balls into k urns. What's the probability the last urn contains j balls

telephone with <u>seven total digits</u>; 432 are the first three digits
four digits remain with 10 potential digits each: 10^4

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= \frac{k!(n-k)!}{n!(j-1)!}$$

12. 26-letter English Alphabet into 8-bit binary words.

$$\binom{2^8}{26} = \frac{2^8!}{26!(2^8-26)!} = 2.74 \times 10^{35}$$

Total possibilities are 10^4.
Chances of four more distinct digits are

$$\frac{7\cdot6\cdot5\cdot4}{10^4} = \frac{840}{10000} = \boxed{8.4\%}$$

13. a) Straight five cards in unbroken sequence: $4 \text{ suits} / \binom{13}{5} = 4/[13!/(13-5)!] = 4\cdot\left(\frac{1}{13}\right)\left(\frac{1}{12}\right)\left(\frac{1}{11}\right)\left(\frac{1}{10}\right)\left(\frac{1}{9}\right)$

b) Four of a kind: $\binom{13}{1}\binom{4}{1}\binom{12}{1}\binom{4}{2} = \frac{13!}{1!\,12!}\cdot\frac{4!}{1!\,3!}\cdot\frac{12!}{1!\,11!}\cdot\frac{4!}{2!\,2!} = 13\cdot4\cdot12\cdot4\cdot3/2 = 3744$

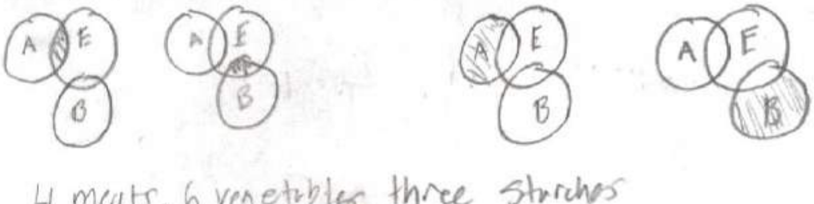
$$\frac{3744}{\binom{52}{5}} = \frac{3744}{2598960} = \boxed{0.05\%}$$

C) A full house (three cards of one value and two of another)
(Probability of three cards of fifty-two) x (Probability of two cards of fifty-two)

$$= \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} = \frac{6}{4165} = \boxed{0.144\%}$$

14. Prove if $P(A|E) \geq P(B|E)$ and $P(A|E^c) \geq P(B|E^c)$
then $P(A) \geq P(B)$

$P(A|E) \geq P(B|E)$ and $P(A|E^c) \geq P(B|E^c)$



E 100
A X 100
B y 100
50 50

$$\frac{x}{50} \geq \frac{y}{50}\ ;\ \frac{1-\frac{x}{100}}{50} \geq \frac{1-\frac{y}{100}}{50}$$

$\boxed{x \geq y}$; $1-\frac{x}{100} \geq 1-\frac{y}{100}$

$\boxed{x \geq y}$

15. 4 meats, 6 vegetables, three starches
$4\cdot6\cdot3 = \boxed{72 \text{ meals}}$

16. Simpsons Paradox:
Black Urn: {3 red and 6 green balls}
White Urn: {5 red and 4 green balls} Set #1
First trial: Black Urn $\left(\frac{3}{9}\right)$; $\boxed{\text{White Urn } \left(\frac{5}{9}\right)}$
Black Urn: {2 red and 6 green balls}
White Urn: {15 red and 1 green balls} Set #2
Second Trial: Black Urn $\left(\frac{2}{8}\right)$; $\boxed{\text{White Urn } \left(\frac{15}{16}\right)}$
Black Urn: {5 red and 12 green balls}
White Urn: {20 red and 5 green balls} Set #3
...al: Black Urn $\left(\frac{5}{12}\right)$; $\boxed{\text{White Urn } \left(\frac{20}{25}\right)}$

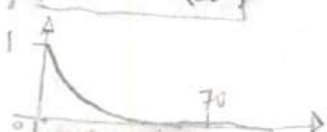
17. Accepts: 4 items of 100
Rejects: 1 item is defective.

$$P(A) = \frac{\binom{100-k}{4}\binom{k}{0}}{\binom{100}{4}} = \frac{\frac{(100-k)!}{4!(100-k-4)!}}{\frac{100!}{4!(96!)!}}$$

$$= \frac{(100-k)(100-k-1)(100-k-2)(100-k-3)(100-k-4)!}{(100-k-4)!}$$

$$\boxed{100\times99\times98\times97}$$

$$= \frac{(100-k)(99-k)(98-k)(97-k)}{100\times99\times98\times97} = \left(1-\frac{k}{100}\right)\left(1-\frac{k}{99}\right)\left(1-\frac{k}{98}\right)\left(1-\frac{k}{97}\right)$$



$\frac{\text{Player one choice}}{\text{Total choices}} = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \boxed{\frac{1}{1296}}$;

19. Five Chicanos, two Asians, three African Americans.

a) $\frac{\binom{5}{1}\cdot\binom{2}{1}\cdot\binom{3}{1}}{\binom{10}{4}} = \frac{\left(\frac{5!}{1!4!}\right)\left(\frac{2!}{1!1!}\right)\left(\frac{3!}{1!2!}\right)}{\frac{10!}{4!6!}} = \frac{5\cdot2\cdot3\,(4!)}{10\cdot9\cdot8\cdot7} = \boxed{1/7}$

b. $\frac{\binom{5}{1}\cdot\binom{2}{1}\cdot\binom{3}{1}}{\binom{10}{5}} = \frac{5\cdot2\cdot3\cdot5!}{10\cdot9\cdot8\cdot7\cdot6} = \boxed{\frac{5}{42}}$

20. Arrangements : Statistically

s's = 2 ; t's = 3; a's = 2;

i's = 2; c's = 1 ; a's = 1; l's = 2

Total Amount = 13 ; y's = 1

$\frac{\binom{2}{1}\binom{3}{1}\binom{2}{1}\binom{2}{1}\binom{1}{1}\binom{1}{1}\binom{2}{1}\binom{1}{1}}{\binom{13}{13}} = 2\cdot3\cdot2\cdot2\cdot1\cdot1\cdot2\cdot1 = \boxed{48}$

21. $\frac{2^2+2^2+2^2}{2^5} = \boxed{\frac{12}{32}}$

22. $\left(\frac{1}{4}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{1}\right) = \boxed{\frac{1}{24}}$

23. $\underbrace{\binom{n}{1}}_{1^{st}\text{ layer}}\underbrace{\binom{n}{2}}_{2^{nd}\text{ layer}}\underbrace{\binom{n-1}{1}}_{\text{Remain one}}(n-2)!$

$\frac{n!}{1!(n-1)!}\cdot\frac{n!}{2!(n-2)!}\cdot\frac{(n-1)!}{1!(n-2)!}(n-2)!$

$\boxed{\binom{n}{2}\cdot n! = \binom{n}{2}(n^n)}$

24. 52 cards ; $\frac{\text{Probability Aces next to each other}}{\text{Total Arrangements}} = \frac{\binom{13}{4}}{\binom{52}{4}} = \frac{13\cdot12\cdot11\cdot10}{4!} \cdot \frac{4!}{52\cdot51\cdot50\cdot49} = 11/4165 = \boxed{26\%}$

25. $\boxed{3!}$

26. n items with K defects: m are selected and inspected.

value of m to be below a probability of 0.90

a) n = 1000 ; k = 10 ; $\frac{\text{Probability of defect}}{\text{Total outcomes}} = \frac{\binom{n-k}{m}\binom{k}{0}}{\binom{n}{m}} = \frac{\binom{n-m}{k}\binom{m}{0}}{\binom{n}{k}} = \frac{\frac{(n-m)!}{k!(n-m-k)!}}{\frac{n!}{k!(n-k)!}} = \frac{(1000-m)!}{1000!}\frac{(n-k)!}{(n-m-k)!}$

b) n = 1000 ; k = 10 ; $0.9\,\frac{(1000!)}{(900)!} = \frac{(1000-m)!}{(900-m)!}$

27. $\frac{\text{Probability of no letters occuring}}{\text{Total Probability}} = \frac{{}_{26}C_5}{26^5} = \frac{\binom{26}{5}}{26^5} = \frac{\frac{26!}{5!(21)!}}{26^5} = \frac{\frac{26\cdot25\cdot24\cdot23\cdot22}{5\cdot4\cdot3\cdot2\cdot1}}{26^5}$

28. 5 players with five cards from 52-card deck. $= \frac{52!}{25!(27!)} = \boxed{4.775\times10^{14}\text{ Ways}}$ $= 5.53\times10^{-3} = 0.55\%$

29.0 Three Spades and Two Hearts:

(a) Discards two hearts and draws two more cards.

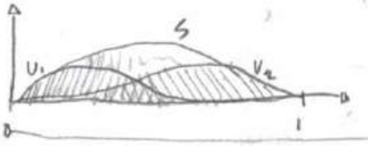
$= \left(\frac{11}{47}\right)\left(\frac{10}{46}\right) = \frac{110}{2162} = \frac{55}{1081} = \boxed{5.1\%}$

30. 60: 2nd graders into two classes of 30 each. Probability of five chosen into same class

$\frac{\binom{2}{1}}{\binom{60}{5}} = \frac{2!}{\frac{60!}{5!55!}} = \frac{2\cdot5!}{60\cdot59\cdot58\cdot57\cdot56} = 0.00004\% =$

$1000-m)(1000-(m+1))(1000-(m+2)) \cdots 1000-(m+10$

43. U_1 & U_2 from $[0,1]$; $S = U_1 + U_2$



44. X & Y $\{0,1,2\}$ $p(0)=\frac{1}{3}$; $p(1)=\frac{1}{3}$; $p(2)=\frac{1}{3}$; Frequency function of $X+Y$.

X	0	0	0	1	1	1	2	2	2
Y	0	1	2	0	1	2	0	1	2
$X+Y$	0	1	2	1	2	3	2	3	4
$P(X+Y)$	1/9	2/9							

$X+Y$	0	1	2	3	4
N	1	2	3	2	1
$P(X+Y)$	1/9	2/9	3/9	2/9	1/9

45 Poisson Distribution:

$p(x) = \frac{\lambda^k}{k!} e^{-\lambda}$

$p_A + p_B = 1$

Prove N_A is a poisson with parameter $p_A \lambda$; $P(N_A = n) = \sum_{i=n}^{\infty} P[N_A = n \mid X_i = i] P[X=i] = \sum_{i=n}^{\infty} \binom{i}{n} p_A^n (1-p_A)^{i-n} \cdot \frac{\lambda^i}{i!} e^{-\lambda}$

$\boxed{\text{Law of Total Probability}}$ $= \frac{e^{-\lambda} \cdot \lambda^n p_A^n}{n!} \sum_{i=n}^{\infty} \frac{[(1-p_A)\lambda]^{i-n}}{(i-n)!} = \frac{e^{-\lambda}\lambda^n p_A^n}{n!} \sum_{j=0}^{\infty} \frac{[(1-p_A)\lambda]^j}{j!} = \frac{e^{-\lambda}\lambda^n p_A^n}{n!} e^{(1-p_A)\lambda}$

$\boxed{= e^{-p_A\lambda} \frac{(p_A \lambda)^n}{n!}}$

46. Let T_1 and T_2 be independent exponentials with λ_1 and λ_2. Find $T_1 + T_2$.

$P(T_1) = \lambda_1 e^{-\lambda_1}$; $P(T_2) = \lambda_2 e^{-\lambda_2}$; $T_1 + T_2 = \lambda_1 e^{-\lambda_1} + \lambda_2 e^{-\lambda_2}$;

$J = \begin{vmatrix} \frac{\partial T_1}{\partial r} & \frac{\partial T_1}{\partial s} \\ \frac{\partial T_2}{\partial r} & \frac{\partial T_2}{\partial s} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$\boxed{f(T_1, T_2) = \lambda_1 \lambda_2 e^{-(\lambda_1 + \lambda_2)}}$

47. $f(x) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}$; $f(y) = \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(x-\mu_2)^2}{2\sigma_2^2}}$; $Z = X + Y = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{(x-\mu_x)^2}{2\sigma_1^2} - \frac{(y-\mu_y)^2}{2\sigma_2^2}}$

Sums and Quotients: $X, Y, Z = X+Y$, then $Y = Z - X$; $p(z) = \sum_{x=-\infty}^{\infty} p(x, z-x)$; $p(x,y) = p_X(x) \cdot p_Y(y)$

$= \sum_{-\infty}^{\infty} p_X(x) p_Y(z-x)$ $\boxed{\text{Convolution}}$ $\boxed{\text{Discrete}}$

$f(z) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-x^2/2} \cdot e^{-\frac{(z-x)^2}{2}} dx$

$= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{\frac{-x^2 + z^2 - 2xz + x^2}{2}} dx = \frac{e^{-\frac{z^2}{2}}}{2\pi} \int_{-\infty}^{\infty} e^{-\left[\frac{2x^2 - 2xz}{2}\right]} dx$

$= \frac{e^{-z^2/2}}{2\pi} \times \sqrt{\pi} e^{z^2/4} = \boxed{\frac{1}{\sqrt{2}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z}{\sqrt{2}}\right)^2}}$

$F(z) = \iint f(x,y) dx\, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f(x,y) dy\, dx$ =

$f(z) = \int_{-\infty}^{\infty} f(x, z-x) dx = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx$

48. $f(N_1) = \frac{\lambda^k}{k!} e^{-\lambda N_1}$; $f(N_2) = \frac{\lambda^k}{k!} e^{-\lambda N_2}$; $f(N) = \int_{-\infty}^{\infty} f(N_1, N - N_1) dN_1 = \int_{-\infty}^{\infty} f(N_1) \cdot f(N - N_1) dN_1 = \int_{-\infty}^{\infty} \frac{\lambda_1^k}{k!} e^{-\lambda_1 N_1} \cdot \frac{\lambda_2^k}{k!} e^{-\lambda_2 (N - N_1)} dN_1$

$= \frac{(\lambda_1 \lambda_2)^k}{(k)!^2} \int_{-\infty}^{\infty} e^{-\lambda_1 N_1 - \lambda_2 N + \lambda_2 N_1} dN_1 = \frac{(\lambda_1 \lambda_2)^k}{k!^2} e^{-\lambda_2 N} \int_{-\infty}^{\infty} e^{-(\lambda_2 - \lambda_1) N_1} dN_1$

$\boxed{= \frac{(\lambda_1 \lambda_2)^k}{(k)!^2} e^{-\lambda_2 N} \cdot \frac{-1}{(\lambda_1 - \lambda_2)}}$

49. $f(x,y) = \begin{cases} \lambda^2 e^{-\lambda y} ; & 0 \le x \le y, \lambda > 0 \\ 0 & \text{elsewhere} \end{cases}$

$f(s) = \int_{s}^{s/2} \lambda^2 e^{-\lambda y} dy = \boxed{\lambda e^{-\lambda s/2} - \lambda e^{-\lambda s}}$

50. X & Y are jointly continuous variables. Find $Z = X - Y$

$\boxed{p(z) = \int_{-\infty}^{\infty} f(x, x-z) dx = \int_{-\infty}^{\infty} f(x) \cdot f(x-z) dx}$

51. $Z = XY$; $f(x,y) = f(x, z/x) = f(\frac{z}{y}, y)$

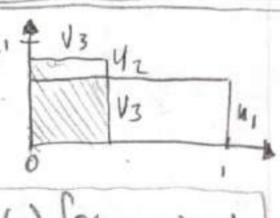
$F(z) = \int_{-\infty}^{\infty} f(x,y) \frac{dx}{dy} = \int_{-\infty}^{\infty} f(\frac{z}{y}, y) \frac{1}{y} dz\, dy$

$\boxed{f(z) = \int_{-\infty}^{\infty} f(\frac{z}{y}, y) \frac{1}{y} dy}$

52. $\boxed{z = x/y}$

$F(z) = \int f(x,y) dx\, dy = \int f(yz, y) y\, dz\, dy$; $f(z) = \int f(yz, y) y\, dy$

53.



$1 \ge \frac{\int_0^1 \int_0^1 p(U_1) p(U_2)}{\int_0^1 \int_0^1 p(V_3)^2}$

$\text{Area}_{12} = U_1 \cdot U_2$; $P(U_1, U_2) = P(U_1) \cdot P(U_2)$;

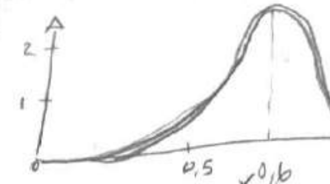
$\text{Area}_{33} = V_3^2$; $P(V_3, V_3) = P(U_3)^2$;

35. $n=20$
$x=$ 𝍸𝍸 𝍸𝍸 ||||

Beta Distribution:
$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}$$

$a=x+1$
$b=n-x+1$
8

$$\frac{\Gamma(21)}{\Gamma(13)\Gamma(8)}\theta^{12}(1-\theta)^{7}$$
$20-14+2$

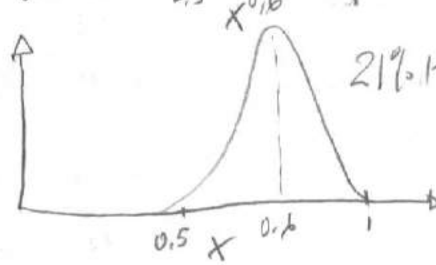


56% H:T

The amount of spins should arise to 50:50.

$n=40$
$x=$ 𝍸𝍸 𝍸𝍸 𝍸𝍸 𝍸𝍸 𝍸𝍸 |
$a=27$
$b=14$

$$\frac{\Gamma(41)}{\Gamma(14)\Gamma(27)}\theta^{26}(1-\theta)^{13}$$



21% H : 79% T

36. Rejection Method $m(x)=\frac{M(x)}{\int_0^{\infty} M(x)\,dx}$; $P(X)=\frac{\sum_1^{i} f(Y)\cdot U}{M(x)}=1$

$\boxed{M(x)=\sum_1^{i} f(y)\cdot U}$

Assuming a geometric random variable: $p\{X=i\}=(1-p)^{i-1}\cdot p$; $\sum_1^{j} P(X=i)=1-P(X>j-1)=1-(1-p)^{j-1}$

By simulating a random variable: $1-(1-p)^{j-1}<U<1-(1-p)^{j}$; $(1-p)^{j}\le 1-U<(1-p)^{j-1}$

$X=\min\{j: j\ge\frac{\log V}{\log(1-p)}\}$; $\boxed{X=1+\frac{\log V}{\log(1-p)}}$

37 $f(x)=6x^2(1-x)^2$; $-1\le x\le 1$ a) Rejection method:

$P(accept)=P(U\le f(T)/M(T))=\frac{6x^2(1-x)^2}{\int_{-1}^{1}\frac{1}{2}\,dx}=\frac{6x^2(1-x)^2}{1}$

$f'(x)=12x(1-x)^2-12x^2(1-x)$
$0=(1-x)-x$
$2x=1$
$x=\frac{1}{2}$

$p=\boxed{\frac{\text{Part}}{\text{Total}}}$

$P(x\le X\le x+dx)=\frac{P(accept\mid x\le T\le x+dx)P(x\le T\le x+dx)}{P(accept)}$

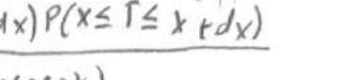
$\boxed{=6x^2(1-x)^2\cdot\int_{-1}^{1}\frac{1}{2}\cdot dt}$

38. $f(x)=(1+\alpha x)/2$ for $-1\le x\le 1$ and $-1\le\alpha\le 1$ "Columbia - Accept Reject Method"

a) 1) Generate Y distributed as G.
2) Generate U (independent from Y)
3) If $U\le\frac{f(Y)}{cg(Y)}$, then set $X=Y$ ("accept") otherwise return ("reject")

b) int val = rand();
int U = rand()
if ($V\le f(Y)/g(Y)$) ; then "accept"; break
else return "reject"

39. $X=\{0,1,2\ldots\}$ with p_0, p_1, p_2. U is a random variable. If $V<p_0$, return $X=0$.
Else $U=U-p_0$.
If $U<p_1$, return $X=1$
Else $U=U-p_1$



40. Suppose X & Y are Discrete random variables: $P_{XY}(X,y)$ a) Generate $X\sim p_X(x)=\int_{-\infty}^{\infty}P_{xy}(x,y)\,dy$

b) Accept X with $p(y\mid X)=\frac{p(x,y)}{p(x)}$

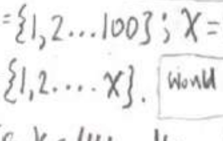
b) Else A.

If X and Y should be continuous random variables, then the process would be rather a function.

$X=\{1,2\ldots 100\}$; $X=x$,
$Y=\{1,2\ldots x\}$.

If $Y=44$, then $P(44\mid x)=\frac{p(x,44)}{p(x)}$

$p(x)=\frac{p(x,44)}{P(44\mid x)}\to$



42. a)

41. T = exponential Random Distribution: $p(x)=\lambda e^{-\lambda x}$; $W=\pm1$ with $p(1)=\frac{1}{2}$; $p(0)=\frac{1}{2}$.
$x=WT$. Show that $f_X(x)=\frac{1}{2}e^{-\lambda|x|}$ is the double exponential distribution

$f_X'(x)=p(W)\cdot p(x)=\boxed{\frac{1}{2}\lambda e^{-\lambda x}}$

b) Prove $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\le ce^{-|x|}$; $e^{-x^2/2+|x|}\le c\sqrt{2\pi}$; $-\frac{x^2}{2}+|x|\le\log c\sqrt{2\pi}$

$-\frac{x^2}{2}+|x|-\log c\sqrt{2\pi}\le 0$; $x^2-2|x|+\log c\le 0$

$x_1, x_2=\frac{+2\pm\sqrt{4-4(1)(\log c)}}{2(1)}$

Real(x) implies C is between 0 to 1.

25 $f(x)$; $P(Y=x)=1/2$; $P(Y=-x)=1/2$; Show Y is symmetric about zero.

Bernouli Distribution: $P(k)=p^{x}(1-p)^{1-x}$

$$P(0)=\int_0^1 p^0(1-p)^1\,dp = 1-1/2 = 1/2$$

$$P(1)=\int_0^1 p^1(1-p)^0\,dp = \frac{1}{2}$$

Conditional Density of a random variable is expressed as:

$$f_{Y|X}(y|x)=F_{Y|X}(x|x)=1/2 \; ; \; f_{Y|X}(y|x)=f_{Y|X}(-x|x)=1/2$$

$$f_{Y|X}(y|x)=\frac{f_{YX}(x,y)}{f(x)} \; ; \; f(x,y)=f_{Y|X}(y|x)f(x)=\boxed{\tfrac{1}{2}f_x(x)}$$

$$f(x,x)=\tfrac{1}{2}f_x(x) ;$$

$$F_{Y|X}(y|x)=\frac{f(x,y)}{f_x(x)} \; ; \; f(x,y)=f_{Y|X}(y|x)f_x(x)=\boxed{\tfrac{1}{2}f_x(-x)}$$

$$f_y(y)=\tfrac{1}{2}f(x)+\tfrac{1}{2}f(x)=\boxed{f_y(-y)}$$

27. Prove x and y are independent if $\boxed{f_{X|Y}(x|y)=f_X(x) = \frac{f(x,y)}{f(y)} = \frac{f(x)f(y)}{f(y)} = f(x)}$

28. Show if $C(u,v)=uv$ is a copula. Why is it called "the independence copula?"

Copula: a joint cumulative distribution of random variables that have uniform marginal distributions. The function $C(u,v)=uv$ is known by the independence copula because independent variables, and marginals, are seperable.

29. Marginal Density $= \lambda e^{-\lambda x}$

Farlie-Morgenstern Copula:

$$H(x,y)=F(x)G(y)\{1+\alpha[1-F(x)][1-G(y)]\} = \lambda_1 e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2 y}\{1+\alpha[1-\lambda_1 e^{-\lambda_1 x}][1-\lambda_2 e^{-\lambda_2 y}]\}$$

$$h(x,y)=\frac{\partial^2}{\partial x\partial y}H(x,y)=\frac{\partial}{\partial x}\Big[\lambda_1 e^{-\lambda_1 x}\lambda_2^2 e^{-\lambda_2 y}\{1+\alpha[1-\lambda_1 e^{-\lambda_1 x}][1-\lambda_2 e^{-\lambda_2 y}]\}$$

$$+\lambda_1 e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2 y}\{1+\alpha[1-\lambda_1 e^{-\lambda_1 x}][1+\lambda_2 e^{-\lambda_2 y}]\}\Big]$$

$$\boxed{\begin{aligned}&=\lambda_1^2 e^{-\lambda_1 x}\cdot\lambda_2^2 e^{-\lambda_2 y}\{1+\alpha[1-\lambda_1 e^{-\lambda_1 x}][1-\lambda_2 e^{-\lambda_2 y}]\}\\&+\lambda_1 e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2 y}\{1+\alpha[1+\lambda_1^2 e^{-\lambda_1 x}][1-\lambda_2 e^{-\lambda_2 y}]\}\\&+\lambda_1^2 e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2 y}\{1+\alpha[1-\lambda_1 e^{-\lambda_1 x}][1+\lambda_2 e^{-\lambda_2 y}]\}\\&+\lambda_1 e^{-\lambda_1 x}\lambda_2 e^{-\lambda_2 y}\{1+\alpha[1+\lambda_2 e^{-\lambda_2 x}][1+\lambda_2 e^{-\lambda_2 y}]\}\end{aligned}}$$

30. For $0\le\alpha\le1$ and $0\le\beta\le1$

Show $C(u,v)=\min(u^{1-\alpha}v, uv^{1-\beta})$ is a copula (Marshall-Olkin)

$$\boxed{\lim_{\alpha\to1}\lim_{\beta\to1} C(u,v)=\min(v,u)=1}$$

Joint Density:

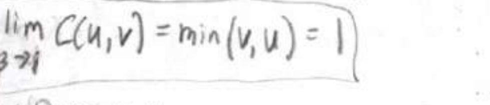
$$h(x,y)=\frac{\partial^2}{\partial u\partial v}H(x,y)=\boxed{\min\left[(1-\alpha)u^{-\alpha},(1-\beta)v^{-\beta}\right]}$$

31. (X,Y) is a uniform disc of radius of 1. $f(x,y)=\begin{cases}\frac{1}{\pi} & x^2+y^2\le1\\ 0, & \text{otherwise}\end{cases}$ x and y are not independent because of the constraint $x^2+y^2=1$.

32. $f_R(r)$; Probability of passing per mesh: Probability Passing $=\frac{\text{Area Square}}{\text{Area mesh}} = \boxed{\frac{\pi f_R^2(r)}{(nW+(n+1)d)^2}}$

33a) Posterior Density [Beta Density]: $f(\theta)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}$; $a=x+1$; $b=n-x+1$

34b)



$$=\frac{\Gamma(N+2)}{\Gamma(2)\Gamma(N)}\theta(1-\theta)^{N-1} = (n+1)(n)\frac{\Gamma(n)}{\Gamma(2)\Gamma(n)}\theta(1-\theta^{N-1})$$

$$=\frac{(n+1)(n)}{\Gamma(2)_{=1}}\theta(1-\theta^{N-1})$$

$$\boxed{=(n+1)(n)\,\theta(1-\theta^{N-1})}$$

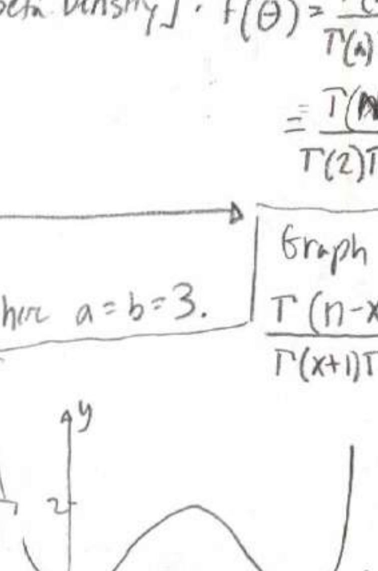
Graph this function:

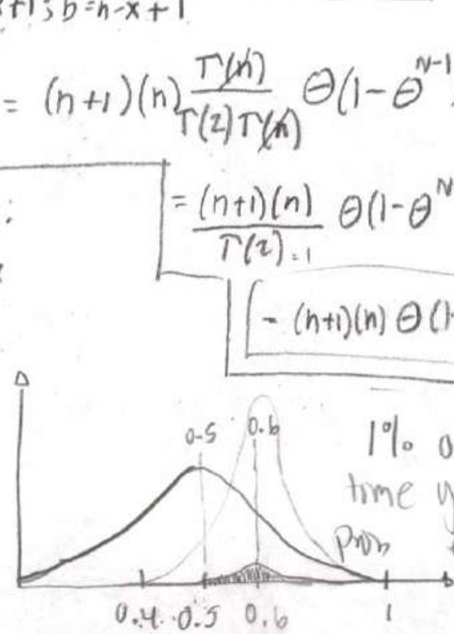
$$\frac{\Gamma(n-x+1)}{\Gamma(x+1)\Gamma(n-x+1)}\theta^x(1-\theta)^{n-x}$$

34. Beta Density: $\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}$; Where $a=b=3$.

Graph the Function:

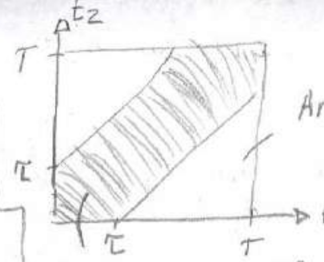
$$\frac{\Gamma(6)}{\Gamma(3)\Gamma(3)}\theta^{3-1}(1-\theta)^{3-1}$$





1% of the time your average prob. errors

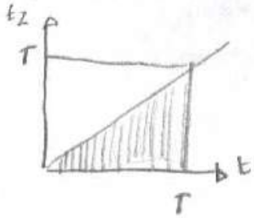
$$P(T_1 > 2T_2) = \int_0^\infty \int_{2T_2}^\infty \alpha e^{-\alpha T_1} \beta e^{-\beta T_2} \, dT_1 dT_2 = \frac{\beta}{(2\alpha+\beta)}$$



Area $(T-\tau)^2/2$

20. Probability of packet collision: $f(t_1,t_2) = \frac{1}{T^2}$ $\boxed{\text{Joint Density}}$ from $[0,T]$

Time between arrivals:



Integral is $f(t_1,t_2) \times \text{Area}$

$$= \frac{1}{T^2}\left(T^2 - (T^2-\tau)^2\right)$$

$$\boxed{= 1 - (1-t/T)^2}$$

Integral is: $f(t_1,t_2) \times \frac{1}{2}\tau \cdot T = \boxed{\frac{\tau}{2}}$

21. Area $T^2-(T-\tau)^2$

$f(x)$ = Present chemical | Y = concentration of a chemical in soil

$R(x)$ = Probability Detected

If $g(y)$ is uniform, then $g(y) = \frac{\text{Probability of Detection} \times \text{Density Function}}{\text{Total outcomes of concentration}}$

$$= R(y) \cdot f(y) \Big/ \int_0^\infty R(x) f(x)\, dx$$

22. Poisson Distribution: $\frac{\lambda^k}{k!} e^{-\lambda} = p(x)$; $N(t_1,t_2)$ = Number of events.

If $t_0 < t_1 < t_2$; find the conditional distribution of $N(t_0,t_1)$ given $N(t_0,t_2)=n$

$$N(t_0|t_1) = \frac{N(t_0,t_1)}{N(t_1)}; \quad N(t_0|t_1) = \frac{N(t_0 t_2)}{N(t_2)} = \frac{n}{N(t_2)}; \quad N(t_1,t_2) \circ p = \lambda; \quad p(\lambda) = \frac{[N(T_1,T_2)p]^k}{k!} \circ e^{-N(T_1,T_2)\cdot p}$$

$$= \frac{[N(T_1)N(T_2)\cdot P_1 \cdot P_2]^k}{k!} e^{-N_{T_1} \cdot N_{T_2} \cdot P_1 P_2}$$

$$P(N(t_i,t_j)) = e^{-\lambda(t_i-t_j)} \frac{[\lambda(t_j-t_i)]^x}{x!}$$

$$P(N(t_0,t_1)=x \mid N(t_0,t_2)=n) = \frac{P(N(t_0,t_1)=x,\, N(t_0,t_1)+N(t_1,t_2)=n)}{P(N(t_0,t_2)=n)}$$

23. $p(N|X) = \frac{p(N,x)}{p(x)}$

$$= \frac{P(N(t_0,t_1)=x,\, N(t_1,t_2)=n-N(t_0,t_1))}{P(N(t_0,t_2)=n)} = \frac{P(N(t_0,t_1)=x,\, N(t_1,t_2)=n-x)}{P(N(t_0,t_2)=n)}$$

Binomial Distribution:

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

N = Trials, p = probability of success

$$= \frac{e^{-\lambda(t_1-t_0)} \frac{[\lambda(t_1-t_0)]^x}{x!} \times e^{-\lambda(t_2-t_1)} \frac{[\lambda(t_2-t_1)]^{n-x}}{(n-x)!}}{e^{-\lambda(t_2-t_0)} \frac{[\lambda(t_2-t_0)]^n}{n!}} = \boxed{\frac{n!}{x(n-x)!} \frac{(t_1-t_0)^x (t_2-t_1)^{n-x}}{(t_2-t_0)^n}}$$

= Binomial random variable with m trials and probability r

$$P(x) = \frac{P(N,x)}{P(N|X)} = \frac{\binom{N}{k} p^k (1-p)^{N-k}}{\binom{m}{k} r^k (1-r)^{N-r}}$$

$$\boxed{= \binom{N}{m} (pr)^m (1-p\cdot r)^{N-m}}$$

24.

Joint Density: $f_{\theta,x}(\theta,x) = f_{x,\theta}(x|\theta) \cdot f(\theta)$

$$= \binom{n}{x} \theta^x (1-\theta)^{n-x} \quad x = 0,1,\dots n; \; 0 \le \theta \le 1$$

Section 3.5.2

Baysian Inference: Conditional: $f_{x|\theta}(x|\theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x} \; x = 0,1,\dots n$

Marginal Density: $f_\theta(\theta) = \int_0^1 \binom{n}{x} \theta^x (1-\theta)^{n-x} d\theta = \int_0^1 \frac{\Gamma(n+1)}{\Gamma(x+1)\Gamma(n-x+1)} \theta^x (1-\theta)^{n-x} d\theta$ by the fact $\Gamma(k) = (k-1)!$

...becomes the beta density $= \int_0^1 \frac{\Gamma(n+1)}{\Gamma(x+1)\Gamma(n-x+1)} \theta^x (1-\theta)^{n-x} d\theta = \frac{\Gamma(n+1)}{\Gamma(x+1)\Gamma(n-x+1)} \frac{\Gamma(x+1)\Gamma(n-x+1)}{\Gamma(n+2)} = \frac{1}{n+1} \frac{\Gamma(n+1)}{\Gamma(n+1)} = \boxed{\frac{1}{n+1}}$

Conditional: $f_{\theta|x}(\theta|x) = \frac{f_{\theta x}(\theta,x)}{f_x(x)} = (n+1)\binom{n}{x} \theta^x (1-\theta)^{n-x} = (n+1) \frac{\Gamma(n+1)}{\Gamma(x+1)\Gamma(n-x+1)} \theta^x (1-\theta)^{n-x} = \frac{\Gamma(n+2)}{\Gamma(x+1)\Gamma(n-x+1)} \theta^x (1-\theta)^{n-x}$

Posterior Density is a β-density with $a = x+1, b = n-x+1$; $g(u) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^{a-1}(1-u)^{b-1}$; $g'(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \left[(a-1) u^{a-2}(1-u)^{b-1} + (b-1) u^{a-1} (1-u)^{b-2}\right] = 0$

26. P is uniform from $[0,1]$, and conditional on $P=p$.
Let X be a Bernoulli Distribution with parameter p.
Find the conditional distribution of P given X.

$$f(x=0) = \int_0^1 p^0 (1-p)^1 dp = 1/2$$

$$f(x=1) = \int_0^1 p^1 (1-p)^0 dp = 1/2$$

Bernouli Distribution: Find $f_{P|X}(P|X) = \frac{f(p,x)}{f(x)} = \frac{p^x (1-p)^{1-x}}{1/2} = \boxed{2p(1-p)^{1-x}}$

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\frac{(a-1)}{(b-1)}(1-u) = u$$

$$\frac{(a-1)}{(b-1)} = u\left[1 + \frac{(a-1)}{(b-1)}\right]$$

$$\boxed{\theta = \frac{(a-1)(b-1)}{(b-1)[(b-1)+(a-1)]}}$$

15. Suppose the joint density $f(x,y) = c\sqrt{1-x^2-y^2}$, $x^2+y^2 \le 1$

a) Find c: $x = r\cos\theta$; $y = r\sin\theta$; $\iint f(x,y)\,dA = \int_0^{2\pi}\int_0^1 c\sqrt{1-r^2}\,r\,dr\,d\theta = c\left(\frac{2}{3}\right)\pi = 1$; $\boxed{c = \left(\frac{3}{2\pi}\right)}$

$P(x^2+y^2) \le 1/2$ is a half of the disk,

$\iint f(x,y)\,dA;\ x^2+y^2 \le 1/2 = \int_0^{\pi/4}\int_0^1 \left(\frac{3}{2\pi}\right)\sqrt{1-r^2}\,r\,dr\,d\theta = \iint_0^{\pi/4,\,1}\left(\frac{3}{2\pi}\right)\sqrt{1-r^2}\,r\,dr\,d\theta = \int_0^{\pi/4}\left(\frac{1}{2}\pi\right)d\theta = \boxed{1/8}$

b) The joint density is an area of decreasing size.

d) $f(x) = \int_{-1}^{1}\sqrt{1-x^2-y^2}\left(\frac{3}{2\pi}\right)dy = \boxed{\left(\frac{3}{4}\right)x^2}$; $f(y) = \int_{-1}^{1}\sqrt{1-x^2-y^2}\left(\frac{3}{2\pi}\right)dx = \boxed{\left(\frac{3}{4}\right)x^2}$

To check independence, $f(x,y) = f_x(x)f_y(y) = \left(\frac{9}{16}\right)x^2y^2 \neq$ independence.

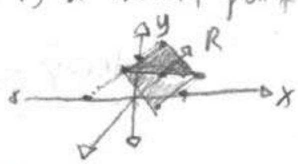
e) Conditional Densities: $\boxed{f(y|x) = f(x,y)/f_x(x);\ f(x|y) = f(x,y)/f(y)}$

16. X_1 is uniform on $[0,1]$, and, conditional on X_1, X_2, from $[0, X_1]$

Find the joint Distributions: $\boxed{f(X_1, X_2) = \int_0^1\int_0^{X_1} dX_2\,dX_1}$ = Marginal Distributions: $\boxed{f(X_1) = \int_0^{X_1} dX_2}$ $\boxed{f(X_2) = \int_0^1 dX_1}$

17. (X,Y) is a random point of the region $R = \{(x,y) : |x|+|y| \le 1\}$

a)

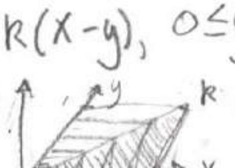


b) $f_x(x) = R$; $f_y(y) = R$

c) $f_{Y|X}(Y|X) = \frac{f(y,x)}{f(x)} = \frac{|x|+|y|}{R} = \boxed{1}$

18 $f(x,y) = k(x-y)$, $0 \le y \le x \le 1$ and zero elsewhere.

a)



b) $1 = \int_0^1\int_0^x k(x-y)\,dy\,dx = \int k\left(x - \frac{y^2}{2}\right)\Big|_0^x = \int_0^1 k\left[\left(x - \frac{x^2}{2}\right) - k(x)\right]dx$

$= \int_0^1 -k\frac{x^2}{2}\,dx = -\frac{k(1)^3}{6} = 1$ $\boxed{k = 6}$

c) $f_x(x) = \int_0^x k(x-y)\,dy = \boxed{k\left(-\frac{x^2}{2}\right)} = \boxed{-3x^2}$; $f_y(y) = \int_0^1 k(x-y)\,dx = k\left(\frac{1}{2} - y\right) + ky = 6\left(\frac{1}{2}\right) = \boxed{3}$

d) $f_{Y|X}(Y|X) = \frac{f(x,y)}{f(x)} = \frac{k(x-y)}{-3x^2}$; $f_{X|Y}(X|Y) = \frac{f(x,y)}{f(y)} = \boxed{\frac{k(x-y)}{3}}$

19. a) Exponentially Distributed lifetimes means: $\lambda e^{-\lambda x} = f(\lambda)$; $f(T_1) = \alpha e^{-\alpha T_1}$; $f(T_2) = \beta e^{-\beta T_2}$

Find $P(T_1 > T_2) = \int_0^{T_1}\beta e^{-\beta T_2}\,dT_2 = -e^{-\beta T_2}\Big|_{-0}^{T_1} = -\left[e^{-\beta T_1} - 1\right] = \boxed{1 + e^{-\beta T_1}}$ Let $S = T_1 + T_2$

b) $P(T_2 > 2T_1) = \boxed{\frac{1 \pm e^{-\alpha(2T_1)}}{2}}$ $P(T_1 > T_2) = \int_0^{T_1}\alpha e^{-\alpha T_1}\cdot\beta e^{-\beta T_2}\,dT_2 = -\alpha e^{-\alpha T_1}\cdot e^{-\beta T_1}\Big|_{-0}^{T_1} = -\alpha e^{-(\alpha+\beta)T_1}$

$P(T_1 > T_2) = \int_{T_2}^{\infty}\int_0^{\infty}\alpha e^{-\alpha T_1}\beta e^{-\beta T_2}\,dT_2\,dT_1 = \int_0^{\infty}\beta e^{-\beta T_2} - \left[e^{0} - e^{-\alpha T_2}\right]dT_2 = \int_0^{\infty}\beta e^{-(\beta+\alpha)T_2}\,dT_2 = \frac{\beta}{(\alpha+\beta)}$

12. $f(x,y)=c(x^2-y^2)e^{-x}$, $0\le x<\infty$, $-x\le y<x$ a) Find c. $P(x,y)=1=\int_0^\infty\int_{-x}^{x} c(x^2-y^2)e^{-x}\,dy\,x$

$=\int_0^\infty c(x^2-\frac{y^3}{3})e^{-x}\Big|_{-x}^{x} dx$

$=\int_0^\infty c(x^2-\frac{x^3}{3})e^{-x} - c(x^2+\frac{x^3}{3})e^{-x}\,dx$

$=\int_0^\infty c(-2\frac{x^3}{3})e^{-x}dx$

$1=-4c$; $\boxed{c=\frac{-1}{4}}$

b) $f(x)=\int_{-x}^{x} c(x^2-y^2)e^{-x}dy = \boxed{\frac{-4x^3}{3}e^{-x}}$ $f(y)=\int_0^\infty 2(x^2-y^2)e^{-x}dx = \boxed{2(2-y^2)e}$

c) $f_{Y|X}(Y|X)=\frac{f(x,y)}{f(x)}=\frac{-2(x^2-y^2)e^{-x}}{\frac{4}{3}x^3e^{-x}}=\boxed{\frac{-3}{2}\frac{(x^2-y^2)}{x^3}}$; $f_{X|Y}(X|Y)=\frac{f(x,y)}{f(y)}=\boxed{\frac{(x^2-y^2)e^{-r}}{(2-y^2)}}$

B. Sample Space: Throws

1	2
H	H
H	T
T	H
T	T

0	1	2
1/2	1/2	1/4

$\boxed{P(0)=1/2 \quad P(2)=\frac{1}{4} \quad P(1)=1/2}$

a) $f(x)=\int_{-\sqrt{y^2+z^2}}^{\sqrt{y^2+z^2}}$

14 Point in a unit sphere $(\mathbb{R}_3)$

$V=\frac{4}{3}\pi r^3=\frac{4}{3}\pi\sqrt{x^2+y^2+z^2}$

$f(x)=\int_{-\sqrt{r^2-x^2-z^2}}^{\sqrt{r^2-x^2-z^2}}\int_{-\sqrt{r^2-x^2-y^2}}^{\sqrt{r^2-x^2-y^2}}\sqrt{\left(\frac{v}{4/3\pi}\right)^2-y^2-z^2}\,dz\,dy$

$=\int_{\sqrt{r^2-x^2-z^2}}^{\sqrt{r^2-x^2-z^2}}\frac{1}{2}\left(z\sqrt{(a)^2-y^2-z^2}+(a^2-y^2)\tan^{-1}\left(\frac{z}{\sqrt{a^2-y^2-z^2}}\right)\right)\Bigg|_{-\sqrt{r^2-x^2-y^2}}^{\sqrt{r^2-x^2-y^2}}$

Density Function of a unit Sphere:

$f(x,y,z)=\begin{cases}k & 0\le x^2+y^2+z^2\le 1\\ 0 & \text{otherwise}\end{cases}$

To find the value of k, such that:

$\iiint_{0\le x^2+y^2+z^2\le1} k\,dx\,dy\,dz=1$; let $x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$; $\rho^2=x^2+y^2+z^2=1$

$0<\rho<1$; $0<\phi<\pi$; $0<\theta<2\pi$

Volume $=\int_0^{2\pi}\int_0^{\pi}\int_0^{1}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\frac{4\pi}{3}$; $k\frac{4\pi}{3}=1$; $k=\frac{3}{4}\pi$

The density function becomes: $\boxed{f(x,y,z)=\begin{cases}\frac{3}{4\pi} & 0<x^2+y^2+z^2\le1\\ 0 & \text{otherwise.}\end{cases}}$

Marginal Densities: 1) Joint Densities:

$f_{x,y}(x,y)=\int_{-\infty}^{\infty}f(x,y,z)\,dz=\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}}\frac{3}{4\pi}dz=\frac{3}{2\pi}\sqrt{1-x^2-y^2}$; $f_{xy}(x,y)\begin{cases}\frac{3}{2\pi}\sqrt{1-x^2-y^2} & ;0\le x^2+y^2\le1\\ 0 & ,\text{otherwise}\end{cases}$

2) $f_x(x)=\int_{-\infty}^{\infty}f_{xy}(x,y)dy=\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\frac{3}{2\pi}\sqrt{1-x^2-y^2}dy$; $y=\sqrt{1-x^2}\sin u$; $dy=\sqrt{1-x^2}\cos u\,du$; $=\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\frac{3}{2\pi}\sqrt{(1-x^2)-\left(\sqrt{1-x^2}\right)^2\sin^2u}\left(\sqrt{1-x^2}\cos u\,du\right)$

$\boxed{=\frac{3}{4}(1-x^2)}$

$\boxed{f_y(y)=\begin{cases}\frac{3}{4}(1-y^2) & -1\le y\le1\\ 0 & ,\text{otherwise}\end{cases} ; f_z(z)=\begin{cases}\frac{3}{4}(1-z^2) & -1\le z\le1\\ 0 & ,\text{otherwise}\end{cases}}$

@ $z=0$ $f_{xy|z}(x,y|z)=\frac{f_{xyz}(x,y,z)}{f_z(z=0)}=\frac{3/4\pi}{\frac{3}{4}(1-(0)^2)}=\frac{1}{\pi}$

c. Conditional Density: $f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \boxed{\frac{\frac{6}{7}(x^2+xy)}{x^2+\frac{x}{2}}}$; $f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(x)} = \boxed{\frac{\frac{6}{7}(x^2+xy)}{\frac{1}{3}+\frac{y}{2}}}$

9. (X,Y) uniformly distributed over: $0 \le y \le 1-x^2$; $-1 \le x \le 1$ Assuming Bivariate Normal Density:

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y}\exp\left(\frac{[x-\mu_x]^2}{\sigma_x^2}+\frac{[y-\mu_y]^2}{\sigma_y^2}\right)$$

a) Find the marginal distribution: $f_X(x) = \int_0^{1-x^2} \frac{1}{2\pi\sigma_x\sigma_y}\exp\left(\frac{[x-\mu_x]^2}{\sigma_x^2}+\frac{[y-\mu_y]^2}{\sigma_y^2}\right)$

$$= \frac{1}{2\pi\sigma_x\sigma_y}\exp\left(\frac{[x-\mu_x]^2}{\sigma_x^2}+\frac{[y-\mu_y]^2}{\sigma_y^2}\right)\left[\frac{1}{\sqrt{2}\sigma_y}\Gamma\left(\frac{1}{2}\right)\right]\left[\frac{2\sigma_y}{\mu_y}\right]\Bigg|_0^{1-x^2}$$

$$\boxed{= \frac{\Gamma(\frac{1}{2})}{\sqrt{2\pi}\sigma_x\mu_y}\exp\left(\left[\frac{[x-\mu_x]^2}{\sigma_x^2}+\frac{[1-x^2-\mu_y]^2}{\sigma_y^2}-\frac{[\mu_y]^2}{\sigma_y^2}\right]\left(-\frac{1}{2}\right)\right)}$$

$= \frac{-1}{2\sigma_y^2}$

b) Find the two conditional densities:

$f_{Y|X}(y|x) = \frac{f_{xy}(x,y)}{f_x(x)}$

$$\boxed{= \frac{\mu_y}{\sqrt{2}\sigma_y\Gamma(\frac{1}{2})}\exp\left[\left(\frac{-1}{2}\right)\frac{[y-\mu_y]^2-[1-x^2-\mu_y]^2+\mu_y^2}{\sigma_y^2}\right]}$$

$f_{X|Y}(x|y) = \frac{f_{x,y}(x,y)}{f_y(y)}$

$$f_y(y) = \int_{-1}^{1}\frac{1}{2\pi\sigma_x\sigma_y}\exp\left[\left(\frac{-1}{2}\right)\left[\frac{(x-\mu_x)^2}{\sigma_x^2}+\frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right]dy$$

$$\boxed{= \frac{\mu_x}{\sqrt{2}\sigma_x\Gamma(\frac{1}{2})}\exp\left[\left(\frac{-1}{2}\right)\frac{(x-\mu_x)^2-(1-\mu_x)^2-\mu_x^2}{\sigma_x^2}\right]}$$

$$\boxed{= \frac{2\Gamma(\frac{1}{2})}{\sqrt{2\pi}\sigma_x\mu_x}\exp\left[\frac{[1-\mu_x]^2-\mu_x^2}{\sigma_x^2}+\frac{(y-\mu_y)^2}{\sigma_y^2}\right]}$$

10. Suppose $f(x,y) = xe^{-x(y+1)}$
$0 \le x \le \infty$; $0 \le y < \infty$

a) Find the marginal density of X and Y. $f(x) = \int_0^\infty xe^{-x(y+1)}dy = \frac{\cancel{x}e^{-x(y+1)}}{-\cancel{x}}\Big|_0^\infty = -e^{-x(\infty+1)} + e^{-x(1)}$ $\boxed{= e^{-x}}$

b) Find the conditional densities.

$f(y) = \int_0^\infty xe^{-x(y+1)}dx$; $\int dvu = uv - \int vdu$ $\quad u = x \quad dv = e^{-x(y+1)}$ $\quad du = dx \quad v = \frac{-e^{-x(y+1)}}{y+1}$

$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{xe^{-x(y+1)}}{e^{-x}} = \boxed{xe^{-x[y+2]}}$

$= \frac{-xe^{-x(y+1)}}{y+1}\Big|_0^\infty + \int_0^\infty \frac{e^{-x(y+1)}}{y+1}dx$

$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{xe^{-x(y+1)}}{\frac{1}{(y+1)^2}} = \boxed{x(y+1)^2 e^{-x(y+1)}}$ $\quad$ $\boxed{\text{Independent}}$

$= 0 - \frac{e^{-x(y+1)}}{(y+1)^2}\Big|_0^\infty = \boxed{\frac{1}{(y+1)^2}}$

11. U_1, U_2, and U_3 independent from $[0,1]$
Find the probability the roots of the quadratic $U_1x^2+U_2x+U_3$ are real.

$0 = U_1x^2+U_2x+U_3$; $x_1, x_2 = \frac{-U_2 \pm \sqrt{U_2^2-4(U_1)(U_3)}}{2(U_1)}$; $P(U_1) = \int_0^1\int_0^1 u_1x^2+u_2x+U_3\,dU_2dU_3 = \int_0^1 u_1x^2+U_2x+\frac{U_3^2}{2}\,dU_2$

$= U_1x^2+\frac{x}{2}+\frac{1}{2}$

~~Extrema and Order Statistics~~

~~$f(U_{(1)},U_{(2)},U_{(3)}\ldots) = n!\prod_{i=1}^{n}f(U_i)$; for $i=3$; $f(U_1,U_2,U_3) = 3!\prod_{i=1}^{3}f(U_i) : 6f(U_1)f(U_2)f(U_3)$~~

$$P(U_2^2 > 4U_1U_3) = P(|U_2| \ge 2\sqrt{U_1U_3}) = \int_0^1\int_0^1\int_{2\sqrt{U_1U_3}}^1 f(u_1,U_2,U_3)\,du_2\,du_3\,du_1 = \int_0^1\int_0^1(1-2\sqrt{u_1u_3})$$

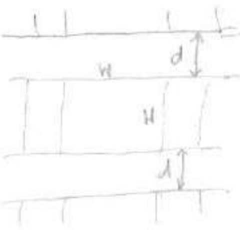
$= 1 - \frac{4}{3}\frac{1^{3/2}}{3/2} = \boxed{1/9}$

3. Three Players play 10 independendent rounds of a game. $P(\text{Player winning}) = \frac{1}{3}$

$$P(\text{Player } A,B,C \,|\, \text{Win}) = \boxed{\binom{10}{x\,y\,z}\left(\frac{1}{3}\right)^x\left(\frac{1}{3}\right)^y\left(\frac{1}{3}\right)^z}$$

Multinomial Distribution: $\boxed{P(X,y,z) = \binom{n}{x\,y\,z} P_1^x P_2^y P_3^z}$

4 Wire diameter $=d$, Holes side length $=W$, spherical particle radius $=r$, What is probability of passing?



Area $= \pi r^2$ Circle

Probability passing per hole $= \frac{\text{Area particle}}{\text{Area square}} = \boxed{\frac{\pi r^2}{W^2}}$

Area $= W^2$ Square

Probability passing per mesh $=$ Probability passing $\cdot \frac{\text{Area square}}{\text{Area mesh}}$

$$= \frac{\pi r^2}{W^2}\frac{W^2}{(nW^2+(n+1)d)^2} = \boxed{\frac{\pi r^2}{(nW+(n+1)d)^2}}$$

Fails to pass through if dropped n-times:

$$P(\text{Failing to pass through}) = \boxed{\left(1-\frac{\pi r^2}{(nW+(n+1)d)^2}\right)^n} \binom{n}{1}$$

5. $\vdash L \dashv$ $\quad a \geq L$

Probability needle crosses line $= 2\left(\frac{\text{Length of Needle}}{\text{Distance of lines}}\right) \times \frac{1 \text{ rot}}{\pi \text{ rad}} = \boxed{\frac{2L}{\pi D}}$

$\leftarrow d \rightarrow \leftarrow d \rightarrow$

6. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$: Marginal Density of X and y coordinates inside an ellipse: Area $= \pi \cdot a \cdot b$

$$F_y(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \frac{1}{\pi a b}\int_{-\sqrt{a^2(1-y^2/b^2)}}^{\sqrt{a^2(1-y^2/b^2)}} dx = \frac{1}{\pi a b}\, 2\sqrt{a^2(1-y^2/b^2)} = \boxed{\frac{2\sqrt{(1-y^2/b^2)}}{\pi b}}$$

$$\boxed{F_x(y) = \frac{2\sqrt{(1-x^2/a^2)}}{\pi a}}$$

7. CDF: $F(X,y) = (1-e^{-\alpha x})(1-e^{-\beta y});\ x \geq 0;\ y \geq 0;\ \alpha > 0;\ \beta > 0$

Joint Density: $F(x,y) = \frac{d}{dx}\frac{d}{dy}(1-e^{-\alpha x})(1-e^{-\beta y})$

$$\boxed{= (1+\alpha e^{-\alpha x})(1+\beta e^{-\beta y})}$$

Marginal Density: $F_x(x) = \int_{-\infty}^{\infty}(1-e^{-\alpha x})(1-e^{-\beta y})\,dy = 2(1-e^{-\alpha x})\left(1+\frac{e^{-\beta y}}{\beta}\right)\Big|_0^{\infty}$

$F_y(y) = \int_{-\infty}^{\infty}(1-e^{-\alpha x})(1-e^{-\beta y}) \quad = 2(1-e^{-\alpha x})(1/\beta)$

$$\boxed{= 2(1-e^{-\beta y})(1/\alpha)}$$

8. $f(x,y) = \frac{6}{7}(x+y)^2,\ 0 \leq x \leq 1,\ 0 \leq y \leq 1$

a) i) Find $P(x>y) = \frac{6}{7}\int_0^1\int_0^x (x^2+xy)\,dy\,dx = \frac{6}{7}\int_0^1\left(x^2 x + \frac{x^3}{2}\right)dx = \frac{6}{7}\left(\frac{x^4}{4}+\frac{x^4}{8}\right)\Big|_0^1 = \frac{6}{7}\left(\frac{1}{4}+\frac{1}{8}\right) = \frac{6}{28}+\frac{6}{48} = \boxed{\frac{11}{56}}$

ii) Find $P(X+Y \leq 1) = \frac{6}{7}\int_0^1\int_0^{1-Y}(x^2+xy)\,dx\,dy = \frac{6}{7}\int_0^1\left(\frac{x^3}{3}+\frac{x^2}{2}y\right)\Big|_0^{1-Y} = \frac{6}{7}\int_0^1\left(\frac{(1-Y)^3}{3}+\frac{(1-Y)^2}{2}y\right)dy$

$X \leq 1-Y$

$$= \frac{6}{7}\left[\frac{-(u)^4}{12}+\frac{y^2}{4}-\frac{2y^3}{2\cdot 3}\right]\Big|_0^1 = \frac{6}{7}\left[\frac{-(1-1)^4}{12}+\frac{1}{4}-\frac{1}{3}+\frac{1}{12}\right] = \boxed{0}$$

$u = 1-y$

iii) $P(X \leq \frac{1}{2}) = \frac{6}{7}\int_0^1\int_0^{1/2}(x^2+xy)\,dx\,dy = \frac{6}{7}\int_0^1\left(\frac{x^3}{3}+\frac{x^2}{2}y\right)\Big|_0^{1/2}dy = \frac{6}{7}\int_0^1 \frac{1}{18}+\frac{1}{8}y\,dy = \frac{6}{7}\left[\frac{1}{18}+\frac{1}{16}\right] = \boxed{17/168}$

$du = -dy$

b) Marginal Densities of X & Y:

$$f_x(x) = F_x'(x) = \int_0^1 (x^2+yx)\,dy = \boxed{x^2+\frac{x}{2}}$$

$$f_y(y) = F_y'(x) = \int_0^1 (x^2+yx)\,dx = \boxed{\frac{1}{3}+\frac{y}{2}}$$

$(1-y)^2 y$

$(1-2y+y^2)y$

$y - 2y^2 + y^3$

69. $p(x) = \lambda e^{-\lambda r}$; $V = \frac{4}{3}\pi R^3$; $R = \sqrt[3]{\frac{3V}{4\pi}}$; $p(R) = \lambda e^{-\lambda \sqrt[3]{\frac{3V}{4\pi}}} \cdot \left| \frac{d}{dV} \lambda e^{-\lambda \sqrt[3]{\frac{3V}{4\pi}}} \right|$

$= \lambda e^{-\lambda \sqrt[3]{\frac{3V}{4\pi}}} \cdot \lambda \left(-\lambda \frac{1}{3} \left(\frac{3V}{4\pi}\right)^{-2/3} \cdot \frac{3}{4\pi} \right) e^{-\lambda \sqrt[3]{\frac{3V}{4\pi}}}$

"Density Function"

70. $p(x) = \lambda e^{-\lambda r}$; $A = \pi r^2$; $r = \sqrt{\frac{A}{\pi}}$;

$f(y) = f_x(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$; $f(y) = \lambda e^{-\lambda r} = \lambda e^{-\lambda\sqrt{\frac{A}{\pi}}} \left| \frac{d}{dA} \lambda e^{-\lambda\sqrt{\frac{A}{\pi}}} \right|$

$= \lambda e^{-\lambda\sqrt{\frac{A}{\pi}}} \left| -\frac{\lambda}{2}\left(\frac{A}{\pi}\right)^{-1/2} e^{-\lambda\sqrt{\frac{A}{\pi}}} \right|$

"Density Function"

71. F = cdf of random variable.

U is uniform from $[0,1]$.

Define $Y = K$ if $F(k-1) < U \leq F(k)$;

$\boxed{\int_{k-1=0}^{k=1} F(k)\,dk = \int_0^1 U\,dx = F}$

$m = 3$;

X	0	2	1	0	2	1

$c = 2$

72. $X_n = (a X_{n-1} + c) \bmod m$ a) $a = 1$ b)

X	1	64069	2	13813	944	

Chapter 3: Joint Distributions:

1. Joint Frequency Function:

y \ X	1	2	3	4	$p_y(y)$
1	0.10	0.05	0.02	0.02	0.19
2	0.05	0.20	0.05	0.02	0.32
3	0.02	0.05	0.20	0.04	0.31
4	0.02	0.02	0.04	0.10	0.18
p(x)	0.19	0.32	0.31	0.18	

= Joint Frequencies

← Marginal Frequencies

A) Find the marginal frequency functions of X and Y. i.e. $p(x)$, and $p(y)$.

b) Find the conditional frequency of X given $Y=1$ and Y given $X=1$.

~~$P(X|Y=1) = \frac{P(X, Y=1)}{P(y)} = \frac{0.10}{0.19} = \boxed{\frac{10}{19}}$; $P(Y|X=1) = \frac{P(Y, X=1)}{P(X)} = \frac{0.10}{0.19} = \boxed{\frac{10}{19}}$~~

$\boxed{p(1|1) = \frac{10}{19};\ p(2|1) = \frac{5}{19};\ p(3|1) = \frac{2}{19},\ p(4|1) = \frac{2}{19}}$

2. p - black balls, q - white balls, r - red balls — Urn; n chosen

a) Find the joint distributions of black, white and red balls.

Multinomial Distribution: $p(n_1, \ldots, n_r) = \binom{n}{n_1, \ldots, n_r} p_1^{n_1} p_2^{n_2} \cdots p_3^{n_r}$

$p(\text{red, white, black}) = \frac{\binom{p}{x}\binom{q}{y}\binom{r}{z}}{\binom{p+q+r}{n}}$

where $n = X + Y + Z$

Total selection outcomes

b) Joint Distribution of black and white balls.

$P(\text{black}(X), \text{white}(Y)) = \frac{\binom{p}{x}\binom{q}{y}\binom{r}{n-x-y}}{\binom{p+q}{n-r}}$

Outcomes

Black + White outcomes

c) $P(Y) = \frac{\binom{q}{y}\binom{p+r}{n-y}}{\binom{p+q+r}{n}}$

Outcomes possible

61. Density of cX when X follows gamma distribution ; Gamma Distribution: $g(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t}$

$g(ct \le \lambda) = \boxed{g(t \le \frac{\lambda}{c}) = \frac{(\frac{\lambda}{c})^\alpha}{\Gamma(\alpha)} t^{\alpha-1} \cdot e^{-\lambda t/c}}$ $\qquad \Gamma(\alpha) = \int_0^\infty u^{\alpha-1} \cdot e^{-u} du$

62. m = mass ; V = random velocity; $\mu = 0$ and σ. Find the density function of Kinetic Energy : $E = \frac{1}{2}mV^2$

Normally Distributed: $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$

$= \frac{1}{\sigma\sqrt{2\pi}} e^{-E^2/2\sigma^2} = \boxed{\frac{1}{\sigma\sqrt{2\pi}} e^{-(\frac{1}{2}mV^2)^2/2\sigma^2}}$

63. Suppose Θ is a uniform distribution; Interval or domain $[-\pi/2, \pi/2]$: Find the CdF and density of tanθ

$\tan\theta = X$; $\theta = \arctan(X)$: $P(\arctan(-X) \le \Theta \le \arctan(X)) = \phi(\arctan(X)) + \phi(\arctan(+X))$

$= \frac{a^2}{x^2+a^2}\phi(\arctan(x)) + \frac{a^2}{x^2+a^2}\phi(\arctan(+x))$

$F_u = \boxed{\frac{1}{\pi(1+x^2)} \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}}$

64. f_X = "Density Function" and $Y = aX + b$, then.

$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$

$f_Y(y) = P(Y \le y) = P(aX + b \le y) = P(X \le \frac{y-b}{a}) = f_X\left(\frac{y-b}{a}\right) = P'(X) = \boxed{f_X\left(\frac{y-b}{a}\right)\frac{1}{|a|}}$

65. $f(x) = \frac{1+\alpha x}{2}$ from $-1 \le x \le 1$ and $-1 \le \alpha \le 1$. $\quad F_X(x) = \int_{-1}^{x} \frac{1+\alpha t}{2} dt = \int_{-1}^{x} \frac{1}{2} + \frac{\alpha}{2} t \, dt$

66 $f(x) = \alpha x^{-\alpha-1}$ for $x \ge 1$ and $f(x) = 0$

$F_X(x) = \frac{1}{2}[x+1] + \frac{\alpha}{4}[x^2 - 1]$

$\int_1^\infty \alpha x^{-\alpha-1} dx = \frac{\alpha x^{-\alpha}}{(-\alpha)} \Big|_1^\infty = -\frac{1}{x^\alpha}\Big|_1^\infty = -\frac{1}{\infty} + \frac{1}{1^\alpha} = 1 = f(x)$

$4F_X(x) = 2[x+1] + \alpha[x^2 - 1]$

$4X = 2[F^{-1}(x) + 1] + \alpha[F^{-1}(x)^2 - 1]$

$F^{-1}(1) = X$ $\quad$ $\boxed{\text{"Cumulative Distribution Functions are pseudo random number generators"}}$

$4X = 2F^{-1}(x) + 2 + \alpha F^{-1}(x)^2 - \alpha$

$\alpha F^{-1}(x)^2 + 2F^{-1}(x) + (2 - \alpha - 4X) = 0$

67. Weibull Cumulative Distribution Function:

$\frac{-2 \pm \sqrt{4 - 4(\alpha)(2 - \alpha - 4X)}}{2\alpha} = F^{-1}(X)$

$F(x) = 1 - e^{-(x/\alpha)^\beta}$, $x \ge 0$, $\alpha > 0$, $\beta > 0$

a) Find the density function. $p(x) = F'(x) = -\left(\frac{x}{\alpha}\right)^\beta e^{-(x/\alpha)^\beta} = -\left(\frac{\beta}{\alpha}\right) x^{\beta-1} e^{-(x/\alpha)^\beta}$

$\frac{-2 \pm \sqrt{4 - 8\alpha + 4\alpha^2 + 8X}}{2\alpha} = F^{-1}(X)$

b) If Weibull is W, then $X = (W/\alpha)^\beta$ is an exponential distribution.

