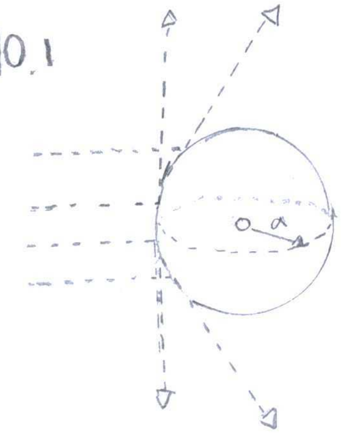
Chapter 10: Scattering and Diffraction:

10.1



"... Scattering cross section... a perfectly conducting sphere"

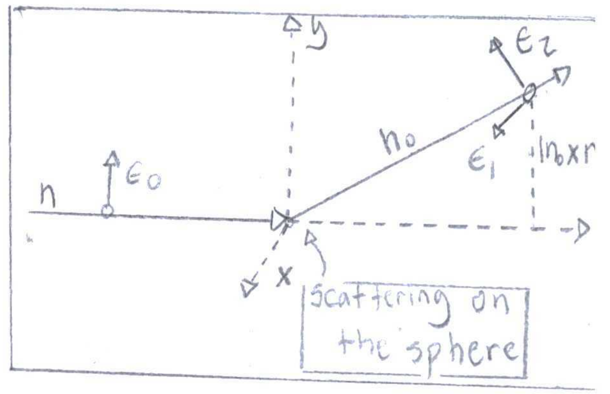
a) (10.4) "Differential Cross Section"

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\epsilon^* \cdot p + \frac{1}{c}(n \times \epsilon^*)\cdot m\right|^2$$

(10.12) "Dipole Moment in Electric and Magnetic fields"

$$p = 4\pi\epsilon_0 a^3 E_0 \epsilon_0 \qquad m = -2\pi a^3 \frac{1}{\mu_0 c} E_0 n_0 \times \epsilon_0$$

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left|\epsilon^* \cdot \epsilon_0 - \frac{1}{2}(n \times \epsilon^*)\cdot(n \times \epsilon_0)\right|^2$$



$$\epsilon_1 = \frac{n \times n_0}{|n \times n_0|} \qquad \epsilon_2 = n \times \epsilon_1 = \frac{n \times (n \times n_0)}{|n \times n_0|} \qquad |n \cdot n_0|^2 + |n \times n_0|^2 = 1$$

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left[\left|\epsilon_1 \cdot \epsilon_0 - \frac{1}{2}(n \times \epsilon_1)\cdot(n_0 \times \epsilon_0)\right|^2 + \left|\epsilon_2 \epsilon_0 - \frac{1}{2}(n \times \epsilon_2)\cdot(n_0 \times \epsilon_0)\right|^2\right]$$

$$= k^4 a^6 \left[\left|\frac{n \times n_0}{|n \times n_0|}\epsilon_0 - \frac{1}{2}\frac{n \times (n \times n_0)}{|n \times n_0|}\cdot(n_0 \times \epsilon_0)\right|^2 + \left|\frac{n \times (n \times n_0)}{|n \times n_0|}\epsilon_0 + \frac{1}{2}\frac{n \times n_0}{|n \times n_0|}(n_0 \times \epsilon_0)\right|^2\right]$$

$$= k^4 a^6 \frac{1}{|n \times n_0|^2}\left[\left|(n \times n_0)\cdot \epsilon_0 - \frac{1}{2}(n \times (n \times n_0))\cdot(n_0 \times \epsilon_0)\right|^2 + \left|(n \times (n \times \epsilon_0))\cdot \epsilon_0 + \frac{1}{2}(n \times n_0)\cdot(n_0 \times \epsilon_0)\right|^2\right]$$

$$= k^4 a^6 \frac{1}{|n \times n_0|^2}\left[|(n \times n_0)\cdot \epsilon_0|^2 + \frac{1}{4}|(n \times (n \times n_0))\cdot(n_0 \times \epsilon_0)|^2\right.$$

$$-\frac{1}{2}((n \times n_0)\cdot \epsilon_0^*)(n \times (n \times n_0))\cdot(n_0 \times \epsilon_0)$$

$$-\frac{1}{2}((n \times n_0)\cdot \epsilon_0)(n \times (n \times n_0))\cdot(n_0 \times \epsilon_0^*)$$

$$+|(n \times (n \times \epsilon_0))\cdot \epsilon_0|^2 + \frac{1}{4}|(n \times n_0)\cdot(n_0 \times \epsilon_0)|^2$$

$$+\frac{1}{2}((n \times (n \times n_0))\cdot \epsilon_0^*)((n \times n_0)\cdot(n_0 \times \epsilon_0))$$

$$+\frac{1}{2}((n \times (n \times n_0))\cdot \epsilon_0)((n \times n_0)\cdot(n_0 \times \epsilon_0^*))]]$$

$$= k^4 a^6 \frac{1}{|n \times n_0|^2}\Big[|n\cdot(n_0 \times \epsilon_0)|^2 + \frac{1}{4}|(n_0\cdot n)|^2 |n\cdot(n_0 \times \epsilon_0)|^2$$

$$-|(n\cdot(n_0 \times \epsilon_0))|^2 (n\cdot n_0) + |n_0\cdot n|^2 |\epsilon_0\cdot n|^2$$

$$-\frac{1}{4}|\epsilon_0\cdot n|^2 - n_0\cdot n\,|\epsilon_0\cdot n|^2\Big]$$

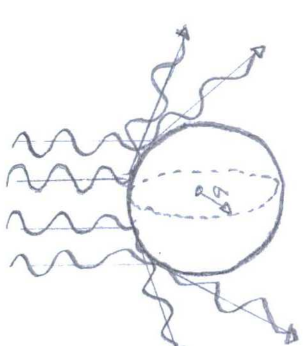
$$= k^4 a^6 \left[\frac{(\frac{5}{4} n_0\cdot n)(|n\cdot(n_0 \times \epsilon_0)|^2 + |\epsilon_0 n|^2 - \frac{1}{4}|n\cdot(n_0 \times \epsilon_0)|^2 - |\epsilon_0\cdot n|^2}{|n \times n_0|^2}\right]$$

Identities:

$$\boxed{\begin{aligned} \hat{b} &= (\hat{b}\cdot n_0)n_0 + (\hat{b}\cdot\epsilon_0)\epsilon_0 \\ &\quad + (b\cdot(n_0 \times \epsilon_0))(n_0 \times \epsilon_0) \\ |\hat{b}|^2 &= |\hat{b}\cdot n_0|^2 + |\hat{b}\cdot\epsilon_0|^2 + \hat{b}\cdot(n_0 \times \epsilon_0)|^2 \\ &= 1 \text{ ... when } b \text{ is a unit vector} \\ |n \times n_0|^2 &= |n\cdot\epsilon_0|^2 + |n\cdot(n_0 \times \epsilon_0)|^2 \end{aligned}}$$

$$= k^4 a^6 \left[\frac{5}{4} - |\epsilon_0\cdot n|^2 - \frac{1}{4}|n\cdot(n_0 \times \epsilon_0)|^2 - n_0\cdot n\right]$$

b) $n\cdot n_0 = \cos\theta$



"...if linearly polarized"

$$\epsilon_0 = \sin\phi\, \epsilon_{0,H} + \cos\phi\, \epsilon_{0,v}$$

$$\frac{d\sigma}{d\Omega}(\epsilon_0, n_0, n) = k^4 a^6 \Big[\frac{5}{4} - |(\sin\phi\, \epsilon_{0,H} + \cos\phi\, \epsilon_{0,v})\cdot n|^2$$

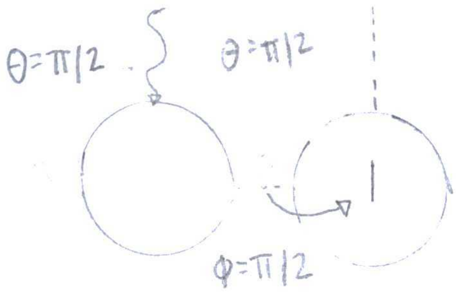
$$- \frac{1}{4}|n\cdot(n_0 \times (\sin\phi\, \epsilon_{0,H} + \cos\phi\, \epsilon_{0,v}))|^2 - n_0\cdot n\Big]$$

$$= k^4 a^6 \left[\frac{5}{4} - \cos^2\phi \sin^2\phi - \frac{1}{4}\sin^2\phi \sin^2\theta - \cos\theta\right]$$

... when $|\epsilon_{0,H}|^2 = |\epsilon_{\ldots}|^2 = n =$

$$= k^4a^6\left[\frac{5}{4} - \left(\frac{1}{2}+\frac{1}{2}\cos 2\phi\right)\sin^2\theta - \frac{1}{4}\left(\frac{1}{2}-\frac{1}{2}\cos 2\phi\right)\sin^2\theta - \cos\theta\right]$$

$$= k^4a^6\left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta - \frac{3}{8}\sin^2\theta\cos 2\phi\right]$$

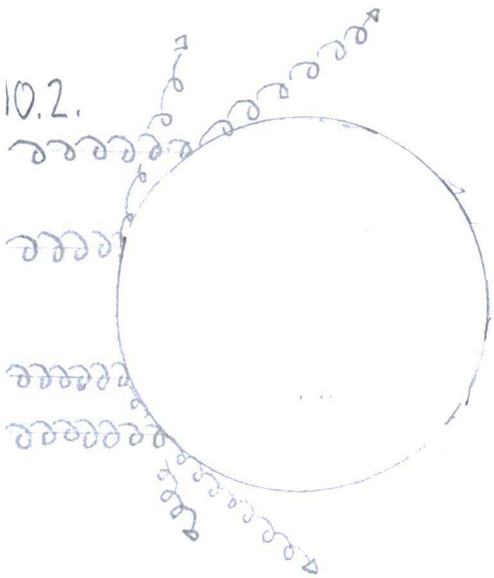


c) $$\frac{\frac{d\sigma_V}{d\Omega}}{\frac{d\sigma_H}{d\Omega}} = \frac{k^4a^6\left[\frac{5}{8}-\frac{3}{8}\right]}{k^4a^6\left[\frac{5}{8}+\frac{3}{8}\right]}. \quad @ \left(\frac{\pi}{2},0\right) \text{ and } \left(\frac{\pi}{2},\frac{\pi}{2}\right)$$

$$= \frac{1}{4}$$

"Ratio of intensities at $(\theta,\phi) = \left(\frac{\pi}{2},0\right), \left(\frac{\pi}{2},\frac{\pi}{2}\right)$"

10.2.



$$\epsilon_\pm = \frac{1}{\sqrt{2}}(\epsilon_1 \pm \epsilon_2)$$

$$\epsilon = \frac{1}{\sqrt{1+r^2}}\left(\epsilon_+ + r e^{i\alpha}\epsilon_-\right)$$

(10.71) "Differential scattering cross section"

$$\frac{d\sigma}{d\Omega} = k^4a^6\left[\frac{5}{6}(1+\cos^2\theta) - \cos\theta - \frac{3}{4}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(2\phi-\alpha)\right]$$

"Electromagnetic radiation with elliptic polarization ... scattered by a perfectly conducting sphere."

(10.55) "Incident Electric and Magnetic Field"

$$E_{inc} = \epsilon E^{(+)}_{lab}$$
$$= \sum_{l=1}^{\infty} i^l \sqrt{\frac{2\pi(2l+1)}{1+r^2}}\left[j_1(kr)\vec{X}_{l,1} + \frac{1}{k}\nabla\times j_l(kr)\vec{X}_{l,1} + re^{i\alpha}\left(j_1(kr)\vec{X}_{l,-1} - \frac{1}{k}\nabla\times j_l(kr)\vec{X}_{l,-1}\right)\right]$$

$$cH_{inc} = \frac{1}{\sqrt{1+r^2}}\left(\epsilon_+ + re^{i\alpha}\epsilon_-\right)\cdot\frac{cB}{\mu_0}$$
$$= c\sum_{l=1}^{\infty} i^l \sqrt{\frac{2\pi(2l+1)}{1+r^2}}\left[-i j_l(kr)\vec{X}_{l,1} - \frac{i}{k}\nabla\times j_l(kr)\vec{X}_{l,1} + re^{i\alpha}\left(-\frac{i}{k}\nabla\times j_l(kr)\vec{X}_{l,-1} + i j_l(kr)\vec{X}_{l,-1}\right)\right]$$

(10.57) "Scattered Electric and Magnetic Field"

$$E_{scatt} = \frac{1}{2}\sum_{l} i^{l}\sqrt{\frac{2\pi(2l+1)}{1+r^2}}\Big[\alpha_+(l)h_l^{(1)}(kr)\vec{X}_{l,1} \pm \frac{\beta_+(l)}{k}\nabla\times h_l^{(1)}(kr)X_{l,1}$$
$$+re^{i\alpha}\big(\alpha_-(l)h_l^{(1)}(kr)X_{l,-1} - \frac{\beta_-(l)}{k}\nabla\times h_l^{(1)}(kr)X_{l,-1}\big)\Big]$$

$$cB_{sc} = \frac{1}{2}\sum_{l=1}^{\infty} i^{l}\sqrt{\frac{2\pi(2l+1)}{1+r^2}}\Big[\mp\beta_+(l)h_l^{(1)}(kr)X_{l,\pm1} - \frac{i\alpha_+(l)}{k}\nabla\times h_l^{(1)}(kr)X_{l,\pm1}\Big]$$

Note: $\alpha_+, \beta_+, \alpha_-, \beta_-$ are coefficients to positive and negative helices in both real or imaginary spaces.

(10.63) "Differential Cross Section for Polarization"

$$\frac{d\sigma_{sc}}{d\Omega} = \frac{\pi}{2k^2(1+r^2)}\Big|\sum_{l}\sqrt{2l+1}\,\big[\alpha_+(l)X_{l,1} + i\beta_+(l)\,n\times X_{l,1}$$
$$+re^{i\alpha}\big(\alpha_-(l)X_{l,-1} - i\beta_-(l)\,n\times X_{l,1}\big)\big]\Big|^2$$

$$\sigma_{sc} = \frac{\pi}{2k^2(1+r^2)}\sum_{l}(2l+1)\Big[|\alpha_+(l)|^2 + |\beta_+(l)|^2 + |\alpha_-(l)|^2 + |\beta_-(l)|^2\Big]$$

Note: The model about differential cross section, e.g. scattering per angle depends on the slope in Bessels.

