frac{dp}{dt} = -i\omega p e^{i(kr-\omega t)}$$

$$\frac{d^2p}{d^2t} = -\omega^2 p e^{i(kr-\omega t)}$$

$$= \frac{ck^4}{12\pi\epsilon_0}|\vec{p}|^2$$

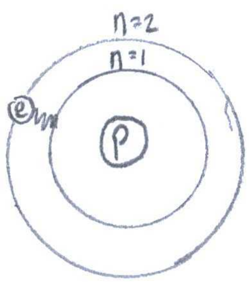
$$\frac{dL}{dt} = \frac{1}{6\pi\epsilon_0 c^3}\left(\frac{\partial p}{\partial t}\times\frac{\partial^2 p}{\partial^2 t}\right)$$

$$= \frac{1}{6\pi\epsilon_0 c^3} i\omega^3\left(\frac{1}{2}\vec{p}\times\vec{p}\right)$$

$$= \frac{ik^3}{12\pi\epsilon_0}\vec{p}\times\vec{p}^{\,*}$$

$$= \frac{k^3}{12\pi\epsilon_0} Im\left[\vec{p}^{\,*}\times\vec{p}\right]$$

9.10.



"The translational charge and current densities for a radiative transition ∘∘∘ in hydrogen."

(9.1) "Charge and current"

$$\rho(X,t) = \rho(X)\cdot e^{-i\omega t}$$

$$J(X,t) = J(X)\, e^{-i\omega t}$$

$$\rho(r,\theta,\phi,t) = \frac{Ze}{\sqrt{6}\,a_0^4}\circ r_0 e^{-3r/2a_0}\circ Y_{00}\,Y_{10}\,e^{-i\omega_0 t}$$

$$J(r,\theta,\phi,t) = \frac{-iV_0}{2}\left(\frac{\hat{r}}{2}+\frac{a_0}{Z}\hat{z}\right)\rho(r,\theta,\phi,t)$$

∘∘∘ where $a_0 = 4\pi\epsilon_0\hbar^2/me^2$

$= 0.529\times10^{-10}\,m$

$\omega_0 = 3e^2/32\pi\epsilon_0\hbar a_0$

$V_0 = e^2/4\pi\epsilon_0\hbar$

$=\alpha c$

$= c/137$

a) (5.53) "Magnetic moment density"

$$M(x) = \frac{1}{2}[X \times J(x)]$$

$$= \frac{1}{2}\left(-\frac{iV_0}{2}\right) r\hat{r} \times \left(\frac{\hat{r}}{2} + \frac{a_0}{Z}\hat{z}\right) \rho(r,\theta,\phi,t)$$

$$= \frac{1}{2}\left(\frac{iV_0}{2}\right)\frac{r_0 a_0}{Z}\sin\theta \cdot \hat{\phi} \cdot \rho(r,\theta,\phi,t)$$

$\hat{r} \times \hat{z} = -\sin\theta\,\hat{\phi}$
$= -\sin\theta(-\hat{x}\sin\phi + \hat{y}\cos\phi)$

$$= \frac{1}{2}\left(\frac{i a_0 V_0}{2}\right)\hat{\phi}\tan\theta\,\rho(r,\theta,\phi,t)$$

$$= -(\hat{x}\sin\phi - \hat{y}\cos\phi)\left(\frac{i a_0 \alpha c}{4}\right)\tan\theta\,\rho(r,\theta,\phi,t)$$

Note: Book moves into, "Power measurements not by $E \times H$ fields, but the individual summation for current and magnetic fields"

(9.169) "Electric multipole coefficient"

$$a_E(l,m) \simeq \frac{c k^{l+2}}{i(2l+1)!!}\left(\frac{l+1}{l}\right)^{1/2}(Q_{lm} + Q'_{lm})$$

(9.170) "Electric Multipole Moments"

$$Q_{lm} = \int r^l Y^*_{lm} \rho\, d^3x$$

$$Q'_{lm} = \frac{-ik}{(l+1)c}\int r^l \cdot Y^*_{lm} \nabla \cdot (r \times M)\, d^3x$$

(9.171) "Magnetic multipole coefficients"

$$a_M(l,m) \simeq \frac{i k^{l+2}}{(2l+1)!!}\left(\frac{l+1}{l}\right)^{1/2}(M_{lm} + M'_{lm})$$

(9.172) "Magnetic Multipole Moments"

$$M_{em} = -\frac{1}{l+1}\int r^{l} \cdot Y_{lm}^{*}\, \nabla\cdot(r\times J)\, d^3x$$

$$M_{em}' = -\int r^{l} Y_{lm}^{*}\, \nabla\cdot M\, d^3x$$

Note: The above approximations are the long wavelength limit.

$$Q_{10} = \int_{\Omega}\int_0^{\infty} r\left(\frac{2e}{\sqrt{6a_0^4}}\, r\exp\left(\frac{-3r}{2a_0}\right) Y_{01} Y_{10}\right) Y_{10}^{*}\, r^2\, dr\, d\Omega$$

$$= \frac{2e}{\sqrt{24\pi\, a_0^4}}\int_0^{\infty} r^4 \exp\left(\frac{-3r}{2a_0}\right) dr$$

$$= \frac{1}{\sqrt{6\pi}}\,\frac{256}{81}\, e\cdot a_0$$

$$Q_{10}' = \frac{-ik}{2c}\int_{\Omega}\int_0^{\infty} r\left(\frac{2e}{\sqrt{6a_0^4}}\, r\exp\left(\frac{-3r}{2a_0}\right) Y_{01} Y_{10}\right) \nabla\cdot(r\times M)\, r^2\, dr\, d\Omega$$

$$= \frac{-ik}{2c}\int_{\Omega}\int_0^{\infty} r\left(\frac{2e}{\sqrt{6a_0^4}}\, r\exp\left(\frac{-3r}{2a_0}\right) Y_{01} Y_{10}\right) \frac{i\alpha c a_0}{4}\, \nabla\cdot(\hat{r}\times\hat{\phi})\, r\tan\theta\, \cdot \rho(r,\theta,\phi;t)\, r^2\, dr\, d\Omega$$

$$= 0$$

b) (9.169) "Electric Multipole Coefficient"

$$a_E(l,m) = \frac{ck^{l+2}}{i(2l+1)!!}\left(\frac{l+1}{l}\right)^{1/2}\left(Q_{lm} + Q_{lm}'\right)$$

Total time-averaged Power:

$$a_E(1,0) = \frac{ck^2}{3i}\sqrt{2}\, Q_{10}$$

(9.154) "Power radiated"

$$P(l,m) = \frac{Z_0}{2k^2}\left|a(l,m)\right|^2$$

$$= \frac{3QR^{\ell}}{2(\ell+3)}\sqrt{\frac{2\ell+1}{4\pi}}\int_0^1 P_\ell(\theta)\, d\cos\theta$$

$$= \frac{3QR^{\ell}}{2(\ell+3)}\sqrt{\frac{2\ell+1}{4\pi}}\int_{-1}^1 P_\ell(x)\,dx$$

$$= \frac{3QR^{\ell}}{(\ell+3)}\sqrt{\frac{2\ell+1}{4\pi}}$$

$\ell = 1$ is the lowest non-zero value and non-infinity value.

$$q_{1,0} = \frac{3QR^{1}}{4}\sqrt{\frac{3}{4\pi}}$$

Note: A dipole formed in a magnetic field by a spherical glob.

