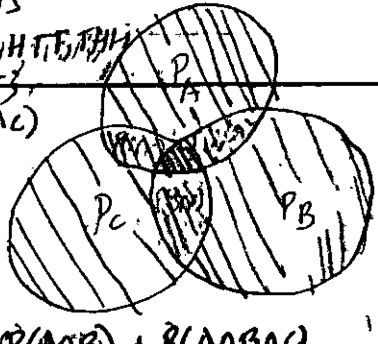
Chapter 1: 1a. Sample Space: {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

b. 1) {HHH, HHT, THH, HTH} 2) ~~HHH, HHT~~ 3) HHT, THT, TTT

c. A^c = "complement": the elements in the space which are not A. {~~HHT, TTT, HTH, THT~~}

$A \cap B$ = "intersection": the event both A and B occur: {HHT, HHH}

$A \cup B$ = "Union": events of A and B, and A or B. {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}



2. a) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

$= P(A \cup B) \cup P(C) = [P(A) + P(B) - P(A \cap B)] \cup P(C)$

Addition Law

$= P(A \cup C) + P(B \cup C) - P(A \cap B) \cup P(C)$

$= P(A) + P(C) - P(A \cap C) + P(B) + \cancel{P(C)} - P(B \cap C) - \cancel{P(C)} - P(A \cap B) + P(A \cap B \cap C)$

$\boxed{= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)}$

3. Semi

3 draws; RRR RRG RRW RGG GGW

RGR RWR GRG GWG WGR

GRR WRR GGB WGG WRG

$n = 6$

$k = 3$

$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{6!}{3!(3!)} = \frac{6 \cdot 5 \cdot 4}{6} = \frac{120}{6} = \frac{20}{6 \cdot 6} = \frac{5}{9}$

$\frac{6!}{1!(5!)} = 6$

Event A: 1 Draw

$\frac{P(R) + P(G) + P(W)}{P(G \cap R \cap W)} = \frac{\binom{3}{1} + \binom{2}{1} + \binom{1}{1}}{\binom{6}{1}} = \frac{\frac{3!}{1!(2!)} + \frac{2!}{1!1!} + \frac{1!}{1!0!}}{\frac{6!}{1!5!}} = \left(\frac{3}{6}\right) + \left(\frac{2}{6}\right) + \left(\frac{1}{6}\right)$

Event B: 2 Draw

$\frac{P(R) + P(G) + P(W)}{P(G \cap R \cap W)}$ = Should write Unions and intersection instead.

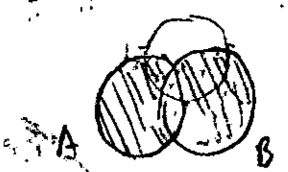
4. Prove $P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$; $P(\bigcup_{i=1}^{n} A_i) = P(A_1) + P(A_2) + \cdots + P(A_n) - P(A_1 \cap A_2) - \cdots - P(A_1 \cap A_n) - P(A_2 \cap A_3) - P(A_2 \cap A_n)$

$\sum_{i=1}^{n} P(A_i) = P(A_1) + P(A_2) + \cdots + P(A_n)$

$\boxed{P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)}$

5. Let (A, not B, or B, not A) be C

$\boxed{C = A \cap B = (A \cap \neg B) \vee (\neg A \cap B) = A + B - A \cup B = A^c \cap B \vee A \cap B^c}$



6. Two six-sided dice are thrown: A) Sample space:

Dice 1:	Dice 2:
1	1
1	2
1	3
1	4
1	5
1	6
2	1
2	2
2	3
2	4
2	5
2	6
3	1
3	2
3	3
3	4
3	5
3	6
4	1
4	2
4	3
4	4
4	5
4	6
5	1
5	2
5	3
5	4
5	5
5	6
6	1
6	2
6	3
6	4
6	5
6	6

B)(1) A = sum of the two value is at least 5.

(1,5) (5,1) (1,6) (6,1) (4,4)
(2,4) (4,2) (2,6) (6,2) (4,5)
(3,3) (3,3) (3,6) (6,3) (5,4)
(2,5) (5,2) (4,6) (6,4)
(3,4) (4,3) (5,6) (6,5)
(3,5) (5,3) (6,6) (6,6)

(2) B = the value of the first dice greater than the second.

(2,1) (3,2) ~~(4,3)~~ (5,4) (6,5)
(3,1) ~~(4,2)~~ (5,3) (6,4)
~~(4,1)~~ (5,2) (6,3)
(5,1) (6,2)
(6,1)

(3) C = the first value is 4

(4,1) (4,4)
(4,2) (4,5)
(4,3) (4,6)

c) $A \cap C = (4,2), (4,3), (4,4), (4,5), (4,6)$

$B \cup C = (2,1), (3,1), (5,1), (6,1), (3,2), (5,2), (6,2), (5,3), (6,3), (5,4), (6,4), (6,5), (4,2), (4,3), (4,4), (4,5), (4,6)$

$A \cap (B \cup C) = (4,2), (4,3), (4,4), (4,5), (4,6)$.

7. Bonferroni's equality: $P(A \cap B) \geq P(A) + P(B) - 1$.

Addition Law: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$

Therefore, $P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1$

$\boxed{P(A \cup B) \leq 1}$

8. De Morgan's Law:

$(A \cup B)^c = A^c \cap B^c$ 

$(A \cap B)^c = A^c \cup B^c$ 

9. Probability of rain on Saturday (25%)
Probability of rain on Sunday (25%)
The probability of consecutive events would be the multiplication of the probability of the events $\left(\frac{1}{4} \cdot \frac{1}{4}\right) = \frac{1}{16} = 12.5\%$, and not 50% proposed

10. n balls into k urns, What's the probability the last urn contains j balls? $\frac{\binom{i}{j}}{\binom{n}{k}} = \frac{\frac{i!}{1!(j-1)!}}{\frac{n!}{k!(n-k)!}} = \boxed{\frac{k!(n-k)!}{n!(j-1)!}}$

11. Telephone with seven total digits; 432 are the first three digits four digits remain with 10 potential digits each: 10^4

Total possibilities: $10 \cdot 10 \cdot 10 \cdot 10^4$

Chances of four more distinct digits are

$\frac{7 \cdot 6 \cdot 5 \cdot 4}{10^4} = \frac{840}{10000} = \boxed{8.4\%}$

12. 26-letter English Alphabet ~~into 8-bit binary words.~~

$\binom{2^8}{26} = \frac{2^8!}{26!(2^8-26)!}$

$= 2.74 \times 10^{35}$

13. a) Straight five cards in unbroken sequence: 4 suits $/\binom{13}{5} = 4/[13!/(13-5)!] = \boxed{4\left(\frac{1}{13}\right)\left(\frac{1}{12}\right)\left(\frac{1}{11}\right)\left(\frac{1}{10}\right)\left(\frac{1}{9}\right)}$

b) Four of a kind: $\binom{13}{1}\binom{4}{1}\binom{12}{1}\binom{4}{2} = \frac{13!}{1!\,12!} \cdot \frac{4!}{1!\,3!} \cdot \frac{12!}{1!\,11!} \cdot \frac{4!}{2!\,2!} = 13 \cdot 4 \cdot 12 \cdot 4 \cdot 3/2 = 3744$

$\frac{3744}{\binom{52}{5}} = \frac{3744}{2598960} = \boxed{0.05\%}$

c) A full house (three cards of one value and two of another)

(Probability of three cards of fifty-two) x (Probability of two cards of fifty-two)

$= \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} = \frac{6}{4165} = \boxed{0.144\%}$

14. Prove if $P(A|E) \geq P(B|E)$ and $P(A|E^c) \geq P(B|E^c)$

then $P(A) \geq P(B)$

$P(A|E) \geq P(B|E)$ and $P(A|E^c) \geq P(B|E^c)$

E 600

A X 600

B Y 600

50 50

$\frac{X}{50} \geq \frac{Y}{100}$; $\frac{1-\frac{X}{100}}{50} \geq \frac{1-\frac{Y}{100}}{50}$

$\boxed{X \geq Y}$; $1 - \frac{X}{100} \geq 1 - \frac{Y}{100}$

$\boxed{X \geq Y}$

15. 4 meats, 6 vegetables, three starches

$\binom{1}{4} \cdot 4 \cdot 6 \cdot 3 = \boxed{72 \text{ meals}}$

16. Simpsons Paradox:

Black Urn: {3 red and 6 green balls}
White Urn: {5 red and 4 green balls} } Set #1

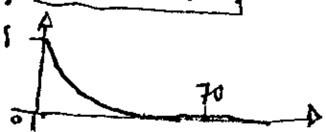
First trial: Black Urn $\left(\frac{3}{9}\right)$; $\boxed{\text{White Urn} \left(\frac{5}{9}\right)}$

Black Urn: {2 red and 6 green balls}
White Urn: {15 red and 1 green balls} } Set #2

Second Trial: Black Urn $\left(\frac{2}{8}\right)$; $\boxed{\text{White Urn} \left(\frac{15}{16}\right)}$

Black Urn: {5 red and 12 green balls}
White Urn: {20 red and 5 green balls} } Set #3

Third Trial: Black Urn $\left(\frac{5}{12}\right)$; $\boxed{\text{White Urn} \left(\frac{20}{25}\right)}$



17. Accepts: 4 items of 100
Rejects: 1 item is defective

$P(A) = \frac{\binom{100-K}{4}\binom{K}{0}}{\binom{100}{4}} = \frac{\frac{(100-K)!}{4!(100-K-4)!}}{\frac{100!}{4!(96!)}}$

$= \frac{\frac{(100-K)(100-K-1)(100-K-2)(100-K-3)(100-K-4)!}{(100-K-4)!}}{(100 \times 99 \times 98 \times 97)}$

$= \frac{(100-K)(99-K)(98-K)(97-K)}{100 \times 99 \times 98 \times 97} = \left(1-\frac{K}{100}\right)\left(1-\frac{K}{99}\right)\left(1-\frac{K}{98}\right)\left(1-\frac{K}{97}\right)$

19. $\dfrac{\text{Player one choice}}{\text{Total choices}} = \dfrac{\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)}{} = \boxed{\dfrac{1}{1296}}$;

19. Five Chicanos, two Asians, three African Americans.

a) $\dfrac{\binom{5}{1}\binom{2}{1}\binom{3}{1}}{\binom{10}{4}} = \dfrac{\left(\frac{5!}{1!4!}\right)\left(\frac{2!}{1!1!}\right)\left(\frac{3!}{1!2!}\right)}{\frac{10!}{4!6!}} = \dfrac{3\cdot 5\cdot 2\cdot 3\,(4!)}{10\cdot 9\cdot 8\cdot 7} = \boxed{1/7}$

b. $\dfrac{\binom{5}{1}\binom{2}{1}\binom{3}{1}}{\binom{10}{5}} = \dfrac{5\cdot 2\cdot 3\cdot 5!}{10\cdot 9\cdot 8\cdot 7\cdot 6} = \boxed{\dfrac{5}{42}}$

20. Arrangements Statistically

s's = 2 ; t's = 3; a's = 2;

i's = 2; c's = 1 ; n's = 1 ; l's = 2

Total Amount = 13 ; y's = 1

$\dfrac{\binom{2}{1}\binom{3}{1}\binom{2}{1}\binom{2}{1}\binom{1}{1}\binom{1}{1}\binom{2}{1}\binom{1}{1}}{\binom{13}{13}} = \dfrac{2\cdot 3\cdot 2\cdot 2\cdot 1\cdot 1\cdot 2\cdot 1}{13} \boxed{= 4\%}$

21. $\dfrac{2^2+2^2+2^2}{2^5} = \boxed{\dfrac{12}{32}}$

22. $\left(\frac{1}{4}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{1}\right) = \boxed{\dfrac{1}{24}}$

23. $\underbrace{\binom{n}{1}}_{1^{st}\text{ player}}\underbrace{\binom{n}{2}}_{2^{nd}\text{ player}}\underbrace{\binom{n-1}{1}}_{\text{Remaining one}}(n-2)!$

$\dfrac{n!}{1!(n-1)!}\cdot\dfrac{n!}{2!(n-2)!}\cdot\dfrac{(n-1)!}{1!(n-2)!}(n-2)!$

$\binom{n}{2}\cdot n! = \binom{n}{2}\binom{n}{1}$

24. 52 cards ; $\dfrac{\text{Probability Aces next to each other}}{\text{Total Arrangements}} = \dfrac{\binom{13}{4}4!}{\binom{52}{4}} = \dfrac{13\cdot 12\cdot 11\cdot 10\cdot 4!}{4!\cdot 52\cdot 51\cdot 50\cdot 49} = 11/4165 = \boxed{26\%}$

25. $\boxed{3!}$

26. n items with K defects; m are selected and inspected.

value of m to be below a probability of 0.90

a) n = 1000 ; K = 10 $\dfrac{\text{Probability of defect}}{\text{Total outcomes}} = \dfrac{\binom{n-K}{m}\binom{K}{0}}{\binom{n}{m}} = \dfrac{\binom{n-K}{m}\binom{K}{0}}{\binom{n}{m}} = \dfrac{\frac{(n-m)!}{m!(n-m-k)!}}{\frac{n!}{m!(n-k)!}} = \dfrac{(1000-m)!\,(n-k)!}{1000!\,(n-m-k)!}$

b) n = 1000 ; k = 10 ; $0.9\,\dfrac{(1000!)}{(990)!} = \dfrac{(1000-m)!}{(990-m)!}$

$(1000-m)(1000-(m+1))(1000-(m+2)) \cdots (1000-(m+10))\cdots$

27. $\dfrac{\text{Probability of no letters occuring}}{\text{Total Probability}} = \dfrac{26C5}{26^5} = \dfrac{\binom{26}{5}}{26^5} = \dfrac{\frac{26!}{5!(21)!}}{26^5} = \dfrac{\frac{26\cdot 25\cdot 24\cdot 23\cdot 22}{5\cdot 4\cdot 3\cdot 2\cdot 1}}{26^5}$

28. 5 players with five cards from 52-card deck. $= \dfrac{(52)!}{25!(27!)} \boxed{= 4.775\times 10^{14}\text{ ways}} = 5.53\times 10^{-3} = 0.55\%$

29.a Three Spades and Two Hearts:

(a) Discards two hearts and draws two more cards.

$= \left(\dfrac{11}{47}\right)\left(\dfrac{10}{46}\right) = \dfrac{110}{2162} = \dfrac{55}{1081} \boxed{5.1\%}$

30. 60 2nd graders into two classes of 30 each. Probability of five chosen into same class

$\dfrac{\binom{2}{1}}{\binom{60}{5}} = \dfrac{2!}{\frac{60!}{5!\,55!}} = \dfrac{2\cdot 5!}{60\cdot 59\cdot 58\cdot 57\cdot 56} = 0.00004\% = \dfrac{\binom{60}{30}}{}\,\dfrac{\binom{60}{5}}{}\;\dfrac{60!}{}$

four students: $\dfrac{\binom{2}{1}}{\binom{60}{4}} = \dfrac{2}{\frac{60!}{4!\,56!}} = \dfrac{2\cdot 4\cdot 3\cdot 2}{60\cdot 59\cdot 58\cdot 57} = \boxed{0.0004\%}$

Marcelle in one class and her friends in another: $\dfrac{\binom{2}{1}\binom{58}{29}}{\binom{60}{30}} = \dfrac{\frac{2\cdot 58!}{29!\,(29)!}}{\frac{60!}{30!\,30!}} = 6\%$ 4%

31. Six Male and Six Female Dancers. ${}_6P_6 \cdot {}_6P_6$ $= \dfrac{6!}{0!} \cdot \dfrac{6!}{0!} = 720^2$

32. $\dfrac{4\binom{13}{n}}{\binom{52}{n}} = \dfrac{4\cdot\frac{13!}{n!(13-n)!}}{\frac{52!}{n!(52-n)!}}$

$= \dfrac{4\cdot 13!\,(52-n)!}{(13-n)!\,52!} = \boxed{\dfrac{1.86\times10^{-57}(52-n)!}{(13-n)!}}$

When is the value 0.5?

$\dfrac{3.7\times10^{-57}\,\Gamma(53-n)}{\Gamma(14-n)} = 1;\ n=3.$

Gamma Identity: $\Gamma(n)=(n-1)!$

$\Gamma(n+1) = n\Gamma(n)$

33. Five people and five floors. Probability of a person per floor
Probability of choosing five.

$= \dfrac{\binom{5}{1}}{\binom{7}{5}} = \dfrac{\frac{5!}{1!(5-1)!}}{\frac{7!}{5!(7-5)!}} = \dfrac{5!}{4!}\cdot\dfrac{5!(2!)}{7!} = \dfrac{5!\,5!\,2!}{4!\cdot 7!} = \boxed{23.8\%}$

34. Prove the following identity:

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m-n}{n-k} = \binom{m}{n}$$

$$\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\,\frac{(m-n)!}{(n-k)!(m-n-n+k)!} = \sum_{k=0}^{n}\frac{n(n-1)(n-2)(n-3)\cdots(n-k)!\,(m-n)!}{k!\,(n-k)!\,(n-k)!\,(m+k)!}$$

$$= \sum_{k=0}^{n}\frac{n(n-1)(n-2)(n-3)\cdots n!\,(m-n)!}{k!\left(\frac{n!}{n(n-1)(n-2)(n-3)\cdots}\right)(m+k)!} = \sum_{0}^{n}\frac{[n(n-1)(n-2)(n-3)\cdots]^2(m+n)!}{k!\;n!\;(m+k)!}$$

Two methods: 1. Pascal's Identity $\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$k! = k\cdot(n-1)(n-2)$

$$\sum_{k=0}^{n}\binom{n}{k} = \sum\left[\binom{n+1}{k+1} - \binom{n}{k+1}\right] = \sum^{n}\binom{n+1}{k+1} - \sum^{n}\binom{n}{k+1}$$

$$= \sum_{k+1}^{n+1}\binom{n}{k+1} - \sum^{n}\binom{n}{k+1} = \binom{n+1}{k+1} - \binom{n}{k+1} = \binom{n+1}{k+1}$$

2. Binomial Theorem: $\displaystyle\sum_{k=0}^{n}\binom{n}{k}\binom{m-n}{n-k} = \sum_{k=0}^{n}(1+k)^n(1+n-k)^{m-n} = ({}^nC_0 k^{n} + {}^nC_1 k^{n-1} + {}^nC_2 k^{n-2} + \cdots)$
$({}^{m-n}C_0(n-k)^{m-n} + {}^{m-n}C_1(n-k)^{m-n-1} + {}^{m-n}C_2(n-k)^{m-n-2})$

"$(1+x)^n$" $= \left(1+nk+\frac{n(n-1)}{2!}k^2+\frac{n(n-1)(n-2)}{3!}k^3+\cdots\right)\left(1+(m-n)(n-k)+\frac{(m-n)(m-n-1)}{2!}(n-k)^2+\cdots\right)$

$= \left(1+mn+\frac{m(m-1)}{2!}n^2+\cdots\right) = \binom{m}{n}$

35. Prove the following identities.

a) $\binom{n}{r} = \binom{n}{n-r} = \dfrac{n!}{r!(n-r)!} = \dfrac{n!}{(n-n+r)!\,(n-r)!} = \boxed{\binom{n}{n-r}}$; An expansion of a binomial is telescoping

b) $\binom{n}{r} = \dfrac{n!}{r!(n-r)!} = \dfrac{(n-1)!\,n}{r!(n-r)!} = \dfrac{(n-1)!}{r!}\left[\dfrac{n-k}{(n-r)!} + \dfrac{k}{(n-r)!}\right] = \dfrac{(n-1)!}{r!(n-1-k)!} + \dfrac{r(n-1)!(n-1+1)}{(r-1)!(n-k)!} = \boxed{\binom{n-1}{k} + \binom{n-1}{k-1}}$

36. R R R G G G — Arrangements: Combinations per type — $\binom{6}{3}\binom{3}{3} = \frac{6!}{3!3!}\frac{3!}{3!0!} = \frac{6\cdot5\cdot4}{3\cdot2} = \frac{120}{6} = \boxed{20}$

6 Blocks

R R R W W W G G G — Combinations per type — $\binom{9}{3}\binom{6}{3}\binom{3}{3}$

$= \frac{9!}{3!\,6!}\frac{6!}{3!\,3!} = \frac{9!}{3!3!3!} = \boxed{1680}$

9 blocks

37. Coefficient of $x^2y^2z^3$ in $(x+y+z)^7$;

Multinomial: $(x_1+x_2+x_3)^n = \sum \binom{n}{n_1 n_2 n_3} x_1^{n_1} x_2^{n_2} x_3^{n_3} \Rightarrow (x+y+z)^7 = \sum \binom{7}{2\,2\,3} x^2y^2z^3$

coefficient: $\binom{7}{2\,2\,3} = \frac{7!}{2!2!3!} = \frac{7\cdot6\cdot5\cdot4\cdot}{24}$

$= \boxed{210}$

38. coefficient of x^3y^4 for $(x+y)^7$;

$(x+y)^7 = \sum \binom{n}{k} x^k y^{7-k} = \binom{7}{0} x^7 y^0 + \binom{7}{6} x^6 y^1 + \cdots$

coefficient: $\binom{7}{3\,4} = \frac{7!}{3!\,4!} = \frac{7\cdot6\cdot5}{3\cdot2} = \frac{210}{6} = \boxed{35}$

39. a. 26 letters choose 6.

$\frac{\text{Probability Handler}}{\text{Total outcomes}} = \frac{\binom{6}{1}}{\binom{26}{6}} = \frac{6!}{5!}\left(\frac{6!(20)!}{26!}\right) = \frac{3}{115115} = \boxed{0.0026\%}$

b. $0.90 = n\left(\frac{3}{115115}\right)$

$\boxed{n = 34534 \text{ monkeys}}$

40. 12 people into three groups; $\binom{12}{4}\binom{8}{4}\binom{4}{4} = \frac{12!}{4!(8)!}\frac{8!}{4!\,4!}\frac{4!}{4!\,1!} = \frac{12!}{4!4!4!} = 34650.$

6 pairs of partners: $\binom{6}{2}\binom{4}{2}\binom{2}{2} = \frac{6!}{2!\,4!}\frac{4!}{2!\,2!}\frac{2!}{2!\,0!} = \frac{6!}{2!2!2!} = 90.$

41. Seven black socks, eight blue socks, and nine green socks. Total: 24.

a) Probability of Matching: $\frac{\binom{7}{2}}{\binom{24}{2}} + \frac{\binom{8}{2}}{\binom{24}{2}} + \frac{\binom{9}{2}}{\binom{24}{2}} = \frac{7!}{5!}\frac{2!\,22!}{24!} + \frac{8!}{6!}\frac{2!\,22!}{24!} + \frac{9!}{7!}\frac{2!\,22!}{24!}$

$= \frac{7}{92} + \frac{7}{69} + \frac{3}{32} = \boxed{27\%}$

b. $7/92 = \boxed{7.61\%}$

42. Number of ways to choose: 11 boys grouped into 4 forwards, 3 midfielders, 3 defenders, 1 goalie.

$\binom{11}{4}\binom{7}{3}\binom{4}{3}\binom{1}{1} = \frac{11!}{4!(7!)}\frac{7!}{3!\,3!}\frac{4!}{3!\,1!}\frac{1!}{1!\,1!} = \boxed{46,200}$

43. Three Jobs; Two Jobs require 3 programmers, the third requires four.

Total of ten programmers. $\binom{10}{3}\binom{7}{3}\binom{4}{4} = \boxed{4,200}$

44. Combinations in Tentacles × Shaking Hands: $\sum_{i=1}^{n=8} 8(8-i)\binom{8}{i+1} =$: Maybe

$= 8\cdot7\binom{8}{2} + 8\cdot6\binom{8}{3} + 8\cdot5\binom{8}{4} + 8\cdot4\binom{8}{5} + 8\cdot3\binom{8}{6} + 8\cdot2\binom{8}{7} + 8\cdot1\binom{8}{8}$

$= 1568 + 2688 + 2800 + 1792 + 672 + 128 + 8 = 9656$

45. Prove $P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2|A_1) P(A_3|A_1 \cap A_2) \cdots P(A_n | A_1 \cap A_2 \cap \cdots \cap A_{n-1})$

Multiplication Law: A & B be events and assume $P(B) \neq 0$, then $P(A \cap B) = P(A|B)P(B)$

$$P(A_n \cap A_{n-1} \cap \cdots \cap A_2 \cap A_1) = P(A_n | A_1 \cap A_2 \cap \cdots \cap A_{n-1}) \cdot P(A_1 \cap A_2 \cap \cdots \cap A_{n-1})$$

$$= P(A_n | A_1 \cap A_2 \cap \cdots \cap A_{n-1}) P(A_{n-1} | A_1 \cap A_2 \cap A_3 \cdots) P(A_1 \cap A_2 \cap A_3 \cdots)$$

$$\boxed{= P(A_n | A_1 \cap A_2 \cap \cdots \cap A_{n-1}) \cdots P(A_3 | A_1 \cap A_2) \cdot P(A_2|A_1) \cdot P(A_1)}$$

48. cont. [Read wrong]

Urn A	Urn B
3xR	2xR
2xW	5xW

A ball from Urn A into B, then a ball is drawn from Urn B.

A) Probability of a red ball:

$$\frac{\binom{3}{1}}{\binom{3}{1}} \cdot \frac{\binom{3}{1}}{\binom{6}{1}} = \left(\frac{3!}{2!}\right)\left(\frac{4!1!}{5!}\right)\left(\frac{3!}{2!}\right)\left(\frac{5!}{6!}\right) = \frac{3}{10}$$

$$\frac{\binom{2}{1}}{\binom{5}{1}} \cdot \frac{\binom{2}{1}}{\binom{6}{1}} = \frac{2}{15}$$

46 a)

Urn A	Urn B
3xR	2xR
2xW	5xW

coin [50/50]:
Heads = Urn A
Tails = Urn B.

$$P(\text{coin} \cap \text{Urn A}) = P(\text{coin} | \text{Urn A}) \cdot P(\text{Urn A}) = \frac{1}{2} \cdot \left(\frac{3}{5}\right) = \frac{3}{10}$$

$$P(\text{coin} \cap \text{Urn B}) = P(\text{coin} | \text{Urn B}) \cdot P(\text{Urn B}) = \frac{1}{2}\left(\frac{2}{5}\right) = \frac{2}{10}$$

$$P(R) = P(\text{coin} \cap \text{Urn A}) + P(\text{coin} \cap \text{Urn B}) = \frac{3}{10} + \frac{2}{10} = \frac{5}{10} = \boxed{\frac{1}{2}}$$

b) $P(R) = P(\text{Heads}) P(\text{Urn A}) + P(\text{Tails}) P(\text{Urn B}) = 1 = P(\text{Heads})\left(\frac{3}{5}\right) + (1 - P(\text{Heads}))\left(\frac{2}{5}\right)$

$$1 = P(\text{Heads})\left[\left(\frac{3}{5}\right) - \left(\frac{2}{5}\right)\right] + \frac{2}{5} ; \quad \boxed{P(\text{Heads}) = \frac{3}{5}}$$

47.

Urn A	Urn B
4xR	2xR
3xB	3xB
2xG	4xB

a) $P(R) = P(R|\text{Urn B} \cap R)P(\text{Urn A} \cap R) + P(R|\text{Urn B} \cap B)P(\text{Urn A} \cap B) + P(R|\text{Urn B} \cap G)P(\text{Urn A} \cap G)$

Multiplication Law

$$= \left(\frac{3}{10}\right)\left(\frac{4}{9}\right) + \left(\frac{2}{10}\right)\left(\frac{3}{9}\right) + \left(\frac{2}{10}\right)\left(\frac{2}{9}\right) = \left(\frac{12}{90}\right) + \left(\frac{6}{90}\right) + \left(\frac{4}{90}\right) = \frac{22}{90} = \boxed{\frac{11}{45}}$$

b) ~~$1 = P(R|\text{Urn B} \cap R)P(\text{Urn A} \cap R) + P(R|\text{Urn B} \cap B)P(\text{Urn A} \cap B) + P(R|\text{Urn B} \cap G)P(\text{Urn A} \cap G)$~~

~~$1 = \left(\frac{3}{x}\right)\left(\frac{4}{9}\right) + \left(\frac{3}{10}\right)\left(\frac{3}{9}\right) + \left(\frac{2}{10}\right)\left(\frac{2}{9}\right) ; \frac{10}{3}\left(1 - \left(\frac{2}{10}\right)\left(\frac{2}{9}\right) - \left(\frac{2}{10}\right)\left(\frac{2}{9}\right)\right) = x$~~

b) Bayes Formula:

$$P[\text{Urn A}(\text{Red}) | \text{Urn B}(R)] = \frac{P[\text{Urn A}(R), \text{Urn B}(\text{Red})]}{P[\text{Urn B}(\text{Red})]} = \frac{P[\text{Urn B}(\text{Red}) | \text{Urn A}(\text{Red})] P[\text{Urn} \ldots}{P(R|\text{Urn B} \cap B)P(\text{Urn A} \cap B) + P(R|\text{Urn B} \cap R)P(R) + P(R|\text{Urn B} \cap G)P(G)}$$

$$\boxed{= 6/11}$$

48.

Urn A
3xR
2xW

1 Draw + 1 Return + Same color Ball. → 2nd Draw

Multiplication Law

a) Probability of White?

$$P(W|\text{Draw \#2}) = P(W | \text{Draw\#2} \cap W) P(W \cap \text{Draw\#1}) + P(W|\text{Draw\#2} \cap R) P(R \cap \text{Draw\#1})$$

$$= \left(\frac{3}{6}\right)\left(\frac{2}{5}\right) + \left(\frac{2}{6}\right)\left(\frac{3}{5}\right) = \left(\frac{6}{30}\right) + \left(\frac{6}{30}\right) = \frac{12}{30} = \frac{4}{10} = \boxed{\frac{2}{5}}$$

b). Bayes Theorem:

$$P(\text{Draw\#2} \cap W | \text{Draw\#2} \cap W) = \frac{P(\text{Draw\#2} \cap W | \text{Draw\#1 W}) P(\text{Draw\#1 W})}{P(W|\text{Draw\#2} \cap W) P(W \cap \text{Draw\#1}) + P(W|\text{Draw\#2} \cap R) P(R \cap \text{Draw\#1})}$$

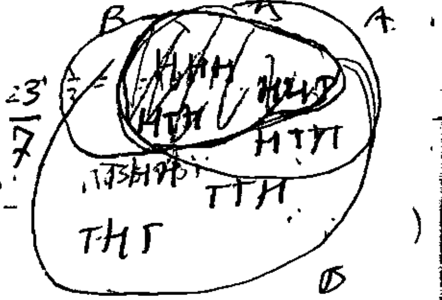
$$= \frac{(3/5)(2/5)}{\left[\left(\frac{3}{6}\right)\left(\frac{2}{5}\right) + \left(\frac{2}{6}\right)\left(\frac{3}{5}\right)\right]} = \boxed{1/2}$$

49. 3 tosses of a coin

a) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(HHH + HHT + HTH) + H}{P(HHH + HHT + HTH + HTT + THH + TTH + THT)} = \frac{3}{7}$

b) $P(T|H) = \frac{HHT + HTH + HTT + THH + THT + TTH}{HHH + HHT + HTH + HTT + THH + TTH + THT} = \boxed{\frac{6}{7}}$

$P(T) = \cancel{(\text{Prob } T)(\text{Prob } T)(\text{Prob } T)} = \frac{1}{2}$



50. Two dice; sum total = 6: $P(6) = \frac{P(6 \cap 3)}{P(6)} = \frac{1/36}{5/36} = \frac{1}{36} \cdot \frac{36}{5} = \boxed{\frac{1}{5}}$

Law of Independent Events

51. Two dice; sum total = 6; $P(<6) = \frac{P(<6 \cap 3)}{P(<6)} = \frac{4}{10} = \boxed{\frac{2}{5}}$

52. $P(G|G) = \frac{P(G \cap A \cap L)}{P(L)} = \boxed{\frac{1}{4}}$; $P(G|G) = \frac{P(G \cap G)}{P(G)} = \boxed{\frac{1}{4}}$

53. High-Risk [0.02] [0.10] 2×10^{-3} High R

Medium Risk [0.01] [0.20] 2×10^{-3}

Low Risk [0.0025] [0.70] 1.75×10^{-3}

(Filing) (People)

0.0325 1.00

$P(\text{High Risk} | \text{People}) = \frac{P(\text{High Risk} \cap \text{People})}{P(\text{Risk} \cap \text{People})} = \boxed{\frac{8}{23}}$

54. R_i = event of rain on day i;

$P(R_i | R_{i-1}) = \alpha_i$; $P(R_i^c | R_{i-1}^c) = \beta$

$P(R_i | R_{i-1} \cap R_{i-2} \cap \cdots \cap R_0) = P(R_i | R_{i-1})$

a) p = probability of rain, What about tomorrow?

$P(R_{i+1} | R_0) = P(R_i | R_{i0}) = P = \alpha$

b) $P(R_2 | R_1 \cap R_0) = P(R_2 | R_{i1}) = \alpha$

c) $P(R_i | R_{i-1} \cap R_{i-2} \cap \cdots \cap R_0) = P(R_i | R_{i-1})$

$\boxed{\lim_{i \to \infty} = P \cdot \prod_{i=1}^{n} \alpha_i}$

55. Upper (U), middle (M), and lower (L)

1 = Father occupation: 2 = Son's occupation.

	U_2	M_2	L_2
U_1	0.45	0.48	0.07
M_1	0.05	0.70	0.25
L_1	0.01	0.50	0.49

$P(U_2 | U_1) = 0.45$

a) $P(M_1 | M_2) = 0.70$; $P(L_1 | L_2) = 0.49$

b)

	U_3	M_3	L_3
U_2	$P(U_3\|U_2)P(U_2)$	$P(M_3\|U_2)P(U_2)$	$P(L_3\|U_2)P(U_2)$
M_2	$P(U_3\|M_2)P(M_2)$	$P(M_3\|M_2)P(M_2)$	$P(L_3\|M_2)P(M_2)$
L_2	$P(U_3\|L_2)P(L_2)$	$P(M_3\|L_2)P(L_2)$	$P(L_3\|L_2)P(L_2)$

	U_3	M_3	L_3
	V_2	42/25	81/100
U_2	0.225	0.3064	0.0364
M_2	0.025	1.176	0.2025
L_2	0.005	0.84	0.3969

$P(U_2) = P(U_2|U_1)P(U_1) + P(U_2|M_1)P(M_1) + P(U_2|L_1)P(L_1) = 0.050$

$P(M_2) = 0.5785$

$P(L_2) = 0.3652$

$P(U_3) = 0.064$

$P(M_3) = 0.614$

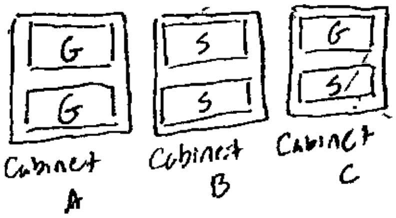
$P(L_3) = 0.322$

56 5 cards of 52 card Deck

1^{st} = King. Law of Independent Events

$\frac{3\binom{51}{3}}{\binom{52}{4}} = \frac{\frac{51}{11.881}}{\frac{511}{41.491}} = \frac{51 \cdot 50 \cdot 49 \cdot 48 \cdot 3 \cdot 2 \cdot 1}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = \frac{1}{2} = \boxed{50\%}$

57. Cabinet A, B, C with two drawers each, inside a coin. $\boxed{\text{Multiplication Law}}$ $P(E) = P(S|S_1)P(S_1) + P(S_2|G_1)P(G_1)$



$P(\text{[illegible]}) = \cancel{P(\text{[illegible]})}$

$(\frac{2}{3})(\frac{1}{2}) = \cancel{P(\text{[illegible]})} \circ P(B) + P(\text{Draw \#}|A) \circ P(A) \frac{1}{2} + P(\text{Draw \#1}|C)\, P(C)$

$$\frac{1}{2} = \frac{1 \cdot \frac{1}{3}}{\frac{1}{3}\cdot\frac{1}{3} + 0 \cdot \frac{1}{3} + (\frac{1}{3})(\frac{1}{2})} = \frac{\frac{1}{3}}{3/6} = \frac{1}{3}\times\frac{6}{3} = \boxed{\frac{2}{3}}$$

58. Drew, Chris, Jason; Two must stay home; one leaves

$\boxed{\text{Drew, Chris, Jason}}$ → 2 remain; 1 leave $(\frac{1}{3})$; 3 slots

The possibilities following Drew asking the teacher contain a relationship known as the multiplicative law. If Chris is chosen $(\frac{1}{3})$ to remain $(\frac{1}{3})$, then there are $(\frac{1}{2})$ possible outcomes. If Jason is chosen $(\frac{1}{3})$ then there is $(\frac{1}{2})$-outcomes. While if Drew is chosen 1 outcome. Thus,

$P(\text{Drew Asking}) = P(\text{outcomes}|\text{Drew})P(\text{Drew}) + P(\text{outcomes}|\text{Chris}) \times P(\text{Chris}) + P(\text{outcomes}|\text{Jason})P(\text{Jason})$

$= \frac{1}{3}(1) + (\frac{1}{2})(\frac{1}{3}) + (\frac{1}{2})(\frac{1}{3}) = \frac{1}{3}$

Asking is no better than's waiting

59. $\boxed{\text{HH, HT, T:T}}$ Box. Random selection. Flip. Heads

a) Probability of a two-headed coin

$\cancel{P(HH|H) = \frac{P(HH|H)P(H)}{P(HH|HH)P(H) + P(HT|HT)P(HT)}}$

$$P(HH|H) = \frac{P(H|HH)P(HH)}{P(H|HH)P(HH) + P(H|HT)P(HT) + P(H|TT)P(T)} = \frac{1 \circ 1/3}{1 \cdot \frac{1}{3} + \frac{1}{2}(\frac{1}{3}) + 0(\frac{1}{3})} = \frac{1/3}{\frac{1}{3}+\frac{1}{6}} = \frac{\frac{2}{6}}{\frac{2}{6}+\frac{1}{6}} = \boxed{\frac{2}{3}}$$

b) $P(H) = P(H|HH)P(HH) + P(H|HT)P(HT) + P(H|TT)P(T) = \frac{3}{6} = \boxed{\frac{1}{2}}$; $P(T) = P(T|HT)P(HT) + P(T|TT)P(T) = (\frac{1}{2})(\frac{1}{3}) + 1 \circ \frac{1}{3} = \frac{3}{6} = \boxed{\frac{1}{2}}$

c) $P(H_2) = P(H|HH)P(HH_1) + P(H|HH_1)P(HT_1)$
$= 1 \cdot (\frac{2}{3}) + \frac{1}{2}(\frac{1}{3}) = \frac{2}{3} + \frac{1}{6} = \frac{4}{6} + \frac{1}{6} = \boxed{\frac{5}{6}}$

60. $P(B) > 0$; $Q(A) = P(A|B)$; $\boxed{\text{Addition Law}}$ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\boxed{P(A \cup B|B) = P(A|B) + P(C|B) - P(A \cap C|B)}$

61. Defect = 0.95 Accuracy; Sound = 0.97 Accuracy

If 0.5% are faulty, what is the probability faulty if sound?

	Defect	Sound
T	0.95	0.97
F	0.05	0.03
	1	1

$P(F|\text{Defect}) = \frac{P(F|\text{Defect})\, P(\text{Defect})}{P(F|\text{Defect})P(\text{Defect}) + P(F|\text{Sound})P(\text{Sound})}$

$= \frac{0.05\,(0.5/200)}{0.05(5/200) + 0.03(195/200)}$; $P(\text{[illegible]}|\text{Defect}) = \frac{P(F \cap \text{Defect})}{P(\text{Defect})} = \frac{P(F|\text{Sound})P(\text{Sound})}{P(F|\text{Sound})P(\text{Sound}) + P(T|\text{Defect})P(\text{Defect})}$

$= \frac{0.03 \cdot 0.995}{0.03\, 0.995 + 0.95(0.005)} \approx 6\%$

62. Four players [13 cards each] $\frac{\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}}{4!} \approx \boxed{8\times 10^{67}}$ =

63. Age >70 = 0.6 ; Age >80 = 0.2. $P(\text{Age}>80 \mid 70) = \frac{P(\text{Age}>80 \cap >70)}{P(70)} = \frac{P(>70)}{0.6} = \boxed{\frac{0.2}{0.6} = \frac{1}{3}}$

64. Three Shifts: 1% of Shift 1 are defective ; 2% of Shift 2 are defective; 5% of shift 3 are defective;

$P(\text{Defective}) = P(\text{Defective} \mid \text{Shift \#1})P(\text{Shift \#1}) + P(\text{Defective} \mid \text{Shift \#2})P(\text{Shift \#2}) + P(\text{Defective} \mid \text{Shift \#3})P(\text{Shift \#3})$

$= 1\%\left(\frac{1}{3}\right) + 2\%\left(\frac{1}{3}\right) + 5\%\left(\frac{1}{3}\right) = \boxed{2.667\%}$; $P(\text{Shift \#3}) \Rightarrow \boxed{P(\text{Defective} \cap \text{Shift \#3})}$

$\overline{\sum P(\text{Defective} \mid \text{Shift \#i})}$

$= \frac{5\%\left(\frac{1}{3}\right)}{2.667\%} = \boxed{62\%}$

65. A and B are independent; A and B^c, A^c and B^c are too.

$P(A \cap B) = P(A)P(B)$; $P(A \cap B^c) = P(A)P(B^c)$; $P(A^c \cap B^c) = P(A^c)P(B^c)$

66 $\emptyset$ independent of A for any A. $P(A \cap \emptyset) = \boxed{P(A) \cdot P(\emptyset) = 0}$

67. If $P(A\cap B) = P(A)P(B)$; then $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B)$

$\boxed{\text{Addition Law}}$ $\boxed{\text{Law of Independence}}$

68. If $p(A \cap B) = p(A)p(B)$ and $p(B\cap C) = p(B)p(C)$; $p(A \cap C) = p(A \cap B \cap C) = p(A\cap B)p(C)$

$= p(A)p(B \cap C) = p(A)p(B)p(C)$

$P(A \cap B) = p(A)\frac{p(B \cap C)}{p(C)} \Rightarrow \boxed{\frac{p(A \cap B)}{p(B\cap C)} = p(A \cap C) = \frac{p(A)}{p(C)}}$

69. If $A \cap C = \emptyset$ "Disjoint", $p(A) = \emptyset \vee p(C) = \emptyset$; thus independent

70. $\boxed{\text{If } A \subset B \text{; then they are not independent.}}$ A B

71. If A, B, C are mutually independent, then $A \cap B$ and C are independent along with $A \cup B$ and C.

$p(A \cap (B \cap C)) = p(A)p(B\cap C) = \boxed{p(A)p(B)p(C)}$

$p(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \mid B \cap C)$

$\boxed{= P(A) + P(B)P(C) - P(A \mid B \cap C)}$

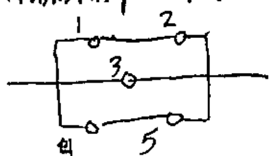
72. $(t = 0, 1, 2 \cdots)$; p, then q. @ t=0, n=1 ;

probability of 0, 1, 2, 3 people at t=2.

73. n independent units, each with p(fails); $P(E)$ = System failure. $P(\text{System}) = (1-p)^n$; $P(\text{Fail}|\text{System}) = 1-(1-p)^n$

$\boxed{= 1-(1-p)^n}$

74. Probability of failure

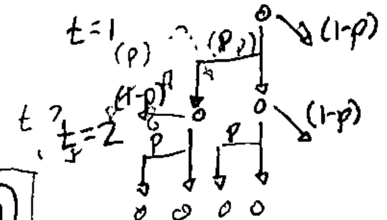


$P(F) = P(A_1 \cap A_2) + P(A_3) + P(A_4 \cap A_5)$

$+ P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)$

$\boxed{= p + 2p^2 - p^5}$

75.



$0.5 = -3p + 2p^2$

$2p^2 - 3p - 0.5 = 0$

$\boxed{p_1 = 1.65}$; $p_2 = -0.15$

$P(\text{Dead}) = 1-(1-p)^3$; $1^3 - a^3 - 3a^2b + 3ab^2 - b^3 = 1-(a-b)^3$

$= 1 - 1 - 3p + 3p^2 - p^2 = \boxed{-3p + 2p^2}$

76. —o—||—o—||—o— n-components. $P(\text{Success}) = \boxed{(1-p)^n}$; $P(\text{Success}) = (1-0.05)^{10} = 0.9975 = \boxed{99.75\%}$

77. $5\% = P(\text{Bulls-Eye})$ $0.5 = (1-0.05)^n$; $\log\frac{1}{2} = n\log(1-0.05)$; $\boxed{n = 13.51}$

78. Pairs of (a or A); AA, Aa, aa, or (Aa or aA). a) Parent #1: AA; Parent #2: Aa; Offspring AA, AA, Aa, Aa $\boxed{AA\ (50\%),\ Aa\ (50\%)}$

b) AA (p); Aa ($2q$); aa (r); $1 = (p+2q+r)^n$; $\boxed{n=2;\ 1=(p+2q+r)^2}$; $n=3$; $\boxed{1=(p+2q+r)^3}$

c) $\boxed{1 = (v+v+w)^2}$; $\boxed{1 = (v+v+w)^3}$: Hardy-Weinberg Law

79. a. aa = Death; Aa = carrier, alive; AA = not carrier, not diseased. $Aa \times Aa = AA + 2Aa + aa$

$\boxed{AA\ (25\%);\ Aa\ (50\%);\ aa\ (25\%)}$

b. $P(\text{Not Diseased} \cap \text{Carrier}) = 50\%$

c. $p(\text{Offspring}) = \boxed{p(AA) + p(Aa)}$; $p\boxed{\tfrac{1}{3}} + \boxed{\tfrac{2}{3}}$

d.

Offspring	Genotypes of Parents			
P(Aa)	AA-AA	AA-Aa	Aa-AA	Aa-Aa
AA	$\frac{1}{3}(1-p)$	$\frac{1}{2}(\frac{1}{3})p$	$\frac{1}{2}(\frac{2}{3})(1-p)$	$\frac{1}{4}\times\frac{2}{3}\times p$
Aa	0	$\frac{1}{2}(\frac{1}{3})\times p$	$\frac{1}{2}(\frac{2}{3})(1-p)$	$\frac{1}{2}\times(\frac{2}{3})\times p$
Aa	0	0	0	$\frac{1}{4}\times\frac{2}{3}\times p$

$P(AA) = \frac{1}{3}(1-p) + \frac{1}{6}p + \frac{2}{6}(1-p) + \frac{p}{6}$

$\boxed{= \frac{2}{3} - \frac{p}{3}}$

$P(Aa) = \frac{p}{6} + \frac{2}{6}(1-p) + \frac{2}{6}p$

$= \frac{p}{6}$

$P(AA)/P(Aa) = \boxed{\dfrac{(2-p)\frac{1}{3}}{\frac{p}{6}}}$

80. Parents Aa (50%); A = child #1 & #2 have the same gene

B = child #1 and #3 have the same gene

C = child #2 & #3 have same gene.

Mutually Independent: $P(A_{i_1} \cap \cdots \cap A_{i_m}) = P(A_{i_1}) \cdots P(A_{i_m})$

Pairwise Independence: $P(A \cap B \cap C) = 0 \neq P(A)P(B)P(C)$.

