Chapter 2:

(Markov's Inequality) (Chebyshev's Inequality)

$P[X \geq t] \leq \frac{E[X]}{t}$ $\qquad P[|X-\mu| \geq t] \leq \frac{Var(X)}{t}$ for $t > 0$

2.1

a) $P[X \geq t] = \sum_{X=t+1}^{n} P[X_i \geq t] = P[t+1 \geq t] + P[t+2 \geq t] + P[t+3 \geq t] + \cdots$

$$= \frac{E[X]}{t}$$

b) $P[|Y-\mu| \geq t] = P[(Y-\mu)^2 \geq c^2] = \sum_{(Y-\mu)^2 = c^2+1}^{n} P[(Y-\mu)^2 \geq c^2] = \frac{E[(Y-\mu)^2]}{c^2} = \frac{\sigma^2}{c^2}$

2.2

a) $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$; $Z \sim N(0,1)$

$\phi'(z) = \frac{-z}{\sqrt{2\pi}} e^{-z^2/2}$ and $\phi' + z\phi(z) = \frac{-z}{\sqrt{2\pi}} e^{-z^2/2} + \frac{z}{\sqrt{2\pi}} e^{-z^2/2}$

$$= 0$$

b) $\phi(z)\left(\frac{1}{z} - \frac{1}{z^3}\right) \leq P[Z \geq z] \leq \phi(z)\left(\frac{1}{z} - \frac{1}{z^3} + \frac{3}{z^5}\right)$

From part a, $\phi'(z) + z\phi(z) = 0$; $z = -\frac{\phi'(z)}{\phi(z)}$

$-\phi(z)\left(\frac{\phi(z)}{\phi'(z)} - \frac{\phi^3(z)}{\phi'(z)^3}\right) \leq -\frac{E[Z]\,\phi(z)}{\phi'(z)} \leq -\phi(z)\left(\frac{\phi(z)}{\phi'(z)} - \frac{\phi^3(z)}{\phi'(z)^3} + \frac{3\phi^5(z)}{\phi'(z)^5}\right)$

$0 \leq \frac{E[Z]}{\phi'(z)} + \frac{\phi(z)}{\phi'(z)} - \frac{\phi^3(z)}{\phi'(z)^3} \leq \frac{-3\phi^5(z)}{\phi'(z)^5}$

(Markov's Upper Bound) (Chernoffs Upper Bound)

$P[(X-\mu) \geq t] := P[e^{\lambda(X-\mu)} \geq e^{\lambda t}] \leq \frac{E[e^{\lambda(X-\mu)}]}{e^{\lambda t}}$ $\qquad \log P[(X-\mu) \geq t] \leq \inf_{\lambda \in [0,b]} \{\log E[e^{\lambda(X-\mu)}] - \lambda t\}$

2.3. $P[|X|^k \geq \delta^k] = \inf \frac{\mathbb{E}[|X|^k]}{\delta^k} := \inf \frac{\mathbb{E}[e^{\lambda X}]}{e^{\lambda\delta}}$

where $\lambda > 0$

2.4. X ; $\mu = E[X]$; $b > a$; $X \in [a,b]$

a) $\psi(\lambda) = \log E[e^{\lambda X}]$

$$\psi(0) = \log(1) = 0$$

$$\psi'(0) = \frac{Xe^{\lambda X}}{\mathbb{E}[e^{\lambda X}]} = \frac{X}{n} = \mu$$

b) $$\psi''(\lambda) = \frac{X^2 e^{\lambda X}\mathbb{E}[e^{\lambda X}] - Xe^{\lambda X}\mathbb{E}[F(X)e^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]^2}$$

$$= \frac{X^2}{n^2} - (e^{\lambda X})' \circ E[F(X)]$$

$$= \mathbb{E}[X^2] - E[X]^2$$

$$\sup|\psi''(\lambda)| = \sup \mathbb{E}[X^2] - E[X]^2$$

$$= \sup \mathbb{E}[X_1^2, X_2^2 \ldots X_{k-1}^2, X] - \mathbb{E}[X_1, X_2 \ldots X_{k-1}]^2$$

c) $X \sim N(\mu, (\frac{b-a}{2})^2)$

$$\mathbb{E}[e^{\lambda X}] = e^{\mu\lambda + \lambda^2 (\frac{b-a}{2})^2}$$

(Chebyshev's Upper bound)

$$\inf[\log \mathbb{E}[e^{\lambda(X-\mu)}] - \lambda t] = \inf\{\frac{\lambda^2}{2}(\frac{b-a}{2})^2 - \lambda t\} \leq \frac{2t^2}{(b-a)^2}$$

2.5.

a) $E[e^{\lambda X}] \leq e^{\frac{\lambda^2\sigma^2}{2} + \lambda\mu}$ for all $\lambda \in \mathbb{R}$

The derivation comes from a Taylor Expansion:

$$E[e^{\lambda X}] = 1 + \sum_{k=1}^{n} \frac{\lambda^k \mathbb{E}[X^k]}{k!}$$

$$= 1 + \frac{\lambda \mathbb{E}[X]}{1!} + \frac{\lambda^2 \mathbb{E}[X^2]}{2!} + \ldots$$

$$= 1 + \lambda\mu + \frac{\lambda^2\sigma^2}{2} + \ldots$$

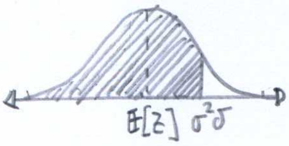
So, $\mathbb{E}[X] = \mu$

b) From part a), $\mathbb{E}[X^2] = \sigma^2$

c. Yes, $var(X) = \sigma^2$ is also the lowest value standard deviation possible. The inequality arises from the removal of "$\underline{1}$" - the first term.

