Quantum Mechanics: Rotating Frame

Scalar Field: $\phi(x^\mu)$ is transformed $\Lambda^\mu_\nu \simeq I^\mu_\nu + \omega^\mu_\nu$

then $\delta\phi(x^\mu) = \phi(x'^\mu = \Lambda^\mu_\nu x^\nu) - \phi(x^\mu) \simeq \frac{i}{2}\omega_{\mu\nu} L^{\mu\nu}\phi(x^\mu)$

where the Lorentz transformation: $L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$

Therefore, the infinitesimal Lorentz Boost in x^i is: $U \simeq I + i\omega_{oi} L^{oi}$

the exact Lorentz Boost U as $U = \exp(i\omega_{oi}(x^o p^i - x^i p^o))$ $\quad \simeq \exp(ikx - i\omega t)$

to become the Galilei Boost with velocity V $\quad U = \exp(it V\cdot\hat{p} - imV\cdot\hat{x})$

Now, implementation of an inertial frame can be implemented: $x' = x - Vt$, $t' = t$

I) U acting on the wave function: $\psi'(x', t') = \exp(-imV\cdot x + i\frac{1}{2}mV^2 t)\psi(x,t)$

II) Acting on the momentum operator: $p' = p$

$$\psi'(x', t') = \psi(x,t)$$

$$\hat{p}' = U^\dagger \cdot \hat{p} \cdot U = \hat{p} - mV;$$

Local Gauge Transformation: $e^{iF(x,t)} = \exp(-imV\cdot x + i\frac{1}{2}mV^2 t)$

Starting from the Schrodinger Equation: $i\frac{\partial}{\partial t}\psi(x,t) = H\psi(x,t)$, $H' = \frac{\vec{p}^2}{2m}$

To transform to F' as: $i(\frac{\partial}{\partial t'} + V\cdot\nabla')\psi(x', t') = H'\psi(x', t')$

$$i(\frac{\partial}{\partial t'} + V\cdot\nabla')\psi(x', t') = H'\psi(x', t') \;;\; H' = U^\dagger H \cdot U = \frac{(p - mV)^2}{2m} = \frac{\hat{p}'^2}{2m}$$

Energy Operator: $\hat{E}' = i(\frac{\partial}{\partial t'} + V\cdot\nabla') = \frac{E - V\cdot p}{(1-V^2)^{1/2}} = E - V\cdot p + \frac{1}{2}mV^2$

Using $p' = \nabla' - mV$; $(i\frac{\partial}{\partial t'} + \frac{1}{2}mV^2)\psi(x', t') = \frac{1}{2m}(\hat{p}' - mV)^2\psi(x', t')$

Minimal coupling of a gauge field: $A^\mu = (-\frac{1}{2}V^2, V)$, by a coupling constant: $p^\mu \to p^\mu - mA^\mu$

Non Relativistic Aspects of a Rotating Frame

$V = \Omega \times x$; $U = \exp(it(\Omega\times\hat{x})\cdot\hat{p}) = \exp(it\Omega\cdot\hat{L})$; $\hat{L} = \hat{x}\times\hat{p}$;

The gauge field A^μ for a rotating frame: $A^\mu(x^\mu) = (A_0(x), A(x)) = (-\frac{1}{2}(\Omega\times x)^2, \Omega\times x)$

$$H = \frac{1}{2m}(\hat{p} - m(\Omega\times x))^2 - \frac{1}{2}m(\Omega\times x)^2 - \Omega\cdot\hat{S}$$

$$\underbrace{m\frac{d^2\langle x\rangle}{dt^2}}_{\text{Force}} = \underbrace{2m\frac{d\langle x\rangle}{dt}\times\Omega}_{\text{Coriolis Force}} + \underbrace{m\Omega\times(\langle x\rangle\times\Omega)}_{\text{Centrifugal Force}}$$

The Sagnac Phase Shift is the same manner as Aharonov-Bohm Effect

$$\delta\phi_{sagnac} = \frac{m}{\hbar}\oint dl\cdot(\Omega\times x) = \frac{2m}{\hbar}\int ds\cdot\Omega = \frac{2mA\cdot\Omega}{\hbar} \;;\; \hat{\Phi}_{spin} = \underbrace{\hat{T}}_{\text{Time ordering operator}}\left[\exp\left(\frac{i}{\hbar}\int dt\,\hat{S}\cdot\Omega\right)\right]$$

$$= \exp\left(\frac{i}{\hbar}\hat{S}\cdot\Omega t\right) = \hat{I}\cos\left(\frac{\Omega t}{2}\right) + i\frac{\hat{\sigma}\cdot\Omega}{|\Omega|}\sin\left(\frac{\Omega t}{2}\right)$$

Schrödinger and Heisenberg Equations of Motion: $\langle \hat{A}(t)\rangle = \langle \psi(t)|\hat{A}|\psi(t)\rangle = \langle \psi(0)|U^\dagger \hat{A} U|\psi(0)\rangle$

$$= \left(\langle \psi(0)|U^\dagger\right)\hat{A}\left(U|\psi(0)\rangle\right)$$

$$= \langle \psi(0)|\left(U^\dagger \hat{A} U\right)|\psi(0)\rangle$$

Schrödinger Picture: $i\hbar \frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle$