$P(B) = P(\text{child \#1} \cap \text{child \#3})$

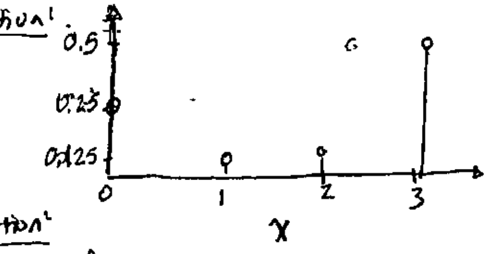
$P(A) = P(\text{child \#1} \cap \text{child \#2})$

$P(C) = P(\text{child \#2} \cap \text{child \#3})$

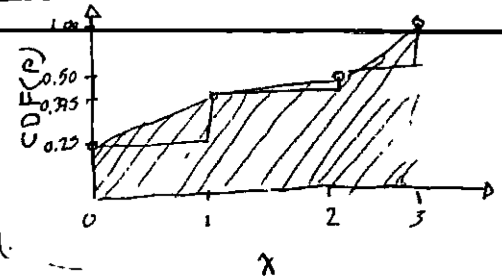
$P(\#1 \cap \#2 \cap \#3) = P(\#1 \cap \#2)P(\#3) = \boxed{P(\#1)P(\#2)P(\#3)}$

$\boxed{P(A \cap B \cap C) \neq P(A)P(B)P(C)} = P(\#1 \cap \#3)P(\#1 \cap \#2)P(\#2 \cap \#3)$

Chapter 2: 1. $P(X=0)=0.25, P(X=1)=0.125, P(X=3)=0.5$, $P(X=2)=0.125$ Frequency Function:

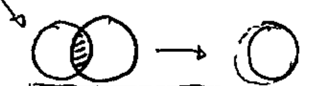


Cumulative Distribution Function:

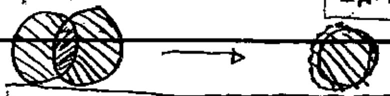


2. $I_{A\cap B} = I_A I_B = \min(I_A, I_B)$

$I_{A\cup B} = \max(I_A; I_b)$



$\boxed{I_A \cap I_B = I_{A\cap B}}$



$\boxed{I_{A\cup B} = \max(I_A, I_B) = I_A + I_B - I_A \cdot I_B}$

3. CDF → Frequency (Δ)

k	F(k)
0	0
1	0.1
2	0.3
3	0.7
4	0.8
5	1.0

CDF

k	P(X)
0	0
1	0.1
2	0.2
3	0.5
4	0.1
5	0.2

Frequency

4. Prove $p(k) = F(k) - F(k-1)$

$CDF = \sum_0^n p(x) = p(k) + \sum_1^n p(k)$

$F(k) = p(k) + F(k-1)$

$\boxed{p(k) = F(k) - F(k-1)}$

5. Show $P(u < X \leq v) = F(v) - F(u)$ a) Discrete $F(k) = \sum_0 p(k) = p(k) + \sum_1 p(k)$

$F(k) = p(k) + F(k-1)$

$\boxed{P(k) = F(k) - F(k-1)}$

b. $F(k) = \int_0^n p(k)\,dk = p(k) + \int_1^n p(k)\,dk = p(k) + F(k-1)$

$\boxed{p(k) = F(k) - F(k-1)}$

6. a) Four tosses: ~~$P(T) = (\frac{1}{2})^{1}(1-\frac{1}{2})^{1-1} = \frac{1}{2}$, $P(T|2) = (\frac{1}{2})^2(1-\frac{1}{2})^{1-2} = \frac{1}{4}(2)$, $P(T|3) = (\frac{1}{2})^3(1-\frac{1}{2})^{1-3} = (\frac{1}{8})4$, $P(T|4) = (\frac{1}{4})^4(1-\frac{1}{2})^{1-4}$~~ $= \frac{8}{16}$

① Sample space ($n=16$)

Outcome	HHHH	HHHT	HHTH	HHTT	HTHH	HTHT	HTTH	HTTT	T x 8
Heads before Tails	4	3	2	2	1	1	1	1	0

② Frequency Function ($n(X)$)

X	0	1	2	3	4
n(X)	8	4	2	1	1
$P(X) = \frac{n(X)}{16}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$
F(X)	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{15}{16}$	$\frac{16}{16}$

① Sample Space: HHHH HHHT HHTH HHTT HTHH HTHT HTTH HTTT THHH THHT THTH THTT TTHH TTHT TTTH TTTT

b) the number of Heads following first tail

② Frequency Function

X	0	1	2	3	4
n(X)	1	1	2	4	8
$P(X) = \frac{n(X)}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$
F(X)	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1

c) (the number of heads) − (number of tails)

② Frequency Function

X	−4	−2	0	2	4
n(X)	1	4	6	4	1
$P(X) = \frac{n(X)}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$
F(X)	$\frac{1}{16}$	$\frac{5}{16}$	$\frac{11}{16}$	$\frac{15}{16}$	$\frac{16}{16}$

d) Frequency Function = (c)

7. Bernoulli Random Variable: $p(1) = p; p(0) = 1-p; p(x) = 0$; $p(x) = \begin{cases} p^x(1-p)^{1-x}; & x=0 \text{ or } x=1 \\ 0 & ; \text{otherwise.} \end{cases}$

CdF of Bernoulli Random Variable: $F(x) = \sum_0^{x+y+z} p(x) = \sum_0^{x} p(0) + \sum_0^{y} p(1) + \sum_0^{z} p(x)$

$= \sum_0^{x} p + \sum_0^{y} 1-p + \sum_0^{z} 0 = \boxed{\sum_0^{x} p + \sum_0^{y} 1-p}$

8. $P(\text{9 Heads in 10 Tosses}) = \frac{\text{chance}}{\text{outcome}} = \frac{10}{2^{10}} = \frac{10}{1024} = \boxed{\frac{5}{512}}$ ✓

$P(\text{18 Heads in 20 Tosses}) = \frac{\binom{20}{2}}{2^{20}} = \frac{201}{2!\,181\ 1048576} = \boxed{\frac{95}{524288}}$

9. $n=3; p = \sum_0^3 \binom{n}{k} p^k (1-p)^{n-k} = \binom{3}{0}p^0(1-p)^3 + \binom{3}{1}p^1(1-p)^2 + \binom{3}{2}p^2(1-p)^1 + \binom{3}{3}p^3(1-p)^0 = (1-p)^3 + 3p(1-p)^2 + 3p^2(1-p)^1 + p^3 = 1$; $p = \frac{M+1}{n-1}$ when

$n=3; p = \sum_0^2 \binom{3}{k} p^k (1-p)^{n-k} = \binom{3}{0}p(1-p)^3 + \binom{3}{1}p(1-p)^2 + \cancel{\binom{3}{2}}(1-\ ) = (1-p)^3 + 3p(1-p)^2 + (1-p)^3 + 3p(1-p)^2 > 1-p$

$p < 0.5$

10. Player A: $p_1 = P(\text{success})$; Player B: $p_2 = P(\text{success})$ a) $P(x) = \cancel{\ldots}$ odd: $p(k) = (1-p_1)^{k/2} \cdot (1-p_2)^{\frac{k}{2}-1} \cdot p_2$

b. $P(\text{Player A Win}) = \dfrac{p_1^n}{1-(1-p_1)(1-p_2)}$ $= \boxed{p_1 \sum [(1-p_1)(1-p_2)]^i}$

11. Binomial Distribution: $p(x) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$; Mode $= p'(x) = 0 = kp^{k-1}(1-p)^{n-k} + (n-k)p^k(1-p)^{n-k-1}$

$= kp^{k-1}(1-p)^{n-k} + n \cdot p^k (1-p)^{n-k} - kp^k(1-p)^{n-k}$

$= k\left(\frac{1}{p}\right)(n-k)$; $k\left(\frac{1}{p}-1\right) + n - k = 0$

$\frac{k}{p} - k + n - k = 0$

$\frac{k}{p} + \left(\frac{n}{k}\right) - 2 = 0$

$p = \frac{k}{n+k - 2p}$

12. Prove $p(k) = \sum_0^1 \binom{1}{k} p^k (1-p)^{n-k} = 1$

$= \binom{1}{0} p^0 (1-p)^1 + \binom{1}{1} p^1 (1-p)^0$

$= (1-p) - p = 1$

13. 20 items [4-choices] - Elimination at one, remainder of three
- Passing is 12 or more, correct.

a. $P(\text{Pass}) = \dfrac{P(\text{correct})}{\text{Total Outcomes}} = \dfrac{1}{\binom{3}{1}\binom{20}{1}} = \dfrac{1}{3 \cdot 20} = \boxed{\dfrac{1}{60}}$

b. $P(\text{Pass}) = \dfrac{P(\text{correct})}{\text{Total Outcomes}} = \dfrac{1}{\binom{2}{1}/\binom{20}{1}} = \dfrac{1}{2 \cdot 20} = \boxed{\dfrac{1}{40}}$

14. 4 bits +3 bits $P(\text{change}) = 0.05$; Mutually Independent.

$P(\text{change} \mid 7 \text{ bits}) = \prod^{7} p(\text{change}_i) = p(\text{change})^7 = \boxed{0.05^7}$

$P(\text{change} \mid 4 \text{ bits}) = \prod^{4} p(\text{change}_i) = p(\text{change})^4 = \boxed{0.05^4}$

$P(\text{correct}) = 1 - P(\text{change})$ $= 1 - 0.05^7$

15. $P(\text{Winning Team A}) = 0.4$; Better advantage at 3 of 5 games or 4 of 7 games. 0.6

$\boxed{P(3 \text{ of } 5) = \prod^{3} p(\text{Winning Game}_i) = (0.4)^3}$; $P(4 \text{ of } 7) = \prod^{4} p(\text{Winning Game}_i) = (0.4)^4$

$P(\text{correct}) = 1 - P(0.05^4)$

16. $n \to \infty$ and $r/n \to p$ and $m = \text{constant}$. Hypergeometric Function: $P(x=k) = \dfrac{\binom{r}{k}\binom{n-r}{m-k}}{\binom{n}{m}}$

$\lim_{n\to\infty} \lim_{r/n \to p} P(x=k) = \lim_{n\to\infty} \lim_{r/n\to p} \dfrac{\frac{r!}{k!(r-k)!} \cdot \frac{(n-r)!}{(m-k)!(n-r-m+k)!}}{\frac{n!}{m!(n-m)!}} = \binom{m}{k} \dfrac{[r(r-1)\cdots(r-k+1)][(n-r)\cdots(n-r-m+k+1)]}{[n(n-1)\cdots(n-m+1)]}$

$= \binom{m}{k}\left(\frac{1}{q}\right)^k \left(\frac{q-1}{q}\right)^{m-k}$; where $q = \frac{1}{p} = \frac{n}{r}$

$= \binom{m}{k} p^k (1-p)^{m-k}$

17. Bernoulli Trials ; $p(\text{success}) = p$; Failures till first round are counted.

Frequency Function: $\boxed{p(\text{Failure}) = p \prod_{i=0}^{n} (1-p)}$

18. Frequency Function $p(\text{Failure}) = p \prod^{r} (1-p) = p(1-p)^n$

19. CDF: $\sum_{x=0}^{\infty} p(1-p)^{x+1} = 1$; $P = \sum_{j=0}^{j=n} \text{cdf}_{i}$

20) Minimum Trials: $p = \frac{1}{2}$ | $P(k+A) = \frac{1}{2}(\dots)$ | $P(k \mid \text{other box}) = \binom{n+1}{n+1} p^{n+1} q^0 + \binom{n+1}{n+1} p^0 q^{n+1}$

Maximum Trials: $2n+1$ | $P(\text{Right}) = \prod p$ | $= p^{(n+1)} + q^{(n+1)} = 2\left(\frac{1}{2}\right)^{(n+1)} = \left(\frac{1}{2}\right)^n$; $n \geq 2$

$P(k-1 \mid \text{other box}) = \binom{n+1}{n} p^n q + \binom{n+1}{n} p q^n = 2\binom{n+1}{n}\left(\frac{1}{2}\right)^{(n+1)}$

21. X = Geometric Random Variable $P(X > n+k-1 \mid X > n-1) = P(X > k)$; $= \binom{n+1}{n}\left(\frac{1}{2}\right)^{(n+1)}$

$$P(\text{Hypergeometric Function}) = \frac{\binom{r}{k}\binom{n-r}{m-k}}{\binom{n}{m}}$$

; $P(X > n+k-1 \mid X > n-1)$; $= (n+1)\left(\frac{1}{2}\right)^n$

$= P(X+1-n > k \mid X+1 > n)$; $P(k = n-r) = \binom{n+r}{n}\left(\frac{1}{2}\right)$

$= P(X-n > k \mid X-n > 0)$; $P = \binom{n+n-k}{n}\left(\frac{1}{2}\right)^{n+n-k}$

$= \boxed{P(X > K \mid X > 0) = P(X > k)}$

22. X : Geometric random variable

$P = 0.5$; $P(X \leq k) \approx 0.99 = 1-(1-p)^k$

$(0.5)^k \approx 0.01$

$k \log(0.5) \approx \log(0.01)$; $k = 6.6438$

$\boxed{k = 7.}$

23) p(success) ; r successes before k^{th} failure ; Binomial $\boxed{p(k) = \binom{n+r-1}{r-1} p^r (1-p)^k}$

Total Number of trials : $(k+r)$ | $p(\text{success}) = \binom{K+r-1}{K-1} p^r (1-p)^{K-1}(1-p)$

Last trial probability : $(1-p)$

Binomial : $\binom{k+r-1}{k-1} p^r (1-p)^{k-1}$ | $\boxed{= \binom{K+r-1}{K-1} p^r (1-p)^K}$

Failures

24) $H = 2^3$; $P(\text{Same}) = P(HHH) + P(TTT)$ $= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

| $P(k) = (1-p)^{k-1}(p) = (0.75)^{k-1}(0.25)$

$\sum_{k=1}^{24} (0.75)^{k-1}(0.25) = 1$; $n! = n(n-1)(n-2)$

$P(X > 3) = 1 - P(X \leq 3)$

$= 1 - \{P(X=1) + P(X=2) + P(X=3)\}$

$= 1 - \{(0.75)^{1-1}(0.25) + (0.75)^{2-1}(0.25) + (0.75)^{3-1}(0.25)\}$

$= 1 - \{0.25 + 0.1875 + 0.140625\}$

$\boxed{\approx 0.4219}$

25. $P(\text{Royal Straight Flush}) = 1.3 \times 10^{-8}$

$n = 100$ hands/week ; 52 weeks/year ; 20 years $= 1.04 \times 10^5$

a) Time Sequence : Poisson Distribution

$\lambda = n \cdot p = 1.04 \times 10^5 \cdot 1.3 \times 10^{-8} = 1.35 \times 10^{-3}$

$p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$; $p(0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-1.35 \times 10^{-3}}$; $\boxed{9.986 \times 10^{-1}}$

① State Space
② Frequency Function
③ $1 - p(1) p(2) p(3)$

b). $2 = \prod p^k (1-p)^{n-k}$; $p(2) = \frac{\lambda^2 e^{-\lambda}}{2!} = \frac{(1.35 \times 10^{-3})^2 e^{-1.35 \times 10^{-3}}}{2!} = \boxed{9.10 \times 10^{-7}}$

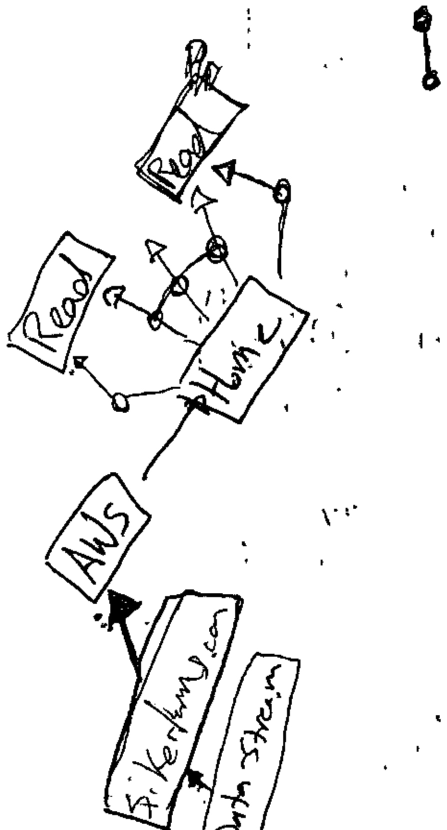
26. $\frac{1}{10{,}000}$ chance of being trapped. $n = 5 \frac{\text{days}}{\text{week}} \cdot \frac{52 \text{ weeks}}{\text{year}} \cdot 10$ years $= 2{,}600$; $\lambda = np = \boxed{\frac{13}{50}}$

$P(0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-13/50} = \boxed{7.71 \times 10^{-1}}$; $P(1) = \frac{\left(\frac{13}{50}\right)^1 e^{-13/50}}{1!} = \boxed{2.0 \times 10^{-1}}$; $p(2) = 0.026$

- Randomize Router [IP]
- Randomize Mac Address ✓
- Randomize external IP. ✓
- VPN through aws
- Tor for browsing

  ↳ IP Scan - ping

- Tor guard -



27. $P(\text{Disease}) = \frac{1}{10{,}000}$; $n = 100{,}000$ people

$k = 0$ cases ; Poisson Distribution

$P(k) = \frac{\lambda^k e^{-\lambda}}{k!}$; $P(0) = \frac{\lambda^0 e^{-10}}{0!} = \boxed{4.54\times10^{-5}}$

$k = 1$ cases : $P(1) = \frac{\lambda^1 e^{-\lambda}}{1!} = \frac{(10)^1 e^{-10}}{1!} = \boxed{4.54\times10^{-4}}$

$k = 2$ cases : $P(2) = \frac{\lambda^2 e^{-\lambda}}{2!} = \frac{(10)^2 e^{-10}}{2!} = \boxed{2.26\times10^{-2}}$

28. $CDF = F(k) = P_0, P_1, \ldots, P_n$; n, and p. $q = 1-p$

Prove the binomial probabilities by $P_0 = q^n$.

$$P_k = \frac{(n-k+1)p}{kq} P_{k-1} \; ; \; k = 1, 2 \ldots n$$

$\boxed{P_0 = (1-p)^n}$; $P_1 = n\frac{p}{(1-p)}(1-p)^n$; $P_2 = \frac{(n-1)p}{2(1-p)} P_1 = \frac{(n-1)}{2}p^2(1-p)^n$

$P_1 = \frac{(n-1+1)p}{1(1-p)} P_0$ $= \boxed{n(p(1-p))^{n-1}}$; $= \frac{n(p)}{2(1-p)} \ldots$ $\boxed{= \frac{n(n-1)}{2} p^2 (1-p)^{n-2}}$

Recursive Binomial Distribution $= \frac{n}{(1-k)} p (1-p)^n$

$$\boxed{= \prod \frac{(n-k+1)}{k} p^n (1-p)^{n-k}}$$

$P(X \le 4)$ for $n = 9000$ and $p = 0.0005$

$$= \frac{(9000-3)(9000-2)(9000-1)(9000)}{4!} (0.0005)^{9000} (1-0.0005)^{9000-4}$$

$\boxed{= 1.89\times10^{-1}}$ $\cong 0.00$

As a Poisson

$n = 9000$; $p = 0.0005$; $np = 9/2$

$P(4) = \frac{(9/2)^4}{4!} e^{-9/2} = \boxed{1.89\times10}$

29. $P_0 = \exp(-\lambda)$

$P_k = \frac{\lambda}{k} P_{k-1}$; $k = 1, 2 \ldots$

$P_0 = \exp(-\lambda)$; $P_1 = \lambda \cdot P_0 = \lambda\exp(-\lambda)$; $P_2 = \frac{\lambda^2}{2}\exp(-\lambda)$; $P_k = \frac{\lambda^k}{k!}\exp(-\lambda)$

$P(X \le 4)$; $\lambda = 4.5$; $P_k = \frac{(4.5)^4}{4!}\exp(-4.5) = \boxed{1.89\times10^{-1}}$

30. Poisson Frequency Function ; $p(k) = \frac{\lambda^k \exp(-\lambda)}{k!}$; $p'(k) = \frac{k \cdot \lambda^{k-1}(-\lambda)\exp(-\lambda)}{k!} = 0$

$-\lambda = 1$

$\boxed{\lambda = 1}$

Sources cite ratio: $\frac{P(X=k+1)}{P(X=k)} = \frac{\lambda^{k+1}\exp(-\lambda)/(k+1)!}{\lambda^k \exp(-\lambda)/k!}$

$\boxed{= \lambda/k+1}$

"Not logical because mix of Poisson changes as ... changes"

There are maximum and minimums to this problem set. $\lambda < 1$, $\lambda > 1$ (int), $\lambda > 1$ (Rational) $\quad \lambda = np = 1.5$

Probability Decreases

Max: $k=0$

31. $\lambda = 2$ per hour a) 10-min shower; p(phone rings) = $\frac{(2)^{1/6}\exp(-2)}{1/6} = \boxed{0.277}$

b) p(phone rings) = 0.5 = $\frac{\delta^0 e^{-\delta}}{0!}$; $\exp(2) \; 69.3$ min

Fractions and Factorial approx to one

Time = $\frac{60 \text{ min}}{2 \text{ phone calls}} \cdot 0.693$

$\boxed{= 20.79 \text{ min}}$

32. $\lambda = 0.33$ per month

a) $k=0$; $p(0) = \frac{(1/3)^0 \exp(-1/3)}{0!} = \exp(-1/3) = \boxed{0.716}$

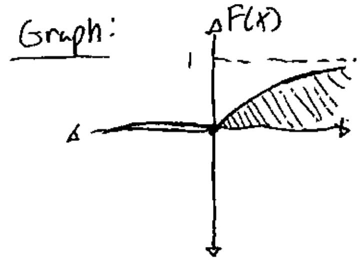
$k=1$; $p(1) = \frac{(1/3)^1 \exp(-1/3)}{1!} = \boxed{0.239}$

$k=2$; $p(2) = \frac{(1/3)^2 \exp(-1/3)}{2!} = \boxed{0.004}$

The most probable number of suicides would be at $k=0$ because $\lambda < 1$ and demonstrates a decreasing probability.

33. $F(x) = 1 - \exp(-\alpha x^\beta)$ for $x \geq 0, \alpha > 0, \beta > 0$, $F(x) = 0$ for $x < 0$.

Graph:

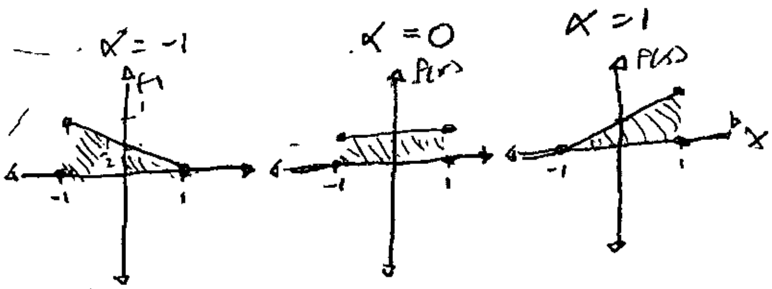


$p(x) = \frac{d}{dx}F(x) = \boxed{\alpha \beta x^{\beta-1} \exp(-\alpha x^\beta)}$

Cumulative Density Function:

$\boxed{\lim_{x \to -\infty} F(x) = 0 \; ; \; \lim_{x \to \infty} F(x) = 1}$

34. $f(x) = (1+\alpha x)/2$ for $-1 \leq x \leq 1$ and $f(x) = 0$



Probability Density Function Requires

$\boxed{P(X=x_i) = p(x_i) \; ; \; \sum_i p(x_i) = 1}$

$F(x) = \sum_{-1}^{1} (1+\alpha x_i)/2 = \int_{-1}^{1} (1+\alpha x)/2 \, dx = \frac{x}{2}\Big|_{-1}^{1} + \frac{\alpha x^2}{4}\Big|_{-1}^{1} = \frac{1}{2} + \frac{1}{2} + \frac{\alpha x^2}{4} - \frac{\alpha x^2}{4} = \boxed{1}$

$\boxed{F(x) = \frac{x}{2} + \frac{\alpha x^2}{4}}$

35. $p(x) = \begin{cases} -1 & -1 \leq x \leq 0 \\ 2 & 0 < x < 1 \end{cases}$

$F(x) = \int_{-1}^{0} -1 \, dx + \int_{0}^{1} 2 \, dx = 3$

36. U is uniform [0,1]. $X = [nU]$, where $[t]$

"Probability Mass Function"

37. $P(X \leq 1/3) = \frac{1}{3}$; $P(X \geq 2/3) = 2/3$

38. $\alpha f + (1-\alpha) g$ where $0 \leq \alpha \leq 1$; $\sum_i P(x_i) = 1$; $= \alpha f + (1-\alpha) g$

max $\frac{d}{d\alpha} \alpha f + (1-\alpha) g = f - g$

$F(\alpha) = \frac{\alpha^2}{2} f + \left(\alpha - \frac{\alpha^2}{2}\right) g$

39. Cauchy Cumulative Distribution: $F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x)$, $-\infty < x < \infty$

a. Cumulative Distribution Requirements: $\lim_{x\to-\infty} F(x) = 0$; $\lim_{x\to\infty} F(x) = \boxed{1}$

$\lim_{x\to-\infty} F(x) = \lim_{x\to-\infty} \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x) = \frac{1}{2} - \frac{1}{2} = \boxed{0}$; $\lim_{x\to\infty} \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x) = \frac{1}{2} + \frac{1}{2} = \boxed{1}$

b. $p(x) = F'(x) = \frac{d}{dx} \ldots \boxed{\frac{1}{(1+x^2)\pi}}$

$\pi \cdot \sin^2(x)$ — $\cos^2 x$; $\sec^2$

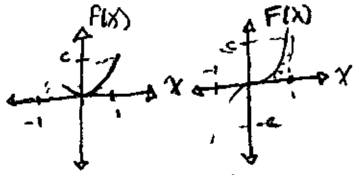
c. $P(X > x) = 0.1$; $0.1\pi(1+x^2) = 1$

$(1+x^2) = \frac{10}{\pi} - 1$

$x^2 = \frac{10-\pi}{\pi}$

$\boxed{x = \sqrt{\frac{10-\pi}{\pi}}}$

40. $f(x) = cx^2$ for $0 \le x \le 1$ and $f(x) = 0$ otherwise



a) $F(1) = c$; $F(0) = 0$; $c = \frac{F(1) - F(0)}{1 - 0} = \Delta F(x)$

b) $F(x) = \frac{cx^3}{3}$ c) $P(0.1 \le X < 0.5) = \frac{[F(0.5) - F(0.1)]/(0.5 - 0.1)}{F(1) - F(0)/(1-0)} = [c(\frac{1}{4}) - c(\frac{1}{10})]/0.4$

$= \frac{6}{40} \cdot \frac{10}{4} = \frac{6}{16} = \boxed{\frac{3}{8}}$

41. Find the upper and lower quartiles of an exponential distribution.

Exponential Distribution: $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & x < 0 \end{cases}$

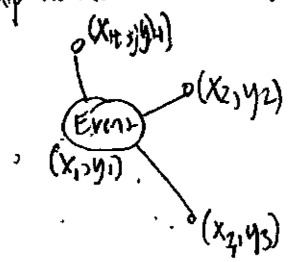
Lower quartile: $P(X) = \frac{1}{4} = \lambda e^{-\lambda x}$; $-\log 4\lambda = -\lambda x$

$\boxed{x = \frac{+\log 4\lambda}{\lambda}}$

Upper quartile: $p(x) = \frac{3}{4} = \lambda e^{-\lambda x}$

$\boxed{x = \log\left(\frac{4\lambda}{3}\right)/\lambda}$

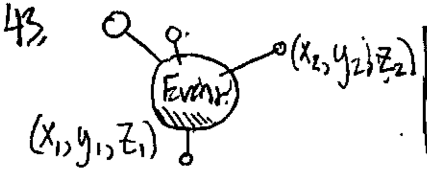
42.



$\boxed{p(x) = \sqrt{\left(\frac{\lambda^{x_2} e^{-\lambda}}{x_2!} - \frac{\lambda^{x_1} e^{-\lambda}}{x_1!}\right)^2 + (x_2 - x_1)^2}}$

$\boxed{P(x) = 2\pi\lambda x e^{-\pi\lambda x^2}}$

43.



$f(x) = \frac{d}{dx} - e^{-4\lambda\pi x^3/3}$ ← volume × lambda

$\boxed{= 4\lambda\pi x^3 e^{-4\lambda\pi x^3/3}}$

Multivariate Poisson Distribution:

$P(X) = \exp(-\sum_{i=1}^{m}\theta_i)\prod_{i}^{m}\frac{\theta_i^{x_i}}{x_i!}\sum_{i=0}^{s}\prod_{j=1}^{m}\binom{x_j}{i} i! \left(\frac{\theta_0}{\prod \theta_i}\right)^i$

44. T: Exponential Random Variable with λ $T = \lambda e^{-\lambda x}$; $k = \lambda e^{-\lambda x}$; $k+1 = \lambda e^{-\lambda x}$

$\boxed{x = \log\left(\frac{\lambda}{T}\right)/\lambda}$

Exponential Random Distribution:

$p(x) = \lambda e^{-\lambda x}$

X: Discrete Random variable; $X = K$; $k < T < k+1$

for $k = 0, 1, \ldots$

45. Exponential Distribution: $p(x)=\lambda e^{-\lambda x}$; $\lambda=0.1$ a) Probability lifetime < 10 years.

$P(\text{lifetime})+P(\text{death})=P(\text{lifetime})+\frac{e^{-10/10}}{10}=1$

$\boxed{P(\text{lifetime})=1-\frac{e^{-1}}{10}}$

b. $\frac{e^{-5/10}}{10}-\frac{e^{-15/10}}{10}$ c) $0.01=\frac{e^{-t/10}}{10}$; $-1=-t/10$; $\boxed{t=10}$

46. <u>Gamma Density</u>: $g(t)=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}t^{\alpha-1}e^{-\lambda t}$, $t\geq 0$

where $\Gamma(x)=\int_0^{\infty}u^{x-1}e^{-u}du$; $x>0$

$$\int_0^{\infty}g(t)dt=\int_0^{\infty}\frac{\lambda^{\alpha}t^{\alpha-1}e^{-\lambda t}}{\Gamma(\alpha)}dt=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-\lambda t}dt \; ; \; t=x/\lambda \; ; \; x=t\lambda \; ; \; =\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty}\left(\frac{x}{\lambda}\right)^{\alpha-1}e^{-x}\frac{dx}{\lambda}$$

$$=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}x^{\alpha-1}e^{-x}dx=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\lambda}=\frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)\lambda}=\boxed{\frac{\alpha}{\lambda}} \; ; \; \lambda=1 \text{ and } \alpha=1.$$

47. $\alpha>1$, Show maximum of Gamma Density : $(\alpha-1)/\lambda$; $\frac{d}{dt}g(t)=0$

$$=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\left[(\alpha-1)t^{\alpha-2}-\lambda t^{\alpha-1}\right]e^{-\lambda t}=0 \; ; \; (\alpha-1)t^{\alpha-2}=\lambda t^{\alpha-1} \; ; \; \boxed{\frac{(\alpha-1)}{\lambda}=t^{\alpha-1-\alpha+2}=t}$$

48. T is an exponential Random variable: $p(x)=\lambda e^{-\lambda x}$, and $P(T<1)=0.05$.
What is λ? $0.05=\lambda e^{-\lambda T}=\lambda\left(1+\lambda+\frac{\lambda^2}{2!}-\dots\right)=\lambda-\lambda^2+\frac{\lambda^3}{2!}$; $\boxed{\lambda=0.0527}$

"Third order Quadratic"

49. a) $\Gamma(1)=1$; <u>Gamma Density</u>: $g(t)=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}t^{\alpha-1}e^{-\lambda t}$

<u>Gamma Function</u>: $\Gamma(x)=\int_0^{\infty}u^{x-1}e^{-u}du$; $\Gamma(1)=\int_0^{\infty}u^{1-1}e^{-u}du=\int_0^{\infty}e^{-u}du=-e^{-u}\Big|_0^{\infty}=\boxed{0-(-1)=1}$

b) $\Gamma(x+1)=x\Gamma(x)$; $\Gamma(x+1)=\int_0^{\infty}u^{x}e^{-u}du$; <u>Integration by Parts</u> $\int u\,dv=uv-\int v\,du$; $u=u^x$, $dv=e^{-u}$, $du=(x)u^{x-1}du$, $v=-e^{-u}$; $=-u^{x}e^{-u}\Big|_0^{\infty}+\int(x)u^{x-1}e^{-u}=\boxed{x\Gamma(x)}$

c) Conclude $\Gamma(n)=(n-1)!$; $n=1,2,3\dots$

<u>Table</u>:

n	$\Gamma(n)$	$(n-1)!$
1	0	0
2	1	1
3	2	2
4	6	6
5	24	24
6	120	120

<u>Graph</u>:

51. <u>Normal Distribution</u>: $p(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$ <u>Standard Normal</u>: $\mu=0,\sigma=1$; $p(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

50. $\Gamma(x)=2\int_0^{\infty}t^{2x-1}e^{-t^2}dt=\int_{-\infty}^{\infty}e^{xt}e^{-e^t}dt$;

$2\int_0^{\infty}e^{-x^2/2}dx=2\sqrt{\frac{\pi}{2}}=\sqrt{2\pi}$; $\int_{-\infty}^{\infty}e^{-x^2/2}dx\cdot\int_{-\infty}^{\infty}e^{-y^2/2}dy=2\pi$

$=\int_{-\infty}^{\infty}t^{2x-1}e^{-t^2}dt=\int_{-\infty}^{\infty}(e^{t/2})^{2x-1}e^{-e^t}dt$; $e^{xt}=t^{2x-1}$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(x^2+y^2)/2}dx\,dy=\int_0^{\infty}e^{-r^2/2}\dots dr$

$e^{t}=t^2$; $\int_{-\infty}^{\infty}e^{xt-t/2}e^{-e^t}dt=\int_{-\infty}^{\infty}e^{xt}e^{-e^t}dt$

$dr=\frac{1}{2}(x^2+y^2)$

52. $\mu=70$ and $\sigma=3$ in. a) What proportion of the population is over 6 ft tall?

$$P(78)=\frac{1}{3\sqrt{2\pi}}e^{-(78-70)^2/2\cdot 3^2}=0.10654;$$ $\boxed{0.35\% \text{ over the height of 6ft tall.}}$

$$P(>78)=\frac{1}{3\sqrt{2\pi}}\int_{78}^{\infty}e^{-(x-70)^2/18}\,dx=0.0038=\boxed{0.38\%}$$

b) Cm: $70\text{in}\times\frac{2.54\text{cm}}{1\text{inch}}=178\text{cm}$; $3\text{in}\times\frac{2.54\text{cm}}{1\text{inch}}=7.5\text{cm}$; m: $70\text{inches}\times\frac{2.54\text{in}}{1\text{inch}}\times\frac{1\text{m}}{100\text{cm}}=1.75\text{cm}$; $7.5\text{in}\times\frac{1\text{m}}{100\text{cm}}=\boxed{0.075}$

53. $\mu=5$, and $\sigma=10$. a) Find $P(X>10)=\frac{1}{10\sqrt{2\pi}}\int_{10}^{\infty}e^{-(x-5)^2/2\cdot 10^2}dx\approx 0.3085=\boxed{30.85\%}$

Z-table: $\frac{10-5}{10}=\frac{1}{2}=Z(30.85\%)$

b) Find $P(-20<X<15)=\frac{1}{10\sqrt{2\pi}}\int_{-20}^{15}e^{-(x-5)^2/2\cdot 10^2}dx=0.8351=\boxed{83.51\%}$ $\left(\frac{15-5}{10}-\frac{-20-5}{10}\right)$

$Z(0.9938)-Z(0.8413)$

$\boxed{84.75\%}$

54. $X\sim N(0,\sigma^2)$; $Y=|X|$

$$=\frac{1}{\sigma^2\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-(x)^2/2\cdot\sigma^2}dx=\frac{\sqrt{\pi\cdot 2\cdot\sigma^2}}{\sqrt{2\pi}}=\boxed{1}$$

55. $X\sim N(\mu,\sigma^2)$ Find c in terms of σ, such that, $P(\mu-c\le X\le\mu+c)=0.95$

$$0.95=\frac{1}{\sigma\sqrt{2\pi}}\int_{\mu-c}^{\mu+c}e^{-(x-\mu)^2/2\sigma^2}dx;\quad \int e^{-ax^2}=\sqrt{\frac{\pi}{a}};\quad \boxed{c\approx 1.95996\sigma}$$

56. $X\sim N(\mu,\sigma^2)$; $P(|X-\mu|\le 0.675\sigma)=0.5$ $\Rightarrow 0.5=\frac{1}{0.675\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-0.675^2\sigma^2/2\sigma^2}\cdot dx=\boxed{0.5}$

57. $X\sim N(\mu,\sigma^2)$; $Y=aX+b$, where $a<0$, show $Y\sim N(a\mu+b,a^2\sigma^2)$

$$p(x)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-(x-\mu)^2/2\sigma^2}dx=P\left(\frac{y-b}{a}\right)=\frac{1}{a\sigma\sqrt{2\pi}}e\left[-\frac{1}{2}\left(\frac{y-b-a\mu}{a\sigma}\right)^2\right]=\frac{1}{a\sigma\sqrt{2\pi}}\exp\left[\frac{-1}{2}\frac{(y-(a\mu+b))^2}{a^2\sigma^2}\right]$$

$\boxed{Y\sim(a\mu+b,a^2\sigma^2)}$

58. $Y=e^Z$, where $Z\sim N(\mu,\sigma^2)$; $\boxed{\text{Lognormal Density}}$

$Y=e^z=e^{N(\mu,\sigma^2)}$; $\log Y=N(\mu,\sigma^2)=0.5$ $\boxed{Y=e}$

59. $U[-1,1]$; Find density function of U^2; $F_u=P(-\sqrt{x}\le z\le\sqrt{x})$

$=\Phi(\sqrt{x})-\Phi(-\sqrt{x})$

$=\frac{1}{2}x^{-1/2}\phi(\sqrt{x})+\frac{1}{2}x^{-1/2}\phi(-\sqrt{x})$

$=x^{-1/2}\phi(\sqrt{x})$; $\boxed{f_x(x)=\frac{x^{-1/2}e^{-x/2}}{\sqrt{2\pi}}}$

60. $U[0,1]$; Find density function of $\sqrt{U}$.

$F_u=P(x^2\le z\le x^2)=\Phi(x^2)-\Phi(0)$

$=2x\,\phi(x^2)$; $\boxed{f(x)=\frac{2xe^{-x/2}}{\sqrt{2\pi}}}$

61. Density of cX when X follows gamma distribution; Gamma Distribution: $g(t) = \frac{\lambda^{\kappa}}{\Gamma(\kappa)} t^{\kappa-1} e^{-\lambda t}$

$\therefore g(ct \le c\lambda) = \boxed{g(t \le \frac{\lambda}{c}) = \frac{(\frac{\lambda}{c})^{\kappa}}{\Gamma(\kappa)} t^{\kappa-1} \cdot e^{-\lambda t/c}}$

$\Gamma(x) = \int_0^{\infty} u^{x-1} \cdot e^{-u} du$

62. m = mass; V = random velocity; $\mu = 0$ and σ. Find the density function of Kinetic Energy: $E = \frac{1}{2} m V^2$

Normally Distributed: $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$

$= \frac{1}{\sigma\sqrt{2\pi}} e^{-E^2/2\sigma^2} = \boxed{\frac{1}{\sigma\sqrt{2\pi}} e^{-(\frac{1}{2}mv^2)^2/2\sigma^2}}$

63. Suppose Θ is a uniform distribution; Interval or domain $[-\pi/2, \pi/2]$: Find the CDF and density of tanΘ

$\tan\Theta = x$; $\Theta = \arctan(x)$; $P(\arctan(-X) \le \Theta \le \arctan(X)) = \Phi(\arctan(X)) + \Phi(\arctan(+X))$

$= \frac{a^2}{x^2+a^2} \Phi(\arctan(x)) + \frac{a^2}{x^2+a^2} \Phi(\arctan(+x))$

64. f_X = "Density Function" and $Y = aX + b$, then.

$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$

$\boxed{F_u = \frac{1}{\pi(1+x^2)} \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}}$

$f_y(y) = P(Y \le y) = P(aX + b \le y) = P(X \le \frac{y-b}{a}) = F_X\left(\frac{y-b}{a}\right) = P'(X) = \boxed{f_X\left(\frac{y-b}{a}\right)\frac{1}{|a|}}$

65. $f(x) = \frac{1+\alpha x}{2}$ from $-1 \le x \le 1$ and $-1 \le \alpha \le 1$.

$F_X(x) = \int_{-1}^{x} \frac{1+\alpha t}{2} dt = \int_{-1}^{x} \frac{1}{2} + \frac{\alpha}{2} t \, dt$

$F_X(x) = \frac{1}{2}[x+1] + \frac{\alpha}{4}[x^2-1]$

$4F_X(x) = 2[x+1] + \alpha[x^2-1]$

$4X = 2[F^{-1}(x)+1] + \alpha[F^{-1}(x)^2 - 1]$

$4X = 2F^{-1}(x) + 2 + \alpha F^{-1}(x)^2 - \alpha$

$\alpha F^{-1}(x)^2 + 2F^{-1}(x) + (2 - \alpha - 4x) = 0$

$\frac{-2 \pm \sqrt{4 - 4(\alpha)(2-\alpha-4x)}}{2\alpha} = F^{-1}(x)$

$\frac{-2 \pm \sqrt{4 - 8\alpha + 4\alpha^2 + 8x}}{2\alpha} = F^{-1}(x)$

$\boxed{\frac{-1 \pm \sqrt{1 - 2\alpha + \alpha^2 + 2x}}{\alpha} = F^{-1}(x)}$

66 $f(x) = \alpha x^{-\alpha-1}$ for $x \ge 1$ and $f(x) = 0$

$\int_1^{\infty} \alpha x^{-\alpha-1} dx = \frac{\alpha x^{-\alpha}}{(-\alpha)} \Big|_1^{\infty} = -\frac{1}{x^{\alpha}} \Big|_1^{\infty} = -\frac{1}{\infty} + \frac{1}{1^{\alpha}} = 1 = F(x)$

$F^{-1}(1) = X$ $\boxed{\text{"Cumulative Distribution Functions are pseudo random number generators"}}$

67. Weibull Cumulative Distribution Function:

$F(x) = 1 - e^{-(x/\alpha)^{\beta}}$, $x \ge 0$, $\alpha > 0$, $\beta > 0$

a) Find the density function. $p(x) = F'(x) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} e^{-(x/\alpha)^{\beta}}$

b) If Weibull is W, then $X = (W/\alpha)^{\beta}$ is an exponential distribution.

$W = \alpha^{\beta} x^{1/\beta}$; $\frac{dW}{dx} = \frac{1}{\beta} \alpha x^{1/\beta - 1}$

$F_X(x) = f_X(\alpha \cdot x^{1/\beta}) \cdot \left|\frac{\partial W}{\partial x}\right| = \frac{\beta}{\alpha} e^{-W} \cdot \left(\frac{\alpha x^{1/\beta}}{\alpha}\right)^{\beta-1} \cdot \frac{\alpha}{\beta} x^{1/\beta - 1} = \boxed{e^{-W}}$

c). $U = e^{-W}$; $\ln U = -W$; $\boxed{W = -\ln U}$;

69, U = Uniform Random Variable. Find $V = U^{-\alpha}$ for: $\alpha > 0$ $U^{\alpha} = V$; $U = V^{1/\alpha}$

$\boxed{\text{Proposition B}}$

$P(U^{1/\alpha}) = P(V^{1/\alpha}) \cdot \left|\frac{d}{dy} V^{1/\alpha}\right| = P(V^{1/\alpha})\left(\frac{1}{\alpha}\right) \cdot V^{(1-\alpha)/\alpha}$

The rates of decrease for the density function is ~~decreased~~ as ... greater rates.

69. $p(x) = \lambda e^{-\lambda x}$; $V = \frac{4}{3}\pi R^3$; $R = \sqrt[3]{\frac{3V}{4\pi}}$; $p(R) = \lambda e^{-\lambda \sqrt[3]{\frac{3V}{4\pi}}}$, $\frac{d}{dV}\left|\lambda e^{-\lambda\sqrt[3]{\frac{3V}{4\pi}}}\right|$

$= \lambda e^{-\lambda\sqrt[3]{\frac{3V}{4\pi}}}, \underbrace{\lambda\left(-\lambda \frac{1}{3}\left(\frac{3V}{4\pi}\right)^{-2/3} \cdot \frac{3}{4\pi}\right) e^{-\lambda\sqrt[3]{\frac{3V}{4\pi}}}}$

"Density Function"

70. $p(x) = \lambda e^{-\lambda r}$; $A = \pi r^2$; $r = \sqrt{\frac{A}{\pi}}$;

$f(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right|$; $f(y) = \lambda e^{-\lambda r} = \lambda e^{-\lambda\sqrt{\frac{A}{\pi}}} \cdot \left|\frac{d}{dA}\lambda e^{-\lambda\sqrt{\frac{A}{\pi}}}\right|$

$= \lambda e^{-\lambda\sqrt{\frac{A}{\pi}}} \underbrace{\left|-\frac{\lambda}{2}\left(\frac{A}{\pi}\right)^{-1/2} e^{-\lambda\sqrt{\frac{A}{\pi}}}\right|}$

"Density Function"

71. F = cdf of random variable.
U is uniform from [0,1].
Define $Y = K$ if $F(k-1) < U \le F(k)$;

$\boxed{\int_{k-1=0}^{k=1} F(k)\,dk = \int_0^1 p(y)\,x\,F(\;)\,F}$

$m = 3$;

X	0	2	2	0	2	1

72. $X_n = (aX_{n-1} + c) \bmod m$ a) $a = 1$ $c = 2$ b)

X	0.064069	2	13813	944	--

Chapter 3: Joint Distributions:

1. Joint Frequency Function:

y \ X	1	2	3	4	$p_Y(y)$
1	0.10	0.05	0.02	0.02	0.19
2	0.05	0.20	0.05	0.02	0.32
3	0.02	0.05	0.20	0.04	0.31
4	0.02	0.02	0.04	0.10	0.18
$p(x)$	0.19	0.32	0.31	0.18	

= Joint Frequencies

Marginal Frequencies

A) Find the marginal frequency function of X and Y. i.e. $p(X)$, and $p(y)$.

b) Find the conditional frequency of X given $Y = 1$ and Y given $X = 1$.

~~$P(X|Y=1) = \frac{P(X, Y=1)}{P(y)} = \frac{0.10}{0.19} = \frac{10}{19}$; $P(Y|X=1) = \frac{P(Y, X=1)}{P(X)} = \frac{0.10}{0.19} = \frac{10}{19}$~~

$\boxed{p(1|1) = \frac{10}{19};\ p(2|1) = \frac{5}{19};\ p(3|1) = \frac{2}{19};\ p(4|1) = \frac{2}{19}}$

2. p - black balls, q - white balls, r - red balls; Urn

n chosen → a) Find the joint distribution of black, white and red balls.

Multinomial Distribution: $p(n_1, n_2, \ldots, n_r) = \binom{n}{n_1, \ldots, n_r} p_1^{n_1} p_2^{n_2} \cdots p_3^{n_r}$

$p(\text{red}, \text{white}, \text{black}) = \frac{\binom{p}{x}\binom{q}{y}\binom{r}{z}}{\binom{p+q+r}{n}}$

b) Joint Distribution of black and white balls.