Long-wavelength limit,

(9.169) "Electric Multipole coefficient

$$a_E(\ell,m) = \frac{ck^{\ell+2}}{i(2\ell+1)!!}\left(\frac{\ell+1}{\ell}\right)^{1/2}(Q_{\ell m}+Q'_{\ell m})$$

$$= \frac{ck^{\ell+2}}{i(2\ell+1)!!}\sqrt{\frac{\ell+1}{\ell}}\, q_{\ell m}$$

$$a_E(1,0) = \frac{-i c\cdot k^3\sqrt{2}}{3}\,\frac{3QR^{1}}{4}\sqrt{\frac{3}{4\pi}}$$

$$= \frac{-i c\cdot k^3 QR}{2^{3/2}}\sqrt{\frac{3}{4\pi}}$$

Angular Distribution radiated per solid angle

(9.21) "Power radiated modified"

$$\frac{dP}{d\Omega} = \frac{1}{2}\mathrm{Re}[r^2 n\cdot E\times H^*]$$

$$= \frac{Z_0}{2k}|a_E(1,0)|^2|X_{1,0}|^2$$

$$= \frac{27\, c^2\cdot k^6\cdot Q^2\cdot R^2}{2048\sqrt{2}\,\pi^3}\sin^4\theta$$

Total Power

$$P = \int \frac{27 c^2 k^6 Q^2\cdot R^2}{2048\sqrt{2}\,\pi^3}\sin^4\theta\, d\Omega$$

$$= \frac{9\sqrt{2}\, c^2 k^6 Q^2 R^2}{1024\pi^2}$$

(Table 9.1) $|X_{20}|^2 = \frac{15}{8\pi}\sin^2\theta\cos^2\theta$

$$\frac{dP}{d\Omega} = \frac{9 Z_0 c^2 k^6 Q^2 R_0^4 \beta^2}{3200\pi^2}\sin^2\theta\cos^2\theta$$

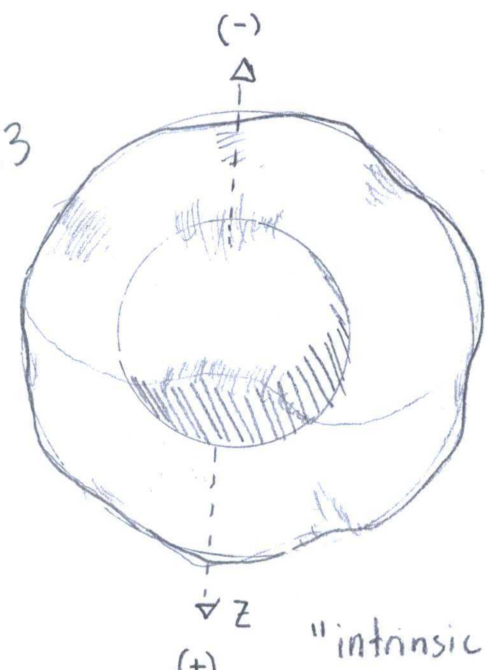
Note: The pages and pages feel archaic, then.... European Space Agency needs a background power signal for their satellites. Professors present on their math from employment in Denmark. Boeing samples antenna before 777 dreamliner commission.

Total Power:

$$P = \int \frac{9 Z_0 c^2 k^6 Q^2 R_0^4 \beta^2}{3200\pi}\sin^2\theta\cos^2\theta\, d\Omega$$

$$= \frac{3\cdot Z_0 c^2 k^6 Q^2 R_0^4 \beta^2}{2000\pi}.$$

9.13



"intrinsic magnetization parallel to the axis"

$$q_{l,0} = \int r^l \cdot Y^*_{l,0}(\theta,\phi)\rho(t)\, d\tau$$

$$= \frac{3Q}{4\pi R^3}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \sqrt{\frac{2l+1}{4\pi}} \cdot P_l(\cos\theta)\int_0^R r^l \cdot r^2\, dr$$

$$= \frac{15Q}{4\pi R_0^3(5+3\beta^2)} \sqrt{\frac{2\ell+1}{4\pi}} \frac{2\pi}{3+\ell} \int_{-1}^{1} (1+\beta P_2(x))^{3+\ell} \cdot P_\ell(x)\,dx$$

If $\ell=0$, then

$$Q_{00} = \frac{15Q}{4\pi R_0^3(5+3\beta^2)} \sqrt{\frac{2\cdot\ell+1}{4\pi}} \frac{2\pi}{3} R_0^3 \frac{2}{5}(3\beta^2+5)$$

$$= \frac{Q}{\sqrt{4\pi}}$$

If $\ell=2$, then

$$Q_{22} = \frac{15Q}{4\pi R_0^3(5+3\beta^2)} \sqrt{\frac{2\cdot 2+1}{4\pi}} \frac{2\pi}{3+2} R_0^5 \frac{2\beta}{1001}(572\beta^2+1001)$$

$$= \sqrt{\frac{9}{20\pi}}\, Q\cdot R_0^2\beta$$

<u>Angular Distribution of Radiation:</u>

(9.151) "Power per solid angle"

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2} |a(\ell,m)|^2 \cdot |X_{\ell m}|^2$$

If $\ell=0$, $\frac{dP}{d\Omega} = \frac{Z_0}{2k^2} |a(0,0)|^2 \circ |X_{00}|^2$

$= 0$ ... because $|X_{00}| = 0$

If $\ell=1$, $\frac{dP}{d\Omega} = \frac{Z_0}{2k^2} |a(1,0)|^2 \circ |X_{10}|^2$

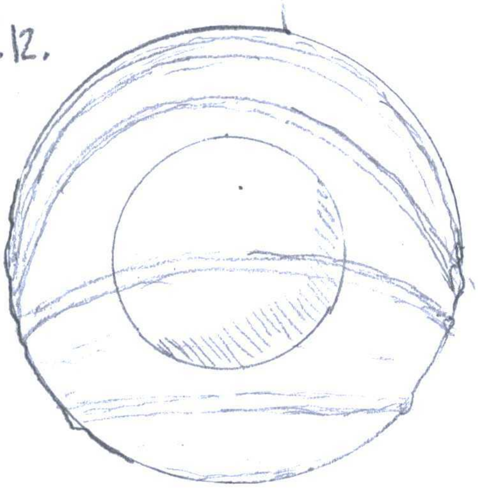
$= 0$ .... because $|a(1,0)|^2 = 0$

If $\ell=2$ $\frac{dP}{d\Omega} = \frac{Z_0}{2k^2} |a(2,0)|^2 \circ |X_{20}|^2$

$$= \frac{Z_0}{2k^2} \left| \frac{ck^4}{15i} \sqrt{\frac{3}{2}} \sqrt{\frac{9}{20\pi}} QR_0^2\beta \right|^2 \circ |X_{20}|^2$$

Power: $P = \frac{Z_0 q^2 \omega^2}{60\pi}(ka)^4$

9.12.



"a spherical surface has inside a uniform volume of charge... surface waves."

$$R(\theta) = R_0\left[1 + \beta P_2(\cos\theta)\right]$$

Nonvanishing Multipole Moments:"

(9.170) "Multipole Moments modified"

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{R(\theta)} \rho r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= 2\pi\rho \int_0^{\pi}\int_0^{R(\theta)} r^2 \sin\theta\, dr\, d\theta$$

$$= \frac{2\pi\rho R^3}{3}\int_0^{\pi}(1+\beta P_2(\cos\theta))^3 \sin\theta\, d\theta$$

$$= \frac{2\pi\rho R_0^3}{3}\int_{-1}^{1}(1+\beta P_2(x))^3 dx$$

$$= \frac{2\pi\rho R_0^3}{3\cdot 2^3}\int_{-1}^{1}(2+3\beta x^2-\beta)^3 dx$$

$$\rho = \frac{6Q}{\pi R_0^3}\left[(2-\beta)^3 + \frac{1}{3}(9\beta(2-\beta)^2) + \frac{1}{5}(27\beta^2(2-\beta)) + \frac{1}{7}\cdot 27\beta^3\right]^{-1}$$

$$\cong \frac{6Q}{\pi R_0^3}\left[\frac{15}{24}\,\frac{1}{(5+3\beta^2)}\right]$$

(9.169.5) "

$$Q_{\ell m} = \int r^\ell Y^*_{\ell m} \rho\, d^3x$$

$$Q_{\ell 0} = \frac{15Q}{4\pi R_0^3(5+3\beta^2)}\sqrt{\frac{2\ell+1}{4\pi}}\,\frac{2\pi}{3+\ell}\int_0^{\pi} r^{3+\ell}\Big|_0^{R(\theta)} \sin\theta\, P_\ell(\cos\theta)\, d\theta$$

(9.41) "Multipole Quadropole"

$$Q_{ij}(t) = \int (3x_i x_j - r^2 \delta_{ij}) \rho(t) d^3x$$

$$Q_{10} = 0$$

$$Q_{11} = -2qa^2\cos^2(\omega t)$$

$$Q_{20} = 0$$

$$Q_{21} = 0$$

$$Q_{22} = -2qa^2\cos^2(\omega t)$$

$$Q_{30} = 0$$

$$Q_{31} = 0$$

$$Q_{32} = 0$$

$$Q_{33} = +4qa^2\cos^2(\omega t)$$

$$= -2qa^2[1+\cos(2\omega t)]$$

$$= Re[-2qa^2(1+e^{-2\omega t})]$$

$$\lim_{t\to\infty} Q_{33} = \lim_{t\to\infty} Re[-2qa^2(1+e^{-2\omega t})]$$

$$= -2qa^2$$

Trigonometric Identity

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

Angular Distribution of Radiation:

$$\frac{dP}{d\Omega} = \frac{e^2 Z_0 k^6}{512\pi^2}|Q_{33}|^2 \sin^2\theta\cos^2\theta$$

$$= \frac{Z_0 q^2}{128\pi^2}(ck)^2(ka)^4\sin^2\theta\cos^2\theta$$

$$= \frac{Z_0 q^2\omega^2}{32\pi^2}(ka)^4\sin^2\theta\cos^2\theta$$ ∘∘∘ when $ck = 2\omega$

