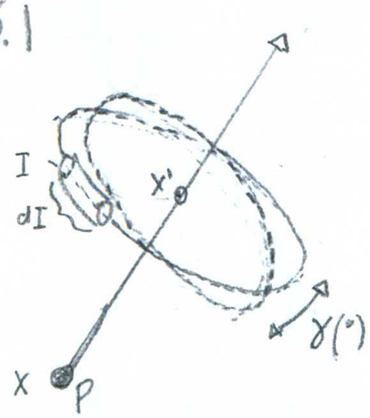
Chapter 5: Magnetostatics:

5.1



Magnetic Induction for a closed loop carrying current

Shape: Loop
Dimension: Area
Charge: q

$$dB = \frac{\mu_0 I}{4\pi} dI \times \frac{(x-x')}{|x-x'|^3}$$

(Equation 5.5) "Magnetic Flux"

$$\frac{4\pi}{\mu_0 I} dB = dI \times \frac{x-x'}{|x-x'|^3}$$

$$\frac{4\pi}{\mu_0 I} \cdot B = \oint dI \times \frac{x-x'}{|x-x'|^3} ds$$

(Equation 1.14.5)
$$\frac{x-x'}{|x-x'|^3} = -\nabla\left(\frac{1}{|x-x'|}\right)$$

$$= \oint dI \times \nabla\left(\frac{1}{|x-x|}\right) ds$$

$$= \int \nabla \times \left[\left(\nabla \frac{1}{|x-x'|}\right) \times \hat{x}\right] \circ \hat{n}\, da'$$

Front Cover
$$\boxed{\nabla \times [a \times b] = a(\nabla \circ b) - b(\nabla \circ a) + (b \circ \nabla)a - (a \circ \nabla)b}$$

$$= \int \left[\left(-\hat{x}\left(\nabla^2 \frac{1}{|x-x'|}\right)\right) + (\hat{x} \circ \nabla)\nabla\left(\frac{1}{|x-x'|}\right)\right] \circ \hat{n}\, da$$

(Equation 1.31)
$$\nabla^2 \frac{1}{|x-x'|} = -4\pi\delta(x-x')$$

$$= -\int \hat{x}(-4\pi\delta(x-x')) \circ \hat{n}\, da + \int (\hat{x} \circ \nabla)\nabla\left(\frac{1}{|x-x'|}\right) \circ \hat{n}\, da$$

$$= \int \frac{\partial}{\partial x'}\left(\nabla\left(\frac{1}{|x-x'|}\right)\right) \circ \hat{n}\, da$$

$$= \frac{-\partial}{\partial x} \int \left(\nabla\left(\frac{1}{|x-x'|}\right)\right) \circ \hat{n}\, da$$

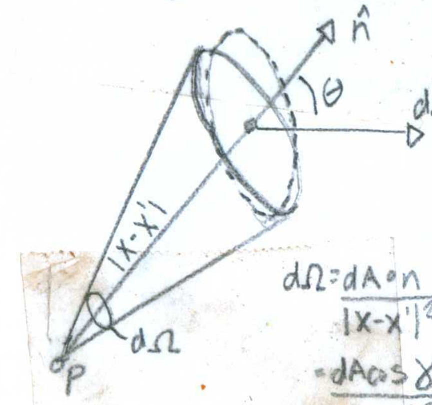
$$= \frac{\partial}{\partial x} \int \left[\frac{\cos\gamma}{|x-x'|^2} \right] da$$

$$= \frac{\partial}{\partial x} \int d\Omega$$

$$= \frac{\partial}{\partial x} \Omega(x)$$

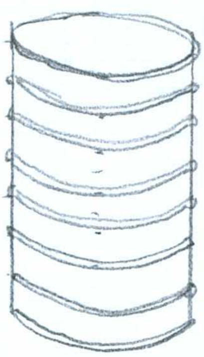
$$B = \frac{\mu_0 I}{4\pi} \nabla \Omega$$

"Solid angle Subtended"



$$d\Omega = \frac{dA \cdot n}{|x-x'|^2} = \frac{dA\cos\gamma}{|x-x'|^2}$$

5.2



A long, cylindrical, ideal solenoid of a large number of loops

Shape: Cylinder
Dimension: Area [2D]
Charge: q

a) From problem 5.1,

$$B = \frac{\mu_0 I}{4\pi} \Omega$$

$$dB = \frac{\mu_0 I}{4\pi} d\Omega$$

$$dB_R = \sum_{i=1}^{N} \frac{\mu_0 I}{4\pi} d\Omega$$

$$= \frac{\mu_0 N I}{4\pi} d\Omega$$

$$B_R = \frac{\mu_0 N I}{4\pi} \int \frac{dA\cos\theta}{r^2}$$

$$= \frac{\mu_0 N I}{4\pi} \int_0^{2\pi} \frac{\rho}{(\rho^2+z^2)^{3/2}} d\phi$$

$$= \frac{\mu_0 N I}{2} \frac{\rho}{(\rho^2+z^2)^{3/2}}$$

$$\lim_{z \to 0} B_R = \frac{\mu_0 N I}{2}$$

$$B_{TOT} = B_L + B_r$$
$$= \mu_0 N I$$
$$= \mu_0 \cdot H$$
$$H = N I$$

(Equation 5.84)

$$B = \mu \cdot H$$

Magnetic Induction — Magnetic permeability — Magnetic Field

b) (Equation 5.25) "Ampere's Law"

$$\oint B \cdot dl = \mu_0 I$$

$$\int_0^{2\pi} B \rho \, d\phi = \mu_0 I$$

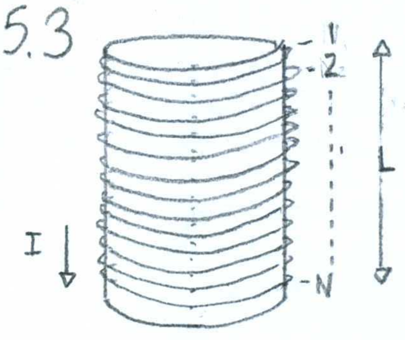
$$B 2\pi\rho = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi\rho}$$

$$= \frac{\mu_0 N I}{2\pi}$$

The book says radius(ρ) is a on a coil and Number loops x radius >>1. So $N \cdot a >> 1$, and $\rho = a = \frac{1}{N}$

5.3



A right-circular Solenoid

Shape: Cylinder

Dimension: Area [2D]

Charge: q

(Equation 5.14) "Biots-Savart law"

$$B(x) = \frac{\mu_0}{4\pi} \int J(x) \times \frac{X - x'}{|X - x'|^3} d^3x$$

Current Density

$$J = I \frac{\delta(r - r')}{r'} \delta(\theta - \theta') \hat{\Phi}$$

$$= \frac{\mu_0}{4\pi} \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{I \cdot \frac{\delta(r-r')}{r'} \delta(\theta - \theta') \hat{\Phi} \times |X - x'| \; r^2 \sin\theta \, dr \, d\theta \, d\phi}{(r^2 + r'^2 - 2rr'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi'))^{3/2}}$$

Spherical Unit Vectors

$$\begin{bmatrix} \hat{r} \\ \hat{\Theta} \\ \hat{\Phi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \times \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

$$= \frac{\mu_0}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{r'^2 \sin\theta' \, I\cdot\delta(r-r')/r \cdot \delta(\theta-\theta')[-\sin\theta\,\hat{i} + \cos\phi\,\hat{j}]}{(r^2 + r'^2 - 2rr'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi'))^{3/2}}$$

$$\times (r\sin\theta\cos\phi - r'\sin\theta'\cos\phi')\hat{i} + (r\sin\theta\cos\phi - r'\sin\theta'\sin\phi')\hat{j} + (r\cos\theta - r'\cos\theta')\hat{k}$$

$$= \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{r\sin\theta\, d\phi\big((r\cos\theta - r_i\cos\theta)(\cos\theta\,\hat{i} + \sin\phi\,\hat{j}) + (r\sin\theta - r\sin\theta\cos(\phi-\phi'))\hat{k}\big)}{(r^2 + r'^2 - 2rr'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')))^{3/2}}$$

$$= \frac{\mu_0 I}{4\pi}\frac{\sin\theta_i}{r_i}\left[-\cos\theta_i\left(\hat{i}\int_0^{2\pi}\cos\phi' d\phi' + \hat{j}\int_0^{2\pi}\sin\phi\, d\phi'\right) + \sin\theta\,\hat{k}\int_0^{2\pi} d\phi\right]$$

Unit Vector Multiplication

$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$

$\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$

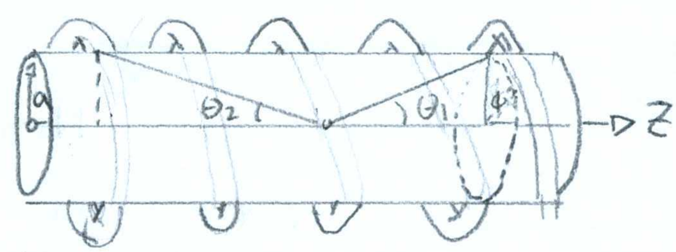
$\hat{j}\times\hat{i} = -\hat{k}$, $\hat{k}\times\hat{j} = -\hat{i}$, $\hat{i}\times\hat{k} = -\hat{j}$

ooo near archaic

$$= \frac{\mu_0 I}{2}\frac{\sin^2\theta}{r_i} \quad @\ r\sin\theta = a$$

$$= \frac{\mu_0 I a^2}{2 r_i^3}$$

$$= \frac{\mu_0 I a^2}{2}\frac{1}{(a^2+z^2)^{3/2}}$$



Problem sets evaluate solenoids as singular rings or continuous loops in polar or spherical coordinates, respectively

$$B_{TOT} = \sum_{i=1}^{N} \frac{\mu_0 I a^2}{2}\frac{1}{(a^2+z^2)^{3/2}}\Delta z$$

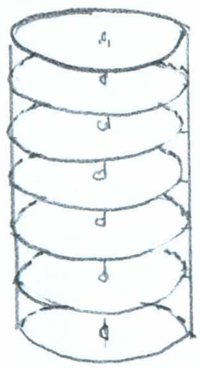
$$= \int_{z_1}^{z_2} \frac{\mu_0 I a^2 \cdot N}{2(a^2+z^2)^{3/2}} dz$$

$$= \frac{\mu_0 I a^2 N}{2}\left[\frac{z_2}{a^2\sqrt{a^2+z^2}} - \frac{z_1}{a^2\sqrt{a^2+z^2}}\right]$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$= \frac{\mu_0 N I}{2}[\cos\theta_2 + \cos\theta_1]$$

5.4.



A current-free region in a uniform, and cylindrical medium

Shape: Cylinder
Dimension: Area [2D]
Charge: q

(Equation 5.17)

$\nabla \circ B = 0$ "Macroscopic Induction, no current"

(Equation 5.26)

$\nabla \times B = \mu_0 J$

$= 0$ "With no magnetic fields"

Cylindrical Divergence

$$\nabla \circ \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \rho} + \frac{\partial A_z}{\partial z}$$

Cylindrical Curl

$$\nabla \times \vec{A} = \begin{vmatrix} \frac{1}{\rho}\hat{\rho} & \hat{\phi} & \frac{1}{\rho}\hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$B_z(\rho, z) = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} b_n(z)$$

$$B_\rho(\rho, z) = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} c_n(z)$$

$$\nabla \circ B = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\rho) + \frac{\partial B_z}{\partial z}$$

$$= \sum_{n=0}^{\infty} \frac{(n+1)\rho^{n-1}}{n!} c_n(z) + \frac{\rho^n}{n!} b_n'(z)$$

$$= \frac{1}{\rho} c_0(z) + \sum_{n=1}^{\infty} \frac{\rho^n}{n!}\left(\frac{n+2}{n+1} c_{n+1}(z) + b_n'(z)\right)$$

$$= 0 \text{, so } c_0(z) = 0 \text{ and } c_{n+1}(z) = \frac{-(n+1)}{n+2} b_n'(z)$$

$$\nabla \times B = \frac{\partial}{\partial z} B_\rho - \frac{\partial}{\partial \rho} B_z$$

$$= \sum_{n=0}^{\infty} \frac{\rho^n}{n!} c_n'(z) + \frac{\rho^{n-1}}{(n-1)!} b_n(z)$$

$$= \sum_{n=0}^{\infty} \frac{\rho^n}{n!} \left(c_n'(z) - b_{n+1}(z) \right)$$

$$= 0 \,, \quad b_{n+1}(z) = c_n'(z) = \frac{-n}{n+1} b_n''(z)$$

$$b_n(z) = (-1)^{n/2} \frac{(n-1)(n-3)\cdots 3\cdot 1}{n(n-2)\cdots 4\cdot 2} b_0^{(n)}(z)$$

$$= \frac{(-1)^{n/2}}{2^n} \frac{n!}{[(n/2)!]^2} b_0^{(n)}(z)$$

$$c_{n+1}(z) = \frac{(-1)^{n/2+1}}{2^n} \frac{(n+1)!}{(n+2)!\,[(n/2)!]^2} b_0^{(n+1)}(z)$$

$$B_z(\rho, z) = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} b_n(z)$$

$$= \sum_{n=0}^{\infty} \frac{\rho^n}{n!} \frac{(-1)^{n/2}}{2^n} \frac{n!}{[(n/2)!]^2} b_0^{(n)}(z)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^k}{4^k} \frac{\rho^{2k}}{(k!)^2} \left[\frac{\partial^{2k} B_z(0,z)}{\partial z^{2k}} \right]$$

$$B_\rho(\rho, z) = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} c_n(z)$$

$$= \sum_{n=0}^{\infty} \frac{\rho^{n+1}}{(n+1)!} \frac{(-1)^{n/2+1}}{2^n} \frac{(n+1)!}{(n+2)[(n/2)!]^2} b_0^{(n+1)}(z)$$

$$= \sum_{n=0}^{\infty} \frac{\rho^{2k+1}}{(2k+2)(k!)^2} \frac{(-1)^{k+1}}{4^k} \left[\frac{\partial^{2k+1} B_z(0,z)}{\partial z^{2k+1}} \right]$$

b) $$\frac{b_{n+2}(z)}{b_n(z)} = \frac{\rho^{n+2} \frac{(-1)^{n/2+1}}{2^{n+2}} \frac{(n+2)!}{[(n/2+1)!]^2} b_0^{(n)}(z)}{\rho^n \frac{(-1)^{n/2}}{2^n} \frac{n!}{[(n/2)!]^2} b_0^{(n)}(z)}$$

The method equates two expansion's and Coefficients; $\nabla \cdot B = \nabla \times B = 0$,

Also, a method found in Physical Chemistry: Virial expansions, along with Nonlinear Dynamics: Poincaré-Lindstedt's method.

$$= \rho^2 \frac{\partial^{n+2} B_z(0,z)/\partial z^{n+2}}{\frac{(n+1)(n+2)}{(n/2+1)\circ 2^2} \partial^n B_z(0,z)/\partial z^n}$$

$$\ll 1$$

$$\rho \ll \sqrt{\frac{(n+1)(n+2)}{(n/2+1)\circ 4} \frac{\partial^n B_z(0,z)/\partial z^n}{\partial^{n+2} B_z(0,z)/\partial z^{n+2}}}$$

5.5 a) From Problem 5.4, $B_{\vec{z}}(\rho,z) = B_z(0,z) - \left(\frac{\rho^2}{4}\right)\left[\frac{\partial^2 B_z(0,z)}{\partial z^2}\right] + \cdots$

$$B_{\vec{\rho}}(\rho,z) = -\left(\frac{\rho}{2}\right)\frac{\partial B_z(0,z)}{\partial z} + \left(\frac{\rho^3}{16}\right)\left[\frac{\partial^3 B_z(0,z)}{\partial z^3}\right] + \cdots$$

From Problem 5.3,

$$B_z(\rho,z) = \frac{\mu_0 I N}{2}\left[\cos\theta_2 + \cos\theta_1\right]$$

$$= \frac{\mu_0 I a^2 N}{2}\left[\frac{z_2}{a^2\sqrt{a^2+z^2}} - \frac{z_1}{a^2\sqrt{a^2+z^2}}\right]$$

$\circ\circ\circ$ When $\rho = a$, ?

@ $\rho = 0$, $z_2 = L/2$, $z_1 = -L/2$

$$B_z(0,z) = \frac{\mu_0 I \cdot N}{2}\left[\frac{L/2}{z} + \frac{L/2}{z}\right]$$

$$= \frac{\mu_0 I \cdot N \cdot L}{2z}$$

$$B_{\vec{z}}(\rho,z) = B_z(0,z) - \frac{\rho^2}{4}\left[\frac{\partial^2 B_z(0,z)}{\partial z^2}\right] + \cdots$$

$$\cong \frac{\mu_0 I N L}{2z}\left[1 - \frac{\rho^2}{2z^2}\right]$$

$$B_{\vec{\rho}}(\rho,z) = -\left(\frac{\rho}{2}\right)\frac{\partial B_z(0,z)}{\partial z} + \left(\frac{\rho^3}{16}\right)\left[\frac{\partial^3 B_z(0,z)}{\partial z^3}\right] + \cdots$$

$$= \frac{\mu_0 I N L}{2 z^2}\left[\frac{\rho^2}{2 z} - \frac{3\rho^3}{8 z}\right]$$

b) (Equation 5.26) "Ampere's Law"

$$\nabla \times B = \mu_0 J$$

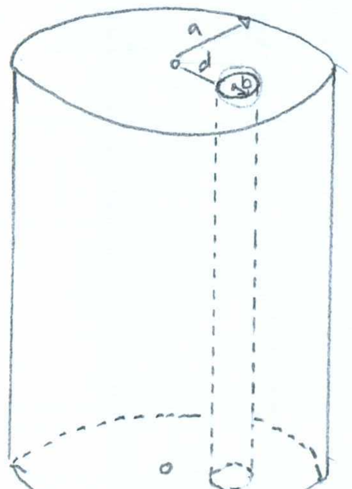
c) $B_{\vec{z}}(z,\rho) \cong \frac{\mu_0 I N L}{2 z}$ ... @ $z = L$

$$\cong \frac{\mu_0 I N}{2}$$

$B_{\vec{\rho}}(z,\rho) \cong \frac{\mu_0 I N L}{2 z^2}\left[\frac{\rho^2}{2}\right]$ ... @ $z = L$

$$\cong \frac{\mu_0 I N \rho^2}{4 L}$$

5.5



A cylindrical Conductor with a parallel bored hole

Shape: Cylinder
Dimension: Volume [3D]
Charge: q

(Equation 5.26) "Amperes Law"

$\nabla \times B = \mu_0 J$ "Differential"

$\oint_c B \cdot dl = \mu_0 \int_s J \cdot n \, da$ "Integral"

$$B\oint_C dl = \mu_0 J_0 \int da$$

$$B\int_0^{2\pi}\int_0^a dr\,d\phi = \mu_0 J_0 \int_0^{2\pi}\int_0^a r\,dr\,d\phi$$

$$B_R = \frac{1}{2}\mu_0 J_0 a\hat{\phi}$$

$$= \frac{1}{2}\mu_0 J_0 a(-y\hat{\imath} + x\hat{\jmath})$$

$$B_L = -\frac{1}{2}\mu_0 J_0 a(-y\hat{\imath} + x\hat{\jmath})$$

$$= -\frac{1}{2}\mu_0 J_0 a(-y\hat{\imath} + (x-d)\hat{\jmath})$$

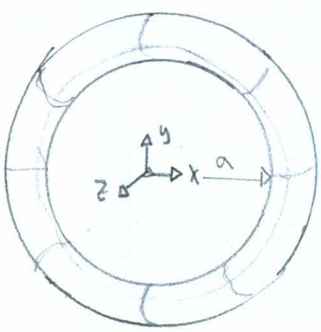
$$B_{TOT} = B_L + B_R$$

$$= -\frac{1}{2}\mu_0 J_0 a(-y\hat{\imath} + (x-d)\hat{\jmath}) + \frac{1}{2}\mu_0 J a(-y\hat{\imath} + x\hat{\jmath})$$

$$= \frac{d}{2}\mu_0 J_0 a\cdot\hat{\jmath} \quad \text{.. as long as } (b+d) < a$$

5.7

a) (Equation 5.4) $dB = kI\dfrac{(dI \times X)}{|X|^3}$ ... $k = \dfrac{\mu_0}{4\pi}$



Compact, circular coil.

Shape: Ring
Dimension: Area [2D]
Charge: q

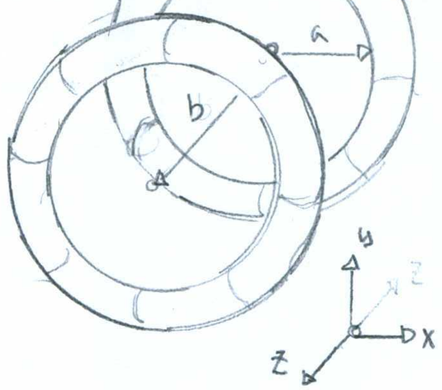
$$B = kI\int \frac{dI}{R^3} \quad \text{... } R = a^2 + z^2$$

$$= \frac{\mu_0 I}{4\pi}\int \frac{dl\,R\sin\alpha}{R^3}$$

$$= \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\int_0^a \frac{a}{R^3}\,dr\,d\phi$$

$$= \frac{\mu_0 I}{2}\cdot\frac{a^2}{(\sqrt{a^2+z^2})^3}$$

b)



If $z = b/2$, $R = d$

$$= a^2 + \frac{b^2}{4}$$

$$B_{TOT} = B_{coil\#1} + B_{coil\#2}$$

$$= \frac{\mu_0 I a^2}{2}\left[\frac{1}{\left(\sqrt{a^2+(z-b/2)^2}\right)^3} + \frac{1}{\left(\sqrt{a^2+(z+b/2)^2}\right)^3}\right]$$

$$= \frac{\mu_0 I a^2}{2}\left[\frac{1}{\left(\sqrt{a^2+b^2/4+z^2-zb}\right)^3} + \frac{1}{\sqrt{a^2+b^2/4+z^2+zb}}\right]$$

$$= \frac{\mu_0 I a^2}{2}\left[\frac{1}{\left(\sqrt{d^2+z^2-zb}\right)^3} + \frac{1}{\left(\sqrt{d^2+z^2+zb}\right)^3}\right]$$

$$= \frac{\mu_0 I a^2}{2\cdot d^2}\left[\frac{1}{\left(\sqrt{1+z^2/a^2-zb/d^2}\right)^3} + \frac{1}{\left(\sqrt{1+z^2/d^2+zb/d^2}\right)^3}\right]$$

$$= \frac{\mu_0 I a^2}{2\cdot d^2}\left[\frac{1}{\left(\sqrt{1+\frac{z^2(a^2+\frac{1}{4}b^2)}{d^4}-\frac{bz}{d^2}}\right)^3} + \frac{1}{\left(\sqrt{1+\frac{z^2(a^2+\frac{1}{4}b^2)}{d^4}+\frac{bz}{d^2}}\right)^3}\right]$$

$$= \frac{\mu_0 I a^2}{2\cdot d^2}\left[1+\frac{3bx}{2}+\frac{3}{2}x^2(b^2-a^2)+\frac{5}{4}x^3(b^3-3a^2b)+\frac{15}{16}x^4(2a^4-6a^2b^2+b^4)\right.$$
$$\left.+1-\frac{3bx}{2}+\frac{3}{2}x^2(b^2-a^2)-\frac{5}{4}x^3(b^3-3a^2b)+\frac{15}{16}x^4(2a^4-6a^2b^2+b^4)\right]$$

... where $x = z/d^2$

$$= \left(\frac{\mu_0 I a^2}{a^3}\right)\left[1+\frac{3(b^2-a^2)z^4}{2d^4}+\frac{15(b^4-6b^2a^2+2a^4)z^4}{16d^8}+\ldots\right]$$

c) I tried a method for magnetostatics, similar to electrostatics. Without Laplaces equation, intermediate steps involve Gauss's Law for magnets and Ampere's Law.