$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle$

He assummed operators are independent of time: $\frac{\partial A}{\partial t} = 0$

Thereby, $\langle A(t)\rangle = \langle \psi|A|\psi\rangle$; $i\hbar\frac{\partial}{\partial t}\langle \hat{A}(t)\rangle = i\hbar\left[\langle \psi|\hat{A}|\frac{\partial \psi}{\partial t}\rangle + \langle\frac{\partial \psi}{\partial t}|\hat{A}|\psi\rangle + \cancel{\langle \psi|\frac{\partial A}{\partial t}|\psi\rangle}\right]$

$= \langle \psi|\hat{A}H|\psi\rangle - \langle \psi|HA|\psi\rangle$

$= \langle[\hat{A}, H]\rangle$

Heisenberg Picture: $A(t) = \langle \psi(t)|\hat{A}|\psi(t)\rangle_s = \langle \psi(t_0)|U^\dagger \hat{A}\cdot U|\psi(t_0)\rangle_s$

$|\psi_s(t)\rangle = U(t,t_0)|\psi_H\rangle$ $= \langle \psi|\hat{A}(t)|\psi\rangle_H$

$\tilde{A}_H(t) = U^\dagger(t,t_0)\hat{A}_s U(t,t_0)$

$|\psi_H\rangle = U^\dagger(t,t_0)|\psi_s(t)\rangle = |\psi_s(t_0)\rangle$ $\tilde{A}_H(t_0) = \hat{A}_s$; Time-Independent $-\ \partial|\psi_H\rangle/\partial t = 0$

$$\hat{A}|\phi_i\rangle_s = a_i|\phi_i\rangle_s \;;\; U^\dagger \hat{A} U U^\dagger|\phi_i\rangle_s = a_i U^\dagger|\phi_i\rangle_s \;;\; \hat{A}_H|\phi_i\rangle_H = a_i|\phi_i\rangle_H$$

Time Evolution Operator: $\frac{\partial A_H}{\partial t} = \frac{\partial}{\partial t}\left(U^\dagger\cdot \hat{A}_s\cdot U\right) = \frac{\partial U^\dagger}{\partial t}\hat{A}_s\cdot U + U^\dagger\cdot \hat{A}_s\frac{\partial U}{\partial t} + U^\dagger \overbrace{\cancel{\frac{\partial A}{\partial t}}}^{=0} U$

$= \frac{i}{\hbar}U^\dagger\cdot H\cdot \hat{A}_s\cdot U - \frac{i}{\hbar}U^\dagger\cdot \hat{A}_s\cdot H\cdot U + \left(\frac{\partial A}{\partial t}\right)$

$= \frac{i}{\hbar}H_H\hat{A}_H - \frac{i}{\hbar}\hat{A}_H\cdot H_H = -\frac{i}{\hbar}[\hat{A}, H]_H$

$i\hbar\frac{\partial}{\partial t}\tilde{A}_H = [\hat{A}, H]_H$

When $H_H = U^\dagger H U$

$U = e^{-iHt/\hbar}$

Classic Equivalence: $H = \frac{p^2}{2m} + V(x)$

$\dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial V(x)}{dx}$

$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$

$$[\hat{x}^n, \hat{p}] = i\hbar n\hat{x}^{n-1} \;;\; [\hat{x}, \hat{p}^n] = i\hbar n\hat{p}^{n-1}$$

$$\langle x(t)\rangle = \frac{\langle p\rangle t}{m} + \langle x(0)\rangle \;;\; m\frac{\partial^2\langle x\rangle}{\partial t^2} = -\langle \nabla V\rangle$$

Equations of Motion in General Relativity and Quantum Mechanics

Dirac Equation: $\psi(x) = e^{-ipx/\hbar}\, U_+(p)$; $\psi = e^{-ipx/\hbar}$ and $U_+(p) = \begin{pmatrix} u_1 \\ u_2\end{pmatrix}$

Tangent Vector Space: $T_p(M)\{\partial_0, \partial_1, \partial_2, \partial_3\}$; $dx^\mu\cdot\partial_\nu = \partial_\nu\cdot x^\mu = \delta^\mu_\nu$

Differential Form:

$ds^2 = g_{\mu\nu}\, dx^\mu\cdot dx^\nu = \eta_{ab}\cdot dx^a\cdot dx^b$

$ds \equiv \gamma_a dx^a$; $ds \otimes ds$

A generalized Dirac Equation: $\partial_s \equiv \gamma^a\cdot\frac{\partial}{\partial x^a}$; $\partial_s\psi\xi \equiv \gamma^a\frac{\partial \psi}{\partial x^a}\xi = \frac{\partial \psi}{\partial s}\xi$

$$\frac{ds}{ds}\tilde{\partial}_s\psi = \frac{1}{2}\left\{\frac{ds}{ds}, \partial_s\psi\right\} + \frac{1}{2}\left[\frac{ds}{ds}, \partial_s\psi\right] = \frac{d\psi}{ds} + \frac{ds_\wedge}{ds}\frac{\partial\psi}{\partial s} \;;\; ds = \gamma^a dx_a$$