"Harmonic frequency" -overlap-

$$= \frac{256}{81^2 9 \cdot 6\pi} Z_0 c^2 k^4 a_0^4 c^2$$

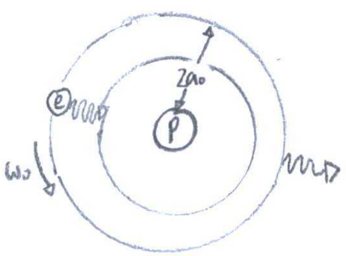
$$\cong 1.02 \times 10^{-9} W$$

c) Transition Probability Rate:

$$\Gamma = \frac{P}{\hbar \omega_0}$$

$$= 6.27 \times 10^{8} 1/s$$

$$\frac{1}{\Gamma} = 1.59 ns$$



"radiated power"

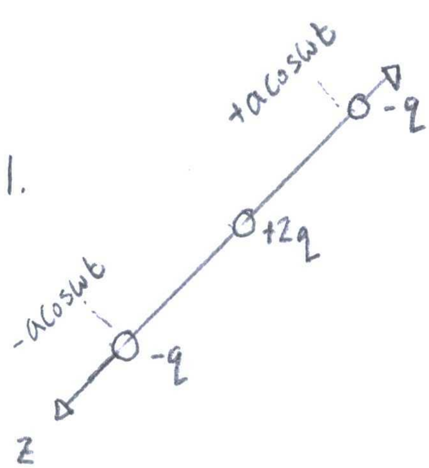
d) Ratio $\dfrac{\text{Magnetic Power radiated}}{\text{Quantum Power radiated}}$

$$= \frac{\left(\frac{32}{2^6}\right) \hbar \omega_0 \left(\frac{\alpha^4 c}{a_0}\right)}{\left(\frac{2}{3}\right)^8 \hbar \omega_0 \left(\frac{\alpha^4 c}{a_0}\right)}$$

$$= \frac{3^{10}}{2^{14}}$$

$$= 3.60$$

9.11.



"three charges along z-axis"

$$\rho = q\left[2\delta(z) - \delta(z - a\cos\omega t) - \delta(z + \cos\omega t)\right]\delta(x)\delta(y)$$

$$J = \hat{z} q a \omega \sin\omega t \left[\delta(z - a\cos\omega t) - \delta(z + a\cos\omega t)\right]\delta(x)\delta(y)$$

(9.17) "Electric dipole moment"

$$p = \int x' \rho(x') d^3x = q$$

(9.34) "Magnetic dipole moment"

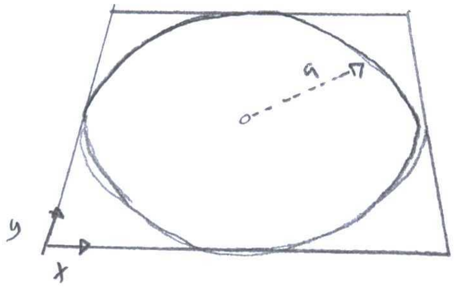
$$m = \frac{1}{2}\int (x \times J) d^3x$$

$$= 0$$

$$p = -q(a\cos\omega t - a\cos\omega t)$$

$$= 0$$

9.14.



"an antenna consists of a circular loop of wire... in x-y plane"

$I = I_0 \cos\omega t = \mathrm{Re}\left[I_0 e^{-i\omega t}\right]$

a) (9.149) "Magnetic Induction in a Radiation Zone"

$$H = \frac{e^{ikr-i\omega t}}{kr} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (-i)^{l+1} \left[a_E(l,m) X_{lm} + a_M(l,m)\, n \times X_{lm}\right]$$

$E = Z_0\, H \times \hat{n}$

Note: $m=0$ because symmetry in x-y plane

(9.168) "Magnetic coefficient"

$$a_M(l,m) = \frac{k^2}{i\sqrt{l(l+1)}} \int Y_{lm}^{*}\, \nabla\cdot(r\times J)\, j_l(kr)\, d^3x$$

$$\nabla\cdot(r\times J) = \nabla\cdot\left(I_0\,\delta(r-a)\,\delta(\cos\theta)\,\hat{\theta}\right)$$

$$= -\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(I_0\,\delta(r-a)\sin\theta\,\delta(\cos\theta)\right)$$

$$= -\frac{I_0}{r}\delta(r-a)\left(\frac{\cos\theta}{\sin\theta}\delta(\cos\theta) + \frac{\partial}{\partial\theta}\delta(\cos\theta)\right)$$

$$= \frac{I_0}{r}\delta(r-a)\sin\theta\,\delta'(\cos\theta)$$

$$a_M(l,0) = \frac{I_0 k^2 2\pi a}{i\sqrt{l(l+1)}} j_l(ka) \int d\theta\, Y_{l0}^{*} \sin^2\theta\, \delta'(\cos\theta)$$

$$= -i I_0 k^2 \sqrt{\pi}\, a \sqrt{\frac{2l+1}{l(l+1)}}\, j_l(ka) \int d\theta\, P_l(\cos\theta)\cdot \sin^2\theta\, \delta'(\cos\theta)$$

$$= i I_0 k^2 \sqrt{\pi}\, a \sqrt{\frac{2l+1}{l(l+1)}}\, j_l(ka) \int dx\, P_l(x)\sqrt{1-x^2}\, \delta'(x)$$

$$= i I_0 k^2 \sqrt{\pi}\, a \sqrt{\frac{2l+1}{l(l+1)}}\, j_l(ka) \left[-\frac{d}{dx}\left(\sqrt{1-x^2}\cdot P_l(x)\right)\right]_{x=0}$$

$$= i I_0 k^2 \sqrt{\pi}\, a \sqrt{\frac{2l+1}{l(l+1)}}\, j_l(ka) \left[\frac{-x P_l(x)}{\sqrt{1-x^2}} + \sqrt{1-x^2}\left(\frac{-l\cdot x P_l(x) + l P_{l-1}(x)}{1-x^2}\right)\right]$$

$$= i I_0 k^2 \sqrt{\pi}\, a \sqrt{\frac{l(l+1)}{l(l+1)}}\, j_l(ka) P_{l-1}(0)$$

$$H = \frac{i e^{i(kr-\omega t)} \cdot I_0 \cdot k\sqrt{\pi}\, a}{r} \sum_{l=odd} \left((-1)^{\frac{l+1}{2}} \sqrt{\frac{l(2l+1)}{l+1}}\, j_l(ka) P_l(0)\, n \times X_{l0} \right)$$

$$E = -Z_0 \frac{i e^{i(kr-\omega t)} \cdot I_0 k\sqrt{\pi} \cdot a}{r} \sum_{l=odd} \left((-1)^{\frac{l+2}{2}} \sqrt{\frac{l(2l+1)}{l+1}}\, j_l(ka) P_{l-1}(0)\, n \times X_{l0} \right)$$

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2} \left| \sum_{l=odd} (-1)^{\frac{l+1}{2}} \cdot i \cdot I_0 \cdot k^2 \sqrt{\pi}\, a \sqrt{\frac{l(2l+1)}{l+1}}\, j_l(ka) P_{l-1}(0) X_{l0} \right|^2$$

$$= \frac{1}{2} Z_0 I_0^2 k^2 \pi a^2 \left| \sum_{l=odd} (-1)^{\frac{l+1}{2}} \sqrt{\frac{l(2l+1)}{l+1}}\, j_l(ka) P_{l-1}(0) X_{l0} \right|^2$$

b) Lowest nonvanishing multipole moment:

$$M_{10} = \frac{3}{\sqrt{2}\, i k^3} a_M(1,0)$$

$$= \frac{3}{\sqrt{2}\, i k^3} (i I_0 k^2 \sqrt{\pi}\, a) \sqrt{\frac{3}{2}}\, j_l(ka) P_0(0)$$

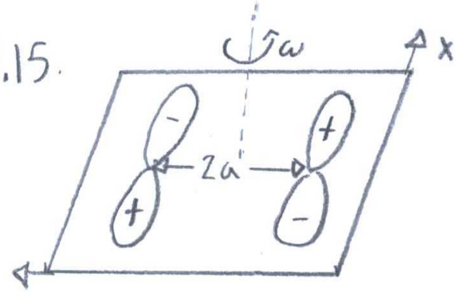
$$= \frac{3\sqrt{3\pi}\, I_0 a}{2k} \left(\frac{\sin(ka)}{k^2 a^2} - \frac{\cos(ka)}{ka} \right)$$

If $ka \ll 1$, then

$$M_{10} = \frac{3\sqrt{3\pi}\, I_0 a}{2k} \left(\frac{1}{ka} - \frac{ka}{6} - \frac{1}{ka} + \frac{ka}{2} \right)$$

$$= \frac{\sqrt{3\pi}\, I_0 a^2}{2}$$

9.15.