(One Sided Bernsteins inequality)

$$\mathbb{E}\left[e^{\lambda(X-\mathbb{E}[X])}\right] \leq \exp\left(\frac{\frac{\lambda^2}{2}\mathbb{E}[X^2]}{1-\frac{b\lambda}{3}}\right), \text{ if } \lambda \in [0, 3/b)$$



2.6. $$P[Z_n \leq \mathbb{E}[Z_n] - \sigma^2\delta] := P\left[e^{\delta(Z-\mathbb{E}[Z])} \leq e^{-\sigma^2\delta}\right]$$

$$= P\left\{e^{\delta\left(1+\mathbb{E}[Z]+\frac{\delta\mathbb{E}[Z^2]}{2}-\mathbb{E}[Z]\right)} \leq e^{-\sigma^2\delta}\right\}$$

$$\leq P\left[e^{\delta\left(1+\mathbb{E}[Z]+\delta\frac{\mathbb{E}[Z^2]}{2}-\mathbb{E}[Z]\right)h(\lambda b)} \leq e^{-\sigma^2\delta}\right]$$

$$= P\left[e^{\delta\left(1+\delta\frac{\mathbb{E}[Z^2]}{2}\right)\sum_{k=2}^{\infty}\left(\frac{\delta b}{3}\right)^{k-2}} \leq e^{-\sigma^2\delta}\right]$$

$$\leq P\left[e^{\delta^2\frac{\mathbb{E}[Z^2]}{2}\left[\frac{1}{1-\delta b/3}\right]} \leq e^{-\sigma^2\delta}\right]$$

$$P[Z_n \leq \mathbb{E}[Z] - \sigma^2\delta] \leq \exp\left(\frac{n\sigma^2\delta^2}{2\,\mathbb{E}[Z^2]}\right)$$

$$\mathbb{E}[Z^2] = \mathbb{E}[X^4] = \int_0^{\infty}[X^4]\cdot P(\|X\|)\,dX$$

$$= 8\sigma^2$$

$$P[Z_n \leq \mathbb{E}[Z] - \sigma^2\delta] \leq \exp\left(\frac{n\delta^2}{16}\right)$$

2.7.

a. $$\log \mathbb{E}[e^{\lambda X_i}] = \left[\sum_{k=0}^{\infty}\frac{\lambda^k\mathbb{E}[X]^k}{k!}\right] = \left[1+\sum_{k=2}^{\infty}\frac{\lambda^k\mathbb{E}[X^2\cdot X^{k-2}]}{k!}\right]$$

$$= \left[1+\sum_{k=2}^{\infty}\frac{\lambda^k\cdot\sigma_i^2\cdot b^{k-2}}{k!}\right] = \left[1+\frac{\sigma^2}{b^2}\sum_{k=2}^{\infty}\frac{\lambda^k b^k}{k!}\right]$$

$$= \left[1+\frac{\sigma^2}{b^2}\left[e^{\lambda b}-1-\lambda b\right]\right]$$

$$\log \mathbb{E}[e^{\lambda X_i}] \leq \sigma_0^2\left\{\frac{e^{\lambda b}-1-\lambda b}{(b^2)}\right\}$$

b. $$P[\Sigma X_i \geq n\delta] \leq e^{n\sigma^2\left\{\frac{e^{\lambda b}-1-\lambda b}{b^2}\right\}-n\delta}$$

$$\underset{\lambda}{\operatorname{argmin}}\left\{n\sigma^2\frac{(e^{\lambda b}-1-\lambda b)}{b^2}-n\delta\right\}$$

$$= \frac{n\sigma^2}{b}(e^{\lambda b} - 1) - n\delta$$

$$= 0$$

$$\lambda^* = \frac{1}{b}\log\left(1 + \frac{b\delta}{\sigma^2}\right)$$

An introduction of maximum probability coefficient:

$$P[\sum X_i \geq t] \leq e^{\frac{n\sigma^2\{e^{\lambda b} - 1 - \lambda b\}}{b^2} - n\delta}$$

$$\leq e^{\frac{n\sigma^2}{b^2}\left[\left(1 + \frac{b\delta}{\sigma^2}\right) - 1 - \log\left(1 + \frac{b\delta}{\sigma^2}\right)\right] - \frac{n}{b}\log\left(1 + \frac{b\delta}{\sigma^2}\right)}$$

$$\leq e^{-\frac{n\sigma^2}{b^2}\left[\left(1 + \frac{b\delta}{\sigma^2}\right)\log\left(1 + \frac{b\delta}{\sigma^2}\right) - \frac{b\delta}{\sigma^2}\right]}$$

$$h(t) = (1+t)\log(1+t) - t \quad \text{for } t \geq 0$$

$$\leq e^{-\frac{n\sigma^2}{b^2}\left[h\left(\frac{b\delta}{\sigma^2}\right)\right]}$$

c. Through a plot, Bernstein's $\approx$ Bennett's

$$\frac{e^{\lambda b} - 1 - \lambda b}{b^2} \approx \frac{\lambda^2}{2(1 - b\lambda/3)}$$

2.8.

a) $P[Z \geq t] \leq C e^{\frac{-t^2}{2(v^2 + bt)}}$

$$\mathbb{E}[Z] = \int P[X \geq t]\,dt = \int_1 P[X \geq t]\,dt + \int_2 P[X \geq t]\,dt$$

$$\leq \int_1 C e^{-t^2/4v^2}\,dt + \int_2 C e^{-t^2/4bt}\,dt$$

Integral #1:

$$\int_1 e^{-t^2/4v^2}\,dt = \int_0^{2v\sqrt{\log C}} dt + \int_{2v\sqrt{\log C}}^{\infty} e^{-t^2/4v^2}\,dt$$

$$= 2v\sqrt{\log C} + \sqrt{\pi}\left(|v| - v\,\mathrm{erf}\left(\sqrt{\log C}\right)\right)$$

$$\leq 2v\left(\sqrt{\pi} + \sqrt{\log C}\right)$$

Integral #2:

$$\int_2 e^{-t^2/4bt}\,dt = \int_0^{4b\log c} dt + \int_{4b\log c}^{\infty} e^{-t/4b}\,dt$$

$$= 4b\log c + 4b \cdot c$$
$$\le 4b(1+\log c)$$

$$\mathbb{E}[Z] \le 2v(\sqrt{\pi}+\sqrt{\log c}) + 4b(1+\log c)$$

(Bernsteins Condition)

$$[\mathbb{E}[(X-\mu)^k]] \le \frac{1}{2}k!\,\sigma^2 b^{k-2} \quad \text{for } k=2,3,4$$

b) $$\mathbb{E}\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right|\right] = \int_0 P[X\ge t]\,dt = \int_1 P[X\ge t]\,dt + \int_2 P[X\ge t]\,dt$$

$$= \int_0^{\frac{2\sigma}{\sqrt{n}}\log c} dt + \int_{\frac{2\sigma\log c}{\sqrt{n}}}^{\infty} e^{-t^2/4\sigma^2}\,dt + \int_0^{\frac{4b}{n}\log c} dt + \int_{\frac{4b\log c}{n}}^{\infty} e^{-t/4b}\,dt$$

$$\le \frac{2\sigma}{\sqrt{n}}(\sqrt{\pi}+\sqrt{\log 2}) + \frac{4b}{n}(1+\log 2)$$

Where $c=2$ and $v=\sigma$.