$P(\text{black}(X), \text{white}(Y)) = \frac{\binom{p}{x}\binom{q}{y}\binom{r}{n-x-y}}{\binom{p+q}{n-r}}$

where $n = X + Y + Z$

Total selection outcomes

c) $P(Y) = \frac{\binom{q}{y}\binom{p+r}{n-y}}{\binom{p+q+r}{n}}$

Outcomes
Black + White outcomes

outcomes problem

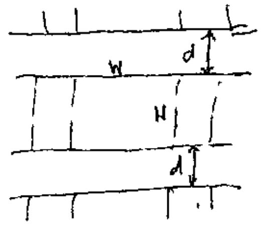
Total selection

3. Three Players play 10 independent rounds at a game. $P(\text{Player} \mid \text{winning}) = \frac{1}{3}$

$$P(\text{Player } A, B, C \mid \text{Win}) = \boxed{\binom{10}{x\,y\,z}\left(\frac{1}{3}\right)^x\left(\frac{1}{3}\right)^y\left(\frac{1}{3}\right)^z}$$

Multinomial Distribution: $P(X, y, z) = \boxed{\binom{n}{x\,y\,z} P_1^x P_2^y P_3^z}$

4 Wire diameter $= d$, Holes side length $= W$, Spherical particle radius $= r$. What is probability of passing?



Area Circle $= \pi r^2$ Probability passing per hole $= \frac{\text{Area particle}}{\text{Area square}} = \boxed{\frac{\pi r^2}{W^2}}$

Area Square $= W^2$, Probability passing per mesh $=$ Probability passing $\cdot \frac{\text{Area square}}{\text{Area mesh}}$

$$= \frac{\pi r^2}{W^2} \frac{W^2}{(nW + (n+1)d)^2} = \boxed{\frac{\pi r^2}{(nW+(n+1)d)^2}}$$

Fails to pass through if dropped n-times:

$$P(\text{Failing to pass through}) = \boxed{\left(1 - \frac{\pi r^2}{(nW+(n+1)d)^2}\right)^n} \binom{n}{1}$$

$n\frac{1}{\cdot} + w$

5. $\vdash L \dashv$ $\quad d \geq L$

Probability needle crosses line $= 2\left(\frac{\text{Length of Needle}}{\text{Distance of lines}}\right) \times \frac{1}{\pi \cdot \text{rad.}} = \boxed{\frac{2L}{\pi D}}$

$|\leftarrow d \rightarrow|\leftarrow d \rightarrow|$

6. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; Marginal Density at X and y coordinates inside an ellipse: Area $= \pi(a b)$

$$f_y(x) = \int_{-\infty}^{\infty} f(x, y)\,dy = \frac{1}{\pi a b}\int_{-\sqrt{b^2(1-y^2/b^2)}}^{\sqrt{a^2(1-y^2/b^2)}} dx = \frac{1}{\pi a b} 2\sqrt{a^2(1-y^2/b^2)} = \boxed{\frac{2\sqrt{1-y^2/b^2}}{\pi b}}$$

$$\boxed{f_x(y) = \frac{2\sqrt{1-x^2/a^2}}{\pi a}}$$

7. CDF: $F(X, y) = (1 - e^{-\kappa x})(1 - e^{-\beta y})$; $x \geq 0$; $y \geq 0$; $\kappa > 0$; $\beta > 0$

Joint Density: $f(x,y) = \frac{d}{dx}\frac{d}{dy}(1-e^{-\kappa x})(1-e^{-\beta y})$

$$\boxed{= (1 + \kappa e^{-\kappa x})(1 + \beta e^{-\beta y})}$$

Marginal Density: $F_x(x) = \int_{-\infty}^{\infty} (1-e^{-\kappa x})(1-e^{-\beta y})\,dy = 2(1-e^{-\kappa x})\left(1 + \frac{e^{-\beta y}}{\beta}\right)\Big|_0^{\infty}$

$F_y(y) = \int_{-\infty}^{\infty} (1-e^{-\kappa x})(1-e^{-\beta y}) \quad (\pm 2(1-e^{-\kappa x})(1/\beta)$

$$\boxed{= 2(1-e^{-\beta y})(1/\kappa)}$$

8. $f(x,y) = \frac{6}{7}(x+y)^2$, $0 \leq x \leq 1$, $0 \leq y \leq 1$

a) i) Find $P(x > y) = \frac{6}{7}\int_0^1 \int_0^x (x^2 + xy)\,dy\,dx = \frac{6}{7}\int_0^1 \left(x^3 + \frac{x^3}{2}\right)dx = \frac{6}{7}\left(\frac{x^4}{4} + \frac{x^4}{8}\right)\Big|_0^1 = \frac{6}{7}\left(\frac{1}{4} + \frac{1}{8}\right) = \frac{6}{28} + \frac{6}{48} = \boxed{\frac{19}{56}}$

ii) Find $P(X + Y \leq 1) = \frac{6}{7}\int_0^1 \int_0^{1-Y} (x^2 + xy)\,dx\,dy = \frac{6}{7}\int_0^1 \left(\frac{x^3}{3} + \frac{x^2}{2}y\right)\Big|_0^{1-Y} = \frac{6}{7}\int_0^1 \left(\frac{(1-Y)^3}{3} + \frac{(1-Y)^2}{2}y\right)dy$

$X \leq 1 - Y$

$$= \frac{6}{7}\left[\frac{(u)^4}{12} + \frac{y^2}{4} - \frac{2y^3}{2 \cdot 3}\right]\Big|_0^1 = \frac{6}{7}\left[-\frac{(1-1)^4}{12} + \frac{1}{4} - \frac{1}{3} + \frac{1}{12}\right] = \boxed{0}$$

iii) $P(X \leq \frac{1}{2}) = \frac{6}{7}\int_0^1 \int_0^{1/2} (x^2 + xy)\,dx\,dy = \frac{6}{7}\int_0^1 \left(\frac{x^3}{3} + \frac{x^2}{2}y\right)\Big|_0^{1/2} dy = \frac{6}{7}\int_0^1 \frac{1}{18} + \frac{1}{8}y\,dy = \frac{6}{7}\left[\frac{1}{18} + \frac{1}{16}\right] = \boxed{17/168}$

$u = 1 - y$, $du = dy$

b) Marginal Densities of X & Y:

$$f_X(x) = F_x'(x) = \int_0^1 (x^2 + yx)\,dy = \boxed{x^2 + \frac{x}{2}}$$

$$f_Y(y) = F_y'(x) = \int_0^1 (x^2 + yx)\,dx = \boxed{\frac{1}{3} + \frac{y}{2}}$$

$(1-y)^2 y$

$(1 - 2y + y^2)y$

$y - 2y^2 + y^3$

c. Conditional Density: $f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \boxed{\frac{\frac{6}{7}y(x^2+xy)}{x^2+\frac{x}{2}}}$; $f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(x)} = \boxed{\frac{\frac{6}{7}(x^2+xy)}{\frac{1}{3}+\frac{y}{2}}}$

9. (X,Y) uniformly distributed over: $0 \le y \le 1-x^2$; $-1 \le x \le 1$ Assuming Bivariate Normal Density:

a) Find the marginal distribution: $F_X(x) = \int_0^{1-x^2} \frac{1}{2\pi\sigma_x\sigma_y} \exp\left(\frac{[x-\mu_x]^2}{\sigma_x^2} + \frac{[y-\mu_y]^2}{\sigma_y^2}\right) f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left(\frac{[x-\mu_x]^2}{\sigma_x^2} + \frac{[y-\mu_y]^2}{\sigma_y^2}\right)$

b) Find the two conditional densities:

$f_{Y|X}(Y|X) = \frac{f_{xy}(x,y)}{f_X(x)}$

$= \frac{1}{2\pi\sigma_x\sigma_y} \exp\left(\frac{[x-\mu_x]^2}{\sigma_x^2} \frac{[y-\mu_y]^2}{\sigma_y^2}\right)\left[\frac{1}{\sqrt{2}\sigma_y}\Gamma\left(\frac{1}{2}\right)\right]\left[\frac{2\sigma_y}{\mu_y^2}\right]\Bigg|^{1-x^2}$

$= \boxed{\frac{\mu_y}{\sqrt{2}\sigma_y\Gamma(\frac{1}{2})} \exp\left[\left(\frac{-1}{2}\right)\frac{[y-\mu_y]^2 - [1-x^2-\mu_y]^2 + \mu_y^2}{\sigma_y^2}\right]}$

$f_X(x) = \boxed{\frac{\Gamma(\frac{1}{2})}{\sqrt{2\pi}\sigma_x\mu_y} \exp\left(\left[\frac{[x-\mu_x]^2}{\sigma_x^2} + \frac{[1-x^2-\mu_y]^2}{\sigma_y^2} - \frac{[\mu_y]^2}{\sigma_y^2}\right]\left(-\frac{1}{2}\right)\right)}$; $= \frac{-1}{2\sigma_y^2}$

$f_{X|Y}(X|Y) = \frac{f_{xy}(x,y)}{f_y(y)}$; $f_y(y) = \int_{-1}^{1} \frac{1}{2\pi\sigma_x\sigma_y} \exp\left(\frac{-1}{2}\right)\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right] dy$

$= \boxed{\frac{\mu_x}{\sqrt{2}\sigma_x\Gamma(\frac{1}{2})} \exp\left[\left(\frac{-1}{2}\right)\frac{(x-\mu_x)^2 - (1-\mu_x)^2 - \mu_x^2}{\sigma_x^2}\right]}$; $= \boxed{\frac{2\Gamma(\frac{1}{2})}{\sqrt{2\pi}\sigma_x\mu_x} \exp\left[\frac{[1-\mu_x]^2 - \mu_x^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right]}$

10. Suppose $f(x,y) = xe^{-x(y+1)}$
$0 \le x \le \infty$; $0 \le y < \infty$

a) Find the marginal density of X and Y. $f(x) = \int_0^\infty xe^{-x(y+1)} dy = \frac{\cancel{x}e^{-x(y+1)}}{-x}\Bigg|_0^\infty = -e^{-x(\infty+1)} + e^{-x(1)}$ $\boxed{= e^{-x}}$

b) Find the conditional densities.

$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{xe^{-x(y+1)}}{e^{-x}} = \boxed{xe^{-x[y+2]}}$; $f(y) = \int_0^\infty xe^{-x(y+1)} dx$; $\int u\,dv = uv - \int v\,du$; $u = x$, $dv = e^{-x(y+1)}$, $du = dx$, $v = \frac{-e^{-x(y+1)}}{y+1}$

$f_{X|Y}(X|Y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{xe^{-x(y+1)}}{\frac{1}{(y+1)^2}} = \boxed{x(y+1)^2 e^{-x(y+1)}}$; $= \frac{-xe^{-x(y+1)}}{y+1}\Bigg|_0^\infty + \int_0^\infty \frac{e^{-x(y+1)}}{y+1} dx$ $\boxed{\text{Independent}}$

$= 0 - \frac{e^{-x(y+1)}}{(y+1)^2}\Bigg|_0^\infty = \boxed{\frac{1}{(y+1)^2}}$

11. U_1, U_2, and U_3 independent from $[0,1]$
Find the probability the roots of the quadratic $U_1x^2 + U_2x + U_3$ are real.

$0 = U_1x^2 + U_2x + U_3$; $x_1, x_2 = \frac{-U_2 \pm \sqrt{U_2^2 - 4(U_1)(U_3)}}{2(U_1)}$; $P(U_1) = \int_0^1\int_0^1 u_1x^2 + u_2x + U_3\, dU_2\, dU_3 = \int_0^1 u_1x^2 + U_2x + \frac{U_3^2}{2} dU_2$

~~Extrema and Order Statistics~~ $= U_1x^2 + \frac{x}{2} + \frac{1}{2}$

$\cancel{f(U_{(1)}, U_{(2)}, U_{(3)} \ldots) = n!\prod_{i=1}^{n} f(U_i)\text{ ; for } i=3 \text{ ; } f(U_1,U_2,U_3) = 3!\prod_{i=1}^{3} f(U_i) \cdot 6f(U_1)f(U_2)f(U_3)}$

$P(U_2^2 > 4U_1U_3) = P(|U_2| \ge 2\sqrt{u_1U_3}) = \int_0^1\int_0^1\int_{2\sqrt{U_1U_3}}^1 f(u_1,U_2,U_3)\, du_2\, du_3\, du_1 = \int_0^1\int_0^1 (1 - 2\sqrt{u_1u_3})$

$= 1 - \frac{4}{3}\frac{1^{3/2}}{3/2} = \boxed{1/9}$

12. $f(x,y) = c(x^2-y^2)e^{-x}$, $0 \le x < \infty$, $-x \le y < x$ a) Find c: $P(x,y) = 1 = \int_0^\infty \int_{-x}^{x} c(x^2-y^2)e^{-x}\,dy\,x$

$= \int_0^\infty c\left(x^2 - \frac{y^3}{3}\right)e^{-x}\Big|_{-x}^{x} dx$

$= \int_0^\infty c\left(x^2 - \frac{x^3}{3}\right)e^{-x} - c\left(x^2 + \frac{x^3}{3}\right)e^{-x}\,dx$

$= \int_0^\infty c\left(-2\frac{x^3}{3}\right)e^{-x}\,dx$

$1 = -4c$; $\boxed{c = \frac{-1}{4}}$

b) $f(x) = \int_{-x}^{x} c(x^2-y^2)e^{-x}\,dy = \boxed{\cancel{2}\frac{4x^{3}}{3}e^{-x}}$ / $f(y) = \int_0^\infty 2(x^2-y^2)e^{-x}\,dx = \boxed{2(2-y^2)}$

c) $f_{Y|X}(Y|X) = \frac{f(x,y)}{f(x)} = \frac{-2(x^2-y^2)\cancel{e^{-x}}}{\frac{4}{3}x^3\cancel{e^{-x}}} = \boxed{\frac{-3}{2}\frac{(x^2-y^2)}{x^3}}$; $f_{X|Y}(x|y) = \frac{f(x,y)}{f(y)}$ $= \boxed{\frac{(x^2-y^2)e^{-x}}{(2-y^2)}}$

13. Sample Space Throws

0	1	2
1/2	1/2	1/4

1	2
H	H
H	T
T	~~H~~
T	~~T~~

$P(0) = 1/2$ $P(2) = \frac{1}{4}$

$P(1) = 1/2$

a) $P(x) = \int_{-\sqrt{y^2+z^2}}^{\sqrt{y^2+z^2}}$

14. Point in a unit sphere ($\mathbb{R}_3$)

$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\sqrt{x^2+y^2+z^2}$

$f(x) = \int_{-\sqrt{r^2-x^2-z^2}}^{\sqrt{r^2-x^2-z^2}} \int_{-\sqrt{r^2-x^2-y^2}}^{\sqrt{r^2-x^2-y^2}} \sqrt{\left(\frac{V}{4/3\pi}\right)^2 - y^2 - z^2}\,dz\,dy$

$u =$

$= \int_{-\sqrt{r^2-x^2-z^2}}^{\sqrt{r^2-x^2-z^2}} \frac{1}{2}\left(z\sqrt{\left(\frac{V}{4/3\pi}\right)^2 - y^2 - z^2} + (a^2-y^2)\tan^{-1}\left(\frac{z}{\sqrt{a^2-y^2-z^2}}\right)\right)\Bigg|_{-\sqrt{r^2-x^2-y^2}}^{\sqrt{r^2-x^2-y^2}}$

Density Function of a unit sphere:

$f(x,y,z) = \begin{cases} k & 0 \le x^2+y^2+z^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$

To find the value of k, such that:

$\iiint_{0 \le x^2+y^2+z^2 \le 1} k\,dx\,dy\,dz = 1$; let $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$; $\rho^2 = x^2+y^2+z^2 = 1$

$0<\rho<1$; $0<\phi<\pi$; $0<\theta<2\pi$

Volume $= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{4\pi}{3}$; $k\frac{4\pi}{3} = 1$; $k = \frac{3}{4}\pi$

The density function becomes: $\boxed{f(x,y,z) = \begin{cases} \frac{3}{4\pi} & 0<x^2+y^2+z^2 \le 1 \\ 0 & \text{otherwise.} \end{cases}}$

Marginal Densities: 1) Joint Densities:

$f_{x,y}(x,y) = \int_{-\infty}^{\infty} f(x,y,z)\,dz = \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \frac{3}{4\pi}\,dz = \frac{3}{2\pi}\sqrt{1-x^2-y^2}$; $f_{xy}(x,y) \begin{cases} \frac{3}{2\pi}\sqrt{1-x^2-y^2} & ; 0 \le x^2+y^2 \le 1 \\ 0 & , \text{otherwise} \end{cases}$

2) $f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{3}{2\pi}\sqrt{1-x^2-y^2}\,dy$; $y = \sqrt{1-x^2}\sin u$; $dy = \sqrt{1-x^2}\cos u\,du$; $= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{3}{2\pi}\sqrt{(1-x^2) - \left(\sqrt{1-x^2}\right)^2\sin^2 u}\left(\sqrt{1-x^2}\cos u\,du\right)$

$\boxed{= \frac{3}{4}(1-x^2)}$

$\boxed{f_y(y) = \begin{cases} \frac{3}{4}(1-y^2) & -1 \le y \le 1 \\ 0 & , \text{otherwise} \end{cases} ; \; f_z(z) = \begin{cases} \frac{3}{4}(1-z^2) & -1 \le z \le 1 \\ 0 & , \text{otherwise} \end{cases}}$

@ $z=0$ $f_{XY|Z}(x,y|z) = \frac{f_{xyz}(x,y,z)}{f_z(z=0)} = \frac{3/4\pi}{\frac{3}{4}(1-(0)^2)} = \boxed{\frac{1}{\pi}}$

15. Suppose the joint density $f(x,y) = c\sqrt{1-x^2-y^2}$; $x^2+y^2 \le 1$

a) Find c: $x = r\cos\theta$; $y = r\sin\theta$; $\iint f(x,y)\,dA = \int_0^{2\pi}\int_0^1 c\sqrt{1-r^2}\,r\,dr\,d\theta = c\left(\frac{2}{3}\right)\pi = 1$; $\boxed{c = \left(\frac{3}{2\pi}\right)}$

b) $P(X^2+Y^2) \le 1/2$ is a half of the disk;

$~~\iint f(x,y)\,dA;\ x^2+y^2 \le 1/2 = \int_0^{\pi/4}\int_0^1 \left(\frac{3}{2\pi}\right)\sqrt{1-r^2}\,r\,dr\,d\theta = \iint_0^{\pi/4,1}\left(\frac{3}{2\pi}\right)\sqrt{1-r^2}\,r\,dr\,d\theta = \int_0^{\pi/4}\left(\frac{1}{2}\pi\right)d\theta = \boxed{1/8}~~$

b) The joint density is an area of decreasing size.

d) $f(x) = \int_{-1}^{1}\sqrt{1-x^2-y^2}\left(\frac{3}{2\pi}\right)dy = \boxed{\left(\frac{3}{4}\right)x^2}$; $f(y) = \int_{-1}^{1}\sqrt{1-x^2-y^2}\left(\frac{3}{2\pi}\right)dx = \boxed{\left(\frac{3}{4}\right)x^2}$

To check independence, $f(x,y) = f_x(x)f_y(y) = \left(\frac{9}{16}\right)x^2y^2 \neq$ independence.

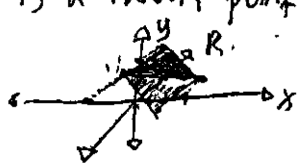
e) Conditional Densities: $\boxed{f(y|x) = f(x,y)/f_x(x);\ f(x|y) = f(x,y)/f(y)}$

16. X_1 is uniform on $[0,1]$, and, conditional on X_1, X_2, from $[0, X_1]$

Find the joint Distributions: $\boxed{f(x_1,x_2) = \int_0^1\int_0^{x_1} dx_2\,dx_1}$ = Marginal Distributions: $\boxed{f(x_1) = \int_0^{x_1} dx_2}$ $\boxed{f(x_2) = \int_0^1 dx_1}$

17. (X,Y) is a random point of the region $R = \{(x,y): |x|+|y| \le 1\}$

a)



b) $f_x(x) = R$; $f_y(y) = R$

c) $f_{Y|X}(Y|X) = \frac{f(y,x)}{f(x)} = \frac{|x|+|y|}{R} = \boxed{1}$

18 $f(x,y) = k(x-y)$, $0 \le y \le x \le 1$ and zero elsewhere.

a)

b) $1 = \int_0^1\int_0^x k(x-y)\,dy\,dx = \int_0^1 k\left(x - \frac{y^2}{2}\right)\Big|_0^x = \int_0^1 k\left[\left(x - \frac{x^2}{2}\right) - k(x)\right]dx$

$= \int_0^1 -k\frac{x^2}{2}\,dx = -\frac{k(1)^3}{6} = 1$ $\boxed{k = 6}$

c) $f_x(x) = \int_0^x k(x-y)\,dy = \boxed{k\left(1 - \frac{x^2}{2}\right)}$; $f_y(y) = \int_0^1 k(x-y)\,dx = k\left(\frac{x^2}{2} - y\right) + ky$ $= 6\left(\frac{1}{2}\right) = \boxed{3}$

$\boxed{= -3x^2}$

d) $f_{Y|X}(Y|X) = \frac{f(x,y)}{f(x)} = \frac{k(x-y)}{-3x^2}$; $f_{X|Y}(X|Y) = \frac{f(x,y)}{f(y)} = \boxed{\frac{k(x-y)}{3}}$

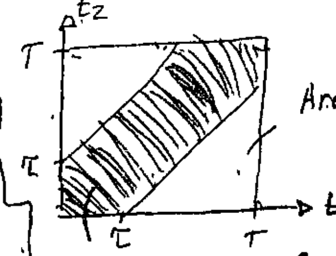
19. a) Exponentially Distributed lifetimes means: $\lambda e^{-\lambda x} = f(\lambda)$; $f(T_1) = \alpha e^{-\alpha T_1}$; $f(T_2) = \beta e^{-\beta T_2}$

Find $P(T_1 > T_2) = \int_{-\infty}^{T_1}\beta e^{-\beta T_2}\,dT_2 = -e^{-\beta T_2}\Big|_{-\infty}^{T_1} = -\left[e^{-\beta T_1} - 1\right] = \boxed{1 + e^{-\beta T_1}}$ Let $S = T_1 + T_2$

b) $\boxed{P(T_2 > 2T_1)} = \boxed{\frac{1 \pm e^{-\alpha(2T_1)}}{2}}$ $P(T_1 > T_2) = \int_0^{T_1}\alpha e^{-\alpha T_1}\cdot\beta e^{-\beta T_2}\,dT_2 = -\int_0^{T_1}\alpha e^{-\alpha T_1}\cdot e^{-\beta T_1} = -\alpha e^{-(\alpha+\beta)T_1}$

$P(T_1 > T_2) = \int_{T_2}^{\infty}\int_0^{\infty}\alpha e^{-\alpha T_1}\beta e^{-\beta T_2}\,dT_2\,dT_1 = \int_0^{\infty}\beta e^{-\beta T_2} - \left[e^{0} e^{-\alpha T_2}\right]dT_2 = \int_0^{\infty}\beta e^{-(\beta+\alpha)T_2}\,dT_2 = \boxed{\frac{\beta}{\alpha+\beta}}$

$P(T_1 > 2T_2) = \int_0^{\infty}\int_{2T_2}^{\infty} \alpha e^{-\alpha T_1} \beta e^{-\beta T_2}\, dT_1\, dT_2 = \frac{\beta}{(2\alpha+\beta)}$

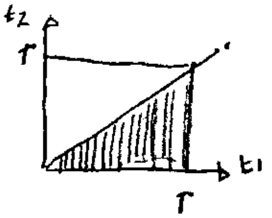


Area $(T-\tau)^2/2$

21\. Area $T^2-(T-\tau)^2$

20\. Probability of packet collision: $f(t_1, t_2) = \frac{1}{T^2}$ $\boxed{\text{Joint Density}}$ from $[0,T]$

Time between arrivals:



Integral is: $f(t_1,t_2) \times \frac{1}{2} T \cdot T = \boxed{\frac{T}{2}}$

Integral is $f(t_1, t_2) \times$ Area
$= \frac{1}{T^2}\left(T^2 - (T-\tau)^2\right)$
$\boxed{= 1 - (1 - \tau/T)^2}$

$f(x)$ = Present chemical
$R(x)$ = Probability Detected

Y = concentration of a chemical in soil

If $g(y)$ is uniform, then $g(y) = \frac{\text{Probability of Detection} \times \text{Density Function}}{\text{Total outcomes of concentration}}$

$= \boxed{R(y)\cdot f(y) \Big/ \int_0^{\infty} R(x) f(x)\, dx}$

22\. Poisson Distribution: $\frac{\lambda^k}{k!}e^{-\lambda} = p(x)$; $N(t_1, t_2)$ = Number of events.

If $t_0 < t_1 < t_2$; find the conditional distribution of $N(t_0, t_1)$ given $N(t_0, t_2) = n$

$N(t_0|t_1) = \frac{N(t_0, t_1)}{N(t_1)}$; $N(t_0|t_1) = \frac{N(t_0, t_2)}{N(t_2)} = \frac{n}{N(t_2)}$; $N(t_1, t_2)\cdot p = \lambda$; $p(x) = \frac{[N(T_1,T_2)p]^k}{k!} e^{-N(T_1,T_2)\cdot p}$

$= \frac{[N(T_1)N(T_2)\cdot P_1 \cdot P_2]^k}{k!} e^{-N_{T_1} \cdot N_{T_2} \cdot P_1 \cdot P_2}$

$P(N(t_i, t_j)) = e^{-\lambda(t_j - t_i)} \frac{[\lambda(t_j - t_i)]^x}{x!}$

$P(N(t_0, t_1) = x \mid N(t_0, t_2) = n) = \frac{P(N(t_0, t_1) = x,\ N(t_0, t_1) + N(t_1, t_2) = n)}{P(N(t_0, t_2) = n)}$

$= \frac{P(N(t_0,t_1) = x,\ N(t_1,t_2) = n - N(t_0,t_1))}{P(N(t_0,t_2) = n)} = \frac{P(N(t_0,t_1) = x,\ N(t_1,t_2) = n-x)}{P(N(t_0,t_2) = n)}$

$= \frac{e^{-\lambda(t_1 - t_0)} \frac{[\lambda(t_1 - t_0)]^x}{x!} \times e^{-\lambda(t_2 - t_1)} \frac{[\lambda(t_2 - t_1)]^{n-x}}{(n-x)!}}{e^{-\lambda(t_2 - t_0)} \frac{[\lambda(t_2 - t_0)]^n}{n!}} = \boxed{\frac{n!}{x!(n-x)!} \frac{(t_1 - t_0)^x (t_2 - t_1)^{n-x}}{(t_2 - t_0)^n}}$

23\. $p(N|X) = \frac{p(N,X)}{p(X)}$

Binomial Distribution:
$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$

N = Trials, p = probability of success

= Binomial random variable with n trials and probability r

$P(X) = \frac{P(N,X)}{P(N|X)} = \frac{\binom{N}{k}p^k(1-p)^{N-k}}{\binom{m}{k}r^k(1-r)^{N-r}}$

$\boxed{= \binom{N}{m}(pr)^m(1-pr)^{N-m}}$

24\.

Section 3.5.2

Baysian Inference: Conditional: $f_{X|\theta}(x|\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$ $x = 0, 1, \dots n$

Joint Density: $f_{\theta,X}(\theta, x) = f_{X,\theta}(x|\theta)\cdot f(\theta)$
$= \binom{n}{x}\theta^x(1-\theta)^{n-x}$ $x = 0, 1, \dots n$; $0 \le \theta \le 1$

Marginal Density: $f_\theta(\theta) = \int_0^1 \binom{n}{x}\theta^x(1-\theta)^{n-x} d\theta = \int_0^1 \frac{\Gamma(n+1)}{\Gamma(x+1)\Gamma(n-x+1)}\theta^x(1-\theta)^{n-x} d\theta$ by the fact $\Gamma(k) = (k-1)!$

...becomes the beta density $= \int_0^1 \frac{\Gamma(n+1)}{\Gamma(x+1)\Gamma(n-x+1)}\theta^x(1-\theta)^{n-x}d\theta = \frac{\Gamma(n+1)}{\Gamma(x+1)\Gamma(n-x+1)} \frac{\Gamma(x+1)\Gamma(n-x+1)}{\Gamma(n+2)} = \frac{1}{n+1}\frac{\Gamma(n+1)}{\Gamma(n+1)} = \boxed{\frac{1}{n+1}}$

Conditional: $f_{\theta|X}(\theta|x) = \frac{f_{\theta X}(\theta, x)}{f_X(x)} = (n+1)\binom{n}{x}\theta^x(1-\theta)^{n-x} = (n+1)\frac{\Gamma(n+1)}{\Gamma(x+1)\Gamma(n-x+1)}\theta^x(1-\theta)^{n-x} = \frac{\Gamma(n+2)}{\Gamma(x+1)\Gamma(n-x+1)}\theta^x(1-\theta)^{n-x}$

Posterior Density is a β-density with $a = x+1, b = n-x+1$; $g(u) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}u^{a-1}(1-u)^{b-1}$; $g'(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\left[(a-1)u^{a-2}(1-u)^{b-1} + (b-1)u^{a-1}(1-u)^{b-2}\right] = 0$

26\. P is uniform from $[0,1]$, and conditional on $P = p$.
Let X be a Bernoulli Distribution with parameter p. Find the conditional distribution of P given X.

$f(x=0) = \int_0^1 p^0(1-p)^1 dp = 1/2$
$f(x=1) = \int_0^1 p^1(1-p)^0 dp = 1/2$

Bernoulli Distribution: $p(k) = \frac{n!}{k!\dots} p^k (1-p)^{n-k}$

Find $f_{P|X}(P|X) = \frac{f(p,x)}{f(x)} = \frac{p^x(1-p)^{1-x}}{1/2} = \boxed{2p(1-p)^{1-x}}$

$\frac{(a-1)}{(b-1)}(1-u) = u$
$\frac{(a-1)}{(b-1)} = u\left[1 + \frac{(a-1)}{(b-1)}\right]$
$\boxed{\theta = \frac{(a-1)(b-1)}{(b-1)[(b-1)+(a-1)]}}$

25 $f(x)$; $P(X=x)=1/2$; $P(Y=-X)=1/2$; Show Y is symmetric about zero.

Bernoulli Distribution: $P(R)=p^x(1-p)^{1-x}$

$p(0)=\int_0^1 p^0(1-p)^1 dp = 1-1/2 = 1/2$

$p(1)=\int_0^1 p^1(1-p)^0 dp = \frac{1}{2}$

Conditional Density of a random variable is expressed as:

$f_{Y|X}(y|x)=f_{Y|X}(x|x)=1/2$; $f_{Y|X}(y|x)=f_{Y|X}(-x|x)=1/2$

$f_{Y|X}(y|x)=\frac{f_{YX}(x,y)}{f(x)}$; $f(x,y)=f_{Y|X}(y|x)f(x)=\boxed{\frac{1}{2}f_X(x)}$

$f(x,x)-\frac{1}{2}f_X(x)$;

$F_{Y|X}(y|x)=\frac{F(X,y)}{f_X(x)}$; $F(x,y)=f_{Y|X}(y|x)f_X(x)=\boxed{\frac{1}{2}f_X(-x)}$ $\quad f_Y(y)=\frac{1}{2}f(x)+\frac{1}{2}f(x)=\boxed{f_Y(-y)}$

27. Prove X and Y are independent if $\boxed{f_{X|Y}(x|y)=f_X(x)=\frac{f(x,y)}{f(y)}=\frac{f(x)f(y)}{f(y)}=f(x)}$

28. Show if $C(u,v)=uv$ is a copula. Why is it called "the independence copula?"

Copula: a joint cumulative distribution of random variables that have uniform marginal distributions. The function $C(u,v)=uv$ is known by the independence copula because independent variables, and marginals, are seperable.

29. Marginal Density $=\lambda e^{-\lambda x}$

Farlie-Morgenstern Copula:

$H(x,y)=F(x)G(y)\{1+\alpha[1-F(x)][1-G(y)]\}=\lambda_1 e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2 y}\{1+\alpha[1-\lambda_1 e^{-\lambda_1 x}][1-\lambda_2 e^{-\lambda_2 y}]\}$

$h(x,y)=\frac{\partial^2}{\partial x\partial y}H(x,y)=\frac{\partial}{\partial x}\lambda_1 e^{-\lambda_1 x}\lambda_2^2 e^{-\lambda_2 y}\{1+\alpha[1-\lambda_1 e^{-\lambda_1 x}][1-\lambda_2 e^{-\lambda_2 y}]\}$
$+\lambda_1 e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2 y}\{1+\alpha[1+\lambda_1 e^{-\lambda_1 x}][1+\lambda_2 e^{-\lambda_2 y}]\}$

30. For $0\le\alpha\le1$ and $0\le\beta\le1$

Show $C(u,v)=\min(u^{1-\alpha}v, uv^{1-\beta})$ is a copula (Marshall-Olkin)

$\boxed{\lim_{\alpha\to1}\lim_{\beta\to0} C(u,v)=\min(v,u)=1}$

Joint Density:

$h(x,y)=\frac{\partial^2}{\partial u\partial v}H(x,y)=\min\left[(1-\alpha)u^{-\alpha},(1-\beta)v^{-\beta}\right]$

$$\boxed{\begin{aligned}&=\lambda_1^2 e^{-\lambda_1 x}\cdot\lambda_2^2 e^{-\lambda_2 y}\{1+\alpha[1-\lambda_1 e^{-\lambda_1 x}][1-\lambda_2 e^{-\lambda_2 y}]\}\\&+\lambda_1 e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2 y}\{1+\alpha[1+\lambda_1^2 e^{-\lambda_1 x}][1-\lambda_2 e^{-\lambda_2 y}]\}\\&+\lambda_1^2 e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2 y}\{1+\alpha[1-\lambda_1 e^{-\lambda_1 x}][1+\lambda_2 e^{-\lambda_2 y}]\}\\&+\lambda_1 e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2 y}\{1+\alpha[1+\lambda_2 e^{-\lambda_2 x}][1+\lambda_2 e^{-\lambda_2 y}]\}\end{aligned}}$$

31. (X,Y) is a uniform disc of radius of 1. $f(x,y)=\begin{cases}\frac{1}{\pi} & x^2+y^2\le1\\ 0, & \text{otherwise}\end{cases}$ X and y are not independent because of the constraint $x^2+y^2=1$.

32. $f_R(r)$; Probability of passing per mesh: Probability Passing $=\frac{\text{Area Square}}{\text{Area mesh}}=\boxed{\frac{\pi f_R^2(r)}{(nW+(n+1)d)^2}}$

33 a) Posterior Density [Beta Density]: $f(\theta)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}$; $a=x+1$; $b=n-x+1$

b)

$=\frac{\Gamma(N+2)}{\Gamma(2)\Gamma(N)}\theta(1-\theta)^{N-1}=(n+1)(n)\frac{\Gamma(n)}{\Gamma(2)\Gamma(n)}\theta(1-\theta^{N-1})$

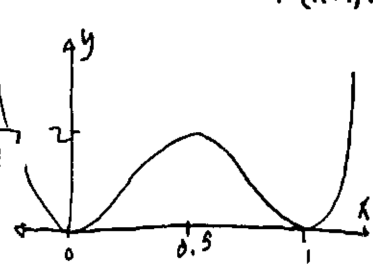
$=\frac{(n+1)(n)}{\Gamma(2)=1}\theta(1-\theta^{N-1})$

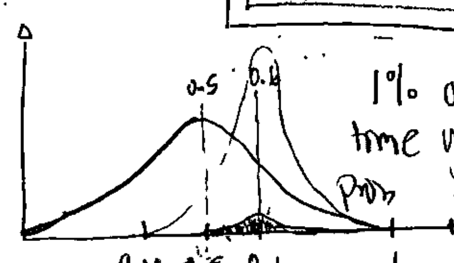
$\boxed{=(n+1)(n)\,\theta(1-\theta^{N-1})}$

34. Beta Density: $\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}$; Where $a=b=3$.

Graph this function: $\frac{\Gamma(n-x+1)}{\Gamma(x+1)\Gamma(n-x+1)}\theta^x(1-\theta)^{n-x}$

Graph the Function: $\frac{\Gamma(6)}{\Gamma(3)\Gamma(3)}\theta^{3-1}(1-\theta)^{3-1}$





1% of the time your average prob errors

35. $n=20$

$x=$ 𝍸𝍸 𝍸𝍸 ||||

Beta Distribution:

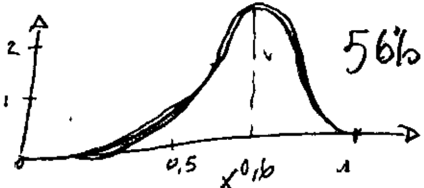
$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}$$

$a=x+1\ 13$

$b=n-x+1\ 8$

$$\frac{\Gamma(21)}{\Gamma(13)\Gamma(8)}\theta^{12}(1-\theta)^{7}$$

$20-14+2$



56% H:T

The amount of spins should arise to 50:50.

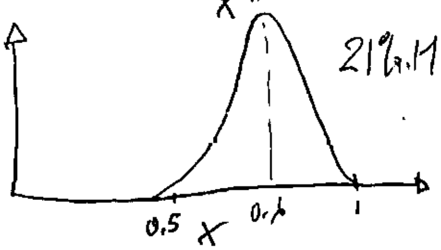
$n=40$

$x=$ 𝍸𝍸 𝍸𝍸 𝍸𝍸 𝍸𝍸 𝍸𝍸 |

$a=27$

$b=14$

$$\frac{\Gamma(41)}{\Gamma(14)\Gamma(27)}\theta^{26}(1-\theta)^{13}$$



21% H : 79% T

36. Rejection Method $m(x)=\frac{M(x)}{\int_a^b M(x)\,dx}$; $P(X)=\frac{\sum_{i=1}^{j} f(Y)U}{M(x)}=1$

$\boxed{M(X)=\sum f(j)\cdot U}$

Assuming a geometric random variable: $p\{X=i\}=(1-p)^{i-1}\cdot p$; $\sum P(X=i)=1-P(X>j-1)=1-(1-p)^{j-1}$

By simulating a random variable: $1-(1-p)^{j-1}<U<1-(1-p)^{j}$; $(1-p)^{j}\leq 1-U<(1-p)^{j-1}$

$X=\min\{j;\ j\geq\frac{\log V}{\log(1-p)}\}$; $\boxed{X=1+\frac{\log V}{\log(1-p)}}$

$P(accept)=P(U\leq f(T)/M(T))=\frac{6x^2(1-x)^2}{\int_{-1}^{1}\frac{1}{2}\,dx}=\frac{6x^2(1-x)^2}{1}$

37. $f(x)=6x^2(1-x)^2$; $-1\leq x\leq 1$ a) Rejection method:

$f'(x)=12x(1-x)^2+12x^2(1-x)$

$0=(1-x)-x$

$2x=1$

$x=\frac{1}{2}$

$\boxed{\frac{\text{Part}}{\text{Total}}}$

$P(x\leq X\leq x+dx)=\frac{P(accept\mid x\leq T\leq x+dx)\,P(x\leq T\leq x+dx)}{P(accept)}$

$\boxed{=6x^2(1-x)^2\cdot\int_{-1}^{1}\frac{1}{2}\cdot dt}$

38. $f(x)=(1+\alpha x)/2$ for $-1\leq x\leq 1$ and $-1\leq\alpha\leq 1$

a) 1) Generate Y distributed as G.

2) Generate U (independent) from Y)

3) If $U\leq\frac{f(Y)}{cg(Y)}$, then set $X=Y$ ("accept") otherwise return ("reject")

"Columbia - Accept Reject Method"

b) int val = rand()

int U = rand()

if $(V\leq f(Y)/g(Y))$; then "accept"; break

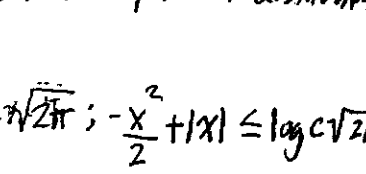
else return "reject"

39. $X=\{0,1,2\ldots\}$ with p_0, p_1, p_2. U is a random variable. If $U<p_0$, return $X=0$.

Else $U=U-p_0$.

If $U<p_1$, return $X=1$.

Else $U=U-p_1$



40. Suppose X & Y are Discrete random variables: 42.

$P_{XY}(X,y)$ a) Generate $X\sim p_X(x)=\int_{-\infty}^{\infty}P_{XY}(x,y)\,dy$ a)

b) Accept X with $p(y\mid X)=\frac{p(x,y)}{p(x)}$

b) Else A.

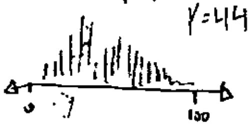
If X and Y should be continuous random variables, then the process would be rather a function

$X=\{1,2\ldots100\}$; $X=X,Y$

$Y=\{1,2\ldots X\}$.

If $Y=44$, then $p(44\mid x)=\frac{p(x,44)}{p(x)}$

$p(x)=\frac{p(x,44)}{p(44\mid x)}$



41. T = exponential Random Distribution : $p(x)=\lambda e^{-\lambda x}$; $W=\pm1$ with $p(1)=\frac{1}{2}$; $p(0)=\frac{1}{2}$

$X=WT$. Show that $f_X(x)=\frac{\lambda}{2}e^{-\lambda|x|}$ is the double exponential distribution

$f_X(x)=p(W)\cdot p(X)=\frac{1}{2}\lambda e^{-\lambda x}$

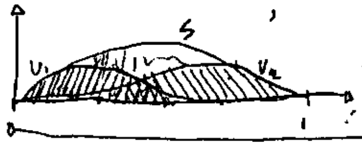
b) Prove $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\leq ce^{-|x|}$; $e^{-x^2/2+|x|}\leq c\sqrt{2\pi}$; $-\frac{x^2}{2}+|x|\leq\log c\sqrt{2\pi}$

$-\frac{x^2}{2}+|x|-\log c\sqrt{2\pi}\leq 0$; $x^2-2|x|+\log c\leq 0$

$$x_1,x_2=\frac{+2\pm\sqrt{4-4(1)(\log c)}}{2(1)}$$

Real (x) implies C is between 0 to 1.

43. U_1 & U_2 from $[0,1]$; $S = U_1 + U_2$



44. X & Y $\{0,1,2\}$ $p(0)=\frac{1}{3}$; $p(1)=\frac{1}{3}$; $p(2)=\frac{1}{3}$; Frequency function of $X+Y$.

X	0	0	0	1	1	1	2	2	2
Y	0	1	2	0	1	2	0	1	2
X+Y	0	1	2	1	2	3	2	3	4
P	1/9	2/9							

X+Y	0	1	2	3	4
N	1	2	3	2	1
P(X+Y)	1/9	2/9	3/9	2/9	1/9

45 Poisson Distribution:

$p(X) = \frac{\lambda^k e^{-\lambda}}{k!}$

$p_A + p_B = 1$

Prove N_A is a poisson with parameter $p_A\lambda$; $P(N_A = n) = \sum_{i=n}^{\infty} P[N_A = n \mid X_i = i] P[X=i] = \sum_{i=n}^{\infty} \binom{i}{n} p_A^n (1-p_A)^{i-n} \cdot \frac{\lambda^i}{i!} e^{-\lambda}$

$\boxed{\text{Law of Total Probability}}$ $= \frac{e^{-\lambda}\cdot\lambda^n p_A^n}{n!} \sum_{i=n}^{\infty} \frac{[(1-p_A)\lambda]^{i-n}}{(i-n)!} = \frac{e^{-\lambda}\lambda^n p_A^n}{n!} \sum_{j=0}^{\infty} \frac{[(1-p_A)\lambda]^j}{j!} = \frac{e^{-\lambda}\lambda^n p_A^n}{n!} e^{(1-p_A)\lambda}$ $\boxed{= e^{-p_A\lambda}\frac{(p_A\lambda)^n}{n!}}$

46. Let T_1 and T_2 be independent exponentials with λ_1 and λ_2. Find $T_1 + T_2$.

$P(T_1) = \lambda_1 e^{-\lambda_1}$; $P(T_2) = \lambda_2 e^{-\lambda_2}$; $T_1 + T_2 = \lambda_1 e^{-\lambda_1} + \lambda_2 e^{-\lambda_2}$;

$J = \begin{vmatrix} \frac{\partial T_1}{\partial r} & \frac{\partial T_1}{\partial s} \\ \frac{\partial T_2}{\partial r} & \frac{\partial T_2}{\partial s} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$\boxed{F(T_1, T_2) = \lambda_1 \lambda_2 e^{-(\lambda_1 + \lambda_2)}}$

47. $P(X) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu)^2}{2\sigma_1^2}}$; $P(Y) = \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(x-\mu)^2}{2\sigma_2^2}}$; $Z = X + Y = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{(X-\mu_x)^2}{2\sigma_1^2} - \frac{(y-\mu_y)^2}{2\sigma_2^2}}$

Sums and Quotients: $X, Y, Z = X+Y$, then $Y = Z - X$; $p(z) = \sum_{x=-\infty}^{\infty} p(x, z-x)$; $p(x,y) = p_X(x) \cdot p_Y(y)$

$= \sum_{-\infty}^{\infty} p_X(x) p_Y(z-x)$ $\boxed{\text{Convolution}}$ $\boxed{\text{Discrete}}$

$f(z) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-x^2/2} \cdot e^{-\frac{(z-x)^2}{2}} dx$

$= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{\frac{-x^2 + z^2 - 2xz + x^2}{2}} dx = \frac{e^{-\frac{z^2}{2}}}{2\pi} \int_{-\infty}^{\infty} e^{-\left[\frac{2x^2 - 2xz}{2}\right]} dx$

$= \frac{e^{-z^2/2}}{2\pi} \times \sqrt{\pi}\, e^{z^2/4}$ $\boxed{= \frac{1}{\sqrt{2}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z}{\sqrt{2}}\right)^2}}$

$F(z) = \iint f(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f(x,y)\,dy\,dx =$

$f(z) = \int_{-\infty}^{\infty} f(x, z-x)\,dx = \int_{-\infty}^{\infty} f_x(x) f_y(z-x)\,dx$

48. $f(N_1) = \frac{\lambda^k}{k!} e^{-\lambda N_1}$; $f(N_2) = \frac{\lambda^k}{k!} e^{-\lambda N_2}$; $f(N) = \int_{-\infty}^{\infty} f(N_1, N - N_1)\,dN_1 = \int_{-\infty}^{\infty} f_x(N_1) \cdot F(N - N_1)\,dN_1 = \int_{-\infty}^{\infty} \frac{\lambda_1^k}{k!} e^{-\lambda_1 N_1} \cdot \frac{\lambda_2^k}{k!} e^{-\lambda_2(N - N_1)}\,dN_1$

$= \frac{(\lambda_1\lambda_2)^k}{(k!)^2} \int_{-\infty}^{\infty} e^{-\lambda_1 N_1 - \lambda_2 N + \lambda_2 N_1}\,dN_1 = \frac{(\lambda_1\lambda_2)^k}{k!^2} e^{-\lambda_2 N} \int_{-\infty}^{\infty} e^{-(\lambda_2 - \lambda_1)N_1}\,dN_1$

$\boxed{= \frac{(\lambda_1\lambda_2)^k}{(k)!^2} e^{-\lambda_2 N} \cdot \frac{-1}{(\lambda_1 - \lambda_2)}}$

49. $f(x,y) = \begin{cases} \lambda^2 e^{-\lambda y}; & 0 \le x \le y,\ \lambda > 0 \\ 0 & \text{elsewhere} \end{cases}$

$F(z) = \int_{s}^{s/2} \lambda^2 e^{-\lambda y}\,dy$ $\boxed{= \lambda e^{-\lambda s/2} - \lambda e^{-\lambda s}}$

50. X & Y are jointly continuous variables. Find $Z = X - Y$

$\boxed{P(z) = \int_{-\infty}^{\infty} F(x, x-z)\,dx = \int_{-\infty}^{\infty} F(x) \cdot F(x-z)\,dx}$

51. $Z = XY$; $f(x,y) = f(x, \frac{z}{x}) = f(\frac{z}{y}, y)$

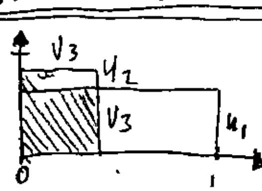
$F(z) = \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_{-\infty}^{\infty} f(\frac{z}{y}, y)\frac{1}{y}\,dz\,dy$

$\boxed{f(z) = \int_{-\infty}^{\infty} f(\frac{z}{y}, y)\frac{1}{y}\,dy}$

52. $Z = X/Y$

$\boxed{F(z) = \int f(x,y)\,dx\,dy = \int f(yz, y)\, y\,dz\,dy;\ f(z) = \int f(yz, y)\, y\,dy}$

53.



