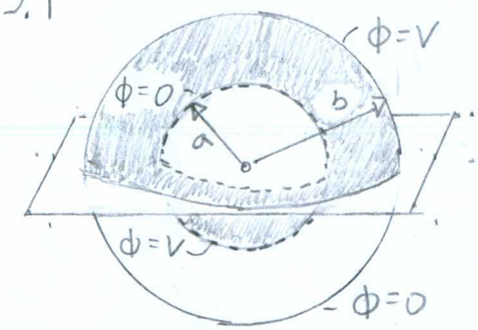
Chapter 3: Boundary-Value Problems in Electrostatics II:

3.1



Two concentric spheres

Shape: Sphere
Dimension: Volume [3D]
Charge: q

Potential Derivation:

① Boundary Conditions:

$$\Phi(\rho = a, \theta > \tfrac{\pi}{2}, \phi = 0) = 0$$

$$\Phi(\rho = b, \theta < \tfrac{\pi}{2}, \phi = 0) = V$$

$$\Phi(\rho = a, \theta < \tfrac{\pi}{2}, \phi = 0) = 0$$

$$\Phi(\rho = b, \theta > \tfrac{\pi}{2}, \phi = 0) = V$$

② Laplace's Equation:

$$\nabla^2 \Phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \Phi}{\partial \phi}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2}$$

$$= 0$$

③ Laplace's Equation Solutions:

Ⓐ Variable separation:

$$\nabla^2 \Phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\Phi^2}$$

$$= 0$$

If, $\Phi = \frac{U(r)}{r} P(\theta) Q(\Phi)$

$$PQ\frac{\partial^2 U}{\partial^2 r} + \frac{UQ}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{UP}{r^2\sin^2\theta}\frac{\partial^2 Q}{\partial\phi^2} = 0$$

Ⓑ Azimuthal Eigenvalues:

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial\phi^2} = -m^2 \; ; \quad Q'' + mQ = 0$$

Ⓒ Angular Eigenvalues:
$$\frac{1}{P\sin^2\theta}\frac{d}{\partial\theta}\left(\sin\theta\frac{dP}{\partial\theta}\right) = \lambda - \ell(\ell+1)$$

$$\frac{1}{P\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{dP}{\partial\theta}\right) + \left[\ell(\ell+1) + \frac{\lambda}{\sin^2\theta}\right]P = 0$$

Ⓓ Radial Eigenvalues:
$$\frac{r^2}{U}\frac{\partial^2 U}{\partial r^2} = -\ell(\ell+1)$$

$$\frac{\partial^2 U}{\partial r^2} - \frac{\ell(\ell+1)}{r^2}U = 0$$

④ General Solution to Laplace's Equation:

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}(A_\ell r^\ell + B_\ell r^{-\ell-1})(A_m + B_m\phi)P_\ell^{m=0}(\cos\theta) \qquad \boxed{m=0}$$

$$+ \sum_{\ell=0}^{\infty}(A_\ell r^\ell + B r^{-\ell-1})(A_m e^{im\phi} + B_m e^{-im\phi})P_\ell^m(\cos\theta) \qquad \boxed{m\,!=0}$$

⑤ Variables by Boundary Conditions:

$A_{m=0}^{m=0}, B_{m=0}^{m=0}$
$$\Phi(r=a,\theta>\tfrac{\pi}{2},\phi=0) = \sum_{\ell=0}^{\infty}(A_\ell r^\ell + B_\ell r^{-\ell-1})(A_m + B_m\cdot 0)P_\ell^{m=0}(\cos\theta)$$

A_m, B_m
$$+\sum_{\ell=0}^{\infty}(A_\ell r^\ell + B r^{-\ell-1})(A_m e^{im\phi} + B_m e^{-im\phi})P_\ell^m(\cos\theta)$$

$= 0$, so $A_m = B_m = B_{m=0} = 0$

$A_{m=0} = e^{-im\phi}$

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}(A_\ell r^\ell + B_\ell r^{-\ell-1})P_\ell(\cos\theta)$$

A_ℓ, B_ℓ
$$\Phi(r=a,\theta<\tfrac{\pi}{2},\phi=0) = \sum_{\ell=0}^{\infty}(A_\ell\cdot a^\ell + B_\ell a^{-\ell-1})P_\ell(\cos\theta)$$

$= V$

$$V[P(\cos\theta)\sin\theta] = \sum_{l=0}^{\infty} (A_l a^{-l} + B_l a^{-l-1}) P_l(\cos\theta)^2 \sin\theta$$

$$\int_0^{\pi} V P(\cos\theta)\sin\theta \, d\theta = \sum_{l=0}^{\infty} (A_l a^{-l} + B_l a^{-l-1}) \int_0^{\pi} P_l(\cos\theta)^2 \sin\theta \, d\theta$$

$$= \sum_{l=0}^{\infty} (A_l a^{-l} + B_l a^{-l-1}) \cdot \frac{2}{2l+1} \cdot \delta_{l \cdot l} \quad \Delta$$

Orthoganality

Legendre Polynomials

$$\int_0^{\pi} P_l(\cos\theta)^2 \sin\theta \, d\theta = \frac{2}{2l+1} \delta_{l \cdot l}$$

$$A_l a^{l} + B_l a^{-l-1} = \frac{(2l+1)V}{2} \int_0^{\pi/2} P_l(\cos\theta)\sin\theta \, d\theta$$

$$= -\frac{(2l+1)}{2} V \int_1^{0} P_l(x) \, dx$$

$$= \frac{2l+1}{2} V \int_0^{1} P_l(x) \, dx$$

$$\phi(r=b, \theta > \tfrac{\pi}{2}, \phi = 0) = \sum_{l=0}^{\infty} (A_l b^{-l} + B_l b^{-l-1}) P_l(\cos\theta)$$

$$= V$$

$$V[P(\cos\theta)\sin\theta] = \sum_{l=0}^{\infty} (A_l b^{-l} + B_l b^{-l-1}) P_l(\cos\theta)^2 \sin\theta$$

$$\int_0^{\pi} V[P(\cos\theta)\sin\theta \, d\theta = \sum_{l=0}^{\infty} (A_l b^{-l} + B_l b^{-l-1}) \int_0^{\pi} P_l(\cos\theta)^2 \sin\theta \, d\theta$$

$$= \sum_{l=0}^{\infty} (A_l b^{-l} + B_l b^{-l-1}) \cdot \frac{2}{2l+1} \delta_{ll}$$

$$A_l b^{-l} + B_l b^{-l-1} = \frac{(2l+1)}{2} V \int_{+\frac{\pi}{2}}^{\pi/2} P_l(\cos\theta) \sin\theta \, d\theta$$

$$= \frac{(2\ell+1)}{2} V \int_1^0 P_\ell(x)\,dx$$

$$A_\ell = \frac{a^{\ell+1} - b^{\ell+1}}{a^{2\ell+1} - b^{2\ell+1}} \frac{(2\ell+1)}{2} V \int_0^1 P_\ell(x)\,dx$$

$$B_\ell = \frac{a^{\ell+1} \cdot b^{\ell+1}(a^\ell - b^\ell)}{a^{2\ell+1} - b^{2\ell+1}} \frac{(2\ell+1)}{2} V \int_0^1 P_\ell(x)\,dx$$

$$\Phi(r,\theta,\phi) = \frac{V}{2} \sum_{\ell=0}^{\infty} (2\ell+1) \int_0^1 P_\ell(x)\,dx \; \frac{(a^{\ell+1} - b^{\ell+1}) r^\ell - ab(a^\ell b^{2\ell} - a^{2\ell} b^\ell) r^{-\ell-1}}{a^{2\ell+1} - b^{2\ell+1}} P(\cos\theta)$$

$@\ell=0$ $\quad \Phi(r,\theta,\phi) = \frac{V}{2} \int_0^1 P_{\ell=0}(x)\,dx$

$$= \frac{V}{2}$$

$\boxed{\text{Legendre Integral}}$

$$\int_0^1 P_\ell(\cos x)\,dx = \frac{1}{2\ell+1}\left[P_{\ell-1}(0) - P_{\ell+1}(0)\right]$$

Only for odd!

$@\ell>0$ $\quad \Phi(r,\theta,\phi) = \frac{V}{2}\left(1 + \sum_{\ell=0}^{\infty} \left[P_{\ell-1}(0) - P_{\ell+1}(0)\right] \frac{(a^{\ell+1} - b^{\ell+1}) r^\ell - ab(a^\ell b^{2\ell} - a^{2\ell} b^\ell) r^{-\ell-1}}{a^{2\ell+1} - b^{2\ell+1}} P(\cos\theta)\right)$

$$\lim_{b\to\infty} \Phi(r,\theta,\phi) = \lim_{b\to\infty} \left[\frac{V}{2}\left(1 + \sum_{\ell=0}^{\infty} \left[P_{\ell-1}(0) - P_{\ell+1}(0)\right] \frac{(a^{\ell+1} - b^{\ell+1}) r^\ell - ab(a^\ell b^{2\ell} - a^{2\ell} b^\ell) r^{-\ell-1}}{a^{2\ell+1} - b^{2\ell+1}} P(\cos\theta)\right)\right]$$

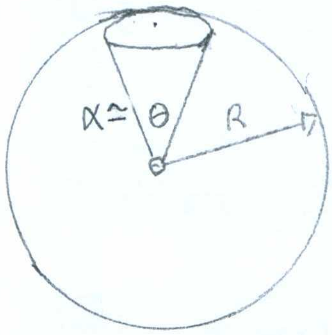
$$= \lim_{b\to\infty} \left[\frac{V}{2}\left(1 + \sum_{\ell=0}^{\infty} \left[P_{\ell-1}(0) - P_{\ell+1}(0)\right] \frac{((a/b)^{\ell+1} + 1)(r/b)^\ell - (1 + (a/b)^\ell)(a/r)^{\ell+1}}{(a/b)^{2\ell+1} - 1} P_\ell(\cos\theta)\right)\right]$$

$$= \frac{V}{2}\left[1 + \sum_{\ell=0}^{\infty} \left[P_{\ell-1}(0) - P_{\ell+1}(0)\right] \left(\frac{a}{r}\right)^{\ell+1} P_\ell(\cos\theta)\right]$$

$$\lim_{a\to\infty}\Phi(r,\theta,\phi)=\lim_{a\to\infty}\left[\frac{V}{2}\left(1+\sum_{\ell=1}^{\infty}[P_{\ell-1}(0)-P_{\ell+1}(0)]\frac{(a^{\ell+1}-b^{\ell+1})r^{\ell}-ab(a^{\ell}b^{2\ell}-a^{2\ell}b^{\ell})r^{-\ell-1}}{a^{2\ell+1}-b^{2\ell+1}}P_\ell(\cos\theta)\right)\right]$$

$$=\frac{V}{2}\left[\left(1+\sum_{\ell=1}^{\infty}[P_{\ell-1}(0)-P_{\ell+1}(0)]\left(\frac{r}{b}\right)^{\ell}P_1(\cos\theta)\right)\right]$$

3.2



A uniform spherical surface with a cap at the North Pole.

Shape: Sphere
Dimension: Area [2D]
Charge: Q

a) Potential Derivation:

① Boundary condition:

$$\Phi(r=R,\theta<\alpha,\phi=0)=0$$

$$\sigma=-\varepsilon_0\nabla\Phi(r=R,\theta>\alpha,\phi=0)$$
$$=\frac{Q}{4\pi R^2}$$

② Laplace's Equation:

$$\nabla^2\Phi=\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

③ Laplace's Equation Solutions:

Ⓐ Variable Separation:

If $\Phi(r,\theta,\phi)=\frac{U(r)}{r}P(\theta)Q(\Phi)$

$$\nabla^2\phi=\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$
$$=P\cdot Q\frac{\partial^2 U}{\partial^2 r}+\frac{U\cdot Q}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)+\frac{UP}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$
$$=0$$

Ⓑ Radial Eigenvalues:

$$\frac{r^2}{U(r)}\frac{\partial^2 U(r)}{\partial r^2}=+\ell(\ell+1)$$

$$\frac{\partial^2 U}{\partial r^2}-\frac{\ell(\ell+1)}{r^2}U=0$$

(C) Angular Eigenvalues:

$$\frac{1}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) = \frac{\lambda}{\sin^2\theta} - \ell(\ell+1)$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) + \left[\ell(\ell+1) - \frac{\lambda}{\sin^2\theta}\right]P = 0$$

(D) Azimuthal Eigenvalues:

$$\frac{1}{Q(\phi)}\frac{\partial^2 Q(\phi)}{\partial\phi^2} = -m \;;\; \frac{\partial^2 Q(\phi)}{\partial\phi^2} + m^2 Q(\phi) = 0$$

(4) General Solution to Laplace's Equation:

Examples show variations in Angular and Azimuthal Eigenvalues.

$$U(r) = A_\ell r^{\ell+1} + B_\ell r^{-\ell}$$

$$P(\theta) = P_\ell^m(\cos\theta)$$

$$Q(\phi) = A_m e^{im\phi} + B_m e^{-im\phi}$$

$$\Phi(r,\theta,\phi) = \sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{U(r)}{r}P(\theta)Q(\phi)$$

$$= \sum_{\ell=0}^{\infty}\left(A_\ell r^{\ell} + B_\ell r^{-\ell-1}\right)\left(A_{m=0} + B_{m=0}\phi\right)P_\ell^{m=0}(\cos\theta)$$

$$+ \sum_{\ell=0}^{\infty}\left(A_\ell r^{\ell} + B_\ell r^{-\ell-1}\right)\left(A_m e^{im\phi} + B_m e^{-im\phi}\right)P_\ell^{m=0}(\cos\theta)$$

(5) Variables by Boundary Conditions:

$A_{m=0}$ $B_{m=0}$

A_m B_m

$$\Phi(r=R, \theta<\alpha, \phi=0) = \sum_{\ell=0}^{\infty}\left(A_\ell r^{\ell} + B r^{-\ell-1}\right)\left(A_{m=0} + B_{m=0}\,0\right)P_\ell^{m=0}(\cos\theta)$$

$$+ \sum_{\ell=0}^{\infty}\left(A_\ell r^{\ell} + B_\ell r^{-\ell-1}\right)\left(A_m e^{im\cdot 0} + B_m e^{-im\cdot 0}\right)P_\ell^{m}(\cos\theta)$$

$$= 0, \text{ so } A_{m=0} = B_{m=0} = B_m = 0$$

$$A_m = e^{-im\phi}$$

$$\int_0^\pi \sigma P_\ell(\cos\theta)\sin\theta\, d\theta = \varepsilon_0 \sum_{\ell=1}^{\infty} A_\ell R^{\ell-1}(2\ell+1)\int_0^\pi P_\ell(\cos\theta)^2 \sin\theta\, d\theta$$

$$= 2\varepsilon_0 A_\ell R^{\ell-1}$$

$$A_\ell = \frac{1}{2\varepsilon_0} R^{-\ell+1}\int_0^\pi \sigma P_\ell(\cos\theta)\sin\theta\, d\theta$$

$$= \frac{1}{2\varepsilon_0} R^{-\ell+1}\frac{Q}{4\pi R^2}\frac{1}{(2\ell+1)}\int_{-1}^{\cos(\alpha)} \frac{d}{dx}\left[P_{\ell+1}(x) - P_{\ell-1}(x)\right]dx$$

$$= \frac{Q}{8\pi\varepsilon_0} R^{-\ell-1}\frac{P_{\ell+1}(\cos(\alpha)) - P_{\ell-1}(\cos(\alpha))}{2\ell+1}$$

$$\Phi(r,\theta,\phi) = [\cos(\alpha)+1]\frac{Q}{8\pi\varepsilon_0 R} + \sum_{\ell=1}^{\infty}\frac{Q}{8\pi\varepsilon_0}\frac{\ell}{(2\ell+1)}\left[P_{\ell+1}(\cos(\alpha)) - P_{\ell-1}(\cos(\alpha))\right]\frac{r^\ell}{R^{\ell+1}}P_\ell(\cos(\alpha))$$

$$= \frac{Q}{8\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\left[P_{\ell+1}(\cos(\alpha)) - P_{\ell-1}(\cos(\alpha))\right]\frac{r^\ell}{R^{\ell+1}}P_\ell(\cos\theta)$$

b) $E = -\nabla\Phi$

$$= \left[-\frac{\partial\Phi}{\partial r} - \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right]_{r=0}$$

$$= -\Bigg[\frac{Q}{8\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\frac{\ell}{2\ell+1}\left[P_{\ell+1}(\cos(\alpha)) - P_{\ell-1}(\cos(\alpha))\right]\frac{r^{\ell-1}}{R^{\ell+1}}P_\ell(\cos\theta)$$

$$+ \frac{Q}{8\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\left[P_{\ell+1}(\cos(\alpha)) - P_{\ell-1}(\cos(\alpha))\right]\frac{r^{\ell-1}}{R^{\ell+1}}\left[\frac{\ell\cos\theta\, P_\ell\cos\theta - \ell P_{\ell-1}(\cos\theta)}{\sin\theta}\right]\Bigg]_{r=0}$$

When $\ell=1$, because the others are zero at $r=0$:

$$E = -\frac{Q}{24\pi\varepsilon_0 R^2}\left[P_2(\cos(\alpha)) - P_0(\cos(\alpha))\right]\left[\cos\theta - \sin\theta\right]$$

$$= \frac{Q^2\sin^2(\alpha)}{16\pi\varepsilon_0 R^2}$$

Identity:
$\hat{z} = \cos\theta - \sin\theta$

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right) P_\ell(\cos\theta)$$

$A_{\ell=0}, B_{\ell=0}$ $\quad \Phi(r=R, \theta > \alpha, \phi = 0) = A_0 R^0 + B_0 R^{-1}$ $\quad$ @ $\ell = 0$

$$= \frac{Q_{TOT}}{4\pi\varepsilon_0 R}, \quad \text{So } A_0 = 0, \; B_0 = \frac{Q_{TOT}}{4\pi\varepsilon_0}$$

$$B_0 = \frac{Q_{TOT}}{4\pi\varepsilon_0}$$

$$= \frac{\frac{Q}{4\pi R^2}\int_0^{2\pi}\int_\alpha^\pi R^2 \sin\theta \, d\theta \, d\phi}{4\pi\varepsilon_0}$$

$$= \frac{Q}{8\pi\varepsilon_0}(\cos(\alpha)+1)$$

B_ℓ $\quad \Phi(r=R, \theta < \alpha, \phi = 0) = \sum_{\ell=1}^{\infty}\left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right) P_\ell(\cos\theta)$ $\quad$ @ $\ell > 1$

$$= 0, \quad B_\ell = A_\ell R^{2\ell+1}$$

A_ℓ $\quad \sigma = \varepsilon_0 \left[-\frac{\partial\phi}{\partial r}\right]_{r=R}$

$$= \varepsilon_0 \left[-(-\cos(\alpha)-1)\frac{Q}{8\pi\varepsilon_0 R^2} - \sum_{\ell=1}^{\infty}\left(A_\ell \ell R^{\ell-1} + A_\ell R^{2\ell+1}(-\ell-1)R^{-\ell-2}\right) P_\ell(\cos\theta)\right]$$

$$= [\cos(\alpha)+1]\frac{Q}{8\pi R^2} + \varepsilon_0 \sum_{\ell=1}^{\infty} A_\ell R^{\ell-1}(2\ell+1) P_\ell(\cos\theta)$$

When $\ell > 0$,

$$\sigma P_\ell(\cos\theta)\sin\theta = \varepsilon_0 \sum_{\ell=1}^{\infty} A_\ell R^{\ell-1}(2\ell+1) P_\ell(\cos\theta)^2 \sin\theta$$

c) $\alpha << 1$ @ $\ell = 0$ $P_{\ell+1}[\cos(\alpha)] - P_{\ell-1}[\cos(\alpha)] = \cos(\alpha) + 1$

$$\cong 1 - \frac{1}{2}\alpha^2 + 1$$

$$\cong 2 - \frac{1}{2}\alpha^2$$

@ $\ell > 1$ $P_{\ell+1}[\cos(\alpha)] - P_{\ell-1}[\cos(\alpha)]$

$$= 1 - \frac{1}{2}\alpha^2 P_{\ell+1}(1) - 1 + \frac{1}{2}\alpha^2 P_{\ell-1}(1)$$

... because $P_\ell(\cos(\alpha)) \cong 1 - \frac{1}{2}\alpha^2 P_\ell(1)$

$$= -\frac{1}{2}\alpha^2 \left[P_{\ell+1}(1) - P_{\ell-1}(1)\right]$$

$$= -\frac{1}{2}\alpha^2 (2\ell + 1)$$

$$\Phi(r,\theta,\phi) = \frac{Q}{8\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} \left[P_{\ell+1}(\cos(\alpha)) - P_{\ell-1}(\cos(\alpha))\right] \frac{r^\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

$$= \frac{Q}{8\pi\varepsilon_0}\left(\left[2 - \frac{1}{2}\alpha^2\right]\frac{1}{R} + \sum_{\ell=1}^{\infty}\left[-\frac{1}{2}\alpha^2(2\ell+1)\right]\frac{r^\ell}{R^{\ell+1}} P_\ell(\cos\theta)\right)$$

$$= \frac{Q}{4\pi\varepsilon_0 R} - \frac{Q\alpha^2}{16\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \frac{r^\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

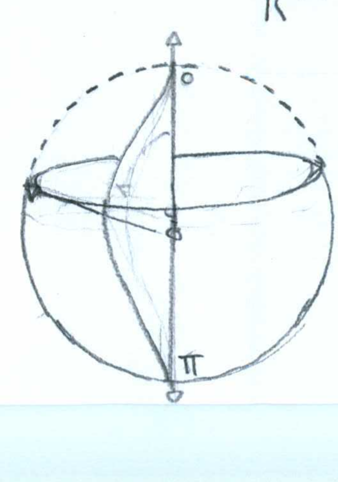
As the spherical cap becomes small, $\alpha << 1$, the potential approaches a regular conducting sphere.

$\pi >> \alpha >> 0$

$$\Phi(r,\theta,\phi) = \frac{Q}{8\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} \left[P_{\ell+1}(\cos(\alpha)) - P_{\ell-1}(\cos(\alpha))\right] \frac{r^\ell}{R^{\ell+1}} P_\ell(\cos(\alpha))$$

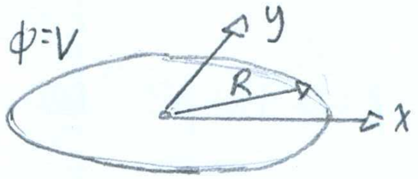
$$= \frac{Q\alpha^2 \sin^2\theta}{16\pi\varepsilon_0 R}$$

The sphere's potential is largest at $\alpha = \frac{\pi}{2}$.