(Kullbeck Leibler Divergence)

$$D(\delta\|\alpha) = \delta\log\left(\frac{\delta}{\alpha}\right) + (1-\delta)\log\frac{(1-\delta)}{(1-\alpha)}$$

2.9.

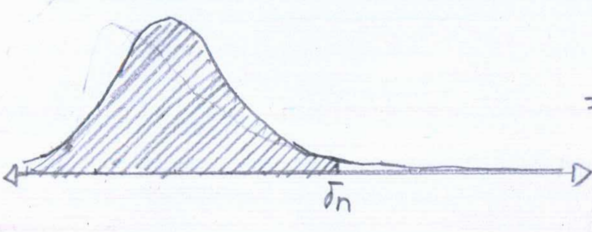
a) $$P[Z_n \le \delta n] := P[e^{\lambda Z_n} \le e^{\lambda\delta n}]$$

$$= P[\mathbb{E}[e^{\lambda Z_n}] \le \mathbb{E}[e^{\lambda\delta n}]]$$

$$= P\left[\sum_k^{\infty} Z_k\, e^{\lambda k} \le \mathbb{E}[e^{\lambda\delta n}]\right]$$

Binomial Moment Generating Function

$$= P\left[\sum_k^{\infty} \binom{n}{k}\alpha^k(1-\alpha)^{n-k} e^{\lambda k} \le \mathbb{E}[e^{\lambda\delta n}]\right]$$



$$= P\left[\sum \binom{n}{k} (\alpha e^{\lambda})^k (1-\alpha)^{n-k} \leq \mathbb{E}[e^{\lambda \delta n}]\right]$$

$$= P\left[(1-\alpha+\alpha e^{\lambda})^n \leq \mathbb{E}[e^{\lambda\delta n}]\right]$$

$$= \frac{(1-\alpha+\alpha e^{\lambda})^n}{e^{\lambda\delta n}}$$

$$\min_{\lambda}\left[\frac{(1-\alpha+\alpha e^{\lambda})^n}{e^{\lambda\delta n}}\right] = -\delta n + \frac{n\alpha e^{\lambda}}{(1-\alpha+\alpha e^{\lambda})} = 0$$

$$\lambda^* = \log \frac{\delta(1-\alpha)}{\alpha(1-\delta)} = \log\frac{(1-\alpha)}{(1-\delta)} - \log\left(\frac{\alpha}{\delta}\right)$$

$$P[Z_n \leq n\delta] \leq e^{\{-n\delta\lambda^* - \log(1-\alpha-\alpha e^{\lambda^*})^n\}}$$

$$\leq e^{\{-n[\delta(\log\frac{(1-\alpha)}{(1-\delta)} - \log\frac{\alpha}{\delta}) - \log(1-\alpha-\alpha\cdot e^{\log\frac{1-\alpha}{1-\delta} - \log\frac{\alpha}{\delta}})]\}}$$

$$\leq e^{-n[(\delta-1)\log\frac{(1-\alpha)}{(1-\delta)} - \delta\log(\frac{\alpha}{\delta})]}$$

$$\leq e^{-nD(\delta\|\alpha)}$$

(Hoeffding Bound)

$$P\left[\sum_{i=1}^{n}(X_i-\mu_i) \geq t\right] \leq \exp\left\{\frac{-t^2}{2\sum_{i=1}^{n}\sigma_i^2}\right\}$$

b) $P[Z_n \leq \delta n] = P[Z_n - \alpha n \leq \delta n - \alpha n]$

$$\leq e^{(-n(\delta-\alpha)^2)}$$ (Similarly, Question 2.3c)

$$\leq e^{\left\{\frac{-t^2}{2\sum_{i=1}^{n}\sigma_i^2}\right\}}$$

When $\delta = \alpha$, then the functions evaluate as one. $e^{-nD(\delta\|\alpha)} \leq e^{-n(\delta-\alpha)^2} \leq e^{-t^2/2\sum\sigma_i^2}$

The second derivative of e

at $\alpha = \delta$.

2.11

a) $\mathbb{E}[Z_n] = \int_0^\infty P[U > \sigma] d\sigma =$

$$= \int_0^{max} d\sigma + n\int_{max}^\infty P[U > x] dx$$

$$\underset{x}{\mathrm{argmax}}\{ne^{-x^2/2\sigma^2}\} = 0$$

$$x^* = \sqrt{2\sigma^2\log(n)}$$

$$\mathbb{E}[Z_n] = \sqrt{2\sigma^2\log(n)} + \int_{\sqrt{2\sigma^2\log n}}^\infty P[U > x] dx$$

$$= \sqrt{2\sigma^2\log(n)} + \frac{1}{\sqrt{2\pi}}\int_{\sqrt{2\sigma^2\log n}}^\infty e^{-x^2/2\sigma^2} dx$$

Supposedly, the book's approximation is a Gaussian "lower tail bound". -Mills Ratio-

$$\leq \sqrt{2\sigma^2\log(n)} + \left(\frac{4\sigma^2}{\sqrt{2\sigma^2\log 2}}\right)\sqrt{\frac{2}{\pi}}\int_{\sqrt{2\sigma^2\log n}}^\infty e^{-\delta^2/2} d\delta$$

$$\leq \sqrt{2\sigma^2\log(n)} + \sqrt{\frac{2}{\pi}}\sqrt{\frac{\pi}{2}}\frac{4\sigma^2}{\sqrt{2\sigma^2\log(n)}} \mathrm{erf}\left(\frac{\delta}{\sqrt{2}}\right)$$

$$\leq \sqrt{2\sigma^2\log(n)} + \frac{4\sigma^2}{\sqrt{2\sigma^2\log(n)}}$$

$$\leq \sqrt{2\sigma^2\log(n)} + \frac{4\sigma}{\sqrt{2\log(n)}}$$

b) $\mathbb{E}[Z, n \geq 5] = \mathbb{E}[\max|X|, n \geq 5]$

$$= \int_0^\infty P[\max|X|, n \geq 5] dx$$

$$= \int_0^{max} dx + \int_{max}^\infty P[\max|X|, n \geq 5] dx$$

Erf-Function:

$$\prod_{i=1}^n e^{-x^2/2\sigma^2} = \left[e^{-x^2/2\sigma^2}\right]^n = \left[1 - \phi(x)\right]^n$$

$$= \int_0^{\sqrt{2\sigma^2\log n}} dx + \int_{\sqrt{2\sigma^2\log n}}^\infty (1 - \phi(x))^n dx$$