$1 \ge \frac{\int_0^1\int_0^1 P(U_1)P(U_2)}{\int_0^1\int_0^1 P(U_3)^2}$

$Area_{12} = U_1 \cdot U_2$; $P(U_1, U_2) = P(U_1) \cdot P(U_2)$;

$Area_{33} = U_3^2$; $P(U_3, U_3) = P(U_3)^2$;

$P(U_3^2 > U_1 U_2)$; $f(u_1,u_2,u_3) = \begin{cases} 1 & 0 \le u_i \le 1,\ i=1,2,3 \\ 0 & \text{Otherwise} \end{cases}$ | To find the required probability consider,

$$P(U_3^2 \ge U_1 U_2) = \int_0^1\int_0^1\int_{\sqrt{u_1 u_2}}^1 f(u_1,u_2,u_3)\, du_3\, du_2\, du_1$$

$$= \int_0^1\int_0^1 (1-\sqrt{u_1}\sqrt{u_2})\, du_2\, du_1 = \int_0^1 \left(1-\tfrac{2}{3}\sqrt{u_1}\right) du_1 \quad \boxed{= 5/9}$$

54. X, Y, and Z be independent $N(0,\sigma^2)$. Let Θ, Φ, R be Spherical coordinates.

$x = r\sin\phi\cos\theta$; $y = r\sin\phi\sin\theta$; $z = r\cos\phi$; $0 \le \phi \le \pi$; $0 \le \theta \le 2\pi$

Find $f(\theta,\phi,R) - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\theta, y, z)\, dx\, dy\, dz = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x) f(y) f(z)\, dx\, dy\, dz$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(2\pi)^{3/2}\sigma^3} e^{-(x^2+y^2+z^2)/2}\, dx\,dy\,dz = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(2\pi)^{3/2}\sigma^3} e^{-r^2/2\sigma^2} r^2 \sin\phi\, dr\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{(2\pi)^{3/2}\sigma^3} e^{-r^2/2\sigma^2} r^2 \sin\phi\, dr\, d\theta\, d\phi = \frac{2\sigma^2}{2^2(2\pi)^{3/2}\sigma^3}\sqrt{\pi\, 2\sigma^2}\int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\theta\, d\phi$$

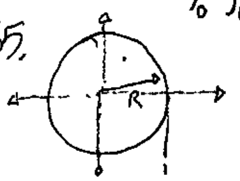
$$= \frac{2\pi}{2(2\pi)^3}\left[-\cos(2\pi) + \cos(0)\right] = \frac{4\pi}{16\pi^2} = \boxed{\frac{1}{4\pi}}$$

$$f(\theta) = \int_0^{\pi}\int_0^{\infty} \frac{1}{(2\pi)^{3/2}\sigma^3} e^{-r^2/2\sigma^2} r^2 \sin\phi\, dr\, d\phi = \int_0^{\pi} \frac{1}{2(2\pi)} \sin\phi\, d\phi = \boxed{\frac{1}{2\pi}}$$

$$f(r) = \int_0^{\pi}\int_0^{2\pi} \frac{1}{(2\pi)^{3/2}\sigma^3} e^{-r^2/2\sigma^2} r^2 \sin\phi\, d\theta\, d\phi = \boxed{\frac{4\pi}{(2\pi)^{3/2}\sigma^3} e^{-r^2/2\sigma^2} \cdot r^2}$$

$$f(\phi) = \int_0^{2\pi}\int_0^{\infty} \frac{1}{(2\pi)^{3/2}\sigma^3} e^{-r^2/2\sigma^2} r^2 \sin\phi\, d\theta\, d\phi = \frac{2\pi \cdot 2\sigma^2}{(2\pi)^{3/2}\sigma^3}\sqrt{2\pi\sigma^2}\, \sin\phi = \boxed{\frac{1}{2\pi^2}\sin\phi}$$

55.



R [0,1]
Θ [0, 2π]

a) $X = R\cos\Theta$, $Y = R\sin\Theta$ a) Find $f(R,\Theta) = f(R\cos\Theta, R\sin\Theta)$ $\boxed{= \frac{r}{2\pi} e^{-r^2/2}}$

b) $f(x) = \int_0^{\infty} \frac{1}{2\pi} e^{-(x^2+y^2)/2}\, dx\, dy = \frac{\sqrt{2\pi}}{2\pi} e^{-x^2/2} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$f(y) = \boxed{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}}$

$A = \pi R^2$; $A = \pi\sqrt{x^2+y^2}^{\,2}$

$\frac{A}{x^2+y^2}$; $x^2+y^2 \le 1$

C) The density is uniform over the disk.

56. Exponential Random variables: $\lambda e^{-\lambda}$; $x = \lambda_x e^{-\lambda|x|}$; $y = \lambda_y e^{-\lambda|y|}$

$f(x,y) = \lambda_x e^{-\lambda x} \cdot \lambda_y e^{-\lambda y} = \lambda_x \lambda_y e^{-\lambda(x+y)} = \boxed{\lambda^2 e^{-\lambda r(\cos\theta + \sin\theta)}}$ r and θ are not independent.

57. $Y_1 = N(0,1)$; $Y_2 = N(0,2)$; $\rho = 1/\sqrt{2}$; Find $X_1 = a_{11}y_1 + a_{12}y_2$ and $X_2 = a_{21}y_1 + a_{22}y_2$

$J(y_1,y_2) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} \cdot a_{22} - a_{12}a_{21} = 1$

$\boxed{X_1 = y_1 ; X_2 = y_2 - y_1 ;}$

Example C: (Section 3.6.2)

$Y_1 = X_1$; $Y_2 = X_1 + X_2$; $J(x,y) = \det\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = 1$

$$f_{Y_1Y_2}(y_1, y_2) = \frac{1}{2\pi}\exp\left[-\frac{1}{2}\left[y_1^2 + (y_2-y_1)^2\right]\right]$$

$$= \frac{1}{2\pi}\exp\left[-\frac{1}{2}(2y_1^2 + y_2^2 - 2y_1y_2)\right]$$

$\sigma_{Y_1}\sigma_{Y_2}\sqrt{1-\rho^2} = 1$

$1 \cdot (2)\sqrt{1-\rho^2} = 1$; $1 - \frac{1}{4} = \rho^2$

58. If X_1 & X_2 are $N(\mu, \sigma^2)$ then $f_{Y_1Y_2}(y_1 y_2)$ is bivariate normal.

$Y_1 = a_1 X_1 + b_1$; $Y_2 = a_2 X_2 + b_2$

58. $J(x,y) = \det\begin{bmatrix}1&0\\0&1\end{bmatrix} = 1$; $\boxed{f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2\pi}e\left[-\frac{1}{2}\left(\left[(y_1-b_1)/a_1\right]^2+\left[(y_2-b_2)/a_2\right]^2\right)\right]}$

59. $Y_1 = a_{11}X_1 + a_{12}X_2 + b_1$; $Y_2 = a_{21}X_1 + a_{22}X_2 + b_2$

① $f(x_1,x_2) = \frac{1}{2\pi}e^{-\frac{1}{2}(x_1^2+x_2^2)}$

② Linear Transformation: $\left(\frac{-a_{22}}{a_{12}}\right)Y_1 = \left(\frac{-a_{22}}{a_{12}}\right)a_{11}X_{11} + \left(\frac{-a_{22}}{a_{12}}\right)a_{12}X_2 + \left(\frac{-a_{22}}{a_{12}}\right)b_1$ $\boxed{x\left(\frac{-a_{22}}{a_{11}}\right)}$

$\left(\frac{-a_{22}}{a_{12}}\right)Y_1 + Y_2 = \left[\left(\frac{-a_{22}}{a_{12}}\right)a_{11} + a_{21}\right]X_1 + \left[\cancel{a_{22}-a_{22}}\right]X_2 + \left(\frac{-a_{22}}{a_{12}}\right)b_1 + b_2$ $\boxed{①+②}$

$\boxed{\text{Solve for } X_1}$

$$X_1 = \frac{\left(\frac{-a_{22}}{a_{11}}\right)Y_1 + Y_2 + \left(\frac{a_{22}}{a_{12}}\right)b_1 + b_2}{\left[\left(\frac{-a_{22}}{a_{12}}\right)a_{11} + a_{21}\right]} = \frac{+a_{22}(Y_1-b_1) - a_{12}(Y_2-b_2)}{(a_{22}a_{11} - a_{21}\cdot a_{12})}$$

③ Solve the Jacobian: $|J| = \begin{vmatrix}\frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2}\\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2}\end{vmatrix} = \begin{vmatrix}\frac{a_{22}}{(a_{22}a_{11}-a_{21}a_{12})} & \frac{-a_{12}}{(a_{22}a_{11}-a_{21}a_{12})}\\ \frac{a_{21}}{(a_{22}a_{11}-a_{21}a_{12})} & \frac{a_{11}}{(a_{22}a_{11}-a_{21}a_{12})}\end{vmatrix} = \frac{1}{(a_{22}a_{11}-a_{21}a_{12})}$

④ Solve for the new bivariate density:

$$f(y_1,y_2) = \frac{1}{2\pi\sigma_{Y_1}\sigma_{Y_2}\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y_1-\mu_{Y_1}}{\sigma_{Y_1}}\right)^2 - 2\rho\frac{(y-\mu_{Y_1})(y-\mu_{Y_2})}{\sigma_{Y_1}\sigma_{Y_2}} + \left(\frac{y_2-\mu_{Y_2}}{\sigma_{Y_2}}\right)^2\right]\right]$$

⑤ Evaluate $X_1^2+X_2^2$

$$x_1^2+x_2^2 = \frac{\{(y_1-b_1)^2(a_{22}^2+a_{21}^2) + (y_2-b_2)^2(a_{12}^2+a_{11}^2) - 2(y_1-b_1)(y_2-b_2)(a_{22}a_{12}+a_{11}a_{21})\}}{(a_{22}a_{11}-a_{21}a_{12})^2}$$

⑥ Since $X_1, X_2 \sim N(0,1)$, then $a_{11}X_1 + a_{22}X_2 + b_1 \sim N(b_1, a_{11}^2+a_{12}^2)$
$a_{21}X_1 + a_{22}X_2 + b_2 \sim N(b_2, a_{21}^2+a_{22}^2)$ $\boxed{\begin{matrix}\mu_{Y_1}=b_1 ; \sigma_{Y_1}^2 = a_{11}^2+a_{12}^2\\ \mu_{Y_2}=b_2 ; \sigma_{Y_2}^2 = a_{21}^2+a_{22}^2\end{matrix}}$

⑦ Reviewing $x_1^2+x_2^2$, ... first term $\left(\left(\frac{(y_1-\mu_{Y_1})}{\sigma_{Y_1}}\right)^2\Big/\frac{(\sigma_{Y_2}\sigma_{Y_1})}{(a_{22}a_{11}-a_{12}a_{21})^2}\right) = \left(\frac{y_1-b_1}{\sqrt{a_{11}^2+a_{12}^2}}\right)^2$

⑧ Solving for $\frac{1}{1-\rho^2} = \frac{\sigma_{Y_1}\sigma_{Y_2}}{(a_{22}a_{11}-a_{21}a_{12})^2}$; $\boxed{\rho = \frac{(a_{22}^2a_{12}^2 + a_{21}^2a_{11}^2)}{\sqrt{(a_{22}^2+a_{21}^2)(a_{11}^2+a_{12}^2)}}}$

⑨ $\boxed{\sigma_{Y_1}\sigma_{Y_2}\sqrt{1-\rho^2} = |(a_{22}a_{11} - a_{21}a_{12})| = J^{-1}}$

60. $\boxed{\text{Pseudorandom variables occur from the previous bivariate normal by a cumulative sum distribution from } -\infty \text{ to } X.}$

61. X & Y are continuous random variables. $U = a + bX$; $V = c + dY$. $f(U,V) = f(x,y)\cdot J^{-1}$
$= f\left(\frac{U-a}{b}, \frac{V-c}{d}\right)\left|\frac{1}{\begin{vmatrix}b&0\\0&d\end{vmatrix}}\right|$

$\boxed{= f\left(\frac{U-a}{b}, \frac{V-c}{d}\right)\frac{1}{bd}}$

62. X & Y are $N(0,1)$; $P(X^2+Y^2 \leq 1) = \frac{1}{2\pi}e^{-(x^2+y^2)/2}$; $\leq \frac{1}{2\pi}e^{-1/2}$ $\boxed{\text{Proposition A}}$

$\boxed{\frac{1}{2\pi} \leq P(X^2+Y^2 \leq 1) \leq \frac{1}{2\pi}e^{-1/2}.}$

63. a) $\boxed{\begin{matrix}X+Y = \cancel{z}U ; f(u,v) = f_{XY}\left(\frac{v+z}{2}, \frac{v-z}{2}\right)\cdot\frac{1}{2}\\ X-Y = V\end{matrix}}$

b) $\boxed{\begin{matrix}XY = Z ; f(u,v) = f_{XY}\left(\sqrt{zv}, \sqrt{z/v}\right)\frac{1}{2|v|}\\ Y/X = V\\ \text{c) } X \sim N(0,1),\ Y \sim N(0,1)\end{matrix}}$

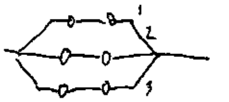
64. $X+Y = Z$, $X|Y = V$; $f_{XY}\left(\frac{vz}{(v+1)}, \frac{z}{(v+1)}\right)\cdot\frac{1}{\begin{vmatrix}1&1\\ \frac{1}{y}&-\frac{x}{y^2}\end{vmatrix}} = f_{XY}\left(\frac{vz}{(v+1)}, \frac{z}{(v+1)}\right)\frac{-y^2}{x+y}$

$= f_{XY}\left(\frac{vz}{(v+1)}, \frac{z}{(v+1)}\right)\frac{-z^2/(v+1)^2}{\frac{vz}{(v+1)} + \frac{z}{(v+1)}}$ $\boxed{f_{XY}\left(\frac{vz}{(v+1)}, \frac{z}{(v+1)}\right)\frac{-z}{(v+1)^2}}$

65. Exponential random variable: $\lambda e^{-\lambda x}$; $f_{xy}(^{x+v}$

65. Exponential Random variable: $p(x) = \lambda e^{-\lambda x}$;

o-o-o-o-o-o, $p(x_1, \dots, x_n) = \prod_{i=1}^{n} \lambda_i e^{-\lambda_i} = \lambda_1 \lambda_2 \cdots \lambda_n \boxed{e^{-\sum \lambda_i}}$

66.  $F(t) = p(1A)p(1B) + p(2A)p(2B) + p(3A)p(3B)$

$= 3\lambda^2 e^{-2\lambda t}$

$f(t) = \frac{d}{dt} F(t) = \boxed{-6\lambda^3 e^{-2\lambda t}}$

67. n-chips; $p(\text{failure} \mid \text{chips} \geq 2)$; Exponential Dist. $p(x) = \lambda e^{-\lambda x}$

Order Statistics:

$f_u(u) = n[F(u)]^{n-1} f(u)$; $u \leq U \leq u + du$

K^{th}-order Statistic?

$f_k(x) = \frac{n!}{(k-1)!(n-k)!} f(x) F^{k-1}(x)[1 - F(x)]^{n-k}$

$f(x) = \lambda e^{-\lambda x}$ $F(x) = \int_0^x f(x) = 1 - e^{-\lambda x}$

$f_k(x) = \frac{n!}{(2-1)!(n-2)!}[\lambda e^{-\lambda x}][1 - e^{-\lambda x}]^{2-1}[1 - 1 + e^{-\lambda x}]^{n-2}$

$= \frac{n!}{(n-2)!}[\lambda e^{-\lambda x}][1 - e^{-\lambda x}][e^{-\lambda x}]^{n-2}$

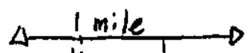
$= n(n-1)\lambda e^{-(n-1)\lambda x}[1 - e^{-\lambda x}]$

$\boxed{= n(n-1)\lambda\left[e^{-(n-1)\lambda x} - e^{-n\lambda x}\right]}$

Notes about order statistics: Multinomial + Differential Argument.

68. U_1, U_2, and U_3 be independent uniform random variables.

a) Find $f(U_1, U_2, U_3) = n! \prod_{i=1}^{n} f(U_i) = 3 \cdot 2 \cdot 1 f(U_1) \cdot f(U_2) \cdot f(U_3)$

b)  $\int\int_{1/3} f(u)\, du = \int_0^{1/3} u\, du = \frac{(1/3)^2}{2} = \frac{1}{18}$

$P(\text{No stations } 1/3 \text{ apart}) = 1 - P(\text{any two stations } 1/3)$

$= 1 - \frac{1}{18}$

$= 0.94.$

70.

Joint Distribution of Order Statistics

$x \leq X_{(i)} \leq x + dx$; $y \leq X_{(n)} \leq y + dy$

$V = X_{(1)}$; $U = X_{(n)}$

$f(u, v) = n(n-1) f(v) f(u) [F(u) - F(v)]^{n-2}$ $u \geq v$

Uniform case: $f(u,v) = n(n-1)(u-v)^{n-2}$, $1 \geq u \geq v \geq 0$

$F(x, y) = \int_{F(y)-F(x)}^{F(y)} F(y)^{n-1} = \boxed{F(y)^n - [F(y) - F(x)]^n}$

69. Weibull Random Variables:

$f(t) = \beta \alpha^{-\beta} t^{\beta-1} e^{-(t/\alpha)^\beta}$

$f_k(x) = \frac{n!}{(k-1)!(n-k)!} f(x) F^{k-1}(x)[1 - F(x)]^{n-k}$

if $k = 1, n = n$

$f_k(t) = \frac{n!}{(1-1)!(n-1)!} f(t) \cdot F(t)^{0} \cdot [1 - F(t)]^{n-1}$

$F(t) = \int_0^t \beta \alpha^{-\beta} t^{\beta-1} e^{-(t/\alpha)^\beta} dt$ $\left| \left(\frac{t}{\alpha}\right)^\beta = u \right.$

$\beta\left(\frac{t}{\alpha}\right)^{\beta-1} \frac{1}{\alpha} dt = du$

$= \int_0^{(t/\alpha)^\beta} \beta \alpha^{-\beta} \left(\frac{\alpha^\beta}{\beta}\right) e^{-u} du$

$\frac{\beta}{\alpha^\beta} t^{\beta-1} dt = du$

$t^{\beta-1} dt = \frac{\alpha^\beta}{\beta} du$

$= 1 - e^{-(t/\alpha)^\beta}$

$f_k(t) = \frac{n\beta}{\alpha^\beta} t^{\beta-1} e^{-(t/\alpha)^\beta} \cdot e^{-(n-1)(t/\alpha)^\beta}$

$\boxed{= \frac{n\beta}{\alpha^\beta} t^{\beta-1} e^{-n\left(\frac{t}{\alpha}\right)^\beta}}$

71. $X_1, \dots, X_n$: $f_{x_1} \dots f_{x_n}$: $f(r) = \frac{\int_{-\infty}^{X_{(n)}} f(v+r, v)^{n-1} dv}{\int_{-\infty}^{\infty} f(v+r, v)^{n-1} dv} = \frac{\int_{-\infty}^{X_{(n)}} f(v+r, v)^{n-1} dv}{\int_{-\infty}^{\infty} f(v+r, v)^{n-1} dv} = \frac{f(X_{(n)}) - F(-\infty)}{F(\infty) - F(-\infty)}$

72. Five numbers $[0,1]$; probability $\left[\frac{1}{4} \leq X_1, X_2 X_3 X_4, X_5 \leq \frac{3}{4}\right] = \int\int\int\int\int f(u) = \int\int\int\int u\, du = \int\int\int \frac{u^2}{2} du = \int\int \frac{u^3}{6} = \int_{1/4}^{3/4} \frac{u^4}{24} = \frac{\left(\frac{3}{4}\right)^5 - \left(\frac{1}{4}\right)^5}{120}$

$= \frac{242/4^5}{120} = \frac{242/1024}{120}$

73. Definition of a independent random variable:

$F(X_1 \dots X_n) = F_{x_1}(x_1) \cdot F_{x_2}(x_2) \cdots F_{x_n}(x_n)$

Definition k^{th} order Statistic:

$f_k(x) = n! f(x_1) f(x_2) f(x_3) \cdots f(x_n)$

4 16 64 256 1024

74. n-servers; $\bar{T} = \lambda e^{-\lambda t}$; $P(\text{Service Time} \geq t) = P(\text{No departure during } t) = P_N(t)$; $P_N'(t) = e^{-\mu t}$

$S(t) = P(T \leq t) = 1 - P(T \geq 1) = 1 - e^{-\mu t}$

75. $\frac{d}{dt} S(t) = \mu e^{-\mu t}$; $S(t) = \begin{cases} \mu e^{-\mu t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$ ‖ Distribution of waiting time $\frac{1}{\mu}$ with a variance $\frac{1}{\mu^2}$.

Find the joint density of $X_{(i)}$ and $X_{(j)}$, $i < j$.

$f_{ij}(u, v) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \times F(u)^{i-1} \times f(u) [F(v) - F(u)]^{j-i-1} \times f(v) [1 - F(v)]^{n-j}$ continued...

76. Prove Theorem A: $f_k(x) = \frac{n!}{(k-1)!(n-k)!} f(x) F^{k-1}(x) [1 - F(x)]^{n-k}$ is derived from interdependent paths, $f(x) F^{k-1}(x) [1 - F(x)]^{n-k} \times$ Multinomial theorem

75. $(i-1)$, $(j-i-1)$; x, $x + dx$

$$\boxed{f_k(x) = \frac{n!}{(k-1)!(n-k)!} f(x) F^{k-1}(x) [1 - F(x)]^{n-k}}$$

$f(x, y) = \lim_{dx \to 0} \lim_{dy \to 0} \frac{P(x \leq X_i \leq x + dx, y \leq X_j \leq y + dy)}{dx\, dy}$

Multinomial Probability Law:

$\frac{n!}{(i-1)!(j-i-1)!(n-j)!} p_1^{i-1} p_2, p_3^{j-i-1} p_4 \cdot p_5 \cdots = F(X)$; $p_2 = P(x \leq X_i \leq x + dx) = F(x + dx) \cdot F(X)$

$\boxed{p_1 = P(X \leq X_i \leq x + dx)}$

$p_3 = P(x + dx < X_i \leq y) = F(y) \cdot F(x + dx)$

$f(x, y) = \lim_{dx \to 0} \lim_{dy \to 0} \frac{P(E)}{dx\, dy}$

$p_4 = P(y < X_i < y + dy) = F(y + dy) \cdot F(y)$

$= \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \times F^{(i-1)} \times \lim_{dx \to 0} \frac{F(x + dx) - F(x)}{dx}$

$p_5 = P(X_i > y + dy) = 1 - P(X \leq y + dy) = 1 - F(y + dy)$

$\times \lim_{dx \to 0} F(y) - F(x + dx)^{j-i-1} \times \lim_{dy \to 0} \frac{F(y + dy) - F(y)}{dy} \times \lim_{dy \to 0} F(y) - F(x + dx)^{j-i-1}$

$\times \lim_{dy \to 0} \frac{F(y + dy) - F(y)}{dy} \times \lim_{dy \to 0} 1 - F(y + dy)$

77. $U_{(k)} - U_{(k-1)}$: $U_{i = 1, \ldots, n}$.

$f(U_k) = n! \prod_{i=1}^{n} f(U_i)$; $f(U_{k-1}) = n! \prod_{i=1}^{n-1} f(U_i)$

$$\boxed{f(U_k) - f(U_{k-1}) = n!\, f(U_1) \cdot f(U_2) \cdots f(U_{n-1}) (f(U_n) - 1)}$$

78. Show $\int_0^1 \int_0^y (y - x)^n dx\, dy = \frac{1}{(n+1)(n+2)}$; $\int_0^1 \frac{-1}{n+1} (y - x)^{n+1} \Big|_0^y dy = \frac{-1}{(n+1)} \int_0^1 [(y - 1)^{n+1} - y^{n+1}] dy = \frac{-1}{(n+1)(n+2)} (y - 1)^{n+2} - (y)^{n+2} \Big|_0^1$

$= \frac{-1}{(n+1)(n+2)} [0 - 1^{n+2} - 1^{n+2} + 0] = \boxed{\frac{1}{(n+1)(n+2)}}$

79. T_1 & T_2 are exponential random variables; $R = T_{(2)} - T_{(1)}$

$F(R, T_2) = \iint f(T_1, T_2) dT_1 dT_2 = \iint f(T_1, R - T_1) dT_1 dR = \iint \lambda e^{-\lambda T_1} \cdot \lambda e^{-\lambda(R - T_1)} dT_1 dR$

$f(z) = \int_0^\infty \lambda_1 \lambda_2 e^{-(\lambda_1 + \lambda_2) T_1 - \lambda_2 R} dT_1$ $\boxed{\begin{matrix} T_2 = R - T_1 \\ dT_2 = dR \end{matrix}}$

$$\boxed{= \frac{\lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)} e^{-(\lambda_2 R)}}$$

80. $U_1 \cdots U_n$, V = uniform, U_i = independent.

a) Find $P(V \leq U_{(n)}) = \int n! \prod U_1 \cdot U_2 \cdots U_n dV = n! \prod f(U_i)^2 / 2 \Big|_0^V = \boxed{n! \prod_{i=1}^{n} f(V)^2 / 2}$

b) $P(U_{(1)} < V < U_{(n)}) = \int_{U_{(1)}}^{U_{(n)}} n! \prod f(V_i) dV_i = n! \left[\prod_{i=1}^{n} f(V_n)^2 - \prod f(U_1)^2 \right] / 2$

$P(V \leq U_{(n)}) = \int_0^{U_{(n)}} n! \prod f(V_i) dV = \boxed{n! \prod f(U_n)^2 / 2}$, $P(U_{(1)} < V < U_{(n)}) = \int_{U_{(1)}}^{U_{(n)}} n! \prod f(U_n) dV = \boxed{\frac{n! \prod f(V)^2 - f(V_1)^2}{2}}$

Chapter 4: 1) Prove if $|X| < M < \infty$, then $E(X)$ exists. $M_i = x_i p(x_i)$; $M_1 + M_2 + \cdots + M_\infty$

$= x_1 p(x_1) + x_2 p(x_2) + \cdots + x_\infty \cdot p(x_\infty)$

2) $F(x) = 1 - x^{-\kappa}$, $x \geq 1$; a) $E(X) = \int_{-\infty}^{\infty} x f(x)dx = \int_{-\infty}^{\infty} x \frac{d}{dx}(1-x^{-\kappa})dx$

$= E(x)$; $\boxed{\text{Definition of Expectation}}$

b) $Var(X) = E\{[X - E(X)]^2\}$

$= \int_1^\infty x(\kappa x^{-(\kappa+1)})dx = \int_1^\infty \kappa x^{-(\kappa)}dx = \frac{-\kappa x^{-(\kappa+1)}}{(\kappa+1)}\Big|_1^\infty = \frac{-\kappa}{(\kappa+1)}(\infty)^{-(\kappa+1)} \frac{1}{} \boxed{\frac{\kappa}{(\kappa+1)}}$

$= \int_1^\infty \left[x - \frac{\kappa}{\kappa+1}\right]^2 dx = \int_1^\infty \left[x^2 - 2x\frac{\kappa}{(\kappa+1)} + \left(\frac{\kappa}{\kappa+1}\right)^2\right]dx = \frac{x^3}{3} - x^2\left(\frac{\kappa}{\kappa+1}\right) + \left(\frac{\kappa}{\kappa+1}\right)^2 x \Big|_1^\infty = \infty.$

$= E(x^2) - \mu^2 = \int_1^\infty x^2(\kappa x^{-(\kappa+1)})dx = \int_1^\infty \kappa x^{-\kappa+1} dx = \kappa \cdot x^{-(\kappa+2)}/(\kappa+2)\Big|_1^\infty = \frac{\pm\kappa}{(\kappa+2)} + \left(\frac{\kappa}{\kappa+1}\right)^2 = \boxed{\frac{\kappa}{(\kappa+2)} - \left(\frac{\kappa}{\kappa+1}\right)^2}$

$-\left(\frac{\kappa}{\kappa+1}\right)^2$ $\quad - \left(\frac{\kappa}{\kappa+1}\right)^2$

3. Find E(x) and Var(x) for Chapter 2: Problem #3.

K	F(K)
0	0
1	0.1
2	0.3
3	0.7
4	0.8
5	1.0

$E(x) = \sum x p(x) = \sum x f(x) = \sum x[F(x_i) - F(x_{i-1})] = 5[1.0-0.8] + 4[0.8-0.7] + 3[0.7-0.3] + 2[0.3-0.1] + 1[0.1-0]$

$= \frac{5\cdot 2}{10} + \frac{4\cdot 1}{10} + \frac{3\cdot 4}{10} + \frac{2\cdot 2}{10} + \frac{1\cdot 1}{10}$

$= \frac{10+4+12+4+1}{10} = \frac{31}{10} = \boxed{3.1}$

$Var(X) = (0.2-3.1)^2(0.2) + (0.1-3.1)^2(0.1) + (0.4-3.1)^2 0.4 + (0.2-3.1)^2 \cdot 0.2 + (0.1-3.1)^2 \cdot 0.1$

$\boxed{= 1.62}$

4. $P(X=k) = 1/n$ for $k = 1, 2, \ldots, n$: Find $E(x)$ and $Var(X)$; $E(X) = 1\cdot\left(\frac{1}{n}\right) + 2\cdot\left(\frac{1}{n}\right) + 3\left(\frac{1}{n}\right) + \cdots + n\left(\frac{1}{n}\right) = \frac{n(n+1)}{2}\cdot\frac{1}{n}$

$Var(X) = \left[\left(1 - \frac{(n+1)}{2}\right)^2 + \left(2 - \frac{(n+1)}{2}\right)^2 + \cdots + \left(n - \frac{(n+1)}{2}\right)^2\right]\frac{(n+1)n}{2}$

5. $f(x) = \frac{1+\alpha x}{2}$; $-1 \leq x \leq 1$; $-1 \leq \alpha \leq 1$

$E(X) = \int_{-1}^{1}\frac{(1+\alpha x)}{2} x\,dx = \frac{x^2}{4} + \frac{\alpha x^3}{6}\Big|_{-1}^{1} = \frac{1}{4} + \frac{\alpha}{6} + \frac{1}{4} + \frac{\alpha}{6} = \frac{\alpha}{3}$; $E(x^2) = \int_{-1}^1 x^2 \frac{(1+\alpha x)}{2}dx = \frac{1}{2}\left[\frac{x^3}{3} + \frac{\alpha x^4}{4}\right]_{-1}^1 = \boxed{\frac{1}{3}}$

$Var(X) = E(x^2) - [E(X)]^2 = \frac{1}{3} - \frac{\alpha^2}{9} = \boxed{\frac{3-\alpha^2}{9}}$

6. $f(x) = 2x$; $0 \leq x \leq 1$

a) $E(X) = \int_0^1 2x^2 dx = \frac{2}{3}$ b) $Y = X^2$; Find $E(Y) = E(X^2) = \int_0^1 2x^3 dx = \boxed{\frac{1}{2}}$

c) $E(X^2) = \int_0^1 2x^3 dx = \frac{2}{3}$; d) $Var(X) = E\{[X - E(X)]^2\} = \int_0^1 (x - \frac{2}{3})^2 2x\,dx = \frac{1}{18}$.

Theorem B: $Var(X) = E(X^2) - E(X)^2 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$

8. Average $= \sum$ weight $\times x_i$

$= \sum p_i \cdot x_i$

7.

X	0	1	2
P(X)	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{8}$

a) $E(X) = \sum x f(x) = 0\left(\frac{1}{2}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{1}{8}\right) = \boxed{5/8}$

b) $Y = X^2$. $E(Y) = 0^2\left(\frac{1}{2}\right) + 1^2\left(\frac{3}{8}\right) + 2^2\left(\frac{1}{8}\right) = 7/8$

c) Theorem A: a) $E(Y) = \sum g(x)p(x) = 0^2\left(\frac{1}{2}\right) + 1^2\left(\frac{3}{8}\right) + 2^2\left(\frac{1}{8}\right) = 7/8$

b) $E(Y) = \int_{-\infty}^{\infty} g(x)f(x)dx$;

9. c = \$ to stock an item

s = \$ to sell an item.

$P(K)$ = Number of items by customer.

$s\sum_{k=n}^{\infty} p(k) > c\sum_{k=1}^{n-1} p(k)$, and $s\sum_{k=n+1}^{\infty} p(k) < c\sum_{k=1}^{n} p(k)$

Selling should be greater than cost

Future sales should be cheaper than cost.

$n+1$

d) $Var(X) = E[(X - E(X))^2] = \frac{1}{2}(0 - 5/8)^2 + \frac{3}{8}(1 - 5/8)^2 + \frac{1}{8}(2 - 5/8)^2 = \frac{31}{64}$.

$Var(X) = E(X^2) - E(X)^2 = 7/8 - (5/8)^2 = \frac{31}{64}$

10. "Random" n-items $E(X)=\sum_{i=1}^{n} p(x_i)x_i = \sum\left(\frac{1}{n}\right)x_i = \boxed{\frac{\sum x_i}{n}}$

Search(n)

11. $E(X)=\sum p_i x_i$; $E(X)=\sum^{n} x_i(1-p_1)(1-p_2)\cdots(1-p_{x-1})p_x$

$p_1 + 2(1-p_1)p_2 + 3(1-p_1)(1-p_2)p_3 + \cdots + n(1-p_1)(1-p_2)(1-p_n)$

12. Suppose $E(X)=\mu$ and $Var(X)=\sigma^2$. Let $Z=(X-\mu)/\sigma$. Show $E(Z)=0$ and $Var(Z)=1$

$E(Z)=E\left(\frac{X-\mu}{\sigma}\right)=\frac{E(X)-E(\mu)}{\sigma}=\frac{\mu-\mu}{\sigma}=0$; $Var(Z)=Var\left(\frac{X-\mu}{\sigma}\right)=\frac{Var(X)}{\sigma^2}=\frac{\sigma^2}{\sigma^2}=\boxed{1}$

13. $E(X)=\int_0^\infty x\cdot f(x)\,dx$; Product Rule: $[xF(x)]' = F(x)+xf(x)$; $1-F(x)=xf(x)$: $E(X)=\int_0^\infty [1-F(x)]\,dx$

$E(X)=\int_0^\infty [1-1+e^{-\lambda x}]\,dx = \frac{-e^{-\lambda x}}{\lambda}\Big|_0^\infty = \boxed{\frac{1}{\lambda}}$

14. $f(x)=2x$; $0\le x\le 1$; $E(X)=\int_0^1 2x^2\,dx = \frac{2}{3}x^3\Big|_0^1 = \boxed{\frac{2}{3}}$ b) $E(X^2)=\int_0^1 2x^3\,dx = \boxed{\frac{1}{2}}$

15. Lottery A n-possible #s ; Lottery B n-possible #'s ; Payoff A = Payoff B

$E(A)=\sum_{1}^{n}\frac{1}{n!}x$; $E(B)=\sum_{1}^{n}\frac{1}{n!}x$; $E(A+B)=\frac{E(A)+E(B)}{2}$; $E(A)=E(B)$; No difference!

$Var(X)=E(X^2)-E(X)^2=E[(X-E(X))^2]$ $=\frac{1}{2}-\left(\frac{2}{3}\right)^2=\boxed{5/9}$

16. $Var(+X)=Var(-X)$; $f_X(5+5)$; $f_X(5-5)$; $Y=X-5$; $f_Y(y)=f_X(y+5)$; $f_Y(y)=f_Y(-y)$

$Y=X-5$; $E(Y)=E(X)-E(5)$; $E(X)=E(5)$

17. k^{th}-order Statistic:

$\frac{n!}{(k-1)!(n-k)!}x^{k-1}(1-x)^{n-k}$; $0\le x\le 1$; $E(X)=\int_0^1 \frac{n!}{(k-1)!(n-k)!}x^k(1-x)^{n-k}\,dx = \frac{n!}{(k-1)!(n-k)!}\cdot\frac{k!(n-k)!}{(n+1)!}$

$=\frac{k!}{(k-1)!(n+1)} = \boxed{\frac{k}{(n+1)}}$

$E(X^2)=\frac{n!}{(k-1)!(n-k)!}\int_0^1 x^{k+1}(1-x)^{n-k}\,dx = \frac{n!}{(k-1)!(n-k)!}\cdot\frac{(k+1)!(n-k)!}{(n+2)!} = \frac{k(k+1)}{(n+1)(n+2)}$

$= k(k+1)$; $Var(X)=\frac{k(k+1)}{(n+1)(n+2)} - \frac{k^2}{(n+1)^2} = \frac{k(k+1)(n+1)-k^2(n+2)}{(n+1)^2(n+2)}$

$=\frac{(k^2+k)(n+1)-k^2n+k^2(2)}{(n+1)^2(n+2)}$

$=\frac{k^2n+k^2+kn+k-k^2n-k^2 2}{(n+1)^2(n+2)}$

$=\boxed{\frac{k(n+1-k)}{(n+1)^2(n+2)}}$

18. $U_1,\dots,U_n$; $E(U_{(n)}-U_{(1)})$; $E(U_{(n)}-U_{(1)})=\sum_{i=1}^{n}(U_{(n)}-U_{(1)})f(u)$

19. $E(U_{(k)}-U_{(k-1)})=\sum_{i=1}^{n}[U_{(k)}-U_{(k-1)}]f(u)$

20. $E[1/(X+1)]$; $X=\frac{\lambda^k}{k!}e^{-\lambda x}$; $E\left[\frac{1}{(x+1)}\right]=\int_0^\infty \frac{\lambda^k}{k!}\frac{1}{(x+1)}e^{-\lambda x}\,dx$

$=\frac{\lambda^k}{k!}\int_0^\infty \frac{e^{-\lambda x}}{(1+x)}\,dx$; $u=1+x$, $du=dx$, $x=(u-1)$; $\frac{\lambda^k}{k!}\int_0^\infty \frac{e^{-\lambda(u-1)}\,du}{u} = \frac{\lambda^k}{k!}e^{\lambda}\int_0^\infty \frac{e^{-\lambda u}}{u}\,du = \frac{\lambda^k}{k!}e^{\lambda}\cdot E_1(\lambda)$ "Exponential Integral"

21. Expected Area $(X^2)=\int_0^1 x^2\,dx = \boxed{\frac{1}{3}}$

$\left(\frac{1}{2}\right)^2+1^2$

22. Expected Area $(X^2)=\int_0^1 x^2\,dx=\boxed{\frac{1}{3}}$

23. Expected Area $(X^2)=\lambda\int_0^\infty x^2 e^{-\lambda x}\,dx$

$E(X)E(Y)=\lambda\int_0^\infty xe^{-\lambda x}\,dx\cdot\lambda\int_0^\infty ye^{-\lambda y}\,dy=\boxed{1/\lambda^2}$

24. Prove Theorem A of Section 4.1.2 ; $Y = a_0 + \sum_{i=1}^{n} b_i X_i$; $E(Y) = a + \sum_{1}^{n} b_i E(X)$

Proof: $E(Y) = \iint (a + b_1 x_1 + b_2 x_2) f(x_1, x_2)\, dx_1 dx_2 = a \iint f(x_1, x_2)\, dx_1 dx_2 + b_1 \iint x_1 f(x_1, x_2)\, dx_1 dx_2$

$+ b_2 \iint x_2 f(x_1, x_2)\, dx_1 dx_2$

$\iint x_1 f(x_1, x_2)\, dx_1 dx_2 = \int x_1 \left[\int f(x_1, x_2)\, dx_2\right] dx_1 = \int x_1 f(x_1)\, dx_1 = E(X_1)$

$E(Y) = a + b_1 E(X_1) + b_2 E(X_2)$; $\iint |a + b_1 x_1 + b_2 x_2| f(x_1, x_2)\, dx_1 dx_2 < \infty$

$|a + b_1 x_1 + b_2 x_2| \le |a| + |b_1||x_1| + |b_2||x_2|$

25. Gamma Distribution:

$f(x) = \frac{\lambda^a}{\Gamma(a)} x^{a-1} e^{-\lambda x}$ Find $R^2 = X_1^2 + X_2^2$; $E(R^2) = \int (x_1^2 + x_2^2) \Gamma(\kappa)\, dx_1 dx_2 = \int_0^\infty x_1^2 \frac{\lambda^\kappa}{\Gamma(\kappa)} x_1^{\kappa-1} e^{-\lambda x_1} dx + \int_0^\infty x_2^2 \frac{\lambda^\kappa}{\Gamma(\kappa)} x_2^{\kappa-1} e^{-\lambda x} dx$

$= \frac{\lambda^\kappa}{\Gamma(\kappa)} \left[\left(\int_0^\infty x_1^{(\kappa+2)-1} e^{-\lambda x} dx\right) + \left(\int_0^\infty x_2^{(\kappa+2)-1} e^{-\lambda x} dx\right) \right]$

$= \frac{\lambda^\kappa}{\Gamma(\kappa)} \frac{\Gamma(\kappa+2)}{\lambda^{\kappa+k}} = \frac{\Gamma(\kappa+2)}{\Gamma(\kappa)\lambda^k} = \frac{(\kappa+2+1)\Gamma(\kappa+2-1)}{\Gamma(\kappa)\lambda^k} + \cdots$

$= \frac{(\kappa+2-1)\cdots\kappa}{\lambda^k} = \frac{(\kappa+1)\kappa}{\lambda^k}$

$\boxed{E(R^2) = \frac{2(\kappa+1)\kappa}{\lambda^k}}$

26. $\frac{\text{Expectation Long Piece}}{\text{Expectation (Shorter Piece)}} = \frac{E(L)}{E(S)} = \frac{\sum L \cdot f(X)}{\sum S \cdot f(X)}$

27. n-men → pile $E(\#\text{ of matches}) = \sum \frac{x}{n} = \frac{n}{n} = 1$

28. $X_i = \begin{cases} 1 : \text{Aircraft } i \text{ is hit} \\ 0 : \text{Otherwise} \end{cases}$; $Z_{ij} = \begin{cases} 1 & \text{if gunner } j \text{ hits } i \\ 0 & \text{otherwise} \end{cases}$

$P(Z_{ij} = 1 | B_i) = p$; $P(Z_{ij} = 1 | \bar{B}_i) = 0$; $P(B) = \frac{1}{n}$

$P(\bar{B}) = 1 - P(B) = 1 - \frac{1}{n}$ "Every gunner selects i"

Addition Rule:

$P(Z_{ij} = 1) = P(Z_{ij} = 1 | B_i) P(B_i) + P(Z_{ij} = 1 | \bar{B}_i) P(\bar{B})$

$= p\left(\frac{1}{n}\right) + 0\left(\frac{1}{n}\right) = \frac{p}{n}$

Since the total probability is equal to 1, this implies:

$P(Z_{ij} = 0) = 1 - P(Z_{ij} = 1) = 1 - \frac{p}{n}$

$E(X) = \sum X_i P(X) = 0(P(X=0)) + 1 \cdot P(X=1)$

$= 1 - P(X=1) = \ldots$

$= 1 - \{P(Z_i = 0) P(Z_{i2} = 0) \cdots P(Z_{im} = 0)\}$

$\boxed{= 1 - \left(1 - \frac{p}{n}\right)^m}$

29. Prove Corollary 4.1.1

$E[g(X) h(Y)] = E[g(X)] E[h(Y)]$

$E[g(X) \cdot h(Y)] = \iint g(x) \cdot h(y)\, dx\, dy = \boxed{\int g(x)\, dx \int h(Y)\, dY}$

30. Coupon Collection: n-distinct types of coupons. P(coupon) are equivalent.

$X_1 = 1, X_2 = 2, X_3 = 3 \cdots X_n = n$; $P_r(\text{success}) = (n-r+1)/n$

$E(X_r) = n/(n-r+1)$

$E(X) = \sum_{r=1}^{n} E(X_r) = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1} = n \sum_{r=1}^{n} \frac{1}{r}$

$\boxed{r = 2r \sum \frac{1}{r} = r}$

31. $X[1, 2]$

$E[1/X] = \int_1^2 x\left(\frac{1}{x}\right) dx$

$= \int_1^2 dx = 1$

$\frac{1}{E(X)} = \frac{1}{\int_1^2 x^2 dx} = \frac{3}{8-1} = \frac{3}{7}$; They are not equal.

32. Gamma Distribution.

$f(x) = \frac{\lambda^a}{\Gamma(a)} x^{a-1} e^{-\lambda x}$

$E\left(\frac{1}{X}\right) = \int_0^\infty \left(\frac{1}{x}\right) \frac{\lambda^a}{\Gamma(a)} x^{a-1} e^{-\lambda x} dx = \frac{\lambda^a}{\Gamma(a)} \int_0^\infty x^{(a-1)-1} e^{-\lambda x} dx$

$= \frac{\lambda^a}{\Gamma(a)} \frac{\Gamma(a-1)}{\lambda^{a-1}} = \frac{\lambda \Gamma(a-1)}{\Gamma(a)} = \boxed{\frac{\lambda}{a}}$

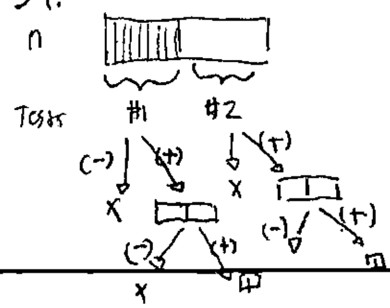
33. Prove Chebyshev's Inequality:

X, μ, σ^2. For $t > 0$,

$P(|X - \mu| > t) \le \frac{\sigma^2}{t^2}$

$E(y) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (X-\mu)^2 f(x) \ge \int_{-\infty}^{\mu-\varepsilon} (x-\mu)^2 f(x)\, dx + \int_{\mu+\varepsilon}^{\infty} (x-\mu)^2 f(x)\, dx = \int_{-\infty}^{\mu-\varepsilon} \varepsilon^2 f(x)\, dx + \int_{\mu-\varepsilon}^{\infty} \varepsilon^2 f(x)\, dx = \boxed{\varepsilon^2 P(|X - \mu| \ge \varepsilon)}$

34.



