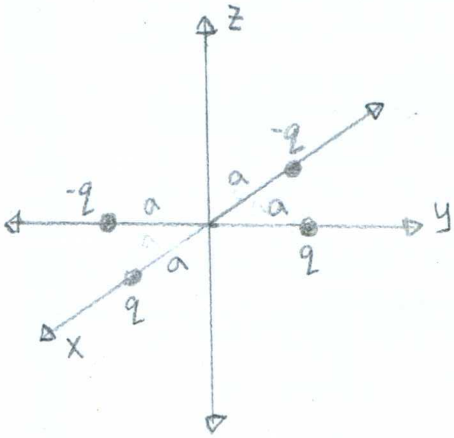
Chapter 4: Multipoles, Electrostatics of Macroscopic Media, Dielectrics

4.1 a) (Equation 4.3) "Multi-pole Moments"

$$q_{\ell m} = \int Y^*_{\ell m}(\theta', \phi') \circ r'^{\ell} \circ \rho(x') d^3x'$$



Quadropole

Shape: Lines
Dimension: Line [1D]
Charge: +q, -q.

$$\rho(x') = \frac{\text{charge}}{\text{volume}} = \frac{Q}{V} = \frac{Q}{\int_{-a}^{a}\int_{-a}^{a} dx\,dy} = \frac{Q}{4a^2} = \frac{q}{4a^2} \circ \delta(x - x')$$

(Expansion 3.116.5)

$$\delta(x-x') = \delta(x_1 - x_1') \circ \delta(x_2 - x_2') \circ \delta(x_3 - x_3')$$

$$\rho(x') = \frac{q}{4a^2} \circ \delta(r-a) \circ [\delta(\phi) - \delta(\phi - 3\pi/2) - \delta(\phi - \pi) + \delta(\phi - \pi/2)]\delta(\cos\theta)$$

$$q_{\ell m} = \int Y^*_{\ell m}(\theta', \phi') \circ r'^{\ell} \circ \rho(x') d^3x$$

$$= \int Y^*_{\ell m}(\theta', \phi') \circ r'^{\ell} \circ \left[\frac{q}{4a^2} \circ \delta(r-a) \circ [\delta(\phi) - \delta(\phi - 3\pi/2) - \delta(\phi - \pi) + \delta(\phi - \pi/2)]\right] \circ \delta(\cos\theta)\, d^3x$$

$$= \frac{q}{4a^2} \int_0^{2\pi}\int_0^{\pi}\int_0^{a} Y^*_{\ell m}(\theta', \phi') \circ r'^{\ell} \circ \left[\delta(r-a) \circ [\delta(\phi) - \delta(\phi - 3\pi/2) - \delta(\phi - \pi) + \delta(\phi - \pi/2)]\right] \circ \delta(\cos\theta)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{q a^{l}}{4} \int_{0}^{2\pi}\int_{0}^{\pi} Y^{*}_{lm}(\theta', \phi') \cdot \left[\delta(\phi) - \delta(\phi - 3\pi/2) - \delta(\phi - \pi) + \delta(\phi - \pi/2)\right] \cdot \delta(\cos\phi) \sin\theta \, d\theta \, d\phi$$

Dirac Delta Function

$$\int_{0}^{2\pi} f(x)\delta(x-a)\,dx = f(a)$$

$$=$$

$$= \frac{q a^{l}}{4} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} P_{l}^{m}(0) \left[\int_{0}^{2\pi} e^{-im\phi} \delta(\phi) \, d\phi - \int_{0}^{2\pi} e^{-im\phi} \delta(\phi - 3\pi/2) \, d\phi - \int_{0}^{2\pi} e^{-im\phi} \delta(\phi - \pi) \, d\phi + \int_{0}^{2\pi} e^{-im\phi} \delta(\phi - \pi/2) \, d\phi \right]$$

$$= \frac{q a^{l}}{4} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} P_{l}^{m}(0) \left[1 + i^{m} - 1 + (-1)^{m} \cdot i^{m} \right]$$

$$= \frac{q a^{l}}{4} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} P_{l}^{m}(0) \left(1 - (-1)^{m}\right)\left(1 - i^{m}\right)$$

$$= \frac{q a^{l}}{2} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} P_{l}^{m}(0) \left(1 - i^{m}\right)$$

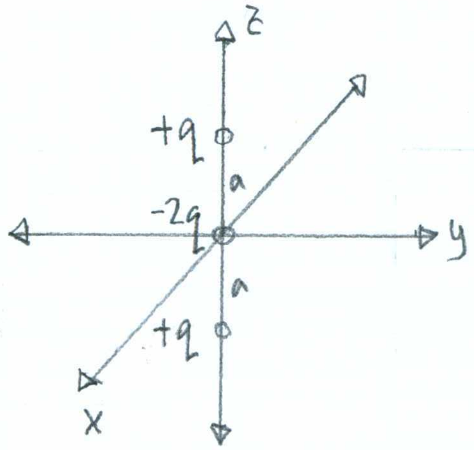
Terms

Monopole	Dipole	Quadrupole	Octapole
			$q_{3,-3} = qa^{3}\sqrt{\frac{35}{16\pi}}(1+i)$
		$q_{2,-2} = 0$	$q_{3,-2} = 0$
	$q_{1,-1} = qa\sqrt{\frac{3}{4\pi}}(1+i)$	$q_{2,-1} = 0$	$q_{3,-1} = qa^{3}\sqrt{\frac{21}{16\pi}}(-1-i)$
$q_{00} = 0$	$q_{1,0} = 0$	$q_{2,0} = 0$	$q_{3,0} = 0$
	$q_{1,1} = qa\sqrt{\frac{3}{4\pi}}(-1+i)$	$q_{2,1} = 0$	$q_{3,1} = qa^{3}\sqrt{\frac{21}{16\pi}}(1-i)$
		$q_{2,2} = 0$	$q_{3,2} = 0$
			$q_{3,3} = qa^{3}\sqrt{\frac{35}{16\pi}}(-1-i)$

Legendre Polynomials

n	Function
0	1
1	x
2	$\frac{1}{2}(3x^{2}-1)$
3	$\frac{1}{2}(5x^{2}-3x)$
4	$\frac{1}{8}(35x^{4}-70x^{3}+3)$

b)



Another Quadropole Type

Shape: Line
Dimension: Line [1D]
Charge: +q, -q

$$\rho_+(x') = \frac{\text{Charge}}{\text{Volume}} = \frac{Q}{V} = \frac{Q}{\int_0^{2\pi}\int_0^{\pi}\int_0^{a} r\sin\theta\, dr\, d\theta\, d\phi}$$

$$= \frac{Q}{2\pi a^2}$$

$$= \frac{q}{2\pi a^2}\delta(r-a)\cdot\left[\delta(\cos\theta-1)+\delta(\cos\theta+1)\right]$$

$$\rho_-(x') = \frac{-q}{2\pi a^2}\delta(r)$$

$$\rho(x) = \rho_+(x') + \rho_-(x')$$

$$= \frac{q}{2\pi a^2}\cdot\delta(r-a)\cdot\left[\delta(\cos\theta-1)+\delta(\cos\theta+1)\right] - \frac{-q}{-2\pi a^2}\delta(r)$$

$$q_{\ell m} = \int Y^*_{\ell m}(\theta,\phi)\, r^\ell \rho(x)\, d^3x$$

$$= \int Y^*_{\ell m}(\theta,\phi)\, r^\ell\left[\frac{q}{2\pi a^2}\cdot\delta(r-a)\cdot\left[\delta(\cos\theta-1)+\delta(\cos\theta+1)\right] - \frac{q}{2\pi a^2}\delta(r)\right] d^3x$$

$$= \frac{q}{2\pi a^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{a} Y^*_{\ell m}(\theta,\phi)\cdot r^\ell\left[\delta(r-a)\cdot\left[\delta(\cos\theta-1)+\delta(\cos\theta+1)\right] - \delta(r)\right] r^2\cdot\sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{qa^\ell}{2\pi}\int_0^{2\pi}\int_0^{\pi} Y^*_{\ell m}(\theta,\phi)\cdot\left[\left(\delta(\cos\theta-1)+\delta(\cos\theta+1)\right)\right]\sin\theta\, d\theta\, d\phi$$

$$= \frac{qa^\ell}{2\pi}\sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell(0)\int_0^{2\pi} e^{-im\phi}\, d\phi$$

$$= \frac{qa^\ell}{2\pi}\sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell(0)\cdot\frac{i(1-e^{-2i\pi m})}{m}$$

...which is zero, unless $m=0$
.... proof by Taylor expansion...

$$= q a^{l} \sqrt{\frac{2l+1}{4\pi}}$$

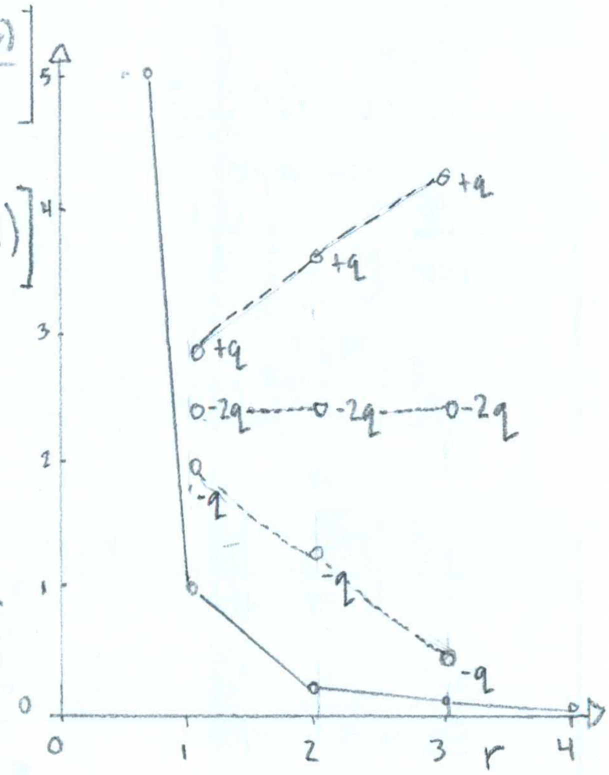
Terms			
Monopole	Dipole	Quadrupole	Octapole
			$q_{3-3} = 0$
		$q_{2-2} = 0$	$q_{3-2} = 0$
	$q_{1-1} = 0$	$q_{2-1} = 0$	$q_{3-1} = 0$
$q_{00} = 0$	$q_{10} = 0$	$q_{20} = \sqrt{\frac{5}{\pi}}\, q a^2$	$q_{30} = 0$
	$q_{11} = 0$	$q_{21} = 0$	$q_{31} = 0$
		$q_{22} = 0$	$q_{32} = 0$
			$q_{33} = 0$

c) (Equation 4.1) "Multipole Expansions"