Side Note:

(9.119) $X_{lm}(\theta,\phi) = \frac{1}{\sqrt{l(l+1)}}L\cdot Y_{lm}(\theta,\phi)$

(9.101) $L = \frac{1}{i}(r\times\nabla)$

Longwavelength limit: $l \geq 1$

(pg 477) $\alpha_+(1) = -\frac{1}{2}\beta_\pm(1) \cong -\frac{2i}{3}(ka)^3$

$$\frac{d\sigma_{sc}}{d\Omega} \cong \frac{2\pi}{3k^2(1+r^2)}(ka)^6\Big|X_{l,1} - 2i\hat{n}\times X_{1,1} + re^{i\alpha}\big(X_{l,-1}$$
$$+2i\hat{n}\times X_{l,-1}\big)\Big|^2$$

$$X_{1\pm 1} = \frac{1}{\sqrt{2}} L \cdot Y_{1,\pm 1}$$

$$L = \frac{r}{i} \times \nabla$$

$$= i\left(\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - \hat{\phi}\frac{\partial}{\partial\theta}\right)$$

$$Y_{1,\pm 1} = \mp\sqrt{3/8\pi}\,\sin\theta\, e^{\pm i\phi}$$

$$X_{1,\pm 1} = \mp\sqrt{\frac{3}{16\pi}}\left(\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - \hat{\phi}\frac{\partial}{\partial\theta}\right)\sin\theta e^{\pm i\phi}$$

$$= -\sqrt{\frac{3}{16\pi}}\left(\hat{\theta} \pm i\hat{\phi}\cos\theta\right)e^{\pm i\phi}$$

$$n \times X_{1,\pm 1} = r \times X_{1,\pm 1}$$

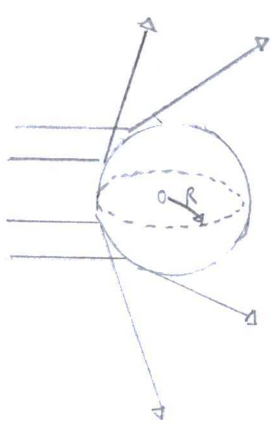
$$= \sqrt{\frac{3}{16\pi}}\left(\hat{\phi} \mp i\theta\cos\theta\right)e^{\pm i\phi}$$

$$\frac{d\sigma_{sc}}{d\Omega} = \frac{(ka)^6}{8k^2(1+r^2)}\left|\left[\hat{\theta}(1-2\cos\theta) + i\hat{\phi}(\cos\theta - 2)\right]e^{i\phi}\right.$$
$$\left. + re^{i(\alpha-\phi)}\left[\hat{\theta}(1-2\cos\theta) - i\phi(\cos\theta-2)\right]\right|^2$$

$$= \frac{(ka)^6}{8k^2(1+r^2)}\left|\hat{\theta}(1-2\cos\theta)(1+re^{i(\alpha-2\phi)})\right.$$
$$\left. + i\hat{\phi}(\cos\theta - 2)(1-re^{i(\alpha-2\phi)})\right|^2$$

$$= k^4a^6\left[\frac{5}{6}(1+\cos^2\theta) - \cos\theta - \frac{3}{4}\left(\frac{r}{1+r^2}\right)\sin^2\theta\cos(2\phi-\alpha)\right]$$

10.3.



"A solid uniform sphere... scatterer of a plane wave beam of unpolarized radiation"

a) Magnetic Field Derivation around a sphere:

① Boundary Conditions:

$$\Psi(r=R, \theta, \phi=0) = 0$$

$$\Psi(r=0, \theta, \phi) = \text{finite}$$

$$\frac{\partial B(r\ r\to ?, \theta, \phi)}{dz} = -B_0 z$$

$\nabla \cdot B = 0$ and $\nabla \times B = 0$

"free space" "no current"

② Laplace's Equation:

$$\nabla^2\Psi = \frac{1}{r}\frac{\partial^2}{\partial^2 r}(r\Psi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}$$

$$= 0$$

③ Laplace's Equation Solutions:

Ⓐ Variable separation:

$$\nabla^2\Psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Psi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial^2\phi}$$

$$= 0$$

If, $\Psi = U(r)\cdot P(\theta)\cdot Q(\phi)$

$$PQ\frac{\partial^2 U}{\partial^2 R} + \frac{U\cdot Q}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{U\cdot P}{r^2\sin^2\theta}\frac{\partial^2 Q}{\partial\phi^2}$$

$$= 0$$

Ⓑ Azimuthal Eigenvalues:

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial\phi^2} = -m^2 \quad ; \quad \frac{\partial^2 Q}{\partial^2\phi} + m^2 Q = 0$$

Ⓒ Angular Eigenvalues:

$$\frac{1}{P\sin^2\theta}\frac{d}{d\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right) = \lambda - \ell(\ell+1)$$

$$\frac{1}{P\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right) + \left[\ell(\ell+1) + \frac{\lambda}{\sin^2\theta}\right]P = 0$$

Ⓓ Radial Eigenvalues:

$$\frac{r^2}{U}\frac{\partial^2 U}{\partial r^2} = -\ell(\ell+1) \quad ; \quad \frac{\partial^2 U}{\partial^2 r} - \frac{\ell(\ell+1)}{r^2}U = 0$$

④ General Solution to Laplace's Equation:

$$U(r) = A_\ell \cdot r^\ell + B_\ell r^{-\ell-1}$$

$$P(\theta) = P_\ell^m(\cos\theta)$$

$$Q(\phi) = A_m e^{im\phi} + B_m e^{-im\phi}$$

$$\Psi(r,\theta,\phi) = \sum_{l=0}^{\infty} U(r)\cdot P(\theta)\cdot Q(\phi)$$

$$= \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-l-1})(A_m e^{im\phi} + B_m e^{-im\phi}) P_l^m(\cos\theta)$$

(5) Variables by Boundary Conditions:

A_m, B_m

$$\Psi(r=R,\theta,\phi=0) = \sum_{l=0}^{\infty} (A_l \cdot R^l + B_l R^{-l-1})(A_m e^{im0} + B_m e^{-im0}) P_l^m(\cos\theta)$$

$$= 0,\ A_m = B_m = 1$$

$$B(r,\theta,\phi) = \sum_{l=0}^{\infty} (A_l \cdot r^l + B_l r^{-l-1})\, P_l^m(\cos\theta)$$

A_l

$$\frac{\partial \Psi(r=\infty,\theta,\phi)}{\partial z} = \sum_{l=0}^{\infty} (A_l(\infty)^l + B_l(\infty)^{-l-1})\, P_l^m(\cos\theta)$$

$$= A_0 + A_1 r\cos\theta + \sum A_l(\infty)^l P_l(\cos\theta)$$

$$= -B_0 \hat{z}$$

$$= -B_0 r\cos\theta,\quad A_l = \begin{cases} 0 & l=0 \\ -B_0 & l=1 \\ 0 & l>1 \end{cases}$$

$$\Psi(r,\theta,\phi) = -B_0 r\cos\theta + \sum_{l=0}^{\infty} B_l \cdot r^{-l-1} \cdot P_l(\cos\theta)$$

B_l

$$\frac{\partial \Psi(r=R,\theta,\phi)}{\partial r} = \left[-B_0\cos\theta + \sum_{l=0}^{\infty} B_l\cdot(-l-1)\cdot R^{-l-2}\cdot P_l(\cos\theta)\right]$$

$$= -B_0\cos\theta + B_0(-1)R^{-2} + B_1(-2)R^{-3}\cos\theta$$

$$+ \sum_{l=2}^{\infty} B_l(-l-1)R^{-l-2}\cdot P_l(\cos\theta)$$

$$= 0, \quad B_\ell = \begin{cases} 0 & \ell = 0 \\ -R^3 B_0 / 2 & \ell = 1 \\ 0 & \ell > 1 \end{cases}$$

⑥ Exact Solution:

$$\Psi(r,\theta,\phi) = -B_0 r\cos\theta - \frac{1}{2} R^3 B_0 r^{-2} \cos\theta$$

$$B = -\nabla \Psi$$
$$= \hat{r} B_0 \cos\theta \left(1 - \left(\frac{R}{r}\right)^3\right) - \hat{\theta} B_0 \sin\theta \left(1 + \frac{1}{2}\left(\frac{R}{r}\right)^3\right)$$

b) Absorption Cross Section:

$$\frac{dP_{loss}}{da} = \frac{\mu_c \omega \delta}{4} |H_{par}|^2$$
$$= \frac{\mu_c \omega \delta}{4\mu_0^2} |B_{par}|^2$$
$$= \frac{\mu_c \omega \delta}{4\mu_0^2} |B \cdot \hat{\theta}|^2$$
$$= \frac{9\mu_c \omega \delta}{16\mu_0^2} B_0^2 \sin^2\theta$$

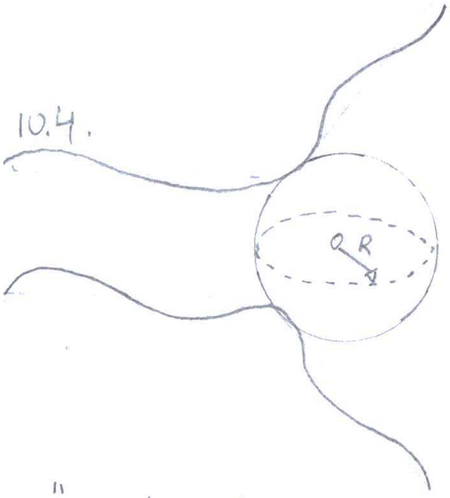
$$P_{loss} = \int_0^R \int_0^\pi \int_0^{2\pi} \frac{9\mu_c \omega \delta}{16\mu_0^2} \sin^2\theta \, r \sin\theta \, d\phi \, d\theta \, dr$$
$$= \frac{3\pi R^2 \mu_c \omega \delta}{2\mu_0^2} B_0^2$$

u-substitution
$\sin^2\theta = 1 - \cos^2\theta$
$u = \cos\theta$
$\frac{du}{d\theta} = -\sin\theta$

$$\sigma_{abs} = \frac{P_{loss}}{I_0}$$
$$= \frac{P_{loss}}{\frac{1}{2}\sqrt{\frac{1}{\epsilon_0 \mu_0^3}} B_0^2}$$
$$= 3\pi R^2 \sqrt{\frac{2\epsilon_0 \omega \mu_c}{\sigma \mu_0}}$$

$$= 3\pi R^2 \sqrt{\frac{2\epsilon_0 \omega}{\sigma}} \quad \circ\circ\circ\circ \; \mu_c = \mu_0 = \text{"non-magnetic"}$$

10.4.



"unpolarized wave... scattered by a slightly lossy uniform isotropic dielectric sphere... much smaller than wavelength"

a) Frequency $= \omega = ck = \frac{1}{\epsilon_0 Z_0} k$

Radius $= R$

Dielectric $= \epsilon_R$

conductivity $= \sigma$

Skin depth $= \delta \gg R$

sphere properties

(10.5) "Electric Dipole Moment"

$$p = 4\pi\epsilon_0 \left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}\right) R^2 \epsilon_0 E_0$$

(4.38) "Electric Susceptibility modified"

$$\frac{\epsilon}{\epsilon_0} = \epsilon_R + i\frac{\sigma}{\omega\epsilon_0}$$

$$p = 4\pi\epsilon_0 \left(\frac{(\epsilon_r - 1) + i Z_0 \sigma / k}{(\epsilon_r + 2) + i Z_0 \sigma / k}\right) R^3 \epsilon_0 E_0$$

(10.4) "Differential Cross Section"

$$\frac{d\sigma_{diff}}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2} \left| \epsilon^* \cdot p + \cancel{(n \times \epsilon^*) \cdot m / c}^{\,0} \right|^2$$

$$= \frac{(\epsilon_r - 1)^2 + (Z_0 \sigma / k)^2}{(\epsilon_r + 2)^2 + (Z_0 \sigma / k)^2} k^4 R^6 \left| \epsilon^* \cdot \epsilon_0 \right|^2$$

(10.11) "Total Scattering Cross Section"

$$\sigma_{Tot} = \int \frac{d\sigma}{d\Omega} d\Omega$$

$$= \int \sum \frac{d\sigma}{d\Omega} d\Omega$$

$$= \int \frac{1}{2} \left[\frac{d\sigma_{HH}}{d\Omega} + \frac{d\sigma_{VH}}{d\Omega} + \frac{d\sigma_{HV}}{d\Omega} + \frac{d\sigma_{VV}}{d\Omega} \right] d\Omega$$

$$= \int \frac{(\epsilon_r - 1)^2 + (Z_0 \sigma / k)^2}{(\epsilon_r + 2)^2 + (Z_0 \sigma / k)^2} k^4 R^6 \frac{1}{2}(1 + \cos^2(\theta)) d\Omega$$

$$= \int_0^{\pi} \int_0^{2\pi} \frac{(\epsilon_r - 1)^2 + (Z_0 \sigma / k)^2}{(\epsilon_r + 2)^2 + (Z_0 \sigma / k)^2} k^4 R^6 \frac{1}{2}(1 + \cos^2(\theta)) \sin\theta \, d\theta \, d\phi$$

$$= \frac{8\pi}{3} \frac{(\epsilon_r - 1)^2 + (Z_0 \sigma / k)^2}{(\epsilon_r + 2)^2 + (Z_0 \sigma / k)^2} k^4 R^6$$

$$= \frac{(\epsilon_r - 1) + \sigma'^2}{(\epsilon_r + 2) + \sigma'^2} \quad \text{∘∘when } \sigma' = Z_0 \sigma / k$$

$$\sigma_{norm} = \frac{\sigma_{unpol}}{\frac{8\pi}{3} k^4 R^6}$$

b) (

$$\sigma_{abs} = \frac{P_{abs}}{\frac{1}{2} I_0}$$

$$= \frac{P_{abs}}{\sqrt{\frac{\epsilon_0}{\mu_0}} \frac{1}{2} |E_0|^2}$$

(10.134) "Scattered Poynting Vector"

$$P_{abs} = -\frac{1}{2\mu_0} \oint_S \mathrm{Re}[E \times B^*] \cdot \hat{n} \, da$$

$$= -\frac{R^2}{2\mu_0} \int_0^{2\pi} \int_0^{\pi} \mathrm{Re}[E \times B^*] \sin\theta \, d\theta \, d\phi$$

$$\sigma = -\frac{cR^2}{|E_0|^2} \int_0^{2\pi} \int_0^{\pi} \mathrm{Re}[E \times B^*] \sin\theta \, d\theta \, d\phi \quad \text{∘∘∘if } E_{inc} = E_0 e^{ik_0 x}$$

$$= -\frac{cR^2}{|E_0|^2} \int_0^{2\pi} \int_0^{\pi} \mathrm{Re}[(E_{inc} + E_{scat}) \times (B_{inc} + B_{scat})] \sin\theta \, d\theta \, d\phi$$

$$E_{scat} = -\frac{k^2}{4\pi\epsilon_0} [\hat{k} \times (\hat{k} \times p)] \frac{e^{ikx}}{r}$$

$$= -E_0 \left(\frac{(\epsilon_r - 1) + i Z_0 \sigma / k}{(\epsilon_r + 2) + i Z_0 \sigma / k}\right) k^2 R^3 (\hat{k} \times \epsilon_0) \frac{e^{ikx}}{r}$$