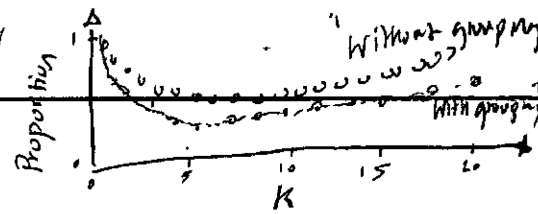
n = Group Samples
m = Groups
k = samples
n = mk

$E(N)=\sum_{1}^{m}\hat{E}(X)=\sum_{1}^{m}p^{k}+(k+1)(1-p^{k})=\sum_{1}^{m}k-kp^{k}+1$

$\boxed{= m(k+1)-mkp^{k}=n\left(1+\frac{1}{k}-p^{k}\right)}$ "Average number of tests performed"

"With no grouping"

$\boxed{E(X)=k+1-kp^{k}=k\left(1+\frac{1}{k}-p^{k}\right)}$



35. Mean of Negative Binomial Random Variable:

$E(K)=\sum_{k=1}^{\infty}k\binom{k-1}{r-1}p^{r}(1-p)^{k-r}=\sum\frac{k(k-1)!}{(r-1)!(r-k)!}p^{r}(1-p)^{k-r}$

$=\frac{k\,T(k)}{T(r)\,T(k-\cancel{k}+1)}\;\frac{T(r+1)\,T(k-\cancel{r}+1)}{T(k+2)}=\frac{kT(k)\,rT(r)}{T(r)(k+1)\,kT(k)}=\boxed{\frac{r}{k+1}}$

36. $X[0,1], Y=\sqrt{X}$; $E(Y)$ a) $E(Y)=P(\sqrt{X})=\sum_{0}^{1}\sqrt{X}\,f(x)=\boxed{\sqrt{X}\cdot p^{x}+\sqrt{X}p(1-p)^{1-x}}=\sqrt{1}=1$

b) $F(n)\;F(\sqrt{x})\;\int_{0}^{1}\sqrt{x}\,dx\;\frac{2}{3}(1)^{3/2}$

$E(Y)=n\left(1+\frac{1}{k}-p^{k}\right)$; $E(X)=n!X$ $E(X)=E(N)$

37. Example C Section 4.1.2.

$n=n\left(1+\frac{1}{k}-p^{k}\right)$

$1=1+\frac{1}{k}-p^{k}$

$\frac{1}{k}=p^{k}$ $\boxed{p=\left(\frac{1}{k}\right)^{1/k}}$

38. $E(Y)=\sum_{k=0}^{n}\binom{n}{k}kp^{k}(1-p)^{n-k}=np$

a) $Y=\sum_{i=1}^{n}X_i$; Length of DNA $=G$, Fragments $=N$ of length $=L$.
$G\sim>100{,}000$; $L\sim>500$

Probability of left end is $1,2\ldots G-L+1$.

What is the probability a particular location $x\in\{L,L+1\ldots G\}$

What is the probability a particular location: $\{1,2\ldots L-1\}$

How many fragments are expected to cover a particular location

What is the chance of $\neq$ covering the left end of L locations $\{x-L+1,x-L,\ldots x\}$

$p=\frac{L}{G-L+1}\approx\frac{L}{G}$; The binomial probability formula, $p(W>0)=1-P(W=0)=1-\left(1-\frac{L}{G}\right)^{N}$

$\lambda=Np=NL/G$

a. Probability that a fragment is the leftmost member of a contig $\frac{L}{G-L+1}$

b. Expected number of fragments left of contig: $E(K)=\sum_{N=0}^{N}K(P(W=0))=\left[1-\left(1-\frac{L}{G}\right)^{N}\right]L$

c. Expected number of contigs: $\boxed{E(K)=Le^{-NL/G}}$

39. DNA Length $=10^{6}$, fragment length $=100$

a) $P(W>0)=0.99=1-\left(1-\frac{100^{2}}{10^{6}}\right)^{N}$; $(1-10^{-4})^{N}=0.01$; $N\approx\frac{\log 10^{-2}}{\log(0.9999)}=4.60\times10^{4}$ fragments.

b) The expected misses: $E(I)=e^{-4.60\times10^{4}\cdot100/10^{6}}\boxed{=0.01}$ 0.999

40. Q, W, E, R, T, Y produces 1000 letters in all. $E(QQQQ)=\sum_{n=1}^{N-q+1}E(I_n)=(N-q+1)\left(\frac{1}{5}\right)^{q}$

$\boxed{n=1000,\;q=4\;\;=(997)\left(\frac{1}{5}\right)^{4}=15.95\text{ times}}$

41. $E(TOY)=\sum_{n=1}^{N-q+1}E(I_n)=(1000-3+1)\left(\frac{1}{5}\right)^{3}=998\left(\frac{1}{5}\right)^{3}$
$=79.84$ times.

Markov's Inequality:

$\frac{79.84}{1000}\approx\frac{1}{125}$, the author would be surprised for the answer to occur.

42. Exponential Random Variable: compare results to the bounds from Chebyshev's inequality

$p(x) = \lambda e^{-\lambda x}$; $P(|X-E(X)| > k\sigma)$ for $k = 2, 3, 4 \ldots$

Chebyshev Inequality: $P(|X-\mu| > t) \leq \frac{\sigma^2}{t}$

$P(|X-E(X)| > k\sigma) \leq \frac{\sigma^2}{k\sigma^2} = \frac{1}{k}$ for $k = 2, 3, 4 \ldots$

43. Show $Var(X-Y)$

$Var(X-Y) = Var(X) + Var(Y) - 2Cov(X,Y)$

$= E([X-Y]^2) - E(X-Y)^2 = E(X^2) - 2E(X \cdot Y) + E(Y^2) - E(X)^2 \ldots E(Y)^2$

Covariance: $Cov(X,Y) = E[(X-\mu_X)(Y-\mu_X)]$ provided the expectation exists

44.

$\boxed{= Var(X) + Var(Y) - 2Cov(X,Y)}$

X&Y have equal variance. Find $Cov(X+Y, X-Y)$

$Cov(X+Y, X-Y) = E[(X+Y-E(X+Y))(X-Y-E(X-Y))] = E[(X-E(X)+Y-E(Y))(X-E(X)-Y+E(Y))]$

45. Find the covariance of N_j and N_i

$= E[(X-E(X))^2] - E[(X-E(X))(Y+E(Y))] + E[(Y-E(Y))(X-E(X))] - E[[Y-E(Y)]^2]$

$= E[(X-E(X))^2] - 2E[(X-E(X))(Y+E(Y))] - E[(X-E(Y))^2]$

where $N_1, N_2, \ldots N_r$ are multinomial random variables. Multinomial Random Variable:

$p(n_1, \ldots, n_r) = \binom{n}{n_1 \ldots n_r} p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$

$Cov(N_i, N_j) = E[(N_i - E(N_i))(N_j - E(N_j))]$

$= E(N_i N_j) - E(N_i)E(N_j)$

$= Pr(N_i = N_j) - Pr(N_i = 1) Pr(N_j = 1) = 0 - p_i p_j \ldots = \boxed{p_i p_k p_r}$

46. U & V; μ and σ^2. $Z = \alpha U + V\sqrt{1-\alpha^2}$. Find $E(Z)$ and ρ_{UZ}.

$E(Z) = E(\alpha U + V\sqrt{1-\alpha^2}) = E(\alpha U) + E(V\sqrt{1-\alpha^2}) = \alpha E(U) + \sqrt{1-\alpha^2} E(V) = \boxed{(\alpha + \sqrt{1-\alpha^2})\mu}$

Correlation coefficient: If X and Y are jointly distributed random variables and the variances and covariances of both X and Y are nonzero, then the correlation of X and Y, denoted by ρ,

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{E(UV) - E(U)E(V)}{\sqrt{Var(U)Var(V)}} = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u \cdot v\, e^{-\frac{(u-\mu)^2}{2\sigma^2} - \frac{(v-\mu)^2}{2\sigma^2}} du\,dv - (\frac{1}{2})^2}{\sqrt{\sigma^2 \cdot \sigma^2}}$$

$$= \frac{(\frac{1}{2})^2 - (\frac{1}{2})^2}{\sigma^2} = \phi$$

$= E[(X-E(X))(Z-E(Z))] = E[XZ] - E[X]E(Z) = E(X(Y-Z)) - E(X)E(Y-Z)$

$\boxed{= E(XY) - E(XZ) - E(X)[E(Y) - E(Z)] =}$

47. $Z = Y - X$; Find $Cov(X,Z) = Cov(X, Y-X) =$

$corr(X,Z) = \rho_{XZ} = \frac{Cov(X,Z)}{\sqrt{Var(X)Var(Z)}} = \boxed{\frac{E(XY) - E(XZ) - E(X)[E(Y) - E(Z)]}{\sqrt{Var(X)Var(Z)}}}$

48. $U = a + bX$; $V = c + dY$ show that $|\rho_{UV}| = |\rho_{XY}|$;

$$\rho = \frac{Cov(U,V)}{\sqrt{Var(U)Var(V)}} = \frac{Cov(a+bX, c+dY)}{\sqrt{Var(a+bX)Var(c+dY)}} = \frac{E[(a+bX)(c+dY)] - E(a+bX)E(c+dY)}{\sqrt{[E[(a+bX)^2] - E(a+bX)^2][E[(c+dY)^2] - E(c+dY)^2]}}$$

$$= \frac{E(ac) + E(adY) + E(bXc) + E(bXdY) - (E(a) + E(bX))(E(c) + E(dY))}{\sqrt{[E(a^2) + 2E(abX) + E(bX)^2 - [E(a) - E(bX)]^2][E(c^2) + 2E(cdY) + E(dY)^2 - [E(c) + E(dY)]^2]}}$$

$\boxed{= \emptyset}$

49. $E(X) = E(Y) = \mu$, but $\sigma_X \neq \sigma_Y$; $Z = \kappa X + (1-\kappa)Y$; where $0 \le \kappa \le 1$

a) Show $E(Z) = \mu$; $E(Z) = E(\kappa X + (1-\kappa)Y) = \kappa E(X) + (1-\kappa)E(Y) = \kappa\mu + (1-\kappa)\mu = \boxed{\mu}$

b) Find κ in terms of σ_X, σ_Y to minimize $Var(Z)$

$Var(Z) = Var(\kappa X + (1-\kappa)Y) = \kappa^2 E[X^2] - \kappa^2 E[X]^2 + (1-\kappa)^2 E[Y^2] + (1-\kappa)^2 E[Y]^2 = \kappa^2[Var(X)] + (1-\kappa)^2[Var(Y)]$

$\frac{d}{d\kappa} Var(Z) = 0 \div (\kappa^2 Var(X) - (1-\kappa)^2 Var(Y)) = (2\kappa\, Var(X) + 2(1-\kappa)(-1)Var(Y)) = 0$

$\boxed{\kappa = \frac{Var(Y)}{Var(X) + Var(Y)}}$ || $Var\left(\frac{X+Y}{2}\right) \le Var(X)$

$\frac{1}{4}[Var(X) + Var(Y)] \le Var(X)$; $\boxed{Var(Y) \le 3\,Var(X)}$

c) When is the average $(X+Y)/2$ better to use than X or Y alone?

V.. when the variance of the average is less than variance of X or Y alone.

50 X_i ; $i = 1 \ldots n$; $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$ $\bar{X} = n^{-1}\sum_{i=1}^{n} X_i$; show $E(\bar{X}) = \mu$ and $Var(\bar{X}) = \sigma^2/n$

$E(\bar{X}) = E(n^{-1}\sum_{i=1}^{n} X_i) = E\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right) = \mu$; $Var(\bar{X}) = E(\bar{X}^2) - E(\bar{X})^2 = \frac{1}{n}E(\sum X_i^2) - \frac{1}{n}E(\sum X_i)^2$

$= \sigma^2/n$

51. Example E: Section 4.3 ; $\mu_1 = \mu_2 = \mu$; $\sigma_{ij} = Cov(R_i, R_j) = 0$; Portfolio $(\pi, 1-\pi)$

Expected Return: $E(R(\pi)) = \pi\mu + (1-\pi)\mu = \mu$: Risk of Return: $Var(R(\pi)) = \pi^2\sigma_1^2 + (1-\pi)^2\sigma_2^2$

Minimizing Risk with respect to π : $\pi_{opt} = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$; $Var(R(\frac{1}{2})) = \frac{\sigma^2}{2}$

When considering unique returns: $E(R(\pi)) = \pi\mu_1 + (1-\pi)\mu_2$

$Var(R(\pi)) = \pi^2\sigma_1^2 + 2\pi(1-\pi)\rho\sigma_1\sigma_2 + (1-\pi)^2\sigma_2^2$

When considering n-total investments: $E(R(\pi)) = \sum \pi_i\mu_i$; $Var(R(\pi)) = \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i\pi_j\sigma$

Problem : n-securities (μ, σ)

uncorrelated. $E(R(\pi)) = \sum_{i=1}^{n} \pi_i\mu$; $Var(R(\pi)) = \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i\pi_j\sigma$; $\boxed{Var = \frac{1}{n}\sigma^2}$; $S.D = \frac{1}{\sqrt{n}}\sigma$

$\mu = n\pi_i\mu$; $\boxed{1 = n\pi_i \mid \frac{1}{n} = \pi_i}$

Risk of one security $= \boxed{\frac{\sigma}{1}}$

52. Two securities $(\mu_1 = 1, \sigma_1 = 0.1)$

$(\mu_2 = 0.8, \sigma_2 = 0.12)$; $\rho = -0.8$;

a) Return #1 $= \mu_1 \pm 0.1 = 0.1 \pm 0.1$; $\boxed{(\mu_1/\sigma_1) = 10}$ greater dollar per risk.

Return #2 $= \mu_2 \pm 0.12 = 0.8 \pm 0.12$ $(\mu_2/\sigma_2) = 6.75$

b) 50% into each stock

$E(R(\pi)) = 0.5 \cdot 1 + 0.8 \cdot 0.5 = 0.9$

$Var(R(\pi)) = 0.5^2(0.1)^2 + 2 \cdot 0.5(1-0.5)(-0.8)\,0.1 \cdot 0.12 + (1-0.5)^2(0.12)^2$

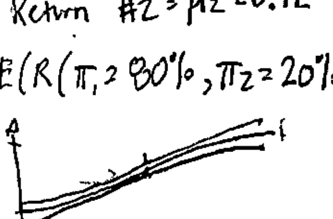
$\boxed{\sigma_\pi = 0.04}$

c) $E(R(\pi_1 = 80\%, \pi_2 = 20\%)) = 0.8 \cdot 1 + 0.2 \cdot 0.8 = \boxed{0.96}$; $Var(R(\pi_1 = 80\%, \pi_2 = 20\%))$

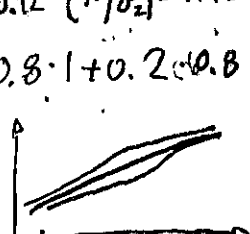
$= 0.8^2(0.1)^2 + 2 \cdot 0.8(0.2)(-0.8) \cdot 0.1 \cdot 0.12 + (0.2)^2(0.12)^2$

$\boxed{\pm 0.06}$

d)



d)



54. X, Y, Z with $\sigma_X^2, \sigma_Y^2, \sigma_Z^2$

Let $U = Z + X$; $V = Z + Y$

$Cov(U,V) = E(UV) - E(U)E(V) = E[(Z+X)(Z+Y)] - E[Z+X]E[Z+Y]$

$= E[Z^2] + E[XZ] + E[ZY] + E[XY] - E[Z^2] - E[Z] - E[XE] - E[XY]$

$= 0$

$Corr(U,V) = \rho_{UV} = \boxed{0}$

53. $Cov(X,Y) \le \sqrt{Var(X)Var(Y)}$;

$Cov(X,Y) = E(XY) - E(X)E(Y) \le \sqrt{[E(X^2) - E(X)^2][E(Y^2) - E(Y)^2]}$

$\le \sqrt{E(X^2)E(Y^2) - E(X^2)E(Y)^2 - E(X)^2E(Y^2) - E(X)^2E(Y)^2}$

55. $T=\sum_{k=1}^{n} kX_k$; X_k are independent random variables with μ, σ^2. Find $E(T)$ and $Var(T)$.

$E(T)=E\left(\sum_1^n kX_k\right)=\frac{n(n+1)}{2}\mu$; $Var(T)=Var\left(\sum_1^n kX_k\right)=E\left[\left(\sum_1^n kX_k\right)^2\right]-E\left[\sum kX_k\right]^2=\frac{n(n+1)(2n+1)\sigma^2}{6}$

56. $S=\sum_{k=1}^{n} X_k$; $Cov(S,T)=E(ST)-E(S)E(T)=E\left(\sum kX_k \sum X_k\right)^2-E\left(\sum kX_k\right)E\left(\sum X_k\right)$

$=\frac{n(n+1)(2n+1)}{6}\mu^2-\frac{n(n+1)}{2}\mu\cdot\mu=\left[\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}\right]\mu^2$

$Corr(S,T)=\rho_{ST}=\frac{Cov(S,T)}{\sqrt{Var(S)Var(T)}}=\frac{\left[\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}\right]\mu^2}{\sqrt{\sigma^2\cdot\frac{n(n+1)(2n+1)}{6}\sigma^2}}=\boxed{\frac{\left[\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}\right]\mu^2}{\sqrt{\frac{n(n+1)(2n+1)}{6}}\,\sigma^2}}$

57. $Var(XY)=E\left[(XY-E(XY))^2\right]$

$=E\left(X^2Y^2-2XYE(XY)+E(XY)^2\right)=E(X^2Y^2)-2E(X)E(Y)E(XY)+E(XY)^2$

$=E(X^2Y^2)-2E(XY)^2+E(XY)^2=E(X^2Y^2)-E(XY)^2$

$=E(X^2)E(Y^2)-\mu_x^2\mu_y^2=\left[Var(X)+E(X)^2\right]\left[Var(Y)+E(Y)^2\right]-\mu_x^2\mu_y^2$

$=Var(X)Var(Y)+Var(X)E(Y)^2+E(X)^2Var(Y)+E(X)^2E(Y)^2-\mu_x^2\mu_y^2$

$\boxed{=\sigma_x^2\sigma_y^2+\sigma_x^2\mu_y^2+\mu_x^2\sigma_y^2}+\mu_x^2\mu_y^2-\mu_x^2\mu_y^2$

58. $X_1=f(x)+\epsilon_1$; $X_2=f(x+h)+\epsilon_2$; $\epsilon_1 \& \epsilon_2(\mu=0,\sigma^2)$; $\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$; $Z=\frac{X_2-X_1}{h}$

a) Find $E(Z)=E\left(\frac{X_2-X_1}{h}\right)=E\left(\frac{f(x+h)+\epsilon_2-f(x)-\epsilon_1}{h}\right)=\frac{1}{h}\left[E(f(x+h))+E(\epsilon_2)-E(f(x))-E(\epsilon_1)\right]$

$=\frac{f(x+h)-f(x)}{h}$ | $=\frac{1}{h^2}\left[Var(f(x+h))+Var(\epsilon_2)+Var(f(x))-Var(\epsilon_1)\right]$ | Mean Squared Error

Find $Var(Z)=Var\left(\frac{f(x+h)+\epsilon_2-f(x)-\epsilon_1}{h}\right)=\frac{(-1)^2}{h^2}\sigma^2+\frac{\sigma^2}{h^2}=\boxed{\frac{2\sigma^2}{h^2}}$

$E\left[(X-X_0)^2\right]$ Mean Squared Error

$=Var\left[(X-X_0)\right]+E\left[(X-X_0)\right]^2$

$\boxed{=\sigma^2+\beta^2}$

In the limit of $E(Z)=\lim_{h\to0}E(Z)=f'(x)$; $\lim_{h\to0}Var(Z)=\boxed{\lim_{h\to0}\frac{2\sigma^2}{h^2}=0}$

b) Mean Squared Error of Z:

$MSE(Z)=E\left[(Z-Z_0)^2\right]=Var(Z-Z_0)+E\left[(Z-Z_0)\right]^2=\frac{2\sigma^2}{h^2}+\frac{f(x+h)-f(x)}{h}$

$\lim_{h\to0}MSE(Z)=f'(x)$ | $X_3=f(x+h+k)+\epsilon_3$; $E(\epsilon_1)=E(\epsilon_2)=E(\epsilon_3)=0$; $Var(\epsilon_1)=Var(\epsilon_2)=Var(\epsilon_3)=\sigma^2$

c) $X_1=f(x)+\epsilon_1$; $X_2=f(x+h)+\epsilon_2$; $Z_1=\frac{1}{h}\left[X_2-X_1\right]$; $Z_2=\frac{1}{h}\left[X_3-X_2\right]$; $Z_3=\frac{1}{h}\left[Z_2-Z_1\right]=\frac{1}{h}\left(\frac{X_3-X_2}{k}-\frac{X_2-X_1}{h}\right)$

$Z_3=\frac{1}{h^2}X_1-\left(\frac{1}{hk}+\frac{1}{h^2}\right)X_2+\frac{1}{hk}X_3$; $\boxed{E(Z_3)=\frac{1}{h^2}f(x)-\left(\frac{1}{hk}+\frac{1}{h^2}\right)f(x+h)+\frac{1}{hk}f(x+h+k)}$

$\boxed{Var(Z_3)=2\sigma^2\left(\frac{1}{h^4}+\frac{1}{h^2k^2}+\frac{1}{h^3k}\right)}$

61. $f(x,y)=2$, $0\le x\le y$; a) $Cov(X,Y)$, $Corr(X,Y)$

59. (X,Y) is a random point on a disk.

Show the $Cov(X,Y)=0$: $E(XY)-E(X)E(Y)$

$=\frac{4}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x\sqrt{1-x^2}\,y\sqrt{1-y^2}\,dx\,dy-\frac{2}{\pi}\int_{-\infty}^{\infty}x\sqrt{1-x^2}\int_{-\infty}^{\infty}y\sqrt{1-y^2}=0$

$f(x,y)=\begin{cases}\frac{1}{\pi r} & x^2+y^2\le r\\ 0 & \text{otherwise}\end{cases}$

60. Y is symmetric about zero. $X=SY$

$S=\pm1$; $P(S=1)=P(S=-1)=\frac{1}{2}$; Show $Cov(X,Y)=0$

$Cov(X,Y)=E(XY)-E(X)E(Y)=E(SY^2)-E(SY)E(Y)=\boxed{SE(Y^2)-SE(Y^2)=0}$

$E(X)=E(SY)=SE(Y)$; $\frac{E(X)}{E(Y)}=S$

$Cov(X,Y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_x)(y-\mu_y)f(x,y)\,dx\,dy$

$=2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_x)(y-\mu_y)\,dx\,dy$

$=2\int_{-\infty}^{\infty}\left(\frac{x^2}{2}\right)(y-\mu_y)\,dy=\infty$

X & Y from $[0,1]$; $f(x,y)=2$; $0\le x\le y\le 1$

$E(x)=\int_0^1 x f(x)\,dx=\int_0^1 x\left[\int_x^1 f(x,y)\,dy\right]dx=\int_0^1 x\left[\int_x^1 2\,dy\right]dx$

$=\int_0^1 x\,[2-2x]\,dx=2\int_0^1 (x-x^2)\,dx=2\left[\frac{y^2}{2}-\frac{y^3}{3}\right]=\boxed{\frac{1}{3}}$

$E(y)=\int_0^1 y f(y)\,dy=\int_0^1 y\left[\int_0^y f(x,y)\,dx\right]dy=\int_0^1 y\,[2y]\,dy=\int_0^1 2y^2\,dy$

$\boxed{=\frac{2}{3}}$

$E(xy)=\int_0^1\int_0^y xy\,f(x,y)\,dx\,dy=2\int_0^1 \frac{y^3}{2}\,dy=\frac{y^4}{4}\Big|_0^1=\boxed{\frac{1}{4}}$

$x^2[2-2x]$

$2[x^2-x^3]$

$2\left[\frac{x^3}{3}-\frac{x^4}{4}\right]$

$\frac{2}{3}-\frac{2}{4}=\frac{8}{12}-\frac{6}{12}=2$

a) $\mathrm{Cov}(X,Y)=E(XY)-E(X)E(Y)$

$=\frac{1}{4}-\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)=\frac{1}{4}-\frac{2}{9}$

$=\frac{9}{36}-\frac{8}{36}=\boxed{\frac{1}{36}}$

$\mathrm{Corr}(X,Y)=\dfrac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$

$=\dfrac{(1/36)}{\sqrt{(E[X^2]-E[X]^2)(E[Y^2]-E[Y]^2)}}=\dfrac{1/36}{\sqrt{(2/12-1/9)(\frac{1}{2}-4/9)}}=\dfrac{1/36}{1/18}=\boxed{\frac{1}{2}}$

b. Find $E(X|Y=y)$ and $E(Y|X=x)$

Conditional Expectations:

if $p_{Y|X}(y|x)$, then $E(Y|X=x)=\sum_y y\,p_{Y|X}(y|x)$

$=\int y\,f_{Y|X}(y|x)\,dy$

More Generally,

$E[h(Y)|X=x]=\int h(y)f_{Y|X}(y|x)\,dy$

$E(Y|X=x)=\sum_y y\,\frac{f(x,y)}{f(x)}\Rightarrow\int_x^1 y\left(\frac{2}{2-2x}\right)dy\quad 2\,\frac{y^2}{4}$

$=\int_x^1 y\cdot 2\left(\frac{1}{1-x}\right)dy=\frac{1}{2}[1-x^2]\left(\frac{1}{1-x}\right)\ \boxed{\frac{(x+1)}{2}}$

$E(X|Y=y)=\sum_x x\cdot p_{X|Y}(x|y)\Rightarrow\int_0^y x\cdot\frac{f(x,y)}{f(y)}\,dy=\int_0^y x\,\frac{2}{2y}\,dx$

$=\frac{y^2}{2y}=\boxed{\frac{y}{2}}$

c) Find $E(X|Y)$ and $E(Y|X)$

$=y/2\qquad=(x+1)/2$

$f_{W_1}(W_1)=2f_y(2W_1)\qquad f_{W_2}(W_2)=2(2W_2-1)$

$=2(2(2W_1))\qquad\boxed{=8(1-W_2)}$

$\boxed{=8W_1}$

d) $\hat{Y}=a+bX$; $\min(E((Y-\hat{Y})^2)$ $\boxed{\text{Predictor}}$

$E(\hat{Y})=a+bE(\hat{X})$; $\mu_y=a+b\mu_x$

$a=\mu_y-b\mu_x=\frac{2}{3}-\rho_{xy}\left(\frac{1}{3}\right)=\frac{2}{3}-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)=\frac{2}{3}-\frac{1}{6}$

$=\frac{12}{18}-\frac{3}{18}=\boxed{\frac{9}{18}}$

Mean Squared Error : $E\left(Y-\frac{1}{2}-\frac{1}{2}X\right)^2=\sigma_y^2(1-\rho^2)$

$=\frac{1}{18}\left(1-\frac{1}{4}\right)=\boxed{1/24}$

e) Predictor of Y in terms of X

MSE = $\boxed{\text{Mean Squared Error}}$ $=E(Y-\hat{Y})^2=E(Y-E(Y|X))^2$

$=E(Y^2)-E(E^2(Y|X))=\frac{1}{2}-E\left(\frac{(X+1)^2}{2}\right)=\frac{1}{2}-\int_0^1\frac{(x+1)^2}{4}(1-x)\,dx$

$\boxed{=1/24}$

$\boxed{\text{Mean Squared Predictor Error}}$

$E(Y|X=x)=\frac{1}{2}-E\left(\frac{(X+1)^2}{4}\right)$

$=\frac{1}{2}-\int_0^1\frac{(x+1)^2}{4}(x-1)\,dx$

$=\boxed{1/24}$

62.

X & Y joint random variables with correlation ρ_{xy}, Define the Standardized random variables $\tilde{X}$ and $\tilde{Y}$

$\tilde{X}=(X-E(X))/\sqrt{\mathrm{Var}(X)}$

$\tilde{Y}=(Y-E(Y))/\sqrt{\mathrm{Var}(Y)}$

Show that $\mathrm{Cov}(\tilde{X},\tilde{Y})=\rho_{xy}$

$\mathrm{Cov}(\tilde{X},\tilde{Y})=E((\tilde{X}-\bar{X})(\tilde{Y}-\bar{Y}))$

$=E(\tilde{X}\tilde{Y})-E(\bar{X}\tilde{Y})-E(\bar{Y}\tilde{X})+E(\bar{X})E(\bar{Y})$

$=E(\tilde{X}\tilde{Y})-E(\tilde{X})E(\tilde{Y})$

$=E\left(\frac{[X-E(X)]}{\sqrt{\mathrm{Var}(X)}}\frac{[Y-E(Y)]}{\sqrt{\mathrm{Var}(Y)}}\right)-\frac{[E(X)-\cancel{E(E(X))}]}{\sqrt{\mathrm{Var}(X)}}\frac{[E(Y)-\cancel{E(E(Y))}]}{\sqrt{\mathrm{Var}(Y)}}=\boxed{\rho_{XY}}$

63.

$f(x,y)=\frac{6}{7}(x+y)^2$

$0\le x\le 1$; $0\le y\le 1$

a) $\mathrm{Cov}(X,Y)=E[(X-\mu_x)(Y-\mu_y)]=E[XY]-E[X]E[Y]$

$=\int_0^1\int_0^1 xy\,f(x,y)\,dx\,dy-\int_0^1 x f(x)\,dx\int_0^1 y f(y)\,dy$

$=\int_0^1\int_0^1 xy\,\frac{6}{7}(x+y)^2\,dx\,dy-\int_0^1 x\left[\int_0^1\frac{6}{7}(x+y)^2\,dy\right]dx\int_0^1 y\left[\int_0^1\frac{6}{7}(x+y)^2\,dx\right]dy$

$$= \int_0^1\int_0^1 xy\,\frac{6}{7}(x^2+2xy+y^2)\,dx\,dy - \int_0^1 x\,\frac{6}{7}\left(x^2+x+\frac{1}{3}\right)dx \int_0^1 y\,\frac{6}{7}\left(\frac{1}{3}+y+y^2\right)dy$$

$$= \int_0^1 y\,\frac{6}{7}\left(\frac{1}{3}+y+y^2\right)dy - \frac{6}{7}\left[\frac{1}{4}+\frac{1}{3}+\frac{1}{6}\right]\frac{6}{7}\left[\frac{1}{6}+\frac{1}{3}+\frac{1}{4}\right] = \frac{6}{7}\left[\frac{1}{6}+\frac{1}{3}+\frac{1}{4}\right] - \frac{36}{49}\left(\frac{3}{4}\right) \doteq \boxed{-0.0085}$$

$\doteq 0.0085$

$$\mathrm{Corr}(X,Y) = \rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = \frac{-19/34}{\sqrt{(E[X^2]-E[X]^2)(E[Y^2]-E[Y]^2)}} = \frac{}{\sqrt{\left(\int_0^1 \frac{6}{7}\left[x^4+x^3+\frac{x^2}{3}\right]dx - \frac{81}{196}\right)\left(\int \frac{6}{7}\left[\frac{y^2}{3}+y^3+y^4\right]dy - \frac{81}{196}\right)}}$$

$$= \frac{-19/34}{\sqrt{\left(\frac{6}{7}\left[\frac{1}{5}+\frac{1}{4}+\frac{1}{9}\right]\right)^2}} = \boxed{0.12516}$$; Miscalculated

b.

Find $E(Y|X=x)$ for $0 \le x \le 1$

Conditional Expectation:

$$E(Y|X=x) = \sum x\, p_{Y|X}(Y|X)$$

$$= \int_0^1 y\,\frac{f(x,y)}{f(x)}\,dy$$

$$= \int_0^1 \frac{y}{(x^2+x+\frac{1}{3})}(x^2+2xy+y^2)\,dy$$

$$\boxed{\doteq \frac{6x^2+8x+3}{4(3x^2+3x+1)}}$$

64.

y \ x	1	2	3	4	$f(y)$
1	0.10	0.05	0.02	0.02	0.19
2	0.05	0.20	[illegible]	0.02	0.32
3	0.02	0.05	0.20	0.04	0.31
4	0.02	0.02	0.04	0.10	0.18
$f(x)$	0.19	0.32	0.31	0.18	

a. $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$

$= \sum xy\, f(x,y) - \sum x f(x) \sum y f(y)$

$= 1^2 \times 0.1 + 2\cdot 1 \times 0.05 + 3 \times 1 \times 0.02 + 4 \times 1 \times 0.02$
$+ 2\cdot 1 \times 0.05 + 4^2 \cdot 0.2 + 6 \times 0.05 + 8 \times 0.02$
$+ 3\cdot 1 \times 0.02 + 6 \times 0.05 + 5^2 \cdot 0.20 + 12 \times 0.04$
$+ 4\cdot 1 + 0.02 + 8 \times 0.02 + 3\cdot 4 \times 0.04 + 4^2 \times 0.10$
$= 6.66$

$E(X) = 1\cdot 0.19 + 2\cdot 0.32 + 3 \times 0.31 + 4 \times 0.18$
$= 2.48 = E(Y)$

$E(X^2) = 1^2 \times 0.19 + 2^2 \cdot 0.32 + 3 \times 0.31 + 4 \times 0.18$
$\doteq 7.14 \doteq E(Y^2)$

$\mathrm{Var}(X) = E(X^2) - E(X)^2$
$= 7.14 - 2.48^2 = 0.9896$

$\mathrm{Var}(Y) = 0.9896$

$\boxed{\mathrm{Cov} = 0.5096 \text{ ; } \mathrm{Corr} = 0.514455}$

b)

Find $E(Y|X=x)$ for $x=1,2,3,4$;

$$E(Y|X=1) = \sum y f(y|x) = \sum y\,\frac{f(x,y)}{f(x)} = \frac{1 \times 0.1}{0.19} + \frac{2 \times 0.05}{0.19} + \frac{3 \times 0.02}{0.19} + \frac{4 \cdot 0.02}{0.19} \;\boxed{= 1.78}$$

$$E(Y|X=2) = \sum y f(y|2) = \sum y\,\frac{f(x,y)}{f(y)} = \frac{1 \cdot 0.05 + 2 \times 0.2 + 3 \cdot 0.05 + 4 \cdot 0.02}{0.32}$$

$E(Y|x) = \{1.78, 2.13, 2.87, 3.22\}$

$$E(Y|X=3) = \sum y f(y|3) = \frac{1 \cdot 0.02 + 2 \cdot 0.05 + 3 \cdot 0.2 + 4 \cdot 0.04}{0.31} \;\boxed{= 2.13}$$

$$E(Y|X=4) = \frac{1 \times 0.02 + 2 \cdot 0.02 + 3 \times 0.04 + 4 \cdot 0.1}{0.18} \; 3.22 \quad \boxed{= 2.87}$$

65. Random Sums:

$$T = \sum_{i=1}^{N} X_i$$

$E(T) = E[E(T|N)] = E[N E(X)] = E(N)E(X) \leftarrow$ Independence

66.

Fast: 1 min, Slow: 3 min ; $E(T) = \sum E(T|E_i)p(E_i) = 1\,\mathrm{min}\left(\frac{2}{3}\right) + 3\,\mathrm{min}\left(\frac{1}{3}\right) \; \boxed{\frac{5}{3}\,\mathrm{min}}$

$p(F) = 2/3$ $\quad p(S) = 1/3$

$E(XH) = E[E(XH|X)] = E[XE(H|X)] = E(X)E(H|X)$
$= \frac{1}{2}E(X^2)$
$= \frac{1}{2}\int_0^1 x^2\,dx = \boxed{\frac{1}{6}}$

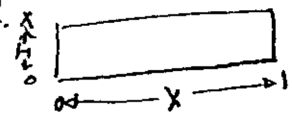
$E(2(X+H)) = E[E(2(X+H)|X)]$
$= E[2X + 2E(H|X)]$

Note:

$= E[2X] + E[2E(H|X)] = 2\left(\frac{a+b}{2}\right) + \left(\frac{a+b}{2}\right) = 1 + \frac{1}{2} = \boxed{\frac{3}{2}}$

$$E(Z) = \int_a^b z f(z)\,dz = \frac{1}{b-a}\int_a^b z\,dz = \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2}$$

67.



$E_{Area} = \sum E(Y|X=x)p_X(X)$

$= \sum_y \sum_x y\, p_{Y|X}(y|x)\, p_X(x)$

68. Show $E[\mathrm{Var}(Y|X)] \le \mathrm{Var}(Y)$

$E[E(Y^2|X)] - E[E(Y|X)^2] \le E[Y^2] - E[Y]^2$

69. $\mu_x = \mu_y = 0$: Sketch $E(Y|X=x)$ and $E(X|Y=y)$ for $\rho = 0, 0.5, 0.9$

$\sigma_x = \sigma_y = 1$;

$E(Y|X) = \mu_y + \rho\frac{\sigma_y}{\sigma_x}(X - \mu_x)$

@$\rho=0$ $= 0 + 0 = 0$

@$\rho=0.5$ $= 0 + \frac{1}{2}\left(\frac{1}{1}\right)(X-\mu_x) = \boxed{(X-\mu_x)/2} = X/2$

@$\rho=0.9$ $= 0 + \boxed{(X-\mu)/9} = X/9$

$E(X|Y) = \mu_x + \rho\frac{\sigma_x}{\sigma_y}(Y-\mu_y)$

@$\rho=0$ $E(X|Y) = 0$

@$\rho=0.5$ $E(X|Y) = Y/2$

@$\rho=0.9$ $E(X|Y) = Y/9$

70. Show $E(X|Y=y) = E(X)$; $E(X|Y=y) = \sum x P_{X|Y}(x|y) = \sum x \frac{f(x,y)}{P(y)} = \sum \frac{x}{P(y)} f(x)f(y) = \sum x f(x) = \boxed{E[X]}$

71. Binomial Random Variable: n-trials, Y-success for m-trials; $m<n$

$p(1) = p$; $p(0) = 1-p$; $p(x) = 0$

$p(x) = \begin{cases} p^x(1-p)^{1-x}, & \text{if } x=0 \text{ or } x=1 \\ 0 & \text{otherwise} \end{cases}$

Find the conditional frequency function of Y given $X=x$, and conditional mean.

$p(x) = \binom{n}{x} p^x q^{n-x}$; $p(y) = \binom{m}{y} p^y q^{m-y}$

$P(Y|X=x) = \frac{\binom{n}{x} p^x q^{n-x} \binom{m}{y} p^y q^{n-y}}{\binom{n}{x} p^x q^{n-x}} p(x) = \frac{\binom{n-m}{x-y} p^{x-y} q^{n-m-x+y}}{\binom{n}{x} p^x q^{n-x}} p(Y)$

Conditional Mean:

$\frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}} = \frac{nK}{N}$ Hypergeometric Fcn

$\boxed{P(Y|X=x) = \frac{\binom{m}{y}\binom{n-m}{x-y}}{\binom{n}{x}}}$

$= m\frac{x}{n} = E[Y|X=x]$

Distribution of a Hypergeometric Fcn

73. n-tosses, N-heads, $+N$ more tosses.

Binomial Random Variable

First N-tosses:

$E[\text{Heads}] = E[\text{Heads}|n] + E[\text{Heads}|E[\text{Heads}|n]]$

$= np + np = \boxed{np(1+p)}$

72.

μ, σ^2

$E[X] = \mu$; $E[\mu] = \mu^2$; $E[\mu^2] = \mu^3$

$Var(X) = \sigma^2$; $Var(\sigma^2) = \sigma^4$; $Var(\sigma^4) = \sigma^6$

$\frac{\binom{x}{k}\binom{n-x}{x-k}}{\binom{n}{m}}$

74. $\leftarrow n \rightarrow$, $\leftarrow p(n) \rightarrow$

$E[X] = \sum_{x=1}^{n} x\, p(x) = \boxed{\frac{n(n+1)}{2} p(x)}$

75. $p(T) = \lambda e^{-\lambda T}$; $E[V]\,E[T] = E[V]\,E[T] = \frac{1}{2} \boxed{\frac{1}{2}\cdot\frac{1}{\lambda}}$; $Var(V \cdot Var(T)) = Var(V)\cdot Var(T) = \boxed{\frac{1}{12}\left(\frac{1}{\lambda^2}\right)}$

76. (X,Y): $x^2+y^2 \le 1$; $y \ge 0$.

$f_{XY}(x,y) = \frac{|A|}{|R|} = \frac{1}{\pi/2} = \frac{2}{\pi}$

$f_X(x) = \int_0^{\sqrt{1-x^2}} f_{XY}(x,y)\,dy = \frac{2}{\pi}\sqrt{1-x^2}$

$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx = \frac{4}{\pi}\sqrt{1-y^2}$

$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{2/\pi}{\frac{2}{\pi}\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$

$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{1}{2\sqrt{1-x^2}}$

$E_{Y|X}(y|x) = \int_0^{\sqrt{1-x^2}} y f_{Y|X}(y|x)\,dy = \boxed{\frac{\sqrt{1-x^2}}{2}}$

$E_{X|Y}(x|y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x f(x|y)\,dx$

$= \emptyset$

77. $f(x,y) = e^{-y}$; $0 \le x \le y$ a) Find $Cov(X,Y) = E[XY] - E[X]E[Y] = \iint xy f(x,y)\,dx\,dy - \int x\left[\int f(x,y)\,dy\right]dx \int y\left[\int f(x,y)\,dx\right]$

$= \int_0^\infty \left[\frac{x^2}{2}\right]_0^y y e^{-y}\,dy - \int_0^y x\left[\int_0^x e^{-y}\,dy\right] \int_0^\infty y\left[\int_0^y e^{-y}\,dx\right]dy$

$\boxed{\int_0^\infty \frac{y^3}{2} e^{-y}\,dy + \int_0^y x e^{-x}\,dx \int y^2 e^{-y}\,dy = 3}$

b) $E(X|Y=y) = \int x f_{X|Y}(x|y)\,dx = \int_0^y x \frac{f(x,y)}{f(y)}\,dx$

$= \int_0^y x \frac{e^{-y}}{y e^{-y}}\,dx = \boxed{\frac{y}{2}}$

$E(Y|X=x) = \int y f_{Y|X}(y|x) = \int_x^\infty y \frac{e^{-y}}{1}\,dy = \boxed{x+1}$

$uv - \int v\,du$

$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\cdot Var(Y)}} = \frac{3}{\sqrt{(2-1)(6-2)}}$

$= 1/\sqrt{2}$

c) Density Function $E(X|Y)$ and $E(Y|X)$

$E[X] = E[E[X|Y]] = E[Y/2] = E[Y]/2$

$E[Y] = E[E[Y|X]] = E[X+1] = E[X]+1$

Skewness - third central moment - asymmetry of a density or frequency about a mean.