3.3

$\Phi = V$



A thin, flat conductor, circular disc

Shape: Disc
Dimension: Area [2D]
Charge: q

a/b) Four possible methods:

- Spherical Harmonics
- Factorials, series from Gauss' law
- Boundary conditions [Regular]
- Green's Theorem

Spherical Harmonics:

$$Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\,e^{im\phi}$$

where
- l, m are integer solutions to Laplaces equation.
- $P_l^m(\cos\theta)$ is Legendres polynomial.
- $e^{im\phi}$ is a complex factor

$$\Phi = \int \frac{\sigma(x)\,ds}{|x-x'|}$$

$$\sigma(x) = \sigma_0 (R^2-\rho^2)^{-1/2}$$

$$\frac{1}{|x-x'|} = \sum_{l=0}^{\infty}\sum_{m=0}^{l} \frac{1}{2l+1}\frac{r^l}{r^{l+1}} Y^*_{lm}(\theta,\phi)\,Y_{l,m}(\theta,\phi)$$

$$\Phi = 4\pi\sigma_0 \sum_{l=0}^{\infty}\sum_{m=0}^{l} \frac{1}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \int_0^{2\pi} Y^*_{lm}\left(\frac{\pi}{2},\phi'\right)d\phi' \int_0^R \rho^{l+1}(R^2-\rho^2)^{-1/2}\,d\rho$$

$$\int_0^{2\pi} Y^*_{ml}\left(\frac{\pi}{2},\phi'\right)d\phi' = 2\pi\sqrt{\frac{2l+1}{4\pi}}\,P_l(0)$$

$$= 2\pi\sqrt{\frac{2l+1}{4\pi}}\,\frac{(-1)^l}{2l+1}\,\frac{(2l+1)!!}{(2l)!!}$$

$$\int_0^R \rho^{l+1}(R^2-\rho^2)^{-1/2}\,d\rho = \int_0^R \rho^{2l+1}(R^2-\rho^2)^{-1/2}\,d\rho \quad @\ l=0$$

$$= -\rho^{2l}\cdot\sqrt{R^2-\rho^2}\Big|_0^R + 2l\int_0^R \rho^{2l-1}(R^2-\rho^2)^{-1/2}\,d\rho$$

$$= \frac{2lR^2}{2l+1}\int_0^R \rho^{2l-1}(R^2-\rho^2)^{-1/2}\,d\rho$$

$$= \frac{(2l)!!}{(2l+1)!!}R^{2l+1}$$

$$\Phi = 4\pi\varepsilon_0\sum_{l=0}^{\infty}\frac{1}{4l+1}\frac{1}{r^{l+1}}\left[\frac{4l+1}{4\pi}P_{2l}(\cos\theta)\right]\left[2\pi\sqrt{\frac{4l+1}{4\pi}}\frac{(-1)^l}{2l+1}\frac{(2l+1)!!}{(2l)!!}\right]\left[\frac{(2l)!!}{(2l+1)!!}R^{2l+1}\right]$$

$$= 2\pi\sigma\frac{R}{r}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}\right)^{2l}P_{2l}(\cos\theta)$$

$$= \frac{2V}{\pi}\left(\frac{R}{r}\right)\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}\right)P_{2l}(\cos\theta) \quad \text{where } \sigma = \frac{V}{\pi^2}$$

Factorials, Series from Gauss' Law:

$$\sigma(r) \propto (R^2-\rho^2)^{-1/2}$$

$$= K(R^2-\rho^2)^{-1/2} \quad \text{where } K \text{ is an arbitrary constant}$$

$$= K\left[1+\frac{1}{2}\left(\frac{r}{R}\right)^2+\frac{3\cdot1}{2!(2\cdot2)}\left(\frac{r}{R}\right)^4+\frac{5\cdot3}{(3!)(2\cdot2\cdot2)}\left(\frac{r}{R}\right)^6+\cdots\right]$$

$$= \frac{K}{R}\sum_{n=0}^{\infty}\frac{(2n-1)!!}{n!\cdot2^n}\left(\frac{r}{R}\right)^{2n}$$

$$E = \frac{\sigma}{\varepsilon_0}$$

$$= -\nabla\Phi$$

$$\sigma = -\varepsilon_0\nabla\Phi$$

$$= -\varepsilon_0 \left[\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta}\right]\phi$$

$$\boxed{\text{A trial: } \phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)}$$

$$= -\varepsilon_0 \left[l \sum_{l=0}^{\infty} A_l r^{r-1} P_l(\cos\theta) + A_l r^{l-1} \frac{d}{d\theta} P_l(\cos\theta) \right]_{\cos\theta = 0}$$

$$-\varepsilon_0 A_l r^{l-1} \frac{d}{d\theta} P_l(\cos\theta) = \frac{K}{R} \sum_{l=0}^{\infty} \frac{(2l-1)!!}{l!\,2^l} \left(\frac{r}{R}\right)^{2l}$$

$$A_{2l+1} = \frac{(-1)^l K r^{l+1}}{(2l+1) R^{2l+1} \varepsilon_0} \quad \ldots \text{at } X = 0,$$

$$\phi(r,\theta) = \frac{K}{\varepsilon_0}\left(\frac{r}{R}\right) \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} \left(\frac{r}{R}\right)^{2l} P_l(\cos\theta)$$

$$= \frac{2V}{\pi}\left(\frac{r}{R}\right) \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} \left(\frac{r}{R}\right)^{2l} P_l(\cos\theta) \quad \text{where } K = \frac{2V}{\pi}$$

Boundary Conditions [Regular Method]

Potential Derivation:

① Boundary conditions: $\phi(r<R, \theta, \phi=0) = 0$

$\phi(r>R, \theta, \phi=0) = V$

② Laplace's Equation:

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2 \Phi}{\partial\phi^2}$$

$$= 0$$

③ Laplace's Equation Solutions:

Ⓐ Variable Separation:

If $\Phi(r,\theta,\phi) = \frac{U(r)}{r} P(\theta) Q(\phi)$

$$\nabla^2\phi = \frac{1}{r}\frac{\partial^2}{\partial^2 r}(r\phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{d\Phi}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial\Phi^2}{\partial\phi^2}$$

$$= P(\theta)Q(\phi)\frac{\partial^2 U(r)}{\partial^2 r} + \frac{U(r)Q(\phi)}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) + \frac{U(r)P(\theta)}{r^2\sin^2\theta}\frac{\partial^2 Q(\phi)}{\partial\phi^2}$$

$$= \frac{r^2\sin^2\theta}{U(r)P(\theta)Q(\phi)}\left[P\cdot Q\frac{\partial^2 U}{\partial^2 r} + \frac{U\cdot Q}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right) + \frac{U\cdot P}{r^2\sin^2\theta}\frac{\partial Q^2}{\partial\phi^2}\right]$$

$$= 0$$

(B) Radial Eigenvalues:

$$\frac{r^2}{U(r)}\frac{\partial^2 U(r)}{\partial^2 r} = \ell(\ell+1)$$

$$\frac{\partial^2 U(r)}{\partial r^2} - \frac{\ell(\ell+1)}{r^2}U(r) = 0$$

(C) Angular Eigenvalues:

$$\frac{1}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) = \frac{\lambda}{\sin^2\theta} - \ell(\ell+1)$$

$$\frac{1}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) + \left(\ell(\ell+1) - \frac{\lambda}{\sin^2\theta}\right)P(\theta) = 0$$

(D) Azimuthal Eigenvalues:

$$\frac{1}{Q(\phi)}\frac{\partial^2 Q(\phi)}{\partial\phi^2} = -m^2$$

$$\frac{\partial^2 Q(\phi)}{\partial\phi^2} + m^2 Q(\phi) = 0$$

(4) General Solution to Laplace's Equation:

$$U(r) = A_\ell r^{\ell+1} + B_\ell r^{-\ell}$$

$$P(\theta) = P_\ell^m(\cos\theta)$$

$$Q(\phi) = A_m e^{im\phi} + B_m e^{-im\phi}$$

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=0}^{l} \frac{U(r)}{r} P(\theta) Q(\phi)$$

$$= \sum_{l=0}^{\infty}\sum_{m=0}^{l} (A_l r^l + B_l r^{-l-1})(A_m e^{im\phi} + B_m e^{-im\phi}) P_l^m(\cos\theta)$$

⑤ Variables by Boundary Conditions:

A_m, B_m $\quad \Phi(r<R, \theta, \phi=0) = \sum_{l=0}^{\infty}\sum_{m=0}^{l} (A_l r^{l+1} + B_l r^{-l-1})(A_m e^0 + B_m e^0) P_l^m(\cos\theta)$

$$= 0, \text{ so } B_m = 0, \; A_m = e^{-im\phi}$$

B_l $\quad \Phi(r<R, \theta, \phi=0) = \sum_{l=0}^{\infty}\sum_{m=0}^{l} (A_l r^l + B_l r^{-l-1}) P_l^m(\cos\theta)$

$$= 0, \text{ so } B_l = A_l r^{2l+1} \;\; @ \; l=1$$

$$\Phi(r,\theta,\phi) = 2\sum_{l=0}^{\infty}\sum_{m=0}^{l} A_l r^l P_l^m(\cos\theta)$$

A_l $\quad \sigma = -\varepsilon_0 \left[+ \frac{\partial\phi}{\partial r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta} \right]$

$$= -2\varepsilon_0 \left[\sum_{l=0}^{\infty}\sum_{m=0}^{l} (A_l \cdot l r^{l-1} P_l^m(\cos\theta) - A_l r^{l-1} \cdot P_l^{m\prime}(\cos\theta) \cdot \sin\theta \right]$$

$$= -2\varepsilon_0 \left[\sum_{l=0}^{\infty}\sum_{m=0}^{l} (A_l \cdot l \cdot r^{l-1} + A_l \cdot r^{l-1} (l+1)) P_l^m(\cos\theta) \right]$$

$$= -2\varepsilon_0 \left[\sum_{l=0}^{\infty}\sum_{m=0}^{l} A_l r^{l-1} (2l+1) P_l^m(\cos\theta) \right]$$

$$\int_0^{\pi} \sigma P_l^m(\cos\theta) \sin\theta \, d\theta = \int_0^{\pi} (-2\varepsilon_0 \sum_{l=0}^{\infty}\sum_{m=0}^{l} A_l r^{l-1} (2l+1) P_l^m(\cos\theta)^2 \sin\theta \, d\theta$$

$$= 2\varepsilon_0 A_l r^{l-1}$$

$$A_\ell = \frac{1}{2\varepsilon_0} r^{-\ell-1} \sigma \frac{\left[P_{\ell+1}(\cos\theta) - P_{\ell-1}(\cos\theta)\right]}{2\ell+1}$$

$$= \frac{1}{2\varepsilon_0} \cdot r^{-\ell-1} \cdot \left(\frac{2\pi\varepsilon_0 R}{\pi^2} V\right) \cdot \frac{(-1)^\ell}{(2\ell+1)}$$

$$= \frac{V}{\pi} \frac{(-1)^\ell}{(2\ell+1)} \left(\frac{R}{r}\right) \cdot r^{-\ell}$$

$$\Phi(r,\theta,\phi) = \frac{2V}{\pi}\left(\frac{R}{r}\right) \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+1)} \left(\frac{R}{r}\right)^\ell P_\ell^m(\cos\theta) \quad \text{when } r_0 = R$$

$$= \frac{2V}{\pi}\left(\frac{R}{r}\right) \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+1)} \left(\frac{R}{r}\right)^{2\ell} P_{2\ell}^m(\cos\theta) \text{ ... because odd-}\ell.$$

Green's Theorem Derivation:

$$G(x,x') = \frac{-1}{4\pi|x-x'|}$$

$$= -\sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}(\theta,\phi') Y_{\ell m}(\theta,\phi')$$

$$\rho(x) d^3x = \frac{Q}{2\pi R} \delta(\cos\theta) \frac{\Theta(R-r)}{r\sqrt{R^2-r^2}} d\cos\theta \, d\phi \, r^2 dr$$

$$\Phi(x) = -\frac{1}{\varepsilon_0}\int G(x,x')\rho(x')d^3x$$

$$= \frac{Q}{4\pi\varepsilon_0 R} \sum_{\ell=0}^{\infty} I_\ell(r) P_\ell(0) P_\ell(\cos\theta)$$

$$\text{Where } I_\ell(r) = \int_0^R \frac{r_<^\ell}{r_>^{\ell+1}} \frac{r' dr}{\sqrt{R^2-r^2}}$$

$$= \left(\frac{R}{r}\right)^{2n+1} \cdot \frac{1}{2} \int \frac{t^n dt}{\sqrt{1-t}}$$

$$= \left(\frac{R}{r}\right)^{2n+1} \frac{(2n)!!}{(2n+1)}$$

$$\Phi(r,\theta) = \frac{2V}{\pi}\left(\frac{R}{r}\right)^{2n+1} \sum_{l=0}^{\infty} \frac{1}{(2n+1)} \left(\frac{R}{r}\right)^{2n+1} P_{2l}(\cos\theta)$$

When $r > R$, the ratio is $\left(\frac{R}{r}\right)$; $r < R$ the ratio is $\left(\frac{r}{R}\right)$.

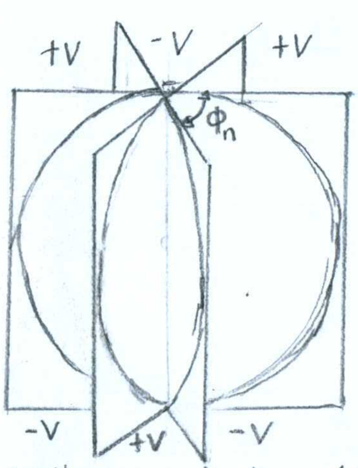
c) $$\sigma = \frac{V}{\pi^2} \frac{1}{(R^2 - r^2)^{1/2}}$$

$$= \frac{2V}{\pi} \int_0^R (R^2 - r^2)^{-1/2} d\rho$$

$$= \frac{2RV}{\pi}$$

$$C = \frac{Q}{V} = \frac{2R}{V}$$

3.4.



Hollow conducting sphere

Shape: Sphere
Dimension: Volume [3D]

a) Potential Derivation:

① Boundary Conditions:

$$\Phi(r=a,\theta,\phi) = (-1)^{2n} \cdot V \; ; \; \Phi(r=0,\theta,\phi) = 0$$

② Laplace's Equation:

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

$$= 0$$

③ Laplace's Equation Solution:

Ⓐ Variable Separation: If $\Phi(r,\theta,\phi) = \frac{U(r)}{r} P(\theta) Q(\phi)$

$$\nabla^2 \Phi = \frac{r^2 \sin^2\theta}{U(r)P(\theta)Q(\phi)}\left[P(\theta)Q \frac{\partial^2 U}{\partial r^2} + \frac{UQ}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right) + \frac{UP}{r^2\sin\theta}\frac{\partial \Phi^2}{\partial\phi^2}\right] = 0$$

Ⓑ Radial Eigenvalues:

$$\frac{r^2}{U(r)}\frac{\partial^2 U(r)}{\partial^2 r} = l(l+1)$$

$$\frac{\partial^2 U(r)}{\partial^2 r} - \frac{l(l+1)U(r)}{r^2} = 0$$

Ⓒ Angular Eigenvalues:

$$\frac{1}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) = \frac{\lambda}{\sin^2\theta} - l(l+1)$$

$$\frac{1}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) + \left(l(l+1) - \frac{\lambda}{\sin^2\theta}\right) = 0$$

Ⓓ Azimuthal Eigenvalues:

$$\frac{1}{Q(\phi)}\frac{\partial^2 Q(\phi)}{\partial\phi^2} = -m^2$$

$$\frac{\partial^2 Q(\phi)}{\partial\phi^2} + m^2 Q(\phi) = 0$$

④ General Solution to Laplace's Equation:

$$U(r) = A_l r^{l+1} + B_l r^{-l}$$

$$P(\theta) = P_l^m(\cos\theta)$$

$$Q(\phi) = A_m e^{im\phi}$$

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{U(r)}{r}P(\theta)Q(\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(A_l r^l + B_l r^{-l-1})(A_m e^{im\phi})P_l^m(\cos\theta)$$

@ $n=1$ $\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} -V\sqrt{\frac{2l-1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}\ \frac{i}{m}\ 4\int_{-l}^{l} P_l^m(\cos\theta)$

$$= \sum_{m=-1}^{1} A_{1m}\left(\frac{r}{a}\right)Y_l(\theta,\phi) + \sum_{m=-2}^{2} A_{2m}\left(\frac{r}{a}\right)^2 Y_{2m}(r,\theta) + \sum_{m=-3}^{3} A_{3m}\left(\frac{r}{a}\right)^3 Y_{3m}$$

$$= \left(\frac{r}{a}\right)\left[A_{1,-1}Y_{1,-1} + A_{1,1}\circ Y_{1,1}\right] + \left(\frac{r}{a}\right)^2\left[A_{2,-1}Y_{2,-1} + A_{21}\ Y_{21}\right]$$

$$+ \left(\frac{r}{a}\right)^3\left[A_{3,-3}\circ Y_{3,-3} + A_{3,-1}Y_{3,-1} + A_{3,1}\circ Y_{3,1} + A_{33}\circ Y_{33}\right]$$

$A_{1,-1} = 0 \qquad A_{11} = 0$

$A_{3,-3} = iV\sqrt{\frac{35\pi}{256}} \qquad A_{33} = iV\sqrt{\frac{35\pi}{256}}$

$A_{3,-1} = iV\sqrt{\frac{21\cdot\pi}{256}} \qquad A_{3,1} = iV\sqrt{\frac{21\cdot\pi}{256}}$

$$\Phi(r,\theta,\phi) = \left(\frac{r}{a}\right)iV\sqrt{\frac{3\pi}{2}}\left[Y_{1,-1}(\theta,\phi) + Y_{11}(\theta,\phi)\right]$$

$$+ \left(\frac{r}{a}\right)^3 iV\sqrt{\frac{35\pi}{256}}\left(Y_{3,-3}(\theta,\phi) + Y_{33}(\theta,\phi)\right)\cdots$$

$$+ \left(\frac{r}{a}\right) \times \sqrt{\frac{21\pi}{256}}\left(Y_{3,-1}(\theta,\phi) + Y_{31}(\theta,\phi)\right) + \cdots$$

$$= \frac{3}{2}V\left(\frac{r}{a}\right)\sin\theta\sin\phi + \left(\frac{r}{a}\right)^3 V\left[\frac{35}{64}\sin^3\theta\sin(3\phi) + \frac{21}{64}\sin\theta(5\cos^2\theta - 1) \times \sin\phi\right]$$

$$= \frac{3}{2}V\left(\frac{r}{a}\right)\cos\theta' - \frac{7}{8}\left(\frac{r}{a}\right)^3 V\left[\frac{5}{2}\cos^3\theta' - \frac{3}{2}\cos\theta' + \cdots\right]$$

when $\sin\theta\sin\phi = \cos\theta'$

$$= \frac{3}{2}V\left(\frac{r}{a}\right)P_1(\cos\theta') - \frac{7}{8}\left(\frac{r}{a}\right)^3 V P_3(\cos\theta')\cdots$$

⑤ Variables by Boundary Conditions:

B_m $\Phi(r=0, \Theta, \Phi) = \sum_{l=0}^{\infty} \sum_{m=l}^{-l} (A_l \cdot 0 + B_l \cdot \frac{1}{0})(A_m e^{im\phi}) P_l^m(\cos\Theta)$

$= 0 \quad , \quad B_l = 0$

$\Phi(r, \theta, \Phi) = \sum_{l=0}^{\infty} \sum_{m=l}^{-l} A_l \cdot r^l \cdot Y_{lm}(\theta, \phi)$ ← Spherical Harmonics

A_l $\Phi(r=a, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=l}^{-l} A_l \cdot a^l Y_{lm}(\theta, \phi)$

$= V$

$\int_0^{2\pi}\int_0^{\pi} V(\phi) Y_{lm}^*(\theta, \phi) \sin\Theta \, d\theta d\phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} a^l \int_0^{2\pi}\int_0^{\pi} Y_{lm}^*(\theta,\phi) Y_{lm}(\theta,\phi) \sin\theta d\theta d\phi$

$= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} a^l \delta_{ll} \cdot \delta_{mm}$

$= A_l a^l$

$A_l = \frac{1}{a^l} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} \int_0^{2\pi} V e^{-im\phi} d\phi \int_0^{\pi} P_l^m \cos\theta \sin\theta d\theta$

$= \frac{1}{a^l} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{l-m}{l+m}} \left(\frac{-i}{m}\right) \sum_{j=0}^{n-1} e^{-im(2j)\frac{\pi}{n}} (e^{-im\pi/n} - 1)^2$

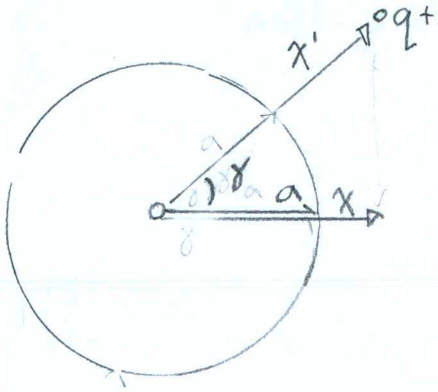
$\times \int_0^{\pi} P_l^m(\cos\theta) \sin\theta d\theta$

$\Phi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A\left(\frac{r}{a}\right) Y_{lm}(\theta, \phi)$

b) (Equation 3.36)

$\Phi(r, \theta) = V\left[\frac{3}{2} \frac{r}{a} P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \frac{11}{16}\left(\frac{r}{a}\right)^5 P_5(\cos\theta) + \cdots\right]$

3.5



Hollow sphere

Shape: Sphere
Dimension: Surface Area [2D]
Charge: q

Potential Derivations:
∘ Green's Theorem
∘ Boundary Conditions

a) Potential Derivation by Green's Theorem

$$G(x,x') = G_1 + G_2$$

$$= \frac{1}{|x-x'|} - \frac{1}{\left(\frac{x'}{a}\right)x - \left(\frac{a}{x'}\right)x'}$$

$$= \frac{1}{\sqrt{x^2+x'^2-2xx'\cos\gamma}} - \frac{1}{\sqrt{\left(\frac{x'}{a}\right)^2 x^2 + a^2 - 2xx'\cos\gamma}}$$

where $\cos\gamma$ derives from the law of cosines or in spherical coordinates as $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$

$$\left.\frac{dG}{da}\right|_{x'=a} = \frac{(x^2-a^2)}{a(x^2+a^2-2xa\cos\gamma)^{3/2}}$$