If $B = \frac{1}{c} \hat{k} \times E$, then

$$B_{inc} = \frac{1}{c} E_0 e^{ik_0 x} \hat{k}_0 \times \epsilon_0$$

$$B_{scat} = \frac{1}{c} E_0 \left(\frac{(\epsilon_r - 1) + i Z_0 \sigma / k}{(\epsilon_r + 2) + i Z_0 \sigma / R}\right) k^2 R^3 \cdot (\hat{k} \times \epsilon_0) \frac{e^{ikx}}{r}$$

$$\sigma_{abs} = -R^2 \int_0^{2\pi} \int_0^{\pi} Re\left[\left(\epsilon_0 e^{ik_0 x} - \left(\frac{(\epsilon_r - 1) + i Z_0 \sigma / k}{(\epsilon_r + 2) + i Z_0 \sigma / k}\right) k^2 R^2 [\hat{k} \times (\hat{k} \times \epsilon_0)\right.\right.$$

$$\cdot e^{ikx} \times \left(e^{-ik_0 x} k_0 \times \epsilon_0 + \left(\frac{(\epsilon_r - 1) + i Z_0 \sigma / k}{(\epsilon_r + 2) + i Z_0 \sigma / k}\right) k^2 R^2 (\hat{k} \times \epsilon_0) e^{-ik_0 x}\right.$$

$$\cdot \sin\theta \, d\theta \, d\phi$$

$$= \frac{-k^2 R^4}{(\epsilon_r + 2)^2 + (Z_0 \sigma / k)^2} \int_0^{2\pi} \int_0^{\pi} Re\left[(\epsilon_0 - \right.$$

$$= \frac{-k^2 R^4}{(\epsilon_r + 2)^2 + (Z_0 \sigma / k)^2} \int_0^{2\pi} \int_0^{\pi} ((((A \cos(kr(\cos\theta - 1))$$

$$+ 3 Z_0 \sigma / k \sin(kr(\cos\theta - 1)))) \cdot (\epsilon_0 \times \hat{k} \times \epsilon_0))$$

$$- ((A \cos(kR(\cos\theta - 1)) + 3 Z_0 \sigma / k \sin(kr(\cos\theta - 1))))$$

$$\cdot [\hat{k} \times (\hat{k} \times \epsilon_0)] \times k_0 \times \epsilon_0) \cdot \sin\theta \, d\theta \, d\phi$$

$\ldots$ where $A = ((\epsilon_r - 1)(\epsilon_r + 2) + (Z_0 \sigma / k)^2$

$$= \frac{-\pi k^2 R^4}{(\epsilon_r+2)^2+(Z_0\sigma/k)^2}\int_0^{\pi}((((A\cos(kr(\cos\theta-1))$$

$$+3Z_0\sigma/k\sin(kR(\cos\theta-1)))))\cdot(1+\cos^2\theta))$$

$$+2((A\cos(kR(\cos\theta-1))$$

$$+3Z_0\sigma/k\sin(kR(\cos\theta-1))))\cos\theta)\sin\theta d\theta$$

$$\therefore \text{ if } x=\cos\theta\, ; \frac{dx}{d\theta}=-\sin\theta$$

$$= \frac{-\pi k^2 R^4}{(\epsilon_r+2)^2+(Z_0\sigma/k)^2}\int_{-1}^{1}((((((A\cos(kR(x-1))$$

$$+3Z_0\sigma/k\cdot\sin(kr(x-1)))))\cdot(1+x^2))+2((A\cos(kr(x-1)$$

$$+3Z_0\sigma/k\cdot\sin(kR(x-1))))x)dx$$

$$= \frac{-\pi k^2 R^4}{(\epsilon_r+2)^2+(Z_0\sigma/k)^2}\left[(A\int_{-2}^{0}x^2\cos(kRx)dx\right.$$

$$+4A\int_{-2}^{0}x\cos(kRx)dx+4A\int_{-2}^{0}\cos(kRx)dx$$

$$+3Z_0\sigma/k\int_{-2}^{0}x^2\sin(kRx)dx+12Z_0\sigma/k\int_{-2}^{0}x\sin(kRx)dx$$

$$+12Z_0\sigma/k\int_{-2}^{0}\sin(kRx)dx$$

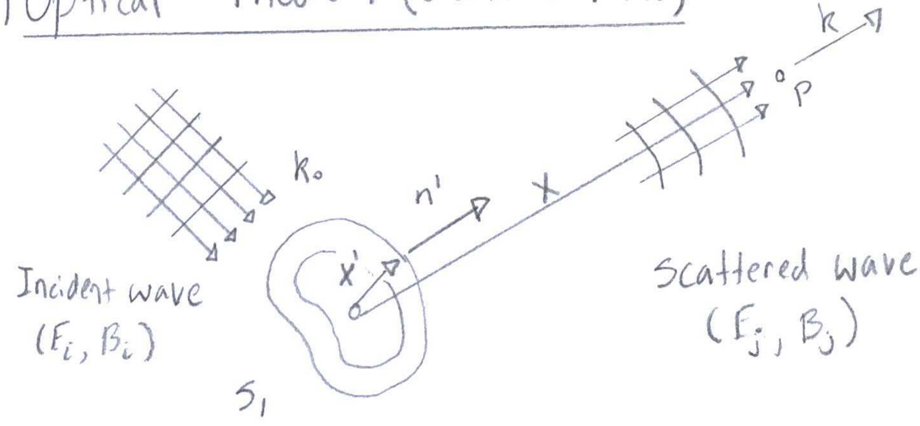
$$= \frac{-\pi k^2 R^4}{(\epsilon_r+2)^2+(Z_0\sigma/k)^2}\left[(-2A\sin(2kR)/kR^3\right.$$

$$\left.+4A/k^2R^2+Z_0\sigma/k\left[6(1-\cos(2kR))/k^3R^3-12/kR\right]\right]$$

$$\cong 12\pi R^2\frac{RZ_0\sigma}{(\epsilon_r+2)^2+(Z_0\sigma/k)^2}$$

$$\sigma_{TOT} \cong \frac{4\pi R^3}{(\epsilon_r+2)^2 + (Z_0\sigma/k)^2}\left[3Z_0\sigma + \frac{2}{3}(\epsilon_r-1)^2 k^4 R^6 + \frac{2}{3} Z_0^2 \sigma^2 k^2 R^2\right]$$

c) Optical Theorem (Section 10.11)



- A single wave absorbs and scatters.
- Also, the coherent waves sum into a total wave.

(10.139) "Optical Theorem"

$$\sigma_t = \frac{4\pi}{k} \operatorname{Im}\left[\epsilon_0^* \cdot f(k = k_0)\right]$$

(10.138) "Normalized Scattering Amplitude"

where $f(k, k_0) = \dfrac{F(k, k_0)}{E_0}$

(10.91) "Forward Scattering Amplitude"

$$F(k, k_0) = \frac{i}{4\pi} \oint_{S_1} e^{-ik x'} \left[\omega(n' \times B_s) + k \times (n' \times E_s) - k(n' \cdot E_s)\right] da'$$

Note: The cross section depends upon a ratio between outward amplitude $(F(k, k_0))$ to initial amplitude (E_0)

$$\sigma_t = -\frac{4\pi}{k} \, Im\left[\epsilon_0^* \left(\frac{(\epsilon_r - 1) + iZ_0\sigma/k}{(\epsilon_r+2) + iZ_0\sigma/k}\right) k^2 R^3 \left[R_0 \times (R_0 \times \epsilon_0)\right]\right]$$

$$= \frac{12\pi R^2 R_0 Z_0 \sigma}{(\epsilon_r+2)^2 + (Z_0\sigma/k)^2}$$

The differential cross section(σ) is possibly accurate to experiments because similar calculations by separate models.

10.5.

a) Magnetic Dipole Moment:

(10.13) "magnetic dipole"

$$m = 2\pi a^3 H_{inc}$$

$$= -2\pi \left(\frac{\epsilon_r - 1 + iZ_0\sigma/k}{(\epsilon_r+2) + iZ_0\sigma/k}\right) R^3 \cdot \frac{B_0}{H}$$

$$\cong \frac{i2\pi\sigma Z_0}{k\mu_0} R^3 B_0$$

∴ book shows a proportional answer, I doubt the logic and relationships.

b) (Problem 10.4)

$$\sigma_{TOT} = 12\pi R^2 \frac{RZ_0\sigma}{(\epsilon_r+2)^2 + (Z_0\sigma/k)^2}$$

$$+ 1i(m)$$

∴ if using books $m = \frac{i4\pi\sigma Z_0}{k\mu_0}(kR)^2\frac{R^3}{30}$

$$= 12\pi R^2 (RZ_0\sigma)\left[\frac{1}{(\epsilon_r+2)^2 + (Z_0\sigma/k)^2} - \frac{(kR)^2}{90}\right]$$

∴ close

10.6

(10.57) "Scattered wave"

$$E_{sc} = \frac{1}{2}\sum_{l=1}^{\infty} i^{l}\sqrt{4\pi(2l+1)}\left[\alpha_{\pm}(l) h_l^{(1)}(kr) X_{l,\pm 1} \pm \frac{\beta_{\pm}(l)}{k} \nabla \times h_l^{(1)} X_{l,\pm}\right.$$

$$cB_{sc} = \frac{1}{2}\sum_{l=1}^{\infty} i^{l}\sqrt{4\pi(2l+1)}\left[\frac{-i\alpha(l)}{k} \nabla \times h_l^{(1)}(kr) X_{l\pm 1} \mp i\beta(l) h_l^{(1)}(kr) X_{l\pm}\right.$$

(10.138) "normalized scattering amplitude"

$$f(k, k_0) = \frac{F(k, k_0)}{E_0}$$

$$= \frac{(\epsilon_1 + i\epsilon_2)}{\sqrt{2} E_0} \cdot E_{sc}$$

$$= \frac{1}{ik}\sqrt{\frac{\pi}{2}} \sum_{l=1}^{\infty} \sqrt{2l+1}\left[\alpha_{\pm}(l) X_{l,\pm 1} \pm i\beta(l)\, n \times X_{l,\pm 1}\right]$$

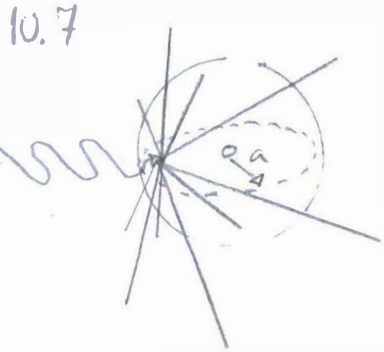
... when $E_0 \equiv \epsilon_1 + i\epsilon_2$

b) (10.139) "Optical Theorem"

$$\sigma_t = \frac{4\pi}{k} \operatorname{Im}\left[\epsilon_0^{*} \cdot f(k = k_0)\right]$$

$$= \frac{\sqrt{8}\,\pi^{3/2}}{k^2} \sum_{l=1}^{\infty} \sqrt{2l+1}\ \alpha_{\pm}(l) X_{l\pm 1}$$

10.7



"... scattering of a plane wave of electromagnetic radiation"

a) (10.57) "scattered Waves"

$$E_{sc} = \frac{1}{2}\sum_{\ell=1}^{\infty} i^{\ell}\sqrt{4\pi(2\ell+1)}\left[\alpha_{\pm}(\ell)\, h_{\ell}^{(1)}(kr)\, X_{\ell,\pm1} \pm \frac{\beta_{\pm}(\ell)}{k}\nabla\times h_{\ell}^{(1)}(kr)\, X_{\ell,\pm1}\right]$$

$$cB_{sc} = \frac{1}{2}\sum_{\ell=1}^{\infty} i^{\ell}\sqrt{4\pi(2\ell+1)}\left[-\frac{i\alpha_{\pm}(\ell)}{k}\nabla\times h_{\ell}^{(1)}(kr)\, X_{\ell,\pm1} \mp i\beta_{\pm}(\ell)\, h_{\ell}^{(1)}(kr)\, X_{\ell\pm1}\right]$$

$$E_{sc} = \sum_{\ell=1}^{\infty} i^{\ell}\sqrt{4\pi(2\ell+1)}\left[a_{M\pm}(\ell)\, j_{\ell}(k'r)\, X_{\ell\pm1} \pm \frac{1}{k'} a_{E\pm}(\ell)\nabla\times j_{1}(k'r)\, X_{\ell,\pm1}\right]$$

$$H = \frac{1}{Z}\sum_{\ell=1}^{\infty} i^{\ell}\sqrt{4\pi(2\ell+1)}\left[-\frac{i a_{M\pm}(\ell)}{k}\nabla\times j_{\ell}(k'r)\, X_{\ell,\pm1} \mp i\, a_{E\pm}(\ell)\, j_{\ell}(k'r)\, X_{\ell\pm1}\right]$$

Identities:

$$\boxed{\begin{aligned} k' &= \omega\sqrt{\mu\epsilon} = \omega\sqrt{\mu_0\epsilon_0}\sqrt{\epsilon_r} \\ &= k\sqrt{\epsilon_r} \\ Z &= \sqrt{\frac{\mu_0}{\epsilon}} = \frac{Z_0}{\sqrt{\epsilon_r}} \end{aligned}}$$

(10.60) "Transverse field"

Perpendicular

$$\nabla\times f_{\ell}(r)\, X_{\ell m} = \frac{i\hat{n}\sqrt{\ell(\ell+1)}}{r} f_{\ell}(r)\, Y_{\ell m} + \frac{1}{r}\frac{\partial}{\partial r}\left[r f_{\ell}(r)\right] n \times X_{\ell m}$$

$$\begin{aligned} r\cdot\nabla\times f_{\ell}(kr)\, X_{\ell m} &= \hat{r}\times\nabla\cdot f_{\ell}(kr)\, X_{\ell m} \\ &= i\vec{L}\cdot f_{\ell}(kr)\, X_{\ell m} \\ &= i f_{\ell}(kr)\cdot\vec{L}\cdot X_{\ell m} \\ &= i\sqrt{\ell(\ell+1)}\, f_{\ell}(kr)\, Y_{\ell m} \end{aligned}$$

Parallel

$$r\times X_{\ell m} \neq 0,\quad r\times\left(\nabla\times f_{\ell}(kr)\, X_{\ell m}\right) = \nabla\left(f_{\ell}(kr)\,\hat{r}\cdot X_{\ell m}\right) - \frac{1}{r} f_{\ell}(kr)\left(X_{\ell m} - \hat{r}(\hat{r}\cdot X_{\ell m}\right) - (\hat{r}\cdot\nabla)\, f_{\ell}(kr)\, X_{\ell m})$$

$$= -\frac{1}{r}\frac{d}{dr}\left(r \cdot f_e(kr)\right) X_{em} \quad \ldots \text{ when } r \cdot X_{em} = 0$$

$$B_\perp : \; a_M \pm (l)\, j_e(x') = j_e(x) + \frac{1}{2}\alpha \pm (l)\, h_e^{(1)}(x)$$

$$H_{\parallel} : \; \sqrt{\epsilon_r}\, a_{E\pm}(l)\, j_e(x') = j_e(x) + \frac{1}{2}\beta_\pm(l)\, h_e^{(1)}(x)$$

$$D_\perp : \; \sqrt{\epsilon_r}\, a_{E\pm}(l)\, j_e(x') = j_e(x) + \frac{1}{2}\beta_\pm(l)\, h_e^{(1)}(x)$$

$$E_{\parallel} : \; a_{M\pm}(l)\, j_e(x') \quad = j_e(x) + \frac{1}{2}\alpha_\pm(l)\, h_e^{(1)}(x)$$

$$a_{E\pm}(l)\frac{d}{dx'}x' j_e(x) = \sqrt{\epsilon_r}\,\frac{d}{dx}x\left(j_e(x) + \frac{1}{2}\beta_\pm(l)\, h_e^{(1)}(x)\right)$$