78. $M(t) = \sum e^{tx} p(x) = e^{tx}/t$ 1st Moment

$M'(t) = E[X] = e^{tx} x$

$M''(t) = E[X^2] = t e^{tx}$ } 2nd Moment

$M'''(t) = e^{tx} + t^2 e^{tx}$
$= (1+t^2)e^{tx}$ } 3rd Moment

$\emptyset = \lim_{t\to\infty} M'''(t) - \lim_{t\to\infty} M''(t)$

79. $p(0)=\frac{1}{2}$; $p(1)=\frac{3}{8}$; $p(2)=\frac{1}{8}$; Find $M(t)$; $M(t)=\sum_x e^{tx}p(x)=e^{tx}\left[\frac{1}{2}+\frac{3}{8}+\frac{1}{8}\right]=e^{tx}$

80. $f(x)=2x$; $0\le x\le 1$

$M(t)=\int_0^1 e^{tx}\cdot 2x\,dx=\frac{2e^{tx}(tx-1)}{t^2}\Big|_0^1=\frac{2e^t(x-1)}{t^2}+\frac{2}{t^2}$

$=\frac{2e^t(x-1)}{t^2}-\frac{2e^0(0-1)}{t^2}$

$M'(0)=xe^{0x}=E[x]$; $M''(0)=x^2e^{0\cdot x}=E(x^2)$; $M''(0)=0$ $\frac{1}{2}$

$M'(0)=\frac{d}{dt}\left[\left[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\dots\right]\left[\frac{2}{t}-\frac{2}{t^2}\right]+\frac{2}{t^2}\right]$

$=2(1)\left(\frac{1}{2!}-\frac{1}{3!}\right)+2(2\times 0)\left(\frac{1}{3!}-\frac{1}{4!}\right)+2(3\times 0)\left(\frac{1}{4!}-\frac{1}{5!}\right)+\dots$

$\boxed{=\frac{2}{3}}$

81. Bernoulli Random Variable.

$p(1)=p$
$p(0)=1-p$
$p(x)=0$

$p(x)=\begin{cases}p^x(1-p)^{1-x}\\0\end{cases}$

$M(t)=\sum_0^1 e^{tx}p(x)=1-p+e^t p$

$M'(0)=p=E(x)$ $\boxed{M''(0)=p}$

$M''(0)=p=E(x^2)$

$\boxed{Var(x)=M'(0)-M'(0)^2=p-p^2}$

82. MGF of a Binomial is $\sum$ Bernoulli

Binomial Random Variable:

$M(t)=\prod_{i=1}^{n}M_{x_i}(t)=(1-p+e^tp)^n$

$M'(0)=n(1-p+p)^{n-1}p=n\cdot p=E[x]$

$M''(0)=n(n-1)(1-p+p)^{n-2}p^2+n(1-p+p)^{n-1}\cdot p=n(n-1)p^2+np=(n^2-n)p^2+np=n^2p^2-np^2+np$

$Var(x)=E[x^2]-E[x]^2=\cancel{n^2p^2}-np^2+np-\cancel{n^2p^2}=\boxed{np(1-p)}$

83. Binomial Distribution:

$p(x)=\binom{n}{x}p^x(1-p)^{n-x}$; $P(X)=\sum_{i=1}^{n}X_i=X_1+X_2+\dots+X_n=\binom{n}{1}p^1(1-p)^{n-1}+\binom{n}{2}p^2(1-p)^{n-2}+\dots+\binom{n}{n}p^n(1-p)^{n-n}=\boxed{\sum_{n=1}^{n}\binom{n}{k}p^k(1-p)^{n-k}}$

84. $=\binom{n}{1}p_1(1-p_1)^{n-1}+\binom{n}{2}p_2(1-p_2)^{n-2}+\dots+\binom{n}{n}p_n(1-p_n)^{n-n}\neq$ Binomial Distribution

$p(k)=p(1-p)^{k-1}$, $k=1,\dots$

85. Moment Generating Function: Geometric Random Variable

Binomial Relationship: $e^t(1-p)^{k-1}-\frac{1}{1-qe^t}$;

$M(t)=\sum_{k=1}^{\infty}e^{tk}\cdot p(1-p)^{k-1}=\frac{e^{tk}p}{1-(1-p)e^{tk}}$

$M'(t)=\frac{ke^{tk}p[1-(1-p)e^{tk}]-e^{tk}p\,k(1-p)e^{tk}}{[1-(1-p)e^{tk}]^2}=\frac{ke^{tk}p-ke^{tk}p(1-p)e^{tk}+e^{tk}pk(1-p)e^{tk}}{[1-(1-p)e^{tk}]^2}=\frac{ke^{tk}p[1+2(1-p)e^{tk}]}{[1-(1-p)e^{tk}]^2}$

$=\frac{pe^t}{(1-qe^t)^2}$

$M'(0)=\frac{1-q}{p^2}=\frac{1}{p}$; $M''(t)=\frac{pe^t(1-qe^t)^2-pe^t(2(1-qe^t)\cdot(-q)e^t)}{(1-qe^t)^4}=\frac{pe^t(1-qe^t)+2pe^t(1-qe^t)qe^t}{(1-qe^t)^4}$

$M''(0)=\frac{p(1-q)+2p(1-q)q}{p^4}=\frac{p-pq+2p-2pq^2}{p^4}=\frac{3-q-2q^2}{p^3}=\frac{3-(1-p)-2(1-p)^2}{p^3}$

$=\frac{2+p-2(1+2p+p^2)}{p^3}=\frac{2+p-2-4p-2p^2}{p^2}$

$Var(x)=E[x^2]-E[x]^2$

$=\frac{1+q}{p^2}-\frac{1}{p^2}=\boxed{\frac{1+p}{p^2}}$ $\boxed{=\frac{1+q}{p^2}}$

86. MGF of a Negative Binomial:

$p(X=k)=\binom{k-1}{r-1}p^r(1-p)^{k-r}$

$M(t)=\prod_{i=1}^{n}M_x(t)=p^n(1-qe^t)^{-n}$

$M'(t)=-np^n(1-qe^t)^{-(n-1)}(-qe^t)$; $M'(0)=+np^n(1-q)^{-(n-1)}q$

$=np^n\cdot p^{-(n-1)}\cdot q=np^{n-n+1}q=\frac{nq}{p}$

$M''(t)=\frac{d}{dt}\left[\frac{+np^nqe^t}{(1-qe^t)^{n-1}}\right]$

$=\frac{nq}{p}+\frac{n^2q^2}{p^2}+\frac{nq^2}{p^2}=\boxed{\frac{nq}{p^2}+\frac{n^2q^2}{p^2}}$

$Var(x)=E(x^2)-E[x]^2=\boxed{\frac{nq}{p^2}}$

87. When is the sum of independent random variables of a binomial also a negative binomial?

Binomial Distribution: $p(k) = \binom{n}{k} p^k (1-p)^{n-k}$

Negative Binomial Distribution: $p(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$

When $k-1 = n$, and $r-1 = k$, when the total trials is the n and successes ... $k!$

88. $X \sim N(0, \sigma^2)$; Prove odd moments of $M_X t$ are zero and even moments are $\mu_{2n} = \boxed{\frac{(2n)!\sigma^{2n}}{2^n (n!)}}$

$\frac{-x^2}{2\sigma^2} + tx = \frac{-1}{2\sigma^2}(x^2 - 2\sigma^2 t x)$

$= \frac{-1}{2\sigma}\left(x^2 - 2\sigma^2 t x + \left(\frac{2\sigma^2 t}{2}\right)^2\right) - (\sigma^2 t)^2$

$= \frac{-1}{2\sigma^2}(x - \sigma^2 t)^2 - (\sigma^2 t)^2$

Even Moments: $M(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tX} N(0, \sigma^2)$

$= \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{tx} e^{-x^2/2\sigma^2} dx = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2} + tx} dx = \frac{e^{(\sigma^2 t)^2}}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma}(x-\sigma^2 t)^2} dx$; $u = x - \sigma t$, $\frac{du}{dt} = -\sigma$

$= \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma}(x - \sigma^2 t)^2 - (\sigma^2 t)^2} dx = \frac{e^{-(\sigma^2 t)^2}}{\sqrt{2\pi\sigma}(\pm\sigma)} \int_{-\infty}^{\infty} e^{-u^2/2\sigma^2} du = \frac{e^{+\frac{(\sigma^2 t)^2}{2}}}{\sqrt{2\pi\sigma}} \sqrt{2\pi\sigma} \cdot \frac{t!}{0} = \frac{t\, e^{+\frac{1}{2}\sigma^2 t^2}}{-\sigma^2}$

$\mu_{2n} = \frac{(2\cdot 0)!\sigma^{2\cdot 0}}{2^0 (0!)} = 1$

$M'(t) = +\sigma^2 t e^{-\sigma^2 t^2/2}$ $\quad \frac{2!\sigma^{2\cdot 1}}{2^1(1)} \quad \sigma^2$

$M''(t) = +\sigma^2 e^{-\sigma^2 t^2/2} + \sigma^2 t(-\sigma^2 t) e^{-\sigma^2 t^2/2} = -\sigma^2 e^{-\sigma^2 t^2/2}\left[1 - \sigma^2 t^2 e^{-\sigma^2 t^2/2}\right] = \boxed{+\sigma^2}$ @ $t = 0$

$\frac{(2\cdot 1)!\sigma^{2n}}{2^1(1!)}$ $\boxed{= \sigma^2}$

89. $X_1, X_2, \ldots X_n = N(\mu_i, \sigma_i^2)$; Prove $Y = \sum_{i=1}^{n} \alpha_i X_i$ where α_i = scalar. Find μ_Y, σ_Y^2

$Y = \sum_{i=1}^{n} \alpha_i X_i = \sum_{i=1}^{n} \alpha_i N(\mu_i, \sigma_i^2)$ = MGF of Y: $M(t) = \sum_{i=1}^{n} e^{tx} \cdot \sum \alpha_i X_i$

$= \sum_{X=1}^{n} e^{tX}\left(\sum \alpha_i N(\mu_i, \sigma_i^2)\right)$

$= \sum_{X=1}^{n} \sum_{i=1}^{n} \alpha_i e^{\cancel{\frac{-1}{2\sigma^2}(X_i - (\mu + \sigma^2 t)^2)^2}} e^{-\frac{(2\mu + \sigma^2 t)}{2}}$

$= e^{-\frac{(2\mu + \sigma^2 t^2)}{2}} \sum_{X=1}^{n} \sum_{i=1}^{n} \alpha_i e^{\cancel{\frac{-1}{2\sigma^2}(X_i - (\mu_i + \sigma_i^2 t))^2}}$

$= \sum_{i=n}^{n} \alpha e^{\cancel{-(2\mu + \sigma^2 t)/2}} \sum_{X=1}^{n} e^{\cancel{-\frac{1}{2\sigma^2}(X_i - (\mu_i + \sigma_i^2 t))^2}}$

$= M_Y(t) = \sum_{X=1}^{n} e^{tX} e^{-y^2/2} = e^{t^2/2}$

$= \sum_{X=1}^{n} e^{tX} \sum_{i=1}^{n} \alpha_i N(\mu_i, \sigma_i^2) = e^{t^2/2} \sum_{i=1}^{N} \alpha_i N(\sigma_i^2 t, \sigma_i^2)$

$\boxed{= e^{\ldots t^2/2}}$

Completing the Square:

$tx - \frac{(x-\mu)^2}{2\sigma^2} = tx - \frac{x^2 + 2\mu x + \mu^2}{2\sigma^2}$

$\frac{-x^2 + (2\mu + 2\sigma^2 t)x}{2\sigma^2} + \frac{\mu^2}{2\sigma^2}$

$\frac{-1}{2\sigma^2}\left(x + (\mu + \sigma^2 t)^2\right)^2 - \frac{(\mu + \sigma^2 t)^2}{2\sigma^2} + \frac{\mu^2}{2\sigma^2}$

$\frac{-1}{2\sigma^2}\left(x - (\mu + \sigma^2 t)\right)^2 - \frac{(2\mu + \sigma^2 t)t}{2} \cancel{+ 2\mu\sigma^2 t + (\sigma^2 t)^2} + \mu^2$

90. $Z = \alpha X + \beta Y$; $M_Z(t) = E(e^{tZ})$

$= E\left(e^{t\alpha X + t\beta Y}\right) = E\left(e^{t\alpha X} \cdot e^{t\beta Y}\right)$

$= E\left(e^{t\alpha X}\right) E\left(e^{t\beta Y}\right)$

$\boxed{= M_X(t) M_Y(t)}$

91. Exponential Distribution:

$p(X) = \lambda e^{-\lambda x}$

$\cancel{M_X(t) = \lambda e^{(t-\lambda)x}}$

$M_{cX}(t) = \lambda e^{(t - c\lambda) X}$

92. Gamma Distribution: ; Poisson Distribution ; With Example:

$g(t) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t}$ $\quad p(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$ $\quad f(\theta) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\lambda\theta}$; $X|\theta = \frac{\theta^x}{x!} e^{-\theta}$

$$P(X=x) = \int_0^{\infty} P(X|\theta) f(\theta)\, d\theta = \int_0^{\infty} e^{-\theta} \frac{\theta^x}{x!} \frac{\lambda^{\alpha}\theta^{\alpha-1} e^{-\lambda\theta}}{\Gamma(\alpha)}\, d\theta = \int_0^{\infty} \frac{\lambda^{\alpha}\theta^{x+\alpha-1} e^{-(\lambda+1)\theta}}{x!(\alpha-1)!}\, d\theta$$

$$= \frac{(x+\alpha-1)!}{x!(\alpha-1)!} \frac{\lambda^{\alpha}}{(\lambda+1)^{x+\alpha}} \int_0^{\infty} \frac{(\lambda+1)^{x+\alpha}\theta^{x+\alpha-1} e^{-(\lambda+1)\theta}}{\Gamma(x+\theta)}\, d\theta$$; Rate $=(\lambda+1)$, Shape $= \alpha + x$

$M_{X|\theta}(t) = E[e^{tX}|\theta] = \exp((e^t-1)\theta)$ and $M_\theta(m) = E[e^{m\theta}] = (1-m/\lambda)^{-\alpha}$

$$M_X(t) = E[e^{tX}] = E[E[e^{tX}|\theta]] = E[M_{X|\theta}(t)] = E[\exp((e^t-1)\theta)] = M_\theta(e^t-1) = \left(1 - \frac{e^t-1}{\lambda}\right)^{-\alpha}$$

93. Geometric Sum: $\quad$ Exponential Random Variable $\quad M_X(t) = \left(\frac{\lambda/(1+\lambda)}{1-e^t(1-\lambda/[1+\lambda])}\right)^{\alpha} \frac{\lambda}{\lambda+1}$

Negative Binomial

$X_r = X_1 + X_2 + \cdots + X_n \quad p(x) = \lambda e^{-\lambda x}$

$$M(t) = \sum_{x=0}^{\infty} \sum_{i}^{n} X(x)_{i,i}\, e^{tx_i} = \sum_{i=0}^{\infty} \lambda e^{-(\lambda+t)x_i} = \boxed{\frac{-\lambda}{(\lambda+t)} e^{-(\lambda+t)x}}$$

94. Probability-Generating Function: $G(s) = \sum_{k=0}^{\infty} s^k p_k$; where $p_k = P(X=k)$

a) Show $p_k = \frac{1}{k!} \frac{d^k}{ds^k} G(s)\Big|_{s=0}$; Fundamental theorem of Calculus: $\int_0^b f(x)\,dx = F(b) - F(a)$; $\frac{d}{dx} F(x)$

b). Show $\frac{dG}{ds} = E(X)$ $\quad \frac{dG}{ds} = k s^{(k-1)} p(k) = E(X) = \boxed{k \cdot p(k)}$

$\frac{d^2G}{ds^2}\Big|_{s=1} = E[X(X-1)]$ $\quad \frac{d^2G}{ds^2} = k(k-1)s^{(k-2)} p(k) \sim \boxed{k(k-1)\,p(k)}$

c) $M(t) = \sum_{s=0}^{\infty}\sum_{s=0}^{\infty} e^{tk} G(s) = \sum_0^{\infty}\sum_0^{\infty} e^{tk} s^k p_k = \boxed{\frac{\sum e^{tk} s^{k+1} p(k)}{k+1}}$

d) $G(s) = \int_0^{\infty} s^k \frac{\lambda^k}{k!} e^{-\lambda}\, dk = e^{-\lambda} \sum_0^{\infty} \frac{(s\lambda)^k}{k!} = e^{-\lambda} e^{\lambda s} = \boxed{e^{(\lambda s - \lambda)}}$

95. Joint Moment Generating Function. $M(t) = \sum_{-\infty}^{\infty} e^{t(x+y)} xy = \sum e^{tx} x\, e^{ty} y = \boxed{M_X(t) M_Y(t)}$

96. $E(XY) = M'(0) = \frac{d}{dt}\left[\sum e^{tx} x\, e^{ty} y\right] = \boxed{xy\, p(x,y)}$ $\quad M(t) = \sum e^{tx+ty} p(x)$; $M'(t) = \sum (x+y) e^{tx+ty} p(x)$

97. $Var(aX+bY) = a^2 Var(X) + b^2 Var(Y)$; $\quad M''(t) = \sum (x+y)^2 e^{tx+ty} p(x)$

$M''(0) = (x+y)^2 p(x) = E[X^2]$; $M'(0) = (x+y)\,p(x) = E[X]$

$$= E[X^2] - E[X]^2 = (x+y)^2 p(x) - (x^2+y)^2 p(x)^2 = (x^2+2xy+y^2)(p(x)-p(x)^2)$$

$$= \boxed{Var(X) + Var(Y)}$$

98. Compound Poisson Distribution

$M_S(t) = \exp[\mu(e^{\lambda(e^t-1)}-1)]$

$M_S'(0) = [\mu(e^{\lambda(e^t-1)}-1)]' \exp[\mu(e^{\lambda(e^t-1)}-1)]$

$= [\lambda(e^t-1)]'[\mu(e^{\lambda(e^t-1)}-1)]\exp[\mu(e^{\lambda(e^t-1)}-1)]$

$= \lambda e^t[\mu(e^{\lambda(e^t-1)}-1)]\exp[\mu(e^{\lambda(e^t-1)}-1)] = \emptyset \neq E[X]$

$M_S''(0) = \lambda[\mu(e^{\lambda(e^t-1)}-1)]\exp[\mu(e^{\lambda(e^t-1)}-1)] + \lambda e^{2t}[\mu(e^{\lambda(e^t-1)}-1)]\exp[\mu(e^{\lambda(e^t-1)}-1)]$
$+ (\lambda e^t[\mu(e^{\lambda(e^t-1)}-1)]\exp[\mu(e^{\lambda(e^t-1)}-1)]$

$= \emptyset$

99. $Y = g(X)$ $\quad$ a) $g(X) = \sqrt{X}$

$E[Y] = \int_0^{\infty} x\sqrt{x}\, dx = x(\tfrac{2}{3})x^{3/2}\Big|_0^{\infty} - \int_0^{\infty} \sqrt{x}\, dx = \infty$

$Var[Y] = \infty$

b) $E[X] = \int_0^\infty x \log x \, dx = x\left(\frac{1}{x}\right)\Big|_{10}^{\infty} - \int_0^\infty \log(x)\,dx = 1 - \frac{1}{x}\Big|_0^\infty = \text{Undefined}$

$Var(X) = \text{Undefined}$

c) $g(x) = \sin^{-1}(x)$; $E[X] = \int_{-\pi/2}^{\pi/2} x\sin^{-1}(x)\,dx$ = Does not converge $Var(X)$ = Does not converge

100. $X[10,20]$; $Y = 1/X$; $E[X] = \int_{10}^{20} \frac{1}{10}\left(\frac{1}{x}\right)dx = \frac{\ln 20 - \ln 10}{10} = 0.0693$; $E[Y^2] = \int_{10}^{20} \frac{1}{10}\left(\frac{1}{x^2}\right)dx = \frac{-x^{-1}}{10}\Big|_{10}^{20} = 0.005$

Exact Method

$Var(X) = E[X^2] - E[X]^2 = 0.005 - 0.0693^2 = 0.000198$

Approximate Method

$Y(X) = \frac{1}{X}$; $Y'(X) = \frac{-1}{X^2}$; $Y''(X) = \frac{-2}{X^3}$; $E(Y) \sim g(\mu_X) + \left(\frac{1}{2}\right)\sigma_X^2 g''(\mu_X) = \frac{1}{15} + \left(\frac{1}{2}\right)8.33(0.0006)$ $\boxed{= 0.0249}$

$Var(Y) \approx \sigma_X^2[g'(\mu_X)]^2 = 0.000161$

$\sigma_X^2 = \frac{(b-a)^2}{12} = 8.33$ $\sigma_Y = 0.0127$

101. $Y = \sqrt{X}$; X = Poisson Distribution

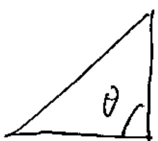
$\frac{\lambda^k}{k!}e^{-\lambda}$; $Y'(X) = \frac{1}{2}(X)^{-1/2}$; $Y''(X) = \frac{-1}{4}(X)^{-3/2}$; $Y'''(X) = \frac{3}{8}(X)^{-5/2}$

$\mu_X = \lambda, \sigma_X^2 = \lambda$

$E(Y) \sim Y(\mu_X) + \frac{1}{2}(\sigma_X^2)g''(\mu_X) \sim \lambda^{1/2} - \frac{1}{2}\lambda \cdot \frac{1}{4}\lambda^{-3/2}$ $\boxed{\sim \sqrt{\lambda} - \frac{1}{8\sqrt{\lambda}}}$

$Var(Y) \approx \sigma_X^2[g'(\mu_X)]^2 \approx \lambda\left[\frac{1}{2}(\lambda)^{-1/2}\right]^2 \approx \boxed{\frac{1}{4}}$

102.



$y_0 = Y$; $E(Y) = y_0$; $Var(X) = Var(Y) = \sigma^2$; $\theta = \tan^{-1}\left(\frac{Y}{X}\right)$; $E(\theta) \sim$

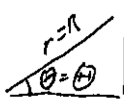
$X_0 = X$, $E(X) = X_0$

$E(\theta) \sim \tan^{-1}\left(\frac{E(Y)}{E(X)}\right) = \boxed{\tan^{-1}\left(\frac{y_0}{x_0}\right)}$

$Var(\theta) \sim \tan^{-1}\left(\frac{Var(Y)}{Var(X)}\right) = \tan^{-1}(1) = \boxed{45°}$

103. $V = \frac{\pi}{6}D^3$; $D = 2mm$; $\sigma_X = 0.01mm$; $V' = \frac{\pi}{2}D^2$; $\sigma_Z^2 \approx \sigma_X^2[g'(\mu_X)]^2$; $\sigma_Z \approx \sigma_Z g'(\mu_X) \approx 0.01mm \cdot \frac{\pi}{2}2mm^2$ $\boxed{\approx 6.28 \times 10^{-2} mm}$

104.



$Y = R\sin\theta$; a) $Var(Y) \sim \sigma_X^2[g'(\mu_X)] \sim \sigma_X^2[R\cos\theta]$

b) $\frac{d\,Var(Y)}{d\theta} = \sin\theta = 0$ $\boxed{90° = \theta}$

Chapter 5: 1) $X_1, X_2 \ldots$; $E(X_i) = \mu$; $Var(X_i) = \sigma_i^2$, Show $n^{-2}\sum_{i=1}^{n}\sigma_i^2 \to 0$; $\bar{X} \to \mu$

Law of Large Numbers: $P(|\bar{X} - \mu| > \epsilon) \leq \frac{Var(\bar{X}_n)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \to 0$

2. $E(X_i) = \mu_i$; if $E(\bar{X}_i) = \frac{1}{n}\sum_1^n E(X_i) - \mu$ $P(|\bar{X} - \mu| > \epsilon) \leq E[X^2] - E[X]^2$; $\bar{X}^2 = \mu^2$; $\boxed{\bar{X} = \mu}$

$= \frac{1}{n}(n E(X_i)) = E(X_i) = \mu$

3. Number of Insurance claims, N, is a Poisson Distribution: $p(x) = \frac{\lambda^k}{k!}e^{-\lambda}$

$E(N) = 10{,}000$. $E(X) = \lambda$

Apply a normal approximation to the Poisson to approximate $P(N > 10{,}200)$. $Var(X) = \lambda$

Standardizing Random Variables

$Z_n = \frac{X_n - E(X_n)}{\sqrt{Var(X_n)}} = \frac{10{,}200 - 10000}{\sqrt{10000}} \to 102 - 100 = 2$

$P(Z_n) = 1 - 0.9772$ $\boxed{= 0.028}$

4. Number of Traffic Accidents (N) is $E(N) = 100$

Find Δ if a poisson $P(100 - \Delta < N < 100 + \Delta) \approx 0.9$

$Z_n = \frac{X_n - E(X_n)}{\sqrt{Var(X)}}$; $P(100 - \Delta) = \frac{100 - \Delta - 100}{\sqrt{100}} = 10 - 10 - \Delta = -\Delta$

$P(N) = \frac{X_n - 100 + \Delta}{\sqrt{100}} = \Delta =$ $\boxed{d\Delta = 1.3}$

Means 1.3 cars more or less for probability of 90%.

5. $n \to \infty, p \to \infty$, and $np = \lambda \to \infty$ Binomial Distribution! $p(x) = \sum_{1}^{\infty} \binom{n}{k} p^k (1-p)^{n-k}$; n and p tend poisson $p(k) = \frac{\lambda^k}{k!} e^{-\lambda}$

Moment Generating Function of Binomial; $M(t) = \int_0^{\infty} e^{tk} \binom{n}{k} p^k (1-p)^{n-k} dk$; if $np = \lambda$

$= \binom{n}{k} \int_0^{\infty} e^{tk} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} dk$; $\boxed{\text{Continuity Theorem}}$ $\lim_{n\to\infty} M_n(t) \to M(t)$; $\lim_{n\to\infty} F_n(x) = F(x)$ $\boxed{\text{Law of Large Numbers}}$

$\lim_{n\to\infty} M_n(t) = \lim_{n\to\infty} \int_0^{\infty} \binom{n}{k} e^{tk} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} dk = \lim_{n\to\infty} \int_0^{\infty} e^{tk} \frac{\lambda^k}{k!} \underbrace{\frac{n!}{(n-k)!} \frac{1}{n^k}}_{=1} \underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{=e^{-\lambda}} \underbrace{\left(1-\frac{\lambda}{n}\right)^{-k}}_{=1} dk = \int_0^{\infty} e^{tk} \frac{\lambda^k}{k!} e^{-\lambda} dk$

$= \underbrace{\int_0^{\infty} e^{tk} \frac{\lambda^k}{k!} e^{-\lambda} dk}_{\text{Poisson Distribution}} = \int_0^{\infty} \frac{(\lambda e^t)^k}{k!} e^{-\lambda} dk = e^{\lambda e^t} \cdot e^{-\lambda} = \boxed{e^{\lambda(e^t - 1)}}$

6. $\alpha \to \infty$ Gamma Distribution: $\frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t}$

$\to M_\alpha(t) = \int_0^{\infty} e^{tx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx$; $\lim_{\alpha\to\infty} M_\alpha(t) = \lim_{\alpha\to\infty} \int_0^{\infty} e^{tx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx$

$= \lim_{\alpha\to\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} e^{x(t-\lambda)} dx = \lim_{\alpha\to\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} \left(\frac{\Gamma(\alpha)}{(\lambda-t)^\alpha}\right) = \left(\frac{\lambda}{\lambda - t}\right)^\alpha = \infty$

If $\mu = \lambda/t$, $\sigma^2 = k/t^2$? $= \left(\frac{\mu t}{\mu t - k/t^2}\right)^k$

$= \left(1 - \frac{t}{\lambda n}\right)^{-\lambda}$

7. $X_n \to c$; g is continuous, then $g(X_n) \to g(c)$

Continuity Theorem $\boxed{\lim_{X_n \to c} g(X_n) = g(c)}$

8. Poisson Cumulative Distribution: a) $\lambda = 10$; $cdf_{Poisson} = \int_0^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} dk = e^{\lambda} \cdot e^{-\lambda}$; 1 b) 1 c) 1 $\approx$ Normal Standard

9. Binomial Cumulative Distribution: a) $n = 20$, $p = 0.2$ $CDF_{Binomial} = \int_0^{\infty} \binom{n}{k} p^k (1-p)^{n-k} dk = \int_0^{\infty} \frac{20!}{k!(n-k)!} 0.2^k (0.8)^{n-k} dk = 20! \, 0.8^{20} \int_0^{\infty} \frac{1}{4^k} \frac{dk}{k!(20-k)!}$

$\boxed{8.9 \times 10^{-5}}$

b) $n = 40$, $p = 0.5$ $= 40! \, 0.5^{40} \int_0^{\infty} \frac{dk}{k!(40-k)!} = 0.99$

$\boxed{\text{The binomial converges to the Normal Standard when correct rule}}$

$\left(\frac{40 - 40 \cdot 0.5}{\sqrt{40 \cdot 0.5(1-0.5)}}\right) x$?

Normal Approximation $\left(\frac{\lambda - E(\lambda)}{\sqrt{Var}}\right) = \frac{0.2 - 20 \cdot 0.2}{\sqrt{20 \cdot 0.2(1-0.2)}}$ $= \emptyset$

10. Six sided dice; $n = 100$; $\left(\frac{X - E(X)}{\sqrt{Var}}\right) = P(X)$; $P(15 < X < 20) = \left(\frac{15 - 100(1/6)}{\sqrt{100(\frac{1}{6})(1-1/6)}} < X < \frac{20 - 100(1/6)}{\sqrt{100(1/6)(1-1/6)}}\right)$ ~~35% to 23%~~

$\approx P(15.5 < X < 19.5) = P\left(\frac{15.5 - 100/6}{\sqrt{100/6(1-1/6)}} < X < \frac{19.5 - 100/6}{\sqrt{100/6(1-1/6)}}\right)$

$\approx P(-0.31 < Z < 0.76) = P(Z < 0.76) - P(Z < -0.31)$

$= P(Z < 0.76) - 1 + P(Z < 0.31) = 0.774 - 1 + 0.6217 = 0.4$

Central Limit Theorem:

$S_n = \sum_{i=1}^{n} X_i$; $\lim_{n\to\infty} P\left(\frac{S_n}{\sigma\sqrt{n}} \le x\right) = \Phi(x)$; $-\infty < x < \infty$

Proof: $Z_n = S_n / (\sigma\sqrt{n})$

$M_{S_n}(t) = [M(t)]^n$; $M_{Z_n}(t) = \left[M\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$

$M(s) = M(0) + sM'(0) + \frac{1}{2}s^2 M''(0) + \epsilon_s$

$\epsilon_s / s^2 \to 0$ as $s \to 0$; $E[X] = 0$, $M'(0) = 0$, $M''(0) = \sigma^2$

As $n \to \infty$, $t/(\sigma\sqrt{n}) \to 0$

$M\left(\frac{t}{\sigma\sqrt{n}}\right) = 1 + \frac{1}{2}\sigma^2 \left(\frac{t}{\sigma\sqrt{n}}\right)^2 + \epsilon_n$

$M_Z(t) = \left(1 + \frac{t^2}{2n} + \epsilon_n\right)^n$; $\lim_{n\to\infty} \left(1 + \frac{a_n}{n}\right)^n = e^a$

$M_{Z_n}(t) = e^{t^2/2}$ $n \to \infty$

$E[X] = \frac{6+1}{2} = 3.5$; $Var(X) = \frac{1}{12}(6^2 - 1) = 2.917$

$E[S] = 100 E[X] = 100(3.5) = 350$; $Var(S) = 100(2.917) = 291.67$

$P(S < 300) \approx P(S < 299.5) = P\left(Z < \frac{299.5 - 350}{\sqrt{291.67}}\right)$

$= P(Z < -2.96) = 1 - P(Z < 2.96) = \boxed{0.00154}$

11. The argument suffices to say $\lambda = 1 = np = \infty \cdot p \Rightarrow \frac{1}{p} = \infty$; The probability must approach $\emptyset$ as $n \to \infty$.

12. Uniform Random Variable $[-\frac{1}{2}, \frac{1}{2}]$. $n = 100$; $P(X > 1) = \text{Prob}\left(\frac{X - \sum E(X_i)}{\sqrt{\sum Var(X)}}\right) = P\left(\frac{X - 100 \cdot 0}{\sqrt{100 \cdot 1/12}} > \frac{1-0}{\sqrt{100/12}}\right) = P\left(\frac{\sqrt{3}}{25} Y > \frac{\sqrt{3}}{25}\right)$

$E[X] = \frac{a+b}{2}$; $Var(X) = \frac{(b-a)^2}{12} = \frac{(\frac{1}{2} - -\frac{1}{2})^2}{12} \cdot \frac{1}{12}$

$= 1 - \phi\left(\frac{\sqrt{3}}{5}\right)$

$= 0.3645.$

b) $P(Y > 2) = P\left(\frac{Y-0}{\sqrt{\frac{25}{3}}} > \frac{2-0}{\sqrt{\frac{25}{3}}}\right) = 1 - \phi\left(\frac{2\sqrt{3}}{5}\right) = 0.2442$

c) $P(Y > 5) = P\left(\frac{Y-0}{\sqrt{\frac{25}{3}}} > \frac{5-0}{\sqrt{\frac{25}{3}}}\right) = 1 - \phi\left(\frac{5\sqrt{3}}{5}\right) = \boxed{0.0416}$

13. $P(\text{North}) = \frac{1}{2}$; Step Length $= 50cm$; $E(X) = \sum X \cdot P(X) = 50(\frac{1}{2}) + (-50)(\frac{1}{2}) = \emptyset$; $Var(X) = E[X^2] - E[X]^2 = \sum x^2 P(x) + \emptyset^2 = 50^2(\frac{1}{2}) + (-50)^2(\frac{1}{2}) = 2500$

$\underbrace{P(\text{South}) = \frac{1}{2}}_{\text{per minute}}$; Approximate probability after 1h.

$E(S) = \sum^{60} E(X) = 60 \cdot \emptyset = \emptyset$; $Var(S) = \sum^{60} Var(X) = 60 \cdot 2500 = 150,000$

Standard Deviation $(\sqrt{Var(S)}) = \sigma_S \cdot \sqrt{150\,000} = 387$

14. $P(\text{North}) = \frac{2}{3}$

$P(\text{South}) = \frac{1}{3}$

$E[X] = \frac{1}{3}(-50) + \frac{2}{3}(+50) = \frac{50}{3}$; $E[X^2] = \frac{1}{3}(-50)^2 + \frac{2}{3}(50)^2 = 2500$; $Var[X] = E[X^2] - E[X]^2 = 2500 - 1000 = 1500$

$E(S) = \sum_0^{60} E[X] = \frac{3000}{3} = 1000$; $E[S^2] = \sum_0^{60} E[X^2] = 60 \cdot 2500 = 150,000$

$Var(S) = \sum_0^{60} Var[X] = 60 \cdot 1500 = 90,000$

Standard Deviation $(\sqrt{Var}) = \sigma = \boxed{300}$

15. $n = 50$; Amount $= \$5$

Probability (Loss ≥ 75).

$P(\text{Win}) = \frac{1}{2}$; $P(\text{Lose}) = \frac{1}{2}$

$\sum_{i=1}^{50} X_i < -75$; $\bar{X} < \frac{-75}{50} = -1.5$; $E(X) = 0$; $Var(X) = E(X^2) - E(X)^2 = \sum x^2 p(x) - 0 = 5^2(\frac{1}{2}) + (-5)^2(\frac{1}{2}) = 25$

$P(\bar{X} < -1.5) = P\left(Z \leq \frac{-1.5 - 0}{\sqrt{1/2}}\right) = P(Z < -2.12) = \phi - \phi(2.12) = 0.017$

16. $X_1, \ldots X_{20}$; $f(x) = 2x$; $0 \leq x \leq 1$; $S = X_1 + \cdots + X_{20}$; $P(S \leq 10)$; $E[X] = \int_0^1 x f(x)\,dx = \int_0^1 x \cdot 2x\,dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$

$P(S \leq 10) = P\left(\frac{X - E[S]}{\sqrt{Var(S)}} < \frac{10 - E[S]}{\sqrt{Var(S)}}\right) = P\left(\frac{X - 40/3}{\sqrt{1.11}} < \frac{10 - 40/3}{\sqrt{1.11}}\right)$

$E[S] = \sum_{i=1}^{20} \frac{2}{3} = \frac{40}{3}$; $E[X^2] = \int_0^1 x^2 \cdot 2x\,dx = \frac{2x^4}{4}\Big|_0^1 = \frac{1}{2}$

$Var(\frac{S}{n}) = \frac{1}{n} \cdot 1.0$

$Var(X) = E[X^2] - E[X]^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$

$= P\left(\frac{S - E[S]}{\sqrt{1.11}} < -3.16\right) = 1 - (1 + \phi(3.16)) = \boxed{0.9992}$

$Var(S) = \sum_0^{20} Var[X] = \frac{20}{18} = 1.11.$

17. $\mu, \sigma^2 = 25$; $P(|\bar{X} - \mu| \leq 1) = P(-1 \leq \bar{X} - \mu < 1) = P\left(-\sqrt{n}/\sigma < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \sqrt{n}/\sigma\right) = 0.95$

$\underbrace{P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \sqrt{n}/\sigma\right) = \frac{1 - 0.95}{2} = 0.025}_{\text{"Describes size of range"}}$; $-\frac{\sqrt{n}}{\sigma} = -1.97$; $n = 97$.

18. $\mu = 15\text{lbs}$; $P(F.B) < 1700\text{lbs}) = P\left(\frac{S - E[S]}{\sigma/\sqrt{n}} < \frac{1700 - n\mu}{\sigma/\sqrt{n}}\right) = P\left(\frac{S - n\mu}{\sigma/\sqrt{n}} < \frac{1700 - 1500}{10/\sqrt{100}}\right)$

$\sigma = 10$;

$= P(2) = 0.9772$

$n = 100$

19. a) $n=100$, $n=1000$; $f(x)=\int_0^1 \cos(2\pi x)\,dx = \frac{\sin(2\pi x)}{2\pi}\Big|_0^1 = \emptyset$; $\hat{I}_{100}(f)=\frac{1}{n}\sum_{i=0}^{1} f(X_i) = \frac{1}{100}\sum_{i=0}^{1}\cos(2\pi x) = \frac{1}{100}+\frac{1}{100}=\frac{2}{100}$

$\hat{I}_{1000}(f)=\frac{1}{1000}\sum_{i=0}^{1} f(X) = \frac{1}{1000}\sum_{0}^{1}\cos(2\pi x) = \frac{1}{1000}+\frac{1}{1000}=\frac{2}{1000}$

b) $I(f)=E[X]=\int_0^1 \cos(2\pi x^2)\,dx =$

$\hat{I}(f)=\frac{1}{n}\sum_{i=0}^{1}\cos(2\pi x^2) = \frac{1}{10^3}\sum_{i=0}^{10^3}\cos(2\pi x^2) = 1\ldots > 0$

Exact Solution ; Cosine integral: $\int \cos(u)\,dx$; $u=2\pi x^2$, $\frac{du}{dx}=2\pi x$; $\int \frac{\cos(u)}{2\pi x}dx = \int \frac{\cos(u)}{2\pi\sqrt{u}/2\pi} du = \int_0^1 \frac{\cos(u)}{\sqrt{u}}du = \int_0^1 \frac{\cos(2u)}{u}du$

$= \frac{C(2)}{2} = \boxed{0.24}$

20. $E(\hat{I}^2(f)) = \left(\frac{1}{1000^3}\right)\left[\left(\frac{1}{\sqrt{2\pi}}\right)\sum_{i=1}^{100} e^{-x^2/2}\right]^2$

$\simeq \frac{1}{1000}\,\frac{1}{2\pi}\sum_{i=1}^{100}\left(e^{-x_i^2}\right) = \frac{0.386}{2\pi\times 10^3} = 6.14\times 10^{-5}$

$Var(\hat{I}(f)) = E[\hat{I}^2(f)] - E[\hat{I}(f)]^2 = \left(\frac{1}{1000^2}\sum_{i=1}^{100}\left(e^{-x^2/2}\right)^2 - \frac{1}{1000^2}\left[\sum_{i=1}^{1000} e^{-x^2/2}\right]^2\right)\frac{1}{2\pi}$

21. $I(f)=\int_a^b f(x)\,dx$; $\hat{I}(f)=\frac{1}{n}\sum_{i=1}^{n}\frac{f(X_i)}{g(X_i)}$ a) Show $E(\hat{I}(f))=I(f)=\int_a^b f(x)/g(x)\,dx$ $\boxed{\frac{1}{n}\sum_{i=1}^{n} f(x)/g(x)}$

b) $Var(\hat{I}(f)) = E[\hat{I}(f)^2] - E[\hat{I}(f)]^2$; $n=100$; $n\to\infty$

c) $E(\hat{I}(f)) = \hat{I}(f) = \int_0^1 \frac{f(x)}{g(x)}dx = \int_0^1 h(x)\,dx = \int_0^1 e^{-x^2/2}dx \to$ Example A: Section 5.2.

22. Find Δ ; $P(|\hat{I}(f)-I(f)|\le\Delta)=0.05$, where $\hat{I}(f)$ is the Monte Carlo Estimate of $\int_0^1\cos(2\pi x)dx$ based upon $n=1000$.

$P\left(\left|\frac{1}{1000}\sum_{i=1}^{100}\cos(2\pi x) - \int_0^1\cos(2\pi x)dx\right| \le \Delta\right) \simeq 0.05$

$P(|1|\le\Delta)=0.05$; $P(\Delta)=0.05-P(1)=0.05-0.84=-0.79$; $\boxed{\Delta=0.81}$

$P(\Delta)-P(1)=0.005$

23. $0\le x\le 1$; $0\le y\le 1$; Random (x,y) ; $Z=1$ if so $Z=0$ otherwise. Prove $E(Z)=A=\sum_0^1\sum_0^1 xy\cdot Z(1) + \sum_0^1\sum_0^1 xy\,Z(0)$ $\boxed{=1.17A}$

24. $\hat{A}$; $E(S)=\sum_{i=1}^{n}E[Z]=\boxed{nE[Z]}$; $\hat{A}=nE[Z]=nA$; $P(|\hat{A}-A|<0.1)\approx 0.99$

$P(|nA-A|<0.1)\approx 0.99$; $P(|(n-1)0.2<0.1)\approx 0.99$; $P(|n|<3/2)\approx 0.99$

$P(3/2)-P(|n|)\approx 0.99$

$P(|n|)\approx P(3/2)-0.99$

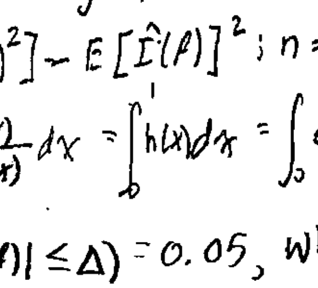
$\approx 0.9332-0.99=-0.0568$

$P(n)=0.0568$

$n=1.58$

25. $f(x)=\frac{3}{2}x^2$ $-1\le x\le 1$;

$\frac{2}{3}\le x\le 1$



$f(x)=\begin{cases}\frac{3}{2}x^2; & -1\le x\le 1\\ 0\end{cases}$

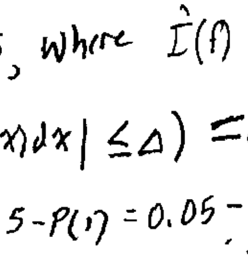
$S=X_1+\ldots+X_{50}$

$E[S]=\sum_{i=1}^{50} S = \sum_{i=1}^{50} X_i$

$=50E[X]=50\cdot\int_1 x\,dx$

$E[X]=\int_{-1}^1 x\,\frac{3}{2}x^2\,dx$

$=0$



$\frac{3}{2}x^2-1$

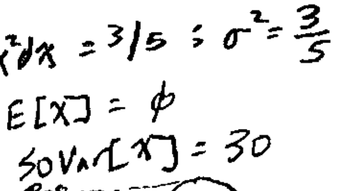
$E[X^2]=\int_{-1}^1 x^2\frac{3}{2}x^2 dx = 3/5$; $\sigma^2=\frac{3}{5}$

$E[S]=\sum_{i=1}^{50} X_i = 50E[X]=\emptyset$

$Var[S]=\sum Var(X) = 50Var[X]=30$

$=50\left[\frac{x^2}{2}-\frac{\left(\frac{3}{2}x^2-1\right)^2}{2}\right]$

$=50\left[\frac{x^2}{2}-\frac{9}{8}x^4+\frac{3}{2}x^2-\frac{1}{2}\right] = 50\left[-\frac{9}{8}x^4+2x^2-\frac{1}{2}\right]=$



26. $P(\text{Shot}) = 0.3$; $E[S] = \sum_{i=1}^{25} x\,p(\text{success} \mid \text{shot}) \cdots 25 \cdot p(x) = 25\left(\frac{3}{10}\right)$; $\underbrace{\frac{5}{25} \geq p(x) > \frac{7}{25}}$ $\underbrace{\geq p(x)\ \frac{9}{25} \geq p(x)\ \frac{11}{25} \geq p(x)}$

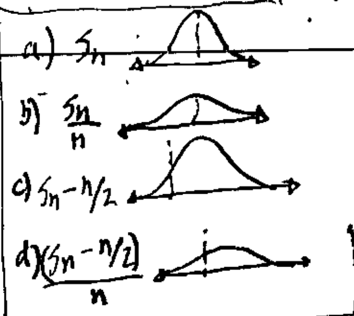
$\leq \frac{3}{10} \leq$ | 30.

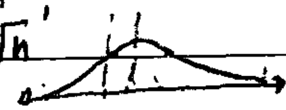
27. Prove $a_n \to a$, then $(1+a_n/n)^n \to e^a$; $\lim_{a_n \to a} (1+a_n/n)^n = \boxed{1 + \frac{1}{2!}\left(\frac{a^2}{n}\right) + \frac{1}{3!}\left(\frac{a^3}{n^2}\right) = e^a}$

28. $f_n(x) = \begin{cases} 1/2 & \text{if } x = \pm(1/2)^n \\ 0 & \text{otherwise} \end{cases}$; $E[X] = \sum_{-\infty}^{\infty} x f(x) = -\frac{1}{2} - \frac{1}{4} \cdots 0 + \frac{1}{4} + \frac{1}{2}$

$F_n(x) = \begin{cases} 1/2 & (1/2)^n/n \\ 0 & \text{otherwise} \end{cases}$ $\boxed{\lim F_n(x) = F_n(x)}$

$U_1, U_2, \ldots U_{1000}$; $S_n = \sum_1^n U_i$; $n = 1 \cdots 1000$





29. $U_1 \ldots U_n$ from $[0,1]$; $V_{(n)}$ = maximum

$\int_0^1 U_{(n)}\,du \doteq \Phi = \frac{U - E[V]}{\sqrt{Var(V)}} = \varphi(n)$

Chapter 6: Distributions derived from Normal Distributions

χ^2 = Chi-squared $V = Z^2$

$X \sim N(\mu, \sigma^2)$; $(X-\mu)/\sigma \sim N(0,1)$; $\left[\frac{X-\mu}{\sigma}\right]^2 \sim \chi_1^2$

$\boxed{\text{Chi-squared Distribution}}$ $[\chi_n^2]$ "n-degrees of freedom" || sum of independent Gamma $\alpha = \frac{n}{2}$; $\lambda = \frac{1}{2}$

$V = U_1 + U_2 + \cdots + U_n$

$$f(v) = \frac{1}{2^{n/2}\Gamma(n/2)} v^{(n/2)-1} e^{-v/2}; \quad v \geq 0$$

Moment Generating Function $M(t) = (1-2t)^{-n/2}$

$E(V) = n$; $Var(V) = 2n$

$\boxed{\text{t-distribution}}$ if $Z \sim N(0,1)$ and $U \sim \chi_n^2$ then $Z/\sqrt{U/n}$ is a t-distribution:

Density Function: $f(t) = \frac{\Gamma[(n+1)/2]}{\sqrt{n\pi}\,\Gamma(n/2)}\left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}$

$\boxed{\text{F-distribution}}$ $W = \frac{U/m}{V/n}$

Density Function: $f(w) = \frac{\Gamma[(m+n)/2]}{\Gamma(m/2)\Gamma(n/2)}\left(\frac{m}{n}\right)^{m/2} w^{m/2-1}\left(1 + \frac{m}{n}w\right)^{-(m+n)/2}$

$E(W) = \frac{n}{(n-2)}$ | If $(X_1 - \bar{X}, X_2 - \bar{X} \cdots X_n - \bar{X})$ are independent

Sample Mean and Variance

Sample: independent random variables. Which are normally distributed

$\underbrace{\bar{X} = \frac{1}{n}\sum X_i}_{\text{Sample mean}}$; $\underbrace{S^2 = \frac{1}{n-1}\sum_1^n (X_i - \bar{X})^2}_{\text{Sample variance.}}$

$E(\bar{X}) = \mu$ $\quad Var(\bar{X}) = \frac{\sigma^2}{n}$

$M(s, t_1 \ldots t_n) = E\{\exp[s\bar{X} + t_1(X_1 - \bar{X}) + \cdots + t_n(X_n - \bar{X})]\}$

$\sum_{i=1}^n t_i(X_i - \bar{X}) = \sum_{i=1}^n t_i X_i - n\bar{X}\bar{t}$

$s\bar{X} + \sum t_i(X_i - \bar{X}) = \sum_{i=1}^n \left[\frac{s}{n} + (t_i - \bar{t})\right] X_i = \sum_{i=1}^n a_i X_i$

$M(s, t_1 \ldots t_n) = \exp\left(\mu s + \frac{\sigma^2}{2n}s^2\right)\exp\left[\frac{\sigma^2}{2}\sum_1^n (t_i - \bar{t})^2\right]$

1. Prove Proposition A of Section 6.2

$\frac{Z}{\sqrt{U/n}} = \frac{N(0,1)}{\sqrt{\chi^2/n}} = \frac{N(0,1)}{\sqrt{\frac{1}{2^{n/2}\Gamma(n/2)} v^{(n/2)-1} e^{-v/2}/n}}$

$f(t^2) = \int \sqrt{\frac{v}{n}}\, N(\sqrt{v/n}\cdot t) f(\chi_n^2)\, dv$

$x = (1 + t^2/n)v/2$

$= \frac{1}{\sqrt{2\pi n}\, 2^{n/2}\Gamma(n/2)} \int_0^\infty v^{(n-1)/2} e^{-(1+t^2/n)v/2}\, dv$

$= \frac{(1+t^2/n)^{-(n+1)/2}}{\sqrt{\pi n}\,\Gamma(n/2)} \int_0^\infty x^{(n+1)/2-1} e^{-x}\, dx$ $\boxed{= \frac{(1+t^2/n)^{-(n+1)/2}}{\sqrt{\pi n}\,\Gamma(n/2)}\Gamma[(n+1)/2]}$

2. Prove Proposition B of Section 6.2

$W = \frac{U/m}{V/n}$

$f(w) = \frac{\Gamma[(m+n)/2]}{\Gamma(m/2)\Gamma(n/2)}\left(\frac{m}{n}\right)^{m/2} w^{m/2-1}\left(1 + \frac{m}{n}w\right)^{-(m+n)/2}$

$$f(w) = \int_0^\infty \frac{1}{2^{n/2}\Gamma(n/2)} x^{(n/2)-1} e^{-x/2} \cdot \frac{m}{2^{m/2}\Gamma(m/2)} (xz)^{m/2-1} \cdot e^{-xz/2} dx = \frac{z^{m/2-1}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})2^{(m+n)/2}} \underbrace{\int_0^\infty x^{(m+n)/2-1} e^{-x(z+1)/2} dx}$$

$$= \frac{z^{m/2-1}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})2^{(m+n)/2}} \frac{1}{\left(\frac{z+1}{2}\right)^{(m+n)/2}} \int_0^\infty t^{(m+n)/2-1} e^{-t} dt$$

"similar to $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$. where $t = x\left(\frac{z+1}{2}\right)$"

$$= \frac{\Gamma(\frac{m+n}{2}) z^{m/2-1}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})(z+1)^{(m+n)/2}} \Rightarrow \boxed{\frac{\Gamma(\frac{m+n}{2})\left(\frac{m}{n}\right)^{m/2} x^{m/2-1}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})\left(1+\frac{m}{n}x\right)^{(m+n)/2}}}$$

3. $n=16$; $\bar{X} = \mu = 0 = \frac{1}{16}\sum_{i=1}^{16} X_i$

$P(|\bar{X}| \le c) = P(-c \le \bar{X} \le c)$
$= P(\bar{X} \le c) - P(\bar{X} \le -c)$
$= P(\bar{X} \le c) - [1 - P(\bar{X} \le c)]$
$= 2 \cdot P(\bar{X} \le c) - 1 = 2 \cdot \Phi\left(\frac{c-0}{\sqrt{1/16}}\right) - 1$
$= 2\cdot\Phi(4c) - 1 = 0.5$
$\Phi(4c) = 0.75$
$4c = \Phi^{-1}(0.75)$
$c = 0.7734/4$
$= 0.19335$

4. T follows a t_7-distribution. Find t_0 such that a) $P(|T| \le t_0) = 0.9$ of a t_7 distribution

$$t_7 = f(t) = \frac{\Gamma[(7+1)/2]}{\sqrt{7\pi}\,\Gamma(7/2)}\left(1+\frac{t^2}{7}\right)^{-(7+1)/2}$$
$$= \frac{\Gamma(4)}{\sqrt{7\pi}\,\Gamma(7/2)}\left(1+\frac{t^2}{7}\right)^{-4}$$

$= P(-t_0 \le T \le t_0)$
$= P(T \le t_0) - P(-t_0 \le T)$
$= P(T \le t_0) - [1 - P(t_0 \le T)]$
$= 2P(T \le t_0) - 1 = 0.9$
$P(T \le t_0) = 0.95$
$\boxed{T = \Phi^{-1}(0.95) = 1.895}$

b) $P(T > t_0) = 0.05$
$1 - P(T < t_0) = 0.05$; $P(T < t_0) = 0.95$
$T = \Phi^{-1}(0.95) = 1.895$

5. If $X \sim F_{n,m}$, then $X^{-1} \sim F_{m,n}$..
$F_{n,m} = W = \frac{U/m}{V/n} = X$; $X^{-1} = \frac{1}{F} = \frac{V/n}{U/m}$ $\boxed{F_{m,n}}$

6. $T \sim t_n$, then $T^2 \sim F_{1,n}$; $T = t_n = Z/\sqrt{V/n}$; $T^2 = Z^2/V/n = \boxed{n\,Z^2/U = F_{1,n}}$

7. Cauchy Distribution:
$$\boxed{f(x) = \frac{1}{\pi}\left(\frac{1}{1+x^2}\right)}$$

t-Distribution:
$$f(t) = \frac{\Gamma[(n+1)/2]}{\sqrt{n\pi}\,\Gamma(n/2)}\left(1+\frac{t^2}{n}\right)^{-(n+1)/2}$$
@ df=1, $f(t_1) = \frac{\Gamma(1)}{\sqrt{\pi}\,\Gamma(1/2)}(1+t^2)^{-1}$
$$\boxed{= \frac{1}{\pi}\left(\frac{1}{1+t^2}\right)}$$

8 Exponential Random Variable:
$f(x) = \lambda e^{-\lambda x}$; $\lambda = 1$; $\frac{X}{Y} = \left(\frac{1}{1}\right)e^{-x+y} = e^{y-x}$

F-Distribution:
$$f(w) = \frac{\Gamma[(m+n)/2]}{\Gamma(m/2)\Gamma(n/2)}\left(\frac{m}{n}\right)^{m/2} w^{m/2-1}\left(1+\frac{m}{n}w\right)^{-(m+n)/2};$$
if $m,n=1$; $f(w) = \frac{\Gamma(1)}{\Gamma(1/2)\Gamma(1/2)}\left(\frac{1}{1}\right)^{1/2}(1+w)^{-1}$
$= 1 + \frac{w^2}{2} + \frac{w^3}{8} + \cdots$
$= e^w = \boxed{e^{y-x}}$

9. Find mean and variance of $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

$$\bar{S}^2 = \frac{1}{n}\sum \frac{1}{(n-1)}(X_i - \bar{X})^2 = \boxed{\sum_{i=1}^{n} X_i \sum_{i=1}^{n}(X_i - \bar{X})^2}$$

$$[S^2]^2 = \frac{1}{(n-1)^2}\sum^{n-4}(X_i - \bar{X})^2$$

$P(a < S^2/\sigma^2 < b) = P(S^2/\sigma^2 < b) - P(a < S^2/\sigma^2)$

$$= \int_a^b f(v)\,dv = 1 = \frac{1}{2^{n/2}\Gamma(n/2)}[\Gamma(b+1) - \Gamma(a+1)]$$
$$= \frac{1}{2^{n/2}\Gamma(n/2)}(b-a)\Gamma(1) = 1;\ \boxed{(b-a) = 2^{n/2}\sqrt{\pi}}$$

10. Chi-Squared Distribution
$$f(v) = \frac{1}{2^{n/2}\Gamma(n/2)} v^{(n/2)-1} e^{-v/2}$$

10. $X_1 \dots X_n ; N(\mu_x, \sigma^2)$; $Y_1 \dots Y_N ; N(\mu_y, \sigma^2)$; Show how a F-distribution can find $P(S_x^2/S_y^2 > c)$

F Distribution:

$$f(w) = \frac{\Gamma[(m+n)/2]}{\Gamma(m/2)\Gamma(n/2)} \cdot \left(\frac{m}{n}\right)^{m/2} \cdot w^{m/2-1} \cdot \left(1+\frac{m}{n}w\right)^{-(m+n)/2} ; \int_{-c}^{c} f(w)\,dw = \int_{-c}^{c} \frac{S_x^2}{S_y^2}\,dx\,dy = 2\int_0^c\int_0^c \frac{S_x^2}{S_y^2}\,dx\,dy$$

$$= \frac{2 \cdot e^{(-2c^2+2\mu_x c+2\mu_y c)/\sigma^2}}{+(1/\sigma^2)^2 \mu_x \mu_y (7)^2} = 1$$

$$= \frac{+\sigma^4 e^{-2(c^2+\mu_x c+\mu_y c)}}{2\mu_x\mu_y} = 1$$

$$\frac{-2(c^2+\mu_x c+\mu_y c)}{\sigma^2} = \ln\left[\frac{2\mu_x\mu_y}{\sigma^4}\right]$$

$$c^2+\mu_x c+\mu_y c = \sigma^2 \ln\left[\frac{\sigma^4}{2\mu_x\mu_y}\right]$$

$$c^2+(\mu_x+\mu_y)c-\sigma^2\ln\left[\frac{\sigma^4}{2\mu_x\mu_y}\right] = 0$$

$$\boxed{c_1, c_2 = \frac{-(\mu_x+\mu_y) \pm \sqrt{(\mu_x+\mu_y)^2+4\sigma^2\ln\left[\frac{\sigma^4}{2\mu_x\mu_y}\right]}}{2}}$$

Chapter 7: Survey Sampling!