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{4\pi}{5} q_{20} \frac{Y_{20}(\theta,\phi)}{r^3}\right]$$

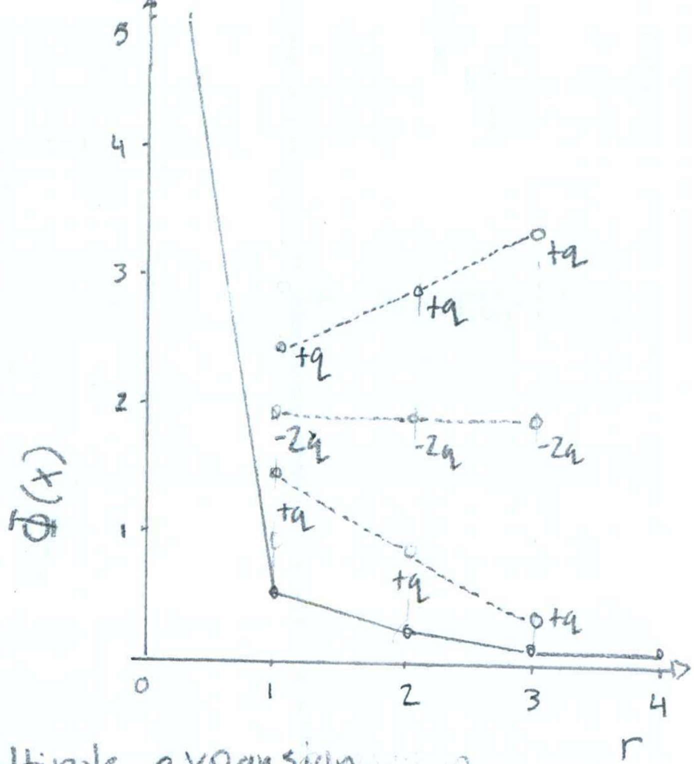
$$= \frac{1}{4\pi\varepsilon_0 a}\left[\left(\frac{a}{r}\right)^3 (3\cos^2\theta - 1)\right]$$



d) $\Phi(r) = \frac{-q}{4\pi\varepsilon_0}\left[\frac{1}{|r+az|} + \frac{1}{|r-az|} + \frac{2}{|r|}\right]$

$$= \frac{2q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2+a^2}} - \frac{1}{r}\right]$$

$$= \frac{-q}{4\pi\varepsilon_0 a}\left[\frac{2}{(r/a)} - \frac{2}{\sqrt{(r/a)^2+1}}\right]$$



A quadropole calculation by a multipole expansion for potential is nearly 100%: off, 200% different, or 2X From a quadropole calculation with Coulomb's law.

4.2 An assumption, $\frac{1}{4\pi\varepsilon_0}\frac{p\cdot(\vec{x}-\vec{x}_0)}{|\vec{x}-\vec{x}_0|^3} = \frac{1}{4\pi\varepsilon_0}\vec{p}\cdot\vec{\nabla}_x\left(\frac{1}{|x-x'|}\right)$

$$= \frac{1}{4\pi\varepsilon_0}\underbrace{\int \vec{p}\cdot\nabla_x\left(\frac{1}{|x-x'|}\right)d^3x}$$

"A requirement for potential is surface charge here"

$$= \frac{1}{4\pi\varepsilon_0}\int \rho(x')\frac{1}{|x-x'|}d^3x$$

where: $-p\cdot\nabla(\delta(x-x_0)) = \rho(x')$

unit check: Polarization (C/m^2) = surface Charge (C/m^2)

4.3. (Equation 4.1) "Multipole Expansion"

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

(Equation 4.3) "Multipole Moments"

$$q_{lm} = \int Y^*_{lm}(\theta', \phi') \cdot r'^{l} \rho(x) d^3x'$$

$$= \int \sqrt{\frac{4\pi}{2l+1}} \cdot P_l(\cos\theta) \cdot r^l \cdot \rho(x') d^3x$$

...because $Y^*_{lm}(\theta', \phi') = 0$ when $m \neq 0$

$$q_{10} = \sqrt{\frac{4\pi}{3}} \int P_1(\cos\theta) \cdot r \cdot \rho(x') d^3x$$

$$q_{20} = \sqrt{\frac{4\pi}{5}} \int P_2(\cos\theta) \cdot r^2 \cdot \rho(x') d^3x$$

$$q_{30} = \sqrt{\frac{4\pi}{7}} \int P_3(\cos\theta) \cdot r^3 \cdot \rho(x') d^3x$$

From the problem, $Q^{(l)}_{\alpha\beta\gamma} = \int x^\alpha \cdot y^\beta \cdot z^\gamma \cdot \rho(x') d^3x$

$$Q^{(1)}_{100} = \int x\, \rho(x') d^3x$$

$$Q^{(2)}_{110} = \int x \cdot y\, \rho(x') d^3x$$

$$Q^{(3)}_{111} = \int x \cdot y \cdot z \cdot \rho(x') d^3x$$

With a 180° (or π rotation) in θ, a monopole, dipole, and quadrupole show invariance to rotation:

$q_{10} \propto Q^{(1)}_{100}$ $\quad q_{20} \propto Q^{(2)}_{110}$ $\quad q_{30} \propto Q^{(3)}_{111}$, at $x = y = z = r$

Only q_{em} is necessary because charge density.

4.4.

a) "Old coordinate system" $Q_{xyz} = \int \rho(x,y,z)\, x \cdot y \cdot z \, dx\,dy\,dz$

"New coordinate system" $Q_{x'y'z'} = \int \rho(x+x', y+y', z+z')(x+x')(y+y')(z+z')\,dx\,dy\,dz$

$$= x' \int \rho(x+x', y+y', z+z')\, y \cdot z \, dx\,dy\,dz$$

$$+ y' \int \rho(x+x', y+y', z+z')\, x \cdot z \, dx\,dy\,dz$$

$$+ z' \int \rho(x+x', y+y', z+z')\, y\, z \, dx\,dy\,dz$$

$$+ \int \rho(x+x', y+y', z+z')\, x y z \, dx\,dy\,dz$$

$$+ x'y'z' \int \rho(x+x', y+y', z+z')\, dx\,dy\,dz$$

$$= x' Q_{yz} + y' Q_{xz} + z' Q_{yz} + Q_{xyz} + x'y'z' Q$$

$Q_{x'y'z'} = Q_{xyz}$ only when $x' = y' = z' = 0$

e.g. original coordinates at the origin.

b) $l = 0$ $\quad q = Q = \int \rho(x,y,z)\, x^0 \cdot y^0 \cdot z^0 \, dx\,dy\,dz$ (Equation 4.9, but 3-D)

$$= \int \rho(x,y,z)\, dx\,dy\,dz$$

$q' = \int \rho(x+x', y+y', z+z')\, dx\,dy\,dz$

$q = q'$ when a monopole

$l=1$ $\quad p_i = \int \rho(x,y,z)\, x_i \, dx\,dy\,dz \qquad$ (Equation 4.9)

$$= \int \rho(x+x', y+y', z+z')(x+x')\, dx\,dy\,dz$$

$$= \int \rho(x+x', y+y', z+z')\, x \, dx\,dy\,dz + x' \int \rho(x+x', y+y', z+z')\, dx\,dy\,dz$$

$$p_i = p_i' + x' q_{em}$$

$$= p' + R\, q_{em}$$

"Polarization translates charge density or charge density translates polarization."

$l=2$ $\quad Q = \int \rho(x,y,z)(3xx - r^2 \delta_{ij})\, dx\,dy\,dz \qquad$ (Equation 4.9)

$$Q' = \int \rho(x+x', y+y', z+z')\left(3(x+x')(y+y') - \left((x+x')^2 + (y+y')^2 + (z+z')^2\right)\delta_{ij}\right) dx\,dy\,dz$$

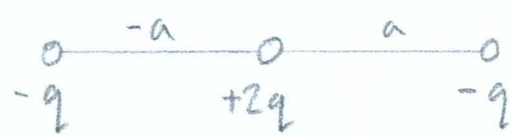
$$= Q' + 3y'p' + 3x'y'q' - R^2 \delta_{ij}\, q - 2\delta_{ij} R \cdot p$$

$$= Q' + 3y'p' + 3xp' - 2\delta_{ij} \cdot R \cdot p + (3x'y' - R^2 \delta_{ij})\, q$$

c) If $q \neq 0$, then $p' = 0$ when $R = \frac{p}{q}$

If $q = 0$, then $p' \neq 0$, and $Q_{ij} = 0$, if and only if $Q = 3y'p + 3x'p - 2\delta_{ij} \cdot R \cdot p$

For the picture,



$$X_1 = \frac{-Q_{23}p_1 + Q_{13}p_2 + Q_{12}p_3}{6p_2p_3} \qquad X_2 = \frac{Q_{23}p_1 - Q_{13}p_2 + Q_{12}p_3}{6p_1p_3} \qquad X_3 = \frac{Q_{23}p_1 + Q_{13}p_2 - Q_{23}p_3}{6p_1p_2}$$

4.5 Total Force:

$$\boxed{E = -\nabla \Phi}$$

a) (Equation 4.21) $W = \int \rho(x)\,\Phi(x)\,d^3x$

(Equation 4.22) $$\Phi(x) = \phi(0) + x \cdot \nabla\phi(0) + \frac{1}{2}\sum_i \sum_j x_i x_j \frac{\partial^2 \phi}{\partial x_i x_j}(0) + \cdots$$

$$= \phi(0) - x \cdot E(0) - \frac{1}{2}\sum_i \sum_j x_i x_j \frac{\partial E}{\partial x_i}(0) + \cdots$$

$$= \phi(0) - x \cdot E(0) - \frac{1}{6}\sum_i \sum_j (3x_i x_j - r^2 \delta_{ij}) \frac{\partial E}{\partial x_i}(0) + \cdots$$

(Equation 4.24) $$W = q\phi(0) - p \cdot E(0) - \frac{1}{6}\sum_i \sum_j Q_{ij} \frac{\partial E}{\partial x_i}(0) + \cdots$$

$$F = qE^{(0)}(0) + \left[\nabla\left[p \cdot E^{(0)}(x)\right]\right] + \left\{\nabla\left[\sum_i \sum_j Q_{ij} \frac{\partial E}{\partial x_i}(x)\right]\right\}_{x=0} + \cdots$$

The work expansion expresses force term by term as the integrable form ; $W = -\int F\,dx$.