$$\partial_s\psi = \gamma^a\frac{\partial \psi}{\partial x_a}$$

Definition: Hamilton-Jacobi: Function $- W = \int_{\sigma(\lambda)}\left(p\frac{dx}{d\lambda} - H\frac{dt}{d\lambda}\right)d\lambda$

$$\frac{dW}{dt} = -H\left(x_1, x_2, x_3, t, \frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_2}, \frac{\partial W}{\partial x_3}\right)$$

$$dW = p\,dx - H\,dt$$

Lemma 1 Let $\mathcal{Z}(W(x^a))$ be a differential W and let $\mathcal{Z}' = \frac{d\mathcal{Z}}{dW}$; Where

$$W = -\int \eta^{ab} p_a dx_b = \int p\,dx - H\,dt$$

$$\mathcal{Z}(W) = \int p^* dx - H^* dt \quad \text{"Hamilton Jacobi"}$$

Proof: $p_a^* = \mathcal{Z}' p_a$; $\mathcal{Z}(W)$, $p_0 = H$; $\frac{d\mathcal{Z}(W)}{dt} = \mathcal{Z}' \frac{\partial W}{\partial t} = -\mathcal{Z}' H(x_1, x_2, x_3, t, p_1, p_2, p_3)$

Haliton - Jacobi

$$= -\mathcal{Z}' H(x_1, x_2, x_3, t, p_1^*/\mathcal{Z}, p_2^*/\mathcal{Z}, p_3^*/\mathcal{Z})$$

$$= -H^*(x_1, x_2, x_3, t, p_1^*, p_2^*, p_3^*)$$

Conversely, $\frac{\partial W}{\partial t} = \frac{\partial \mathcal{Z}(W)}{dt} / \mathcal{Z}' = -H^*(x_1, x_2, x_3, t, p_1^*, p_2^*, p_3^*)/\mathcal{Z}'$

Hamilton - Jacobi

$$= -H(x_1, x_2, x_3, t, p_1^*/\mathcal{Z}', p_2^*/\mathcal{Z}', \cancel{p_3^*/\mathcal{Z}'})$$

$$= -H(x_1, x_2, x_3, t, p_1, p_2, p_3)$$

Lemma 2: $\mathcal{Z}(W(x,t))$ is a differential and $\sigma(\lambda)$ a family of curves on the manifold.

With unit tangent vectors $\frac{ds}{ds}$, with respect to a local tetrad, then, $\tilde{d}s\, \partial_s \mathcal{Z}(W)$.

$\mathcal{Z}(W)$ is a Hamiltonian-Jacobi Function such that $p_a^* = \frac{\partial \mathcal{Z}}{\partial s}\frac{\partial x_a}{dt} = \frac{\partial \mathcal{Z}_{11}(W)}{\partial x}$

Proof: $\frac{\tilde{d}s}{ds} \partial_s \mathcal{Z}(W)$; Exact Differential $\left[\frac{ds}{ds}, \frac{\partial \mathcal{Z}(W)}{\partial s}\right]$. Where $\mathcal{Z}(W) = \mathcal{Z}_{11}(W) + \mathcal{Z}_{\perp}(W)$. This means $\mathcal{Z}(W) = \mathcal{Z}_{11}(W) + \mathcal{Z}_{\perp}(W)$

Where $\left[\frac{\tilde{d}s}{ds}, \frac{d\mathcal{Z}_{11}(W)}{ds}\right] = 0$

and $\mathcal{Z}_{\perp}(W) = c_0 t + c_1 x_1 + c_2 x_2 + c_3 x_3$

Also, $\left[\frac{\tilde{d}s}{ds}, \frac{\partial \mathcal{Z}_{11}(W)}{\partial s}\right] = 0$, means $\frac{dx^a}{d\lambda}$ / such that $\frac{\partial \mathcal{Z}_{11}}{ds}$, λ_{11}

and $\exists g(\lambda)$ such that $p_{11a}^* = g(\lambda)\frac{dx_a}{d\lambda} = \frac{\partial \mathcal{Z}_{11} W}{\partial x^a}$

Also given $\frac{ds}{ds} \cdot \partial_s \mathcal{Z}_{11}(W)$ is an exact differential denoting $x_0 = t$

$$\frac{\tilde{d}s}{ds} \cdot \partial_s \mathcal{Z}_{11}(W) = \frac{\partial \mathcal{Z}_{11}(W)}{dx^1}\frac{\partial x^1}{\partial s} + \frac{\partial \mathcal{Z}_{11}(W)}{dx^2}\frac{\partial x^2}{ds} + \frac{\partial \mathcal{Z}_{11}(W)}{dx^3}\frac{dx^3}{ds}$$

$$+ \frac{\partial \mathcal{Z}_{11}(W)}{dt}\frac{dt}{ds} = \frac{d\mathcal{Z}_{11}(W)}{ds}$$

Or substituting, $g(s)\frac{dx_a}{ds} = \frac{\partial \mathcal{Z}_{11}(W)}{dx^a}$; $\frac{dx^a}{ds}\frac{dx_a}{ds} = 1$; $g(s) = \frac{\partial \mathcal{Z}_{11}(W)}{ds}$

$$\left(\frac{\partial \mathcal{Z}_{11}(W)}{\partial t}\right)^2 = (p_{11}^{*1})^2 + (p_{11}^{*2})^2 + (p_{11}^{*3})^2 + \left(\frac{\partial \mathcal{Z}_{11}(W)}{\partial s}\right)^2 = \left(H_0^*(x^a, p_{11}^{*1}, p_{11}^{*2}, p_{11}^{*3})\right)^2$$