Magnetic Induction by Boundary Conditions

① Boundary Conditions:

(Equation 5.17) $\nabla \cdot B = 0$ "Gauss' Law" -No current-

(Equation 5.22) $\nabla \times B = \mu_0 J$ "Ampere's Law" -

$= 0$ -No magnetic fields-

$$\left.\begin{array}{l} B_z(0,z) = b(z) \\ B_\rho(0,z) = 0 \end{array}\right\}$$ Magnetic induction points in the z-direction by an amount.

② Series Expansion:

$$B_z(\rho,z) = \sum_{n=0}^{\infty} b_n(z)\rho^n \qquad B_\rho(\rho,z) = \sum_{n=0}^{\infty} c_n(z)\rho^n$$

③ General Solution to Series Expansion:

$b_0(z)$ $\quad B_z(0,z) = \sum_{n=0}^{\infty} b_n(z)\rho^n$

$$= b_0(z) + b_1(z)\rho + b_2(z)\rho^2$$

$$= b(z), \text{ so } b_0(z) = b(z)$$

$c_0(z)$ $\quad B_\rho(0,z) = \sum_{n=0}^{\infty} c_n(z)\rho^n$

$$= c_0(z) + c_1(z)\rho + c_2(z)\rho^2 + \cdots$$

$$= 0, \text{ so } c_0(z) = 0$$

$$B_z(\rho,z) = b(z) + \rho b_1(z) + \rho^2 b_2(z) + \cdots$$

$$B_\rho(\rho,z) = \rho c_1(z) + \rho^2 c_2(z) + \cdots$$

$c_1(z), c_2(z)$ $\quad \nabla\cdot B = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho B_\rho + \frac{\partial}{\partial z}B_z$

$$= 2c_1(z) + 3\rho c_2(z) + b'(z) + \rho b_1'(z) + \cdots$$

$$= 0 \text{, so } c_1(z) = -\frac{1}{2}b'(z) \text{ and } c_2(z) = \frac{-1}{3}b_1'(z)$$

$b_1(z), b_2(z)$ $\quad \nabla\times B = \frac{\partial}{\partial z}B_\rho - \frac{\partial}{\partial\rho}B_z$

$$= \rho c_1'(z) - b_1(z) - 2\rho b_2(z) + \cdots$$

$$= 0 \text{, so } b_1(z) = 0 \text{ and } b_2(z) = \frac{1}{2}c_1'(z)$$

$$B_z(\rho,z) = b(z) - \frac{1}{4}\rho^2 b''(z) + \cdots$$

$$B_\rho(\rho,z) = -\frac{1}{2}\rho b'(z) + \cdots$$

If $b(z) = \sigma_0 + \sigma_2 z^2$, then

$$B_z(\rho,z) = \sigma_0 + \sigma_2\left(z^2 - \frac{1}{2}\rho^2\right) + \cdots$$

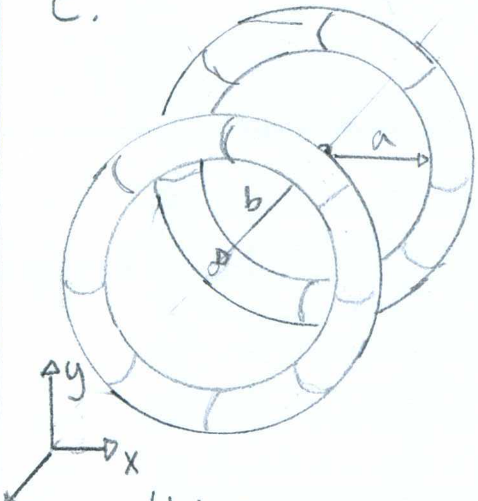
$$B_\rho(\rho,z) = -\sigma_2 \rho z$$

d) From part b,

$$B_z(\rho,z) = \frac{\mu_0 I a^2}{2|z|^3}\left[\frac{1}{\left(\sqrt{1-b/|z|+(a^2+b^2/4)/|z|^2}\right)^3} - \frac{1}{\left(\sqrt{1+b/|z|+(a^2+b^2/4)/|z|^2}\right)^3}\right]$$

…when $d = |z|$

e.



Helmholtz Coils

Shape: Rings
Dimension: Area [2D]
Charge: q

$\sigma_z = 0$

$$B_z(\rho, z) = \frac{\mu_0 I a^2}{d^3}\left[1 + \frac{3(b^2-a^2)z^2}{2d^4} + \frac{15(b^4-6b^2a^2+2a^4)z^4}{16d^8}\right]$$

$\circ\circ$ when $d = a^2 + (z + b^2/4)^2$

and $b = a$

$$= \frac{\mu_0 I a^2}{\left(\frac{8}{5\sqrt{5}} a^3\right)}\left[1 + \frac{45 a^4 z^4}{16\left(\frac{256}{625} a^8\right)}\right]$$

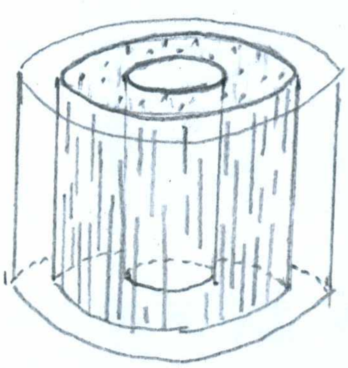
The second order term defines dB/B, so that

$$\frac{dB}{B} = \frac{343}{50}\left(\frac{z}{a}\right)^4$$

A Helmholtz coil magnetic field shifts by 1 part in 10,000 with a $z/a < 6.10\%$.

While, coils separation near $z/a \sim 19.5\%$ changes the magnetic field by 1 part in 100.

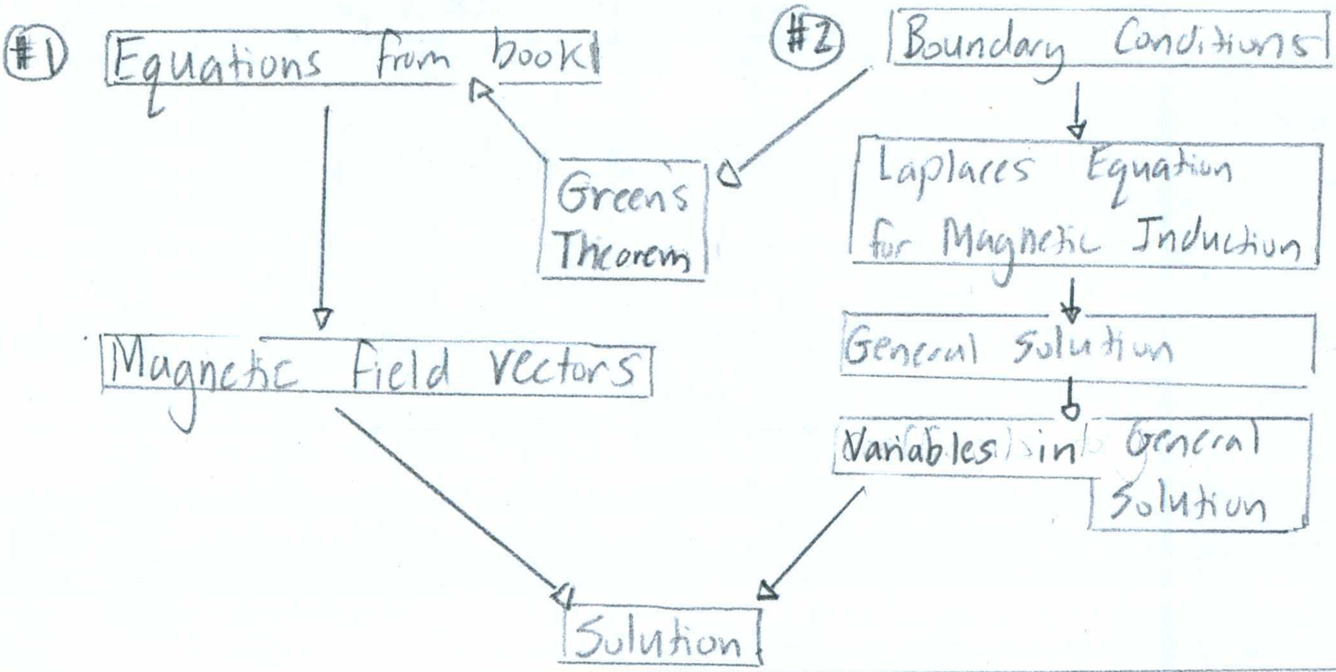
5.8.



A localized cylindrically symmetric current distribution

Shape: Cylinder
Dimension: Area [2D]
Charge: q

methods possible: vectors or boundary conditions



The left column (#1) is new to chapter 5, while the right (#2) in chapters 1, 2, 3, and 4. Problem 3.1 spits an image about general solution regularity and right column (#2).

The vector method has symmetry arguments:

a) (Equation 5.16) $B(x) = \frac{\mu_0}{4\pi} \nabla \times \int \frac{\vec{J}(x')}{|x - x'|} d^3x$

(Equation 5.27) $B(x) = \nabla \times A(x)$

(Equation 5.32) $A(x) = \frac{\mu_0}{4\pi} \nabla \times \int \frac{\vec{J}(x')}{|x - x'|} d^3x$

(Equation 3.70) $\frac{1}{|x - x'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi)$

$$A(x) = \frac{\mu_0}{4\pi} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \int \frac{r_<^l}{r_>^{l+1}} \vec{J}(r, \theta', \phi') Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) r'^2 dr' d\phi' d\cos\theta'$$

$$= \frac{\mu_0}{4\pi} \sum \sum \frac{4\pi}{2l+1} \int \frac{r_<^l}{r_>^{l+1}} \hat{\phi} \vec{J}(r, \theta) Y_{lm}^*(\theta', 0) Y_{lm}(\theta, 0) \cos(m(\phi - \phi')) r'^2 dr' d\phi' d\cos\theta'$$

$$\text{Spherical unit vectors}$$

$$\begin{bmatrix}\hat{r}\\ \hat{\theta}\\ \hat{\phi}\end{bmatrix} = \begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\phi\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\phi\\ -\sin\phi & \cos\phi & 0\end{bmatrix} \times \begin{bmatrix}\hat{i}\\ \hat{j}\\ \hat{k}\end{bmatrix}$$

$$A(x) = \frac{\mu_0}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{4\pi}{2l+1}\int \frac{r_<^l}{r_>^{l+1}}\vec{J}(r,\theta)\,Y_{lm}(\theta',0)\,Y_{lm}^*(\theta,0)\,r'^2\,dr\,d\cos\theta' \int_0^{2\pi}\hat{\phi}'\cos(m(\phi-\phi'))\,d\phi'$$

$$\boxed{\hat{\phi}' = \sin(\phi-\phi')\sin\theta\,\hat{r} + \sin(\phi-\phi')\cos\theta\,\hat{\theta} + \cos(\phi-\phi')\,\hat{\phi}}$$

$$= \frac{\mu_0}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\int \frac{r_<^l}{r_>^{l+1}}\vec{J}(r,\theta)\,Y_{lm}(\theta',0)\,Y_{lm}^*(\theta,0)\,r^2\,dr\,d\cos\theta' \int_0^{2\pi}\left(\sin(\phi-\phi')\sin\theta\,\hat{r} + \sin(\phi-\phi')\cos\theta\,\hat{\theta} + \cos(\phi-\phi')\,\hat{\phi}\right)\times \cos(m(\phi-\phi'))\,d\phi$$

$$= \frac{\mu_0}{4\pi}\sum_{l=0}^{\infty}\frac{4\pi}{2l+1}\int \frac{r_<^l}{r_>^{l+1}}\vec{J}(r,\theta)\,Y_{lm}(\theta',0)\,Y_{lm}^*(\theta,0)\,r^2\,dr\,d\cos\theta' \left[\pi\left(\delta_{m,1}+\delta_{m,-1}\right)\hat{\phi}\right]$$

$$= \frac{\mu_0}{4\pi}\sum_{l=0}^{\infty}\frac{4\pi}{2l+1}2\pi\hat{\phi}\int \frac{r_<^l}{r_>^{l+1}}\vec{J}(r,\theta)\,Y_{lm}(\theta',0)\,Y_{lm}^*(\theta,0)\,r^2\,dr\,d\cos\theta' \qquad @\,m=1$$

$$= \frac{\mu_0}{4\pi}\sum_{l=0}^{\infty}\frac{P_l^1(\cos\theta)}{l(l+1)}\int \frac{r_<^l}{r_>^{l+1}}\vec{J}(r,\theta)\,P_l^1(\cos\theta')\,d^3x$$

Interior:

$$A(x) = \frac{\mu_0}{4\pi}\sum_{l=0}^{\infty}\frac{r_<^l}{l(l+1)}P_l^1(\cos\theta)\int \frac{\vec{J}(r,\theta)}{r_>^{l+1}}P_l^1(\cos\theta)\,d^3x$$

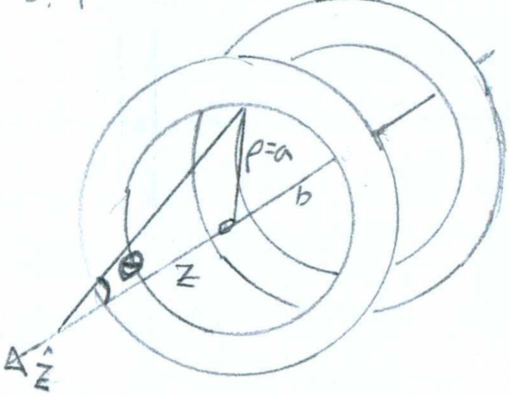
Exterior:

$$A(x) = \frac{\mu_0}{4\pi}\sum_{l=0}^{\infty}\frac{P_l^1(\cos\theta)}{l(l+1)\,r_>^{l+1}}\int r_<^l\cdot\vec{J}(r,\theta)\,P_l^1(\cos\theta)\,d^3x$$

b)

$$m_l = \frac{-1}{l(l+1)}\int \frac{P_l^1(\cos\theta)}{r^{l+1}}\vec{J}(r,\theta)\,d^3x \qquad \mu_l = \frac{-1}{l(l+1)}\int r^l P_l^1(\cos\theta)\,\vec{J}(r,\theta)\,d^3x$$

5.9



$$\vec{J} = \hat{\phi}\cdot I\cdot\delta(\rho-a)\left[\delta(z-b/2)+\delta(z+b/2)\right]$$

a) $$m_\ell = \frac{-1}{\ell(\ell+1)}\int \frac{1}{r_>^{\ell+1}} P_\ell^1(\cos\theta)\,\vec{J}(r,\theta)\,d^3x$$

$$= \frac{-1}{\ell(\ell+1)}\int \frac{1}{(\rho^2+z^2)^{(\ell+1)/2}} P_\ell^1\left(\frac{z}{\sqrt{\rho^2+z^2}}\right)\cdot I\cdot\delta(\rho-a)\left[\delta(z-b/2)+\delta(z+b/2)\right] d^3x$$

$$= \frac{-I}{\ell(\ell+1)d^{\ell+1}}\int_0^a \delta(\rho-a)\rho\, d\rho \int_0^{2\pi} d\phi \int_0^{b/2} P_\ell^1\left(\frac{z}{\sqrt{\rho^2+z^2}}\right)\left[\delta(z-b/2)+\delta(z+b/2)\right] dz$$

$$= \frac{-2\pi\cdot a\cdot I}{\ell(\ell+1)d^{\ell+1}}\left[P_\ell^1(b/2d)+P_\ell^1(-b/2d)\right]$$

$$\cos\theta = \frac{z}{\sqrt{z^2+\rho^2}}$$

$$r = \sqrt{z^2+\rho^2}$$

$$a=\rho,\; d=a^2+\frac{b^2}{4}$$

$$z=b/2$$

$$\mu_\ell = \frac{-1}{\ell(\ell+1)}\int r^\ell P_\ell^1(\cos\theta)\,\vec{J}(r,\theta)\,d^3x$$

$$= \frac{-1}{\ell(\ell+1)}\int (\rho^2+z^2)^{\ell} P\left(\frac{z}{\sqrt{\rho^2+z^2}}\right)\cdot I\,\delta(\rho-a)\cdot\left[\delta(z-b/2)+\delta(z+b/2)\right] d^3x$$

$$= \frac{-I}{\ell(\ell+1)}\int_0^a \delta(\rho-a)\rho\, d\rho \int_0^{2\pi} d\phi \int_0^{b/2} P_\ell^1\left(\frac{z}{\sqrt{\rho^2+z^2}}\right)\left[\delta(z-b/2)+\delta(z+b/2)\right] dz$$

$$= \frac{-2\pi I a}{\ell(\ell+1)} d^\ell \left[P_\ell^1(b/2d)+P_\ell^1(-b/2d)\right]$$

Associated Legendre Polynomials

$$P_\ell^m(x) = (-1)^m (1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x)$$

Note: The associated Legendre Polynomials consider the symmetry. An $m=0$ defines the common legendre polynomials.

ℓ	1	2	3	4	5
m_ℓ	$\frac{2\pi I a^2}{d^3}$	0	$\frac{2\pi I a^2}{d^5}\,\frac{b^2-a^2}{4d^2}$	0	$\frac{2\pi I a^2}{d^7}\,\frac{b^4-6a^2b^2+2a^4}{16d^4}$
μ_ℓ	$2\pi I a^2$	0	$2\pi I a^2\,\frac{b^2-a^2}{4}$	0	$2\pi I a^2\,\frac{b^4-6a^2b^2+2a^4}{16}$

b) $A_\phi = \frac{-\mu_0}{4\pi}\left[m_1 r P_1'(\cos\theta) + m_3 r^3 P_3'(\cos\theta) + m_5 r^5 P_5'(\cos\theta)\right]$

$$= \frac{\mu_0}{4\pi}\sin\theta\left[m_1 r + m_3 r^3 \frac{3}{2}(5\cos^2\theta - 1) + m_5 r^5 \frac{15}{8}(21\cos^4\theta - 14\cos^2\theta + 1) + \cdots\right]$$

$B = \nabla \times A_\phi$

$$= \hat{r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \hat{\theta}\frac{1}{r}\frac{\partial}{\partial r}(r A_\phi)$$

$$= \frac{\mu_0}{2\pi}\Big[\hat{r}\cos\theta\Big(m_1 + 3m_3 r^2(5\cos^2\theta - 3) + \frac{15}{8}m_5 r^4(63\cos^4\theta - 70\cos^2\theta + 15) + \cdots\Big)$$

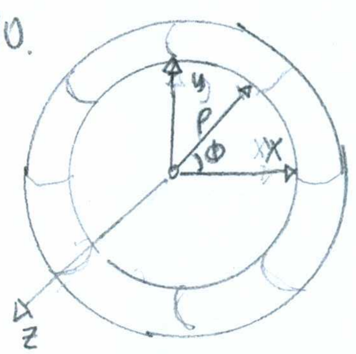
$$- \hat{\theta}\sin\theta\Big(m_1 + 3m_3 r^2(5\cos^2\theta - 1) + \frac{45}{8}m_5 r^4(21\cos^4\theta - 14\cos^2\theta + 1) + \cdots\Big]$$

$\cdots$ when $\theta = 0°$

$$= \frac{\mu_0}{2\pi}(m_1 + 6m_3 z^2 + 15 m_5 z^4 + \cdots)$$

$$= \frac{\mu_0}{d^3}\left[1 + \frac{3}{2}\frac{b^2 - a^2}{d^2}\left(\frac{z}{d}\right)^2 + \frac{15}{16}\frac{b^4 - 6a^2b^2 + 2a^4}{d^4}\left(\frac{z}{d}\right)^4 + \cdots\right]$$

5.10.



A circular current loop

Shape: Ring
Dimension: Area [2D]
Charge: q

a) $\boxed{\text{Magnetic Induction by Boundary Conditions:}}$

① Boundary Conditions:

$A(\rho = 0, \Phi, z) = \text{finite}$

$A(\rho = \infty, \Phi, z) = 0$

$A(\rho = a, \Phi = 0, z) = \mu_0 I$

② Laplace's Equation:

$$\nabla A = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\Phi^2} + \frac{\partial^2\Phi}{\partial^2 z}$$

$$= 0$$

③ Laplace's Equation Solutions:

If $A(\rho, \phi, z) = R(\rho)Q(\phi)Z(z)$,

Ⓐ Variable Separation:

$$\nabla A = \frac{1}{\rho}\frac{\partial}{\partial r}\left(\rho \cdot \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial^2 z}$$

$$= \frac{Q(\phi)Z(z)}{\rho}\frac{\partial}{\partial \rho}\left(\rho \cdot \frac{\partial R(\rho)}{\partial \rho}\right) + \frac{R(\rho)Z(z)}{\rho^2}\frac{\partial^2 Q(\phi)}{\partial^2 \phi} + R(\rho)Q(\phi)\frac{\partial^2 Z(z)}{\partial^2 z}$$

$$= \frac{1}{R(\rho)\cdot \rho}\frac{\partial}{\partial r}\left(\rho \cdot \frac{\partial R(\rho)}{\partial \rho}\right) + \frac{1}{Q(\phi)\rho^2}\frac{\partial^2 Q(\phi)}{\partial^2 \phi} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z}$$

$$= 0$$

Ⓑ Radial Eigenvalues:

$$\frac{\rho}{R}\frac{\partial R}{\partial \rho} + \frac{\rho^2}{R}\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z} = \lambda \rho^2$$

$$\frac{\rho}{R}\frac{\partial R}{\partial \rho} + \frac{\rho^2}{R}\frac{\partial^2 R}{\partial \rho^2} - \lambda \rho^2 = \frac{-1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 \phi}$$

$$\frac{\rho}{R}\frac{\partial R}{\partial \rho} + \frac{\rho^2}{R}\frac{\partial^2 R}{\partial \rho^2} - \lambda \rho^2 = k^2$$ ... if $r = x$ and $\lambda = \frac{1}{r^2}$, then

$$\frac{\partial^2 R}{\partial^2 x} + \frac{1}{x}\frac{\partial R}{\partial x} - \left(1 + \frac{k^2}{x^2}\right)R = 0$$

Ⓒ Angular Eigenvalues:

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial^2 \phi} = m^2 \quad : \quad \frac{\partial^2 Q}{\partial^2 \phi} + m^2 Q = 0$$

Ⓓ Vertical Eigenvalues:

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = k^2 \qquad \frac{\partial^2 Q}{\partial^2 \phi} - k^2 Z = 0$$

Ⓔ General Solution:

$$A(\rho, \phi, z) = \sum_{k,m} R(\rho)Q(\phi)Z(z)$$

④ General Solution to Laplace's Equation:

$R(\rho) = g_m(k, \rho, \rho')$ "Radial solution, Wronskian" (Equation 3.142)