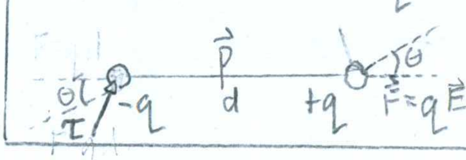
b) Total Torque:

Regular Torque:

$\tau = r \times F$

$= rF\sin\theta$

Electrostatic Torque:



$$N_i = \int x \cdot \rho(x) \cdot E(x)\,d^3x$$

$$= \int p \times E\,d\tau$$

$$N_1 = N_3 - N_2$$

$$= \int \rho(x)\left[xE_3 - xE_2\right]d^3x$$

$$E_2^{(0)} = E_2^{(0)}(x) + \sum_j x_j \frac{\partial}{\partial x_j} E_2^{(0)}(x) + \cdots$$

$$E_3^{(0)} = E_3^{(0)}(x) + \sum_j x_j \frac{\partial}{\partial x_j} E_3^{(0)}(x) + \cdots$$

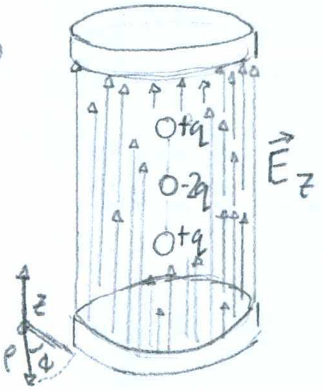
$$N_1 = \left[E_3^{(0)}(x)\int \rho(x)\,x_2\,d^3x - E_2^{(0)}(x)\int \rho(x)\,x_3\,d^3x + \sum_j \frac{\partial}{\partial x_j} E_3^{(0)} \int \rho(x)\,x_2 x_j\,d^3x - \sum_j \frac{\partial}{\partial x_i} E_2^{(0)} \int \rho(x)\,x_3 x_j\,d^3x\right.$$

$$= E_3^{(0)}(X)p_2 - E_2^{(0)}p_3 + \frac{1}{3}\sum \frac{\partial}{\partial x_i} E_3^{(0)} Q_{21} - \frac{1}{3}\sum \frac{\partial}{\partial x} E_2^{(0)} Q_3 + \cdots$$

$$= [p \times E^{(0)}(0)] + \frac{1}{3}\left[\frac{\partial}{\partial x_3}\left(\sum Q_{21}E_j^{(0)}\right) - \frac{\partial}{\partial x_2}\left(\sum Q_{3j}E_j^{(0)}\right)\right] + \cdots$$

the 1/3 is a grain of salt, drop in water, or ____ of air.

4.6



a) $Q_{11} = Q_{22} = -\frac{1}{2}Q_{33}$

$= -\frac{1}{2}eQ$ and $Q_{33} = eQ$

From (Equation 4.25),

$$Q_{JM} = \frac{1}{e}\int (3z^2 - r^2)\rho_{JM}(x)\,d^3x$$

Further (Equation 4.24),

$$W = \underbrace{q\Phi(0)}_{\text{Monopole term}} - \underbrace{p\cdot E(0)}_{\text{Dipole term}} - \underbrace{\frac{1}{6}\sum_i\sum_j Q_{ij}\frac{\partial E_j}{\partial x_i}(0) + \cdots}_{\text{Quadrupole term}}$$

A quadrupole nucleus finds itself in a gradient along the z-axis

Shape: Cylindrical
Dimension: Volume [3D]
Charge: +q, -q

With the quadrupole term,

$$\text{Work} = -\frac{1}{6}\sum_i\sum_j Q_{ij}\frac{\partial E_j}{\partial x_i}(0)$$

$$= -\frac{1}{6}\left[Q_{11}\left[\frac{\partial E}{\partial x} + \frac{\partial E}{\partial y}\right] + Q_{22}\left[\frac{\partial E}{\partial x} + \frac{\partial E}{\partial y}\right] + Q_{33}\left[\frac{\partial E}{\partial z}\right]\right]$$

$$= -eQ\left[-\frac{1}{2}\frac{\partial E_x}{\partial x} - \frac{1}{2}\frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right]$$

$$= -\frac{eQ}{4}\,\frac{\partial E}{\partial z}$$

Divergence in an Electric Field
$\nabla\cdot E = \frac{\partial E}{\partial x} + \frac{\partial E}{\partial y} + \frac{\partial E}{\partial z} = 0$
"in Cylinders" $\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y} = -\frac{1}{2}\frac{\partial E_z}{\partial z}$

b) $Q = 2\times10^{-28}\,m^2$

$\frac{W}{h} = 10\,MHz$

$h = 6.626\times10^{-34}\,J\cdot s$

$a_0 = 4\pi\varepsilon_0 \hbar^2/me^2$

$= 0.529\times10^{-10}\,m$

$$\frac{\partial E}{\partial z} = \frac{-4W}{eQ}$$

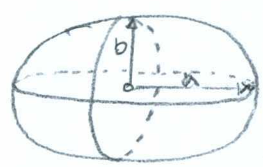
$$= \frac{-4h}{ea_0^2}\frac{(W/h)}{(Q/a_0^2)}$$

$$\frac{\partial E}{\partial z}\Big/\left(\frac{e}{4\pi\varepsilon_0 a^3}\right) = \frac{-16\pi h\varepsilon_0 a_0}{e^2}\frac{(W/h)}{(Q/a_0^2)}$$

$$= \frac{-8\pi}{\frac{e}{4\pi\varepsilon_0\hbar c}}\frac{a_0}{c}\frac{(W/h)}{(Q/a_0^2)}$$

$= 0.085$ meters

c)



Nuclear Charge Distribution "Spheroidal Volume"

(Equation 4.25) "Traceless Quadrupole Moment"

$$Q = \int (3x_ix_j - r^2\delta_{ij})\rho\, d^3x$$

$$x = a\eta\cos\phi\sin\theta$$

$$y = a\eta\sin\phi\sin\theta$$

$$z = b\eta\cos\theta$$

$$J\left(\frac{x,y,z}{\eta,\theta,\phi}\right) = a^2b\eta^2\sin\theta$$

Technically, a moment for a square quadrupole

+q o o -q

-q o o +q

$$Q = \int (3xx - r^2\delta_{ij})\rho\, d^3x$$

$$= \int (3xy - (x^2+y^2)\delta_{ij})\rho\, d^3x$$

$$= \rho\int (2b^2\eta^2\cos^2\theta - a^2\eta^2\sin^2\theta)a^2b\eta^2\sin\theta\, d\eta\, d\phi\, d\theta$$

$$= \rho\int_0^{\pi}\int_0^{2\pi}\int_0^1 (2b^2\eta^2\cos^2\theta - a^2\eta^2\sin^2\theta)a^2b\eta^2\sin\theta\, d\eta\, d\phi\, d\theta$$

$$= \frac{8\pi a^2 b^3 \rho}{15} - \frac{8\pi a^4 b \rho}{15}$$

$$= \frac{8\pi a^2 b \rho}{15}(b^2 - a^2)$$

$$= \frac{2}{5} Ze (b-a)(b+a)$$

$$= \frac{4}{5} Z R (b-a)$$

Element
Eu^{153}
Sphere Volume
$V = \frac{4}{3}\pi R^3$
Charge Density
$\rho = \frac{Ze}{V}$

$$\frac{(a-b)}{R} = \frac{5}{4}\frac{Q}{ZR^2}$$

$$= \frac{5 \times 2.5 \times 10^{-20}}{4 \times 63 \times (7 \times 10^{-13})^2}$$

$$= 0.101$$

$\cong 10\%$ "Charge difference between major and minor axes"

Identity:
$x'^2 = \frac{2}{3}P_2(x') + \frac{1}{3}P_0(x')$

b) $$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(x')}{|x - x'|}dx'$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{64\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{r^2 e^{-r}\sin^2\theta}{|x - x'|} r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{64\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r^4 e^{-r}\sin^2\theta \cdot 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y^*(\theta,\phi) Y(\theta,\phi)\sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{64\pi}\sum_{l=0}^{\infty} P_l(\cos\theta)\int_{-1}^{1} P_l(x')(1 - x'^2)dx \int_0^{\infty} r^4 e^{-r}\frac{r^l}{r^{l+1}}dr$$

... when $m = 0$ because $Y(\theta,\phi) \neq 0$.

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{32}\sum_{l=0}^{\infty} P_l(\cos\theta)\left[\frac{-2}{3}\int_{-1}^{1} P_l(x')P_2(x')dx + \frac{2}{3}\int_{-1}^{1} P_l(x')P_0(x')dx\right] \cdot \int_0^{\infty} r^4 e^{-r}\frac{r^l}{r^{l+1}}dr$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{32}\left[\frac{4}{3}\int_0^{\infty} r^4 e^{-r}\frac{1}{r}dr - \frac{4}{15}P_2(\cos\theta)\int_0^{\infty} r^4 e^{-r}\frac{r^2}{r^3}dr\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{32}\left[\frac{4}{3}\frac{1}{r}\int_0^r r^4 e^{-r}dr + \frac{4}{3}\int_r^\infty r^3 e^{-r}dr - \frac{4}{15}P_2(\cos\theta)\frac{1}{r^3}\int_0^r r^6 e^{-r}dr - \frac{4}{15}P_2(\cos\theta)r^2\int_r^\infty r e^{-r}dr\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{32}\left[\frac{4}{3}\frac{1}{r}\left[e^{-r}(-r^4-4r^3-12r^2-24r-24)\right]_0^r + \frac{4}{3}\left[e^{-r}(-r^3-3r^2-6r-6)\right]_r^\infty\right.$$

$$\left. - \frac{4}{15}P_2(\cos\theta)\left(\frac{1}{r^3}\left[e^{-r}(-r^6-6r^5-30r^4-120r^3-360r^2-720r-720)\right]_0^r + r^2\left[-e^{-r}(r+1)\right]\right)\right]$$

$$\Phi = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{24}\left[-e^{-r}(r^2+6r+18+\frac{24}{r}) + \frac{24}{r}\right] - P_2(\cos\theta)\frac{1}{120}\left[e^{-r}(-5r^2-30r-120 - \frac{360}{r} - \frac{720}{r^2} - \frac{720}{r^3}) + \frac{720}{r^3}\right]\right]$$

$$\approx \frac{1}{4\pi\varepsilon_0}\frac{1}{24}\left[-r^2-6r-18-\frac{24}{r}+6r^2+18r+24-9r^2 -12r+4r^2+\frac{24}{r}\right] - P_2(\cos\theta)\frac{1}{120}\left[-5r^2 -30r+30r^2-120+120r-120\frac{1}{2}r^2-\frac{360}{r} +360-180r+60r^2-\frac{720}{r^2}+\frac{720}{r}-360 +120r-30r^2-\frac{720}{r^3}+\frac{720}{r^2}-\frac{360}{r}+120-30r+6r^2+\frac{720}{r^3}\right]\Big]$$