(Equation 1.44)

$$\Phi(x) = \underbrace{\frac{1}{4\pi\varepsilon_0}\int_V \rho(x')\,G_D(x,x')\,d^3x'}_{=0} - \frac{1}{4\pi}\oint_S \Phi(x')\,\frac{\partial G_D}{\partial n'}\,da'$$

$$\Phi(x) = \frac{-1}{4\pi}\int \Phi(x')\,\frac{\partial G_D}{\partial n'}\,da'$$

$$= \frac{-1}{4\pi}\int \Phi(x')\,\frac{\partial G_D}{\partial n'}\,a^2\,d\Omega \qquad @\ \Phi(x') = V(\theta',\phi')$$

$$= \frac{a(a^2-r^2)}{4\pi}\int \frac{V(\theta',\phi')}{(r^2+a^2-2ra\cos\gamma)^{3/2}} \qquad @\ r = x$$

b) Potential Derivation by Boundary Conditions:

① Boundary Conditions: $\phi(r=a,\theta,\phi)=V$

$\phi(r=0,\theta,\phi)=0$

② Laplaces Equation: $\nabla^2\Phi=\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial\phi^2}{\partial\phi^2}$

$=0$

③ Laplace's Equation Solutions:

Ⓐ Variable Separation: If $\Phi(r,\theta,\phi)=\frac{U(r)}{r}P(\theta)Q(\phi)$

$$\nabla^2\Phi=\frac{r^2\sin^2\theta}{U(r)P(\theta)Q(\phi)}\left[P(\theta)Q(\phi)\frac{\partial^2 U}{\partial r^2}+\frac{U(r)Q(\theta)}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)+\frac{U(r)P(\theta)}{r^2\sin\theta}\frac{\partial\Phi^2}{\partial\Phi^2}\right]$$

$=0$

Ⓑ Radial Eigenvalues: $\frac{r^2}{U(r)}\frac{\partial^2 U(r)}{\partial^2 r}=\ell(\ell+1)$

$$\frac{\partial^2 V(r)}{\partial^2 r}-\frac{\ell(\ell+1)U(r)}{r^2}=0$$

Ⓒ Angular Eigenvalues:

$$\frac{1}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right)=\frac{\lambda}{\sin^2\theta}-\ell(\ell+1)$$

$$\frac{1}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right)+\left(\ell(\ell+1)-\frac{\lambda}{\sin^2\theta}\right)=0$$

Ⓓ Azimuthal Eigenvalues:

$$\frac{1}{Q(\phi)}\frac{\partial^2 Q(\phi)}{\partial\phi^2}=-m^2$$

$$\frac{\partial^2 Q(\phi)}{\partial\phi^2}+m^2 Q(\phi)=0$$

④ General Solution to Laplace's Equation:

$$U(r) = A_\ell r^{\ell+1} + B_\ell r^{-\ell}$$

$$P(\theta) = P_\ell^m(\cos\theta)$$

$$Q(\phi) = A_m e^{im\phi}$$

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \frac{U(r)}{r} P(\theta) Q(\phi)$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (A_\ell r^{\ell} + B_\ell r^{-\ell-1}) P_\ell^m(\cos\theta) e^{im\phi}$$

⑤ Variables by Boundary Conditions:

B_ℓ $\quad \Phi(r=0,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (A_\ell \cdot 0^{\ell} + B_\ell \cdot 0^{-\ell-1}) P_\ell^m(\cos\theta) e^{im\phi}$

$$= 0, \quad \text{so} \quad B_\ell = 0$$

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_\ell r^{\ell} P_\ell^m(\cos\theta) e^{im\phi}$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_\ell \cdot r^{\ell} Y_\ell^m(\theta,\phi)$$

A_ℓ $\quad \Phi(r=a,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_\ell \cdot a^{\ell} Y_\ell^m(\theta,\phi)$

$$= V$$

$$\int_0^{2\pi}\int_0^{\pi} Y_\ell^m(\theta,\phi) V(\theta,\phi) \sin\theta \, d\theta \, d\phi = \int_0^{2\pi}\int_0^{\pi} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_\ell \cdot a^{\ell} \cdot Y_\ell^m(\theta,\phi) Y_\ell^m(\theta,\phi) \sin\theta \, d\theta \, d\phi$$

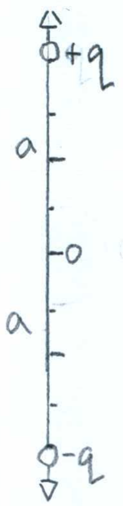
$$= A_\ell \cdot a^{\ell} \cdot \delta_{\ell\ell} \cdot \delta_{mm}$$

$$A_e = \frac{1}{a^e}\int_0^{2\pi}\int_0^{\pi} Y_e^m(\theta,\phi)\,V(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A\left(\frac{r}{a}\right)^l Y_e^m(\theta,\phi)$$

$$\text{where } A = \int_0^{2\pi}\int_0^{\pi} Y_e^m(\theta,\phi)\,V(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

3.6



Two Point Charges

Shape: Line
Dimension: Line
Charge: +q, -q

a) Potential Derivation:

① Boundary Conditions:

$$\Phi(r=0,\theta,\phi) = 0 \quad ; \quad \Phi(r=a,\theta,\phi) = V$$

② Laplaces Equation:

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

$$= 0$$

③ Laplaces Equation Solution:

Ⓐ Variable Separation: If $\Phi(r,\theta,\phi) = \frac{U(r)}{r}P(\theta)Q(\phi)$

$$\nabla^2\Phi = \frac{r^2\sin^2\theta}{U(r)P(\theta)Q(\phi)}\left[P(\theta)Q(\phi)\frac{\partial^2 U(r)}{\partial r^2} + \frac{U(r)Q(\phi)}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) + \frac{U(r)P(\theta)}{r^2\sin\theta}\frac{\partial\Phi^2}{\partial\phi^2}\right]$$

$$= 0$$

Ⓑ Radial Eigenvalues:

$$\frac{r^2}{U(r)}\frac{\partial^2 U(r)}{\partial^2 r} = l(l+1)$$

$$\frac{\partial^2 U(r)}{\partial^2 r} - \frac{l(l+1)\,U(r)}{r^2} = 0$$

Ⓒ Angular Eigenvalues:

$$\frac{1}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) = \frac{\lambda}{\sin^2\theta} - l(l+1)$$

$$\frac{-1}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) + \left(l(l+1) - \frac{\lambda}{\sin^2\theta}\right) = 0$$

③ Azimuthal Eigenvalues:
$$\frac{1}{Q(\phi)}\frac{\partial^2 Q(\phi)}{\partial\phi^2} = -m^2$$
$$\frac{\partial^2 Q(\phi)}{\partial\phi^2} + m^2 Q(\phi) = 0$$

④ General Solution to Laplaces Equation:

$$U(r) = A_\ell r^{\ell+1} + B_\ell r^{-\ell}$$

$$P(\theta) = P_\ell^m(\cos\theta)$$

$$Q(\phi) = A_m e^{im\phi}$$

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{U(r)}{r} P(\theta)\, Q(\phi)$$
$$= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \left(A_\ell r^{\ell} + B_\ell r^{-\ell-1}\right) Y_\ell^m(\theta,\phi)$$

⑤ Variables by Boundary Conditions:

B_ℓ $$\Phi(r=0,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(A_\ell \cdot 0 + B_\ell \cdot \frac{1}{0}\right) Y_\ell^m(\theta,\phi)$$
$$= 0, \text{ so } B_\ell = 0$$

A_ℓ $$\Phi(r=a,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_\ell \cdot a^\ell \cdot Y_\ell^m(\theta,\phi)$$
$$= V$$

$$\int_0^\pi V\cdot Y_\ell^m(\theta,\phi)\sin\theta\, d\theta = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_\ell a^\ell \int_0^\pi Y^*(\theta,\phi)\, Y_\ell^m(\theta,\phi)\sin\theta\, d\theta$$
$$= 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_\ell \cdot a^\ell \cdot \delta_{\ell\ell'}\,\delta_{mm'}$$

$$A_\ell = \frac{q}{\varepsilon_0}\left(\frac{1}{a^\ell}\right)\frac{1}{(2\ell+1)} Y_\ell^m(\theta,\phi)$$

$$\Phi(r,\theta,\phi)=\frac{q}{\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left(\frac{r}{a}\right)^l Y_l^m(\theta,\phi)\cdot Y_l^m(\theta,\phi)$$

- or -

$$\phi(r,\theta,\phi)=\frac{q}{\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left(\frac{a}{r}\right)^l Y_l^m(\theta,\phi)\, Y_l^m(\theta,\phi)$$

- or -

$$\phi(r,\theta,\phi)=\frac{q}{\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left(\frac{r}{a}\right)^l Y_l^m(\theta,\phi)\left[Y_l^m(0,0)-Y_l^m(\pi,0)\right]$$

$$=\frac{2q}{2\pi\varepsilon_0}\sum_{l=0}^{\infty}\left(\frac{r}{a}\right)^l P_l(\cos\theta)$$

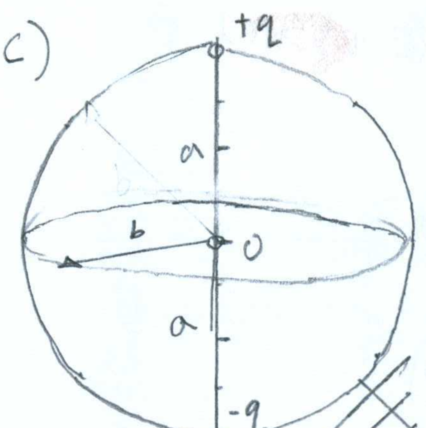
Total potential
$$\phi(q_{TOT})=\phi(q)+\phi(-q)$$

b) $qa=\frac{p}{2}$ $\quad \lim_{a\to 0}\phi(r,\theta,\phi)=\lim_{a\to 0}\frac{q}{2\pi\varepsilon_0}\sum_{l=0}^{\infty}\left(\frac{a}{r}\right)^l P_l(\cos\theta)$

$$=\frac{p}{4\pi\varepsilon_0}\left[2+\left(\frac{1}{r}\right)\cos\theta+\left(\frac{1}{6}\right)\frac{1}{r^2}\left(3\cos^2\theta-1\right)+\cdots\right]$$

where $q=\sum q_i$ and $p=\sum_i p_i$

Note: A general solution without the $\frac{1}{r}$ in $\Phi(r,\theta,\phi)=\frac{U(r)}{r}P(\theta)Q(\phi)$ is another solution.

c) 

A grounded spherical shell

Shape: sphere
Dimension: Surface [20]
Charge: +q, -q

$$\Phi(x)=\phi(\text{line})+\phi(\text{shell})$$

$$\cong\frac{p}{4\pi\varepsilon_0 r^2}\cos\theta+\sum_{l=0}^{\infty}A_l r^l P_l^m(\cos\theta)$$

$$=0$$

$$A_l=\frac{-p}{4\pi\varepsilon_0}\frac{1}{b^3}\quad @\ l=1,\ r=b$$

$$\phi=\frac{p\cos\theta}{4\pi\varepsilon_0 b^2}\left[\frac{b^3}{r^2}-\left(\frac{r}{b}\right)\right]$$

In the case $a<r$:

$$\Phi(r,\theta,\phi) = \frac{q}{2\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{a^{l+1}}P_l(\cos\theta) + \sum_{l=0}^{\infty}A_l\left(\frac{r}{a}\right)^l P_l(\cos\theta)$$

$$A_l = \frac{-q}{4\pi\varepsilon_0}\left[\sum_{l=0}^{\infty}\left(\frac{a}{r}\right)^l\right]P_l(\cos\theta)$$

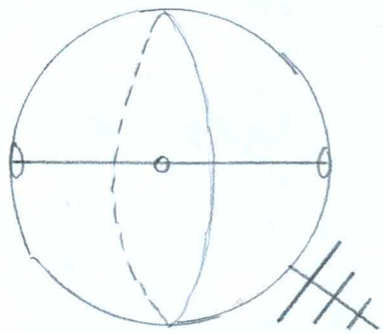
$$\Phi(r,\theta,\phi) = \frac{1}{2\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{a^{l+1}}\left[1-\frac{a}{r}\right]P_l(\cos\theta)$$

The limit as $a\to 0$ for $r<a$:

$$\lim_{a\to 0}\Phi(r,\theta,\phi) = \lim_{a\to 0}\left[\frac{q}{2\pi\varepsilon_0}\sum_{l=2}^{\infty}\left(\frac{a}{b}\right)^l\left[\frac{b^l}{r^{l+1}}-\frac{r^l}{b^{l+1}}\right]P_l(\cos\theta)\right]$$

$$= \frac{Q}{2\pi\varepsilon_0}\frac{1}{r^3}\left[1-\frac{r^5}{b^5}\right]P_2(\cos\theta)$$

3.8



Grounded sphere with uniformly Charged wire

Shape: Sphere
Dimension: Area [2D]
Charge: Q

a) (Equation 3.136)

$$\Phi(x) = \frac{Q}{4\pi\varepsilon_0 b}\left\{\ln\left(\frac{b}{r}\right)+\sum_{j=1}^{\infty}\frac{4j+1}{2j(2j+1)}\left[1-\left(\frac{r}{b}\right)^{2j}\right]P_{2j}(\cos\theta)\right\}$$

(Equation 3.10) "Legendre Polynomial"

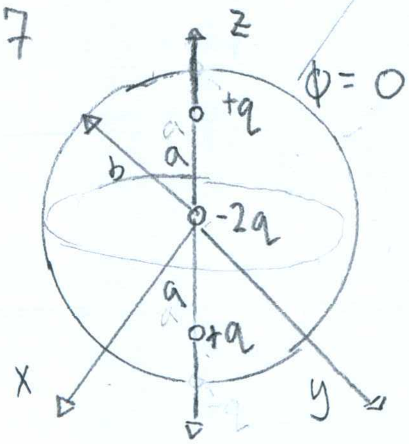
$$\frac{d}{dx}\left[(1-x^2)\frac{\partial P}{\partial x}\right]+l(l+1)P = 0$$

... is a Rodriguez formula.

$$P_n(x) \equiv \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

... multiple reoccurrence relations Pg 100

3.7



Grounded Conductor Spherical shell

Shape: Sphere
Dimension: Surface [2D]
Charge: +q, -q

a) Potential Derivation without a Grounded Spherical Shell:

From problem 3.6,

$$\phi(r,\theta,\phi) = \frac{1}{2\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{a^l}{r^{l+1}}P_l(\cos\theta)$$

$$\lim_{a\to\infty}\phi(r,\theta,\phi) = \lim_{a\to\infty}\left[\frac{1}{2\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{a^l}{r^{l+1}}P_l(\cos\theta)\right]$$

$$= \frac{1}{2\pi\varepsilon_0}\left[\frac{a^2}{r^3}P_2(\cos\theta) + \cdots\right]$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{a^2}{r^3}(3\cos^2\theta - 1)\right]$$

b) Potential Derivation with a Grounded Spherical Shell:

From problem 3.6,

$$\phi(r,\theta,\phi) = \frac{Q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{a^l}{r^{l+1}}P_l(\cos\theta) + \sum_{l=0}^{\infty}A_l b^l P_l(\cos\theta)$$

$$= 0$$

$$A_l = \frac{-q}{2\pi\varepsilon_0}\frac{a^l}{b^{2l+1}}$$

$$\phi(r,\theta,\phi) = \frac{q}{2\pi\varepsilon_0}\sum_{l=2}^{\infty}\left(\frac{a}{b}\right)^l\left[\frac{b^l}{r^{l+1}} - \frac{r^l}{b^{l+1}}\right]P_l(\cos\theta)$$

$$(\ell+1)P_{\ell+1} - (2\ell+1)xP_\ell + \ell \cdot P_{\ell-1} = 0$$

$\downarrow$ ... the book evaluated separate integrals

$$\int (\ell+1)P_{\ell+1} \cdot P_\ell \, dx - \int (2\ell+1) x P_\ell \cdot P_\ell \, dx + \int x \cdot P_{\ell-1} \cdot P_\ell \, dx$$

$\downarrow$ ... the basic integral displayed

$$\int P_\ell \cdot P_\ell \cdot dx = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!} \delta_{\ell\ell}$$

$\downarrow$ ... as spherical harmonics

$$\int Y^*_{\ell m}(\theta,\phi) Y_{\ell m}(\theta,\phi) \, dx = \frac{2\ell+1}{4\pi} \cdot \frac{(\ell-m)!}{(\ell+m)!} \cdot P^m_\ell(\cos\theta)^2 \cdot e^{2im\phi}$$

$$= \frac{2\ell+1}{4\pi} \quad @ \ m=0$$

$\boxed{\text{(Equation 3.38) "Green's Theorem}}$

$$\frac{1}{|x-x'|} = \sum_{\ell=0}^{\infty} \frac{r^\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

$$= 4\pi \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} \frac{r^\ell}{r^{\ell+1}} Y^*_{\ell m}(\theta,\phi) Y_{\ell m}(\theta,\phi)$$

$\boxed{\text{(Equation 2.16) "Greens Theorem for Images"}}$

... when $x = r$

$$G(x,x') = \frac{1}{|x-x'|} - \frac{a}{x\left|x - \frac{a^2}{x'^2}x\right|}$$

$$= 4\pi \cdot \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} \left[\frac{r^\ell}{r^{\ell+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{\ell+1}\right] Y^*_{\ell m}(\theta,\phi) Y_{\ell m}(\theta,\phi)$$

$\boxed{\text{(Equation 3.130) "Potential"}}$

$$\phi(x) = \frac{1}{4\pi\varepsilon_0}\int \rho(x)\cdot G(x,x')\,d^3x$$

$$= \frac{Q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty} P_l(0)\int_0^b \frac{4\pi}{2l+1}\left[\frac{r^l}{r^{l+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1} dr'\right] Y^*_{lm}(\theta,\phi)\,Y_{lm}(\theta,\phi)$$

$$= \frac{Q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty} P_l(0)\,r^l\left[\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right] P_l(\cos\theta)$$

... when $r=\rho\sin\theta$

$\boxed{\text{(Equation 3.126) "Greens Surface Integral}}$

$$\Phi(x) = \frac{-1}{4\pi}\oint \phi(x)\frac{d\sigma}{dn}\,da$$

$$= \frac{Q}{8\pi\varepsilon_0 b}\sum_{l=0}^{\infty}\left[P_l(1)+P_l(-1)\right]P_l(\cos\theta)\int_0^b \rho^l\left[\frac{1}{\rho^{l+1}} - \frac{\rho^l}{b^{2l+1}}\right]da$$

$$= \frac{Q}{8\pi\varepsilon_0 b}\left\{\ln\left(\frac{b}{\rho}\right) + \sum_{l=1}^{\infty}\frac{2l+1}{2l(2l+1)}\left[1-\left(\frac{\rho}{b}\right)^l\right]P_{2l}(\cos\theta)\right\}$$

$$= \frac{Q}{8\pi\varepsilon_0 b}\left\{\ln\left(\frac{b}{r\sin\theta}\right) + \sum_{j=1}^{\infty}\frac{4j+1}{4j(4j+1)}\left[1-\left(\frac{r\sin\theta}{b}\right)^{2j}\right]P_{2l}(\cos\theta)\right\}$$

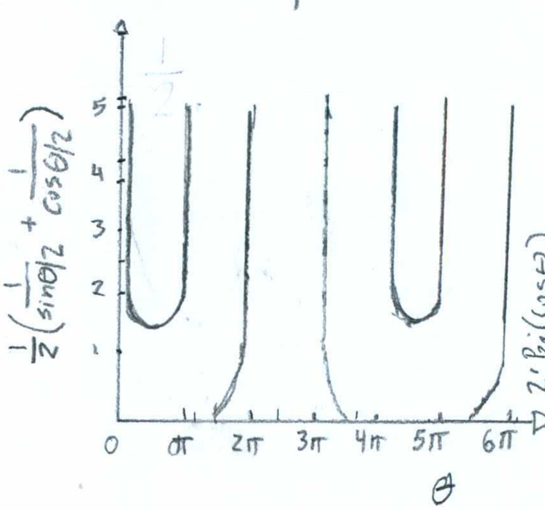
... the book's answer.

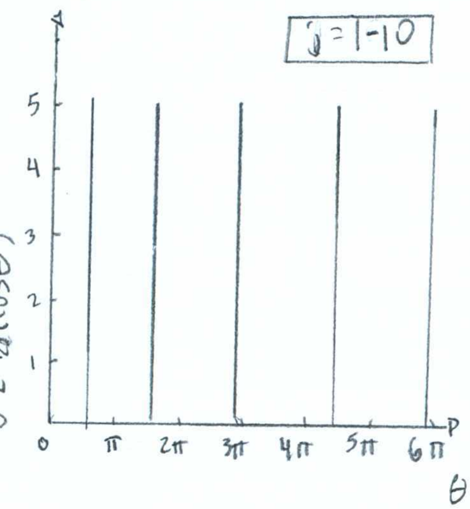
$$= \frac{Q}{8\pi\varepsilon_0 b}\left\{\ln\left(\frac{2b}{r\sin\theta}\right) - 1 - \sum_{l=1}^{\infty}\frac{4j+1}{4j(4j+1)}\left[\left(\frac{r}{b}\right)^{2j}\right]P_{2l}(\cos\theta)\right\}$$

b). Hypothesis: Question 3.9b contradicts an equality in the derivation

$$\frac{1}{2}\left(\frac{1}{\sin\theta/2} + \frac{1}{\cos\theta/2}\right) = 2\sum_{j=1}^{\infty} P_{2j}(\cos\theta)$$

Proof by Deduction:





If the derivation were correct, then

(Equation 3.38) $\dfrac{1}{|x - x'|} = \sum_{l=0}^{\infty} \dfrac{r^l}{r^{2l+1}} P_l(\cos\theta)$

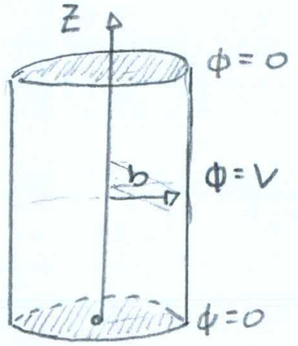
(Equation 2.16) $G(x, x') = \dfrac{1}{|x + x'|} - \dfrac{a}{x\left|x - \frac{a^2}{x'^2} x\right|}$

(Equation 3.130) $\Phi(x) = \dfrac{1}{4\pi\varepsilon_0} \int P(x)\, G(x, x')\, d^3x$

(Equation 3.126) $\Phi(x) = \dfrac{-1}{4\pi} \oint \Phi(x) \dfrac{dG}{\partial n}\, da$

$$= \frac{Q}{4\pi\varepsilon_0 b}\left\{\ln\left(\frac{b}{r}\right)\frac{1}{2}\left\{\frac{1}{\sin\theta/2} + \frac{1}{\cos\theta/2}\right\} + \sum_{j=0}^{\infty} \frac{(4j+1)}{2j(2j+1)}\left[1 - \left(\frac{r}{b}\right)^{2j}\right] P_{2l}(\cos\theta)\right]$$

3.9.