$Q(\phi) = Ae^{im\phi}$

$Z(z) = Be^{-|k|z}$

⑤ Variables by Boundary Conditions:

$$A(\rho, \phi, z) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} g_m(k, \rho, \rho') A e^{im\phi} \cdot B e^{-|k|z}$$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A g_m(k, \rho, \rho') e^{im\phi} e^{-|k|z}$$

$$A \quad A(\rho = a, \phi = 0, z) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A g_m(k, \rho, \rho') e^{im \cdot 0} \cdot e^{-|k|z}$$

$$= \mu_0 I$$

$$\int_0^{\infty}\int_0^{2\pi}\int_0^{\infty} \mu_0 I \, dz \, d\phi \, d\rho = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} 2 \cdot A \int_0 g(k, \rho, \rho') d\rho \int_0^{\infty} \cos(kz) \, dz \int_0^{2\pi} d\phi$$

$$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} 2 \cdot A \cdot \pi \, (2\pi) \cdot g(k, \rho, \rho')$$

$$A = \frac{\mu_0 I a}{2\pi} \int_0^{\infty} dz$$

(Equation 3.142)

$$\boxed{\begin{aligned} g_m(k, \rho, \rho') &= \psi_1(k\rho) \psi_2(k\rho) \\ &= I_m(k\rho) K(k\rho) \end{aligned}}$$

$$A(\rho, \phi, z) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{\mu_0 I a}{2\pi} \int_0 e^{-|k|z} \cdot g_m(k, \rho, \rho') \, dz$$

$$= \frac{\mu_0 I a}{\pi} \int_0^{\infty} \cos(kz) I_1(k\rho) K_1(k\rho) \, dk$$

b) From part a)

$$A_\phi(\rho, z) = \frac{\mu_0 I a}{\pi}\int_0^\infty dk \cos(kz)\, I_1(k\rho)\, K_1(k\rho)$$

$$= \frac{\mu_0 I a}{\pi}\int_0^\infty dk \frac{e^{-k|z|}}{2}\cdot(i\, J_1(ik\rho))\left(\frac{\pi}{2} i^2 \cdot H_1^{(1)}(ix)\right)$$

$$= \frac{\mu_0 I a}{2}\int_0^\infty dk \cdot e^{-k|z|}\cdot J_1(k\rho)\cdot(J_1(k\rho) + i N(k\rho))$$

$$= \frac{\mu_0 I a}{2}\int_0^\infty dk\, e^{-k|z|} J_1(k\rho)\, J_1(k\rho)$$

(Equation 3.100)

$I_\nu(x) = i^\nu J_\nu(ix)$

(Equation 3.101)

$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix)$

(Equation 3.86)

$H_\nu^{(1)}(x) = J_\nu(x) + i N_\nu(x)$

@ $J_1(ix) = -i J(x)$

@ $N_1(ix) = 0$

c) $B = \nabla \times A$

$$= -\frac{\partial A_\rho}{\partial z} + \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi)$$

$$= B_\rho \hat{\rho} + B_z \hat{z}$$

$$B_\rho(\rho, z) = \frac{-\partial A}{\partial z}$$

$$= \frac{\mu_0 I a}{\pi}\int_0^\infty dk \cdot k \sin(kz)\, I_1(k\rho)\cdot K_1(k\rho)$$

$$B_\rho(0, z) = \frac{\mu_0 I a}{\pi}\int_0^\infty dk \cdot k \cdot \sin(kz)\, I_1(k\cdot 0)\cdot K_1(k\cdot 0)$$

$$= 0$$

$$B_z(\rho, z) = \frac{\partial A_\phi}{\partial \rho} + \frac{1}{\rho} A_\phi$$

$$= \frac{\mu_0 I a}{\pi}\int_0^\infty dk \cos(kz)\left\{\frac{\partial}{\partial \rho} + \frac{1}{\rho}\right\} I_1(k\rho)\, K_1(k\rho)$$

$$B_z(0, z) = \frac{\mu_0 I a}{\pi}\int_0^\infty dk \cos(kz)\left\{\frac{\partial}{\partial \rho} + \frac{1}{\rho}\right\} I_1(k0)\, K_1(k\cdot 0)$$

$$= \frac{\mu_0 I a^2}{2}\frac{a\pi}{2}\frac{1}{(a^2+z^2)^{3/2}}$$

$$= \frac{\mu_0 I a}{2}\frac{1}{(a^2+z^2)^{3/2}}$$

$$B_\rho(\rho,z) = \frac{-\partial A}{\partial \rho}$$

$$= \frac{\mu_0 I a}{2}\int_0^\infty dk\, J_1(ka)\, J_1(k\rho)\frac{\partial}{\partial z} e^{-k|z|}$$

$$B_\rho(0,z) = \frac{\mu_0 I a}{2}\int_0^\infty dk\, J_1(k\cdot a) J_1(k\cdot 0)\;\frac{\partial}{\partial z} e^{-k|z|}$$

$$= 0$$

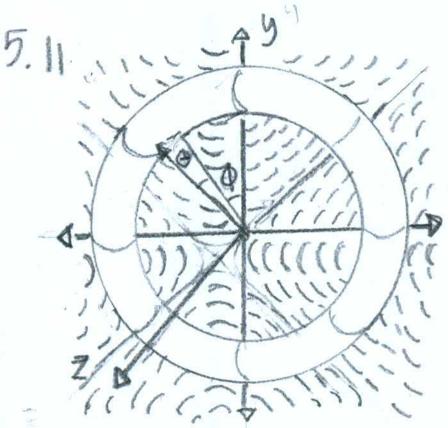
$$B_z(\rho,z) = \frac{\partial A}{\partial \rho} + \frac{1}{\rho}A_\phi$$

$$= \frac{\mu_0 I a}{2}\int_0^\infty dk\, e^{-k|z|} J_1(ka)\left\{\frac{\partial}{\partial \rho} + \frac{1}{\rho}\right\} J_1(k\rho)$$

$$B_z(0,z) = \frac{\mu_0 I a}{2}\int_0^\infty dk\, e^{-k|z|} J_1(k\cdot a)\left\{\frac{\partial}{\partial \rho} + \frac{1}{\rho}\right\} J_1(k\cdot 0)$$

$$= \frac{\mu_0 I a}{2}\int_0^\infty dk\cdot k e^{-k|z|} J_1(k\cdot a)$$

5.11



A circular loop of wire in a magnetic field

Shape: Ring
Dimension: Area [2D]
Charge: q

a) (Equation 5.69)

$$F = (m \times \nabla) \times B$$

$$= \nabla(m\cdot B) - m(\nabla\cdot B) \quad \ldots \text{When } \nabla\cdot B = 0 \text{ "No current"}$$

$$= \nabla(m\cdot B)$$

(Equation 5.57)

$$|m| = I \times |\text{Area}|$$

$$= I\cdot\pi a^2\cdot(\hat{x}+\hat{y}+\hat{z})$$

$$= I\cdot\pi\cdot a^2\cdot(\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k})$$

Spherical to Cartesian

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{bmatrix}$$

$$F = \nabla(m \cdot B)$$

$$= \nabla(I\pi a^2(\sin\theta\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{k}) \cdot (B_x + B_y))$$

$$= I\pi a^2\,\nabla(\sin\theta\cos\phi\, B_0(1+\beta y)\hat{\imath} + \sin\theta\sin\phi\, B_0(1+\beta x)\hat{\jmath})$$

$$F_x = I\pi a^2 B_0 \beta \sin\theta\sin\phi$$

$$F_y = I\pi a^2 B_0 \beta \sin\theta\cos\phi$$

$$F_z = 0$$

b) (Equation 5.71)

$$N = m \times B(0)$$

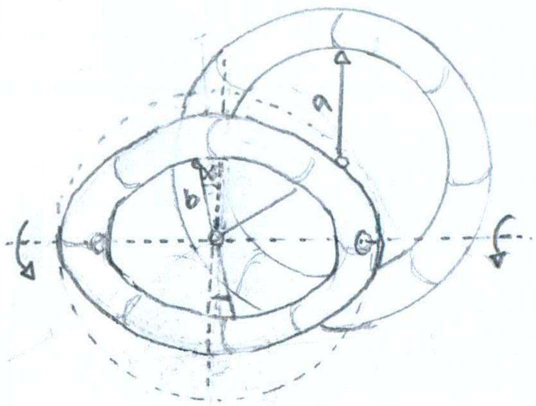
$$= \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\phi \\ B_x(0) & B_y(0) & 0 \end{bmatrix}$$

$$= -\cos\theta(B_0(1+\beta\cdot 0))\hat{\imath} + \cos\theta(B_0(1+\beta\cdot 0))\hat{\jmath} + [\sin\theta\cos\phi(B_0(1+\beta 0)) - \sin\theta\sin\phi(B_0(1+\beta\cdot 0))]\hat{k}$$

$$= -B_0\cos\theta\,\hat{\imath} + B_0\cos\theta\,\hat{\jmath} + B_0[\sin\theta\cos\phi - \sin\theta\sin\phi]\hat{k}$$

Higher order terms in the magnetic induction as with (equation 5.65), $B_k(x) = B_k(0) + x \cdot \nabla B_k(0)$, approaches zero by the initial magnetic induction $B_x = B_0(1+\beta y)$ and $B_y = B_0(1+\beta x)$.

5.12



(Equation 5.70)

$$N=\int x'\times[J\times B(0)]\,d^3x$$

$$=\int[(x'\cdot B)J-(x'\cdot J)B]\,d^3x$$

$$=\int(x\cdot B)J\,d^3x$$ ... because $x'\cdot J=0$ from the loculized current distribution in Equation 5.52.

Two concentric circular loops with an angle between their planes

Shape: Rings

Dimension: Area [2D]

Charge: q

$$J(r)=I\cdot\delta(r-b)\cdot\delta(\theta-\pi/2)[-\sin\theta\,\hat{i}+\cos\phi\,\hat{j}]$$

$$N=\int_0^b\int_0^{\pi/2}\int_0^{2\pi} r\,I\cdot\delta(r-b)\cdot\delta(\theta-\pi/2)\,[-\sin\phi\,\hat{i}+\cos\phi\,\hat{j}]\,\vec{B}\,d\phi\,d\theta\,dr$$

Now $\vec{B}$ is necessary, from boundary conditions:

Magnetic Induction Derivation by Boundary Conditions:

① Boundary Conditions:

$$A(\rho=0,\theta,\phi)=0 \qquad A(\rho=r,\theta,\phi)=\mu_0 I$$

② Laplace's Equation:

$$\nabla^2 A=\frac{1}{\rho}\frac{\partial^2}{\partial\rho^2}(\rho A)+\frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A}{\partial\theta}\right)+\frac{1}{\rho^2\sin\theta}\frac{\partial^2 A}{\partial\phi^2}$$

$$=0$$

③ Laplace's Equation Solutions:

Ⓐ Variable Separation: If $A(\rho,\theta,\phi)=U(\rho)P(\theta)Q(\phi)$

$$\nabla^2 A=\frac{1}{\rho}\frac{\partial^2}{\partial\rho^2}(\rho A)+\frac{1}{\rho^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A}{\partial\theta}\right)+\frac{1}{\rho^2\sin\theta}\frac{\partial^2 A}{\partial\phi^2}$$

$$=\frac{\rho^2\sin^2\theta}{U(\rho)P(\theta)Q(\phi)}\left[\frac{P(\theta)Q(\phi)}{\rho}\frac{\partial^2U(\rho)}{\partial\rho^2}+\frac{U(\rho)Q(\phi)}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right)+\frac{U(\rho)P(\theta)}{\rho^2\sin\theta}\frac{\partial^2Q(\phi)}{\partial\phi^2}\right]=0$$

(B) Radial Eigenvalues:

$$\frac{\rho^2\sin^2\theta}{U(\rho)}\frac{\partial^2 U(\rho)}{\partial^2\rho} + \frac{\sin^2\theta}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) = m$$

$$\frac{\rho^2}{U(\rho)}\frac{\partial^2 U(\rho)}{\partial^2\rho} + \frac{1}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) = \frac{m}{\sin^2\theta}$$

$$\frac{\rho^2}{U(\rho)}\frac{\partial^2 U(\rho)}{\partial^2\rho} = \frac{m}{\sin^2\theta} - \frac{1}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right)$$

$$= \ell(\ell+1)$$

$$\frac{\partial^2 U(\rho)}{\partial^2\rho} - \frac{\ell(\ell+1)U(\rho)}{\rho^2} = 0$$

(C) Angular Eigenvalues:

$$\frac{m}{\sin^2\theta} - \frac{1}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) = \ell(\ell+1)$$

$$\frac{1}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) + \left(\ell(\ell+1) - \frac{m}{\sin^2\theta}\right) = 0$$

(D) Azimuthal Eigenvalues:

$$\frac{1}{Q(\phi)}\frac{\partial^2 Q(\phi)}{\partial\phi^2} = -m^2$$

$$\frac{\partial^2 Q(\phi)}{\partial\phi^2} + m^2 Q(\phi) = 0$$

(4) General Solution to Laplace's Equation:

$$U(\rho) = A\rho^{\ell+1} + B\rho^{-\ell}$$

$$P(\theta) = C\cdot P_\ell^m(\cos\theta)$$

$$Q(\phi) = De^{im\phi}$$

$$A(\rho,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} U(r)\cdot P(\theta)\cdot Q(\phi)$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_\ell \rho^{\ell+1} + B_\ell \rho^{-\ell-1}\right) \cdot C P_\ell^m(\cos\theta) \cdot D e^{im\phi}$$

Spherical Harmonics

$$\boxed{Y_\ell^m(\theta,\phi) = P_\ell^m(\cos\theta) e^{im\phi}}$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_\ell \rho^{\ell+1} + B_\ell \rho^{-\ell-1}\right) Y_\ell^m(\theta,\phi)$$

(5) Variables by Boundary Conditions:

B_ℓ $$A(\rho=0,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_\ell \cdot 0 + B_\ell \frac{1}{0}\right) \cdot Y_\ell^m(\theta,\phi)$$

$$= 0 \qquad , \text{ so } B_\ell = 0$$

A_ℓ $$A(\rho=r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_\ell \cdot r^{\ell+1} \cdot Y_\ell^m(\theta,\phi)$$

$$= \mu_0 I$$

$$\int_0^a \int_0^{2\pi} \int_0^{\pi} \mu_0 I \, Y_\ell^m(\theta,\phi) \sin\theta \, d\theta \, d\phi \, dr = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_\ell \cdot r^{\ell+1} \int_0^a \int_0^{2\pi} \int_0^{\pi} Y_{\ell m}(\theta,\phi) Y_{\ell m}^*(\theta,\phi) \sin\theta \, d\theta \, d\phi \, dr$$

$$= (2\ell+1) \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_\ell \cdot r^{\ell+1} \delta_{\ell\ell} \cdot \delta_{mm} \delta_{rr}$$

$$A_\ell = \mu_0 I \frac{1}{r^{\ell+1}} \frac{1}{2\ell+1} \int_0^a \int_0^{2\pi} \int_0^{\pi} Y_\ell^m(\theta,\phi) \sin\theta \, d\theta \, d\phi \, dr$$

$$A(\rho,\theta,\phi) = 2\pi \mu_0 I a \sum_{\ell=0}^{\infty} \frac{Y_{\ell,1}(\theta,0)}{2\ell+1} \frac{r^\ell}{r^{\ell+1}} \left[Y_{\ell,1}\left(\frac{\pi}{2}, 0\right) \right]$$

$$= \frac{-\mu_0 I a}{4} \sum_{0=0}^{\infty} \frac{r^{2n+1}}{r^{2n+2}} \frac{(-1)^n}{(2n+1)(2n+2)} \frac{\Gamma(n+3/2)}{\Gamma(n+1)\Gamma(3/2)} P_{2n+1}(\cos\theta)$$

Gamma Relationships:

$$\Gamma(n+3/2) = \frac{(2n+3)!}{(2n+3)!}$$

even:

$$\Gamma\left(\frac{n}{2}\right) = \left(\frac{n}{2}-1\right)!$$

odd:

$$\Gamma\left(\frac{n}{2}\right) = \sqrt{\pi} \frac{(n-1)!}{2^{n-1}\left(\frac{n-1}{2}\right)!}$$

$$= -\frac{\mu_0 I a}{4} \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{2^n (n+1)!} \frac{r^{2n+1}}{r^{2n+2}} P_{2n+1}(\cos\theta)$$

⑥ Magnetic Induction:

(Equation 5.38)

$$B_r = \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta A_\phi)$$

$$\boxed{\begin{aligned} &\ldots \text{ when } \sin\theta = \sqrt{1-x^2} \\ &\text{(Equation 5.47)} \\ &\frac{d}{dx}\left[\sqrt{1-x^2}\, P_l'(x)\right] = l(l+1) P_l'(x) \end{aligned}}$$

$$= \frac{1}{r\sin\theta} \frac{-\mu_0 I' a}{4} \sum_{l=0}^{\infty} \frac{(-1)^n l(l+1)(2n-1)!!}{2^n (n+1)!} \frac{r_<^{2n+1}}{r_>^{2n+2}} P_{2n+1}^1 (\cos\theta)$$

$$= \frac{\mu_0 I' a}{2r} \sum_{l=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n n!} \frac{r_<^{2n+1}}{r_>^{2n+2}} P_{2n+1}(\cos\theta)$$

Back to torque,

$$N = \int_0^b \int_0^{\pi/2} \int_0^{2\pi} r \cdot I \cdot \delta(r-b) \cdot \delta(\theta - \pi/2) \left[-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right] \frac{\mu_0 I a}{2r} \sum_{l=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n \cdot n!} \frac{r^{2n+1}}{r^{2n+2}} P_{2n+1}(\cos\theta)\, d\phi\, d\theta\, dr$$

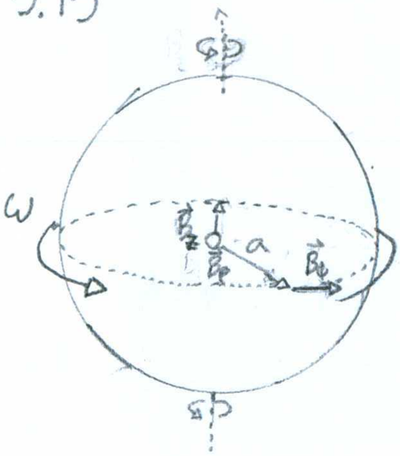
$$= I b^2 \int_0^{2\pi} \frac{\mu_0 I' a}{2r} \sum \frac{(-1)(2n+1)!!}{2^n \cdot n!} \frac{r^{2n+1}}{r^{2n+2}} \left\{ P_{2n+1}(0) P_{2n+1}(\cos\alpha) + 2 \sum P_{2l+1}^m(0) P_{2l+1}^m(\cos\alpha) \cos m\phi \right\}$$

$$\times \left[-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right] d\phi$$

$$= \frac{\pi \mu_0 I I' b^2}{a} \sum_{l=0}^{\infty} \frac{(-1)^l (2l+1)!!}{2^n \cdot l!} \left(\frac{b}{a}\right)^{2l} P_{2l+1}^1(0) P_{2l+1}^1(\cos\alpha)$$

$$= \frac{\pi \mu_0 I I' b}{a} \sum_{l=0}^{\infty} (l+1)^2 \left[\frac{\Gamma(l+3/2)}{\Gamma(l+2)\,\Gamma(3/2)}\right]^2 \left(\frac{b}{a}\right)^{2l} P_{2l+1}^1(\cos\alpha)$$

5.13



A rotating sphere

Shape: sphere
Dimension: Area [2D]
Charge: σ

(Equation 5.57.5)

$$\vec{J} = \sum_i q_i v_i \delta(x - x_i)$$

$$= \sigma \cdot \omega a \sin\theta \delta(r-a)\hat{\Phi}$$

Angular Velocity
$\omega = \frac{v}{r}$

(Equation 5.32)

$$A(x) = \frac{\mu_0}{4\pi}\int \frac{J(x')}{|x - x'|} d^3x$$

$$= \frac{\mu_0}{4\pi}\int \frac{\sigma\omega a \sin\theta \delta(r-a)\hat{\Phi}}{|x - x'|} d^3x$$

(Equation 3.70)... Green's theorem.

$$\frac{1}{|x - x'|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta,\phi) Y_{lm}(\theta,\phi)$$

Note: The full steps are in Problem 5.12, the previous problem.

$$A(x) = \frac{\mu_0}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \sigma\omega a \sin\theta \delta(r-a)\hat{\phi}\cdot 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\phi) Y_{lm}(\theta,\phi) r^2 \sin^2\theta \, dr\, d\theta\, d\phi$$

$$= \mu_0\sigma\omega \sum\sum \frac{Y_{lm}(\theta,\phi)}{2l+1}\int_0^a \frac{r_<^l}{r_>^{l+1}} \cdot r^2 dr \int_0^{2\pi}\int_0^{\pi} (-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}) Y_{lm}^*(\theta,\phi)\sin^2\theta \, d\theta\, d\phi$$

Inside: $r = r_>$

Outside: $r = r_>$

Inside: $$A(x) = \mu_0\sigma\omega \sum\sum \frac{1}{2l+1}\frac{r^l}{a^{l-2}} Y_{lm}(\theta,\phi)\int_0^{2\pi}\int_0^{\pi} (-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}) Y_{lm}(\theta,\phi)\sin^2\theta \, d\theta\, d\phi$$

Outside: $$A(x) = \mu_0\sigma\omega \sum\sum \frac{1}{2l+1}\frac{a^{l+3}}{r^{l+1}} Y_{lm}(\theta,\phi)\int_0^{2\pi}\int_0^{\pi} (-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}) Y_{lm}(\theta,\phi)\sin^2\theta \, d\theta\, d\phi$$

$$\int_0^{2\pi}\int_0^{\pi}(-\sin\phi\,\hat{i}+\cos\phi\,\hat{j})\,Y_{\ell m}(\theta,\phi)\sin^2\theta\,d\theta\,d\phi$$

$$=\sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell\mp1)}{(\ell\pm1)}}\,\pi\left(\pm i\hat{i}+\hat{j}\right)\int_0^{\pi}P_1^{\pm1}(\cos\theta')\sin^2\theta\,d\theta$$

$x=\cos\theta \quad dx=-\sin\theta\,d\theta$

$\sin\theta=\sqrt{1-x^2}=P_\ell^1(x)=-2P_1^{-1}(x)$

$$=\sqrt{\frac{2\pi}{3}}\left(i\hat{i}\pm\hat{j}\right) \quad \ldots \text{when } \ell=\pm1$$

Inside:

$$A(x)=\frac{\mu_0\sigma\omega}{3}\,a r\sin\theta\,\hat{\phi}$$

$$B=\nabla\times A$$

$$=\hat{r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)-\frac{1}{r}\frac{\partial}{\partial r}(r\cdot A_\phi)\,\hat{\phi}$$

$$=\frac{2}{3}\mu_0\sigma\omega\,\hat{z}$$

Outside:

$$A(x)=\frac{\mu_0\sigma\omega}{3}\frac{a^4}{r^2}\sin\theta\,\hat{\phi}$$

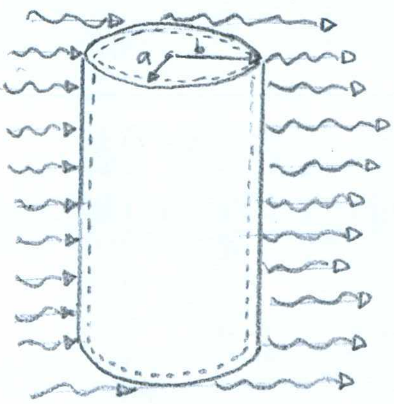
$$B=\nabla\times A$$

$$=\hat{r}\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)-\frac{1}{r}\frac{\partial}{\partial r}(r\cdot A_\phi)\,\hat{\phi}$$

$$=\frac{\mu_0\sigma\omega}{3}\frac{a^4}{r^3}\left[3\cos\theta\,\hat{r}-\hat{z}\right]$$

5.14.