$\ldots$ where $x = ka$, $x' = k'a = x\sqrt{\epsilon_r}$

$$\alpha_\pm(l) + 1 = -\frac{h_l^{(2)}(x)\frac{d}{dx}x' \cdot j_e(x') - j_e(x)\frac{d}{dx}\, x \cdot h_e^{(2)}(x)}{h_e^{(1)}(x)\frac{d}{dx}x' j_e(x') - j_e(x')\frac{d}{dx}\, x\, h_e^{(1)}(x)}$$

$$\beta_\pm(l) + 1 = -\frac{h_e^{(2)}(x)\frac{d}{dx}x' \cdot j_e(x') - \epsilon_r \cdot j_e(x')\frac{d}{dx} \cdot x \cdot h_e^{(2)}(x)}{h_e^{(1)}(x)\frac{d}{dx'}x' \cdot j_e(x') - \epsilon_r \cdot j_e(x')\frac{d}{dx}\, x \cdot h_e^{(1)}(x)}$$

Phase shift:

$$\tan\delta_e = a/b$$

where $$e^{2i\delta_e} = -\frac{a - ib}{a + ib}$$

$$= \alpha_\pm(l) + 1$$

$$e^{2i\delta_e'} = \beta_\pm(l) + 1$$

when $$\tan\delta_e = \frac{j_e(x)\frac{d}{dx}x' \cdot j_e(x') - j_e(x')\frac{d}{dx} \cdot x \cdot j_e(x)}{h_e(x)\frac{d}{dx'}x \cdot j_e(x) - j_e(x')\frac{d}{dx}\, x \cdot h_e(x)}$$

$$\tan\delta_\ell' = \frac{j_\ell(x)\frac{d}{dx'}x' j_\ell(x') - \epsilon_r j_\ell(x')\frac{d}{dx}x\, j_\ell(x)}{h_\ell(x)\frac{d}{dx}x' j_\ell(x') - \epsilon_r j_\ell(x')\frac{d}{dx}x\cdot j_\ell(x)}$$

$$\tan\delta = \frac{x\cdot j_\ell'(x) - B_\ell\cdot j_\ell(x)}{x\cdot n_\ell'(x) - B_\ell\cdot n_\ell(x)}$$

$$\tan\delta' = \frac{x\cdot j_\ell'(x) - B_\ell'\cdot j_\ell(x)}{x\cdot n_\ell'(x) - B_\ell'\cdot n_\ell(x)}$$

when $B_\ell = \dfrac{j_\ell'(x)}{j_\ell(x)}$ and $B_\ell' = \dfrac{1}{\epsilon_r}\left(x'\dfrac{j_\ell'(x')}{j_\ell(x')} + 1 - \epsilon_r\right)$

b) Long-wavelength limit:

$$\tan\delta_1 = \frac{1}{45}x^3(x'^2 - x^2) = \frac{1}{45}(\epsilon_r - 1)(ka)^5$$

$$\tan\delta_1' = \frac{2}{3}\frac{\epsilon_r - 1}{\epsilon_r + 2}x^3 = \frac{2}{3}\frac{\epsilon_r - 1}{\epsilon_r + 2}(ka)^3$$

Spherical Bessels Approx:

$$j_1(x) = \frac{x}{3}\left(1 - \frac{x^2}{10} + \cdots\right)$$

$$n_1(x) = \frac{-1}{x^2}\left(1 + \frac{x^2}{2} + \cdots\right)$$

$$\alpha_\pm(1) = e^{2i\delta_1} - 1 \cong 2i\delta_1 = \frac{2i}{45}(\epsilon_r - 1)(ka)^5$$

$$\beta_\pm(1) = e^{2i\delta_1'} - 1 \cong 2i\delta_1 = \frac{4i}{3}\frac{\epsilon_r - 1}{\epsilon_r + 2}(ka)^3$$

(10.63) "Scattering Cross Section - Incident Polarization"

$$\frac{d\sigma}{d\Omega} = \frac{\pi}{2k^2}\left|\sum_\ell \sqrt{2\ell+1}\left(\alpha_\pm(\ell)X_{\ell,\pm1} \pm i\beta_\pm(\ell)\hat{n}\times X_{\ell,\pm1}\right)\right|^2$$

$$\cong \frac{\pi}{2k^2}\left|\sqrt{3}\,\frac{4i}{3}\frac{\epsilon_r - 1}{\epsilon_r + 2}(ka)^3\,\hat{n}\times X_{1,\pm1}\right|^2$$

$$= \frac{16\pi}{6k^2}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2 (ka)^6 \left| X_{1,\pm1} \right|^2$$

$$= \frac{1}{2} k^4 a^6 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)^2 (1+\cos^2\theta)$$

(10.10) "Scattered Polarization"

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left| \frac{\epsilon_r - 1}{\epsilon_r + 2} \right|^2 \frac{1}{2}(1+\cos^2\theta)$$

The derivation and equation equal.

c) $\lim_{\epsilon_r \to \infty} j_l(x') = \frac{1}{x'} \sin\left(x' - \frac{l\pi}{2}\right)$ $\therefore$ where $x' = \sqrt{\epsilon_r}\, x$

$$\lim_{\epsilon_r \to \infty} B_l = \lim_{\epsilon \to \infty} \frac{x' \cdot j_l'(x)}{j_l(x')}$$

$$= \lim_{\epsilon \to \infty} \left[x' \cdot \cot\left(x' - \frac{l\pi}{2}\right) - 1 \right]$$

$$= \infty$$

$$\lim_{\epsilon_r \to \infty} B_l' = \lim_{\epsilon \to \infty} \frac{1}{\epsilon_r}\left(x' \frac{j_l'(x')}{j_l(x')} + 1 - \epsilon_r \right)$$

$$= \lim_{\epsilon_r \to \infty} \left[\frac{x'}{\epsilon_r} \cot\left(x' - \frac{l\pi}{2}\right) - 1 \right]$$

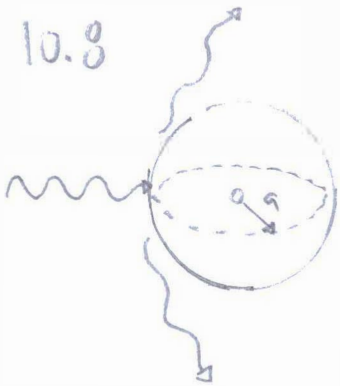
$$= -1$$

$$\tan\delta_l = \frac{j_l(x)}{n_l(x)} \quad \text{and} \quad \tan\delta_l' = \frac{x j_l'(x) + j_l(x)}{x n_l'(x) + n_l(x)}$$

$$= \frac{\frac{d}{dx} x \cdot j_l(x)}{\frac{d}{dx} x \cdot n_l(x)}$$

(10.69) "Scattering Phase Shifts"

10.8



"Scattering of a plane wave by a nonpermeable sphere"

Longwavelength limit: $ka \ll 1$

Skin depth: $\delta < a$

a) (8.11) "Just outside small tangential field"

$$E_{\parallel} = -\sqrt{\frac{\mu_c \omega}{2\sigma}}(1-i)(n \times H_{\parallel})$$

If $Z_s = \sqrt{\frac{\mu_c \omega}{2\sigma}}(1-i)$

$$= \frac{\delta \cdot \mu_c \cdot \omega}{2}(1-i)$$

Skin Depth: $\delta = \sqrt{2/\mu_0 \sigma \omega}$

$$\frac{Z_s}{Z_0} = \frac{k\delta}{2} \cdot \frac{\mu_c}{\mu_0}(1-i)$$

$$= \frac{k\delta}{2}(1-i) \quad \ldots \text{ in a nonpermeable conductor}$$

b) (10.65) "Bessels at Boundary condition"

$$j_\ell + \frac{\alpha_\pm(\ell)}{2} h_\ell^{(1)} = i\left(\frac{Z_s}{Z_0}\right)\frac{1}{x}\frac{d}{dx}\left[x\left(j_\ell + \frac{\alpha_\pm(\ell)}{2} h_\ell^{(1)}\right)\right]$$

$$j_\ell + \frac{\beta_\pm(\ell)}{2} h_\ell^{(1)} = i\left(\frac{Z_0}{Z_s}\right)\frac{1}{x}\frac{d}{dx}\left[x\left(j_\ell + \frac{\beta_\pm(\ell)}{2} h_\ell^{(1)}\right)\right]$$

If $\ell = 1$, then

(10.69) "Real scattering coefficients"

$$\alpha_\pm(\ell) \simeq \frac{-2i(ka)^{2\ell+1}}{(2\ell+1)[(2\ell+1)!!]}\left[\frac{x - i(\ell+1) Z_s/Z_0}{x + i \cdot \ell \cdot Z_s/Z_0}\right]$$

$$\alpha_\pm(1) \simeq -\frac{2i}{3}(ka)^3\left[\frac{x - i(2)\frac{k\delta}{2}(1-i)}{x + i\frac{k\delta}{2}(1-i)}\right]$$

$$\simeq -\frac{2i}{3}(ka)^3\left[\frac{(1-\delta/a) - i\delta/a}{1 + \delta/2a + i\delta/2a}\right] \quad \ldots \text{if } x = ka$$

$$\beta_\pm(1) \cong \frac{4i}{3}(ka)^3 \left| \frac{(1-\delta/a) + i\,\delta/a}{1+\delta/2a + i\delta/2a} \right|$$

$$\cong 2\,\alpha_\pm(1)$$

$$\alpha_\pm(l) = -2\,\frac{j_l - i\frac{Z_s}{Z_0}\frac{1}{x}\frac{d}{dx}(x j_l)}{h_l^{(1)} - i\frac{Z_s}{Z_0}\frac{1}{x}\frac{d}{dx}(x h_l^{(1)})}$$

$$\alpha_\pm(1) \cong -2\,\frac{\frac{x}{3} - i\frac{2}{3}\frac{Z_s}{Z_0}}{\frac{-i}{x^2} + \frac{1}{x^2}\frac{Z_s}{Z_0}}$$

Identity:

$$j_1 = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$h_1 = -\frac{e^{ix}}{x}\left(1+\frac{i}{x}\right)$$

If $x \ll 1$, then

$$j_1 \approx \frac{x}{3}\ ;\ \frac{1}{x}\frac{d}{dx}(x j_1) \approx \frac{2}{3}$$

$$h_1 \approx \frac{-1}{x^2}\ ;\ \frac{1}{x}\frac{d}{dx}(x \cdot h_1) \approx \frac{i}{x}$$

$$\cong -\frac{2ix^3}{3}\frac{(x - 2i\,Z_s/Z_0)}{(x + i\,Z_s/Z_0)}$$

$$\beta_\pm(1) \cong -\frac{2i}{3}x^3\,\frac{x - 2i\,Z_s/Z_0}{x + i\,Z_s/Z_0}$$

$$\cong \frac{4i}{3}x^3\,\frac{1 + i/2 \cdot Z_s/Z_0}{1 - i \cdot Z_s/Z_0}$$

$$\cong \frac{4i}{3}(ka)^3$$

c) (10.63) "Scattering cross section - Incident"

$$\frac{d\sigma}{d\Omega} = \frac{\pi}{2k^2}\left|\sum_i^1 \sqrt{2l+1}\left(\alpha_\pm(l)\,X_{l,\pm1} \pm i\beta_\pm(l)\,\hat{n}\times X_{l,\pm1}\right)\right|^2$$

$$= \frac{3\pi}{2k^2}\left|\alpha_\pm(1)\,X_{l,\pm1} \pm i\,\beta_\pm(l)\,\hat{n}\times X_{l,\pm1}\right|^2$$

$$\frac{d\sigma_{scat}}{d\Omega} = \frac{9}{32k^2}\left|(\alpha_\pm(1) + \beta_\pm(1)\cos\theta)\hat{\theta} \pm i(\beta_\pm(1) + \alpha_\pm(1)\cos\theta)\cdot\hat{\phi}\right|^2$$

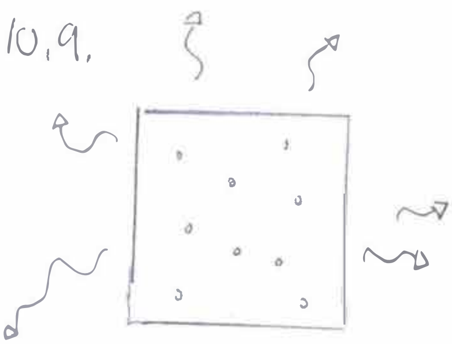
$$= \frac{9}{32k^2}\Big[\left(|\alpha_\pm(1)|^2 + |\beta_\pm(1)|^2\right)(1+\cos^2\theta) + 4\mathrm{Re}(\alpha_\pm(1)\cdot\beta_\pm(1)^*)\cos\theta\Big]$$

$$\cong 3\pi(k\delta)a^2 \quad \ldots \text{if } \delta \simeq 0$$

$$\cong 3\pi(k\delta)\, a^2 \left(\frac{2}{5}\right) \quad \ldots \text{ if } \delta \approx a$$

When penetration depth is the entire radius, then absorption scales by $\frac{2}{5}$

10.9.