$\mathcal{Y}(W) \doteq \mathcal{Y}_{\parallel}(W) + \mathcal{Y}_{\perp}(W)$; with $p_a^* = p_{\parallel a}^* + C_a$; $p_a^* = \frac{d\mathcal{Y}}{ds}\frac{dx_a}{dt} + C_a = \frac{\partial \mathcal{Y}_{\parallel}(W)}{dx^a} + \frac{\partial \mathcal{Y}_{\perp}(W)}{\partial x^a} \doteq \frac{\partial_s \mathcal{Y}(W)}{dx^a}$

$p_a^* = p_{\parallel a}^* + C_a$

Conversely, $p_a^* = \frac{d\mathcal{Y}}{ds}\frac{dx_a}{dt} + C_a = \frac{\partial \mathcal{Y}_{\parallel}(W)}{dx^a} + \frac{\partial \mathcal{Y}_{\perp}(W)}{\partial x^a} = \frac{\partial \mathcal{Y}(W)}{dx^a}$; $[d_s, \partial_s \mathcal{Y}_{\parallel}] = 0$

$d_s \partial_s \mathcal{Y}_{\parallel}(W) = d\mathcal{Y}_{\parallel}(W)$; $\left[\frac{d_s}{ds}, \frac{\partial \mathcal{Y}_{\parallel}(W)}{\partial s}\right]$

Corollary, $\mathcal{Y}(W) = W = -m_0 s + k$; $g(s) \equiv m_0$; $H^2 = p_1^2 + p_2^2 + p_3^2 + m_0^2$

Corrolary, $ds^2 = dx^a dx_a$; $\sigma(\lambda) = x^a(\lambda)$; $W = \int_r m \frac{ds}{d\lambda} ds = \int p\,dx - m\frac{dt}{d\lambda}dt$

λ = "Smooth curve"

Where $p^a = m\frac{dx^a}{d\lambda}$; $m = m_0 \frac{d\lambda}{ds}$

Proof ; Principle of Equivalence ; $p^a = m\frac{dx^a}{d\lambda}$; $m = m(s)\frac{d\lambda}{ds}$ "Rest Mass" $= \frac{\partial W}{ds} = m_0$

; $m(s) = m_0$; $dW = m\left(\frac{ds}{d\lambda}\right)^2 d\lambda = m_0 ds$

$dW = m\left(\frac{ds}{d\lambda}\right)^2 d\lambda = m_0 d\sigma$

$W = m_0 s + k = m_0\sqrt{x_1^2 + x_2^2 + x_3^2 - t^2} + k$

Therefore, $\widehat{\partial}_s W = \gamma^a \frac{\partial W}{\partial x^a}$; and $dW = \widehat{d}_s \partial_s \mathcal{Y}(W)$

Example : Equation of Motion for a projectile in the Minkowski space ;

$F = -m_0 g\hat{j}$; or $\frac{dx}{ds} = \dot{x}$; Where $m\ddot{x} = 0$; $m\ddot{y} = -m_0 g$

$x = x_0 + u_x s$; $y = y_0 + u_y s - \frac{1}{2}g s^2$; However,

$-ds^2 = dx^2 + dy^2 + dt^2$

$-m_0 ds = m_0 \dot{x}\,dx + m_0 \dot{y}\,dy - m_0 t\,dt$

$-m_0 ds = m_0 u_x dx + m_0(u_y - gs)dy - m_0 t\,dt$

Now let, $dW = -m_0 ds$ passing through (x_0, y_0, t_0)

$W = m_0 u_x x + m_0(u_y - gs)y - m_0 c^2 \mathring{t}\, t + W_0$

Indeed $\frac{\partial W}{\partial x} = m_0 u_x = p_x$; $\frac{\partial W}{\partial y} = m_0(u_y - gs) = p_y$; $\left(\frac{\partial W}{\partial t}\right) = -m_0 \hat{t}$

and $\left(\frac{\partial W}{\partial t}\right)^2 = m_0^2 + p_1^2 + p_2^2 + p_3^2 = H^2$ "Hamilton-Jacobi"

Dual operator $\widetilde{\partial}_s$; Metric operator $\widetilde{d}_s$

Lemma 3 ; If $\mathcal{Y}(W)$, Hamilton-Jacobi, such that $[\widetilde{\partial}_s W, d_s] = 0$

then there is a simultaneous eigenfunction ξ : $(\partial_s \mathcal{Y})\xi(p) = \partial\mathcal{Y}\xi(p)$

Where $\partial\mathcal{Y}_s = \frac{\partial\mathcal{Y}}{\partial s}$; reduces to $\partial_s \mathcal{Y} = \frac{\partial\mathcal{Y}}{\partial s}$ Where $\mathcal{Y} \in \mathcal{Y}_s \xi$

Corollary 3: $ds^2 = dx^a \cdot dx_a$; $p^a = m(s)\frac{dx^a}{ds}$; Hamilton-Jacobi $\psi(W)$ and vector