At large n-values, $\left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$

$$= \sqrt{2\sigma^2\log n} + \sqrt{2\sigma^2\log n}\left(1 - \frac{1}{n}\right)^n$$

$$= (1 - 1/e)\sqrt{2\sigma^2\log n}$$

2.10

a) $\frac{1}{n} \log P[Z_n \le \delta n] = \frac{1}{n} P[\log Z_n \le \log \delta n]$

$= \frac{1}{n} P[\log \sum X_i \le \log \delta n]$

$= \frac{1}{n} P[\log \sum \binom{n}{m} \alpha^m (1-\alpha)^{n-m} \le \log \delta n]$

$= P[\frac{1}{n} \log \binom{n}{m} + \frac{m}{n} \log \alpha + \frac{(n-m)}{n} \log(1-\alpha) \le \log \delta n]$

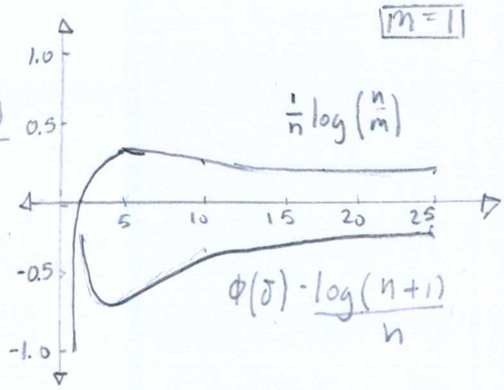
$= P[\frac{1}{n} \log \binom{n}{m} + \bar{\delta} \log \alpha + (1-\bar{\delta}) \log(1-\alpha) \le \log \delta n]$

$\le \frac{1}{n} \log \binom{n}{m} + \bar{\delta} \log \alpha + (1-\bar{\delta}) \log(1-\alpha) - \log \delta n$

b) $\frac{1}{n} \log \binom{n}{m} \ge \phi(\bar{\delta}) - \frac{\log(n+1)}{n}$

① The $\lim_{n \to \infty} \frac{1}{n} \log \binom{n}{m} \ge \phi(\bar{\delta}) - \frac{\log(n+1)}{n}$

② By graph, the inequality remains true.



c) $P[Z_n \le \delta n] := P[e^{\lambda Z_n} \le e^{\lambda \delta n}]$

$= P[\mathbb{E}[e^{\lambda Z_n}] \le \mathbb{E}[e^{\lambda \delta n}]]$

$= P[\sum_k^{\infty} Z e^{\lambda k} \le \mathbb{E}[e^{\lambda \delta n}]]$

$= P[\sum_k^{\infty} \binom{n}{k} \alpha^k (1-\alpha)^{n-k} e^{\lambda k} \le \mathbb{E}[e^{\lambda \delta n}]]$

$= P[\sum_n^{\infty} \binom{n}{k} (e^{\lambda} \alpha)^k (1-\alpha)^{n-k} \le \mathbb{E}[e^{\lambda \delta n}]]$

$= P[(1-\alpha + \alpha e^{\lambda})^n \le \mathbb{E}[e^{\lambda \delta n}]]$

$\operatorname{argmin}_{\lambda} [(1-\alpha+\alpha e^{\lambda})^n / e^{\lambda \delta n}] = 0$

$\lambda^* = \log \frac{\delta(1-\alpha)}{\alpha(1-\delta)} = \log \frac{(1-\alpha)}{(1-\delta)} - \log\left(\frac{\alpha}{\delta}\right)$

$P[e^{\lambda Z n} \le e^{\lambda n \delta}] = e^{-n\delta\lambda^* - \log(1-\alpha-\alpha e^{\lambda^*})^n}$

$= e^{-n[(\delta-1)\log\frac{(1-\alpha)}{(1-\delta)} - \delta \log\left(\frac{\alpha}{\delta}\right)]}$

$= e^{-nD(\delta \| \alpha)} \ge e^{-nD(\delta \| \alpha)} / (n+1)$

2.12. a) $\mathbb{E}[\max_{i=1..n} X_i] = \sum \mathbb{E}[e^{\lambda X}] \leq n e^{-\lambda^2\sigma^2/2}$

$$\log \mathbb{E}[\max e^{\lambda X}] \leq \log n - \frac{\lambda^2\sigma^2}{2} \approx 0$$ because an upper tail bound

$$\lambda^* = \sqrt{2\log n}/\sigma$$

$$\mathbb{E}[\max X_i] \leq \log n - \lambda^2\left(\frac{\sigma^2}{2}\right)$$

Authors across the pond scale λ at the beginning.

$$\frac{\mathbb{E}[\max X_i]}{\lambda} \leq \frac{\log n}{\lambda} - \lambda\left(\frac{\sigma^2}{2}\right)$$

$$\leq \frac{\sigma\sqrt{\log n}}{\sqrt{2}} - \frac{\sigma\sqrt{\log n}}{\sqrt{2}}$$

$$\leq \sqrt{2\sigma^2\log n}$$

b) $$\frac{\mathbb{E}[\max X_i]}{\lambda} \leq \sqrt{2\sigma^2\log n}$$

$$\mathbb{E}[\max X_i] \leq 2\sqrt{\log n} < 2\sqrt{\sigma^2\log n}$$

2.13 a) $$P[X_1+X_2] = \mathbb{E}[e^{\lambda(X_1+X_2)}]$$

$$= \mathbb{E}\left[e^{\lambda[(1+\mathbb{E}[X_1]+\frac{\lambda\mathbb{E}[X_1^2]}{2})+(1+\mathbb{E}[X_2]+\frac{\lambda\mathbb{E}[X_2^2]}{2})]}\right]$$

$$\leq \mathbb{E}[e^{\lambda^2(\mathbb{E}[X_1^2]+\mathbb{E}[X_2^2])/2}]$$

$$\leq \mathbb{E}[e^{\lambda^2(\sigma_1^2+\sigma_2^2)/2}]$$

$$\leq e^{\lambda^2(\sqrt{\sigma_1^2+\sigma_2^2})/\sqrt{2}}$$

b) $$P[X_1+X_2] = \mathbb{E}[e^{\lambda(X_1+X_2)}]$$

$$\leq e^{\lambda(\sigma_1^2+\sigma_2^2)/2}$$

$$\leq e^{\lambda\sqrt{2}\sqrt{\sigma_1^2+\sigma_2^2}/2}$$

c) $$P[X_1+X_2] = \mathbb{E}[e^{\lambda(X_1+X_2)}]$$

$$\leq e^{\lambda(\sigma_1^2+\sigma_2^2)/2}$$

d) $$P[X_1 \circ X_2] = \mathbb{E}[e^{\lambda(X_1 \circ X_2)}]$$

$$= \mathbb{E}\left[e^{(1+\lambda\mathbb{E}[X_1] + \lambda^2\mathbb{E}[X_1^2]/2)(1+\lambda\mathbb{E}[X_2]+\lambda^2\mathbb{E}[X_2^2]/2)}\right]$$

$$\leq \mathbb{E}\left[e^{\left(\frac{\lambda^2\mathbb{E}[X_1^2]}{2}\right)\left(\frac{\lambda^2\mathbb{E}[X_2^2]}{2}\right)}\right]$$

$$\leq \mathbb{E}\left[e^{\lambda^4\sigma_1^2\sigma_2^2/4}\right] \qquad (V, b) \text{ parameters: } e^{\lambda v^2/b}$$

$$(v, b) = (\sqrt{2}\sigma_1\sigma_2, \sqrt{2}\sigma_1\sigma_2)$$

2.14.