"Two fixed electric dipoles ... in x-y plane ... rotating"

Nonrelativistic Motion: $\left(\frac{\omega a}{c} \ll 1\right)$

"Rotation speed not close to light"

a) (9.41) Quadropole moment tensor

$$Q_{ij} = q_k \int (3 x x - r^2 \delta_{ij}) dx^3$$

$$Q_{xx} = q_k \int \ldots$$

$$Q_{ij} = \begin{pmatrix} Q_{xx} & Q_{xy} & Q_{xz} \\ Q_{yx} & Q_{yy} & Q_{yz} \\ Q_{zx} & Q_{zy} & Q_{zz} \end{pmatrix}$$

$$= q_R \begin{pmatrix} 2x_R^2 - y_R^2 & 0 & 0 \\ 0 & 2y_R^2 - x_R^2 & 0 \\ 0 & 0 & -(x_R^2 + y_R^2) \end{pmatrix}$$

$$Q_{ij}(t) = q \begin{pmatrix} \cos(2\omega t) & \sin(2\omega t) & 0 \\ \sin(2\omega t) & -\cos(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

b) (9.44) Magnetic Induction

$$H = -\frac{ick^3}{24\pi} \frac{e^{ikr}}{r} n \times Q(n)$$

Note: Problem suggests "apart from overall phase factor"

$$H(\phi) = -\frac{ick^3}{24\pi} \frac{e^{ikr}}{r} n \times Q(n) \times e^{i\phi} \hat{z}$$

$$= \frac{cq}{2\pi} k^3 \left[(x + i\hat{y}) \cos\theta - \hat{z} \sin\theta e^{i\phi} \right] \cos\theta \frac{e^{ikr}}{r}$$

c) Angular Distribution radiated.

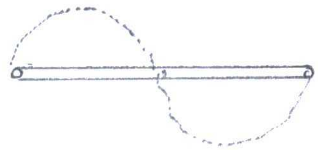
$$\frac{dP}{d\Omega} = \frac{Z_0}{2k} |a_E(2,1)|^2 |X_{2,\pm 1}|^2$$

$$\propto |X_{1,\pm}|^2$$

$$\propto (\cos^4\theta + 2\cos^2\theta + 1) \qquad \text{(Table 9.1)}$$

$$P = \frac{4}{15\pi \epsilon_0} \cdot c \cdot k^6 \cdot q^2$$

9.16.



"linear antenna ...sinusoidal current"

a) Power Radiated per unit solid angle:

(9.1) "Current"

$$J(x,t) = J(x) \circ e^{-i\omega t}$$

$$= I \sin\left(\frac{2\pi z}{d}\right)\delta(x)\cdot\delta(y)\,\hat{z} \quad \ldots \text{for } |z| \leq \frac{1}{2}d$$

(9.19) "Magnetic Field modified"

$$H(x,t) = \frac{i\omega}{4\pi c r} e^{i(kr-\omega t)}\, n \times \int d^3x\, J(x)\, e^{-ikx\cdot n}$$

$$E(x,t) = Z_0\, H(x,t) \times \hat{n}$$

(9.21) "Power radiated per solid angle"

$$\frac{dP}{d\Omega} = \frac{1}{2}\mathrm{Re}\left[r^2 \cdot n \circ E \times H^*\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[r^2 \circ n \circ (Z_0 \cdot H(x,t) \times \hat{n}) \times H^*\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[r^2 \cdot n \cdot (Z_0(-H^* \times (H \times n)))\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[r^2 \cdot Z_0 \cdot |\hat{H}|^2\right]$$

$$= \frac{i\omega}{4\pi c r} e^{i(kr-\omega t)} \cdot \hat{n} \times \int_{-d/2}^{d/2} I \sin(kz)\, e^{-ikz\cos\theta} \cdot \hat{z}\; dz$$

$$= \frac{i\omega}{4\pi c r} e^{i(kr-\omega t)} \cdot \hat{n} \times \cdot I \left.\frac{(-ik\cos\theta \sin(kz) - k\cos kz)}{(-ik\cos\theta)^2 + k^2}\right|_{-\pi/k}^{\pi/k} \hat{z}$$

$$= \frac{i\omega}{4\pi c r} e^{i(kr-\omega t)} \cdot \hat{n} \times \cdot I \left(\frac{e^{-i\pi\cos\theta} - e^{i\pi\cos\theta}}{k \cdot \sin^2\theta}\right) \cdot \hat{z}$$

$$= \frac{i\omega}{4\pi c r} e^{i(kr-\omega t)} \cdot \hat{n} \times (-2Ii) \frac{\sin(\pi\cos\theta)}{k\sin^2\theta} \cdot \hat{z}$$

Integral Identity:

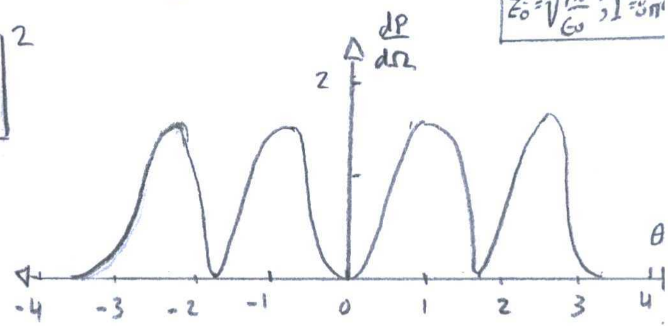
$$\int e^{az} \cdot \sin(kz)\, dz = e^{az}\frac{(a\sin(kz) - k\cos(kz))}{a^2 + k^2}$$

$$= \frac{I\omega}{2\pi k c r} e^{i(kr-\omega t)} \frac{\sin(\pi\cos\theta)}{\sin^2\theta} \hat{n}\times\hat{z}$$

$$= \frac{I\omega}{2\pi k c r} e^{i(kr-\omega t)} \cdot \frac{\sin(\pi\cos\theta)}{\sin\theta} \hat{\Phi} \qquad \hat{n}\times\hat{z} = -\sin\theta\,\hat{\Phi}$$

$\boxed{Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} ; I = }$

$$\frac{dP}{d\Omega} = \frac{Z_0 I^2}{8\pi^2}\left[\frac{\sin(\pi\cos\theta)}{\sin\theta}\right]^2$$



b) $$P = \int \frac{Z_0 I^2}{8\pi^2}\left[\frac{\sin(\pi\cos\theta)}{\sin\theta}\right]^2 d\Omega$$

$$= \frac{Z_0 I^2}{4\pi}\int_{-1}^{1} \frac{\sin^2(\pi x)}{1-x^2}dx \quad \therefore \text{ when } x=\cos\theta$$

$$= \frac{Z_0 I^2}{8\pi}\int_{-1}^{1} \frac{1-\cos(2\pi x)}{1-x^2}dx$$

$\boxed{\text{Identity: } \frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1-x}+\frac{1}{1+x}\right)}$

$$= \frac{Z_0 I^2}{8\pi}\int_{-1}^{1} \frac{1-\cos(2\pi x)}{1+x}dx$$

If $t = 1+x$, then

$$= \frac{Z_0 I^2}{8\pi}\int_{0}^{4\pi} \frac{1-\cos(t)}{t}dt$$

$\boxed{\text{Cosine Integral} \quad Ci(x) = -\int_x^{\infty} \frac{\cos t}{t}dt}$

$$= \frac{Z_0 I^2}{8\pi}\Big[\gamma + \ln(x) - Ci(x)\Big]_0^{4\pi}$$

$\therefore$ where γ = Euler constant $= 0.5772$

$$= \frac{Z_0 I^2}{8\pi}\left[\gamma + \ln(4\pi) - Ci(4\pi)\right]$$

$$\simeq \frac{Z_0 I^2}{8\pi}(3.114)$$

Radiative Resistance:

(pg 412) "Radiative Resistance"

$$P = \frac{1}{2} I_0 \cdot R_{rad} \quad \ldots \text{some general language}$$

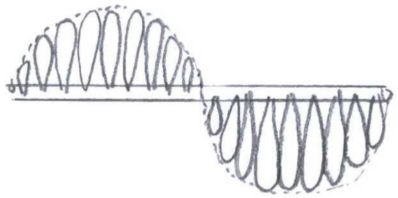
$$= \frac{Z_0 I_0^2}{8\pi} (3.114)$$

$$R_{rad} \cong \frac{Z_0}{4\pi}$$

$$\cong 93.3 \text{ Ohms} \ldots \text{when } Z_0 = 376.7 \text{ Ohms (pg 297)}$$

"Impedance of free space"

9.17.