Hollow right Cylinder

Shape : Cylinder
Demension : Volume [3D]
Charge : Q

Potential Derivation:

① Boundary Condition:

$$\phi(r,\phi,z=0)=\phi(r,\phi,z=L)=0$$

$$\Phi(r=b,\phi,z)=V$$

$$\Phi(r=0,\phi,z)=\text{finite}$$

② Laplace's Equation:

$$\nabla^2\Phi=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\phi^2}+\frac{\partial^2\phi}{\partial z^2}$$
$$=0$$

③ Laplace's Equation Solution:

If $\Phi(r,\phi,z)=R(r)\,Q(\phi)\,Z(z)$

Ⓐ Variable Separation:

$$\nabla^2\Phi=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\phi^2}+\frac{\partial^2\phi}{\partial z^2}$$

$$=\frac{Q(\phi)Z(z)}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)+\frac{R(r)Z(z)}{r^2}\frac{\partial^2 Q(\phi)}{\partial^2\phi}+R(r)Q(\phi)\frac{\partial^2 Z(z)}{\partial z^2}$$

$$=\frac{1}{R(r)\cdot r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)+\frac{1}{Q(\phi)r^2}\frac{\partial^2 Q(\phi)}{\partial^2\phi}+\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2}$$

$$=0$$

Ⓑ Radial Eigenvalues:

$$\frac{r}{R}\frac{\partial R}{\partial r}+\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2}+\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z}=\lambda r^2$$

$$\frac{r}{R}\frac{\partial R}{\partial r}+\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2}-\lambda r^2=\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z}$$

$$\frac{r}{R}\frac{\partial R}{\partial r} + \frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} - \lambda r^2 = K$$

If $r = X$ and $\lambda = \frac{1}{r}$, then

$$\frac{\partial^2 R}{\partial X^2} + \frac{1}{X}\frac{\partial R}{\partial X} - \left(1 + \frac{k}{X^2}\right)R = 0$$

© Angular Eigenvalues:

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial \phi^2} = -m^2 \quad ; \quad \frac{\partial^2 Q}{\partial \phi^2} + m^2 Q = 0$$

Ⓓ Vertical Eigenvalues:

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial \phi^2} = k^2 \quad ; \quad \frac{\partial^2 Z}{\partial \phi^2} - k^2 Z = 0$$

Ⓔ General Solution:

$$\phi(r, \phi, z) = \sum_{m,k} R(r)\, Q(\phi)\, Z(z)$$

④ General Solution to Laplace's Equation:

$$R(r) = E\, J_v(kX) + F\, Y_v(kX)$$

where $J_v = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{-1}{2} kX\right)^{v+2m}}{m!\,(m+v)!}$

$$Y_v = \frac{\cos v\pi\, J_v(kX)\, J_{-v}(kX)}{\sin v\pi}$$

$$Q(\phi) = A \circ e^{im\phi} + B \circ e^{-im\phi}$$

$$Z(z) = C\sinh(kz) + D\cosh(kz)$$

⑤ Variables by Boundary Conditions:

D $\Phi(r,\phi,z=0) = \sum_{m,k} [E\cdot J_m(kx) + F Y_m(kz)][Ae^{im\phi} + Be^{-im\phi}] \times [C\sinh(k\cdot 0) + D\cdot\cosh(k\cdot 0)]$

$= 0$, so $D = 0$

F $\Phi(r=0,\phi,z) = \sum_{m,k} [E\cdot J_m(k\cdot 0) + F\cdot Y_m(k\cdot 0)][Ae^{im\phi} + Be^{-im\phi}] \times [C\sinh(kz)]$

$=$ Finite, so $F = 0$

k $\Phi(r=b,\phi,z) = \sum_{m,k} [E\cdot J_m(k\cdot b)][Ae^{im\phi} + Be^{-im\phi}][C\sinh(kz)]$

$= 0$, so $k = i\frac{n\pi}{L}$ where $n = 1, 2, 3\ldots$

A, B $\Phi(r,\phi,z) = \sum_{l=0}^{\infty}\sum_{m=0}^{l} J_m \sinh(kz)\{A\cdot e^{im\phi} + Be^{-im\phi}\}$

$= V$

Two identities: $\sinh(iz) = i\sin(z)$

$\boxed{i\cdot J_m = I_m \text{ (Equation 3.100)}}$

$$\int_0^{2\pi}\int_0^{L} V\sin\left(\frac{n\pi z}{L}\right)e^{-im\phi}\,d\phi\,dz = A\int_0^{2\pi}\int_0^{L}\sum_{l=0}^{\infty}\sum_{m=0}^{l} i\, J_m(kz)\sin^2\left(\frac{n\pi z}{L}\right)dz\cdot d\phi$$

$$= A\sum_{l=0}^{\infty}\sum_{m=0}^{l} I_m(k\cdot z)\cdot L\cdot\pi$$

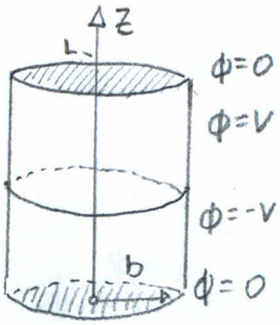
$$A = \frac{1}{L\pi\cdot I_m(k\cdot z)}\int_0^{2\pi}\int_0^{L} V\sin\left(\frac{n\pi z}{L}\right)e^{-im\phi}\,dz\,d\phi$$

A similar integral yields:

$$B = \frac{1}{\pi L I_m(kx)} \int_0^{2\pi}\int_0^{L} V \sin\left(\frac{n\pi z}{L}\right) e^{-im\phi}\, dz\, d\phi$$

$$\Phi(r,\phi,z) = \sum_{l=0}^{\infty}\sum_{m=0}^{l} \frac{2\, I_m\left(\frac{n\pi r}{L}\right)}{\pi L \cdot I_m\left(\frac{n\pi b}{L}\right)} \left[\int_0^{2\pi}\int_0^{L} V \sin\left(\frac{n\pi z}{L}\right) e^{-im\phi}\, dz\, d\phi\right]\left(e^{im\phi} + e^{-im\phi}\right)\sin\left(\frac{n\pi z}{L}\right)$$

3.10.



Two equal half cylinders

Shape: Cylinder
Dimension: Volume [3D]
Charge: q, -q

Potential Derivation:

① Boundary Condition:

$$\phi(r,\phi,z) = V(\phi,z) = \begin{cases} V & -\pi/2 < \phi < \pi/2 \\ -V & \pi/2 < \phi < 3\pi/2 \end{cases}$$

$$\phi(r,\phi,z=0) = 0$$

$$\phi(r,\phi,z=L) = 0$$

② Laplace's Equation:

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \phi^2} + \frac{\partial^2 \phi}{\partial z^2}$$
$$= 0$$

③ Laplaces Equation Solutions:

If $\Phi(r,\phi,z) = R(r)\,Q(\phi)\,Z(z)$

④ Variable Separation:

$$\nabla^2 \Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \phi^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \frac{Q(\phi)Z(z)}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{R(r)Z(z)}{r^2}\frac{\partial^2 Q(\phi)}{\partial^2 \phi} + R(r)Q(\phi)\frac{\partial^2 Z(z)}{\partial z^2}$$

$$= \frac{1}{R(r)\cdot r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{Q(\phi)r^2}\frac{\partial^2 Q(\phi)}{\partial^2 \phi} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2}$$
$$= 0$$

(B) Radial Eigenvalues:

$$\frac{r}{R}\frac{\partial R}{\partial r} + \frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z} = \lambda r^2$$

$$\frac{r}{R}\frac{\partial R}{\partial r} + \frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} - \lambda r^2 = \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z}$$

$$\frac{r}{R}\frac{\partial R}{\partial r} + \frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} - \lambda r^2 = K$$

If $r = X$ and $\lambda = \frac{1}{r^2}$,

$$\frac{\partial^2 R}{\partial^2 X} + \frac{1}{X}\frac{\partial R}{\partial x} - \left(1 + \frac{k}{X^2}\right)R = 0$$

(C) Angular Eigenvalues:

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial \phi^2} = -m^2 ; \qquad \frac{\partial^2 Q}{\partial \phi^2} + m^2 Q = 0$$

(D) Vertical Eigenvalues:

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = k^2 ; \qquad \frac{\partial^2 Z}{\partial z^2} - k^2 Z = 0$$

(E) General Solution:

$$\Phi(r,\phi,z) = \sum_{l,m} R(r)\,Q(\phi)\,Z(z)$$

(4) General Solution to Laplaces Equation:

$$R(r) = E\,J_v(kX) + F\,Y_v(kX)$$

Where $$J_v = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{-1}{2}kx\right)^{v+2m}}{m!\,(m+v)!}$$

$$Y_v = \frac{\cos(v\pi)J_v(kx) - J_{-v}(kx)}{\sin v\pi}$$

$$Q(\phi) = A\sin(m\pi\phi) + B\cos(m\pi\phi)$$

$$Z(z) = C\sinh(kz) + D\cosh(kz)$$

⑤ Variables by Boundary Conditions:

$$D \quad \Phi(r,\phi,z=0) = \sum_{m,k}\left[E\cdot J_m(kx) + F\,Y_m(kx)\right]\left[A\circ e^{im\phi} + B\circ e^{-im\phi}\right]$$

$$\times\left[C\sinh(k\cdot 0) + D\cosh(k\cdot 0)\right]$$

$$= 0, \text{ so } D = 0$$

$$F \quad \Phi(r=0,\phi,z) = \sum_{m,k}\left[E\cdot J_m(k\cdot 0) + F\,Y_m(k\cdot 0)\right]\left[A\circ e^{im\phi} + B\circ e^{-im\phi}\right]$$

$$\times\left[C\sinh(k\cdot z)\right]$$

$$= \text{finite, so } F = 0$$

$$K \quad \Phi(r=b,\phi,z) = \sum_{m,k}\left[E\circ J_m(kb)\right]\left[A\circ e^{im\phi} + B\circ e^{-im\phi}\right]\left[C\sinh(kz)\right]$$

$$= 0, \quad k = \frac{n\pi}{L} \quad \text{where } n = 1, 2, 3\ldots$$

$$A, B \quad \Phi(r,\phi,z) = \sum_{k=0}^{\infty}\sum_{m=0}^{k} J_m(kx)\sinh(kz)\left[A\circ e^{im\phi} + B\circ e^{-im\phi}\right]$$

$$= V$$

$$\int_{-\pi/2}^{\pi/2}\int_{0}^{L} V\cdot\sin\left(\frac{n\pi z}{L}\right)e^{-im\phi}\,dz\,d\phi = A_+\int_{-\pi/2}^{\pi/2}\int_{0}^{L}\sum_{l=0}^{\infty}\sum_{m=0}^{l} I_m(kx)\sin^2\left(\frac{n\pi z}{L}\right)dz\,d\phi$$

$$= A_+\sum_{l=0}^{\infty}\sum_{m=0}^{l} I_m(kx)\cdot L\cdot\pi$$

$$A_+ = \frac{1}{\pi\cdot L\cdot\sum_{l=0}^{\infty}\sum_{m=0}^{l} I_m(kx)}\int_{-\pi/2}^{\pi/2}\int_{0}^{L} V\cdot\sin\left(\frac{n\pi z}{L}\right)e^{-im\phi}\,dz\,d\phi$$

$$A_- = \frac{+1}{\pi L\cdot\sum_{l=0}^{\infty}\sum_{m=0}^{l} I_m(kx)}\int_{\pi/2}^{3\pi/2}\int_{0}^{L} V\sin\left(\frac{n\pi z}{L}\right)e^{-im\phi}\,dz\,d\phi$$

$$A = A_+ + A_-$$

$$= \frac{1}{\pi L \cdot I_m(kx)}\left[\int_{-\pi/2}^{\pi/2}\int_0^{L} V\sin\left(\frac{n\pi z}{L}\right)e^{-im\phi}\,dz\,d\phi - \int_{\pi/2}^{3\pi/2}\int_0^{L} V\sin\left(\frac{n\pi z}{L}\right)e^{-im\phi}\,dz\,d\phi\right]$$

$$= \frac{iV}{\pi^2 m \cdot I_m(kx)}\left[1-(-1)^n\right]\circ\left(1-(-1)^m\right)^2\circ(-1)e^{im\pi/2}$$

$$= \frac{8 \cdot V}{m \cdot n \cdot \pi^2 I_m(kx)}(-1)^{(m-1)/2}$$

Similarly, the B integral yields:

$$B = \frac{8V}{mn\pi^2 I_m(kx)}(-1)^{(m-1)/2}$$

Eulers Formula

$$\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$$

$$\Phi(r,\phi,z) = \frac{16V}{\pi^2}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{1}{m \cdot n}\frac{I_m(n\pi r/L)}{I_m(n\pi b/L)}(-1)^{(m-1)/2}\sin\left(\frac{n\pi z}{L}\right)\cos(m\phi)$$

b) When $z = L/2$, then

$$\Phi(r,\phi,z=L/2) = \frac{16V}{\pi^2}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{1}{mn}\frac{I_m(n\pi r/L)}{I_m(n\pi b/L)}(-1)^{(m-1)/2}\cdot(-1)^{(n-1)/2}\cdot\cos(m\phi)$$

When $L \gg b$

$$= \frac{4V}{\pi}\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{r^m}{b^m}\left(\frac{1}{m}\right)(-1)^{(m-1)/2}\cos(m\phi)$$

$$= \frac{4V}{\pi}\tan^{-1}\left(\frac{\rho}{b}\circ e^{im\phi}\right)$$

$$= \frac{2V}{\pi}\left[\tan^{-1}\left(\frac{\rho\cos\phi}{b+\rho\sin\phi}\right)+\tan^{-1}\left(\frac{\rho\cos\phi}{b-\rho\sin\phi}\right)\right]$$

$$= \frac{2V}{\pi}\tan^{-1}\left[\frac{2b\rho\cos\phi}{b^2-\rho^2}\right]$$

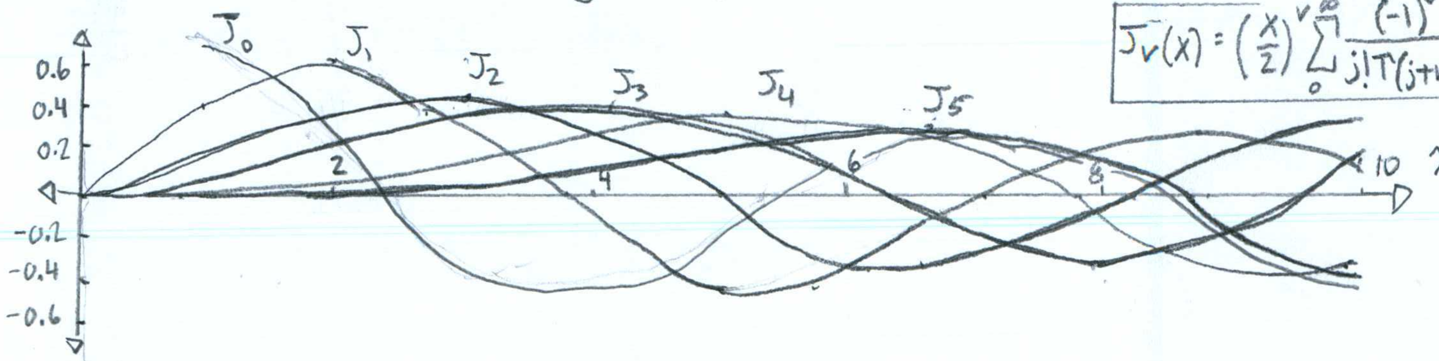
Trigonometric Identity:

$$\tan^{-1}(x) = \sum_{m=1,\text{odd}}(-1)^{(m-1)/2}\circ\frac{x^m}{m}$$

$$\tan(x+iy) = \frac{\tan^{-1}\left(\frac{x}{1+y}\right)+\tan^{-1}\left(\frac{x}{1-y}\right)}{2}$$

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

3.11. Bessel Function Orthogonality, and Normalization:



$$\boxed{J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(j+\nu+1)}\left(\frac{x}{2}\right)^{2j}}$$

a) $$x\frac{dJ_1(x)}{dx} + \lambda J_1(x) = 0 \rightarrow x\frac{dJ_1(x)}{dx}J_2(x) + \lambda J_1(x)J_2(x) = 0$$

$$x\frac{dJ_2(x)}{dx} + \lambda J_2(x) = 0 \rightarrow x\frac{dJ_2(x)}{dx}J_1(x) + \lambda J_2(x)J_1(x) = 0$$

$$x\left[\frac{dJ_1(x)}{dx}J_2(x) - \frac{dJ_2(x)}{dx}J_1(x)\right] = 0$$

$$\frac{dJ_1(x)}{dx}J_2(x) - \frac{dJ_2(x)}{dx}J_1(x) = 0$$

b) $$f(\rho) = \sum_{n=1}^{\infty} A_n J_\nu\left(\frac{y\rho}{a}\right)$$

$$= A_n\left[\left(1-\frac{\nu}{y^2}\right)J_\nu\left(\frac{y\rho}{a}\right) + \frac{\partial J_\nu\left(\frac{y\rho}{a}\right)}{dy}\right]$$

Differential Form:

$$x^2y'' + xy' + (x^2+\nu^2)y = 0$$

$$\frac{d}{dx}(x\cdot y') + \left(x + \frac{\nu^2}{x}\right)y = 0$$

$$y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0$$

$$\int_0^a f(\rho)\rho\, J_\nu\left(\frac{y\rho}{a}\right)d\rho = A_n\left[\left(1-\frac{\nu}{y^2}\right)J_\nu\left(\frac{y\rho}{a}\right)J_\nu\left(\frac{y\rho}{a}\right) + \frac{\partial J_\nu\left(\frac{y\rho}{a}\right)}{dy}J_\nu\left(\frac{y\rho}{a}\right)\right]$$

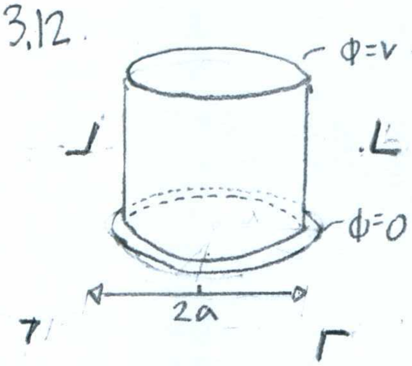
Bessel's Identity:

$$J_\nu(x)J_\nu'(x) = \frac{\rho^2}{2}|J'|^2$$

$$= A_n\cdot\frac{a^2}{2}\left[\left(1-\frac{\nu}{y^2}\right)J_\nu^2\left(\frac{y\rho}{a}\right) + \left(\frac{\partial J_\nu\left(\frac{y\rho}{a}\right)}{\partial y}\right)^2\right]$$

$$A_n = \frac{2}{a^2}\left[\left(1-\frac{v^2}{y^2}\right)J_v^2\left(\frac{y\rho}{a}\right)+\left(\frac{\partial J_v\left(\frac{y\rho}{a}\right)}{\partial y_n}\right)^2\right]^{-1}\int_0^a f(\rho)\,\rho\, J_v\left(\frac{y\rho}{a}\right)d\rho$$

3.12.



An infinite, thin plane sheet with a circular hole and disc inside.