Taylor Expansion:

$$e^{-r} = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= 1 - r + \frac{r^2}{2} + \frac{r^4}{24} - \frac{r^5}{120} + \ldots$$

$$\Phi \approx \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{4} - \frac{r^2}{120}P_2(\cos\theta)\right]$$

c) (Equation 4.21)

$$W = \int \rho(x)\,\Phi(x)\,d^3x$$

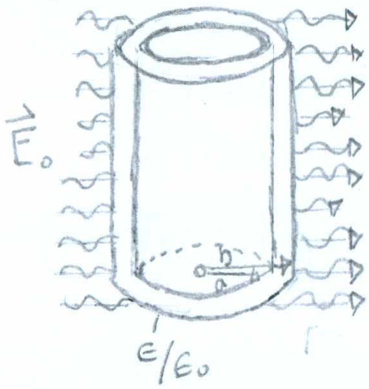
$$= \int \rho(x)\frac{-e}{4\pi\varepsilon_0 a_0}\left[\frac{1}{4} - \frac{(r/a_0)^2}{120}P_2(\cos\theta)\right]d^3x$$

$$= \frac{-e}{4\pi\varepsilon_0 a_0}\left[\frac{1}{4}q_{nuc} - \frac{1}{240 a_0^2}\int \rho_{nuc}(3z^2-r^2)d^3x\right]$$

$$= \frac{-e}{4\pi\varepsilon_0 a_0}\left[\frac{1}{4}q_{nuc} - \frac{e}{240 a_0^2}Q_{zz,nuc}\right]$$

$$\frac{W}{h} \approx \frac{e^2}{480\pi\varepsilon_0 a_0^3 h}Q_{zz,nuc} \approx 3\times10^6\,Hz$$

4.8



"A very long, right circular, cylindrical shell"

Shape: Cylinder
Dimension: Volume [3D]
Charge: q

a) $\boxed{\text{Potential Derivation:}}$

① Boundary Conditions:

$\Phi(\rho=0, \phi, z) = \text{finite}$

$\Phi(\rho=a, \phi, z) = 0$

$\Phi(\rho=b, \phi, z) = E_0 \rho \cos\phi$

$\epsilon_0 \cdot E_{In} = \epsilon \cdot E_{mid}$

$\epsilon \cdot E_{mid} = \epsilon_0 \cdot E_{out}$

$\boxed{\text{Two new boundary condition types.}}$

② Laplace's Equation:

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\cdot\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$
$$= 0$$

③ Laplaces Equation Solutions:

If $\Phi(\rho, \phi, z) = R(\rho)\cdot Q(\phi) Z(z)$

Ⓐ Variable Separation:

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\cdot\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$= \frac{Q(\phi)\cdot Z(z)}{\rho}\frac{\partial}{\partial\rho}\left(\rho\cdot\frac{\partial R(\rho)}{\partial\rho}\right) + \frac{R(\rho)Z(z)}{\rho^2}\frac{\partial^2 Q(\phi)}{\partial^2\phi^2} + R(\rho)Q(\phi)\frac{\partial^2 Z(z)}{\partial^2 z}$$

$$= \frac{1}{R(\rho)\cdot\rho}\frac{\partial}{\partial\rho}\left(\rho\cdot\frac{\partial R(\rho)}{\partial\rho}\right) + \frac{1}{Q(\phi)\rho^2}\frac{\partial^2 Q(\phi)}{\partial^2\phi^2} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z}$$

$$= 0$$

Ⓑ Radial Eigenvalue:

$$\frac{1}{R\cdot\rho}\frac{\partial}{\partial\rho}\left(\rho\cdot\frac{\partial R}{\partial\rho}\right) + \frac{1}{Q\cdot\rho^2}\frac{\partial^2 Q}{\partial\phi^2} = -k^2$$

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\cdot\frac{\partial R}{\partial\rho}\right)+\rho^2k^2=\frac{-1}{Q\cdot\rho^2}\frac{\partial^2 Q}{\partial\phi^2}$$

$$\frac{\rho}{R}\left(\rho\frac{\partial^2 R}{\partial^2\rho}+\frac{\partial R}{\partial\rho}\right)+\rho^2k^2=m^2$$

$$\frac{\partial^2 R}{\partial p^2}+\frac{1}{p}\frac{\partial R}{\partial p}+\left(k^2-\frac{m^2}{p^2}\right)R=0$$

(C) Azimuthal Eigenvalue:

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial^2\phi^2}+m^2=0$$

$$\frac{\partial^2 Q}{\partial^2\phi^2}+m^2Q=0$$

(D) Vertical Eigenvalues:

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}-k^2=0$$

$$\frac{\partial^2 Z}{\partial z^2}-k^2Z=0$$

(E) General Solution:

$$\Phi(\rho,\phi,z)=R(\rho)Q(\phi)Z(z)$$

(4) General Solution to Laplaces Equation:

$$R(\rho)=A\rho^m+B\rho^{-m}$$

$$Q(\phi)=Ce^{im\phi}+De^{-im\phi}$$

$$Z(z)=E\sinh(kz)+F\cosh(kz)$$

$$\phi(\rho,\phi,z)=(A\rho^m+B\rho^{-m})(Ce^{im\phi}+De^{-im\phi})(E\sinh(kz)+F\cosh(kz))$$

(5) Variables by Boundary Conditions:

F $\Phi(\rho=a,\phi,z=0)=(Aa^{m}+Ba^{-m})(Ce^{im\phi}+De^{-im\phi})(E\sinh(k\cdot 0)+F\cosh(k\cdot 0))$

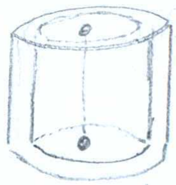
$=0$, so $F=0$

E $\Phi(\rho=b,\phi,z=0)=(Ab^{m}+Bb^{-m})(Ce^{im\phi}+De^{-im\phi})(E\cdot\sinh(k\cdot 0))$

$=V$, so $E=0$

D $\Phi(\rho,\phi,z)=(A\cdot\rho^{m}+B\rho^{-m})(Ce^{im\phi}+De^{-im\phi})$

Inside

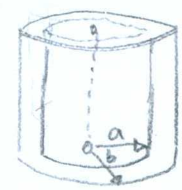


$\Phi(\rho=0,\phi,z)=(A\cdot 0^{m}+B\cdot 0^{-m})(Ce^{in\phi}+De^{-im\phi})$

$=$ finite, so $B=0$

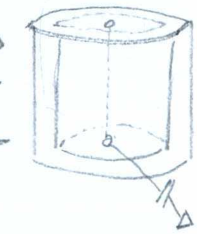
$\Phi(\rho,\phi,z)=A\rho^{-m}(Ce^{im\phi}+De^{-im\phi})$

Middle



$\Phi(a<\rho<b,\phi,z)=(A\rho^{l}+B\rho^{-l})(Ce^{im\phi}+De^{-im\phi})$

Outside



$\Phi(\rho=\infty,\phi,z)=(A\cdot\infty^{m}+B\infty^{-m})(Ce^{im\phi}+De^{-im\phi})$

$=-E_0\rho\cos\phi$, $m=1$, $C=D$

for a solution

$-E_0\rho\cos\phi=(A\cdot\infty)\cdot 2\cdot C\cos\phi$

$-E_0=A\cdot 2\cdot C$

$\Phi(\rho,\phi,z)=(-E_0\rho+B_1\rho^{-1})\cos\phi$

Between outside and middle regions:

C $\quad \epsilon_0 E_{out}=\epsilon\cdot E_{mid}$

$$\epsilon_0\frac{\partial\Phi_{out}}{\partial\rho}=\epsilon\cdot\frac{\partial\Phi_{mid}}{\partial\rho}$$

$$\epsilon_0 \frac{\partial}{\partial\rho}(-E_0\rho + B_1\rho^{-1})\cos\phi = \epsilon\frac{\partial}{\partial\rho}(A\rho^m + B\rho^{-m})(Ce^{im\phi}+De^{-im\phi})$$

@ $\rho=b$, $m=1$

$$\epsilon_0\left(-E_0 - B_1\frac{1}{\rho^2}\right)\cos\phi = \epsilon\left(A - B\frac{1}{\rho^2}\right)\cdot 2\cdot C\cdot\cos\phi$$

$$C = -\frac{\epsilon_0\left(E_0+B_1\frac{1}{\rho^2}\right)}{2\cdot\epsilon\cdot\left(1-B\frac{1}{\rho^2}\right)}$$

$$\Phi(a<\rho<b,\phi,z) = -(\rho + B\rho^{-1})\frac{\epsilon_0\left(E_0+B_1\frac{1}{\rho^2}\right)}{\epsilon(1-B\rho^{-2})}\cos\phi$$

B $\frac{\partial\Phi_{out}}{\partial\phi} = \frac{\partial\Phi_{mid}}{\partial\phi}$

$$\frac{\partial}{\partial\phi}\left((-E_0\rho + B_1\rho^{-1})\cos\phi\right) = \frac{\partial}{\partial\phi}\left(-(\rho+B\rho^{-1})\frac{\epsilon_0(E_0+B_1\frac{1}{\rho^2}}{\epsilon(1-B\rho^{-2})}\cos\phi\right)$$

@ $\rho=b$

$$\left(E_0 - B_1\frac{1}{b^2}\right)\epsilon\left(1-B\frac{1}{b^2}\right) = \left(1+B\frac{1}{b^2}\right)\epsilon_0\left(E_0+B_1\frac{1}{b^2}\right)$$

$$B = b^2\frac{B_1(\epsilon+\epsilon_0) - E_0 b^2(\epsilon-\epsilon_0)}{B_1(\epsilon-\epsilon_0)-E_0b^2(\epsilon+\epsilon_0)}$$

$$\Phi(a<\rho<b,\phi,z) = \left((B_1(\epsilon-\epsilon_0) - E_0b^2(\epsilon+\epsilon_0))\rho + b^2(B_1(\epsilon+\epsilon_0)-E_0b^2(\epsilon-\epsilon_0))\rho^{-1}\right)\frac{\cos\phi}{2\epsilon b^2}$$