Magnetic Flux by Boundary Conditions:



A long, hollow, right circular cylinder

Shape: cylinder
Dimension: Area [2D]
Charge: q

① Boundary Conditions:

$$A(\rho=0,\phi)=\text{finite} \qquad A(\rho=\infty,\phi=0)=-B\rho\cos\phi$$

② Laplace's Equation:

$$\nabla A^2 = \frac{\partial^2 A}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 A}{\partial \phi^2}$$

$$=0$$

③ Laplace's Equation Solutions:

Ⓐ Variable Separation: If $A(\rho,\phi)=R(\rho)Q(\phi)$, then

$$Q(\phi)\frac{\partial^2 R(\rho)}{\partial^2 \rho} + \frac{R(\rho)}{\rho^2}\frac{\partial^2 Q(\phi)}{\partial^2 \phi} = 0$$

$$R(\rho)\,\rho^2\frac{\partial^2 R(\rho)}{\partial^2 \rho} + \frac{1}{Q(\phi)}\frac{\partial^2 Q(\phi)}{\partial^2 \phi} = 0$$

Ⓑ Radial Eigenvalues:

$$\frac{\partial^2 R(\rho)}{\partial^2 \rho} - \frac{k^2}{\rho^2}R(\rho) = 0$$

Ⓒ Angular Eigenvalues:

$$\frac{\partial^2 Q(\phi)}{\partial^2 \phi} + k^2 Q(\phi) = 0$$

Ⓓ General Solution:

$$A(\rho,\phi)=\sum_{m=1}^{\infty} R(\rho)Q(\phi)$$

④ General Solution to Laplace's Equation:

$R(\rho)=A\rho^{k}+B\rho^{-k}$

$Q(\phi)=Ce^{im\phi}+De^{-im\phi}$

$$A(\rho,\phi)=\sum_{m=1}^{\infty}(A\rho^{m}+B\rho^{-m})(Ce^{im\phi}+De^{-im\phi})$$

⑤ Variables by Boundary Conditions:

Inner ($\rho<a$)

B $A(\rho=0,\phi)=\sum_{m=1}^{\infty}(A0^{m}+B0^{-m})(Ce^{im\phi}+De^{-im\phi})$

$=$ finite, so $B=0$

$$A_I(\rho,\phi)=\sum_{m=1}^{\infty}\rho^{m}(Ce^{im\phi}+De^{-im\phi})$$

Middle ($a<\rho<b$)

$$A_M(\rho,\phi)=\sum_{m=1}^{\infty}(A\rho^{m}+B\rho^{-m})(Ce^{im\phi}+De^{-im\phi})$$

Outer: ($b<\rho$)

A, B $A(\rho=\infty,\phi=0)=\sum_{m=1}^{\infty}(A\infty^{m}+B\infty^{-m})(Ce^{im\phi}+De^{-im\phi})$

$=-B_0\infty\cos(0)$

If $m=1$, $C_1=D_1=-\dfrac{B_0}{2A_1}$

If $m>1$, $A_c=0$

$$A_o(\rho,\phi)=-B_0\rho\cos\phi+\sum_{m=1}^{\infty}\rho^{-m}(Ce^{im\phi}+De^{-im\phi})$$

$$\frac{\partial A_o}{\partial\rho}=\frac{\partial A_M}{\partial\rho}\quad @\rho=b,\ \phi=0$$

$$-B_0b\cos\phi+\sum_{m=1}^{\infty}(-m)b^{-m-1}(C_oe^{im\phi}+D_oe^{-im\phi})=\sum_{m=1}^{\infty}(A_M\cdot m b^{m-1}-B_M m b^{-m-1})(C_Me^{im\phi}+D_Me^{-im\phi})$$

$@m=1$ $\quad \dfrac{C_o}{b^2}+(A_M-b^{-2})C_M=-\dfrac{B_0}{2}$ $\quad$ and $C_o=D_o$, $C_M=D_M$

$@m>1$ $\quad C_o+(A_Mb^{2m}-1)C_M=0$

$$D_o+(A_Mb^{2m}-1)D_M=0$$

$$\frac{\partial A_o}{\partial\phi}=\frac{1}{\mu_r}\frac{\partial A_M}{\partial\phi}\quad @\rho=b,\ \phi=0$$

$$B_0\cdot b\cdot\sin\phi+\sum_{m=1}^{\infty}b^{-m}(C_oime^{im\phi}+imD_oe^{-im\phi})=\frac{1}{\mu_r}\sum_{m=1}^{\infty}(A\rho^{m}+B\rho^{-m})(C_m\cdot ime^{im\phi}-D_M\cdot ime^{-im\phi})$$

$@m=1$ $\quad \dfrac{C_o}{b^2}-\dfrac{1}{\mu_r}(A_M+b^{-2})C_M=\dfrac{B_0}{2}$

$@m>1$ $\quad C_o-\dfrac{1}{\mu_r}(A_mb^{2m}+1)C_m=0$

$$D_o-\frac{1}{\mu_r}(A_mb^{2m}+1)D_m=0$$

$$\frac{\partial A_m}{\partial \rho} = \frac{\partial A_I}{\partial \rho} \quad @\ \rho = a$$

$$\sum_{m=1}^{\infty} (A_m m a^{m-1} - m B_m a^{-m-1})(C_m e^{im\phi} + D_m e^{-im\phi}) = \sum_{m=1}^{\infty} m a^{m-1} (C_I e^{im\phi} + D_I e^{-im\phi})$$

$@m=1 \ (A_m - a^{-2}) C_m = C_I \quad C_I = D_I$ and $B_m = 1$

$@m>1 \ (A_m - a^{-2m}) C_m = C_I$ and $B_m = 1$

$(A_m - a^{-2m}) D_m = D_I$ and $B_m = 1$

$$\frac{1}{\mu_r}\frac{\partial A_m}{\partial \phi} = \frac{\partial A_I}{\partial \phi}$$

$$\frac{1}{\mu_r} \sum_{m=1}^{\infty} (A_m a^m + B_m a^{-m})(C_m i m e^{im\phi} - D_m i m e^{-im\phi}) = \sum_{m=1}^{\infty} a^m (C_I i m e^{im\phi} - D_I i m e^{-im\phi})$$

$@m=1 \ \frac{1}{\mu_r}(A_m - a^{-2}) C_m = C_I$

$@m>1 \ \frac{1}{\mu_r}(A_m + a^{-2m}) C_m = C_I$

$\frac{1}{\mu_r}(A_m + a^{-2m}) D_m = D_I$

$C_0 = 0,\ D_0 = 0,\ C_m = 0,\ D_m = 0\ C_I = 0,\ D_I = 0 \quad @m>1$

$@m=1$

$$A_m = -\frac{1}{a^2}\left(\frac{1+\mu_r}{1-\mu_r}\right) \qquad C_I = -2B_0 \frac{b^2}{a^2} \frac{\mu_r}{\frac{b^2}{a^2}(1+\mu_r)^2 - (1-\mu_r)^2}$$

$= D_m$

$$C_m = -\frac{1}{a^2}\left(\frac{1+\mu_r}{1-\mu_r}\right) \qquad C_0 = -B_0 \frac{b^2}{a^2} \frac{\mu_r}{\frac{b^2}{a^2}(1+\mu_r)^2 - (1-\mu_r)^2}$$

$= D_m \qquad\qquad = D_0$

$$A_o(\rho,\phi) = -B_0\rho\cos\phi - \frac{(1-\mu_r^2)(b^2-a^2)}{2\mu_r ab}\,\frac{2\mu_r ab}{b^2(1+\mu_r)^2-a^2(1-\mu_r)^2}\,\frac{b}{a}B_0 b\cos\phi$$

$$A_m(\rho,\phi) = -\frac{2\mu_r ab}{b^2(1+\mu_r)^2-a^2(1-\mu_r)^2}\left[(1+\mu_r)\frac{\rho}{a}-(1-\mu_r)\frac{a}{\rho}\right]B_0 b\cos\phi$$

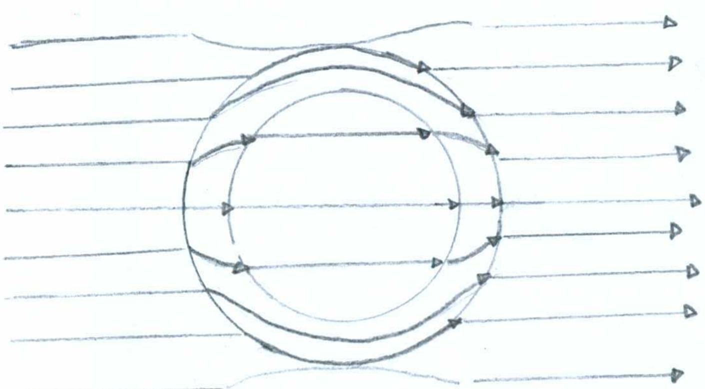
$$A_I(\rho,\phi) = -\frac{b}{a}\,2\cdot\frac{2\mu_r ab}{b^2(1+\mu_r)^2-a(1-\mu_r)^2}\,B_0\rho\cos\phi$$

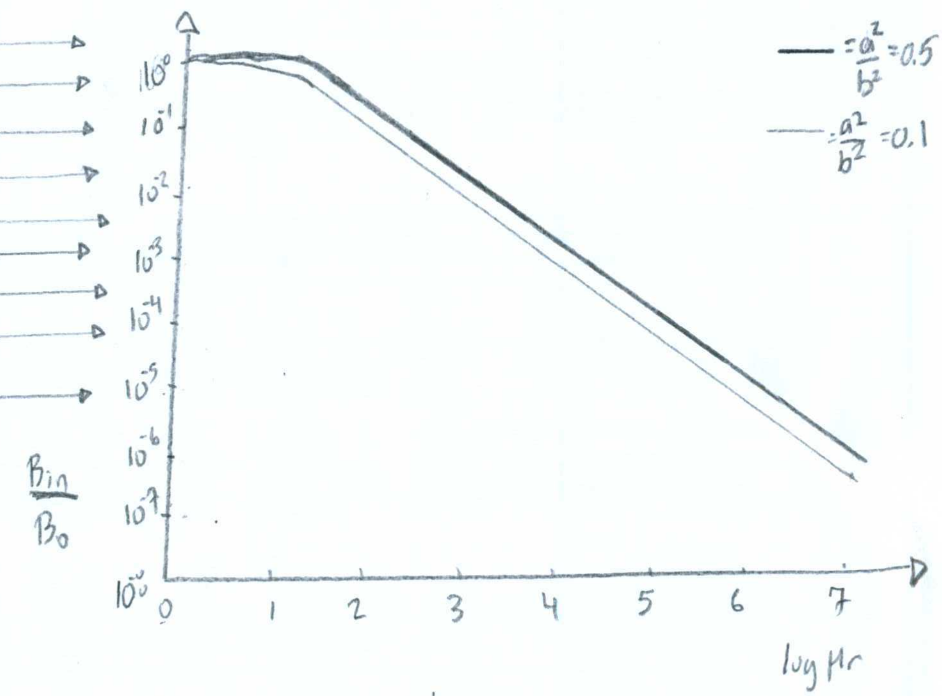
$$B = -\hat{\rho}\frac{\partial A}{\partial\rho} - \hat{\phi}\frac{1}{\rho}\frac{\partial A}{\partial\phi}$$

$$B_{out} = B_0\hat{x} - \frac{(1-\mu_r^2)(b^2-a^2)}{2\mu_r ab}\,\frac{2\mu_r ab}{b^2(1+\mu_r)^2-a^2(1-\mu_r)^2}\,\frac{b^2}{a^2}B_0\left[\hat{x}+2\hat{\phi}\sin\phi\right]$$

$$B_m = \frac{2\mu_r ab}{b^2(1+\mu_r)^2-a^2(1-\mu_r)^2}\,B_0\frac{b}{a}(1+\mu_r)\hat{x} + B_0(1-\mu_r)\frac{ab}{\rho^2}\left[\hat{x}+2\hat{\phi}\sin\phi\right]$$

$$B_I = \frac{b}{a}\,2\cdot\frac{2\mu_r ab}{b^2(1+\mu_r)^2-a^2(1-\mu_r)^2}\,B_0\hat{x}$$

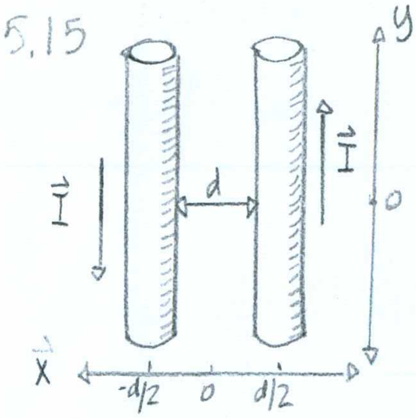




$$\frac{B_{in}}{B_0} = \frac{4\mu_r}{(1+\mu_r)^2-\frac{a^2}{b^2}(1-\mu_r)^2}$$

The magnetic flux never changes until $\mu_r \geq 1$, with logarithmic dropoff.

5.15



Two long, straight wires parallel

$\boxed{\begin{array}{l}\text{Shape: Rod} \\ \text{Dimension: Area [2D]} \\ \text{Charge: q}\end{array}}$

a) (Equation 5.82)

$$\nabla \times H = J \quad \text{"Magnetic Field"}$$

$$\int_s (\nabla \times H) \cdot \hat{n}\, da = \int_s J \cdot \hat{n}\, da$$

$$\oint_c H \cdot dl = I_{ac}$$

$$H \oint dl = I_{ac}$$

$$H = \frac{I}{\int_0^{2\pi}\int_0^{\rho} d\rho\, d\phi}$$

$$= \frac{I}{2\pi\rho}\hat{\phi}$$

(Equation 5.93)

$$H = -\nabla \Phi$$

$$= \frac{-1}{\rho}\frac{\partial \Phi}{\partial \rho}$$

$$= \frac{I}{2\pi\rho}\hat{\phi}$$

$$\Phi = \frac{-I}{2\pi}\phi$$

$$\Phi_T = \Phi_L - \Phi_R$$

$$= \frac{-I}{2\pi}\arctan\left(\frac{y}{x - d/2}\right) + \frac{I}{2\pi}\arctan\left(\frac{y}{x + d/2}\right)$$

$$= \frac{I}{2\pi}\arctan\left(\frac{\rho\sin\phi}{\rho\cos\phi - d/2}\right) + \arctan\left(\frac{-\rho\sin\phi}{\rho\cos\phi + d/2}\right)$$

$$= \frac{-I}{2\pi} \tan^{-1}\left(\frac{\sin\phi \, \frac{d}{\rho}}{1 - \frac{1}{4}\left(\frac{d}{\rho}\right)^2} \right)$$

Trigonometric Identity:

$$\tan^{-1} A + \tan^{-1} B = \tan^{-1}\left[\frac{A+B}{1-AB}\right]$$

$$= \frac{-I}{2\pi}\left[\frac{\sin\phi\left(\frac{d}{\rho}\right)}{1-\frac{1}{4}\left(\frac{d}{\rho}\right)^2} - \left(\frac{\sin\phi \, \frac{d}{\rho}}{1-\frac{1}{4}\left(\frac{d}{\rho}\right)^2} \right)^3 \Big/ 3 + \ldots \right]$$

Taylor Series

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

$$= -\frac{I d \sin\phi}{2\pi\rho} \quad \ldots \text{when } d \ll \rho$$

c) Conditions: $\mu_r \gg 1$ and $b = a + t$ and $t \ll b$

$\mu_r = 200$, $b = 1.25\text{cm}$, $t = 3\text{mm}$ @ 60Hz, 110 or 220V

$$F = \frac{4\mu_r b^2}{(\mu_r+1)^2 b^2 - (\mu_r-1)a^2}$$

$$= \frac{4\mu_r b^2}{(\mu_r+1)^2 - (\mu_r+1)\left(\frac{a}{b}\right)^2}$$

$$= \frac{4\mu_r}{(\mu_r+1)^2 - (\mu_r-1)^2\left(1 - 2\left(\frac{t}{b}\right) + \left(\frac{t}{b}\right)^2\right)}$$

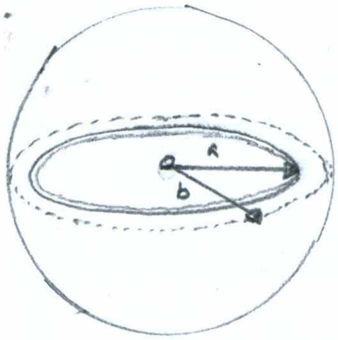
$$= \frac{1}{1 + \frac{(\mu_r-1)^2}{4\mu_r}\left[2\left(\frac{t}{b}\right) - \left(\frac{t}{b}\right)^2\right]} \quad \ldots \text{when } t/b \ll 1 \text{ and } \mu_r \gg 1$$

$$\cong \frac{1}{1 + \frac{(\mu_r-1)^2}{2\mu_r}\left(\frac{t}{b}\right)}$$

$$\cong \frac{1}{1 + \frac{\mu_r \cdot t}{2b}}$$

$$\cong \frac{1}{1 + \frac{(200)(0.3\text{cm})}{2\cdot(1.25\text{cm})}} \approx 0.04$$

5.16.



A circular loop in a spherical Cavity

Shape: sphere, loop
Dimension: Volume [3D]
Charge: q

a) (Equation 5.32)

$$A(x) = \frac{\mu_0}{4\pi}\int \frac{J(x)}{|x-x'|} d^3x$$

$$= \frac{\mu_0}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{\hat{\phi} I \delta(\theta-\pi/2)\delta(r-a)/a}{|x-x'|} r'^2 \sin\theta dr' d\theta d\phi'$$

$$= \frac{\mu_0}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{\hat{\phi} I \delta(\theta-\pi/2)\delta(r-a)}{4\pi a} \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y(\theta,\phi) Y(\theta,\phi)$$

$$\times r^2 \sin\theta dr d\theta d\phi$$

$$= \mu_0 I a \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} \int_0^{2\pi} \hat{\phi} Y_{lm}^*(\pi/2,\phi) Y_{lm}(\theta,\phi') d\phi$$

$$= \frac{\mu_0 I a}{2}\sum_{l=0}^{\infty} \frac{1}{l(l+1)}\frac{r_<^l}{r_>^{l+1}} P_1'(0) P_1'(\cos\theta)\hat{\phi}$$

$$A_{TOTAL} = A_{Inside} + A_{middle}$$

$$= \frac{\mu_0 I a}{2}\sum_{l=0}^{\infty} \frac{1}{l(l+1)}\frac{r_<^l}{r_>^{l+1}} P_1'(0) P_1'(\cos\theta)\hat{\phi} + \sum_{l=0}^{\infty} A_l r^l P_1'(\cos\theta)\hat{\phi}$$

$$B = \nabla \times A$$

$$= -\frac{1}{r}\frac{\partial}{\partial r}(rA)\Big|_{r_<=a, r_>=b}$$

$$= \sum_{l=0}^{\infty}\left[\frac{\mu_0 I a}{2}\frac{-1}{(l+1)}\frac{a^l}{b^{l+2}} P_1'(0) + A_l(l+1) b^{l-1}\right] P_1'(\cos\theta)$$

$$= 0$$

When $A(l+1) = \frac{\mu_0 I a}{2} \frac{1}{(l+1)^2} \frac{a^l}{b^{2l+1}} P_1'(0)$

$$A = \frac{\mu_0 I a}{2} \sum_{l=0}^{\infty} \frac{1}{l+1} \left[\frac{1}{l} \frac{r_<^l}{r_>^{l+1}} + \frac{1}{l+1} \frac{a^l}{b^l} \frac{r^l}{b^{l+1}} \right] P_1'(0) P_1'(\cos\theta) \hat{\phi}$$

$$= \frac{\mu_0 I}{2} \sum_{l=0}^{\infty} \frac{1}{l(l+1)} \left[1 + \frac{1}{l+1} \left(\frac{a}{b}\right)^{2l+1} \right] \frac{r^l}{a^l} P_1'(0) P_1'(\cos\theta) \hat{\phi}$$

When $l=1$, the factor appears:

$$= \frac{\mu_0 I}{2} \sum_{l=0}^{\infty} \frac{1}{2} \left[1 + \frac{1}{2} \left(\frac{a}{b}\right)^{3} \right] \frac{r}{a} P_1'(0) P_1'(\cos\theta) \hat{\phi}$$

b) When $c > b$,

$$A = \frac{\mu_0 I}{2} \sum_{l=0}^{\infty} \frac{1}{l(l+1)} \left[\frac{1}{a^l} + \frac{1}{c^l} \right] r^l P_1'(0) P_1'(\cos\theta) \hat{\phi}$$

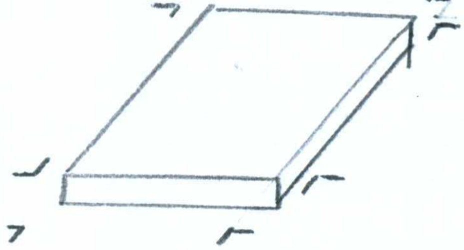
$$= \frac{\mu_0 I}{2} \sum \frac{1}{l(l+1)} \left[1 + \frac{1}{l+1} \left(\frac{a}{b}\right)^{2l+1} \right] \frac{r^l}{a^l} P_1'(\cos\theta) P_1'(\cos\theta) \hat{\phi}$$

$$C = \left[\frac{l+1}{1} \frac{b^{2l+1}}{a^{l+1}} \right]$$

When $l=1$

$$= 2\left(\frac{b^3}{a^2}\right)$$

2.17.