$\boxed{\frac{-1 \pm \sqrt{1 - 2\alpha + \alpha^2 + 2X}}{\alpha} = F^{-1}(X)}$

$W = \alpha^\beta x^{1/\beta}$; $\frac{dW}{dX} = \frac{1}{\beta}\alpha x^{1/\beta - 1}$

$f_X(W) = f_X(\alpha \cdot x^{1/\beta}) \cdot \left|\frac{\partial W}{\partial x}\right| = \frac{\beta}{\alpha} e^{-W} \left(\frac{\alpha x^{1/\beta}}{\alpha}\right)^{\beta-1} \cdot \frac{\alpha}{\beta}(x)^{\frac{1}{\beta}-1} = \boxed{e^{-W}}$

c). $U = e^{-W}$; $\ln U = -W$; $\boxed{W = -\ln U}$;

68. U = Uniform Random Variable. Find $V = U^{-\alpha}$ for $\alpha > 0$ $\quad U^\alpha = V$; $U = V^{1/\alpha}$

$P(U^{1/\alpha}) = P(V^{1/\alpha}) \cdot \left|\frac{d}{dV} V^{-1/\alpha}\right| = P(V^{-1/\alpha}) \circ \left(\frac{1}{\alpha}\right) \cdot V^{(1-\alpha)/\alpha}$ $\quad$ $\boxed{\text{Proposition B}}$

The rates of decrease for the density function is described as of greater rates.

52. $\mu = 70$ and $\sigma = 3$in. a) What proportion of the population is over 6 ft tall?

$$p(78) = \frac{1}{3\sqrt{2\pi}} e^{-(78-70)^2/2\cdot 3^2} = 0.1065 \text{ ; } \boxed{0.35\% \text{ over the height of 6ft tall.}}$$

$$p(>78) = \frac{1}{3\sqrt{2\pi}} \int_{78}^{\infty} e^{-(x-70)^2/18} dx = 0.00038 = \boxed{0.38\%}$$

b) cm: $70\text{in} \times \frac{2.54\text{cm}}{1\text{inch}} = 175\text{cm}$; $3\text{in} \times \frac{2.54\text{cm}}{1\text{inch}} = 7.5\text{cm}$; m: $70\text{inches} \times \frac{2.54\text{in}}{1\text{inch}} \times \frac{1\text{m}}{100\text{cm}} = 1.75\text{cm}$; $7.5\text{in} \times \frac{1\text{m}}{100\text{cm}} = \boxed{0.075\text{m}}$

53. $\mu = 5$, and $\sigma = 10$. a) Find $P(X>10) = \frac{1}{10\sqrt{2\pi}} \int_{10}^{\infty} e^{-(x-5)^2/2\cdot 10^2} dx = 0.3085 = \boxed{30.85\%}$ Z-table: $\frac{10-5}{10} = \frac{1}{2} = Z(30.85\%)$

b) Find $P(-20 < X < 15) = \frac{1}{10\sqrt{2\pi}} \int_{-20}^{15} e^{-(x-5)^2/2\cdot 10^2} dx = 0.8351 = \boxed{83.51\%}$ $\left(\frac{15-5}{10} - \frac{-20-5}{10}\right)$

$Z(0.9938) - Z(0.8413)$

$\boxed{84.75\%}$

54. $X \sim N(0, \sigma^2)$; $Y = |X|$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x)^2/2\cdot\sigma^2} \cdot dx = \frac{\sqrt{\pi \cdot 2 \cdot \sigma^2}}{\sigma\sqrt{2\pi}} = \boxed{1}$$

55. $X \sim N(\mu, \sigma^2)$ find c in terms of σ, such that, $P(\mu - c \leq X \leq \mu + c) = 0.95$

$$0.95 = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu-c}^{\mu+c} e^{-(x-\mu)^2/2\sigma^2} \cdot dx \text{ ; } \int e^{-ax^2} = \sqrt{\frac{\pi}{a}} \text{ ; } \boxed{c \approx 1.95996\sigma}$$

56. $X \sim N(\mu, \sigma^2)$; $P(|X-\mu| \leq 0.675\sigma) = 0.5 \Rightarrow 1 - \frac{1}{0.675\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-0.675\sigma^2/2\sigma^2} \cdot dx = \boxed{0.5}$

57. $X \sim N(\mu, \sigma^2)$; $Y = aX + b$, where $a < 0$, show $Y \sim N(a\mu + b, a^2\sigma^2)$

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2} dx = P\left(\frac{y-b}{a}\right) = \frac{1}{a} \frac{1}{\sigma\sqrt{2\pi}} e\left[-\frac{1}{2}\left(\frac{y-b-a\mu}{a\sigma}\right)^2\right] = \frac{1}{a\sigma\sqrt{2\pi}} \exp\left[\frac{-1}{2} \frac{(y-(a\mu+b))^2}{a^2\sigma^2}\right]$$

$\boxed{Y \sim (a\mu + b, a^2\sigma^2)}$

58. $Y = e^Z$, where $Z \sim N(\mu, \sigma^2)$; $\boxed{\text{Lognormal Density}}$

$$Y = e^z = e^{N(\mu,\sigma^2)} \text{ ; } \log Y = N(\mu, \sigma^2) = 1 \text{ ; } \boxed{Y = e}$$

59. $U[-1,1]$; Find density function of U^2; $F_v = P(-\sqrt{x} \leq z \leq \sqrt{x})$

$$= \Phi(\sqrt{x}) - \Phi(-\sqrt{x})$$

$$= \frac{1}{2} x^{-1/2} \phi(\sqrt{x}) + \frac{1}{2} x^{-1/2} \phi(-\sqrt{x})$$

$$= x^{-1/2} \phi(\sqrt{x}) \text{ ; } \boxed{f_x(x) = \frac{x^{-1/2} e^{-x/2}}{\sqrt{2\pi}}}$$

60. $U[0,1]$; Find density function of $\sqrt{U}$.

$$F_u = P(x^2 \leq z \leq x^2) = \Phi(x^2) + \Phi(0)$$

$$= 2x\, \phi(x^2) \text{ ; } \boxed{f(x) = \frac{2x e^{-x/2}}{\sqrt{2\pi}}}$$

45. Exponential Distribution: $p(x)=\lambda e^{-\lambda x}$; $\lambda=0.1$ a) Probability lifetime <10 years.

b. $\frac{e^{-5/10}}{10}-\frac{e^{-15/10}}{10}$ c) $0.01=\frac{e^{-t/10}}{10}$; $-1=-t/10$; $\boxed{t=10}$

$P(\text{lifetime})+P(\text{death})=P(\text{lifetime})+\frac{e^{-10/10}}{10}=1$

$\boxed{P(\text{lifetime})=1-\frac{e^{-1}}{10}}$

46. Gamma Density: $g(t)=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}t^{\alpha-1}e^{-\lambda t}$, $t\geq 0$

where $\Gamma(x)=\int_0^{\infty}u^{x-1}e^{-u}du$; $x>0$

$\int_0^{\infty}g(t)dt=\int_0^{\infty}\frac{\lambda^{\alpha}t^{\alpha-1}e^{-\lambda t}}{\Gamma(\alpha)}dt=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-\lambda t}dt$; $t=x/\lambda$; $x=t\lambda$; $=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\int_0^{\infty}\left(\frac{x}{\lambda}\right)^{\alpha-1}e^{-x}\frac{dx}{\lambda}$

$=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}x^{\alpha-1}e^{-x}dx=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\lambda}=\frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)\lambda}=\boxed{\frac{\alpha}{\lambda}}$; $\lambda=1$ and $\alpha=1$.

47 $\alpha>1$, Show maximum of Gamma Density : $(\alpha-1)/\lambda$; $\frac{d}{dt}g(t)=0$

$=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\left[(\alpha-1)t^{\alpha-2}-\lambda t^{\alpha-1}\right]e^{-\lambda t}=0$; $(\alpha-1)t^{\alpha-2}=\lambda t^{\alpha-1}$; $\boxed{\frac{(\alpha-1)}{\lambda}=t^{\alpha-1-\alpha+2}=t}$

48, T is an exponential Random variable: $p(x)=\lambda e^{-\lambda x}$, and $P(T<1)=0.05$.

What is λ? $0.05=\lambda e^{-\lambda T}=\lambda\left(1-\lambda+\frac{\lambda^2}{2!}-\cdots\right)=\lambda-\lambda^2+\frac{\lambda^3}{2!}$; $\boxed{\lambda=0.0527}$

"Third order Quadratic"

49. a) $\Gamma(1)=1$; Gamma Density: $g(t)=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}t^{\alpha-1}\cdot e^{-\lambda t}$

Gamma Function: $\Gamma(x)=\int_0^{\infty}u^{x-1}e^{-u}du$; $\Gamma(1)=\int_0^{\infty}u^{1-1}e^{-u}du=\int_0^{\infty}e^{-u}du=-e^{-u}\Big|_0^{\infty}=\boxed{0-(-1)=1}$

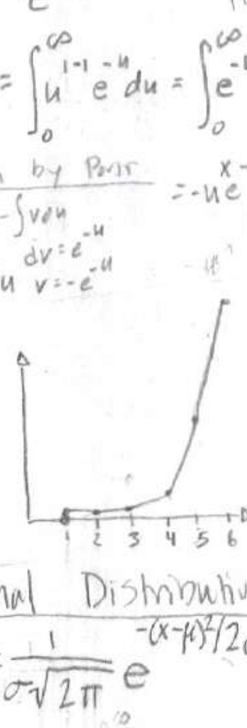
b) $\Gamma(x+1)=x\Gamma(x)$; $\Gamma(x+1)=\int_0^{\infty}u^{x}e^{-u}du$; Integration by Parts $\int u\,dv=uv-\int v\,du$; $u=u^x$, $dv=e^{-u}$; $du=(x)u^{x-1}du$, $v=-e^{-u}$; $=-u^xe^{-u}\Big|_0^{\infty}+\int(x)u^{x-1}e^{-u}=\boxed{x\Gamma(x)}$

c) Conclude $\Gamma(n)=(n-1)!$; $n=1,2,3\ldots$

Table:

n	$\Gamma(n)$	$(n-1)!$
1	0	0
2	1	1
3	2	2
4	6	6
5	24	24
6	120	120

Graph:



51. Normal Distribution $p(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$

Standard Normal: $\mu=0,\sigma=1$ $p(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$

$2\int_0^{\infty}e^{-x^2/2}dx=2\sqrt{\frac{\pi}{2}}=\sqrt{2\pi}$; $\int_{-\infty}^{\infty}e^{-x^2/2}dx\cdot\int_{-\infty}^{\infty}e^{-y^2/2}dy=2\pi$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(x^2+y^2)/2}dx\,dy=\boxed{\int_0^{\infty}e^{-r^2/2}(r)dr}$

50. $\Gamma(x)=2\int_0^{\infty}t^{2x-1}e^{-t^2}dt=\int_{-\infty}^{\infty}e^{xt}\cdot e^{-e^t}dt$;

$=\int_{-\infty}^{\infty}t^{2x-1}\cdot e^{-t^2}dt=\int_{-\infty}^{\infty}(e^{t/2})^{2x-1}\cdot e^{-e^t}dt$

$e^t=t^2$ $=\int_{-\infty}^{\infty}e^{xt-t/2}\cdot e^{-e^t}dt=\boxed{\int_{-\infty}^{\infty}e^{xt}e^{-e^t}dt}$ $e^{xt}=t^{2x}$

39. Cauchy Cumulative Distribution:

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x), \quad -\infty < x < \infty$$

a. Cumulative Distribution Requirements: $\lim_{x\to-\infty} F(x) = 0$; $\lim_{x\to\infty} F(x) = \boxed{1}$

$$\lim_{x\to-\infty} F(x) = \lim_{x\to-\infty} \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x) = \frac{1}{2} - \frac{1}{2} = \boxed{0} ; \lim_{x\to\infty} \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x) = \frac{1}{2} + \frac{1}{2} = \boxed{1}$$

b. $$p(x) = F'(x) = \frac{d}{dx}\frac{1}{\pi}\tan^{-1}\left(\frac{x}{1}\right) = \boxed{\frac{1}{(1+x^2)\pi}}$$

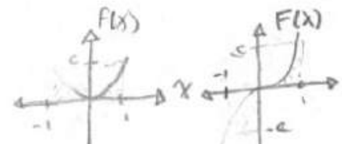
c. $P(X > x) = 0.1$; $0.1 \cdot (1+x^2)\pi x = 1$

$$(1+x^2) = \frac{10}{\pi}$$

$$x^2 = \frac{10-\pi}{\pi}$$

$$\boxed{x = \sqrt{\frac{10-\pi}{\pi}}}$$

40. $f(x) = cx^2$ for $0 \le x \le 1$ and $f(x) = 0$ otherwise



a) $F(1) = c$; $f(0) = 0$; $c = \frac{f(1) - f(0)}{1-0} = \Delta f(x)$

b) $F(x) = \frac{cx^3}{3}$ c) $P(0.1 \le X < 0.5) = \frac{[F(0.5) - F(0.1)]/(0.5-0.1)}{F(1) - F(0)/(1-0)} = \frac{[c(\frac{1}{4}) - c(\frac{1}{10})]/0.4}{c}$

$$= \frac{6}{40} \cdot \frac{10}{4} = \frac{6}{16} = \boxed{\frac{3}{8}}$$

41. Find the upper and lower quartiles of an exponential distribution.

Exponential Distribution: $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & x < 0 \end{cases}$

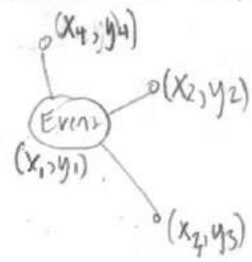
Lower quartile: $P(X) = \frac{1}{4} = \lambda e^{-\lambda x}$; $-\log 4\lambda = -\lambda x$

$$\boxed{x = \frac{+\log 4\lambda}{\lambda}}$$

Upper quartile: $p(x) = \frac{3}{4} = \lambda e^{-\lambda x}$

$$\boxed{x = \log\left(\frac{4\lambda}{3}\right)/\lambda}$$

42.



$$\boxed{p(x) = \sqrt{\left(\frac{\lambda^{x_2} e^{-\lambda}}{x_2!} - \frac{\lambda^{x_1} e^{-\lambda}}{x_1!}\right)^2 + (x_2 - x_1)^2}}$$

$$\boxed{f(x) = 2\pi\lambda x e^{2-\pi\lambda x^2}}$$

43.

$$f(x) = \frac{d}{dx} - e^{-4\lambda\pi x^3/3} \quad \text{volume} \times \text{lambda}$$

$$\boxed{= 4\lambda\pi x^3 e^{-4\lambda\pi x^3/3}}$$

Multivariate Poisson Distribution:

$$P(X) = \exp\left(-\sum_{i}^{m}\theta_i\right)\prod_{i}^{m}\frac{\theta_i^{x_i}}{x_i!}\sum_{i=0}^{s}\prod_{j=1}^{m}\binom{x_j}{i} i! \left(\frac{\theta_0}{\prod \theta_i}\right)^i$$

44. T: Exponential Random Variable with λ, $T = \lambda e^{-\lambda x}$; $k = \lambda e^{-\lambda x}$; $k+1 = \lambda e^{-\lambda x}$

$$\boxed{x = \log\left(\frac{\lambda}{T}\right)/\lambda}$$

Exponential Random Distribution:

$$p(x) = \lambda e^{-\lambda x}$$

X: Discrete Random variable; $X = K$; $k < T < k+1$ for $k = 0, 1, \ldots$

30. Poisson Frequency Function ; $p(k) = \frac{\lambda^k \exp(-\lambda)}{k!}$; $p'(k) \doteq \frac{k \cdot \lambda^{k-1}(-\lambda)\exp(-\lambda)}{k!} = 0$

Sources cite ratio: $\frac{P(X=k+1)}{P(X=k)} = \frac{\lambda^{k+1}\exp(-\lambda)/(k+1)!}{\lambda^k \exp(-\lambda)/k!}$

$-\lambda = 1$

$\boxed{\lambda = 1}$

$\boxed{= \lambda/k+1}$

"Not logical because max of Poisson changes" as n changes.

There are maximum and minimums to this problem set. Probability Decisions

$\lambda < 1$, $\lambda > 1\,(\text{int})$, $\lambda > 1\,(\text{Rational})$ $\quad \lambda = np = 1$ s

Max : $k=0$

31. $\lambda = 2$ per hour a) 10-min shower ; $p(\text{phone rings}) = \frac{(2)^{1/6}\exp(-2)}{1/6} = \boxed{0.27}$

Fractions and Factorial approx to one

b) $p(\text{phone rings}) = 0.5 = \frac{\delta^0 e^{-\delta}}{0!} = e^{-\delta}$; $\delta = 0.693$ min

$\text{Time} = \frac{60\text{min}}{2\text{ phone calls}} \cdot 0.693$

$\boxed{= 20.79\text{ min}}$

32. $\lambda = 0.33$ per month

a) $k=0$; $p(0) = \frac{(1/3)^0 \exp(-1/3)}{0!} = \exp(-1/3) = \boxed{0.716}$

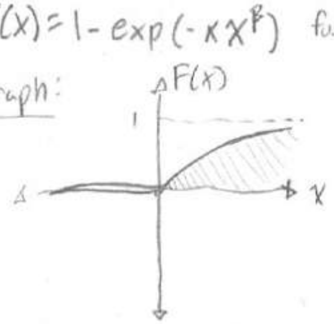
$k=1$; $p(1) = \frac{(1/3)^1 \exp(-1/3)}{1!} = \boxed{0.238}$

$k=2$; $p(2) = \frac{(1/3)^2 \exp(-1/3)}{2!} = \boxed{0.004}$

The most probable number of suicides would be at $k=0$ because $\lambda < 1$ and demonstrates a decreasing probability.

33. $F(x) = 1 - \exp(-\alpha x^\beta)$ for $x \geq 0, \alpha > 0, \beta > 0$, $F(x) = 0$ for $x < 0$.

Graph:

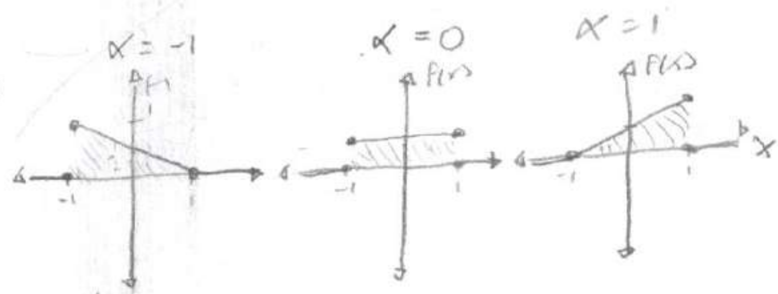


$p(x) = \frac{d}{dx}F(x) = \alpha\beta x^{\beta-1}\exp(-\alpha x^\beta)$

Cumulative Density Function.

$\lim_{x \to -\infty} F(x) = 0$; $\lim_{x \to \infty} F(x) = 1$

34. $f(x) = (1+\alpha x)/2$ for $-1 \leq x \leq 1$ and $f(x) = 0$



Probability Density Function Requires

$P(X=x_i) = p(x_i)$; $\sum_i p(x_i) = 1$

$F(x) = \sum_{-1}^{1}(1+\alpha x)/2 = \int_{-1}^{1}(1+\alpha x)/2\,dx = \frac{x}{2}\Big|_{-1}^{1} + \frac{\alpha x^2}{4}\Big|_{-1}^{1} = \frac{1}{2} + \frac{1}{2} + \frac{\alpha x^2}{4} - \frac{\alpha x^2}{4} = \boxed{1}$

$\boxed{F(x) = \frac{x}{2} + \frac{\alpha x^2}{4}}$

35. $p(x) = \begin{cases} -1 & -1 \leq x \leq 0 \\ 2 & 0 < x < 1 \end{cases}$

$F(x) = \int_{-1}^{0} -1\,dx + \int_{0}^{1} 2\,dx = 1$

36. U is uniform $[0,1]$. $X = [nU]$, where $[t]$

"Probability Mass Function"

37. $p(X \leq 1/3) = \frac{1}{3}$;

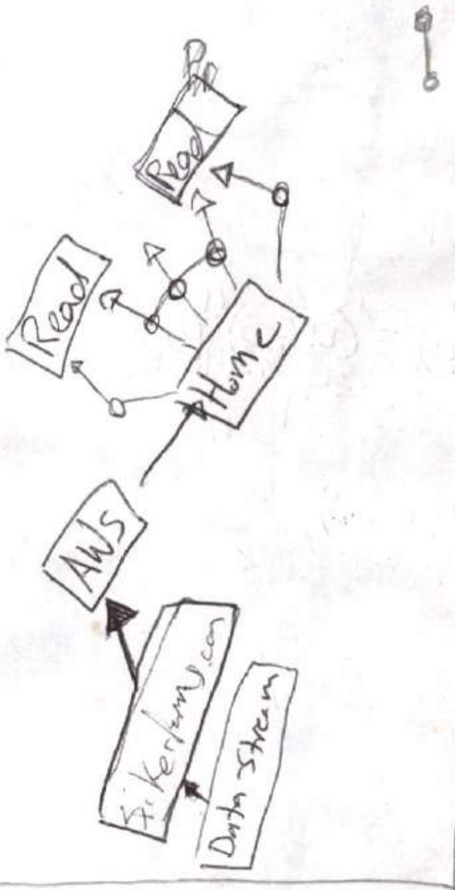
$p(X \geq 2/3) = 2/3$

38. $\alpha f + (1-\alpha)g$; $\sum P(x) = 1 = \alpha f + (1-\alpha)g$

where $0 \leq \alpha \leq 1$ $\quad$ Max $\frac{d}{d\alpha}(\alpha f + (1-\alpha)g) = f - g$

$F(x) = \frac{\alpha^2}{2}f + (\alpha - \frac{\alpha^2}{2})g$

- Randomize Router [IP] ☐ ✓
- Randomize Mac Address ✓
- Randomize external IP. ✓
- VPN through aws
- Tor for browsing
  - ↳ IP Scan - ping
- Tor guard.



27. $P(\text{Disease}) = \frac{1}{10{,}000}$; $n = 100{,}000$ people

Poisson Distribution

$k=0$ cases: $p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$; $p(0) = \frac{\lambda^0 e^{-10}}{0!} = \boxed{4.54\times10^{-5}}$

$k=1$ cases: $p(1) = \frac{\lambda^1 e^{-\lambda}}{1!} = \frac{(10)\cdot e^{-10}}{1!} = \boxed{4.54\times10^{-4}}$

$k=2$ cases: $p(2) = \frac{\lambda^2 e^{-\lambda}}{2!} = \frac{(10)^2 e^{-10}}{2!} = \boxed{2.26\times10^{-2}}$

28. $CDF = F(k) = p_0, p_1, \dots, p_n$; n, and p. $q = 1-p$

Prove the binomial probabilities by $p_0 = q^n$.

$$P_k = \frac{(n-k+1)p}{kq} P_{k-1} \;;\; k = 1, 2, \dots n$$

$\boxed{P_0 = (1-p)^n}$; $p_1 = n\frac{p}{(1-p)}(1-p)^n$; $p_2 = \frac{(n-1)p}{2(1-p)} p_1 = \frac{(n-1)p^2}{2(1-p)(1-p)}(1-p)^n$

$\boxed{= n(p(1-p))^{n-1}}$ $\boxed{= \frac{(n-1)}{2} p^2 (1-p)^{n-2}}$

Recursive Binomial Distribution $= \frac{n}{(1-p)} p(1-p)^n$

$$\boxed{= \prod \frac{(n-k+1)}{k!} p^n (1-p)^{n-k}}$$

$p(X \le 4)$ for $n = 9000$ and $p = 0.0005$

$$= \frac{(9000-3)(9000-2)(9000-1)(9000)}{4!}(0.0005)^{9000}(1-0.0005)^{9000-4}$$

$\boxed{= 1.89\times10^{-1}}$ $\cong 0.00$

As a Poisson

$n = 9000$; $p = 0.0005$; $np = 9/2$

$P(4) = \frac{(9/2)^4}{4!} e^{-9/2} = \boxed{1.89\times10^{-1}}$

29. $p_0 = \exp(-\lambda)$

$p_k = \frac{\lambda}{k} p_{k-1}$; $k = 1, 2 \dots$

$p_0 = \exp(-\lambda)$; $p_1 = \lambda \cdot p_0 = \lambda\exp(-\lambda)$; $p_2 = \frac{\lambda^2}{2}\exp(-\lambda)$; $p_k = \frac{\lambda^k}{k!}\exp(-\lambda)$

$P(X \le 4)$; $\lambda = 4.5$; $p_k = \frac{(4.5)^4}{4!}\exp(-4.5) = \boxed{1.89\times10^{-1}}$

20. Minimum Trials: $n+1$ | $P(\text{Left}) = \frac{1}{2}$ | $P(k \mid \text{other box}) = \binom{n+1}{n+1} p^{n+1} q^0 + \binom{n+1}{n+1} p^0 q^{n+1}$
Maximum Trials: $2n+1$ | $P(\text{Right}) = 1-p = \frac{1}{2}$ | $= p^{(n+1)} + q^{(n+1)} = 2\left(\frac{1}{2}\right)^{(n+1)} = \left(\frac{1}{2}\right)^n$

$P(k-1 \mid \text{other box}) = \binom{n+1}{n} p^n q + \binom{n+1}{n} p q^n = 2\binom{n+1}{n}\left(\frac{1}{2}\right)^{n+2}$
$= \binom{n+1}{n}\left(\frac{1}{2}\right)^{(n+1)}$
$= (n+1)\left(\frac{1}{2}\right)^{(n+1)}$

21. X = Geometric Random Variable $\quad P(X > n+k-1 \mid X > n-1) = P(X > k)$

$P(\text{Hypergeometric Function}) = \dfrac{\binom{r}{k}\binom{n-r}{m-k}}{\binom{n}{m}}$; $P(X > n+k-1 \mid X > n-1)$
$= P(X+1-n > k \mid X+1 > n)$ $\quad P(k=n-r) = \binom{n+r}{n}\left(\frac{1}{2}\right)^{n+r}$
$= P(X-n > k \mid X-n > 0)$ $\quad P = \binom{n+n-k}{n}\left(\frac{1}{2}\right)^{n+n-k}$
$= \boxed{P(X > K \mid X > 0) = P(X > k)}$

22. X: Geometric random variable
$p = 0.5$; $P(X \le k) \approx 0.99 = 1-(1-p)^k$
$(0.5)^k = 0.01$
$k\log(0.5) \approx \log(0.01)$; $k = 6.643$;
$\boxed{k = 7.}$

23. $p(\text{success})$; r successes before k^{th} failure ; Binomial $\boxed{p(k) = \binom{k+r-1}{r} p^r (1-p)^k}$
Total Number of trials : $(k+r)$
Last trial probability : $(1-p)$
Binomial : $\boxed{\binom{k+r-1}{k-1} p^r (1-p)^{k-1}}$
Failures

$p(\text{success}) = \binom{k+r-1}{k-1} p^r (1-p)^{k-1} (1-p)$
$\boxed{= \binom{k+r-1}{k-1} p^r (1-p)^k}$

24. $n = 2^3$; $P(\text{Same}) = P(HHH) + P(TTT)$
$= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

$P(k) = (1-p)^{k-1}(p) = (0.75)^{k-1}(0.25)$
$\sum_{k=1}^{\infty} (0.75)^{k-1}(0.25) = 1$
$P(X>3) = 1 - P(X \le 3)$
$= 1 - \{P(X=1) + P(X=2) + P(X=3)\}$
$= 1 - \{(0.75)^{1-1}(0.25) + (0.75)^{2-1}(0.25) + (0.75)^{3-1}(0.25)\}$
$= 1 - \{0.25 + 0.1875 + 0.140625\}$
$\boxed{\approx 0.4219}$

25. $P(\text{Royal Straight Flush}) = 1.3 \times 10^{-8}$
$n = 100$ hands/week ; 52 weeks/year ; 20 years $= 1.04 \times 10^5$
a) Time Sequence: Poisson Distribution
$\lambda = n \cdot p = 1.04 \times 10^5 \cdot 1.3 \times 10^{-8} = 1.35 \times 10^{-3}$
$p(k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$; $p(0) = \dfrac{\lambda^0 e^{-\lambda}}{0!} = e^{-1.35 \times 10^{-3}} = \boxed{9.986 \times 10^{-1}}$

① State Space
② Frequency Function
③ $1 - p(1)p(2)p(3)$

b). $2 = \prod p^k (1-p)^{n-k} = P(2) = \dfrac{\lambda^2 e^{-\lambda}}{2!} = \dfrac{(1.35 \times 10^{-3})^2 e^{-1.35 \times 10^{-3}}}{2!} = \boxed{9.10 \times 10^{-7}}$

26. $\frac{1}{10{,}000}$ chance of being trapped. $n = 5\frac{\text{days}}{\text{week}} \cdot 52\frac{\text{weeks}}{\text{year}} \cdot 10\text{years} = 2{,}600$; $\lambda = np = \boxed{\frac{13}{50}}$
$P(0) = \dfrac{\lambda^0 e^{-\lambda}}{0!} = e^{-13/50} = \boxed{7.71 \times 10^{-1}}$; $P(1) = \left(\frac{13}{50}\right)^1 e^{-13/50} = \boxed{2.0 \times 10^{-1}}$; $p(2) = 0.026$

10. Player A: $p_1 = P(\text{Success})$; Player B: $p_2 = P(\text{success})$ a) $P(x)$: even: $p(k) = (1-p_1)^{k-1/2} \cdot (1-p_2)^{(k-1)/2} \cdot p_1$

b. $P(\text{Player A: Win}) = \dfrac{p_1}{1-(1-p_1)(1-p_2)} = \boxed{p_1 \sum_i [(1-p_1)(1-p_2)]^i}$ odd: $p(k) = (1-p_1)^{k/2} \cdot (1-p_2)^{\frac{k}{2}-1} \cdot p_2$

11. Binomial Distribution: $p(x) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$; Mode $= p'(x) = 0 = kp^{k-1}(1-p)^{n-k} + (n-k)p^k(1-p)^{n-k-1}$

12. Prove $p(k) = \sum_0^1 \binom{n}{k} p^k (1-p)^{n-k} = 1$

$= \binom{1}{0} p^0 (1-p)^1 + \binom{1}{1} p^1 (1-p)^0$

$\boxed{= (1-p) - p = 1}$

$= kp^{k-1}(1-p)^{n-k} + n \cdot p^k (1-p)^{n-k} - kp^k(1-p)^{n-k}$

$p^k(1-p)$

$= k\left(\frac{1}{p}\right)(1-p) k + (n-k)$; $k\left(\frac{1}{p}-1\right) + n - k = 0$

$\frac{k}{p} - k + n - k = 0$

$\frac{1}{p} + \left(\frac{n}{k}\right) - 2 = 0$

$\left(\frac{1}{p}\right) - 2 = \left(\frac{n}{k}\right) + 2$

$\frac{1-2p}{p} = \frac{n-2k}{k}$

$\boxed{k = \frac{p \cdot n}{1-2p}}$

13. 20 items [4-choices] - Elimination of one, remainder of three
- Passing is 12 or more, correct.

a. $P(\text{Pass}) = \dfrac{P(\text{Correct})}{\text{Total Outcomes}} = \dfrac{1}{\binom{3}{1}\binom{20}{1}} = \dfrac{1}{3 \cdot 20} = \boxed{\dfrac{1}{60}}$

b. $P(\text{Pass}) = \dfrac{P(\text{correct})}{\text{Total outcomes}} = \dfrac{1}{\binom{2}{1}/\binom{20}{1}} = \dfrac{1}{2 \cdot 20} = \boxed{\dfrac{1}{40}}$

14. $\underbrace{\square\square\square\square}_{4 \text{ bits}}\underbrace{\square\square\square}_{+3 \text{ bits}}$ $P(\text{Change}) = 0.05$; Mutually Independent.

$P(\text{change} | 7 \text{ bits}) = \prod_i^7 p(\text{change}_i) = p(\text{change})^7 = \boxed{0.05^7}$

$P(\text{change} | 4 \text{ bits}) = \prod_i^4 p(\text{change}_i) = p(\text{change})^4 = \boxed{0.05^4}$

$P(\text{correct}) = 1 - P(\text{change})$
$= 1 - 0.05^7$

15. $P(\text{Winning} | \text{Team A}) = 0.4$; Better advantage of 3 of 5 games or 4 of 7 games. 0.6

$\boxed{\boxed{P(\text{correct}) = 1 - 0.05^4}}$

$P_{\text{correct}} + P(\ldots)$

$\boxed{P(3 \text{ of } 5) = \prod_{i=1}^{3} p(\text{Winning} | \text{Game}_i) = (0.4)^3}$; $P(4 \text{ of } 7) = \prod_1^4 p(\text{Winning} | \text{Game}_i) = (0.4)^4$

16. $n \to \infty$ and $r/n \to p$ and $m = \text{constant}$. Hypergeometric Function: $P(x=k) = \dfrac{\binom{r}{k}\binom{n-r}{m-k}}{\binom{n}{m}}$

$\lim_{n \to \infty} \lim_{r/n \to p} P(x=k) = \lim_{n \to \infty} \lim_{r/n \to p} \dfrac{\frac{r!}{k!(r-k)!} \frac{(n-r)!}{(m-k)!(n-r-m+k)!}}{\frac{n!}{m!(n-m)!}} = \binom{m}{k} \dfrac{[r(r-1)\cdots(r-k+1)][(n-r)\cdots(n-r-m+k+1)]}{[n(n-1)\cdots(n-m+1)]}$

$= \binom{m}{k}\left(\frac{1}{q}\right)^k \left(\frac{q-1}{q}\right)^{m-k}$; where $q = \frac{1}{p} = \frac{n}{r}$

$= \binom{m}{k} p^k (1-p)^{m-k}$

17. Bernoulli Trials ; $p(\text{success}) = p$; Failures till first round are counted.

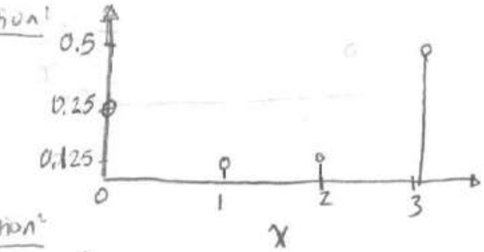
Frequency Function: $\boxed{p(\text{Failure}) = p \prod_i^n (1-p)^n}$

18. Frequency Function $p(\text{Failure}) = p \prod^r (1-p) = p(1-p)^n$

19. CDF: $\sum_x^{\infty} p(1-p)^{j} = 1$; $P = (1-p)^{j} \sum_1^{j} = \text{cdf} =$

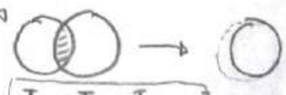
Chapter 2: 1. $P(x=0)=0.25$, $P(x=1)=0.125$, $P(x=3)=0.5$, $P(x=2)=0.125$

Frequency Function:



2. $I_{A\cap B} = I_A I_B = \min(I_A, I_B)$

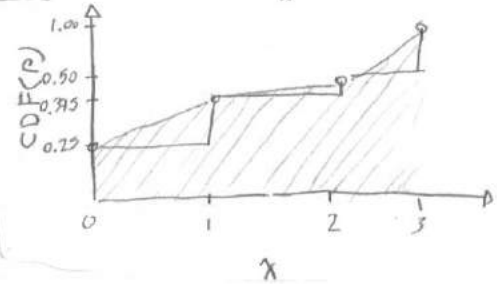
$I_{A\cup B} = \max(I_A, I_b)$



$\boxed{I_A \cap I_B = I_{A\cap B}}$

$\boxed{I_{A\cup B} = \max(I_A, I_b) = I_A + I_B - I_A \cdot I_B}$

Cumulative Distribution Function:



3. CDF → Frequency (Δ)

k	F(k)
0	0
1	0.1
2	0.3
3	0.7
4	0.8
5	1.0

CDF

k	P(X)
0	0
1	0.1
2	0.2
3	0.5
4	0.1
5	0.2

Frequency

4. Prove $p(k) = F(k) - F(k-1)$

$CDF = \sum_0^n p(k) = p(k) + \sum_1^n p(k)$

$F(k) = p(k) + F(k-1)$

$\boxed{p(k) = F(k) - F(k-1)}$

5. Show $P(u < X \leq v) = F(v) - F(u)$ a) Discrete $F(k) = \sum_0 p(k) = p(k) + \sum_1 p(n)$

$F(n) = p(k) + F(k-1)$

$\boxed{P(k) = F(k) - F(k-1)}$

b. $F(k) = \int_0^n p(k)\,dk = p(k) + \int_1^n p(k)\,dk = p(k) + F(n-1)$

$\boxed{p(n) = F(k) - F(k-1)}$

6. a) Four tosses: ~~$P(T) = (\frac{1}{2})^1(1-\frac{1}{2})^{1-1} = \frac{1}{2}$; $P(T|2) = (\frac{1}{2})^2(1-\frac{1}{2})^{1-2} = \frac{1}{4}(2)$; $P(T|3) = (\frac{1}{2})^3(1-\frac{1}{2})^{1-3} = (\frac{1}{8})4$; $P(T|4) = (\frac{1}{4})^4(1-\frac{1}{2})^{1-4}$~~ $= \frac{8}{16}$

① Sample space (n=16)

Outcome	HHHH	HHHT	HHTH	HHTT	HTHH	HTHT	HTTH	HTTT	T x8
Heads before Tails	4	3	2	2	1	1	1	1	0

② Frequency Function (n(x))

X	0	1	2	3	4
n(X)	8	4	2	1	1
$P(X) = \frac{n(X)}{16}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$
F(X)	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{15}{16}$	$\frac{16}{16}$

Sample Space: HHHH HHHT HHTH HHTT HTHH HTHT HTTH HTTT THHH THHT THTH THTT TTHH TTHT TTTH TTTT

b) the number of Heads following first tail

② Frequency Function

X	0	1	2	3	4
n(x)	1	1	2	4	8
$P(x) = \frac{n(x)}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$
F(x)	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1

c) (the number of heads) − (number of tails)

② Frequency Function

X	−4	−2	0	2	4
n(X)	1	4	6	4	1
$P(X) = \frac{n(X)}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$
F(X)	$\frac{1}{16}$	$\frac{5}{16}$	$\frac{11}{16}$	$\frac{15}{16}$	$\frac{16}{16}$

d) Frequency Function = (c)

7. Bernoulli Random Variable: $p(1) = p$; $p(0) = 1-p$; $p(x) = 0$; $p(x) = \begin{cases} p^x(1-p)^{1-x}; & x=0 \text{ or } x=1 \\ 0 & ; \text{otherwise.} \end{cases}$

CdF of Bernoulli Random Variable: $F(x) = \sum_0^{x+y+z} p(x) = \sum_0^x p(0) + \sum_0^y p(1) + \sum_0^z p(x)$
$= \sum_0^x p + \sum_0^y 1-p + \sum_0^z 0 = \boxed{\sum_0^x p + \sum_0^y 1-p}$

8. $P(9 \text{ Heads in } 10 \text{ Tosses}) = \frac{\text{chance}}{\text{outcome}} = \frac{10}{2^{10}} = \frac{10}{1024} = \boxed{\frac{5}{512}}$ ✓

$P(18 \text{ Heads in } 20 \text{ Tosses}) = \frac{\binom{20}{2}}{2^{20}} = \frac{\frac{20!}{2!\,18!}}{1048576} = \boxed{\frac{95}{524288}}$

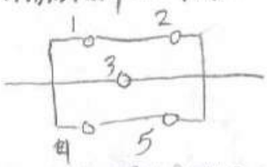
9. $n=3$; $p = \sum_0^3 \binom{n}{k} p^k (1-p)^{n-k} = \binom{3}{0}p^0(1-p)^3 + \binom{3}{1}p^1(1-p)^2 + \binom{3}{2}p^2(1-p)^1 + \binom{3}{3}p^3(1-p)^0 = (1-p)^3 + 3p(1-p)^2 + 3p^2(1-p)^1 + 6p^3 = 1$; $p = \frac{All}{n-1}$ when

$n=3$; $p = \sum_0^1 \binom{3}{k} p^k (1-p)^{n-k} = \binom{3}{0}p^0(1-p)^3 + \binom{3}{1}p^1(1-p)^2 = (1-p)^3 + 3p(1-p)^2$; $(1-p)^3 + 3p(1-p)^2 > 1-p$

$p < 0.5$

73. n independent units, each with p = Fail; $P(R^c)$ = System failure. $P(\text{System}) = (1-p)^n$; $P(\text{Fail}|\text{System}) = 1-(1-p)^n$ $\boxed{= 1-(1-p)^n}$

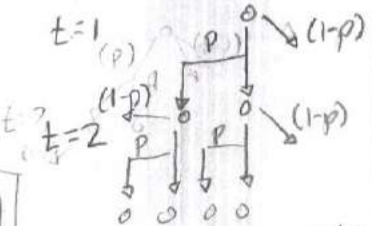
74. Probability of failure



$$P(F) = P(A_1 \cap A_2) + P(A_3) + P(A_4 \cap A_4) + P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)$$

$\boxed{= p + 2p^2 - p^5}$

75.



$t=3$

0.5

$0.5 = -3p + 2p^2$

$2p^2 - 3p - 0.5 = 0$

$\boxed{p_1 = 1.65}$; $p_2 = -0.15$

$P(\text{Dead}) = 1-(1-p)^3 = 1^3 - a^3 - 3a^2b + 3ab^2 - b^2 = 1-(a-b)^3$

$= 1 - 1 - 3p + 3p^2 - p^2 = \boxed{-3p + 2p^2}$

76. —o—||—o—||—o— n-components $P(\text{Success}) = (1-p_1p_2)^n$; $P(\text{Success}) = (1-0.05^2)^{10} = 0.9975 = \boxed{99.75\%}$

77. $5\% = P(\text{Bulls-Eye})$ $0.5 = (1-0.05)^n$; $\log\frac{1}{2} = n\log(1-0.05)$; $\boxed{n = 13.51}$

78. Pairs of (a or A); AA, Aa, aa, or (Aa or aA). a) Parent #1: AA; Parent #2: Aa; Offspring AA, AA, Aa, Aa $\boxed{AA\ (50\%),\ Aa\ (50\%)}$

b) AA(p); Aa$(2q)$; aa(r); $1 = (p+2q+r)^n$; $\boxed{n=2;\ 1=(p+2q+r)^2}$, $n=3$; $\boxed{1=(p+2q+r)^3}$

c) $\boxed{1 = (u+v+w)^2}$; $\boxed{1 = (u+v+w)^3}$ Hardy-Weinberg Law

79. a. aa = Dead; Aa = carrier, alive, AA = not carrier, not diseased. Aa×Aa = AA + 2Aa + aa $\boxed{AA\ (25\%);\ Aa\ (50\%);\ aa\ (25\%)}$

b. $P(\text{Not Diseased} \cap \text{Carrier}) = 50\%$

c. $P(\text{Offspring}) = [P(AA) + P(Aa)] = \boxed{\frac{1}{3}} + \boxed{\frac{2}{3}}$

d.

Offspring	Genotypes of Parents			
P	AA-AA	AA-Aa	Aa-AA	Aa-Aa
AA	$\frac{1}{3}(1-p)$	$\frac{1}{2}(\frac{1}{3})\ p$	$\frac{1}{2}(\frac{2}{3})(1-p)$	$\frac{1}{4} \times \frac{2}{3} \times p$
Aa	0	$\frac{1}{2}(\frac{1}{3}) \times p$	$\frac{1}{2}(\frac{2}{3})(1-p)$	$\frac{1}{2} \times (\frac{2}{3}) \times p$
Aa	0	0	0	$\frac{1}{4} \times \frac{2}{3} \times p$

$P(AA) = \frac{1}{3}(1-p) + \frac{1}{6}p + \frac{2}{6}(1-p) + \frac{p}{6}$

$P(AA) \boxed{= \frac{2}{3} - \frac{p}{3}}$

$P(Aa) = \frac{p}{6} + \frac{2}{6}(1-p) + \frac{2}{6}p$

$= \frac{p}{6}$

$P(AA) / P(1-Aa) = \boxed{\dfrac{(2-p)\frac{1}{3}}{1 - \frac{p}{6}}}$

80. Parent Aa (50%); A = child #1 & #2 have the same gene

B = child #1 and #3 have the same gene

C = child #2 & #3 have same gene.

Mutually Independence: $P(A_{i_1} \cap \cdots \cap A_{i_m}) = P(A_{i_m}) \cdots P(A_{i_m})$

Pairwise Independence: $P(A \cap B \cap C) = 0 \neq P(A)P(B)P(C)$.

$P(B) = P(\text{child}\#1 \cap \text{child}\#3)$

$P(A) = P(\text{child}\#1 \cap \text{child}\#2)$

$P(C) = P(\text{child}\#2 \cap \text{child}\#3)$

$P(\#1 \cap \#2 \cap \#3) = P(\#1 \cap \#2)\,P(\#3) = \boxed{P(\#1)P(\#2)P(\#3)}$

$\boxed{P(A \cap B \cap C) \neq P(A) \cdot P(B) P(C)} = P(\#1 \cap \#3)P(\#1 \cap \#2)P(\#2 \cap \#3)$

56. a) A survey of household expenditures in a city. [Stratification of expenditure type or district]

b) Examination of lead concentration in a large plot of land [concentration ranges]

c) Surveying the number of people who use elevators in a large building [Time of day]

d) Surveying television by time of day [Stratification or seasons]

57. Sample Pool: $\{1,2,2,4, \text{and } 8\} \rightarrow (1,2,2)$ and $(4,8)$: $\bar{X}_{s1} = \frac{1}{3}(1) + \frac{1}{3}(2) + \frac{1}{3}(2) = \frac{5}{3}$; $\bar{X}_{so} = \frac{3}{5}\left(\frac{5}{3}\right) + \frac{2}{5}(6)$

$\bar{X}_{s2} = \frac{1}{2}(4) + \frac{1}{2}(8) = 6$: $\quad = 1 + \frac{12}{5} = 1 + 2.4$ $\boxed{=3.4}$

$\sigma_{\bar{e}1} = \sqrt{\frac{(1-5/3)^2 + (2-5/3)^2 + (2-5/3)^2}{3}} = 0.47$

$\sigma_{\bar{e}2} = \sqrt{\frac{(4-6)^2 + (8-6)^2}{2}} = 2.00$

$Var(\bar{X}_s) = \left(\frac{3}{5}\right)^2\left(\frac{1}{3}\right)\left(1 - \frac{3-1}{5-1}\right) \cdot 0.47^2 + \left(\frac{2}{5}\right)^2\left(\frac{1}{2}\right)\left(1 - \frac{2-1}{5-1}\right)2^2$

$\boxed{= 1.02}$

58. $[n_1\, n_2 \cdots\cdots n_{12}]$
$[n_1, n_2, \cdots\cdots n_{12}]$
$[n_1, n_2 \cdots\cdots n_{12}]$
$\vdots$
$[n_1, n_2 \cdots\cdots n_{12}]$ $\Big\}\times 100$

59.

61. $\mu(t) = \alpha + \beta t$

Sample size = n

@ time = 1, 2 and 3

a) Find W_1, W_2 and W_3, such that $\hat{\beta} = w_1\bar{X}_1 + w_2\bar{X}_2 + w_3\bar{X}_3$ is unbiased estimate of β. X_i denotes sample mean.

$\boxed{\frac{d}{dX}\hat{\beta} = w_1 + w_2 + w_3 = 0}$

and

$X \propto$ time: $\boxed{w_1 + w_2(2) + w_3(3) = 1}$

b) $w_1 = -w_2 - w_3$; $w_1 = 1 - 2w_2 - 3w_3$; $-w_2 - w_3 = 1 - 2w_2 - 3w_3$

$w_2 = 1 - 2w_3$

$w_1 + 1 - 2w_3 + w_3 = 0$

$w_1 + 1 - w_3 = 0$; $w_1 - w_3 = -1$; $\boxed{w_1 = -\frac{1}{2} ; w_3 = \frac{1}{2} ; w_2 = 0}$

60. $n_H = 100{,}000$, $\sigma_{\bar{e}H} = 20$

$n_L = 500{,}000$ $\sigma_{\bar{e}L} = 12$

Sample size = 100

a) Optimal Allocation

Population Mean:

$\boxed{\bar{X}_{so} = \sum_{l=1}^{L} W_l \cdot \bar{X}_l =}$

b) $\mu_H - \mu_L = W_H\bar{X}_H - W_L\bar{X}_L$

$= \frac{100}{100{,}000}\bar{X}_H - \frac{100}{500{,}000}\bar{X}_L$

62. Example B: Section 7.5.2

$\bar{X}_1 = 240.6$	$s_1^2 = 6827.6$
$\bar{X}_2 = 507.4$	$s_2^2 = 23790.7$
$\bar{X}_3 = 865.1$	$s_3^2 = 42573.0$
$\bar{X}_4 = 1716.5$	$s_4^2 = 152{,}099.6$

$\bar{X}_s = 832.5$

$s_{\bar{X}}^2 = \frac{1}{10}\sum_{l=1}^{4} W_l^2\left(1 - \frac{n-1}{N-1}\right)s_l^2 = 1282.0$

$s_{\bar{X}} = 35.8$

$T_s = 393\bar{X}_s = 327{,}172.$

$S_{T_s} = 393\, s_{\bar{X}} = 14{,}069$

The estimate of $s_x = 35.8$ has a standard error of

$\sigma_{\bar{X}} = \sqrt{\frac{s_x^2}{10}\left(1 - \frac{10}{393}\right)} = 11.2$

which demonstrates the sample size of $n=10$ is small compared to the population size of $N=393$.

63.

$X = [0,1]$

$f(x) = X_0$: Vertical line test.

$\hat{I}(f) = \frac{1}{n}\sum_{i=1}^{n} f(x)$: Mean.

Standard Error:

$\sigma^2 = E[f(x)^2] - E[f(x)]^2$

$= \frac{1}{n}\left[\sum f(x)^2\right] - \left[\sum f(x)\right]^2$

$= \frac{1}{n^2}\left[n\sum f(x)^2 - \left(\sum f(x)\right)^2\right] =$

$\sigma_X = \sqrt{\frac{\sigma^2}{n}\left(1 - \frac{n}{N}\right)} = \sqrt{\frac{\frac{1}{1000^2}\left(\frac{1}{2\pi}\right)\left[2\pi\sum_{i=1}^{1000} e^{-X_i^2} - \sum e^{-X_i^2/2}\right]}{n}\left(1 - \frac{n}{N}\right)}$ where $n = 1000$

The population stratification would be best exampled as σ-intervals of seperation.

64.

Stratum	N_e	μ_e	σ_e
\$1000+	70	3000	1250
\$200-1000	500	500	100
\$1-200	10,000	90	30

a) Mean · S.D.

Proportional Allocation $\bar{X} = \sum_{\ell=1}^{L} W_\ell X_\ell$ $\quad var(\bar{X}_s) = \frac{1}{n}\sum W_\ell\left(1-\frac{n_\ell-1}{N-1}\right)\sigma_\ell^2$

Optimal Allocation: $\mu = \frac{1}{N}\sum_{\ell=1}^{L} N_\ell \mu_\ell$ $\quad var(\bar{X}_s) = \sum_{\ell=1}^{L} W_\ell^2\left(\frac{1}{n_\ell}\right)\left(1-\frac{n_\ell-1}{N_\ell-1}\right)\sigma_\ell^2$

Relative Sampling: $\frac{n_\ell}{N_\ell} = \frac{n}{N}$; $n_\ell = N_\ell \frac{n}{N} = nW_\ell$

$\boxed{\text{Proportional Allocation}}$

\$1-200: $n_\ell = n\left(\frac{10000}{10570}\right) = 0.95n$

\$200-1000: $n_\ell = n\left(\frac{500}{10570}\right) = 0.47n$

\$1000: $n_\ell = n\left(\frac{70}{10570}\right) = 0.66n$

b) Two methods exist to compare the differences of population mean based upon proportional allocation and optimal allocation.

$n_\ell = n\frac{W_\ell \sigma_\ell}{\sum_{\ell=1}^{L} W_\ell \sigma_\ell}$ $=$

$\boxed{\text{Optimal Allocation}}$

\$1-200: $\frac{28.5n}{900.5} = 0.03n = n_\ell$

\$200-1000: $0.05n = n_\ell$

\$1000: $0.09n = n_\ell$

$Var(\bar{X}_{sp}) - Var(\bar{X}_{so}) = \frac{1}{n}\sum_{\ell=1}^{L} W_\ell(\sigma_\ell - \bar{\sigma})^2$

$= \frac{1}{n}\left[\sum_{\ell=1}^{L} W_\ell \sigma_\ell^2 - \left(\sum_{\ell=1}^{L} W_\ell \sigma_\ell\right)^2\right]$ and $\frac{Var(\bar{X}_{sp})}{Var(\bar{X}_{s\cdot})} = 1 + \frac{\sum W_\ell(\sigma_\ell-\bar{\sigma})^2}{\left(\sum W_\ell \sigma_\ell\right)^2}$

65.
time: 1950-1960
Adult White Female Population 1960, 301 counties, North Carolina, South Carolina, Georgia.

a) Histogram: Bins by year b) Mean $= \frac{1}{N}\sum X_i$; Total cancer Mortality: $T = N$ variance: $\sqrt{\frac{\sum(X_i-\bar{X})^2}{N}}$, SD: $\sqrt{\frac{\sum(X_i-\bar{X})^2}{N}}$

c) $n = 25$

d) Mean $= \frac{1}{n}\sum X_i$; $T = N\bar{X}$ e) $Var(\bar{X}) = \sqrt{\frac{\sum(X_i-\bar{X})^2}{n}}$

f) $\bar{X} \pm 1.96 s_{\bar{X}}$, $T = N s_{\bar{X}}$ g) See (d-f) h) Ratio Estimator$(r) = \frac{\sum_{i=1}^{n}\bar{X}}{\sum_{i=1}^{N}\bar{X}}$ A ratio estimator would be effective to compare counties

i) see c) j) see d) k) --- l) Seperate and stratify by

66. The sampling procedure would be regions of the beach vs # of people. Stratified means would be calculated with proportional allocation for variance.

$n_\ell = n\frac{W_\ell \sigma_\ell}{\sum W_\ell \sigma_\ell}$ $\boxed{\text{Optimal Allocation}}$ or $n_\ell = nW_\ell$ $\boxed{\text{Proportional Allocation}}$

n) Further stratum may resolve or represent the data better.

67. a) i) Proportion of female-headed Families $[n = 500]$; $\bar{X} = \sum_{\ell=1}^{L} W_\ell X_\ell$; $Var(\bar{X}_s) = \frac{1}{500}\sum(W_\ell)\left(1-\frac{500-1}{43086-1}\right)\sigma^2$

ii) The average number of children per family $s_x = \sqrt{\frac{\sigma_{xs}^2}{n}\left(1-\frac{n}{N}\right)}$; $\bar{X} \pm 1.96 s_x$

representation would be best demonstrated by region of Cyberville. $\bar{X}_R = \sum_{\ell=1}^{L} W_\ell X_\ell$

$Var(\bar{X}_{SR}) = \frac{1}{500}\sum(W_\ell)\left(1-\frac{500-1}{43086-1}\right)\sigma^2$

$\bar{X} \pm 1.96 s_x$

iii) The proportion of heads of household Who did not recieve a high scool diploma

$\bar{X} \pm 1.96\sqrt{\frac{\sigma^2}{n}\left(1-\frac{n}{N}\right)}$

62. Four players [13 cards each] ; $52! \doteq \boxed{8.0 \times 10^{67}}$ =

63. Age > 70 = 0.6 ; Age > 80 = 0.2. $P(\text{Age}>80|70) = \frac{P(\text{Age}>80 \cap >70)}{P(70)} = \frac{(0.2)}{0.6} = \boxed{\frac{1}{3}}$

64. Three Shifts. 1% of Shift 1 are defective ; 2% of Shift 2 are defective ; 5% of shift 3 are defective ;

$P(\text{Defective}) = P(\text{Defective}|\text{Shift \#1})P(\text{Shift \#1}) + P(\text{Defective}|\text{Shift \#2})P(\text{Shift \#2}) + P(\text{Defective}|\text{Shift \#3})P(\text{Shift \#3})$

$= 1\%\left(\frac{1}{3}\right) + 2\%\left(\frac{1}{3}\right) + 5\%\left(\frac{1}{3}\right) = \boxed{2.667\%}$; $P(\text{Shift \#3}) = \frac{P(\text{Defective} \cap \text{Shift \#3})}{\sum P(\text{Defective}|\text{Shift \#i})} = \frac{5\%\left(\frac{1}{3}\right)}{2.667\%} \doteq \boxed{62\%}$

65. A and B are independendent ; A and B^c, A^c and B^c are too.

$P(A \cap B) = P(A)P(B)$; $P(A \cap B^c) = P(A)P(B^c)$; $P(A^c \cap B^c) = P(A^c)P(B^c)$

66 $\emptyset$ independendent of A for any A. $P(A \cap \emptyset) = \boxed{P(A) \cdot P(\emptyset) = 0}$

67. If $P(A \cap B) = P(A)P(B)$; then $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B)$

Addition Law | Law of Independence

68. If $p(A \cap B) = p(A)p(B)$ and $p(B \cap C) = p(B)p(C)$; $p(A \cap C) = p(A \cap B \cap C) = p(A \cap B)\,p(c)$
$= p(A)p(B \cap C) = p(a)p(b)p(c)$

$P(A \cap B) = p(A)\frac{p(B \cap C)}{p(C)} \Rightarrow \boxed{\frac{p(A \cap B)}{p(B \cap C)} = p(A \cap C) = \frac{p(A)}{p(C)}}$

69. If $A \cap C = \emptyset$ "Disjoint", $p(A) = \emptyset \vee p(C) = \emptyset$; thus independendent.

70. If $A \subset B$; then they are not independendent.

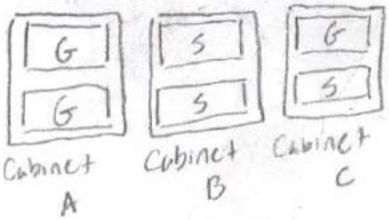
71. If A, B, C are mutually independendent, then $A \cap B$ and C are independendendent along with $A \cup B$ and C.

$p(A \cap (B \cap C)) = \boxed{p(A)\,p(B \cap C) = p(A)p(B)p(C)}$

$p(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A|B \cap C)$
$\boxed{= p(A) + P(B)P(C) - P(A|B \cap C)}$

72. ($t = 0, 1, 2 \cdots$) ; p, then q. @ $t=0, n=1$;
Probability of 0,1,2,3 people at $t=2$.

57. Cabinet A, B, C with two drawers each, inside a coin. $\boxed{\text{Multiplicative Law}}$ $P(S_2) = P(S_2|S_1)P(S_1) + P(S_2|G_1)P(G_1)$



$$P(\text{Draw}\#2|\text{Select}\ 2) = \frac{P(\text{Draw}\#1|B)\cdot P(B)}{P(\text{Draw}\#1|B)\cdot P(B) + P(\text{Draw}\#1|A)\cdot P(A) + P(\text{Draw}\#1|C)P(C)}$$

$$= \frac{1\cdot\frac{1}{3}}{1\cdot\frac{1}{3} + 0\cdot\frac{1}{3} + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)} = \frac{1/3}{3/6} = \frac{1}{3}\times\frac{6}{3} = \boxed{\frac{2}{3}}$$

58. Drew, Chris, Jason; Two must stay home; one leave.

Drew, Chris, Jason → 2 remain, 1 leave $\left(\frac{1}{3}\right)$ 3 sides

The possibilities following Drew asking the teacher contain a relationship known as the multiplicative law. If Chris is chosen to remain $\left(\frac{2}{3}\right)$, then there are $\left(\frac{1}{2}\right)$ possible outcomes. If Jason is chosen $\left(\frac{1}{3}\right)$ then there is $\left(\frac{1}{2}\right)$ outcomes. While if Drew is chosen 1-outcome. Thus,

$$P(\text{Drew Asking}) = P(\text{Outcomes}|\text{Drew})P(\text{Drew}) + P(\text{Outcomes}|\text{Chris}) \times P(\text{Chris}) + P(\text{Outcomes}|\text{Jason})P(\text{Jason})$$

$$= \frac{1}{3}(1) + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{1}{3}$$

Asking is no better than waiting.

59.

Box	
H:H	Random choice
H:T	Flip
T:T	Heads

a) Probability of a two-headed coin

$$P(H|HH) = \frac{P(\cancel{HH|HH})P(H)}{P(\cancel{HH|HH})P(H) + P(\cancel{HT|HT})P(HT)}$$

$$P(HH|H) = \frac{P(H|HH)P(HH)}{P(H|HH)P(HH) + P(H|HT)P(HT) + P(H|TT)P(T)} = \frac{1\cdot 1/3}{1\cdot\frac{1}{3} + \frac{1}{2}\left(\frac{1}{3}\right) + 0\left(\frac{1}{3}\right)} = \frac{1/3}{\frac{1}{3}+\frac{1}{6}} = \frac{\frac{2}{6}}{\frac{2}{6}+\frac{1}{6}} = \boxed{\frac{2}{3}}$$

b) $P(H) = P(H|HH)P(HH) + P(H|HT)P(HT) + P(H|TT)P(T) = \frac{3}{6} = \boxed{\frac{1}{2}}$; $P(T) = P(T|HT)P(HT) + P(T|TT)P(TT)$
$= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + 1\cdot\frac{1}{3} = \frac{3}{6} = \boxed{\frac{1}{2}}$

c) $P(H_2) = P(H|HH)P(HH_1) + P(H|HH_1)P(HT_1)$
$= 1\cdot\left(\frac{2}{3}\right) + \frac{1}{2}\left(\frac{1}{3}\right) = \frac{2}{3} + \frac{1}{6} = \frac{4}{6} + \frac{1}{6} = \boxed{\frac{5}{6}}$

60. $P(B) > 0$; $Q(A) = P(A|B)$; $\boxed{\text{Addition Law}}$ $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$$\boxed{P(A\cup C|B) = P(A|B) + P(C|B) - P(A\cap C|B)}$$

61. Defect = 0.95 Accuracy Sound = 0.97 Accuracy

If 0.5% are faulty, what is the probability faulty is sound?

	Defect	Sound
T	0.95	0.97
F	0.05	0.03
	1	1

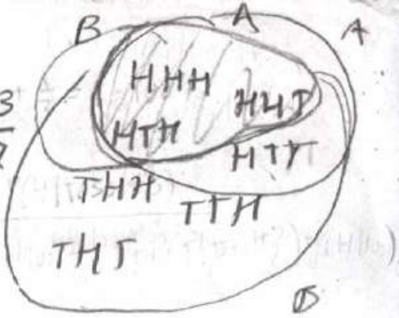
$$P(F|\text{Defect}) = \frac{P(F|\text{Defect})\cdot P(\text{Defect})}{P(F|\text{Defect})P(\text{Defect}) + P(F|\text{Sound})P(\text{Sound})}$$

$$= \frac{0.05\,(5/200)}{0.05(5/200) + 0.03(195/200)}$$; $P(F|\text{Defect}) = \frac{P(F\cap \text{Defect})}{P(\text{Defect})} = \frac{P(F|\text{Sound})P(\text{Sound})}{P(F|\text{Sound})P(\text{Sound}) + P(T|\text{Defect})P(\text{Defect})}$

$$= \frac{0.03\cdot 0.995}{0.03\,0.995 + 0.95(0.005)} = 86\%$$

49. 3 tosses of a coin

a) $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(HHH + HHT + HTH + H}{P(HHH + HHT + HTH + HTT + THH + TTH + THT} = \frac{3}{7}$



b) $P(T|H) = \frac{HHT + HTH + HTT + THH + THT + TTT}{HHH + HHT + HTH + HTT + THH + TTH + THT} = \boxed{\frac{6}{7}}$

$P(T) = \cancel{(\text{Prob } T) + (\text{Prob } T)(\text{Prob } T)} = \frac{1}{2}$

50. Two dice; sum total $= 6$: $P(6) = \frac{P(6\cap 3)}{P(6)} = \frac{1/36}{5/36} = \frac{1}{36}\cdot\frac{36}{5} = \boxed{\frac{1}{5}}$

$\boxed{\text{Law of Independent Events}}$

51. Two dice; Sum total $= 6$; $P(<6) = \frac{P(<6 \cap 3)}{P(<6)} = \frac{4}{10} = \boxed{\frac{2}{5}}$

52. $P(G|G) = \frac{P(G\cap C)}{P(C)} = \boxed{\frac{1}{4}}$; $P(G|G) = \frac{P(G\cap G)}{P(G)} = \boxed{\frac{1}{4}}$

53.

	0.0325	1.00	
High-Risk	[0.02]	[0.10]	2×10^{-3}
Medium Risk	[0.01]	[0.20]	2×10^{-3}
Low Risk	[0.0025]	[0.70]	1.75×10^{-3}
	(Filing)	(People)	

$P(\text{High Risk}|\text{People}) = \frac{P(\text{High Risk} \cap \text{People})}{P(\text{Risk}\cap\text{People})} = \boxed{\frac{8}{23}}$

54. R_i = event of rain on day i;

$P(R_i|R_{i-1}) = \alpha$; $P(R_i^c|R_{i-1}^c) = \beta$

$P(R_i|R_{i-1}\cap R_{i-2}\cap\cdots\cap R_0) = P(R_i|R_{i-1})$

a) p = probability of rain, What about tomorrow?

$P(R_1|R_0) = P(R_1|R_0) = P = \alpha$

b) $P(R_2|R_1\cap R_0) = P(R_2|R_1) = \alpha$

c) $P(R_i|R_{i-1}\cap R_{i-2}\cap\cdots\cap R_0) = P(R_i|R_{i-1})$

$\boxed{\lim_{i\to\infty} P\cdot\prod_{i=1}^{n}\alpha_i}$

55. Upper (U), middle (M), and lower (L)

1 = Father occupation: 2 = Son's occupation.