1. Sample: 1, 2, 2, 4, and 8;

$E[X] = \bar{X} = \frac{1}{5}\sum_{i=1}^{5} X_i = \frac{(1+2+2+4+8)}{5} = 17/5$

$E[X^2] = \frac{1}{5}\sum_{i=1}^{5} X_i^2 = \frac{(1^2+2^2+2^2+4^2+8^2)}{5} = \frac{89}{5}$

$Var(X) = \frac{89}{5} - \left(\frac{17}{5}\right)^2 = \frac{89-(10+7)(10+7)}{5} = \frac{89-\cancel{289}}{5} \dots = \frac{20.00}{5}$ $\boxed{1.44}$

Sampling Distribution

Sample size = 2

$T = N\bar{X} = 2(17/5) = \boxed{1.7}$

$E(T) = E(N\bar{X}) = 1.7$

$Var(N) = \frac{4.20}{5}\left(\frac{5-2}{5-1}\right) = \frac{12.60}{20} = \frac{1.26}{2} = \boxed{0.63}$

(2) $n = 2$; $x = 4, 8$; $T = N\bar{X} = NE(X) = 1.7$; $Var(\bar{x}) = 0.63$

(3) a) A population mean [No] b) Population Size: [No] c) Sample Size [No]
d) The Sample Mean [Yes] e) Variance of Sample mean [No] f) The largest value of Data [Yes]
g) Population Variance [No] h) Estimated variance of sample mean [Yes]

(4) Population I: n_1, σ_1 ; Population II: $n_2 = 2n_1$, $\sigma_2 = 2\sigma_1$

Accuracy is better approximated by a large n-value. The law of large numbers states a sequence of independent values converge to $E(\bar{X}_n)$ as $n \to \infty$.

(5) A random variable is defined as a variable which can take on only a finite number of values. The sample mean is a random variable because of the finite form.

(6) $N_1 = 100{,}000$; $N_2 = 10{,}000{,}000$; $n = 25$. $var(\bar{X}) = \frac{\sigma^2}{25}\left(\frac{100{,}000-25}{100000-1}\right) = 0.04\sigma^2$

Yes, it is substantially easier to measure the smaller size because of finite solutions to sample mean. $var(\bar{x}) = \frac{\sigma^2}{25}\left(\frac{10^7-25}{10^7-1}\right) = 0.04\sigma^2$

(7). Standard Error: $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$; $Var(\bar{X}) = \frac{\sigma^2}{n}\left(\frac{15}{100}\right) \Rightarrow \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}\frac{\sqrt{15}}{10} = 0.02$; $\frac{\phi}{\sqrt{n}} = \frac{0.2}{\sqrt{15}}$; $\frac{15}{0.04} = n = 375$

8. $n=100$; $p=1/5$ a) Find δ such that $P(|\hat{p}-p|\geq\delta)=0.025$

Sample proportion $\hat{p}$ = $\sigma_{\hat{p}}$ standard error $=\sqrt{\frac{p(1-p)}{n}}=\sqrt{\frac{0.2(1-0.2)}{50}}=0.057$ || $P\left(\frac{|p-\hat{p}|}{\sigma_p}\geq\frac{\delta}{\sigma_p}\right)=0.025$

$P\left(\frac{|p-\hat{p}|}{\sigma_p}\geq\frac{\delta}{\sigma_p}\right)=1-0.025$

$2P\left(z\leq\frac{\delta}{0.057}\right)-1=0.975$

$P\left(z\leq\frac{\delta}{0.04}\right)=\frac{1.975}{2}$

$\Phi\left(\frac{\delta}{0.04}\right)=0.9875$

$\frac{\delta}{0.04}=\Phi^{-1}(0.9875)$

$\delta=0.08964$

b) $\hat{p}=0.25$; 95% ;

$p=\hat{p}\pm z(0.025)s_{\hat{p}}=0.25\pm1.96\sqrt{\frac{0.25(1-0.25)}{100-1}}$

$=0.25\pm0.0853$

$=(0.1647,0.3353)$

The original $p=0.2$ is within the range of proportion and at the true population.

9. $n=1{,}500$ voters, 55% planned to vote a particular proposition. $\boxed{\text{Standard Error}}$ $\sigma_{\bar{x}}\approx\frac{\sigma}{\sqrt{n}}$

45% planned to vote against a proposition.

Margin of victory [10%]. $\boxed{\text{Confidence Interval}}$

$\bar{X}=50\%$; $\bar{X}\pm z(0.025)\sigma_{\bar{x}}=10\%\pm1.96(2.66\%)=(9.49\%,10.51\%)\approx\frac{100\%}{\sqrt{1500}}=2.26\%$

$=(4.90\%,15.10\%)$

10. False, as a population grows $(n\to\infty)$, then a distinct mean (μ) and standard deviation (σ) become more distinct, and possibly less normal and more dichotomous.

11. $n=4$; X_1,X_2,X_3,X_4 a) $\binom{n}{k}=\binom{4}{2}=\frac{4!}{2!(4-2)!}=\frac{4\cdot3}{2}=\boxed{6}$ b) $\{X_1,X_2\},\{X_2,X_3\},\{X_3,X_4\},\{X,X_4\}$

Mean Square error = variance + bias^2 ; $E[X]=\frac{1}{6}$; $E[X^2]=\frac{1}{6}$; $\sigma=\sqrt{\frac{1}{6}-\frac{1}{6^2}}=\sqrt{\frac{5}{36}}=\frac{\sqrt{5}}{6}$

$\frac{1}{n}\sum_{i}(X-\mu)^2=\sigma^2+\beta^2$ || This case shows the sample mean is unbiased because the choices are 1 2 3 4 ; $\frac{1}{8}\sum X=\frac{12}{8}=\frac{1+2+3+1+4+4}{4}$

12. Random Sampling with replacement

$s^2=\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$ is the unbiased parameter of σ^2. $\boxed{\text{Variance of a Biased Sample}}$

$Var(\bar{X})=\frac{1}{n^2}\sum_i\sum_j Cov(X_i,X_j)=\frac{1}{n^2}\sum_{i=1}^{n}Var(X_i)+\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j\neq i}Cov(X_i,X_j)=\frac{\sigma^2}{n}-\frac{1}{n^2}n(n-1)\frac{\sigma^2}{N-1}$

$=\frac{\sigma^2}{n}\left[1-\frac{(n-1)}{N-1}\right]$

$=\frac{\sigma^2}{n}\left[\frac{N-n}{N-1}\right]$

$\boxed{\text{Expected Variance of a population}}$

$\hat{\sigma}^2=\frac{1}{n}\sum_{i=1}^{n}X_i^2-\bar{X}^2$; $E(\hat{\sigma}^2)=\frac{1}{n}\sum_{i=1}^{n}E(X_i^2)-E(\bar{X}^2)$

$=\frac{1}{n}\sum_i\left[Var(X_i)+E(X)^2\right]-\left[Var(\bar{X})+E(\bar{X})^2\right]$

$=\frac{1}{n}\sum_i[\sigma^2+\mu^2]-\left[\frac{\sigma^2}{n}\left(1-\frac{n-1}{N-1}\right)+\mu^2\right]$

$=\frac{\sigma^2}{n}\sum_i\left[1-\left(1-\frac{n-1}{N-1}\right)\frac{1}{n}\right]=\frac{\sigma^2}{n}\left[1-\frac{1}{n}+\frac{(n-1)}{n(N-1)}\right]=\frac{\sigma^2}{n}\left[\frac{n(N-1)-(N-1)+(n-1)}{n(N-1)}\right]$

$=\frac{\sigma^2}{n}\left[\frac{nN-n-N+1+n+1}{n(N-1)}\right]=\boxed{\frac{\sigma^2}{n}\left[\frac{(n-1)N}{n(N-1)}\right]}$

$$s_{\bar{x}}^2 = \frac{\sigma^2}{n}\left(\frac{n}{n+1}\right)\left(\frac{N-1}{N}\right)\left(\frac{N-n}{N-1}\right) + \left[\frac{\sigma^2}{n}\left(\frac{N-n}{N-1}\right)\right]^2 \left(\frac{1}{n}\right)\left(\frac{n}{n-1}\right)\left(\frac{N-1}{N}\right)\left(\frac{N-n}{N-1}\right)$$

$$= \frac{\sigma^4}{n^2}\left(\frac{N-n}{N-1}\right)^2\left(\frac{1}{n}\right)\left(\frac{n}{n-1}\right)\left(\frac{N-1}{N}\right)\left(\frac{N-n}{N-1}\right) = \frac{\sigma^4}{n^3}\left(\frac{N-n}{N-1}\right)\left(\frac{1}{n-1}\right)\left(\frac{N-n}{N}\right)$$

12 ?

(12) $s^2 = \frac{1}{n-1}\sum(x_i - \bar{x})^2$; $E(s^2) = \frac{1}{n-1}E\left(\sum_i^n x_i^2 - n\bar{x}^2\right) = \frac{n}{n-1}\left[\sum E(x^2) - E(x)^2\right]$

$$= \frac{1}{n-1}\left[\sum\left[Var(x_i) + E(y_i)^2\right] - n\left[Var(y) + E(y)^2\right]\right]$$

$$= \frac{1}{n-1}\left[n\sigma^2 + n\bar{y}^2 - \sigma^2 - ny^2\right] = \frac{(n-1)}{(n-1)}\sigma^2 = \sigma^2$$

b) $$E(s) = E\left(\sqrt{\frac{\sum(x-\bar{x})^2}{n-1}}\right) = \frac{1}{\sqrt{n-1}}E\left(\sqrt{\frac{\sum(x-\bar{x})^2}{\sigma}}\,\sigma\right) = \frac{\sigma^2}{\sqrt{n-1}}E\left(\sqrt{\frac{\sum(x-\bar{x})^2}{\sigma^2}}\right) = \frac{\sigma^2}{\sqrt{n-1}}E(\sqrt{y})$$

$\boxed{E(s) \neq \sigma \text{ ; It is not an unbiased estimate of } \sigma}$

c) Show $\frac{s^2}{n}$ is an unbiased estimate of $\sigma_{\bar{x}}$;

$$E\left(\frac{s^2}{n}\right) = E\left(\frac{1}{n(n-1)}\left(\sum x_i^2 - n\bar{x}^2\right)^2\right) = \frac{1}{n(n-1)}\sum E(x^2) - E(x)^2 =$$

$$= \frac{1}{n(n-1)}\left[\sum\left[Var(X_i) + E(y_i)^2\right] - n\left[Var(y) + E(y)^2\right]\right]$$

$$= \frac{1}{n(n-1)}\left[n\sigma^2 + n\cancel{\bar{y}^2} - \sigma^2 - \cancel{ny^2}\right] = \frac{\sigma^2(n-1)}{n(n-1)}$$

Yes, $E\left(\frac{s^2}{n}\right)$ is an unbiased estimate, being $\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n}$; $\boxed{\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}}$

d) $E\left(\frac{N^2 s^2}{n}\right) = \frac{N^2\sigma_T^2}{n}$

$\boxed{\sigma_T^2 = N^2\sigma_x^2}$

These are expected estimates of the sample and population, separately.

e) $$E\left(\frac{\hat{p}(1-\hat{p})}{(n-1)}\right) = \frac{n}{n-1}\left[E(\hat{p}) - E(\hat{p}^2)\right] = \frac{1}{n-1}\left[p - \left[Var(\hat{p}) + E(\hat{p})^2\right]\right]$$

$$= \frac{1}{n-1}\left[p - \left[\frac{p(1-p)}{n} + \hat{p}^2\right]\right] = \frac{1}{n-1}\left[p(1-p) - \frac{p(1-p)}{n}\right] = \frac{n-1}{n-1}\,\frac{p(1-p)}{n}$$

13.

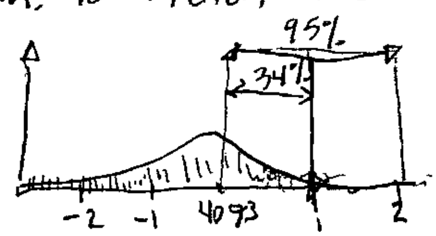
Example A 7.2:

Herksen (1976): $N = 393$; x_i = number of patients discharged from i^{th} hospital January 1968.

$\boxed{\neq \sigma_p^2}$

Suppose Total $[\tau]$ is an estimate of size 50.

Denoting estimate T by the Central Limit theorem to sketch the probability density of the error $T - \tau$



$T = N\bar{X}$, $\sigma_T^2 = N^2\sigma_x^2$ $s_T = N^2 s_x^2$

$\bar{x} = 814.6$; $\sigma_x^2 = \frac{\sigma^2}{n} = \frac{590^2}{393} = 886$

$s_x = 81.19$

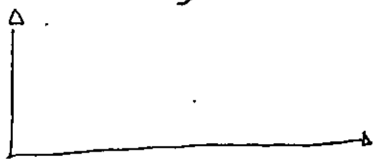
$T = 50 \cdot 814.6$; $\sigma_T^2 = 221{,}500$; $s_T = 1647{,}954$
$= 4073$

Huge Standard error with small total population

14. $p = 0.654$ Total Number (<1000) discharges is from $n = 25$. Apply central limit theorem to the distribution

$n_{total} = 393$, $\mu = 814.6$, $\sigma = 590$

$\sigma = \sqrt{\frac{0.654(1-0.654)}{25}} = \boxed{0.095}$



15. a) n = simple random sample.

a) Sketch $P(|\bar{X}-\mu| > 200)$; $20 < n \le 100$

$P(-200 < \bar{X}-\mu < 200) = 1 - P(|\bar{X}-\mu| \le 200)$

$= 1 - P(-200 \le \bar{X}-\mu \le 200)$

$= 1 - P\left(\frac{-200}{\sigma_{\bar{X}}} \le \frac{\bar{X}-\mu}{\sigma_{\bar{X}}} \le \frac{200}{\sigma_{\bar{X}}}\right)$

$= 1 - \left[\Phi\left(\frac{200}{\sigma_{\bar{X}}}\right) - \Phi\left(\frac{-200}{\sigma_{\bar{X}}}\right)\right]$

$= 1 - \left[\Phi\left(\frac{200}{\sigma_{\bar{X}}}\right) - 1 + \Phi\left(\frac{200}{\sigma_{\bar{X}}}\right)\right]$

$= 2 - \left[2\Phi\left(\frac{200}{\sigma_{\bar{X}}}\right)\right] = 2\left[1 - \Phi\left(\frac{200}{\sigma_{\bar{X}}}\right)\right]$

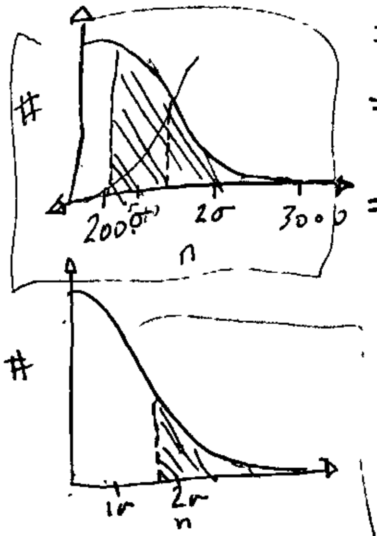
$= 2\left[1 - \Phi\left(\frac{200}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-1}{N-n}}}\right)\right]$

$= 2\left[1 - \Phi\left(6.71\sqrt{\frac{n}{393-n}}\right)\right]$

@ $n = 20$; $= 2\left[1 - \Phi\left(6.71\sqrt{\frac{20}{393-20}}\right)\right]$

$= 2[1 - \Phi(1.55)] = 0.12$

@ $n = 100$ $= 2[1 - \Phi(3.92)] = 0.002$



$P(-200 < \bar{X}-\mu < 200)$

b) For $n = 20, 40$, and 80. Find Δ such that $P(|\bar{X}-\mu| > \Delta) \approx 0.10$

$n = 20$; $P(|\bar{X}-\mu| > \Delta) \approx 0.1$

$P(\Delta < \bar{X}-\mu < \Delta) \approx 0.1$

$P(\bar{X}-\mu < \Delta) - [1 - P(\bar{X}-\mu < \Delta)] \approx 0.1$

$2\left[1 - \Phi\left(\frac{\Delta}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-1}{N-n}}}\right)\right] \approx 0.1$

$\Phi\left(\frac{\Delta}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-1}{N-n}}}\right) \approx 0.95$

$\frac{\Delta}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-1}{N-n}}} = 1.65$

$\Delta_{0.1} = \frac{590}{\sqrt{20}}\sqrt{\frac{393-1}{393-20}}(1.65) = 222$

$\Delta_{0.5} = \frac{590}{\sqrt{20}}\sqrt{\frac{393-1}{393-20}}(0.68) = 91.9$

$n = 40$ $P(|\bar{X}-\mu| > \Delta) \approx 0.10 = 1 - P(|\bar{X}-\mu| < \Delta)$

$= 1 - P(-\Delta < |\bar{X}-\mu| < \Delta) = 1 - [\Phi(\bar{X}-\mu < \Delta) - \Phi(-\Delta < \bar{X}-\mu)] = 1 - [\Phi(\bar{X}-\mu < \Delta) - 1 + \Phi(-\Delta < \bar{X}-\mu)]$

$= 2 - [2\Phi(\bar{X}-\mu < \Delta)] = 2[1 - \Phi(\bar{X}-\mu < \Delta)] = 0.1$; $\Phi\left(\frac{\bar{X}-\mu}{\sigma_{\bar{X}}} < \frac{\Delta}{\sigma_{\bar{X}}}\right) = 0.95$

$\Delta_{0.1} = \frac{590}{\sqrt{40}}\sqrt{\frac{393-1}{393-40}}(1.65) = 162$

$\Delta_{0.5} = \frac{590}{\sqrt{40}}\sqrt{\frac{393-1}{393-40}}(0.68) = 66.8$

$n = 80$

$\Delta_{0.1} = 122$

$\Delta_{0.5} = 50$

16. a) The mean is a random variable.

b) True, a 95% confidence interval contains the mean and has a total probability of 0.95

c) True, a 95% confidence interval contains 95% of the population

d) True, 95% of a 100 is 95.

17. A 90% confidence interval $(1-\alpha)$ for average number of children per household is (min = 0.7, max = 2.1). Yes a confidence interval describes a random interval from a lower and upper bound that contains the mean.

18. 90% Confidence Interval: $P(\text{Mean}|\text{Interval}) = \sum_{0.9}^{n} p^n = 90\%^2 = \boxed{81\%}$

$1 - P(\text{Mean}|\text{Interval}) = P(\text{No Mean}|\text{Interval}) = 1 - 0.81 = \boxed{19\%}$

19. One-sided Confidence Interval

k be chosen $(-\infty, \bar{X} + ks_{\bar{X}})$ that a 90% confidence interval for μ

$P(-\infty \le \mu \le \bar{X} + ks_{\bar{X}}) = 90\%$; $P(\mu \le \bar{X} + ks_{\bar{X}})$; $1.28 = \bar{X} + ks_{\bar{X}}$; $\boxed{k = \frac{1.28 - \bar{X}}{s_{\bar{X}}}}$

$P(\bar{X} - ks_{\bar{X}} \le \mu) = 0.95$; $\bar{X} - ks_{\bar{X}} = 1.65$; $k = \frac{\bar{X} - 1.65}{s_{\bar{X}}}$

20. $N = 8000$ condominium units; $n = 100$ sample size; $\bar{X} = 1.6$ motor vehicles; $\sigma_{\bar{X}} = 0.8$

$s_{\bar{X}} = \frac{s}{\sqrt{n}}\sqrt{1 - \frac{n}{N}} = \frac{0.8}{10}\sqrt{1 - \frac{100}{8000}} = 0.08$; Confidence interval $\bar{X} \pm z(0.025)s_{\bar{X}} = \bar{X} \pm 1.96(0.08) = (1.44, 1.76)$

- Total Number of Motor Vehicles $T = 8000 \times 1.6 = 12,800$; $s_T = Ns_{\bar{X}} = 640$ $= (11546, 14054)$

- 12% respondents planned $\hat{p} = 0.12$ with a proportion p.

Standard Error: $s_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n-1}}\sqrt{1 - \frac{100}{1000}} = 0.03$

$\hat{p} \pm 1.96 s_{\hat{p}} = (0.06, 0.18)$

$T = N\hat{p} = 960$; $s_T = Ns_{\hat{p}} = 240$

$T \pm 1.96 s_T$; $(490, 1450)$

At 95%, confidence intern suggests another sample size of 100 would contain a mean between (1.44 and 1.76)

21. To halve the width of a 95% confidence interval

$\frac{\bar{X} - \mu}{\sigma_{\bar{X}}} = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = 95\%$; $\frac{\bar{X} - \mu}{\sigma/\sqrt{4}} = 95\%$; $\frac{\bar{X} - \mu}{\sigma/2} = \frac{\bar{X} - \mu}{\sigma}\left(\frac{1}{2}\right) = 95\%/2 = 47.5\%$

22. $\bar{X} \pm s_{\bar{X}} = \bar{X} \pm 1 s_{\bar{X}} \Rightarrow z(\alpha/2) = 1$; $\alpha = 2(1 - 0.8413)$; Confidence Interval: $\boxed{1 - \alpha = 0.682}$

23 a) Show $\sigma_{\bar{X}}$ is largest when $p = \frac{1}{2}$; $\frac{d}{dp}s_{\hat{p}} = \frac{d}{dp}\sqrt{\frac{\hat{p}(1-\hat{p})}{n-1}}\sqrt{1 - \frac{100}{8000}} = 0$

$= \frac{1}{2} \cdot \frac{1 - 2p}{\sqrt{\hat{p}(1-\hat{p})/(n-1)}} = \frac{1}{2}(1 - 2p) = \frac{1}{2} - p$

$\boxed{p = 1/2}$

b) $s_{\hat{p}}^2 = \frac{\hat{p}(1-\hat{p})}{n-1}\left(1 - \frac{n}{N}\right)$ $\boxed{\text{"Unbiased Estimate of } Var(\hat{p})\text{"}}$

$s_{\hat{p}}^2 = \frac{\frac{1}{2}(1 - 1/2)}{n-1}\left(1 - \frac{n}{N}\right)$

$s_{\hat{p}}^2 = \frac{1}{4}\left(\frac{N-n}{N(n-1)}\right)$; $s_{\hat{p}} = \sqrt{\frac{1}{4}\left(\frac{N-n}{N(n-1)}\right)}$ $\boxed{\frac{1}{2}\sqrt{\left(\frac{N-n}{N(n-1)}\right)}}$

c) $\hat{p} \pm \sqrt{\frac{N-n}{N(n-1)}} \rightarrow \Phi(z) = \Phi(0.05)$

$P\left(\hat{p} - \sqrt{\frac{N-n}{N(n-1)}} < \frac{\bar{X} - \mu}{\sigma_{\bar{X}}} < \hat{p} + \sqrt{\frac{N-n}{N(N-1)}}\right) \rightarrow \Phi(z)$

$\lim_{n \to \infty} P\left(\hat{p} - \sqrt{\frac{N-n}{N(n-1)}} < \frac{\bar{X} - \mu}{\sigma \cdot \sqrt{n}} < \hat{p} + \sqrt{\frac{N-n}{N(n-1)}}\right)$

$P(\hat{p} < 0 < \hat{p}) = \Phi(z) = 2P(0 \le \hat{p}) - 1 = \Phi(z)$; $z = \boxed{0.9990}$

24. Sample size $= n$; Population size $= N$; Estimate of $\mu = \bar{X}_c = \sum_{i=1}^{n} c_i X_i$

a) Find the condition $[c_i]$ such that the estimate is unbiased.

$$\bar{X} = E\left[\sum_{i=1}^{n} c_i X_i\right] = \sum_{i=1}^{n} c_i E(X_i) = \mu \sum_{i=1}^{n} c_i = \mu(1) \quad \boxed{\sum c_i = 1}$$

b) $Var(\bar{X}_c) = Var\left(\sum c_i X_i\right) = \sum c_i^2 Var(X) = \sum c_i^2 \sigma^2 = \sigma^2 \sum c_i^2$

Applying a LaGrangian Multiplier: $L(c_1 \ldots c_n, \lambda) = \sigma^2 \sum_{i=1}^{n} c_i^2 + \lambda\left(\sum c_i - 1\right)$

$\frac{\partial L}{\partial c_i} = 0$; $\frac{\partial}{\partial c_i}\left[\sigma^2 \sum_{i=1}^{n} c_i^2 + \lambda\left(\sum_{i=1}^{n} c_i - 1\right)\right] = 0$; $\frac{d}{dc_i}\left[\sigma^2 \sum c_i^2\right] + \frac{d}{dc_i}\left[\lambda\left(\sum c_i - 1\right)\right] = 0$

$\sigma^2 \frac{\partial}{\partial c_i}\left[\sum c_i^2\right] + \lambda \frac{\partial}{\partial c_i} \sum c_i = 0$

Therefore, $\sum c_i = 1$; $\sum \frac{-\lambda}{2\sigma^2} = \frac{-n\lambda}{2\sigma^2} = 1$

$2\sigma^2 \sum c_i^{2-1} + \lambda = 0$; $2\sigma^2 c_i + \lambda = 0$

$\boxed{\lambda = \frac{-2\sigma^2}{n}}$

$2\sigma^2 c_i = -\lambda$

$c_i = \frac{-1}{2\sigma^2}\left(\frac{-2\sigma^2}{n}\right)$ $\boxed{= 1/n}$

$Cov(X_i, X_j) = -\sigma^2/(N-1)$

$c_i = \frac{-\lambda}{2\sigma^2}$

25. Lemma B

Section 7.3.2:

$= E(X_i X_j) - E(X_i)E(X_j)$

$E(X_i X_j) = \sum_{k=1}^{m} \sum_{l=1}^{m} \zeta_k \zeta_l P(X_i = \zeta_k \text{ and } X_j = \zeta_l) = \sum_{1}^{m} \zeta_k P(X_i = \zeta_k) \sum_{1}^{m} \zeta P(X_j \mid X_i)$

where $P(X_j \mid X_i) = \begin{cases} n_l/(N-1) & k \neq l \\ (n_l - 1)/(N-1) & k = l \end{cases}$

$$\sum \zeta_l P(X_j \mid X_i) = \sum_{l \neq k} \zeta_l \frac{n_l}{N-1} + \zeta_k \frac{n_k - 1}{N-1}$$

$$= \sum_{l=1}^{m} \zeta_l \frac{n_l}{N-1} - \zeta_k \frac{1}{N-1}$$

$$E(X_i X_j) = \sum_{k=1}^{m} \zeta_k \frac{n_k}{N}\left(\sum \zeta_l \frac{n_l}{N-1} - \frac{\zeta_k}{N-1}\right) = \frac{1}{N(N-1)}\left(\tau^2 - \sum_{k=1}^{m} \zeta^2 n_k\right)$$

$$= \frac{\tau^2}{N(N-1)} - \frac{1}{N(N-1)} \sum_{k=1}^{m} \zeta^2 n_k = \frac{N\mu^2}{N-1} - \frac{1}{N-1}(\mu^2 + \sigma^2)$$

$$= \mu^2 - \frac{\sigma^2}{N-1}$$

26. $U_i = 1$; i^{th} population member

$Cov(X_i X_j) = \mu^2 - \frac{\sigma^2}{N-1} - \mu^2 = \frac{-\sigma^2}{N-1}$

$Cov(Y_i, Y_j) = E(Y_i Y_j) - E(Y_i) \cdot E(Y_j)$

$= E(Y_i Y_j) - \sqrt{E(Y_i^2) - Var(Y_i)} \sqrt{E(Y_j^2) - Var(Y_j)}$

a) show $\bar{X} = n^{-1} \sum_{i=1}^{N} U_i X_i$; $E(\bar{X}) = \frac{1}{n} \sum_{i=1}^{n} B$ $\frac{1}{n} \sum_{i=1}^{N} U_i X_i$

b) $P(U_i = 1) = n/N$; $E(U_i) = \frac{1}{N} \sum_{i=1}^{n} U_i P(U_i = 1) = \frac{n}{N}(1) = \boxed{\frac{n}{N}}$

c) $Var(U_i) = E[U_i^2] - E[U_i]^2 = \frac{1}{N^2} \sum_{i=1}^{n} U_i^2 P(U_i = 1) - \left[\frac{1}{N} \sum U_i P_0\right]^2$

$= \frac{n}{N} - \left(\frac{n}{N}\right)^2 = \boxed{\frac{n}{N}\left(1 - \frac{n}{N}\right)}$

d) $E(U_i U_j) = \frac{\sum_i^n \sum_j^n U_i U_j P(U_i, U_j)}{N^2} = \frac{n^2}{N^2}$

e) $Cov(U_i, U_j) = E(U_i U_j) - E(U_i)E(U_j) = \frac{n^2}{N^2} - \left(\frac{n}{N}\right)\left(\frac{n}{N}\right) = 0$

f) $Var(\bar{X}) = E(\bar{X}^2) - E(\bar{X})^2 = \frac{n}{N^2} - \frac{n}{N^2} = 0$

27. Population size (N) is unknown ; $n \le N$. Show will generate a simple random sample.

a) List $\{X_1, X_2 \dots X_n \dots X_N\}$ b) For $k = 1, 2 \dots$ i) $X_{(n+k)} = X_{n-1+k} + X_i$; $X_{i=1} = X_1$

$n = \{X_1, X_2 \dots X_n\}$ $k < N$ ii) $\frac{n}{(n+k)} = \frac{X_1, X_2 \dots X_n}{X_1, X_2 \dots X_{n+k}}$

$N = \{X_1, X_2 \dots X_n\}$

28. Randomized Response: $\underbrace{\text{Spin an Arrow - Draw ball from urn.}}_{\text{Action}}$ $\underbrace{\text{Choice A | choice B}}_{\text{Result}}$ } Interview

Records Response } Interview

R = Proportion Yes ; $p = P(\text{Response}_{\text{Yes}} | \text{Statement \#1})$ "Randomized Device"

$r = P(\text{Yes})$ q = proportion of Characteristic A. $= P(\text{Statement \#1})$

a) Show $r = (2p-1)q + (1-p)$; Hint: $P(\text{yes}) = P(\text{yes} | \text{Statement \#1}) \times P(\text{Statement \#1}) + P(\text{yes} | \text{Statement \#2}) P(\#2)$

$r = p \cdot q + (1-p)(1-q)$

b) If r, what is q?

$$\boxed{q = \frac{r - 1 + p}{(2p-1)}}$$

$2pq - q + 1 - p$

c) $E(R) = r$ and propose Q, for q. Show the expected estimator is unbiased.

$E(R) = \sum_{i=1}^{2} P(\text{yes} | \text{Statement \#}i) P(\text{Statement \#}i) = P(\text{yes} | \text{Statement \#1}) P(\text{Statement \#1}) + P(\text{yes} | \text{Statement \#2}) P(\#2) = r$

$E(Q) = E\left(\frac{R - (1-p)}{2p-1}\right) = \frac{E(R) - (1-p)}{2p-1} = \frac{r - 1 + p}{2p-1} = q$

d) Show $Var(R) = \frac{r(1-r)}{n} = E[R^2] - E[R]^2 = \frac{1}{(2p-1)^2} Var(R - (1-p)) = \frac{1}{(2p-1)^2} \frac{r(1-r)}{n}$

e) $Var(Q) = E[Q^2] - E[Q]^2 = \frac{1}{(2p-1)^2} \frac{r(1-r)}{n}$

29. a. $P(\text{yes} | \text{Statement \#3})$ $P(\text{Statement \#2})$

$r = qp + t(1-p)$ … $+ P(\text{yes} | \dots) P(\#3)$

b) $E(Q) = \frac{E(R) - t(1-p)}{p} = q$

c) $Var(Q) = \frac{r(1-r)}{np^2} = \frac{[qp + t(1-p)](1 - qp + t(1-p))}{np^2} = \frac{qp - q^2p^2 + qpt(1-p) + t(1-p) - qpt(1-p) + t^2(1-p)^2}{np^2}$

30. Problem #28 ; $Var(Q) = \frac{r(1-r)}{(2p-1)^2 n}$ } Difference

~~Problem #29~~ $Var(Q) = \frac{r(1-r)}{np^2}$

31. $N = 8000$ condominium units ; $n = 100$ sample size ; $\bar{X} = 1.6$ motor vehicles

$s_{\bar{X}} = \frac{s}{\sqrt{n}}\sqrt{1-\frac{n}{N}} = \frac{0.8}{\sqrt{100}}\sqrt{1-\frac{100}{8000}} = 0.08$ } Standard error ; $\sigma = 0.8$ motor vehicles

confidence interval { $\bar{X} \pm 1.96 s_{\bar{X}} = (1.44, 1.76)$; $T = 8000 \times 1.6 = 12{,}800$ } Total

$\underbrace{s_T = N s_{\bar{X}} = 640}_{\text{Total standard error}}$; $T \pm 1.96 s_T$ $(11546, 14054)$ } Interval of total.

12% planned to sell their condo $[\hat{p} = 0.12]$; $s_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n-1}}\sqrt{1-\frac{100}{8000}} = 0.03$

$\hat{p} \pm 1.96 s_p (0.06, 0.18)$; $T = N\hat{p} = 960$; $s_T = N s_p = 240$; $T \pm 1.96 s_T$ $(490, 1430)$

What is the sample size for 95% confidence interval to have 500 width (ΔT) of 500?

$T + 1.96 s_T - T + 1.96 s_T = 2 \cdot 1.96 s_T = 500$; $s_T = 500/2 \cdot 1.96 = 127.55$

$= N s_p = 8000 s_p$; $s_p = \frac{127.55}{8000} = 1.59 \times 10^{-2} = \sqrt{\frac{0.12(1-0.12)}{n-1}}\sqrt{1-\frac{n}{8000}}$

$2.53 \times 10^{-4} = \frac{0.12(0.88)}{n-1}\left(\frac{8000-n}{8000}\right) \Rightarrow 2.53 \times 10^{-4} n - 2.53 \times 10^{-4} = \frac{66}{625}\left(\frac{8000-n}{8000}\right)$

$2.53 \times 10^{-4} n - 2.53 \times 10^{-4} = \frac{66}{625} - \frac{33}{250000} n$; $2.66 \times 10^{-4} n = 0.1058$; $\boxed{n = 397.9}$

32. $N = 12{,}000$ units ; $n = 200$ sample size ; $\hat{p} = 0.18$

a) What is $s_{\hat{p}}$? $s_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n-1}}\sqrt{1-\frac{n}{N}} = \sqrt{\frac{0.18(1-0.18)}{200-1}}\sqrt{1-\frac{200}{12000}} = 0.027$

$P\left(\frac{|p-\hat{p}|}{s_p} < z(\alpha/2)\right) = 1 - \alpha = 0.90$; $\alpha = 0.05$; $z(0.1/2) = z(0.05) = z(1-0.05) = z(0.95)$
$= 1.65$

$P(|p-\hat{p}| \le s_{\hat{p}} \cdot 1.96) = 0.95$; $P(-1.96 s_{\hat{p}} \le |p - \hat{p}| \le 1.96 s_{\hat{p}}) = P(\hat{p} - 1.96 s_p \le p \le \hat{p} + 1.96 s_{\hat{p}})$

$= P(p \le \hat{p} + 1.96 s_{\hat{p}}) - P(\hat{p} - 1.96 s_p \le p) = P(p \le \hat{p} + 1.96 s_{\hat{p}}) - (1 - P(p \le \hat{p} + 1.96 s_p))$

$= 2P(p \le \hat{p} + 1.96 s_p) - 1 = 0.95$; $P(p \le \hat{p} + 1.96 s_p) = \frac{1.95}{2} = 0.975$ → solve

... or ... $(\hat{p} - 1.65 s_p, \hat{p} + 1.65 s_p) = (0.18 - 1.65 \cdot 0.027, 0.18 + 1.65 \cdot 0.027)$
$\boxed{= (0.135, 0.225)}$

b) $\hat{p}_1 = 0.12$, $\hat{p}_2 = 0.18$; $\hat{d} = \hat{p}_1 - \hat{p}_2$

$Var(\hat{d}) = s_p^2 = E[\hat{d}^2] - E[\hat{d}]^2 = 1$

$= E[(\hat{p}_1 - \hat{p}_2)^2] - E[\hat{p}_1 - \hat{p}_2]^2$

$= E[\hat{p}_1^2] - 2E[\hat{p}_1\hat{p}_2] + E[\hat{p}_2^2] - E[\hat{p}_1 - \hat{p}_2]^2$

$= \hat{p}_1 - 2\hat{p}_1\hat{p}_2 + \hat{p}_2 - (\hat{p}_1 - \hat{p}_2)^2$

$= \hat{p}_1 - 2\hat{p}_1\hat{p}_2 + \hat{p}_2 - \hat{p}_1^2 + \cancel{2\hat{p}_1\hat{p}_2} - \hat{p}_2^2$

$\boxed{= p_1(1-\hat{p}_1) + p_2(1-\hat{p}_2)}$

Standard Error (unbiased):

$s_{\bar{X}}^2 = \frac{1}{n}\sum_i^n E(X_i^2) - E(\bar{X}^2) = \frac{1}{n}\sum[Var(X) + E(X)^2] - [Var(\bar{X}) + E(\bar{X})^2]$

$= \frac{1}{n}\left[p_1(1-p_1) + p_2(1-p_2) + E(\hat{d})^2\right] - \left[\frac{1}{n^2}\sum Var(X_i) + \frac{1}{n^2}\sum\sum Cov(X_i X_j) + E(X)^2\right]$

$= \frac{p_1(1-p_1) + p_2(1-p_2)}{n}\left[1 - \frac{1}{n^2}[E(p_1 p_2) - E(p_1)E(p_2)]\right]$

$= \frac{p_1(1-p_1) + p_2(1-p_2)}{n}\left[1 - \frac{1}{n^2}\frac{N}{n^2}\frac{(n-1)}{(N-1)}\right]$

$\boxed{= \frac{p_1(1-p_1) + p_2(1-p_2)}{n}\left[1 - \frac{N}{n^4}\frac{(n-1)}{(N-1)}\right]}$

d. 99% confidence Interval: $\hat{d} + \hat{z}\left(\frac{1+(1-0.01)}{2}\right) s_x < \hat{d} < \hat{d} + \hat{z}\left(\frac{1-0.01}{2}\right) s_x$

$$\hat{d} - \hat{z}^{x} 2.57\left(\sqrt{\frac{0.12(1-0.12)+0.18(1-0.18)}{100}}\left[1-\frac{400}{100^4}\frac{(100-1)}{(400-1)}\right]\right)$$

$$-0.06 - 2.57(5.03\times10^{-2}) < d < \frac{3}{50} + 2.57(5.03\times10^{-2})$$

$$-\frac{3}{25} < d < 6.9\times10^{-2}$$

95% confidence Interval:

$$-0.06 - 1.96(5.03\times10^{-2}) < d < -0.06 + 1.96(5.03\times10^{-2})$$

$$-0.159 < d < 3/25$$

90% confidence Interval:

$$-0.06 - 1.65(5.03\times10^{-2}) < d < -0.06 + 1.65(5.03\times10^{-2})$$

$$-0.143 < d < 0.023$$

No, there is little difference for a 99% confidence interval ranging from $\left(\frac{-3}{25}, \frac{6.9}{100}\right)$

33. n = simple random samples, two proportions, $\hat{p}_1$ and $\hat{p}_2 \approx 0.5$

What should the sample size be for $\hat{p}_1 - \hat{p}_2 < 0.02$?

standard error of proportions: $s_{\hat{p}_1,\hat{p}_2} = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}} < 0.02$

$$p_1(1-p_1) + p_2(1-p_2) = 0.0004 \cdot n$$

$$\frac{2p_1(1-p_1)}{0.0004} = \frac{2\cdot0.5\cdot0.5}{0.0004} = \boxed{1.25\times10^3}$$

34. P(Problem #1) = 3% population

P(Problem #2) = 40% population

a) $s_{\hat{p}_1,\hat{p}_2} = \sqrt{\frac{p_1(1-p_1)}{n_2} + \frac{p_2(1-p_2)}{n_2}}$

Same population size

$$s_{\hat{p}_1,\hat{p}_2} = 0.01 > \sqrt{\frac{0.03(0.97)+0.4(0.6)}{n}}$$

$$n > \frac{2.691\times10^{-1}}{1\times10^{-4}}$$

$$\boxed{n > 2691}$$

b) $n > \frac{2.691\times10^{-1}}{0.01}$

$n > 26.91$

c)

35. N = Population size = 2000

with n = 25 values.

104	109	111	109	87
86	80	119	88	122
91	103	99	108	96
104	98	98	83	107
79	87	94	92	97

a) Calculate unbiased estimate of population mean.

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{25}\left[104 + \cdots + \cdots + \cdots + 97\right] = \boxed{98}$$

b) Calculate unbiased estimate of population variance

$$\sigma_{\bar{x}}^2 = \left(1-\frac{1}{N}\right)s^2 = \left(1-\frac{1}{2000}\right)\frac{1}{2}\left\{E[X^2] - E[X]^2\right\}$$

$$= \frac{1999}{2000}\left[\frac{1}{n}\sum_{i=1}^{25} x_i^2 + \left[\frac{1}{n}\sum(x_i)\right]^2\right] \cdot \frac{1999}{48000}\left[\frac{243505}{25} - \left(\frac{2450}{25}\right)^2\right]$$

$$= \boxed{136} \quad \frac{\sigma_s^2}{n}\left(1-\frac{n}{N}\right) = \boxed{5.37}$$

c) Approximate a 95% confidence interval.

$$P\left(\left|\frac{X-\bar{X}}{\sigma_s}\right| < z\left(1-\frac{\alpha}{2}\right)\right) = 0.95 \;;\; X - \sigma_s z\left(1-\frac{\alpha}{2}\right)$$

$$2P\left(x-\bar{x} < \sigma_s z\left(1-\frac{\alpha}{2}\right)\right) = 0.95 \;;\; = \sigma_s 1.96$$

$\boxed{98 \pm 4.54}$, $\boxed{196{,}000 \pm 23{,}323}$

36. Simple Random Sampling: $\bar{x}^2$ is unbiased estimate of μ^2. Simple random sampling is an unbiased estimate of $\bar{\mu}^2$ when there is true <u>random</u> sampling, for example, each value has equal probability. Otherwise, the $\bar{x}^2$ is not random, and biased.