$(\partial_s \psi)\xi(p) = (\partial_s \psi)\xi(\rho)$

Proof by Cor 2: $p_a = \frac{\partial W}{\partial x^a}$; and solve for W:

$$[\tilde{\partial}_s W, \hat{\partial}_s] = \left[\gamma^a \frac{\partial W}{\partial x^a}, \gamma^a \cdot dx_a\right] = [\gamma^a p_a, \gamma^a \cdot dx_a] = 0$$

Remark: $\psi = Ae^{KW}$; A arbitrary constant $K = i/\hbar$

Corollary 4: $\psi = Ae^{KW}$ where $W = -ms + K$ and $K = i/\hbar$

Corollary ... : $\gamma^a \cdot \frac{\partial \psi}{\partial x} = -\frac{i}{\hbar} m\psi$; Proof: $\gamma^a \frac{\partial \psi}{\partial x} = \frac{-i}{\hbar} m\psi$; This can be written in Conventional:

$-i\hbar\gamma^0$, $\alpha^0 = \gamma^0$, $\alpha^a = \gamma^0\gamma^a$, $E = i\hbar H^* = i\hbar\frac{\partial \psi}{\partial t}$

$$\left[-i\hbar\left(\alpha_1\frac{\partial}{\partial x_1} + \alpha_2\frac{\partial}{\partial x_2} + \alpha_3\frac{\partial}{\partial x_3}\right) + \alpha_0 m\right]\psi = E\psi$$

where $\psi = \psi\xi$

Corollary 5: $W = -m_0 s + c_1x^1 + c_2x^2 + c_3x^3 + c_0x^0 + d = W_{\parallel} + W_{\perp}$

Let $s_a \equiv \frac{\partial W}{\partial x^a} = c_a$ (called spin); $d\sigma = \gamma_a dx^a + \gamma_a(c^a/m_0)ds$

$\partial_\sigma W \equiv \gamma^a \frac{\partial W}{\partial x^a} = \gamma^a(p_a + c_a)$; therefore exists a function

such that $\partial_\sigma \psi\xi(p) = (\partial_\sigma\psi)\xi(\rho)$

By construction $\left[\gamma^a\frac{\partial W}{\partial x^a}, d\tilde{\sigma}\right] = 0$; therefore $(\partial_\sigma\psi)\xi(\rho) = (\partial_\sigma\psi)\xi(\rho)$

Theorem 1: Lie Derivative + Dirac Equation.

$\{\sigma(s)\}$: $u^a = \frac{dx^a}{ds}$; $ds^2 = dx^a \cdot dx_a$; then the Lie Derivative.

$\mathcal{L}(p) = 0$; iff there exists a Hamiltonian-Jacobi $\psi(W)$

Such that $\left[\gamma^a\frac{\partial W}{\partial x^a}, ds\right] = 0$; and $\tilde{\partial}\psi\xi(p_{\parallel}) = \partial\psi(\xi p_{\parallel})$ long ... +

Proof: $\mathcal{L}_u(p) = 0$; such that $u^a = \delta_0^a$ and $p_0^a = 0$; Lorentz Transforming

$p^a = m_0u^a + m_0v^a$

$$\frac{\partial v^a}{\partial x^b}\frac{\partial x^b}{\partial s} = 0$$

Where the assumption is $m_0 u^a = p_{\parallel}^a = \frac{\partial W}{\partial x^a}$; $\psi = \psi(W)$, $[\partial_s\psi, ds] = 0$

$(\partial_s\psi)\xi(p_{\parallel}) = \partial_s\psi\xi(p_{\parallel})$

Therefore $\frac{\partial \psi}{\partial x^a} = \psi'\frac{\partial W}{\partial x^a}$

The Hamiltan Jacobi derived is $ds^2 = u\,dx - \mathring{t}\,dt$
The motion is described as a generalized Dirac equation with eigen vector

Solution: $\psi(W) = A\exp(k(u_0 x - \mathring{t}\,t))$

k = dimensional constant.

An detector at distance X_0 is modeled with an exponential dist
$\theta\exp(-\theta t)$ and mean $\frac{1}{\theta}$. Moreover, $x = X_0$, $\psi^2_{x_0}(W) = A\exp(2ku_0 X_0)\exp(2k\mathring{t}\,t)$
With this exponential distribution: $\theta = 2k\mathring{t} \doteq A\exp(2ku_0X_0)$
the ith decay at time t_i is: $\frac{1}{\theta} = \frac{1}{n}\sum_i (t_i - t_{i-1}) = \frac{t_n - t_0}{n}$.