Semi-infinite Slab

Shape : Block
Dimension: Volume [3D]
Charge : q

a) $J_1^*(x,y,z) = A_1 J_1(x,y,-z)$

(Equation 5.16)

$$B(x) = \frac{\mu_0}{4\pi}\int J(x')\times\frac{x-x'}{|x-x'|^3}d^3x$$

Method of Images:

$$B(x) = \frac{\mu_0}{4\pi}\int \frac{J(x')+J^*(x')}{|x-x'|^3}\cdot(x-x')\,d^3x$$

$$= \frac{\mu_0\mu_r a}{4\pi}\int\frac{J(x')\times(x-x')}{|x-x'|^3}d^3x$$

$$[J(x)+J^*(x)](x-x') = \mu_r a J(x')(x-x')$$

$$[J_x - J_y + J_x^* - J_y^*][x-x^*+y-y^*] = \mu_r a[J_x - J_y][x-x^*+y-y^*]$$

$J_y^* = (\mu_r a - 1)J_y$ and $J_x^* = (\mu_r a - 1)J_x$

In the case of no free current,

$$[J(x)+J^*(x)](x-x') = a J(x')(x-x')$$

$$[J_x - J_y - J_z + J_x^* - J_y^* - J_z^*](x-x') = a[J_x - J_y - J_z](x-x')$$

$J_z^* = (a-1)J_z$

$J_x^* = (1-a)J_x$

$J_y^* = (1-a)J_y$

$$a = \frac{2}{\mu_r + 1}$$

$$\left[\frac{\partial}{\partial r}(r \cdot A_{\phi,upper})\right] = \left[\frac{\partial}{\partial r}(r \cdot A_{\phi,lower})\right]_{\theta=\pi/2}$$

$$\mu_0(I + I') = \mu I''$$

Boundary Condition #2

$$n \times H_{upper} = n \times H_{lower}$$

$$\frac{1}{\mu_0}\hat{\theta} \times (\nabla \times A_{upper}) = \frac{1}{\mu}\hat{\theta} \times (\nabla \times A_{lower})$$

$$\frac{1}{\mu_0}\frac{\partial}{\partial \theta}(\sin\theta \cdot A_{upper}) = \frac{1}{\mu}\frac{\partial}{\partial \theta}(\sin\theta \cdot A_{lower})$$

$$I - I' = I''$$

$$I' = \frac{\mu - \mu_0}{\mu + \mu_0} I \qquad I'' = \frac{2\mu_0}{\mu + \mu_0} I$$

$$A_{upper} = \frac{\mu_0 I a}{4\pi}\int_0^{2\pi} d\phi \cos\phi \left[\frac{1}{\sqrt{a^2 + d^2 + r^2 - 2r\cos\theta d - 2ar\sin\theta\cos\phi}} + \left(\frac{\mu - \mu_0}{\mu + \mu_0}\right)\frac{1}{\sqrt{a^2 + d^2 + r^2 + 2r\cos\theta d - 2ar\sin\theta\cos\phi}}\right]$$

$$= \frac{\mu_0 I a}{4\pi}\int_0^{2\pi} d\phi \cos\phi \left[\frac{1}{\sqrt{a^2 + d^2 + \rho^2 + z^2 - 2zd - 2a\rho\cos\phi}} + \left(\frac{\mu - \mu_0}{\mu + \mu_0}\right)\frac{1}{\sqrt{a^2 + d^2 + \rho^2 + z^2 + 2zd - 2a\rho\cos\phi}}\right]$$

(Equation 5.7)

$$dF = I(d\vec{I} \times \vec{B})$$

$$F = I\oint dI \times B$$

$$J_x^* = \left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_x(x, y, -z)$$

$$J_y^* = \left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_y(x, y, -z)$$

$$J_z^* = -\left(\frac{\mu_r - 1}{\mu_r + 1}\right) J_z(x, y, -z)$$

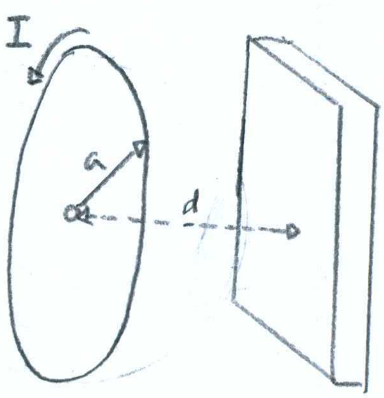
b) When $z < 0$,

$$B(x) = \frac{\mu_0 \mu_{ra}}{4\pi} \int \frac{J(x') \cdot (x - x')}{|x - x'|^3} d^3x$$

$$= \frac{\mu_0 \mu_r}{4\pi} \left(\frac{2}{\mu_r + 1}\right) \int \frac{J(x') \cdot (x - x')}{|x - x'|^3} d^3x$$

$$= \frac{\mu_0}{4\pi} \int \frac{J_{eff}(x') \cdot (x - x')}{|x - x'|^3} d^3x \quad \text{where } J_{eff} = \left(\frac{2\mu_r}{\mu_r + 1}\right)$$

5.18



A circular loop parralel a distance from a slab,

Shape: Wire loop
Dimension: Volume
Charge: q

(Equation 5.16)

$$B(x) = \frac{\mu_0}{4\pi} \nabla \times \int \frac{J(x')}{|x-x'|} d^3x$$

(Equation 5.19)

$$\nabla \times B(x) = \frac{\mu_0}{4\pi} \nabla \int J(x') \cdot \nabla \left(\frac{1}{|x-x'|} \right) d^3x - \frac{\mu_0}{4\pi} \int J(x') \nabla^2 \left(\frac{1}{|x-x'|} \right) d^3x$$

$$= \nabla^2 A$$

$$A = \frac{\mu_0}{4\pi} \int \frac{J(x')}{|x-x'|} d^3x$$

Problem #2.7 is an example about the Cartesian to law of cosines.

$$= \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \frac{\cos\phi' \, d\phi'}{\sqrt{a^2 + r^2 - 2ar\sin\theta\cos\phi}}$$

$$= \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} d\phi \cos\phi' \frac{}{\sqrt{a^2 + x^2 + y^2 + z^2 - 2a\sqrt{x^2+y^2}\cos\phi'}}$$

$$A_{upper} = \frac{\mu_0 a}{4\pi} \int_0^{2\pi} d\phi \cos\phi' \left[I \frac{}{\sqrt{a^2 + d^2 + r^2 - 2r\cos\theta d - 2ar\sin\theta\cos\phi}} + I' \frac{}{\sqrt{a^2 + d^2 - r^2 - 2r\cos\theta d - 2ar\sin\theta\cos\phi}} \right]$$

$$A_{lower} = \frac{\mu_0 a}{4\pi} \int_0^{2\pi} d\phi' \cos\phi' I'' \frac{}{\sqrt{a^2 + d^2 + r^2 - 2r\cos\theta d - 2ar\sin\theta\cos\phi}}$$

<u>Boundary Condition #1:</u>

$$B_{upper} \cdot n = B_{lower} \cdot n$$

$$[(\nabla \times A_{upper})] \cdot \hat{\theta} = [(\nabla \times A_{lower})] \cdot \hat{\theta} \qquad @\ \theta = \pi/2$$

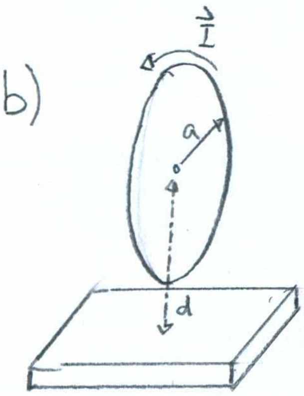
$$= 2\pi a I \times \vec{B}$$

$$= 2\pi a I \times (\nabla \times A)$$

$$= 2\pi a I \left[\frac{\partial}{\partial z} A_\phi\right]_{\rho=a,\, z=d}$$

$$= -\mu_0 I^2 a^2 d \left(\frac{\mu-\mu_0}{\mu+\mu_0}\right) \int_0^{2\pi} \frac{\cos\phi}{(4d^2 + 2a^2(1-\cos\phi))^{3/2}}\, d\phi$$

b)



A circular loop perpendicular to a slab face

Shape: Ring
Dimension: Volume [3D]
Charge: q

$$A = \frac{\mu_0 I}{4\pi} \hat{\phi} \int_0^{2\pi} \frac{\cos\phi}{\sqrt{a^2 + d^2 - 2ar\sin\theta\cos\phi}}\, d\phi$$

$$= \frac{\mu_0 I a (y\hat{i} + x\hat{j})}{4\pi \sqrt{x^2+y^2}} \int_0^{2\pi} \frac{\cos\phi}{\sqrt{a^2 + x^2 + y^2 + z^2 - 2a\sqrt{x^2+y^2}\cos\phi}}\, d\phi$$

$$= \frac{\mu_0 I a}{4\pi} \frac{(-y\hat{i} + (x-2d)\hat{j})}{\sqrt{(x-2d)^2 + y^2}} \int_0^{2\pi} \frac{\cos\phi \quad d\phi}{\sqrt{a^2 + (x-2d)^2 + y^2 + z^2 - 2a\sqrt{(x-2d)^2 + y^2}\cos\phi}}$$

$$= \frac{\mu_0 a}{4\pi} \frac{(-y\hat{i} + (x-2d)\hat{j})}{\sqrt{(x-2d)^2 + y^2}} \frac{(\mu-\mu_0)}{(\mu+\mu_0)} I \circ\circ\circ$$

$$\times \int_0^{2\pi} \frac{\cos\phi \quad d\phi}{\sqrt{a^2 + (x-2d)^2 + y^2 + z^2 - 2a\sqrt{(x-2d)^2 + y^2}\cos\phi}}$$

$$= \frac{\mu_0 a}{4\pi} \frac{(1-2d\sin\phi)\hat{\rho} + (\rho - 2d\cos\phi)\hat{\phi}}{\sqrt{\rho^2 + 4a^2 - 4d\cos\phi}} \frac{(\mu-\mu_0)}{(\mu+\mu_0)} I \times \int_0^{2\pi} \frac{\cos\phi \quad d\phi}{a^2 - 4d^2 - \rho^2 - 4d\rho\cos\phi - z^2 - 2a\sqrt{\rho^2 + 4d^2 - 4d\rho\cos\phi}\cdot\cos\phi}$$

(Equation 5.7)

$$dF = I(d\vec{I} \times d\vec{B})$$

$$F = I \oint d\vec{I} \times B$$

$$= Ia \int_0^{2\pi} d\phi (B_z \cdot \hat{\rho} - B_\rho \hat{z})$$

$$= Ia\int_0^{2\pi} d\phi\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho A_\phi - \frac{1}{\rho}\frac{\partial}{\partial\phi}A_\rho\right)\hat{\rho} - \left(-\frac{\partial}{\partial z}A_\phi\right)\hat{z}$$

$$= Ia\int_0^{2\pi} d\phi\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{1}{\rho}\frac{\partial}{\partial\phi}A_\rho\right)(\cos\phi\,\hat{i} + \sin\theta\,\hat{j}) - \left(-\frac{\partial}{\partial z}A_\phi\right)\hat{z}$$

c) When $d \gg a$, then $d/a \gg 1$ and...

$$F = -\mu_0 I^2 (a/d)^2 \frac{1}{8}\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right)\int_0^{2\pi}\frac{\cos\phi}{(1+\tfrac{1}{2}(a/d)^2(1-\cos\phi'))^{3/2}}\,d\phi$$

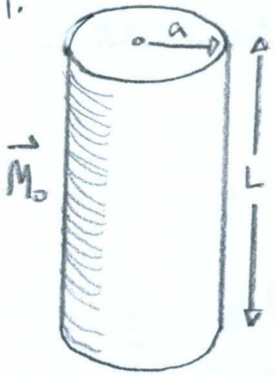
$$= -\mu_0 I^2 (a/d)^2 \frac{1}{8}\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right)\int_0^{2\pi}\cos\phi'\left(1-\tfrac{3}{4}(a/d)^2(1-\cos\phi')\,d\phi\right)$$

$$= -\frac{3\pi}{32}\mu_0 I^2\left(\frac{\mu-\mu_0}{\mu+\mu_0}\right)\left(\frac{a}{d}\right)^4$$

Taylor Expansion:

$$(1+\tfrac{1}{2}(a/d)^2(1-\cos\phi'))^{-3/2} = 1-\tfrac{3}{4}(a/d)^2(1-\cos\phi')$$

5.19.



Hard Material Right Circular Cylinder.

Shape: Cylinder
Dimension: Volume
Charge: q

Various Boundary Conditions:

$\nabla \times H = 0$ "No free currents"

$M = 0$ "Magnetization outside equals zero"

(Equation 5.97)

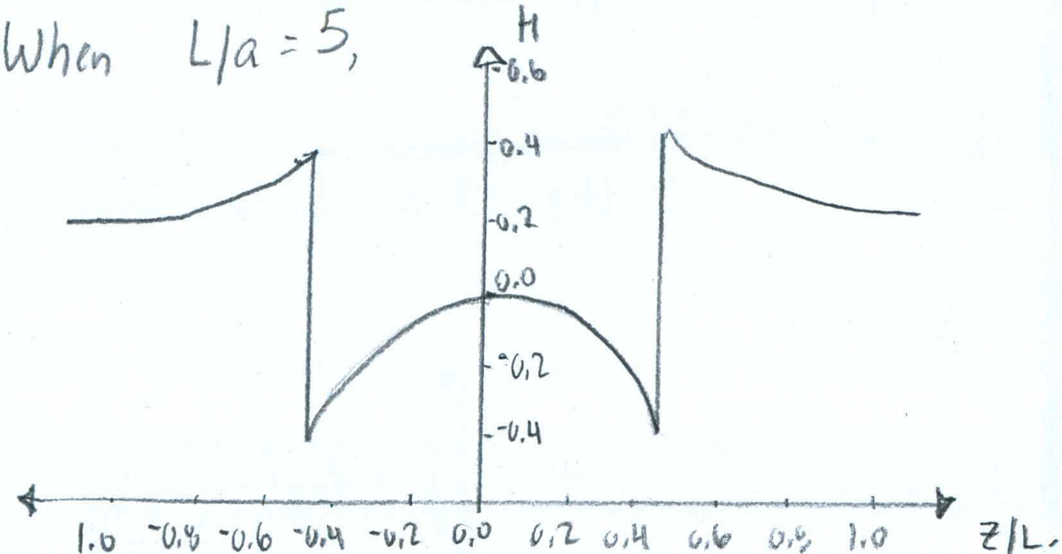
$$A = \frac{-1}{4\pi}\int\frac{\nabla\cdot M(x')}{|x-x'|}d^3x$$

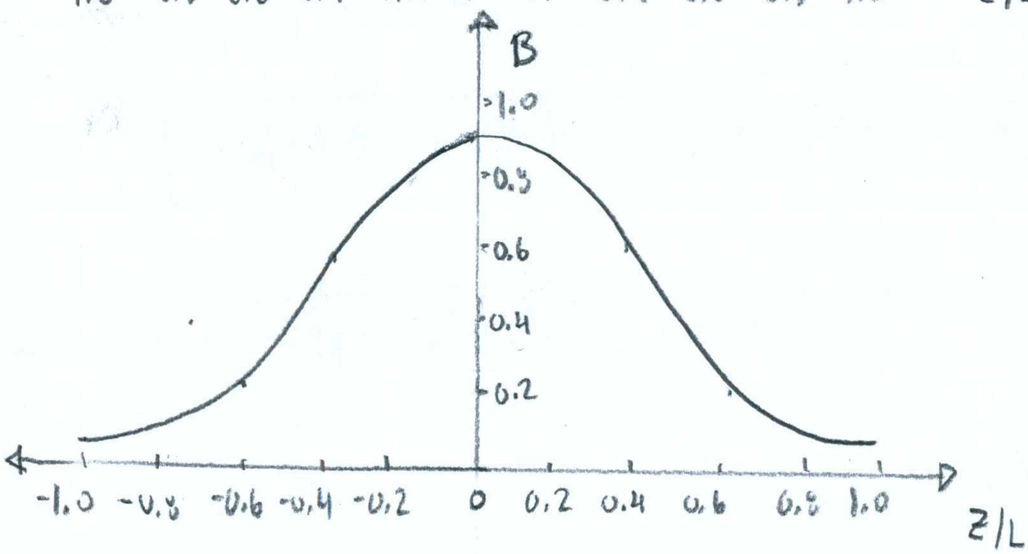
$$= \frac{1}{4\pi}\int\frac{\rho_m}{|x-x'|}d^3x$$ ...because Equation 5.96 $\rho = -\nabla\cdot M$

$$= \frac{1}{4\pi}\int\frac{\rho_m\,\rho\,d\rho\,d\phi\,dz}{\sqrt{\rho^2-\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z-z')^2}}$$

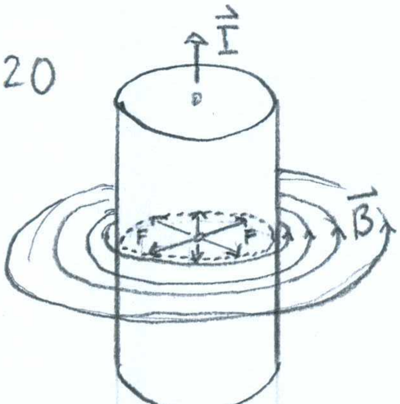
$$= \mu_0 \left(-\frac{M_0}{2}\right)\left[\frac{z - L/2}{\sqrt{a^2 + (z - L/2)^2}} + \frac{z + L/2}{\sqrt{a^2 + (z + L/2)^2}}\right]$$

b) When $L/a = 5$,





5.20



Right circular cylinder with magnetic forces

a) (Equation 5.12)

$$F = \int J(x) \times B(x)\, d^3x$$

(Equation 5.79)

$$J_m(x) = \nabla \times M$$

$$F = \int (\nabla \times M) \times B(x)\, d^3x$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{a}\int_{-L/2}^{L/2} \frac{M_0(\delta(z-L/2) + \delta(z'+L/2))}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z-z')^2}}\rho' d\rho' d\phi' dz'$$

$$= \frac{M_0}{4\pi}\left[\int_0^{2\pi}\int_0^{a} \frac{\rho' d\rho' d\phi}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z-L/2)^2}} - \frac{\rho' d\rho' d\phi}{\sqrt{\rho^2-\rho'^2-2\rho\rho'\cos(\phi-\phi')+(z+L/2)^2}}\right]$$

$\therefore$ when $\rho=0$

$$= \frac{M_0}{4\pi}\left[\int_0^{2\pi}\int_0^{a} \frac{\rho' d\rho' d\phi'}{\sqrt{\rho'^2+(z-L/2)^2}} - \int_0^{2\pi}\int_0^{a} \frac{\rho' d\rho d\phi}{\sqrt{\rho'^2+(z+L/2)^2}}\right]$$

$$= \frac{M_0}{4\pi}\left[\sqrt{\rho'^2+(z-L/2)^2} - \sqrt{\rho'^2+(z+L/2)^2}\right]_0^a$$

$$= \frac{M_0}{4\pi}\left[\sqrt{a^2+(z-L/2)^2} - \sqrt{a^2+(z+L/2)^2} - (z-L/2) + (z+L/2)\right]$$

$H = -\nabla A$

$$= -\frac{\partial A}{\partial z}$$

$$= -\frac{M_0}{2}\left[\frac{z-L/2}{\sqrt{a^2+(z-L/2)^2}} - \frac{z+L/2}{\sqrt{a^2+(z+L/2)^2}}\right]$$

(Equation 5.100.5)

$\nabla \times H = \nabla \times (B/\mu_0 - M) = 0$

$B = \mu_0 H + \mu_0 M$

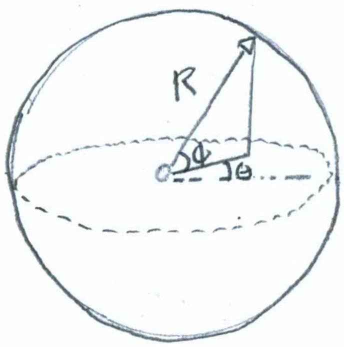
$= \mu_0 H$ when $M=0$ i.e. no magnetization outside the bar.

$$\boxed{\begin{aligned}&\text{Front Page:}\\ &\nabla\times(a\times b)=a(\nabla\cdot b)-b(\nabla\cdot a)\\ &\qquad+(b\cdot\nabla)a-(a\cdot\nabla)b\end{aligned}}$$

$$=\underbrace{\int M(\nabla\cdot B(X))\,dX}_{=0,\ \nabla\cdot B=0}-\underbrace{\int B(X)(\nabla\cdot M)\,d^3X}_{\int_V \nabla\cdot A\,d^3x=\int_S A\cdot n\,da}+\underbrace{\int (B(X)\cdot\nabla)M(X)\,d^3X}_{=0,\ B\cdot\nabla=0}+\int (M\cdot\nabla)B(X)\,d^3X$$

$$=-\int (M\cdot\nabla)B\,dV+\oint (M\cdot n)B\,da$$

b)



A sphere with a uniform magnetization

Shape: sphere
Dimension: Volume [3D]
Charge: q

$$\sigma_M = M\cdot n$$

$$= M_0\cdot\hat{r}\cdot\hat{r}$$

$$= M_0\left(\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos\phi\cos\phi_0 + \sin\theta_0\sin\phi\sin\phi_0\right)$$

If $X_0=\sin\theta_0\cos\phi_0$

$y_0=\sin\theta_0\sin\phi_0$

$z_0=\cos\theta_0$, then

$$\sigma_M = M_0\left(z_0\cos\theta+X_0\sin\theta\cos\phi+y_0\sin\theta\sin\phi\right)$$

$$\vec{B}_X = B_0(1+B_y)$$

$$B_y = B_0(1+B_x)$$

$$B=B_0\left[(1+B_y)\hat{\imath}+(1+B_x)\hat{\jmath}\right]$$

$$=B_0\left[(\hat{\imath}+\hat{\jmath})+\beta r\sin\theta(\hat{\imath}\sin\phi+\hat{\jmath}\cos\phi)\right]$$

$$F = -\int (M \cdot \nabla) B\,dV + \oint (M \cdot n) B\,da$$

$$= \int \rho B\,dV - \oint \sigma \cdot B a$$

$$= \oint \sigma_m \cdot B\,da$$

$$= \int_0^{2\pi}\int_0^{\pi} M_0 (z_0 \cos\theta + x_0 \sin\theta\cos\theta + y_0 \sin\theta \sin\phi)$$
$$\times B_0 \left[(\hat{\imath} + \hat{\jmath}) + \beta R \sin\theta (\hat{\imath}\sin\phi + \hat{\jmath}\cos\phi)\right] R^2 \sin\theta\, d\theta\, d\phi$$

$$= M_0 B_0 \beta \frac{4}{3}\pi R^3 (y_0 \hat{\imath} + x_0 \hat{\jmath})$$

$$= M_0 B_0 \beta \cdot V \cdot (\sin\theta_0 \sin\phi\, \hat{\imath} + \sin\theta_0 \cos\phi_0\, \hat{\jmath})$$

5.21 a) (Equation 5.147) $\delta W = \int H \cdot \delta B\, d^3x$

$$= \int \left[\underbrace{H \cdot (\nabla \times \delta A)}_{\nabla \cdot B = 0} + \underbrace{\nabla \cdot (H \times \delta A)}_{=0}\right] d^3x$$

$$= 0$$

$\int H \cdot B\, d^3x = 0$ "No angle between dipole and Magnetic induction"

b) (Equation 5.72) $U = -m \cdot B$

(Equation 5.150) $W = -\frac{1}{2}\int M \cdot B\, d^3x$

(Equation 5.101) $\nabla \times H = \nabla \times (B/\mu_0 - M)$

$= 0$

So, $W_{HH} = -\frac{1}{2}\int B\cdot H\, d^3x + \frac{\mu_0}{2}\int H\cdot H\, d^3x$

$= \frac{\mu_0}{2}\int H\cdot H\, d^3x$

$W_{MH} = -\frac{\mu_0}{2}\int M\cdot M\, d^3x - \frac{\mu_0}{2}\int M\cdot H\, d^3x$

$= \frac{\mu_0}{2}\int M\cdot H\, d^3x$

$W_{HH} = W_{MH}$

$\frac{\mu_0}{2}\int H\cdot H\, d^3x = -\frac{\mu_0}{2}\int M\cdot H\, d^3x$

Magnetic field is the magnetization.

5.22. $\sigma_M = n\cdot M$

$= -M$

$\sigma' = n\cdot M$

$= M$

$B_{out} = \mu_0 M \hat{z}$

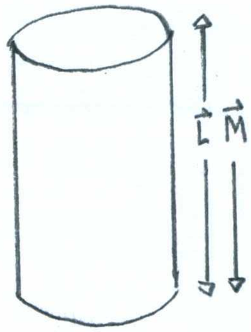
$F = qB$

$= \int \sigma B\, da$

$= \int (-M)(\mu M \hat{z})\, da$

$= -z\mu_0 A M^2$

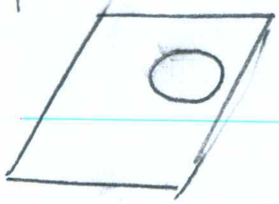
5.23.



A right circular cylinder with a uniform magnetization

Shape: cylinder
Dimension: Volume
Charge: q

5.24



A conducting plane with a circular hole.