Between inner and middle:

A_1, B_1 $\quad \epsilon\cdot E_{mid} = \epsilon_0\cdot E_{in}$

$$\epsilon\cdot\frac{\partial\Phi_{mid}}{\partial\rho} = \epsilon_0\cdot\frac{\partial\Phi_{in}}{\partial\rho} \quad @\ \rho=a$$

$$\left((B_1(\epsilon-\epsilon_0) - E_0b^2(\epsilon+\epsilon_0)) - b^2(B_1(\epsilon+\epsilon_0)-E_0b^2(\epsilon-\epsilon_0)a^{-2}\right)\cdot\frac{1}{2b^2} = \epsilon_0 A_1$$

$$\frac{\partial \Phi_{mid}}{d\phi} = \frac{\partial \Phi_{in}}{\partial \phi} \quad @\ \rho = a$$

$$\left(\left(B_1(\epsilon-\epsilon_0) - E_0 b^2(\epsilon+\epsilon_0)\right)a + b^2\left(B_1(\epsilon+\epsilon_0) - E_0 \cdot b^2(\epsilon-\epsilon_0)\right)a^{-1}\right)\frac{1}{2\epsilon a b^2} = A_1$$

$$B_1 = E_0 b^2 \frac{(b^2-a^2)(\epsilon^2-\epsilon_0^2)}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}$$

$$A_1 = \frac{-4b^2 E_0 \cdot \epsilon \cdot \epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}$$

$$\Phi_{in}(\rho,\phi,z) = \left(\frac{-4b^2 \cdot \epsilon \cdot \epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\right) E_0 \cos\phi\, \rho$$

$$\Phi_{mid}(\rho,\phi,z) = \frac{-2ab^2\epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\left((\epsilon+\epsilon_0)\frac{\rho}{a} + (\epsilon-\epsilon_0)\frac{a}{\rho}\right)$$

$$\Phi_{out}(\rho,\phi,z) = \left(-\rho + \frac{(b^2-a^2)(\epsilon^2-\epsilon_0^2)}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\frac{b^2}{\rho^2}\right)E_0 \cos\phi$$

$$E = -\hat{\rho}\frac{\partial \Phi}{\partial \rho} - \hat{\phi}\frac{1}{\rho}\frac{\partial \Phi}{\partial \phi}$$

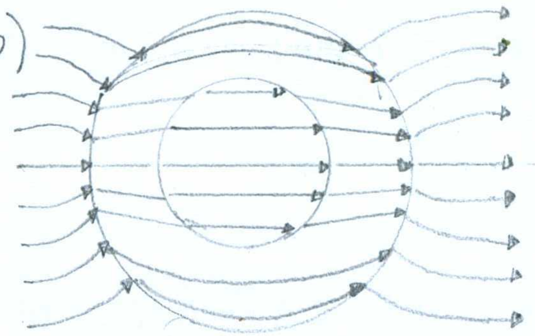
$$E_{in} = \left(\frac{-4b^2 \cdot \epsilon \cdot \epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\right) E_0\left[\hat{\rho}\cos\phi - \hat{\phi}\sin\phi\right]$$

$$E_{mid} = \left(\frac{-2ab^2\epsilon_0}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\right)E_0\left[\left((\epsilon+\epsilon_0) - (\epsilon-\epsilon_0)\frac{a^2}{\rho^2}\right)(\hat{\rho}\cos\phi - \hat{\phi}\sin\phi) - 2\cdot\hat{\phi}\cdot(\epsilon-\epsilon_0)\frac{a^2}{\rho^2}\sin\phi\right]$$

$$E_{out} = \left(-\rho + \frac{(b^2-a^2)(\epsilon^2-\epsilon_0^2)}{b^2(\epsilon+\epsilon_0)^2 - a^2(\epsilon-\epsilon_0)^2}\frac{b^2}{\rho^2}\right)E_0\left(\hat{\rho}\cos\phi + \hat{\phi}\sin\phi\right)$$

$$+ E_0\left(\hat{\rho}\cos\phi - \hat{\phi}\sin\phi\right)$$

b)



Electric field lines for the inside, middle, and outside regions "bending" in medium.

c) Solid cylinder [lim a -> 0]:

$$\lim_{a\to 0} E_{in} = \lim_{a\to 0}\left[\frac{-4b^2\epsilon_0\epsilon}{b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2}E_0\left[\left((\epsilon+\epsilon_0)-(\epsilon-\epsilon_0)\frac{a^2}{\rho^2}\right)(\hat{\rho}\cos\phi-\hat{\phi}\sin\phi) - 2\hat{\phi}(\epsilon-\epsilon_0)\frac{a^2}{\rho^2}\sin\phi\right]\right]$$

$$= \frac{2\epsilon_0}{(\epsilon+\epsilon_0)}E_0(\hat{\rho}\cos\phi-\hat{\phi}\sin\phi)$$

$$\lim_{a\to 0} E_{out} = \lim_{a\to 0}\left[\left(\frac{(b^2-a^2)(\epsilon^2-\epsilon_0^2)}{b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2}\frac{b^2}{\rho^2}\right)E_0(\hat{\rho}\cos\phi+\hat{\phi}\sin\phi) + E_0(\hat{\rho}\cos\phi-\hat{\phi}\sin\phi)\right]$$

$$= \frac{(\epsilon-\epsilon_0)}{(\epsilon+\epsilon_0)}\frac{b^2}{\rho^2}E_0(\hat{\rho}\cos\phi+\hat{\phi}\sin\phi) + E_0(\hat{\rho}\cos\phi-\hat{\phi}\sin\phi)$$

Cylindrical Cavity [lim b -> 0]:

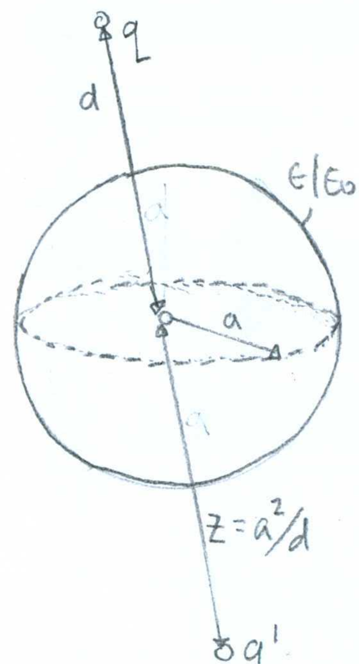
$$\lim_{b\to\infty} E_{in} = \lim_{b\to\infty}\left[\frac{-4b^2\epsilon_0\epsilon_0}{b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2}E_0\left[\left((\epsilon+\epsilon_0)-(\epsilon-\epsilon_0)\frac{a^2}{\rho^2}\right)(\hat{\rho}\cos\phi-\hat{\phi}\sin\phi) - 2\hat{\phi}(\epsilon-\epsilon_0)\frac{a^2}{\rho^2}\sin\phi\right]\right]$$

$$= \frac{4b^2\epsilon\epsilon_0}{b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2}E_0(\hat{\rho}\cos\phi-\hat{\phi}\sin\phi)$$

$$\lim_{b\to\infty} E_{mid} = \lim_{b\to\infty}\left[\left(\frac{-2ab^2\epsilon_0}{b^2(\epsilon+\epsilon_0)^2-a^2(\epsilon-\epsilon_0)^2}\right)E_0\left[\left((\epsilon+\epsilon_0)-(\epsilon-\epsilon_0)\frac{a^2}{\rho^2}\right)(\hat{\rho}\cos\phi-\hat{\phi}\sin\phi)\right.\right.$$
$$\left.\left. -2\hat{\phi}(\epsilon-\epsilon_0)\frac{a^2}{\rho^2}\sin\phi\right]\right.$$

$$=\frac{2\epsilon_0}{(\epsilon+\epsilon_0)^2}E_0\left[(\epsilon+\epsilon_0)(\hat{\rho}\cos\phi+\hat{\phi}\sin\phi)-(\epsilon-\epsilon_0)\frac{a^2}{\rho^2}(\hat{\rho}\cos\phi+\hat{\phi}\sin\phi)\right]$$

4.9



A point charge a distance from a dielectric sphere

Shape: Sphere
Dimension: Volume [3D]
Charge: q

a) Much like Problem 3.6, but with further steps for a solution to charges.

$\boxed{\text{Potential Derivation by Boundary Conditions:}}$

① Boundary Conditions:

$\Phi(r=0,\theta,\phi)=0 \qquad \Phi(r=d,\theta,\phi)=V$

$\epsilon_0 E_{out}=\epsilon\cdot E_{in}$

② Laplace's Equation:

$$\nabla^2\Phi=\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$
$$=0$$

③ Laplace's Equation Solutions:

Ⓐ Variable Separation: If $\Phi(r,\theta,\phi)=U(r)\cdot P(\theta)\cdot Q(\phi)$

$$\nabla^2\Phi=\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$
$$=\frac{r^2\sin^2\theta}{U(r)P(\theta)Q(\phi)}\left[\frac{P(\theta)Q(\phi)}{r}\frac{\partial^2 U(r)}{\partial r^2}+\frac{U(r)Q(\phi)}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right)+\frac{U(r)P(\theta)}{r^2\sin\theta}\frac{\partial^2 Q(\phi)}{\partial\phi^2}\right]$$
$$=0$$

Ⓑ Radial Eigenvalues:

$$\frac{r^2\sin^2\theta}{U(r)}\frac{\partial^2 U(r)}{\partial^2 r}+\frac{\sin^2\theta}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right)=m$$

$$\frac{r^2}{U(r)}\frac{\partial^2 U(r)}{\partial^2 r} + \frac{1}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) = \frac{m}{\sin^2\theta}$$

$$\frac{r^2}{U(r)}\frac{\partial^2 U(r)}{\partial^2 r} = \frac{m}{\sin^2\theta} - \frac{1}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right)$$

$$= l(l+1)$$

$$\frac{\partial^2 U(r)}{\partial^2 r} - \frac{l(l+1)\,U(r)}{r^2} = 0$$

Ⓒ <u>Angular Eigenvalues:</u>

$$\frac{m}{\sin^2\theta} - \frac{1}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) = l(l+1)$$

$$\frac{1}{P(\theta)\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) + \left(l(l+1) - \frac{m}{\sin^2\theta}\right) = 0$$