a) $P[X - \mathbb{E}[X] \geq t] := P[e^{\lambda(X-\mathbb{E}[X])} \geq e^{\lambda t}]$

$$= \mathbb{E}\left[e^{\lambda(1+\mathbb{E}[X] + \lambda\mathbb{E}[X^2]/2) - \lambda\mathbb{E}[X]} \geq e^{\lambda t}\right]$$

$$\leq \mathbb{E}\left[e^{\lambda^2\mathbb{E}[X^2]/2} \geq e^{\lambda t}\right]$$

$$\leq e^{\lambda^2\sigma^2/2 - \lambda t}$$

$$\underset{\lambda}{\operatorname{argmax}}\{\lambda^2\sigma^2/2 - \lambda t\} = 0$$

$$\lambda^* = \frac{t}{\sigma^2}$$

$$P[X - \mathbb{E}[X] \geq t] \leq e^{\lambda^2\sigma^2/2 - \lambda t}$$

$$\leq e^{-t^2/2\sigma^2}$$

$c_1 = 1$; $c_2 = 1/2\sigma^2$; $var(X) \leq \frac{c_1}{c_2}$; $\sigma \leq \sqrt{2}\sigma$

b) $\mathbb{E}[|X - m_X|] = \int_{-\infty}^{m_X} P[X \leq m_X]\,dx + \int_{m_X}^{\infty} P[X \geq m_X]\,dx$

Uniqueness of X:

$$\mathbb{E}[|X - m_X|] = 0 = \int_{-\infty}^{m_X} P[X \leq m_X]\,dx + \int_{m_X}^{\infty} P[X \geq m_X]\,dx$$

$$\int_{-\infty}^{m_X} P[X \leq m_X]\,dx = \int_{m_X}^{\infty} P[X \leq m_X]\,dx$$

Uniqueness of another X:

$$\mathbb{E}[|X-m_X|]=c=\int_{-\infty}^{m_X} P[X\leq m_X]\,dx+\int_{m_X}^{\infty} P[X\geq m_X]\,dx$$

$$\int_{-\infty}^{m_X} P[X\leq m_X]\,dx=c+\int_{-\infty}^{m_X} P[X\leq m_X]\,dx$$

c) $P[|X-m_X|\geq t]=P[|X-m_X|^2\geq t^2]$

$$=P\left[\frac{c_2}{8}|X-m_X|^2\geq \frac{c_2}{8}t^2\right]$$

$$\leq e^{-\frac{c_2}{8}t^2}\cdot\mathbb{E}\left[\frac{c_2}{8}|X-m_X|^2\right]$$

$$\leq e^{-\frac{c_2}{8}t^2}\cdot\int_t^{\infty} X\cdot e^{-\frac{c_2}{8}X^2}\,dX$$

I removed an absolute sign at $m_X=0$.

$$\leq \frac{4}{c_2}\cdot e^{-\frac{c_2}{8}x^2}$$

$$\leq \frac{4c_1}{c_2}e^{-\frac{c_2}{8}t^2}$$

$$\leq \frac{c_3}{c_2}e^{-c_4t^2}$$ where $c_3=4c_1$ and $c_4=\frac{c_2}{8}$

$$\leq c_3e^{-c_4t^2}$$

d) $P[|X-m_X|\geq t]=P[|X-m_X|\geq t]$

$$=P\left[\frac{c_4}{4}|X-m_X|\geq \frac{c_4}{4}t\right]$$

$$=P\left[\frac{c_4}{4}|X-m_X|\geq \frac{c_4}{4}\left(\frac{t}{2}\right)+\frac{c_4}{4}\left(\frac{t}{2}\right)\right]$$

$$|X-\mathbb{E}[X]|=|X-m_X|+\frac{c_4}{4}\left(\frac{t}{2}\right)$$

$$P[|X-m_X|^2\geq t^2]=P\left[\frac{c_4}{4}|X-E[X]|^2\geq \frac{c_4}{4}\left(\frac{t^2}{2}\right)\right]$$

$$=e^{-\frac{c_4}{4}\left(\frac{t^2}{2}\right)}\cdot\int_{\frac{t}{2}}^{\infty} X e^{-\frac{c_4}{4}X}\,dx$$

$$\le \frac{2}{c_4} \cdot e^{-\frac{c_4}{8}t^2} \quad \text{where } c_2 = \frac{c_4}{8} \text{ and } c_1 = 2c_3$$

$$\le \frac{c_3}{c_4} \cdot e^{-c_2 t^2/2}$$

$$\le c_3 e^{-c_2 t^2/2} \quad \text{when } c_4 \ge 1$$

2.15

a) $\hat{f}_n(x) := \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right)$; $\int_{-\infty}^{\infty} K(t)\,dt = 1$; $h > 0$

$$P[\|\hat{f}_n - f\|] = \frac{1}{nh}\int_{-\infty}^{\infty} K\left(\frac{x-\hat{X}_i}{h}\right) - K\left(\frac{x - X_i}{h}\right) dx$$

$$= \frac{2}{n}\int_{-\infty}^{\infty} K\left(\frac{x - X_i}{h}\right) dx$$

$$= \frac{2}{n}$$

$$P[\|\hat{f}_n - f_n\| \ge \mathbb{E}[\|f_n - f\| + \delta] = P[\|\hat{f}_n - f_n\| - \mathbb{E}[\|\hat{f}_n - f_n\|] \ge \delta]$$

$$= P[e^{\lambda(\|\hat{f}_n - f_n\| - \mathbb{E}[\|\hat{f}_n - f_n\|])} \ge e^{\lambda\delta}]$$

$$\le e^{\frac{\lambda^2\sigma^2}{n}} / e^{\lambda\delta}$$

$$\operatorname{argmin}_\lambda \left\{ \frac{\lambda^2\sigma^2}{n} - \lambda\delta \right\} = 0$$

$$\lambda^* = \frac{\delta n}{2\sigma^2}$$

$$P[\|\hat{f}_n - f_n\| - \mathbb{E}[\|f_n - f_n\|] \ge \delta] \le e^{\left(\frac{\delta^2 n^2}{2\sigma^2}\right)^2\left(\frac{\sigma^2}{n}\right)} / e^{\left(\frac{\delta n}{2\sigma^2}\right)\delta}$$

$$\le e^{\frac{-n\delta^2}{4}}$$

$$\le e^{-n\delta^2/8}$$

(Hibert Space)