"light by a gas ... scattered ... to be "white" "

(Citation: Lord Rayleigh, 1914)

a) (10.31) "Scattering amplitude"

$$\frac{\epsilon^* \cdot A_{sc}^{(1)}}{D_0} = \frac{k^2}{4\pi}\int d^3x\, e^{iqx} \left\{ \epsilon^* \cdot \epsilon_0 \frac{\delta \epsilon(x)}{\epsilon_0} + (n \times \epsilon^*) \cdot (n_0 \times \epsilon_0) \frac{\delta \mu(x)}{\mu_0} \right\}$$

where $q = k(n_0 - n)$

$$q^2 = k^2(2 - 2\cos\theta)$$
$$= (2k)^2 \cdot \sin^2\theta/2$$

$$\frac{\delta\mu}{\mu_0} = 0, \quad \frac{\delta \epsilon}{\epsilon_0} = \begin{cases} \epsilon_r - 1 & r < a \\ 0 & r > a \end{cases}$$

$$\frac{\epsilon^* \cdot A_{sc}^{(1)}}{D_0} = \frac{k^2}{4\pi}(\epsilon_r - 1)(\epsilon^* - \epsilon_0)\int_{r<a} e^{iqx} \cdot d^3x$$

$$= \frac{k^2}{4\pi}(\epsilon_r - 1)(\epsilon^* - \epsilon_0)\int_{r<a} e^{iqr\cos\gamma} \cdot r^2\, dr\, d\cos\gamma\, d\phi$$

$$= \frac{k^2}{4\pi}(\epsilon_r - 1)(\epsilon^* - \epsilon_0)\int_0^{2\pi}\int_0^{a}\int_{-1}^{1} e^{iqr\cos\gamma} \cdot r^2\, dr\, d\cos\gamma\, d\phi$$

$$= \frac{k^2}{4\pi}(\epsilon_r - 1)(\epsilon^* - \epsilon_0)\frac{4\pi}{q}\int_0^a r \sin(qr)\, dr$$

$$= \frac{k^2}{4\pi}(\epsilon_r - 1)(\epsilon^* - \epsilon_0)\frac{4\pi}{q}\left[\sin(qa) - qa\cos(qa)\right]$$

$$\alpha_{\pm}(1) \cong \frac{-2i}{3}(ka)^3\left(1 - \frac{3\delta}{2a}(1+i)\right)$$

$$\beta_{\pm}(1) \cong \frac{4i}{3}(ka)^3$$

$$\frac{d\sigma}{d\Omega} \cong \frac{1}{8k^2}(ka)^6\left[\left(\left|1 - \frac{3\delta}{2a}(1+i)\right|^2 + 4\right)(1+\cos^2\theta) - 8\cdot \mathrm{Re}\left(1 - \frac{3\delta}{2a}(1+i)\right)\right]$$

If $\delta = 0$, $= k^4 a^6\left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta\right]$

∴ this is equation 10.16 with zero penetration depth.

d) (10.6) "absorption cross section"

$$\sigma_{sc} = \frac{\pi}{2k^2}\sum_l (2l+1)\left[|\alpha(l)|^2 + |\beta(l)|^2\right]$$

$$\sigma_{abs} = \frac{\pi}{2k^2}\sum_l (2l+1)\left[2 - |\alpha(l)+1|^2 - |\beta(l)+1|^2\right]$$

$$= -\frac{3\pi}{2k^2}\left[\mathrm{Re}\left(2\alpha(1) + 2\beta(1)\right) + |\alpha(1)|^2 + |\beta(1)|^2\right]$$

Longwavelength limit $(ka \ll 1)$:

$$= -\frac{3\pi}{k^2}\left(\alpha(1) + \beta(1)\right)$$

$$\cong -2\pi k a^3 \frac{1 - \frac{\delta}{\alpha} - i\frac{\delta}{\alpha}}{1 + \frac{\delta}{2\alpha} + i\frac{\delta}{2\alpha}}$$

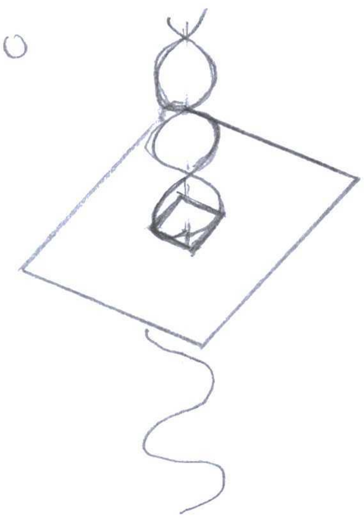
$$\cong 3\pi k\delta a^2\left[1 + \frac{\delta}{a} + \frac{\delta^3}{2a^2}\right]^{-1}$$

$$= \frac{k^4 a^6 |\epsilon_r - 1|^2}{2} \frac{2\pi}{(ka)^2} \int_0^{\infty} j_1(\xi) \frac{d\xi}{\xi}$$

$$= \frac{k^4 a^6 |\epsilon_r - 1|^2}{2} \cdot \frac{\pi}{2(ka)^2}$$

$$= \frac{\pi a^2}{4} (ka)^2 |\epsilon_r - 1|^2$$

10.10



"aperature in a perfectly conducting plane.. dipole radiation ... incident plane wave"

(10.101) "Diffracted Electric Field"

$$E_{diff}(x) = \frac{1}{2\pi} \nabla \times \int_{aperature} (n \times E) \frac{e^{ikR}}{R} da'$$

$$\cong \frac{1}{2\pi} \nabla \times \int_{aperature} (n \times E) \frac{e^{ikr}}{r} \cdot e^{-ikx} da'$$

$$\cong \frac{1}{2\pi} \nabla \times \int_{aperature} (n \times E) \frac{e^{ikr}}{r} (1 - ikx) da$$

Exponential:

$$e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \ldots$$

$$\cong \frac{ik}{2\pi} \frac{e^{ikr}}{r} k \times \int (n' \times E)(1 - ikx) da$$

(9.72) "Effective Electric and Magnetic Dipole Moments"

$$p = \epsilon n \int (x \cdot E_{tan}) da$$

$$m = \frac{2}{i\omega\mu} \int (n \times E_{tan}) da$$

$$E_{diff}(x) \cong \frac{-Z_0}{4\pi} k^2 \frac{e^{ikr}}{r} (k \times m) + \frac{1}{2\pi} \frac{e^{ikr}}{r} k \times \int (n' \times E)(k_0 x) da'$$

$$\cong \frac{-Z_0}{4\pi} k^2 \frac{e^{ikr}}{r} (k \times m) - \frac{1}{4\pi} \frac{e}{r} k \times (n \times k) \int x \cdot E da$$

$$\cong \frac{-Z_0}{4\pi} k^2 \frac{e^{ikr}}{r} (k \times m) - \frac{1}{4\pi} \frac{e^{ikr}}{r} k \times (k \times n) \int x' \cdot E da$$

$$= \frac{(ka)^2}{q}(\epsilon_r - 1)(\epsilon - \epsilon_0)\frac{\sin(qa) - qa\cos(qa)}{(qa)^2}$$

$$= \frac{(ka)^2}{q}(\epsilon_r - 1)(\epsilon - \epsilon_0) \circ j_1(qa)$$

Third time identity

$$j_1(qa) = \frac{\sin(qa)}{(qa)^2} - \frac{\cos(qa)}{qa}$$

(10.28) "Correspondence with differential cross section"

$$\frac{d\sigma}{d\Omega} = \frac{|\epsilon^* \circ A_{sc}|^2}{|D_0|^2}$$

$$= k^4 a^6 |\epsilon_r - 1|^2 \left(\frac{j_1(qa)}{qa}\right)^2 |\epsilon \circ \epsilon_0|^2$$

$$= k^4 a^6 |\epsilon_r - 1|^2 \left(\frac{j_1(qa)}{qa}\right)^2 \frac{1 + \cos^2\theta}{2}$$

If $ka \ll 1$, $qa \ll 1$ and $j_1(qa) \approx \frac{qa}{3}$

$$= \frac{k^4 a^6 |\epsilon_r - 1|^2}{9} \frac{1 + \cos^2\theta}{2}$$

If $qa \approx 0$, then

$$\frac{d\sigma}{d\Omega} = k^4 a^6 |\epsilon_r - 1|^2 \left(\frac{j_1(qa)}{qa}\right)^2 |\epsilon \circ \epsilon_0|^2$$

$$\sigma = k^4 a^6 |\epsilon_r - 1|^2 \int \left(\frac{j_1(qa)}{qa}\right)^2 \frac{1 + \cos^2\theta}{2} \sin\theta \, d\Omega$$

$$= \frac{k^4 a^6 |\epsilon_r - 1|^2}{2} \int_0^\pi \left(\frac{j_1(qa)}{qa}\right)^2 \theta \, d\Omega$$ ... small angle identity.

$$\simeq \frac{-Z_0}{4\pi} k^2 \frac{e^{ikr}}{r}(k \times m) - \frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr}}{r} k \times (k \times p)$$

b) The problem wants magnetic fields rather than electric:

(1.1a) "Faradays Law"

$$\nabla \times E + \frac{\partial B}{\partial t} = 0$$

$$n' \cdot (\nabla \times E) = i\omega (n' \cdot B)$$

$$i\omega \int x \cdot (n \cdot B) da' = \int x [n' \cdot (\nabla \times E)] da'$$

$$= \int x \epsilon_{ijk} n' \frac{\partial}{\partial j} E_k da'$$

$$= -\int \frac{\partial}{\partial i}(x') \epsilon_{ijk} n' E_k da$$

$$= \int n' \times E da'$$

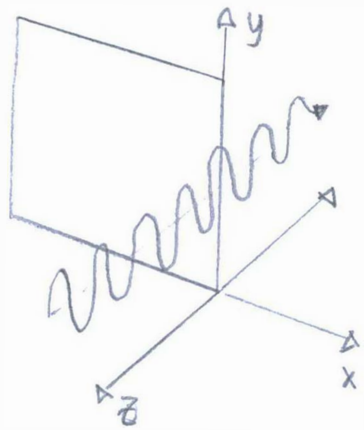
(9.72) "Effective Electric Moment"

$$m = \frac{2}{i\omega\mu} \int (n \times E_{tan}) da$$

$$= \frac{2}{i\mu\varphi} \int x'(n' \cdot B) da'$$

The Smythe-Kirchoff equation is unique to diffraction at an operature in the same dimension. Fourier transforms works at an operature, but between space and time domain.

10.11



"A perfectly conducting flat screen occupies half the x-y plane"

$\therefore$ a plane wave incident

a) Kirchoff's Approximation (pg 480)

If light (ψ) after incidence

1) ψ and $d\psi/dn$ vanish everywhere except on S, except in openings. (Light disappears everywhere but the aperature)

2) The values ψ and $d\psi/dn$ in the openings are equal to the values of incident wave in the absence of any obstacle. (The wave continues)

(10.109) "Smythe-Kirchoff Equation"

$$E = \frac{i e^{ikr}}{2\pi r} \, k \times \int_{S_1} n' \times E(x') e^{-ik \cdot x} da$$

$z >> x$, $\sqrt{kz} >> 1$

(10.108) "Diffracting System"

$$\psi(x) = -\frac{e^{ikr}}{4\pi r} \int_{S_1} e^{-ik \cdot x'} \left[n \cdot \nabla \psi(x') + ik \cdot n \psi(x') \right] da'$$

(10.79) "Kirchoff approximation generalized"

$$\psi(x) = -\frac{1}{4\pi} \int_{S_1} \frac{e^{ikr}}{R} n' \cdot \left[\nabla' \psi + ik\left(1 + \frac{i}{kR}\right) \frac{k}{R} \psi \right] da'$$

$$= \frac{k}{4\pi i} \int_{S_1} \psi(x') \frac{e^{ikR}}{R} \left(1 + \frac{i}{kR}\right) \frac{n \cdot R}{R} da'$$

$\therefore$ when $\psi = \sqrt{I_0}$; $\nabla \psi = 0$

b) $I = |\psi|^2$

$$= \frac{I_0}{2} i \cdot \left(\cos(kz-\omega t) + i\sin(kz-\omega t)\right)^2$$

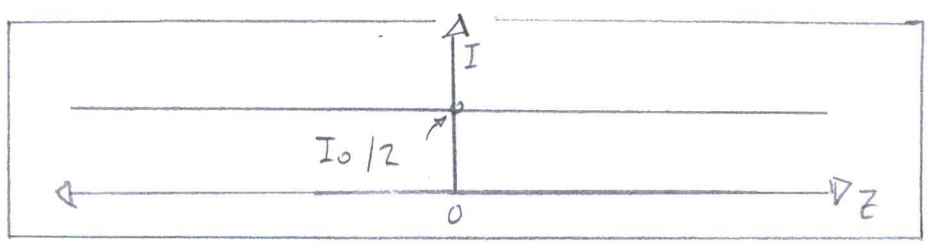
$$\doteq \frac{I_0}{2} i \left[C(\infty) + iS(\infty) - C(0) - S(0) + C(0) + iS(0) + C(\xi) - iS(\xi) \right]^2$$

$$= \frac{I_0}{2} \left[\left(C(\xi) + \tfrac{1}{2}\right)^2 + \left(S(\xi) + \tfrac{1}{2}\right)^2 \right]$$

At large ξ, the intensity in the previous function approches zero.

If $X = 0$, then $I = |\psi(0,0,z,t)|^2$

$$= \frac{I_0}{2}$$

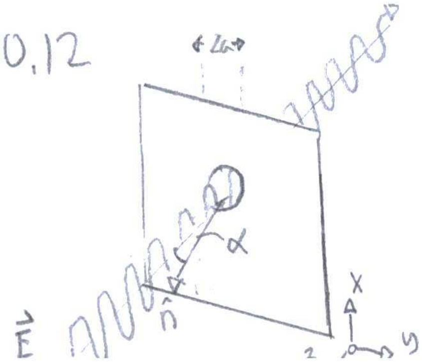


c) Same as part (a, b), except

$$t' = \sqrt{\frac{k}{2\pi}}\, Y' \ ; \ dY' = \sqrt{\frac{2\pi}{k}}\, dt'$$

$$t = \sqrt{\frac{k}{2\pi}}\, X' \ ; \ dX' = \sqrt{\frac{2\pi}{k}}\, dt$$

10.12



"linearly polarized light incident on a circular opening... making an angle α with the normal to the screen"

$$= \frac{k}{4\pi i}\int \sqrt{I_0}\,\frac{e^{ikR}}{R}\cdot da'$$

$\circ\circ\circ$ when $R = \sqrt{(x'-x)^2 + Y'^2 + Z^2} \cong \frac{(x'-x)^2}{2Z} + \frac{Y'^2}{2Z} + Z$

$$\Psi(X,y,Z,t) = \sqrt{I_0}\, e^{-i\omega t}\cdot \frac{k}{4\pi i}\int \frac{e^{ikR}}{R}\, da'$$

$$\Psi(X,0,Z,t) = \sqrt{I_0}\, e^{-i\omega t}\,\frac{k}{4\pi i}\int_0^{\infty}\int_{-\infty}^{\infty} \frac{e^{ik[Z+(x'-x)^2/2Z + Y'^2/2Z]}}{Z + \frac{(x'-x)^2}{2Z} + \frac{Y'^2}{2Z}}\, dY' d$$

$$= \sqrt{I_0}\, e^{ikZ - i\omega t}\left(\frac{1+i}{2i}\right)\int_{\xi}^{\infty} e^{\frac{ikx'^2 - 2ikx'x + ikx^2}{2Z}}\, dx'$$

$$\cong \sqrt{I_0}\, e^{ikZ - i\omega t}\left(\frac{1+i}{2i}\right)\int_{\xi}^{\infty} e^{it^2}$$

"Book has a $\sqrt{\frac{2}{\pi}}$ coefficient"

Math above by:

<u>Fresnel Integral</u>: $\int_0^a e^{i2\pi t^2}\, dt$

$$= \int_0^a \cos(2\pi t^2) + i\sin(2\pi t^2)\, d$$

$$= C(u) + i\, S(u)$$

$$C(\infty) = 1/2 \;;\; S(\infty) = 1/2$$

<u>U-substitution</u>: $t' = \sqrt{\frac{k}{4Z\pi}}\, Y' \;;\; dY' = \sqrt{\frac{4Z\pi}{k}}\, dt$

$$t = \sqrt{\frac{k}{4Z\pi}}\, X' \;;\; dX' = \sqrt{\frac{4Z\pi}{k}}\, dt$$

$$\xi = (k/2Z)^{1/2} X$$

$$= 0$$

a) (10.101) "Kirchoff Equation"

$$E_{diff}(X) = \frac{1}{2\pi}\nabla \times \int_{aperture} (n\times E)\frac{e^{ikR}}{R}\, da'$$

becomes...