Consider n identical particles: $W = \sum_i m_0 s_i$
in the Minkowski space: $\psi(s_1, s_2 .. s_n) = \prod_{i=1}^{n}\psi(s_i) = A\exp(k\sum_{i=1}^{n}(t^2 - p_1^2 - p_2^2 - p_3^2))$

$$= A\exp(knt^2)\exp(-k\sum(p_1^2 + p_2^2 + p_3^2))$$
$$= A\exp(knt^2)\exp(-k\sum p_i^2)$$

A suitable choice for A and B and k where $p_i = p_1^2 + p_2^2 + p_3^2$
is with $\epsilon_i = \frac{p_i^2}{2m_0}$, the squared wavefunction.

$$\psi^2(W) = A(N)\psi^2(t)\exp\left(\frac{-\beta}{2m_0}\sum p_i^2\right) = A(N)\psi^2(t)\exp(-\beta\sum\epsilon_i)$$

Such that

$$\psi^2(W(p_1 \cdots p_n)) \equiv A(N)\exp\left(-\frac{\beta}{2m_0}\sum p_i^2\right) = A(N)\exp(-\beta\sum\epsilon_i)$$

is independent of t. and center of mass frame.
In addition this also requires $\epsilon_i = p_i^2 \in \{h^2k^2/2m_0\}$ $[l = 0, 1, 2 ..]$
Then the wavefunction becomes $E = \sum n_i\epsilon_i \doteq \sum n_i = n_i$

$$\psi^2(W(p_1 \cdots p_n)) = A(N, n_1, n_2 -)\exp(-\beta\sum n_i\epsilon_i)$$

<u>Non-Geodesic Motion and Hamiltonian:</u> $W = \int(p\,dx - H\,dt)$
Hamilton Equation of Motion: $W;\ \frac{dW}{ds} = \mathring{W}$; $\left\|\frac{dx^a}{ds}\right\| = 1$; requires $\mathring{W} = \frac{\partial W}{\partial s} = -H(s)$
With the derivation of Hamilton-Jacobi:
$-H(s) = p\dot{x} - H\mathring{t}$
With the understanding: $\mathring{W} = \frac{\partial W}{\partial s} = -H(s)$
$\frac{\partial W}{\partial x} = p_a$; $\frac{\partial W}{\partial t} = -H$

Now $[\gamma^a \frac{\partial w}{\partial x}, ds] = 0$; impries $\gamma^a \frac{\partial W}{\partial x^a} = m(s)\frac{ds}{ds}$; But $(\partial_s \psi)\xi(p_{\parallel}) = (\partial_s \psi)\xi(p_{\parallel})$.

So $p_{\parallel}^a = \frac{\partial W}{\partial x^a} = m_0 \frac{dx^a}{ds}$; $p^a = p_{\parallel}^a + m_0 v^a$ is a Killing vector v^a $\mathcal{L}(v) = 0$ $m(s) = \frac{\partial W}{\partial s} = m_0$

Follows, $\mathcal{L}_u(p^a) = \mathcal{L}_u(p_{\parallel}) + \mathcal{L}_u(mv^a) = (p_{\parallel}^a) u^b - u_b^a p_{\parallel}^b = mu^a u^b - m^a u^b = 0$

- Final Observations: $\tilde{\partial}\psi = \partial_s\psi_{\parallel} + \partial_s\psi_{\perp}$, such that $[\partial\psi_{\parallel}, ds] = 0$ and $\{ds\psi_{\parallel}, ds\} = 0$

$d_s\psi_{\parallel}$ is projected along 'ds' satisfies $(\partial_s\psi)_{\parallel}\xi(p) = (\partial_s\psi)_{\parallel}\xi(p)$ $p_a = \frac{\partial W}{\partial x^a}$

where $d_s\xi(p) = d_s\xi(p)$; $\partial_s\psi_{\perp}$ defines spin along ds.

- The Hamilton Jacobi can be written in a covariant form for a general coordinate system: $dW = g^{\mu\nu} p_\mu \cdot dx_\nu$; with the corresponding wave operator

$$\gamma^\mu \frac{\partial\psi}{\partial x^\mu}\xi = \gamma^\mu p_\mu \psi \xi$$

With an associated curve: $2g^{\mu\nu} = \gamma^\mu\cdot\gamma^\nu + \gamma^\nu\cdot\gamma^\mu$

- The generalized Dirac Equation: where $\gamma^\mu = \frac{\partial x^\mu}{\partial x^a}\cdot\gamma^a$

$$\gamma^\mu \frac{\partial\psi}{\partial x}\xi = \frac{\partial\psi}{\partial s}\xi$$

- Gauge Potentials of the form A^μ are introduced by defining "Lorentzian is invariant" $p_\mu = m_0\frac{dx_\mu}{d\tau} = \partial_\mu W - eA_\mu$.

However, in general, $\oint A^\mu \cdot dx_\mu \neq 0$, and therefore is not exact;

and $\tilde{d}_s\partial_s W \neq \frac{d\psi}{ds}ds$

- Principle of Complimentary
- Kinematics and not dynamics of motion. i.e. Geodesic Motion.