"linear antenna ... by multipole expansion"

a) (9.63.5) "Electric dipole Moment"

$$p = \frac{i}{\omega} \int J(x) d^3x$$

$$= \frac{i}{\omega} \int I \sin\left(\frac{2\pi z}{d}\right) \delta(x) \delta(y) dz$$

$$= -\frac{iI}{c} \cos(kz) \delta(x) \delta(y) \Theta(d/2 - |z|)$$

(9.17) "Electric Dipole"

$$p = \int x \rho d^3x = -z \frac{iI}{c} \int_{-d/2}^{d/2} z \cos(kz) dz = 0$$

(9.34) "Magnetic Dipole"

$$m = \frac{1}{2} \int x \times J d^3x = \frac{I}{2} \int_{-d/2}^{d/2} \sin(kz) (z\hat{z}) \times \hat{z} dz = 0$$

(4.9) "Electric Quadropole

$$Q_{ij} = \int (3x_i x_j - r^2 \delta_{ij}) \rho d^3x$$

$$= \frac{-iI}{c} \int_{-d/2}^{d/2} (3(z\delta_{i3})(z\delta_{j3}) - z^2 \delta_{ij}) \cos(kz) dz$$

$$= \frac{8\pi i I}{ck^3}$$

Exact Multipole

$$\rho = \frac{1}{i\omega}\int J(x)\,d^3x$$

$$= \frac{-iI}{c}\,\delta(x)\delta(y)\,\Theta(d/2-|z|)$$

(9.17) "Electric Dipole"

$$p = \int x\rho\,d^3x = -\frac{Z_0 i I}{c}\int z\,dz = 0$$

(9.34) "Magnetic Dipole"

$$m = \frac{1}{2}\int x\times J\,d^3x = \frac{I}{2}\int z\,\hat{z}\times\hat{z}\,dz = 0$$

(9.41) "Electric Quadrupole Moment"

$$Q_{ij} = \int(3x_i x_j - r^2\delta_{ij})\rho\,d^3x$$

$$Q_{10} = Q_{20} = Q_{21} = Q_{31} = Q_{32} = 0$$

$$Q_{33} = -2Q_{11} = -2Q_{22} = \frac{-8\pi}{ck^3}\frac{(kd)^3}{48\pi} = \frac{-iId^3}{6c}$$

b) Angular Distribution of radiated Power:

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^6}{512\pi^2}|Q_0|^2\sin^2\theta\cos^2\theta \quad \ldots \text{where } Q_0 = Q_{33}, Q_{11}, Q_{22}$$

$$= Z_0\frac{|I|^2}{8}\sin^2\theta\cos^2\theta$$

c) Total Power "Exact":

$$P = \frac{Z_0|I|^2}{8}\,2\pi\int_{-1}^{1}\sin^2\theta\cos^2\theta\,d\cos\theta$$

$$= Z_0|I|^2\pi/15$$

Total Power "Multipole moments":

( )

$$P = \frac{Z_0}{2k^2}\sum^{\infty}\sum^{l}\left[|a_E(l,m)|^2 + |a_M(l,m)|^2\right]$$

Long-Wavelength Limit

$$= \frac{Z_0}{2k^2} \sum_{l=0} \sum_{m=-l} |a_E(l,0)|^2$$

$$a_E(l,0) = \frac{I}{\pi d} \sqrt{\frac{4\pi(2l+1)}{l(l+1)}} \left(\frac{kd}{2}\right)^2 j_1\left(\frac{kd}{2}\right)$$

$$a_E(2,0) = \frac{I}{\pi d} \sqrt{\frac{4\pi(2\cdot 2+1)}{2(2+1)}} \left(\pi\right)^2 j_1(\pi)$$

$$= \frac{Ik}{2} \sqrt{\frac{30}{\pi^3}}$$

$$\frac{dP}{d\Omega} = \frac{Z_0}{2k^2} \left|a_E(2,0)\right|^2 \frac{15}{8\pi} \sin^2\theta \cos^2\theta$$

$$= \frac{Z_0 I^2}{8} \left(\frac{15}{2\pi^2}\right)^2 \sin^2\theta \cos^2\theta$$

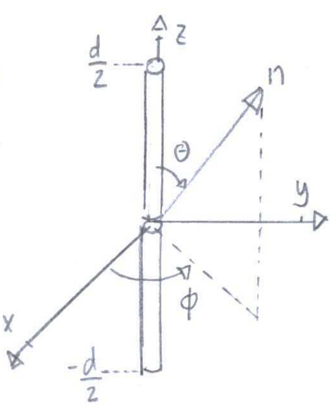
$$P = \int \frac{Z_0 I^2}{8} \left(\frac{15}{2\pi^2}\right)^2 \sin^2\theta \cos^2\theta$$

$$= \frac{1}{2} I_0^2 \cdot R_{rad}$$

$$R_{rad} = 91.2\,\Omega$$

In problem 9.16, 93 Ohms is larger than problem 9.17's 91.2Ω because no higher order modes.

9.18



"Short, center-fed, linear antenna"

(9.18) "Electric dipole fields"

$$H = \frac{ck^2}{4\pi} (n \times p) \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr}\right)$$

$$E = \frac{1}{4\pi\epsilon_0} \left\{ k^2 (n \times p) \times n \frac{e^{ikr}}{r} + [3n(n \cdot p) - p] \left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{ikr} \right\}$$

$$\int \left[\epsilon |E|^2 - \mu_0 |H|^2\right] d\Omega$$

$$= \int \epsilon_0 \left| \frac{1}{4\pi\epsilon_0} \left\{ k^2 (n \times p) \times n \frac{e^{ikr}}{r} + [3n(n\cdot p) - p] \left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{ikr} \right\} \right|^2 - \mu \left| \frac{ck^2}{4\pi} (n \times p) \frac{e^{ikr}}{r} \left(1 - \frac{1}{ikr}\right) \right|^2 d\Omega$$

$$= \frac{1}{2\pi\epsilon_0} \frac{|p|^2}{r^6}$$ ∘∘∘ when $(n \times p) \times n = n(n \cdot p) = (n \times p) = 0$

b) (6.140) Inductance

$$X = \frac{4\omega}{|I_i|^2} \int_V (w_m - w_e) d^3x$$

(6.133) "Electric and magnetic densities"

$$w_E = \frac{1}{4}(E \cdot D^*) \qquad w_m = \frac{1}{4}(B \cdot H^*)$$

$$X = \frac{4\omega}{|I_i|^2} \int_V \left(\frac{\epsilon_0}{4}|E|^2 - \frac{\mu_0}{4}|H|^2 \right) d^3x$$

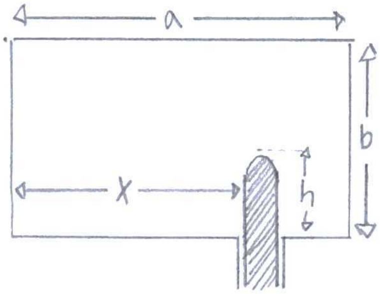
$$= -\frac{\omega |p|^2}{6\pi\epsilon_0 |I_i|^2 a^2}$$ ∘∘ approximation not apparent, the guess is $r \gg 1$

c) $X_a \approx -d^2/24\pi\epsilon_0 \omega a^3$

Capacitance $= 24\pi\epsilon_0 a^3/d^2$ ∘∘ when $a = d/2$

The problem asks about no solution. Rather, the approximations are not exact in statements from other authors through experimental data on short antennas. Possibly because of an induced field, background, the probe, temperature or moisture.