Ⓓ <u>Azimuthal Eigenvalues:</u>

$$\frac{1}{Q(\phi)}\frac{\partial^2 Q(\phi)}{\partial\phi^2} = -m^2$$

$$\frac{\partial^2 Q(\phi)}{\partial\phi^2} + m^2 Q(\phi) = 0$$

④ <u>General Solution to Laplace's Equation:</u>

$$U(r) = A r^{l+1} + B r^{-l}$$

$$P(\theta) = C \cdot P_l^m(\cos\theta)$$

$$Q(\phi) = D \cdot e^{im\phi}$$

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} U(r)\cdot P(\theta)\, Q(\phi)$$

$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \left(A r^{l+1} + B r^{-l-1}\right)\cdot C\, P_l^m(\cos\theta)\cdot D e^{im\phi}$$

$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \left(A r^{l+1} + B r^{-l-1}\right)\cdot Y_l^m(\theta,\phi)$$

<u>Spherical Harmonics:</u>

$$\boxed{Y_l^m(\theta,\phi) = P_l^m(\cos\theta)\, e^{im\phi}}$$

⑤ Variables by Boundary Conditions:

B_ℓ $\quad \Phi(r=0,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(A_\ell\cdot 0 + B_\ell\frac{1}{0}\right)\circ Y_\ell^m(\theta,\phi)$

$= 0 \quad , \text{ so } \quad B_\ell = 0$

A_ℓ $\quad \Phi(r=d,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_\ell \cdot d^{\ell+1}\, Y_\ell^m(\theta,\phi)$

$= V$

$$\int_0^{2\pi}\int_0^{\pi} V\cdot Y_\ell^m(\theta,\phi)\sin\theta\, d\theta\, d\phi = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_\ell d^{\ell+1}\int_0^{2\pi}\int_0^{\pi} Y^*(\theta,\phi)\, Y(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

$$= (2\ell+1)\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_\ell\cdot d^{\ell+1}\cdot\delta_{\ell\ell}\cdot\delta_{mm}$$

$$\int_0^{\pi}\int_0^{2\pi} Y^*(\theta,\phi)\,Y(\theta,\phi)\sin\theta\, d\theta\, d\phi = \underbrace{\frac{2\ell+1}{4\pi}}_{\text{coefficient}}\circ\underbrace{4\pi\,\delta_{\ell\ell}\cdot\delta_{mm}}_{\text{Integral leftovers}}$$

$$A_\ell = \frac{q}{4\epsilon_0}\frac{1}{d^{\ell+1}}\frac{1}{(2\ell+1)}Y_\ell^m(\theta,\phi)$$

$$\Phi_{in}(r,\theta,\phi) = \frac{q}{4\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r^\ell}{d^{\ell+1}}Y_\ell^m(\theta,\phi)\,Y_\ell^m(\theta,\phi)$$

$$= \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{r^\ell}{d^{\ell+1}}P_\ell^m(\cos\theta)$$

Idiosyncrasy from problem 3.6: $d^{\ell+1}$, d^ℓ, $r^{\ell+1}$, and r^ℓ. The exact answer remains circular.

Other Boundary Solutions generate:

	q	q'
$r<d$	$\Phi(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0}\sum\frac{r^\ell}{d^{\ell+1}}$	$\Phi(r,\theta,\phi) = \frac{q'}{4\pi\epsilon_0}\sum\frac{a^{2\ell}}{d^\ell r^{\ell+1}}$
$r>d$	$\Phi(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0}\sum\frac{d^\ell}{r^{\ell+1}}$	$\Phi(r,\theta,\phi) = \frac{q'}{4\pi\epsilon_0}\sum\frac{a^{2\ell}}{d^\ell r^{\ell+1}}$

q, q' $\quad \epsilon_0 \cdot E_{out} = \epsilon \cdot E_{in}$

$$\epsilon_0 \cdot \frac{\partial \Phi_{out}}{\partial r} = \epsilon \frac{\partial \Phi_{in}}{\partial r}$$

$$\epsilon_0 \cdot \frac{\partial}{\partial r}\left[\frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty} P_l(\cos\theta)\left(q\frac{r^l}{d^{l+1}} + q'\frac{a^{2l}}{d^l \cdot r^{l+1}}\right)\right] = \epsilon \frac{\partial}{\partial r}\left[\frac{1}{4\pi\epsilon}\sum_{l=0}^{\infty} q''\frac{r^l}{d^{l+1}} P_l(\cos\theta)\right]$$

$$q \cdot l + q'(-l-1)\frac{d}{a} = q'' \cdot l \quad @\ r=a \quad \boxed{\text{Equation \#1}}$$

$$\epsilon_0 E_{out} = \epsilon \cdot E_{in}$$

$$\frac{\partial \Phi_{out}}{\partial \theta} = \frac{\partial \Phi_{in}}{\partial \theta}$$

$$\frac{\partial}{\partial \theta}\left[\frac{1}{4\pi\epsilon_0}\sum P_l(\cos\theta)\left(q\frac{r^l}{d^{l+1}} + q'\frac{a^{2l}}{d^l \cdot r^{l+1}}\right)\right] = \frac{\partial}{\partial r}\left[\frac{1}{4\pi\epsilon_0}\sum q''\frac{r^l}{d^{l+1}} P_l(\cos\theta)\right]$$

$$\left(\frac{\epsilon}{\epsilon_0}\right) q + \left(\frac{\epsilon}{\epsilon_0}\right) q' \frac{d}{a} = q'' \quad @\ r=a \quad \boxed{\text{Equation \#2}}$$

$$q' = \frac{\left(\frac{\epsilon_0}{\epsilon}\right) - 1}{d\left(\frac{\epsilon_0}{\epsilon}(l+1) + 1\right)} q \cdot l \cdot a$$

$$q'' = \frac{2l+1}{\frac{\epsilon_0}{\epsilon}(l+1) + l} \cdot q$$

$$\Phi_{in}(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0 d}\sum_{l=0}^{\infty}\frac{2l+1}{\frac{\epsilon_0}{\epsilon}(l+1) + l}\left(\frac{r}{d}\right)^l P_l(\cos\theta)$$

$$\Phi_{out}(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0 d}\sum_{l=0}^{\infty} P_l(\cos\theta)\frac{d^{l+1}}{r^{l+1}}\left(1 + \frac{\left(\left(\frac{\epsilon_0}{\epsilon}\right) - 1\right) l}{\frac{\epsilon_0}{\epsilon_0}(l+1) + l}\left(\frac{a}{d}\right)^{2l+1}\right) \quad \boxed{r > d}$$

$$\Phi_{out}(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0 d}\sum_{l=0}^{\infty} P_l(\cos\theta)\left(\left(\frac{r}{a}\right)^{l} + \frac{\left(\frac{\epsilon_0}{\epsilon} - 1\right) l}{\left(\frac{\epsilon_0}{\epsilon}(l+1) + l\right)}\left(\frac{a}{d}\right)^l\left(\frac{a}{r}\right)^{l+1}\right) \quad \boxed{r < d}$$

b) $r/d \ll 1$

$$\Phi_{in}(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0 d}\left[1 + \frac{3}{1+2\epsilon_0/\epsilon d}\,\frac{r}{d}\cos\theta\right]$$

$$\Phi_{in}(r,\theta,\phi) = \frac{q}{4\pi\epsilon_0 d}\left[1 + \frac{3}{2+\epsilon/\epsilon_0}\,\frac{z}{d}\right]$$

$$E = -\nabla\Phi$$

$$= -\frac{q}{4\pi\epsilon_0 d^2}\left[\frac{3}{2+\epsilon/\epsilon_0}\right]\hat{z}$$

c) $$\lim_{\epsilon/\epsilon_0\to\infty}\Phi_{in} = \lim_{\epsilon/\epsilon_0\to\infty}\frac{q}{4\pi\epsilon_0 d}\left[1 + \sum_{l=0}^{\infty}\frac{2l+1}{(l+1)+l\epsilon/\epsilon_0}\left(\frac{r}{d}\right)^l P_l(\cos\theta)\right]$$

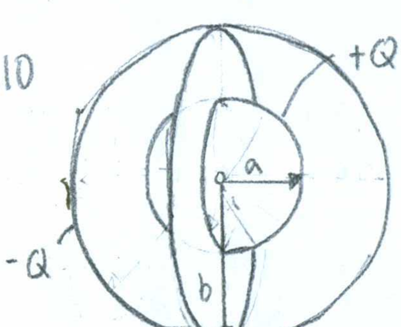
$$= \frac{q}{4\pi\epsilon_0 d}$$

$$\lim_{\epsilon/\epsilon_0\to\infty}\Phi_{out} = \lim_{\epsilon/\epsilon_0\to\infty}\left[\frac{q}{4\pi\epsilon_0 d}\sum_{l=0}^{\infty}P_l(\cos\theta)\left(\frac{d}{r}\right)^{l+1}\left(1 + \frac{\left(\left(\frac{\epsilon_0}{\epsilon}\right)-1\right)l}{\left(\frac{\epsilon_0}{\epsilon}(l+1)+l\right)}\left(\frac{a}{d}\right)^{2l+1}\right)\right] \quad \boxed{r > d}$$

$$= \frac{q(a/d)}{4\pi\epsilon_0 r} + \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}P_l(\cos\theta)\frac{d^l}{r^{l+1}} - \frac{q}{4\pi\epsilon_0 d}\sum_{l=0}^{\infty}P_l(\cos\theta)\frac{a^{l+1}}{r^{l+1}}\left(\frac{a}{d}\right)^l$$

$$\lim_{\epsilon/\epsilon_0\to\infty}\Phi_{out} = \lim_{\epsilon/\epsilon_0\to\infty}\frac{q}{4\pi\epsilon_0 d}\sum_{l=0}^{\infty}P_l(\cos\theta)\left(\left(\frac{r}{d}\right)^l + \frac{\left(\frac{\epsilon_0}{\epsilon}-1\right)l}{\left(\frac{\epsilon_0}{\epsilon}(l+1)+l\right)}\left(\frac{a}{d}\right)^l\left(\frac{a}{r}\right)^{l+1}\right) \quad \boxed{r < d}$$

4.10



Shape: Sphere
Dimension: Volume [3D]
Charge: +Q, -Q

Two concentric conducting spheres with a half-filled hemisphere.