$\mathbb{V}$ $\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$

1) $\langle f, g \rangle_{\mathbb{V}} = \langle g, f \rangle_{\mathbb{V}}$ For all $f, g \in \mathbb{V}$

2) $\langle f, f \rangle_{\mathbb{V}} \ge 0$ for all $f \in \mathbb{V}$, with equality iff $f = 0$

3) $\langle f + \alpha g, h \rangle_{\mathbb{V}} = \langle f, h \rangle_{\mathbb{V}} + \alpha \langle g, h \rangle$ for all $f, g, h \in \mathbb{V}$ and $\alpha \in \mathbb{R}$

Also, Hilbert Space ($\mathbb{H}$) is an inner product $(\langle \circ, \circ \rangle_{\mathbb{H}}, \mathbb{H})$ in which every Cauchy sequence $(f_n)_{n=1}^{\infty}$ in $\mathbb{H}$ converges to $f^* \in \mathbb{H}$.

2.16.

a) $\delta \geq 0$; $S_n = \| \sum X_i \|_{\mathbb{H}}$; $b^2 = \frac{1}{n} \sum_{i=1}^{n} b_i^2$; $\| X_i \|_{\mathbb{H}} \leq b_i$

$$P[|S - \mathbb{E}[S_n]| \geq n\delta] := \mathbb{E}[e^{\lambda|S - \mathbb{E}[S_n]|} \geq e^{\lambda n \delta}]$$

$$= \mathbb{E}[e^{\lambda(1 + \mathbb{E}[S_n] + \lambda \mathbb{E}[S_n^2]/2 - \lambda \mathbb{E}[S_n])} \geq e^{\lambda n \delta}]$$

$$\leq \mathbb{E}[e^{\lambda^2 \mathbb{E}[S_n^2]/2} \geq e^{\lambda n \delta}]$$

$$\leq \mathbb{E}[e^{\lambda^2 b^2/2} \geq e^{\lambda n \delta}]$$

$$\underset{\lambda}{\operatorname{argmin}} \{ \lambda^2 b^2/2 - \lambda n \delta \} = 0$$

$$\lambda^* = \frac{n\delta}{b^2}$$

$$P[|S - \mathbb{E}[S_n]| \geq n\delta] \leq \frac{e^{\lambda^2 b^2/2}}{e^{\lambda n \delta}}$$

$$\leq e^{\frac{-n\delta}{2b^2}}$$

$$\leq 2 \circ e^{\frac{-n\delta}{8b^2}}$$

b) $P[\frac{S_n}{n} \geq a + \delta] := P[e^{\lambda \frac{S_n}{n}} \geq e^{\lambda(a+\delta)}]$

$$= P[e^{\lambda(1 + \mathbb{E}[\frac{S_n}{n}] + \lambda \mathbb{E}[\frac{S_n^2}{n^2}]/2)} \geq e^{\lambda(a+\delta)}]$$

If $a = \sqrt{\frac{1}{n^2} \sum \mathbb{E}[\|X\|_{\mathbb{H}}^2]}$

$$P[\frac{S_n}{n} - a \geq \delta] \leq P[e^{\lambda^2 \mathbb{E}[S_n^2/n]/2} \geq e^{\lambda \delta}]$$

$$\underset{\lambda}{\operatorname{argmin}} \{ \lambda^2 \mathbb{E}[S_n^2/n^2]/2 - \lambda \delta \} = 0$$

$$\lambda^* = \delta / \mathbb{E}[S_n^2/n^2]$$

$$\le \frac{e^{\left(\frac{n\delta}{\mathbb{E}[S_n^2]}\right)^2 \frac{\mathbb{E}[S_n^2]}{2}}}{e^{\left(\frac{n\delta}{\mathbb{E}[S_n^2]}\right)\cdot\delta}}$$

$$\le e^{\frac{-n\delta^2}{2b^2}}$$

$$\le e^{\frac{-n\delta^2}{8b^2}}$$

(Hansen-Wright Inequality) 2.17

$$P[|X^T A X - \mathbb{E}[X^T A X]| > t] \le 2\cdot e^{\left[-c\cdot\min\left(\frac{t^2}{K^4\|A\|_F^2}, \frac{t}{K^2\|A\|_2}\right)\right]}$$

where $c>0$, $\|X\| \le K$, $\|A\|_F^2 = \sqrt{\sum A^2}$, $\|A\|_2 = \sup\|Ax\|_2$

2.17.

$$Z = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} X_i X_j$$

$$P[|X^T A X - \mathbb{E}[X^T A X]| > t] = P[|Z - \mathbb{E}[Z]| \ge t]$$

$$\le P[e^{\lambda^2 \mathbb{E}[Z^2]/2} \ge e^{\lambda t}]$$

$$\le e^{-\frac{t^2}{2\sigma^2}}$$

(v,b) parameters : $e^{-\frac{\lambda v^2}{b}}$; $(v,b) = (t, 2\sigma^2\lambda)$

$$P[|X^T A X - \mathbb{E}[X^T A X]| > t] = e^{-\min\left(\frac{\lambda^2\sigma^2}{2} - \lambda t\right)}$$

$$= e^{-\min\left(\left(\frac{v}{b}\right)^2, \left(\frac{v}{b}\right)\right)}$$

$$= e^{-\min\left(\frac{t^2}{4\cdot\lambda^2\cdot\sigma^4}, \frac{t}{2\lambda\sigma^2}\right)}$$

$$\le 2\cdot e^{-c\min\left(\frac{t^2}{K^4\|A\|_F^2}, \frac{t}{K^2\|A\|_2}\right)}$$

(Orlicz Norm) 2.18

$$\|X\|_\psi = \inf\{t>0 \mid \mathbb{E}[\psi(t^{-1}|X|)] \le 1\}, \quad \psi(0)=0$$

2.18.