Shape: Plane
Dimension: Area [2D]
Charge: q

Potential Derivation by Boundary Conditions

① Boundary conditions:

$$\phi(r=a, \phi=\pi/2, z)=0$$

$$\phi(r, \phi, z=0)=H_0 a$$

$$\phi(r=0, \phi, z)=\text{finite}$$

② Laplace's Equation

$$\nabla^2\Phi=\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial^2\phi}+\frac{\partial^2\Phi}{\partial^2 z}$$

$$=0$$

③ Laplace's Equation Solutions:

If $\Phi(r,\phi,z)=R(r)Q(\phi)Z(z)$, then

Ⓐ Variable Separation:

$$\nabla^2\Phi=\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial^2\phi}+\frac{\partial^2\Phi}{\partial^2 z}$$

$$=\frac{Q(\phi)Z(z)}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial R(r)}{\partial r}\right)+\frac{R(r)Z(z)}{r^2}\frac{\partial^2 Q(\phi)}{\partial^2\phi}+R(r)\,Q(\phi)\frac{\partial^2 Z(z)}{\partial^2 z}$$

$$=\frac{1}{R(r)\cdot r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial R(r)}{\partial r}\right)+\frac{1}{Q(\phi)\,r^2}\frac{\partial^2 Q(\phi)}{\partial^2\phi}+\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z}$$

$$=0$$

Ⓑ Radial Eigenvalues:

$$\frac{r}{R}\frac{\partial R}{\partial r}+\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2}+\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z}=\lambda r^2$$

$$\frac{r}{R}\frac{\partial R}{\partial r}+\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2}-\lambda r^2=\frac{-1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z}$$

$$\frac{r}{R}\frac{\partial R}{\partial r} + \frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} - \lambda r^2 = k^2$$

If $r = x$ and $\lambda = \frac{1}{r^2}$, then:

$$\frac{\partial^2 R}{\partial^2 x} + \frac{1}{x}\frac{\partial R}{\partial x} - \left(1 + \frac{k^2}{x^2}\right)R = 0$$

(C) Angular Eigenvalues:

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial^2 \phi} = m^2 \quad ; \quad \frac{\partial^2 Q}{\partial^2 \phi} + m^2 Q = 0$$

(D) Vertical Eigenvalues:

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial^2 Z} = k^2 \quad ; \quad \frac{\partial^2 Z}{\partial Z^2} - k^2 Z = 0$$

(E) General Solutions:

$$\Phi(r,\phi,Z) = \sum_{k,m} R(r)\cdot Q(\phi)\, Z(Z)$$

(4) General Solution to Laplaces Equation:

$$R(r) = E\cdot J_v(kx)$$

where $$J_v(kx) = \sum_{m=0}^{\infty} \frac{(-1)^m\left(-\frac{1}{2}kx\right)^{v+2m}}{m!\,(m+v)!}$$

$$Q(\phi) = C\cdot \sin(m\phi)$$

$$Z(Z) = A e^{-kZ}$$

(5) Variables by Boundary Conditions:

$$\Phi(r,\phi,z) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}[E\cdot J_v(kx) + F\, Y_v(kx)][C\sin(m\phi)][De^{-kz}]$$

F $$\Phi(r=0,\phi,z) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}[E\cdot J_v(k\cdot 0) - F\,Y_v(k\cdot 0)][C\sin(m\phi)][De^{-kz}]$$

$$= 0 \quad , \text{so } F=0$$

E $$\Phi(r,\phi,z=0) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} E\, J_v(k\,x)\sin(m\phi)\,e^{-k\cdot 0}$$

$$= Ha$$

$$\int_0^{\infty}\int_0^{\pi} H\cdot a\cdot J_v(kx)\sin(m\phi)\,x\,d\phi\,dx = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{\infty}\int_0^{\pi} E\cdot J_v^2(kx)\sin^2(m\phi)\,x\,d\phi\,dx$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{E\cdot\pi}{2k}\delta(k-k')$$

$$E = \frac{4k}{\pi}\int_0^{\infty} H\cdot a\, J_v(kx)\cdot x\cdot d\phi\, dx$$

$$\Phi(r,\phi,z) = \frac{4kH_0 a}{\pi}\int_0^{\infty} J_v(kx)\,J_v(kx)\,x\sin(m\phi)\,dx$$

$$= \frac{2H_0a^2}{\pi}\int_0^{\infty} dk\, j_1(k\cdot a)\,J_1(k\rho)\sin\phi$$

$$= \frac{2H_0a^2}{\pi}\int_0^{\infty} dk\left[\frac{\sin(ka)}{(ka)^2} - \frac{\cos(ka)}{ka}\right]J_1(k\rho)\sin\phi$$

Identities from: Gradshteyn and Ryzhim:

$\boxed{\text{Eqn } 6.693\ \#1}$ $$\int_0^{\infty}\frac{J_v(ax)\sin(bx)\,dx}{x^2} = \frac{a^2-b^2\sin(v\arcsin\frac{b}{a})}{v^2-1} - \frac{b\cos[v\arcsin(b/a)]}{v(v^2-1)}$$

... when $0<b<a$

$$= \frac{-a^{\nu}\cos\left(\frac{\nu\pi}{2}\right)\left[b + \nu\sqrt{b^2 - a^2}\right]}{\nu(\nu^2-1)\left[b + \sqrt{b^2 - a^2}\right]^{\nu}}$$

... when $0 < a < b$

$$\int_0^{\infty} \frac{J_\nu(\alpha x)\cos\beta x}{x} = \frac{1}{\nu}\cos\left(\nu\arcsin\left(\frac{\beta}{\alpha}\right)\right) \qquad \alpha < \beta$$

Eqn 6.693 #2

$$= \frac{\alpha^{\nu}\cos\frac{\nu\pi}{2}}{\nu\left(\beta + \sqrt{\beta^2 - \alpha^2}\right)^{\nu}} \qquad \beta > \alpha$$

$$\Phi(r,\varphi,z) = \frac{2H_0 a^2}{\pi}\left(\rho\sin^{-1}\left(\frac{a}{\rho}\right) - a\sqrt{1 - \frac{a^2}{\rho^2}}\right)\sin\phi$$

$$H_\rho = -\frac{\partial\Phi}{\partial\rho}$$

$$= \frac{H_0}{\pi}\left[\frac{a^3}{\sqrt{1 - a^2/\rho^2}\cdot\rho^3} + \frac{a}{\sqrt{1 - a^2/\rho^2}\,\rho} - \sin^{-1}\left(\frac{a}{\rho}\right)\right]\sin\phi$$

$$H_\varphi = \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho}$$

$$= \frac{H_0}{\pi}\left[\frac{a}{\rho}\sqrt{1 - \frac{a^2}{\rho^2}} - \sin^{-1}\left(\frac{a}{\rho}\right)\right]\cos\phi$$

$$H_x = H_\rho\cos\phi - H_\phi\sin\phi$$

$$= \frac{H_0}{\pi}\frac{a^3}{\rho^2}\frac{\sin 2\phi}{\sqrt{\rho^2 - a^2}}$$

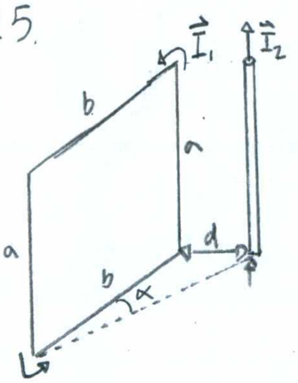
$$= \frac{2H_0}{\pi}\frac{a^3}{\rho^4}\frac{xy}{\sqrt{\rho^2 - a^2}}$$

$$H_y = H_\rho\sin\phi + H_\phi\cos\phi$$

$$= \frac{H_0}{\pi}\left[\frac{a}{\sqrt{\rho^2 - a^2}} - \sin^{-1}\left(\frac{a}{\rho}\right) - \frac{a^3}{\rho^2}\frac{\cos 2\phi}{\sqrt{\rho^2 - a^2}}\right]$$

b) $(K_x, K_y) = \left(-H_0 - H_y^{(1)}, H_{(x)}^{(1)}\right)$

5.25



A rectangular loop interacting with a wire

Shape: Rectangle, line
Dimension: Area [20]
Charge: q

a) (Equation 5.144) $dW = \int \delta A \cdot J d^3x$

(Equation 5.24, 5.25) $J = \frac{I}{A}$, $J = \mu_0 I$

$$\text{Current Density} = \frac{\text{Current}}{\text{Area}} = \text{constant} \times \text{current}$$

(Equation 5.8) $$W = \frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{dI_1 \times (dI_2 \times X_{12})}{|X_{12}|^2} dV$$

$$= \frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \left(-dI_1 \cdot dI_2 \frac{X_{12}}{|X_{12}|^2} + dI_2 \cdot dI_1 \frac{X_{12}}{|X_{12}|^2}\right) dV$$

$$= -\frac{\mu_0}{4\pi} I_1 I_2 \left[A(b/2, 0) - A(-b/2, 0)\right]$$

$$= \frac{\mu_0}{4\pi} I_1 I_2 a \ln \left[\frac{(-b/2 - d\cos\alpha)^2 - (-d\sin\alpha)^2}{(b/2 - d\cos\alpha)^2 - (-d\sin\alpha)^2}\right]$$

$$= \frac{\mu_0}{4\pi} I_1 I_2 a \ln \left[\frac{4d^2 + b^2 + 4db\cos\alpha}{4d^2 + b^2 - 4db\cos\alpha}\right]$$

b) $$B_x(X, 0) = -\left.\frac{\partial A_z(x, 0)}{\partial z}\right|_{z=0}$$

$$= \frac{-d\sin\alpha}{(X - d\cos\alpha)^2 + d^2\sin^2\alpha}$$

$\Rightarrow$ $$F_x = I_1 \left[B_x(b/2, 0) - B_x(-b/2, 0)\right]$$

$$= \frac{2\mu_0 I_1 I_2 ab(4d^2\cos(2\alpha) - b^2)}{\pi(b^4 - 8d^2\cos(2\alpha)b^2 + 16d^4)}$$

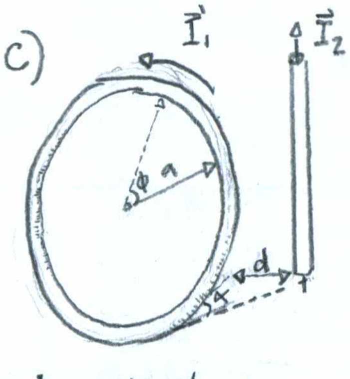
$$B_z(X, 0) = -\left.\frac{\partial A_z(x, 0)}{\partial X}\right|_{z=0}$$

$$= \frac{(X - d\cos\alpha)}{(X - d\cos\alpha)^2 + d^2\sin^2\alpha}$$

$\Rightarrow$ $$F_z = -I_2\left[B_x(b/2, 0) - B_x(-b/2, 0)\right]$$

$$= \frac{-8\mu_0 I_1 I_2 ab d^2 \sin(2\alpha)}{\pi(b^4 - 8d^2\cos(2\alpha)b^2 + 16d^4)}$$

c)



A circular loop interacting with a wire

Shape: Circle, line
Dimension: Area [2D]
Charge: q

(Equation 5.144) $dW = \int dA \cdot J \, d^3x$

If $x = a\cos\phi$ and $dl = a\cos\phi \, d\phi$,

then $dW = \frac{\mu_0}{4\pi} I_1 I_2 \int_0^{2\pi} a\cos\phi \, A_z(a\cos\phi, 0)\, d\phi$

$$= \frac{\mu_0}{4\pi} I_1 I_2 a \left(\frac{\cos(\alpha) a}{2d} + \frac{\cos(3\alpha) a^3}{8d^3} + \frac{\cos(5\alpha) a^5}{16 d^5} + \frac{5\cos(7\alpha) a^7}{128 d^7} + \frac{7\cos(9\alpha) a^9}{256 d^9} + \cdots \right)$$

$$= \frac{\mu_0}{4\pi} I_1 I_2 a \cdot \mathrm{Re}\left[\frac{z}{2} + \frac{z^3}{8} + \frac{z^5}{16} + \frac{5z^7}{128} + \frac{7z^9}{256} + \cdots \right]$$

where $z = \frac{a}{d} e^{i\alpha} = \frac{1-\sqrt{1-z^2}}{z}$

$$= \frac{\mu_0}{4\pi} I_1 I_2 a \, \mathrm{Re}\left[\frac{1-\sqrt{1-z^2}}{z} \right]$$

$$= \frac{\mu_0}{4\pi} I_1 I_2 a \, \mathrm{Re}\left[e^{i\alpha} - \sqrt{e^{2i\alpha} - a^2/d^2} \right]$$

Force:

$$F = I_1 \int_0^{2\pi} a\cos\phi \, B_z(a\cos\phi, 0)\, d\phi$$

$$F_x = \mu_0 I_1 I_2 \left(\frac{\cos(2\alpha) a^2}{2d^2} + \frac{3\cos(4\alpha) a^4}{8d^4} + \frac{5\cos(6\alpha) a^6}{16 d^6} + \frac{35\cos(8\alpha) a^8}{128 d^8} + \frac{63\cos(10\alpha) a^{10}}{256 d^{10}} + \cdots \right)$$

$$F_y = \mu_0 I_1 \cdot I_2 \left(\frac{\sin(2\alpha) a^2}{2d^2} + \frac{3\sin(4\alpha) a^4}{8d^4} + \frac{5\sin(6\alpha) a^6}{16 d^6} + \frac{35\sin(8\alpha) a^8}{128 d^8} + \frac{63\sin(10\alpha) a^{10}}{256 d^{10}} + \cdots \right)$$

$$\boxed{\text{Identity: } G(z) = \frac{1}{\sqrt{1-z^2}} = 1 + \frac{z^2}{2} + \frac{3z^4}{8} + \frac{5z^6}{16} + \frac{35z^8}{128} + \frac{63z^{10}}{256} + \cdots)}$$

$$F_x = \mu_0 I_1 I_2 \, Re\left[\frac{a}{d} e^{i\alpha}\right]$$

$$F_y = \mu_0 I_1 I_2 \, Re\left[\frac{a}{d} e^{i\alpha}\right]$$

d) Rectangular:

$$W_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \, a \ln\left[\frac{4d^2+b^2+4db\cos\alpha}{4d^2+b^2-4db\cos\alpha}\right] \quad \text{∘∘if } d \gg a,b$$

$$= \frac{\mu_0}{4\pi} I_1 I_2 \frac{ab\cos\alpha}{d}$$

$$= \left(\frac{I_1 ab}{\pi 2}\right) \cdot \left(\frac{\mu_0 I_2}{2\pi d}\cos\alpha\right)$$

$$= m_1 \circ B_z$$

Circular:

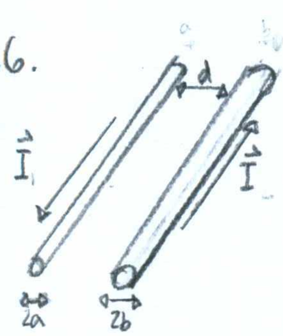
$$W_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \, a\left[\frac{\cos(\alpha)a}{2d} + \frac{\cos(3\alpha)a^3}{8d^3} + \frac{\cos(5\alpha)a^5}{16d^5} + \circ\circ\circ\right]$$

$$\simeq \mu_0 I_1 I_2 a \frac{\cos(\alpha)a}{2d}$$

$$\simeq (I_1 \pi a^2) \circ \left(\frac{\mu_0 I_2}{2\pi d}\cos\alpha\right)$$

$$\simeq m_1 \circ B_z$$

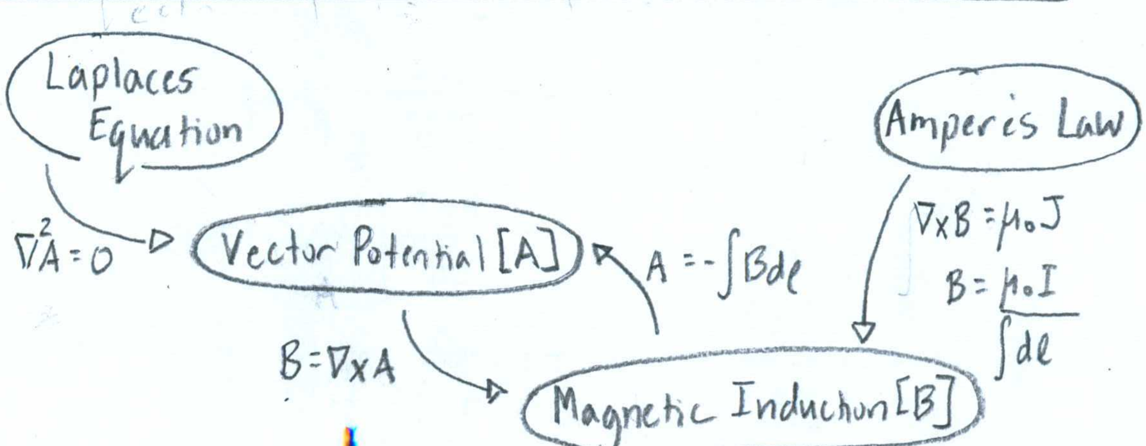
5.26.



Two wire
Transmission line

Shape: Wire
Dimension: volume [3D]
Charge: q

$\boxed{\text{Two methods toward Vector Potential math}}$



"outside" (Equation 5.25) $\oint B\cdot dl = \mu_0 I$ "Ampere's Equation" in a static electric field

$$B\oint dl = \mu_0 I$$

$$B = \frac{\mu_0 I}{\int_0^{2\pi}\int_0^{\rho} d\rho\, d\phi}\hat{z}$$

$$= \frac{\mu_0 I}{2\pi\rho}\hat{z}$$

"inside" (Equation 5.24) $\oint B\cdot dl = \mu_0 \int J\cdot n\, da$

$$B\oint dl = \mu_0 I \int n\, da$$

$$B = \mu_0 I \frac{\int n\, da}{\oint dl}$$

$$= \mu_0 I \frac{\int_0^{2\pi}\int_0^{\rho} d\rho\, d\phi}{\int_0^{2\pi}\int_0^{R} d\rho\, d\phi}$$

$$= \frac{\mu_0 I}{2\pi}\frac{\rho}{R^2}$$

(Equation 5.27) $B = \nabla \times A$

$$A = -\int B\, d\rho$$

$$= -\frac{\mu_0 I}{2\pi}\int_{\rho}^{R}\frac{1}{\rho'}\, d\rho'$$

$$= -\frac{\mu_0 I}{2\pi}\left[\log R - \log \rho\right]$$

$$= +\frac{\mu_0 I}{4\pi}\log\left(\frac{\rho^2}{R^2}\right)$$

Outside

Inside

$$A = -\int B\,d\rho$$

$$= -\frac{\mu_0 I}{2\pi R^2}\int_0^{\rho} \rho\,d\rho$$

$$= -\frac{\mu_0 I}{4\pi}\,\frac{\rho^2}{R^2}$$

(Equation 5.149) $$W = \frac{1}{2}\int \vec{J}\cdot\vec{A}\,dx$$

$$= \frac{1}{2}\int \vec{J}^a\cdot\vec{A}^a\,dx^a + \frac{1}{2}\int \vec{J}^b\cdot\vec{A}^b\,dx^b$$

$$= \frac{1}{2}\left(\frac{I}{\pi a^2}\right)\int_0^{2\pi}\int_0^{a} A_z(\rho_a)\,\rho_a\,d\rho_a\,d\phi + \frac{1}{2}\left(\frac{I}{\pi b^2}\right)\int_0^{2\pi}\int_0^{b} A_z(\rho_b)\,\rho_b\,d\rho_b\,d\phi$$

$$= \frac{1}{2}\left(\frac{I}{\pi a^2}\right)\frac{\mu_0 I}{4\pi}\int_0^{2\pi}\int_0^{a}\left[\log\left(\frac{\rho_b^2}{b^2}\right) - \frac{\rho_a^2}{a^2}\right]\rho_a\,d\rho_a\,d\phi$$

$$+ \frac{1}{2}\left(\frac{I}{\pi b^2}\right)\frac{\mu_0 I}{4\pi}\int_0^{2\pi}\int_0^{b}\left[\log\left(\frac{\rho_a^2}{a^2}\right) - \frac{\rho_b^2}{b^2}\right]\rho_a\,d\rho_b\,d\phi$$

$$= \frac{1}{2}\left(\frac{I}{\pi a^2}\right)\frac{\mu_0 I}{4\pi}\int_0^{2\pi}\int_0^{a}\left[\log\frac{\rho_a^2+d^2-2d\rho_a\cos\phi_1}{b^2} - \frac{\rho_a^2}{a^2}\right]\rho\,d\rho\,d\phi$$

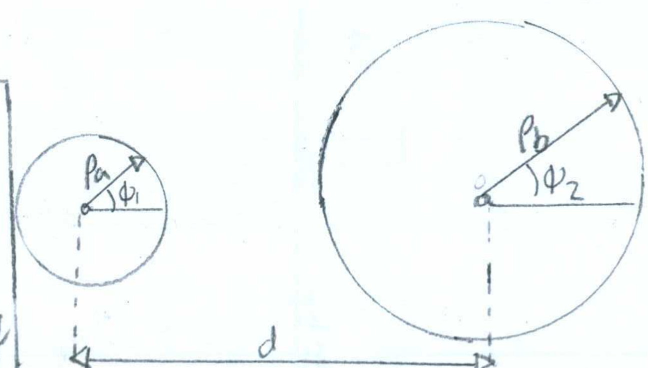
$$+ \frac{1}{2}\left(\frac{I}{\pi b^2}\right)\frac{\mu_0 I}{4\pi}\int_0^{2\pi}\int_0^{b}\left[\log\frac{\rho_b^2+d^2-2d\rho_b\cos\phi_2}{a^2} - \frac{\rho_b^2}{b^2}\right]\rho\,d\rho\,d\phi$$

Law of Cosines

$$\rho_a^2 = \rho_b^2 + d^2 - 2d\rho_b\cos\phi$$

-or-

$$\rho_b^2 = \rho_a^2 + d^2 - 2d\rho_a\cos\phi$$



$$\underline{\text{Identities:}} \quad \int_0^{2\pi} \log(s-\cos\phi)\,d\phi = 2\pi\log\left(\frac{s+\sqrt{s^2-1}}{2}\right)$$

$$\int_0^{2\pi} \log(a^2+b^2-2ab\cos\phi)\,d\phi = 2\pi\log(2ab) + \int_0^{2\pi}\log\left(\frac{a^2+b^2}{2ab}-\cos\phi\right)d\phi$$

$$= 2\pi\log(2ab) + 2\pi\log\left(\frac{1}{2}\left[\frac{a^2+b^2}{2ab} + \sqrt{\left(\frac{a^2+b^2}{2ab}\right)^2-1}\,\right]\right)$$

$$= 2\pi\log\left(\frac{1}{2}\left(a^2+b^2+\sqrt{(a^2+b^2)^2-4a^2b^2}\right)\right)$$

$$= 2\pi\log\left(\frac{1}{2}\left(a^2+b^2+\sqrt{(a^2-b^2)^2}\right)\right)$$

$$= 2\pi\log(\max(a^2, b^2))$$

So, $$W = \frac{1}{2}\left(\frac{I}{\pi a^2}\right)\frac{\mu_0 I}{4\pi}\int_0^{2\pi}\int_0^{a}\left[\log\left(\frac{d^2}{b^2}\right) - \frac{\rho_b^2}{a^2}\right]\rho_b\,d\rho_b\,d\phi$$

$$+ \frac{1}{2}\left(\frac{I}{\pi b^2}\right)\frac{\mu_0 I}{4\pi}\int_0^{2\pi}\int\left[\log\left(\frac{d^2}{a^2}\right) - \frac{\rho_a^2}{b^2}\right]\rho_a\,d\rho_a\,d\phi$$

$$= \frac{1}{2}\frac{\mu_0 I^2}{4\pi}\left(2\log\left(\frac{d}{b}\right)+\frac{1}{2}\right) + \frac{1}{2}\frac{\mu_0 I^2}{4\pi}\left(2\log\left(\frac{d}{a}\right)+\frac{1}{2}\right)$$

$$= \frac{1}{2}\frac{\mu_0 I^2}{4\pi}\left(1+2\log\left(\frac{d^2}{ab}\right)\right)$$

(Equation 5.152)

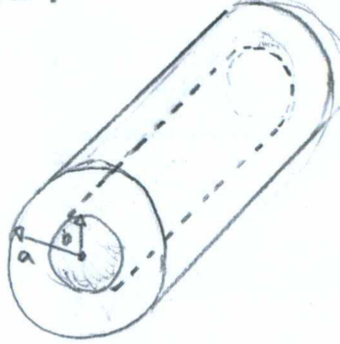
$$W = \frac{1}{2}\sum_{i=1}^{N} L_i I_i^2 + \sum_{i=1}^{N}\sum_{j>i}^{N} M_{ij}\cdot I_i I_j$$

$$= \frac{1}{2}L\cdot I^2$$

$$L = \frac{2\cdot W}{I^2}$$

$$= \frac{\mu_0}{4\pi}\left(1+2\log\left(\frac{d^2}{ab}\right)\right)$$

5.27.



A long thin conducting shell or thin hollow tube

Shape: cylinder
Dimension: Area [2D]
Charge: q

Two methods exist in the magnetic induction derivations: Ampere's Law or Laplace's equation.