a) $\boxed{\text{Potential Derivation by Boundary Conditions:}}$

① Boundary Conditions:

$$\Phi(r<a, \theta, \phi) = 0$$

② Laplace's Equation:

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$
$$= 0$$

③ Laplace's Equation Solutions:

Ⓐ Variable Separation: If $\Phi(r,\theta,\phi) = \frac{U(r)}{r}\cdot P(\theta)\cdot Q(\phi)$

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2\Phi}{\partial\Phi}$$

$$= \frac{r^2\sin^2\theta}{U(r)P(\theta)Q(\phi)}\left[\frac{P(\theta)Q(\phi)}{r}\frac{\partial^2 U(r)}{\partial r^2} + \frac{U(r)Q(\phi)}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) + \frac{U(r)P(\theta)}{r^2\sin\theta}\frac{\partial\Phi^2}{\partial\phi^2}\right]$$

$$= 0$$

Ⓑ Radial Eigenvalues:

$$\frac{r^2\sin\theta}{U(r)}\frac{\partial^2 U(r)}{\partial^2 r} + \frac{\sin^2\theta}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) = m$$

$$\frac{r^2}{U(r)}\frac{\partial^2 U(r)}{\partial^2 r} + \frac{1}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) = \frac{m}{\sin^2\theta}$$

$$\frac{r^2}{U(r)}\frac{\partial^2 U(r)}{\partial^2 r} = \frac{m}{\sin^2\theta} - \frac{1}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right)$$
$$= \ell(\ell+1)$$

$$\frac{\partial^2 U(r)}{\partial^2 r} - \frac{\ell(\ell+1)U(r)}{r^2} = 0$$

(C) Angular Eigenvalues:

$$\frac{m}{\sin^2\theta} - \frac{1}{P(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) = \ell(\ell+1)$$

$$\frac{1}{P(\theta)\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right) + \left(\ell(\ell+1) - \frac{m}{\sin^2\theta}\right) = 0$$

(D) Azimuthal Eigenvalues:

$$\frac{1}{Q(\phi)}\frac{\partial^2 Q(\phi)}{\partial\phi^2} = -m^2$$

$$\frac{\partial^2 Q(\phi)}{\partial\phi^2} + m^2 Q(\phi) = 0$$

(4) General Solution to Laplace's Equation:

$$U(r) = Ar^{\ell+1} + Br^{-\ell}$$

$$P(\theta) = C \cdot P_\ell^m(\cos\theta)$$

$$Q(\phi) = De^{im\phi}$$

$$\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{U(r)}{r} P(\theta) Q(\phi)$$

$$= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} (Ar^{\ell} + Br^{-\ell-1})(C \cdot P_\ell^m(\cos\theta))(De^{im\phi}) \qquad @\, m=0$$

$$= \sum_{\ell=0}^{\infty} (Ar^{\ell} + Br^{-\ell-1}) P_\ell(\cos\theta)$$

(5) Variables by Boundary Conditions:

$$\Phi(r,\theta,\phi) = (Ar^{l} + Br^{-\ell-1}) P_\ell(\cos\theta) \qquad @\, m=0$$

$$E = -\nabla\Phi$$

$$= -\frac{\partial}{\partial r}\left[Ar^{l} + Br^{-l-1}\right] \quad @\, l=0$$

$$= -\frac{\partial}{\partial r}\left[A + Br^{-1}\right]$$

$$= \frac{B}{r^2}$$

$$D = \epsilon\frac{B}{r^2} \quad \text{-or-} \quad D = \epsilon_0\frac{B}{r^2}$$

(Equation 4.37)

"Electric Susceptibility"

$$D = \epsilon_0 E$$

Gauss' Law: $\nabla \cdot E = \frac{\rho}{\epsilon_0}$

$$\int E\,ds = \frac{Q}{\epsilon_0}$$

$$\int D \cdot ds = Q$$

$$\int_0^{2\pi}\int_0^{\pi/2}(D)r^2\sin\theta\, d\theta\, d\phi + \int_0^{2\pi}\int_{\pi/2}^{\pi}(D)\cdot r^2\sin\theta\, d\theta\, d\phi = Q$$

$$\epsilon B_0\int_0^{2\pi}\int_0^{\pi/2}\sin\theta\, d\theta\, d\phi + \epsilon_0 B_0\int_0^{2\pi}\int_{\pi/2}^{\pi}\sin\theta\, d\theta\, d\phi = Q$$

$$B_0 = \frac{Q}{2\pi(\epsilon+\epsilon_0)}$$

$$D = \frac{\epsilon_0 Q}{(\epsilon+\epsilon_0)2\pi r^2} = \frac{\epsilon \cdot Q}{(\epsilon+\epsilon_0)2\pi r^2}$$

$$E = \frac{Q}{(\epsilon+\epsilon_0)2\pi r^2}$$

b) Surface-charge:

$$\sigma = \frac{\epsilon_0 Q}{(\epsilon + \epsilon_0) 2\pi a^2} \quad @ r = a$$

c) Induced Polarization:

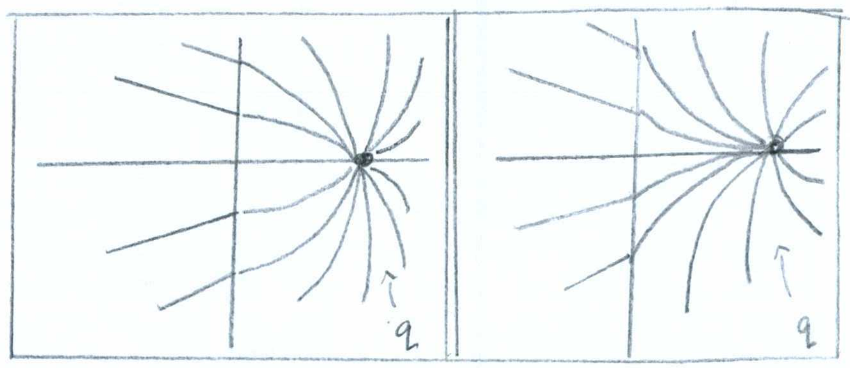
$$\sigma = -(P_2 - P_1) \cdot n$$
$$= P_1 \quad \ldots \text{because } P_2 = 0$$
$$P_1 = (\epsilon - \epsilon_0) E$$
$$= \frac{(\epsilon - \epsilon_0) E}{(\epsilon_0 + \epsilon) \cdot 2\pi a^2}$$

(Equation 4.46)

$$\sigma = -(P_2 - P_1) n_{21}$$

(Equation 4.46.5)

$$P = (\epsilon - \epsilon_0) E$$



$\epsilon_2 > \epsilon_1$ $\qquad$ $\epsilon_2 < \epsilon_1$

A charge's electric field "refracted" in medium, toward-or-away a material, and not a line at all, but volume.

4.11

Air at 292K		
Pressure (atm)	ϵ / ϵ_0	Density (g/mL)
20	1.0108	0.998435
40	1.0218	0.998435
60	1.0333	0.998435
80	1.0439	0.998435
100	1.0548	0.998435

Source: American Institute of Physics Handbook Gray. (1972)

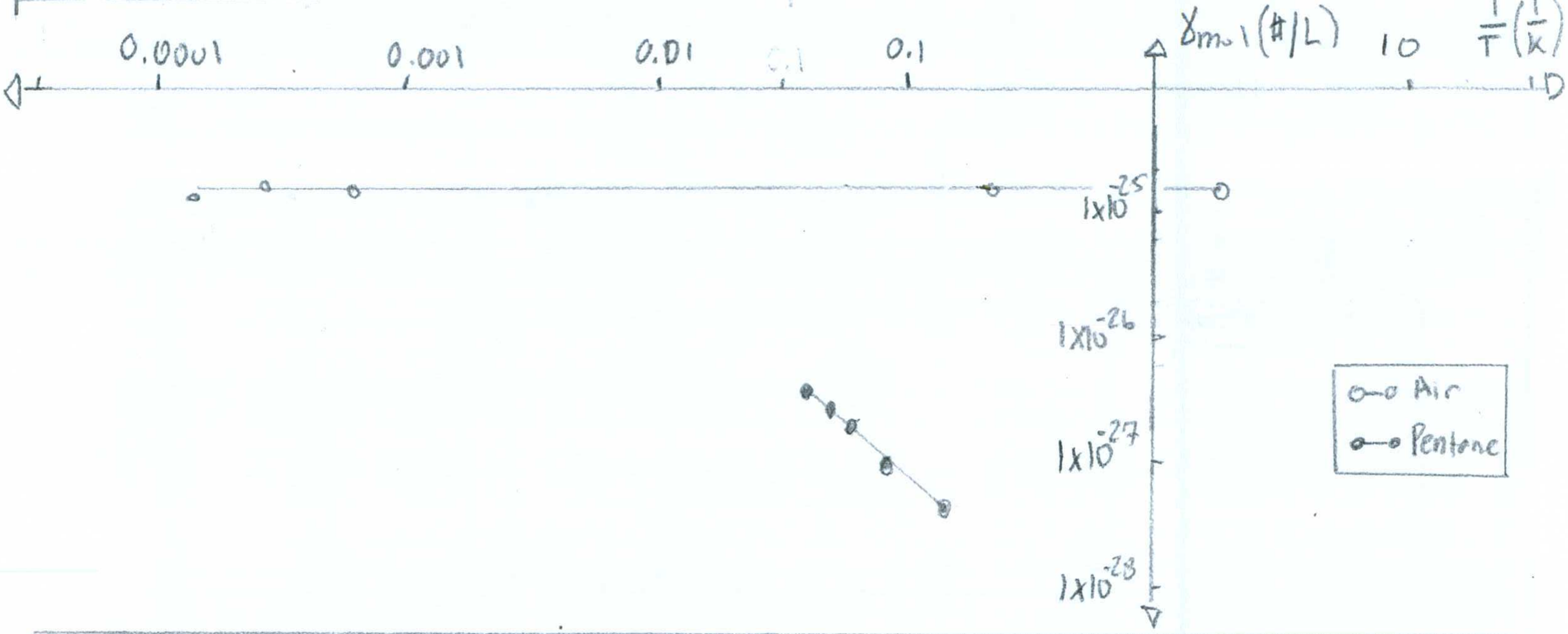
Pg 2-152, Table 2l-2

Pentane (C_5H_{12}) at 303K		
Pressure (atm)	ϵ/ϵ_0	Density (g/cm^3)
1	1.82	0.613
10^1	1.96	0.701
4×10^3	2.12	0.796
8×10^3	2.24	0.865
12×10^3	2.33	0.907