Shape: Disc
Dimension: Volume
Charge: Q

a) $\boxed{\text{Potential Derivation:}}$

① Boundary Conditions:

$$\phi(r=a,\phi,z=0)=0$$

$$\phi(r,\phi,z=0)=V$$

$$\phi(r=0,\phi,z)=\text{finite}$$

② Laplace's Equation:

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial^2 z}$$

$$=0$$

③ Laplace's Equation Solutions:

If $\Phi(r,\phi,z)=R(r)Q(\phi)Z(z)$

Ⓐ Variable Separation:

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial^2 z}$$

$$=\frac{Q(\phi)Z(z)}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial R(r)}{\partial r}\right)+\frac{R(r)Z(z)}{r^2}\frac{\partial^2 Q(\phi)}{\partial^2\phi}+R(r)Q(\phi)\frac{\partial^2 Z(z)}{\partial^2 z}$$

$$=\frac{1}{R(r)\cdot r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial R(r)}{\partial r}\right)+\frac{1}{Q(\phi)\,r^2}\frac{\partial^2 Q(\phi)}{\partial^2\phi}+\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z}$$

$$=0$$

Ⓑ Radial Eigenvalues:

$$\frac{r}{R}\frac{\partial R}{\partial r}+\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2}+\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z}=\lambda r^2$$

$$\frac{r}{R}\frac{\partial R}{\partial r}+\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2}-\lambda r^2=\frac{-1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2\phi}$$

$$\frac{r}{R}\frac{\partial R}{\partial r} + \frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} - \lambda r^2 = k^2$$

If $r = x$ and $\lambda = \frac{1}{r^2}$, then,

$$\frac{\partial^2 R}{\partial^2 x} + \frac{1}{x}\frac{\partial R}{\partial x} - \left(1 + \frac{k^2}{x^2}\right)R = 0$$

Ⓒ Angular Eigenvalues:

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial^2 \phi} = m^2 \quad ; \quad \frac{\partial^2 Q}{\partial^2 \phi} + m^2 Q = 0$$

Ⓓ Vertical Eigenvalues:

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial Z^2} = k^2 \quad ; \quad \frac{\partial^2 Z}{\partial Z^2} - k^2 Z = 0$$

Ⓔ General Solutions:

$$\Phi(r,\phi,z) = \sum_{k,m} R(r)\,Q(\phi)\,Z(z)$$

④ General Solution to Laplaces Equation:

$R(r) = E \cdot J_v(kx) + F \cdot Y_v(kx)$ when $r = x$

Where $$J_v(kx) = \sum_{m=0}^{\infty} \frac{(-1)^m\left(-\frac{1}{2}kx\right)^{v+2m}}{m!(m+v)!}$$

$$Y_v(kx) = \sum_{m=0}^{\infty} \frac{\cos(v\pi)\cdot J_v(kx) - J_{-v}(kx)}{\sin(v\pi)}$$

$$Q(\phi) = A\sin(m\phi)$$

$$Z(\phi) = B \cdot e^{-kz}$$

⑤ Variables by Boundary Conditions:

$$\Phi(r,\phi,z) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left[E\cdot J_v(kx) + F\cdot Y_v(kx)\right]\left[A\sin(m\phi)\right]\left[B\cdot e^{-kz}\right]$$

K $\Phi(r=a,\phi,z=0) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left[E\cdot J_v(k\cdot a)+F\cdot Y_v(k\cdot a)\right]\left[A\sin(m\phi)\right]\left[Be^{-k\cdot 0}\right]$

$= 0$, so $k = \frac{n\pi}{L}$

F $\Phi(r=0,\phi,z) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left[E\cdot J_v(k\cdot 0)+F\cdot Y_v(k\cdot 0)\right]\sin(m\phi)e^{-kz}$

$=$ finite , so $F=0$

E $\Phi(r,\phi,z=0) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} E\cdot J_v(kx)\sin(m\phi)\,e^{-k\cdot 0}$

$= V$

$$\int_0^{\infty}\int_0^{2\pi} V\cdot J_v(kx)\sin(m\phi)\,x\,d\phi\,dx = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{\infty}\int_0^{2\pi} E\cdot J_v^2(kx)\sin^2(m\phi)\cdot x\,d\phi\cdot dx$$

$$\int_0^{\infty}\int_0^{2\pi} V\cdot J_v(kx)\sin(m\phi)\,x\,d\phi\,dx = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{E\cdot\pi}{k}\delta(k-k') \quad \text{(Equation 3.108)}$$

$$E = \frac{k}{\pi}\int_0^{\infty}\int_0^{2\pi} V\cdot J_v(kx)\sin(m\phi)\,x\,d\phi\,dx$$

$$= \frac{k}{\pi}\cdot J_{v+1}(kx)\int_0^{2\pi}\sin(m\phi)\,d\phi$$

$$\Phi(r,\phi,z) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{k}{\pi}V\cdot J_{v+1}(kx)\cdot J_v(kx)\int_0^{2\pi}\sin^2(m\phi)e^{-kz}\,d\phi$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} k\cdot V\cdot J_{v+1}(kx)\cdot J_v(kx)\,e^{-kz}$$

b) Potential at a Perpendicular Distance:

$$E = \frac{k}{\pi}\int_0^{2\pi}\int_0^a V_0 \sin(m\phi) J_v(kx)\, x\, dx\, d\phi$$

$$= \frac{1}{\pi}\int_0^{2\pi} V_0 \sin(m\phi) \cdot J_{v+1}(k\cdot a)\, a\, d\phi$$

$$\Phi(r,\phi,z) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{V_0 a}{\pi} J_{v+1}(k\cdot a)\cdot J_v(kx)\int_0^{2\pi} \sin^2(m\phi)\, d\phi\, e^{-kz}$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} V_0 a\, J_{v+1}(k\cdot a)\cdot J_v(kx)\, e^{-kz}$$

At an infinite distance from center:

$$\Phi(r=0,\phi,z) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{\infty} V_0 a \cdot J_{v+1}(k\cdot a) J_v(k\cdot 0)\, e^{-kz}\, dk$$

$$= V_0 \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{\infty} J_{v+1}(k\cdot a)\cdot e^{-kz}\cdot a\, dk$$

$\boxed{ka = x \,;\, k = \frac{x}{a} \quad dk = \frac{dx}{a}}$

$$= V_0 \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{\infty}\left[\frac{1}{2\pi i}\int_0^{2\pi} e^{i(x\cos\phi + \phi) - xz/a}\, d\phi\right] dx$$

Integral Representation of a Bessel:

$$J_n(x) = \frac{1}{2\pi i}\int_0^{2\pi} e^{i(x\cos\phi + n\phi)}\, d\phi$$

$$= \frac{V_0}{2\pi}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{2\pi} \frac{e^{i\phi}}{(\cos\phi + iz/a)}\, d\phi$$

$$= \frac{V}{2\pi}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left[\int_0^{2\pi} \frac{a(a\cos^2\phi + z\sin\phi)}{a^2\cos^2\phi + z^2}\, d\phi + \int_0^{2\pi} \frac{i a\cos(\phi)(a\sin(\phi) - z)}{a^2\cos^2\phi + z^2}\, d\phi\right]$$

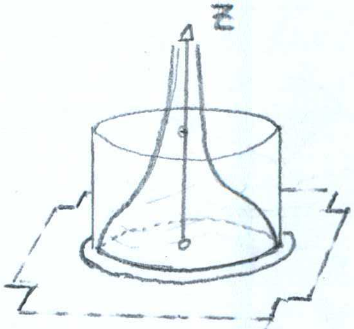
$$= \frac{V}{2\pi}\int_0^{2\pi} \frac{\cos^2\phi}{\cos^2\phi + z^2/a^2}\, d\phi$$

$$= \frac{V}{\pi}\left[\pi - \frac{2}{\pi}\frac{z^2}{a^2}\int_0^{\pi/2}\frac{1}{(\cos^2\phi + z^2/a^2)}\,d\phi\right]$$

$$u = \frac{1}{\sqrt{1+a^2/z^2}} \;;\; \cos^2\theta = \frac{1}{1+(1+a^2/z^2)u^2} \;;\; d\theta = \frac{\sqrt{1+a^2/z^2}}{1+(1+a^2/z^2)u^2}$$

$$= V\left[1 - \frac{2}{\pi}\frac{z}{\sqrt{z^2+a^2}}\left(\tan^{-1}(\infty) - \tan^{-1}(0)\right)\right]$$

$$= V\left[1 - \frac{z}{\sqrt{a^2+z^2}}\right]$$



c) $$\Phi(r=a,\phi,z) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{\infty} V\cdot a\cdot J_{\nu+1}(k\cdot a)\, J_\nu(k\cdot a)\, e^{-kz}\, dk$$

$$J_1(x) = \frac{1}{2\pi i}\int_0^{2\pi} e^{i(x\cos\theta+\theta)}\,d\theta$$

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta'}\,d\theta'$$

$$\Phi(r=a,\phi,z) = \frac{1}{4\pi}\frac{1}{2\pi i}\cdot V\int_0^{2\pi}\int_0^{2\pi} e^{i\theta}\int_0^{\infty} e^{(i(\cos\theta+\cos\theta') - z/a)x}\,dx\,d\theta'\,d\theta$$

$$= \frac{V}{4\pi^2}\int_0^{2\pi} e^{i\theta}\int_0^{2\pi}\frac{1}{\cos\theta' + \cos\theta + z\cdot i/a}\,d\theta'\,d\theta$$

$$= \frac{V}{\pi^2}\int_0^{\pi}\int_0^{\pi}\frac{\cos\theta\cos\theta' + \cos^2\theta}{(\cos\theta'+\cos\theta)^2 + z^2/a^2}\,d\theta'\,d\theta$$

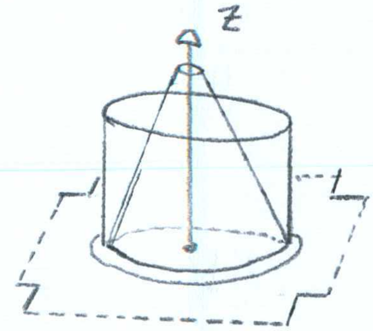
$$= \frac{V}{2}\left[1 - \frac{1}{\pi}\left[\int_0^{\pi}\frac{1}{\pi}\int_0^{\pi}\frac{\cos\theta\cos\theta' + \cos^2\theta}{(\cos\theta'+\cos\theta)^2 + z^2/a^2}\,d\theta'\,d\theta\right]\right]$$

$$= \frac{V}{2}\left(1 - \frac{1}{\pi}\int_0^{\pi}\frac{d\theta}{\sqrt{1+4(a/z)^2\cos^2\theta}}\,d\theta\right) \qquad k = \frac{2}{\sqrt{(z/a)^2+4}}$$

$$= \frac{V}{2}\left(1 - \frac{kz}{\pi a}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\right.$$ "Elliptic Integral"

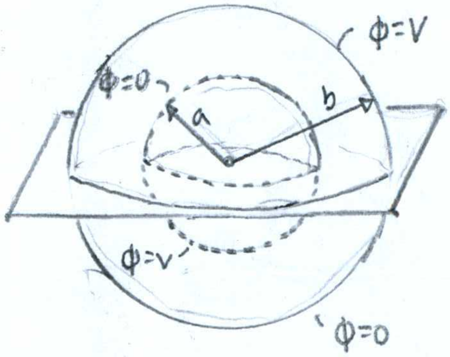
$$\Phi(r,\phi,z) = \frac{V}{2}\left(1 - \frac{kz}{\pi a}K(k)\right)$$

where $K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$



"edges"

3.13



Two concentric spheres

Shape: Sphere
Dimension: Area [2D]
Charge: Q

Potential Derivation by Green's Theorem:

$$\frac{1}{|x-x'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$$ "sphere"

$$\frac{1}{|x-x'|} = 4\pi\sum\sum\frac{Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)}{(2l+1)\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]}\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)$$

"concentric spheres"

Note: Both the upper (Equation 3.70) and lower (Equation 3.125) derive from boundary conditions, Laplace equation, and variable separation. Why not?

$$\Phi(x) = -\frac{1}{4\pi}\int\phi(x)\frac{\partial G}{\partial n}a^2 da - \frac{1}{4\pi}\int\phi(x)\frac{\partial G}{\partial n}b^2 da$$

$$\left.\frac{dG}{dn}\right|_{r_<'=a} = \frac{-4\pi}{a^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)}\left[\left(-\frac{a}{r}\right)^{l+1} - \left(\frac{a}{b}\right)^{l+1}\left(\frac{r}{b}\right)^{l}\right]Y^*_{lm}(\theta,\phi)Y_{lm}(\theta,\phi)$$

$$\left.\frac{\partial G}{\partial n}\right|_{r_>=b} = \frac{-4\pi}{b^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)}\left[\left(\frac{r}{b}\right)^{l} - \left(\frac{a}{b}\right)^{l}\left(\frac{a}{r}\right)^{l+1}\right]Y^*_{lm}(\theta,\phi)Y_{lm}(\theta,\phi)$$

$$\Phi(x) = \sum_{l,k} \left[\int V \cdot Y_{em}(\theta,\phi)\, d\phi\, d\theta \right] \frac{1}{1-\left(\frac{a}{b}\right)^{2l+1}} \left[\left(\frac{a}{r}\right)^{l+1} - \left(\frac{a}{b}\right)^{l+1} \left(\frac{r}{b}\right)^{l} \right] Y_{em}(\theta,\phi)$$

$$+ \sum_{l,k} \left[\int V \cdot Y_{em}(\theta,\phi)\, d\phi\, d\theta \right] \frac{1}{1-\left(\frac{a}{b}\right)^{2l+1}} \left[\left(\frac{a}{r}\right)^{l+1} - \left(\frac{a}{b}\right)^{l} \left(\frac{a}{r}\right)^{l+1} \right] Y_{em}(\theta,\phi)$$

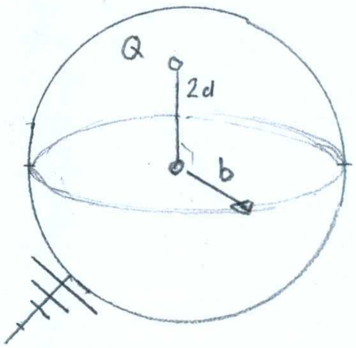
$$= \sum_{l=0}^{\infty} \frac{(2l+1) \cdot V \cdot \int_0^1 P_e(\cos\theta)\, d\cos\theta}{2\left(1-\left(\frac{a}{b}\right)^{2l+1}\right)} \left[\left(1+(-1)^{l+1}\left(\frac{a}{b}\right)^{l}\right)\left(\frac{a}{r}\right)^{l+1} + (-1)^{l}\left(1+(-1)^{l+1}\left(\frac{a}{b}\right)^{l+1}\right)\left(\frac{r}{b}\right)^{l} \right] P_e(\cos\theta)$$

$$\int_0^1 P_e(\cos\theta)\, d\cos\theta = \begin{cases} 1 & l=0 \\ (-1)^{j+1} \dfrac{T(j-1/2)}{2\sqrt{\pi}\, j!} & l=2j-1 \end{cases}$$

$$= \frac{V}{2} + V \sum_{j=1}^{\infty} \frac{(-1)^{j+1} \cdot (4j+1) T(j+1/2)}{4\sqrt{\pi}\, j! \left(1-\left(\frac{a}{b}\right)^{4j+1}\right)} \left[\left(1+\left(\frac{a}{b}\right)^{2j+1}\right)\left(\frac{a}{r}\right)^{2j} - \left(1+\left(\frac{a}{b}\right)\right)\left(\frac{r}{b}\right)^{2j+1} \right] P_{2j+1}(\cos\theta)$$

... when $l = 2j$

3.14.



A grounded, conducting spherical shell centered at a line charge.

Shapes: Sphere, line
Dimension: Volume [3D]
Charge: Q

The advice about a line charge indicates a potential derivation with Green's Theorem, specifically Equation 3.130. How and what formed the equation? Potential by Boundary conditions; a line charge's potential added to a spheres potential.

(Equation 3.130)

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0} \int \rho(x') G(x,x')\, d^3x$$

(Equation 3.116.5)

$$\delta(x-x') = \frac{\delta(r-r')}{r^2} \circ \delta(\phi-\phi') \circ \delta(\cos\theta - \cos\theta')$$

$$= \frac{(d^2-r^2)}{r^2} \circ \rho_o \circ \delta(\phi-\phi') \circ \left[\delta(\cos\theta-1)+\delta(\cos\theta+1)\right]$$

(Equation 1.6)

$$\rho(x) = \sum_{i=1}^{n} q_i \cdot \delta(x-x')$$

(Equation 1.10, 1.11 [Right side])

$$Q = \sum q_i = \int \rho(x) d^3x$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_0^{d} \frac{\rho_o}{r^2}(d^2-r^2)\left[\delta(\cos\theta-1)+\delta(\cos\theta+1)\right] r^2 dr\, d\phi\, d(\cos\theta)$$

$$= 4\pi \cdot \rho_o \int_0^{d} (d^2-r^2)\, dr$$

$$= \frac{8\pi}{3} \rho_o d^3$$

$$\rho_o = \frac{3Q}{8\pi d^3}$$

(Equation 3.130)

$$G(x,x') = \frac{4\pi}{2\ell+1} \sum_{\ell=0}^{\infty} \cdot r_<^{\ell} \left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^{\ell}}{b^{2\ell+1}}\right) Y_\ell^{m*}(\theta,\phi)\, Y_\ell^{m}(\theta,\phi)$$

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\rho(x')G(x,x')r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{P}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\rho_0\frac{(d^2-r^2)}{r^2}\left[\delta(\cos\theta-1)+\delta(\cos\theta+1)\right]\cdot\frac{4\pi}{2l+1}\sum_{l=0}^{\infty}r^l\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)$$

$$\times(2l+1)P_l(\cos\theta)P_l(\cos\theta)$$

$$= \frac{3Q}{8\pi\varepsilon_0 d^3}\sum_{l=0}^{\infty}P_l(\cos\theta)\int_0^d(d^2-r^2)r^l\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)dr$$

Potential when $r>d$, e.g. beyond radius of wire

$$\Phi(x) = \frac{3Q}{8\pi\varepsilon_0 d^3}\sum_{l=0}^{\infty}P_l(\cos\theta)\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)\int_0^d(d^2-r^2)r^l dr$$

$$= \frac{3Q}{8\pi\varepsilon_0 d^3}\sum_{l=0}^{\infty}P_l(\cos\theta)\frac{2d^2}{(l+1)(l+3)}\left(\frac{d}{r}\right)^{l+1}\left(1-\left(\frac{r}{b}\right)^{2l+1}\right)$$

Potential when $r<d$ e.g up to radius of wire

$$\Phi(x) = \frac{3Q}{8\pi\varepsilon_0 d^3}\sum_{l=0}^{\infty}P_l(\cos\theta)\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)\int_0^r(d^2-r^2)r^l dr + r^l\int_r^d(d^2-r^2)\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right)dr$$

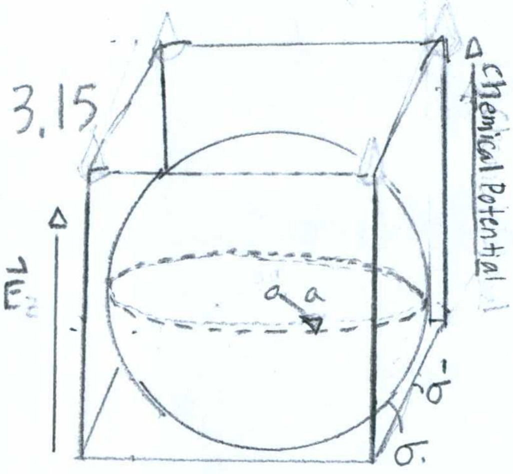
$$= \frac{3Q}{8\pi\varepsilon_0 d^3}\sum_{l=0}^{\infty}P_l(\cos\theta)\left[d\frac{2l+1}{l(l+1)}+r^2\left(\frac{2l+1}{(2-l)(3+l)}\right)+r^l\left(\frac{1}{d^{l-2}}\left(\frac{2}{l(l-2)}\right)-\left(\frac{2}{(l+3)(l+1)}\right)\frac{d^{l+3}}{b^{2l+1}}\right)\right.$$

b) $\sigma = -\varepsilon_0 \left.\frac{\partial\phi}{\partial n}\right|_{r=b \text{ and } r>d}$

$$= \varepsilon_0\frac{3Q}{4\pi\varepsilon_0}\frac{\partial}{\partial r}\sum_{l=0}^{\infty}P_l(\cos\theta)\frac{d^l}{(l+1)(l+3)b^{2l+1}}\left[\left(\frac{b}{r}\right)^{l+1}-\left(\frac{r}{b}\right)^{l}\right]_{r=b}$$

$$= -\frac{3Q}{4\pi b^2}\sum P_l(\cos\theta)\frac{(2l+1)}{(l+1)(l+3)}\left(\frac{d}{b}\right)^l$$

C) The math states, "A hollow, grounded sphere with a wire inside emits a proportional frequency at any angle, unless a super small length. Then, no frequency passes beyond the sphere."



"Spherical cow" model

Shape: Sphere
Dimension: Volume [3D]
Charge: Q

a) $\boxed{\text{Electric Field Derivation:}}$

① Boundary Conditions:

$$\Phi(r=0, \theta, \phi=0) = 0$$

$$\Phi(r=\infty, \theta, \phi=0) = 0$$

② Laplaces Equation:

$$\nabla^2 \Phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

③ Laplace's Equation Solutions:

Ⓐ Variable Separation: If $\Phi(r,\theta,\phi) = \frac{U(r)P(\theta)Q(\phi)}{r}$

$$\nabla^2 \Phi = \frac{r^2 \sin^2\theta}{U(r)P(\theta)Q(\phi)}\left[P(\theta)Q(\phi)\frac{\partial^2 U}{\partial r^2} + \frac{U(r)Q(\phi)}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial \theta}\right) + \frac{U(r)P(\theta)}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2}\right]$$

$$= 0$$

Ⓑ Radial Eigenvalues:

$$\frac{r^2}{U(r)}\frac{\partial^2 U(r)}{\partial^2 r} = \ell(\ell+1)$$

$$\frac{\partial^2 U(r)}{\partial^2 r} - \frac{\ell(\ell+1)U(r)}{r^2} = 0$$

Ⓒ Angular Eigenvalues:

$$\frac{1}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) = \frac{\lambda}{\sin^2\theta} - \ell(\ell+1)$$

$$\frac{1}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) + \left(\ell(\ell+1) - \frac{\lambda}{\sin^2\theta}\right) = 0$$

Ⓓ Azimuthal Eigenvalues:

$$\frac{1}{Q(\phi)}\frac{\partial^2 Q(\phi)}{\partial\phi^2} = -m^2$$

$$\frac{\partial^2 Q(\phi)}{\partial\phi^2} + m^2 Q(\phi) = 0$$

④ General Solution to Laplace's Equation:

$$U(r) = A_\ell r^{\ell+1} + B_\ell \cdot r^{-\ell}$$

$$P(\theta) = P_\ell^m(\cos\theta)$$

$$Q(\phi) = A_m e^{im\phi}$$

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{U(r)}{r} P(\theta)Q(\phi)$$

$$= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} (A_\ell r^\ell + B_\ell r^{-\ell-1}) P_\ell^m(\cos\theta) e^{im\phi}$$

⑤ Variables by Boundary Conditions:

The book compares internal-and-external electrical systems. Up to this point, the boundary conditions derived a single equation. Here are two equations as pieces and shambles. Spherical antennas with a wire inside have a term for the sphere and another for the wire, and both in a single equation.