(10.109) "Smythe-Kirchoff Formula"

$$E(X) = \frac{i e^{ikr}}{2\pi r}\, k \times \int_{S_1} n\times E(x')\, e^{-ik x'}\, da$$

$$k_0 = k_x + k_z$$

$$= k(\hat{x}\sin\alpha + \hat{z}\cos\alpha)$$

$$E(X) = \frac{ie^{ikr}}{2\pi r}\, k\times \int_{S_1} z\times E(x)\, e^{-ikx}\, da$$

$$= \frac{ie^{ikr}}{2\pi r}\, k\times\int_{S_1} -E_0 \hat{x}\, e^{ik(x'\sin\alpha + z'\cos\alpha)} \cdot e^{-ikx}\, da$$

$$= -\frac{iE_0 e^{ikr}}{2\pi r}(k\times\hat{x})\int e^{ikx\sin\alpha}\cdot e^{-i(k_x x + k_y y)}\cdot da$$

is (10.112):

$$= -\frac{iE_0e^{ikr}}{2\pi r}(k\times\hat{x})$$

$$\times \int_0^a\int_0^{2\pi} \rho\, e^{ik\rho[\sin\alpha\cos\beta - \sin\theta\cos(\phi-\beta)]}\, d\beta\, d\rho$$

Polar coordinates

$x' = \rho\cos\beta$

$y' = \rho\sin\beta$

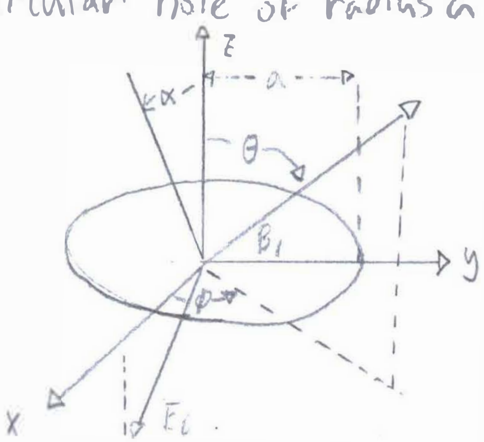
Spherical Wavevector

$k_x = k\sin\theta\cos\phi$

$k_y = k\sin\theta\sin\phi$

$k_z = k\cos\theta$

Figure 10.13: Diffraction by a circular hole of radius a



$$= \frac{ie^{ikr}}{r} a^2 E_0 \cos\alpha (k \times \epsilon_2) \frac{J_1(ka\xi)}{ka\xi} \quad \therefore (10.113)$$

$\therefore$ when $\xi = (\sin^2\theta + \sin^2\alpha - 2\sin\theta\sin\alpha\cos\phi)^{1/2}$

$\therefore$ by $\frac{1}{2\pi}\int_0^{2\pi} d\beta' e^{-ik\rho\xi\cos\beta'} = J_0(k\rho\xi)$

$$(k \times \hat{x}) = \hat{y} k_z - z k_y$$

$$= k(y\cos\theta - z\sin\theta\sin\phi)$$

$$E = -\frac{iE_0 e^{ikr}}{r}(ka^2)(y\cos\theta - z\sin\theta\sin\phi)\frac{J_1(ka\xi)}{ka\xi}$$

(10.114) "time-averaged differential power per unit solid angle"

$$\frac{dP}{d\Omega} = \frac{r^2}{2Z_0}|E|^2$$

$$= \frac{|E_0|^2 a^2}{8 Z_0}(ka)^2(\cos^2\theta + \sin^2\theta\sin^2\phi)\left|\frac{2J_1(ka\xi)}{ka\xi}\right|^2$$

$$= P_i \cos\alpha \frac{(ka)^2}{4\pi}(\cos^2\theta + \sin^2\theta\sin^2\phi)\left|\frac{2J_1(ka\xi)}{ka\xi}\right|^2$$

$\therefore$ when $P_i = \left(\frac{E_0}{2Z_0}\right)\pi a^2 \cos\alpha$

b) If $\epsilon_{\parallel}$, $\frac{dP}{d\Omega} = P_i \cos\alpha \frac{(ka)^2}{4\pi}(\cos^2\theta + \sin^2\theta\cos^2\phi)$

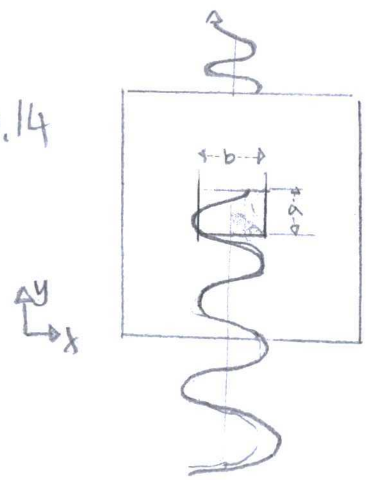
$$\cdot \left|\frac{2J_1(ka\xi)}{ka\xi}\right|^2$$

In diffraction,

$$\frac{dP}{d\Omega} = P_i \cos\alpha \frac{(ka)^2}{4\pi}\left(\frac{\cos\alpha+\cos\Theta}{2\cos\alpha}\right)^2\left|\frac{2J_1(ka\xi)}{ka\xi}\right|^2$$

The equations from parallel, perpendicular, and differential light show similar characteristics, except angles.

10.14



"a rectangular opening... a plane wave incident with long edge"

a) (10.109) "Smythe-Kirchoff relation"

$$E = \frac{ie^{ikr}}{2\pi r} k \times \int_{S_1} n' \times E(x') e^{-ikx} da$$

... when $E(x) = \epsilon_0 E_0 e^{ikz}$

$$= E_0(x\sin\beta + y\cos\beta) e^{ikz}$$

At $z=0$, $\hat{n} = \hat{z}$,

$$E = \frac{iE_0 e^{ikr}}{2\pi r} k \times \int z \times (x\sin\beta + y\cos\beta) e^{-ikx'} da'$$

$$= \frac{iE_0 e^{ikr}}{2\pi r} k \times \int_{-a/2}^{a/2} dx' \int_{-b/2}^{b/2} dy' (\hat{y}\sin\beta - \hat{x}\cos\beta) e^{-ikx'} da'$$

$$= \frac{iE_0 e^{ikr}}{2\pi r} (k_x + k_y + k_z)(y\sin\beta - x\cos\beta)$$

$$\times \int_{-a/2}^{a/2} dx' e^{-ik_x x} \int_{-b/2}^{b/2} dy\, e^{-ik_y y}$$

Unit vector Products
$x \times y = z$
$y \times z = x$
$z \times x = i$
$y \times x = -z$
$x \times z = -y$
$z \times y = -x$

$$= \frac{2iE_0 e^{ikr}}{\pi r}\Big[-\hat{x}k_z\sin\beta - \hat{y}k_z\cos\beta$$

$$+\hat{z}(k_x\sin\beta + k_y\cos\beta)\Big]\cdot\frac{\sin(k_x a/2)\sin(k_y b/2)}{k_x k_y}$$

$$\boxed{\begin{array}{l}\underline{\text{Spherical}}\\ \underline{\text{Wavevectors}}\\ k_x = k\sin\theta\cos\phi\\ k_y = k\sin\theta\sin\phi\\ k_z = k\cos\theta\end{array}}$$

$$= \frac{2iE_0 e^{ikr}}{\pi k r}\Big[-x\cos\theta\sin\beta - y\cos\theta\cos\beta$$

$$+z\sin\theta\sin(\phi+\beta)\Big]\frac{\sin\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{\sin\theta\sin\phi}$$

$$\cdot\frac{\sin\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\sin\theta\sin\phi}$$

(9.21) "Time-averaged power radiated per unit solid angle"

$$\frac{dP}{d\Omega} = \frac{1}{2}\mathrm{Re}\left[r^2 n\cdot E\times H^*\right] \quad \ldots \text{if} \quad H = E/Z_0$$

$$= \frac{1}{2Z_0}\,\frac{4|E_0|^2}{\pi^2k^2}\left[\cos^2\theta + \sin^2\theta\sin^2(\phi+\beta)\right]$$

$$\times\frac{\sin^2\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{(\sin\theta\cos\phi)^2}\cdot\frac{\sin^2\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{(\sin\theta\sin\phi)^2}$$

$$\ldots \text{if} \quad P_i = \frac{|E_0|^2}{2Z_0}ab$$

$$= \frac{P_i}{\pi^2}\left[\cos^2\theta + \sin^2\theta\sin^2(\phi+\beta)\right]$$

$$\times\frac{\sin^2\left(\frac{ka}{2}\sin\theta\cos\phi\right)}{\frac{ka}{2}(\sin\theta\cos\phi)^2}\cdot\frac{\sin^2\left(\frac{kb}{2}\sin\theta\sin\phi\right)}{\frac{kb}{2}(\sin\theta\cos)^2}$$

$$\cong \frac{P_i}{\pi^2}\frac{ka}{2}\frac{kb}{2}\left[\cos^2\theta + \sin^2\theta \sin^2(\phi+\beta)\right]$$

b) (10.108) Kirchoffs Approximation... by scalars

$$\psi = -\frac{e^{ikr}}{4\pi r}\int_{S_1}\left[n'\cdot\nabla\psi + ik\cdot n'\psi\right]e^{-ikx}da$$

If $\psi(x') = \psi_0 e^{ikx}$

$$n\cdot\nabla\psi = \hat{z}\cdot\nabla\psi$$
$$= \frac{\partial}{\partial z}\psi$$
$$= ik\psi_0 e^{ikx}$$

$$\text{So, } \psi = -\frac{e^{-ikr}}{4\pi r}\int(ik\psi_0 + ik_z\psi_0)e^{ikx'}da'$$

$$= -\frac{i\psi_0 e^{ikr}}{4\pi r}(k+k_z)\int_{-a/2}^{a/2}dx' e^{-ik_x\cdot x}\int_{-b/2}^{b/2}dy' e^{-ik_y y}$$

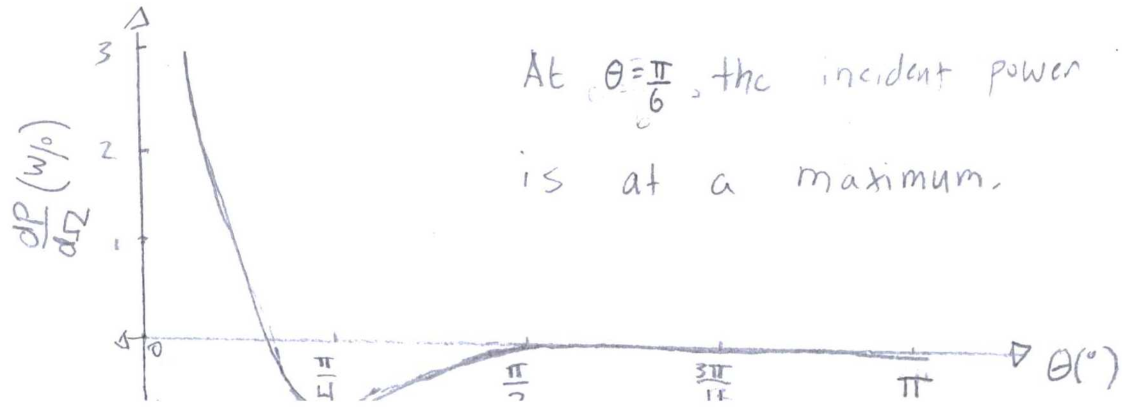
$$= -\frac{i\psi_0 e^{ikr}}{\pi k r}(1+\cos\theta)\frac{\sin^2(\frac{ka}{2}\sin\theta\cos\phi)}{\frac{ka}{2}(\sin\theta\cos\phi)^2}\frac{\sin^2(\frac{kb}{2}\sin\theta\sin\phi)}{\frac{kb}{2}(\sin\theta\sin\phi)^2}$$

$$= -\frac{i\psi_0 e^{ikr}}{\pi k r}(1+\cos\theta)\sinh\left(\frac{ka}{2}\sin\theta\cos\phi\right)\cdot\sinh\left(\frac{kb}{2}\sin\theta\cos\phi\right)$$

c) When $b=a$, $\beta=45^\circ$, $ka=4\pi$, $\phi=0$,

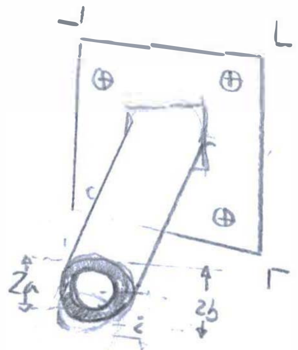
$$\left.\frac{dP}{d\Omega}\right|_{vector} = \frac{P_i}{\pi^2}\left[\frac{1}{2}(1+\cos^2\theta)\right]\frac{\sin^2(2\pi\sin\theta)}{\sin^2\theta}$$

$$\left.\frac{dP}{d\Omega}\right|_{scalar} = \frac{P_i}{\pi^2}\left[\cos^4(\theta/2)\right]\frac{\sin^2(2\pi\sin\theta)}{\sin^2\theta}$$



At $\theta=\frac{\pi}{6}$, the incident power is at a maximum.