	U_2	M_2	L_2
U_1	0.45	0.48	0.07
M_1	0.05	0.70	0.25
L_1	0.01	0.50	0.49

$P(U_2|U_1) = 0.45$

a) $P(M_1|M_2) = 0.70$; $P(L_1|L_2) = 0.49$

b)

	U_3	M_3	L_3
U_2	$P(U_3\|U_2)P(U_2)$	$P(M_3\|U_2)P(U_2)$	$P(L_3\|U_2)P(U_2)$
M_2	$P(U_3\|M_2)P(M_2)$	$P(M_3\|M_2)P(M_2)$	$P(L_3\|M_2)P(M_2)$
L_2	$P(U_3\|L_2)P(L_2)$	$P(M_3\|L_2)P(L_2)$	$P(L_3\|L_2)P(L_2)$

	U_3	M_3	L_3
U_2	0.225	0.3064	0.0367
M_2	0.025	1.176	0.2025
L_2	0.005	0.34	0.3969

$P(U_2) = P(U_2|U_1)P(U_1) + P(U_2|M_1)P(M_1) + P(U_2|L_1)P(L_1) = 0.070$

$P(M_2) = 0.578$

$P(L_2) = 0.352$

$P(U_3) = 0.064$

$P(M_3) = 0.614$

$P(L_3) = 0.322$

56 5 cards of 52 card Deck

1^{st} = King. $\boxed{\text{Law of Independent Events}}$

$\frac{3\binom{51}{3}}{\binom{51}{4}} = \frac{\frac{51!}{1!\,48!}}{\frac{51!}{4!\,47!}} = \frac{51\cdot50\cdot49\cdot4\cdot3\cdot2\cdot1}{51\cdot50\cdot49\cdot48} = \frac{1}{2} = \boxed{50\%}$

45. Prove $P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2|A_1) P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1})$

Multiplication Law: A & B be events and assume $P(B) \neq 0$, then $P(A \cap B) = P(A|B)P(B)$

$$P(A_n \cap A_{n-1} \cap \cdots \cap A_2 \cap A_1) = P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1}) \cdot P(A_1 \cap A_2 \cap \cdots \cap A_{n-1})$$

$$= P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1}) P(A_{n-1}|A_1 \cap A_2 \cap A_3 \cdots) P(A_1 \cap A_2 \cap A_3 \cdots)$$

$$\boxed{= P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1}) \cdots P(A_3|A_1 \cap A_2) \cdot P(A_2|A_1) \cdot P(A_1)}$$

48. cont. [Read wrong]

3xR 2xW	2xR 5xW
Urn A	Urn B

A ball from Urn A into B, then a ball is drawn from Urn B.

A) Probability of a red ball: $\dfrac{\binom{3}{1}}{\binom{5}{1}} \dfrac{\binom{3}{1}}{\binom{6}{1}} = \left(\dfrac{3!}{2!}\right)\left(\dfrac{4!1!}{5!}\right)\left(\dfrac{3!}{2!}\right)\left(\dfrac{5!}{6!}\right) = \dfrac{3}{10}$

$\dfrac{\binom{2}{1}}{\binom{5}{1}} \dfrac{\binom{2}{1}}{\binom{6}{1}} = \dfrac{2}{15}$

46 a)

3xR 2xW	2xR 5xW
Urn A	Urn B

coin [50/50]:

Heads = Urn A

Tails = Urn B

$$P(\text{coin} \cap \text{Urn A}) = P(\text{Coin}|\text{Urn A}) \cdot P(\text{Urn A}) = \frac{1}{2}\left(\frac{3}{5}\right) = \frac{3}{10}$$

$$P(\text{coin} \cap \text{Urn B}) = P(\text{coin}|\text{Urn B}) \cdot P(\text{Urn B}) = \frac{1}{2}\left(\frac{2}{5}\right) = \frac{2}{10}$$

$$P(R) = P(\text{coin} \cap \text{UrnA}) + P(\text{coin} \cap \text{UrnB}) = \frac{3}{10} + \frac{2}{10} = \frac{5}{10} = \boxed{\frac{1}{2}}$$

b) $P(R) = P(\text{Heads})P(\text{UrnA}) + P(\text{Tails})P(\text{Urn B}) = 1 = P(\text{Heads})\left(\frac{3}{5}\right) + (1 - P(\text{Heads}))\left(\frac{2}{5}\right)$

$1 = P(\text{Heads})\left[\left(\frac{3}{5}\right) - \left(\frac{2}{5}\right)\right] + \frac{2}{5}$; $\boxed{P(\text{Heads}) = \frac{3}{5}}$

47.

4xR 3xB 2xG	2xR 3xB 4xG
Urn A	Urn B

a) $P(R) = P(R|\text{Urn B} \cap R)P(\text{UrnA} \cap R) + P(R|\text{Urn B} \cap B)P(\text{Urn A} \cap B) + P(R|\text{Urn B} \cap G)P(\text{Urn A} \cap G)$

Multiplication Law

$= \left(\frac{3}{10}\right)\left(\frac{4}{9}\right) + \left(\frac{2}{10}\right)\left(\frac{3}{9}\right) + \left(\frac{2}{10}\right)\left(\frac{2}{9}\right) = \left(\frac{12}{90}\right) + \left(\frac{6}{90}\right) + \left(\frac{4}{90}\right) = \frac{22}{90} = \boxed{\frac{11}{45}}$

b) $1 = \cancel{P(R|\text{Urn B} \cap R)P(\text{Urn A} \cap R) + P(R|\text{Urn B} \cap B)P(\text{Urn A} \cap B) + P(R|\text{Urn B} \cap G)P(\text{Urn A} \cap G)}$

$\cancel{1 = (X)\left(\frac{4}{9}\right) + \left(\frac{2}{10}\right)\left(\frac{3}{9}\right) + \left(\frac{2}{10}\right)\left(\frac{2}{9}\right)}$; $\cancel{\frac{10}{3}\left(1 - \left(\frac{2}{10}\right)\left(\frac{2}{9}\right) - \left(\frac{2}{10}\right)\left(\frac{2}{9}\right)\right)} = X$

b) Bayes Formula: $P[\text{Urn A(Red)}|\text{Urn B(R)}] = \dfrac{P[\text{Urn A(R), Urn B(Red)}]}{P[\text{Urn B(Red)}]} = \dfrac{P[\text{Urn B(Red)}|\text{Urn A(Red)}]P[\text{Urn}}{P(R|\text{Urn B} \cap B)P(\text{Urn A} \cap B) + P(R|\text{Urn B} \cap R)P(R) + P(R|\text{Urn B} \cap G)P(G)}$

$P(W|\text{Urn A})$

$\boxed{= 6/11}$

48.

3xR 2xW
Urn A

1 Draw + 1 Return + Same Color Ball → 2nd Draw

a) Probability of White?

Multiplication Law

$$P(W|\text{Draw \#2}) = P(W|\text{Draw\#2} \cap W)P(W \cap \text{Draw\#1}) + P(W|\text{Draw \#2} \cap R)P(R \cap \text{Draw \#1})$$

$$= \left(\frac{3}{6}\right)\left(\frac{2}{5}\right) + \left(\frac{2}{6}\right)\left(\frac{3}{5}\right) = \left(\frac{6}{30}\right) + \left(\frac{6}{30}\right) = \frac{12}{30} = \frac{4}{10} = \boxed{\frac{2}{5}}$$

b) Bayes Theorem:

$$P(\text{Draw\#1} \cap W|\text{Draw \#2} \cap W) = \frac{P(\text{Draw \#2} \cap W|\text{Draw\#1 W})P(\text{Draw\#1 W})}{P(W|\text{Draw\#2} \cap W)P(W \cap \text{Draw\#1}) + P(W|\text{Draw \#2} \cap R)P(R \cap \text{Draw \#1})}$$

$$= (2/5)(2/5) \Big/ \left[\left(\frac{3}{6}\right)\left(\frac{2}{5}\right) + \left(\frac{2}{6}\right)\left(\frac{3}{5}\right)\right] = \boxed{1/2}$$

36. [R|R|R|G|G|G] Arrangements; Combinations per type - $\binom{6}{3}\binom{3}{3} = \frac{6!}{3!3!}\frac{3!}{3!0!} = \frac{6\cdot5\cdot4}{3\cdot2} = \frac{120}{6} = \boxed{20}$

6 Blocks

[R|R|R|W|W|W|G|G|G] Combinations per type - $\binom{9}{3}\binom{6}{3}\binom{3}{3}$

$= \frac{9!}{3!6!}\frac{6!}{3!\cdot3!} = \frac{9!}{3!3!3!} = \boxed{1680}$

9 blocks

37. Coefficient of $x^2y^2z^3$ in $(x+y+z)^7$;

Multinomial: $(x_1+x_2+x_3)^n = \sum \binom{n}{n_1, n_2, n_3} x_1^{n_1}\cdot x_2^{n_2}\cdot x_3^{n_3}$; $(x+y+z)^7 = \sum \binom{7}{223} x^2y^2z^3$

Coefficient: $\binom{7}{223} = \frac{7!}{2!2!3!} = \frac{7\cdot6\cdot5\cdot4}{4} = \boxed{210}$

38. coefficient of x^3y^4 for $(x+y)^7$;

$(x+y)^7 = \sum \binom{n}{k} x^k y^{7-k} = \binom{7}{07} x^7 y^0 + \binom{7}{61} x^6 y^1 + \cdots$

Coefficient: $\binom{7}{34} = \frac{7!}{3!\,4!} = \frac{7\cdot6\cdot5}{3\cdot2} = \frac{210}{6} = \boxed{35}$

39. a. 26 letters choose 6.

$\frac{\text{Probability Hamlet}}{\text{Total outcomes}} = \frac{\binom{6}{1}}{\binom{26}{6}} = \frac{6!}{5!}\left(\frac{6!(20)!}{26!}\right) = \frac{3}{115115} = \boxed{0.0026\%}$

b. $0.90 = n\left(\frac{3}{115115}\right)$

$\boxed{n = 34534 \text{ monkeys}}$

40. 12 people into three groups; $\binom{12}{4}\binom{8}{4}\binom{4}{4} = \frac{12!}{4!(\cancel{8})!}\frac{\cancel{8!}}{4!\,\cancel{4!}}\frac{4!}{4!\,1!} = \frac{12!}{4!4!4!} = 34650.$

6 pairs of partners: $\binom{6}{2}\binom{4}{2}\binom{2}{2} = \frac{6!}{2!\,4!}\frac{4!}{2!\,2!}\frac{2!}{2!\,0!} = \frac{6!}{2!2!2!} = 90.$

41. Seven black socks, eight blue socks, and nine green socks. Total: 24.

a) Probability of Matching: $\frac{\binom{7}{2}}{\binom{24}{2}} + \frac{\binom{8}{2}}{\binom{24}{2}} + \frac{\binom{9}{2}}{\binom{24}{2}} = \frac{7!}{5!}\frac{2!22!}{24!} + \frac{8!}{6!}\frac{2!22!}{24!} + \frac{9!}{7!}\frac{2!22!}{24!}$

$= \frac{7}{92} + \frac{7}{69} + \frac{3}{32} = \boxed{27\%}$

b. $7/92 = \boxed{7.61\%}$

42. Number of ways to choose 11 boys grouped into 4 forwards, 3 midfielders, 3 defenders, 1 goalie.

$\binom{11}{4}\binom{7}{3}\binom{4}{3}\binom{1}{1} = \frac{11!}{4!(7!)}\frac{7!}{3!\,3!}\frac{4!}{3!\,1!}\frac{1!}{1!\,1!} = \boxed{46,200}$

43. Three Jobs; Two Jobs require 3 programmers, the third requires four.

Total of ten programmers. $\binom{10}{3}\binom{7}{3}\binom{4}{4} = \boxed{4,200}$

44. Combinations of Tentacles x Shaking Hands: $\sum_{i=1}^{n=8} 8(8-i)\binom{8}{i+1} =$: Mann

$= 8\cdot7\binom{8}{2} + 8\cdot6\binom{8}{3} + 8\cdot5\binom{8}{4} + 8\cdot4\binom{8}{5} + 8\cdot3\binom{8}{4} + 8\cdot2\binom{8}{3} + 8\cdot1\binom{8}{1}$

$= 1568 + 2688 + 2800 + 1792 + 672 + 128 + 8 = 9656$

four students: $\dfrac{\binom{2}{1}}{\binom{60}{4}} = \dfrac{2}{\frac{60!}{4!\,56!}} = \dfrac{2\cdot4\cdot3\cdot2}{60\cdot59\cdot58\cdot57} = \boxed{0.0004\%}$

Murelle in one class and her friends in another: $\dfrac{\binom{2}{1}\binom{55}{29}}{\binom{60}{30}} = \dfrac{2\frac{55!}{29!(26)!}}{\frac{60!}{30!\,30!}} = 6\%$

31. Six Male and Six Female Dancers. $_6P_6 \cdot {}_6P_6 = \dfrac{6!}{0!}\cdot\dfrac{6!}{0!} = \boxed{720^2}$

32. $\dfrac{4\cdot\binom{13}{n}}{\binom{52}{n}} = \dfrac{4\cdot\frac{13!}{n!(13-n)!}}{\frac{52!}{n!(52-n)!}}$

When is the value 0.5?

$= \dfrac{4!\cdot13!\,(52-n)!}{(13-n)!\,52!} = \boxed{\dfrac{1.85\times10^{-57}(52-n)!}{(13-n)!}} = \underbrace{\dfrac{3.7\times10^{-57}\cdot\Gamma(53-n)}{\Gamma(14-n)}}_{} = 1;\ n=3.$

Gamma Identity: $\Gamma(n)=(n-1)!$

$\Gamma(n+1)=n\cdot\Gamma(n)$

33. Five people and five floors. $\dfrac{\text{Probability of a person per floor}}{\text{Probability of choosing five.}}$

$= \dfrac{\binom{5}{1}}{\binom{7}{5}} = \dfrac{\frac{5!}{1!(5-1)!}}{\frac{7!}{5!(7-5)!}} = \dfrac{5!}{4!}\,\dfrac{5!(2!)}{7!} = \dfrac{5!\,5!\,2!}{4!\cdot7!} = \boxed{23.8\%}$

34. Prove the following identity:

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m-n}{n-k} = \binom{m}{n}$$

$$\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\,\frac{(m-n)!}{(n-k)!(m-n-n+k)!} = \sum_{k=0}^{n}\frac{n\cdot(n-1)(n-2)(n-3)\cdots(\cancel{n-k})!\,(m-n)!}{k!(n-k)!\,(\cancel{n-k})!\,(m+k)!}$$

$$= \sum_{k=0}^{n}\frac{n(n-1)(n-2)(n-3)\cdots\ (m-n)!}{k!\left(\frac{n!}{n\cdot(n-1)(n-2)(n-3)\cdots}\right)(m+k)!} = \sum_{0}^{n}\frac{[n\cdot(n-1)(n-2)(n-3)\cdots]^2(m-n)!}{k!\ n!\ (m+k)!}$$

Two methods: 1. Pascal's Identity: $\binom{n+1}{r} = \binom{n}{r}+\binom{n}{r-1}$; $\binom{n}{k} = \binom{n-1}{k-1}+\binom{n-1}{k}$

$$\sum_{k=0}^{n}\binom{n}{k} = \sum^{n}\left[\binom{n+1}{k+1}-\binom{n}{k+1}\right] = \sum^{n}\binom{n+1}{k+1} - \sum^{n}\binom{n}{k+1}$$

$$= \sum_{k+1}^{n+1}\binom{n}{k+1} - \sum^{n}\binom{n}{k+1} = \binom{n+1}{k+1} - \binom{\cancel{k}}{\cancel{k}+1} = \binom{n+1}{k+1}\ \ldots$$

2. Binomial Theorem: $\displaystyle\sum_{k=0}^{n}\binom{n}{k}\binom{m-n}{n-k} = \sum_{k=0}^{n}(1+k)^n(1+n-k)^{m-n} = \left({}^nC_0k^{n-1}+{}^nC_1k^{n-2}+{}^nC_2k^{n-3}+\ldots\right)$

$\left({}^{m-n}C_0(n-k)^{m-n-1}+{}^{m-n}C_1(n-k)^{m-n-2}+{}^{m-n}C_2(n-k)^{m-n-3}\ldots\right)$

"$(1+x)^n$" $= \left(1+nk+\dfrac{n(n-1)}{2!}k^2+\dfrac{n(n-1)(n-2)}{3!}k^3+\ldots\right)\left(1+(m-n)(n-k)+\dfrac{(m-n)(m-n-1)}{2!}(n-k)^2+\ldots\right)$

$= \left(1+mn+\dfrac{m(m-1)}{2!}n^2+\ldots\right) = \binom{m}{n}$

35. Prove the following identities.

a) $\binom{n}{r} = \binom{n}{n-r} = \dfrac{n!}{r!(n-r)!} = \dfrac{n!}{(n-n+r)!(n-r)!} = \boxed{\binom{n}{n-r}}$; An expansion of a binomial is equivalent to telescoping

b) $\binom{n}{r} = \dfrac{n!}{r!(n-r)!} = \dfrac{(n-1)!\,n}{r!(n-r)!} = \dfrac{(n-1)!}{r!}\left[\dfrac{n-r}{(n-r)!}+\dfrac{r}{(n-r)!}\right] = \dfrac{(n-1)!}{r!(n-1-r)!} + \dfrac{(n-1)!}{(r-1)!(n-k)!} = \boxed{\binom{n-1}{k}+\binom{n-1}{k-1}}$

$P(U_3^2 > U_1 U_2)$; $f(u_1,u_2,u_3) = \begin{cases} 1 & 0 \le u_i \le 1, i=1,2,3 \\ 0 & \text{otherwise} \end{cases}$

To find the required probability consider,

$$P(U_3^2 \ge U_1 U_2) = \int_0^1 \int_0^1 \int_{\sqrt{u_1 u_2}}^1 f(u_1,u_2,u_3)\, du_3 du_2 du_1$$

$$= \int_0^1 \int_0^1 (1 - \sqrt{u_1}\sqrt{u_2})\, du_2 du_1 = \int_0^1 \left(1 - \tfrac{2}{3}\sqrt{u_1}\right) du_1$$

$$\boxed{= 5/9}$$

54. X, Y, and Z be independent $N(0,\sigma^2)$. Let Θ, Φ, R be spherical coordinates.

$x = r\sin\phi\cos\theta$; $y = r\sin\phi\sin\theta$; $z = r\cos\phi$; $0 \le \phi \le \pi$; $0 \le \theta \le 2\pi$

Find $f(\theta,\phi,R) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\, dx dy dz = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)f(y)f(z)\, dx dy dz$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(2\pi)^{3/2}\sigma^3} e^{-(x^2+y^2+z^2)/2} dx dy dz = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(2\pi)^{3/2}\sigma^3} e^{-r^2/2\sigma^2} \cdot r^2 \sin\phi\, dr d\theta d\phi$$

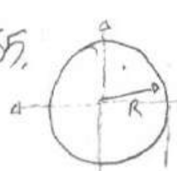
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{(2\pi)^{3/2}\sigma^3} e^{-r^2/2\sigma^2} r^2 \sin\phi\, dr d\theta d\phi = \frac{2\sigma^2}{2^2 (2\pi)^{3/2}\sigma^3} \sqrt{\pi 2\sigma^2} \int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\theta d\phi$$

$$= \frac{2\pi}{2(2\pi)^3}\left[-\cos(2\pi) + \cos(0)\right] = \frac{4\pi}{16\pi^2} = \boxed{\frac{1}{4\pi}}$$

$$f(\theta) = \int_0^{\pi}\int_0^{\infty} \frac{1}{(2\pi)^{3/2}\sigma^3} e^{-r^2/2\sigma^2} r^2 \sin\phi\, dr d\phi = \int_0^{\pi} \frac{1}{2(2\pi)} \sin\phi\, d\phi = \boxed{\frac{1}{2\pi}}$$

$$f(r) = \int_0^{\pi}\int_0^{2\pi} \frac{1}{(2\pi)^{3/2}\sigma^3} e^{-r^2/2\sigma^2} r^2 \sin\phi\, d\theta d\phi = \boxed{\frac{4\pi}{(2\pi)^{3/2}\sigma^3} e^{-r^2/2\sigma^2} \cdot r^2}$$

$$f(\phi) = \int_0^{2\pi}\int_0^{\infty} \frac{1}{(2\pi)^{3/2}\sigma^3} e^{-r^2/2\sigma^2} r^2 \sin\phi\, dr d\theta = \frac{2\pi \cdot 2\sigma^2}{(2\pi)^{3/2}\sigma^3}\sqrt{2\pi\sigma^2}\, \sin\phi = \boxed{\frac{1}{2\pi^2}\sin\phi}$$

55. 

$R\,[0,1]$ a) $X = R\cos\Theta$, $Y = R\sin\Theta$ a) Find $f(R,\Theta) = R f(R\cos\Theta, R\sin\Theta)$

$\Theta\,[0,2\pi]$ b) $f(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-(x^2+y^2)/2} dx dy = \frac{\sqrt{2\pi}}{2\pi} e^{-x^2/2} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ $\boxed{= \frac{r}{2\pi} e^{-r^2/2}}$

$\boxed{f(y) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}}$

$A = \pi R^2$; $A = \pi \sqrt{x^2+y^2}^{\,2}$

c) The density is uniform over the disk.

$\frac{A}{x^2+y^2}$; $x^2+y^2 \le 1$

56. Exponential Random variables: $\lambda e^{-\lambda}$; $x = \lambda_x e^{-\lambda|x|}$, $y = \lambda_y e^{-\lambda|y|}$

$f(x,y) = \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda y} = \lambda_x \lambda_y e^{-\lambda(x+y)}$ $\boxed{= \lambda^2 e^{-\lambda r(\cos\theta + \sin\theta)}}$ r and θ are not independent.

57. $Y_1 = N(0,1)$; $Y_2 = N(0,2)$; $\rho = 1/\sqrt{2}$; Find $X_1 = a_{11}y_1 + a_{12}y_2$ and $X_2 = a_{21}y_1 + a_{22}y_2$

$J(y_1,y_2) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}\cdot a_{22} - a_{12}a_{21} = 1$

Example C: (Section 3.6.2)

$Y_1 = X_1$; $Y_2 = X_1 + X_2$; $J(x,y) = \det\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = 1$

$$f_{Y_1 Y_2}(y_1,y_2) = \frac{1}{2\pi}\exp\left[-\frac{1}{2}\left[y_1^2 + (y_2 - y_1)^2\right]\right]$$

$$= \frac{1}{2\pi}\exp\left[-\frac{1}{2}(2y_1^2 + y_2^2 - 2y_1 y_2)\right]$$

$\sigma_{Y_1}\sigma_{Y_2}\sqrt{1-\rho^2} = 1$

$1 \cdot (2)\sqrt{1-\rho^2} = 1$; $1 - \frac{1}{4} = \rho^2$

$\boxed{X_1 = y_1 \,;\, X_2 = y_2 - y_1}$

If X_1 & X_2 are $N(\mu,\sigma^2)$ then $f_{Y_1 Y_2}(y_1,y_2)$ is bivariate normal.

$Y_1 = a_1 X_1 + b_1$; $Y_2 = a_2 X_2 + b_2$

58. $J(x,y)=\det\begin{bmatrix}1&0\\0&1\end{bmatrix}=1$; $f_{Y_1Y_2}(y_1,y_2)=\boxed{\frac{1}{2\pi}e\left[-\frac{1}{2}\left([(Y_1-b_1)/a_1]^2+[(Y_2-b_2)/a_2]^2\right)\right]}$

59. $Y_1=a_{11}X_1+a_{12}X_2+b_1$; $Y_2=a_{21}X_1+a_{22}X_2+b_2$

① $f(x_1,x_2)=\frac{1}{2\pi}e^{-\frac{1}{2}(x_1^2+x_2^2)}$

② Linear Transformation: $\left(\frac{-a_{22}}{a_{12}}\right)Y_1=\left(\frac{-a_{22}}{a_{12}}\right)a_{11}X_{11}+\left(\frac{-a_{22}}{a_{12}}\right)a_{12}X_2+\left(\frac{-a_{22}}{a_{12}}\right)b_1$ $\quad \boxed{\times\left(\frac{-a_{22}}{a_{11}}\right)}$

$\left(\frac{-a_{22}}{a_{12}}\right)Y_1+Y_2=\left[\left(\frac{-a_{22}}{a_{12}}\right)a_{11}+a_{21}\right]X_1+\left[a_{22}-\cancel{a_{22}}\right]X_2+\left(\frac{-a_{22}}{a_{12}}\right)b_1+b_2$ $\quad \boxed{①+②}$

$\boxed{\text{Solve for } X_1}$

$$X_1=\frac{\left(\frac{-a_{22}}{a_{11}}\right)Y_1+Y_2+\left(\frac{a_{22}}{a_{12}}\right)b_1-b_2}{\left[\left(\frac{-a_{22}}{a_{12}}\right)a_{11}+a_{21}\right]}=\frac{+a_{22}(Y_1-b_1)-a_{12}(Y_2-b_2)}{(a_{22}a_{11}-a_{21}\cdot a_{12})}$$

③ Solve the Jacobian: $|J|=\begin{vmatrix}\frac{\partial x_1}{\partial y_1}&\frac{\partial x_1}{\partial y_2}\\\frac{\partial x_2}{\partial y_1}&\frac{\partial x_2}{\partial y_2}\end{vmatrix}=\begin{vmatrix}\frac{a_{22}}{(a_{22}a_{11}-a_{21}a_{12})}&\frac{-a_{12}}{(a_{22}a_{11}-a_{21}a_{12})}\\\frac{a_{21}}{(a_{22}a_{11}-a_{21}a_{12})}&\frac{a_{11}}{(a_{22}a_{11}-a_{21}a_{12})}\end{vmatrix}=\frac{1}{(a_{22}a_{11}-a_{21}a_{12})}$

④ Solve for the new bivariate density:

$$f(y_1,y_2)=\frac{1}{2\pi\sigma_{Y_1}\sigma_{Y_2}\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y_1-\mu_{Y_1}}{\sigma_{Y}}\right)^2-2\rho\frac{(y-\mu_{Y_1})(y-\mu_{Y_2})}{\sigma_{Y_1}\sigma_{Y_2}}+\left(\frac{y_2-\mu_{Y_2}}{\sigma_{Y_2}}\right)^2\right]\right]$$

⑤ Evaluate $x_1^2+x_2^2$

$$x_1^2+x_2^2=\frac{\{(y_1-b_1)^2(a_{22}^2+a_{21}^2)+(y_2-b_2)^2(a_{12}^2+a_{11}^2)-2(y_1-b_1)(y_2-b_2)(a_{22}a_{12}+a_{11}a_{21})\}}{(a_{22}a_{11}-a_{21}a_{12})^2}$$

⑥ Since $X_1,X_2\sim N(0,1)$, then $a_{11}X_1+a_{22}X_2+b_1\sim N(b_1,a_{11}^2+a_{12}^2)$
$a_{21}X_1+a_{22}X_2+b_2\sim N(b_2,a_{21}^2+a_{22}^2)$ $\quad \boxed{\mu_{Y_1}=b_1;\ \sigma_{Y_1}^2=a_{11}^2+a_{12}^2 \quad \mu_{Y_2}=b_2;\ \sigma_{Y_2}^2=a_{21}^2+a_{22}^2}$

⑦ Reviewing $x_1^2+x_2^2$, ... first term $\left(\frac{(y_1-\mu_{Y_1})}{\sigma_{Y}}\right)^2\Big/\frac{(\sigma_{Y_2}\sigma_{Y_1})}{(a_{22}\sigma_Y-a_{12}\sigma_{2})^2}=\left(\frac{y_1-b_1}{\sqrt{a_{11}^2+a_{12}^2}}\right)^2$

⑧ Solving for $\frac{1}{1-\rho^2}=\frac{\sigma_{Y_1}\sigma_{Y_2}}{(a_{22}a_{11}-a_{21}a_{12})^2}$; $\boxed{\rho=\frac{(a_{22}a_{12}^2+a_{21}^2a_{11})}{\sqrt{(a_{22}^2+a_{21}^2)(a_{11}^2+a_{12}^2)}}}$

⑨ $\boxed{\sigma_{Y_1}\sigma_{Y_2}\sqrt{1-\rho^2}=|(a_{22}a_{11}-a_{21}a_{12})|=J^{-1}}$

60. Pseudorandom variables occur from the previous bivariate normal by a cumulative sum distribution from $-\infty$ to X.

61. X & Y are continuous random variables. $U=a+bX$; $V=c+dY$. $f(u,v)=f(x,y)\cdot J^{-1}$
$\boxed{\text{Proposition A}}$ $=f\left(\frac{u-a}{b},\frac{v-c}{d}\right)\left|\begin{matrix}b&0\\0&d\end{matrix}\right|^{-1}$ $\boxed{=f\left(\frac{u-a}{b},\frac{v-c}{d}\right)\frac{1}{bd}}$

62. X & Y are $N(0,1)$: $P(X^2+Y^2\leq 1)=\frac{1}{2\pi}e^{-(x^2+y^2)/2}$; $\leq\frac{1}{2\pi}e^{-1/2}$
$\boxed{\frac{1}{2\pi}\leq P(X^2+Y^2\leq 1)\leq\frac{1}{2\pi}e^{-1/2}}$

63. a. $X+Y=Z$; $f(u,v)=f_{XY}\left(\frac{v+z}{2},\frac{v-z}{2}\right)\cdot\frac{1}{2}$
$X-Y=V$

b) $XY=Z$; $f(u,v)=f_{xy}\left(\sqrt{zv},\sqrt{z/v}\right)\frac{1}{2|v|}$
$Y/X=V$

c) $X\sim N(0,1)$, $Y\sim N(0,1)$

64. $X+Y=Z$, $X|Y=V$; $f_{XY}\left(\frac{vz}{(v+1)},\frac{z}{(v+1)}\right)\cdot\frac{1}{\begin{vmatrix}1&1\\\frac{1}{Y}&-\frac{X}{Y^2}\end{vmatrix}}=f_{XY}\left(\frac{vz}{(v+1)},\frac{z}{(v+1)}\right)\frac{-Y^2}{X+Y}$

$=f_{XY}\left(\frac{vz}{(v+1)},\frac{z}{(v+1)}\right)\frac{-z^2/(v+1)^2}{\frac{vz}{(v+1)}+\frac{z}{(v+1)}}$ $\boxed{f_{xy}\left(\frac{vz}{(v+1)},\frac{z}{(v+1)}\right)\frac{-z}{(v+1)^2}}$

65. Exponential random variable: $\lambda e^{-\lambda x}$; $f_{xy}($

65. Exponential Random variable: $p(x) = \lambda e^{-\lambda x}$;

o-o-o-o-o-o, $p(x_1, \dots, x_n) = \prod_{i=1}^{n} \lambda_i e^{-\lambda_i} = (\lambda_1 \lambda_2 \cdots \lambda_n) \boxed{e^{-\sum \lambda_i}}$

66.  $F(t) = p(1A)p(1B) + p(2A)p(2B) + p(3A)p(3B)$

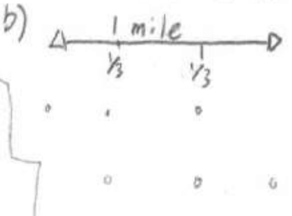
$= 3\lambda^2 e^{-2\lambda t}$

$f(t) = \frac{d}{dt} F(t) = \boxed{-6\lambda^3 e^{-2\lambda t}}$

Notes about order statistics: Multinomial + Differential Argument.

68. U_1, U_2, and U_3 be independent uniform random variables.

a) Find $f(U_1, U_2, U_3) = n! \prod_{i=1}^{n} f(U_i) = 3 \cdot 2 \cdot 1 \, f(U_1) \cdot f(U_2) \cdot f(U_3)$

b)  $\int \int f(u)\, du = \int_0^{1/3} u \, du = \frac{(1/3)^2}{2} = \frac{1}{18}$

$P(\text{No stations } 1/3 \text{ apart}) = 1 - P(\text{any two stations } 1/3)$

$= 1 - \frac{1}{18}$

$= 0.94.$

70.

Joint Distribution of Order Statistics

$x \le X_{(1)} \le x + dx$; $y \le X_{(n)} \le y + dy$

$V = X_{(1)}$; $U = X_{(n)}$

$f(u, v) = n(n-1) f(v) f(u) [F(u) - F(v)]^{n-2} \quad u \ge v$

Uniform case: $f(u,v) = n(n-1)(u-v)^{n-2}$, $1 \ge u \ge v \ge 0$

$F(x,y) = \int_{F(y)-F(x)}^{F(y)} F(y)^{n-1} = \boxed{F(y)^n - [F(y) - F(x)]^n}$

67. n-chips ; $p(\text{failure} \mid \text{chips} \ge 2)$; Exponential Dist. $p(x) = \lambda e^{-\lambda x}$

Order Statistics:

$f_u(u) = n[F(u)]^{n-1} f(u)$; $u \le U \le u + du$

k^{th}-order Statistic?

$$f_k(x) = \frac{n!}{(k-1)!(n-k)!} f(x) F^{k-1}(x)[1 - F(x)]^{n-k}$$

$f(x) = \lambda e^{-\lambda x} \quad F(x) = \int_0^x f(x) = 1 - e^{-\lambda x}$

$$f_k(x) = \frac{n!}{(2-1)!(n-2)!} [\lambda e^{-\lambda x}][1 - e^{-\lambda x}]^{2-1}[1 - 1 + e^{-\lambda x}]^{n-2}$$

$$= \frac{n!}{(n-2)!} [\lambda e^{-\lambda x}][1 - e^{-\lambda x}][e^{-\lambda x}]^{n-2}$$

$$= n(n-1)\lambda e^{-(n-1)\lambda x} [1 - e^{-\lambda x}]$$

$$\boxed{= n(n-1)\lambda [e^{-(n-1)\lambda x} - e^{-n\lambda x}]}$$

69. Weibull Random Variables:

$f(t) = \beta \alpha^{-\beta} t^{\beta - 1} e^{-(t/\alpha)^\beta}$

$f_k(x) = \frac{n!}{(k-1)!(n-k)!} f(x) F^{k-1}(x)[1 - F(x)]^{n-k}$ if $k = 1, n = n$

$$f_k(t) = \frac{n!}{(1-1)!(n-1)!} f(t) \cdot F(t)^0 \cdot [1 - F(t)]^{n-1}$$

$F(t) = \int_0^t \beta \alpha^{-\beta} t^{\beta-1} e^{-(t/\alpha)^\beta} dt \quad \left(\frac{t}{\alpha}\right)^\beta = u$

$\beta \left(\frac{t}{\alpha}\right)^{\beta-1} \frac{1}{\alpha} dt = du$

$= \int_0^{(t/\alpha)^\beta} \beta \alpha^{-\beta} \left(\frac{\alpha^\beta}{\beta}\right) e^{-u} du$

$\frac{\beta}{\alpha^\beta} t^{\beta-1} dt = du$

$= 1 - e^{-(t/\alpha)^\beta}$

$t^{\beta-1} dt = \frac{\alpha^\beta}{\beta} du$

$f_k(t) = \frac{n\beta}{\alpha^\beta} t^{\beta-1} e^{-(t/\alpha)^\beta} \cdot e^{-(n-1)(t/\alpha)^\beta}$

$$\boxed{= \frac{n\beta}{\alpha^\beta} t^{\beta-1} e^{-n(t/\alpha)^\beta}}$$

71. $X_1, \dots, X_n$: $f_{x_1} \dots f_{x_n}$; $f(r) = \dfrac{\int_{-\infty}^{X_{(n)}} f(v+r, v)^{n-1} dv}{\int_{-\infty}^{\infty} f(v+r, v)^{n-1} dv} = \dfrac{\int_{-\infty}^{X_{(n)}} f(v+r, v)^{n-1} dv}{\int_{-\infty}^{\infty} f(v+r, v)^{n-1} dv} = \dfrac{F(X_{(n)}) - F(-\infty)}{F(\infty) - F(-\infty)}$

72. Five numbers $[0,1]$; probability$[\frac{1}{4} \le X_1, X_2, X_3, X_4, X_5 \le \frac{3}{4}] = \int\int\int\int\int f(u) = \int\int\int\int u \, du = \int\int\int \frac{u^2}{2} du = \int\int \frac{u^3}{6} = \int_{1/4}^{3/4} \frac{u^4}{24} = \frac{(\frac{3}{4})^5 - (\frac{1}{4})^5}{120}$

$= \frac{242/45}{120} = \frac{242/1024}{120}$

4 16 64 256 1024 $\quad$ 192 $\quad$ 120 $\quad$ 273

73. Definition of a independent random variable:

$F(x_1, \dots X_n) = F_{x_1}(x_1) \cdot F_{x_2}(x_2) \cdots F_{x_n}(x_n)$

Definition k^{th} order Statistic:

$\boxed{f_k(x) = n! \, f(x_1) f(x_2) f(x_3) \cdots f(x_n)}$

74. n-servers; $T=\lambda e^{-\lambda t}$; $P(\text{Service Time} \geq t) = P(\text{No departure during } t) = P_N(t)$; $P_N(t) = e^{-\mu t}$

$$S(t) = P(T \leq t) = 1 - P(T \geq 1) = 1 - e^{-\mu t}$$

75. $\frac{d}{dt}S(t) = \mu e^{-\mu t}$; $s(t) = \begin{cases} \mu e^{-\mu t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$ || Distribution of waiting time $\frac{1}{\mu}$ with a variance $\frac{1}{\mu^2}$.

Find the joint density of $X_{(i)}$ and $X_{(j)}$, $i<j$

$$f_W(u,v) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \times F(u)^{i-1} \times f(u)\left[F(v) - F(u)\right]^{j-i-1} \times f(v)\left[1-F(v)\right]^{n-j}$$ ... continued...

76. Prove Theorem A: $f_K(x) = \frac{n!}{(k-1)!(n-k)!} f(x) F^{k-1}(x)\left[1-F(x)\right]^{n-k}$ is derived from

75. $\leftarrow (i-1) \rightarrow$, $\leftarrow (j-i-1) \rightarrow$; x , $x+dx$

interdependent paths, $f(x)F^{k-1}(x)\left[1-F(x)\right]^{n-k}$ x Multinomial theorem

$$f(x,y) = \lim_{dx\to 0}\lim_{dy \to 0} \frac{P(x \leq X_i \leq x+dx,\ y \leq X_j \leq y+dy)}{dx\,dy}$$

$$\boxed{f_k(x) = \frac{n!}{(k-1)!(n-k)!} f(x)F^{k-1}(x)\left[1-F(x)\right]^{n-k}}$$

Multinomial Probability Law:

$$\frac{n!}{(i-1)!(j-i-1)!(n-j)!} P_1^{i-1} P_2, P_3^{j-i-1} P_4 \cdot P_5 \cdots = F(X) \ ; \ P_2 = P(x \leq X_i \leq x+dx) = F(x+dx)\cdot F(X)$$

$$\boxed{P_1 = P(x \leq X_i \leq x+dx)}$$

$$f(x,y) = \lim_{dx\to 0}\lim_{dy\to 0} \frac{P(E)}{dx\,dy}$$

$P_3 = P(x+dx < X_i \leq y) = F(y)\cdot F(x+dx)$

$P_4 = P(y < X_i < y+dy) = F(y+dy)\cdot F(y)$

$P_5 = P(X_i > y+dy) = 1 - P(X \leq y+dy) = 1 - F(y+dy)$

$$= \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \times F^{(i-1)} \times \lim_{dx\to 0}\frac{F(x+dx)-F(x)}{dx}$$

$$\times \lim_{dx \to 0} F(y) - F(x+dx)^{j-i-1} \times \lim_{dy\to 0} \frac{F(y+dy)-F(y)}{dy} \times \lim_{dy \to 0} F(y) - F(x+dx)^{j-i-1}$$

$$\times \lim_{dy\to 0}\frac{F(y+dy)-F(y)}{dy} \times \lim_{dy \to 0} 1 - F(y+dy)$$

77. $U_{(k)} - U_{(k-1)}$; $V_{i=1,\dots,n}$.

$F(U_k) = n!\prod_{i=1}^{n} F(V_i)$; $F(U_{k-1}) = n!\prod_{i=1}^{n-1} F(U_i)$

$$\boxed{F(U_k) - F(U_{k-1}) = n!\, f(U_1)\cdot f(U_2) \cdots f(U_{n-1})\left(f(U_i) - 1\right)}$$

78. Show $\int_0^1\int_0^y (y-x)^n\,dx\,dy = \frac{1}{(n+1)(n+2)}$; $\int_0^1 \frac{-1}{n+1}(y-x)^{n+1}\Big|_0^y dy = \frac{-1}{(n+1)}\int_0^1 \left[(y-1)^{n+1} - y^{n+1}\right] dy = \frac{-1}{(n+1)(n+2)}(y-1)^{n+2} - (y)^{n+2}\Big|_0^1$

$$= \frac{-1}{(n+1)(n+2)}\left[0 - 1^{n+2} - 1^{n+2} + 0\right] = \boxed{\frac{1}{(n+1)(n+2)}}$$

79. T_1 & T_2 are exponential random variables; $R = T_{(2)} - T_{(1)}$

$$F(R) = \iint f(T_1, T_2)\,dT_1\,dT_2 = \iint f(T_1, R-T_1)\,dT_1\,dR = \iint \lambda e^{-\lambda T_1} \cdot \lambda e^{-\lambda(R-T_1)}\,dT_1\,dR$$

$$f(z) = \int_0^\infty \lambda_1\lambda_2 e^{-(\lambda_1+\lambda_2)T_1 - \lambda_2 R}\,dT_1$$

$T_2 = R - T_1$, $dT_2 = dR$

$$\boxed{= \frac{\lambda_1\lambda_2}{(\lambda_1 - \lambda_2)} e^{-\lambda_2 R}}$$

80. $U_1 \dots U_n$, V = uniform, U_i = independent

a) Find $P(V \leq U_{(n)}) = \int_0^V n!\prod U_1\cdot U_2 \cdots U_n\,dV = n!\prod_{i=1}^{n} f(U_i)^2/2\Big|_0^V = \boxed{n!\prod_{i=1}^{n} F(V)^2/2}$

b) $P(U_{(1)} < V < U_{(n)}) = \int_{U_{(1)}}^{U_{(n)}} n!\prod f(V_i)\,dV_i = n!\left[\prod_{i=1}^{n} f(V_n)^2 - \prod f(u_i)^2\right]/2$ 81

$$P(V \leq U_{(n)}) = \int_0^{U_{(n)}} n!\prod f(V_i)\,dV = \boxed{n!\prod f(U_n)^2/2}, \quad P(V_{(1)} < V < V_{(n)}) = \int_{V_{(1)}}^{U_{(n)}} n!\prod f(U_n)\,dV = \boxed{\frac{n!\prod f(V_i)^2 - f(V_i)^2}{2}}$$

Chapter 4: 1) Prove if $|X| < M < \infty$, then $E(X)$ exists. $M_i = x_i p(x_i)$; $M_1 + M_2 + \cdots + M_\infty$
$= x_1 p(x_1) + x_2 p(x_2) + \cdots + x_\infty \cdot p(x_\infty) = E(x)$; $\boxed{\text{Definition of Expectation}}$

2) $F(x) = 1 - x^{-\alpha}$, $x \geq 1$; a) $E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_{-\infty}^{\infty} x \frac{d}{dx}(1 - x^{-\alpha}) dx$

$= \int_1^\infty x(\alpha x^{-(\alpha+1)}) dx = \int_1^\infty \alpha x^{-(\alpha)} dx = \frac{-\alpha x^{-(\alpha+1)}}{(\alpha+1)}\Big|_1^\infty = \frac{-\alpha}{(\alpha+1)}(1)^{-(\alpha+1)} = \boxed{\frac{\alpha}{(\alpha+1)}}$

b) $Var(x) = E\{[X - E(x)]^2\}$

$= \int_1^\infty \left[x - \frac{\alpha}{\alpha+1}\right]^2 dx = \int_1^\infty \left[x^2 - 2x\frac{\alpha}{(\alpha+1)} + \left(\frac{\alpha}{\alpha+1}\right)^2\right] dx = \frac{x^3}{3} - x^2\left(\frac{\alpha}{\alpha+1}\right) + \left(\frac{\alpha}{\alpha+1}\right)^2 x \Big|_1^\infty = \infty$

$= E(x^2) - \mu^2 = \int_1^\infty x^2(\alpha x^{-(\alpha+1)}) dx - \left(\frac{\alpha}{(\alpha+1)}\right)^2 = \int_1^\infty \alpha x^{-\alpha+1} dx - \left(\frac{\alpha}{\alpha+1}\right)^2 = \alpha \cdot x^{-(\alpha+2)} / -(\alpha+2) \Big|_1^\infty = \frac{\pm\alpha}{(\alpha+2)} + \left(\frac{\alpha}{\alpha+1}\right)^2 = \boxed{\frac{\alpha}{(\alpha+2)} + \left(\frac{\alpha}{\alpha+1}\right)^2}$

3. Find E(x) and Var(x) for Chapter 2: Problem #3

K	F(K)
0	0
1	0.1
2	0.3
3	0.7
4	0.8
5	1.0

$E(x) : \sum^n p(x) = \sum x f(x) = \sum x[F(x_i) - F(x_{i-1})] = 5[1.0-0.8] + 4[0.8-0.7] + 3[0.7-0.3] + 2[0.3-0.1] + 1[0.1-0]$

$= \frac{5 \cdot 2}{10} + \frac{4 \cdot 1}{10} + \frac{3 \cdot 4}{10} + \frac{2 \cdot 2}{10} + \frac{1 \cdot 1}{10}$

$= \frac{10+4+12+4+1}{10} = \frac{31}{10} = \boxed{3.1}$

$Var(x) = (0.2-3.1)^2(0.2) + (0.1-3.1)^2(0.1) + (0.4-3.1)^2 0.4 + (0.2-3.1)^2 \cdot 0.2 + (0.1-3.1)^2 \cdot 0.1$

$\boxed{= 1.62}$

4. $P(X=k) = 1/n$ for $k = 1, 2, \ldots, n$: Find $E(x)$ and $Var(x)$; $E(X) = 1\cdot\left(\frac{1}{n}\right) + 2\cdot\left(\frac{1}{n}\right) + 3\left(\frac{1}{n}\right) + \cdots + n\left(\frac{1}{n}\right) = \frac{n(n+1)}{2}\cdot\frac{1}{n}$

$Var(X) = \left[\left(1 - \frac{(n+1)}{2}\right)^2 + \left(2 - \frac{(n+1)}{2}\right)^2 + \cdots + \left(n - \frac{(n+1)}{2}\right)^2\right]\frac{(n+1)n}{2}$

5. $f(x) = \frac{1+\alpha x}{2}$; $-1 \leq x \leq 1$; $-1 \leq \alpha \leq 1$

$E(X) = \int_{-1}^{1}\int_{-1}^{1} \frac{(1+\alpha x)}{2} x \, dx \, d\alpha = \frac{x^2}{4} + \frac{\alpha x^3}{6}\Big|_{-1}^{1} = \frac{1}{4} + \frac{\alpha}{6} - \frac{1}{4} + \frac{\alpha}{6} = \frac{\alpha}{3}$; $E(x^2) = \int_{-1}^{1} x^2 \frac{(1+\alpha x)}{2} dx = \frac{1}{2}\left[\frac{x^3}{3} + \frac{\alpha x^4}{4}\right]_{-1}^{1} = \boxed{\frac{1}{3}}$

$Var(X) = E(x^2) - [E(x)]^2 = \frac{1}{3} - \frac{\alpha^2}{9} = \boxed{\frac{3-\alpha^2}{9}}$

6. $f(x) = 2x$; $0 \leq x \leq 1$

a) $E(x) = \int_0^1 2x^2 dx = \boxed{\frac{2}{3}}$ b) $Y = X^2$; Find $E(Y) = E(X^2) = \int_0^1 2x^3 dx = \boxed{\frac{1}{2}}$

c) $E(X^2) = \int_0^1 2x^3 dx = \frac{1}{2}$; d) $Var(X) = E\{[X - E(X)]^2\} = \int_0^1 (x - \frac{2}{3})^2 2x \, dx = \boxed{\frac{1}{18}}$

Theorem B: $Var(X) = E(x^2) - E(x)^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$

7.

X	0	1	2
P(x)	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{8}$

a) $E(X) = \sum x f(x) = 0\left(\frac{1}{2}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{1}{8}\right) = \boxed{5/8}$

b) $Y = X^2$: $E(Y) = 0^2\left(\frac{1}{2}\right) + 1^2\left(\frac{3}{8}\right) + 2^2\left(\frac{1}{8}\right) = \boxed{7/8}$

c) Theorem A: a) $E(Y) = \sum g(x) p(x) = 0^2\left(\frac{1}{2}\right) + 1^2\left(\frac{3}{8}\right) + 2^2\left(\frac{1}{8}\right) = \boxed{7/8}$

b) $E(Y) = \int_{-\infty}^{\infty} g(x) f(x) dx$;

d) $Var(x) = E[(X - E(x))^2] = \frac{1}{2}(0 - 5/8)^2 + \frac{3}{8}(1 - 5/8)^2 + \frac{1}{8}(2 - 5/8)^2 = \frac{31}{64}$.

$Var(X) = E(x^2) - E(x)^2 = 7/8 - (5/8)^2 = \frac{31}{64}$

8. Average: $\sum$ weight $\times x_i$
$= \boxed{\sum p_i \cdot x_i}$

9. c = \$ to stock an item
s = \$ to sell an item.

$P(k)$ = Number of items by customer.

$\underbrace{s\sum_{k=n}^{\infty} p(k) > c\sum_{k=1}^{n-1} p(k)}_{\text{Selling should be greater than cost}}$ and $\underbrace{s\sum_{k=n+1}^{\infty} p(k) < c\sum_{k=1}^{n} p(k)}_{\text{Future sales should be cheaper than cost.}}$

n+1

10. n-items $E(X)=\sum_{i=1}^{n} p(x_i)x_i = \sum \left(\frac{1}{n}\right) x_i = \boxed{\frac{\sum x_i}{n}}$

"Random"

Search(n)

11. $E(X)=\sum p_i x_i$; $E(X)=\sum x_i (1-p_1)(1-p_2)\cdots(1-p_{x-1})p_x$

$\boxed{p_1 + 2(1-p_1)p_2 + 3(1-p_1)(1-p_2)p_3 + \cdots + n(1-p_1)(1-p_2)(1-p_n)}$

12. Suppose $E(X)=\mu$ and $Var(X)=\sigma^2$. Let $Z=(X-\mu)/\sigma$. Show $E(Z)=0$ and $Var(Z)=1$

$E(Z)=E\left(\frac{x-\mu}{\sigma}\right)=\frac{E(x)-E(\mu)}{\sigma}=\frac{\mu-\mu}{\sigma}=0$; $Var(Z)=Var\left(\frac{x-\mu}{\sigma}\right)=\frac{1}{\sigma^2}[Var(X)-Var(\mu)]=\frac{\sigma^2}{\sigma^2}=\boxed{1}$

13. $E(X)=\int_0^\infty X\cdot f(x)\,dx$; Product Rule: $[xF(x)]'=F(x)+xf(x)$; $1-F(x)=xf(x)$: $E(X)=\int_0^\infty [1-F(x)]\,dx$

$E(X)=\int_0^\infty [1+1+e^{-\lambda x}]\,dx = -\frac{e^{-\lambda x}}{\lambda}\Big|_0^\infty = \boxed{\frac{1}{\lambda}}$

14. $f(x)=2x$; $0\le x\le 1$: a) $E(X)=\int_0^1 2x^2\,dx = \frac{2}{3}x^3\Big|_0^1=\boxed{\frac{2}{3}}$ b) $E(X^2)=\int_0^1 2x^3\,dx = \boxed{\frac{1}{2}}$

15. Lottery A — n-possible #s; Lottery B — n-possible #'s

Payoff A = Payoff B

$E(X)=\sum_{1}^{n}\frac{1}{n!}\cdot X$; $E(A)=\sum_{1}^{n}\frac{1}{n!}X$

$E(A+B)=\frac{E(A)+E(B)}{2}$; $=E(A)=E(B)$

No difference:

$Var(X)=E(X^2)-E(X)^2=E[(X-E(X))^2]$

$=\frac{1}{2}-\left(\frac{2}{3}\right)^2=\boxed{5/9}$

16. $Var(+X)=Var(-X)$; $f_X(\xi+s)=f_X(\xi-s)$; $Y=X-\xi$; $f_Y(y)=f_X(y+\xi)$; $f_Y(y)=f_Y(-y)$

$Y=X-\xi$; $E(Y)=E(X)-E(\xi)$; $E(X)=E(\xi)$

17. k^{th}-order Statistic:

$\frac{n!}{(k-1)!(n-k)!}x^{k-1}(1-x)^{n-k}$; $0\le x\le 1$; $E(X)=\int_0^1 \frac{n!}{(k-1)!(n-k)!}x^k(1-x)^{n-k}\,dx = \frac{n!}{(k-1)!(n-k)!}\frac{k!(n-k)!}{(n+1)!}$

$=\boxed{\frac{k!}{(k-1)!(n+1)!}=\frac{k}{(n+1)}}$

$E(X^2)=\frac{n!}{(k-1)!(n-k)!}\int_0^1 x^{k+1}(1-x)^{n-k}\,dx = \frac{n!}{(k-1)!(n-k)!}\frac{(k+1)!(n-k)!}{(n+2)!}=\frac{k(k+1)}{(n+1)(n+2)}$

$=k(k+1)$; $Var(X)=\frac{k(k+1)}{(n+1)(n+2)}-\frac{k^2}{(n+1)^2}=\frac{k(k+1)(n+1)-k^2(n+2)}{(n+1)^2(n+2)}$

$=\frac{(k^2+k)(n+1)-k^2n-k^2(2)}{(n+1)^2(n+2)}$

$=\frac{k^2n+k^2+kn+k-\cancel{k^2n}-k^2 2}{(n+1)^2(n+2)}$

$=\boxed{\frac{k(n+1-k)}{(n+1)^2(n+2)}}$

18. $U_1,\dots,U_n$; $E(U_{(n)}-U_{(1)})$; $E(U_{(n)}-U_{(1)})=\sum_{i=1}^{n}(U_{(n)}-U_{(1)})f(U)$

19. $\boxed{E(U_{(k)}-U_{(k-1)})=\sum_{i=1}^{n}[U_{(k)}-U_{(k-1)}]f(U)}$

20. $E[1/(X+1)]$; $X=\frac{\lambda^k}{k!}e^{-\lambda x}$; $E\left[\frac{1}{(x+1)}\right]=\int_0^\infty \frac{\lambda^k}{k!}\frac{1}{(x+1)}e^{-\lambda x}\,dx$

$=\frac{\lambda^k}{k!}\int_0^\infty \frac{e^{-\lambda x}}{(1+x)}\,dx$; $u=1+x$, $du=dx$, $x=(u-1)$; $\frac{\lambda^k}{k!}\int_0^\infty \frac{e^{-\lambda(u-1)}}{u}\,du=\frac{\lambda^k}{k!}e^{\lambda}\int_0^\infty \frac{e^{-\lambda u}}{u}\,du=\frac{\lambda^k}{k!}e^{2\lambda}\cdot E_1(n)$ "Exponential Integral"

21. Expected Area $(X^2)=\int_0^1 x^2\,dx=\boxed{\frac{1}{3}}$

$\left(\frac{1}{2}\right)^2+1^2$

22. Expected Area $(X^2)=\int_0^1 x^2\,dx=\frac{1}{3}$

23. Expected Area $(X^2)=\lambda\int_0^\infty x^2e^{-\lambda x}\,dx$

$=2/\lambda^2$

$E(X)E(Y)=\lambda\int_0^\infty xe^{-\lambda x}\,dx\cdot\lambda\int_0^\infty ye^{-\lambda y}\,dy=\boxed{1/\lambda^2}$

24. Prove Theorem A of Section 4.1.2 ; $Y = a + \sum_{i=1}^{n} b_i X_i$; $E(Y) = a + \sum_{1}^{n} b_i E(X)$

Proof: $E(Y) = \iint (a + b_1 x_1 + b_2 x_2) f(x_1, x_2)\,dx_1 dx_2 = a \iint f(x_1, x_2)\,dx_1 dx_2 + b_1 \iint x_1 f(x_1, x_2)\,dx_1 dx_2$
$+ b_2 \iint x_2 f(x_1, x_2)\,dx_1 dx_2$

$\iint x_1 f(x_1, x_2)\,dx_1 dx_2 = \int x_1 \left[\int f(x_1, x_2)\,dx_2\right] dx_1 = \int x_1 f(x_1)\,dx_1 = E(X_1)$

$E(Y) = a + b_1 E(X_1) + b_2 E(X_2)$; $\iint |a + b_1 x_1 + b_2 x_2| f(x_1, x_2)\,dx_1 dx_2 < \infty$

$|a + b_1 x_1 + b_2 x_2| \leq |a| + |b_1||x_1| + |b_2||x_2|$

25. Gamma Distribution:

$f(x) = \frac{\lambda^a}{T(a)} x^{a-1} e^{-\lambda x}$ Find $R^2 = X_1^2 + X_2^2$; $E(R^2) = \int (X_1^2 + X_2^2) T(x)\,dx_1 dx_2 = \int_0^\infty x_1^2 \frac{\lambda^\alpha}{T(\alpha)} x_1^{\alpha-1} e^{-\lambda x_1} dx + \int_0^\infty x_2^2 \frac{\lambda^\alpha}{T(\alpha)} x_2^{\alpha-1} e^{-\lambda x_2} dx$

$= \frac{\lambda^\alpha}{T(\alpha)} \left[\underbrace{\left(\int_0^\infty x_1^{(\alpha+2)-1} e^{-\lambda x} dx\right)} + \left(\int_0^\infty x_2^{(\alpha+2)-1} e^{-\lambda x} dx\right) \right]$

$= \frac{\lambda^\alpha}{T(\alpha)} \frac{T(\alpha + k)}{\lambda^{\alpha+k}} = \frac{T(\alpha+k)}{T(\alpha)\lambda^k} = \frac{(\alpha+k+1)T(\alpha+k-1)}{T(\alpha)\lambda^k} + \cdots$

$= \frac{(\alpha+k-1)\cdots\alpha}{\lambda^k} = \frac{(\alpha+1)\alpha}{\lambda^k}$

$\boxed{E(R^2) = \frac{2(\alpha+1)\alpha}{\lambda^k}}$

26. $\frac{\text{Expectation Long Piece}}{\text{Expectation (Shorter Piece)}} = \frac{E(L)}{E(S)} = \boxed{\frac{\sum L \cdot f(x)}{\sum S \cdot f(x)}}$

27. n-men → pile $E(\#\text{ of matches}) = \sum \frac{x}{n} = \frac{n}{n} = 1$

28. $X_i = \begin{cases} 1 : \text{Aircraft } i \text{ is hit} \\ 0 : \text{Otherwise} \end{cases}$; $Z_{ij} = \begin{cases} 1 & \text{if gunner } j \text{ hits } i \\ 0 & \text{otherwise} \end{cases}$

$P(Z_{ij} = 1 | B_i) = p$; $P(Z_{ij} = 1 | B_i) = 0$; $P(B) = \frac{1}{n}$

$P(B) = 1 - P(B) = 1 - \frac{1}{n}$ "Every gunner selects i"

Addition Rule:

$P(Z_{ij} = 1) = P(Z_{ij} = 1 | B_i) P(B_i) + P(Z_{ij} = 1 | B_i) P(B)$

$= p\left(\frac{1}{n}\right) + 0\left(\frac{1}{n}\right) = \frac{p}{n}$

Since the total probability is equal to 1, this implies.

$P(Z_{ij} = 0) = 1 - P(Z_{ij} = 1) = 1 - \frac{p}{n}$

$E(X) = \sum x_i P(x) = 0(PX=0) + 1 \cdot P(X=1)$

$= 1 - P(X=1)$

$= 1 - \{P(Z_i = 0) P(Z_{i2} = 0) \cdots P(Z_{im} = 0)\}$

$\boxed{= 1 - \left(1 - \frac{p}{n}\right)^m}$

29. Prove Corollary 4.1.1

$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$

$E[g(x) \cdot h(Y)] = \iint g(x) \cdot h(Y)\,dx\,dY = \boxed{\int g(x)\,dx \int h(Y)\,dY}$

30. Coupon Collection: n-distinct types of coupons.

P(coupon) are equivalent.

$X_1 = 1, X_2 = 2, X_3 = 3 \cdots X_n = n$; $p(\text{success}) = (n-r+1)/n$

$E(X_r) = n/(n-r+1)$

$E(X) = \sum_{r=1}^{n} E(X_r) = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1} = n \sum_{r=1}^{n} \frac{1}{r}$

$\boxed{r = 2r \sum \frac{1}{r} = r}$

31. $X[1, 2]$

$E[1/x] = \int_1^2 x\left(\frac{1}{x}\right) dx$

$= \int_1^2 dx = 1$

$\frac{1}{E(x)} = \frac{1}{\int_1^2 x^2 dx} = \frac{3}{8-1} = \frac{3}{7}$; they are not equal.

32. Gamma Distribution:

$f(x) = \frac{\lambda^a}{T(a)} x^{a-1} e^{-\lambda x}$

$E\left(\frac{1}{x}\right) = \int_0^\infty \left(\frac{1}{x}\right) \frac{\lambda^a}{T(a)} x^{a-1} e^{-\lambda x} dx = \frac{\lambda^a}{T(a)} \int_0^\infty x^{(a-1)-1} e^{-\lambda x} dx$

$= \frac{\lambda^a}{T(a)} \frac{T(a-1)}{\lambda^{a-1}} = \frac{\lambda T(a-1)}{T(a)} = \boxed{\frac{\lambda}{a}}$

33. Prove Chebyshev's Inequality:

X, μ, σ^2. For $t > 0$,

$P(|X - \mu| > t) \leq \frac{\sigma^2}{t^2}$

$E(Y) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) \geq \int_{-\infty}^{\mu-\varepsilon} (x-\mu)^2 f(x)\,dx + \int_{\mu+\varepsilon}^{\infty} (x-\mu)^2 f(x)\,dx = \int_{-\infty}^{\mu-\varepsilon} \varepsilon^2 f(x)\,dx + \int_{\mu+\varepsilon}^{\infty} \varepsilon^2 f(x)\,dx = \boxed{\varepsilon^2 P(|X-\mu| \geq \varepsilon)}$

34.

n = Group Samples
m = Groups
k = samples
$n = mk$

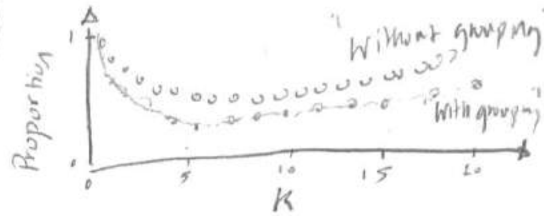
$E(N) = \sum_1^m E(X) = \sum_1^m p^k + (k+1)(1-p^k) = \sum_1^m k - kp^k + 1$

$\boxed{= m(k+1) - mkp^k = n\left(1 + \frac{1}{k} - p^k\right)}$

"Average number of tests performed"

"With no grouping"

$\boxed{E(X) = k + 1 - kp^k = k\left(1 + \frac{1}{k} - p^k\right)}$



35. Mean of Negative Binomial Random Variable.

$$E(k) = \sum_{k=1}^{\infty} k \binom{k-1}{r-1} p^r (1-p)^{k-r} = \sum \frac{k(k-1)!}{(r-1)!(r-k)!} p^r (1-p)^{k-r}$$

$$= \frac{k\,\Gamma(k)}{\Gamma(r)\,\Gamma(k-\cancel{r}+1)} \; \frac{\Gamma(r+1)\,\Gamma(k-\cancel{r}+1)}{\Gamma(k+2)} = \frac{k\Gamma(\cancel{k})\, r\Gamma(\cancel{r})}{\Gamma(\cancel{r})(k+1)\,k\,\Gamma(\cancel{k})} = \boxed{\frac{r}{k+1}}$$

36. $X[0,1], Y = \sqrt{X}$; $E(Y)$ a) $E(Y) = E(\sqrt{X}) = \sum_0^1 \sqrt{X} f(x) + \boxed{\sqrt{X} \cdot p^x + \sqrt{X}(1-p)^{1-x}} = \sqrt{1} = 1$

b) $E(Y) = E(\sqrt{x}) = \int_0^1 \sqrt{x}\,dx = \frac{2}{3}(1)^{3/2}$

37. Example C : Section 4.1.2 - $E(N) = n\left(1 + \frac{1}{k} - p^k\right)$; $E(X) = n$; $E(X) = E(N)$

$n = n\left(1 + \frac{1}{k} - p^k\right)$

$1 = 1 + \frac{1}{k} - p^k$

$\frac{1}{k} = p^k$

$\boxed{p = \left(\frac{1}{k}\right)^{1/k}}$

38. $E(Y) = \sum_{k=0}^{n} \binom{n}{k} k p^k (1-p)^{n-k} = np$

a) $Y = \sum_{i=1}^{n} X_i$; Length of DNA = G, Fragments = N of length = L.
$G \sim > 100{,}000$ $\quad L \sim > 500$

Probability of left end is $1, 2 \cdots G-L+1$.

What is the probability a particular location $x \in \{L, L+1 \ldots G\}$

How many fragments are expected to cover a particular location ; $\{1, 2 \ldots L-1\}$

What is the chance of x covering the left end of L locations $\{x-L+1, x-L, \ldots x\}$

$$p = \frac{L}{G-L+1} \approx \frac{L}{G} \text{ ; The binomial probability formula, } p(W>0) = 1 - P(W=0) = 1 - \left(1 - \frac{L}{G}\right)^N$$

$\lambda = Np = NL/G$

a. Probability that a fragment is the leftmost member of a contig $\frac{L}{G-L+1}$

b. Expected number of fragments left of contig : $E(k) = \sum_{N=0}^{N} L\,(P(W=0)) = \left[1 - \left(1 - \frac{L}{G}\right)^N\right] L$

c. Expected number of contigs : $\boxed{E(L) = L e^{-NL/G}}$

39. DNA Length = 10^6, fragment length = 100

a) $P(W>0) = 0.99 = 1 - \left(1 - \frac{10^2}{10^6}\right)^N$; $(1 - 10^{-4})^N = 0.01$; $N = \frac{\log 10^{-2}}{\log(0.9999)} = 4.60 \times 10^4$ fragments.

b) The expected misses $E(I) = e^{-4.60\times10^4 \cdot 100/10^6} \boxed{= 0.01}$

40. Q, W, E, R, T, Y produces 1000 letters in all. $E(QQQQ) = \sum_{n=1}^{N-q+1} E(I_n) = (N - q + 1)\left(\frac{1}{5}\right)^q$

$\boxed{n = 1000,\; q = 4 \quad = (997)\left(\frac{1}{5}\right)^4 = 15.95 \text{ times}}$

41. $E(Troy) = \sum_{n=1}^{N-q+1} E(I_n) = (1000 - 3 + 1)\left(\frac{1}{5}\right)^3 = 998\left(\frac{1}{5}\right)^3$
$= 79.84$ times.

Markov's Inequality:

$\frac{79.84}{1000}$; Yes, the author would be surprised for the answer to occur.