a) $P[|X| > t]$

$$P[|X|^q > t^q] = P[e^{\lambda|X|^q} > e^{\lambda t^q}]$$

$$= P\left[e^{\lambda(1+\mathbb{E}[X]+\lambda\mathbb{E}[X^2]/2)^q} > e^{\lambda t^2}\right]$$

$$\leq \frac{e^{\lambda^2\mathbb{E}[X^2]^q/2^2}}{e^{\lambda t^q}}$$

$$\operatorname{argmin}\{\lambda^2\mathbb{E}[X^2]^q/_2 - \lambda t^q\} = 0$$

$$\lambda^* = \frac{t^q}{\mathbb{E}[X^2]^q} = \frac{t^q}{\|X\|^q}$$

$$P[|X|^q > t^q] \leq e^{\frac{-t^q}{2\|X\|^q}}$$

$$\leq 2^\circ e^{\frac{-t^q}{2\|X\|^q}}$$

$$\leq c_1 e^{-c_2 t^q}$$ where $c_1 = 2$ and $c_2 = \|X\|^{-q}$

b) If $1 \geq c_1 e^{-c_2 t^q}$

$$e^{1/c_2 t^q} - 1 \geq 0$$

$$\mathbb{E}[e^{1/c_2 t^2} - 1] = \int_0^\infty P(e^{1/c_2 t^q} - 1 > u)\,du$$

$$\leq \int_0^\infty P(c_2 > t^q \log(1+u)^{1/}\,)\,du$$

$$\leq c_1 \int_0^\infty e^{-c_2 t^q \log(1+u)}\,du$$

$$\leq c_1 \int_0^\infty e^{-c_2 t^2 v}(1+v)\,dv$$ when $v = \log(1+u)$

$$= \frac{c_1(1-c_2 t^q)}{c_2^2 t^{2q}}$$

$$\lim_{t\to\infty} \frac{c_1(1-c_2 t^q)}{c_2^2 t^{2q}} = 0$$ and less than infinity.

2.19. An example about inverse functions:

$$\psi\left(\frac{\mathbb{E}[\max_{i=1..n}|X_i|]}{\sigma}\right) \le \mathbb{E}\left[\psi\left(\frac{\max_{i=1..n}|X_i|}{\sigma}\right)\right]$$

$$\le \sum_{i=1}^{n} \mathbb{E}\left[\frac{\psi(|X|)}{\sigma}\right]$$

$$\le n$$

A function and inverse function both strictly increase when positive.

$$\psi\left(\frac{\mathbb{E}[\max_{i=1..n}|X_i|]}{\sigma}\right) \le n$$

$$\frac{\mathbb{E}[\max_{i=1..n}|X_i|]}{\sigma} \le \psi^{-1}(n)$$

$$\mathbb{E}[\max_{i=1..n}|X_i|] \le \sigma\psi^{-1}(n)$$

(Rosenthal's Inequality)

$$\mathbb{E}\left[\left(\sum_{i=1}^{n} X_i\right)^{2m}\right] \le R_m\left\{\sum_{i=1}^{n}\mathbb{E}[X_i^{2m}] + \left(\sum_{i=1}^{n}\mathbb{E}[X_i^{2}]\right)^{2m}\right\}$$

2.20. $\|X_i\|_{2m} = (\mathbb{E}[X^{2m}])^{1/2m} \le C_m$

$$P\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| \ge \delta\right]$$

$$P\left[\left|\frac{1}{n}\sum X_i\right|^{2m} \ge \delta^{2m}\right] := P\left[e^{\lambda(|\frac{1}{n}\sum X_i|)^{2m}} \ge e^{\lambda\delta^{2m}}\right]$$

$$= P\left[e^{\lambda(1+\mathbb{E}[X_i^{2m}] + \frac{\lambda\mathbb{E}[X_i^{2m}]^2}{2})/n^{2m}} \ge e^{\lambda\delta^{2m}}\right]$$

$$\le P\left[e^{\frac{\lambda^2\mathbb{E}[X_i^2]^{2m}}{2n^{2m}}} \ge e^{\lambda\delta^{2m}}\right]$$

$$\operatorname{argmin}_{\lambda}\{\lambda^2\mathbb{E}[X_i^2]^{2m}/2n^{2m} - \lambda\delta^{2m}\} = 0$$

$$\lambda^* = \left(\frac{\delta n}{\mathbb{E}[X_i^2]}\right)^{2m} \le \left(\frac{\delta n}{C_m}\right)^{2m}$$

$\leq 2^{n(R-D(\delta\|\alpha))}$ where $N = 2^{nR}$

b) i) $R - D(\delta\|\alpha) > 0$

$$P[2^{V} \geq 2^{\delta}] = 2^{n(R-D(\delta\|\alpha))}$$

$$\geq \frac{\mathbb{E}[V]^2}{\mathbb{E}[V^2]}$$ where Bernoulli mean $= \alpha$

variance $= \alpha(1-\alpha)$

$$\geq \frac{\alpha^2}{\alpha(1-\alpha)}$$ at $\alpha = 1/2$

$$\geq 1$$

ii) $\lim_{n\to\infty} P[2^{V} \geq 2^{\delta}] = \lim_{n\to\infty} 2^{n(R-D(\delta\|\alpha))}$

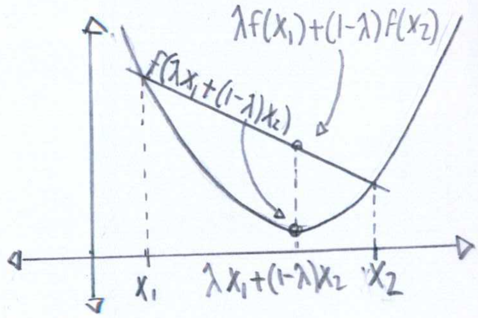
$$= \infty$$

2.22.