9.19.



"..long rectangular waveguide with... a coaxial line extending vertically... into the interior"

$I(y) = I_0 \sin[(\omega/c)(h-y)]$

a) Amplitude from the multipole expansion:

Chapter 8
TE/TM Modes
∘ Boundary Conditions
∘ Roots of Bessel

Chapter 9
Electric pole
magnetic pole
∘ power per angle

E,H field

(9.69) "Multipole Expansion"

$$A_\lambda^{(\pm)} = i\frac{\omega Z_\lambda}{2}\Big\{p\cdot E_\lambda^{(\mp)}(0) - m\cdot B_\lambda^{(\mp)}(0) + \frac{1}{6}\sum_{\alpha,\beta}\Big[Q_{\alpha\beta}\frac{\partial E_{\lambda\alpha}^{(\mp)}}{\partial X_\beta}(0) - Q_{\alpha\beta}^{M}\frac{\partial B_{\lambda\alpha}^{(\mp)}}{\partial X_\beta}(0)\Big] + \cdots\Big]$$

(8.130) "Waves propagating in lossless guides"

$$E_\lambda^{(-)} = [E_\lambda - E_{z\lambda}]e^{-ik_\lambda z}$$

$$H_\lambda^{(-)} = [-H_\lambda + H_{z\lambda}]e^{-ik_\lambda z}$$

Current Density:

$$J = \hat{y}\, I_0 \sin\left[\frac{\omega}{c}(h-y)\right]\delta(x)\delta(y)\delta(z)\Theta(h-y)$$

Electric Dipole Moment:

(9.63.5)

$$p = \frac{i}{\omega}\int J(x)\,d^3x$$

$$= \frac{i I_0}{\omega}\hat{y}\int \sin\left[\frac{\omega}{c}(h-y)\right]dy$$

$$= \frac{i c I_0}{\omega^2}\left(1-\cos\left(\frac{\omega h}{c}\right)\right)$$

$$= \frac{2 i c I_0}{\omega^2}\sin^2\left(\frac{\omega h}{2c}\right)\hat{y}$$

Magnetic Dipole Moment:

(9.32) "Magnetization"

$$M = \frac{1}{2}(x \times J)$$

$$= \frac{1}{2}(x,y,z) \times \hat{y}\, I_0 \sin\left[\frac{\omega}{c}(h-y)\right]\delta(x)\delta(y)\delta(z)\Theta(h-y)$$

$$= \frac{1}{2} I_0(-z,0,x)\sin\left[\frac{\omega}{c}(h-y)\right]\delta(x)\delta(y)\delta(z)\Theta(h-y)$$

$$= 0$$

Electric Quadropole Moment:

(9.41) "Multipole Moment"

Note: Many names, electric, magnetic, and traceless... from chapter 4:

$$Q_{\alpha\beta} = \int (3x_\alpha x_\beta - \delta_{\alpha\beta} r^2)\rho\, d^3x$$

$$\ldots \text{where } \rho = -\frac{i}{\omega}\nabla\cdot J = \frac{iI_0}{c}\cos\left[\frac{\omega}{c}(h-y)\delta(x)\delta(z)\Theta(h-y)\right]$$

$$Q_{xx} = Q_{zz} = \frac{1}{2}Q_{yy}$$

$$= -\int y^2 \rho\, d^3x$$

$$= -\frac{iI_0}{c}\int_0^h y^2\cos\left[\frac{\omega}{c}(h-y)\right]dy$$

$$= -\frac{2iI_0c}{\omega^2}\left(h - \frac{c}{\omega}\sin\frac{\omega h}{c}\right)$$

b) $H_z = H_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$ ... where $H_0 = \frac{2i\gamma_{mn}}{\mu_0\omega\sqrt{ab}}$

(9.5) "Electric Field"

$$E_t = -\frac{i\mu_0\omega}{\gamma_{mn}^2}\hat{z}\times\nabla_t H_z \quad \ldots \text{when } \gamma^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$E_x = -\frac{i\mu_0\omega}{\gamma_{mn}^2}\frac{n\pi}{b}H_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \quad \ldots \text{as } a = \begin{cases} a & \text{else} \\ 2a & m=0 \end{cases}$$

$$E_y = \frac{i\mu_0\omega}{\gamma_{mn}^2}\frac{m\pi}{a}H_0\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \qquad b = \begin{cases} b & \text{else} \\ 2b & m=0 \end{cases}$$

$$E_z = 0$$

$$A_{mn}^{(\pm)} = \frac{i\omega Z_{mn}}{2}\left[p\cdot E_{mn}^{(\mp)}(x,0,0) - m\, B_{mn}^{(\mp)}(x,0,0) + \frac{1}{6}Q_{\alpha\beta}\partial_\beta E_{mn}^{(\mp)}(x,0,0) - \frac{1}{6}Q_{\alpha\beta}\partial_\beta B_{mn}^{(\mp)}(x,0,0)\right]$$

$$= \frac{i\mu_0\omega^2}{2k}\left[p_y E_y + \frac{1}{6}Q_{xx}(\partial_x E_x - 2\partial_y E_y + \partial_z E_z) + \dots\right]$$

... when $Z_{mn} = \mu_0\omega/k_{mn}$

$$E_y = \frac{i\mu_0\omega}{\gamma_{mn}^2}\frac{m\pi}{a}H_0\sin\left(\frac{m\pi x}{a}\right)$$

$$A_{mn}^{(\pm)} = \frac{2\mu_0 c I_0}{k_{mn}\gamma_{mn}\sqrt{ab}}\cdot\frac{m\pi}{a}\cdot\sin\left(\frac{m\pi x}{a}\right)\cdot\sin^2\left(\frac{\omega h}{2c}\right)$$
$$\times\left(1-\frac{\sin^2(n\pi h/(2b))}{\sin^2(\omega h/2c)}\right)\cdot\left(1-\left(\frac{n\pi c}{b\omega}\right)^2\right)^{-1}$$

$$A_{1,0}^{(\pm)} = \frac{2\mu_0 c I_0}{k_{10}\sqrt{2ab}}\sin\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{\omega h}{2c}\right)$$

$$P_{1,0} = \frac{\mu_0 c^3}{\omega\cdot k_{1,0}ab}|I_0|^2\sin^2\left(\frac{\pi x}{a}\right)\sin^4\left(\frac{\omega h}{2c}\right)$$

The electric fields are independent of x and z, so, y-modes appear.

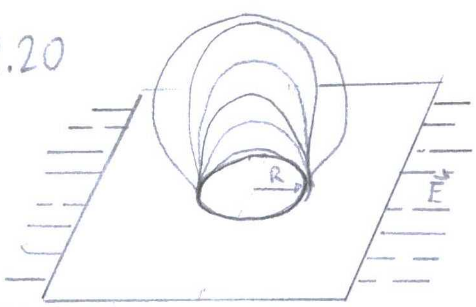
$$E_x = 0$$

$$E_y = \frac{i\mu_0\omega}{\gamma_{m0}^2}\frac{m\pi}{a}H_0\sin\left(\frac{m\pi x}{a}\right)$$

$$E_z = 0$$

When $n \neq 0$, the higher order modes contribute to the lowest multiple.

9.20



"... circular opening in a flat conducting plane."

(3.186) "tangential electric field in the opening of a radial field"

$$E_{tan}(\rho,0) = \frac{(E_0-E_1)}{\pi}\frac{\rho}{\sqrt{a^2-\rho^2}}$$

(9.72) "Effective Dipole Moment"

$$p_{eff} = \epsilon n \int (x \circ E_{tan})\, du$$

$$m_{eff} = \frac{Z}{i\mu w} \int (n \times E_{tan})\, du$$

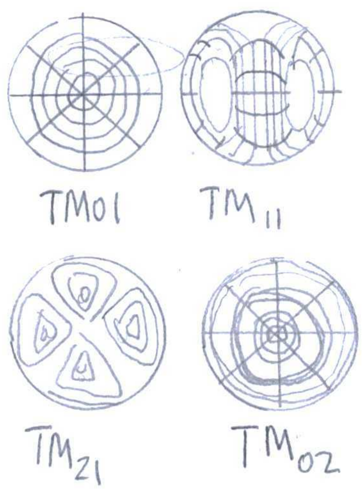
(9.75) "Effective Dipole Moment in Circular opening"

$$p_{eff} = -\frac{4\epsilon_0 R^3}{3} E_0$$

$$m_{eff} = \frac{8R^3}{3} H_0$$

Note: University of Washington beat me to it.