(Equation 4.70)

$$\gamma_{mol} = \frac{3}{N}\left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}\right)$$

"Claussius-Mossoti Relation"
(1874) (1850)



A "Flat"-slope indicates a nonpolar solution or medium. The experiment corretates polarization with temperature

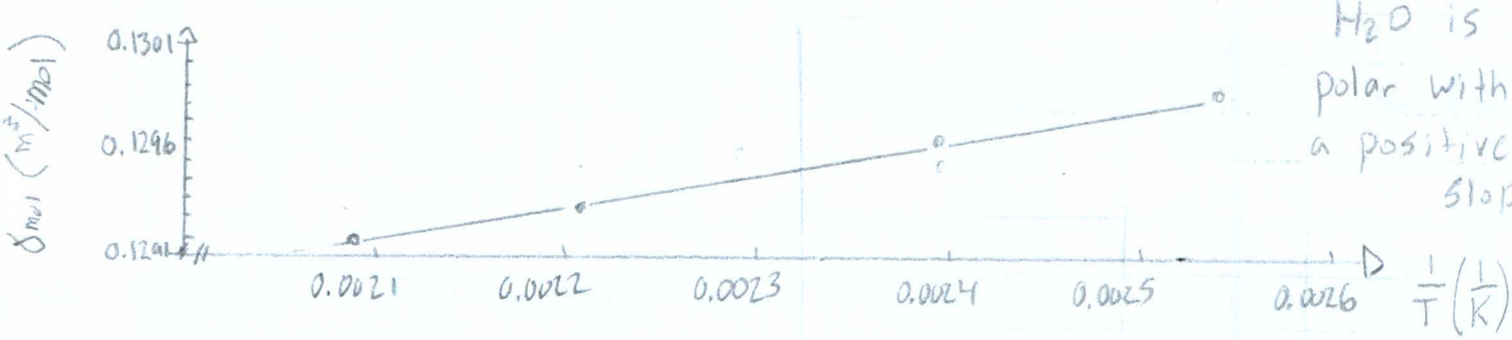
Another experiment, "cruders" relation justifies $\epsilon/\epsilon_0 \propto$ density. The slope is positive from the pentane table. In other terms, lights traversal in highly dense medium sharply refracts at the interface.

4.12

T(K)	Pressure (cm Hg)	$(\epsilon/\epsilon_0 - 1)\times10^5$
393	56.49	400.2
423	60.93	371.7
453	65.34	348.8
483	69.75	328.7

H_2O data.

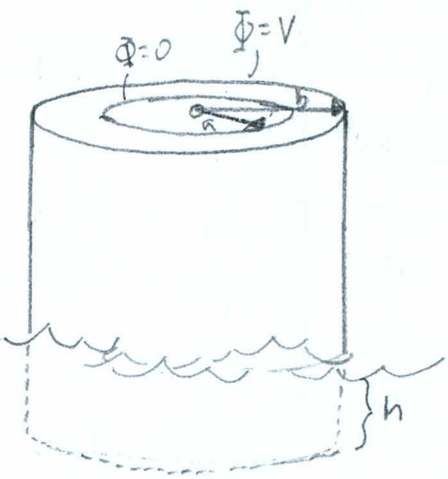
I guess data collection from refraction of light through water with Snell's Law for index of refraction, then dielectric constant.



H_2O is polar with a positive slope.

The polarization of light rotates more, even more, and none when in medium.

4.13



Two long, coaxial, cylindrical conducting surfaces lowered into liquid dielectric.

Shape: Cylinder
Dimension: Volume [3D]
Charge: q

Potential Derivation:

① Boundary Conditions:

$$\Phi(\rho=a, \phi, z) = 0$$

$$\Phi(\rho=b, \phi, z) = V \qquad \phi(\rho=b, \phi, z=h) = V$$

② Laplaces Equation:

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial^2 z}$$

$$= 0$$

③ Laplaces Equation Solutions:

If $\Phi(\rho, \phi, z) = R(\rho)\cdot Q(\phi)\cdot Z(z)$

Ⓐ Variable Separation:

$$\nabla^2\Phi = \frac{Q(\phi)Z(z)}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)}{\partial\rho}\right) + \frac{R(\rho)Z(z)}{\rho^2}\frac{\partial^2 Q(\phi)}{\partial^2\phi} + R(\rho)Q(\phi)\frac{\partial^2 Z(z)}{\partial^2 z}$$

$$= \frac{1}{R(\rho)\cdot\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)}{\partial\rho}\right) + \frac{1}{Q(\phi)\rho^2}\frac{\partial^2 \phi(\phi)}{\partial^2\phi} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial^2 z}$$

$$= 0$$

Ⓑ Radial Eigenvalues:

$$\frac{1}{R\cdot\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{1}{Q\,\rho^2}\frac{\partial^2 Q}{\partial\phi^2} = -k^2$$

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)+\rho^2k^2=\frac{-1}{Q\cdot\rho^2}\frac{\partial^2 Q}{\partial\phi^2}$$

$$\frac{\rho}{R}\left(\rho\frac{\partial^2 R}{\partial^2\rho}+\frac{\partial R}{\partial\rho}\right)+\rho^2k^2=m^2$$

$$\frac{\partial^2 R}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial R}{\partial\rho}+\left(k-\frac{m^2}{\rho^2}\right)R=0$$

C) <u>Azimuthal Eigenvalues:</u>

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial^2\phi}+m^2=0$$

$$\frac{\partial^2 Q}{\partial^2\phi}+m^2Q=0$$

D) <u>Vertical Eigenvalues:</u>

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}-k^2=0$$

$$\frac{\partial^2 Z}{\partial z^2}-k^2Z=0$$

E) <u>General Solution:</u>

$$\Phi(\rho,\phi,z)=R(\rho)Q(\phi)Z(z)$$

4) <u>General Solution to Laplaces Equation:</u>

$$R(\rho)=A\rho^m+B\rho^{-m} \quad \text{-or-} \quad R(\rho)=A_0+B_0\ln\rho$$

$$Q(\phi)=Ce^{im\phi}+De^{-im\phi}$$

$$Z(z)=E\sinh(kz)$$

$$\Phi(\rho,\phi,z)=(A\rho^m+B\rho^{-m})(Ce^{im\phi}+De^{-im\phi})(E\sinh(kz))$$

$$+A_0+B_0\ln\rho$$

<u>Note:</u> Two solutions with the radial equation as shown by the linear equation of constant coefficients.

⑤ Variables by Boundary Conditions:

E $\Phi(\rho=a, \phi, z=0) = (A \cdot a^{m} + B \cdot a^{-m})(Ce^{im\phi} + De^{-im\phi})(E\sinh(k\cdot 0))$

$+ A_0 + B_0 \ln a$

$= 0$, so $A_0 = -B_0 \ln a$

> Note: Prior arguments about general solutions integrated the angular coordinates from 0 to π (or 0 to 2π). The past arguments involved a multiplier ($e^{im\phi}$, $P_\ell^m(\cos\theta)$, $Y_\ell^m(\cos\theta)$, $\sinh(kz)$). Here is a case when the derivative of potential equates to zero. An "azimuthally symmetric" method.

C, D $\dfrac{d\Phi(\rho, \phi)}{d\phi} = \dfrac{d}{d\phi}\left[B_0 \ln(\rho/a) + (A\rho^{m} + B \cdot \rho^{-m})(Ce^{im\phi} + De^{-im\phi})\right]$

$= (A\rho^{m} + B\rho^{-m})(im \cdot Ce^{im\phi} - im\,De^{-im\phi})$

$= 0$, so $C = D = 0$

$\Phi(\rho, \phi) = B_0 \ln(\rho/a)$

B_0 $\Phi(\rho=b, \phi) = B_0 \ln(b/a)$

$= V$

$B_0 = \dfrac{V}{\ln(b/a)}$

$\Phi(\rho, \phi) = \dfrac{V \ln(\rho/a)}{\ln(b/a)}$

$E = -\nabla \Phi$

$$= -\frac{\partial}{\partial \rho}\left[\frac{V \cdot \ln(\rho/a)}{\ln(b/a)}\right]$$

$$= \frac{-V}{\ln(b/a)} \frac{1}{\rho}$$

(Equation 4.89)

$$W = \frac{1}{2}\int E \cdot D \, d^3x$$

Work = $\frac{1}{2}$ x Energy x "Diverging" angle in medium

(Equation 4.37)

$D = \epsilon \cdot E$

"Electric susceptibility is a medium coefficient x Electric Field"

ΔW = "Amount in air + Amount in water" subtracted by "All amounts in air"

$$= \frac{1}{2}\left[(L-h) \cdot \int E_{Air} \cdot D_{Air} \cdot d^3x + h\int E_{liq} \cdot D_{liq} \, d^3x - L\int E_{Air} \cdot D_{Air} \, d^3x\right]$$

$$= \frac{1}{2}\left[(L-h) \cdot \epsilon_0 \int E_{Air}^2 \, d^3x + h \cdot \epsilon_0 (1+\chi)\int E_{liq}^2 \, d^3x - L \cdot \epsilon_0 \int E_{Air}^2 \, d^3x\right]$$

$$= -\frac{1}{2} h \epsilon_0 \int E_{Air}^2 \, d^2x + \frac{1}{2}\epsilon_0 (1+\chi) h \int E_{liq}^2 \, d^2x$$

$$= \frac{h \epsilon_0 \chi}{2} \int E^2 d^2x$$

$$= \frac{h \epsilon_0 \chi}{2} \int_0^{2\pi} \int_a^b E^2 \rho \, d\rho \, d\phi$$

$$= \frac{h \epsilon_0 \chi}{2} \int_0^{2\pi} \int_a^b \left(V^2 \frac{1}{\ln(b/a)^2} \frac{1}{\rho^2}\right) \rho \, d\rho \, d\phi$$

$$= \frac{\pi h \epsilon_0 \chi \cdot V^2}{\ln(b/a)}$$

$$W = m \cdot g \cdot h$$

$$= \rho \cdot h \cdot A \cdot g \cdot h$$

$$= \rho \pi (b^2 - a^2) g h^2$$

$$= \frac{\pi h \epsilon_0 \chi \cdot V^2}{\ln(b/a)}$$

$$\chi = \frac{(b^2 - a^2) \rho g h \ln(b/a)}{\epsilon_0 \cdot V^2}$$