(Ising Model)

$$P_\theta(x_1, \ldots, x_d) = \exp\left\{\frac{1}{\sqrt{d}} \sum_{i\neq j} \theta_{jk} x_j x_k - F_d(\theta)\right\}$$

$$F_d(\theta) = \log\left(\sum_{i=1}^{n} \exp\left\{\frac{1}{\sqrt{d}} \sum \theta_{jk} x_j x_k\right\}\right)$$



(convex Function)

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

a) $F_d(\theta) = \log\left(\sum_{i=1}^{n} \exp\left\{\frac{1}{\sqrt{d}} \sum \theta_{jk} x_j x_k\right\}\right)$

$$= \log\left(\sum_{i=1}^{n} \exp\left\{\frac{1}{\sqrt{d}} [\theta x_1 + (1-\theta) x_2]\right\}\right)$$

$$\leq \theta \log\left(\sum_{i=1}^{n} \exp\left\{\frac{x_1}{\sqrt{d}}\right\}\right) + (1-\theta) \log\left(\sum_{i=1}^{n} \exp\left\{\frac{x_2}{\sqrt{d}}\right\}\right)$$

b) $\frac{\|F_d(\theta) - F_d(\theta')\|}{\|\theta - \theta'\|} = \|\nabla_\theta F_d(\theta)\|$

$$= \frac{1}{\sqrt{d}} x_j x_k$$ where $|x_j x_k| = 1$

$$\leq \sqrt{d}$$ for the set $\{1, 2, \ldots, d\}$

So, $\|F_d(\theta) - F_d(\theta')\| \leq \sqrt{d}\,\|\theta - \theta'\|$

$$P\left[\left|\frac{1}{n}\sum X_i\right|^{2m} \geq \delta^{2m}\right] \leq e^{\left(\frac{n\delta}{c_m}\right)^{2m} \alpha \left(\frac{c_m}{2n}\right)^{2m} - \left(\frac{n\delta}{c_m}\right)^{2m} \cdot \delta^{2m}}$$

$$\leq e^{-\left(\frac{n\delta}{2c_m}\right)^{2m}} \qquad c_m = \frac{n^{3/2}\delta^3}{2}$$

$$\leq e^{-\left(\frac{1}{\sqrt{n}\delta}\right)^{2m}}$$

$$\leq e \qquad \beta_m = e$$

$$\leq \beta_m \left(\frac{1}{\sqrt{n}\delta}\right)^{2m}$$

(Rescaled Hamming Distortion)

$$d(X) = \min_{j=1..N} \rho_H(X, z^j) = \min_{j=1..N} \left\{ \frac{1}{n} \sum_{i=1}^{n} [X_i \neq z_i^j] \right\}$$

2.21.

a) $R < D_2(\delta \| 1/2) = \delta \log_2 \frac{\delta}{1/2} + (1-\delta) \log_2 \frac{1-\delta}{1/2}$

If $V = \sum_{j=1}^{n} V^j$, then $d(X) = \min_{j=1..N} \{ \frac{1}{n} V \}$

$$P[d(X) \leq \delta] = P\left[\min_{j=1..N} \frac{1}{n} V \leq \delta\right] ; P\left[\min_{j=1..N} V \leq n\delta\right]$$

$$= N \circ P[\lambda V \leq \lambda n \delta]$$

$$N \circ P[2^{\lambda V} \leq 2^{\lambda n \delta}] = N \circ P[\mathbb{E}[2^{\lambda V}] \leq \mathbb{E}[2^{\lambda n \delta}]]$$

$$= N \circ P\left[\sum_{k}^{\infty} \chi 2^{\lambda k} \leq \mathbb{E}[2^{\lambda n \delta}]\right]$$

$$= N \circ P\left[\sum_{k}^{\infty} \binom{n}{k} \alpha^k (1-\alpha)^{n-k} 2^{\lambda k} \leq \mathbb{E}[2^{\lambda n \delta}]\right]$$

$$= N \circ P\left[\sum_{k}^{\infty} \binom{n}{k} (\alpha 2^{\lambda})^k (1-\alpha)^{n-k} \leq \mathbb{E}[2^{\lambda n \delta}]\right]$$

$$= N \circ P[(1-\alpha+\alpha 2^{\lambda})^n \leq \mathbb{E}[2^{\lambda n \delta}]]$$

$$\arg\min_{\lambda} \{(1-\alpha+\alpha 2^{\lambda})^n - \lambda n \delta\} = 0$$

$$\lambda^* = \log \frac{\delta(1-\alpha)}{\alpha(1-\delta)} = \log \frac{(1-\alpha)}{(1-\delta)} - \log\left(\frac{\alpha}{\delta}\right)$$

$$N \circ P[2^{\lambda V} \leq {}^{\lambda n \delta}] = N \circ 2^{-n\delta\lambda^* - \log(1-\alpha-\alpha e^{\lambda^*})^n}$$

$$= N\, 2^{-nD(\delta \| \alpha)}$$

c) $\Theta_{jk} \sim N(0, \beta^2)$

From the hint: $\mathbb{E}[F_d(\theta)] = \lim_{d \to \infty} \frac{F_d(\theta)}{d} = \log 2 + \frac{\beta^2}{4}$

$$P\left[\frac{F_d(\theta)}{d} \geq \log 2 + \frac{\beta^2}{4} + t\right] = P\left[\frac{F_d(\theta)}{d} \geq \mathbb{E}[F_d(\theta)] + t\right]$$

$$= P\left[\lambda \frac{F_d(\theta)}{d} \geq \lambda \mathbb{E}[F_d(\theta)] + \lambda t\right]$$

$$P\left[e^{\lambda\left(\frac{F_d(\theta)}{2} - \mathbb{E}[F_d(\theta)]\right)} \geq e^{\lambda t}\right] = P\left[e^{\frac{\lambda}{d}(1 + \mathbb{E}[F_d(\theta)] + \frac{\lambda \mathbb{E}[F_d(\theta)^2]}{2} - \mathbb{E}[F_d(\theta)]} \geq e^{\lambda t}\right]$$

$$\leq P\left[e^{\frac{\lambda^2 \beta^2}{2d}} \geq e^{\lambda t}\right]$$

$$\leq \frac{e^{\frac{\lambda^2 \beta^2}{2d}}}{e^{\lambda t}}$$

$$\underset{\lambda}{\operatorname{argmin}} \left\{ \frac{\lambda^2 \beta^2}{2d} - \lambda t \right\} = 0$$

$$\lambda^* = \frac{dt}{\beta^2}$$

$$P\left[e^{\lambda\left(\frac{F_d(\theta)}{d} - \mathbb{E}[F_d(\theta)]\right)} \geq e^{\lambda t}\right] \leq e^{-\beta d t^2/2}$$

$$\leq 2 \cdot e^{-\beta d t^2/2}$$

$\boxed{\text{Notes:}}$ 1) Symbol ($\doteq$) implies a similar argument, although not truly equivalent.

2) Two like-methods from the chapter are:

A) $P[z > t] = P[\lambda z \geq \lambda t]$

$$P[e^{\lambda z} \geq e^{\lambda t}] \leq \frac{e^{\lambda z}}{e^{\lambda t}}$$

B) $P[z > t] = P[\lambda z \geq \lambda t]$

$$\mathbb{E}[P[z > t]] \doteq \frac{\int_t^{\infty} P[z > t]\, dz}{e^{\lambda t}}$$