42. Exponential Random Variable:

$p(x) = \lambda e^{-\lambda x}$; $P(|X-E(X)| > k\sigma)$ for $k = 2, 3, 4 \dots$

compare results to the bounds from Chebyshev's inequality

Chebyshev's Inequality: $P(|X-\mu| > t) \le \frac{\sigma^2}{t}$

$P(|X-E(X)| > k\sigma) \le \frac{\sigma^2}{k\sigma^2} = \frac{1}{k}$ for $k = 2, 3, 4 \dots$

43. Show Var(X-Y)

$$Var(X-Y) = Var(X) + Var(Y) - 2Cov(X,Y)$$

$$= E([X-Y]^2) - E(X-Y)^2 = E(X^2) - 2E(X \cdot Y) + E(Y^2) - E(X)^2 - E(Y)^2$$

$$\boxed{= Var(X) + Var(Y) - 2Cov(X,Y)}$$

covariance: $Cov(X,Y) = E[(X-\mu_X)(Y-\mu_X)]$ provided the expectation exists

44. X & Y have equal variance. Find $Cov(X+Y, X-Y)$

$$Cov(X+Y, X-Y) = E[(X+Y-E(X+Y))(X-Y-E(X-Y))] = E[(X-E(X)+Y-E(Y))(X-E(X)-Y+E(Y))]$$

$$= E[(X-E(X))^2] - E[(X-E(X))(Y+E(Y))] + E[(Y-E(Y))(X-E(X))] - E[[Y-E(Y)]^2]$$

$$= E[(X-E(X))^2] - 2E[(X-E(X))(Y+E(Y))] - E[(X-E(Y))^2]$$

45. Find the covariance of N_j and N_i where $N_1, N_2, \dots, N_r$ are multinomial random variables

Multinomial Random Variable:

$$p(n_1, \dots, n_r) = \binom{n}{n_1 \dots n_r} P_1^{n_1} P_2^{n_2} \cdots P_r^{n_r}$$

$$Cov(N_i, N_j) = E[(N_i - E(N_i))(N_j - E(N_j))]$$

$$= E(N_i N_j) - E(N_i)E(N_j)$$

$$= Pr(N_i = N_j) - Pr(N_i = 1)Pr(N_j = 1) = 0 - np_ip_k = \boxed{np_ip_k}$$

46. U & V; μ and σ^2; $Z = \alpha U + V\sqrt{1-\alpha^2}$ Find $E(Z)$ and ρ_{UZ}.

$$E(Z) = E(\alpha U + V\sqrt{1-\alpha^2}) = E(\alpha U) + E(V\sqrt{1-\alpha^2}) = \alpha E(U) + \sqrt{1-\alpha^2}E(V) = \boxed{(\alpha + \sqrt{1-\alpha^2})\mu}$$

Correlation coefficient: If X and Y are jointly distributed random variables and the variances and covariances of both X and Y are nonzero, then the correlation of X and Y, denoted by ρ, is

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{E(UV) - E(U)E(V)}{\sqrt{Var(U)Var(V)}} = \frac{\int_{\infty}^{\infty}\int_{\infty}^{\infty} u \cdot v\, e^{-\frac{(u-\mu)^2}{2\sigma^2} - \frac{(v-\mu)^2}{2\sigma^2}} uv - \left(\frac{1}{2}\right)^2}{\sqrt{\sigma^2 \cdot \sigma^2}}$$

$$= \frac{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}{\sigma^2} = \phi$$

47. $Z = Y - X$; Find $Cov(X,Z) = Cov(X, Y-X) =$

$$\boxed{= E[(X-E(X))(Z-E(Z))] = E[XZ] - E[X]E(Z) = E(X(Y-X)) - E(X)E(Y-X)}$$

$$\boxed{= E(XY) - E(XZ) - E(X)[E(Y) - E(Z)] =}$$

$$Corr(X,Z) = \rho_{XZ} = \frac{Cov(X,Z)}{\sqrt{Var(X)Var(Z)}} = \boxed{\frac{E(XY) - E(XZ) - E(X)[E(Y) - E(Z)]}{\sqrt{Var(X)Var(Z)}}}$$

48. $U = a + bX$; $V = c + dY$ show that $|\rho_{UV}| = |\rho_{XY}|$;

$$\rho = \frac{Cov(U,V)}{\sqrt{Var(U)Var(V)}} = \frac{Cov(a+bX, c+dY)}{\sqrt{Var(a+bX)Var(c+dY)}} = \frac{E[(a+bX)(c+dY)] - E(a+bX)E(c+dY)}{\sqrt{[E[(a+bX)^2] - E(a+bX)^2][E[(c+dY)^2] - E(c+dY)^2]}}$$

$$= \frac{E(ac) + E(adY) + E(bXc) + E(bXdY) - (E(a) + E(bX))(E(c) + E(dY))}{\sqrt{[E(a^2) + 2E(abX) + E[(bX)^2] - [E(a) - E(bX)]^2][E(c^2) + 2E(cdY) + E(dY)^2 - [E(c) + E(dY)]^2]}}$$

$$\boxed{= \emptyset}$$

49. $E(X)=E(Y)=\mu$, but $\sigma_x \neq \sigma_y$; $Z=\alpha X+(1-\alpha)Y$; where $0\le \alpha \le 1$

a) Show $E(Z)=\mu$; $E(Z)=E(\alpha X+(1-\alpha)Y)=\alpha E(X)+(1-\alpha)E(Y)=\alpha\mu+(1-\alpha)\mu=\boxed{\mu}$

b) Find α in terms of σ_x, σ_y to minimize $Var(Z)$

$Var(Z)=Var(\alpha X+(1-\alpha)Y)=\alpha^2E[X^2]-\alpha^2E[X]^2+(1-\alpha)^2E[Y^2]-(1-\alpha)^2E[Y]^2=\alpha^2[Var(X)]+(1-\alpha)^2[Var(Y)]$

$\frac{d}{d\alpha}Var(Z)=0=(\alpha^2 Var(X)-(1-\alpha)^2Var(Y))=(2\alpha Var(X)+2(1-\alpha)(-1)Var(Y))=0$

$\boxed{\alpha=\frac{2\,Var(Y)}{Var(X)+Var(Y)}}$ || $var\left(\frac{X+Y}{2}\right)\le var(X)$

$\frac{1}{4}[Var(X)+Var(Y)]\le Var(X)$; $\boxed{Var(Y)\le 3\,Var(X)}$

c) When is the average $(X+Y)/2$ better to use than X or Y alone?

When the variance of the average is less than variance of X or Y alone.

50 X_i; $i=1\ldots n$; $E(X_i)=\mu$ and $Var(X_i)=\sigma^2$ $\bar{X}=n^{-1}\sum_{i=1}^{n}X_i$; show $E(\bar{X})=\mu$ and $Var(\bar{X})=\sigma^2/n$

$E(\bar{X})=E(n^{-1}\sum_{i=1}^{n}X_i)=E\left(\frac{X_1+X_2+\cdots+X_n}{n}\right)=\mu$; $Var(\bar{X})=E(\bar{X}^2)-E(\bar{X})^2=\frac{1}{n}E(\sum X_i^2)-\frac{1}{n}E(\sum X_i)^2$
$=\sigma^2/n$

51. Example E: Section 4.3; $\mu_1=\mu_2=\mu$; $\sigma_{ij}=Cov(R_i,R_j)=0$; Portfolio $(\pi,1-\pi)$

Expected Return: $E(R(\pi))=\pi\mu+(1-\pi)\mu=\mu$; Risk of Return: $Var(R(\pi))=\pi^2\sigma_1^2+(1-\pi)^2\sigma_2^2$

Minimizing Risk with respect to π: $\pi_{opt}=\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$; $Var(R(\tfrac{1}{2}))=\frac{\sigma^2}{2}$

When considering unique returns: $E(R(\pi))=\pi\mu_1+(1-\pi)\mu_2$

$Var(R(\pi))=\pi^2\sigma_1^2+2\pi(1-\pi)\rho\sigma_1\sigma_2+(1-\pi)^2\sigma_2^2$

When considering n-total investments: $E(R(\pi))=\sum\pi_i\mu_i$; $Var(R(\pi))=\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_i\pi_j\sigma$

Problem: n-securities (μ,σ)

uncorrelated. $E(R(\pi))=\sum_{i=1}^{n}\pi_i\mu$; $Var(R(\pi))=\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_i\pi_j\sigma$; $\boxed{Var=\frac{1}{n}\sigma^2}$; $S.D=\frac{1}{\sqrt{n}}\sigma$

$\mu=n\pi_i\mu$; $1=n\pi_i$; $\boxed{\frac{1}{n}=\pi_i}$;

Risk of one security $=\boxed{\frac{\sigma}{1}}$

52. Two securities $(\mu_1=1,\sigma_1=0.1)$

$(\mu_2=0.8,\sigma_2=0.12)$; $\rho=-0.8$;

a) Return #1 $=\mu_1\pm0.1=0.1\pm0.1$; $\boxed{(\mu_1/\sigma_1)=10}$ Greater dollar per risk.

Return #2 $=\mu_2\pm0.12=0.8\pm0.12$ $(\mu_2/\sigma_2)=6.75$

b) 50% into each stock

$E(R(\pi))=0.5\cdot1+0.8\cdot0.5=0.4$

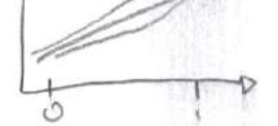
$Var(R(\pi))=0.5^2(0.1)^2+2\cdot0.5(1-0.5)(-0.8)0.1\cdot0.12+(1-0.5)^2(0.12)^2$

$\boxed{\sigma_{12}=0.04}$

c) $E(R(\pi_1=80\%,\pi_2=20\%))=0.8\cdot1+0.2\cdot0.8=\boxed{0.96}$; $Var(R(\pi_1=80\%,\pi_2=20\%))$
$=0.8^2(0.1)^2+2\cdot0.8(0.2)(-0.8)\cdot0.1\cdot0.12+(0.2)^2(0.12)^2$
$\boxed{\pm0.06}$

d)

e)



54. X, Y, Z with $\sigma_x^2, \sigma_y^2, \sigma_z^2$

Let $U=Z+X$; $V=Z+Y$

$Cov(U,V)=E(UV)-E(U)E(V)=E[(Z+X)(Z+Y)]-E[Z+X]E[Z+Y]$
$=E[Z^2]+E[XZ]+E[ZY]+E[XY]-E[Z^2]-E[ZY]-E[XZ]-E[XY]$
$=0$

$Corr(U,V)=\rho_{UV}=\boxed{0}$

53. $Cov(X,Y)\le\sqrt{Var(X)Var(Y)}$;

$Cov(X,Y)=E(XY)-E(X)E(Y)\le\sqrt{[E(X^2)-E(X)^2][E(Y^2)-E(Y)^2]}$
$\le\sqrt{E(X^2)E(Y^2)-E(X^2)E(Y)^2-E(X)^2E(Y^2)-E(X)^2E(Y)^2}$

55. $T=\sum_{k=1}^{n} kX_k$; X_k are independent random variables with μ, σ^2. Find $E(T)$ and $Var(T)$

$E(T)=E\left(\sum_1^n kX_k\right)=\frac{n(n+1)}{2}\mu$; $Var(T)=Var\left(\sum_1^n kX_k\right)=E\left[\left(\sum_1^n kX_k\right)^2\right]-E\left[\sum kX_k\right]^2=\frac{n(n+1)(2n+1)\sigma^2}{6}$

56. $S=\sum_{k=1}^{n} X_k$; $Cov(S,T)=E(ST)-E(S)E(T)=E\left(\sum kX_k \sum X_k\right)-E\left(\sum kX_k\right)E\left(\sum X_k\right)$

$=\frac{n(n+1)(2n+1)}{6}\mu^2-\frac{n(n+1)}{2}\mu\cdot\mu=\left[\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}\right]\mu^2$

$Corr(S,T)=\rho_{ST}=\frac{Cov(S,T)}{\sqrt{Var(S)Var(T)}}=\frac{\left[\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}\right]\mu^2}{\sqrt{\sigma^2\cdot\frac{n(n+1)(2n+1)}{6}\sigma^2}}=\boxed{\frac{\left[\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}\right]\mu^2}{\sqrt{\frac{n(n+1)(2n+1)}{6}}\sigma^2}}$

57. $Var(XY)=E\left[(XY-E(XY))^2\right]$

$=E\left(X^2Y^2-2XYE(XY)+E(XY)^2\right)=E(X^2Y^2)-2E(X)E(Y)E(XY)+E(XY)^2$

$=E(X^2Y^2)-2E(XY)^2+E(XY)^2=E(X^2Y^2)-E(XY)^2$

$=E(X^2)E(Y^2)-\mu_x\mu_y=\left[Var(X)+E(x)^2\right]\left[Var(Y)+E(y)^2\right]-\mu_x^2\mu_y^2$

$=Var(X)Var(Y)+Var(X)E(y)^2+E(x)^2Var(Y)+E(x)^2E(y)^2-\mu_x^2\mu_y^2$

$=\boxed{\sigma_x^2\sigma_y^2+\sigma_x^2\mu_y^2+\mu_x^2\sigma_y^2}+\mu_x^2\mu_y^2-\mu_x^2\mu_y^2$

58. $X_1=f(x)+\epsilon_1$; $X_2=f(x+h)+\epsilon_2$; $\epsilon_1 \& \epsilon_2(\mu=0,\sigma^2)$; $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$; $Z=\frac{X_2-X_1}{h}$

a) Find $E(Z)=E\left(\frac{X_2-X_1}{h}\right)=E\left(\frac{f(x+h)+\epsilon_2-f(x)-\epsilon_1}{h}\right)=\frac{1}{h}\left[E(f(x+h))+\cancel{E(\epsilon_2)}-E(f(x))-\cancel{E(\epsilon_1)}\right]$

$=\frac{f(x+h)-f(x)}{h}$ $\quad = \frac{1}{h^2}\left[Var(f(x+h))+Var(\epsilon_2)-Var(f(x))-Var(\epsilon_1)\right]$

Find $Var(Z)=Var\left(\frac{f(x+h)+\epsilon_2-f(x)-\epsilon_1}{h}\right)=\frac{(-1)^2}{h^2}\sigma^2+\frac{\sigma^2}{h^2}=\boxed{\frac{2\sigma^2}{h^2}}$

In the limit or $E(Z)=\lim_{h\to 0}E(Z)=f'(x)$; $\lim_{h\to 0}Var(Z)=\boxed{\lim_{h\to 0}\frac{2\sigma^2}{h^2}=0}$

Mean Squared Error

$E\left[(X-X_0)^2\right]$ Mean Squared Error

$=Var\left[(X-X_0)\right]+E\left[(X-X_0)\right]^2$

$=\sigma^2+\beta^2$

b) Mean Squared Error of Z:

$MSE(Z)=E\left[(Z-Z)^2\right]=Var(Z-Z_0)+E\left[(Z-Z_0)\right]^2=\frac{2\sigma^2}{h^2}+\frac{f(x+h)-f(x)}{h}$

$\lim_{h\to 0}MSE(Z)=f'(x)$ | $X_3=f(x+h+k)+\epsilon_3$; $E(\epsilon_1)=E(\epsilon_2)=E(\epsilon_3)=0$; $Var(\epsilon_1)=Var(\epsilon_2)=Var(\epsilon_3)=\sigma^2$

c) $X_1=f(x)+\epsilon_1$; $X_2=f(x+h)+\epsilon_2$; $Z_1=\frac{1}{h}\left[X_2-X_1\right]$; $Z_2=\frac{1}{h}\left[X_3-X_2\right]$; $Z_3=\frac{1}{h}\left[Z_2-Z_1\right]=\frac{1}{h}\left(\frac{X_3-X_2}{k}-\frac{X_2-X_1}{h}\right)$

$Z_3=\frac{1}{h^2}X_1-\left(\frac{1}{hk}+\frac{1}{h^2}\right)X_2+\frac{1}{hk}X_3$; $\boxed{E(Z_3)=\frac{1}{h^2}f(x)-\left(\frac{1}{hk}+\frac{1}{h^2}\right)f(x+h)+\frac{1}{hk}f(x+h+k)}$

$\boxed{Var(Z_3)=2\sigma^2\left(\frac{1}{h^4}+\frac{1}{h^2k^2}+\frac{1}{h^3k}\right)}$ Show the $Cov(X,Y)=0$: $E(XY)-E(X)E(Y)$

$=\frac{4}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\sqrt{1-x^2}\,y\sqrt{1-y^2}\,dx\,dy-\frac{2}{\pi}\int_{-\infty}^{\infty}x\sqrt{1-x^2}\int_{-\infty}^{\infty}y\sqrt{1-y^2}=0$

61. $f(x,y)=2$; $0\le x\le y$

a) $Cov(x,y)$ $Corr(x,y)$

59. (X,Y) is a random point on a disk.

$\boxed{f(x,y)=\begin{cases}\frac{1}{\pi r} & x^2+y^2\le r\\ 0 & \text{otherwise}\end{cases}}$

60. Y is symmetric about zero. $X=SY$

$S=\pm 1$; $P(S=1)=P(S=-1)=\frac{1}{2}$; Show $Cov(X,Y)=0$

$Cov(X,Y)=E(XY)-E(X)E(Y)=E(SY^2)-E(SY)E(Y)=\boxed{SE(Y)^2-SE(Y)^2=0}$

$E(X)=E(SY)=SE(Y)$; $\frac{E(X)}{E(Y)}=S$

$Cov(x,y)=\int_{-\infty}^{\infty}\int_{0}^{x}(x-\mu_x)(y-\mu_y)f(x,y)\,dx\,dy$

$=2\int_{-\infty}^{\infty}\int_{0}^{x}(x-\mu_x)(y-\mu_y)\,dy\,dy$

$=2\int_{-\infty}^{\infty}\left(\frac{x^2}{2}\right)(y-\mu_y)\,dy=\infty$

X & Y from $[0,1]$; $f(x,y)=2$; $0 \le x \le y \le 1$

a) $Cov(X,Y) = E(XY) - E(X)E(Y)$;

$= \frac{1}{4} - \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{1}{4} - \frac{2}{9}$

$= \frac{9}{36} - \frac{8}{36} = \boxed{\frac{1}{36}}$

$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$

$= \frac{(1/36)}{\sqrt{(E[X^2]-E[X]^2)(E[Y^2]-E[Y]^2)}} = \frac{1/36}{\sqrt{(2/12 - 1/9)(\frac{1}{2} - 4/9)}} = \frac{1/36}{1/18} = \boxed{\frac{1}{2}}$

$E(x) = \int_0^1 x f(x)\,dx = \int_0^1 x\left[\int_x^1 f(x,y)\,dy\right]dx = \int_0^1 x\left[\int_x^1 2\,dy\right]dx$

$= \int_0^1 x\,[2-2x]\,dx = 2\int_0^1 x - x^2\,dx = 2\left[\frac{1}{2} - \frac{1}{3}\right] = \boxed{\frac{1}{3}}$

$E(y) = \int_0^1 y f(y)\,dy = \int_0^1 y\left[\int_0^y f(x,y)\,dx\right]dy = \int_0^1 y\,[2y]\,dy = \int_0^1 2y^2\,dy$

$= \frac{2}{3}$

$E(xy) = \int_0^1\int_0^y xy f(x,y)\,dx\,dy = 2\int_0^1 \frac{y^3}{2}\,dy = \frac{y^4}{4}\Big|_0^1 = \boxed{\frac{1}{4}}$

$x^2[2-2x]$

$2[x^2 - x^3]$

$2\left[\frac{x^3}{3} - \frac{x^4}{4}\right]$

$\frac{2}{3} - \frac{2}{4} = \frac{8}{12} - \frac{6}{12}$

$= 2$

b.

Find $E(X|Y=y)$ and $E(Y|X=x)$

Conditional Expectations:

if $p_{Y|X}(y|x)$, then $E(Y|X=x) = \sum_y y\, p_{Y|X}(y|x)$

$= \int y\, f_{Y|X}(y|x)\,dy$

More Generally,

$E[h(Y)|X=x] = \int h(y) f_{Y|X}(y|x)\,dy$

$E(Y|X=x) = \sum_y y \frac{f(x,y)}{f(x)} \Rightarrow \int_x^1 y\left(\frac{2}{2-2x}\right)dy \quad \frac{y^2}{2}\ \frac{y^4}{4}$

$= \int_x^1 y \cdot 2\left(\frac{1}{1-x}\right)dy = \frac{1}{2}[1-x^2]\left(\frac{1}{1-x}\right) = \boxed{\frac{(x+1)}{2}}$

$E(X|Y=y) = \sum_x x \cdot p_{X|Y}(x|y) \Rightarrow \int_0^y x \frac{f(x,y)}{f(y)}\,dx = \int_0^y x \frac{2}{2y}\,dx$

$= \frac{y^2}{2y} = \boxed{\frac{y}{2}}$

c) Find $E(X|Y)$ and $E(Y|X)$

$= y/2 \qquad = (x+1)/2$

$f_{W_1}(W_1) = 2f_y(2W_1)$

$= 2(2(2W_1))$

$= \boxed{8W_1}$

$f_{W_2}(W_2) = 2(2W_2 - 1)$

$= \boxed{8(1-W_2)}$

d) $\hat{Y} = a + bX$; $\min(E((Y-\hat{Y})^2))$ $\boxed{\text{Predictor}}$

$E(\hat{Y}) = a + bE(\hat{X})$; $\mu_y = a + b\mu_x$

$a = \mu_y - b\mu_x = \frac{2}{3} - \rho_{xy}\left(\frac{1}{3}\right) = \frac{2}{3} - \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{2}{3} - \frac{1}{6}$

$= \frac{12}{18} - \frac{3}{18} = \boxed{\frac{9}{18}}$

Mean Squared Error : $E\left(Y - \frac{1}{2} - \frac{1}{2}x\right)^2 = \sigma_y^2(1-\rho^2)$

$= \frac{1}{18}\left(1 - \frac{1}{4}\right) = \boxed{1/24}$

e) Predictor of Y in terms of X

MSE = $\boxed{\text{Mean Squared Error}}$ $= E(Y-\hat{Y})^2 = E(Y - E(Y|X))^2$

$= E(Y^2) - E(E(Y|X))^2 = \frac{1}{2} - E\left(\frac{(x+1)^2}{2}\right) = \frac{1}{2} - \int_0^1 \frac{(x+1)^2}{4}(1-x)\,dx$

$\boxed{= 1/24}$

$\boxed{\text{Mean Squared Predictor Error}}$

$E(Y|X=x) = \frac{1}{2} - E\left(\frac{(x+1)^2}{4}\right)$

$= \frac{1}{2} - \int_0^1 \frac{(x+1)^2}{4}(x-1)\,dx$

$= \boxed{1/24}$

62.

X & Y joint random variables with correlation ρ_{xy}.

Define the Standardized random variables $\tilde{X}$ and $\tilde{Y}$

$\tilde{X} = (X - E(X))/\sqrt{Var(X)}$

$\tilde{Y} = (Y - E(Y))/\sqrt{Var(Y)}$

Show that $Cov(\tilde{X},\tilde{Y}) = \rho_{xy}$

$Cov(\tilde{X},\tilde{Y}) = E((\tilde{X}-\bar{X})(\tilde{Y}-\bar{Y}))$

$= E(\tilde{X}\tilde{Y}) - E(\tilde{X}\bar{Y}) - E(\tilde{Y}\bar{X}) + E(\bar{X})E(\bar{Y})$

$= E(\tilde{X}\tilde{Y}) - E(\tilde{X})E(\tilde{Y})$

$= E\left(\frac{[X-E(X)]}{\sqrt{Var(X)}}\frac{[Y-E(Y)]}{\sqrt{Var(Y)}}\right) - \frac{[E(X) - \cancel{E(E(X))}]}{\sqrt{Var(X)}}\frac{[E(Y) - \cancel{E(E(Y))}]}{\sqrt{Var(Y)}} = \rho_{xy}$

63.

$f(x,y) = \frac{6}{7}(x+y)^2$

$0 \le x \le 1$; $0 \le y \le 1$

a) $Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = E[XY] - E[X]E[Y]$

$= \int_0^1\int_0^1 xy f(x,y)\,dx\,dy - \int_0^1 x f(x)\,dx \int_0^1 y f(y)\,dy$

$= \int_0^1\int_0^1 xy \frac{6}{7}(x+y)^2\,dx\,dy - \int_0^1 x\left[\int_0^1 \frac{6}{7}(x+y)^2\,dy\right]dx \int_0^1 y\left[\int_0^1 \frac{6}{7}(x+y)^2\,dx\right]dy$

$$= \int_0^1\int_0^1 xy\,\frac{6}{7}(x^2+2xy+y^2)\,dx\,dy - \int_0^1 x\,\frac{6}{7}(x^2+x+\tfrac{1}{3})\,dx \int_0^1 y\,\frac{6}{7}(\tfrac{1}{3}+y+y^2)\,dy$$

$$= \int_0^1 y\,\frac{6}{7}(\tfrac{1}{3}+y+y^2)\,dy - \frac{6}{7}\left[\frac{1}{4}+\frac{1}{3}+\frac{1}{6}\right]\frac{6}{7}\left[\frac{1}{6}+\frac{1}{3}+\frac{1}{4}\right] = \frac{6}{7}\left[\frac{1}{6}+\frac{1}{3}+\frac{1}{4}\right] - \frac{36}{49}\left(\frac{3}{4}\right) = \boxed{-0.0085}$$

$$\text{Corr}(X,Y) = \rho_{XY} = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{-19/34}{\sqrt{(E[X^2]-E[X]^2)(E[Y^2]-E[Y]^2)}} = \frac{-0.0085}{\sqrt{\left(\int_0^1 \frac{6}{7}[x^4+x^3+\frac{x^2}{3}]dx - \frac{81}{196}\right)\left(\int_0^1 \frac{6}{7}[\frac{y^2}{3}+y^3+y^4]dy - \frac{81}{196}\right)}}$$

$$= \frac{-19/34}{\sqrt{\left(\frac{6}{7}\left[\frac{1}{5}+\frac{1}{4}+\frac{1}{9}\right]\right)^2}} = \boxed{0.1256}$$ Miscalculated

b.

Find $E(Y|X=x)$ for $0 \le x \le 1$

Conditional Expectation:

$$E(Y|X=x) = \sum x\,P_{Y|X}(Y|X)$$

$$= \int_0^1 y\,\frac{f(x,y)}{f(X)}\,dx$$

$$= \int_0^1 \frac{y}{(x^2+x+1/3)}(x^2+2xy+y^2)\,dy$$

$$\boxed{= \frac{6x^2+8x+3}{4(3x^2+3x+1)}}$$

64.

		x: 1	2	3	4	$f(y)$
y	1	0.10	0.05	0.02	0.02	0.19
	2	0.05	0.20	0.05	0.02	0.32
	3	0.02	0.05	0.20	0.04	0.31
	4	0.02	0.02	0.04	0.10	0.18
	$f(x)$	0.19	0.32	0.31	0.18	

a. $\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$

$= \sum xy\,f(x,y) - \sum x f(x) \sum y f(y)$

$= 1^2\times0.1 + 2\cdot1\times0.05 + 3\times1\times0.02 + 4\times1\times0.02$
$+ 2\cdot1\times0.05 + 4^2\cdot0.2 + 6\times0.05 + 8\times0.02$
$+ 3\cdot1\times0.02 + 6\times0.05 + 3^2\cdot0.20 + 12\times0.04$
$+ 4\cdot1\times0.02 + 8\times0.02 + 3\cdot4\times0.04 + 4^2\times0.10$
$= 6.66$

$E(X) = 1\cdot0.19 + 2\cdot0.32 + 3\times0.31 + 4\times0.18$
$= 2.48 = E(Y)$

$E(X^2) = 1^2\times0.19 + 2\cdot0.32 + 3\times0.31 + 4\times0.18$
$= 7.14 = E(Y^2)$

$\text{Var}(X) = E(X^2) - E(X)^2$
$= 7.14 - 2.48^2 = 0.9896$

$\text{Var}(Y) = 0.9896$

Cov $= 0.5096$; Corr $= 0.514455$

b)

Find $E(Y|X=x)$ for $x = 1,2,3,4$;

$E(Y|X=1) = \sum y f(y|x) = \sum y \frac{f(x,y)}{f(x)} = \frac{1\times0.1}{0.19} + \frac{2\times0.05}{0.19} + \frac{3\times0.02}{0.19} + \frac{4\cdot0.02}{0.19}$ $\boxed{= 1.78}$

$E(Y|X=2) = \sum y f(y|2) = \sum y \frac{f(x,y)}{f(y)} = \frac{1\cdot0.05 + 2\times0.2 + 3\cdot0.05 + 4\cdot0.02}{0.32}$ $\boxed{= 2.13}$

$E(Y|X=3) = \sum y f(y|3) = \frac{1\cdot0.02 + 2\cdot0.05 + 3\cdot0.2 + 4\cdot0.04}{0.31}$ $\boxed{= 2.87}$

$E(Y|X=4) = \frac{1\times0.02 + 2\cdot0.02 + 3\times0.04 + 4\cdot0.1}{0.18} = \boxed{3.22}$

$E(Y|x) = \{1.78, 2.13, 2.87, 3.22\}$

65. Random Sums:

$$T = \sum_{i=1}^{N} X_i$$

$E(T) = E[E(T|N)] = E[NE(X)] = E(N)E(X) \leftarrow$ Independence

66. $\boxed{\text{Fast}}$ $\boxed{\text{Slow}}$; $E(T) = \sum E(T|E_i)P(E_i) = 1\,\text{min}\left(\frac{2}{3}\right) + 3\,\text{min}\left(\frac{1}{3}\right) = \boxed{\frac{5}{3}\text{ min}}$

1 min 3 min

$P(F) = 2/3$ $P(S) = 1/3$

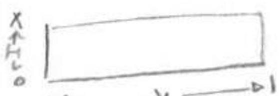
$E(XH) = E[E(XH|X)] = E[XE(H|X)] = E(X)E(H|X)$
$= \frac{1}{2}E(X^2)$
$= \frac{1}{2}\int_0^1 x^2\,dx = \boxed{\frac{1}{6}}$

$F(2(X+H)) = E[E(2(X+H)|X)]$
$= E[2X + 2E(H|X)]$

Note: $= E[2X] + E[2E(H|X)] = 2\left(\frac{a+b}{2}\right) + \left(\frac{a+b}{2}\right) = 1 + \frac{1}{2} = \boxed{\frac{3}{2}}$

$$E(Z) = \int_a^b z f(z)\,dz = \frac{1}{b-a}\int_a^b z\,dz = \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2}$$

67.



$E_{\text{Area}} = \sum E(Y|X=x)p_X(x)$

$= \sum_0^1 \sum_0^x y\,P_{Y|X}(y|x)\,P_X(x)$

68. Show $E[\text{Var}(Y|X)] \le \text{Var}(Y)$

$E[E(Y^2|X)] - E[E(Y|X)^2] \le E[Y^2] - E[Y]^2$

69. $\mu_X = \mu_Y = 0$; Sketch $E(Y|X=x)$ and $E(X|Y=y)$ for $\rho = 0, 0.5, 0.9$

$\sigma_X = \sigma_Y = 1$;

$E(Y|X) = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(X-\mu_X)$

@$\rho=0$ $= \cancel{0} + \cancel{0} = 0$

@$\rho=0.5$ $= \cancel{0} + \frac{1}{2}\left(\frac{1}{1}\right)(X-\mu_X) = \boxed{(X-\mu_X)/2} = X/2$

@$\rho=0.9$ $= \cancel{0} + \boxed{(X-\mu)/9} = X/9$

$E(X|Y) = \mu_X + \rho\frac{\sigma_Y}{\sigma_X}(Y-\mu_Y)$

@$\rho=0$ $E(X|Y) = 0$

@$\rho=0.5$ $E(X|Y) = Y/2$

@$\rho=0.9$ $E(X|Y) = Y/9.$

70. Show $E(X|Y=y) = E(X)$; $E(X|Y=y) = \sum x P_{X|Y}(x|y) = \sum x \frac{f(x,y)}{f(y)} = \sum \frac{x}{f(y)} f(x)f(y) = \sum x f(x) = \boxed{E[X]}$

71. Binomial Random Variable: n-trials, Y-success for m-trials ; $m<n$

$p(1) = p$; $p(0) = 1-p$; $p(x) = 0$

$$p(x) = \begin{cases} p^x(1-p)^{1-x}, & \text{if } x=0 \text{ or } x=1 \\ 0 & \text{otherwise} \end{cases}$$

Find the conditional frequency function of Y given $X=x$. and conditional mean.

$$p(x) = \binom{n}{x} p^x q^{n-x} ; \; p(y) = \binom{m}{y} p^y q^{m-y}$$

$$P(Y|X=x) = \frac{\binom{n}{x} p^x q^{n-x} - \binom{m}{y} p^y q^{n-y}}{\binom{n}{x} p^x q^{n-x}} p(X) = \frac{\binom{n-m}{x-y} p^{x-y} q^{n-m-x+y}}{\binom{n}{x} p^x q^{n-x}} p(Y)$$

Conditional Mean:

$$\frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}} = \frac{nK}{N}$$

Hypergeometric Fxn

$$\boxed{P(Y|X=x) = \frac{\binom{m}{y}\binom{n-m}{x-y}}{\binom{n}{x}}}$$

$= m\frac{x}{n} = E[Y|X=x]$

Distribution of a Hypergeometric Fxn

72. μ, σ^2 ; $E[X] = \mu$; $E[\mu] = \mu^2$; $E[\mu^2] = \mu^3$

$Var(X) = \sigma^2$; $Var(\sigma^2) = \sigma^4$; $Var(\sigma^4) = \sigma^6$

$$\frac{\binom{x}{k}\binom{n-x}{x-k}}{\binom{n}{m}}$$

73. n-tosses, N-heads, $+N$ more tosses.

Binomial Random Variable

First N-tosses:

$E[Heads] = E[Heads|n] + E[Heads|E[Heads|n]]$

$= np + np \cdot p = \boxed{np(1+p)}$

74. $\leftarrow n \rightarrow$

$o - p(n) \longrightarrow$ $E[X] = \sum_{i=1}^{n} x\,p(x) = \boxed{\frac{n(n+1)}{2} p(x)}$

75. $P(T) = \lambda e^{-\lambda T}$; $E[U \cdot E[T]] = E[U] \cdot E[T] = \boxed{\frac{1}{2} \cdot \frac{1}{\lambda}}$; $Var(U \cdot Var(T)) = Var(U) \cdot Var(T) = \boxed{\frac{1}{12}\left(\frac{1}{\lambda^2}\right)}$

76. $(X,Y): x^2+y^2 \le 1$; $y \ge 0$.

$f_{XY}(x,y) = \frac{|A|}{|R|} = \frac{1}{\pi/2} = \frac{2}{\pi}$

$f_X(x) = \int_0^{\sqrt{1-x^2}} f_{XY}(x,y)\,dy = \frac{2}{\pi}\sqrt{1-x^2}$

$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx = \frac{4}{\pi}\sqrt{1-y^2}$

$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{2/\pi}{\frac{2}{\pi}\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$

$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{1}{2\sqrt{1-x^2}}$

$E_{Y|X}(y|x) = \int_0^{\sqrt{1-x^2}} y\, f_{Y|X}(y|x)\,dy = \boxed{\frac{\sqrt{1-x^2}}{2}}$

$E_{X|Y}(x|y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x f(x|y)\,dx$

$= \emptyset$

77. $f(x,y) = e^{-y}$; $0 \le x \le y$ a) Find $Cov(X,Y) = E[XY] - E[X]E[Y] = \int_0^y\int_0^y xy f(x,y)\,dx\,dy - \int x\left[\int f(x,y)dy\right]dx \int y\left[\int f(x,y)dx\right]$

$= \int_0^\infty \left[\frac{x^2}{2}\right]_0^y y e^{-y} dy - \int_0^y x\left[\int_0^x e^{-y} dy\right] \int_0^\infty y\left[\int_0^y e^{-y} dx\right] dy$

$= \boxed{\int_0^\infty \frac{y^3}{2} e^{-y} dy + \int_0^y x e^{-x} dx \int_0^\infty y^2 e^{-y} dy = 3}$

b) $E(X|Y=y) = \int x f_{X|Y}(x|y)\,dx = \int_0^y x \frac{f(x|y)}{f(y)}\,dx$

$= \int_0^y x \frac{e^{-y}}{y e^{-y}}\,dx = \boxed{\frac{y}{2}}$

$E(Y|X=x) = \int y f_{Y|X}(y|x) = \int_x^\infty y \frac{e^{-y}}{1}\,dy = \boxed{x+1}$

$uv - \int v du$

$$Corr(X,Y) = \frac{cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{3}{\sqrt{(2-1)(6-2)}} = 1/\sqrt{2}$$

c) Density Function $E(X|Y)$ and $E(Y|X)$

$E[X] = E[E[X|Y]] = E[Y/2] = E[Y]/2$

$E[Y] = E[E[Y|X]] = E[X+1] = E[X]+1$

Skewness - third central moment - asymmetry of a density of frequency about a mean.

78. $M(t) = \sum e^{tx} p(x) = e^{tx}/t$ } 1st Moment

$M'(t) = E[X] = e^{tx}$

$M''(t) = E[X^2] = t e^{tx}$ } 2nd Moment

$M'''(t) = e^{tx} + t^2 e^{tx}$
$= (1+t^2)e^{tx}$ } 3rd Moment

$\emptyset = \lim_{t \to -\infty} M'''(t) - \lim_{t\to\infty} M'''(t)$

79. $p(0)=\frac{1}{2}$; $p(1)=\frac{3}{8}$; $p(2)=\frac{1}{8}$; Find $M(t)$; $M(t)=\sum_x e^{tx}p(x) = e^{tx}\left[\frac{1}{2}+\frac{3}{8}+\frac{1}{8}\right] = e^{tx}$

80. $f(x)=2x$; $0\le x\le 1$

$M'(0) = xe^{0x} = E[x]$; $M''(0) = x^2 e^{0\cdot x} = E(x^2)$ $\quad \frac{1}{2}$

$$M(t) = \int_0^1 e^{tx}\cdot 2x\,dx = \frac{2e^{tx}(tx-1)}{t^2}\Big|_0^1 = \frac{2e^{t}(x-1)}{t^2} + \frac{2}{t^2}$$

$$= \frac{2e^{t}(x-1)}{t^2} - \frac{2e^{0}(0-1)}{t^2} \qquad M''(0)=0$$

81. Bernoulli Random Variable.

$p(1)=p$
$p(0)=1-p$
$p(x)=0$

$p(x) = \begin{cases} p^x(1-p)^{1-x} \\ 0 \end{cases}$

$$M'(0) = \frac{d}{dt}\left[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right]\left[\frac{2}{t}-\frac{2}{t^2}\right]+\frac{2}{t^2}$$

$$= 2(1)\left(\frac{1}{2!}-\frac{1}{3!}\right)+2(2\times 0)\left(\frac{1}{3!}-\frac{1}{4!}\right)+2(3\times 0)\left(\frac{1}{4!}-\frac{1}{5!}\right)+\cdots$$

$$\boxed{=\frac{2}{3}}$$

$$M(t) = \sum_0^1 e^{tx}p(x) = 1-p+e^{t}p$$

$\boxed{M'(0)=p=E(x)}$ $\quad \boxed{M''(0)=p}$

$\boxed{M''(0)=p=E(x^2)}$

$\boxed{Var(x) = M'(0)-M''(0)^2 = p-p^2}$

82. Binomial Random Variable:

MGF of a Binomial is $\sum$ Bernoulli

$$M(t) = \prod_{i=1}^{n} M_x(t) = (1-p+e^{t}p)^n$$

$M'(0) = n(1-p+p)^{n-1}\cdot p = n\cdot p = E[x]$

$M''(0) = n(n-1)(1-p+p)^{n-2}p^2 + n(1-p+p)^{n-1}\cdot p = n(n-1)p^2+np = (n^2-n)p^2+np = n^2p^2-np^2+np$

$Var(x) = E[x^2]-E[x]^2 = \cancel{n^2p^2}-np^2+np-\cancel{n^2p^2} = \boxed{np(1-p)}$

83. Binomial Distribution:

$p(x)=\binom{n}{x}p^x(1-p)^{n-x}$; $P(x)=\sum_{i=1}^{n}X_i = X_1+X_2+\cdots+X_n = \binom{n}{1}p^1(1-p)^{n-1}+\binom{n}{2}p^2(1-p)^{n-2}+\cdots+\binom{n}{n}p^n(1-p)^{n-n} = \boxed{\sum_{n=1}^{n}\binom{n}{k}p^k(1-p)^{n-k}}$

84. $=\binom{n}{1}p_1(1-p_1)^{n-1}+\binom{n}{2}p_2(1-p_2)^{n-2}+\cdots+\binom{n}{n}p_n(1-p_n)^{n-n} \neq$ Binomial Distribution

85. Moment Generating Function: Geometric Random Variable $-$ $p(k)=p(1-p)^{k-1}$, $k=1,\ldots$

$$M(t)=\sum_{1}^{\infty}e^{tk}\cdot p(1-p)^{k-1} = \frac{e^{tk}p}{1-(1-p)e^{t}} \quad \text{Binomial Relationship} = e^{t}(1-p)^{k-1} - \frac{1}{1-qe^{t}};$$

$$M'(t) = \frac{ke^{t}p[1-(1-p)e^{t}] - e^{tk}p\cdot(1-p)e^{t}}{[1-(1-p)e^{t}]^2} = \frac{ke^{t}p-ke^{t}p(1-p)e^{t}+e^{t}p\cdot(1-p)e^{tk}}{[1-(1-p)e^{tk}]^2} = \frac{ke^{t}p[1+2(1-p)e^{tk}]}{[1-(1-p)e^{t}]^2}$$

$$= \frac{pe^{t}}{(1-qe^{t})^2}$$

$$M'(0) = \frac{p}{p^2} = \frac{1}{p}; \quad M''(t) = \frac{pe^{t}(1-qe^{t})^2 - pe^{t}(2(1-qe^{t})\cdot(-q)e^{t}}{(1-qe^{t})^4} = \frac{pe^{t}(1-qe^{t})+2pe^{t}(1-qe^{t})qe^{t}}{(1-qe^{t})^4}$$

$$M''(0) = \frac{p(1-q)+2p(1-q)q}{p^4} = \frac{p-pq+2p-2pq^2}{p^4} = \frac{3-q-2q^2}{p^3} = \frac{3-(1-p)-2(1-p)^2}{p^3}$$

$$= \frac{2+p-2(1+2p+p^2)}{p^3} = \frac{\cancel{2}+p-\cancel{2}-4p-2p^2}{p^3}$$

$Var(x)=E[x^2]-E[x]^2$

$$= \frac{1+q}{p^2} - \frac{1}{p^2} = \boxed{\frac{1-p}{p^2}}$$

86. MgF of a Negative Binomial

$p(X=k) = \binom{k-1}{r-1}p^r(1-p)^{k-r}$

$M(t) = \prod_{i=1}^{n} M_x(t) = p^n(1-qe^{t})^{-n}$

$M'(t) = -np^n(1-qe^{t})^{-(n-1)}(-qe^{t})$; $M'(0) = +np^n(1-q)^{-(n-1)}q$

$= np^n\cdot p^{-(n-1)}\cdot q = np^{n-n+1}\cdot q = \boxed{np^1q}$

$$\boxed{=\frac{1+q}{p^2}}$$

$$M''(t) = \frac{d}{dt}\left[\frac{+np^nqe^{t}}{(1-qe^{t})^{n-1}}\right] =$$

$$= \frac{nq}{p}+\frac{n^2q^2}{p^2}+\frac{nq^2}{p^2} = \boxed{\frac{nq}{p^2}+\frac{n^2q^2}{p^2}}$$

$$Var(x) = E(x^2)-E[x]^2 = \boxed{\frac{nq}{p^2}}$$

87. When is the sum of independent random variables of a binomial also a negative binomial?

Binomial Distribution: $p(k) = \binom{n}{k} p^k (1-p)^{n-k}$

Negative Binomial Distribution: $P(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$

When $k-1 = n$, and $r-1 = k$.
- When the total trials is the n and successes is k.

88. $X \sim N(0, \sigma^2)$; Prove odd moments of Mgf are zero and even moments are $\mu_{2n} = \frac{(2n)!\sigma^{2n}}{2^n (n!)}$

$$\frac{-x^2}{2\sigma^2} + tx = \frac{-1}{2\sigma^2}(x^2 - 2\sigma^2 t x)$$
$$= \frac{-1}{2\sigma}\left(x^2 - 2\sigma^2 t x + \left(\frac{2\sigma^2 t}{2}\right)^2\right) - (\sigma t)^2$$
$$= \frac{-1}{2\sigma^2}(x - \sigma^2 t)^2 - (\sigma^2 t)^2$$

Even Moments: $M^0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} N(0, \sigma^2)$

$$= \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{tx} e^{-x^2/2\sigma^2} dx = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2} + tx} dx = \frac{e^{(\sigma^2 t)^2}}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{\frac{-1}{2\sigma}(x - \sigma^2 t)^2} dx \; ; \quad u = x - \sigma t, \; \frac{du}{dt} = -1$$

$$= \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{\frac{-1}{2\sigma}(x-\sigma^2 t)^2 - (\sigma^2 t)^2} dx = \frac{e^{-(\sigma^2 t)^2}}{\sqrt{2\pi\sigma}(+\sigma)} \int_{-\infty}^{\infty} e^{-u^2/2\sigma^2} du = \frac{e^{+\frac{(\sigma^2 t)^2}{2}}}{\sqrt{2\pi}\,\cancel{\sigma}} \sqrt{2\pi\sigma} \cdot \frac{+1}{\sigma} = \frac{+e^{+\frac{1}{2}\sigma^2 t^2}}{-\sigma^2}$$

$$\mu_{2n} = \frac{(2\cdot 0)!\sigma^{2\cdot 0}}{2^0 (0!)} = 1$$

$$M'(t) = +\sigma^2 t e^{-\sigma^2 t^2/2} \quad \cancel{=0}$$

$$M''(t) = +\sigma^2 e^{-\sigma^2 t^2/2} + \sigma^2 t(-\sigma^2 t) e^{-\sigma^2 t^2/2} = -\sigma^2 e^{-\sigma^2 t^2/2}\left[1 - \sigma^2 t^2 e^{-\sigma^2 t^2/2}\right] = \boxed{+\sigma^2} \text{ @ } n = 1$$

$$\frac{(2\cdot 1)!\sigma^{2n}}{2^1(1!)} \quad \boxed{= \sigma^2}$$

89. $X_1, X_2 \ldots X_n = N(\mu_i, \sigma_i^2)$; Prove $Y = \sum_{i=1}^{n} \alpha_i X_i$; where α_i = scalar. Find μ_Y, σ_Y^2

$Y = \sum_{i=1}^{n} \alpha_i X_i = \sum_{i=1}^{n} \alpha_i N(\mu_i, \sigma_i^2)$; MGF of Y: $M(t) = \sum_{i=1}^{n} e^{tx} \cdot \sum \alpha_i X_i$

Completing the Square:

$$tx - \frac{(x-\mu)^2}{2\sigma^2} = tx - \frac{x^2 + 2\mu x + \mu^2}{2\sigma^2}$$

$$\frac{-x^2 + (2\mu + 2\sigma^2 t)x}{2\sigma^2} + \frac{\mu^2}{2\sigma^2}$$

$$\frac{-1}{2\sigma^2}(x - (\mu + \sigma^2 t))^2 - \frac{(\mu + \sigma^2 t)^2}{2\sigma^2} + \frac{\mu^2}{2\sigma^2}$$

$$\frac{-1}{2\sigma^2}(x - (\mu + \sigma^2 t)^2)^2 - \frac{(2\mu + \sigma^2 t^2)}{2}$$

$$= \sum_{x=1}^{n} e^{tx}\left(\sum \alpha_i N(\mu_i, \sigma_i^2)\right)$$

$$= \sum_{x=1}^{n} \sum_{i=1}^{n} \alpha_i e^{\frac{-1}{2\sigma^2}(x_i - (\mu + \sigma^2 t)^2)^2} \cdot e^{-\frac{(2\mu + \sigma^2 t^2)}{2}}$$

$$= e^{-\frac{(2\mu + \sigma^2 t^2)}{2}} \cdot \sum_{x=1}^{n} \sum_{i=1}^{n} \alpha_i e^{\frac{-1}{2\sigma^2}(x_i - (\mu_i + \sigma_i^2 t)^2)^2}$$

$$= \sum_{i=n}^{n} \alpha e^{-(2\mu + \sigma^2 t^2)/2} \cdot \sum_{x=1}^{n} e^{\frac{-1}{2\sigma^2}(x_i - (\mu_i + \sigma_i^2 t)^2)^2}$$

$$= M_Y(t) = \sum_{x=1}^{n} e^{tx} e^{-y^2/2} = e^{t^2/2}$$

$$= \sum_{x=1}^{n} e^{tx} \sum_{i=1}^{n} \alpha_i N(\mu_i, \sigma_i^2) = e^{t^2/2} \sum \sum \alpha_i N(\sigma_i^2 t, \sigma_i^2) \quad \boxed{= e^{t^2/2}}$$

90. $Z = \alpha X + \beta Y$; $M_Z(t) = E(e^{tZ})$

$$= E(e^{t\alpha X + t\beta Y}) = E(e^{t\alpha X} \cdot e^{t\beta Y})$$
$$= E(e^{t\alpha X}) E(e^{t\beta Y})$$
$$\boxed{= M_X(t) M_Y(t)}$$

91. Exponential Distribution:

$$p(x) = \lambda e^{-\lambda x}$$
$$M_X(t) = \lambda e^{(t-\lambda)x}$$
$$M_{cX}(t) = \lambda e^{(t - c\lambda)x}$$

92. Gamma Distribution: ; Poisson Distribution ; With Example:

$$g(t)=\frac{\lambda^\alpha}{\Gamma(\alpha)}t^{\alpha-1}e^{-\lambda t} \quad p(x=k)=\frac{\lambda^k}{k!}e^{-\lambda} \quad f(\theta)=\frac{\lambda^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\lambda\theta} ; X|\theta=\frac{\theta^x}{x!}e^{-\theta}$$

$$P(X=x)=\int_0^\infty P(X|\theta)f(\theta)d\theta=\int_0^\infty e^{-\theta}\frac{\theta^x}{x!}\frac{\lambda^\alpha\theta^{\alpha-1}e^{-\lambda\theta}}{\Gamma(\alpha)}d\theta=\int_0^\infty\frac{\lambda^\alpha\theta^{x+\alpha-1}e^{-(\lambda+1)\theta}}{x!(\alpha-1)!}d\theta$$

$$=\frac{(x+\alpha-1)!}{x!(\alpha-1)!}\frac{\lambda^\alpha}{(\lambda+1)^{x+\alpha}}\int_0^\infty\frac{(\lambda+1)^{x+\alpha}\theta^{x+\alpha-1}e^{-(\lambda+1)\theta}}{\Gamma(x+\theta)}d\theta ;$$ Rate $=(\lambda+1)$ Shape $=\alpha+x$

$$M_{X|\theta}(t)=E[e^{tX}|\theta]=\exp((e^t-1)\theta) \text{ and } M_\theta(m)=E[e^{m\theta}]=(1-m/\lambda)^{-\alpha}$$

$$M_X(t)=E[e^{tX}]=E[E[e^{tX}|\theta]]=E[M_{X|\theta}(t)]=E[\exp((e^t-1)\theta)]=M_\theta(e^t-1)=\left(1-\frac{e^t-1}{\lambda}\right)^{-\alpha}$$

93. Geometric Sum: Exponential Random Variable

$$M_X(t)=\left(\frac{\lambda/(1+\lambda)}{1-e^t(1-\lambda/[1+\lambda])}\right)^\alpha \underbrace{\frac{\lambda}{\lambda+1}}_{\text{Negative Binomial}}$$

$$X_r=X_1+X_2+\cdots+X_n \quad p(x)=\lambda e^{-\lambda x}$$

$$M(t)=\sum_{x=0}^{\infty}\sum_{1}^{n}X(x)_{i,j,t}e^{tx}=\sum_{x=0}^{\infty}\lambda e^{-(\lambda+t)x_i}=\boxed{\frac{-\lambda}{(\lambda+t)}e^{-(\lambda+t)x}}$$

94. Probability Generating Function: $G(s)=\sum_{k=0}^{\infty}s^k p_k$; where $p_k=P(X=k)$

a) Show $p_k=\frac{1}{k!}\frac{d^k}{ds^k}G(s)\Big|_{s=0}$; Fundamental theorem of Calculus: $\int_a^b f(x)dx=F(b)-F(a)=\frac{d}{dx}F(x)$

b). Show $\frac{dG}{ds}=E(X)$ $\quad \frac{dG}{ds}=ks^{(k-1)}p(k)=E(X)=\boxed{k\cdot p(k)}$

$\frac{d^2G}{ds^2}\Big|_{s=1}=E[X(X-1)]$ $\quad \frac{d^2G}{d^2s}=k(k-1)s^{(k-2)}p(k)=\boxed{k(k-1)p(k)}$

c) $$M(t)=\sum_{s=0}^{\infty}\sum_{s=0}^{\infty}e^{tk}G(s)=\sum_0^\infty\sum_0^\infty e^{tk}s^k p_k=\boxed{\sum\frac{e^{tk}s^{k+1}p(k)}{k+1}}=$$

d) $$G(s)=\int_0^\infty s^k\frac{\lambda^k}{k!}e^{-\lambda}dk=e^{-\lambda}\sum_0^\infty\frac{(s\lambda)^k}{k!}=e^{-\lambda}e^{\lambda s}=\boxed{e^{(\lambda s-\lambda)}}$$

95. Joint Moment Generating Function $M(t)=\sum_{-\infty}^{\infty}e^{t(x+y)}xy=\sum e^{tx}x\,e^{ty}y=\boxed{M_x(t)M_y(t)}$

96. $E(XY)=M'(0)=\frac{d}{dt}[\sum e^{tx}e^{ty}p]=xy\,p(x,y)$ $\quad \boxed{M(t)=\sum e^{tX+tY}p(x) ; M'(t)=\sum(x+y)e^{tX+tY}p(x)}$

97. $Var(aX+bY)=a^2Var(X)+b^2Var(Y)$; $\quad M''(t)=\sum(x+y)^2e^{tx+ty}p(x)$

$$M''(0)=(x+y)^2p(x)=E[X^2] ; M'(0)=(x+y)p(x)=E[X]$$

$$=E[X^2]-E[X]^2=(x+y)^2p(x)-(x^2+y^2)p(x)^2=(x^2+2xy+y^2)(p(x)-p(x)^2)$$

$$=\boxed{Var(X)+Var(Y)}$$

98. Compound Poisson Distribution

$$M_S(t)=\exp[\mu(e^{\lambda(e^t-1)}-1)]$$

$$M_S'(0)=[\mu(e^{\lambda(e^t-1)}-1)]'\exp[\mu(e^{\lambda(e^t-1)}-1)]$$

$$=[\lambda(e^t-1)]'[\mu(e^{\lambda(e^t-1)}-1)]\exp[\mu(e^{\lambda(e^t-1)}-1)]$$

$$=\lambda e^t[\mu(e^{\lambda(e^t-1)}-1)]\exp[\mu(e^{\lambda(e^t-1)}-1)]=\emptyset=E[X]$$

$$M_S''(0)=\lambda[\mu(e^{\lambda(e^t-1)}-1)]\exp[\mu(e^{\lambda(e^t-1)}-1)]+\lambda^2e^{2t}[\mu(e^{\lambda(e^t-1)}-1)]\exp[\mu(e^{\lambda(e^t-1)}-1)]$$
$$+(\lambda e^t[\mu(e^{\lambda(e^t-1)}-1)]\exp[\mu(e^{\lambda(e^t-1)}-1)]$$

$$=\emptyset$$

99. $Y=g(X)$ a) $g(x)=\sqrt{x}$

$$E[Y]=\int_0^\infty x\sqrt{x}\,dx=x(\tfrac{2}{3})x^{3/2}\Big|_0^\infty-\int_0^\infty\sqrt{x}\,dx=\infty$$

$Var[Y]=\infty$

b) $E[X] = \int_0^\infty x \log x \, dx = x\left(\frac{1}{x}\right)\Big|_0^\infty - \int_0^\infty \log(x)\,dx = 1 - \frac{1}{x}\Big|_0^\infty = \text{Undefined}$

$Var(X) = \text{Undefined}$

c) $g(x) = \sin^{-1}(x)$; $E[X] = \int_{-\pi/2}^{\pi/2} x \sin^{-1}(x)\,dx$ = Does not converge $Var(X)$ = Does not converge

100. $X[10, 20]$; $Y = 1/X$; $E[X] = \int_{10}^{20} \frac{1}{10}\left(\frac{1}{x}\right)dx = \frac{\ln 20 - \ln 10}{10} = 0.0693$; $E[Y^2] = \int_{10}^{20} \frac{1}{10}\left(\frac{1}{x^2}\right)dx = \int_{10}^{20} \frac{1}{10x^2}\,dx = \frac{-x^{-1}}{10}\Big|_{10}^{20} = 0.005$

$\boxed{\text{Exact Method}}$

$Var(x) = E[X^2] - E[X]^2 = 0.005 - 0.0693 = 0.000198$

$\boxed{\text{Approximate Method}}$

$Y(x) = \frac{1}{x}$; $Y'(x) = \frac{-1}{x^2}$; $Y''(x) = \frac{-2}{x^3}$; $E(Y) \sim g(\mu_X) + \left(\frac{1}{2}\right)\sigma_x^2 g''(\mu_X) = \frac{1}{15} + \left(\frac{1}{2}\right)8.33(0.0006)$ $\boxed{= 0.0249}$

$Var(Y) \approx \sigma_x^2 [g'(\mu_X)]^2 = 0.000161$

$\sigma_x^2 = \frac{(b-a)^2}{12} = 8.33$

$\sigma_x = 0.0127$

101. $Y = \sqrt{X}$; X = Poisson Distribution

$\frac{\lambda^k}{k!} e^{-\lambda}$; $Y'(x) = \frac{1}{2}(x)^{-1/2}$

$Y''(x) = -\frac{1}{4}(x)^{-3/2}$

$Y'''(x) = \frac{-3}{8}(x)^{-5/2}$

$\mu_X = \lambda, \sigma_x^2 = \lambda$

$E(Y) \sim Y(\mu_X) + \frac{1}{2}(\sigma_x^2) g''(\mu_X)$

$\sim \lambda^{1/2} - \frac{1}{2}\lambda \cdot \frac{1}{4}\lambda^{-3/2}$

$\boxed{\sim \sqrt{\lambda} - \frac{1}{8\sqrt{\lambda}}}$

$Var(Y) \approx \sigma_x^2 [g'(\mu_X)]^2$

$\approx \lambda\left[\frac{1}{2}(\lambda)^{-1/2}\right]^2$

$\approx \boxed{\frac{1}{4}}$

102.

$X_0 = X$

$E(X) = X_0$

$y_0 = Y$; $E(Y) = y_0$; $Var(X) = Var(Y) = \sigma^2$; $E(\theta) \sim \tan^{-1}\left(\frac{E(Y)}{E(X)}\right) = \boxed{\tan^{-1}\left(\frac{y_0}{x_0}\right)}$

$\theta = \tan^{-1}\left(\frac{Y}{X}\right)$; $E(\theta) \sim$

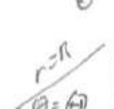
$Var(\theta) \sim \tan^{-1}\left(\frac{Var(Y)}{Var(X)}\right) = \tan^{-1}(1) = \boxed{45°}$

103. $V = \frac{\pi}{6} D^3$; $D = 2\text{mm}$; $\sigma_x = 0.01\text{mm}$; $V' = \frac{\pi}{2} D^2$; $\sigma_z^2 \approx \sigma_x^2 [g'(\mu_X)]^2$; $\sigma_z \approx \sigma_z g'(\mu_X)$

$\approx 0.01\text{mm} \cdot \frac{\pi}{2} 2^2 \text{mm}^2$

$\boxed{\approx 6.28 \times 10^{-2} \text{mm}^3}$

104.



$Y = R\sin\theta$; a) $Var(Y) \sim \sigma_x^2 [g'(\mu_X)] \sim \boxed{\sigma_x^2 [R\cos\theta]}$

b) $\frac{dVar(Y)}{d\theta} = \sin\theta = 0$; $\boxed{90° = \theta}$

Chapter 5: 1) $X_1, X_2 \ldots$; $E(X_i) = \mu$; $Var(X_i) = \sigma_i^2$. Show $n^{-2}\sum_{i=1}^{n} \sigma_i^2 \to 0$; $\bar{X} \to \mu$

Law of Large Numbers: $P(|\bar{X} - \mu| > \epsilon) \leq \frac{Var(\bar{X}_n)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \to 0$

$P(|\bar{X} - \mu| > \epsilon) \leq \emptyset = E[X^2] - E[X]^2$; $\bar{X}^2 = \mu^2$; $\boxed{\bar{X} = \mu}$

2. $E(X_i) = \mu_i$; if $E(X_i) = \frac{1}{n}\sum_1^n E(X_i) - \mu$

$= \frac{1}{n}(n E(X_i)) = E(X_i) = \mu$

3. Number of Insurance claims, N, is a Poisson Distribution: $p(x) = \frac{\lambda^k}{k!} e^{-\lambda}$

$E(N) = 10{,}000$.

$E(X) = \lambda$

Apply a normal approximation to the Poisson to approximate $P(N > 10{,}200)$.

$Var(X) = \lambda$

Standardizing Random Variables:

$Z_n = \frac{X_n - E(X_n)}{\sqrt{Var(X_n)}} = \frac{10{,}200 - 10000}{\sqrt{10000}} = 102 - 100 = 2$

$P(Z_n) = 1 - 0.9772 = \boxed{0.028}$

4. Number of Traffic Accidents (N) is $E(N) = 100$

Find Δ if a poisson $P(100 - \Delta < N < 100 + \Delta) \approx 0.9$

$Z_n = \frac{X_n - E(X_n)}{\sqrt{Var(X)}}$; $P(100 - \Delta) = \frac{100 - \Delta - 100}{\sqrt{100}} = 10 - 10 - \Delta = -\Delta$

$P(N) = \frac{X_n - 100 + \Delta}{\sqrt{100}} = \Delta$; $\boxed{\Delta = 1.3}$

Means 1.3 cars more or less for probability of 90%

5. $n \to \infty, p \to \infty$, and $np = \lambda \to \infty$ Binomial Distribution: $p(x) = \binom{n}{k} p^k (1-p)^{n-k}$; n and p tend poisson $p(k) = \frac{\lambda^k}{k!} e^{-\lambda}$

Moment Generating Function of Binomial: $M(t) = \int_0^\infty e^{tk} \binom{n}{k} p^k (1-p)^{n-k} dk$; if $np = \lambda$

$= \binom{n}{k} \int_0^\infty e^{tk} \cdot \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} dk$; $\boxed{\text{Continuity Theorem}}$ $\lim_{n\to\infty} M_n(t) \to M(t)$; $\lim_{n \to \infty} F_n(x) = F(x)$ $\boxed{\text{Law of Large Numbers}}$

$$\lim_{n\to\infty} M_n(t) = \lim_{n\to\infty} \int_0^\infty \binom{n}{k} e^{tk} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k} dk = \lim_{n\to\infty} \int_0^\infty e^{tk} \frac{\lambda^k}{k!} \underbrace{\frac{n!}{(n-k)!} \frac{1}{n^k}}_{=1} \underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{=e^{-\lambda}} \underbrace{\left(1-\frac{\lambda}{n}\right)^{-k}}_{=1} dk = \int_0^\infty e^{tk} \frac{\lambda^k}{k!} e^{-\lambda} dk$$

$$= \underbrace{\int_0^\infty e^{tk} \frac{\lambda^k}{k!} e^{-\lambda} dk}_{\text{Poisson Distribution}} = \int_0^\infty \frac{(\lambda e^t)^k}{k!} e^{-\lambda} dk = e^{\lambda e^t} \cdot e^{-\lambda} = \boxed{e^{\lambda(e^t - 1)}}$$

6. $\alpha \to \infty$ Gamma Distribution: $\frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t}$

$\to M_\alpha(t) = \int_0^\infty e^{tx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx$; $\lim_{\alpha\to\infty} M_\alpha(t) = \lim_{\alpha\to\infty} \int_0^\infty e^{tx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx$

$= \lim_{\alpha\to\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{x(t-\lambda)} dx = \lim_{\alpha\to\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} \left(\frac{\Gamma(\alpha)}{(\lambda - t)^\alpha}\right) = \left(\frac{\lambda}{\lambda - t}\right)^\alpha = \infty$

If $\mu = \lambda/t$, $\sigma^2 = k/t^2$? $= \left(\frac{\mu t}{\mu t - k/t^2}\right)^k$

$\boxed{= \left(1 - \frac{t}{\lambda n}\right)^{-n}}$

7. $X_n \to c$; g is continuous, then $g(X_n) \to g(c)$

Continuity Theorem $\boxed{\lim_{X_n \to c} g(X_n) = g(c)}$

8. Poisson Cumulative Distribution: a) $\lambda = 10$; $cdf_{Poisson} = \int_0^\infty \frac{\lambda^k}{k!} e^{-\lambda} dk = e^{\lambda} \cdot e^{-\lambda}$; 1 b) 1 c) 1 ≈ Normal Standard

9. Binomial Cumulative Distribution: a) $n = 20$, $p = 0.2$ $CDF_{Binomial} = \int_0^\infty \binom{n}{k} p^k (1-p)^{n-k} dk = \int_0^\infty \frac{20!}{k!(n-k)!} 0.2^k (0.8)^{n-k} dk = 20!\,0.8^{20} \int_0^\infty \frac{1}{4^k} \frac{dk}{k!(20-k)!}$

$\boxed{= 8.89 \times 10^{-1}}$

b) $n = 40$, $p = 0.5$ $= 40!\, 0.5^{40} \int_0^\infty \frac{dk}{k!(40-k)!} = 0.99$

The binomial converges to the normal standard when correct value

$\left(\frac{40 - 40 \cdot 0.5}{\sqrt{40 \cdot 0.5 (1-0.5)}}\right) \approx$?