Inside:

$$B \quad \Phi(r=0,\theta,\phi=0)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(A_\ell\cdot 0^\ell+B_\ell\cdot\frac{1}{0}\right)P_\ell^m(\cos\theta)\,e^{im\cdot 0}$$

$$= \text{finite}, \text{ so } B_\ell = 0$$

$$\Phi(r,\theta,\phi)_{Int} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_\ell r^\ell \cdot P_\ell^m(\cos\theta)$$

Outside:

$$A \quad \Phi(r=\infty,\theta,\phi=0)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(A_\ell\cdot\infty+B_\ell\cdot\frac{1}{\infty}\right)P_\ell^m(\cos\theta)\,e^{im\phi}$$

$$=0, \text{ so } A_\ell=0$$

$$\Phi(r,\theta,\phi)_{Ext} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} B_\ell r^{-\ell-1}\cdot P_\ell^m(\cos\theta)$$

6 Electric Field from Potential:

$$\boxed{\text{Gauss' Law: } E=\sigma/\varepsilon_0}$$

$$(E_{ext}-E_{int})=\sigma/\varepsilon_0$$

$$(-\nabla\Phi_{ext}+\nabla\Phi_{int})=\sigma/\varepsilon_0$$

$$\left(-\frac{\partial\phi(r,\theta,\phi)_{ext}}{\partial r}+\frac{\partial\phi(r,\theta,\phi)_{int}}{\partial r}\right)=-\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}B_\ell(-\ell-1)\,r^{-\ell-2}P_\ell^m(\cos\theta)+\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}A_\ell\cdot\ell\cdot r^{\ell-1}P_\ell^m(\cos\theta)$$

$$=\frac{C_1\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}P_\ell^m(\cos\theta)}{\varepsilon_0}$$

When $\frac{\partial\phi_{int}}{\partial\theta}=\frac{\partial\Phi_{ext}}{\partial\theta}$, then $B_\ell = A_\ell a^{2\ell+1}$

Such as, Inside: $A_\ell=\dfrac{C_1}{\varepsilon_0 r^{\ell-1}(2\ell+1)}$

Outside: $B_\ell=\dfrac{C_1 r^{\ell+2}}{\varepsilon_0(2\ell+1)}$

A solution for C_1, $\vec{E} = E \cdot r\cos\theta$

$$= -\Phi$$

$$= -\frac{\sigma}{\varepsilon_0} \sum \sum \frac{C_l}{2l+1} \left(\frac{r}{r}\right)^l P_l^m(\cos\theta)$$

@ $l=1$, $C_1 = -3\varepsilon_0 \vec{E}$

	Inside	Outside
Potential	$\Phi(r,\theta,\phi) = \sum A_l \cdot r^l P_l^m(\cos\theta)$ $= -E r\cos\theta$	$\phi(r,\theta,\phi) = \sum B_l \cdot r^{-l-1} \cdot P_l^m(\cos\theta)$ $= -E \frac{a^3}{r^2} \cos\theta$
Electric Field	$E_{in} = \vec{E}_{int}$	$E = (1 - 3r\cos\theta) \vec{E}_{int} \frac{a^3}{r^3}$
Surface Charge Density	$\sigma_s = -3\varepsilon_0 E_{int} \cos\theta$	$\sigma_s = -3\varepsilon_0 E_{int} \cos\theta$
Force	$J_{int} = \sigma(E_{int} + F)$ $= \sigma(E_{int} + F)$ @ $\theta = 0°$ $= \frac{2\sigma\sigma'}{\sigma + 2\sigma'} F$	$J_{ext} = \sigma E_{ext}$ $= \sigma(1 - 3r\cos\theta) E_{int} \frac{a^3}{r^3}$ @ $\theta = 0°$ $= \frac{2 \cdot \sigma' \cdot \sigma}{\sigma + 2\sigma'} F \frac{a^3}{r^3}$

If $\sigma = \frac{1}{4\pi\varepsilon_0}$, then $\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{p}{r^3} [3\cos\theta - 1]$

Where $p = 4\pi\varepsilon_0 F a^3 \frac{\sigma}{\sigma + 2\sigma'}$

b) $\boxed{\text{Total Current}}$

$$I = \int_0^{\pi/2}\int_0^{2\pi} J \cdot a^2 \sin\theta \, d\theta \, d\phi$$

$$= a^2 F \frac{2\sigma\sigma'}{\sigma + 2\sigma'} \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{2\sigma\sigma'}{\sigma+2\sigma'} \pi a^2 F$$

$\boxed{\text{Total Power Dissipation}}$

$$P_{int} = \int J \cdot E \, dV$$

$$= \frac{1}{\sigma} \int J_{int}^2 \, dV$$

$$= \frac{16\sigma\sigma'^2}{3(\sigma+2\sigma')^2} F^2 \pi^2 a^3$$

$$\int J \cdot E \, dV$$

$$= \frac{1}{\sigma} \int J_{int}^2 \, dV$$

$$= \frac{\sigma'^2 \sigma^2}{(\sigma+2\sigma')^2} \frac{F^2 a^6}{\sigma'} \int_0^{2\pi}\int_0^{\pi}\int_a^{\infty} \frac{(3\cos^2\theta + 1)}{r^4} \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{8\sigma'\sigma^2}{3(\sigma+2\sigma')^2} F^2 \pi a^3$$

$$P_{Diss} = P_{ext} - P_{int}$$

$$= \frac{8\sigma\sigma'}{3(\sigma+2\sigma')} F^2 \pi a^3$$

$\boxed{\text{Effective Internal Resistance}}$

$$R_{ext} = \frac{P}{I_{ext}^2}$$

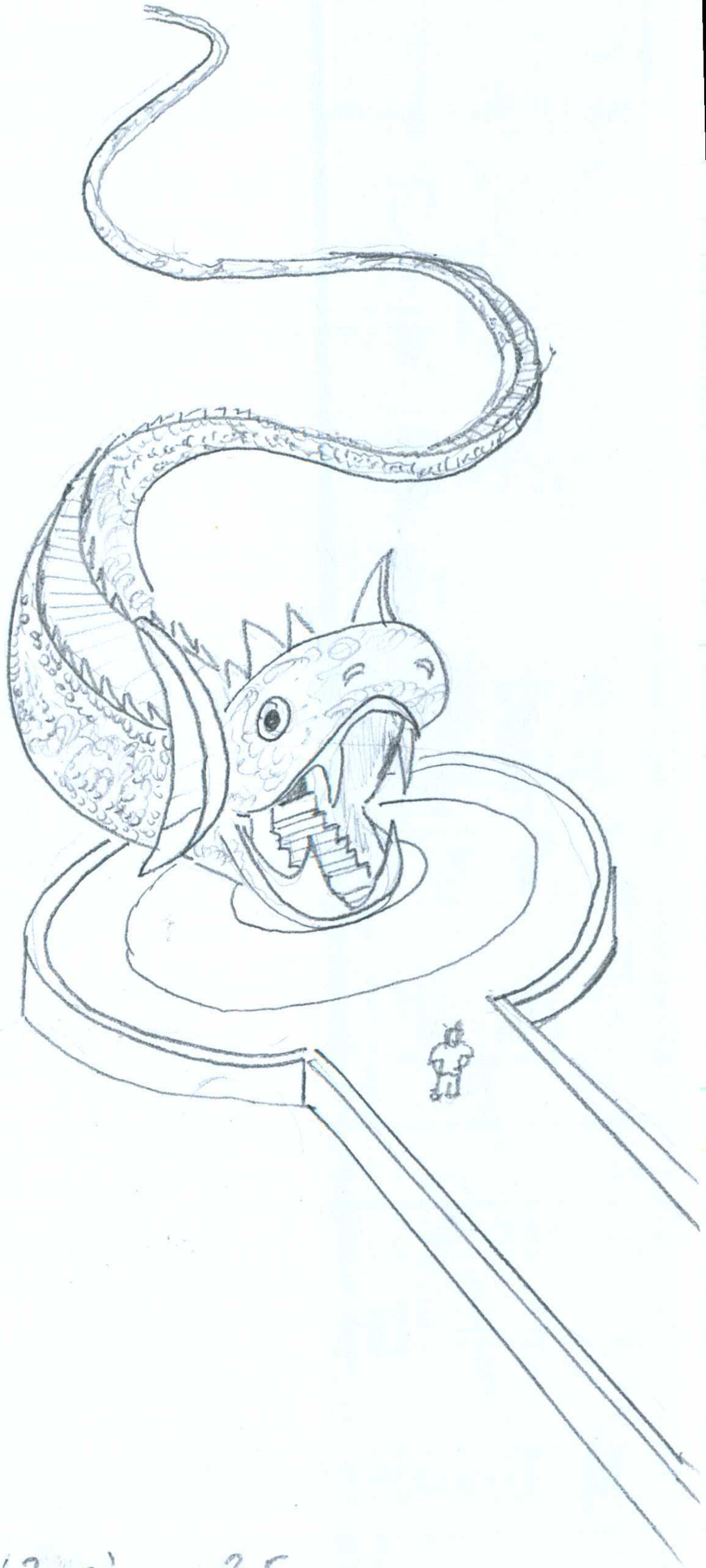
$$= \frac{2}{3\sigma' \pi a}$$

Effective External Voltage

$$V_{ext} = \frac{P_{ext}}{I_{ext}} = \frac{4\sigma}{3(\sigma + 2\sigma')} F \cdot a$$

c) Power Dissipation Internal

$$P_{int} = \frac{16\sigma\sigma'^2}{3(\sigma+2\sigma')^2} F^2 \cdot \pi \cdot a^3$$



Effective Internal Resistance

$$R_{int} = \frac{P_{int}}{I_{int}^2} = \frac{4}{3\sigma\pi a}$$

Effective Internal Voltage

$$V_{int} = \frac{P_{int}}{I} = \frac{8\sigma'}{3(\sigma+2\sigma')} F \cdot a$$

d) Total Voltage

$$V = (R_{ext} + R_{int}) \cdot I$$

$$= \left(\frac{2}{3\sigma'\pi a} + \frac{4}{3\sigma\pi a}\right) \cdot \left(\frac{2\sigma\sigma'}{(\sigma+2\sigma')} \cdot \pi \cdot a^2 \cdot F\right)$$

$$= \frac{4}{3} \cdot a \cdot F$$

$$V = V_{ext} + V_{int}$$

$$= \frac{4}{3} a \cdot F$$

Total Power Supply

$$P_T = I \cdot V = \frac{8\sigma\sigma'}{3(\sigma+2\sigma')} \pi a^3 F^2$$

3.16

a) <u>Orthogonality of Bessel Functions:</u>

$$x^2y''+xy'+(\alpha^2x^2-v^2)y=0 \quad -\text{or}- \quad \frac{d}{dx}\left(x\frac{dy}{dx}\right)+\left(\alpha^2x-\frac{v^2}{x}\right)y=0$$

$$\frac{d}{dx}\left(x\frac{J_1}{dx}\right)+\left(k_1^2x-\frac{v}{x}\right)J_1=0$$

$$\frac{d}{dx}\left(x\frac{J_{2}}{dx}\right)+\left(k_2^2x-\frac{v}{x}\right)J_2=0$$

$$(k_1^2-k_2^2)J_1\circ J_2\circ x=\frac{d}{dx}\left(x\frac{dJ_1}{dx}\right)J_2+\frac{d}{dx}\left(x\frac{dJ_2}{dx}\right)J_1$$

$$(k_1^2-k_2^2)\int_0^\ell J_1\circ J_2\circ x\,dx=-\int_0^\ell\frac{d}{dx}\left(x\frac{dJ_1}{dx}\right)J_2\,dx+\int_0^\ell\frac{d}{dx}\left(x\frac{dJ_2}{dx}\right)J_1\,dx$$

$$(k_1^2-k_2^2)\int_0^\ell J_1\circ J_2\,x\,dx=\ell\circ\frac{dJ_2}{dx}\bigg|_{x=\ell}\circ J_1-\ell\frac{dJ_1}{\partial x}\bigg|_{x=\ell}J_2 \quad \circ\circ\circ \text{ when } k_1=\mu_n/\ell,\; k_2=\mu_k/\ell$$

$$\frac{(\mu_n^2-\mu_k^2)}{\ell^2}\int_0^\ell x\,J_v\left(\mu_v\frac{x}{\ell}\right)\circ J_v\left(\mu_n\frac{x}{\ell}\right)dx=\mu_k J_v(\mu_n)J_v'(\mu_k)-\mu_n J_v(\mu_k)J_v(\mu_n')$$

$\circ\circ\circ$ if $\mu_n=\mu_k$, then the righthand is zero, otherwise one.

$$\frac{1}{\rho}\delta(\rho-\rho')=\int\rho\,J_v(k\rho)J_v(k\rho')$$

b) (Equation 3.116.5)

$$\delta(x-x')=\delta(r-r')\circ\delta(\phi-\phi')\circ\delta(z-z')$$

(Equation 1.31)

$$\nabla^2\left|\frac{1}{|x-x'|}\right|=-4\pi\delta(x-x')$$

$$\left|\frac{1}{|x-x'|}\right|=-4\pi\sum_{m=-\infty}^{\infty}\int_0^\infty\delta(\rho-\rho')\circ\delta(\phi-\phi')\,\delta(z-z')\,d\Omega$$

$$= -4\pi \sum_{m=-\infty}^{\infty} \int_0^{\infty} J_\nu(k\rho) J_\nu(k\rho') e^{im(\phi-\phi')} \cdot e^{-k(z-z')} dk$$

c) $$\int_0^{\infty} e^{k \cdot z} \cdot J_0(k\rho) dk = \int_0^{\infty} e^{kz} \cdot \left[\frac{1}{2\pi} \int_0^{2\pi} e^{ik\rho\cos\theta} d\theta \right] dk$$

$$\boxed{J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix\cos\theta} d\theta}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{(i\rho\cos\theta + z)k} \cdot dk$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\rho\cos\theta - iz}$$

$$\boxed{\cos\theta = \sum_{i=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}$$

$$= \frac{1}{\rho - iz}$$

$$= \frac{1}{\sqrt{(\rho - iz)(\rho - iz)}}$$

$$= \frac{1}{\sqrt{\rho^2 + z^2}}$$

$$\sum_{m=-\infty}^{\infty} e^{im\phi} J_m(k\rho) J_m(k\rho') = J_0\left(k\sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos\phi}\right)$$

when $m = 0$ and $\nu = 0$

because $\dfrac{1}{|x - x'|} = \dfrac{1}{\sqrt{x^2 + x'^2 - 2xx'\cos\gamma}}$

$$e^{ik\rho\cos\phi} = e^{ix\sin(\phi + \pi/2)}$$

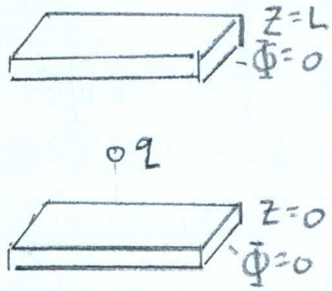
$$= \sum_{n=-\infty}^{\infty} a_n \cdot e^{in(\phi + \pi/2)}$$

$$= \sum_{n=-\infty}^{\infty} i^n \cdot J_n \cdot e^{in\phi}$$

Jacob-Anger Expansion -or- Fourier Transform Pairs

d) $J_m(x) = \frac{1}{2\pi i^m}\int_0^{2\pi} e^{ix\cos\phi - im\phi}\, d\phi$ with a $\frac{1}{2\pi}$ because Fourier periodicity.

3.17



Parallel Conducting Plates

Shape: "Cylindrical"
Dimension: Volume [3D]
Charge: q

a) $\boxed{\text{Greens Theorem from Potential}}$

① Boundary Conditions:

$\phi(\rho, \phi, z=0) = 0 \qquad \phi(\rho=\rho', \phi=0, z=z') = V$

$\phi(\rho, \phi, z=L) = 0$

$\phi(\rho=0, \phi, z) = \text{finite}$

② Laplaces Equation:

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$
$$= 0$$

③ Laplaces Equation Solutions:

If $\Phi(\rho,\phi,z) = R(\rho)\,Q(\phi)\,Z(z)$, then

Ⓐ Variable Separation:

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$= \frac{Q(\phi)Z(z)}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial R(\rho)}{\partial r}\right) + \frac{R(\rho)\,Z(z)}{r^2}\frac{\partial^2 Q(\phi)}{\partial^2\phi} + R(\rho)Q(\phi)\frac{\partial^2 Z(z)}{\partial z^2}$$

$$= \frac{1}{R(\rho)\cdot r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial R}{\partial r}\right) + \frac{1}{Q(\phi)r^2}\frac{\partial^2 Q(\phi)}{\partial^2\phi} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2}$$

$$= 0$$

Ⓑ Radial Eigenvalues:

$$\frac{r}{R}\frac{\partial R}{\partial r} + \frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z} = \lambda r^2$$

$$\frac{r}{R}\frac{\partial R}{\partial r} + \frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} - \lambda r^2 = \frac{1}{Q(\phi)}\frac{\partial^2 Q(\phi)}{\partial^2 \phi}$$

$$\frac{r}{R}\frac{\partial R}{\partial r} + \frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} - \lambda r^2 = k$$

If $r = x$ and $\lambda = 1/r^2$

$$\frac{\partial^2 R}{\partial^2 x} + \frac{1}{x}\frac{\partial R}{\partial x} - \left(1 + \frac{n}{x^2}\right)R = 0$$

Ⓒ Angular Eigenvalues:

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial \phi^2} = -m^2 \quad ; \quad \frac{\partial^2 Q}{\partial \phi^2} + m^2 Q = 0$$

Ⓓ Vertical Eigenvalues:

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = k^2 \quad ; \quad \frac{\partial^2 Z}{\partial z^2} - k^2 Z = 0$$

Ⓔ General Solution:

$$\Phi(\rho, \phi, z) = \sum_{k,m} R(\rho)\, Q(\phi)\, Z(z)$$

④ General Solution to Laplaces Equation:

$$R(\rho) = E \cdot J_v(kx) + F \cdot Y_v(kx)$$

where $J_v = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{-1}{2}kx\right)^{v+2m}}{m!\,(m+v)!}$

$$Y_v = \frac{\cos(v\pi)\, J_v(kx) - J_{-v}(kx)}{\sin(v\pi)}$$

$$Q(\phi) = A e^{im\phi}$$

$$Z(z) = B\sinh(kz) + C\cosh(kz)$$

⑤ Variables by Boundary Conditions:

$C \quad \Phi(\rho,\phi,z=0)=\sum_{k,m}\left[E\cdot J_m(kx)+F\cdot Y_m(kx)\right]\left[Ae^{im\phi}\right]\left[B\sinh(k\cdot 0)+C\cosh(k\cdot 0)\right]$

$=0$, so $C=0$

$F \quad \Phi(\rho=0,\phi,z)=\sum_{k,m}\left[E\cdot J_m(k\cdot 0)+F\cdot Y(k\cdot 0)\right]\left[Ae^{im\phi}\right]\left[B\sinh(kz)\right]$

$=$ finite, so $F=0$

$k \quad \Phi(\rho,\phi,z=L)=\sum_{k,m}\left[E\cdot J(k\cdot x)\right]e^{im\phi}\cdot\sinh(k\cdot L)$

$=0$, so $k=\frac{n\pi}{L}$

$E \quad \Phi(\rho=\rho',\phi=0,z')=\sum_{k,m} E\cdot J_\nu(k\rho')\,e^{im\phi}\cdot\sinh(kz')$

$=V$

$E=\frac{V}{L\cdot\pi}J_\nu(k\cdot\rho')\sinh(kz')$ and $V=1\text{Volt}$

$$\Phi(\rho,\phi,z)=\frac{1}{L\pi}\sum_{k=0}^{\infty}\sum_{m=-\infty}^{\infty}e^{im\phi}\cdot J_\nu(k\rho')\cdot J_\nu(k\rho)\cdot\sinh(kz)\cdot\sinh(kz)$$

$$=\frac{1}{L\pi}\sum_{n=0}^{-\infty}\sum_{m=-\infty}^{\infty}e^{im\phi}\cdot\sinh\left(\frac{n\pi z}{L}\right)\cdot\sinh\left(\frac{n\pi z'}{L}\right)I_m\left(\frac{n\pi}{L}\rho\right)K_m\left(\frac{n\pi}{L}\rho'\right)$$

∴ with the identities

$\sinh(x)=i\cdot\sin(x)$

$I_\nu(x)=i\cdot J_\nu(x)$

⑥ Greens Function:

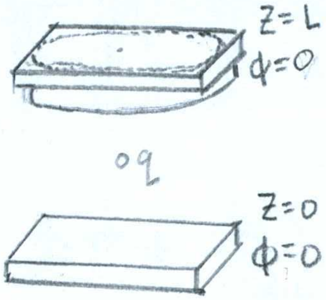
$$G(x,x') = 4\pi \cdot \Phi(\rho,\phi,z)$$

$$= \frac{4}{L}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} e^{im\phi} \cdot \sin\left(\frac{n\pi z}{L}\right)\cdot \sin\left(\frac{n\pi z'}{L}\right)\cdot I_m\left(\frac{n\pi}{L}\cdot\rho\right) K_m\left(\frac{n\pi}{L}\cdot\rho'\right)$$

b) Equations 3.165 to 3.168 example another derivation.

$$= 2\cdot\sum_{m=-\infty}^{\infty}\int_0^{\infty} e^{im\phi}\cdot J_m(k\rho)\cdot J_m(k\rho)\cdot \frac{\sinh(kz)\cdot\sinh(k(L-z))}{\sinh(kL)}dk$$

3.18



Two parallel planes, one with a disc insert

Shapes: Disc, plane
Dimension: Volume [3D]
Charge: Q

a) Potential Derivation:

① Boundary conditions:

$\Phi(\rho,\phi,z=0)=0$ $\quad\Phi(\rho=\rho',\phi=0,z')=V$

$\Phi(\rho,\phi,z=L)=0$

$\Phi(\rho=0,\phi,z)=$ Finite

② Laplace's Equation:

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$=0$$

③ Laplace's Equation Solutions:

If $\Phi(\rho,\phi,z)=R(\rho)\,Q(\phi)\,Z(z)$, then

Ⓐ Variable Separation:

$$\nabla\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$= \frac{Q(\phi)Z(z)}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial R(\rho)}{\partial r}\right) + \frac{R(\rho)Z(z)}{r^2}\frac{\partial^2 Q(\phi)}{\partial^2\phi} + R(\rho)Q(\phi)\frac{\partial^2 Z(z)}{\partial^2 z}$$

$$= \frac{1}{R(\rho)\cdot r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial R}{\partial r}\right) + \frac{1}{Q(\phi)\cdot r^2}\frac{\partial^2 Q(\phi)}{\partial^2\phi} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2}$$

$$= 0$$

Ⓑ Radial Eigenvalues:

$$\frac{r}{R}\frac{\partial R}{\partial r} + \frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z} = \lambda r^2$$

$$\frac{r}{R}\frac{\partial R}{\partial r} + \frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} - \lambda r^2 = \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2}$$

If $r = x$ and $\lambda = 1/r^2$,

$$\frac{\partial^2 R}{\partial^2 x} + \frac{1}{x}\frac{\partial R}{\partial x} - \left(1 + \frac{k}{x^2}\right)R = 0$$

Ⓒ Angular Eigenvalues:

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial\phi^2} = -m \quad ; \quad \frac{\partial^2 Q}{\partial\phi^2} + mQ = 0$$

Ⓓ Vertical Eigenvalues:

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial^2 z} = k \quad : \quad \frac{\partial^2 Z}{\partial^2 z} - kZ = 0$$

Ⓔ General Solution:

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} R(\rho)\,Q(\phi)\,Z(z)$$

④ General Solution to Laplace's Equation:

$$R(\rho) = E\cdot J_v(kx) + F\cdot Y_v(kx)$$

where $$J_v = \sum_{m=0}^{\infty}\frac{(-1)^m\left(-\frac{1}{2}kx\right)^{v+2m}}{m!\,(m+v)!}$$

$$Y_v = \frac{\cos(v\pi)\,J_v(kx) - J_{-v}(kx)}{\sin(v\pi)}$$

$$Q(\phi) = Ae^{im\phi}$$

$$Z(z) = B\sinh(kz) + C\cosh(kz)$$

⑤ <u>Variables by Boundary conditions:</u>

C $$\Phi(\rho, \phi, z=0) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}[E\circ J_m(kx) + F\circ Y_m(kx)][Ae^{im\phi}][B\sinh(k0) + C\cosh(k0)]$$

$$= 0 \text{, so } C = 0$$

F $$\Phi(\rho=0, \phi, z) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}[E\circ J_m(k\cdot 0) + FY_m(k\cdot 0)][Ae^{im\phi}][B\sinh(kz)]$$

$$= \text{finite} \text{, so } F = 0$$

k $$\Phi(\rho, \phi, z=L) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} E\circ J(kx)\circ e^{im\phi}\circ\sinh(k\cdot L)$$

$$= 0 \text{, so } k = \frac{n\pi}{L}$$

E $$\Phi(\rho=\rho, \phi=0, z=z') = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} E\circ J_v(k\rho)\circ e^{im\cdot 0}\circ\sinh(k\cdot z')$$

$$= V$$

$$V\int_0^{\infty}\rho J_v(k\rho)\,d\rho = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} E\circ\int_0^{\infty} J_v^2(k\rho)\sinh(kL)\,d\rho$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} E\circ\delta(\rho-\rho')\circ\sinh(k\cdot L)$$

$$E = \frac{V\circ J_1(k\rho)\circ}{\sinh(k\circ L)}$$

$$\Phi(\rho,\phi,z) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} V_0\, J_1(k\rho)\, J_0(k\rho)\, \frac{\sinh(kz)}{\sinh(kL)}$$

$$\Phi(\rho,z) = V\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \int_0^{\infty} J_1(\lambda)\cdot J_0(\lambda\rho/a)\cdot \frac{\sinh(\lambda z/a)}{\sinh(\lambda L/a)}\, d\lambda$$

b) If $J_0(x) = 1 - \frac{1}{4}x^2 + \cdots$

and $\frac{\sinh(x)}{\sinh(y)} = \frac{x + \frac{1}{6}x^3 + \cdots}{y + \frac{1}{6}y^3 + \cdots} = \frac{x}{y}\left[1 + \frac{1}{6}(x^2 - y^2)\right] + O(x^4)$

$$J_0(\lambda\rho/a)\cdot\frac{\sinh(\lambda z/a)}{\sinh(\lambda L/a)} \cong \left(1 - \frac{1}{4}\left(\frac{\lambda\rho}{a}\right)^2\right)\left(\frac{z}{L}\right)\left(1 + \frac{1}{6}(z^2 - L^2)\right)$$

$$\cong \frac{z}{L}\left[1 - \left(\frac{\lambda}{a}\right)^2\left(\frac{1}{6}(L^2 - z^2) + \frac{1}{4}\rho^2\right) + \cdots\right]$$

$$\Phi(\rho,z) = \frac{Vz}{L}\left[\int_0^{\infty} J_1(\lambda)\, d\lambda - \frac{1}{a^2}\left[\frac{1}{6}(L^2 - z^2) + \frac{1}{4}\rho^2\right]\int_0^{\infty}\lambda^2 J_1(\lambda)\, d\lambda + \cdots\right.$$

$$\lim_{a\to\infty}\Phi(\rho,z) = \frac{V_0 z}{L}$$

"Linear and general equation for two plates"

c) $\lim_{L\to\infty}$ $\Phi(\rho,z) = V\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{\infty} J_1(\lambda)\circ J_0(\lambda\rho/a)\circ\frac{\sinh(k(L-z'))}{\sinh(k\cdot L)}\, d\lambda$

$$\lim_{L\to\infty}\frac{\sinh(k(L-z'))}{\sinh(k\cdot L)} = \frac{\sinh(k\circ L)\cosh(-kz') + \cosh(k\cdot L)\circ\sinh(-kz)}{\sinh(k\circ L)}$$

$= \cosh(k\cdot z) - \coth(kL)\circ\sinh(kz)$ ... because L is huge!