<u>Relationship between Quantum Mechanics and Classical Mechanics</u>

Example: Consider particle m_0, having uniform velocity u_0, with respect to Minkowski space

The Hamilton Jacobi function is $W = m_0 u_x x - m_0 \overset{\circ}{t} t = -m_0 s$

$\approx p\cdot x - Ht$;

when $s=0$; $x = u_0 s$; $u_0 = \frac{dx}{ds}$

The Dirac-Delta Functions: $\psi(W) = \delta(W) \equiv W(s)$; in terms

"Lab Frame" $\psi(W(x,t)) = \delta(x - u_x t)$

In other words, the probability of particle at position $x = u_x t$ is 1

$\dot{W} = \frac{\partial W}{\partial s} = -H(s)$; $\frac{\partial W}{\partial x^a} = p_a$; $\frac{\partial W}{\partial t} = -H$; Written as covariant form $p^\mu = g^{\mu\nu}\frac{\partial W}{\partial x^\nu}$

Instead we work with tetrads: Differentiating $\frac{\partial H}{\partial s} = \dot{H}(s)$; $\dot{p}^a = -\frac{\partial H(s)}{\partial x^a}$

To derive the remaining equation of motion:

$$\dot{p}_a = \frac{\partial}{\partial x^a}\left(\frac{\partial W}{\partial s}\right) = \frac{\partial}{\partial x^a}\left(\frac{\partial W}{\partial x^b}\dot{x}^b + \frac{\partial W}{\partial t}\dot{t}\right) = \frac{\partial^2 W}{\partial x^a \partial x^b}\dot{x}^b + \frac{\partial^2 W}{\partial x^a \partial t}\dot{t}$$

Also, $$\frac{\partial}{\partial x^a}\left(\frac{\partial W}{\partial t}\right)\dot{t} = -\frac{\partial}{\partial x^a}\left(H\left(x^b, \frac{\partial W}{\partial x^b}, t\right)\right)\dot{t} = -\frac{\partial H}{\partial x^a}\dot{t} - \frac{\partial H}{\partial p^b}\frac{\partial^2 W}{\partial x^a \partial x^b}\dot{t}$$

$$= -\frac{\partial H(s)}{\partial x^a} - \frac{\partial H}{\partial p^b}\frac{\partial^2 W}{\partial x^a \partial x^b}\dot{t}$$

Combining the two yield: $\frac{\partial^2 W}{\partial x^a \partial x^b}\left(\dot{x}^b - \frac{\partial H(s)}{\partial p^b}\right) = 0$

Therefore, $\dot{x}^b = \frac{\partial H}{\partial p^b}$; provided $\det\left(\frac{\partial W}{\partial x^a \partial x^b}\right) \neq 0$

In terms of tetrad summation, notation, these can be written;

$$\frac{\partial x^a}{\partial s} = \eta^{ab}\frac{\partial H(s)}{\partial p^b} \; ; \; \frac{dp^a}{\partial s} = -\eta^{ab}\frac{\partial H(s)}{\partial x^b}$$

Covariant Form:

$$\frac{dx^\mu}{d\tau} = g^{\mu\nu}\frac{\partial K}{\partial p^\nu} \; ; \; \frac{Dp^\mu}{\partial \tau} = -g^{\mu\nu}\frac{\partial K}{\partial x^\nu}$$; where $\frac{Dp^\mu}{\partial\tau} = p^\mu{}_{;\nu}u^\nu = \dot{p} + \Gamma^\mu_{\nu\lambda}\dot{x}^\mu \dot{x}^\lambda$

Oscillator Dynamics - Heisenberg Picture: $X(0) = X$; $P(0) = P$

To compute $X(t)$ and $P(t)$; $X(t) = e^{i/\hbar Ht} \cdot X(0) e^{-i/\hbar Ht}$; $P(t) = e^{i/\hbar Ht} \cdot P(0) e^{-i/\hbar Ht}$

$$H = H(t) = \frac{P^2(t)}{2m} + \frac{1}{2}m\omega^2 X^2(t) = H(0) = \frac{P^2(0)}{2m} + \frac{1}{2}m\omega^2 X^2(0)$$

- A possible solution is to expand the exponentials, then deduce a closed form expression.
- A second solution is to solve the Heisenberg Equations.

$[X(t), P(t)] = i\hbar I$, we have $i\hbar \frac{d}{dt}X(t) = [X(t), H] = i\hbar \frac{P(t)}{m}$

$$i\hbar \frac{d}{dt}P(t) = [P(t), H] = -i\hbar m\omega^2 X(t)$$

So, that $\frac{d}{dt}X(t) = \frac{P(t)}{m}$; $\frac{d}{dt}P(t) = -m\omega^2 X(t)$

$$X(t) = (\cos\omega t)X(0) + \left(\frac{1}{m\omega}\sin\omega t\right)P(0)$$

$$P(t) = (\cos\omega t)\,P(0) - (m\omega\sin\omega t)X(0)$$

Means: $\langle X\rangle(t) = (\cos\omega t)\langle X\rangle(0) + \frac{1}{m\omega}\sin\omega t\langle P\rangle(0) =$

$$\langle P\rangle(t) = (\cos\omega t)\langle P\rangle(0) - (m\omega\sin\omega t)\langle X\rangle(0)$$

Where $\langle X\rangle(t) = \langle X\rangle(0) = 0$ and $\langle P\rangle(t) = \langle P\rangle(0) = 0$

Charged Particle in an Electric Field: $\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$, $\vec{B} = \nabla\times A$

"Dynamics of a Particle" $H = \frac{1}{2m}\left(\vec{P} - \frac{q}{c}A(\vec{X}, t)\right)^2 + q\phi(\vec{X}, t)$

(I) The Coulomb Field ; $\phi = \frac{K}{|\vec{X}|}$, $\vec{A} = 0$ "Stationary"