Normal Approximation $\left(\frac{\lambda - E(\lambda)}{\sqrt{Var}}\right) = \frac{0.2 - 20 \cdot 0.2}{\sqrt{20 \cdot 0.2 (1-0.2)}}$

$= \varnothing$

10. Six sided dice n = 100 ; $\left(\frac{X - E(X)}{\sqrt{Var}}\right) = P(x)$; $P(15 < X < 20) = \left(\frac{15 - 100(1/6)}{\sqrt{100(\frac{1}{6})(1 - 1/6)}} < X < \frac{20 - 100(1/6)}{\sqrt{100(1/6)(1-1/6)}}\right)$ ~~= 35% to 23%~~

$\approx P(15.5 < X < 19.5) = P\left(\frac{15.5 - 100/6}{\sqrt{100/6(1 - 1/6)}} < X < \frac{19.5 - 100/6}{\sqrt{100/6(1-1/6)}}\right)$

$\approx P(-0.31 < Z < 0.76) = P(Z < 0.76) - P(Z < -0.31)$

$= P(Z < 0.76) - 1 + P(Z < 0.31) = 0.774 - 1 + 0.6217 = 0.4$

Central Limit Theorem:

$S_n = \sum_{i=1}^n X_i$; $\lim_{n\to\infty} P\left(\frac{S_n}{\sigma\sqrt{n}} \le x\right) = \Phi(x)$; $-\infty < x < \infty$

Proof: $Z_n = S_n / (\sigma\sqrt{n})$

$M_{S_n}(t) = [M(t)]^n$; $M_{Z_n}(t) = \left[M\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$

$M(s) = M(0) + sM'(0) + \frac{1}{2}s^2 M''(0) + \epsilon_s$

$\epsilon_s / s^2 \to 0$ as $s \to 0$; $E[X] = 0$, $M'(0) = 0$, $M''(0) = \sigma^2$

As $n \to \infty$, $t/(\sigma\sqrt{n}) \to 0$

$M\left(\frac{t}{\sigma\sqrt{n}}\right) = 1 + \frac{1}{2}\sigma^2 \left(\frac{t}{\sigma\sqrt{n}}\right)^2 + \epsilon_n$

$M_Z(t) = \left(1 + \frac{t^2}{2n} + \epsilon_n\right)^n$; $\lim_{n\to\infty} \left(1 + \frac{a_n}{n}\right)^n = e^a$

$M_{Z_n}(t) = e^{t^2/2}$ $n \to \infty$

$E[X] = \frac{6+1}{2} = 3.5$; $Var(X) = \frac{1}{12}(6^2 - 1) = 2.917$

$E[S] = 100\,E[X] = 100(3.5) = 350$; $Var(S) = 100(2.917) = 291.67$

$P(S < 300) \approx P(S < 299.5) = P\left(Z < \frac{299.5 - 350}{\sqrt{291.67}}\right)$

$= P(Z < -2.96) = 1 - P(Z < 2.96) = \boxed{0.00154}$

11. The argument suffices to say $\lambda = 1 = np = \infty \cdot p \Rightarrow \frac{1}{p} = \infty$; The probability must approach $\emptyset$ as $n \to \infty$.

12. Uniform Random Variable $[-\frac{1}{2}, \frac{1}{2}]$. $n=100$; $P(X>1) = Prob\left(\frac{X - \sum E(X_i)}{\sqrt{\sum Var(X)}}\right) = P\left(\frac{X-100\cdot 0}{\sqrt{100 \cdot 1/12}} > \frac{1-0}{\sqrt{100/12}}\right) = P\left(\frac{\sqrt{3}}{25}Y > \frac{\sqrt{3}}{25}\right)$

$E[X] = \frac{a+b}{2}$; $Var(X) = \frac{(b-a)^2}{12} = \frac{(\frac{1}{2} - -\frac{1}{2})^2}{12}, \frac{1}{12}$

$= 1 - \phi\left(\frac{\sqrt{3}}{5}\right)$

$= 0.3645.$

b) $P(Y>2) = P\left(\frac{Y-0}{\sqrt{\frac{25}{3}}} > \frac{2-0}{\sqrt{\frac{25}{3}}}\right) = 1 - \phi\left(\frac{2\sqrt{3}}{5}\right) = 0.2442$

c) $P(Y>5) = P\left(\frac{Y-0}{\sqrt{\frac{25}{3}}} > \frac{5-0}{\sqrt{\frac{25}{3}}}\right) = 1 - \phi\left(\frac{5\sqrt{3}}{5}\right) = \boxed{0.0416}$

13.
$P(North) = 1/2$; Step Length $= 50cm$ $E(X) = \sum X \cdot P(X) = 50(\frac{1}{2}) + (-50)(\frac{1}{2}) = \emptyset$; $Var(X) = E[X^2] - E[X]^2 = \sum x^2 P(x) + \emptyset^2 = 50^2(\frac{1}{2}) + (-50)^2(\frac{1}{2}) = 2500$

$\underbrace{P(South) = 1/2}_{\text{per minute}}$; Approximate probability after 1h.

$E(S) = \sum^{60} E(X) = 60 \cdot \emptyset = \emptyset$ $Var(S) = \sum_1^{60} Var(X) = 60 \cdot 2500 = 150{,}000$

Standard Deviation $(\sqrt{Var(s)}) = \sigma_s = \sqrt{150000} = 387$

14. $P(North) = \frac{2}{3}$

$P(South) = \frac{1}{3}$

$E[X] = \frac{1}{3}(-50) + \frac{2}{3}(+50) = \frac{50}{3}$; $E[X^2] = \frac{1}{3}(-50)^2 + \frac{2}{3}(50)^2 = 2500$; $Var[X] = E[X^2] - E[X]^2 = 2500 - 1000 = 1500$

$E(S) = \sum_0^{60} E[X] = \frac{3000}{3} = 1000$;

$E[S^2] = \sum_0^{60} E[X] = 60 \cdot 2500 = 150{,}000$

$Var(S) = \sum_0^{60} Var[X] = 60 \cdot 1500 = 90{,}000$

Standard Deviation $(\sqrt{Var}) = \sigma = \boxed{300}$

15. $n = 50$; Amount $= \$5$

Probability (Loss > \$75).

$P(Win) = \frac{1}{2}$; $P(Lose) = 1/2$

$\sum_1^{50} X_i < -75$; $\bar{X} < \frac{-75}{50} = -1.5$; $E(X) = 0$; $Var(X) = E(X^2) - E(X)^2 = \sum x^2 p(x) - 0 = 5^2(\frac{1}{2}) + (-5)^2(\frac{1}{2}) = 25$

$P(\bar{X} < -1.5) = P\left(Z \le \frac{-1.5-0}{\sqrt{1/2}}\right) = P(Z < -2.12) = 1 - P(2.12) = 0.017$

16. $X_1, \ldots, X_{20}$; $f(x) = 2x$; $0 \le x \le 1$; $S = X_1 + \cdots + X_{20}$: $P(S \le 10)$; $E[X] = \int_0^1 x f(x) dx = \int_0^1 x \cdot 2x dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$

$E[S] = \sum^{20} \frac{2}{3} = \frac{40}{3}$; $E[X^2] = \int_0^1 x^2 2x dx = \frac{2x^4}{4}\Big|_0^1 = \frac{1}{2}$

$P(S \le 10) = P\left(\frac{S - E[S]}{\sqrt{Var(S)}} < \frac{10 - E[S]}{\sqrt{Var(S)}}\right) = P\left(\frac{X - 40/3}{\sqrt{1.11}} < \frac{10 - 40/3}{\sqrt{1.11}}\right)$

$Var(X) = E[X^2] - E[X]^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$

$= P\left(\frac{S - 40/3}{\sqrt{1.11}} < -3.16\right) = 1 - (1 + \phi(3.16)) = \boxed{0.9992}$

$Var(S) = \sum_0^{20} var[X] = \frac{20}{18} = 1.11.$

17.
$\mu, \sigma^2 = 25$; $P(|\bar{X} - \mu| < 1) = P(-1 < \bar{X} - \mu < 1) = P\left(-\sqrt{n}/\sigma < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \sqrt{n}/\sigma\right) = 0.95$

$\underbrace{P\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < \sqrt{n}/\sigma\right) = \frac{1 - 0.95}{2} = 0.025}_{\text{"Describes slice of range"}}$; $-\frac{\sqrt{n}}{\sigma} = -1.97$; $n = 97$.

18. $\mu = 15 lbs$; $P(\Sigma S < 1700 lbs) = P\left(\frac{S - E[S]}{\sigma/\sqrt{n}} < \frac{1700 - n\mu}{\sigma/\sqrt{n}}\right) = P\left(\frac{S - E[S]}{\sigma/\sqrt{n}} < \frac{1700 - 1500}{10/\sqrt{100}}\right)$

$\sigma = 10$;

$= P(2) = 0.9772$

$n = 100$

19. a) $n=100$, $n=1000$; $f(x)=\int_0^1 \cos(2\pi x)dx = \frac{\sin(2\pi x)}{2\pi}\Big|_0^1 = \emptyset$; $\hat{I}_{100}(f)=\frac{1}{n}\sum_{i=0}^{1} f(x_i) = \frac{1}{100}\sum_{i=0}^{1}\cos(2\pi x) = \frac{1}{100}+\frac{1}{100}=\frac{2}{100}$

$\hat{I}_{1000}(f)=\frac{1}{1000}\sum_{i=0}^{1} f(x) = \frac{1}{1000}\sum_{0}^{1}\cos(2\pi x) = \frac{1}{1000}+\frac{1}{1000}=\frac{2}{1000}$

b) $I(f)=E[X]=\int_0^1 \cos(2\pi x^2)\,dx =$

$\hat{I}(f) = \frac{1}{n}\sum_{i=0}^{1}\cos(2\pi x^2) = \frac{1}{10^3}\sum_{i=0}^{10^3}\cos(2\pi x^2) = 1 \to 0$

Exact Solution : Cosine integral: $\int \cos(u)dx$; $u=2\pi x^2$, $\frac{du}{dx}=2\pi x$ $\int \frac{\cos(u)}{2\pi x}dx = \int \frac{\cos(u)}{2\pi\sqrt{u}/2\pi}du = \int_0^1 \frac{\cos(u)}{\sqrt{u}}du = \int_0^1 \frac{\cos(2u)}{u}du$

$= \frac{C(2)}{2} = \boxed{0.24}$

20. $E(\hat{I}^2(f)) = \left(\frac{1}{1000}\right)\left[\left(\frac{1}{\sqrt{2\pi}}\right)\sum_{i=1}^{1000} e^{-x^2/2}\right]^2$

$= \frac{1}{1000}\,\frac{1}{2\pi}\sum_{i=1}^{1000}\left(e^{-x_i^2}\right) = \frac{0.386}{2\pi\times10^3} = 6.14\times10^{-5}$

$Var(\hat{I}(f)) = E[\hat{I}^2(f)] - E[\hat{I}(f)]^2 = \left(\frac{1}{1000^2}\sum_{i=1}^{1000}(e^{-x^2/2})^2 - \frac{1}{1000^2}\left[\sum_{i=1}^{1000} e^{-x^2/2}\right]^2\right)\frac{1}{2\pi}$

21. $I(f)=\int_a^b f(x)dx$; $\hat{I}(f) = \frac{1}{n}\sum_{i=1}^{n}\frac{f(x_i)}{g(x_i)}$ a) Show $E(\hat{I}(f)) = I(f) = \int_a^b f(x)/g(x)\,dx$ $\boxed{\frac{1}{n}\sum_{i=1}^{n} f(x)/g(x)}$

b) $Var(\hat{I}(f)) = E[\hat{I}(f)^2] - E[\hat{I}(f)]^2$; $n=100$; $n\to\infty$

c) $E(\hat{I}(f)) = \hat{I}(f) = \int_0^1 \frac{f(x)}{g(x)}dx = \int_0^1 h(x)dx = \int_0^1 e^{-x^2/2}dx \to$ Example A: Section 5.2.

22.

Find Δ ; $P(|\hat{I}(f)-I(f)| \le \Delta) = 0.05$, where $\hat{I}(f)$ is the Monte Carlo Estimate of $\int_0^1 \cos(2\pi x)dx$ based upon $n=1000$

$P\left(\left|\frac{1}{1000}\sum_{i=1}^{1000}\cos(2\pi x) - \int_0^1 \cos(2\pi x)dx\right| \le \Delta\right) \simeq 0.05$

$P(|1| \le \Delta) = 0.05$; $P(\Delta) = 0.05 - P(1) = 0.05 - 0.84 = -0.79$; $\boxed{\Delta = 0.81}$

23. $P(\Delta) - P(1) = 0.05$

$0\le x\le 1$; $0\le y\le 1$; Random (x,y) ; $Z=1$ if so, $Z=0$ otherwise. Prove $E(Z)=A=\sum_0^1\sum_0^1 xy\cdot Z(1) + \sum_0^1\sum_0^1 xy\,Z(0)$

24. $\hat{A}$; $E(S)=\sum_{i=1} E[Z] = \boxed{nE[Z]}$; $\hat{A}=nE[Z]=nA$; $P(|\hat{A}-A|<0.1)\approx 0.99$ $\boxed{= 1.1A}$

$P(|nA-A|<0.1)\approx 0.99$; $P(|n-1)0.2<0.1)\approx 0.99$; $P(|n|<3/2)\approx 0.99$

$P(3/2) - P(|n|) \approx 0.99$

$P(|n|) \approx P(3/2) - 0.99$

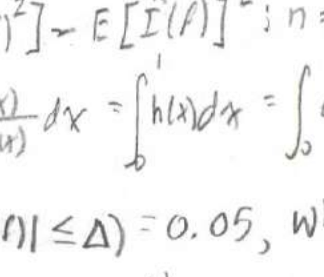
$\approx 0.9332 - 0.99 = -0.0568$

$P(n) = 0.0568$

$n = 1.58$

25. $f(x) = \frac{3}{2}x^2$ $-1\le x\le 1$;

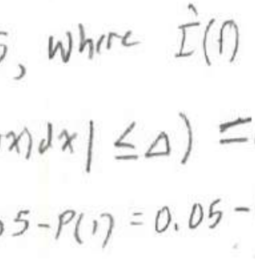
$\frac{2}{3}\le x\le 1$



$f(x) = \begin{cases}\frac{3}{2}x^2; -1\le x\le 1\\ 0\end{cases}$

$S = X_1 + \dots + X_{50}$

$E[S] = \sum_{i=1}^{50} S = \sum_{i=1}^{50} X_i$



$E[X] = \int_{-1}^{1} x\,\frac{3}{2}x^2\,dx$

$= \cancel{0}$

$E[X^2] = \int_{-1}^{1} x^2\,\frac{3}{2}x^2\,dx = 3/5$; $\sigma^2 = \frac{3}{5}$

$= 50E[X] = 50\cdot\int_{-1}^{1} x\,dx$

$\frac{3}{2}x^2 - 1$

$E[S] = \sum_{i=1}^{50} X_i = 50E[X] = \phi$

$Var[S] = \sum_1^{50} Var(X) = 50 Var[X] = 30$

$= 50\left[\frac{x^2}{2} - \frac{\left(\frac{3}{2}x^2-1\right)^2}{2}\right]$

$= 50\left[\frac{x^2}{2} - \frac{9}{8}x^4 + \frac{3}{2}x^2 - \frac{1}{2}\right] = 50\left[-\frac{9}{8}x^4 + 2x^2 - \frac{1}{2}\right] =$

26. $P(\text{Shot}) = 0.3$; $E[S] = \sum_{i=1}^{25} x\, p(\text{success}|\text{shot}) = 25 \cdot p(x) = 25\left(\frac{3}{10}\right)$; $\underbrace{\frac{5}{25} \geq p(x) > \frac{7}{25} \geq p(x)\ \frac{9}{25} \geq p(x)\ \frac{11}{25} \geq p(x)}_{\leq\ \frac{3}{10}\ \leq}$

27. Prove $a_n \to a$, then $(1+a_n/n)^n \to e^a$; $\lim_{a_n \to a} (1+a_n/n)^n = 1 + \frac{1}{2}\left(\frac{a}{2}\right) + \frac{1}{2}\left(\frac{a^2}{2^2}\right) = e^a$

28. $f_n(x) \begin{cases} 1/2 & x = \pm(1/2)^n \\ 0 & \text{otherwise} \end{cases}$; $E[X] = \sum_{-\infty}^{\infty} x f(x) = -\frac{1}{2} - \frac{1}{4} \cdots 0 + 1/4 + 1/2$

$F_n(x) = \begin{cases} 1/2 & (1/2)^{n+1}/n \\ 0 & \text{otherwise} \end{cases}$ $\boxed{\lim F_n(x) = F_n(x)}$

29. $U_1 \ldots U_n$ from $[0,1]$; $U_{(n)}$ = maximum

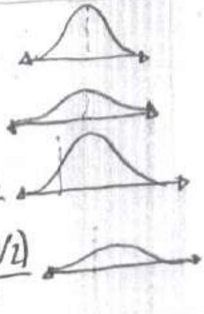
$\int_0^1 U_{(n)} du = 0 = \frac{U - E[U]}{\sqrt{Var(U)}} = U_{(n)}$

a) S_n

b) $\frac{S_n}{n}$

c) $S_n - n/2$

d) $\frac{(S_n - n/2)}{n}$



30. $U_1, U_2, \ldots U_{1000}$; $S_n = \sum_1^n U_i$; $n = 1 \cdots 1000$

e) $(S_n - n/2)/\sqrt{n}$

Chapter 6: Distributions derived from Normal Distribution

χ^2 = Chi-squared $V = Z^2$

$X \sim N(\mu, \sigma^2)$; $(X-\mu)/\sigma \sim N(0,1)$; $\left[\frac{X-\mu}{\sigma}\right]^2 \sim \chi_1^2$

$\boxed{\text{Chi-squared Distribution}}$ $[\chi_n^2]$ "n-degrees of freedom"

$V = U_1 + U_2 + \cdots + U_n$

sum of independent Gamma $\alpha = \frac{n}{2}$; $\lambda = \frac{1}{2}$

$$f(v) = \frac{1}{2^{n/2}\Gamma(n/2)} v^{(n/2)-1} e^{-v/2} \ ; \ v \geq 0$$

Moment Generating Function $M(t) = (1-2t)^{-n/2}$

$E(V) = n$; $Var(V) = 2n$

$\boxed{t\text{-distribution}}$ if $Z \sim N(0,1)$ and $U \sim \chi_n^2$

then $Z/\sqrt{U/n}$ is a t-distribution:

Density Function: $f(t) = \frac{\Gamma[(n+1)/2]}{\sqrt{n\pi}\,\Gamma(n/2)}\left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}$

$\boxed{F\text{-distribution}}$ $W = \frac{U/m}{V/m}$

Density Function: $f(w) = \frac{\Gamma[(m+n)/2]}{\Gamma(m/2)\Gamma(n/2)}\left(\frac{m}{n}\right)^{m/2} w^{m/2-1}\left(1 + \frac{m}{n}w\right)^{-(m+n)/2}$

$E(W) = \frac{n}{(n-2)}$

Sample Mean and Variance

Sample: independent random variables which are normally distributed

$\underbrace{\bar{X} = \frac{1}{n}\sum X_i}_{\text{Sample Mean}}$; $\underbrace{S^2 = \frac{1}{n-1}\sum_i^n (X_i - \bar{X})^2}_{\text{Sample Variance}}$

$E(\bar{X}) = \mu$ $\quad Var(\bar{X}) = \frac{\sigma^2}{n}$

If $(X_1 - \bar{X}, X_2 - \bar{X} \cdots X_n - \bar{X})$ are independent

$M(s, t_1 \ldots t_n) = E\{\exp[s\bar{X} + t_1(X_1 - \bar{X}) + \cdots + t_n(X_n - \bar{X})]\}$

$\sum_{i=1}^n t_i(X_i - \bar{X}) = \sum_{i=1}^n t_i X_i - n\bar{X}\bar{t}$

$s\bar{X} + \sum t_i(X_i - \bar{X}) = \sum_{i=1}^n \left[\frac{s}{n} + (t_i - \bar{t})\right] X_i = \sum_{i=1}^n a_i X_i$

$M(s, t_1 \ldots t_n) = \exp\left(\mu s + \frac{\sigma^2}{2n}s^2\right) \exp\left[\frac{\sigma^2}{2}\sum_1^n (t_i - \bar{t})^2\right]$

1. Prove Proposition A of Section 6.2

$\frac{Z}{\sqrt{U/n}} = \frac{N(0,1)}{\sqrt{\chi^2/n}} = \frac{N(0,1)}{\sqrt{\frac{1}{2^{n/2}\Gamma(n/2)} v^{(n/2)-1} e^{-v/2} / n}}$

$f(t^2) = \int \sqrt{\frac{v}{n}}\, N(\sqrt{v/n}\cdot t)\, f(\chi_n^2)\, dv$

$x = (1 + t^2/n)v/2$

$= \frac{1}{\sqrt{2\pi n}\, 2^{n/2}\,\Gamma(n/2)} \int_0^\infty v^{(n-1)/2} e^{-(1+t^2/n)v/2} dv$

$= \frac{(1+t^2/n)^{-(n+1)/2}}{\sqrt{\pi n}\,\Gamma(n/2)} \int_0^\infty x^{(n+1)/2-1} e^{-x} dx$ $\boxed{= \frac{(1+t^2/n)^{-(n+1)/2}}{\sqrt{\pi n}\,\Gamma(n/2)} \Gamma[(n+1)/2]}$

2. Prove Proposition B of Section 6.2

$W = \frac{U/m}{V/n}$

$f(w) = \frac{\Gamma[(m+n)/2]}{\Gamma(m/2)\Gamma(n/2)}\left(\frac{m}{n}\right)^{m/2} w^{m/2-1}\left(1 + \frac{m}{n}w\right)^{-(m+n)/2}$

$$f(w) = \int_0^{\infty} \frac{1}{2^{n/2}\Gamma(n/2)} x^{(n/2)-1} e^{-x/2} \cdot \frac{1}{2^{m/2}\Gamma(m/2)} (xz)^{m/2-1} \cdot e^{-xz/2} dx = \frac{z^{m/2-1}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})2^{(m+n)/2}} \underbrace{\int_0^{\infty} x^{(m+n)/2-1} e^{-x(z+1)/2} dx}$$

"similar to $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$ where $t = x\left(\frac{z+1}{2}\right)$

$$= \frac{z^{m/2-1}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})2^{(m+n)/2}} \frac{1}{\left(\frac{z+1}{2}\right)^{(m+n)/2}} \int_0^{\infty} t^{(m+n)/2-1} e^{-t} dt$$

$$= \frac{\Gamma\left(\frac{m+n}{2}\right) z^{m/2-1}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})(z+1)^{(m+n)/2}} \Rightarrow \boxed{\frac{\Gamma\left(\frac{m+n}{2}\right)\left(\frac{m}{n}\right)^{m/2} x^{m/2-1}}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})\left(1+\frac{m}{n}x\right)^{(m+n)/2}}}$$

3. $n = 16$; $\bar{x} = \mu = 0 = \frac{1}{16}\sum_{i=1}^{16} x_i$

$P(|\bar{X}| \le c) = P(-c \le \bar{X} \le c)$
$= P(\bar{X} \le c) - P(\bar{X} \le -c)$
$= P(\bar{X} \le c) - [1 - P(\bar{X} \le c)]$
$= 2 \cdot P(\bar{X} \le c) - 1 = 2 \cdot \Phi\left(\frac{c-0}{\sqrt{1/16}}\right) - 1$
$= 2 \cdot \Phi(4c) - 1 = 0.5$

$\Phi(4c) = 0.75$
$4c = \Phi^{-1}(0.75)$
$c = 0.7734/4$
$= 0.19335$

4. T follows a t_7-distribution.

Find t_0 such that a) $P(|T| < t_0) = 0.9$ of a t_7 distribution

$$t_7 = f(t) = \frac{\Gamma[(7+1)/2]}{\sqrt{7\pi}\,\Gamma(7/2)} \left(1 + \frac{t^2}{7}\right)^{-(7+1)/2}$$

$$= \frac{\Gamma(4)}{\sqrt{7\pi}\,\Gamma(7/2)} \left(1 + \frac{t^2}{7}\right)^{-4}$$

$= P(-t_0 \le T \le t_0)$
$= P(T \le t_0) - P(-t_0 \le T)$
$= P(T \le t_0) - [1 - P(t_0 \le T)]$
$= 2P(T \le t_0) - 1 = 0.9$
$P(T \le t_0) = 0.95$
$\boxed{T = \Phi^{-1}(0.95) = 1.895}$

b) $P(T > t_0) = 0.05$

$1 - P(T < t_0) = 0.05$; $P(T < t_0) = 0.95$
$T = \Phi^{-1}(0.95) = 1.895$

5. If $X \sim F_{n,m}$, then $X^{-1} \sim F_{m,n}$.

$F_{n,m} = W = \frac{U/m}{V/n} = X$; $X^{-1} = \frac{1}{F} = \frac{V/n}{U/m} = \boxed{F_{m,n}}$

6.
$T \sim t_n$, then $T^2 \sim F_{1,n}$; $T = t_n = Z/\sqrt{U/n}$; $T^2 = Z^2/U/n = \boxed{nZ^2/U = F_{1,n}}$

7. Cauchy Distribution:

$$\boxed{f(x) = \frac{1}{\pi}\left(\frac{1}{1+x^2}\right)}$$

t-Distribution:

$$f(t) = \frac{\Gamma[(n+1)/2]}{\sqrt{n\pi}\,\Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}$$

@ df = 1, $f(t_1) = \frac{\Gamma(1)}{\sqrt{\pi}\,\Gamma(1/2)}(1+t^2)^{-1}$

$$\boxed{= \frac{1}{\pi}\left(\frac{1}{1+t^2}\right)}$$

8. Exponential Random Variables:

$f(x) = \lambda e^{-\lambda x}$; $\lambda = 1$; $\frac{X}{Y} = \left(\frac{1}{1}\right) e^{-x+y} = e^{y-x}$

F-Distribution:

$$f(w) = \frac{\Gamma[(m+n)/2]}{\Gamma(m/2)\Gamma(n/2)} \left(\frac{m}{n}\right)^{m/2} w^{m/2-1} \left(1 + \frac{m}{n}w\right)^{-(m+n)/2};$$

if $m, n = 1$; $f(w) = \frac{\Gamma(1)}{\Gamma(1/2)\Gamma(1/2)}\left(\frac{1}{1}\right)^{1/2}(1+w)^{-1}$
$= 1 + \frac{w^2}{2} + \frac{w^3}{8} + \cdots$
$= e^{w} = \boxed{e^{y-x}}$

9. Find mean and variance of $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

$$\bar{S}^2 = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{(n-1)}(X_i - \bar{X})^2 = \boxed{\frac{\sum_{i=1}^{n} X_i \sum_{i=1}^{n}(X_i - \bar{X})^2}{2}} =$$

$$[S^2]^2 = \frac{1}{(n-1)^2}\sum^{n-4}(X_i - \bar{X})^4$$

$P(a < S^2/\sigma^2 < b) = P(S^2/\sigma^2 < b) - P(a < S^2/\sigma^2)$

$$= \int_a^b f(v)\,dv = 1 = \frac{1}{2^{n/2}\Gamma(n/2)}[\Gamma(b+1) - \Gamma(a+1)]$$

$$= \frac{1}{2^{n/2}\Gamma(n/2)}(b-a)\Gamma(1) = 1; \boxed{(b-a) = 2^{n/2}\sqrt{\pi}}$$

10. Chi-Squared Distribution

$$f(v) = \frac{1}{2^{n/2}\Gamma(n/2)} v^{(n/2)-1} e^{-v/2}$$

10. $X_1 \dots X_n : N(\mu_x, \sigma^2)$; $Y_1 \dots Y_N ; N(\mu_y, \sigma^2)$; Show how a F-distribution can find $P(S_x^2/S_y^2 > c)$

F Distribution:

$$f(w) = \frac{\Gamma[(m+n)/2]}{\Gamma(m/2)\Gamma(n/2)} \cdot \left(\frac{m}{n}\right)^{m/2} \cdot w^{m/2-1} \cdot \left(1+\frac{m}{n}w\right)^{-(m+n)/2} ; \int_{-c}^{c} f(w)\,dw = \int_{-c}^{c} \frac{S_x^2}{S_y^2}\,dx\,dy = 2\int_0^c\int_0^c \frac{S_x^2}{S_y^2}\,dx\,dy$$

$$= \frac{2 \cdot e^{(-2c^2+2\mu_x c+2\mu_y c)/\sigma^2}}{+(1/\sigma^2)^2 \mu_x \mu_y (2)^2} = 1$$

$$= \frac{+\sigma^4 e^{-2(c^2+\mu_x c+\mu_y c)/\sigma^2}}{2\mu_x\mu_y} = 1$$

$$\frac{-2(c^2+\mu_x c+\mu_y c)}{\sigma^2} = \ln\left[\frac{2\mu_x\mu_y}{\sigma^4}\right]$$

$$c^2+\mu_x c+\mu_y c = \sigma^2 \ln\left[\frac{\sigma^4}{2\mu_x\mu_y}\right]$$

$$c^2+(\mu_x+\mu_y)c - \sigma^2\ln\left[\frac{\sigma^4}{2\mu_x\mu_y}\right] = 0$$

$$\boxed{c_1, c_2 = \frac{-(\mu_x+\mu_y) \pm \sqrt{(\mu_x+\mu_y)^2+4\sigma^2\ln\left[\frac{\sigma^4}{2\mu_x\mu_y}\right]}}{2}}$$

Chapter 7: Survey Sampling:

1. Sample: 1, 2, 2, 4, and 8;

$$E[X] = \bar{X} = \frac{1}{5}\sum_{i=1}^{5} X_i = \frac{(1+2+2+4+8)}{5} = 17/5$$

$$E[X^2] = \frac{1}{5}\sum_{i=1}^{5} X_i^2 = \frac{(1^2+2^2+2^2+4^2+8^2)}{5} = \frac{89}{5}$$

$$Var(X) = \frac{89}{5} - \left(\frac{17}{5}\right)^2 = \frac{89-(10+7)(10+7)}{5} = \frac{89-57.9}{5} \approx \frac{20.1}{5} = \boxed{1.42}$$

Sampling Distribution

Sample size = 2

$T = N\bar{X} = 2(17/5) = \boxed{1.7}$

$E(T) = E(N\bar{X}) = 1.17$

$$Var(\bar{X}) = \frac{4.20}{5}\left(\frac{5-2}{5-1}\right) = \frac{12.60}{20} = \frac{1.26}{2} = \boxed{0.63}$$

(2) $n = 2$; $X_a = 4, 8$; $T = N\bar{X} = NE(X) = 1.7$; $var(\bar{x}) = 0.63$

(3) a) A population mean [No] b) Population Size: [No] c) Sample Size [No]

d) The sample Mean [Yes] e) Variance of Sample mean [No] f) The largest value of Data [Yes]

g) Population Variance [No] h) Estimated variance of sample mean [Yes]

(4) Population I: (n_1, σ_1) Population II: $(n_2 = 2n_1, \sigma_2 = 2\sigma_1)$ Accuracy is better approximated by a large n-value. The law of large numbers states a sequence of independent values converge to $E(\bar{X}_n)$ as $n \to \infty$.

(5) A random variable is defined as a variable which can take on only a finite number of values. The sample mean is a random variable because of the finite form.

(6) $N_1 = 100{,}000$; $N_2 = 10{,}000{,}000$; $n = 25$. $var(\bar{X}) = \frac{\sigma^2}{25}\left(\frac{100{,}000-25}{100000-1}\right) = 0.04\sigma^2$

$var(\bar{x}) = \frac{\sigma^2}{25}\left(\frac{10^7-25}{10^7-1}\right) = 0.04\sigma^2$

Yes, it is substantially easier to measure the smaller size because of finite solutions to sample mean.

(7). Standard Error: $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$; $Var(\bar{X}) = \frac{\sigma^2}{n}\left(\frac{15}{100}\right) \Rightarrow \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}\frac{\sqrt{15}}{10} = 0.02$; $\frac{\sigma}{\sqrt{n}} = \frac{0.2}{\sqrt{15}}$; $\frac{15}{0.04} = n = 375$

8. $n=100$; $p=1/5$ a) Find δ such that $P(|\hat{p}-p|\geq\delta)=0.025$

Sample proportion → $\hat{p}=\sigma_{\hat{p}}=\sqrt{\frac{p(1-p)}{n}}=\sqrt{\frac{0.2(1-0.2)}{50}}=0.057$ ← standard error

$P\left(\frac{|p-\hat{p}|}{\sigma_p}\geq\frac{\delta}{\sigma_p}\right)=0.025$

$P\left(\frac{|p-\hat{p}|}{\sigma_p}\geq\frac{\delta}{\sigma_p}\right)=1-0.025$

$2P\left(z\leq\frac{\delta}{0.057}\right)-1=0.975$

$P\left(z\leq\frac{\delta}{0.04}\right)=\frac{1.975}{2}$

$\Phi\left(\frac{\delta}{0.04}\right)=0.9875$

$\frac{\delta}{0.04}=\Phi^{-1}(0.9875)$

$\delta=0.08964$

b) $\hat{p}=0.25$; 95%;

$p=\hat{p}\pm z(0.025)s_{\hat{p}}=0.25\pm1.96\sqrt{\frac{0.25(1-0.25)}{100-1}}$

$=0.25\pm0.0853$

$=(0.1647,\ 0.3353)$

The original $p=0.2$ is within the range of proportion and of the true population.

9. $n=1{,}500$ voters, 55% planned to vote a particular proposition. 45% planned to vote against a proposition. Margin of victory [10%] $\boxed{\text{Confidence Interval}}$

$\boxed{\text{Standard Error}}$ $\sigma_x\cong\frac{\sigma}{\sqrt{n}}$

$\bar{X}=50\%$; $\bar{X}\pm z(0.025)\sigma_{\bar{X}}=10\%\pm1.96(2.66\%)$ $=(19.91\%, 50.51\%)$ $\cong\frac{10\%}{\sqrt{1500}}=0.26\%$

$\boxed{=(4.90\%,\ 15.16\%)}$

10. False, as a population grows $(n\to\infty)$, then a distinct mean (μ) and standard deviation (σ) become more distinct, and possibly less normal and more dichotomous.

11. $n=4$; X_1, X_2, X_3, X_4 a) $\binom{n}{k}=\binom{4}{2}=\frac{4!}{2!(4-2)!}=\frac{4\cdot3}{2}=6$ b) $\{x_1,x_2\},\{x_2,x_3\},\{x_3,x_4\},\{x_1,x_4\}$

Mean Square error = variance + bias²; $E[X]=\frac{1}{6}$; $E[X^2]=\frac{1}{6}$; $\sigma=\sqrt{\frac{1}{6}-\frac{1}{6^2}}=\sqrt{\frac{5}{36}}=\frac{\sqrt{5}}{6}$

$\frac{1}{n}\sum(X-\mu)^2=\sigma^2+\beta^2$ ‖ This case shows the sample mean is unbiased because the choices are $1\ 2\ 3\ 4$; $\frac{1}{8}\sum X=\frac{12}{8}=\frac{1+2+3+4+4}{4}$

12. Random Sampling with replacement

$s^2=\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$ is the unbiased parameter of σ^2. $\boxed{\text{Variance of a Biased Sample}}$

$Var(\bar{X})=\frac{1}{n^2}\sum\sum Cov(X_i,X_j)=\frac{1}{n^2}\sum_{i=1}^{n}Var(X_i)+\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j\neq i}Cov(X_i,X_j)=\frac{\sigma^2}{n}-\frac{1}{n^2}n(n-1)\frac{\sigma^2}{N-1}$

$=\frac{\sigma^2}{n}\left[1-\frac{(n-1)}{N-1}\right]$

$=\frac{\sigma^2}{n}\left[\frac{N-n}{N-1}\right]$

$\boxed{\text{Expected Variance of a population}}$

$\hat{\sigma}^2=\frac{1}{n}\sum_{i=1}^{n}X_i^2-\bar{X}^2$; $E(\hat{\sigma}^2)=\frac{1}{n}\sum_{i=1}^{n}E(X_i^2)-E(\bar{X}^2)$

$=\frac{1}{n}\sum\left[Var(X_i)+E(X)^2\right]-\left[Var(\bar{X})+E(\bar{X})^2\right]$

$=\frac{1}{n}\sum[\sigma^2+\mu^2]-\left[\frac{\sigma^2}{n}\left(1-\frac{n-1}{N-1}\right)+\mu^2\right]$

$=\frac{\sigma^2}{n}\sum\left[1-\left(1-\frac{n-1}{N-1}\right)\frac{1}{n}\right]=\frac{\sigma^2}{n}\left[1-\frac{1}{n}+\frac{(n-1)}{n(N-1)}\right]=\frac{\sigma^2}{n}\left[\frac{n(N-1)-(N-1)+(n-1)}{n(N-1)}\right]$

$=\frac{\sigma^2}{n}\left[\frac{nN-\cancel{n}-N+1+\cancel{n}+1}{n(N-1)}\right]=\boxed{\frac{\sigma^2}{n}\left[\frac{(n-1)N}{n(N-1)}\right]}$

$$S_x^2 = \frac{\sigma^2}{n}\left(\frac{n}{n-1}\right)\left(\frac{N-1}{N}\right)\left(\frac{N-n}{N-1}\right) \mp \left[\frac{\sigma^2}{n}\left(\frac{N-n}{N-1}\right)\right]^2\left(\frac{1}{n}\right)\left(\frac{n}{n-1}\right)\left(\frac{N-1}{N}\right)\left(\frac{N-n}{N-1}\right)$$

$$= \frac{\sigma^4}{n^2}\left(\frac{N-n}{N-1}\right)^2\left(\frac{1}{n^2}\right)\left(\frac{n}{n-1}\right)\left(\frac{N-1}{N}\right)\left(\frac{N-n}{N-1}\right) = \frac{\sigma^4}{n^3}\left(\frac{N-n}{N-1}\right)\left(\frac{1}{n-1}\right)\left(\frac{N-n}{N}\right) =$$

12 $s^2 = \frac{1}{n-1}\sum(x_i-\bar{x})^2$; $E(s^2) = \frac{1}{n-1}E\left(\sum^n x_i^2 - n\bar{x}^2\right) = \frac{1}{n-1}\left[\sum E(x^2) - E(x)^2\right]$

$$= \frac{1}{n-1}\left[\sum\left[Var(x)+E(y_i)^2\right] - n\left[Var(y)+E(y)^2\right]\right]$$

$$= \frac{1}{n-1}\left[n\sigma^2 + n\bar{y}^2 - \sigma^2 - n\bar{y}^2\right] = \frac{(n-1)}{(n-1)}\sigma^2 = \sigma^2$$

b) $E(s) = E\left(\sqrt{\frac{\sum(x-\bar{x})^2}{n-1}}\right) = \frac{1}{\sqrt{n-1}}E\left(\sqrt{\frac{\sum(x-\bar{x})^2}{\sigma}}\,\sigma\right) = \frac{\sigma^2}{\sqrt{n-1}}E\left(\sqrt{\frac{\sum(x-\bar{x})^2}{\sigma^2}}\right) = \frac{\sigma^2}{\sqrt{n-1}}E(\sqrt{y})$

$\boxed{E(s) \neq \sigma \text{; It is not an unbiased estimate of } \sigma}$

c) Show $\frac{s^2}{n}$ is an unbiased estimate of $\sigma_{\bar{x}}$;

$$E\left(\frac{s^2}{n}\right) = E\left(\frac{1}{n(n-1)}\left(\sum x_i^2 - n\bar{x}^2\right)^2\right) = \frac{1}{n(n-1)}\sum E(x^2) - E(x)^2 =$$

d) $E\left(\frac{N^2 s^2}{n}\right) = \frac{N^2\sigma_T^2}{n}$

$$= \frac{1}{n(n-1)}\left[\sum\left[Var(X_i)+E(y_i)^2\right] - n\left[Var(y)+E(y)^2\right]\right]$$

$\boxed{\sigma_T^2 = N^2\sigma_x^2}$

$$= \frac{1}{n(n-1)}\left[n\sigma^2 + \cancel{n\bar{y}^2} - \sigma^2 - \cancel{n\bar{y}^2}\right] = \frac{\sigma^2(n-1)}{n(n-1)}$$

These are expected estimates of the sample and population, seperately.

Yes, $E\left(\frac{s^2}{n}\right)$ is an unbiased estimate. being $\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n}$; $\boxed{\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}}$

e) $E\left(\frac{\hat{p}(1-\hat{p})}{(n-1)}\right) = \frac{1}{n-1}\left[E(\hat{p}) - E(\hat{p}^2)\right] = \frac{1}{n-1}\left[\hat{p} - \left[Var(\hat{p}) + E(\hat{p})^2\right]\right]$

13. $$= \frac{1}{n-1}\left[p - \left[\frac{p(1-p)}{n} + \hat{p}^2\right]\right] = \frac{1}{n-1}\left[p(1-p) - \frac{p(1-p)}{n}\right] = \frac{n-1}{n-1}\frac{p(1-p)}{n}$$

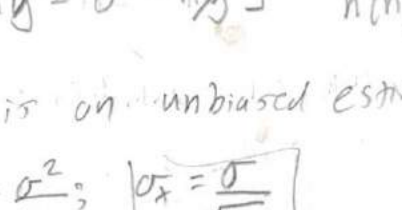
$\boxed{= \sigma_p^2}$

Example A: 7.2:

Herksen (1976); $N = 393$; X_i = number of patients discharged from i^{th} hospital January 1968.

Suppose Total $[\tau]$ is an estimate of size 50.

Denoting estimate T by the Central Limit theorem to sketch the probability density of the error $T - \tau$



$T = N\bar{X}$, $\sigma_T^2 = N^2\sigma_x^2$ $s_T = N^2 s_x^2$

$\bar{x} = 814.6$; $\sigma_x^2 = \frac{\sigma^2}{n} = \frac{590^2}{393} = 886$

$s_x = 81.19$

$T = 50 \cdot 814.6$; $\sigma_T^2 = 221,500$; $s_T = 1647,954$

$= 4073$

Huge Standard error with small total population

14. $p = 0.654$ Total Number (<1000) discharges is from $n = 25$. Apply central limit theorem to the distribution

$n_{Total} = 393$, $\mu = 814.6$, $\sigma = 590$

$\sigma = \sqrt{\dfrac{0.654(1-0.654)}{25}} = \boxed{0.095}$

15. a) n = simple random sample.

a) Sketch $P(|\bar{X}-\mu| > 200)$; $20 < n \le 100$

$P(-200 < X - \mu < 200) = 1 - P(|\bar{X}-\mu| \le 200)$

$= 1 - P(-200 \le \bar{X} - \mu \le 200)$

$= 1 - P\left(\dfrac{-200}{\sigma_{\bar{X}}} \le \dfrac{\bar{X}-\mu}{\sigma_{\bar{X}}} \le \dfrac{200}{\sigma_{\bar{X}}}\right)$

$= 1 - \left[\Phi\left(\dfrac{200}{\sigma_{\bar{X}}}\right) - \Phi\left(\dfrac{-200}{\sigma_{\bar{X}}}\right)\right]$

$= 1 - \left[\Phi\left(\dfrac{200}{\sigma_{\bar{X}}}\right) - 1 + \Phi\left(\dfrac{200}{\sigma_{\bar{X}}}\right)\right]$

$= 2 - \left[2\Phi\left(\dfrac{200}{\sigma_{\bar{X}}}\right)\right] = 2\left[1 - \Phi\left(\dfrac{200}{\sigma_{\bar{X}}}\right)\right]$

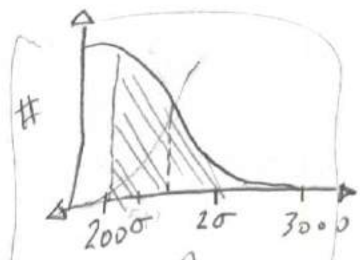
$= 2\left[1 - \Phi\left(\dfrac{200}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-1}{N-n}}}\right)\right]$

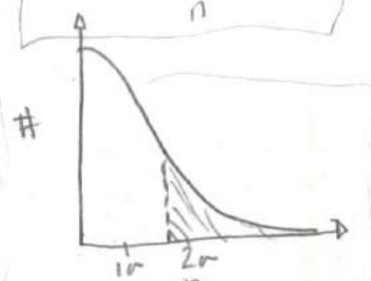
$= 2\left[1 - \Phi\left(6.71\sqrt{\dfrac{n}{393-n}}\right)\right]$

@ $n = 20$; $= 2\left[1 - \Phi\left(6.71\sqrt{\dfrac{20}{393-20}}\right)\right]$

$= 2[1 - \Phi(1.55)] = 0.12$

@ $n = 100$ $= 2[1 - \Phi(3.92)] = 0.002$





$P(-200 < \bar{X} - \mu < 200)$

b) For $n = 20, 40,$ and 80. Find Δ such that $P(|\bar{X}-\mu| > \Delta) \approx 0.10$

$n = 20$; $P(|\bar{X}-\mu| > \Delta) \approx 0.1$

$P(\Delta < \bar{X} - \mu < \Delta) \approx 0.1$

$P(\bar{X}-\mu < \Delta) - [1 - P(\bar{X}-\mu < \Delta)] \approx 0.1$

$2\left[1 - \Phi\left(\dfrac{\Delta}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-1}{N-n}}}\right)\right] \approx 0.1$

$\Phi\left(\dfrac{\Delta}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-1}{N-n}}}\right) \approx 0.95$

$\dfrac{\Delta}{\frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-1}{N-n}}} = 1.65$

$\Delta_{0.1} = \dfrac{590}{\sqrt{20}}\sqrt{\dfrac{393-1}{393-20}}(1.65) = 222$

$\Delta_{0.5} = \dfrac{590}{\sqrt{20}}\sqrt{\dfrac{393-1}{393-20}}(0.68) = 91.9$

$n = 40$ $P(|\bar{X}-\mu| > \Delta) \approx 0.10 = 1 - P(|\bar{X}-\mu| < \Delta)$

$= 1 - P(-\Delta < |\bar{X}-\mu| < \Delta) = 1 - [\Phi(\bar{X}-\mu < \Delta) - \Phi(-\Delta < \bar{X}-\mu)] = 1 - [\Phi(\bar{X}-\mu < \Delta) - 1 + \Phi(-\Delta > \bar{X}-\mu)]$

$= 2 - [2\Phi(\bar{X}-\mu < \Delta)] = 2[1 - \Phi(\bar{X}-\mu < \Delta)] = 0.1$; $\Phi\left(\dfrac{\bar{X}-\mu}{\sigma_{\bar{X}}} < \dfrac{\Delta}{\sigma_{\bar{X}}}\right) = 0.95$

$\Delta_{0.1} = \dfrac{590}{\sqrt{40}}\sqrt{\dfrac{393-1}{393-40}}(1.65) = 162$

$\Delta_{0.5} = \dfrac{590}{\sqrt{40}}\sqrt{\dfrac{393-1}{393-40}}(0.68) = 66.8$

$n = 80$

$\Delta_{0.1} = 122$
$\Delta_{0.5} = 50$

16.
a) The mean is a random variable.
b) True, a 95% confidence interval contains the mean and has a total probability of 0.95
c) True, a 95% confidence interval contains 95% of the population
d) True, 95% of a 100 is 95.

17. A 90% confidence interval $(1-\alpha)$ for average number of children per household is (min = 0.7, max = 2.1). Yes a confidence interval describes a random interval from a lower and upper bound that contains the mean.

18. 90% Confidence Interval: $P(\text{Mean} \mid \text{Interval}) = \sum_{0.9}^{n} p^{n} = 90\%^{2} = \boxed{81\%}$

$1 - P(\text{Mean} \mid \text{Interval}) = P(\text{No Mean} \mid \text{Interval}) = 1 - 0.81 = \boxed{19\%}$

19. One-sided Confidence Interval

k be chosen $(-\infty, \bar{X} + k s_{\bar{x}})$ that a 90% confidence interval for μ.

$P(-\infty \le \mu \le \bar{X} + k s_{\bar{x}}) = 90\%$; $P(\mu \le \bar{X} + k s_{\bar{x}})$; $1.28 = \bar{X} + k s_{x}$; $\boxed{k = \frac{1.28 - \bar{X}}{s_x}}$

$P(\bar{X} - k s_{\bar{x}} \le \mu) = 0.95$; $\bar{X} - k s_{\bar{x}} = 1.65$; $k = \frac{\bar{X} - 1.65}{s_{\bar{x}}}$

20. $N = 8000$ condominium units; $n = 100$ sample size; $\bar{X} = 1.6$ motor vehicles; $\sigma_{\bar{x}} = 0.8$

- $s_{\bar{x}} = \frac{s}{\sqrt{n}} \sqrt{1 - \frac{n}{N}} = \frac{0.8}{10} \sqrt{1 - \frac{100}{8000}} = 0.08$; confidence interval $\bar{X} \pm z(0.025) s_{\bar{x}} = \bar{X} \pm 1.96(0.08) = (1.44, 1.76)$
- Total Number of Motor Vehicles $T = 8000 \times 1.6 = 12{,}800$; $s_T = N s_{\bar{x}} = 640$ $= (11546, 14054)$
- 12% respondents planned $\hat{p} = 0.12$ with a proportion p.

Standard Error: $s_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n-1}} \sqrt{1 - \frac{100}{1000}} = 0.03$

$\hat{p} \pm 1.96 s_p = (0.06, 0.18)$

$T = N\hat{p} = 960$; $s_T = N s_p = 240$

$T \pm 1.96 s_T$; $(490, 1430)$

At 95% confidence interval suggests another sample size of 100 would contain a mean between (1.44 and 1.76.

21. To halve the width of a 95% confidence interval

$\frac{\bar{X} - \mu}{\sigma_{\bar{x}}} = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = 95\%$; $\frac{\bar{X} - \mu}{\sigma/\sqrt{4}} = 95\%$; $\frac{\bar{X} - \mu}{\sigma/2} = \frac{\bar{X} - \mu}{\sigma}\left(\frac{1}{2}\right) = 95\%/2 = 47.5\%$

22. $\bar{X} \pm s_{\bar{x}} = \bar{X} \pm 1 \cdot s_{\bar{x}} \Rightarrow z(\alpha/2) = 1$; $\boxed{\alpha = 2(1 - 0.8413)}$; Confidence Interval: $\boxed{1 - \alpha = 0.682}$

23 a) Show $\sigma_{\bar{x}}$ is largest when $p = \frac{1}{2}$; $\frac{d}{dp} s_{\hat{p}} = \frac{d}{dp} \sqrt{\frac{\hat{p}(1-\hat{p})}{n-1}} \sqrt{1 - \frac{100}{8000}} = 0$

$= \frac{1}{2} \frac{1 - 2p}{\sqrt{\hat{p}(1-\hat{p})/(n-1)}} = \frac{1}{2}(1 - 2p) = \frac{1}{2} - p$

$\boxed{p = 1/2}$

b) $s_{\hat{p}}^2 = \frac{\hat{p}(1-\hat{p})}{n-1}\left(1 - \frac{n}{N}\right)$ $\boxed{\text{"Unbiased Estimate of } Var(\hat{p})\text{"}}$

$s_{\hat{p}}^2 = \frac{\frac{1}{2}(1 - 1/2)}{n-1}\left(1 - \frac{n}{N}\right)$

$s_{\hat{p}}^2 = \frac{1}{4}\left(\frac{N-n}{N(n-1)}\right)$; $s_{\hat{p}} = \sqrt{\frac{1}{4}\left(\frac{N-n}{N(n-1)}\right)} = \boxed{\frac{1}{2}\sqrt{\left(\frac{N-n}{N(n-1)}\right)}}$

c) $\hat{p} \pm \sqrt{\frac{N-n}{N(n-1)}} \rightarrow \Phi(z) = \Phi(0.05)$

$P\left(\hat{p} - \sqrt{\frac{N-n}{N(n-1)}} < \frac{\bar{X} - \mu}{\sigma_{\bar{x}}} < \hat{p} + \sqrt{\frac{N-n}{N(N-1)}}\right) \rightarrow \Phi(z)$

$\lim_{n \to \infty} P\left(\hat{p} - \sqrt{\frac{N-n}{N(n-1)}} < \frac{\bar{X} - \mu}{\sigma \cdot \sqrt{n}} < \hat{p} + \sqrt{\frac{N-n}{N(n-1)}}\right)$

$P(\hat{p} < 0 < \hat{p}) = \Phi(z) = 2P(0 < \hat{p}) - 1 = \Phi(z)$; $z = \boxed{0.9990}$

24. Sample size $= n$; Population size $= N$; Estimate of $\mu = \bar{X}_c = \sum_{i=1}^{n} c_i X_i$

a) Find the condition $[c_i]$ such that the estimate is unbiased.

$$\bar{X} = E\left[\sum_{i=1}^{n} c_i X_i\right] = \sum_{i=1}^{n} c_i E(X_i) = \mu \sum_{i=1}^{n} c_i = \mu(1) \quad \boxed{\sum c_i = 1}$$

b) $Var(\bar{X}_c) = Var(\sum c_i X_i) = \sum c_i^2 Var(X) = \sum c_i^2 \sigma^2 = \sigma^2 \sum c_i^2$

Applying a LaGrangian Multiplier: $L(c_1 \dots c_n, \lambda) = \sigma^2 \sum_{i=1}^{n} c_i^2 + \lambda(\sum c_i - 1)$

$\frac{\partial L}{\partial c_i} = 0$; $\frac{\partial}{\partial c_i}\left[\sigma^2 \sum_{i=1}^{n} c_i^2 + \lambda(\sum_{i=1}^{n} c_i - 1)\right] = 0$; $\frac{d}{dc_i}[\sigma^2 \sum c_i^2] + \frac{d}{dc_i}[\lambda(\sum c_i - 1)] = 0$

$\sigma^2 \frac{\partial}{\partial c_i}[\sum c_i^2] + \lambda \frac{\partial}{\partial c_i} \sum c_i = 0$

$2\sigma^2 \sum c_i^{2-1} + \lambda = 0$; $2\sigma^2 c_i + \lambda = 0$

$2\sigma^2 c_i = -\lambda$

$c_i = \frac{-\lambda}{2\sigma^2}$

Therefore, $\sum c_i = \sum \frac{-\lambda}{2\sigma^2} = \frac{-n\lambda}{2\sigma^2} = 1$

$\boxed{\lambda = \frac{-2\sigma^2}{n}}$

$\boxed{c_i = \frac{-1}{2\sigma^2}\left(\frac{-2\sigma^2}{n}\right) = 1/n}$

25. Lemma B Section 7.3.2:

$Cov(X_i, X_j) = -\sigma^2/(N-1)$

$= E(X_i X_j) - E(X_i)E(X_j)$

$E(X_i X_j) = \sum_{k=1}^{m} \sum_{l=1}^{m} \zeta_k \zeta_l P(X_i = \zeta_k \text{ and } X_j = \zeta_l) = \sum_1^m \zeta_k P(X_i = \zeta_k) \sum_1^m \zeta P(X_j | X_i)$

where $P(X_j | X_i) = \begin{cases} n_l/(N-1) & k \neq 1 \\ (n_1 - 1)/(N-1) & k = 1 \end{cases}$

$$\sum \zeta_l P(X_j | X_i) = \sum_{l \neq k} \zeta_l \frac{n_l}{N-1} + \zeta_k \frac{n_k - 1}{N-1}$$

$$= \sum_{l=1}^{m} \zeta_l \frac{n_l}{N-1} - \zeta_k \frac{1}{N-1}$$

$$E(X_i X_j) = \sum_{k=1}^{m} \zeta_k \frac{n_k}{N}\left(\sum \zeta_l \frac{n_l}{N-1} - \frac{\zeta_k}{N-1}\right) = \frac{1}{N(N-1)}\left(\tau^2 - \sum_{k=1}^{m} \zeta^2 n_k\right)$$

$$= \frac{\tau^2}{N(N-1)} - \frac{1}{N(N-1)} \sum_{k=1}^{m} \zeta^2 n_k = \frac{N\mu^2}{N-1} - \frac{1}{N-1}(\mu^2 + \sigma^2)$$

$$= \mu^2 - \frac{\sigma^2}{N-1}$$

26. $U_i = 1$; i^{th} population member

$Cov(X_i X_j) = \mu^2 - \frac{\sigma^2}{N-1} - \mu^2 = \frac{-\sigma^2}{N-1}$

$Cov(Y_i, Y_j) = E(Y_i Y_j) - E(Y_i) \cdot E(Y_j)$

$= E(Y_i Y_j) - \sqrt{E(Y_i^2) - Var(Y_i)} \sqrt{E(Y_j^2) - Var(Y_j)}$

a) show $\bar{X} = n^{-1} \sum_{i=1}^{N} U_i X_i$; $E(\bar{X}) = \frac{1}{n} \sum_{i=1}^{n} B$ $\boxed{\frac{1}{n} \sum_{i=1}^{N} U_i X_i}$

b) $P(U_i = 1) = n/N$; $E(U_i) = \frac{1}{N} \sum_{i=1}^{n} U_i P(U_i = 1) = \frac{n}{N}(1) = \boxed{\frac{n}{N}}$

c) $Var(U_i) = E[U_i^2] - E[U_i]^2 = \frac{1}{N^2} \sum_{i=1}^{n} U_i^2 P(U_i = 1) - \left[\frac{1}{N} \sum U_i P_U\right]^2$

$= \frac{n}{N} - \left(\frac{n}{N}\right)^2 = \boxed{\frac{n}{N}\left(1 - \frac{n}{N}\right)}$

d) $E(U_i U_j) = \frac{\sum^n \sum^n U_i U_j P(U_i, U_j)}{N^2} = \frac{n^2}{N^2}$

e) $Cov(U_i, U_j) = E(U_i U_j) - E(U_i) E(U_j) = \frac{n^2}{N^2} - \left(\frac{n}{N}\right)\left(\frac{n}{N}\right) = 0$

f) $Var(\bar{X}) = E(\bar{X}^2) - E(\bar{X})^2 = \frac{n}{N^2} - \frac{n}{N^2} = 0$

27. Population size (N) is unknown ; $n \leq N$, Show will generate a simple random sample.

a) List $\{X_1, X_2 \ldots X_n \ldots X_N\}$ b) For $k = 1, 2 \ldots$ i) $X_{(n+k)} = X_{n-1+k} + X_i \; ; \; X_{i=1} = X_1$

$n = \{X_1, X_2 \ldots X_n\}$ $\quad k < N \quad$ ii) $\dfrac{n}{(n+k)} = \dfrac{X_1, X_2 \ldots X_n}{X_1, X_2 \ldots X_{n+k}}$

$N = \{X_1, X_2 \ldots X_n\}$

28. Randomized Response: $\underbrace{\text{Spin an Arrow - Draw ball from urn.}}_{\text{Action}}$ $\underbrace{\text{Choice A | Choice B}}_{\text{Result}}$ } Interviewee

Records Response } Interviewer

R = Proportion Yes ; $p = P(\text{Response}_{\text{Yes}} | \text{Statement \#1})$ "Randomized Device"

$r = P(\text{Yes})$ $\quad$ q = proportion of Characteristic A. $= P(\text{Statement \#1})$

a) Show $r = (2p-1)q + (1-p)$; Hint: $P(\text{yes}) = P(\text{yes} | \text{Statement \#1}) \times P(\text{Statement \#1}) + P(\text{yes} | \text{Statement \#2}) P(\#2)$

$$r = p \cdot q + (1-p)(1-q)$$

b) If r, what is q?

$$\boxed{q = \frac{r-1+p}{(2p-1)}}$$

$2pq - q + 1 - p$

c) $E(R) = r$ and propose Q, for q. Show the expected estimator unbiased.

$$E(R) = \sum_{i=1}^{2} P(\text{yes} | \text{Statement \#}i) P(\text{Statement \#}i) = P(\text{yes} | \text{Statement \#1}) P(\text{Statement \#1}) + P(\text{yes} | \text{Statement \#2}) P(\#2) = r$$

$$E(Q) = E\left(\frac{R-(1-p)}{2p-1}\right) = \frac{E(R)-(1-p)}{2p-1} = \frac{r-1-p}{2p-1} = q$$

d) Show $\operatorname{var}(R) = \frac{r(1-r)}{n} = E[R^2] - E[R]^2 = \frac{1}{(2p-1)^2} \operatorname{Var}(R-(1-p)) = \frac{1}{(2p-1)^2} \frac{r(1-r)}{n}$

e) $\operatorname{Var}(Q) = E[Q^2] - E[Q]^2 = \frac{1}{(2p-1)^2} \frac{r(1-r)}{n}$

29. a. $P(\text{yes} | \text{Statement \#3})$ $\quad$ $P(\text{Statement \#2})$

$$\boxed{r = qp + t(1-p)}$$

b) $E(Q) = \boxed{\frac{E(R) - t(1-p)}{p} = q}$

c) $$\operatorname{Var}(Q) = \frac{r(1-r)}{np^2} = \frac{[qp + t(1-p)](1 - qp + t(1-p))}{np^2} = \boxed{\frac{qp - q^2p^2 + qpt(1-p) + t(1-p) - qpt(1-p) + t^2(1-p)^2}{np^2}}$$

30. Problem #28 : $\operatorname{Var}(Q) = \frac{r(1-r)}{(2p-1)^2 n}$ } Difference

Problem #29 $\operatorname{Var}(Q) = \frac{r(1-r)}{np^2}$

7. Poisson Distribution:

$$P(x) = \frac{\lambda^x}{x!} e^{-\lambda}$$

Likelihood Ratio: $\Lambda_o = P(\lambda = \lambda_o | H_o) / P(\lambda = \lambda_o | H_1) = \left(\frac{\lambda_o}{\lambda_1}\right)^x e^{-(\lambda_o - \lambda_1)}$

$$\Lambda_n = \left(\frac{\lambda_o}{\lambda_1}\right)^{\sum X} e^{-n(\lambda_o - \lambda_1)} ; \ \ln \Lambda_n = \sum x \ln\left(\frac{\lambda_o}{\lambda_1}\right) - n(\lambda_o - \lambda_1)$$

$$-2\ln\Lambda = -2\sum x \ln\left(\frac{\lambda_o}{\lambda_1}\right) - \phi = 2\sum x \ln\left(\frac{\lambda_1}{\lambda_o}\right)$$

$$-2\ln\Lambda = \sum \frac{(\lambda - \lambda_o)^2}{\lambda_o} = X^2$$

$\boxed{\text{Pearson's Chi-square statistic}}$

Taylor Series: $\ln(x) = \sum_{i=1}^{n} \frac{(x-x_o)^i}{i} x^i(0) = (x - x_o)\ln(0) + \frac{1}{2}(x-x_o)^2 \frac{1}{x_o}$

$$= (\lambda_1 - \lambda_o) + \frac{1}{2}(\lambda - \lambda_o)^2 \frac{1}{\lambda_o}$$

$$-2\ln\Lambda = 2\underbrace{\sum(\lambda - \lambda_o)}_{=0} + \sum \frac{(\lambda_1 - \lambda_o)^2}{\lambda_o}$$

$$-2\ln\Lambda = \sum \frac{(\lambda - \lambda_o)^2}{\lambda_o}$$

$\boxed{\text{Poisson Dispersion Test}}$

"Measure of clustering"

$$\boxed{P\left(\frac{\sum (\lambda - \lambda_o)^2 - \lambda_o}{\lambda_o \sqrt{2 \cdot \nu}} > z(\alpha/2)\right) = \alpha \ ; \ \nu = \text{degrees of Freedom}}$$

$\boxed{\text{Normal Z-table}}$

8. $\lambda = \lambda_o$ $P\left(\frac{1}{\sqrt{2 \cdot \nu}} > z(\alpha/2)\right) = \alpha$ $\boxed{\text{Simple hypothesis}}$

vs.

$\lambda > \lambda_o$ $P\left(\frac{\sum(\lambda - \lambda_o)^2 - \lambda_o}{\lambda_o \sqrt{2 \cdot \nu}} > z(\alpha/2)\right) = \alpha$ $\boxed{\text{composite hypothesis}}$

9. Normal Distribution $\sigma^2 = 100$ $\mu_o = 0.0$ $\alpha = 0.10$ $\mu_A = 1.5$

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$\Lambda = \frac{P(x|H_o)}{P(x|H_A)} = e^{\frac{-\sum(x-\mu_o)^2 + \sum(x-\mu_1)^2}{2\sigma^2}}$$

$$= e^{\frac{2n\bar{x}(\mu_o - \mu_1) + n\mu_1^2 - n\mu_o^2}{2\sigma^2}} ; \ \Lambda >> 1 : \mu_o - \mu_1 > 0 ; \bar{x} << 1$$

$$\Lambda << 1 : \mu_o - \mu_1 < 0 ; \bar{x} >> 1$$

$$P(\bar{X} > x_o) = P\left(\frac{\bar{X} - \mu_o}{\sigma/\sqrt{n}} > \frac{x_o - \mu_o}{\sigma/\sqrt{n}}\right) ; \ \frac{x_o - \mu_o}{\sigma/\sqrt{n}} = z(\alpha) ; \ x_o = \frac{10}{\sqrt{n}} z(0.1) = \frac{10 \cdot 1.28}{\sqrt{n}} = \boxed{2.56}$$

$\boxed{\text{Unfavorable for } H_o}$

$\alpha = 0.01$; $\Lambda = \frac{P(x|H_o)}{P(x|H_1)} = e^{\frac{2n\bar{x}(\mu_o - \mu_1) + n\mu_1^2 - n\mu_o^2}{2\sigma^2}}$

$$z = \frac{x_o - \mu_1}{\sigma/\sqrt{n}} = \frac{2.56 - 1.5}{10/5} = 0.53 ; \ P(z > 2.56) = 1 - \beta = 0.7019$$

β $\boxed{= 0.2981}$

10. $P(\bar{x} > x_o) = P\left(\frac{\bar{x} - \mu_o}{\sigma/\sqrt{n}} > \frac{x_o - \mu_o}{\sigma/\sqrt{n}}\right) ; \ \frac{x_o - \mu_o}{\sigma/\sqrt{n}} = z(\alpha) ; \ x_o = \frac{10}{5} z(0.01) = 2(0.5040) = \boxed{1.080}$

$$z = \frac{x_o - \mu_1}{\sigma/\sqrt{n}} = \frac{1.09 - 1.5}{2} = -0.21 ; \ P(z > 1.23) = 1 - \beta = 0.8907$$

β $\boxed{= 0.1093}$

Suppose $X_1 \ldots X_n$

$f(x|\theta)$, T = sufficient statistic

Likelihood: $\Lambda = \frac{F(x|\theta_o)}{F(x|\theta_1)}$

$\boxed{\text{The rejection region is determined from normalizing the threshold mean of the numerator } [F(x|\theta_o)], \text{ such that rejection value is greater than the significance level}}$

11. $n = 25$

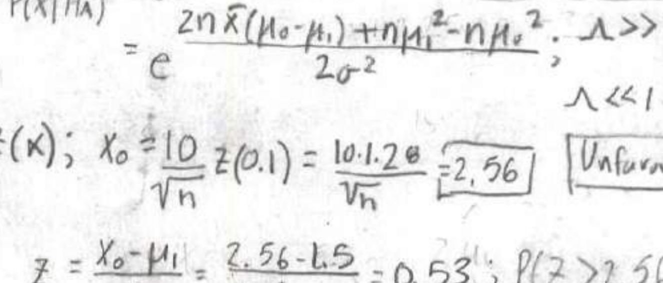
$\sigma^2 = 100$

Normal Distribution

$$F(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

$H_o : \mu = 0$

$H_A : \mu \neq 0$



12. Let $X_1, \ldots X_n$

Exponential Distribution

$$F(x) = \theta e^{-\theta x}$$

$$\Lambda = \frac{f(x|H_o)}{f(x|H_A)} = \frac{F(x|\theta_o)}{F(x|\theta \neq \theta_o)}$$

$$= \left(\frac{\theta_o}{\theta_A}\right) e^{(\theta_A - \theta_o)x} ;$$

$$; \Lambda = (\theta\bar{x}) e^{-n(\bar{x}\theta - 1)}$$

$$\boxed{= \theta e^n \left[\bar{x} e^{-n\bar{x}\theta}\right]}$$

13. $\theta_o = 1$, $n = 10$, $\alpha = 0.05$

a) Show $\boxed{\{\bar{x} \le x_o\} \cup \{\bar{x} \ge x_1\} = \{\bar{x} e^{-10\bar{x}} \le \chi^2\}}$

b) The rejection region is chosen by a threshold value of c at a significance level.

c) The exponential distribution relates to the gamma through a summation and $\theta = \frac{1}{\theta_o}$

d) Generating outputs from an exponential may be graphically similar to a gamma.