10.15



"Cylindrical coaxial transmission line ... infinite copper flange"

a) (Problem 8.2) $E = \frac{V}{\ln(b/a)} \frac{\hat{\rho}}{\rho}$ $\quad cB = \frac{V}{\ln(b/a)} \frac{\hat{\Phi}}{\rho}$

(10.109) "Vector expression"

$$E(x) = \frac{ie^{ikr}}{2\pi r} k \times \int_{S_1} n \times E(x') e^{-ik x'} da'$$

$$= \frac{ie^{ikr}}{2\pi r} \frac{V}{\ln(b/a)} k \times \int \frac{e^{ikr}}{\rho} \hat{\phi} \, da'$$

$$k = k(\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z})$$

$$\hat{r} = \rho'(\cos\phi'\,\hat{x} + \sin\phi\,\hat{y})$$

$$k \cdot r = k\rho'\sin\theta\cos(\phi' - \phi) \qquad \phi' = -\sin\phi' X + \cos\phi' y$$

$$E(r) = \frac{ie^{ikr}}{2\pi r} \frac{V}{\ln(b/a)} k \times \int_a^b d\rho' \int_0^{2\pi} d\phi' \cdot e^{-ik\rho'\sin\theta\cos(\phi' - \phi)} \cdot (-\sin\phi X + \cos\phi y)$$

Citation about Identities	Digital Library of Mathematical Functions [dlmf.nist.gov]
	$2\pi i^m J_m(x) = \int_0^{2\pi} d\phi\, e^{i(x\cos\phi - m\phi)}$
	$J_{-m}(x) = (-1)^m J_m(x)$; $J_m(-x) = (-1)^m J_m(x)$

$$E(r) = \frac{ie^{ikr}}{2\pi r} \frac{V}{\ln(b/a)} k \times \int_a^b d\rho' \Big[\int_0^{2\pi} e^{ik\rho'\sin\theta\cos\phi} \cdot \sin(\phi + \phi')\, d\phi$$

$$+ \int_0^{2\pi} e^{ik\rho\sin\theta\cos\phi} \cdot \cos(\phi + \phi')\, d\phi' \Big]$$

(8.47) "Poynting Vector"

$$\langle S \rangle = \frac{|E|^2}{2Z_0}\hat{k} = \frac{V^2}{8Z_0 \ln^2(b/a)} \frac{\{J_0(kb\sin\theta) - J_0(ka\sin\theta)\}^2}{\sin^2\theta} \frac{\hat{k}}{r^2}$$

(9.21) "time-averaged Power radiated ... by oscillating dipole"

$$\frac{dP}{d\Omega} = r^2 \langle S \rangle \cdot \hat{k}$$

$$= \frac{V^2}{8Z_0 \ln^2(b/a)} \frac{\{J_0(kb\sin\theta) - J_0(ka\sin\theta)\}^2}{\sin^2\theta}$$

Power:

$$P = \int \frac{V^2}{8Z_0 \ln^2(b/a)} \frac{\{J_0(kb\sin\theta) - J_0(ka\sin\theta)\}^2}{\sin^2\theta} d\Omega$$

$$= \frac{V^2}{8Z_0 \ln^2(b/a)} \int_0^{\pi/2} \int_0^{2\pi} \frac{\{J_0(kb\sin\theta) - J_0(ka\sin\theta)\}^2}{\sin^2\theta} \sin\theta \, d\phi \, d\theta$$

$$= \frac{\pi V^2}{4Z_0 \ln^2(b/a)} \int_0^{\pi/2} \frac{\{J_0(kb\sin\theta) - J_0(ka\sin\theta)\}^2}{\sin^2\theta} \sin\theta$$

If $kb \ll 1$,

$$P_{rad} \cong \frac{k^4 V^2 (b^2 - a^2)^2}{96 Z_0 \ln^2(b/a)}$$

$$P_{trans} \cong \frac{1}{2} \int (E \times H) \cdot z \, da$$

$$\cong \frac{\pi V^2}{Z_0 \ln^2(b/a)}$$

$$\frac{P_{rad}}{P_{trans}} = \frac{k^4 (b^2 - a^2)^2}{96\pi \ln(b/a)} \ll 1$$

$$= \frac{i e^{ikr}}{2\pi r} \frac{V \cdot k}{\ln(b/a)} \times \int_a \Bigg[\int_0^{2\pi} \frac{1}{2i} e^{-ik\rho \sin\theta \cos\phi} \left\{ e^{i(\phi+\phi')} - e^{-i(\phi+\phi')} \right\} d\phi$$

$$+ \int_0^{2\pi} \frac{1}{2} e^{ik\rho \sin\theta \cos\phi} \left\{ e^{i(\phi+\phi')} + e^{-i(\phi+\phi')} \right\} d\phi$$

$$= \frac{i e^{ikr}}{2\pi r} \frac{V \cdot k}{\ln(b/a)} \times \int_a^b \Bigg[\int_0^{2\pi} \frac{1}{2i} \Big\{ (2\pi) i^{-1} e^{i\phi} J_{-1}(-k\rho' \sin\theta)$$

$$- (2\pi) i^{1} e^{-i\phi} J_1(-k\rho' \sin\theta) \Big\} d\phi$$

$$+ \int_0^{2\pi} \frac{1}{2} \Big\{ (2\pi) i^{-1} e^{i\phi} J_{-1}(-k\rho' \sin\theta)$$

$$+ (2\pi) i \cdot e^{-i\phi} J_1(-k\rho' \sin\theta) \Big\} d\phi' \Bigg]$$

$$= \frac{i e^{ikr}}{2\pi r} \frac{V \cdot k}{\ln(b/a)} \times \int_a^b \Bigg[\int_0^{2\pi} -2\pi i \sin\phi \, J_1(k\rho' \sin\theta) - 2\pi i \cos\phi \, J_1(-k\rho' \sin\theta) \Bigg] d\phi$$

...when $\hat{\phi} = -\sin\theta \hat{x} + \cos\theta \hat{y}$

$$\int_a^b J_1(x)dx = \tfrac{1}{2}\left(J_0(b) - J_0(a)\right)$$

$$E(r) = \frac{i e^{ikr}}{2\pi r} \frac{V}{\ln(b/a)} k \times \int_0^a d\rho' (-2\pi i) J_1(k\rho' \sin\theta)(-\sin\phi \hat{x} - \cos\phi \hat{y})$$

$$= \frac{e^{ikr}}{r} \frac{V}{\ln(b/a)} k \times \hat{\phi} \int_a^b d\rho' J_1(k\rho' \sin\theta)$$

$$= \frac{e^{ikr}}{r} \frac{V}{\ln(b/a)} k \times \hat{\phi} \frac{J_0(kb\sin\theta) - J_0(ka\sin\theta)}{2\sin\theta}$$

$$= \frac{e^{ikr}}{r} \frac{V}{\ln(b/a)} \hat{r} \times \hat{\phi} \frac{J_0(kb\sin\theta) - J_0(ka\sin\theta)}{2\sin\theta}$$

$$= -\frac{e^{ikr}}{r} \frac{V}{\ln(b/a)} \frac{J_0(kb\sin\theta) - J_0(ka\sin\theta)}{2\sin\theta} \hat{\theta}$$

<u>Angular Distribution of Radiation:</u>

If $kb >> 1$, then

$$P = \frac{\pi r^2}{4 Z_0 \ln^2(b/a)} \int_0^{2\pi} \frac{\{J_0(kb\sin\theta) - J_0(ka\sin\theta)\}^2}{\sin\theta} d\theta$$

10.16 (10.125) "Shadow contribution to scattering"

$$\epsilon^{*} \cdot F_{sh} \cong \frac{ik}{2\pi} E_0 (\epsilon^{*} \cdot \epsilon_0) \int_{sh} e^{-ik_{\perp} \cdot x_{\perp}} d^2 x_{\perp}$$

$$\frac{d\sigma}{d\Omega} = |\epsilon^{*} \cdot f_{sh}|^2$$

$$= \frac{k^2}{(2\pi)^2} |\epsilon^{*} \cdot \epsilon_0|^2 \int_{sh} e^{ik_{\perp} \cdot x_{\perp}} \cdot d^2 x_{\perp} \cdot \int_{sh} e^{-ik_{\perp} \cdot x_{\perp}} \cdot d^2 x_{\perp}$$

$$= \frac{1}{(2\pi)^2} \int_{sh} d^2 x_{\perp} \cdot \int_{sh} d^2 x_{\perp}' \cdot e^{ik_{\perp}(x_{\perp}' - x_{\perp})} |\epsilon^{*} \cdot \epsilon_0|^2 k^2$$

Three aperature concepts:

<u>Shadow region</u>: dark region not illuminated on an aperature at <u>small</u> wavelength

<u>Smyth-Kirchoff</u>: edge of an aperature not centered at <u>large</u> wavelength

<u>Babinet's Principle</u>: Dimensions in an aperature correlate to diffraction.

Now simply "diffraction theory"

$$\sigma_{sh} = \frac{1}{(2\pi)^2}\int_{sh} d^2x_\perp \int_{sh} d^2x'_\perp \int_{sh} e^{ik(x_\perp - x_\perp')} |\epsilon^* \cdot \epsilon_0|^2 k^2 d\Omega$$

At $\theta \simeq 0^\circ$, $|\epsilon^* \cdot \epsilon_0| \simeq 1$

$$\sigma_{sh} = \frac{1}{(2\pi)^2}\int_{sh} d^2x_\perp \int_{sh} d^2x_\perp \int e^{ik_\perp(x_\perp - x'_\perp)} d^2k_\perp$$

$$= \frac{1}{(2\pi)^2}\int_{sh} d^2x_\perp \int_{sh} d^2x_\perp \cdot (2\pi)^2$$

$$= \int_{sh} d^2x_\perp \int_{sh} d^2x_\perp$$

= Area

b) (10.139) "Optical Theory"

$$\sigma_t = \frac{4\pi}{k} \text{Im}\left[\epsilon_0^* \cdot f(k = k_0)\right]$$

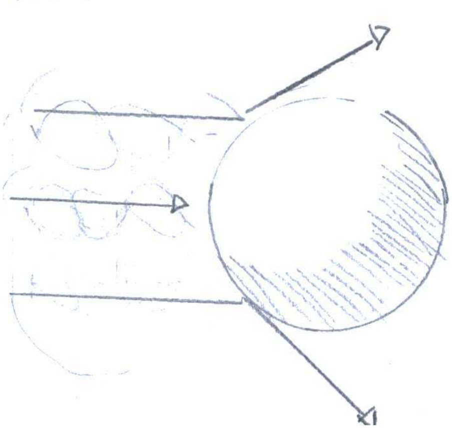
$$\cong \frac{4\pi}{k} \text{Im}\left[\epsilon_0^* \cdot f_{sh}(k_\perp = 0)\right]$$

$$\cong \frac{4\pi}{k} \text{Im}\left[\cancel{\epsilon_0^{k*}} \frac{ik}{2\pi}(\epsilon^* \cdot \epsilon_0)\int_{sh} d^2x_\perp\right]$$

$$\cong \frac{4\pi}{k} \text{Im}\left[\cancel{\epsilon_0^{i*}}(i\,\epsilon_0)\frac{ik}{2\pi} A_{sh}^{\perp}\right]$$

$$\cong 2 \cdot \epsilon_0 \cdot A_{sh}^{\perp}$$

10.17



"Linearly polarized light on a perfectly conducting sphere... in a long wavelength limit" a)(10.2) "Scattered fields"

$$E_{sc} = \frac{1}{4\pi\epsilon_0} k^2 \frac{e^{ikr}}{r} \left[(n \times p) \times n - n \times m/c \right]$$

$$H_{sc} = n \times E_{sc} / Z_0$$

So, $B_{sc} = n \times E_{sc}$

(10.1) "Incident fields"

$$E_{inc} = \epsilon_0 E_0 e^{ikn_0 \cdot x}$$

$$H_{inc} = n_0 \times E_{inc} / Z_0$$

So, $B_{inc} = n_0 \times E_{inc}$

(10.3) "Differential Scattering Cross section"

$$\frac{d\sigma}{d\Omega}(n, \epsilon, n_0, \epsilon_0) = \frac{r^2 \frac{1}{2Z_0} |\epsilon^* \cdot E_{sc}|^2}{\frac{1}{2Z_0} |\epsilon_0^* \cdot E_{inc}|^2}$$

(10.4) "Differential Cross Section"

$$\frac{d\sigma}{d\Omega}(n, \epsilon, n_0, \epsilon_0) = \frac{k}{(4\pi\epsilon_0 E_0)^2} |\epsilon^* \cdot p + (n \times \epsilon^*) \cdot m/c|^2$$

$$= k^4 a^6 |\epsilon^* \cdot \epsilon_0 - \tfrac{1}{2}(n \times \epsilon^*) \cdot (n_0 \times \epsilon_1)|^2$$

(10.14) "Parallel and Perpendicular Differential Scattering Cross Section"

$$\frac{d\sigma_{\parallel}}{d\Omega} = \frac{(ka)^4 a^2}{2} \Big\{ |\epsilon_1^* \epsilon_{01} - \tfrac{1}{2}(n \times \epsilon_1^*)(n_0 \times \epsilon_{01})|^2$$

$$+ |\epsilon_1 \cdot \epsilon_{02} - \tfrac{1}{2}(n \times \epsilon_1^*)(n_0 \times \epsilon_{02})|^2 \Big\}$$

$$= \frac{(ka)^4 a^2}{2} \Big\{ |\cos\theta - \tfrac{1}{2}|^2 - 0 \Big\}$$

$$\frac{d\sigma}{d\Omega} = \frac{(ka)^4 a^2}{2} \Big\{ |\epsilon_1^* \epsilon_{02} - \tfrac{1}{2}(n \times \epsilon_1^*)(n_0 \times \epsilon_{02})|^2$$

$$+ |\epsilon_1 \cdot \epsilon_{01} - \tfrac{1}{2}(n \times \epsilon_1^*)(n_0 \times \epsilon_{01})|^2 \Big\}$$

$$= \frac{(ka)^4 a^2}{2} \Big\{ 0 - |1 - \tfrac{1}{2}\cos\theta|^2 \Big\}$$

$$\frac{d\sigma_{TOT}}{d\Omega} = \frac{(ka)^4 a^2}{2} (\cos^2\theta - \cos\theta + \tfrac{1}{4} + 1 - \cos\theta$$

$$+ \cos^2 \frac{\theta}{4} \Big\}$$

$$\frac{d\sigma}{d\Omega}(E\text{plane}) \cong a^2 \Big[\cot^2\theta \, J_1^2(ka\sin\theta) + \frac{1}{4}$$

$$- 4\cot^2 J_1(ka\sin\theta)\sin\left(2ka\sin\frac{\theta}{2}\right) \Big]$$

$$\cong \frac{a^2}{4} \Big[4\cot^2\theta \, J_1^2(ka\sin\theta) + 1$$

$$- 4\cot^2\theta \, J_1(ka\sin\theta)\sin\left(2ka\sin\frac{\theta}{2}\right) \Big]$$

$\therefore$ from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$

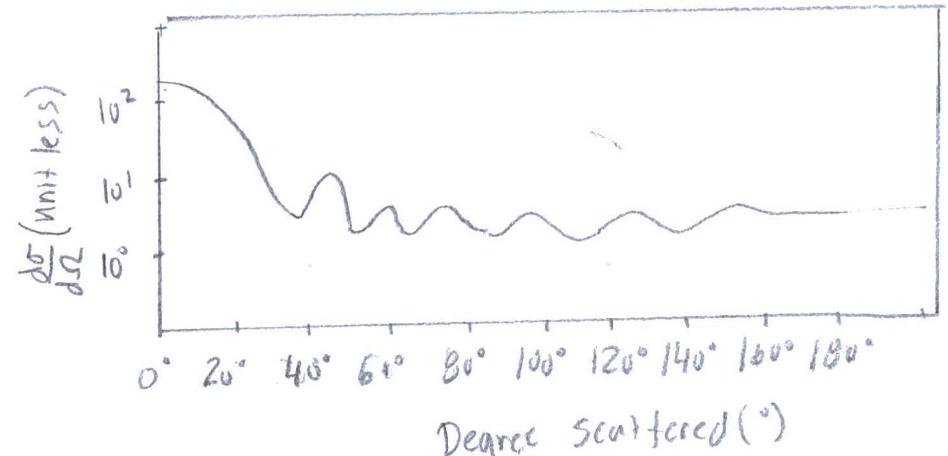
$$\frac{d\sigma}{d\Omega}(H\text{plane}) \cong \frac{a^2}{4} \Big[4\operatorname{cosec}^2\theta \, J_1^2(ka\sin\theta) + 1$$

$$- 4\operatorname{cosec}^2\theta \, J_1(ka\sin\theta)\sin\left(2ka\,\frac{\theta}{2}\right) \Big]$$