Method #1: Magnetic induction by Ampere's Law

(Equation 5.25) $\oint_C B \cdot dl = \mu_0 I$

$$B \oint ds = \mu_0 I$$

$$B = \frac{\mu_0 I}{\int_0^{2\pi} \int_0^r dr\, d\phi}$$

$$= \frac{\mu_0 I}{2\pi r}$$

... when $b < \rho < a$

(Equation 5.24) $\oint B \cdot dl = \mu_0 \oint J \cdot n\, da$

$$B \oint ds = \mu_0 I \oint da$$

$$B = \frac{\mu_0 I \int_0^r dr}{\int_0^{2\pi} \int_0^b r\, dr\, d\phi}$$

$$= \frac{\mu_0 I r}{\pi b^2} \quad \therefore \rho < b$$

Method #2 Magnetic Induction by Boundary Conditions

... I tired.

(Equation 5.152) $W = \sum_{i=1}^{N} L_i I_i^2$

$$L = \frac{W'}{I^2}$$

$$= \frac{1}{I^2}\int \frac{B \cdot B}{\mu} d^3x$$

$$=$$

$$= \frac{1}{I^2}\int_0^{2\pi}\int_0^{l}\left[\int_0^{b} \frac{\mu_0^2 I^2 r^2}{\pi^2 \mu b^4} r\,dr + \int_b^{a} \frac{\mu_0^2 I^2}{\pi^2 \mu_0 r^2} r\,dr\right] dz\,d\theta$$

$$= \frac{1}{I^2}\int_0^{2\pi}\int_0^{l}\left[\frac{\mu_0^2 I^2 b^3}{3\pi^2 \mu b^4} + \frac{\mu_0^2 I^2}{\pi^2 \mu_0}\ln\left(\frac{a}{b}\right)\right] dz\,d\theta$$

$$= \frac{1}{I^2} 2\pi l \mu_0^2 I^2 \left[\frac{1}{3\pi^2 \mu b} + \frac{1}{\pi^2 \mu_0}\ln\left(\frac{a}{b}\right)\right]$$

$$= \frac{2\mu_0^2 \cdot l}{\pi}\left[\frac{1}{3\mu b} + \frac{1}{4\mu_0}\ln\left(\frac{a}{b}\right)\right]$$

Hollow: $L = \frac{2\mu_0^2 l}{\pi}\left[\frac{1}{4\mu_0}\ln\left(\frac{a}{b}\right)\right]$

$$\frac{L}{l} = \frac{\mu_0}{2\pi}\ln\left(\frac{a}{b}\right)$$

Derivation of 5.152

Power = voltage × current

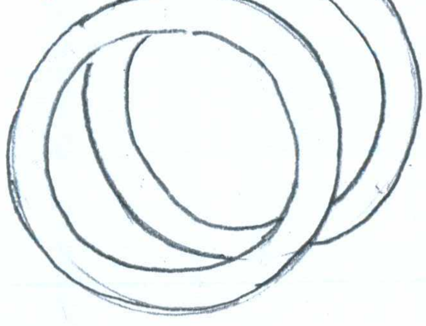
$= \frac{\Delta \text{Work}}{\Delta \text{time}}$

Work = Voltage · current · Δtime

$= V \cdot I \cdot dt$

$= L \cdot \frac{dI}{dt} \cdot I \cdot dt$

$= \frac{1}{2} L I^2$

5.28. 

Mutual inductance of two circular loops in homogeneous medium.

Shape: Rings

Dimension: Area [2D]

Charge: q

Cite #1: Mutual Inductance of Two Coaxial circles.
Bulletin of the Bureau of Standards.
Vol. # 2, 3. pg. 360

Cite #2: A treatise of Electricity and Magnetism.
Maxwell, James Clerk. (1875). Chapter XIV.
Vol. II, pg 307.

Note: The first citation modifies Maxwells equation because the situation when two coaxial cables with similar radii.

Maxwell's Derivation:

$$M = \iint \frac{\cos \epsilon}{r} \, ds \, d\bar{s}$$

Where $r^2 = A^2 + a^2 + b^2 - 2Aa\cos(\phi - \phi')$

$\epsilon = \phi - \phi'$

$ds = a\,d\phi$

$ds' = A\,d\phi'$

$$M = \int_0^{2\pi}\int_0^{2\pi} \frac{Aa\cos(\phi-\phi')\,d\phi\,d\phi'}{\sqrt{A^2+a^2+b^2-2Aa\cos(\phi-\phi')}}$$

$$= 2\pi\sqrt{Aa}\left\{\left(c - \frac{2}{c}\right)F + \frac{2}{c}E\right\}$$

Where $c = \dfrac{\sqrt{Aa}}{\sqrt{(A+a)^2+b^2}}$

F, E = Elliptic Integrals

Mutual Inductance is symmetric

$$M_{12} = \frac{\mu_0}{4\pi} M = -\frac{\mu_0}{4\pi} M$$

The modified mutual induction [cite #]

$$M_{12} = \mu_0 \sqrt{ab}\left[\left(\frac{2}{k} - k\right)K(k) - \frac{2}{k}E(k)\right]$$

where $k^2 = \frac{4ab}{(a+b^2)+d^2}$ and K, E = Elliptic Integrals

If $d \ll a,b$ and $a \cong b$

$$M_{12} = \mu_0 a\left[\left(\frac{1}{2} - \frac{1}{4}\right)K(4) - \frac{1}{2}E(4)\right]$$

$$\cong \frac{4\mu_0 a}{10}$$

Elliptic table citations:

1. American Mathematical Society
   "Guide to Tables of Elliptic Functions"

2. National Aeronautics and Space Agency
   "Tables of Elliptic Integrals"

3. National Institute of Standards and Technology
   "Tables of complete elliptic integrals"

$$K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2\phi)^{-1/2}\, d\phi$$

$$E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2\phi)^{1/2}\, d\phi$$

5.29 Inductance:

(Equation 5.152) $$W = \frac{1}{2}\sum_{i=1}^{N} L_i \cdot I_i^2 + \sum_{i=1}^{N}\sum_{j>1}^{N} M_{ij} I_i I_j$$

$$= 0$$

$$L = 2 \cdot M$$

$$= 2 \cdot \frac{F}{I}$$

$$= \frac{\mu}{I}\iint B \circ d\vec{a}_s$$

$$= \frac{\mu}{I}\int_a^{a'} B \cdot n\, dl$$

$$= -\mu \frac{\int_a^{a'} B \circ n\, dl}{\oint B \cdot dl}$$

Capacitance:

(Equation 1.61) $$Q_i = \sum_{j=1}^{n} C_{ij} V_j$$

$$C = \frac{Q}{V}$$

$$= \frac{\epsilon}{V}\iint E \circ d\vec{a}$$ "Gauss' Law"

$$= \frac{\epsilon}{V}\oint \vec{E} \circ dl$$

$$= \epsilon \frac{\oint_c E \circ n\, dl}{-\int_a^{a'} \vec{E} \cdot dl}$$

$$C \cdot L = -\mu \frac{\int_a^{a'} \vec{B} \circ \vec{n}\, dl}{\oint_c \vec{B} \circ dl} \cdot \epsilon \frac{\oint_c \vec{E} \cdot \vec{n}\, dl}{-\int_a^{a'} \vec{E} \circ dl}$$

$$= \mu\epsilon \frac{\int_a^{a'} (\hat{z} \times E) \circ n\, dl}{\oint_c (\hat{z} \times E) \circ dl} \frac{\oint \vec{E} \circ \vec{n}\, dl}{\int_a^{a'} \vec{E} \cdot dl}$$

$$= \mu\epsilon.$$

5.30.

a) Surface current density: $K(\phi) = I \cos\phi / 2R$

Initial Uniform Magnetic Induction: $B_0 = \mu_0 I / 4R$

(Equation 5.32) $A = \frac{\mu_0}{4\pi} \int \frac{J(x')}{|x-x'|} d^3x$

$$= \frac{\mu_0}{4\pi} \int \frac{I \cos\phi'}{|x-x'|} d^3x$$

(Equation 3.148) $\frac{1}{|x-x'|} = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk\, e^{im(\phi-\phi')} \cos[k(z-z')] I_m(k\rho) K_m(k\rho)$

$$A(x) = \frac{\mu_0}{4\pi} \int \frac{I\cos\phi'}{2R} \left[\frac{2}{\pi} \sum_{m=-\infty} \int_0^{\infty} dk\, e^{im(\phi-\phi')} \cos[k(z-z')] I_m(k\rho) K_m(k\rho) \right]$$

$$= \frac{\mu_0 I}{4\pi^2 R} \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{2\pi} \left[\left[\cos(m(\phi-\phi')) + i \sin(m(\phi-\phi')) \right] \cos\phi' \cos[k(z-z')] I_m(k\rho) K_m(k\rho) \right]$$

$$\times R \cdot d\phi\, dk\, dz$$

$= \frac{\mu_0 I}{4\pi^2}$

$$\boxed{\begin{array}{l} \underline{\text{Trigonometric}} \quad \underline{\text{Identity:}} \\ \cos(\phi-\phi') = \cos\phi\cos\phi' + \sin\phi\sin\phi' \end{array}}$$

$$\boxed{\begin{array}{l} \underline{\text{Integral}} \quad \underline{\text{Identity:}} \\ \int_0^{2\pi} d\phi \cos\phi \cos(\phi-\phi') = \cos\phi \int_0^{2\pi} d\phi' \cos^2\phi' = \pi\cos\phi \end{array}}$$

$$W_{TOT} = \frac{1}{2\mu_0}\int_{\rho<R} B\cdot B\, d^3x + \frac{1}{2\mu_0}\int_{\rho>R} B\cdot B\, d^3x$$

$$= \frac{1}{2\mu_0}\int_{\rho<R}\left(\frac{-\mu_0 I\pi}{2R}\right)^2 d^3x + \frac{1}{2\mu_0}\int_{\rho>R}\left(\frac{-\mu_0 I\pi}{2}\frac{R}{\rho^2}\right)^2 d^3x$$

$$= \frac{\mu_0 I^2\pi^2}{8R^2}\int_{\rho<R} d^3x + \frac{\mu_0 I^2\pi^2}{8}\int_0^{2\pi}\rho\, d\phi\int_0^{l} dz\int_R^{\infty} d\rho\, \frac{\rho}{\rho^4}$$

$$= \frac{\mu_0\pi^2 l I^2}{4}$$

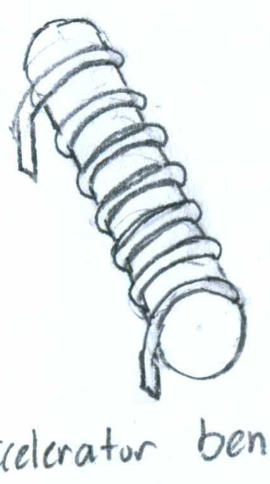
c) (Equation 5.152) $W = \sum_{i=1}^{N} L_i I_i^2$

$$L = \frac{W}{I^2}$$

$$= \frac{\mu_0\pi^2 l}{4}$$

$$\frac{L}{l} = \frac{\mu_0\pi^2}{4}$$

5.31.

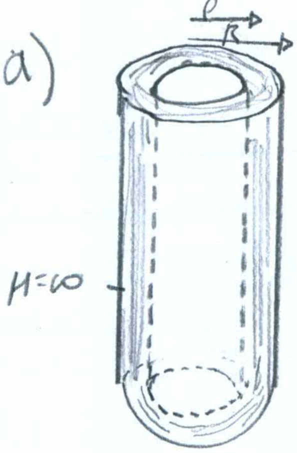


Accelerator bending magnet of Superconducting cable

Shape: Rings
Dimension: Area [2D]
Charge: q

$$J_z(\rho,\phi) = \left(\frac{NI}{2R}\right)\cos\phi\cdot\delta(\rho-R)$$

a)



Hollow iron cylinder

Shape: cylinder
Dimension: Volume[3D]
Charge: q

a) $J_z(\rho,\phi) = \left(\frac{NI}{2R}\right)\cos\phi\,\delta(\rho-R)$

(Equation 5.32)

$$A(x) = \frac{\mu_0}{4\pi}\int d^3x \times \frac{\vec{J}(x)}{|x-x'|}$$

(Equation 3.148)

$$\frac{1}{|x-x'|} = \frac{2}{\pi}\sum_{m=-\infty}^{\infty} dk\, e^{im(\phi-\phi')}\cos[k(z-z')]\, I_m(k\rho_<)K_m(k\rho_>)$$

$$A(x) = \frac{\mu_0}{4\pi}\int d^3x\left(\frac{NI}{2R}\right)\cos\phi\,\delta(\rho-R)\,\frac{2}{\pi}\int dk\, e^{im(\phi-\phi')}\cos[k(z-z')]\, I_m(k\rho_<)K_m(k\rho_>)$$

$$Re(A) = \frac{\mu_0 NI}{4\pi^2 R}\int_0^{2\pi}\cos\phi\cos(\phi-\phi')\cdot R\,d\phi\int_{-\infty}^{\infty}\int_0^{\infty}\cos[k(z-z')]\, I_m(k\rho_<)K_m(k\rho_>)\,dz\,dk$$

$$= \frac{\mu_0 NI}{2\pi R}\cos\phi\int_0^{\infty} e^{ikz}\, I(k\rho_<)K_m(k\rho_>)\,\delta(\rho-R)\,dk$$

$$= \frac{\mu_0 NI}{2}\cos\phi\left[\int_0^{\infty} e^{ikz}\frac{1}{\Gamma(2)}\left(\frac{k\rho_<}{2}\right)\frac{\Gamma(1)}{2}\left(\frac{2}{k\rho_>}\right)\right]$$

$$= \frac{\mu_0 NI}{4}\cos\phi\,\frac{\rho_<}{\rho_>}$$

(Equation 5.27)

$$B = \nabla\times A$$

$$= \hat\rho\,\frac{1}{\rho}\frac{\partial A}{\partial\phi} - \hat\phi\,\frac{\partial A}{\partial\rho}$$

$$= \frac{\mu_0 NI}{4}\left[-\frac{\hat\rho\sin\phi}{\rho}\frac{\rho_<}{\rho_>} - \cos\phi\begin{cases}\frac{1}{R} & \rho<R\\ -\frac{R}{\rho^2} & \rho>R\end{cases}\right.$$

@ $\phi = \pm\pi/2$

$$= +\frac{1}{4}\mu_0 NI\begin{cases}\frac{\hat\rho}{R}\\ \frac{R}{\rho^2}\hat\rho\end{cases}$$ where $$B_0 = \frac{\mu_0 NI}{4R}\left[1+\frac{R^2}{\rho^2}\right]$$

b) Inside (with iron core):

(Equation 5.148)

$$W = \frac{1}{2}\int H \cdot B\, d^3x$$

$$= \frac{1}{2\mu_0}\int H \cdot B\, d^3x$$

$$W_T = \frac{1}{2\mu_0}\int d^3x\, B \cdot B + \frac{1}{2\mu_0}\int d^3x\, B \cdot B$$

$$= \frac{1}{2\mu_0}\int_{\rho<R} d^3x \left(\frac{\mu_0 N I}{4R}\right)^2 + \frac{1}{2\mu_0}\int_{\rho>R} d^3x \left(\frac{\mu_0 N I R}{4\rho}\right)^2$$

$$= \frac{\mu_0 N^2 I^2}{16 R^2}\int_{\rho<R} d^3x + \frac{\mu_0 N^2 I^2 R^2}{16}\int_0^{2\pi} \rho\, d\phi \int_0^{\ell} dz \int_R^{\infty} d\rho\, \frac{1}{\rho^4}$$

$$= \frac{\mu_0 N^2 I^2}{16 R^2}\pi R^2 \ell + \frac{\mu_0 N^2 I^2 R^2}{16} 2\pi \ell \left[-\frac{1}{2\rho^2}\right]_R^{\infty}$$

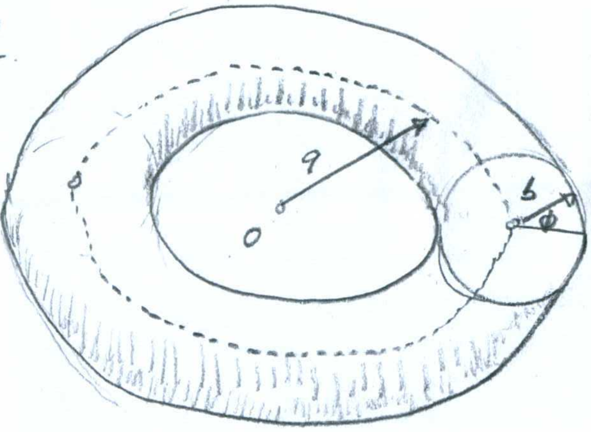
$$= \frac{\mu_0 \pi N^2 I^2 \ell}{16} + \frac{\mu_0 \pi \ell I^2 N^2}{16}$$

vs only the outside $\frac{\mu_0 \pi \ell I^2}{16}$

c) $\frac{dL}{dz} = \frac{1}{I^2}\frac{\partial W}{\partial \ell}$

$$= \frac{\mu_0 \cdot \pi \cdot N^2}{8} \cdot \left[1 + \frac{R^2}{R'^2}\right]$$

5.32



A circular loop

Shape: Torus
Dimension: Volume [3D]
Charge: q

a) (Equation 5.37)

$$A_\phi(r,\theta) = \frac{\mu_0}{4\pi} \frac{4Ia}{\sqrt{a^2+r^2+2ar\sin\theta}} \left[\frac{(2-k^2)K(k)-2E(k)}{k^2} \right]$$

Where $k^2 = \frac{4ar\sin\theta}{a^2+r^2+2ar\sin\theta}$

$$= \frac{\mu_0}{4\pi} \frac{I}{\sqrt{1+\frac{r^2}{a^2}+\frac{r}{a}\cos\phi}} \left[\frac{(2-\bar{k}^2)K(k)-2E(k)}{\bar{k}^2} \right]$$

where

$$\bar{k}^2 = \frac{1}{a} \frac{4r\sin\theta}{(1+\frac{r^2}{a^2}+\frac{r}{a}\cos\phi)}$$

$$= \frac{\mu a I}{2\pi\sqrt{r\sin\theta}} \left[\frac{(2-k^2)K(k)-2E(k)}{k} \right]$$

If $k=1$, then $E(1)=1$,

and $\frac{K(r)}{\sqrt{r}} \cong \ln\left(8\frac{a}{x}\right)$ between $0.20 \le \frac{a}{b} \le 0.80$

at $a=1$ and $\phi = n2\pi$.

$$= \frac{\mu_0 I}{2\pi} \left[\ln\left(8\frac{a}{r}\right) - 2 \right]$$ between a $\frac{\text{diameter}}{\text{radius}}$ ratio

from 0.20 to 0.80a

b) Inside: $A_\phi = A_{\phi,\text{inside}} + A_{\phi,\text{outside}}$

$$= \frac{\mu_0 I}{4\pi}\left(1 - \frac{\rho^2}{b^2}\right) + \left(\frac{\mu_0 I}{2\pi}\right)\left(\ln\left(\frac{9a}{b}\right) - 2\right)$$

c) (Equation 5.149) $W = \frac{1}{2}\int J_0 \cdot A\, d^3x$ ... when $\frac{J_0 A}{2} = LI^2$

$$= L \cdot I^2$$

(Equation 5.27) $B = \nabla \times A_\phi$

$$= \frac{\mu_0 I}{2\pi}\frac{\rho}{\partial a}\frac{\partial a}{\rho^2}$$

$$= \frac{\mu_0 I}{2\pi}\frac{1}{\rho}$$

$$L = \frac{1}{I^2}\int \frac{B^2}{\mu} dV$$

$$= \frac{1}{I^2 \mu}\int\left(\frac{\mu_0 I}{2\pi}\frac{1}{\rho}\right)^2 dV$$

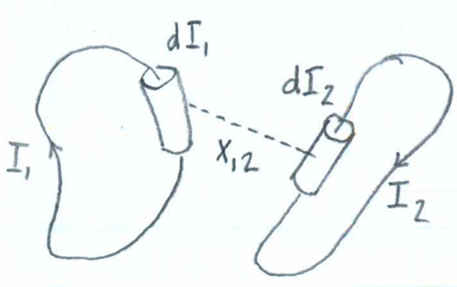
$$= \frac{\mu_0}{4\pi^2}\int_0^{2\pi}\int_0^{l}\int_b^{a}\frac{1}{\rho^2}\rho\, d\rho\, d\phi\, dz$$

$$= \frac{\mu_0}{4\pi^2} 2\pi \cdot l\left[\ln(a) - \ln(b)\right]$$

$$= \frac{\mu_0}{2\pi} l \ln\left(\frac{a}{b}\right)$$

$$\frac{dL}{dl} = \frac{\mu_0}{2\pi}\ln\left(\frac{a}{b}\right)$$

5.33



Two Amperian Loops

Shape: Lines
Dimension: line [1D]
Charge: q

(Equation 5.10) $$F_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint\oint \frac{(dI_1 \circ dI_2) X_{12}}{|X_{12}|^3}$$

$$\text{IF } M_{12}(R) = \frac{\mu_0}{4\pi} \oint\oint \frac{dI_1 \cdot dI_2}{|X_1 - X_2 + R|}$$

$$\text{then } \nabla M_{12}(R) = \frac{\mu_0}{4\pi} \oint\oint \nabla \frac{dI_1 \cdot dI_2}{|X_1 - X_2 + R|}$$

$$= -\frac{\mu_0}{4\pi} \oint\oint \left(\frac{(X_1 - X_2 + R}{|X_1 - X_2 + R|^3} \right) dI_1 \cdot dI_2$$

$$= -\frac{\mu_0}{4\pi} \oint\oint \left(\frac{X_{12}}{|X_{12}|^3} \right) dI_1 \cdot dI_2$$

So, $F_{12} = I_1 I_2 \nabla M_{12}(\vec{R})$

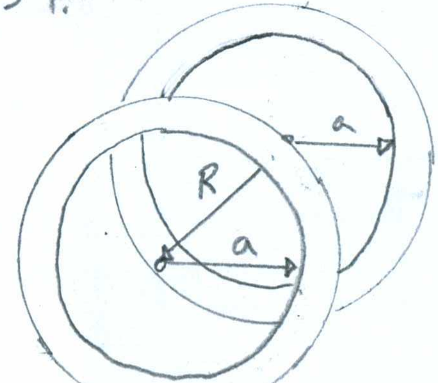
b) $$\nabla^2 M_R(R) = \frac{\mu_0}{4\pi} \oint\oint \nabla^2 \left(\frac{dI_1 \cdot dI_2}{|X_1 - X_2 + R|} \right)$$

$$= \frac{\mu_0}{4\pi} \oint\oint (-4\pi \delta(X_1 - X_2 + R) dl_1 dl_2$$

$$= \mu_0 \oint\oint \delta(X_1 - X_2 + R) dI_1 \cdot dI_2$$

$$= 0$$

5.34.