Chapter 9: Goodness of Fit

1. $P(H|X) = 0.50$; $P(T|X) = 0.50$

a) $P(\text{reject}|0) + P(\text{reject}|10) =$
$\binom{10}{0}0.5^0 0.5^{10} + \binom{10}{10}0.5^{10}0.5^0 = 0.002$

b) Power of Test $[1-\beta] = \binom{10}{0}(0.1)^0 0.9^{10} + \binom{10}{10}0.1^{10}0.9^0$
$\boxed{= 0.3487}$

2. a. X is uniform on [0, 1]
$\boxed{\text{Simple}}$

Simple Hypothesis: only null and alternative hypotheses.
Composite Hypothesis: when a probability distribution is not specified

b. A die is unbiased
$\boxed{\text{Composite}}$

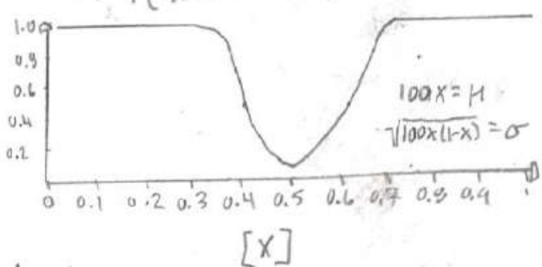
c. X follows a normal with mean 0 and var $\sigma^2 > 10$
$\boxed{\text{Simple}}$ $\boxed{\text{Simple}}$

d. X follows a normal with mean $\mu = 0$

3. $X \sim bin(100, p)$; H_0 is rejected when: $p = 0.5$
H_a is favored when: $p \neq 0.5$ for $|X-50| > 10$

a) $\alpha = P(|X-50| > 10) \doteq P\left(\frac{|X-50|}{5} > 2\right) = 2P\left(\frac{X-50}{5} < -2\right) \simeq 2\Phi(-2)$
$= 2(1-\Phi(2))$
$\boxed{= 0.0456}$

b) Power as a function of $p \doteq 1-\beta = P(|X-50| > 10)$
$= 1 - P(40 < X < 60)$



4.

X	H_0	H_a
x_1	0.2	0.1
x_2	0.3	0.4
x_3	0.3	0.1
x_4	0.2	0.4

a) Likelihood Ratio:

X_1	X_2	X_3	X_4
2	0.75	3	0.5

	X_3	X_1	X_2	X_4
Λ	3.0	2.0	0.75	0.5

b) $\alpha = 0.2$: 3 degrees of freedom

$\Lambda = \frac{P(x|H_0)}{P(x|H_1)} = \kappa$;

$\boxed{P(\Lambda \le \lambda_0 | H_0) = 0.2 \text{ or } \Lambda > \chi^2_{df=3,\,0.2} = 4.642}$

$\boxed{P(\Lambda \le \lambda_0 | H_0) = 0.5 \quad \Lambda > \chi^2_{df=3,\,0.5} = 2.366}$

d) $\boxed{\Lambda_{x_3} \text{ corresponds to the decision rules for prior probabilities.}}$

c) Prior $P(H_0) = P(H_A)$;

$\boxed{H_0 \text{ is accepted for } \alpha = 0.5, \text{ but not } \alpha = 0.2}$

5. a) False, the significance level of a statistical test is equal to the probability the likelihood is less than a threshold.

b) False, the power $[1-\beta]$ is described by the null hypothesis rejection, while significance level is denoted as the threshold of rejection

c) False, the probability that the null hypothesis is true is $[1-\alpha]$

d) False, the probability the null hypothesis is falsely rejected is not rejected when is false.

e) False, a type 1 error occurs when the statistic crosses the significance level,

The test statistic relates two hypothesis likelihood.

f) False, a type II error tends to be less important than type I.

g) False, the power of a test is determined by the alternative hypothesis.

h) True, the likelihood is a random variable.

6. a)

X	0	1	2	3	4	5	6	7	8	9	10	
Coin 1	0.001	0.0098	0.0439	0.1172	0.2051	0.2461	0.2051	0.1172	0.0439	0.0098	0.0010	$P(x|H_0) = 0.5$
Coin 2	0.000	0.001	0.0014	0.0090	0.0368	0.1029	0.2001	0.2668	0.2335	0.1211	0.0282	$P(x|H_1) = 0.7$
$\frac{P(x|H_1)}{P(x|H_0)} = \frac{Geo(x|p_1)}{Geo(x|p_0)}$	/	1.4	0.84	0.50	0.30	0.19	0.11	0.07	0.04	0.02	0.01	

$\boxed{\frac{P(X|H_1)}{P(x|H_0)} < 1 : \text{Favors } H_0 \qquad \frac{P(X|H_1)}{P(X|H_0)} > 1 : \text{Favors } H_1}$

b) $\boxed{\text{If } P(H_0)|P(H_1) = 10 \text{ then each of the outcomes favor } H_0.}$

c) Significance Level of H_0 if $X \ge 8$

$\alpha = P(\text{Reject } H_0 | H_0) = P(X \ge 8 | H_0)$
$= 1 - P(X \le 7 | 0.5) = 1 - \sum_1^7 (1-0.5)^{x-1}(0.5)$
$= 1 - \sum_1^7 0.5^i = \boxed{0.0078}$

d) The power of the test $[1-\beta] = P(\text{Reject } H_0 | H_1)$
$= P(X \ge 8 | H_1) = 1 - P(X \le 7 | 0.7) = 1 - \sum_1^7 (1-0.7)^{i-1}(0.7)$
$\boxed{= 0.0002}$

68 λ = mean ; $T = \sum_{i=1}^{n} X_i$ a) $\mathbb{P}(x) = \prod_{i=1}^{n} \frac{\lambda^{x_i}}{x_i!} e^{-\lambda} = \frac{\lambda^{\sum x_i}}{\prod x_i!} e^{-n\lambda}$; $\frac{\mathbb{P}(x)}{\mathbb{P}(y)} = \frac{\lambda^{\sum x_i}}{\prod x_i!} e^{-n\lambda} \frac{\prod y_i!}{\lambda^{\sum y_i}} e^{n\lambda} = \lambda^{\sum x_i - \sum y_i} \frac{\prod y_i!}{\prod x_i!}$

Poisson Distribution:

$P(x) = \frac{\lambda^x}{x!} e^{-\lambda}$

Independent if and only if $\sum x_i - \sum y_i = 0$

$\boxed{\sum x_i = \sum y_i = T}$

b) $\frac{P(x_1)}{P(y_1)} = \lambda^{x_1 - y_1} \frac{y_1}{x_1}$; $\boxed{x_1 = y_1 \text{ which is not independent}}$

c) Theorem A: Section 8.8.1:

$f(x_1 \dots x_n|\theta) = g[T(x_1 \dots x_n), \theta] h(x_1 \dots x_n)$; $\boxed{g[T(x_1 \dots x_n), \theta] = \lambda^T = \lambda^{\sum x_i}}$ $\boxed{h(x_1, \dots x_n) = \frac{e^{-\lambda}}{\prod x!}}$

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69. Geometric Distribution: Theorem A: Section 8.8.1 ; $f(x_1 \dots x_n|\theta) = g[T(x_1 \dots x_n), \theta] h(x_1 \dots x_n)$

$P(X=k) = (1-p)^{k-1} p$

$\mathbb{P}(x) = \prod_{i=1}^{n} (1-p)^{n(k_i+1)} p = (1-p)^{\sum k_i - n} \cdot p^n = (1-p)^{\sum x_i - n} \cdot p^n$

$f(x_1 \dots x_n|\theta) = (1-p)^{T-n} \cdot p^n$; $\boxed{g[T(x_1 \dots x_n), \theta] = (1-p)^{T-n} ; h(x_1 \dots x_n) = p^n}$

70. Factorization Theorem

$f(x_1 \dots x_n|\theta) = g[T(x_1 \dots x_n), \theta] h(x_1 \dots x_n)$; Exponential Distribution: $P(x) = \lambda e^{-\lambda x}$; $\mathbb{P}(x) = \prod \lambda e^{-\lambda x_i}$

$= \lambda e^{-\lambda \sum x_i}$

$f(x_1 \dots x_n|\theta) = \lambda e^{-\lambda \sum x_i}$; $\boxed{g[T(x_1 \dots x_n), \theta] = e^{-\lambda \sum x_i}}$

$\boxed{h(x_1 \dots x_n) = \lambda}$

71. $F(x|\theta) = \frac{\theta}{(1+x)^{\theta+1}}$;

$\frac{P(x|\theta)}{P(y|\theta)} = \frac{n\theta}{\prod(1+x)^{\theta+1}} \frac{\prod(1+y)^{\theta+1}}{n\theta}$; $\boxed{\prod_{i=1}^{n}(1+x_i) = \prod_{i=1}^{n}(1+y_i) = T}$

72. Gamma Distribution: $\mathbb{P}(X) = \prod \frac{b^a}{\Gamma(a)} x_i^{a-1} e^{-b x_i}$; $\frac{\mathbb{P}(x)}{\mathbb{P}(y)} = \frac{b^a}{\Gamma(a)} \prod x_i^{a-1} e^{-b x_i} \cdot \frac{\Gamma(a)}{b^a} \frac{1}{\prod y_i^{a-1} e^{-b y_i}}$

$P(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}$;

$\prod_{i=1}^{n} x_i^{a-1} e^{-b x_i} = \prod_{i=1}^{n} y_i^{a-1} e^{-b y_i}$

73. Rayleigh Density: $f(x|\theta) = \frac{x}{\theta^2} e^{-x^2/(2\theta^2)}$

$\mathbb{P}(x|\theta) = \frac{\prod x_i}{\theta^2} e^{-\sum x_i^2/(2\theta^2)}$; $\frac{\mathbb{P}(x|\theta)}{\mathbb{P}(y|\theta)} = \frac{\prod x_i}{\theta^2} e^{-\sum x_i^2/(2\theta^2) + \sum y_i^2/(2\theta^2)} \cdot \left(\frac{\theta}{\prod y_i}\right)$ $\boxed{\prod x_i \text{ or } \sum x_i}$

$\prod x_i e^{-\sum x_i^2/2\theta^2} = \prod x_i e^{-\sum y_i^2/2\theta^2}$; sufficient statistic $\boxed{\prod x_i \text{ or } \sum x_i}$

74. Binomial Distribution:

$P(k) = \sum_{i=1}^{n} \binom{n}{k} p^k (1-p)^{n-k}$

$\mathbb{P}(k) = \exp\left[\log \sum \binom{n}{k} + \sum k \log p + (n - \sum k) \log(1-p)\right]$

$\boxed{\approx \exp[d(k) + \sum T_i(k) c_i(\theta) + S(k)]}$

75. Gamma Distribution

$P(X) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}$

$\mathbb{P}(x) = \exp\left[n \log \frac{b}{\Gamma(a)} + (a-1)\sum \log x - b \sum x\right]$

$\boxed{\approx \exp[d(\theta) + \sum c_i(\theta) T_i(x) + S(x)]}$

Chapter 9: Goo

67. Negative Binomial Distribution:

$P(X=k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$

Poisson Distribution:

$P(x) = \frac{\lambda^x}{x!} e^{-\lambda}$

Frequency 1st Data Set:
500 contiguous $20cm^2$ quad

Frequency 2nd Data Set:
4½ rows wide and 96ft long
2304 sampling units of 2ft length

1st Data Set: Glaux maritima			
Count	Frequency	Poisson	Negative Binomial
0	1	1	2
1	15	9	11
2	27	26	29
3	42	50	51
4	77	72	70
5	77	83	77
6	89	79	73
7	57	65	60
8	48	47	44
9	24	30	29
10	14	17	18
11	16	9	10
12	9	4	5
13	3	2	2
14	1	1	1
Total	500		
Mean	5.76		
S.D.	2.53		
λ	5.76		
r	50.29		
p	0.1028		

The negative binomial shows a goodness of fit with greater accuracy.

2nd Data Set: Potato Beetles			
Count	Frequency	Poisson	Negative Binomial
0	190	20	195
1	264	95	254
2	304	226	57
3	260	357	52
4	294	423	44
5	219	401	36
6	193	316	29
7	150	214	23
8	104	126	17
9	90	67	13
10	60	32	10
11	46	14	7
12	29	5	5
13	36	2	4
14	19	1	3
15	12	0	2
16	11	0	2
17	6	0	1
18	10	0	0
19	2	0	0
20	4	0	0
21	1	0	0
22	3	0	0
23	4	0	0
24	1	0	0
25	1	0	0
26	0	0	0
27	0	0	0
28	1	0	

Method of Moments:

Negative Binomial: $y = k - r$

$$E[x] = \sum_{k=1}^{\infty} k \binom{k-1}{r-1} p^r (1-p)^{k-r} = \sum_{k=1}^{\infty} \binom{y+r-1}{r-1} k p^r (1-p)^{k-r}$$

$$E[y] = \sum_{k=1}^{\infty} \binom{y+r-1}{y} y p^r (1-p)^y = \sum \frac{(y+r-1)!}{(y-1)!(r-1)!} p^r (1-p)^y$$

$$= \sum_{k=1}^{\infty} \frac{r(1-p)}{p} \frac{p}{r(1-p)} \frac{(y+r-1)!}{(y-1)!(r-1)!} p^r (1-p)^y$$

$$= \frac{r(1-p)}{p} \sum_{k=1}^{\infty} \frac{(y+r-1)!}{(y-1)!\, r!} p^{r+1} (1-p)^{y-1}$$

Let $y - 1 = z$; $y = z + 1$

$y = 1$; $z = 0$

$$= \frac{r(1-p)}{p} \sum_{k=1}^{\infty} \frac{(z+1+r-1)!}{z!\, r!} p^{r+1} (1-p)^z :$$ Probability Mass Function

$$= \frac{r(1-p)}{p} \sum_{k=1}^{\infty} \binom{r+1+z-1}{z} p^{r+1} (1-p)^z \quad 1$$

$$\boxed{E[y] = \frac{r(1-p)}{p}} \; ; \; \boxed{p = 1 - \frac{E[y]}{Var[y]}}$$

Poisson $r = \frac{p}{(1-p)} E[y]$

Poisson: $E[x] = \int_0^{\infty} x \frac{\lambda^x}{x!} e^{-\lambda} dx = e^{-\lambda} \int_0^{\infty} \frac{\lambda^x}{(x-1)!} dx$; $x - 1 = t$

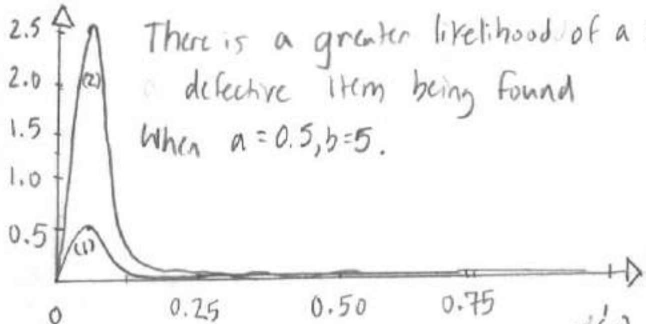
$$= e^{-\lambda} \int_0^{\infty} \frac{\lambda^{t+1}}{t!} dt = \lambda e^{-\lambda} \int_0^{\infty} \frac{\lambda^t}{t!} dt = \lambda e^{-\lambda} e^{\lambda} = \boxed{\lambda}$$

The goodness of fit for the second data set did not accurately represent the data. χ^2-values described by $\frac{(x-E)}{\sigma}$ were large for both Poisson and Negative Binomial data.

63. $n = 100$; $N = 3$ defective items; Beta Distribution: $f(x) = \underbrace{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}}_{\text{Prior}}$ $E[x] = \frac{a}{a+b}$

1) $a=b=1$ 2) $a = 0.5$, $b = 5$

$Var[x] = \frac{ab}{(a+b)^2(a+b+1)}$



There is a greater likelihood of a second defective item being found When $a = 0.5, b = 5$.

Likelihood

$$f(x) = \sum_{i=1}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

Posterior: $f(x|\alpha,\beta) = \binom{n}{k}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{(k+\alpha-1)}(1-x)^{(n-k+\beta-1)}$

64. $X = 0$ or 1; 1) $a=b=1$ $f(x|a,b) = \binom{100}{1}\frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}\int_0^1 x^{(4)}(1-x)^{(99)}\,dx$; $f(x|a,b) = \binom{99}{1}\frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}\int_0^1 x^4(1-x)^{98}dx$

2) $a = 0.5, b = 5$ $\underbrace{f(x|a,b) = \binom{100}{1}\frac{\Gamma(11/2)}{\Gamma(\frac{1}{2})\Gamma(5)}\int_0^1 x^{7/2}(1-x)^{103}\,dx}_{\#1 \text{ draw}}$; $\underbrace{f(x|a,b) = \binom{99}{1}\frac{\Gamma(11/2)}{\Gamma(\frac{1}{2})\Gamma(5)}\int_0^1 x^{7/2}(1-x)^{102}dx}_{\#2 \text{ Draw}}$

65. $n = 20$
$\mu = ? \rightarrow \bar{x} = 10$
$\sigma^2 = 1$

Normal Distribution: $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Prior: $\mu = ?$ $\sigma = ?$

Posterior: $\sigma^2 = 0.1$ $\mu = 15$

Prior $\mu \sim N(\mu_0, \sigma_0^2) \propto \exp\left(\frac{-\sum(x-\mu)^2}{2\sigma^2}\right)$

$= \exp\left(-\frac{(\theta-\bar{x})^2}{2\sigma^2/n}\right)$

$\sum(x_i-\mu)^2 = \sum[x_i - \bar{x} - (\mu-\bar{x})]^2 = \sum(x-\bar{x})^2 + n(\theta-\bar{x})^2$

Posterior = Likelihood × Prior:

$$\boxed{\propto e^{\frac{-\sum(x-\mu)^2}{2\sigma^2}} \circ e^{\frac{-(\mu-\mu_0)^2}{2\sigma_0^2}}}$$

$$\propto e^{-\frac{(\mu-\bar{x})^2}{2\sigma^2/n} - \frac{(\mu-\mu_0)^2}{2\sigma_0^2}}$$

$$\propto e^{\frac{-\mu^2+2\mu\bar{x}+\bar{x}^2}{2\sigma^2/n} - \frac{\mu^2+2\mu_0\mu-\mu_0^2}{2\sigma_0^2}}$$

$$\propto e^{-\left(\frac{1}{2\sigma^2/n}-\frac{1}{2\sigma_0^2}\right)\mu^2 + \frac{2}{2}\left(\frac{\mu_0}{\sigma_0^2}+\frac{\bar{x}}{\sigma^2/n}\right)\mu - \frac{\bar{x}^2}{2\sigma^2/n} - \frac{\mu_0^2}{2\sigma_0^2}}$$

$$\propto e^{-\frac{1}{2}\left(\frac{1}{\sigma^2/n}-\frac{1}{\sigma_0^2}\right)\left[\mu - \frac{\frac{\mu_0}{\sigma_0^2}+\frac{\bar{x}}{\sigma^2/n}}{\left(\frac{1}{\sigma^2/n}-\frac{1}{\sigma_0^2}\right)}\right]^2 + \left[\frac{\frac{\mu_0}{\sigma_0^2}+\frac{\bar{x}}{\sigma^2/n}}{\left(1/\sigma^2/n - 1/\sigma_0^2\right)}\right]^2 - \frac{\bar{x}^2}{2\sigma^2/n} - \frac{\mu_0^2}{2\sigma_0^2}}$$

$$\boxed{\propto e^{-\frac{1}{2}\frac{(\mu-\bar{\mu})^2}{\sigma_0^2}}}$$

Posterior $p(x) \propto \exp\left\{\frac{-\sum(x-\mu)^2}{2\sigma^2} - \frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right\}$

$\propto \exp\left\{-\frac{(\mu-\bar{x})^2}{2\sigma^2/n} - \frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right\}$

$\propto \exp\left\{-\frac{(\mu-\mu_n)^2}{2\sigma_n^2}\right\}$

$\mu_n = \frac{\frac{1}{\sigma_0^2}\mu_0 + \frac{n}{\sigma^2}\bar{x}}{\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}}$

$\frac{1}{\sigma_n^2} = \frac{1}{\sigma_0^2} + \frac{1}{\sigma^2/n}$

$$\frac{1}{\sigma_n^2} = \frac{1}{\sigma_0^2} + \frac{1}{\sigma^2/n} = \frac{1}{0.1^2} = \frac{1}{\sigma_0^2} + \frac{1}{1/20}$$

$$\frac{1}{\sigma_0^2} = 100 + 20 = 120; \boxed{\sigma_0 = \sqrt{1/120}}$$

$$\mu_n = \frac{\frac{1}{1/120}\circ\mu_0 + \frac{20}{0.1^2}10}{\frac{1}{1/120}+\frac{20}{0.1^2}} = 15 = \frac{120\mu_0 + 20{,}000}{2120}; \quad \mu_0 = \frac{15\cdot 2120 - 20000}{120} = 98.33$$

66 θ is uniform $[0,1]$

$P(x) = \frac{1}{b-a}$

$P(\text{success}) = \theta$

a) Posterior Density

$$\boxed{P(x) = \frac{1}{b-a}\left(\frac{1}{b-a}\right) = \theta}$$

b) Probability of a third shot would be θ.

59. $P(x|M) = 50\%$; $P(x|F) = 50\%$; $P(a|M) = P(b|M) = \alpha$: $P(a|F) = P(b|F) = \alpha$

a) $P(MM) = P(I)\cdot P(I|M) + P(F)P(F|FF) = \frac{1}{2}\cdot\alpha + \frac{1}{4}(1-\alpha)$

$= \frac{1}{2}\alpha + \frac{1}{4} - \frac{\alpha}{4} = \frac{1}{4}\alpha + \frac{1}{4} = \boxed{\frac{1+\alpha}{4}}$

$P(MF) = 1 - P(MM) - P(FF) = \boxed{\frac{(1-\alpha)}{2}}$

$\boxed{\text{Maternal}}$
α

	M	F
M	1/2	
F		1/2

Identical

$\boxed{\text{Fraternal}}$
$(1-\alpha)$

	M	F
M	$1/4\,\alpha$	1/4
F	1/4	1/4

b) $n_1 = MM$; $n_2 = FF$; $n_3 = MF$

Maximum Likelihood Estimation: $\boxed{\text{Multinomial}}$

$\ln P(x|M/F) = \ln n! - \ln \prod x_i + (n_1+n_2)\log(1+\alpha) + n_3 \log(1-\alpha)$

$\ln' P(x|M/F) = \frac{n_1+n_2}{1+\alpha} - \frac{n_3}{1-\alpha} = 0$; $\boxed{\alpha = \frac{n_1+n_2-n_3}{n_1+n_2+n_3}}$

Variance of a Multinomial: $Var(\theta) = n\alpha(1-\alpha) = (n_1+n_2-n_3)\left(1 - \frac{n_1-n_2+n_3}{n_1+n_2+n_3}\right) = \boxed{\frac{(n_1+n_2-n_3)(2n_3)}{n_1+n_2+n_3}}$

60. Exponential Distribution: a) Maximum Likelihood Estimate:

$f(x|\tau) = \frac{1}{\tau} e^{-x/\tau}$

$\ln f(x|\tau) = -\ln\tau - \frac{x}{\tau}$; $\frac{d\ln f(x|\tau)}{d\tau} = \frac{-1}{\tau} + \frac{x}{\tau^2} = 0$; $1 = \frac{x}{\tau}$; $\boxed{\tau = x}$

b) Sampling Distribution of the mle: $f(x|\hat{\tau}) = \frac{1}{\hat{\tau}} e^{-x/\hat{\tau}} = \frac{1}{x} e^{-1} = \boxed{\frac{1}{ex}}$

$u-1 = -1$; $\frac{x}{\tau} = y$; $x = \tau y$; $dx = \tau dy$

c) Central Limit Theorem: $\lim_{n\to\infty} P\left(\frac{S}{\sigma\sqrt{n}} \le x\right) = \Phi(x)$

where $S = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} \frac{1}{e}$; $\boxed{\lim_{n\to\infty} P\left(\frac{1}{e\sigma\sqrt{n}} < \frac{1}{e}\right) = \Phi(x)}$

d) Bias $E[\tau] = \int_0^\infty \frac{x}{\tau} e^{-x/\tau} dx = \frac{1}{\tau}\int_0^\infty x^{2-1} e^{-x/\tau} dx = \frac{1}{\tau}\int_0^\infty (\tau y)^{u-1} e^{-y} \tau dy = \tau\Gamma(2)$

$\to$ Bias $= E[\tau] - \tau = \tau\Gamma(2) - \tau = \boxed{\emptyset}$

e) Asymptotic variance: $I(\tau) = -E\left[\frac{\partial^2}{\partial\tau^2}[\ln f(x|\tau)]\right] = -E\left[\frac{\partial}{\partial\tau}\left[\frac{-1}{\tau} + \frac{x}{\tau^2}\right]\right] = -E\left[\frac{1}{\tau^2} - \frac{2x}{\tau^3}\right] = \frac{2E[x]}{\tau^3} - \frac{1}{\tau^2} = \frac{2\tau}{\tau^3} - \frac{1}{\tau^2} = \frac{1}{\tau^2}$; $\boxed{Var(\hat{\tau}) = \frac{\tau^2}{n}}$

Method of Moments: $E[x] = \tau$; $E[x^2] = 2\tau^2$; $\boxed{\text{The method of moments estimate shows a similar unbiased estimate of variance.}}$

f) Confidence Interval for τ:

$\hat{\tau} \pm 1.96\sqrt{Var(\hat{\tau})} = \boxed{\hat{\tau} \pm \frac{1.96\tau}{\sqrt{n}}}$ g) The exact confidence interval for $\hat{\tau}$ would be $\boxed{\bar{x} \pm 1.96\tau}$

61. $\lim_{n\to\infty} \frac{(n+1)}{(n+2)} = 1$; Laplace's rule of succession suggests the probability approaches 100% success.

62. Gamma Distribution Exponential Distribution: Average time to serve = 5.1 minutes ; $\lambda = 20/5.1 = 3.92 \frac{\text{customers}}{\text{min}}$

$n = 20$ customers

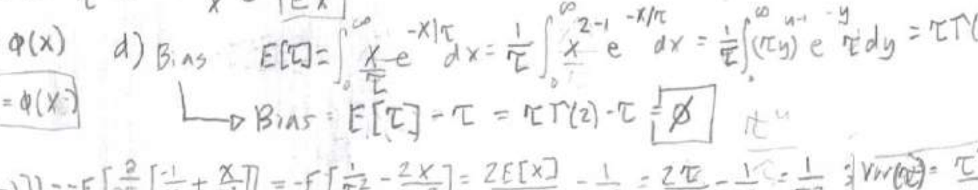
$\underbrace{T(x) = \int_0^\infty \frac{b^a x^{a-1} e^{-bx}}{\Gamma(a)} dx}_{\text{Conjugate prior}}$ $\underbrace{f(x|\lambda) = \frac{1}{\lambda} e^{-x/\lambda}}_{\text{Posterior}}$ $f(x|\tau)\cdot T(x)$

1)

$\bar{X}_{Gam} = 0.5$	$\sigma_{Gam} = 1$
$\alpha = 0.25$	$b = 0.5$

2)

$\bar{X}_{Gam} = 10$	$\sigma_{Gam} = 20$
$a = 5$	$b = 0.5$



Posterior Means: $\frac{x}{\lambda_{post}} = bx + \frac{x}{\lambda_{prior}}$

$\lambda_{post} = \frac{\lambda_{prior}}{b + \lambda_{prior}}$

1) $\lambda_{post} = 0.887$

2) $\lambda_{post} = 0.887$

The posterior means represent exact average customers per minute, although with different priors, waiting times for 1-2 or 4-8 customers in the restaurant are shifted.

55.

Type	Count	Probability
Starchy Green	1997	$0.25(2+\theta)$
Starchy White	906	$0.25(1-\theta)$
Sugary Green	904	$0.25(1-\theta)$
Sugary White	32	0.25θ

$\underbrace{\qquad\qquad}_{\text{Linkage Factors.}}$

a) Multinomial Distribution: $P(\theta) = \dfrac{n!}{X_1 \cdot X_2 \cdot X_3 \cdot X_4} 0.25(2+\theta)^{X_1} \cdot 0.25(1-\theta)^{X_2+X_3} \cdot 0.25\theta^{X_4}$

$P(X) = \dfrac{n!}{\prod X_i} \prod p(X_i)^{n_i}$

$\ln P(\theta) = \ln n! - \sum X_i + X_1 \ln \frac{(2+\theta)}{4} + (X_2+X_3)\ln\frac{(1-\theta)}{4} + X_4 \ln \frac{\theta}{4}$

$\ln' P(\theta) = \dfrac{4X_1}{2+\theta} - \dfrac{4(X_2+X_3)}{1-\theta} + \dfrac{4X_4}{\theta} = 0 ; [4X_1\theta + 4X_4(2+\theta)](1-\theta) = 4(X_2+X_3)(2+\theta)\theta$

$(4X_1\theta + 8X_4 + 4X_4\theta) - 4X_1\theta^2 - 8X_4\theta - 4X_4\theta^2$

$= (4X_2\theta + 4X_3\theta)(2+\theta) = 8X_2\theta + 8X_3\theta + 4X_2\theta^2 + 4X_3\theta^2$

$-4(X_1+X_2+X_3+X_4)\theta^2 + 4(X_1 - 2X_2 - 2X_3 + X_4)\theta - 2X_4 = 0$

$3839\theta^2 + 1655\theta - 64 = 0$

$\boxed{\hat\theta = 0.0357}$

$I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\ln p(\theta)\right]$

$= -E\left[\dfrac{4X_1}{(2+\theta)^2} - \dfrac{4(X_2+X_3)}{(1-\theta)^2} + \dfrac{4X_4}{\theta^2}\right]$

$= \dfrac{-4n(2+\theta)}{4(2+\theta)^2} - \dfrac{[4(1-\theta)+4(1-\theta)]n}{4(1-\theta)^2} - \dfrac{4n_4\theta}{4\theta^2} = \dfrac{+n_1}{(2+\theta)} + \dfrac{n_2+n_3}{2(1-\theta)} + \dfrac{n_4}{\theta} = \dfrac{n_1+n_2+n_3+n_4}{(2+\theta)2(1-\theta)\theta}$

$Var(\hat\theta) = \dfrac{1}{nI(\theta)} = \dfrac{1}{5670} = \boxed{3.24\times10^{-5}}$

b) 95% confidence Interval:

$\boxed{0.0357 \pm 1.11\times10^{-2}}$

c) Actual $\theta = 4\left(\frac{32}{3839}\right) = 3.33\times10^{-2}$; $SD = \sqrt{np(1-p)} = \sqrt{32 \cdot 0.25(3.33\times10^{-2})(1-0.25(3.33\times10^{-2}))} = \boxed{5.13\times10^{-3}}$

d) $\boxed{3.33\times10^{-2} \pm 1.01\times10^{-2}}$

56. 1) $\bar{X} = n(2+\theta)/4$

$\boxed{\hat\theta_1 = \dfrac{4\bar{X}}{n} - 2}$

Bias: $E[\hat\theta_1] = E\left[\frac{4\bar X}{n} - 2\right] = \frac{4}{n}E[\bar X] - 2 = \boxed{\dfrac{4\bar X}{n} - 2}$

Variance: $\boxed{Var(\hat\theta_1) = \frac{1}{n}\sum(X_i - \bar X) - \frac{4\bar X}{n} - 2}$

Standard Error: $\boxed{\sigma_{\hat\theta_1} = \dfrac{\sqrt{\frac{1}{n}\sum(X_i-\bar X) - \frac{4\bar X}{n} + 2}}{\sqrt{n}}}$

2) $\bar X = n\theta_4/4$

$\boxed{\theta_4 = \dfrac{4\bar X}{n}}$

$E[\hat\theta_2] = E\left[\frac{4\bar X}{n}\right] = \frac{4}{n}E[\bar X] = \boxed{\dfrac{4}{n}\bar X}$

$\boxed{Var(\hat\theta_2) = \frac{1}{n}\sum(X_i - \bar X) - \frac{4}{n}\bar X}$

$\boxed{\sigma_{\theta_2} = \dfrac{\sqrt{\frac{1}{n}\sum(X_i - \bar X) - \frac{4}{n}\bar X}}{\sqrt{n}}}$

57. $\dfrac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$ a) Which of the following is unbiased? $s^2 = \frac{1}{n-1}\sum(X_i - \bar X)^2$; $\hat\sigma^2 = \frac{1}{n}\sum(X_i - \bar X)^2$

$MSE = Var + Bias^2$

$\boxed{Bias_{s^2} = E[s^2] = E\left[\frac{n}{n-1}\hat\sigma^2\right] = \frac{n}{n-1}E[\hat\sigma^2] = \frac{n}{n-1}\hat\sigma^2 =}$

b) $MSE_{s^2} = Var(s^2) + Bias(s^2)^2 = \boxed{\dfrac{2\sigma^4}{(n-1)} + 0}$

$MSE_{\hat\sigma^2} = Var(\hat\sigma^2) + Bias(\hat\sigma^2)^2 = Var\left(\frac{n-1}{n}s^2\right) + \left[E(\hat\sigma^2) - \sigma^2\right]^2$

$Bias_{\hat\sigma^2} = E[\hat\sigma^2] = E\left[\frac{(n-1)}{n}s^2\right] = \frac{(n-1)}{n}E[s^2] = \frac{n-1}{n}\sigma^2$

$= Var\left(\frac{n-1}{n}s^2\left(\frac{\sigma^2}{\sigma^2}\right)\right) + \frac{(n-1)}{n}\sigma^2 - \sigma^2 = \frac{\sigma^4}{n^2}Var\left(\frac{n-1}{\sigma^2}s^2\right) - \frac{1}{n}\sigma^2 = \boxed{\dfrac{(2n-1)\sigma^4}{n^2}}$ Smallest

c) $Y = \rho\sum(X_i - \bar X)^2$; $E[Y] = E[\rho(n-1)s^2] = \rho(n-1)E[s^2] = \rho(n-1)\sigma^2$

$Var(Y) = V(\rho(n-1)s^2) = \rho^2(n-1)^2V(s^2) = 2\rho^2(n-1)\sigma^4$; $MSE[Y] = V[Y] + Bias^2(Y) = \sigma^4[2\rho^2(n-1) + (\rho n - \rho - 1)^2]$

$MSE'[Y] = 2(n-1)\sigma^4(\rho n + \rho - 1) = 0$; $\rho = \frac{1}{n+1}$

$MSE''[Y] = 2(n-1)\sigma^4(n+1) = 0$

$\boxed{\text{MSE is minimized for } \tfrac{1}{n+1}\sum(X_i - \bar X)^2}$

58. $P(AA) = (1-\theta)^2$; $P(Aa) = 2\theta(1-\theta)$; $P(aa) = \theta^2$ a) Maximum Likelihood Estimate:

Haptoglobin Type		
Hp1-1	Hp1-2	Hp2-2
10	68	112

Total: 200

a) $P(\theta) = \dfrac{n!}{\prod X_i!}\prod_i p(X_i)^{X_i}$; $\ln P(\theta) = \ln n! - \sum \ln X_i + \sum X_i\, p(X_i)$

$\ln P(\theta) = \ln n! - \sum_{i=1}^{3}\log X_i! + X_1\log((1-\theta)^2) + X_2\log(2\theta(1-\theta)) + X_3\log\theta^2$

$\ln' P(\theta) = \dfrac{-2X_1}{(1-\theta)} + \dfrac{2X_2}{2\theta} - \dfrac{X_2}{1-\theta} + \dfrac{2X_3}{\theta} = \dfrac{(X_2+2X_3)}{\theta} - \dfrac{(X_2+2X_1)}{(1-\theta)} = 0$

b) $(X_2+2X_3)(1-\theta) = (X_2+2X_1)\theta$; $\dfrac{+X_2+2X_3}{X_2+2X_1+X_2+2X_3} = \boxed{\dfrac{+X_2+2X_3}{2X_1+2X_3+2X_2}} = \dfrac{+68+2(112)}{2(10)+2(112)+2(68)} = 0.768$

Asymptotic Variance:

$I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\left[\ln n! - \sum\log X_i! + X_1\log((1-\theta)^2) + X_2\log(2\theta(1-\theta)) + X_3\log\theta^2\right]\right]$

$= -E\left[\frac{\partial}{\partial\theta}\left[\frac{-2X_1}{(1-\theta)} + \frac{2X_2}{2\theta} - \frac{X_2}{1-\theta} + \frac{2X_3}{\theta}\right]\right] = -E\left[\frac{-2X_1}{(1-\theta)^2} - \frac{X_2}{\theta^2} - \frac{X_2}{(1-\theta)^2} - \frac{2X_3}{\theta^2}\right] = \left[\frac{2n(1-\theta)^2}{(1-\theta)^2} - \frac{n2\theta(1-\theta)}{\theta^2} - \frac{2n2\theta(1-\theta)}{(1-\theta)} - \frac{2\theta^2 n}{\theta^2}\right]$

$= \dfrac{2n}{\theta(1-\theta)}$

$Var(\hat\theta) = \dfrac{1}{nI(\theta)} = \dfrac{\theta(1-\theta)}{2n^2} = \boxed{2.23\times10^{-6}}$

c) 99% Confidence Interval:

$\hat\theta \pm 2.575\cdot\sqrt{Var(\hat\theta)}$

$\boxed{0.768 \pm 3.85\times10^{-3}}$

d) $\hat\theta^2 = \frac{112}{200}$; $\boxed{\hat\theta = 0.748}$ $\boxed{SD = 1.53\times10^{-3}}$

vs

mle : $SD = 1.43\times10^{-3}$

e) 99% Confidence:

$\boxed{0.748 \pm 3.94\times10^{-3}}$

50. Rayleigh Distribution

a) $f(x|\theta) = \frac{x}{\theta^2} e^{-x^2/2\theta^2}$

a) Method of Moments Estimate: $x^2 = u$; $dx = \frac{1}{2} u^{-1/2} du$

$$E[X] = \int_0^{\infty} \frac{x^2}{\theta^2} e^{-x^2/2\theta^2} dx = \frac{1}{\theta^2}\int_0^{\infty} u\, e^{-u/2\theta^2} \frac{du}{2\sqrt{u}} = \frac{1}{2\theta^2}\int_0^{\infty} u^{1/2} e^{-u/2\theta^2} du = \frac{1}{2\theta^2}\int_0^{\infty} u^{3/2-1/2} e^{-u/2\theta^2} du$$

$$= \frac{\Gamma(3/2)}{2\theta^2}(2\theta^2)^{3/2} = \sqrt{2}\theta\, \Gamma(3/2) = \sqrt{2}\theta \cdot \frac{1}{2}\cdot \Gamma(1/2) = \theta\sqrt{\frac{\pi}{2}}; \boxed{\hat{\theta} = \sqrt{\frac{2}{\pi}} E[X]}$$

b) Maximum Likelihood Estimate:

$$\frac{d\sum \ln f(x|\theta)}{d\theta} = \frac{-2n}{\theta} + \frac{\sum x^2}{\theta^3} = 0$$

$$-2n + \frac{\sum x^2}{\theta^2} = 0; \frac{\sum x^2}{2n} = \theta^2; \boxed{\hat{\theta} = \sqrt{\frac{1}{2n}\sum x_i}}$$

c) Asymptotic Variance of Maximum Likelihood:

$$I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\ln f(x|\theta)\right] = -E\left[\frac{\partial}{\partial\theta}\left[-2n + \frac{x^2}{\theta^2}\right]\right] = +\frac{2E[x^2]}{\theta^3}; \quad var(\hat{\theta}) \approx \frac{1}{nI(\hat{\theta})} = \boxed{\frac{\hat{\theta}^3}{2E[x^2]}} = \boxed{\frac{\hat{\theta}}{4}}$$

51. Double Exponential Distribution:

$$f(x|\theta) = \frac{1}{2} e^{-|x-\theta|}; -\infty < x < \infty \quad ; n = 2m+1$$

$$\frac{d\sum \ln f(x|\theta)}{d\theta} = \sum_i \frac{x-\theta}{|x-\theta|} = 0; \boxed{\hat{\theta} = \sum_i x_i}$$

$$\hat{\theta} = \sum_{i=1}^{n} x_i = \sum_{i=1}^{2m+1} x_i = \underbrace{\sum_{i=1}^{m+1} x_i}_{1/2} + \underbrace{\sum_{m+1}^{2m+1} x_i}_{1/2}$$

52. $f(x|\theta) = (\theta+1)x^{\theta}$; $0 \le x \le 1$

a) Method of Moments Estimate:

$$E[X] = \int_0^1 (\theta+1)x^{\theta+1}dx = \frac{(\theta+1)}{(\theta+2)} x^{\theta+2}\Big|_0^1 \Rightarrow \frac{(\theta+1)}{(\theta+2)}; \theta(E[X]-1) = 1-2E[X]$$

$$\boxed{\hat{\theta} = \frac{1-2E[X]}{E[X]-1}}$$

b) Maximum Likelihood Estimate:

$$\frac{d\sum \ln f(x|\theta)}{d\theta} = \frac{1}{\theta+1} + \sum \ln x_i = 0; \boxed{\hat{\theta} = \frac{1-\sum \ln x_i}{\sum \ln x_i}}$$

c) Asymptotic Variance of MLE:

$$I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\ln f(x|\theta)\right] = -E\left[\frac{\partial}{\partial\theta}\left[\frac{1}{\theta+1} + \ln x_i\right]\right] = \frac{1}{(\theta+1)^2}; \quad Var(\hat{\theta}) = \frac{1}{nI(\hat{\theta})} = \frac{(1+\hat{\theta})^2}{n} = \frac{\left(1 + \frac{1-\sum \ln x_i}{\sum \ln x_i}\right)^2}{n}$$

$$\boxed{= \frac{1}{n(\sum \ln x_i)^2}}$$

d) Sufficient Statistic: $\boxed{X^{\theta}}$

53. $X_1, \ldots X_n$ uniform on $[0,\theta]$ a) Find the method of moments estimate of θ, mean, and variance.

Uniform Distribution:

$$P(x) = \frac{1}{b-a}\ [a,b]$$

$$= \frac{1}{\theta - 0}\ [0,\theta]$$

$$E[X] = \int_0^{\theta} \frac{x}{\theta} dx = \frac{\theta^2}{2\theta} = \frac{\theta}{2}; \boxed{\hat{\theta} = 2E[X]}; E[X^2] = \frac{\theta^2}{3}; var(\hat{\theta}) = \frac{\hat{\theta}^2}{12} = \boxed{\frac{\theta^2}{3}}$$

b) Maximum Likelihood Estimate: $\frac{d\sum \ln p(x)}{d\theta} = \frac{-1}{\theta} = 0$; $\boxed{\hat{\theta} = 0}$

Thus $\boxed{\theta_{max} = max(X_1, \ldots X_n)}$

c) $f(x_n|\theta) = n\frac{x^{n-1}}{\theta^{n}}$; $E[X] = \int_0^{\theta} \frac{nx^n}{\theta^n}dx = \frac{n}{\theta^n}\int_0^{\theta} x^n dx = \frac{n}{(n+1)}\theta$

$$E[X^2] = \int_0^{\theta} \frac{n}{\theta} x^{n+1} dx = \boxed{\frac{n\theta^2}{n+2}}; Var(X) = \boxed{\frac{n\theta^2}{n+2} - \frac{n^2\theta^2}{(n+1)^2}}$$

$$= \frac{n\theta^2}{(n+1)^2(n+2)}$$

Bias of Maximum Likelihood Estimate:

$$Bias = E[X - \theta] = \theta\left[\frac{n}{n+1} - 1\right] = \boxed{\frac{-\theta}{(n+1)}}$$

Bias of Method of Moments:

$$Bias = E[X - \theta] = \boxed{0}; \boxed{\text{Bias of MOM} < \text{MLE}}$$

d) $\hat{\theta} = ?\ max(X_i)$

$$E[\hat{\theta}] = E[max(X_i)] = \frac{n\theta}{(n+1)}$$

$$\boxed{\theta = \frac{n+1}{n} max\ X_i}$$

54. $n = 15$; $\mu = 10$; $s^2 = 25$

90% confidence Interval:

$P(|\frac{X-\mu}{\sigma_{\bar{x}}}| < z(\alpha/2)) = 0.90$

$P(0.05 < \frac{X-\mu}{\sigma_{\bar{x}}} < 0.95) = 0.90$

$\mu \pm 1.65\sigma_{\bar{x}}$

$\boxed{10 \pm 2.13}$

$$P\left(\chi^2_{n-1}(1-\alpha/2) < \frac{n\hat{\sigma}^2}{\sigma^2} < \chi^2_{n-1}(\alpha/2)\right) = 0.90$$

$$P\left(\frac{n\hat{\sigma}^2}{\chi^2_{14,0.95}} < \sigma^2 < \frac{n\hat{\sigma}^2}{\chi^2_{14,0.05}}\right) = 0.90$$

$\boxed{(16 < \sigma^2 < 57)}$

47. Pareto Distribution:

$f(x|x_0,\theta) = \theta x_0^{\theta} x^{-\theta-1}$; $x \geq x_0$, $\theta > 1$

a) Method of Moments Estimate for θ

$$E[x] = \int_{x_0}^{\infty} x\,\theta x_0^{\theta} x^{-\theta-1}\,dx = \frac{\theta x_0^{\theta} x^{-\theta+1}}{-\theta+1}\Big|_{x_0}^{\infty} = \left(\frac{\theta}{1-\theta}\right) x_0 x_0^{\theta} x_0^{-\theta+1} = \boxed{\left(\frac{\theta}{1-\theta}\right) x_0}$$

$$E[x^2] = \int_{x_0}^{\infty} x^2 \theta x_0^{\theta} x^{-\theta-1}\,dx = \frac{\theta x_0^{\theta} x^{-\theta+2}}{-\theta+2}\Big|_{x_0}^{\infty} = \frac{\theta x_0^{\theta}}{-\theta+2} x_0^{-\theta+2} = \left(\frac{\theta}{1-\theta}\right) x_0^2$$

$$Var[x] = \left(\frac{\theta}{1-\theta}\right) x_0^2 - \left(\frac{\theta}{1-\theta}\right)^2 x_0^2 = \left(\frac{\theta-\theta^2-\theta^2}{(1-\theta)^2}\right) x_0^2 = \frac{(1-2\theta)\theta}{(1-\theta)^2} x_0^2$$

$$\boxed{\hat{\theta} = \frac{E[x]}{(x_0 + E[x])}}$$

b) Maximum Likelihood Estimate:

$\ln f(x|x_0,\theta) = \ln\theta + \theta\ln x_0 - (\theta+1)\ln x$

$\sum \ln f(x|x_0,\theta) = n\ln\theta + n\theta\ln x_0 - (\theta+1)\sum \ln x_i$

$$\frac{d\sum \ln f(x|x_0,\theta)}{d\theta} = \frac{n}{\theta} + n\ln x_0 - \sum \ln x_i = 0$$

$$\boxed{\hat{\theta} = \frac{n}{\sum \ln x_i - n\ln x_0}}$$

c) Asymptotic Variance: $I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\ln f(x|x_0,\theta)\right] = -E\left[\frac{\partial}{\partial\theta}\left(\frac{n}{\theta} + n\ln x_0 - \sum \ln x_i\right)\right] = +\frac{n}{\theta^2}$

$$Var(\hat{\theta}) \approx \frac{1}{n I(\hat{\theta})} = \frac{\hat{\theta}^2}{n^2} = \boxed{\frac{1}{(\sum \ln x_i - n\ln x_0)^2}}$$

d) Sufficient Statistic $\boxed{x \geq x_0 \text{ and } \theta > 2.}$

48. Observation: $p_0 = P(X=0) = e^{-\lambda}$ Method of Propagation Error: ① Expansion of $F(x)$ about the mean

② First term is the mean

③ Second term is variance

Poisson Distribution: Note: $Y \sim bin(n, p_0)$

$$P(x) = \frac{\lambda^x}{x!} e^{-\lambda} \qquad \hat{\lambda} = -\log(Y/n);\ p_0 = e^{-\lambda} = \frac{Y}{n};\ E[Y] = np_0$$

Approximate Expression of Variance: $\sigma_f^2 = \sum_{i=1}^{n} \left|\frac{df}{dx}\right|^2 \sigma_i^2 = \sum \frac{1}{y^2}\lambda = \boxed{\frac{n\lambda}{y^2}}$

Bias: $\sum \frac{\partial f}{\partial x_i} E[x_i] - \lambda = \boxed{-\frac{n\lambda}{y} - \lambda}$;

Maximum Likelihood Estimate: $\frac{d\sum \ln p(\lambda)}{d\lambda} = \frac{\sum x_i}{\lambda} - n = 0$; $\boxed{\hat{\lambda}_{mle} = \frac{\sum x_i}{n} = \bar{x}}$

Variance Maximum likelihood Estimate: $Var(\hat{\lambda}_{mle}) = \frac{\lambda}{n}$ Efficiency: $\frac{Var(\hat{\lambda}_{mle})}{Var(\hat{\lambda}_{mop})} = \frac{\frac{\lambda}{n}}{\frac{n\lambda}{y^2}} = \boxed{\frac{y^2}{n^2}}$

49. Muon Decay Binomial Distribution Method of Moments:

a) $f(x|\alpha) = \frac{1+\alpha x}{2}$ $\quad p(k) = \binom{n}{k} p^k (1-p)^{n-k}$ $\quad$ Binomial(n,p) = Bin$\left(n, \frac{2+\alpha}{4}\right)$

$-1 \leq x \leq 1$

$-1 \leq \alpha \leq 1$

$$\int_0^1 f(x|\alpha)\,dx = \int_0^1 \frac{1+\alpha x}{2}\,dx = \left[\frac{1}{2} + \frac{\alpha}{4}\right] = \frac{2+\alpha}{4} = p$$

$E[\hat{p}] = E\left[\frac{x}{n}\right] = \frac{2+\alpha}{4} = p$; $Var(4p-2) = 4^2 Var(p) = \boxed{16p(1-p)}$

$\boxed{\alpha = 4p-2}$ $\qquad Var\left(4\frac{X}{n} - 2\right) = \frac{4^2}{n^2} Var(p) = \boxed{\frac{16}{n^2} p(1-p)}$

$E[x] = \mu = \frac{\alpha}{3}$

b) Binomial Variance: $Var[k] = np(1-p)$; Muon Decay Variance

K	Var Bin	Var Binn	Var Mom = Var Binn / n^2
0	0	3.99	$3.99/n^2$
0.1	0.53	3.96	$3.96/n^2$
0.2	0.55	3.91	$3.91/n^2$
0.3	0.58	3.84	$3.84/n^2$
0.4	0.60	3.75	$3.75/n^2$
0.5	0.63	3.64	$3.64/n^2$
0.6	0.65	3.51	$3.51/n^2$
0.7	0.68	3.51	$3.51/n^2$
0.8	0.70	3.36	$3.36/n^2$
0.9	0.73	3.19	$3.19/n^2$

Maximum Likelihood Estimate

$$\frac{d\sum \ln p(x)}{dp} = \frac{\sum x}{p} - \frac{n - \sum x}{1-p} = 0$$

$(1-p)\sum x = (n - \sum x)p$

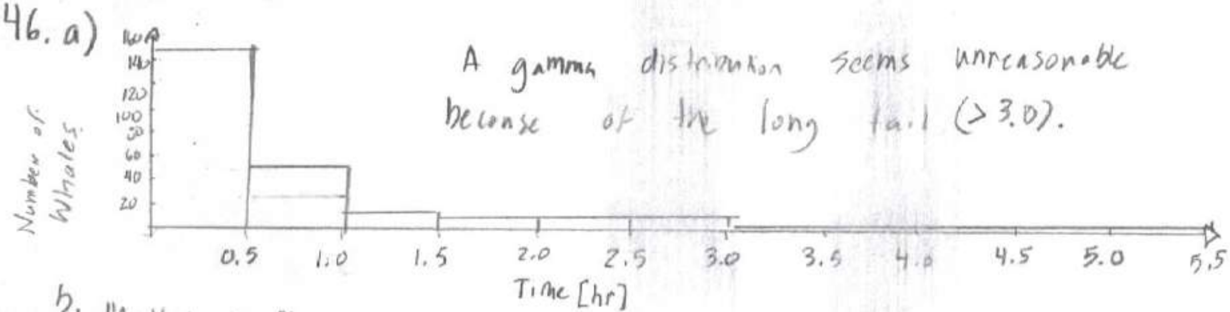
$\sum x - p\sum x = np - p\sum x$

$\boxed{p = \frac{\sum x}{n}}$

$Var(4p-2)$ $\boxed{= 16p(1-p)}$

Method of Moments was more efficient than Binomial Bootstrap or Maximum Likelihood.

46. a)



A gamma distribution seems unreasonable because of the long tail (>3.0).

b. Method of Moments

$\alpha = k-1 \;\; ; \; k = \alpha+1$

$$E[x] = \int_0^\infty x \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-x\beta} dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-x\beta} dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}} = \frac{\alpha}{\beta} \; ; \; \alpha = \beta E[x] \; ;$$

$$E[x^2] = \int_0^\infty x^2 \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-x\beta} dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+1} e^{-x\beta} dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{(\alpha+1)\alpha\Gamma(\alpha)}{\beta^{\alpha+2}} = \frac{(\alpha+1)\alpha}{\beta^2} \; ; \; Var[x] = E[x^2] - E[x]^2$$

$\alpha+1 = k-1$

$$= \frac{(\alpha+1)\alpha}{\beta^2} - \frac{\alpha^2}{\beta^2} = \frac{\alpha}{\beta^2}$$

From data:

Mean = 0.6060,	Variance = 0.4595
Shape (α) = 0.7991	Rate = 1.3198

$$Var[x] = \frac{\cancel{\beta} E[x]}{\beta^{\cancel{2}}} \; ; \; \boxed{\beta = \frac{E[x]}{Var[x]}} \quad \boxed{\alpha = \frac{E[x]^2}{Var[x]}}$$

c. Maximum Likelihood Estimate:

$$\ln f(x|a,b) = \alpha \ln\beta - \ln\Gamma(\alpha) + (\alpha-1)\ln x - \beta x$$

$$\sum \ln f(x|a,b) = n\alpha\ln\beta - n\ln\Gamma(\alpha) + (\alpha-1)\sum \ln x_i - \beta \sum x_i$$

$$\frac{\sum \ln f'(x|a,b)}{d\alpha} = n\ln\beta - n\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \sum \ln x_i = 0$$

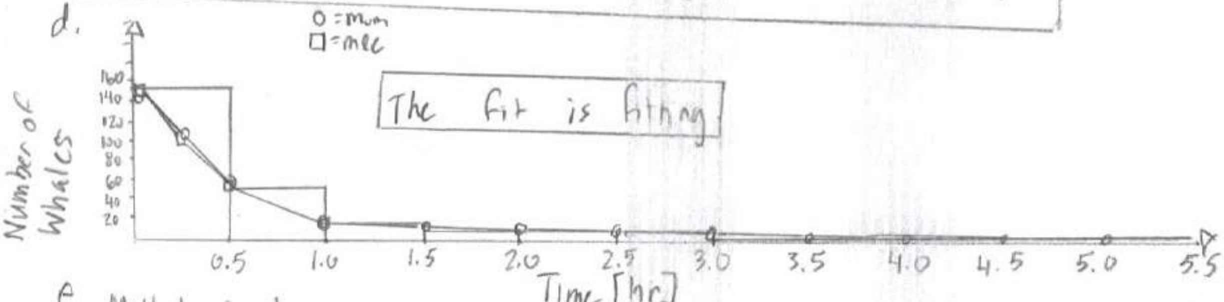
$\boxed{\text{Solve with roots}}$ $n = 210$; $E[x] = 0.6060$

$$\frac{\sum \ln f'(x|a,b)}{d\beta} = \frac{n\alpha}{\beta} - \sum x_i = 0 \; ; \; \boxed{\frac{\alpha}{E[x]} = \beta} \; ; \; \boxed{n\ln\alpha - n\ln E[x] - n\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \sum \ln x_i = 0}$$

$\boxed{\alpha = 0.7569 \; ; \; \beta = 1.249}$

The values from M.O.M in part b) are similar to the mle of scale(α) and shape(β).

d.



The fit is fitting

e. Method of Moments:

Variance: 0.4595; Standard Deviation: 0.6779; $n = 210$; Standard Error: $\boxed{\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = 4.68 \times 10^{-2}}$

Sampling Distribution: $P\left(\frac{\bar{x}-\mu}{\sigma_{\bar{x}}} < Z_{upper}\right) = 0.95$. $\boxed{Z_{upper} = 1.556}$

f. Maximum Likelihood/Mom:

Variance $= \frac{E[x]^2}{\alpha} = \frac{0.6060^2}{0.7569} = 0.4852$; Standard Deviation $= 0.6966$; Standard Error $= \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = 4.81 \times 10^{-2}$

Sampling Distribution: $P\left(\frac{\bar{x}-\mu}{\sigma_{\bar{x}}} < Z_{upper}\right) = 0.95$; $\boxed{Z_{upper} = 1.66}$ Similar results to part C

g. Confidence Interval: 95%; $\boxed{0 < \bar{x} < 0.700}$

45. a. Maximum Likelihood Estimate: Rayleigh Distribution: $f(r|\theta) = \frac{r}{\theta^2}\exp\left(\frac{-r^2}{2\theta^2}\right)$

Log Rayleigh Distribution: $\ln f(r|\theta) = \sum \ln r_i - 2n\ln\theta - \frac{\sum r^2}{2\theta^2}$

MLE Estimate: $\frac{\ln f(r|\theta)}{d\theta} = -\frac{2n}{\theta} + \frac{\sum r_i^2}{\theta^3}$; $\boxed{\hat{\theta} = \left(\frac{1}{2n}\sum r_i^2\right)^{1/2}}$

b. Method of Moments Estimate:

$E[r] = \int_0^\infty \frac{r^2}{\theta^2}\exp\left(\frac{-r^2}{2\theta^2}\right)dr = \frac{1}{\theta^2}\int_0^\infty r^2\exp\left(\frac{-r^2}{2\theta^2}\right)dr = \frac{1}{\theta^2}\int_0^\infty u\exp\left(\frac{-u}{2\theta^2}\right)\frac{du}{2\sqrt{u}} = \frac{1}{2\theta^2}\int_0^\infty u^{\frac{3}{2}-\frac{2}{2}}\exp\left(\frac{-u}{2\theta^2}\right)du$ $\quad u=r^2;\ dr=\frac{1}{2\sqrt{u}}du$

$= \frac{\Gamma(3/2)}{2\theta^2}(2\theta^2)^{3/2} = \sqrt{2}\,\theta\,\Gamma\left(\frac{3}{2}\right) = \sqrt{2}\,\theta \times \left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \theta\left(\frac{\sqrt{\pi}}{\sqrt{2}}\right)$

$\boxed{\hat{\theta} = \frac{\sqrt{2}}{\sqrt{\pi}}E[r]}$

c. Approximate Variance of MLE:

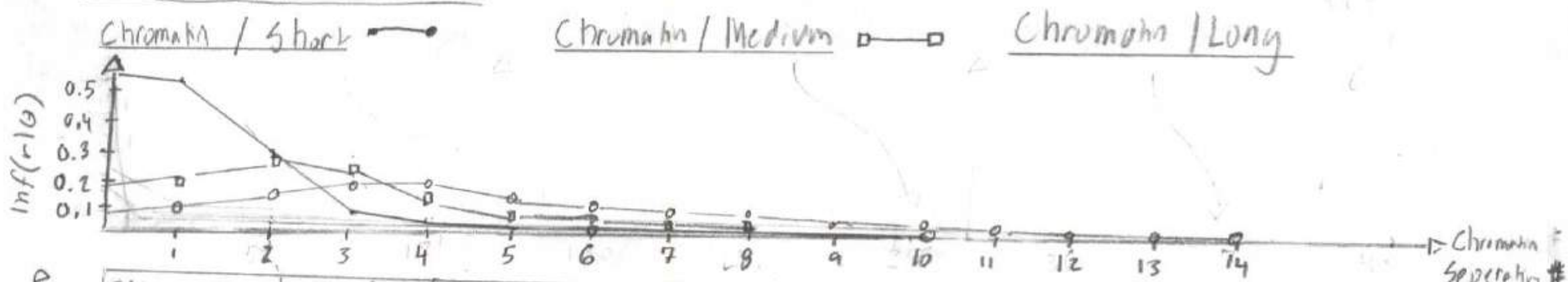
$I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\ln f(r|\theta)\right]$

$= -E\left[\frac{\partial}{\partial\theta}\left[-2\left(\frac{1}{\theta}\right) + \frac{r^2}{\theta^3}\right]\right] = -E\left[\frac{+2}{\theta^2} - \frac{3r^2}{\theta^4}\right] = \frac{3E[r^2]}{\theta^4} - \frac{2}{\theta^2} = \frac{6\theta^2}{\theta^4} - \frac{2}{\theta^2} = \boxed{\frac{4}{\theta^2}}$ $\quad var(\theta) \approx \frac{1}{nI(\theta)} = \boxed{\frac{\hat{\theta}^2}{4n}}$ $\boxed{\frac{\sum r_i^2}{8n^2}}$

Approximate variance of Method of Moments:

$I(\theta) = \frac{4}{\theta^2}$; $Var(\hat{\theta}) = \frac{1}{nI(\hat{\theta})} = \frac{\hat{\theta}^2}{4n} = \boxed{\frac{\sum r_i^2}{2n^2\pi}}$

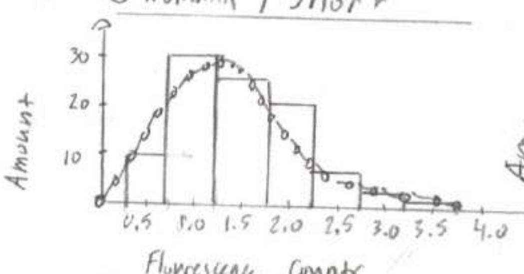
d. Plot of Likelihood Functions



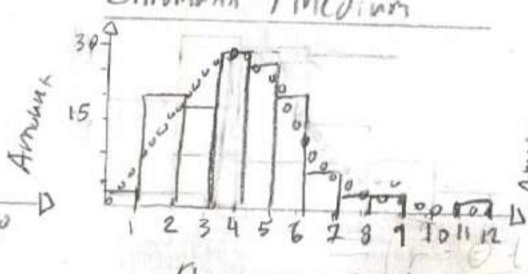
e.

Chromatin	MLE	MLE Asymptotic Variance	MoM	MoM Asymptotic Variance
Short	1.12	3.27×10^{-3}	1.17	3.04×10^{-3}
Medium	3.31	2.08×10^{-2}	3.39	2.18×10^{-2}
Long	2.08	4.34×10^{-3}	2.07	4.30×10^{-3}

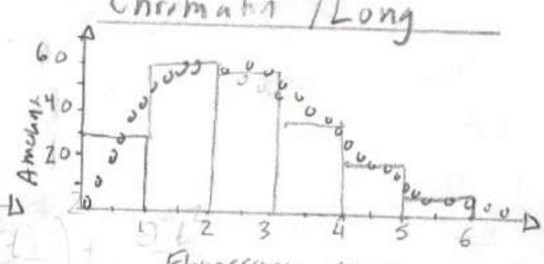
f. Chromatin / Short



Chromatin / Medium



Chromatin / Long



The distributions for the domain [0, 12] fit similarly to datasets.

g. Both short and long strands of DNA show flourescence signals with a mle of ~1.2, and medium strands ~3.01.

44.

Gender	Mean Temperature (°F)	Standard Deviation (°F)	Mean Rate (beats/min)	Standard Deviation (beats/min)
Male	98.1	0.69	73.37	5.83
Female	98.39	0.74	74.15	8.04

95% - Confidence Interval

$\boxed{\text{Male: } 98.1°F \pm 0.18}$ $\boxed{\text{Male: } 73.37 \text{ beats/min} \pm 1.4}$

$\boxed{\text{Female: } 98.39°F \pm 0.18}$ $\boxed{\text{Female: } 74.15 \text{ beats/min} \pm 1.97}$

$\boxed{\text{Folklore of } 98.6°F \text{ does not fit inside the confidence interval for males or females}}$

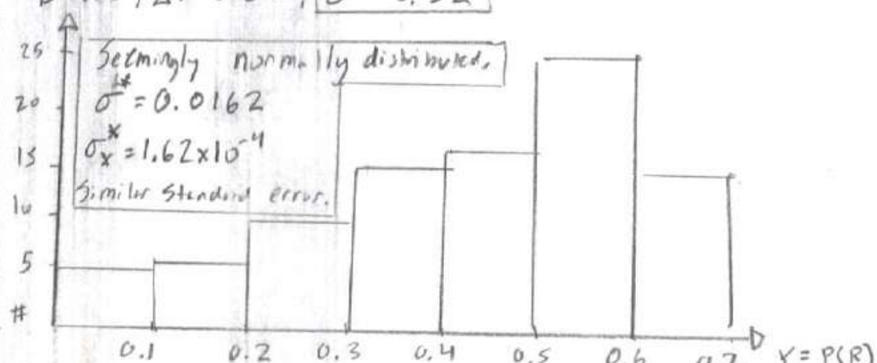
45. Began on otherside:

h. Proposition D: Section 2.3: Of the domain $[0,1]$ let $X = F^{-1}(U)$, then the cdf of X is F.

$\boxed{\text{Pseudorandom variables were generated of the domain } [0,1] \text{ with } \Delta r = 0.1}$

r	$\theta=1$, P(r)	CDF
0.0	0	0
0.1	0.17	0.17
0.2	0.39	0.56
0.3	0.46	1.02
0.4	0.29	1.31
0.5	0.60	1.91
0.6	0.12	2.03
0.7	0.55	2.58
0.8	0.56	3.14
0.9	0.24	3.38
1.0	0.60	3.99

$B=100$, $\Delta r = 0.01$, $\boxed{\theta^* = 0.32}$



i. $B=1000$, $\theta^* = 0.43$. A value generated from large sample theory.