9.21



"Transverse magnetic waves propagating in a cylinder"

$$E_z = J_m(\gamma r) e^{im\phi} e^{i\beta z - wt} \qquad H_z = 0$$

$$E_\phi = \frac{-m\beta}{\gamma^2} \frac{E_z}{r} \qquad H_\phi = -\frac{K}{Z_0 \beta} E_\phi$$

$$E_r = \frac{i\beta}{\gamma^2} \frac{\partial E_r}{\partial r} \qquad H_r = \frac{K}{Z_0 \beta} E_r$$

... where $\gamma^2 = k^2 - \beta^2$ and $J_m(\gamma R) = 0$

(9.102) "L components"

$$L_z = -i \frac{\partial}{\partial \phi}$$

$$\vec{L}_z = -i \frac{\partial}{\partial \phi} E_z$$

$$= m J_m(\gamma r) e^{im\phi} e^{i\beta z - wt}$$

(9.134) "Energy density modified"

$$U = \epsilon_0 \left|\frac{E}{2}\right|^2 + \mu_0 \left|\frac{H}{2}\right|^2 = \frac{\epsilon_0 J_m^2(\gamma r) e^{2im\phi} e^{2(i\beta z - wt)}}{2}$$

(9.135) "Energy modified"

$$U = \int u \cdot |X_{em}|^2 r^2 dr\, d\theta = \frac{\epsilon_0 J_m'(X_{em})^2 \cdot e^{2im\phi} \cdot e^{2(i\beta z - wt)}}{4} \circ R^3$$

$$\frac{L_z}{U} = \frac{4m}{-}\left(\frac{m}{\epsilon_0}\right)\left(\frac{J_m(\gamma r)}{J_m'(x_{\ell m})^2}\right)\frac{e^{-im\phi} \cdot e^{-i(\beta z - \omega t)}}{R^3}$$

TM Modes:

(9.118)

$$H_{\ell m}^{(E)} = f_\ell(kr) \cdot L \cdot Y_{\ell m}(\theta, \phi)$$

$$E_{\ell m} = \frac{iZ_0}{k} \nabla \times H_{\ell m}^{(E)}$$

$$(H_{\ell m})_r = 0 \qquad (E_{\ell m})_r = \frac{-Z_0}{kr} \ell(\ell+1) Y_{\ell m} j_\ell(kr)$$

$$(H_{\ell m})_\theta = i \frac{1}{\sin\theta} \frac{\partial}{\partial \phi} Y_{\ell m} j_\ell(kr) \qquad (E_{\ell m})_\theta = \frac{-Z_0}{kr} \frac{\partial}{\partial \theta} Y_{\ell m} \frac{\partial}{\partial r}(r j_\ell(kr$$

$$(H_{\ell m})_\phi = -i \frac{\partial}{\partial \theta} Y_{\ell m} \cdot j_\ell(kr) \qquad (E_{\ell m})_\phi = \frac{-Z_0}{kr} \frac{1}{\sin\theta} \frac{\partial}{\partial \phi} Y_{\ell m} \frac{\partial}{\partial r}(r j_\ell$$

Cutoff Frequencies:

TE Modes: $\omega^{TE} = \frac{c}{a} x_{\ell m}$

TM Modes: $\omega^{TM} = \frac{c}{a} y_{\ell m}$

b) Four Lowest Modes:

$$\lambda = \frac{2\pi}{x} a$$

n \ m	1	2	3
1	4.5	5.8	6.85
2	7.64	0	0
3	0	0	0

$\lambda_{11} = 1.4a$, $\lambda_{21} = 1.1a$, $\lambda_{31} = 0.9a$, $\lambda_{12} = 0.8a$

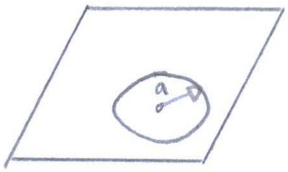
c) $k_{11} = \frac{2\pi}{\lambda_{11}}$
$= 4.5a$

$$(H_{10})_r = \sqrt{\frac{3}{\pi}} \frac{1}{(4.5r/a)} j_1(4.5r/a)\cos\theta$$

$$(H_{10})_\theta = -\sqrt{\frac{3}{4\pi}} \frac{1}{4.5r/a} \frac{\partial}{\partial r}(r \cdot j_1(4.5r/a))\sin\theta$$

$$(H_{10})_\phi = 0$$

9.22.



"a spherical hole ... in a conducting medium"

a) (9.122) "General solution to Maxwell Equations"

$$H = \sum\sum\left[a_E(l,m)\cdot j_l(kr)X_{lm} - \frac{i}{k}a_M(l,m)\cdot \nabla \times j_l(kr)\cdot X_{lm}\right]$$

(9.122) "General solution to Maxwell equations"

$$E = Z_0 \cdot \sum\sum\left[\frac{i}{k}a_E(l,m)\nabla \times j_l(kr)X_{lm} + a_M(l,m)j_l(kr)X_{lm}\right]$$

$$@r=0,\quad \hat{r}\cdot H = -\frac{i}{k}\sum\sum a_M(l,m)\hat{r}\cdot\nabla\times j_l(kr)X_{lm}$$

$$= \sum\sum \frac{1}{kr}a_M(l,m)\cdot j_l(kr)\vec{L}\cdot X_{lm}$$

$$= \sum\sum \frac{\sqrt{l(l+1)}}{kr}a_M(l,m)\cdot j_l(kr)\cdot Y_{lm}$$

$$\hat{r}\cdot E = -Z_0\sum\sum\frac{\sqrt{l(l+1)}}{kr}a_E(l,m)\cdot j_l(kr)Y_{lm}$$

TE Modes:

(9.116)

$\ell \neq 0$

$$H^{(M)}_{lm} = -\frac{i}{kZ_0}\nabla\times E^{(M)}_{lm}$$

$$E^{(M)}_{lm} = Z_0 j_l(kr)\cdot L\cdot Y_{lm}(\theta,\phi)$$

$\ell = 0$

$$(H_{lm})_r = \frac{1}{kr}l(l+1)Y_{lm}j_l(kr) \qquad (E_{lm})_r = 0$$

$$(H_{lm})_\theta = \frac{1}{kr}\frac{\partial}{\partial\theta}Y_{lm}\frac{\partial}{\partial r}(r\cdot j_l(kr)) \quad (E_{lm})_\theta = iZ_0\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}Y_{lm}j_l(kr)$$

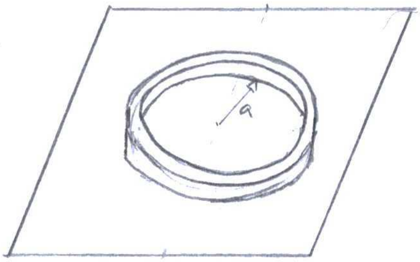
$$(H_{lm})_\phi = \frac{1}{kr}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}Y_{lm}\frac{\partial}{\partial r}(r\cdot j_l(kr)) \quad (E_{lm})_\phi = -iZ_0\frac{\partial}{\partial\theta}Y_{lm}j_l(kr)$$

$$(E)_{10r} = 0$$

$$(E)_{10\theta} = 0$$

$$(E)_{\phi} = i\sqrt{\frac{3}{4\pi}} Z_0 j_1(4.5r/a)\sin\theta$$

1.23



"spherical resonant cavity... has nonpermeable walls"

TE Modes:

(9.116) "Magnetic Induction"

$$H = -\frac{i}{k} \nabla \times j_e(kr) X_{em}$$

$$E = Z_0 j_e(kr) X_{em}$$

(9.134) "Energy density modified"

$$U = \epsilon_0 \left|\frac{E}{2}\right|^2 + \mu_0 \left|\frac{H}{2}\right|^2$$

$$\simeq \frac{\epsilon_0}{2}|E|^2 \ \ldots \text{ when } E = H$$

$$\simeq \frac{\mu_0}{2} j_e(kr)^2 |X_{em}|^2$$

(9.135) "Energy modified"

$$U = \frac{\mu_0}{2} \int j_e(kr)^2 |X_{em}|^2 r^2 dr\, d\Omega$$

$$= \frac{\mu_0}{2} \int_0^a j_e(kr)^2 r^2 dr\, d\Omega$$

$$= \frac{\mu_0 a^3}{4} j_1'(X_{em})^2$$

$$= \frac{1}{2\sigma\delta} \int |r \times H|^2 da$$

( )

$$P = \frac{1}{2\sigma\delta}\int |r \times H|^2 da$$

$$= \frac{1}{2\sigma\delta}\int \left(\frac{1}{kr}\frac{d}{dr} r\, j_l(kr)\right)^2 |X_{em}|^2 r^2 d\Omega$$

$$= \frac{1}{2\sigma\delta k^2}\left(|r\, j_l(kr)|\right)^2\Big|_{r=a}$$

$$= \frac{1}{2\sigma\delta k^2}\left(j_l(ka) + ka\, j_l'(ka)\right)^2$$

$$= \frac{a^2}{2\sigma\delta} j_l'(X_{em})^2 \quad \ldots \text{when } ka = X_{en}, \text{ so } j_l(X_{em}) =$$

$$Q_{emn} = \omega \frac{U_{emn}}{P_{emn}}$$

$$= \frac{\mu_0 \sigma \omega \delta a}{2}$$

$$= \frac{a}{\delta} \quad \ldots \text{when } \delta = \sqrt{2/\mu_0\sigma\omega}$$

Otherwise, if $p = 1$ in TM modes:

$$U = \frac{\mu_0}{2}\int_0^a j_l(kr)^2 r^2 da$$

$$= \frac{\mu_0 a^3}{4}\left(1 - \frac{l(l+1)}{y_{lm}^2}\right) j_l(y_{lm})^2$$

$$P = \frac{1}{2\sigma\delta}\int |r \times H|^2 da$$

$$= \frac{1}{2\sigma\delta}\int j_l(kr)^2 |r \times X_{em}^2| r^2 d\Omega$$

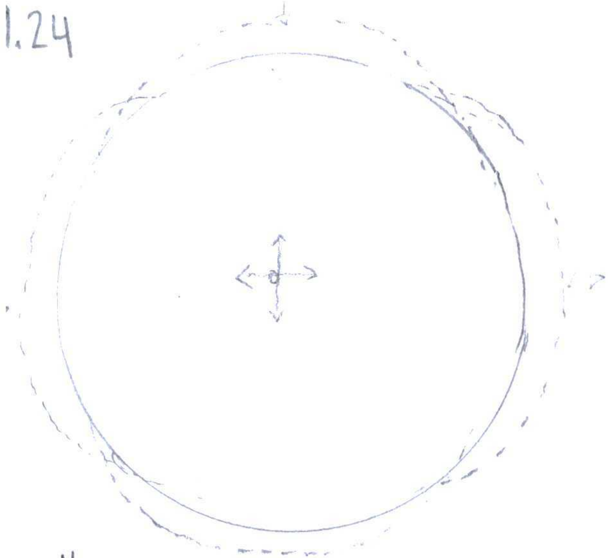
$$= \frac{a^2}{2\sigma\delta} j_l(y_{mn})^2$$

$$Q_{lmn} = \omega \frac{U_{emn}}{P_{emn}}$$

$$= \frac{\mu_0\sigma\omega\delta a}{2}\left(1 - \frac{l(l+1)}{y_{lm}^2}\right)$$

$$= \frac{a}{\delta}\left(1 - \frac{l(l+1)}{y_{lm}^2}\right)$$

1.24



"Normal modes of oscillation of a perfectly conducting sphere"

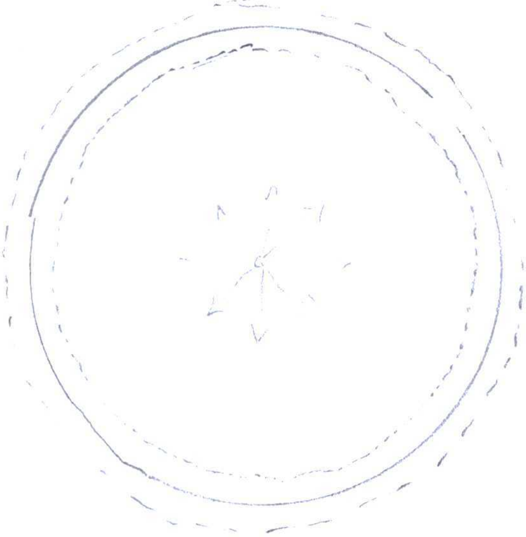
a) (9.133) "Magnetic and Electric Multipoles"

$$H = \sum a_E(l,m)\, X_{em}\, h_e^{(1)}(kr)\, e^{-i\omega t}$$

$$E = \frac{i}{k} Z_0 \nabla \times H$$

..where $h_l^{(1)}(kr)$ = Hankel function

Hankel Functions
$l=1 \quad h_1^{(1)}(\xi) = \frac{-e^{i\xi}}{\xi}\left(1+\frac{i}{\xi}\right)$
$l=2 \quad h_2^{(1)}(\xi) = \frac{ie^{i\xi}}{\xi}\left(1+\frac{3i}{\xi}-\frac{3}{\xi^2}\right)$



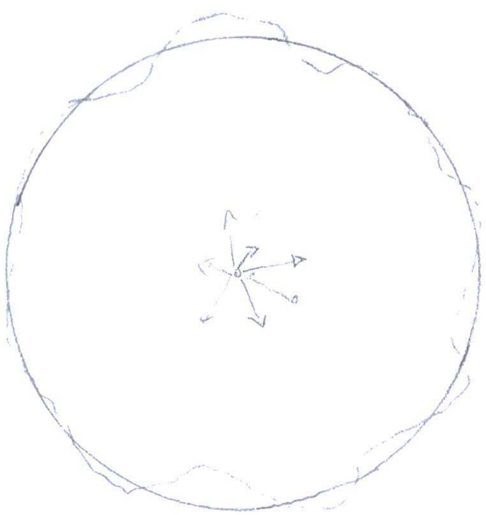
$$H = h_1^{(1)}(kr)\, X_{em}$$

$$E = \frac{i}{k} Z_0 \nabla \times H$$

Transverse Electric frequencies:

$$\omega_{nlm} = \frac{x_{ln}\, c}{a} \quad \text{for} \quad h_1^{(1)}(x_{ln}) = 0$$

at $l \geq 1$, $|m| \leq 1$, and $n = 1, 2 \ldots l$



Transverse Magnetic frequencies:

$$\omega_{nlm} = \frac{y_{ln}\, c}{a} \quad \text{for} \quad \frac{d}{dx}\left[x\, h_1^{(1)}(x)\right] = 0$$

at $l \geq 1$, $|m| \leq 1$, and $n = 1, 2 \ldots l+1$

..where $h_1^{(1)}(x) = \left(\frac{\pi}{2x}\right)\left[J_{l+1/2}(x) \pm i N_{l+1/2}(x)\right]$

Bessel Function

Neumann Function

b) Eigenfrequencies for $l=1$ and $l=2$

$$h_1^{(1)} = -\frac{e^{i\xi}}{\xi}\left(1+\frac{i}{\xi}\right)$$

$$=0 \quad , \xi = -i$$

$$h_2^{(1)} = \frac{ie^{i\xi}}{\xi}\left(1-\frac{3i}{\xi}-\frac{3}{\xi^2}\right)$$

$$=0 \quad , \xi = \pm\frac{\sqrt{3}}{2}-\frac{3i}{2}$$

$$[\xi h_1]' = 0, \quad \xi = \pm\frac{\sqrt{3}}{2}-\frac{i}{2}$$

$$[\xi h_2]' = 0, \quad \xi = -1.596i, \pm 1.807 - 0.702i,$$

Mode	ω	λ/a	$\tau/(a/c)$
TE_{11}	$-i$	Imaginary	$1/2$
TE_{12}	$\frac{\sqrt{3}}{2}-\frac{3i}{2}$	$4\pi/\sqrt{3}$	$1/3$
TM_{11}	$\frac{\sqrt{3}}{2}-\frac{i}{2}$	$4\pi/\sqrt{3}$	1
TM_{12}	$-1.596i$	Imaginary	0.313
TM_{22}	$\pm 1.807-0.702i$	3.476	0.712

when $\omega = \frac{2\pi}{\lambda} - \frac{i}{2i}$