37. Population Mean $= \mu$; <u>Survey #1</u> · <u>Survey #2</u>

$\bar{X}_1 = \text{Mean}$, $\bar{X}_2 = \text{Mean}$ } Unbiased. $X = \alpha \bar{X}_1 + \beta \bar{X}_2$

$\sigma_{\bar{X}_1} =$ Standard Error $\sigma_{\bar{X}_2} =$ Standard Error

a) Find conditions for α and β which are an unbiased combination.

$$Var(\mu) = E[\mu^2] - E[\mu]^2 = \frac{(\alpha\bar{X}_1 + \beta\bar{X}_2)^2}{n} - \left[\frac{(\alpha\bar{X}_1 + \beta\bar{X}_2)}{n}\right]^2 = \frac{(\alpha\bar{X}_1)^2 n + 2(\alpha\beta\bar{X}_1\bar{X}_2)n + (\beta\bar{X}_2)^2 n - (\alpha\bar{X}_1)^2 - 2(\alpha\beta\bar{X}_1\bar{X}_2) - (\beta\bar{X}_2)^2}{n^2}$$

$$= \frac{(\alpha\bar{X}_1)^2(n-1) + 2(\alpha\beta\bar{X}_1\bar{X}_2)(n-1) + (\beta\bar{X}_2)^2(n-1)}{n^2} = \frac{n-1}{n^2}\left[\alpha\bar{X}_1 + \beta\bar{X}_2\right]^2$$

$$Var(\mu) = Var(\alpha\bar{X}_1 + \beta\bar{X}_2) = \alpha^2 Var(\bar{X}_1) + \beta^2 Var(\bar{X}_2) \quad \frac{(n-1)^2}{n^2}\left[\alpha\bar{X}_1 + \beta\bar{X}_2\right]^2$$

$$E(X) = \alpha E(\bar{X}_1) + \beta E(\bar{X}_2) \qquad = \frac{n-1}{n^2}\left[(\alpha\bar{X}_1)^2 + 2\alpha\beta\bar{X}_1\bar{X}_2 + (\beta\bar{X}_2)^2\right]$$

$$= (\alpha+\beta)\mu$$

$\boxed{(\alpha+\beta)=1}$ "unbiased"

$$\alpha^2 Var(\bar{X}_1) + \beta^2 Var(\bar{X}_2) = \alpha^2[\sigma_{\bar{X}_1}^2] + (1-\alpha)^2[\sigma_{\bar{X}_2}^2]$$

$$\boxed{= \alpha^2[\sigma_{\bar{X}_1}^2] + \sigma_{\bar{X}_2}^2 + \alpha^2[\sigma_{\bar{X}_2}^2] - 2\alpha[\sigma_{\bar{X}_2}^2]}$$

b) $\frac{d}{d\alpha}[\alpha^2[\sigma_{\bar{X}_1}^2] + \sigma_{\bar{X}_2}^2 + \alpha^2[\sigma_{\bar{X}_2}^2] - 2\alpha[\sigma_{\bar{X}_2}^2]]$

$$\boxed{\alpha = \frac{\sigma_{\bar{X}_2}^2}{[\sigma_{\bar{X}_1}^2 + \sigma_{\bar{X}_2}^2]}} \; ; \; \boxed{\beta = \frac{\sigma_{\bar{X}_1}^2}{\sigma_{\bar{X}_1}^2 + \sigma_{\bar{X}_2}^2}}$$

38. $X_1 \dots X_n$ be random sample. Show $\frac{1}{n}\sum_{i=1}^{n} X_i^3$ is an unbiased estimate of $\frac{1}{N}\sum_{i=1}^{N} X_i^3$

$E[X^3] = \frac{1}{N}\sum_{i=1}^{N} X_i^3$ ← No β to incorporate, no bias and no shift.

39. N = Population of items. How large should a sample be to find a defective item?

k = Defective items.

Assuming $p = 0.95$; n = sample size ; $k = 1$; $P\left(1 < \frac{X-M}{\sigma_x} < k\right) = 0.95$

$$1 - \frac{N-k}{N} \times \frac{N-k-1}{N-1} \times \cdots \times \frac{N-k-n+1}{N-n+1} > 0.95 \quad \text{if } k = 1$$

$$1 - \frac{N-1}{N} \times \frac{N-1-1}{N-1} \times \cdots \times \frac{N-n}{N-n+1} > 0.95 \; ;$$

40. i^{th} member has $P(i) = \frac{n - n_i}{N - i + 1}$

$$= \frac{(N-n)!n!}{N!} = \boxed{\frac{N-n}{N-(i+1)}} = \frac{1}{\binom{N}{n}}$$

$$1 - \left(\frac{N-n}{N}\right)^k > 0.95$$

$$\left(\frac{N-n}{N}\right)^k < 0.05$$

$$\log(N-n) - \log(N) \approx \frac{\log(0.05)}{k}$$

$$n \approx N - e^{\left(\frac{\log(0.05)}{k} + \log(N)\right)}$$

$$\approx 50\%$$

41. D = Audited value − Book value

$N\bar{D}$, where N = Population size.

Inventory value.

a) Prove unbiased estimate

$$\boxed{E[N\bar{D}] = \frac{N}{n}\sum^{N} D = NE[D]}$$

b.) Variance of Estimate: $Var(N\bar{D}) = E[N\bar{D}^2] - E[N\bar{D}]^2 = N^2E[\bar{D}^2] - N^2E[\bar{D}]^2 = N^2[E[\bar{D}^2] - E[\bar{D}]^2]$

c) Population Parameter $[\tau]$; Estimate $T = N\bar{X}$; Variance of Estimate $\sigma_T^2 = N^2\sigma_{\bar{X}}^2 = N^2[E[\bar{X}^2] - E[\bar{D}]^2]$

The proposed method would be as accurate. $= N^2\sigma_{\bar{D}}^2 = N^2[E[\bar{D}^2] - E[\bar{D}]^2]$

d) Estimation of Ratio: $r = \frac{\sum y_i}{\sum x_i} = \frac{\mu_D}{\mu_X}$

A ratio estimate would provide advantage to a differently sized pool of population. In the listed case of part a, b, or c, then there would be no difference.

42. Population Correlation Coefficient: $\rho = \frac{\sigma_{xy}}{\sigma_x \cdot \sigma_y} = \frac{\frac{1}{N}\sum_i (x_i - \mu_x)(y_i - \mu_y)}{\frac{1}{N}\sum_i (x_i - \mu_x)^2 \frac{1}{N}\sum_i (y_i - \mu_y)^2}$

43. Example D: Section 7.3.3

$\boxed{= \frac{1}{N}\left(\frac{1}{\sum (x - \mu_x)\sum (y - \mu_y)}\right)}$

44. $\bar{X} = 2.2$, $\sigma = 0.7$, $\rho[\text{\# occupancy} \mid \text{\# Motor vehicles}] = 0.85$

Estimate Population ratio of # Motor-Vehicles per occupant + S.E.

Population Ratio: $\frac{\mu_2}{\mu_1} = \frac{\bar{X}_2}{\bar{X}_1} = \frac{\text{2.2 motor vehicle per car}}{\text{2.2 motor vehicle per car}} = 0.727$

Standard Error: $Var(r) = \frac{1}{n}\left(1 - \frac{n-1}{N-1}\right)\frac{1}{\mu_1^2}\left(r^2\sigma_1^2 + \sigma_2^2 - 2r\sigma_{12}\right)$

$= \frac{1}{n}\left(1 - \frac{n-1}{N-1}\right)\frac{1}{\mu_1^2}\left(r^2\sigma_1^2 + \sigma_2^2 - 2r\rho\sigma_x\sigma_y\right)$

$= \frac{1}{100}\left(1 - \frac{100-1}{8000-1}\right)\frac{1}{2.2^2}\left(0.727^2 0.7^2 + 0.8^2 - 2(0.727)(0.85)(0.8)(0.7)\right)$

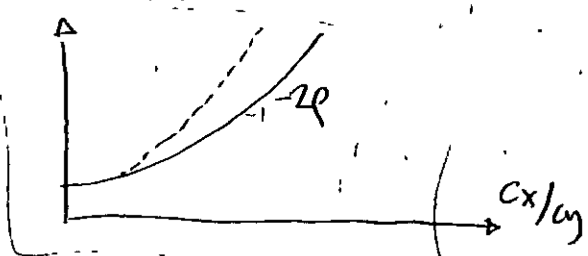
$= 4.225 \times 10^{-4}$; $\boxed{SE = 2.05 \times 10^{-2}}$

Confidence Interval (95%): $0.95 = P\left(\frac{X - \bar{X}}{\sigma_{\bar{x}}} < z(1 - \frac{\alpha}{2})\right)$; $0.727 \pm 1.96(\sigma_{\bar{x}})$

$2.05 \pm 1.96(2.05 \times 10^{-2})$

$\boxed{0.727 \pm 0.04}$

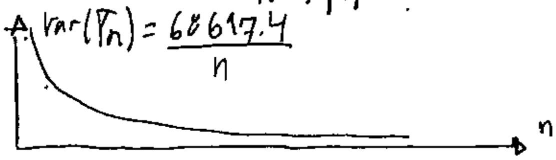
44. $\frac{Var(\bar{Y}_R)}{Var(\bar{Y})} \approx 1 + \frac{C_x}{C_y}\left(\frac{C_x}{C_y} - 2\rho\right) = 1 + x^2 + x$



45. $\rho = 0.91$ $Var(\bar{Y}_R)$; Plot $Var(\bar{Y}_R)$ for $n = 64$

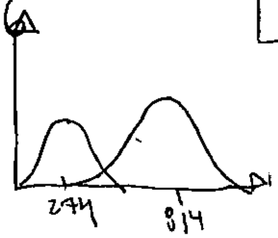
$-1 < \rho < 1$

$Var(\bar{Y}_R) = \frac{1}{n}\left(1 - \frac{n-1}{N-1}\right)\frac{1}{\mu_1^2}\left(r^2\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_2\sigma_1\right)$



46. $\sigma_{\bar{Y}_R} \approx 32.76$

$\sigma_{\bar{Y}} = 66.3$



47. $n=64$. Corollary B of Section 7.4: Approximate Bias of the ratio estimate of μ_y

$$E(\bar{Y}_R)-\mu_y \approx \frac{1}{n}\left(1-\frac{n-1}{N-1}\right)\frac{1}{\mu_x}\left(r\sigma_x^2-\rho\sigma_x\sigma_y\right)$$

$$E(\bar{Y}_R)-\mu_y \approx \frac{1}{64}\left(1-\frac{64-1}{393-1}\right)\frac{1}{274.8}\left(2.96\cdot 213.2^2-0.91\times 213.2\times 559.7\right)$$

$= 0.96$

$n=128$

$$\approx \frac{1}{128}\left(1-\frac{64-1}{393-1}\right)\frac{1}{274.8}\left(2.96\times 213.2^2-0.91\times 213.2\times 559.7\right)$$

$= 0.59$

48. $n=100$ Households; # people in Household [X]; Weekly Expenditure for Food [Y]

Total Number of Households $=100{,}000$

$\sum X_i = 320$: Total sum # people in Household

$\sum Y_i = 10{,}000$: Total Weekly Expenditure for Food

$\sum X_i^2 = 1250$

$\sum Y_i^2 = 1{,}100{,}000$

$\sum X_i Y_i = 36{,}000$

a) Estimate the ratio $r=\frac{\mu_y}{\mu_x} = \frac{(\sum Y_i)/N}{(\sum X_i)/N} = \frac{10{,}000}{3200} = \boxed{\frac{125}{4}}$

b) Confidence Interval (95%): $r \pm 1.96\,\sigma_r$

$$\frac{125}{4}\pm 1.96\cdot\left(\frac{1}{n}\left[1-\frac{n-1}{N-1}\right]\frac{1}{\mu_x^2}\left(r^2\sigma_x^2+\sigma_y^2-2r\rho\sigma_x\sigma_y\right)\right)$$

$$\frac{125}{4}\pm 1.96\left(\frac{1}{10^2}\left[1-\frac{10^2-1}{10^5-1}\right]\frac{1}{\left(\frac{320}{100}\right)^2}\left(\left[\frac{125}{4}\right]^2\left(\frac{1250}{100}-\left(\frac{320}{100}\right)^2\right)+\left(\frac{1{,}100{,}000}{100}-\left(\frac{10000}{100}\right)^2\right)-2\left(\frac{125}{4}\right)\left(\frac{36{,}000}{100}-\frac{320\cdot 10000}{100^2}\right)\right)\right)$$

$$\boxed{\frac{125}{4}\pm 1.34}$$

c. $\tau = N\bar{Y} = 100000\cdot\frac{10000}{100} = 10^7$; $\left(1-\frac{1}{10^5}\right)s^2 \cdot \left(\frac{1}{99995}\right)\frac{1}{100-1}\left(\frac{1{,}100{,}000}{100}-\left(\frac{10{,}000}{100}\right)^2\right) = 1.66\times 10^{-4}$

$100 \pm 1.65\cdot 1.66\times 10^{-4}$

$\boxed{100 \pm 0.00027}$

49. $N=1000$ squares

$n=50$ sampled

$Y=$ # of birds

$X=$ Area covered by vegetation

$\sum X_i = 3{,}000$

$\sum Y_i = 150$

$\sum X_i^2 = 225{,}000$

$\sum Y_i^2 = 650$

$\sum X_i Y_i = 11{,}000$

$P\left(\left|\frac{\bar{X}-\bar{Y}}{\sigma_x}\right| < z(\alpha/2)\right) = 0.90$

a) $r = \frac{\sum Y_i/n}{\sum X_i/n} = \frac{150}{3000} = \frac{1}{20}$

b) Standard Error:

$$s_R = \sqrt{Var(R)} = \sqrt{\frac{1}{n}\left(1-\frac{n-1}{N-1}\right)\frac{1}{\mu_x^2}\left(r^2\sigma_x^2+\sigma_y^2-2r\rho\sigma_x\sigma_y\right)}$$

$\sigma_x = \sqrt{\frac{1}{50-1}\left[225{,}000-50\cdot 60^2\right]} = 30$; $\sigma_{xy} = \frac{1}{50-1}\left[11{,}000-50\cdot 60\cdot 3\right] = 40.8$

$\sigma_y = \sqrt{\frac{1}{50-1}\left[650-50\cdot 3^2\right]} = 2.1$; $\rho = 0.68$

$$s_R = \sqrt{\frac{1}{50}\left(1-\frac{50-1}{1000-1}\right)\frac{1}{\left(\frac{3000}{50}\right)^2}\left(\left(\frac{1}{20}\right)^2 30^2+2.1^2-2\left(\frac{1}{20}\right)\cdot 30\cdot 2.1\cdot 0.68\right)}$$

$= 7.34\times 10^{-4}$

90% Confidence Interval: $r \pm 1.65\,s_R = \frac{1}{20}\pm 1.65\cdot 7.34\times 10^{-4} = 0.05 \pm 0.001$

c) Total Number of Birds:

$\tau = N\cdot\bar{Y} = 1000\cdot\frac{150}{50} = 3{,}000$

95% Confidence Interval

$\bar{T} \pm 1.96\, s_T = 3000 \pm 1.96\cdot 275 = 300 \pm 540$

Standard Error:

$$s_T = s_T\sqrt{\frac{N(N-n)}{n}} = N s_{\bar{X}} = 1000\sqrt{\frac{s_y^2}{n}\left(1-\frac{50}{1000}\right)} = 1000\sqrt{\frac{4}{50}\left(1-\frac{50}{1000}\right)} = 275$$

d) $T_R = \frac{\bar{Y}}{\bar{X}} \tau_x = R\tau_x$

$= \frac{150}{3000} 1000 \cdot \left(\frac{3000}{50}\right) = \boxed{3000}$

$S_{TR} = \sqrt{\frac{N^2}{n}\left(1-\frac{n-1}{N-1}\right)(R^2 s_x^2 + s_y^2 - 2R s_{xy})} = N S_{Rn}$

$= \sqrt{\frac{1000^2}{50^2}\left(1-\frac{50-1}{1000-1}\right)\left(\left(\frac{1}{20}\right)^2 30^2 + 2^2 - 2\left(\frac{1}{20}\right)40.8\right)}$

$= 28.7$ the standard error is much worse.

50. Standard Error of a Ratio Estimate:

$$\frac{|E(R)-r|}{\sigma_R} \le \frac{\sigma_{\bar{X}}}{\mu_x} = \frac{\sigma_x}{\mu_x}\sqrt{\frac{1}{n}\left(1-\frac{n-1}{N-1}\right)}$$

a) $Cov(R,\bar{X}) = E\left(\frac{\bar{Y}}{\bar{X}}\bar{X}\right) - E\left(\frac{\bar{Y}}{\bar{X}}\right)E(\bar{X})$

$= \frac{Cov(\bar{X},\bar{Y})}{\bar{X}}$

b $\frac{Var(\bar{Y}_R)}{Var(\bar{Y})} = 1 + \frac{C_x}{C_y}\left(\frac{C_x}{C_y} - 2\rho\right)$

$$\boxed{\left|\frac{E(R)-\bar{Y}_R}{\sigma_{\bar{X}}/\mu_x}\right| = 1 + \frac{C_x}{C_y}\left(\frac{C_x}{C_y}-2\rho\right)}$$

$\frac{\sigma_{XY}}{\bar{X}} = E[Y] - E(R)E[\bar{X}]$

$-\frac{\sigma_y \sigma_x \rho}{\bar{X}} = \frac{E(R)E[\bar{X}] - E[Y]}{\rho \cdot \sigma_y}$

$-\frac{\sigma_x}{\bar{X}} = \frac{E[R]E[\bar{X}] - E[Y]}{\sigma_y \rho}$

$$\boxed{\frac{\sigma_x}{\bar{X}} = \left|\frac{E(R)E[\bar{X}] - E[Y]}{\sigma_y \cdot \rho}\right| = \frac{|E(R) - E\left[\frac{\bar{Y}}{\bar{X}}\right]|}{\sigma_R}}$$

51. $E(\hat\theta) = \hat\theta + \frac{b_1}{n} + \frac{b_2}{n^2} + \cdots$; $\hat\theta$ = estimate of θ

$\frac{1}{n} \cdots \frac{1}{n}$ θ_j

$n = mp$ For $(j=1\cdots p)$

Estimate $\hat\theta_j$ from $m(p-1)$

$E(\hat\theta) = \hat\theta + \frac{b_1}{m(p-1)} + \frac{b_2}{[m(p-1)]^2} + \cdots$

P-pseudovalues: $V_j = p\hat\theta - (p-1)\hat\theta_j$ Prove $\theta_j = \frac{1}{p}\sum_{j=1}^{p} V_j$

$\hat\theta_j = \frac{1}{p}\sum_{j=1}^{p} V_j$

$E(\hat\theta_j) = \hat\theta + \frac{b_1}{m(p-1)} + \frac{b_2}{[m(p-1)]^2} + \cdots = \frac{1}{p}\sum_{j=1}^{p} V_j + \frac{b_1}{m(p-1)} + \frac{b_2}{[m(p-1)]^2} + \cdots = p\hat\theta - (p-1)\theta_j + \frac{b_1}{m(p-1)} + \frac{b_2}{[m(p-1)]^2} + \cdots$

$\frac{dE(\hat\theta_j)}{dp} = \hat\theta - \theta_j + \frac{b_1}{m(p-1)^2} - \frac{2b_2}{[m(p-1)]^3} + \cdots$

$\frac{d^2E(\hat\theta_j)}{d^2p} = \frac{+2b_1}{m(p-1)^2} + \frac{6b_2}{m(p-1)^4} + \cdots = 0$

$b_1 + \frac{3b_2}{m(p-1)^4} + \cdots = 0$

$b_1 = \frac{-3b_2}{m(p-1)^4}$

$E(\hat\theta) = \hat\theta - \frac{3b_2}{n(p-1)^4} + \frac{b_2}{n^2} + \cdots$

$= \hat\theta + b_2\left(\frac{1}{n^2} - \frac{3}{n(p-1)^4}\right) + \cdots$

52. $N_1 = N_2 = 1000$

$N_3 = 500$

10 observations

Stratum #1: 94 99 106 106 101 102 122 104 97 97

Stratum #2: 183 183 179 211 178 179 192 192 201 177

Stratum #3: 343 302 286 317 289 284 357 288 314 276

$\bar{X}_1 = 103$	$\sigma_1 = 7.4$
$\bar{X}_2 = 188$	$\sigma_2 = 11$
$\bar{X}_3 = 278$	$\sigma_3 = 38$

$\tau_1 = N \cdot \bar{X}_1 = 103000$; $\tau_2 = N\bar{X}_2 = 188000$; $\tau_3 = N\bar{X}_3 = 139,000$

103 ± 12; 188 ± 18; 278 ± 63

53.

a. $n=100$ Sample size Methods of Allocation. Sample size $\left(n_\ell = n \frac{W_\ell \sigma_\ell}{\sum_{k=1} W_k \sigma_k}\right)$

$\sum W_k \sigma_k = \frac{394}{2010} 8.3 + \frac{461}{2010} \cdot 13.3 + \frac{391}{2010} \cdot 15.1 + \frac{334}{2010} \cdot 19.8 + \frac{169}{2010} \cdot 24.5 + \frac{113}{2010} 26 + \frac{140}{2010} 35.2$

$= 1.63 + 3.05 + 2.94 + 3.29 + 2.06 + 1.46 + 2.59$

$\boxed{= 17}$ $Var(\bar{X}_{so}), Var(\bar{X}_{sp}), Var(\bar{X}_{SRS})$

Optimal Allocation

Farm Size	n_ℓ
0-40	9.631
41-80	17.9
81-120	17.3
121-160	19.4
161-200	12
201-240	8.6
241+	15.2

b. $Var(\bar{X}) = \frac{1}{n_\ell}\left(1 - \frac{n_\ell - 1}{N_\ell - 1}\right)\sigma_\ell^2$ "a scaled variance"

Farm Size	$Var(\bar{X})$
0-40	7.01
41-80	9.51
81-120	12.62
121-160	19.09
161-200	43.23
201-240	73.27
241+	73.64

"A scaled sample size to the true value."

c.

Farm Size:	$E(\bar{X}_s) = \sum W_\ell E(\bar{X}_\ell)$
0-40	1.06
41-80	3.74
81-120	4.73
121-160	5.73
161-200	3.54
201-240	2.81
241+	4.70

"Population Expectation"

$Var(\bar{X}_s) = \sum_{\ell=1}^{L} W_\ell^2 \left(\frac{1}{n_\ell}\right)\left(1 - \frac{n_\ell - 1}{N_\ell - 1}\right)\sigma_\ell^2$

0-40	1.98×10^{-2}
41-80	7.30×10^{-2}
81-120	6.44×10^{-2}
121-160	7.61×10^{-2}
161-200	1.74×10^{-2}
201-240	2.48×10^{-3}
240+	2.19×10^{-2}

"Population Variance"

Proportional Allocation

d. $n=10$ farms

Farm Size:	$Var(\bar{X}_s)$:
0-40	2.59×10^{-1}
41-80	9.12×10^{-1}
81-120	8.42×10^{-1}
121-160	1.05
161-200	4.02×10^{-1}
201-240	1.96×10^{-1}
241+	6.30×10^{-1}

"smaller sample size, variance"

e) $n = 70$ samples

Farm Size:	$Var(\bar{X}_s)$:
0-40	2.69×10^{-1}
41-80	5.00×10^{-1}
81-120	4.78×10^{-1}
121-160	5.27×10^{-1}
161-200	3.06×10^{-1}
201-240	2.32×10^{-1}
241+	3.99×10^{-1}

b) $Var(\bar{X}_{so}) = \frac{\left(\sum_{\ell=1} W_\ell \sigma_\ell\right)^2}{n}$ "Stratified Optimal Allocation" to stratified population

$Var(\bar{X}_{sp}) = \sum_{\ell=1}^{L} W_\ell^2 Var(\bar{X}_\ell) = \sum_{\ell=1}^{L} W_\ell^2 \frac{\sigma_\ell^2}{n_\ell}$ "Stratified Proportional Allocation" to total stratified population

$Var(\bar{X}_{SRS}) = \sum_{\ell=1}^{L} W_\ell^2 \frac{\sigma_\ell^2}{n_\ell} + \frac{\sum W_\ell (\mu_\ell - \mu)^2}{n_\ell}$ "Stratified Random Sampling" increases precision for diverse values of population.

$= \sum_{\ell=1}^{L} \frac{W_\ell^2 \left(\sigma_\ell^2 + (\mu_\ell - \mu)^2\right)}{n_\ell}$

$Var(\bar{X}_{so}) =$

54 a) $C = C_0 + C_1 n$; L Strata ; Find a function which minimizes the variance.

Start-up Cost (C_0); cost per observation (C_1)

Lagrangian Multiplier: $L(n_1, \dots n_L, \lambda) = \sum_{l=1}^{L} \frac{W_l^2 \sigma_l^2}{n_l} + \lambda\left(\sum_{l=1}^{L} n_l - n\right)$

$$\frac{\partial L}{\partial n_l} = -\frac{W_l^2\sigma^2}{n_l^2} + \lambda \; ; \; n_l = \frac{W_l \sigma_l}{\sqrt{\lambda}}$$

$$n = \frac{1}{\sqrt{\lambda}} \sum_{l=1}^{L} W_l \sigma_l \qquad \frac{1}{\sqrt{\lambda}} = \frac{n}{\sum_{l=1}^{L} W_l \sigma_l} \qquad n_l = n \frac{W_l \sigma_l}{\sum_{l=1}^{L} W_l \sigma_l}$$

$$L(n_1, \dots n_L, \lambda) = \sum_{l=1}^{L} \frac{W_l^2 \sigma_l^2}{n_l} + \lambda\left(\sum C_l n_l - C_1 n\right)$$

$$\frac{\partial L}{\partial n_l} = -\frac{W_l^2 \sigma_l^2}{n_l^2} + \lambda C_l = 0 \; ; \; n_l = \frac{W_l \sigma_l}{\sqrt{\lambda C_l}} \; ; \; n = \frac{1}{\sqrt{\lambda}} \sum_{l=1}^{L} \frac{W_l \sigma_l}{\sqrt{C_l}}$$

$$\frac{1}{\sqrt{\lambda}} = n \sum_{l=1}^{L} \frac{\sqrt{C_l}}{W_l \sigma_l} \qquad n_l = n W_l \sigma_l \sum_{l=1}^{L} \frac{\sqrt{C_l}}{W_l \sigma_l}$$

$$Var(\bar{X}_{st}) = \sum_{l=1}^{L} W_l^2 \left(\frac{\cancel{n_l}}{\cancel{n_l}}\right)\left(1 - \frac{\cancel{n_l - 1}}{\cancel{N_l - 1}}\right) \sigma_l^2$$

Neglecting finite population correction

$$= \sum_{l=1}^{L} \frac{W_l^2 \sigma_l^2}{n_l} = \frac{\sum W_l^2 \sigma_l^2}{n W_l \sigma_l} \sum \frac{W_l \sigma_l}{\sqrt{C_l}} = \frac{\sum_{l=1}^{L} (W_l \sigma_l)^2 / \sqrt{C_l}}{n}$$

b) $\boxed{Var(\bar{X}_{st}) = \frac{\sum (W_l \sigma_l)^2 / \sqrt{C_l}}{n}}$

c) $\boxed{n_l = n W_l \sigma_l \sum \frac{\sqrt{C_l}}{W_l \sigma_l}}$

55. a) Proportional Allocation of a population mean $\bar{X}_{sp} = \sum_{l=1}^{L} W_l \bar{X}_l = \sum W_l \left(\frac{1}{n_l} \sum X_{il}\right) = \frac{1}{n} \sum_{l=1}^{L} \sum_{i=1}^{n_l} X_{il}$ is utilized when W_L is large representation.

Optimal Allocation of a population mean $\bar{X}_{st} = \sum_{l=1}^{L} W_l \bar{X}_l$ is utilized when ~~a sample of each stratum is taken.~~

Being that ($H_s = 100$, $H_T = 100{,}000$) and ($L_s = 200$, $L_T = 500{,}000$) then optimal allocation of a population ~~seems best fit model~~ $\frac{1}{6}\bar{X}_H + \frac{5}{6}\bar{X}_L = \bar{X}_T$

b) $\sigma_H = 20$, $\sigma_L = 10$; Standard Error ; $S_{\bar{X}H} = \sqrt{\frac{20^2}{100}\left(1 - \frac{100}{100{,}000}\right)} = \boxed{2.00}$ $S_T = 0.6775$

c) A 95% confidence interval $S_{\bar{X}L} = \sqrt{\frac{10^2}{200}\left(1 - \frac{200}{500{,}000}\right)} = \boxed{0.71}$ would be best to determine allocation error. The currents allocation ($H_s = 100$, $L_T = 200$) provides an interval of (± 3.92, ± 1.39) by increasing the allocation to ($H_s = 200$, $L_s = 100$) would shift the interval to (± 2.77, ± 1.96). Also, the standard error of the population would shift from 0.6775 to 0.8886 and be of greater error.

d) Proportional Allocation provides a standard error of:

$\sqrt{\sigma_{\bar{X}}}$ $(= \sigma_{sp})$ 1.25 which is $\dots$ worse, $\frac{1}{4{,}000}$ $S_T =$

56. a) A survey of household expenditures in a city. [Stratification of expenditure type or district]
b) Examination of lead concentration in a large plot of land [concentration ranges]
c) Surveying the number of people who use elevators in a large building [Time of day]
d) Surveying television by time of day [Stratification of seasons]

57. Sample Pool: {1, 2, 2, 4, and 8} → (1, 2, 2) and (4, 8): $\bar{X}_{s1} = \frac{1}{3}(1) + \frac{1}{3}(2) + \frac{1}{3}(2) = \frac{5}{3}$; $\bar{X}_s = \frac{3}{5}\left(\frac{5}{3}\right) + \frac{2}{5}(6)$

$\bar{X}_{s2} = \frac{1}{2}(4) + \frac{1}{2}(8) = 6$. $= 1 + \frac{12}{5} = 1 + 2.4$ $\boxed{= 3.4}$

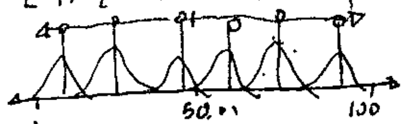
$\sigma_{\bar{X}_{s1}} = \sqrt{\frac{(1-5/3)^2 + (2-5/3)^2 + (2-5/3)^2}{3}} = 0.47$

$\sigma_{\bar{X}_{s2}} = \sqrt{\frac{(4-6)^2 + (8-6)^2}{2}} = 2.00$

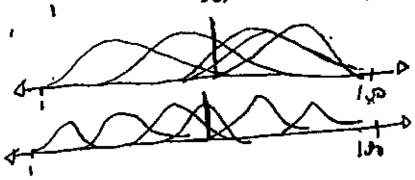
$Var(\bar{X}_s) = \left(\frac{3}{5}\right)^2\left(\frac{1}{3}\right)\left(1 - \frac{3-1}{5-1}\right) \cdot 0.47^2 + \left(\frac{2}{5}\right)^2\left(\frac{1}{2}\right)\left(1 - \frac{2-1}{5-1}\right)2^2$

$= 1.02$

58. $[n_1, n_2 \cdots\cdots n_{12}]$
$[n_1, n_2 \cdots\cdots n_{12}]$
$[n_1, n_2 \cdots\cdots n_{12}]$ ×100
$\vdots$
$[n_1, n_2 \cdots\cdots n_{12}]$



59.



60. $n_H = 100{,}000$, $\sigma_{\bar{X}_H} = 20$
$n_L = 500{,}000$ $\sigma_{\bar{X}_L} = 12$
Sample Size = 100
a) Optimal Allocation
Population Mean:
$\bar{X}_{so} = \sum_{l=1}^{L} W_l \cdot \bar{X}_l$
b) $\mu_H - \mu_L = W_H \bar{X}_H - W_L \bar{X}_L$
$= \frac{100}{100{,}000}\bar{X}_H - \frac{100}{500{,}000}\bar{X}_L$

61. $\mu(t) = \alpha + \beta t$
Sample size = n
@ times = 1, 2 and 3
a) Find W_1, W_2 and W_3, such that $\hat{\beta} = w_1\bar{X}_1 + w_2\bar{X}_2 + w_3\bar{X}_3$ is unbiased estimate of β. X_i denotes sample mean.

$\boxed{\frac{d}{dX}\hat{\beta} = W_1 + W_2 + W_3 = 0}$

and

$X \propto$ time: $\boxed{W_1 + W_2(2) + W_3(3) = 1}$

$-W_2 - W_3 = 1 - 2W_2 - 3W_3$

b) $W_1 = -W_2 - W_3$; $W_1 = 1 - 2W_2 - 3W_3$; $W_2 = 1 - 2W_3$

$W_1 + 1 - 2W_3 + W_3 = 0$

$W_1 + 1 - W_3 = 0$; $W_1 - W_3 = -1$; $\boxed{W_1 = -\frac{1}{2} \; ; \; W_3 = \frac{1}{2} \; ; \; W_2 = 0}$

62. Example B: Section 7.5.2

$\bar{X}_1 = 240.6$ $s_1^2 = 6827.6$
$\bar{X}_2 = 507.4$ $s_2^2 = 23790.7$
$\bar{X}_3 = 865.1$ $s_3^2 = 42573.0$
$\bar{X}_4 = 1716.5$ $s_4^2 = 152{,}099.6$

$\bar{X}_s = 832.5$
$s_{\bar{X}}^2 = \frac{1}{10}\sum_{l=1}^{4} W_l^2\left(1 - \frac{n-1}{N-1}\right)s_l^2 = 1282.0$

$s_{\bar{X}} = 35.8$

$T_s = 393\bar{X}_s = 327{,}172.$

$s_{T_s} = 393 s_{\bar{X}} = 14{,}069$

The estimate of $s_{\bar{X}} = 35.8$ has a standard error of

$\sigma_{\bar{X}} = \sqrt{\frac{s_X^2}{10}\left(1 - \frac{10}{393}\right)} = 11.2$

which demonstrates the sample size of $n = 10$ is small compared to the population size of $N = 393$.

63.
$X = [0, 1]$
$f(X) = X_0$: Vertical line test.
$\hat{I}(f) = \frac{1}{n}\sum_{i=1}^{n} f(x)$: Mean.

Standard Error:
$\sigma^2 = E[f(x)^2] - E[f(x)]^2$
$= \frac{1}{n}\left[\sum f(x)^2\right] - \left[\sum f(x)\right]^2$
$= \frac{1}{n^2}\left[n\sum f(x)^2 - \left(\sum f(x)\right)^2\right] =$

The actual

$\sigma_X = \sqrt{\frac{\sigma^2}{n}\left(1 - \frac{n}{N}\right)} = \sqrt{\frac{\frac{1}{1000^2}\left[\frac{1}{2\pi}\right]\left[2\pi\sum_{i=1}^{1000} e^{-X_i^2} - \sum e^{-X_i^2/2}\right]}{n}\left(1 - \frac{n}{N}\right)}$ where $n = 1000$

The population stratification would be best exampled as σ-intervals of separation.

64.

Stratum	N_ℓ	μ_ℓ	σ_ℓ
\$1000+	70	3000	1250
\$200-1000	500	500	100
\$1-200	10,000	90	30

a) Proportional Mean S.D.

Proportional Allocation $\bar{X} = \sum_{\ell=1}^{L} W_\ell X_\ell$ $Var(\bar{X}_s) = \frac{1}{n} \sum W_\ell \left(1 - \frac{n_\ell - 1}{N-1}\right) \sigma_\ell^2$

Optimal Allocation: $\mu = \frac{1}{N} \sum_{\ell=1}^{L} N_\ell \mu_\ell$ $var(\bar{X}_s) = \sum_{\ell=1}^{L} W_\ell^2 \left(\frac{1}{n_\ell}\right)\left(1 - \frac{n_\ell - 1}{N_\ell - 1}\right) \sigma_\ell^2$

Relative Sampling: $\frac{n_\ell}{N_\ell} = \frac{n}{N}$; $n_\ell = N_\ell \frac{n}{N} = n W_\ell$; $\boxed{\text{Proportional Allocation}}$

\$1-200: $n_\ell = n \left(\frac{10000}{10570}\right) = 0.945 n$

\$200-1000: $n_\ell = n \left(\frac{500}{10570}\right) = 0.47 n$

\$1000: $n_\ell = n \left(\frac{70}{10570}\right) = 0.66 n$

b) Two methods exist to compare the differences of population mean based upon proportional allocation and optimal allocation.

$n_\ell = n \frac{W_\ell \sigma_\ell}{\sum_{\ell=1}^{L} W_\ell \sigma_\ell}$ = $\boxed{\text{Optimal Allocation}}$

\$1-200: $\frac{28.5n}{900.5} = 0.03n = n_\ell$

\$200-1000: $0.05n = n_\ell$

\$1000: $0.09n = n_\ell$

$Var(\bar{X}_{sp}) - Var(\bar{X}_{so}) = \frac{1}{n} \sum_{\ell=1}^{L} W_\ell (\sigma_\ell - \bar{\sigma})^2$

$= \frac{1}{n}\left[\sum_{\ell=1}^{L} W_\ell \sigma_\ell^2 - \left(\sum_{\ell=1}^{L} W_\ell \sigma_\ell\right)^2\right]$ and $\frac{Var(\bar{X}_{sp})}{Var(\bar{X}_{so})} = 1 + \frac{\sum W_\ell (\sigma_\ell - \bar{\sigma})^2}{\left(\sum W_\ell \sigma_\ell\right)^2}$

65.

Time: 1950-1960

Adult White Females Population 1160,301 counties, North Carolina, South Carolina, Georgia.

a) Histogram: Bins by year b) Mean $= \frac{1}{N}\sum X_i$; Total Cancer Mortality: $T = N\bar{x}$

variance: $\frac{\sum (X_i - \bar{X})^2}{N}$; SD $= \sqrt{\frac{\sum (X_i - \bar{X})^2}{N}}$

c) $n = 25$

d) Mean $= \frac{1}{n}\sum X_i$; $T = N\bar{X}$ e) $Var(\bar{X}) = \sqrt{\frac{\sum (X_i - \bar{X})^2}{n}}$

f) $\bar{x} \pm 1.96 s_{\bar{x}}$, $T = N s_{\bar{x}}$ g) See (d-f) h) Ratio Estimator $(r) = \frac{\sum_{i=1}^{n} \bar{X}}{\sum_{i=1}^{N} \bar{X}}$ A ratio estimator would be effective to compare [illegible]

i) see c) j) see d) k) --- l) Separate and stratify by

66. The sampling procedure would be regions of the beach vs # of people. $n_\ell = n \frac{W_\ell \sigma_\ell}{\sum W_\ell \sigma_\ell}$ $\boxed{\text{Optimal Allocation}}$ or $n_\ell = n W_\ell$ $\boxed{\text{Proportional Allocation}}$

n) Further strata may resolve or represent the data better.

Stratified means would be calculated with proportional allocation for variance.

67. a) i) Proportion of female-headed Families $[n = 500]$; $\bar{X} = \sum_{\ell=1}^{L} W_\ell X_\ell$; $Var(\bar{X}_s) = \frac{1}{500} \sum (W_\ell)\left(1 - \frac{500-1}{43086-1}\right) \sigma^2$

ii) The average number of children per family. $s_x = \sqrt{\frac{\sigma_{xs}^2}{n}\left(1 - \frac{n}{N}\right)}$; $\bar{X} \pm 1.96 s_x$

representation would be best demonstrated by region of Cyberville. $\bar{X}_R = \sum_{\ell=1}^{L} W_\ell X_\ell$

$Var(\bar{X}_{sR}) = \frac{1}{500} \sum (W_\ell)\left(1 - \frac{500-1}{43086-1}\right) \sigma^2$

$\bar{X} \pm 1.96 s_x$

iii) The proportion of heads of household who did not receive a high school diploma

$\bar{X} \pm 1.96 \sqrt{\frac{\sigma^2}{n}\left(1 - \frac{n}{N}\right)}$

iv) Average Family income with a sample size of 500 could be represented as a proportional distribution by region. $\bar{X} = \frac{1}{n}\sum_i X_i$; $Var(\bar{X}_{sp}) = \frac{1}{n}\sum W_e \left(1 - \frac{n_e - 1}{N - 1}\right)\sigma^2$

b) i) 100 samples of $n = 400$; Average Family income: $\mu = \frac{1}{N}\sum_{e=1}^{L}\sum_{i=1}^{N_e} X_{ie}$

ii) $SO\,[\sigma] \cdot \sqrt{\frac{\sum (X_i - \bar{X})^2}{n}}$ iv) $CDF = \int_{-\infty}^{n} f(x)\,dx = n\sum f(x)^2 - (\sum f(x))^2$

iii) Not applicable

vi) $\bar{X} \pm 1.96 s_{\bar{x}}$ .. vii) --- c) Boxplot:

Histogram:

d) i) see c) ii) iii) see c) e) f) --- see c)

Chapter 8: Estimation of Parameters and Fitting of Probability Distribution

1. Poisson $[f(\lambda) = \frac{\lambda^k}{k!}e^{-\lambda}]$; $\lambda = \frac{\text{counts}}{\text{total}} = \frac{10,220}{12,169} = 0.8398$

n	Observed	Poisson
0	5267	5254
1	4436	4413
2	1800	1853
3	534	519
4	111	109
5+	21	957
	12,169	

The expected counts do match the observed counts.

2. $\lambda = \frac{1168}{300} = 3.89$

n	Frequency	Expected
0	14	6
1	30	24
2	36	46
3	68	60
4	43	59
5	43	46
6	30	30
7	14	16.5
8	10	8.05
9	6	3.5
10	4	1.36
11	1	0
12	1	
13+	0	
	300	

3. a) Estimate λ for each dataset.

Total 1: ~~400~~ Total 2: ~~400~~ Total 3: ~~400~~ Total 4: 480

$\boxed{\lambda_1 = 0.6825 \quad \lambda_2 = 1.3225 \quad \lambda_3 = 1.80 \quad \lambda_4 = 4.65}$

b) Concentration #1: $E_1(X) = \lambda_1$; $Var_1(X) = E_1[X^2] - E_1[X]^2 = 1.28 - 0.46 = 0.82$; $SE = \sqrt{\frac{\sigma^2}{n}} = 0.045$

95% confidence Interval: $P\left(\left|\frac{X - \bar{X}}{\sigma_{\bar{x}}}\right| \le z(\alpha/2)\right) = 0.95$; $\bar{X} \pm 1.96\sigma_{\bar{x}}$ $\boxed{0.6825 \pm 0.088}$

Concentration #2: $E_2(X) = \lambda_2$; $Var_2(X) = E_2[X^2] - E_2[X]^2 = 3.03 - 1.3225 = 1.71$; $SE = \sqrt{\frac{\sigma^2}{n}} = 0.064$

95% Confidence Interval: $P\left(\left|\frac{X - \bar{X}}{\sigma_{\bar{x}}}\right| \le z(\alpha/2)\right) = 0.95$; $\bar{X} \pm 1.96\sigma_{\bar{x}}$ $\boxed{1.33 \pm 0.125}$

Concentration #3: $E_3(X) = \lambda_3$; $Var_3(X) = E_3[X^2] - E[X]^2 = 5.20$ $SE = \sqrt{\frac{\sigma^2}{n}} = 0.114$

95% Confidence Interval: $P\left(\left|\frac{X - \bar{X}}{\sigma_{\bar{x}}}\right| \le z(\alpha/2)\right) = 0.95$; $\bar{X} \pm 1.96\sigma_{\bar{x}}$ $\boxed{1.80 \pm 0.223}$

Concentration #4: $E_4(X) = \lambda_4$; $Var_4(X) = E_4[X^2] - E_4[X]^2 = 34.5 - 4.65^2 = 12.86$; $SE = \sqrt{\frac{\sigma^2}{n}} = 0.179$

95% Confidence Interval: $P\left(\left|\frac{X - \bar{X}}{\sigma_{\bar{x}}}\right| \le z(\alpha/2)\right) = 0.95$; $\bar{X} \pm 1.96\sigma_{\bar{x}}$ $\boxed{4.65 \pm 0.351}$

c)

n	Observed	Expected	Observed	Expected	Observed	Expected	Obs.	Exp.
0	213	214	103	106	75	66	0	4
1	128	134	143	141	103	119	20	18
2	37	42	98	93	121	109	43	41
3	18	9	42	41	54	64	53	65
4	3	1.36	8	14	30	29	86	75
5	1	0	4	4	13	10	70	69
6	0	0	2	1	2	3	54	54
7	0	0	0	0	1	1	37	36
8	0	0	0	0	0	0	18	21
9	0	0	0	0	1	0	10	11
10	0	0	0	0	0	0	5	5
11	0	0	0	0	0	0	2	2
12	0	0	0	0	0	0	2	1

The expected and observed counts are fitting.

1. Suppose X is a discrete Random Variable: $P(X=0) = \frac{2}{3}\theta$ where $0 \le \theta \le 1$

$P(X=1) = \frac{1}{3}\theta$, $n=10$ observations

$P(X=2) = \frac{2}{3}(1-\theta)$, $(3,0,2,1,3,2,1,0,2,1)$

$P(X=3) = \frac{1}{3}(1-\theta)$

a) Find the method of moments estimator of θ

① Parameter Estimate

$\mu_k = E(X^k)$; $\mu_1 = E(X^1) = \frac{1}{10}[3+0+2+1+3+2+1+0+2+1] = \frac{3}{2}$

$\mu_2 = E(X^2) = \frac{1}{10}[3^2+0^2+2^2+1^2+3^2+2^2+1^2+0^2+2^2+1^2] = 3.3$

② Moment Estimates

$E(X^1) = \frac{1}{10}[3\cdot 1p(X=1) + 3\cdot 2p(X=2) + 2\cdot 3\cdot p(X=3)] = \frac{1}{10}[\theta + 4(1-\theta) + 2(1-\theta)]$

$= \frac{1}{10}[\theta + 4 - 4\theta + 2 - 2\theta] = \frac{6+5\theta}{10} = \frac{3}{2}$

$E(X^2) = \frac{1}{10}[3\cdot 1^2 p(X=1) + 3\cdot 2^2 p(X=2) + 2\cdot 3^2 p(X=3)]$

③ Estimates in terms of Moments

$= \frac{1}{10}[\theta + 8(1-\theta) + 6(1-\theta)] = \frac{1}{10}[\theta + 8 - 8\theta + 6 - 6\theta]$

$= \frac{1}{10}[14 - 13\theta] = 3.3$

Therefore,

$\mu_1 = \frac{3}{2} = \frac{6+5\theta}{10}$

$\mu_2 = \frac{33}{10} = \frac{14-13\theta}{10}$

b) Standard Error of Estimates

$\sigma_x = \sqrt{\frac{\sigma^2}{n}} = \sqrt{\frac{E[X^2] - E[X]^2}{n}} = \sqrt{\frac{33/10 - 9/4}{10}} = \sqrt{\frac{\frac{132}{40} - \frac{90}{40}}{10}} = \sqrt{\frac{11/10}{10}} = \sqrt{0.11}$

c) Maximum Likelihood of Estimate: $lik(\theta) = f(X_1, X_2, \dots X_n|\theta) = \prod_{i=1}^{n} P(X_i|\theta)$

$lik(\theta) = \prod^{3} P(X=i) = \frac{2}{3}\theta \cdot \frac{1}{3}\theta \cdot \frac{2}{3}(1-\theta) \cdot \frac{1}{3}(1-\theta) = \frac{4}{81}[\theta(1-\theta)]^2$; $\frac{d\,lik(\theta)}{d\theta} = \frac{4}{81}[2\theta(1-\theta)^2 + \theta^2(2[1-\theta])(-\theta)]$

$= \frac{8}{81}[\theta(1-2\theta+\theta^2) + \theta^3 - \theta^4]$

$= \frac{8}{81}[\theta - 2\theta^2 + \theta^2 + \theta^3 - \theta^4]$

$0 = \frac{8}{81}[\theta - \theta^2 + \theta^3 - \theta^4]$

$0 = 1 - \theta + \theta^2 - \theta^3$

d) Standard ~~Error of~~ Likelihood Estimate.