42. Poisson Distribution:

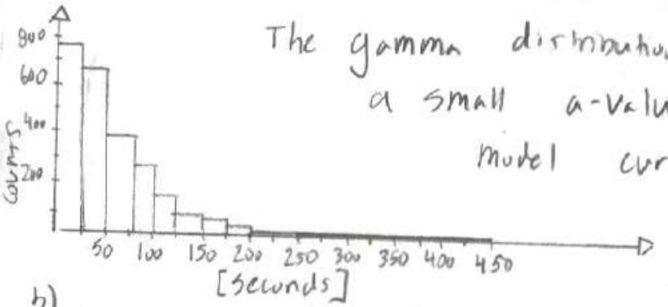
$$P(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

$\boxed{\lambda = 0.004207 \text{ counts/sec} \quad \sigma_x = \sqrt{\frac{\lambda}{n}} = 0.00004209 \text{ counts/sec}}$

An informal determination that emission rate is constant would be sampling the dataset for similar values.

43.

a) Histogram of Interarrival Times:



The gamma distribution would fit if a small a-value represented model curve.

$\boxed{\bar{X} = 79.3522}$

$\boxed{\sigma^2 = 6313.291}$

The posterior distribution = likelihood × prior = poisson × gamma.

$$= \prod_{k=0}^{n} \frac{\lambda^k}{k!} e^{-\lambda} \cdot \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda}$$

$$= \frac{\lambda^{\Sigma k}}{\Pi k!} e^{-\lambda} \cdot \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda}$$

$$= \frac{\lambda^{(a-1)+\Sigma k} \cdot b^a}{\Pi k! \, \Gamma(a)} e^{-(b+1)\lambda}$$

b) Method of Moments

Gamma Distribution: $P(x|a,b) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}$

$$M(t) = \int_0^\infty e^{tx} \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} dx = \frac{b^a}{\Gamma(a)} \left(\frac{\Gamma(a)}{(b-t)^a}\right) = \left(\frac{b}{b-t}\right)^a$$

$$M'(0) = E[X] = \frac{a}{b}$$

$$M''(0) = E[X^2] = \frac{a(a+1)}{b^2}$$

$$Var(x) = \frac{a(a+1)}{b^2} - \frac{a^2}{b^2} = a/b^2$$

$$a = b \cdot E[X]$$

$$var(x) = \frac{b \cdot E[x]}{b^2}$$

$\boxed{b = \frac{E[X]}{Var(x)}} = 1.012$

$\boxed{a = \frac{E[X]^2}{Var(x)}} = 78.980$

The method of moments does not fit, and decays to zero by sight.

Maximum Likelihood

$$\ln P(x|a,b) = \ln b^a - \ln\Gamma(a) + (a-1)\ln x - bx$$

$$\prod \ln P(x|a,b) = n \ln b^a - n\ln\Gamma(a) + (a-1)\Sigma \ln x - b\Sigma x$$

$$\frac{d(\Pi \ln P(x|a,b))}{db} = \frac{na}{b} - \Sigma x = 0 \; ; \; b = \frac{a}{\bar{x}}$$

$$\frac{d(\Pi \ln P(x|a,b))}{da} = n\ln a - \ln\bar{x} - n\frac{\Gamma'(a)}{\Gamma(a)} + \Sigma \ln x = 0$$

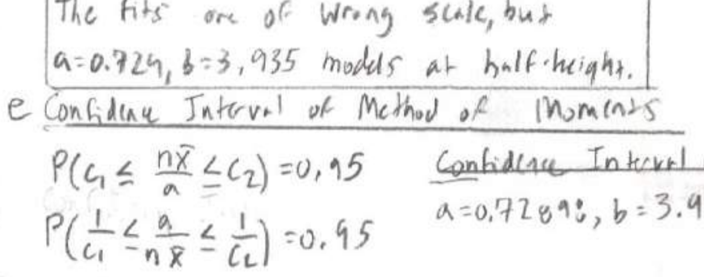
"Solving for roots"

$\boxed{a = 0.72898, \; 3.93466}$

$\boxed{b = 6.00912, \; 0.049223}$

The mle shows two solutions because of two roots, when graphically evaluating shape(a). A fit is of the wrong scale.

c) Plot of the fittings:



The fits are of wrong scale, but $a = 0.729, b = 3.935$ models at half-height.

d) Bootstrap Estimate of S.E

Method of Moments S.E.

$\boxed{Shape(a) = \bar{x}^2/\sigma^2}$

$\boxed{Scale(b) = \bar{x}/\sigma^2}$

The variance does not change for parameters, so $S.E = \sqrt{\frac{Var(x)}{n}} = 1.27$

Maximum Likelihood S.E

Shape(a) = Solved numerically

Scale(b) = solved via bootstrap.

The variance of parameters depends on precision of mean and standard deviation. S.E. = 1.27

e Confidence Interval of Method of Moments

$$P\left(c_1 \le \frac{n\bar{x}}{a} \le c_2\right) = 0.95$$

$$P\left(\frac{1}{c_1} \le \frac{a}{n\bar{x}} \le \frac{1}{c_2}\right) = 0.95$$

$$P\left(\frac{n\bar{x}}{c_1} \le a \le \frac{n\bar{x}}{c_2}\right) = 0.95$$

$\boxed{\min\left[\int_{\frac{n\bar{x}}{c_1}}^{\frac{n\bar{x}}{c_2}} Gam(x|a,b)dx = 0.95\right]}$ for c_1 & c_2

Confidence Interval of Maximum Likelihood Estimate:

$a = 0.72898, b = 3.93466$ $\boxed{\min\left[\int_{\frac{n\bar{x}}{c_1}}^{\frac{n\bar{x}}{c_2}} Gam(x|a,b)dx = 0.95\right]}$

$a = 3.93466, b = 0.049223$ $\boxed{\min\left[\int_{\frac{n\bar{x}}{c_1}}^{\frac{n\bar{x}}{c_2}} Gam(x|a,b)dx = 0.95\right]}$

F. See plot of (part c)

35. $U_1, U_2, \ldots, U_{1029}$. $X_1 = U_i < 0.331$; $X_2 = 0.331 < U_i < 0.820$; $X_3 = 0.820 < U_i$; Why X_1, X_2 and X_3 multinomial with probabilities 0.331, 0.489, and 0.180 and $n = 1029$? Multinomial Distribution:

The example of Section 8.5.1 described gene frequencies modeled with Hardy-Weinberg Equations. If the maximum likelihood estimate is $\hat{\theta} = 0.4247$, then each probability of M, MN, and N representing θ, $2\theta(1-\theta)$, and $(1-\theta)^2$, respectively, show probabilities of existence 0.331, 0.489, and 0.180.

$$p(n_1 \ldots n_r) = \binom{n}{n_1 \cdots n_r} p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$$

$$1 = [(\theta + (1-\theta))]^2 = \theta^2 + 2\theta(1-\theta) + (1-\theta)^2.$$

36. The 90% confidence intervals of Example E: section 8.5.3 were determined to be $(\hat{\alpha} - \bar{\delta}, \hat{\alpha} - \underline{\delta}) = (0.404, 0.523)$, along with $\lambda_0 = (1.462, 2.321)$; however, a normal approximation generated $S\alpha = (0.407, 0.443)$ and $S\lambda = \left(\frac{\sqrt{(n-1)s}}{\chi^2_{0.05,df}}, \frac{\sqrt{(n-1)s^2}}{\chi^2_{0.95,df}} \right) = (\,?\,,\,?\,)$.

37. Lower Bound $\underline{\theta}$, Upper Bound $\bar{\theta}$ } Quantiles of Distribution θ^*. Prove the bootstrap confidence interval is $(2\hat{\theta} - \bar{\theta}, 2\hat{\theta} - \underline{\theta})$.

$\underline{\theta} = \underline{\theta}^* - \hat{\theta}$; $\bar{\theta} = \bar{\theta}^* - \hat{\theta}$;

$$(\hat{\theta} - \underline{\theta}, \hat{\theta} - \bar{\theta}) = (\hat{\theta} - \underline{\theta}^* + \hat{\theta}, \hat{\theta} - \bar{\theta}^* + \hat{\theta}) = \boxed{(2\hat{\theta} - \underline{\theta}^*, 2\hat{\theta} - \bar{\theta}^*)}$$

38. $P(\underline{\theta} \le \theta^* \le \bar{\theta}) \Rightarrow \boxed{(\underline{\theta}, \bar{\theta})}$

39. If the distribution were considered $\hat{\theta}/\theta$, then the argument would proceed with the a bootstrap interval of $\left(\frac{\hat{\theta} - \underline{\theta}}{\theta}, \frac{\hat{\theta} - \bar{\theta}}{\theta} \right)$.

40. a) $P(|\hat{\theta} - \theta_0| > 0.01) = 1 - P(-0.01 < \theta - \theta_0 < 0.01) = 0.5080$

Sample the probability distribution 1000 times and determine if 50.80% of the values are found inside the interval -0.01 to 0.01.

b). $E(|\hat{\theta} - \theta|) = \frac{1}{n}\sum_i^n (|\hat{\theta} - \theta|)\, p(|\hat{\theta} - \theta_0|) = \frac{1}{n}\sum_i (|\hat{\theta} - \theta|)(|\hat{\theta} - \theta_0| - (1 - |\hat{\theta} - \theta_0|))^2$ if sampled 1000 times would generate a substitute expectation for the mle estimate.

c) $P(|\hat{\theta} - \theta_0| > \Delta) = 0.5$ would be determined by sampling 1000 times and checking if the points fit between 0.25 and 0.75.

41. Efficiency: Given two estimates, $\hat{\theta}$ and $\tilde{\theta}$, the ratio of variances $var(\tilde{\theta})/var(\hat{\theta})$

8.4 Example C: Gamma Distribution:

$$\mu_1 = \frac{\alpha}{\lambda} \; ; \; \mu_2 = \frac{\alpha(\alpha+1)}{\lambda^2} = \mu_1^2 + \frac{\mu_1}{\lambda}$$

$$\lambda = \frac{\mu_1}{\mu_2 - \mu_1^2} \; ; \; \alpha = \lambda\mu_1 = \frac{\mu_1^2}{\mu_2 - \mu_1^2}$$

$$\hat{\lambda} = \frac{\bar{X}}{\hat{\sigma}^2} \qquad \hat{\alpha} = \frac{\bar{X}^2}{\hat{\sigma}^2}$$

Method of Moments

8.5: Example C: Gamma Distribution

$$f(x \mid \alpha, \lambda) = \frac{1}{\Gamma(\alpha)} \lambda^\alpha x^{\alpha - 1} e^{-\lambda x}$$

$$l(\alpha, \lambda) = \sum_{i=1}^n [\alpha \log \lambda + (\alpha - 1)\log X_i - \lambda X_i - \log \Gamma(\alpha)]$$

$$\frac{\partial l}{\partial \alpha} = n \log \lambda + \sum \log X_i - n \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$$

$$\frac{\partial l}{\partial \lambda} = \frac{n\alpha}{\lambda} - \sum X_i$$

$$\hat{\lambda} = \frac{\hat{\alpha}}{\bar{X}} \; ; \; \hat{\alpha} = \bar{X}\hat{\lambda}$$

Maximum Likelihood Estimate

Efficiency

$eff(\hat{\lambda}) = 0.85$

$eff(\hat{\alpha}) = 0.44.$

28. The intervals on the left panel represent 20-trials of a sample size of $n=11$, each with a unique μ and confidence interval.

29. Yes, variance estimates of 20 trials are represented in Figure 8.8b from a sample sizes of $n=11$. The intervals are short and long because of individual trials, which each have their own confidence interval. Variance with smallest span is trial #4, and largest trial #10.

30. $f(x;\lambda) = \lambda e^{-\lambda x}$ and $E(x) = \lambda^{-1}$; $F(x) = P(X \le x) = 1 - e^{-\lambda x}$; $x_1 = 5$, $x_2 = 3$, $x_3 > 10$.

a) The likelihood function is: $\boxed{\ln f(x;\lambda) = \ln \lambda - \lambda x}$ b) the mle is $\frac{d \ln f(x;\lambda)}{d\lambda} = \frac{1}{\lambda} - x$; $\boxed{\lambda = \frac{1}{\bar{x}}}$

31.

Trial	#1	#2	#3
Side	T	T	T

George

Trial	#1	#2	#3	#4
Side	T	T	T	H

Hilary

a) $p(x|\theta) = \binom{n}{x} \theta^{x} (1-\theta)^{n-x}$

$\boxed{p(0|\theta) = \binom{3}{0}\theta^0(1-\theta)^{3-0} = (1-\theta)^3}$

$p(Y|\theta) = (1-\theta)^{k-1} \cdot \theta$

$\boxed{p(4|\theta) = (1-\theta)^3\theta}$

$p(X,Y|\theta) = P(X=0|\theta) \cap P(Y=4|\theta) = (1-\theta)^3(1-\theta)^3\theta = \boxed{(1-\theta)^6\theta}$

b) The mle of $\hat{\theta}$ is $\frac{d \ln p(x,y|\theta)}{d\theta} = \frac{-6}{(1-\theta)} + \frac{1}{\theta} = 0$

$\boxed{\theta = \frac{1}{7}}$

32.

a. mean $[\mu] = \frac{\sum x_i}{n} = \frac{57.77}{16} = \boxed{3.61}$; variance $[\sigma^2] = E[x^2] - E[x]^2 = \frac{260}{16} - 3.61^2 = \boxed{3.21}$

b)
$90\%: 3.61 \pm 1.65\sigma_{\bar{x}} = 3.61 \pm 0.74$

$95\%: 3.61 \pm 1.96\sigma_{\bar{x}} = 3.61 \pm 0.88$

$99\%: 3.61 \pm 2.58\sigma_{\bar{x}} = 3.61 \pm 1.26$

$\frac{(n-1)\sigma^2}{\chi^2_{\alpha/2, df}} < \sigma^2 < \frac{(n-1)\sigma^2}{\chi^2_{1-(\alpha/2), df}}$

$\frac{(16-1)\sigma^2}{\chi^2_{\frac{0.10}{2}, 16-1}} = \frac{15 \times 3.21}{7.26} = 6.63 < \sigma^2 < \frac{(16-1)\sigma^2}{\chi^2_{1-\frac{0.10}{2}, 16-1}} = \frac{15 \times 3.21}{25} = 1.92$

$\boxed{p = 0.10}$ $\boxed{\sigma^2 \pm 1.29}$

$\frac{(16-1)\sigma^2}{\chi^2_{1-\frac{0.05}{2}, 16-1}} = \frac{15 \times 3.21}{27.49} = 1.79 < \sigma^2 < \frac{(16-1)\sigma^2}{\chi^2_{\frac{0.05}{2}, 16-1}} = \frac{15 \times 3.21}{6.26} = 7.69$

$\boxed{p = 0.05}$ $\boxed{\sigma^2 \pm 4.48}$

$\frac{(16-1)\sigma^2}{\chi^2_{1-\frac{0.01}{2}, 16-1}} = \frac{15 \times 3.21}{32.80} = 1.47 < \sigma^2 < \frac{(16-1)\sigma^2}{\chi^2_{\frac{0.01}{2}, 16-1}} = \frac{15 \times 3.21}{4.60} = 10.47$

$\boxed{p = 0.01}$ $\boxed{\sigma^2 \pm 7.25}$

c)
$90\%: \sigma \pm 1.34$

$95\%: \sigma \pm 2.12$

$99\%: \sigma \pm 2.69$

d) $\frac{1}{2} 0.74 = 1.65\sigma_{\bar{x}}$

$= 1.65\sqrt{\frac{\sigma^2}{n}}$

$n = \frac{1.65^2 \sigma^2 \cdot 2^2}{0.74^2}$

$\boxed{n = 205.}$

33. $X_1, X_2 \ldots X_n$ i.d.d $N(\mu, \sigma^2)$. How should c be chosen for $(-\infty, \bar{X} + c)$ to be 95% confidence interval for μ; so that $P(-\infty < \mu \le \bar{X} + c) = 0.95$;

$\boxed{c = z(1 - \frac{\alpha}{2})\sigma = 1.96\sigma}$

34. $X_1, X_2 \ldots X_n$ i.i.d. $N(\mu, \sigma^2)$

a) The bootstrap estimate would be $N(\hat{\mu}, \frac{\hat{\sigma}^2}{n})$, because this methods distribution is representative of the sampling distribution

b) The bootstrap estimate of $\hat{\mu} - \mu_0$ is $N(0, \hat{\sigma}^2/n)$, due to the fact $\hat{\mu} = \bar{X}$ and $\mu_0 = \bar{X}$, therefore $\hat{\mu} - \mu_0 = 0$.

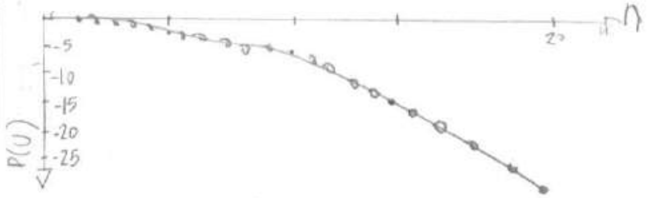
c) $\boxed{\hat{\mu} \pm z(1-\alpha/2)\hat{\sigma} = \pm z(1 - \frac{\alpha}{2})\hat{\sigma}}$

$\boxed{\mu \pm z(1-\alpha/2)\hat{\sigma}}$

$n = 20$

Trial	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Side	D	D	D	D	D	D	D	D	D	D	D	D	D	D	D	D	D	D	D	U

$P(U) = \frac{1}{20}$, $P(D) = \frac{19}{20}$, $\pi = 0.05$



$n = 5$

Trials	1	2	3	4	5
Side	D	D	D	D	U

$P(U) = \frac{1}{5}$, $P(D) = \frac{4}{5}$, $\pi = 0.20$

$\log P(k) = \log \binom{20}{k} + k \log p(u) + (n-k)\log(1-p(u))$

Posterior Distribution of a Binomial:

Posterior $\propto$ Likelihood $\times$ Prior where Prior

$$P(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}$$

likelihood

$$\boxed{\propto \binom{n}{k}\theta^{k}(1-\theta)^{n-k}\theta^{a-1}(1-\theta)^{b-1}} \quad p(k|\theta) = \binom{n}{k}\theta^{k}(1-\theta)^{n-k}$$

1:60
1:60.2
1:6.14

Posterior Mean: $E[\theta|k] = \int_0^1 \theta\, p(k|\theta)\, d\theta = \int_0^1 \theta \binom{n}{k}\theta^{k}(1-\theta)^{n-k}\, d\theta = \int_0^1 \text{Beta}(k+1, n+1-k)\, \theta^{k}(1-\theta)^{n-k}\, d\theta$

$= \frac{k+1}{n+2}$; $\boxed{k=1, E[\theta|k] = \frac{2}{7},\ k=2, E[\theta|k] = \frac{3}{7},\ k=3, E[\theta|k] = \frac{4}{7},\ k=4, E[\theta|k] = \frac{5}{7},\ k=5, E[\theta, k] = \frac{6}{7}}$

$Var[\theta|k] = \frac{(1+k)(n+1-k)}{(n+2)^2(n+3)}$; $k = 5, n = 20$, $\boxed{Var[\theta|k] = 0.0081}$

If tossed 20 more times, then $\theta = p$ would change and shift the distribution.

26.

$n_1 = 100$

$n_2 = 50$, $p(\text{tagged}) = \frac{20}{50}$

$$P(\text{tagged}) = \frac{\binom{n_1}{t}\binom{n-n_1}{n_2-t}}{\binom{n}{n_2}} = \frac{\binom{100}{20}\binom{n-100}{30}}{\binom{n}{20}}$$

Maximum Likelihood

$$\frac{L_n}{L_{n-1}} = \frac{\binom{n_1}{t}\binom{n-n_1}{n_2-t}}{\binom{n}{n_2}} \cdot \frac{\binom{n-1}{n_2}}{\binom{n_1}{t}\binom{n-n_1-1}{n_2-t}} = \frac{(n-n_1)(n-n_2)}{n(n-n_1-n_2+t)}; \quad \frac{n_2 n_1}{n_2 - t} = \frac{5000}{30} = \boxed{166}$$

The assumptions about the capture and recapture process include dependant data sets, and no bias to solution.

27.

$f(t|\tau) = \frac{e^{-t/\tau}}{\tau}$; $t \geq 0$

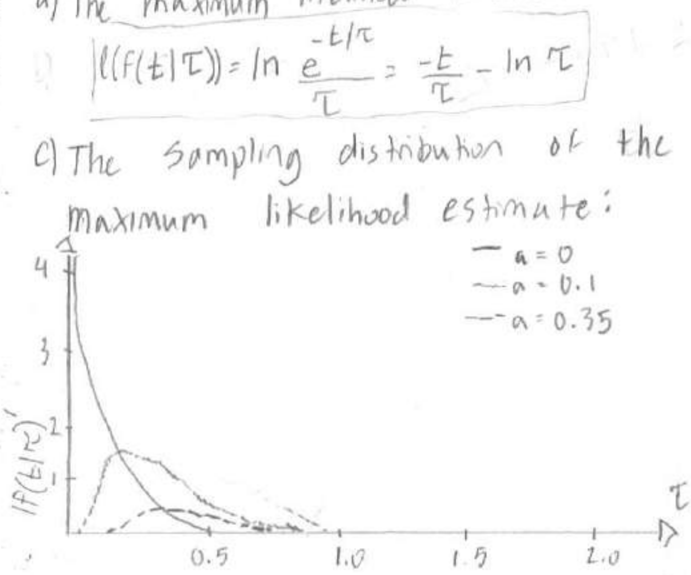
Five components are tested, the first fails in 100 days.

a) The maximum likelihood function of τ is

$$\boxed{l(f(t|\tau)) = \ln \frac{e^{-t/\tau}}{\tau} = \frac{-t}{\tau} - \ln \tau}$$

b) $\frac{d\,l(t|\tau)}{d\tau} = \frac{t}{\tau^2} - \ln\tau = 0$; $\boxed{t = \ln \tau^{2\tau}}$

c) The sampling distribution of the maximum likelihood estimate:



d) Standard Error of Maximum Likelihood Estimate:

$\sigma_\tau = \sqrt{\frac{\sigma^2}{n}} = \sqrt{\frac{E[\tau^2] - E[\tau]^2}{n}}$; seemingly impossible to solve.

Asymptotic Variance: $Var(\tau) \cong \frac{1}{nI(\tau)} = \frac{1}{-nE\left[\frac{\partial^2}{\partial\tau^2} l(t|\tau)\right]}$

$I(\tau) = -E\left[-\frac{3t}{\tau^3} - \frac{1}{\tau}\right] = \frac{3\Gamma(2)}{\tau^3} - \frac{1}{\tau}$

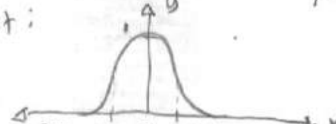
$$\boxed{\sigma_\tau = \frac{1}{n}\sqrt{\frac{\tau^3}{3\Gamma(2) - \tau^2}}}$$

b. $l(\mu,\sigma^2) = \prod_{i=1}^{n} \log \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$; From part a) $\hat{\sigma}^2 = \sqrt{\frac{1}{n}\sum_i (x-\mu)^2}$

c. An unbiased estimate of μ would be $\hat{\mu} = \frac{\sum_i X_i}{n} = \bar{X}$ $\quad n\hat{\sigma}^2 = \sum_i (x-\mu) = \sum_i x_i - \sum_i \mu_i$

$\hat{\mu} = \frac{\sum_i x_i - n\hat{\sigma}^2}{n} = \boxed{\bar{X} - \sigma^2}$

20. $X_1, X_2 \ldots X_{25} \to N(\mu,\sigma^2)$, $\mu = 0$, $\sigma = 10$. Plot:



21. $f(x|\theta) = e^{-(x-\theta)}$, $x \geq \theta$, $f(x|\theta) = 0$ otherwise.

a) Method of Moments Estimate of θ.

$\mu_1 = E(x) = \int_\theta^\infty x e^{-(x-\theta)} dx = \int_\theta^\infty x^{2-1} e^{-(x-\theta)} dx = e^{\theta}\int_\theta^\infty x^{2-1} e^{-x} dx = e^{\theta}(\theta+1)e^{-\theta} = \boxed{\theta+1}$; $\boxed{\hat{\theta} = \mu_1 - 1}$

$\mu_2 = E(x^2) = \int_\theta^\infty x^2 e^{-(x-\theta)} dx = e^{\theta}\int_\theta^\infty x^2 e^{-x} dx = \boxed{(\theta^2 + 2\theta + 2)}$

b) Maximum Likelihood Estimate: $F(x|\theta) = e^{-(x-\theta)}$; $\frac{dl(x|\theta)}{d\theta} = 1$; $\boxed{\text{Undefined solution. Analytically search for } \min(X_i)}$

c) The sufficient statistic for the function $f(x|\theta)$ is the minimal value of $\boxed{\frac{F(x|\theta)}{f(\theta)} = \min(X_i)}$

22. Weibull Distribution

Cumulative: $F(x) = 1 - e^{-(x/\alpha)^\beta}$, $x \geq 0$, $\alpha > 0$, $\beta > 0$

Weibull Distribution: $f(x) = \frac{\beta}{\alpha} e^{-(x/\alpha)^\beta}$

The Weibull distribution would fit a lifetime by approximating the decay of a process, where α is the decay constant and β is the scaling factor.

To find the standard error of the Weibull fitting, then one must solve the second-moment (or variance), and compare observed to predicted model.

23. $1 \ldots N$, where N = # objects manufactured. A random object is selected with serial number 888.

3:36
3:40
3:145

$\mu_1 = \sum_{k=1}^{n} k P(N) = \sum_{k=1}^{n} k \frac{1}{N} = \frac{1}{N}\frac{N(N+1)}{2} = \boxed{\frac{N+1}{2}}$

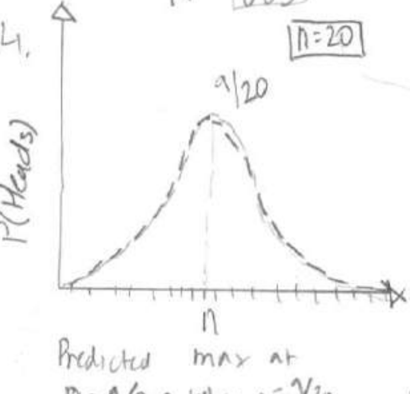
$\hat{N} = 2\mu_1 - 1 = \boxed{1775}$

MLE $= \frac{1}{N} = \boxed{\frac{1}{888}}$

Trial	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Face	T	T	T	H	T	H	T	T	H	H	T	H	H	H	T	T	H	T	H	T

n-heads = 9, n-tails = 11, $\pi = \frac{9}{20}$

24.



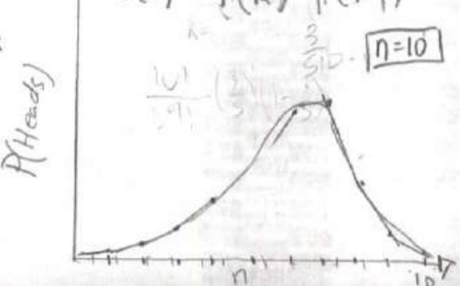
Predicted max at $n = 9/20$, where $p = 9/20$

$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$

Trial	1	2	3	4	5	6	7	8	9	10
Face	H	H	H	T	H	T	T	H	T	H

n-heads = 6, n-tails = 4, $\pi = \frac{3}{5}$

$P(k) = \binom{n}{k} \cdot p^k (1-p)^{n-T}$



25.

Log likelihood of a thumbtack

$\log(P(k)) = \log\binom{n}{k} + n\log p + (n-k)\log(1-p)$

a. Find the sufficient statistic. Sufficient statistic $[T]$ is the limit of knowledge for $x_1 \dots x_n$

$\boxed{\text{Possibly when the variance of } Var(\theta) = \frac{\sigma^2}{n} \text{ is set to } 1 \text{ (e.g. } n = \sigma^{2}).}$

17. $f(x|\alpha) = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}[x(1-x)]^{\alpha-1}$. where $\alpha > 0$. $E(x) = \frac{1}{2}$; $Var(x) = \frac{1}{4(2\alpha+1)}$

a. The shape of the density depends on the α-variable through adjustment of the continuity.

b. The method of moments will aid the estimation of α through the variance of the second-moment.

c. Maximum Likelihood estimates denote the equation $\boxed{\log f'(x|\alpha) = \frac{(\alpha-1)(2x-1)}{x(x-1)} + 2\Psi^{(0)}(2x)}$

d. The asymptotic variance $I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\log f(x|\alpha)\right] = -E\left[\frac{\partial}{\partial\alpha}\left[\frac{(\alpha-1)(2x-1)}{(x-1)(x)} + 2\Psi^{(0)}(2x)\right]\right]$

$$= -E\left[4\Psi^{(1)}(2x) - \frac{(\alpha-1)(2x^2-2x+1)}{(x^2)(x-1)^2}\right]$$

$$= \underbrace{-2\left[\int_0^\infty 4\Psi^{(1)}(2x)\frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}x(x-1)^{\alpha-1}dx - \int_0^\infty \frac{(\alpha-1)(2x^2-2x+1)}{x^2(x-1)^2}\frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}x(x-1)^{\alpha-1}dx\right]}_{b}$$

e). The sufficient statistic is :

$$\frac{f(x,\alpha)}{f(\alpha)} = \frac{\Gamma(2\alpha)/\Gamma(\alpha)^2[x(1-x)]^{\alpha-1}}{\Gamma(2\alpha)/\Gamma(\alpha)^2\binom{n}{s}[x(1-x)]^{\alpha-1}}$$

$$\boxed{= 1/\binom{n}{s}}$$

$$\boxed{Var(\hat{\theta}-\theta_0) = \frac{1}{4(2(\hat{\alpha}-x_0)-1)} \cong \frac{1}{nI(\theta)} = \frac{1}{nb}}$$

18. Suppose

$$f(x|\alpha) = \frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)}x^{\alpha-1}(1-x)^{2\alpha-1} ; \alpha > 0$$

$E(X) = \frac{1}{3}$; $Var(X) = \frac{2}{9(3\alpha+1)}$

a) Method of Moments Estimate for α.

$$\mu_1 = \frac{1}{3} ; \mu_2 = \sigma^2 + \mu_1^2 = \frac{2}{9(3\alpha+1)} + [1/3]^2$$

$$\mu_2 - \frac{1}{9} = \frac{2}{9(3\alpha+1)} ; 9(3\alpha+1) = \frac{2\cdot 9}{9\mu_2 - 1}$$

$$27\alpha + 9 = \frac{18}{9\mu_2 - 1}$$

$$\boxed{\alpha = \frac{2}{3}\left(\frac{1}{9\mu_2-1}\right) - \frac{1}{3}}$$

$$\boxed{\hat{\alpha} = \frac{2}{3}\left[\frac{1}{9(\hat{\sigma}^2+\bar{x}^2)-1}\right] - \frac{1}{3}}$$

b) What is the maximum log likelihood?

$$l(\alpha) = \log\prod f(x|\alpha) = \sum_i \log f(x|\alpha) = \sum \log\frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)}x^{\alpha-1}(1-x)^{2\alpha-1}$$

$$= n\log\frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)} + (\alpha-1)\sum_i \log x_i + (2\alpha-1)\sum_i \log(1-x)$$

$$\frac{\partial l(\alpha)}{d\alpha} = \left[\frac{n\Gamma(3\alpha)'}{\Gamma(3\alpha)}(3) - \frac{n\Gamma(\alpha)'}{\Gamma(\alpha)} - \frac{n\Gamma(2\alpha)'}{\Gamma(2\alpha)}(2)\right] + \sum_i \log x_i + 2\sum_i \log(1-x_i) = 0$$

$\boxed{\alpha = \text{Solve for alpha.}}$

c) Compare the asymptotic variance. $I(\alpha) = -E\left[\frac{\partial^2}{\partial\alpha^2}l(\alpha)\right] = \alpha_0$; $\frac{1}{nI(\alpha_0)} = \boxed{\frac{1}{n\alpha_0}}$

d) Find the sufficient statistic for α. $L = \prod f(x|\alpha) = \left[\frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)}\right]^n\left[\prod x_i\right]^{\alpha-1}\left[\prod[1-x_i]\right]^{2\alpha-1}$

19. The sufficient statistics are $\frac{f(x|\alpha)}{f(\alpha)} = \frac{\boxed{\sum_i \log x_i + \sum_i \log(1-x_i)}}{\sum_i x_i \mid \sum_i \log(1-x_i)}$

a) Suppose $N(\mu, \sigma^2)$. If μ is unknown, what is the mle of σ

$$l(\mu,\sigma) = \prod_{i=1}^n \log\frac{1}{\sqrt{2\pi\sigma^2}}\exp(-(x-\mu)^2/2\sigma^2) = -\frac{n}{2}\log 2\pi\sigma^2 - \sum_{i=0}^n (x-\mu)^2/2\sigma^2$$

$$\frac{dl(\mu,\sigma)}{d\sigma^2} = -\frac{n4\pi\sigma}{2(2\pi\sigma^2)} + \sum_{i=1}^n (x-\mu)/\sigma^3 = -n + \sum_{i=1}^n (x-\mu)/\sigma^2 = 0 ; \boxed{\hat{\sigma} = \sqrt{\frac{1}{n}\sum_i (x-\mu)}}$$

b) Approximate 95% confidence Interval: $\boxed{\text{Likelihood}}$ [redacted] $p(1-p)^{x_i-1}$

$\boxed{\text{Log likelihood}}$ $\log L(p) = \sum_{i=1}^{n} \log p + (x_i-1)\log(1-p) = n\log p - n\log(1-p) + \log(1-p)\sum x_i$

$\boxed{\text{Maximum log Likelihood}}$ $\log L(p)' = \frac{n}{p} + \frac{n}{1-p} - \frac{\sum x_i}{(1-p)}$; $\bar{p} = \frac{1}{\sum x_i}$ $\boxed{\text{Mean}}$

$$\log L(p)'' = \frac{-n}{p^2} + \frac{n}{(1-p)^2} - \frac{\sum x_i}{(1-p)^2} \quad @ \ \bar{p} = \frac{1}{\sum x_i}$$

$$= \frac{n}{p^2(1-p)^2}\left[-(1-p)^2 + p^2 - \bar{x}p^2\right] = \frac{-n}{\bar{p}^2(1-p)}$$ Variance $v = \frac{1}{\log L(p)''} @ \bar{p}$ $\boxed{= \frac{\bar{p}^2(1-p)}{17}}$

$\boxed{\text{95\%-Confidence Interval}}$ $\boxed{95\%\,CI = \bar{p} \pm 1.96\sigma = 0.36 \pm 0.5}$

c) Goodness of Fit (Comparison)

# Hops	Frequency	Fit
1	48	47
2	31	30
3	20	19
4	9	12
5	6	8
6	5	5
7	4	3
8	2	2
9	1	1
10	1	1
11	2	1
12	1	0

$P(x) = n\bar{p}(1-\bar{p})^{k-1}$

$\sum x p(x) = 327$

$n = 129$

$\bar{x} = 2.53$

$\bar{p} = \frac{1}{\bar{x}} = 0.394$

$(O-E)/\sqrt{E}$	$(O-E)^2/E$
0.15	0.21
0.18	0.33
0.23	0.53
-0.86	0.75
-0.71	0.25
0	0
0.57	0.33
0	0
0	0
0	0
1	1
Undefined	Undefined

χ^2 cdf: 1.71

Degree Freedom: 5

$p \sim 0.8$

d) Posterior Distribution, Posterior Mean, and Standard Deviation.

$\boxed{\text{Prior}}$ $Beta(1,1) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} = x(1-x)$

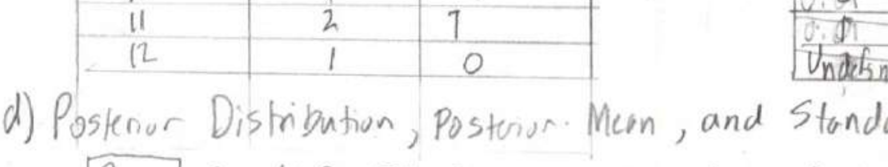
$\boxed{\text{Likelihood}}$ $p(x|\theta) = (1-\theta)^{x-1}\theta$ $\boxed{\text{Posterior}}$ $\pi(\theta|x,a,b) = (1-\theta)^{x-1}\theta \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$

$\boxed{\text{Posterior Mean}}$ $\frac{a}{a+b}$ $\boxed{\text{Posterior S.D.}}$ $\sqrt{\frac{ab}{(a+b)^2(a+b+1)}}$ $= (1-0.394)^{x-1}(0.394)\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$

9\. $\lambda = E(X)$; $\hat{\mu}_1 = \hat{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$; $\hat{\lambda} = \bar{X}$ ... The mean of a probability distribution is considered a random variable while the standard deviation is not.

10\. Normal Approximation of a Poisson Distribution $P(\hat{\lambda} = v) = \frac{(n\lambda_0)^{nv} e^{-n\lambda_0}}{(nv)!}$

What is $P(|\lambda_0 - \hat{\lambda}| > \delta)$ for $\delta = 0.5, 1, 1.5, 2,$ and 2.5.



$P(|\lambda_0 - \hat{\lambda}| > \delta) = P(|\lambda_0 - \bar{x}| > \delta) = P\left(\frac{\sqrt{n}|\lambda_0 - \bar{x}|}{\sqrt{\lambda_0}} > \sqrt{n}\frac{\delta}{\sqrt{\lambda_0}}\right)$

$\approx P\left(|N(0,1)| > \sqrt{n}\frac{\delta}{\sqrt{\bar{x}}}\right)$

$\cong P\left(|N(0,1)| > \frac{\sqrt{23}}{\sqrt{24.9}}\delta\right)$

Standard Error: $S_{\bar{x}} = \sqrt{\frac{\sigma^2}{n}} = \sqrt{\frac{\bar{x}}{n}} = 1.04$. $= 2(1 - \phi(\sqrt{23}/24.9 \times \delta))$

$\delta = 0.5$: 63% ; $\delta = 1$: 33% ; $\delta = 1.5$: 24% ; $\delta = 2$: 2.5% ; $\delta = 2.5 = 0.4\%$

11. $s_{\hat{\lambda}} = \sqrt{\frac{\hat{\lambda}}{n}}$; Poisson Distribution : [redacted] $n = 23$, $\lambda = 24.9$

The bootstrap method of sampling a large population, then averaging the estimates will approach s_λ because $\boxed{\lim_{B \to \infty} \frac{1}{B} \sum_{i=1}^{B} \left[\sum_{i=1}^{n} x_i / n \right] = \frac{\sum x_i}{N} = \bar{x}}$

12. The method of moments is best when n is small, but approaches maximum likelihood estimates as $n \to \infty$. The answer of choice depends on n-amounts.

13. Example D: 8.4: . $f(x|\alpha) = \frac{1 + \alpha x}{2}$, $-1 \le x \le 1$, $-1 \le \alpha \le 1$; $x = \cos\theta$ $\exists |\alpha| \le \frac{1}{3}$

a) $\mu = \int_{-1}^{1} x \frac{1 + \alpha x}{2} dx = \frac{\alpha}{3}$; Thus, $\hat{\alpha} = 3\bar{x}$

$E[x^2] = \int_{-1}^{1} x^2 \frac{1 + \alpha x}{2} dx = \int_{-1}^{1} \frac{x^2}{2} + \frac{\alpha x^3}{2} dx = \left. \frac{x^3}{6} \right|_{-1}^{1} + \left. \frac{\alpha x^4}{8} \right|_{-1}^{1}$

$= \frac{1}{6} + \frac{1}{6} + \frac{\alpha}{8} - \frac{\alpha}{8} = \frac{1}{3}$

Show $E(\hat{\alpha}) = \alpha$; $E(\hat{\alpha}) = E(3\bar{x}) = 3E(\bar{x}) = 3 \cdot \frac{\alpha}{3} = \boxed{\alpha}$

b) Show $Var(\hat{\alpha}) = (3 - \alpha^2)/n$; $var(\hat{\alpha}) = Var(3\bar{x})$

$= 9\,var(\bar{x}) = \frac{9}{n}\left[E[X^2] - E[X]^2 \right] = \frac{9}{n}\left[\frac{1}{3} - \alpha^2 \right]$

$\boxed{= (3 - \alpha^2)/n}$

c) $n = 25$, $\alpha = 0$. $P\left(\frac{|x - \bar{x}|}{\sigma_{\bar{x}}} > 0.5\right) = P\left(\frac{|x - \bar{x}|}{(3 - \alpha^2)/n^{1/2}} > 0.5\right) = P\left(\frac{|x - \bar{x}|}{3/25^{1/2}} > 0.5\right) = P(|x - \bar{x}| > 0.012) = 0.5040$

14. Example C: Section 8.5:

a) $P(|\hat{\alpha} - \alpha_0| > 0.05)$

Through comparing the probability to a normal table.

b) $E(|\hat{\alpha} - \alpha_0|)$

By comparing the expectation, variance, and standard error.

c) $P(|\hat{\alpha} - \alpha_0| > \Delta) = 0.5$

By comparing the probability of the norm to $= \Delta$

15. $F(q_{0.25}) = 0.75$. Gamma Distribution : $g(t) = \frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha - 1} e^{-\lambda t}$; Upper quantile is dependent on $q(\alpha, \lambda)$.

Example C of Section 8.5:

To estimate the standard error of $q(\alpha, \lambda)$ with the bootstrap method, then $q(\hat{\alpha} - \alpha_0, \hat{\lambda} - \lambda_0)$ should be evaluated.

16. $f(x|\sigma) = \frac{1}{2\sigma} \exp\left(\frac{-|x|}{\sigma}\right)$

a) Find the method of moments of σ

$\mu_2 = E[x|\sigma] = \int_{-\infty}^{\infty} \frac{|x|^2}{2\sigma} \exp\left(\frac{-|x|}{\sigma}\right) dx = \int_{0}^{\infty} \frac{|x|^2}{\sigma} \exp\left(\frac{-|x|}{\sigma}\right) dx = 2\sigma^2$

$2\sigma^2 = \frac{1}{n} \sum (x_i - \bar{x})^2$

$\boxed{\hat{\sigma} = \sqrt{\frac{1}{2n} \sum (x_i - \bar{x})^2}}$

b) Find the maximum liklihood estimate σ

$\log \prod f(x|\sigma) = \log \prod \frac{1}{2\sigma} \exp\left(\frac{-|x|}{\sigma}\right) = \sum \log \frac{1}{2\sigma} \sum \exp\left(\frac{-|x|}{\sigma}\right)$

$= n \log\left(\frac{1}{2\sigma}\right) + \sum \frac{|x_i|}{\sigma} = -n \log 2\sigma - \sum \frac{|x_i|}{\sigma}$

$\frac{\partial \log L(\sigma|x)}{\partial \sigma} = \frac{-2n}{2\sigma} + \frac{\sum |x_i|}{\sigma^2} = 0$; $\boxed{\hat{\sigma} = \frac{\sum |x_i|}{n}}$

"Asymptotic Unbiased"

c) Find the asymptotic variance of mle.

$I(\theta) = -E\left[\frac{\partial^2}{\partial \theta^2} \log F(x|\theta)\right]^2 = -E\left[\frac{\partial}{\partial \sigma}\left[\frac{n}{\sigma} + \frac{\sum |x_i|}{\sigma^2}\right]\right] = -E\left[\frac{-n}{\sigma^2} - \frac{2\sum |x_i|}{\sigma^3}\right] = -2\int_0^\infty \frac{-n}{\sigma^2}\left(\frac{1}{2\sigma^2}\right)\exp\left(\frac{-|x|}{\sigma}\right) - \frac{2}{\sigma^3}\int_0^\infty \sum x_i \frac{\exp\left(\frac{-|x|}{\sigma}\right)}{2\sigma}$

$= \int \frac{n}{\sigma^2}$; $\frac{1}{n I(\theta_0)} = \boxed{\frac{\sigma^2}{n^2}}$

b. Maximum likelihood of p: $P(X=k) = p(1-p)^{k-1}$; $\frac{dp(x=k)}{dp} = (1-p)^{k-1} + (k-1)p(1-p)^{k-2} = 0$

$(1-p)^{k-1} = (k-1)(p)(1-p)^{k-2}$

$(1-p) = (k-1)(p)$; $1 - p(1-(k-1)) = 0$

$1 - pk = 0$

$\boxed{\frac{1}{k} = p}$

C. Asymptotic variance of mle:

$$I(\theta) = E\left[\frac{\partial}{\partial\theta}\log f(X|\theta)\right]^2;\ I(X) = E\left[\frac{d}{dp}\log p + (X-1)\log(p-1)\right]$$

$$= E\left[\frac{1}{p} - \frac{X-1}{(1-p)}\right]^2 = E\left[\frac{1}{p^2} - \frac{2(X-1)}{p(1-p)} + \frac{(X-1)^2}{(1-p)^2}\right] = \boxed{n/p^2(1-p)}$$

$$I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\log f(X|\theta)\right];\ I(X) = -E\left[\frac{\partial^2}{\partial p^2}\log p(1-p)^{k-1}\right] = -E\left[\frac{-1}{p^2} - \frac{X-1}{(1-p)^2}\right]$$

$$n I(\theta)\ \boxed{= -n/p^{-2}(1-p)}\ ; \infty$$

d) Let p be uniform from $[0,1]$. What is the posterior distribution? What is the posterior mean?

$\boxed{\text{Posterior Distribution}}$: $f_{\theta|X}(\theta|X) = \frac{f_{X,\theta}(X,\theta)}{P_X(X)} = \frac{f_{X|\theta}(X|\theta)f(\theta)}{\int f_{X|\theta}(X|\theta)f_\theta(\theta)d\theta}$; Posterior $\propto$ likelihood $\times$ prior.

Posterior Mean: Most probable value of the posterior mode [Requires calculation]

Prior of a Geometric Function $[f(\theta)]$: $\beta eta(1,1) = \frac{1}{\beta(1,1)}p^{1-1}(1-p)^{1-1} = 1$

Likelihood of a Geometric Function $[f_{X|\theta}(X|\theta)]$: $p(x|\theta) = (1-\theta)^{x-1}\theta$

Posterior Distribution $[f_{\theta|X}(\theta|X)]$: $X = 1$, $\boxed{f_{\theta|X}(\theta|X) \propto \theta(1-\theta)^{x-1}}$

$X = 2$, Prior: $\beta eta(2,x) = \theta^{2-1}(1-\theta)^{x-1}$

likelihood: $p(x|\theta) = \prod \theta(1-\theta)^{x-1} = \theta^2(1-\theta)^{2-1}$

Posterior: $\theta^1(1-\theta)^1\theta(1-\theta)^{x-1} = \theta^3(1-\theta)^x = \boxed{\beta(a+2, b+2)}$

Posterior Mean: Expectation of a Beta Distribution $-\ \boxed{E(X) = \frac{a}{a+b}}$

8.

Number of Hops	Frequency	p	$P(X=k)$		σ	$\sigma_{\bar{x}}$
1	48	0.37	0.37		0.62	0.18
2	31	0.24	0.18		1.05	0.303
3	20	0.15	0.11		1.77	0.51
4	9	0.07	0.06		3.97	1.15
5	6	0.05	0.04		5.62	1.62
6	5	0.04	0.03		7.07	2.04
7	4	0.03	0.02		9.47	2.73
8	2	0.02	0.02		14.29	4.13
9	1	0.01	0.01		28.72	8.28
10	1	0.01	0.01		28.72	8.28
11	2	0.02	0.02		14.29	4.13
12	1	0.01	0.01		28.72	8.28

$P(X=k) = p(1-p)^{x-k}$

95% confidence interval for p.

$E(X) = 1\cdot0.37 + 2\cdot0.24 + 3\cdot0.15 + 4\cdot0.07 + 5\cdot0.05 + 6\cdot0.04 + 7\cdot0.03 + \ldots + 9\cdot0.01 + 10\cdot0.01 + \ldots + 12\cdot0.01 = 14.29$

$E(X^k) = 2.47$

Total: 130

a) Fit a geometric distribution to the data

b) Find a 95% confidence interval

$$E(X) = \frac{1}{130}[1\cdot48 + 2\cdot31 + 3\cdot20 + 4\cdot9 + 5\cdot6 + 6\cdot5 + 7\cdot4 + 8\cdot2 + 9 + 10 + 11\cdot2 + 12\cdot1]$$
$$= 2.79$$

$$\bar{p} = \frac{1}{\bar{x}} = 0.36.$$

5. $P(X=1)=\theta; P(X=2)=1-\theta \parallel x_1=1, x_2=2, x_3=2 \parallel$ a) Find the method of moment estimator.

a) $E[X']=\sum_{i=1}^{2} x_i P(X=i) = 1\theta + 2(1-\theta) = 2$; $E[X_i] = \frac{1}{3}\sum x_i = (\frac{1}{3}) + (\frac{2}{3})2 = \boxed{\frac{5}{3}}$

$E[X^2] = \sum_{i=1}^{2} x_i P(X=i) = 1^2\theta + 2^2(1-\theta) = \theta + 4 - 4\theta = 4 - 3\theta$; $E[X_i^2] = \frac{1}{n}\sum x_i^2 = \frac{1}{3}(1) + (\frac{2}{3})2^2 = \boxed{9/3}$; $4-3\theta = \frac{9}{3}$; $\boxed{\theta = \frac{1}{3}}$

b) $l(x_1, x_2 | p_1, p_2) = \frac{n!}{\prod_{i=1}^{2} x_i} \prod_{i=1}^{2} p_i^{x_i} = \left| \frac{2!}{2!} (\theta)^1 (1-\theta)^2 = \boxed{\theta(1-\theta)^2} \right.$

c) $\frac{dl(x_1, x_2 | p_1, p_2)}{d\theta} = (1-\theta)^2 - 2\theta(1-\theta) = (1-\theta)(1-\theta-2\theta) = (1-\theta)(1-3\theta) = 0$: $\boxed{\hat{\theta} = 1 \text{ or } 1/3}$

d) $a=2, b=3$, $f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$; $f(x) = \frac{\Gamma(5)}{\Gamma(2)\Gamma(3)} \cdot x^1(1-x)^2 = \frac{12}{\Gamma(2)} x^1 (1-x)^2$

$5 = \frac{12}{\Gamma(2)}$ $\boxed{B(2,3)}$

6. Suppose $X \sim bin(n,p)$ a) Show mle of p is $\hat{p} = X/n$

a) $P(x) = \binom{n}{x} \prod_{i=1}^{2} p_i^{x_i}$ $\log P(X) = \log n! - \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} x_i \log p_i$; $1 = \sum_{i=1}^{2} p_i$; $\boxed{\text{Lagrangian}}$

$= \log n! - \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{2} x_i \log p_i + \lambda(\sum_{i=1}^{2} p_i - 1)$

b) Cramér-Rao Lower Bound:

$\frac{d \log p(x)}{dp} = \sum_{i=1}^{2} \frac{x_i}{p_i} + \lambda = 0$; $\lambda = -\sum_{i=1}^{2} \frac{x_i}{p_i}$; $\sum_{i=1}^{n} p_i = \frac{-\sum_{i=1}^{2} x_i}{\lambda}$; $1 = \frac{-n}{\lambda}$

Measure of concentration:

$\lambda = -n$

$MSE(\theta) = E(\hat{\theta} - \theta)^2$

Cramer-Rao Inequality $var(T) \geq \frac{1}{n I(\theta)}$

$= var(\hat{\theta}) + (E(\hat{\theta}) - \theta_0)^2$

$\boxed{P_j = \frac{X}{n}}$

Efficiency: $eff(\hat{\theta}, \tilde{\theta}) = \frac{Var(\tilde{\theta})}{var(\hat{\theta})}$

Moment estimate is similar to mle when α tends to 1.

$\boxed{\text{Cramer-Rao Lower Bound of } X \sim bin(n,p)}$; $var(T) \geq \frac{1}{n I(\theta)}$ where $I(\theta) = E\left[\frac{\partial}{\partial\theta} \log bin(n,p)\right]^2$

c. $n=10$, $X=5$ plot $\log lik(x_1 \dots x_n | p_1 \dots p_n)$

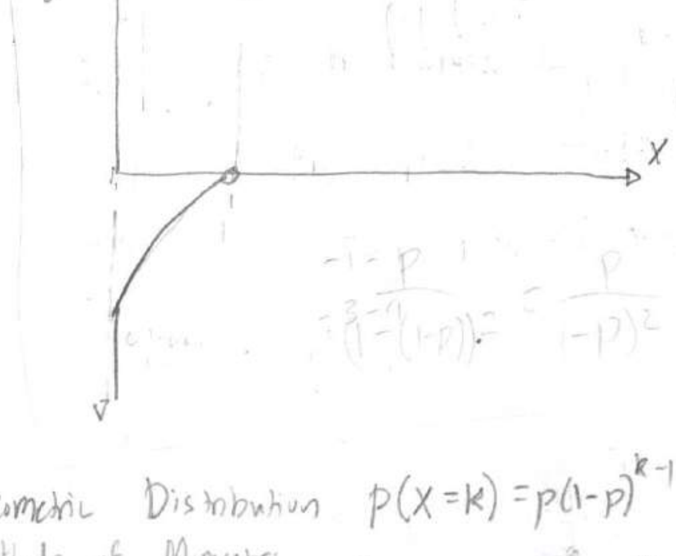
$I(x) = E\left[\frac{\partial}{\partial p} \log\left[\frac{n!}{\prod x_i} \sum_{i=1}^{2} p_i^{x_i}\right]\right]^2$ $x - xp - pn + pX$

$= E\left[\frac{x}{p} - \frac{(n-x)}{(1-p)}\right]^2 = E\left[\frac{x(1-p) - p(n-x)}{p(1-p)}\right]^2 = E\left[\frac{x-pn}{p(1-p)}\right]^2$

$p(x) = \frac{10!}{5!(5)!} p^5 (1-p)^5$

$\log p(x) = \log \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} + 5\log(p - p^2)$

$= \frac{E[X-pn]^2}{p^2(1-p)^2} = \frac{var(X)}{p^2(1-p)^2} = \boxed{\frac{n}{p(1-p)}}$



7a. Geometric Distribution $p(X=k) = p(1-p)^{k-1}$ $\mu_2 = \frac{p}{n} \sum_{k=1}^{\infty} (1-p)^{2(k-1)} =$

Methods of Moments:

$\mu_1 = \frac{1}{n} \sum_{k=1}^{\infty} p(1-p)^{k-1} = \frac{p}{n} \sum_{k=1}^{\infty} (1-p)^{k-1} = \frac{p}{n} \sum_{k=0}^{\infty} (1-p)^k = \frac{p^2}{n} \frac{d}{dR} \sum_{k=0}^{\infty} k(1-p)^{k-1} = \frac{p^2}{n} \frac{d}{dp}\left(\frac{1-p}{p}\right) = \frac{p^2}{p^2} \frac{-1}{1-p}$

4. Suppose X is a discrete Random Variable : $P(X=0)=\frac{2}{3}\theta$ where $0 \leq \theta \leq 1$

a) Find the method of moment estimates of θ

$P(X=1)=\frac{1}{3}\theta$ $n=10$ observations

$P(X=2)=\frac{2}{3}(1-\theta)$ $(3,0,2,1,3,2,1,0,2,1)$

$P(X=3)=\frac{1}{3}(1-\theta)$

① Parameter Estimate

$\mu_k = E(X^k)$; $\mu_1 = E(X^1) = \frac{1}{10}[3+0+2+1+3+2+1+0+2+1] = \frac{3}{2}$

$\mu_2 = E(X^2) = \frac{1}{10}[3^2+0^2+2^2+1^2+3^2+2^2+1^2+0^2+2^2+1^2] = 3.3$

② Moment Estimates

$E(X^1) = \sum_{i=0}^{3} x P(x) = 0(\frac{2}{3}\theta) + 1(\frac{1}{3}\theta) + 2(\frac{2}{3}(1-\theta)) + 3(\frac{1}{3}(1-\theta)) = \frac{1}{3}\theta + \frac{7}{3}(1-\theta) = \boxed{-2\theta + 7/3}$

$E(X^2) = \sum_{i=0}^{3} x^2 P(x) = 0^2(\frac{2}{3}\theta) + 1^2(\frac{1}{3}\theta) + 2^2(\frac{2}{3}(1-\theta)) + 3^2(\frac{1}{3}(1-\theta)) = \frac{1}{3}\theta + \frac{8}{3}(1-\theta) + 3(1-\theta) = \boxed{\frac{-16\theta}{3} + \frac{17}{3}}$

③ Estimates in terms of Moments

If $\frac{3}{2} = -2\theta + 7/3$; $\theta_1 = \frac{5}{12}$; then $\boxed{\hat{\theta}_1 = \frac{7}{6} - \frac{\hat{\mu}_1}{2}}$

If $\frac{33}{10} = -\frac{16\theta}{3} + \frac{17}{3}$; $\theta_2 = \frac{71}{240}$; then $\boxed{\hat{\theta}_2 = \frac{17}{16} - \frac{3}{16}\hat{\mu}_2}$

b) Standard Error of Estimates

$Var(\hat{\theta}) = Var(\frac{7}{6} - \frac{1}{2}\hat{\mu}_1) = (-\frac{1}{2})^2 \cdot Var(\hat{\mu}_1) = \frac{1}{4}\frac{Var(X)}{n} = \frac{1}{4}\frac{1}{10}[\frac{-16\theta}{3} + \frac{17}{3} - (-2\theta + 7/3)^2]$

$= \frac{1}{40}[\frac{-16\theta}{3} + \frac{17}{3} - 4\theta^2 + \frac{28}{3}\theta - 49/9] = \frac{1}{40}[-4\theta^2 + 4\theta + 2/9]$

$\sigma_x = \sqrt{\frac{\sigma^2}{n}} = \sqrt{\frac{E[X^2]-E[X]^2}{n}} = \sqrt{\frac{1}{40}[-4\theta^2+4\theta+\frac{2}{9}]} = \sqrt{\frac{1}{40}[-4(\frac{5}{12})^2+4(\frac{5}{12})+\frac{2}{9}]} = \sqrt{\frac{43}{1440}} = \boxed{0.173}$

c) Maximum Likelihood of Estimate

Multinomial because multiple X and $p(x)$. $f(x_1, \ldots x_m | p_1 \ldots p_m) = \frac{n!}{\prod_{i=1}^{m} x_i!} \prod_{i=1}^{m} p_i^{x_i}$

d) Standard Error of Likelihood Estimate

$\log[f(x_1 \ldots x_n | p_1 \ldots p_n)] = \log n! - \sum_{i=1}^{m} \log x_i + \sum_{i=1}^{m} x_i p_i$

$f(x_0 \ldots x_3 | p_0(\theta) \ldots p_3(\theta)) = [\prod_{i=0}^{3} p(X=i)^{p_i} = (\frac{2}{3}\theta)^2 (\frac{1}{3}\theta)^3 (\frac{2}{3}(1-\theta))^3 (\frac{1}{3}(1-\theta))^2] \frac{n!}{3!}$

$\log f(x_0 \ldots x_3 | p_0(\theta) \ldots p_3(\theta)) = 2\log(\frac{2}{3}\theta) + 3\log(\frac{1}{3}\theta) + 2\log(\frac{2}{3}(1-\theta)) + \frac{2}{3}\log\frac{1}{3}(1-\theta) + \log(\frac{n!}{3!})$

$\frac{d \log f(x_0 \ldots x_3 | p_0(\theta) \ldots p_3(\theta))}{d\theta} = \frac{2(\frac{3}{2})(\frac{2}{3})}{\theta} + \frac{3(\frac{3}{1})(\frac{1}{3})}{\theta} + \frac{2(\frac{3}{2})(\frac{-2}{3})}{(1-\theta)} + \frac{\frac{2}{3}(\frac{3}{1})(\frac{-1}{3})}{(1-\theta)} = 0$; $\frac{5}{\theta} - \frac{8}{3}\frac{1}{1-\theta} = 0$

$5(1-\theta) = \frac{8}{3}\theta$

$\sigma_x(\hat{\theta}) = \sqrt{\frac{1}{40}[-4(\frac{15}{23})^2 + 4(\frac{15}{23}) + \frac{2}{9}]} = \boxed{0.168}$

$15(1-\theta) = 8\theta$

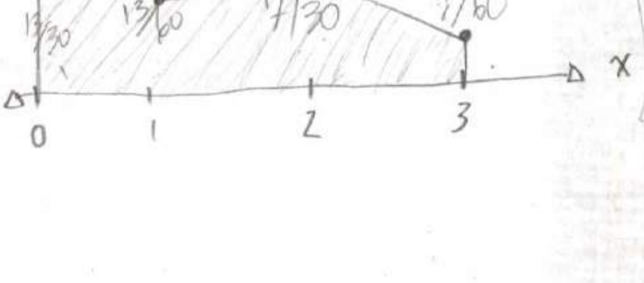
$15 - 15\theta = 8\theta$

$15 = 23\theta$

$\frac{15}{23} = \theta$

$\boxed{= 0.65}$

e)



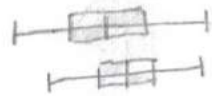
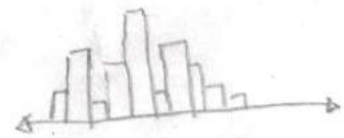
The mode of the posterior seems to be at $x=1$

iv) Average Family income. With a sample size of 500 could be represented as a proportional distribution by region. $\bar{x} = \frac{1}{n}\sum_i x_i$; $Var(\bar{X}_{sp}) = \frac{1}{n}\sum_l W_e \left(1 - \frac{n_e - 1}{N - 1}\right)\sigma^2$

b) i) 100 samples of $n = 400$; Average Family income: $\mu = \frac{1}{N}\sum_{e=1}^{L}\sum_{i=1}^{N_L} x_{ik}$

ii) $SD[\sigma] \cdot \sqrt{\frac{\sum(x_i - \bar{x})^2}{n}}$ iv) $CDF = \int_{-\infty}^{n} f(x)\,dx = n\sum f(x)^2 - \left(\sum f(x)\right)^2$

iii) Not applicable

vi) $\bar{x} \pm 1.96 s_x$ .. vii) .... c) Boxplot:  Histogram: 

d) i) see (c) ii) iii) see c) e) f) .. see c)

Chapter 8: Estimation of Parameters and Fitting of Probability Distribution

1. Poisson $[F(\lambda) = \frac{\lambda^k}{k!}e^{-\lambda}]$; $\lambda = \frac{\text{counts}}{\text{total}} = \frac{10{,}220}{12{,}169} = 0.8398$

n	Observed	Poisson
0	5267	5254
1	4436	4413
2	1800	1853
3	534	519
4	111	109
5+	21	957
	12,169	

The expected counts do match the observed counts.

2. $\lambda = \frac{1168}{300} = 3.89$

n	Frequency	Expected
0	14	6
1	30	24
2	36	46
3	68	60
4	43	58
5	43	46
6	30	30
7	14	16.5
8	10	8.05
9	6	3.5
10	4	1.36
11	1	0
12	1	0
13+	0	
	300	

3. a) Estimate λ for each dataset.

Total 1: 400 Total 2: 400 Total 3: 400 Total 4: 400

$\boxed{\lambda_1 = 0.6825 \quad \lambda_2 = 1.3225 \quad \lambda_3 = 1.80 \quad \lambda_4 = 4.65}$

b) Concentration #1: $E_1(X) = \lambda_1$; $Var_1(X) = E_1[X^2] - E_1[X]^2 = 1.28 - 0.46 = 0.82$; $SE = \sqrt{\frac{\sigma^2}{n}} = 0.045$

95% confidence Interval: $P\left(\left|\frac{X - \bar{X}}{\sigma_x}\right| \le Z(\alpha/2)\right) = 0.95$; $\bar{x} \pm 1.96\sigma_x$ $\boxed{0.6825 \pm 0.088}$

Concentration #2: $E_2(X) = \lambda_2$; $Var_2(X) = E_2[X^2] - E_2[X]^2 = 3.03 - 1.3225 = 1.71$; $SE = \sqrt{\frac{\sigma^2}{n}} = 0.064$

95% Confidence Interval: $P\left(\left|\frac{X - \bar{X}}{\sigma_x}\right| \le Z(\alpha/2)\right) = 0.95$; $\bar{X} \pm 1.96\sigma_x$ $\boxed{1.33 \pm 0.125}$

Concentration #3: $E_3(X) = \lambda_3$; $Var_3(X) = E_3[X^2] - E[X]^2 = 5.20$ $SE = \sqrt{\frac{\sigma^2}{n}} = 0.114$

95% Confidence Interval: $P\left(\left|\frac{X - \bar{X}}{\sigma_x}\right| \le Z(\alpha/2)\right) = 0.95$; $\bar{X} \pm 1.96\sigma_x$ $\boxed{1.80 \pm 0.223}$

Concentration #4: $E_4(X) = \lambda_4$; $Var_4(X) = E_4[X^2] - E_4[X]^2 = 34.5 - 4.65^2 = 12.86$; $SE = \sqrt{\frac{\sigma^2}{n}} = 0.179$

95% Confidence Interval: $P\left(\left|\frac{x - \bar{x}}{\sigma_x}\right| \le Z(\alpha/2)\right) = 0.95$; $X \pm 1.96\sigma_x$ $\boxed{4.65 \pm 0.351}$

c)

n	Observed	Expected	Observed	Expected	Observed	Expected	Obs.	Exp.
0	213	214	103	106	75	66	0	4
1	128	134	143	141	103	119	20	18
2	37	42	98	93	121	109	43	41
3	18	9	42	41	54	64	53	65
4	3	1.36	8	14	30	29	86	75
5	1	0	4	4	13	10	70	69
6	0	0	2	1	2	3	54	54
7	0	0	0	0	1	1	37	36
8	0	0	0	0	0	0	18	21
9	0	0	0	0	1	0	10	11
10	0	0	0	0	0	0	5	5
11	0	0	0	0	0	0	2	2
12	0	0	0	0	0	0	2	1

The expected and observed counts are fitting.

54a) $C = C_0 + C_i n$; L Strata ; Find a function which minimizes the variance.