Two Identical Circular loops

Shape: Rings
Dimension: Volume[3D]
Charge: q

(Equation 5.10b)

$$W_{12} = \int d^3x\, J_1 \circ A_2$$

$$A_\phi(\rho, z) = \frac{\mu_0 I a}{2}\int_0^\infty dk\, e^{-k|z|} J_1(ka) J_1(k\rho)$$

$$W_{12} = \int_0^{2\pi} d^3x\, J_1 \frac{\mu_0 I_a}{2}\int_0^\infty dk\, e^{-k|z|} J_1(k\rho) J_2(k\rho)$$

$$= \mu_0 I_1 I_2 \pi a^2 \int_0^\infty dk\, e^{-k|R|} J_1(ka) J_1(ka) \quad \ldots \text{ when } z = R$$

$$M_{12} = \mu_0 \pi a^2 \int_0^\infty e^{-k|R|} J_1^2(ka)\, dk$$

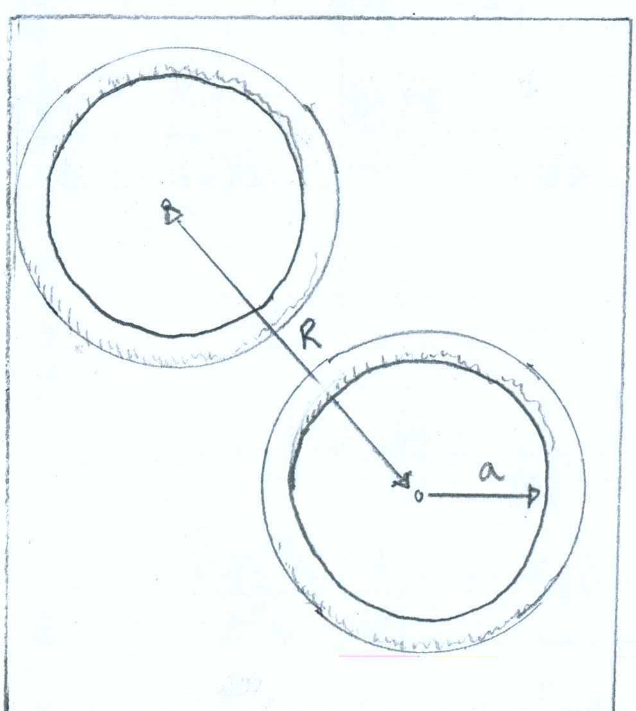
b) $$|J_1(ka)|^2 = \frac{a^2k^2}{4} - \frac{a^4k^4}{16} + \frac{5a^6k^6}{768} - \frac{7a^8k^8}{18432}$$

$$M_{12} = \mu_0 \pi a^2 \int_0^\infty e^{-k|R|} J_1^2(ka)\, dk$$

$$= \mu_0 \pi a^2 \int_0^\infty e^{-k|R|}\left[\frac{a^2k^2}{4} - \frac{a^4k^4}{16} + \frac{5a^6k^6}{768} - \frac{7a^8k^8}{18432}\right] dk$$

$$= \frac{\mu_0 a \pi}{2}\left(\frac{a^3}{R^3} - \frac{3a^5}{R^5} - \frac{75a^7}{8R^7} - \frac{245a^9}{8R^9}\right)$$

c)



Two coplanar identical circular loops

Shape: Rings
Dimension: Volume [3D]
Charge: q

A similar solution exists in Problem 3.12b.

Potential Derivation by Boundary Conditions:

① Boundary Conditions:

$$A(r=a, \phi, z=0) = 0$$

$$A(r, \phi, z=0) = \frac{\mu_0 I}{2}$$

$$A(r=0, \phi, z) = \text{finite}$$

② Laplace's Equation:

$$\nabla^2 A = \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial A}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A}{\partial \phi^2} + \frac{\partial^2 A}{\partial z}$$
$$= 0$$

③ Laplace's Equation Solutions:

If $A(r,\phi,z) = R(r)\,Q(\phi)\,Z(z)$, then

Ⓐ Variable Separation:

$$\nabla^2 A = \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial A}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A}{\partial \phi^2} + \frac{\partial^2 A}{\partial^2 z}$$

$$= \frac{Q(\phi)Z(z)}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial R(r)}{\partial r}\right) + \frac{R(r)Z(z)}{r^2}\frac{\partial^2 Q(\phi)}{\partial^2 \phi} + R(r)Q(\phi)\frac{\partial^2 Z(z)}{\partial^2 z}$$

$$= \frac{1}{R(r)\cdot r}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right) + \frac{1}{Q(\phi)\, r^2}\frac{\partial^2 Q(\phi)}{\partial^2 \phi} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z}$$

$$= 0$$

Ⓑ Radial Eigenvalues:

$$\frac{r}{R}\frac{\partial R}{\partial r} + \frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z} = \lambda r^2$$

$$\frac{r}{R}\frac{\partial R}{\partial r} + \frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} - \lambda r^2 = \frac{-1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 \phi}$$

$$\frac{r}{R}\frac{\partial R}{\partial r} + \frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} - \lambda r^2 = k^2$$

If $r = x$ and $\lambda = \frac{1}{r^2}$, then

$$\frac{\partial^2 R}{\partial^2 x} + \frac{1}{x}\frac{\partial R}{\partial x} - \left(1 + \frac{k^2}{x^2}\right) R = 0$$

© Angular Eigenvalues:

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial^2 \phi} = m^2 ; \qquad \frac{\partial^2 Q}{\partial^2 \phi} + m^2 Q = 0$$

(D) Vertical Eigenvalues:

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = k^2 ; \qquad \frac{\partial^2 Z}{\partial^2 z} - k^2 Z = 0$$

(E) General Solutions:

$$A(r,\phi,z) = \sum_{k,m} R(r)\, Q(\phi)\, Z(z)$$

(4) General Solution to Laplace's Equation:

$$R(r) = E \cdot J_v(kx) + F \cdot Y_v(kx) \qquad \text{when } r = x$$

where

$$J_v(kx) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(-\frac{1}{2}kx\right)^{v+2m}}{m!\,(m+v)!}$$

$$Y_v(kx) = \sum_{m=0}^{\infty} \frac{\cos(v\pi) \cdot J_v(kx) - J_{-v}(kx)}{\sin(v\pi)}$$

$$Q(\phi) = A \sin(m\phi)$$

$$Z(z) = B e^{-kz}$$

(5) Variables by Boundary Conditions:

$$A(r,\phi,z) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \left[E \cdot J_v(kx) + F\, Y_v(kx)\right]\left[A \sin(m\phi)\right]\left[B e^{-kz}\right]$$

$$k \quad A(r=a,\phi,z=0)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}[E\cdot J_v(ka)+FY_v(ka)][A\sin(m\phi)][Be^{-k0}]$$

$$=0\ ,\ \text{so}\ \ k=\frac{n\pi}{L}$$

$$F \quad A(r=0,\phi,z)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}[E\cdot J_v(k\cdot 0)+F\cdot Y_v(k\cdot 0)]\sin(m\phi)e^{-kz}$$

$$=\text{finite}\ ,\ \text{so}\ \ F=0$$

$$E \quad A(r,\phi,z=0)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}E\cdot J_v(kx)\sin(m\phi)e^{-k\cdot 0}$$

$$=\frac{\mu_0 I}{2}$$

$$\int_0^a\int_0^{2\pi}\frac{\mu_0 I}{2}\cdot J_v(kx)\cdot\sin(m\phi)\,x\,d\phi\,dx=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\int_0\int_0 E\cdot J_v^2(kx)\sin^2(m\phi)\,x\,d\phi\,dx$$

$$E=\frac{k}{\pi}\int_0^a\int_0^{2\pi}\frac{\mu_0 I}{2}\sin(m\phi)J_v(kx)\,x\,d\phi\,dx$$

$$=\frac{\mu_0 I}{2\pi}\int_0^{2\pi}\sin(m\phi)\,J_{v+1}(ka)\cdot a\,d\phi$$

$$A(r,\phi,z)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\mu_0 I a}{2\pi}J_{v+1}(ka)\cdot J_v(kx)\int_0 \sin^2(m\phi)\,d\phi\, e^{-kz}$$

$$=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\mu_0 I a}{2}J_{v+1}(k\cdot a)\cdot J_v(k\cdot x)\,e^{-kz}$$

At an infinite distance from the center:

$$A(r=0,\phi,z)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{\infty}\frac{\mu_0 I}{2}a\cdot J_{v+1}(ka)\,J_v(k\cdot 0)\,e^{-kz}\,dk$$

$$=\frac{\mu_0 I a}{2}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{\infty}J_{v+1}(k\cdot a)\cdot e^{-kz}\,dk$$

$$\boxed{\begin{aligned}ka&=x\\ k&=\frac{x}{a}\\ dk&=\frac{dx}{a}\end{aligned}}$$

$$=\frac{\mu_0 I a}{2}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{\infty}\left[\frac{1}{2\pi i}\int_0^{\infty}e^{i(x\cos\phi+\phi)-xz/a}\,d\phi\,dx\right]$$

Integral Representation of a Bessel Function:

$$J_m(x)=\frac{1}{2\pi i}\int e^{i(x\cos\phi+n\phi)}\,d\phi$$

$$=\frac{\mu_0 I a}{4\pi}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{\infty}\frac{e^{i\phi}}{(\cos\phi+iz/a)}\,d\phi$$

$$=\frac{\mu_0 I a}{4\pi}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left[\int_0^{\infty}\frac{a(a\cos^2\phi+z\sin\phi)}{a^2\cos^2\phi+z^2}\,d\phi+\int_0^{\infty}\frac{i\,a\cos(\phi)(a\sin(\phi)-z)}{a^2\cos^2\phi+z^2}\,d\phi\right]$$

$$=\frac{\mu_0 I a}{2\pi}\int_0^{2\pi}\frac{\cos^2\phi}{\cos^2\phi+z^2/a^2}\,d\phi$$

$$=\frac{\mu_0 I a}{2\pi}\left[\pi-\frac{2}{\pi}\frac{z^2}{a^2}\int_0^{\pi/2}\frac{d\phi}{(\cos^2\phi+z^2/a^2)}\right]$$

$$u=\frac{1}{\sqrt{1+a^2/z^2}}\;;\;\cos^2\theta=\frac{1}{1+(1+a^2/z^2)u^2}\;;\;d\theta=\frac{\sqrt{1+a^2/z^2}}{1+(1+a^2/z^2)u^2}$$

$$=\frac{\mu_0 I a}{2}\left[1-\frac{2}{\pi}\frac{z}{\sqrt{z^2+a^2}}\left(\tan^{-1}(\infty)-\tan^{-1}(0)\right)\right]$$

$$\cong\frac{\mu_0 I a}{2}\left[1-\frac{z}{\sqrt{z^2+a^2}}\right]$$

$$=\frac{\mu_0 I a}{2}\left(\frac{a^2}{2z^2}-\frac{3a^4}{8z^4}+\frac{5a^6}{16z^6}-\frac{35a^8}{128z^8}+\frac{63a^{10}}{256z^{10}}\right)$$

$$= \frac{\mu_0 I a}{2}\left(\frac{a^2}{2r^2}P_1(\cos\theta) - \frac{3a^4}{8r^4}P_3(\cos\theta) + \frac{5a^6}{16r^6}P_5(\cos\theta)\right.$$
$$\left. - \frac{35a^8}{128r^6}P_7(\cos\theta) + \frac{63a^{10}}{256r^{10}}P_9(\cos\theta) + \cdots\right)$$

$$B_z(r, \pi/2) = \frac{1}{r}\frac{\partial\phi}{\partial\theta}\Big|_{\theta=\pi/2}$$
$$= \frac{\mu_0 I_1 a}{2}\left(\frac{a^2}{2r^3} + \frac{9a^4}{8r^5} + \frac{375a^6}{128r^7} + \frac{1225a^8}{2048r^9} + \frac{19845a^{10}}{32768r^{11}}\right)$$

$$\Phi_{12} = \int B d^3x$$
$$= \int_0^a \rho\, d\rho \int_0^{2\pi} B_z\, d\phi$$
$$= \frac{\mu_0 I \pi a}{4}\left(\frac{a^3}{R^3} + \frac{9a^5}{4R^5} + \frac{375a^7}{64R^7} + \frac{8575a^9}{512R^9}\right)$$

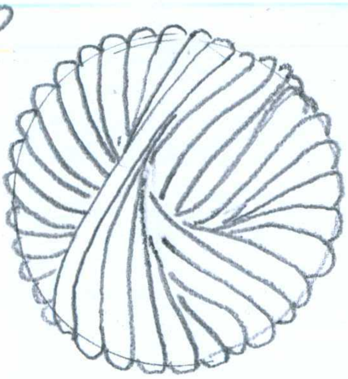
$$\Phi = M_{12} I$$
$$M_{12} = \frac{\mu_0 \pi a}{4}\left(\left(\frac{a}{R}\right)^3 + \frac{9}{4}\left(\frac{a}{R}\right)^3 + \frac{375}{64}\left(\frac{a}{R}\right)^7 + \cdots\right)$$

d) (Equation 5.12) $F = \int J(x) \times B(x) d^3x$
$$= I_2 \int_0^R B_z(\rho=a, \phi)\, dr$$
$$= \frac{\mu_0 I_1 I_2 \pi}{2}\left(\frac{3a^4}{2R^4} + \frac{45a^6}{8R^6} + \frac{2625a^8}{128R^8} + \frac{97175a^{10}}{1024R^{10}}\right)$$

In a coplanar configuration:
$$F = |2 \cdot F_{12}|$$

$$= I_1 I_2 \mu_0 \pi \left(\frac{3a^4}{2R^4} + \frac{45a^6}{8R^6} + \frac{2625a^8}{128R^8} + \frac{77175a^{10}}{1024R^{10}} \right)$$

5.35



An insulated spheroidal coil

Shape: sphere
Dimension: Volume [3D]
Charge: q

a) $\boxed{\text{Vector Potential by Greens Theorem:}}$

(Equation 5.103)

$$A(x) = \underbrace{\frac{\mu_0}{4\pi}\int \frac{\nabla' \times M(x')}{|x-x'|} d^3x}_{=0,} + \frac{\mu_0}{4\pi}\oint \frac{M(x') \times n'}{|x-x'|} da'$$

$$= \frac{\mu_0}{4\pi}\oint \frac{M(x') \times n'}{|x-x'|} da'$$

$$M(x) \times n' = M_0 \sin\theta \, \epsilon_\phi$$

$$= M_0 \sin\theta' (-\sin\phi' \epsilon_1 + \cos\phi' \epsilon_2)$$

$$= M_0 \sin\theta' \cos\theta'$$

$$A(x) = \frac{\mu_0 M_0 a^2}{4\pi} \int da' \frac{\sin\theta' \cos\theta'}{|x-x'|}$$

$$= \frac{\mu_0 M_0 a^2}{4\pi} \int -\sqrt{\frac{8\pi}{3}} \frac{Re[Y_{1,1}(\theta, \phi')]}{|x-x'|} d\Omega$$

(Equation 3.70)

$$\frac{1}{|x-x'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta', \phi') Y_{lm}(\theta, \phi)$$

$$A(x) = \frac{\mu_0 M a^2}{4\pi} \int -\sqrt{\frac{8\pi}{3}} - \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\phi} \cdot 4\pi \left(\frac{1}{3}\right)\left(\frac{r_<}{r_>^2}\right) \quad @ \, l=1$$

$$= \frac{\mu_0}{3} \frac{r_<}{r_>^2} M_0 a^2 \sin\theta \, \hat{\phi}$$

(Equation 5.105) $B_0 = \frac{2\mu_0}{3} M_0$

$A = \frac{B_0 a^3}{2} \frac{r_<}{r_>^2} \sin\theta \, \hat{\phi}$

The equation derives from line 5.53, 5.54 and 5.62

b) (Equation 5.160) $\nabla^2 A = \mu_0 \sigma \frac{\partial A}{\partial t}$

(Equation 3.113) $A(r) = \int \bar{A}(k) j_\ell(kr)\, dk$

where $\bar{A}(k) = \frac{2k^2}{\pi} \int_0^\infty r^2 A(r) j_1(kr)\, dr$

$\nabla^2 A = \mu_0 \sigma \frac{\partial A}{\partial t}$

$-\int_0^\infty dk\, A(k,t) J_1(k\rho) k^2 = \mu_0 \sigma \int_0^\infty dk \frac{\partial}{\partial t} A(kt) J(kr) \sin\theta$

$-k^2 A(k,t) = \mu_0 \sigma \frac{\partial}{\partial t} A(k,t)$

$A(k,t) = e^{-(k^2 z/\mu_0\sigma)t}$ as a general solution

Equation 3.113 becomes...

$A(r) = \int \bar{A}(k) j_\ell(kr)\, dk$

$= \frac{2k^2}{\pi} \frac{B_0 a^3}{2} \frac{r_<}{r_>^2} \sin\theta \int_0^\infty e^{-(k^2 z/\mu_0\sigma)t} j_1(k) j_1\left(\frac{kr}{a}\right) dk$

$= \frac{B_0 a}{\pi} \cdot \sin\theta \int_0^\infty e^{-vtk^2} \cdot j_1(k) \cdot j_1\left(\frac{kr}{a}\right) dk$

when $v = \frac{1}{\mu_0 \sigma}$ and $\frac{r_<}{r_>} = \frac{1}{a}$

Identities: $j_l'(x) = \frac{l}{x} j_l(x) - j_{l+1}(x)$

$$\int_0^\infty dx\, j_l(kx)\, j_l(Kx) = \frac{\pi}{2(2l+1)} \frac{k_<^l}{k_>^{l+1}}$$

$$\int_0^\infty dx\, j_1(ux)\, j_1(vx) = \frac{1}{u^2} F(u|v)$$

A generic function.

Generic mathematical Principles:

$$\frac{d}{du}\left[\frac{1}{u^2} F(u|v)\right] = -\frac{2}{u^3} F(u|v) + \frac{1}{u^2 v} F'(u/v)$$ Product Rule

$$= \int_0^\infty dx\, x^2 \left[\frac{j_1(ux)}{ux} - j_2(ux)\right] j_2(ux)$$

$$= \frac{1}{u^3} F(u|v) - \frac{\pi}{2v^3} \delta(u|v - 1)$$ Substitution

$$F'\left(\tfrac{u}{v}\right) - \frac{3}{(u|v)} F\left(\tfrac{u}{v}\right) = -\frac{\pi}{2} \delta\left(\tfrac{u}{v} - 1\right)$$ Arrangement

$$F\left(\tfrac{u}{v}\right) = \left(\tfrac{u}{v}\right)^3 A \quad \text{when } \tfrac{u}{v} < 1$$

$$\left(\tfrac{u}{v}\right)^3 (A - \pi/2) \quad \text{when } \tfrac{u}{v} > 1$$ Dirac

So, $A = \frac{1}{3} \int_0^\infty dx\, x^2 j_2(x)$ Equality

$= \pi/2$

$$F(u|v) = \left(\frac{u}{v}\right)^3 \left(\frac{\pi}{2}\right) \delta\left(\frac{u}{v} - 1\right)$$ Answer

c) (Equation 5.148) $W = \frac{1}{2}\int H \circ B d^3x$

$$= \frac{1}{2\mu}\int B \circ B d^3x$$

$$W_M = W_r + W_\theta$$

$$= \frac{1}{2\mu}\int B_r \circ B_r d^3x + \frac{1}{2\mu}\int B_\theta \circ B_\theta d^3x$$

$$= \frac{\pi}{\mu} B_0^2 \int_0^\infty dr\, r^2 \int_0^\pi d\theta \sin\theta \, (B_r^2 + B_\theta^2)$$

$$= \frac{\pi}{\mu} B_0^2 \int_0^\infty dr\, r^2 \int_0^\pi d\theta \sin\theta \left(\frac{4a^2}{r^2}\cos^2\theta + \frac{a'^2 \sin^2\theta}{r^2} + a'^2 \sin^2\theta + \frac{2aa'\sin^2\theta}{r}\right)$$

$$= \frac{\pi}{\mu} B_0^2 \circ x \, i \left(a^2 + \frac{1}{3} r^2 a'^2 + \frac{2raa'}{3}\right)$$

$$= \frac{9B_0^2}{\pi\mu} \int_0^\infty ds\, e^{-s\nu t} j_1(\sqrt{s}) \int_0^\infty ds' e^{-s'\nu t} j_1(\sqrt{s'})$$

$$\times \int_0^\infty dr \left[\frac{j_1(\sqrt{s}r)\, j_1(\sqrt{s'}r)}{\sqrt{ss'}} + \frac{r^2}{3} j_1'(\sqrt{s}r)\, j_1'(\sqrt{s'}r) + \frac{2r}{3\sqrt{s}} j_1(\sqrt{s}r)\, j_1'(\sqrt{s'}r) \right]$$

$\circ\circ\circ$ when $a = e^{-s\nu t} j_1(\sqrt{s}) j_1(\sqrt{s}r)$

Identity citation: J. K. Bloomsfield, S.H.P. Face, Z. Moss, Indefinite Integrals of Spherical Bessel Functions.

Thus, $\int_0^\infty dx \circ x \circ j_1(ux) j_2(vx) = \frac{\pi u}{2v^3}\Theta\left(\frac{u}{v}-1\right)$

$$W_M = \frac{9B_u^2}{\pi\mu}\int_0^\infty ds\, e^{-s\nu t} j_1(\sqrt{s}) \int_0^\infty ds'\, e^{-s'\nu t} j_1(\sqrt{s'}) \circ \left[\frac{\pi}{3s_>^{3/2}} + \frac{\pi}{3\sqrt{s}}\delta(s-s') - \frac{2\pi}{3s'\sqrt{s}}\theta(s-s')\right]$$

$$= \frac{3B_0^2}{\mu}\int_0^\infty ds\, e^{-s\nu t} j_1(\sqrt{s}) \int_0^\infty ds'\, e^{-s'\nu t} j_1(\sqrt{s'}) \circ \left[\frac{1}{s_>^{3/2}} + \frac{1}{\sqrt{s}}\delta(s-s') - \frac{1}{s\sqrt{s}}\theta(s-s') - \frac{1}{s'\sqrt{s'}}\theta(s-s')\right]$$

$$= \frac{3B_0^2}{\mu}\int_0^\infty ds\, e^{-s\nu t} j_1(\sqrt{s}) \int_0^\infty ds'\, e^{-s'\nu t} j_1(\sqrt{s'}) \frac{1}{\sqrt{s}}\delta(s-s')$$

$$= \frac{3B_0^2}{\mu}\int_0^\infty \frac{ds}{\sqrt{s}} e^{-2s\nu t} j_1^2(\sqrt{s})$$

$$= \frac{3B_0^2}{\mu}\int_0^\infty dk\, e^{-2\nu k^2 t} j_1^2(k)$$

When $\nu t \gg 1$, $W_M \approx \frac{3B_0^2}{\mu}\int \frac{ds}{\sqrt{s}} e^{-2s\nu t}\left(\frac{s}{9}\right)$

$$\approx \frac{1}{3}\circ(2t)^{-3/2}\cdot\Gamma(3/2)\frac{B_0^2}{\mu}$$

$$\approx \frac{\sqrt{\pi}\, B_0^2}{24\mu(\nu t)^{3/2}}$$

d) <u>Vector Potential:</u>

$$A_\theta = \frac{3B_0 a}{\pi}\sin\theta\int_0^\infty e^{-\nu t k^2} j_1(k) j_1\left(\frac{kr}{a}\right) dk$$

$$\cong \frac{B_0 a}{\pi}\left(\frac{1}{3}\right)\circ(2\nu t)^{-3/2}\circ\Gamma(3/2)$$

$$\cong \frac{B_0 a}{24\sqrt{\pi}(\nu t)^{3/2}}$$

Magnetic Induction:

$$B = \frac{3B_0 a}{\pi}\int_0^\infty ds \frac{e^{-s\nu t}}{s} j_1(\sqrt{s})\, j_1\left(\frac{kr}{a}\right)$$

$$= \frac{B_0 a}{6\sqrt{\pi}(\nu t)^{3/2}}$$

6.36 a) (Equation 5.168) $J = \sigma E_y$

(Equation 5.168.5) $K_y = \int_0^\infty J_y(z,t)\,dz$

$$E_y = \frac{1}{\sigma}\frac{dk}{dz}$$

b) $P = -\frac{\partial W_m}{\partial t}$

$$= -\frac{\partial}{\partial t}\left[\frac{6B_0^2 a^3}{\mu}\int_0^\infty e^{-2\nu t k^2} j_1^2(k)\,dk\right]$$

$$= \frac{12 B_0^2 a^3 \nu}{\mu}\int_0^\infty e^{-2\nu t k^2} k\, j_1^2(k)\,dk$$

c) $P = -\frac{\partial}{\partial t} W_T$

$$= -\frac{\partial}{\partial t}\left[W_e + W_m\right]$$

$$= \frac{12 B_0^2 a^3 \nu}{\mu}\int_0^\infty e^{-2\nu t k^2} k\, j_1^2(k)\,dk$$