Note: Book has possible typo and above $\cot^2\theta$

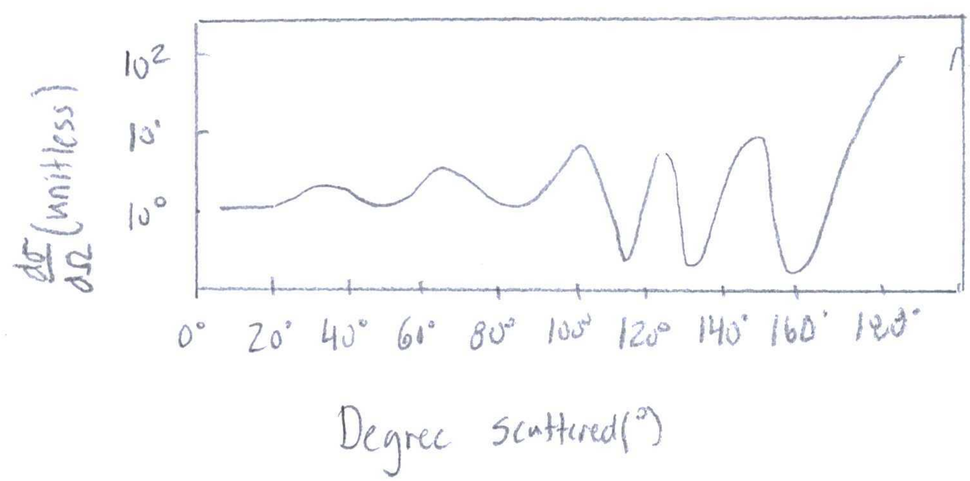
b) Citation: Kung and Wu

Figure 10.16: Experimental Results

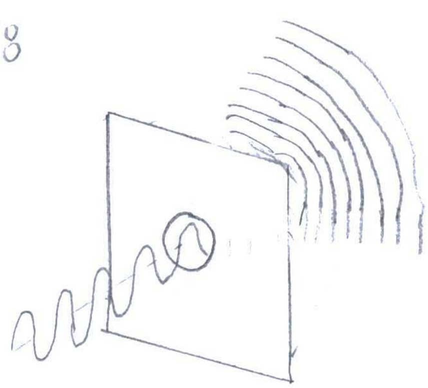


Citation: Semor, Uslenght

Figure 10.23g, pg 403



10.18



"diffraction due to a small, circular hole"

a) $E = E_0 e^{-i\omega t}$

$B = B_0 e^{-i\omega t}$

Fraunhofer zone:

(9.75) "Effective Electric and Magnetic Dipole Moments"

$$P_{eff} = \frac{4\epsilon_0 a^3}{a} E_0 \qquad m_{eff} = -\frac{8a^3}{3} H_0$$

b) If $ka \ll 1$, then

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle = \left|\frac{\delta\epsilon}{\epsilon}\right|^2 \frac{k^4 a^4 c^2}{32} \left(1+\cos^2\theta\right) \int_0^{\pi/2} \left[\frac{2J_1(q_\perp a)}{q_\perp a} \frac{\sin(qL/2)}{qL/2}\right] \sin\alpha \, d\Omega$$

$$= \left|\frac{\delta\epsilon}{\epsilon}\right|^2 \frac{k^4 a^4 c^2}{32} \left(1+\cos^2\theta\right) \left[\frac{2}{qL} \int_0^X \frac{\sin x}{x} - \left(\frac{\sin(qL/2)}{qL/2}\right)^2\right]$$

$$= \left|\frac{\delta\epsilon}{\epsilon}\right|^2 \frac{k^4 a^4 c^2}{32} \left(1+\cos^2\theta\right) \left[\frac{2}{qL} S_i(qL) - \left(\frac{\sin(qL/2)}{qL/2}\right)^2\right]$$

c) If $kL \gg 1$, then

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle = \left|\frac{\delta\epsilon}{\epsilon}\right| \frac{k^4 a^4 c^2}{32} \left(1+\cos^2\theta\right) \left[\frac{2}{qL} S_i(qL) - \left(\frac{\sin(qL/2)}{qL/2}\right)^2\right]$$

$$\sigma = \int_0^{\pi/2} \int_0^{2\pi} \left|\frac{\delta\epsilon}{\epsilon}\right| \frac{k^4 a^4 c^2}{32} \left(1+\cos^2\theta\right) \left[\frac{2}{qL} S_i(qL) - \left(\frac{\sin(qL/2)}{qL/2}\right)^2\right] \sin\theta \, d\phi \, d\theta$$

$$\cong \frac{11\pi^2}{60} \left|\frac{\delta\epsilon}{\epsilon}\right|^2 k^3 a^4 L$$

(10.2) "Electric and Magnetic fields"

$$E = -\frac{k^2}{4\pi\epsilon_0}\frac{e^{ikr}}{r}\left[\hat{n}\times(n\times p_{eff}) + n\times m_{eff}/c\right]$$

$$H = \frac{1}{Z_0}\hat{n}\times E$$

$$\text{So, } E = -\frac{k^2 a^3}{3\pi}\frac{e^{ikr}}{r}\left[\hat{n}\times(E_0\times\hat{n}) + 2c\,\hat{n}\times(\mu_0 H_0)\right]$$

$$= -\frac{k^2 a^3}{3\pi}\frac{e^{i(kr-wt)}}{r}\left[\frac{\vec{k}}{k}\times(E_0\times\frac{\hat{k}}{k}) + 2c\,\frac{\vec{k}}{k}\times B_0\right]$$

... when $B_0 = \mu_0 H_0$, $\hat{n} = \hat{k} = \vec{k}/k$

b) $$\frac{dP}{d\Omega} = r^2\frac{dP}{da}$$

$$= r^2\hat{n}\cdot S$$

$$= \frac{r^2}{2}\hat{n}\cdot(E\times H^*)$$

$$= \frac{r^2}{2Z_0}|E|^2$$

$$= \frac{k^4a^6}{18\pi^2 Z_0}|\hat{n}\times(E_0\times\hat{n} + 2cB_0)|^2$$

$$= \frac{k^4a^6}{18\pi^2 Z_0}\left(|\hat{n}\times\hat{n}| + |2\cdot\hat{n}\times(E_0\hat{n}+2cB_0)| + |E_0\times n + 2cB_0|^2\right)$$

$$= \frac{k^4a^6}{18\pi^2 Z_0}\left(|E_0\times n + 2cB_0|^2 - 4c^2|\hat{n}\cdot B_0|^2\right)$$

$$= \frac{k^4a^6}{18\pi^2 Z_0}\left(|E_0|^2 - |n\cdot E_0|^2 + 4c^2(|B_0|^2 - |n\cdot B_0|^2) + 4c\,\mathrm{Re}(B_0\cdot(E_0\times n))\right)$$

When $E_0 = E_0\hat{z}$, then $B_0 = B_0\hat{x}$

$$\hat{n} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$$

$$n\cdot E = E_0\cos\theta\ ,\quad n\cdot B_0 = B_0\sin\theta\cos\phi$$

$$B_0^*\cdot(E_0\times n) = -B_0E_0\sin\theta\sin\phi$$

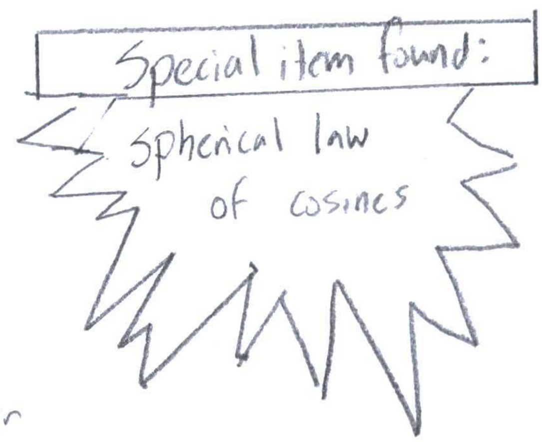
$$\frac{dP}{d\Omega} = \frac{k^4a^6}{18\pi^2 Z_0}\Big(|E_0|^2\sin^2\theta + 4c^2|B_0|^2(1-\sin^2\theta\cos^2\phi) - 4c\,\mathrm{Re}[E_0\cdot B_0^*]\sin\theta\sin\phi\Big)$$

$$P = \int\frac{k^4a^6}{18\pi^2 Z_0}\Big(|E_0|^2\sin^2\theta + 4c^2|B_0|^2(1-\sin^2\theta\cos^2\phi) - 4c\,\mathrm{Re}[E_0\cdot B_0^*]\sin\theta\sin\phi\Big)d\Omega$$

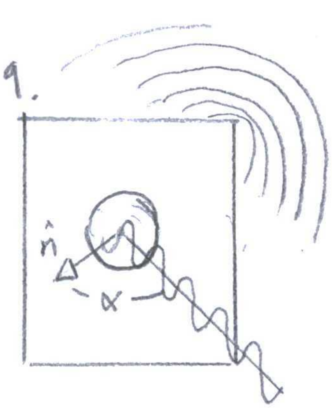
$$= \frac{k^4a^6}{18\pi^2 Z_0}\int_0^1 d\cos\theta\int_0^{2\pi}d\phi\Big(|E_0|^2\sin^2\theta + 4c^2|B_0|^2\cdot(1-\sin^2\theta\cos^2\phi) - 4c\,\mathrm{Re}(E_0\cdot B_0^*)\sin\theta\sin\phi\Big)$$

$$= \frac{k^4a^6}{9\pi Z_0}\int_0^1 d\cos\theta\Big(|E_0|^2\sin^2\theta + 2c^2|B_0|^2(2-\sin^2\theta)\Big)$$

$$= \frac{2k^4a^6}{27\pi Z_0}\left(|E_0|^2 + 4c^2|B_0|^2\right)$$



10.19.



"diffraction of a plane wave... by a small circular hole ... at an angle α"

a) (10.114) "time averaged diffracted power per unit solid angle"

$$\frac{dP}{d\Omega} = \frac{r^2}{2Z_0}|E_0|^2$$

$$= \frac{r^2}{2Z_0}\left|\frac{e^{i(kr-\omega t)}}{3\pi r}k^2a^3\left[2c\frac{\hat{k}}{k}\times B_0 + \frac{\hat{k}}{k}\times\left(E_0\times\frac{\hat{k}}{k}\right)\right]\right|^2$$

If $B_0 = E_0$, then

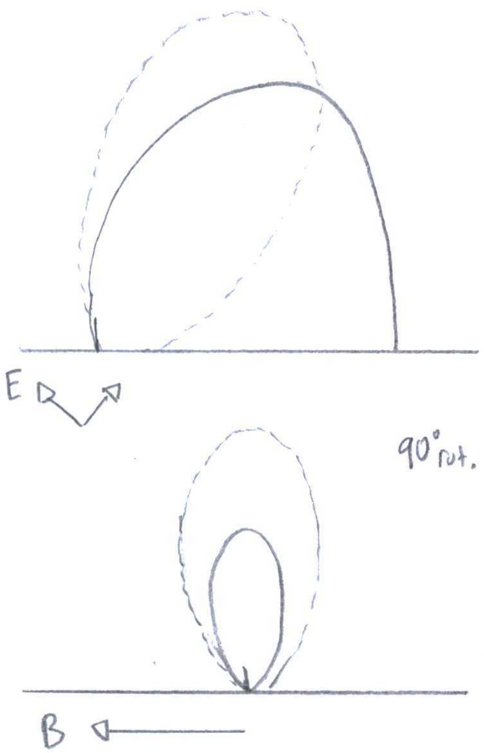
$$\frac{dP}{d\Omega} = \frac{E_0^2}{2Z_0}\pi a^2 \frac{144(ka)^2}{4\pi^3}(k_z^2+k_y^2)a^2\left[4c^2\frac{\hat{k}}{k^2}+4c\frac{\hat{k}}{k^3}+\frac{\hat{k}}{k^3}\right]$$

$$= \frac{P_i}{\cos\alpha}\,\frac{144(ka)^2}{4\pi^2}(\cos^2\theta+\sin^2\theta\sin^2\phi)\left[4c^2+(4c+1)|k\right]$$

when $ka \ll 1$, $P_i = \left(\frac{E_0^2}{2Z_0}\right)\pi a^2\cos\alpha$

vs Problem 10.14

$$= P_i\cos\alpha\,\frac{(ka)^2}{4\pi}(\cos^2\theta+\sin^2\theta\sin^2\phi)\left[\frac{2J_1(ka\xi)}{ka\xi}\right]^2$$



"Fraunhofer diffraction pattern for a circular opening."

b) Protractor Values:

Figure 10.14a:

θ (°)	P/Pmax
0	0.19
10	0.38
20	0.62
30	0.83
40	0.97
50	0.99
60	0.85
70	0.89
80	0.59
90	0.30

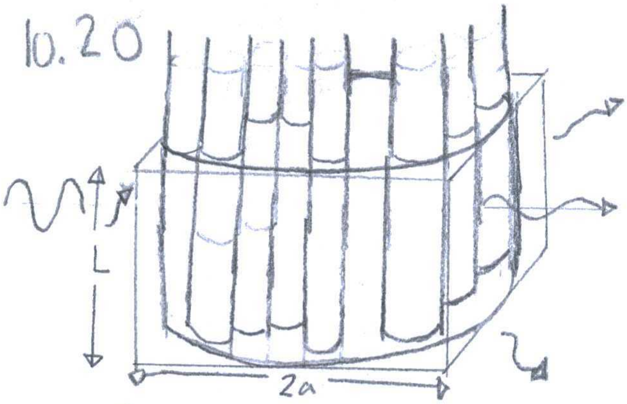
Figure 10.14b:

α (°)	P/Pmax
0°	0.00
10°	0.06
20°	0.07
30°	0.09
40	0.09
50°	0.14
60°	0.30
70°	0.53
80	0.90
90	1.00

c) (10.116) "Transmission Coefficient"

$$T=\frac{\cos\alpha}{\pi}\int_0^{2\pi}d\phi\int_0^{\pi/2}(\cos^2\theta+\cos^2\phi\sin^2\theta)\left|\frac{J_1(ka\xi)}{\xi}\right|^2\sin\theta\,d\theta$$

10.20



"Suspension of transparent fibers in a clear liquid"

a) Section 10.1d : Collection of Scatterers

(10.18)

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\sum_j\left[\epsilon^*\cdot p_j + (n\times\epsilon^*)\, m_j/c\right]e^{iq x_j}\right|^2$$

∴ where $q = kn_i - kn_j$

(10.19) "Structure Function"

$$F(q) = \left|\sum_j e^{iqx_j}\right|^2$$

(pg 461) "Total differential cross section is cross section × structure factor"

$\boxed{\text{Citation: Fournet (1949) "Small-angle Scattering of X-rays"}}$

$$P_{11}(q) = \int_0^{\pi/2}\left[\frac{2J_1(qr\sin\alpha)}{qk\sin\alpha}\,\frac{\sin(qL\cos\alpha/2)}{qL\cos\alpha/2}\right]^2\sin\alpha\, d\alpha$$

$$\frac{d\sigma}{d\Omega} = \underbrace{\left|\frac{\delta\epsilon}{\epsilon}\right|^2}_{\text{"differs by } \left|\frac{\delta\epsilon}{\epsilon}\right|\text{"}}\ \underbrace{\frac{(ka)^4L^2}{32}(1+\cos^2\theta)}_{\text{"normal differential cross section"}}\ \underbrace{\left[\frac{2J_1(q_\perp a)}{q_\perp a}\,\frac{\sin(qL/2)}{qL/2}\right]^2}_{\text{"Structure factor"}}$$