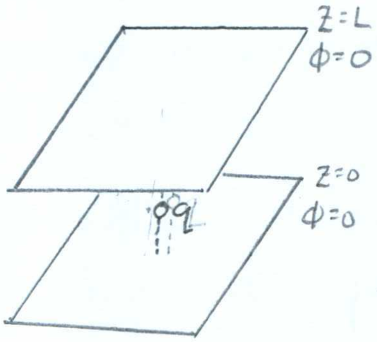
$= \cosh(kz) - \sinh(kz)$

$= e^{-kz}$

$$\Phi(\rho,z) = V\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{\infty} J_1(\lambda)\cdot J_0(\lambda\rho/a)\circ e^{-kz}\, d\lambda$$

3.19.

a) $\Phi(z_a, \rho) = V \int_0^\infty d\lambda\, J_1(\lambda) J_0(\lambda \rho/a) \frac{\sinh(\lambda z/a)}{\sinh(\lambda L/a)}$



Two infinite parallel conducting plates

Shape: Plate
Dimension: Area[2D]
Charge: q

$$Q_t(a) = -\frac{q}{V} \cdot \Phi(z_a, \rho)$$

$$= -q \int_0^\infty d\lambda\, J_1(\lambda) \cdot J_0(\lambda \cdot 0/a) \cdot \frac{\sinh(\lambda z_a/a)}{\sinh(\lambda \cdot L/a)}$$

$$= -q \int_0^\infty d\lambda\, J_1(\lambda) \left[\left(\frac{z}{L}\right)\left(1 + \dots\right) \right]$$

$$= -q \frac{z}{L}$$

b) $\sigma(\rho) = \frac{Q}{2\pi} \cdot \frac{-q}{V}$

$$= \frac{-q}{2\pi V} \Phi(z_0, \rho)$$

$$= \frac{-q}{2\pi} \int_0^\infty dk \frac{\sinh(k \cdot z_0)}{\sinh(k \cdot L)} k J_0(k\rho)$$

c) $\sigma(0) = \frac{-q}{2\pi} \int_0^\infty dk \frac{\sinh(k \cdot z_0)}{\sinh(k \cdot L)} k$

$$= \frac{-q}{2\pi} \int_0^\infty \frac{e^{kz_0} - e^{-kz_0}}{e^{kL} - e^{-kL}} k\, dk \qquad k = \frac{n\pi}{L}$$

$$= \frac{-q}{2\pi L^2}$$

3.20

a) From Problem 3.17,

$$G(x, x') = \frac{4}{L} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} e^{im(\phi-\phi')} \cdot \sin\left(\frac{n\pi z}{L}\right) \cdot \sin\left(\frac{n\pi z'}{L}\right) \cdot I_m\left(\frac{n\pi}{L}\rho_<\right) \cdot K_m\left(\frac{n\pi}{L}\rho_>\right)$$

$$\Phi(z, \rho) = \Phi(0) \cdot \frac{dG}{d\rho_<}\Big|_{\rho=0}$$

$$= V_0 \sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{n\pi Z}{L}\right) \cdot K_0\left(\frac{n\pi \rho_>}{L}\right)$$

$$= \frac{-q}{\pi \varepsilon_0 L} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{L}\right) \cdot \sin\left(\frac{n\pi z}{L}\right) \cdot K_0\left(\frac{n\pi \rho_>}{L}\right)$$

b) $\sigma_L(\rho) = -\varepsilon_0 \nabla \Phi \big|_{z=L}$

$$= -\varepsilon_0 \frac{\partial}{\partial z_0}\left[\frac{q}{\pi \varepsilon_0 L} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi z_0}{L}\right) \sin\left(\frac{n\pi z}{L}\right) K_0\left(\frac{n\pi \rho}{L}\right)\right]$$

$$= \frac{q}{L^2} \sum_{n=1}^{\infty} (-1)^n \cdot n \cdot \sin\left(\frac{n\pi z_0}{L}\right) \cdot K_0\left(\frac{n\pi \rho}{L}\right)$$

$$= \sigma_0(\rho)$$

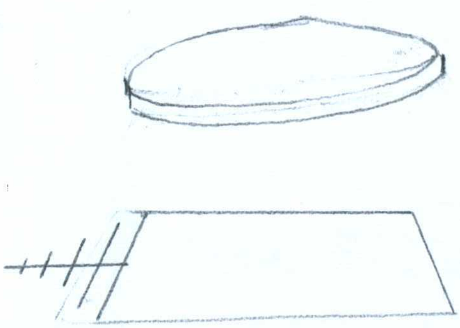
The solution in 3.19b encompasses $n=0$ with complex ~~im~~math.

c) $Q_T = \frac{-q}{V} \Phi(z_a, 0)$

$$= -q \sum_{n=1}^{\infty} \sin\left(\frac{n\pi z_0}{L}\right) \cdot \sin\left(\frac{n\pi z_a}{L}\right) K_0\left(\frac{n\pi 0}{L}\right)$$

$$= -q \sum_{n=1}^{\infty} \sin\left(\frac{n\pi z_0}{L}\right) \cdot \sin\left(\frac{n\pi z_a}{L}\right)$$

3.21



A flat, thin, circular disc parallel to a grounded plate

Shape: Disc, plate

Dimension: Volume [3D]

Charge: q

a) $G(x,x') = 2\sum_{m=0}^{\infty} \int_0^{\infty} dk\, e^{im(\phi-\phi')} \cdot J_m(k\rho) \cdot J_m(k\rho) \cdot \frac{\sinh(kz) \cdot \sinh(k(L-z))}{\sinh(k \cdot L)}$

$\Phi(z,\rho) = \Phi(0) \cdot \frac{dG}{dn}$

$= 2 \cdot V \sum_{m=0}^{\infty} \int_0^{\infty} dk\, \rho\, J_m(k\rho) \cdot J_m(k\rho) \cdot \frac{\sinh(kz) \cdot \sinh(k(L-z))}{\sinh(k-L)}$

$Q_T = -\frac{q}{V} \cdot \Phi(z_a, \rho)$

$= -q \cdot 2 \sum_{m=0}^{\infty} \int_0^{\infty} dk\, \rho\, J_m(k\rho) \cdot J_m(k\rho) \cdot \frac{\sinh(kz) \cdot \sinh(k(L-z))}{\sinh(k-L)}$

$C = \frac{Q_T}{V} = \frac{-q}{V} \cdot 2 \sum_{m=0}^{\infty} \int_0^{\infty} dk\, \rho\, J_m(k\rho)\, J_m(k\rho) \cdot \frac{\sinh(kz) \cdot \sinh(k(L-z))}{\sinh(k-L)}$

$= \dfrac{4 \int_0^{\infty} \rho\, J_0(k\rho)\, \frac{q}{L^2} \sum_{m=0}^{\infty} \sin(kz) \cdot k_0(k\rho)\, dk}{\int_0^R \rho\, \sigma(\rho)\, d\rho \,/\, 4\pi\varepsilon_0 \quad \cdot \sinh(k-L)}$

$\lim_{L\to\infty} C = \frac{16\pi\varepsilon_0}{L^2} \int_0^{\infty} dk \frac{\int_0^R \rho\, J_0(k\rho)\, \sigma(\rho)\, d\rho}{\int_0^R \rho\, \sigma(\rho)\, d\rho} \frac{\sinh(kL)\cosh(-kz) + \cosh(k\cdot L)\sinh(-kz)}{\sinh(k\cdot L)}$

$= 16\pi\varepsilon_0 \int_0^{\infty} dk \frac{\int_0^R \rho\, J_0(k\rho)\, \sigma(\rho)\, d\rho}{\int_0^R \rho\, \sigma(\rho)\, d\rho} \cdot e^{-kz}$

$\frac{16\pi\varepsilon_0}{C} = \int_0^{\infty} dk \cdot e^{kz} \frac{\int_0^R \rho\, \sigma(\rho)\, d\rho}{\int_0^R \rho\, J_0(k\rho)\, \sigma(\rho)\, d\rho}$ "An answer"

$\frac{4\pi\varepsilon_0}{C} = \int_0^{\infty} dk \cdot (1 - e^{-2kd}) \frac{\left[\int_0^R \rho\, J_0(k\rho)\, \sigma(\rho)\, d\rho\right]^2}{\left[\int_0^R \rho \cdot \sigma(\rho)\, d\rho\right]^2}$ "Book's goal"

b) $$\frac{4\pi\varepsilon_0}{C} = \int_0^\infty dk\left(1 - e^{-2kd}\right) \cdot \frac{\left[\int_0^R \rho J_0(k\rho)\,\sigma(\rho)\,d\rho\right]^2}{\left[\int_0^R \rho\,\sigma(\rho)\,d\rho\right]^2}$$

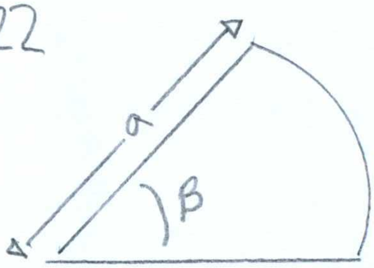
$$= \int_0^\infty dk\left(\frac{\sin(k\rho)}{k\rho}\right)^2$$

$$= \frac{1}{R}\int_0^\infty \frac{\sin^2(t)}{t^2}\,dt \quad \ldots \text{when } k\rho = t$$

$$= \frac{\pi}{2R}$$

c)

3.22



$\boxed{\text{Greens Function from Potential}}$

Ⓐ Boundary Conditions:

$\Phi(\rho, \phi = 0) = 0$ $\quad$ $\Phi(\rho = a, \phi) = 0$

$\Phi(\rho, \phi = \beta) = 0$ $\quad$ $\Phi(\rho = \rho', \phi) = \Phi$

Ⓑ Laplaces Equation:

$$\frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

Ⓒ Laplaces Equation Solutions:

Ⓐ Variable Separation:

If $\Phi(\rho, \phi) = R(\rho)\,Q(\phi)$, then

$$\frac{Q(\phi)}{} \frac{\partial^2 R(\rho)}{\partial^2 \rho} + \frac{R(\rho)}{\rho^2} \frac{\partial^2 Q(\phi)}{\partial^2 \phi} = 0$$

$$R(\rho)\rho^2 \frac{\partial^2 R(\rho)}{\partial^2 \rho} + \frac{1}{Q(\phi)} \frac{\partial^2 Q(\phi)}{\partial^2 \phi} = 0$$

Ⓑ Radial Eigenvalues:

$$\frac{\partial^2 R(\rho)}{\partial^2 \rho} - \frac{k^2 R(\rho)}{\rho^2} = 0$$

Ⓒ Angular Eigenvalues:

$$\frac{\partial^2 Q(\phi)}{\partial^2 \phi} + k^2 Q(\phi) = 0$$

Ⓓ General Solution:

$$\Phi(\rho, \phi) = \sum_{k=0}^{\infty} R(\rho) Q(\phi)$$

④ General Solution to Laplaces Equation:

$$R(\rho) = A\rho^k + B\rho^{-k}$$

$$Q(\phi) = C\sin(k\phi) + D\cos(k\phi)$$

⑤ Variables by Boundary Conditions:

D $\quad \Phi(\rho, \phi=0) = \sum_{k=0}^{\infty} (A\rho^k + B\rho^{-k})(C\sin(k\cdot 0) + D\cos(k\cdot 0))$

$= 0$, so $D = 0$

A, B $\quad \Phi(\rho=a, \phi) = \sum_{k=0}^{\infty} (Aa^k + Ba^{-k})\, C\cdot\sin(k\cdot\phi)$

$= 0$, so $\quad A = C\cdot a^{-l} \qquad B = -C\cdot a^{l}$

$$\Phi(\rho, \phi) = \sum_{k=0}^{\infty} C\left[\left(\frac{\rho}{a}\right)^k + \left(\frac{a}{\rho}\right)^k\right] \sin(k\phi)$$

$$k \quad \Phi(\rho, \phi=\beta) = \sum_{k=0}^{\infty} C\left[\left(\frac{\rho}{a}\right)^{k} - \left(\frac{a}{\rho}\right)^{k}\right] \sin(k \cdot \beta)$$

$$= 0 \quad , \text{so} \quad k = \frac{m\pi}{\beta}$$

$$C \quad \Phi(\rho=\rho', \phi) = \sum_{k=0}^{\infty} C \cdot \left[\left(\frac{\rho}{a}\right)^{\frac{m\pi}{\beta}} - \left(\frac{a}{\rho}\right)^{m\pi/\beta}\right] \sin\left(\frac{m\pi\phi}{\beta}\right)$$

$$= \sum_{k=0}^{\infty} C \cdot \left[\left(\frac{\rho'}{a}\right)^{m\pi/\beta} - \left(\frac{a}{\rho'}\right)^{m\pi/\beta}\right] \sin\left(\frac{m\pi\phi}{\beta}\right)$$

$$C = \rho'^{\,m\pi/\beta} \text{ or } \rho^{m\pi/\beta}$$

$$\Phi(\rho,\phi) = \sum_{k=0}^{\infty} \rho'^{\,m\pi/\beta}\left[\left(\frac{\rho}{a}\right)^{m\pi/\beta} - \left(\frac{a}{\rho}\right)^{m\pi/\beta}\right] \sin\left(\frac{m\pi\phi}{\beta}\right)$$

⑥ Greens Theorem:

$$\nabla^2 G = \left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right] G$$

$$= \left[\frac{\partial^2}{\partial\rho^2}\left(\sum_{k=0}^{\infty} \rho'^{\,m\pi/\beta}\left[\left(\frac{\rho}{a}\right)^{m\pi/\beta} - \left(\frac{a}{\rho}\right)^{m\pi/\beta}\right]\right) + \left(\frac{m\pi}{\beta\rho}\right)^2\left(\sum_{k=0}^{\infty} \rho^{m\pi/\beta}\left[\left(\frac{\rho}{a}\right)^{m\pi/\beta} - \left(\frac{a}{\rho}\right)^{m\pi/\beta}\right]\right)\right]$$

$$\times \sin\left(\frac{m\pi\phi}{\beta}\right)$$

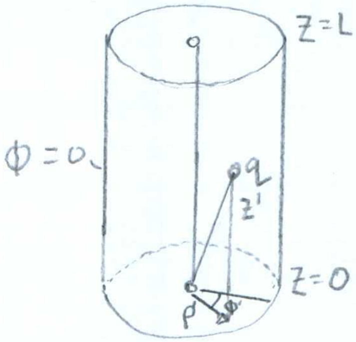
$$= \lambda\left[\sum_{k=0}^{\infty} \rho'^{\,m\pi/\beta}\left[\left(\frac{\rho}{a}\right)^{m\pi/\beta} - \left(\frac{a}{\rho}\right)^{m\pi/\beta}\right]\right]$$

$$\lambda = \frac{1}{2m\pi} a^{-m\pi/\beta} \cdot \sin\left(\frac{m\pi}{\beta}\phi'\right)$$

$$G(\rho,\phi,\rho',\phi') = \lambda \cdot \Phi(\rho,\phi)$$

$$= \sum \frac{1}{2\pi m} \rho'^{\,m\pi/\beta}\left[\left(\frac{\rho^{m\pi/\beta}}{a^{2m\pi/\beta}}\right) - \left(\frac{1}{\rho^{m\pi/\beta}}\right)\right] \sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right)$$

3.23.



Grounded Cylindrical box

Shape: Cylinder
Dimension: Volume [3D]
Charge: q

$\boxed{\text{Potential Derivation}}$

① **Boundary Conditions:**

$\Phi(\rho = \frac{L}{2}, \phi, z=0) = 0 \qquad \Phi(\rho=\rho', \phi=\phi', z=z') = V$

$\Phi(\rho, \phi, z=L) = 0$

$\Phi(\rho=a, \phi, z=0) = 0$

$\Phi(\rho=0, \phi, z) = \text{finite}$

② **Laplaces Equation:**

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial \phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$
$$= 0$$

③ **Laplace's Equation solutions:**

If $\Phi(\rho, \phi, z) = R(\rho) \cdot Q(\phi) \cdot Z(z)$, then,

Ⓐ **Variable Separation:**

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial \phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$= \frac{Q(\phi)Z(z)}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial R(\rho)}{\partial r}\right) + \frac{R(\rho)Z(z)}{r^2}\frac{\partial^2 Q(\phi)}{\partial^2 \phi} + R(\rho)Q(\phi)\frac{\partial^2 Z}{\partial z^2}$$

$$= \frac{1}{R(\rho)\cdot\rho}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial R(\rho)}{\partial r}\right) + \frac{1}{Q(\phi)r^2}\frac{\partial^2 Q(\phi)}{\partial^2 \phi} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2}$$

$$= 0$$

Ⓑ **Radial Eigenvalues:**

$$\frac{r}{R}\frac{\partial R}{\partial r} + \frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial^2 z} = \lambda r^2$$

$$\frac{r}{R}\frac{\partial R}{\partial r} + \frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} - \lambda r^2 = \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}$$

If $r = x$ and $\lambda = 1/r^2$

$$\frac{\partial^2 R}{\partial^2 x} + \frac{1}{x}\frac{\partial R}{\partial x} - \left(1+\frac{k}{x^2}\right)R = 0$$

Ⓒ Angular Eigenvalues:

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial \phi^2} = -m^2 \; ; \; \frac{\partial^2 Q}{\partial \phi^2} + m^2 Q = 0$$

Ⓓ Vertical Eigenvalues:

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial^2 z} = k^2 \; ; \; \frac{\partial^2 Z}{\partial^2 z} - k^2 Z = 0$$

Ⓔ General Solution:

$$\Phi(\rho, \phi, z) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} R(\rho)\, Q(\phi)\, Z(z)$$

④ General Solution to Laplace's Equation:

$$R(\rho) = E \cdot J_v(kx) + F\, Y_v(kx)$$

Where $J_v = \sum_{m=0}^{\infty} \frac{(-1)^m \left(-\frac{1}{2}kx\right)^{v+2m}}{m!\,(m+v)!}$

$$Y_v = \frac{\cos(v\pi)\, J_v(kx) - J_{-v}(kx)}{\sin(v\pi)}$$

$$Q(\phi) = Ae^{im\phi}$$

$$Z(z) = C\sinh(kz) + D\cosh(kz)$$

⑤ Variables by Boundary Conditions:

F $\Phi(\rho=0, \phi, z) = \sum_{m=-\infty}^{\infty} [E \cdot J_v(k \cdot 0) + F\, Y_v(k \cdot 0)][A \circ e^{im\phi}][C\sinh(kz) + D\cosh(kz)]$

$= \text{finite, so } F = 0$

D $\quad \Phi(\rho = \frac{L}{2}, \phi, z=0) = \sum_{m=0}^{\infty} \left[E J_v(k \tfrac{L}{2})\right]\left[A \circ e^{im\phi}\right]\left[C\sinh(kz) + D\cosh(kz)\right]$

$= 0$, so $D = 0$

K $\quad \Phi(\rho = a, \phi, z=0) = \sum_{m=-\infty}^{\infty} E \circ J_v(k \cdot a)\, e^{im\phi} \circ \sinh(k \cdot 0)$

$= 0$, so $k = \frac{x}{a}$

E $\quad \Phi(\rho = \rho', \phi = \phi', z = z') = \sum_{m=-\infty}^{\infty} E \circ J_v\left(\frac{x \cdot \rho'}{a}\right) e^{im\phi'} \cdot \sinh\left(\frac{x \cdot z'}{a}\right)$