(II) Uniform Magnetic Field $\vec{B}$, where ; $\phi = 0$, $\vec{A} = \frac{1}{2}B\times\vec{X}$

(III) An electromagnetic Plane Wave ; $\phi = 0$; $\vec{A} = A_0\cos(\vec{k}\cdot\vec{X} - kct)$, $K\cdot A_0 = 0$

Models

Assume Time-Independence: $H = \frac{1}{2m}\left(P - \frac{q}{c}A(x)\right)^2 + q\phi(\vec{X})$

$$\frac{d}{dt}X(t) = \frac{1}{i\hbar}[X(t), H] = \frac{1}{m}\left\{P(t) - \frac{q}{c}A\,X(t)\right\}$$

$$P(t) = m\frac{dX(t)}{dt} + \frac{q}{c}A\,(X(t))$$

Canonical Momentum: $\Pi = m\frac{dX(t)}{dt} = P - \frac{q}{c}A(X(t))$

<u>Non-Commutative Operators:</u> $\psi = \psi^i e_i$; spinor manifold. ψ^i = scalar function

$\psi = \psi \xi$; ψ = scalar field; ξ = spinor

$$\underbrace{\partial_s \psi(W) = \gamma^a \frac{\partial \psi(W)}{\partial x^a}}_{\text{"Hamilton-Jacobi"}} ; \underbrace{\partial_s \psi(H) = \gamma^a \frac{\partial \psi(H)}{\partial x^a}}_{\text{"Hamiltonian"}}$$

$$[\partial_s \psi(W), \partial_s \psi(H)] \neq 0$$

$$\partial \psi(W) = \gamma^a \frac{\partial \psi(W)}{\partial x^a} = \gamma^a \dot{p}_a \psi'(W) ; \partial \psi(H) = \gamma^a \frac{\partial \psi(H)}{\partial x^a} = \gamma^a \dot{p}_a \psi'(H); \dot{p}^a = g(t) p^a$$

$$x^a = \exp\left(\int g(w) dw\right)$$

$$\partial \psi(H) \xi = \gamma^a \frac{\partial \psi(H)}{\partial x^a} \xi = \frac{dm}{ds} \xi ; \frac{dm}{ds} = \| f^a \| ; f^a$$

<u>Statistical Mechanics and Ideal Gases</u>; $W = -ms$; $H = \frac{\partial W}{\partial s} = m$; $\psi = \psi(H)$

$$\frac{\partial \psi}{\partial x^a} = -\psi' \dot{p}^a, \text{ where } \dot{p} = \frac{dp}{ds}$$

which is $\gamma^a \frac{\partial \psi}{\partial x^a} = -\gamma^a \psi' \dot{p}^a$

Taking the dot product with $m \frac{dx^a}{ds} \left(\frac{1}{2} \gamma^a \frac{\partial \psi(H)}{\partial x^a}, \gamma^a p_a \right)$

$$\frac{\partial \psi(H)}{\partial s} = -\psi'(H) \dot{p}^a p_a$$

Remember $\psi'(H) = \frac{\partial \psi(H)}{\partial H}$; $2 \dot{p}_a p^a = \frac{d}{ds}(p_a p^a)$

$$H = \frac{1}{2} p_a p^a$$

Solving when $\psi' = k \psi$; $\therefore \psi = A e^{\frac{k}{2} p^a p_a}$

If k is real and time dependent

$$\psi(t, x) = A \psi(t) \psi(x|t) = A \exp(k t^2) \cdot \exp(-k(p_1^2 + p_2^2 + p_3^2))$$

Let's assume product of n independent eigenfunctions; $\psi = e^{\frac{k}{2} \sum^n p^a p_a}$

$$\psi = C \exp\left[\frac{k}{2} m \sum (\dot{t}_1^2 - \dot{x}_1^2 - \dot{x}_2^2 - \dot{x}_3^2)\right] = e^{\frac{k}{2} nm} ; T = \frac{1}{k_B k} ; \psi = \psi(t) \psi(x_1, x_2, x_3)$$

Where $\psi(x_1, x_2, x_3 | t) = C \exp\left(\frac{-\frac{1}{2} m}{k_B T} \sum (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)\right)$

$$\psi = \psi(t) \psi(x_1, x_2, x_3)$$

Variance: $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \frac{k_B T}{2m}$; $\psi = e^{\int k(\tau) p_\mu \dot{p}^\mu d\tau}$

Conclusion:

$$\begin{tikzcd}
\text{General Relativity} \arrow[rr, leftrightarrow] & & \text{Quantum Mechanics} \\
\text{Metric Structure of Spacetime} \arrow[rr, leftrightarrow] \arrow[dr] & & \text{Hamilton-Jacobi} \arrow[dl] \\
& \text{Generalized Dirac} &
\end{tikzcd}$$

General Form $\gamma^\mu \frac{\partial \psi}{\partial x^\mu} = \phi$; where ϕ is physics of problem.

Maxwell Equations: Minkowski Space: $i\kappa^a \frac{\partial \psi}{\partial x^a} = -4\pi\phi$

Where $\phi_0 = \rho$ = charge Density; and $\phi_a = j_a$, $a \in \{1,2,3\}$ Current Density.

$\phi_0 = 0$; $\phi_a = H_a - i E_a$