$= V$

$$V \circ \int_0^a \rho' J_m\left(\frac{x\rho'}{a}\right) \sinh\left(\frac{xz'}{a}\right) dz\, d\rho = E \sum_{m=-\infty}^{\infty} \int_0^a J_m^2\left(\frac{x\rho'}{a}\right) d\rho \int_0^{\pi/2} \sinh^2\left(\frac{xz'}{a}\right) dz \circ e^{im\phi'}$$

$$= E \sum_{m=-\infty}^{\infty} \left(\frac{a^2}{2}\right) \circ J_{m+1}^2\left(\frac{x\rho'}{a}\right) \circ \left(\frac{\pi}{2}\right) \circ e^{im\phi'}$$

$$E = \frac{\frac{4 \circ V}{a^2\pi^2} \int_0^a \rho J_m\left(\frac{x\rho'}{a}\right) \int_0^{\pi/2} \sinh\left(\frac{xz'}{a}\right) dz\, d\rho\, e^{-im\phi'}}{J_{m+1}^2\left(\frac{x\rho'}{a}\right)}$$

$$\Phi(\rho, \phi, z) = \frac{4V}{a^2\pi^2} \sum_{m=-\infty}^{\infty} \frac{\int_0^a \rho' J_m\left(\frac{x\rho'}{a}\right) \cdot J_m\left(\frac{x\rho}{a}\right) d\rho \int_0^{\pi/2} \sinh\left(\frac{xz}{a}\right) \sinh\left(\frac{xz'}{a}\right) dz \circ e^{im(\phi - \phi')}}{J_{m+1}^2\left(\frac{x\rho}{a}\right)}$$

Two identities:

- Fundamental Theorem of Calculus

  $$\int_c^d a(x)\,dx = \frac{d}{dx} a(x)\Big|_{x=d} - \frac{d}{dx} a(x)\Big|_{x=c}$$

- $$\int_0^x J_n(t) J_{n+1}(t)\, dt = \sum_{k=n+1}^{\infty} J_k^2(x)$$

$$\int_0^{\pi/2} \sinh^2\left(\frac{\chi z}{a}\right)dz = C\left[\frac{d}{dz}\sinh^2\left(\frac{\chi z}{a}\right) - \frac{d}{dz}\sinh^2\left(\frac{\chi z}{a}\right)\right] \quad \text{(Equation 3.144)}$$

$$= C\left[\sinh\left(\frac{\chi z}{a}\right)\cosh\left(\frac{\chi}{a}(L-z)\right) + \cosh\left(\frac{\chi z}{a}\right)\cdot\sinh\left(\frac{\chi}{a}(L-z)\right)\right]$$

$$= C\left(\frac{\chi}{a}\right)\sinh\left(\frac{\chi}{a}\cdot L\right)$$

$$= \pi$$

$$C = \frac{a\pi}{\chi}\cdot\frac{1}{\sinh\left(\frac{\chi L}{a}\right)}$$

Equation #1

$$\Phi(\rho,\phi,z) = \frac{4V}{a^2\pi^2}\left(\frac{a\pi}{\chi}\right)\sum_{k=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{im(\phi-\phi')}\cdot J_v\left(\frac{\chi\rho'}{a}\right)\cdot J_v\left(\frac{\chi\rho}{a}\right)\sinh\left(\frac{\chi z'}{a}\right)\sinh\left(\frac{\chi}{a}(L-z)\right)}{J_{m+1}^2\left(\frac{\chi\rho'}{a}\right)\ \sinh\left(\frac{\chi L}{a}\right)}$$

$$= \frac{q}{\pi\varepsilon_0 a}\sum_{k=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{im(\phi-\phi')}\cdot J_v\left(\frac{\chi\rho'}{a}\right)\cdot J_v\left(\frac{\chi\rho}{a}\right)\cdot\sinh\left(\frac{\chi z'}{a}\right)\cdot\sinh\frac{\chi}{a}(L-z)}{\chi\cdot J_{m+1}^2\left(\frac{\chi\rho}{a}\right)\cdot\sinh\left(\frac{\chi L}{a}\right)}$$

Equation #2

$$\Phi(\rho,\phi,z) = \frac{q}{\pi\varepsilon_0 a}\sum_{k=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{e^{im(\phi-\phi')}\cdot J_v\left(\frac{\chi\rho}{a}\right)\cdot J_v\left(\frac{\chi\rho}{a}\right)\cdot\sinh\left(\frac{\chi z}{a}\right)\cdot\sinh\left(\frac{\chi}{a}(L-z)\right)}{\chi\cdot J_{m+1}^2\left(\frac{\chi\rho}{a}\right)\cdot\sinh\left(\frac{\chi L}{a}\right)}$$

$$= \frac{q}{\pi\varepsilon_0 a}\sum_{k=1}^{\infty}\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')}\sin\left(\frac{n\pi z}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\cdot\frac{I_m\left(\frac{n\pi\rho}{a}\right)}{J_m\left(\frac{n\pi\rho}{a}\right)}\ \frac{I_m\left(\frac{n\pi\rho}{a}\right)}{\rho\cdot J_m\left(\frac{n\pi\rho}{a}\right)}$$

Hankel Transform:

$$M(\chi) = 2\pi\left[\frac{1}{\chi^3}\int_0^{\pi} J_0(\chi)\,d\chi - \frac{1}{\chi^2}J_0(\chi)\right]$$

$$= \frac{\pi^2}{\chi^2}\left[J_1 H_0(\chi) - J_0 H_1(\chi)\right]$$

$$\sinh(x) = i\cdot\sin(x)$$

$$\frac{1}{x}\int_0^{\pi} dx - 1 = \frac{\pi}{2}\left[J_1(x)H_0(x) - J_0(x)H_1(x)\right]$$

$$\cong \frac{q}{\pi \varepsilon_0 a} \sum_{k=1}^{\infty} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \sin\left(\frac{n\pi z}{L}\right) \cdot \sin\left(\frac{n\pi z}{L}\right) \frac{J_m\left(\frac{n\pi\rho}{a}\right)}{J_m\left(\frac{n\pi\rho}{a}\right)} \frac{i}{\rho} \int_0^{\pi} 1\, dx$$

$$\cong \frac{q}{\pi \varepsilon_0 L} \sum_{k=0}^{\infty} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} \sin\left(\frac{n\pi z}{L}\right) \cdot \sin\left(\frac{n\pi z}{L}\right) \frac{J_m\left(\frac{n\pi\rho}{a}\right)}{J_m\left(\frac{n\pi\rho}{a}\right)} \cdot$$

$$\left[J_m\left(\frac{n\pi a}{L}\right) \cdot K_m\left(\frac{n\pi\rho}{L}\right) - K_m\left(\frac{n\pi a}{L}\right) \cdot I_m\left(\frac{n\pi\rho}{L}\right) \right]$$

$\boxed{\text{Equation 3}}$

(Equation 3.169)

$$\frac{\sinh(kz) \cdot \sinh(k(c-z))}{k \cdot \sinh(kc)} = \frac{2}{c} \sum_{k=1}^{\infty} \frac{\sin\left(\frac{n\pi z}{c}\right)}{k^2 + \left(\frac{n\pi}{c}\right)^2} \sin\left(\frac{n\pi z}{c}\right)$$

The equation above derives from Greens Function.

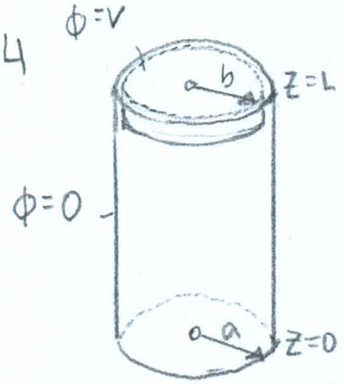
and $\nabla^2 G = k^2 \cdot G$; $G = \nabla^2 G / k^2$; $\Phi(r) = \int G \cdot \rho\, ds$
$= \int \frac{\nabla^2 G \rho}{k^2} ds$

From Equation 1,

$$\Phi(\rho, \phi, z) = \frac{q}{\pi \varepsilon_0 a} \sum_{m=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{e^{im(\phi-\phi')} \cdot J_v\left(\frac{x\rho}{a}\right) \cdot J_v\left(\frac{x\rho}{a}\right) \cdot \sinh\left(\frac{xz}{a}\right) \sinh\left(\frac{x}{a}(L-z)\right)}{x \cdot J_{m+1}^2\left(\frac{x\rho}{a}\right) \cdot \sinh\left(\frac{xL}{a}\right)}$$

$$= \frac{2q}{\pi \varepsilon_0 L a^2} \sum_{m=-\infty}^{\infty} \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{e^{im(\phi-\phi)} \sin\left(\frac{k\pi z}{L}\right) \sin\left(\frac{k\pi z}{L}\right) J_m\left(\frac{x\rho}{a}\right) \cdot J_m\left(\frac{x\rho}{a}\right)}{\left[\left(\frac{x}{a}\right)^2 + \left(\frac{k\pi}{L}\right)^2\right] J_{m+1}^2(x)}$$

3.24



Conducting cylinder except for upper disc.

Shape: Cylinder
Dimension: Volume [3D]
Charge = q

a/b)

$$\Phi(x,x') = \frac{q}{\pi \varepsilon_0 a} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{e^{im(\phi-\phi')} \cdot J_m\left(\frac{x}{a}\left(\frac{a}{2}\right)\right) \cdot J_m\left(\frac{x \cdot 0}{a}\right) \cdot \sinh\left(\frac{x}{a}\left(\frac{L}{2}\right)\right) \sinh\left(\frac{x}{a}\left(L - L/2\right)\right)}{x \cdot J_{m+1}^2(x) \cdot \sinh\left(\frac{x \cdot L}{a}\right)}$$

$$= \frac{q}{\pi \varepsilon_0 a} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{e^{im(\phi-\phi')} \cdot J_m\left(\frac{x}{2}\right)}{x \cdot J_{m+1}^2(x)} \cdot \frac{\tanh\left(\frac{xL}{2a}\right)}{2}$$

∴ when $\rho = 0$, $z = L/2$, and $b = a/2$.

∴ the book wants two significant figures of Bessel ratios. What's the x-distance from the disc?

$$\Phi(x,x') = \frac{q}{\pi \varepsilon_0 L} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} e^{im(\phi-\phi')} \cdot \sin\left(\frac{n\pi}{2}\right) \cdot \sin\left(\frac{n\pi}{2}\right) \frac{I_m\left(\frac{n\pi a}{2L}\right)}{I_m\left(\frac{n\pi a}{L}\right)} \times$$

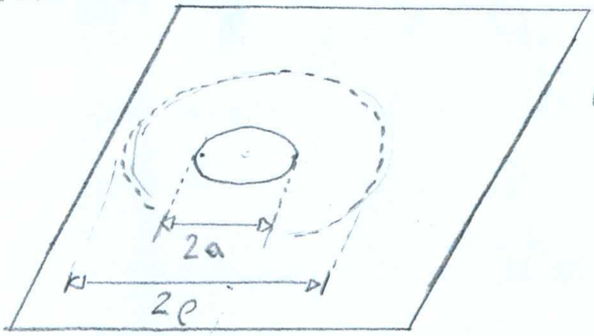
$$\cdot \left[I_m\left(\frac{n\pi a}{L}\right) K_m\left(\frac{n\pi a}{2L}\right) - K_m\left(\frac{n\pi a}{L}\right) I_m\left(\frac{n\pi a}{2L}\right)\right]$$

$$= \frac{q}{\pi \varepsilon_0 L} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} e^{im(\phi-\phi')} I_m(1/2) \cdot \left[I_m\left(\frac{n\pi a}{L}\right) K_m\left(\frac{n\pi a}{2L}\right) - K_m\left(\frac{n\pi a}{L}\right) I_m\left(\frac{n\pi a}{2L}\right)\right]$$

$$\Phi(x,x') = \frac{q}{\pi \varepsilon_0 a} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{e^{im(\phi-\phi')} \cdot \sin\left(\frac{k\pi L}{L \cdot 2}\right) \cdot \sin\left(\frac{k\pi L}{L \cdot 2}\right) J_m\left(\frac{x \cdot 0}{a}\right) \cdot J_m\left(\frac{x \cdot a}{2 \cdot a}\right)}{\left[\left(\frac{x}{a}\right)^2 + \left(\frac{k\pi}{L}\right)^2\right] J_{m+1}^2(x)}$$

$$= \frac{q}{\pi \varepsilon_0 a} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{e^{im(\phi-\phi')} \cdot J_m\left(\frac{x}{2}\right)}{\left[\left(\frac{x}{a}\right)^2 + \left(\frac{k\pi}{L}\right)^2\right] J_{m+1}^2(x)}$$

3.25.



Conducting plane with a circular hole

Shape: Plane
Dimension: Line
Charge: q

a) (Equation 3.170)

$$\Phi = \begin{cases} E_0 z + \phi^{(1)} & z > 0 \\ E_1 z + \phi^{(1)} & z < 0 \end{cases}$$

(Equation 3.184)

$$\Phi^{(1)}(\rho, z) = \frac{(E_0 - E_1)}{\pi} a^2 \int_0^\infty dk\, J_1(ka) e^{-kz} J_0(k\rho)$$

$$= \frac{(E_0 - E_1)}{\pi} a^2 \left[\frac{1}{a\sqrt{\rho^2 - a^2}} - \frac{\sin^{-1}\left(\frac{a}{\rho}\right)}{a} \right]$$

Gauss' Law

$$E = \frac{\sigma}{\varepsilon_0}$$

Potential

$$E = \nabla \Phi$$

$$\sigma = \varepsilon_0 \nabla \Phi$$

$$\sigma = \begin{cases} -\varepsilon_0 E z + \Delta\sigma(\rho) \\ \varepsilon_0 E_1 + \Delta\sigma(\rho) \end{cases}$$

where $\Delta\sigma = -\frac{\varepsilon_0 (E_0 - E_1)}{\pi} \left[\frac{a}{\sqrt{\rho^2 - a^2}} - \sin^{-1}\left(\frac{a}{\rho}\right) \right]$

b)

The function suggests a large ρ as zero surface charge, across infinite space, and much more surface charge when ρ<a e.g. a wire around the hole.

b. The integral was a challenge.

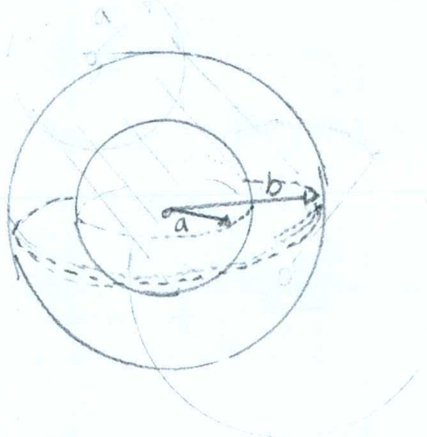
$$\lim_{R \to \infty} \left[2\pi \int_0^R d\rho\, \rho (\sigma_+ + \sigma_-) + 2\pi \varepsilon_0 \int_0^R d\rho\, \rho (E_0 - E_1) \right]$$

The final limit never converges. Although with pages 130, $E_0 = 0$ and $E_1 = 0$ in this "contrived" problem.

3.26

a) (Equation 1.46)

$$\Phi(x) = \langle \Phi \rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \rho(x') G_n(x,x') d^3x + \frac{1}{4\pi}\oint_S \frac{\partial \Phi}{\partial n} G_n \, da'$$



(Equation 1.45)

$$\frac{\partial G_n}{\partial n'}(x, x') = -\frac{4\pi}{S}$$ "Neumanns Boundary"

$$G(x,x') = \sum g(r,r') \cdot P_\ell(\cos\gamma)$$

where $g(r,r') = \frac{r^\ell}{r^{\ell+1}} + f_\ell(r,r')$

Volume between Concentric Spheres

Shape: Spheres
Dimension: Volume
Charge: q

Again, Dirichlets boundary condition suggests, "A balloon's surface changed is work and work is the surface change. Neumanns boundary condition states, "The volume (or pressure) change is work, or work is the volume change."

$$\frac{\partial G_n(x,x')}{\partial n'} = -\frac{4\pi}{S}$$

$$= -\frac{4\pi}{4\pi(a^2+b^2)} \circ \delta(x,x')$$

$$= \frac{-1}{a^2+b^2} \circ \delta(x,x')$$

$$= \sum \frac{dg(r,r')}{dn} P_\ell(\cos\gamma)$$

$$= \sum \frac{d}{dn}\left[\frac{r_<^\ell}{r_>^{\ell+1}} + A_\ell r^\ell + B_\ell \frac{1}{r^{\ell+1}}\right] P_\ell(\cos\gamma)$$

Inner sphere:

$r_< = a$; $$\frac{d}{dn_a}\left[\frac{a^\ell}{r_>^{\ell+1}} + A_\ell \circ a^{\ell} + B_\ell \frac{1}{r^{\ell+1}}\right]$$

$$= \frac{\ell \circ a^{\ell-1}}{r^{\ell+1}} + \ell \circ A_\ell \circ a^{\ell-1} - (\ell+1) B_\ell \frac{1}{a^{\ell+2}}$$

$$= \frac{1}{a^2+b^2}$$

Outer sphere:

$r_> = b$ $$\frac{d}{dn_b}\left[\frac{r^\ell}{b^{\ell+1}} + A_\ell \circ a + B_\ell \frac{1}{r^{\ell+1}}\right]$$

$$= -(\ell+1)\frac{r^\ell}{b^{\ell+2}} + \ell A_\ell b^{\ell-1} - (\ell+1) B_\ell \frac{1}{b^{\ell+2}}$$

$$= \frac{-1}{a^2+b^2}$$

$$\begin{pmatrix} l a^{2l+1} & -(l+1) \\ l b^{2l+1} & -(l+1) \end{pmatrix} \begin{pmatrix} A_l \\ B_l \end{pmatrix} = \begin{pmatrix} -l a^{2l+1}/r^{l+1} \\ (l+1) r^l \end{pmatrix}$$

$$\boxed{\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \frac{1}{AD-BC} \begin{bmatrix} D & -B \\ -C & A \end{bmatrix}}$$

$$\begin{pmatrix} A_l \\ B_l \end{pmatrix} = \frac{1}{l(l+1)(b^{2l+1} - a^{2l+1})} \begin{pmatrix} -(l+1) & l+1 \\ -l b^{2l-1} & l a^{2l+1} \end{pmatrix} \begin{pmatrix} -l a^{2l+1}/r^{l+1} \\ (l+1) r^l \end{pmatrix}$$

$$= \frac{r^l}{b^{2l+1} - a^{2l+1}} \begin{pmatrix} (a/r)^{2l+1} + (l+1)/l \\ a^{2l+1} + l/(l+1)(ab/r)^{2l+1} \end{pmatrix}$$

$$g(r, r') = \frac{r_<^l}{r_>^{l+1}} + \frac{r^l}{b^{2l+1} - a^{2l+1}} \left[\left(\left(\frac{a}{r}\right)^{2l+1} + \frac{l+1}{l} \right) r^l \right.$$

$$\left. + \left(a^{2l+1} + \frac{l}{l+1} \left(\frac{ab}{r}\right)^{2l+1} \right) \cdot \frac{1}{r^{l+1}} \right]$$

b) Earlier, $\frac{l a^{l-1}}{r^{l+1}} + l A_l a^{l-1} - (l+1) B_l \frac{1}{a^{l+2}} = \frac{1}{a^2+b^2}$

When $l=0$, $\frac{-B_0}{a^2} = \frac{1}{a^2+b^2}$

$$B_0 = \frac{-a^2}{a^2+b^2}$$

$$g(r, r') = \frac{r_<^l}{r_>^{l+1}} + A_l \cdot r^l + B_l \frac{1}{r^{l+1}}$$

$$= \frac{1}{r} + A_l - \frac{a^2}{a^2+b^2} \frac{1}{r^1} \quad \circ\circ\circ \text{ when } A_l = f(r)$$

$$= \frac{1}{r} - \frac{a^2}{a^2+b^2} \frac{1}{r} + f(r)$$

3.27 a) (Equation 1.46)

$$\Phi(x) = \frac{1}{4\pi}\oint_S \frac{d\Phi(x)}{dn} G(x,x')$$

$$= \frac{-1}{4\pi}\int_{r=b} E_r(\Omega')G(x,x')\, b^2 d\Omega$$

$$= \frac{E_0 b^2}{4\pi}\int G(x,x')\cos\Theta' d\Omega'$$

$$= \frac{E_0 b^2}{4\pi}\sum_{\ell=0}^{\infty}\int_{r=b} g_\ell(r,r') P_\ell(\cos\Theta)\cos\Theta' d\Omega'$$

$$P_\ell(\cos\theta) = P_\ell(\cos\theta)$$

$$= P_\ell(\cos\theta)\, e^{-im\phi} e^{im\phi}$$

$$= \frac{4\pi}{2\ell+1}\sum_{\ell,m}^{\infty} Y_\ell^m(\Omega) Y_\ell^{m*}(\Omega')$$

$$Y_\ell^m(\Omega) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta)$$

$$\cos(\theta) = \sqrt{\frac{4\pi}{3}} Y_1^0(\Omega)$$

$$= E_0 b^2\sqrt{\frac{4\pi}{3}}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\ell}\frac{g_\ell(r,b) Y_\ell^m(\Omega)}{2\ell+1}\int Y_\ell^m(\Omega) Y_1^0(\Omega) d\Omega'$$

$$= E_0 b^2\sqrt{\frac{4\pi}{3}}\frac{g_\ell(r,b) Y_\ell^m(\Omega)}{3}$$

@ $\ell = 1$

$$= \frac{E_0 b^2\cos\Theta}{3}\frac{2}{b^3 - a^3}\left(r + \frac{a^3}{2r^2}\right)\frac{3b}{2}$$

$$= \frac{E_0 r\cos\Theta}{1-(a/b)^3}\left(1 + \frac{a^3}{2r^3}\right)$$

$$E_r = -\frac{\partial \Phi}{\partial r} = \frac{-E_0 \cos\theta}{1-p^3}\left(1 - \frac{a^3}{r^3}\right) \quad ; \quad E_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = \frac{E_0 \sin\theta}{1-p^3}\left(1 + \frac{a^3}{2r^3}\right)$$

b) $$E_z = -\frac{d\Phi}{dz} = \frac{-d}{dz}\left[\frac{E_0}{1-p^3}\left(1 + \frac{a^3}{2(\rho^2+z^2)^{3/2}}\right)z\right]$$

$$= \frac{-E_0}{1-p^3}\left[1 + \frac{a^3(\hat{\rho}^2 - 2\hat{z}^2)}{r^3}\right]$$ ...where $\hat{\rho} = \rho/r$

$\hat{z} = z/r$

$p = a/b$

$$E_\rho = -\frac{d\Phi}{d\rho} = -\frac{d}{d\rho}\left[\frac{E_0}{1-p^3}\left(1 + \frac{a^3}{2(\rho^2+z^2)^{3/2}}\right)z\right]$$

$$= \frac{-E_0}{1-p^3}\left[\frac{3a^3\hat{\rho}\hat{z}}{2r^3}\right]$$

Plot of the "Force field" e.g. $\vec{F} = q\vec{E}$