Start-up Cost / cost per observation

Lagrangian Multiplier: $L(n_1, \dots n_L, \lambda) = \sum_{\ell=1}^{L} \frac{W_\ell^2 \sigma_\ell^2}{n_\ell} + \lambda\left(\sum_{\ell=1}^{L} n_\ell - n\right)$

$L(n_1, \dots n_L, \lambda) = \sum_{\ell=1}^{L} \frac{W_\ell^2 \sigma_\ell^2}{n_\ell} + \lambda\left(\sum C_\ell n_\ell - C_i n\right)$

$\frac{\partial L}{\partial n_\ell} = -\frac{W_\ell^2 \sigma^2}{n_\ell^2} + \lambda$; $n_\ell = \frac{W_\ell \sigma_\ell}{\sqrt{\lambda}}$

$\frac{\partial L}{\partial n_\ell} = -\frac{W_\ell^2 \sigma_\ell^2}{n_\ell^2} + \lambda C_\ell = 0$; $n_\ell = \frac{W_\ell \sigma_\ell}{\sqrt{\lambda C_\ell}}$; $n = \frac{1}{\sqrt{\lambda}} \sum_{\ell=1}^{L} \frac{W_\ell \sigma_\ell}{\sqrt{C_\ell}}$

$n = \frac{1}{\sqrt{\lambda}} \sum_{\ell=1}^{L} W_\ell \sigma_\ell$

$\frac{1}{\sqrt{\lambda}} = \frac{n}{\sum_{\ell=1}^{L} W_\ell \sigma_\ell}$

$n_\ell = n \frac{W_\ell \sigma_\ell}{\sum_{\ell=1}^{L} W_\ell \sigma_\ell}$

$Var(X_{\bar{S}0}) = \sum_{\ell=1}^{L} W_\ell^2 \left(\frac{1}{n_\ell}\right)\left(1 - \frac{n_\ell - 1}{N_\ell - 1}\right)\sigma_\ell^2$

$\frac{1}{\sqrt{\lambda}} = n \sum_{\ell=1}^{L} \frac{\sqrt{C_\ell}}{W_\ell \sigma_\ell}$

$n_\ell = n W_\ell \sigma_\ell \sum_{\ell=1}^{L} \frac{\sqrt{C_\ell}}{W_\ell \sigma_\ell}$

Neglecting "finite population" correction

$= \sum_{\ell=1}^{L} \frac{W_\ell^2 \sigma_\ell^2}{n_\ell} = \frac{\sum W_\ell^3 \sigma_\ell^2}{n W_\ell \sigma_\ell} \sum \frac{W_\ell \sigma_\ell}{\sqrt{C_\ell}}$

$= \frac{\sum_{\ell=1}^{L} (W_\ell \sigma_\ell)^2 / \sqrt{C_\ell}}{n}$

c) $n_\ell = n W_\ell \sigma_\ell \sum \frac{\sqrt{C_\ell}}{W_\ell \sigma_\ell}$

b) $Var(X_{\bar{S}0}) = \frac{\sum (W_\ell \sigma_\ell)^2 / \sqrt{C_\ell}}{n}$

55. a) Proportional Allocation of a population mean $\bar{X}_{SP} = \sum_{\ell=1}^{L} W_\ell \bar{X}_\ell = \sum W_\ell \left(\frac{1}{n_\ell} \sum X_{i\ell}\right)$
is utilized when W_L is large representation. $= \frac{1}{n} \sum_{\ell=1}^{L} \sum_{i=1}^{n_\ell} X_{i\ell}$

Optimal Allocation of a population mean $\bar{X}_{S0} = \sum_{\ell=1}^{L} W_\ell \bar{X}_\ell$
is utilized when a sample of each stratum is taken.

Being that $(H_S = 100, H_T = 100{,}000)$ and $(L_S = 200, L_T = 500{,}000)$ then optimal allocation of a population seems best for model $\frac{1}{6}\bar{X}_H + \frac{5}{6}\bar{X}_L = \bar{X}_T$

b) $\sigma_H = 20$, $\sigma_L = 10$; Standard Error ; $S_{\bar{X}H} = \sqrt{\frac{20^2}{100}\left(1 - \frac{100}{100{,}000}\right)} = \boxed{2.00}$ $S_T = 0.677$

$S_{\bar{X}L} = \sqrt{\frac{10^2}{200}\left(1 - \frac{200}{500{,}000}\right)} = \boxed{0.71}$

c) A 95% confidence interval would be best to determine allocation error. The current allocation $(H_S = 100, L_T = 200)$ provides an interval of $(\pm 3.92, \pm 1.39)$ by increasing the allocation to $(H_S = 200, L_S = 100)$ would shift the interval to $(\pm 2.77, \pm 1.96)$. Also, the standard error of the population would shift from 0.677 to 0.866 and be of greater error.

d) Proportional Allocation provides a standard error of:
$\sigma_S = 0.925$, Which is worse.

53.

a. n=100 Sample size   Methods of Allocation: Sample size $\left(n_\ell = n \dfrac{W_\ell \sigma_\ell}{\sum_{k=1} W_k \sigma_k}\right)$

$\sum W_k \sigma_k = \frac{394}{2010} 8.3 + \frac{461}{2010} \cdot 13.3 + \frac{391}{2010} \cdot 15.1 + \frac{334}{2010} \cdot 19.8 + \frac{169}{2010} \cdot 24.5 + \frac{113}{2010} 26 + \frac{148}{2010} 35.2$

$= 1.63 + 3.05 + 2.94 + 3.29 + 2.06 + 1.46 + 2.59$

$\boxed{= 17}$ $Var(\bar{X}_{so})$, $Var(\bar{X}_{sp})$, $Var(\bar{X}_{SRS})$

Optimal Allocation

Farm Size	n_ℓ
0-40	9.61
41-80	17.9
81-120	17.3
121-160	19.4
161-200	12.1
201-240	8.6
241+	15.2

"A scaled sample size to the true value."

b. $Var(\bar{X}) = \frac{1}{n_\ell}\left(1 - \frac{n_\ell - 1}{N_\ell - 1}\right)\sigma_\ell^2$ "a scaled variance"

Farm Size	$Var(\bar{X})$
0-40	7.01
41-80	9.51
81-120	12.62
121-160	19.09
161-200	43.23
201-240	73.27
241+	73.64

c.

Farm Size:	$E(\bar{X}_s) = \sum_i W_i E(\bar{X}_i)$
0-40	1.06
41-80	3.74
81-120	4.73
121-160	5.73
161-200	3.54
201-240	2.81
241+	4.70

"Population Expectation"

	$Var(\bar{X}_s) = \sum_{\ell=1}^{L} W_\ell^2 \left(\frac{1}{n_\ell}\right)\left(1 - \frac{n_\ell - 1}{N_\ell - 1}\right)\sigma_\ell^2$
0-40	1.98×10^{-2}
41-80	7.30×10^{-2}
81-120	6.44×10^{-2}
121-160	7.61×10^{-2}
161-200	1.74×10^{-2}
201-240	2.48×10^{-3}
240+	2.19×10^{-2}

"Population Variance"

Proportional Allocation

d. n=10 farms

Farm Size:	$Var(\bar{X}_s)$:
0-40	2.59×10^{-1}
41-80	9.12×10^{-1}
81-120	8.42×10^{-1}
121-160	1.05
161-200	4.02×10^{-1}
201-240	1.96×10^{-1}
241+	6.30×10^{-1}

"smaller sample size variance"

e) n = 70 samples

Farm Size:	$Var(\bar{X}_s)$:
0-40	2.69×10^{-1}
41-80	5.00×10^{-1}
81-120	4.78×10^{-1}
121-160	5.27×10^{-1}
161-200	3.06×10^{-1}
201-240	2.32×10^{-1}
241+	3.99×10^{-1}

b) $Var(\bar{X}_{so}) = \frac{\left(\sum_{\ell=1}^{L} W_\ell \sigma_\ell\right)^2}{n}$ "Stratified Optimal Allocation" to stratified population

$Var(\bar{X}_{sp}) = \sum_{\ell=1}^{L} W_\ell^2 Var(\bar{X}_\ell) = \sum_{\ell=1}^{L} W_\ell^2 \frac{\sigma_\ell^2}{n_\ell}$ "Stratified Proportional Allocation" to total stratified population

$Var(\bar{X}_{SRS}) = \sum_{\ell=1}^{L} W_\ell^2 \frac{\sigma_\ell^2}{n_\ell} + \frac{\sum W_\ell^2 (\mu_\ell - \mu)^2}{n_\ell}$ "Stratified Random Sampling" increases precision for diverse values of population.

$= \sum_{\ell=1}^{L} \frac{W_\ell^2 \left(\sigma_\ell^2 + (\mu_\ell - \mu)^2\right)}{n_\ell}$

$Var(\bar{X}_{so}) =$

d) $T_R = \frac{\bar{Y}}{\bar{X}}\tau_X = R\tau_X$ $\qquad S_{TR} = \sqrt{\frac{N^2}{n}\left(1-\frac{n-1}{N-1}\right)\left(R^2 s_x^2 + s_y^2 - 2R s_{xy}\right)} = N S_{\bar{Y}_R}$

$= \frac{150}{3000} 1000 \cdot \left(\frac{3000}{50}\right) = \boxed{3000}$ $\qquad = \sqrt{\frac{1000^2}{50^2}\left(1-\frac{50-1}{1000-1}\right)\left(\left(\frac{1}{20}\right)^2 30^2 + 2^2 - 2\left(\frac{1}{20}\right)40.8\right)}$

$= 28.7$ the standard error is much worse.

50. Standard Error of a Ratio Estimate:

$$\frac{|E(R)-r|}{\sigma_R} \le \frac{\sigma_{\bar{X}}}{\mu_X} = \frac{\sigma_X}{\mu_X}\sqrt{\frac{1}{n}\left(1-\frac{n-1}{N-1}\right)}$$

b $$\frac{Var(\bar{Y}_R)}{Var(\bar{Y})} = 1 + \frac{c_x}{c_y}\left(\frac{c_x}{c_y} - 2\rho\right)$$

$$\boxed{\left|\frac{E(R)-r}{\sigma_{\bar{X}}/\mu_X}\right| = 1 + \frac{c_x}{c_y}\left(\frac{c_x}{c_y}-2\rho\right)}$$

a) $$Cov(R,\bar{X}) = E\left(\frac{\bar{Y}}{\bar{X}}\bar{X}\right) - E\left(\frac{\bar{Y}}{\bar{X}}\right)E(\bar{X})$$

$$= \frac{1}{\bar{X}}\left[E(\bar{Y}\bar{X}) - E(\bar{Y})E(\bar{X})\right]$$

$$= \frac{Cov(\bar{X},\bar{Y})}{\bar{X}}$$

$$\frac{\sigma_{XY}}{\bar{X}} = E[\bar{Y}] - E(R)E[\bar{X}]$$

$$-\frac{\sigma_y \sigma_x \rho}{\bar{X}} = E(R)E[\bar{X}] - E[\bar{Y}]$$

$$-\frac{\sigma_x}{\bar{X}} = \frac{E[R]E[\bar{X}] - E[\bar{Y}]}{\sigma_y \rho}$$

$$\boxed{\frac{\sigma_x}{\bar{X}} = \left|\frac{E(R)E[\bar{X}] - E[\bar{Y}]}{\sigma_y \cdot \rho}\right| = \frac{|E(R) - E[\frac{\bar{Y}}{\bar{X}}]|}{\sigma_R}}$$

51. $E(\hat{\theta}) = \hat{\theta} + \frac{b_1}{n} + \underbrace{\frac{b_2}{n^2}}_{\theta_j} + \dots$; $\hat{\theta}$ = estimate of θ

$\frac{1}{n} \cdots \frac{1}{n}$

$\boxed{m} \cdots \boxed{n}$ $n = mp$ $\quad$ For ($j = 1 \dots p$)

p $\quad$ Estimate $\hat{\theta}_j$ from $m(p-1)$

$$E(\hat{\theta}) = \hat{\theta} + \frac{b_1}{m(p-1)} + \frac{b_2}{[m(p-1)]^2} + \dots$$

P-pseudovalues: $V_j = p\hat{\theta} - (p-1)\hat{\theta}_j$ $\qquad$ Prove $\theta_j = \frac{1}{p}\sum_{j=1}^{p} V_j$

$\hat{\theta}_j = \frac{1}{p}\sum_{j=1}^{p} V_j$

$$E(\hat{\theta}_j) = \hat{\theta} + \frac{b_1}{m(p-1)} + \frac{b_2}{[m(p-1)]^2} + \dots = \frac{1}{p}\sum_{j=1}^{p} V_j + \frac{b_1}{m(p-1)} + \frac{b_2}{[m(p-1)]^2} + \dots = p\hat{\theta} - (p-1)\theta_j + \frac{b_1}{m(p-1)} + \frac{b_2}{[m(p-1)]^2} + \dots$$

$$\frac{dE(\hat{\theta}_j)}{dp} = \hat{\theta} - \theta_j - \frac{b_1}{m(p-1)^2} - \frac{2b_2}{[m(p-1)]^3} + \dots$$

$$\frac{d^2E(\hat{\theta}_j)}{d^2p} = \frac{+2b_1}{m(p-1)^2} + \frac{6b_2}{m(p-1)^4} + \dots = 0$$

$$b_1 + \frac{3b_2}{m(p-1)^4} + \dots = 0$$

$$b_1 = \frac{-3b_2}{m(p-1)^4}$$

$$E(\hat{\theta}) = \hat{\theta} - \frac{3b_2}{n(p-1)^4} + \frac{b_2}{n^2} + \dots$$

$$= \hat{\theta} + \frac{b_2}{n}\left(\frac{1}{n^2} - \frac{3}{n(p-1)^4}\right) + \dots$$

52. $N_1 = N_2 = 1000$

$N_3 = 500$

10 observations

Stratum #1: 94 99 106 106 101 102 122 104 97 97 $\quad \boxed{\bar{X}_1 = 103 \quad \sigma_1 = 7.4}$

Stratum #2: 183 183 179 211 178 179 192 192 201 177 $\quad \boxed{\bar{X}_2 = 188 \quad \sigma_2 = 11}$

Stratum #3: 343 302 286 317 289 284 357 288 314 296 $\quad \boxed{\bar{X}_3 = 278 \quad \sigma_3 = 38}$

$\boxed{\tau_1 = N \cdot \bar{X}_1 = 103{,}000;\ \tau_2 = N\bar{X}_2 = 188{,}000;\ \tau_3 = N\bar{X}_3 = 139{,}000}$

$\boxed{103 \pm 12 \quad 188 \pm 18 \quad 278 \pm 63}$

47. $n=64$; Corollary B of Section 7.4 : Approximate Bias of the ratio estimate of μ_y

$$E(\bar{Y}_R)-\mu_y \approx \frac{1}{n}\left(1-\frac{n-1}{N-1}\right)\frac{1}{\mu_x}\left(r\sigma_x^2-\rho\sigma_x\sigma_y\right)$$

$E(\bar{Y}_R)-\mu_y \approx \frac{1}{64}\left(1-\frac{64-1}{393-1}\right)\frac{1}{274.8}\left(2.96\cdot 213.2^2-0.91\times 213.2\times 539.7\right)$

$= 0.96$

$n=128$

$\approx \frac{1}{128}\left(1-\frac{64-1}{393-1}\right)\frac{1}{274.8}\left(2.96\times 213.2^2-0.91\times 213.2\times 539.7\right)$

$= 0.39.$

48. $n=100$ Households ; # people in Household [X] ; Weekly Expenditure for Food [Y]

Total Number of Households = 100,000

a) Estimate the ratio $r=\frac{\mu_y}{\mu_x}$

$= \frac{(\sum Y_i)/N}{(\sum X_i)/N}$

$\boxed{= \frac{10,000}{3200} = \frac{125}{4}}$

$\sum X_i = 320$: Total sum # people in Household

$\sum Y_i = 10,000$: Total Weekly Expenditure for Food

$\sum X_i^2 = 1250$

$\sum Y_i^2 = 1,100,000$

$\sum X_i Y_i = 36,000$

b) Confidence Interval (95%): $r \pm 1.96\sigma_x$

$\frac{125}{4} \pm 1.96\cdot\left(\frac{1}{n}\left[1-\frac{n-1}{N-1}\right]\frac{1}{\mu_x^2}\left(r^2\sigma_x^2+\sigma_y^2-2r\rho\sigma_x\sigma_y\right)\right)$

$\frac{125}{4} \pm 1.96\left(\frac{1}{10^2}\left[1-\frac{10^2-1}{10^5-1}\right]\frac{1}{\left(\frac{320}{100}\right)^2}\left(\left[\frac{125}{4}\right]^2\left(\frac{1250}{100}-\left(\frac{320}{100}\right)^2\right)+\left(\frac{1,100,000}{100}-\left(\frac{10000}{100}\right)^2\right)-2\left(\frac{125}{4}\right)\left(\frac{36,000}{100}-\frac{320\cdot 10000}{100^2}\right)\right)\right)$

$\boxed{\frac{125}{4} \pm 1.34}$

C. $\tau = N\bar{Y} = 100000\cdot\frac{10000}{100} = 10^7$; $\left(1-\frac{1}{10^5}\right)s^2 = \left(\frac{1}{99999}\right)\frac{1}{100-1}\left(\left(\frac{1,100,000}{100}\right)^2-\left(\frac{10,000}{100}\right)^2\right) = 1.66\times 10^{-4}$

$P\left(\left|\frac{\bar{x}-\bar{Y}}{\sigma_x}\right| < z(1-\frac{\alpha}{2})\right) = 0.90$

$100 \pm 1.65\cdot 1.66\times 10^{-4}$

$\boxed{100 \pm 0.00027}$

49. $N=1000$ squares

$n=50$ sampled

Y = # of birds

X = Area covered by vegetation

a) $r = \frac{\sum Y_i/n}{\sum X_i/n} = \frac{150}{3000} = \frac{1}{20}$

$\sum X_i = 3,000$

$\sum Y_i = 150$

$\sum X_i^2 = 225,000$

$\sum Y_i^2 = 650$

$\sum X_i Y_i = 11,000$

b) Standard Error:

$s_R = \sqrt{Var(R)} = \sqrt{\frac{1}{n}\left(1-\frac{n-1}{N-1}\right)\frac{1}{\mu_x^2}\left(r^2\sigma_x^2+\sigma_y^2-2r\rho\sigma_x\sigma_y\right)}$

$\sigma_x = \sqrt{\frac{1}{50-1}\left[225,000-50\cdot 60^2\right]} = 30$ $\quad \sigma_{xy} = \frac{1}{50-1}\left[11,000-50\cdot 60\cdot 3\right] = 40.8$

$\sigma_y = \sqrt{\frac{1}{50-1}\left[650-50\cdot 3^2\right]} = 2$ $\quad \rho = 0.68$

$s_R = \sqrt{\frac{1}{50}\left(1-\frac{50-1}{1000-1}\right)\frac{1}{\left(\frac{3000}{50}\right)^2}\left(\left(\frac{1}{20}\right)^2 30^2+2^2-2\left(\frac{1}{20}\right)\cdot 30\cdot 2\cdot 0.68\right)}$

$= 7.34\times 10^{-4}$

90% Confidence Interval: $r \pm 1.65 s_R = \frac{1}{20} \pm 1.65\cdot 7.34\times 10^{-4} = 0.05 \pm 0.001$

c) Total Number of Birds: 95% Confidence Interval

$T = N\cdot\bar{Y} = 1000\cdot\frac{150}{50}$

$= 3,000$

Standard Error:

$s_T = s_T\sqrt{\frac{N(N-n)}{n}} =$

$= N s_{\bar{x}} = 1000\cdot\sqrt{\frac{s_y^2}{n}\left(1-\frac{50}{1000}\right)} = 1000\sqrt{\frac{4}{50}\left(1-\frac{50}{1000}\right)} = 275$

$T \pm 1.96\cdot s_T = 3,000 \pm 1.96\cdot 275 = 3000 \pm 540$

b.) Variance of Estimate: $Var(N\bar{D}) = E[N\bar{D}^2] - E[N\bar{D}]^2 = N^2E[\bar{D}^2] - N^2E[\bar{D}]^2 = N^2\left[E[\bar{D}^2] - E[\bar{D}]^2\right]$

c) Population Parameter $[\tau]$; Estimate $T = N\bar{X}$; Variance of Estimate $\sigma_T^2 = N^2\sigma_{\bar{x}}^2 = N^2\left[E[\bar{x}^2] - E[\bar{D}]^2\right]$

The proposed method would be as accurate. $\boxed{= N^2\sigma_{\bar{D}}^2 = N^2\left[E[\bar{D}^2] - E[\bar{D}]^2\right]}$

d) Estimation of Ratio: $r = \frac{\sum_i y_i}{\sum_i x_i} = \frac{\mu_y}{\mu_x}$

A ratio estimate would provide advantage to a differently sized pool of populations. In the listed case of part a, b, or c, then there would be no difference

42. Population Correlation Coefficient: $\rho = \frac{\sigma_{xy}}{\sigma_x \cdot \sigma_y} = \frac{\frac{1}{N}\sum_i (x_i - \mu_x)(y_i - \mu_y)}{\frac{1}{N}\sum_i (x - \mu_x)^2 \frac{1}{N}\sum_i (y - \mu_y)^2}$ $\boxed{= \frac{1}{N}\left(\frac{1}{\sum_i (x - \mu_x)\sum_i (y - \mu_y)}\right)}$

43. Example D: Section 7.3.3.

44. $\bar{X} = 2.2$, $\sigma = 0.7$, $\rho(\text{\#occupancy} | \text{\#Motor vehicles}) = 0.85$

Estimate population ratio of # Motor Vehicles per occupant + S.E.

Population Ratio: $\frac{\mu_2}{\mu_1} = \frac{\bar{X}_1}{\bar{X}_2} = \frac{1.6 \text{ motor vehicle per car}}{2.2 \text{ motor vehicle per car}} = 0.727$

Standard Error: $Var(r) = \frac{1}{n}\left(1 - \frac{n-1}{N-1}\right)\frac{1}{\mu_1^2}\left(r^2\sigma_1^2 + \sigma_2^2 - 2r\sigma_{12}\right)$

$= \frac{1}{n}\left(1 - \frac{n-1}{N-1}\right)\frac{1}{\mu_1^2}\left(r^2\sigma_1^2 + \sigma_2^2 - 2r\rho\sigma_x\sigma_y\right)$

$= \frac{1}{100}\left(1 - \frac{100-1}{9000-1}\right)\frac{1}{2.2^2}\left(0.727^2 \cdot 0.7^2 + 0.8^2 - 2(0.727)(0.85)(0.8)(0.7)\right)$

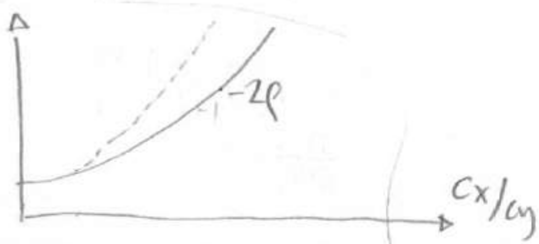
$= 4.225 \times 10^{-4}$; $\boxed{SE = 2.05 \times 10^{-2}}$

Confidence Interval (95%): $0.95 = P\left(\frac{x - \bar{x}}{\sigma_{\bar{x}}} < z(1 - \frac{\alpha}{2})\right)$; $0.727 \pm 1.96(\sigma_{\bar{x}})$

$0.727 \pm 1.96(2.05 \times 10^{-2})$

$\boxed{0.727 \pm 0.04}$

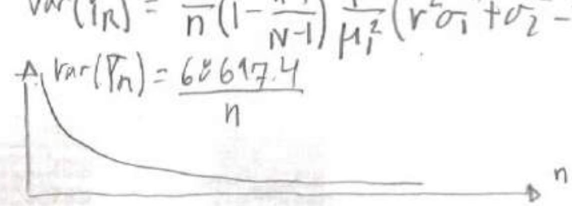
44. $\frac{Var(\bar{Y}_R)}{Var(\bar{Y})} \approx 1 + \frac{C_x}{C_y}\left(\frac{C_x}{C_y} - 2\rho\right) = 1 + x^2 + x$



45. $\rho = 0.91$ $Var(\bar{Y}_R)$; Plot $Var(\bar{Y}_R)$ for $n = 64$

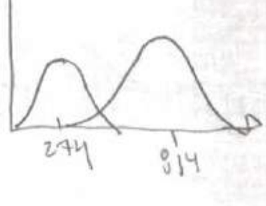
$-1 < \rho < 1$

$Var(\bar{Y}_R) = \frac{1}{n}\left(1 - \frac{n-1}{N-1}\right)\frac{1}{\mu_1^2}\left(r^2\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_2\sigma_1\right)$



46. $\boxed{\sigma_{\bar{Y}_R} \approx 30.0}$ ~~32.76~~

$\boxed{\sigma_{\bar{Y}} = 66.3}$



36. Simple Random Sampling: $\bar{X}^2$ is unbiased estimate of μ^2. Simple random sampling is an unbiased estimate of $\bar{\mu}^2$ when there is true <u>random</u> sampling, for example, each value has equal probability. Otherwise, the $\bar{X}^2$ is not random, and biased.

37. Population Mean $=\mu$; <u>Survey #1</u>   <u>Survey #2</u>

$\bar{X}_1 = \text{Mean}$ $\bar{X}_2 = \text{Mean}$ } Unbiased. $X = \alpha\bar{X}_1 + \beta\bar{X}_2$

$\sigma_{\bar{X}_1}$ = Standard Error $\sigma_{\bar{X}_2}$ = Standard Error

a) Find conditions for α and β which are an unbiased combination.

$$Var(\mu) = E[\mu^2] - E[\mu]^2 = \frac{(\alpha\bar{X}_1+\beta\bar{X}_2)^2}{n} - \left[\frac{(\alpha\bar{X}_1+\beta\bar{X}_2)}{n}\right]^2 = \frac{(\alpha\bar{X}_1)^2 n + 2(\alpha\beta\bar{X}_1\bar{X}_2)n + (\beta\bar{X}_2)^2 n - (\alpha\bar{X}_1)^2 - 2(\alpha\beta\bar{X}_1 X_2) - (\beta\bar{X}_2)^2}{n^2}$$

$$= \frac{(\alpha\bar{X}_1)^2(n-1) + 2(\alpha\beta\bar{X}_1\bar{X}_2)(n-1) + (\beta\bar{X}_2)^2(n-1)}{n^2} = \frac{n-1}{n^2}\left[\alpha\bar{X}_1 + \beta\bar{X}_2\right]^2$$

$$var(\mu) = Var(\alpha\bar{X}_1 + \beta\bar{X}_2) = \alpha^2 Var(\bar{X}_1) + \beta^2 Var(\bar{X}_2) + \frac{(n-1)}{n^2}\left[\alpha\bar{X}_1 + \beta\bar{X}_2\right]^2$$

$$= \frac{n-1}{n^2}\left[(\alpha\bar{X}_1)^2 + 2\alpha\beta\bar{X}_1\bar{X}_2 + (\beta\bar{X}_2)^2\right]$$

$E(X) = \alpha E(\bar{X}_1) + \beta E(\bar{X}_2)$

$= (\alpha+\beta)\mu$

$\boxed{(\alpha+\beta) = 1}$

"unbiased"

$$\alpha^2 Var(\bar{X}_1) + \beta^2 Var(\bar{X}_2) = \alpha^2[\sigma_x^2] + (1-\alpha)^2[\sigma_x^2]$$

$$\boxed{= \alpha^2[\sigma_{x_1}^2] + \sigma_{x_2}^2 + \alpha^2[\sigma_{x_2}^2] - 2\alpha[\sigma_{x_2}^2]}$$

b) $\frac{d}{d\alpha}[\alpha^2[\sigma_{x_1}^2] + \sigma_{x_2}^2 + \alpha^2[\sigma_{x_2}] - 2\alpha[\sigma_{x_2}^2]]$

$$\boxed{\alpha = \frac{\sigma_{x_2}^2}{[\sigma_{x_1}^2 + \sigma_{x_2}^2]}} \quad ; \quad \boxed{\beta = \frac{\sigma_{x_1}^2}{\sigma_{x_1}^2 + \sigma_{x_2}^2}}$$

38. $X_1 \ldots X_n$ be random sample. Show $\frac{1}{n}\sum_{i=1} X_i^3$ is an unbiased estimate of $\frac{1}{N}\sum_{i=1}^{N} X_i^3$

$E[X^3] = \frac{1}{N}\sum_{i=1}^{N} X_i^3$ ← No β to incorporate, no bias and no shift.

39. N = Population of items   How large should a sample be to find a defective item?

k = Defective items.

Assuming $p = 0.95$; n = sample size ; $k=1$; $P\left(1 < \frac{X-\mu}{\sigma_x} < k\right) = 0.95$

$$1 - \frac{N-k}{N} \times \frac{N-k-1}{N-1} \times \cdots \times \frac{N-k-n+1}{N-n+1} > 0.95 \quad \text{if } k=1$$

$$1 - \frac{N-1}{N} \times \frac{N-1-1}{N-1} \times \cdots \times \frac{N-n}{N-n+1} > 0.95;$$

$$1 - \left(\frac{N-n}{N}\right)^k > 0.95$$

$$\left(\frac{N-n}{N}\right)^k < 0.05$$

$$\log(N-n) - \log(N) \approx \frac{\log(0.05)}{k}$$

$$n \approx N - e^{\left(\frac{\log(0.05)}{k} + \log(N)\right)}$$

$$\boxed{\approx 501}$$

40. i^{th} member has $P(i) = \frac{n-n_i}{N-i+1}$

$$= \frac{(N-n)!n!}{N!} = \boxed{\frac{N-n}{N-(i-1)}} = \frac{1}{\binom{N}{n}}$$

41. D = Audited value − Book value

$\underbrace{N\bar{D}}_{\text{Inventory value.}}$, where N = population size.

a) Prove unbiased estimate

$$\boxed{E[N\bar{D}] = \frac{N}{n}\sum_{i=1}^{N} D = NE[D]}$$

d. 99% confidence Interval: $\hat{d} - z(1-\frac{0.01}{2}) \cdot s_x < \hat{d} < \hat{d} + \hat{z}(\frac{1-0.01}{2}) s_x$

$\hat{d} - 2.57 \left(\sqrt{\frac{0.12(1-0.12)+0.18(1-0.18)}{100}} \left[1 - \frac{4000}{1000^4}\frac{(100-1)}{(4000-1)}\right]\right.$

$-0.06 - 2.57(5.03\times10^{-2}) < d < \frac{3}{50} + 2.57(5.03\times10^{-2})$

$-\frac{3}{25} < d < 6.927\times10^{-2}$

95% confidence Interval:

$-0.06 - 1.96(5.03\times10^{-2}) < d < -0.06 + 1.96(5.03\times10^{-2})$

$-0.159 < d < -3/25$

90% confidence Interval:

$-0.06 - 1.65(5.03\times10^{-2}) < d < -0.06 + 1.65(5.03\times10^{-2})$

$-0.143 < d < 0.029$

No, there is little difference for a 99% confidence interval ranging from $\left(\frac{-3}{25}, \frac{69}{100}\right)$.

33. n = simple random samples, two proportions, $\hat{p}_1$ and $\hat{p}_2 \approx 0.5$
What should the sample size be for $\hat{p}_1 - \hat{p}_2 < 0.02$?

standard error of proportions: $s_{\hat{p}_1,\hat{p}_2} = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}} < 0.02$

$p_1(1-p_1) + p_2(1-p_2) = 0.0004 \cdot n$

$\frac{2p_1(1-p_1)}{0.0004} = \frac{2 \cdot 0.5 \cdot 0.5}{0.0004} = \boxed{1.25\times10^3}$

34. P(Problem #1) = 3% population
P(Problem #2) = 40% population

a) $s_{\hat{p}_1,\hat{p}_2} = \sqrt{\frac{p_1(1-p_1)}{n_2} + \frac{p_2(1-p_2)}{n_2}}$

Same population size

$s_{\hat{p}_1,\hat{p}_2} = 0.01 > \sqrt{\frac{0.03(0.97)+0.4(0.6)}{n}}$

2 $\quad n > \frac{2.691\times10^{-1}}{1\times10^{-4}}$

$\boxed{n > 2,691}$

b) $n > \frac{2.691\times10^{-1}}{0.01}$

$n > 26.91$

c)

35. N_1 = Population Size = 2000
with n = 25 values.

104	109	111	109	87
86	80	119	88	122
91	103	99	108	96
104	98	98	83	107
79	87	94	92	97

a) Calculate unbiased estimate of population mean.

$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{25}[104 + \cdots + \cdots + \cdots + 97] = \boxed{98}$

b) Calculate unbiased estimate of population variance

$\sigma_{\bar{x}}^2 = \left(1 - \frac{1}{N}\right)s^2 = \left(1 - \frac{1}{2000}\right)(E[X^2] - E[X]^2)$

$= \frac{1999}{2000}\left[\frac{1}{n}\sum_{i=1}^{25} x_i^2 + \left[\frac{1}{n}\sum(x_i)\right]^2\right] \cdot \frac{1999}{2000}\left[\frac{243505}{25} - \left(\frac{2450}{25}\right)^2\right]$

$= \boxed{136}$ $\quad \frac{\sigma_{\bar{x}}^2}{n}\left(1 - \frac{n}{N}\right) = \boxed{5.37}$

c) Approximate a 95% confidence interval.

$P\left(\left|\frac{x - \bar{x}}{\sigma_{\bar{x}}}\right| < z(1-\frac{\alpha}{2})\right) = 0.95$; $\sigma_{\bar{x}} z(1-\frac{\alpha}{2})$

$2P(x - \bar{x} < \sigma_{\bar{x}} z(1-\frac{\alpha}{2})) = 1.95$; $= \sigma_{\bar{x}} 1.96$

$98 \pm 1.96\sqrt{5.37}$ $\quad T = N\bar{x}_s = 196{,}000$

$\boxed{98 \pm 4.54}$ $\quad \boxed{196{,}000 \pm 23{,}323}$

31. N = 8000 condominium units ; n = 100 sample size ; $\bar{X} = 1.6$ motor vehicles

$s_{\bar{X}} = \frac{s}{\sqrt{n}}\sqrt{1-\frac{n}{N}} = \frac{0.8}{\sqrt{100}}\sqrt{1-\frac{100}{8000}} = 0.08$ } Standard error ; $\sigma = 0.8$ motor vehicles

confidence interval { $\bar{X} \pm 1.96 s_{\bar{X}} = (1.44, 1.76)$; $T = 8000 \times 1.6 = 12{,}800$ } Total

$\underbrace{s_T = N s_{\bar{X}} = 640}_{\text{Total standard error}}$; $T \pm 1.96 s_T$ $(11546, 14054)$ } Interval of total

12% planned to sell their condo $[\hat{p} = 0.12]$; $s_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n-1}}\sqrt{1-\frac{100}{8000}} = 0.03$

$\hat{p} \pm 1.96 s_p (0.06, 0.18)$; $T = N\hat{p} = 960$; $s_T = N s_p = 240$; $T \pm 1.96 s_T$ $(490, 1430)$

What is the sample size for 95% confidence interval to have 500 width (ΔT) of 500?

$T + 1.96 s_T - T + 1.96 s_T = 2 \cdot 1.96 s_T = 500$; $s_T = 500/2 \cdot 1.96 = 127.55$

$= N s_p = 8000 s_p$; $s_p = \frac{127.55}{8000} = 1.59 \times 10^{-2} = \sqrt{\frac{0.12(1-0.12)}{n-1}} \cdot \sqrt{1-\frac{n}{8000}}$

$2.53 \times 10^{-4} = \frac{0.12(0.88)}{n-1}\left(\frac{8000-n}{8000}\right) \Rightarrow 2.53 \times 10^{-4} n - 2.53 \times 10^{-4} = \frac{66}{625}\left(\frac{8000-n}{8000}\right)$

$2.53 \times 10^{-4} n - 2.53 \times 10^{-4} = \frac{66}{625} - \frac{33}{2500000} n$; $2.66 \times 10^{-4} n = 0.1058$; $\boxed{n = 397.9}$

32. N = 12,000 units ; n = 200 sample size ; $\hat{p} = 0.18$

a) What is $s_{\hat{p}}$? $s_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n-1}}\sqrt{1-\frac{n}{N}} = \sqrt{\frac{0.18(1-0.18)}{200-1}}\sqrt{1-\frac{200}{12000}} = 0.027$

$P\left(\frac{|p-\hat{p}|}{s_p} < z(\alpha/2)\right) = 1-\alpha = 0.90$; $\alpha = 0.05$; $z(0.1/2) = z(0.05) = z(1-0.05) = z(0.95) = 1.65$

$P(|p-\hat{p}| \le s_{\hat{p}} \cdot 1.96) = 0.95$; $P(-1.96 s_{\hat{p}} \le p - \hat{p} \le 1.96 s_{\hat{p}}) = P(\hat{p} - 1.96 s_p \le p \le \hat{p} + 1.96 s_{\hat{p}})$

$= P(p \le \hat{p} + 1.96 s_{\hat{p}}) - P(\hat{p} - 1.96 s_p \le p) = P(p \le \hat{p} + 1.96 s_{\hat{p}}) - (1 - P(p \le \hat{p} + 1.96 s_p))$

$= 2P(p \le \hat{p} + 1.96 s_p) - 1 = 0.95$; $P(p \le \hat{p} + 1.96 s_p) = \frac{1.95}{2} = 0.975 \rightarrow$ solve

... or ... $(\hat{p} - 1.65 s_p, \hat{p} + 1.65 s_p) = (0.18 - 1.65 \cdot 0.027, 0.18 + 1.65 \cdot 0.027)$

$\boxed{= (0.135, 0.225)}$

b) $\hat{p}_1 = 0.12$, $\hat{p}_2 = 0.18$; $\hat{d} = \hat{p}_1 - \hat{p}_2$

$Var(\hat{d}) : s_p^2 = E[\hat{d}^2] - E[\hat{d}]^2 =$

$= E[(\hat{p}_1 - \hat{p}_2)^2] - E[\hat{p}_1 - \hat{p}_2]^2$

$= E[\hat{p}_1^2] - 2E[\hat{p}_1\hat{p}_2] + E[\hat{p}_2^2] - E[\hat{p}_1 - \hat{p}_2]^2$

$= \hat{p}_1 - 2\hat{p}_1\hat{p}_2 + \hat{p}_2 - (\hat{p}_1 - \hat{p}_2)^2$

$= \hat{p}_1 - \cancel{2\hat{p}_1\hat{p}_2} + \hat{p}_2 - \hat{p}_1^2 + \cancel{2\hat{p}_1\hat{p}_2} - \hat{p}_2^2$

$\boxed{= p_1(1-\hat{p}_1) + p_2(1-\hat{p}_2)}$

Standard Error (unbiased):

$s_{\bar{X}}^2 = \frac{1}{n}\sum_i^n E(X_i^2) - E(\bar{X}^2) = \frac{1}{n}\sum_i [Var(X) + E(X)^2] - [Var(\bar{X}) + E(X)^2]$

$= \frac{1}{n}[p_1(1-p_1) + p_2(1-p_2) + E(\hat{d})^2] - [\frac{1}{n^2}\sum_i Var(X_i) + \frac{1}{n^2}\sum_i\sum_j Cov(X_i, Y_j) + E(X)^2]$

$= \frac{p_1(1-p_1) + p_2(1-p_2)}{n}\left[1 - \frac{1}{n^2}[E(p_1p_2) - E(p_1)E(p_2)]\right]$

$= \frac{p_1(1-p_1) + p_2(1-p_2)}{n}\left[1 - \frac{1}{n^2}\frac{N}{n^2}\frac{(n-1)}{(N-1)}\right]$

$\boxed{= \frac{p_1(1-p_1) + p_2(1-p_2)}{n}\left[1 - \frac{N}{n^4}\frac{(n-1)}{(N-1)}\right]}$

14. $P(x|H_0) = N(0,\sigma^2)$ $\quad \dfrac{P(H_0|x)}{P(H_1|x)} = \dfrac{P(H_0)}{P(H_1)} \dfrac{P(x|H_0)}{P(x|H_1)} = 2e^{\frac{(x-1)^2 - x^2}{2\sigma^2}} = 2e^{\frac{-2x+1}{2\sigma^2}}$; $\ln(\frac{1}{2}) = \dfrac{-2x+1}{2\sigma^2}$; $x = \dfrac{2\sigma^2 \ln(\frac{1}{2}) - 1}{-2}$

$P(x|H_1) = N(1,\sigma^2)$

$P(H_0) = 2 \times P(H_1)$

a)

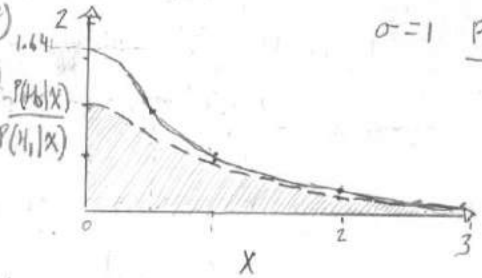
σ^2	0.1	0.5	1.0	5.0
$x: H_0 > H_1$	0.57	0.95	1.19	3.96

b) $\boxed{2/3}$

15. $P(x|H_0) = N(0,\sigma^2)$

$P(x|H_1) = N(1,\sigma^2)$

$P(H_0) = P(H_1)$



$\sigma = 1$ $\quad \dfrac{P(H_0|x)}{P(H_1|x)} = \dfrac{P(H_0)}{P(H_1)} \cdot \dfrac{P(x|H_0)}{P(x|H_1)} = e^{\frac{-2x+1}{2\sigma^2}}$

The p-value of $P(H_0|x)/P(H_1|x)$ and $P(H_0)$ show similarly symmetric graphs, so their p-values are similarly symmetric. Another σ will show equivalent results because of scaled proportions.

16. $\alpha = 0.05$

$\dfrac{P(H_0|x)}{P(H_1|x)} = e^{\frac{-2x+1}{2\sigma^2}} > 1$; $\boxed{x > \frac{1}{2}}$; $\dfrac{P(H_0|x)}{P(H_1|x)} = e^{\frac{-2x+1}{2\sigma^2}} < 1$; $\boxed{x < \frac{1}{2}}$

17. $P(x|H_0) = N(0,\sigma_0^2)$

$P(x|H_1) = N(0,\sigma_1^2)$

$\sigma_1 > \sigma_0$

a) $\Lambda = \dfrac{P(x|H_0)}{P(x|H_1)} = \dfrac{N(0,\sigma_0^2)}{N(0,\sigma_1^2)} = e^{-x^2/2\sigma_0^2 + x^2/2\sigma_1^2} = \boxed{e^{-\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)\frac{x^2}{2}} \cdot \sigma_1/\sigma_0}$ $\quad$ H_0 Favored: $\boxed{x > \sqrt{\dfrac{2\ln \sigma_1/\sigma_0}{\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)}}}$

The rejection region of a level α test: $\ln\Lambda = -\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)\frac{x^2}{2} + \ln \sigma_1/\sigma_0$

$-2\ln\Lambda = \left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)x^2 - 2\ln\sigma_1/\sigma_0$

$-2\ln\Lambda = \sum\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)x_i^2 - 2n\ln\sigma_1/\sigma_0$

$\chi^2 < \left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)\sum x_i^2 - 2n\ln\sigma_1/\sigma_0$

$\left|\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right| > \dfrac{2z(\alpha/2) + 4n\ln\sigma_0/\sigma_1}{\sum x_i^2}$

$P(\chi^2 > \chi^2(\alpha)) = \alpha$

$P\left(\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)\sum x_i^2 - 2n\ln\sigma_1/\sigma_0 > \chi^2(\alpha)\right) = \alpha$

Assuming largest term is $\frac{1}{\sigma_0^2}\sum x_i^2$ because $\sigma_1 >> \sigma_0$.

$P\left(\frac{1}{\sigma_0^2}\sum x_i^2 > \chi^2(\alpha)\right) = \alpha$; $\boxed{\sigma_0^2 \chi^2 < X^2}$

b. $x_1, x_n \ldots x_n$

$P\left(\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)\sum x_i^2 - 2n\ln\sigma_1/\sigma_0 > \chi^2(\alpha)\right) = \alpha$; $\boxed{\sigma_0^2 \chi_n^2(\alpha) < \sum X_i^2}$

c. Yes, because the rejection level is a simple hypothesis.

18. $x_1, x_2 \ldots x_n$ i.i.d.

Double Exponential Distribution

$f(x) = \frac{1}{2}\lambda \exp(-\lambda|x|)$

Maximum Likelihood:

$\dfrac{d\ln f(x)}{d\lambda} = \dfrac{n}{\lambda} - \sum|x|$

$\hat{\lambda} = \dfrac{1}{\bar{x}}$

Likelihood Ratio Test: $\Lambda = \dfrac{f(x|\lambda_0)}{f(x|\lambda_1)} = \left(\dfrac{\lambda_0}{\lambda_1}\right)\exp^{\left(\frac{1}{x_i} - \frac{1}{\bar{x}}\right)|\bar{x} - x_i|}$

$-\ln\Lambda = \sum\ln\left(\frac{\lambda_0}{\lambda_1}\right) + \dfrac{\sum(\bar{x} - x_i)^2}{x_i\bar{x}}$ $= 0$

$\ln\Lambda = \sum\ln\left(\frac{x_i}{\bar{x}_i}\right) = \sum \underbrace{(x_i - \bar{x})}_{=0} + \frac{1}{2}(x_i - \bar{x})^2\frac{1}{\bar{x}}$

$\boxed{2\ln\Lambda = 2\dfrac{\sum(x_i - \bar{x})^2}{\bar{x}} = 2X^2}$

a) $P(H_0) = P(H_1)$

$\dfrac{P(H_0|x)}{P(H_1|x)} = \dfrac{P(H_0)}{P(H_1)} \dfrac{P(x|H_0)}{P(x|H_1)} = \dfrac{F_0(x)'}{F_1(x)'} = \dfrac{2x}{3x^2} = \dfrac{2}{3x} > 1$; $\boxed{x < \frac{2}{3}}$

b) $\boxed{\Lambda = \dfrac{2}{3x}}$

The test is uniformly most powerful because of the squared terms. $\lambda > \lambda_0$ vs $\lambda_1 > \lambda_0$ are equivalent outcomes.

19. $H_0: F_0(x) = x^2$

$0 \le x \le 1$

$H_1: F_1(x) = x^3$

$0 \le x \le 1$

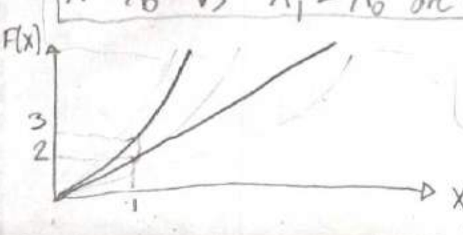
c) $P(x: 2/3 > z(\alpha/2)) = 1 - \alpha$

$x^2 = 1 - \alpha$

$\boxed{x = \sqrt{1-\alpha}}$

d) $P(x: 2/3 > z(\alpha/2)) = 1 - \beta$

$\boxed{\beta = 1 - \dfrac{(1-\alpha)^3}{2}}$



20. $[0,1]$: $\underbrace{f_0(x)=1}_{H_0}$, $\underbrace{f_1(x)=2x}_{H_1}$; $\alpha = 0.10$; $P\left(\frac{f_0(x)}{f_1(x)} \le c\right) = P\left(\frac{1}{2x} \le c\right) = P\left(x \ge \frac{1}{2c}\right)$

$\boxed{\text{Null Hypothesis}} = \int_{1/2c}^{1} f_0(x)\,dx = \int_{1/2c}^{1} 1\,dx = 1 - \frac{1}{2c} = 0.1$; $c = 5/9$

$P\left(\frac{f_0(x)}{f_1(x)} < c\right) = P\left(\frac{1}{2x} \le c\right) = P\left(x \ge \frac{1}{2c}\right) = P(x \ge 0.9) = \int_{0.9}^{1} f_1(x)\,dx$

$= 1 - 0.9^2 = \boxed{0.19}$

21. $[0,\theta]$; $H_0 : \theta = 1$ vs $H_1 : \theta = 2$ Uniform

a. $\alpha = 0$. What is its power? $f(\theta) = \frac{1}{\theta}$

$P\left(\frac{f_0(\theta|H_0)}{f_1(\theta|H_1)} < c\right) = P(2 < c) = \int_c^\theta d\theta = \theta - c = 0$
$c = \theta$

$P\left(\frac{f_0(\theta|H_0)}{f_1(\theta|H_1)} < c\right) = P(2 < c) = P(2 > c) = P(2 > 1) = \int_1^\theta \frac{1}{2} d\theta = \frac{1}{2}(\theta - 1) = \boxed{\frac{1}{2}}$

b. $0 < \alpha < 1$; $X \in [0, \alpha]$; $P(0 < X < \alpha) = P(X < \alpha) - P(0 < X) = \int_x^\alpha d\theta - \int_0^x d\theta = \alpha - x - x = \boxed{\alpha}$

$P(1 < X < 1-\alpha) = P(X < 1-\alpha) - P(1 < X) = \int_x^{1-\alpha} \frac{1}{2} d\theta - \int_1^x \frac{1}{2} d\theta = \frac{1}{2}(1 - \alpha - x) - \frac{1}{2}(x - 1) = \boxed{1 - \frac{\alpha}{2}}$

c. $X \in [1-\alpha, 1]$; $P(1-\alpha < X < 1) = P(X < 1) - P(1-\alpha < X)$

d $\boxed{x \in [(1-\alpha)/2 \le x \le (1+\alpha)/2]}$ $= \int_1^x d\theta - \int_{1-\alpha}^x d\theta = x - 1 - x + 1 + \alpha = \boxed{\alpha}$; $P(\alpha < X < 1) = P(X > 1) - P(\alpha < X) = \int_1^x \frac{1}{2} d\theta - \int_\alpha^x \frac{1}{2} d\theta$

$\boxed{1 - \frac{\alpha}{2}}$

e. $\boxed{\text{The likelihood ratio test determines unique rejection for } \alpha > 0.}$

f. $H_0 = \theta = 2$; $H_1 : \theta = 1$; $P\left(\frac{f_0(\theta|H_0)}{f_1(\theta|H_1)} < c\right) = P\left(\frac{1}{2} < c\right) = \int_c^\theta \frac{1}{2} d\theta = \frac{\theta - c}{2} = 0$; $c = \theta$

$P\left(\frac{f_0(\theta|H_0)}{f_1(\theta|H_1)} < c\right) = P\left(\frac{1}{2} < c\right) = P\left(\frac{1}{2} < 1\right) = \int_1^\theta d\theta = \theta - 1 = 0$

$\boxed{\text{The rejection region is not capable of being determined.}}$

22. Example A: Section 8.5.3

$\left(\frac{n\hat{\sigma}^2}{\chi^2_{n-1}(\alpha/2)}, \frac{n\hat{\sigma}^2}{\chi^2_{n-1}(1-\alpha/2)}\right)$ $P(X|H_0) = P(x|\sigma_0^2)$

Theorem B: Section 9.3 Significance level: α

$P[\theta_0 \in C(X) | \theta = \theta_0] = 1 - \alpha$ $X_1, X_2, \ldots, X_n$

$A(\theta_0) = \{X | \theta_0 \in C(X)\}$ $\sigma_0 = 1, n = 15, \alpha = 0.05$

$A(\sigma_0^2) = \{X | \sigma_0^2 \in C(X)\}$

$= \left\{X \,\middle|\, \frac{n}{n-1} \frac{\sum_1^n (X_i - \bar{X})^2}{\chi^2_{n-1}(\alpha/2)} > \sigma_0^2 > \frac{n}{n-1} \frac{\sum_1^n (X_i - \bar{X})^2}{\chi^2_{n-1}(1-\alpha/2)}\right\}$

$= \left\{X \,\middle|\, \frac{15}{14} \frac{\sum_1^{15} (X_i - \bar{X})^2}{\chi^2_{14}(0.05/2)} > \sigma_0^2 > \frac{15}{14} \frac{\sum_1^{15} (X_i - \bar{X})^2}{\chi^2_{14}(1 - 0.05/2)}\right\}$

$P(\sigma_0^2 \in C(x) | \sigma^2 = \sigma_0^2) = 1 - \alpha$

$\boxed{P\left(\frac{19}{100} \sum_1^{15} (X_i - \bar{X})^2 > 1 > \frac{4}{100} \sum_1^{15} (X_i - \bar{X})^2\right) = 0.95}$ $\boxed{= \left\{X \,\middle|\, \frac{19}{100} \sum_1^{15} (X_i - \bar{X})^2 > \sigma_0^2 > \frac{4}{100} \sum_1^{15} (X_i - \bar{X})^2\right\}}$

23. 99% Confidence Interval: μ

Normal Distribution:

$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$; $\mu \in (-2.0, 3.0) = P(-\sigma^2 z(\alpha/2) < \mu < \sigma^2 z(\alpha/2)) = 1 - \alpha = 0.99$

$\alpha = 0.01$; $z(0.005) = 2.57$ $\sigma^2 = 0.195$;

$\Lambda = e^{\frac{+(x-\mu_1)^2 - (x-\mu_0)^2}{2\sigma^2}} = e^{\frac{-2\mu_1 x + \mu_1^2 + 2\mu_0 x + \mu_0^2}{2\sigma^2}} = e^{\frac{-2\mu_1 x + \mu_1^2 - 6x + 9}{2(0.195)}} = e^{\frac{-(2\mu_1 + 6)x + \mu_1^2 + 9}{0.39}}$

$P\left(\frac{-(2\mu_1 + 6)x + \mu_1^2 + 9}{0.39} > 0\right) = P\left(x > \frac{\mu_1^2 + 9}{(2\mu_1 + 6)}\right)$; $\boxed{\text{Yes, the rejection region demonstrates } H_A : \mu_1 \ne -3 \text{ does not suffice.}}$

24. n = trials, p = prob success

Binomial Random Variable

$p(1) = p$

$p(0) = 1 - p$

$p(x) = 0$; $x \ne 0, x \ne 1$

a) $H_0 : p = 0.5$

$H_A : p \ne 0.5$

$\Lambda = \frac{p(x|H_0)}{p(x|H_A)} = \frac{P_0^n (1 - P_0)^{n-x}}{P_A^n (1 - P_A)^{n-x}} = \boxed{\frac{(0.5)^{n-x}}{\left(\frac{x}{n}\right)^x \left(1 - \frac{x}{n}\right)^{n-x}}}$

b. Rejection Region: $\Lambda = \frac{(1/2)^n}{\left(\frac{x + n/2}{n}\right)^{x + n/2} \left(1 - \frac{x + n/2}{n}\right)^{n/2 - x}}$

$= \frac{(1/2)^n}{\left(\frac{1}{2} + \frac{x}{n}\right)^{x + n/2} \left(\frac{1}{2} - \frac{x}{n}\right)^{n/2 - x}}$

$k, 4|x|$; $0 < \alpha < \alpha < 0$

25. Example B: Section 9.5

Number per Square	0	1	2	3	4	5	6	7	8	9	10	19
Frequency	56	104	80	62	42	27	9	9	5	3	2	1

Likelihood Ratio for a Poisson

$$\Lambda = \frac{\prod \hat{\lambda}^{x_i} e^{-\bar{x}} / x_i!}{\prod \tilde{\lambda}^{x_i} e^{-\tilde{\lambda}_i} / x_i!} = \prod_{i=1}^{n} \left(\frac{\bar{x}}{x_i}\right)^{x_i} e^{x_i - \bar{x}} \; ; \; \bar{X} = \frac{0\times56 + 1\times104 + 2\times80 + \cdots + 19\times1}{400} = 2.44$$

Number Per Square	0	1	2	3	4	5	6	7, 8, 9, 10, 19
log likelihood	266	21	-20	-19	49	2	0	27, 726

$= 29$

$$-2\log\Lambda = -2\sum x_i \log\left(\frac{\bar{x}}{x_i}\right) + (x_i - \bar{x}) = -2\sum x_i \log\left(\frac{x_i}{\bar{x}}\right) \approx -2\sum (x_i - \bar{x}) - \frac{(x_i-\bar{x})^2}{2\bar{x}} + \cdots$$

$$\approx \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{\bar{x}} = 58$$

$\boxed{\Lambda = 2.54\times10^{-13}}$

The similarities of likelihood $[\Lambda]$ and chi-square $[\chi^2]$ are within 23% of each other indicating error of measurement from maximum likelihood estimate applied to observed vs expected values.

26. a) False, the generalized likelihood ratio statistic favors or rejects at a boundry of less than or greater than 1

b) True, the corresponding test would reject at $\alpha = 0.02$

c) False, the p-value is 0.06 and not less than.

d) False, p-value is the smallest value at which the test would be rejected.

e) False, p-value is a threshold for rejection and simple hypothesis depend on a single value.

f) False, the p-value would be greater than 0.05

27. df = 7 $\boxed{{}_4\chi^2_{0.9} = 12.02}$; $\boxed{{}_4\chi^2_{0.95} = 14.07}$; $\boxed{{}_4\chi^2_{0.975} = 16.01}$

$\boxed{{}_4\chi^2_{0.99} = 18.48}$ $\boxed{{}_4\chi^2_{0.995} = 20.28}$

28a) $P(T > t_0 | H_0) = \alpha$; $P(1.5 > t_0 | H_0)$; $1.5 = z(\alpha)$

$\boxed{0.9332 = \alpha}$

b) $\boxed{\alpha > 0.9901}$

29. Yes, the monotone increasing function $s > g(t_0)$ is a test.

30a) Show $V = 1 - F(T)$; $F(t) = \int_{-\infty}^{\alpha} f(t)\,dt = P(T > \alpha | H_0) = 1 - V = 1 - \alpha$

b) $P(V \le z) = P(T \le F^{-1}(z)) = F(F^{-1}(z)) = z$

c) $\boxed{0.1}$ d) $P(V < \alpha) = P(F(T) < \alpha) = P(T < F^{-1}(\alpha))$ $P(F^{-1}(1-\alpha) < T)$

$= F(F^{-1}(\alpha)) = z = \alpha$

31. ${}_1\chi^2_{0.1} = 0.0158$

$$-2\log\Lambda = \frac{n}{\sigma^2}(\bar{x} - \mu_0)^2 > {}_1\chi^2_{0.1} = 0.0158$$

${}_{df}\chi^2_p$	${}_1\chi^2_{0.1}$	${}_5\chi^2_{0.1}$	${}_{10}\chi^2_{0.1}$	${}_{20}\chi^2_{0.1}$
Λ	2.6×10^{-1}	9.9×10^{-3}	3.4×10^{-4}	6.8×10^{-7}

24. continued...

c) $n = 100$, $k = 2$

$$P(X - 5 > 2) = P(X = 0, 1, 2, 8, 9, 10) = \frac{7}{64} = 0.1094$$

d) $n = 100$, $k = 2$

$P(X - 50 > 10) = P(X - 50 \ge 11)$

$E(X) = 100 \times 0.5 = 50$

$V(X) = 100 \times 0.5 \times 0.5 = 25$

32. Object A: $\mu_A = 100$; $\sigma = 25$; Object B: $\mu_B = 125$; $\sigma = 25$; $X = 120$

a) $\Lambda = \dfrac{e^{-(x-\mu_A)^2/2\sigma^2}}{e^{-(x-\mu_B)^2/2\sigma^2}} = e^{\frac{-(120-100)^2+(120-125)^2}{2\cdot 25^2}} = \boxed{0.74}$

b) $P(A)=P(B)=\frac{1}{2}$; $\dfrac{P(H_A|X)}{P(H_B|X)} = \dfrac{P(H_A)}{P(H_B)}\dfrac{P(X|H_A)}{P(X|H_B)} = \boxed{0.74}$

c) $P(X>125|H_0) = P(X>125) = \alpha$; $0 = z(\alpha/2)\sigma$; $z(\alpha) = 0$; $\boxed{\alpha = 0.5}$

d) Power of Test; $\beta = \int_{125}^{\infty} \dfrac{e^{-(x-\mu_B)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,dx = \boxed{0.5}$

e) $X=120$; $\frac{X-125}{\sigma} = z(\alpha)$; $\frac{-5}{25} = z(\alpha)$; $\frac{-1}{5} = z(\alpha)$; $\frac{1}{5} = 1-z(\alpha)$; $z(\alpha) = 0.8$; $\boxed{\alpha = 0.7881}$

33.

	Jewish	Chinese & Japanese
Deaths Before Holiday	922	418
Deaths After Holiday	997	434
Pearsons [X^2]	2.93	3.00
Likelihood [Λ]	2.93	3.004
p-value (df = 1)	0.05-0.10	0.05-0.10

No evidence for capability of pospaning death.

34. Problem #55: Chapter 8 0.0357

Type	O	E	Likelihood [Λ]	Chi-Squared [X^2]
Starchy Green	1997	1953		
Starchy White	906	925	5.98	1.97
Sugary Green	904	925		
Sugary White	32	34		

p-value ~ 0.1 pvalue ~ 0.9

${}_3X^2_{0.9} = 6.25$ ${}_3X^2_{0.1} = 0.584$

There is consideration to reject null hypothesis.

35.

	Hp I-1	Hp 1-2	Hp 2-2
O_i	10	68	112
	$(1-\theta)^2$	$2\theta(1-\theta)$	θ^2
E_i	10	68	112

Haptoglobin Type

$n = 190$

$\hat{\theta} = 0.768$

$\Lambda = 0$ $X^2 = 0$

df: 2; p-value < 0.01

${}_2X^2_{0.995} = 7.88$

Cannot reject null hypothesis.

36.

Month	# Suicides	Days/Month	Probability
Jan	1867	31	0.085
Feb	1789	28	0.077
Mar	1944	31	0.085
Apr.	2094	30	0.082
May	2097	31	0.085
June	1981	30	0.082
July	1887	31	0.085
Aug	2024	31	0.085
Sep	1928	30	0.082
Oct	2032	31	0.085
Nov	1978	30	0.082
Dec	1859	31	0.085

$X^2 = \dfrac{(1867-1957)^2 + \cdots + (1859-1957)^2}{}$; $p(\theta|H_0) = \frac{1}{12}$; $p(\theta|H_1) = \sum p(\theta_i) = 1$

$n = 23480$; $E = np = 1957$

$= 51.79$

${}_{11}X^2_{0.995} = 26.76$; p-value < 0.005

Since $X^2 > {}_{11}X^2_{0.9995}$ a rejection of the null hypothesis occurs and suicide is not constant. at p-value = 0.005.

Multinomial: $f(\theta) = \dfrac{n!}{x_1!\cdots x_m!} p_1(\theta)^{x_1}\cdots p_m(\theta)^{x_m}$

mle $= x = np_i$; $p_i = \frac{x}{n}$

Likelihood Ratio:

$$\Lambda = \dfrac{\frac{n!}{x_1!\cdots x_m!} p_1(\theta)^{x_1}\cdots p_m(\theta)^{x_m}}{\frac{n!}{x_1!\cdots x_m!} p_1^{x_1}\cdots p_m^{x_m}} = \prod_{i=1}^{n}\left(\frac{p_i(\theta)}{p_i}\right)^{x_i}$$

$$-2\log\Lambda = -2\sum x_i\left(\frac{p_i(\theta)}{p_i}\right)$$

$$= -2n\sum p_i\left(\frac{p_i(\theta)}{p_i}\right)$$

$$= 2n\sum p_i\left(\frac{p_i}{p_i(\theta)}\right)$$

37.

Month	# of Deaths
Jan	1668
Feb	1407
Mar.	1370
Apr	1309
May	1341
June	1338
July	1406
Aug	1446
Sep.	1332
Oct	1363
Nov	1410
Dec	1526

$P(H_0)$ = Same "rate" of deaths $= \frac{1}{12}$

$P(H_1)$ = Different "rate" of Deaths $= \sum_i P(x_i) = 1$

$$\chi^2 = \frac{(1668-1410)^2 + \cdots + (1526-1410)^2}{1410}$$

$\boxed{= 79}$

$\chi^2_{11,\,0.995} = 26.76$; $\chi^2 > \chi^2_{11,\,0.995}$ which is a rejection of null hypothesis at p-value ≈ 0.005.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Male	3755	3251	377									
Female												

38.

Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec
Male	3755	3251	3777	3706	3717	3660	3669	3626	3481	3590	3605	3392
Female	1362	1244	1496	1452	1448	1376	1370	1301	1337	1351	1416	1226

$n = 43229$; $E[x] = \bar{x} = 3602$; $\chi^2 = 74.6$; $\chi^2_{11,\,0.995} = 26.76$; Reject H_0

$n = 16379$; $E[x] = \bar{x} = 1364.9$; $\chi^2 = 20.4$; $\chi^2_{11,\,0.975} = 21.92$: Accept H_0 at p-value of 0.025.

$P(H_0)$ = Same "rate" of sickness $= 1/12$

$P(H_1)$ = Different "rate" of sickness $= \sum_i P(x_i) = 1$

39.

Lunar Day	16,17,18	19,20,21	22,23,24	25,26,27	28,29,1	2,3,4	5,6,7	8,9,10	11,12,13	14,15
# of Bites	137	150	163	201	269	155	142	146	148	110

$E[x] = \bar{x} = 162.1$; $\chi^2 = 107$; $\chi^2_{9,\,0.995} = 23.59$; Reject H_0

$P(H_0)$ = Same "rate" of bites $= 1/10$

No, temporal trends

$P(H_1)$ = Different "rate" of bites $= \sum_i P(x_i) = 1$

40. Multinomial Distribution:

$$f(\theta) = \frac{n!}{x_1! \cdots x_n!} p_1(\theta)^{x_1} \cdots p_n(\theta)^{x_n}$$

Observation
X_1
X_2

Pearson Chi-Squared:

$$\sum_{i=1}^{2} \frac{(X_i - np_i)^2}{np_i} = \frac{(X_1-np_1)^2}{np_1} + \frac{(X_2-np_2)^2}{np_2}$$

$$= \frac{(X_1-np_1)^2}{np_1} + \frac{(X_2-n(1-p_1))^2}{n(1-p_1)}$$

$$= \frac{(X_1-np_1)^2 (1-p_1)}{np_1(1-p_1)} + \frac{(X_2-n(1-p_1))^2 p_1}{np_1(1-p_1)}$$

$$= \frac{X_1^2 - 2np_1 + n^2p_1^2 - X_1^2 + 2np_1^2 - np_1^3 + X_2^2 p_1 - 2n(1-p_1)p_1 + n^2(1-p_1)^2 p_1}{np_1(1-p_1)}$$

$$\boxed{= \frac{(X_1^2 - np_1)^2}{np_1(1-p_1)}}$$

Relationships: $p_1 X_T = X_1$; $np = \frac{X_1+X_2}{X_1+X_2} = 1$

$p_2 X_T = X_2$

42. $n = 230$

a) mle of p: $f(\theta) = \frac{n!}{\prod^2 x!} \prod^2 p_i^{x}$; $\ln f(\theta,\lambda) = \ln n! - \sum_i \ln x! + \sum_i x_i \log p_i + \lambda(\sum_i p_i - 1)$

$\frac{d \ln f(\theta,\lambda)}{dp} = \sum_i \frac{x_i}{p_i} + \lambda$; $\boxed{\hat{p}_i = \frac{-x_i}{\lambda}}$; $1 = \frac{-n}{\lambda}$; $\lambda = n$

$\boxed{p_i = \frac{x_i}{n}}$

b)

Breaks	Frequency
0	157
1	69
2	35
3,4,5	19

$\boxed{= 0.71}$

$$\chi^2 = \frac{(0-0.71)^2 + (1-0.71)^2 + (2-0.71)^2 + (3-0.71)^2}{0.71}$$

$= 1.06$; $\chi^2_{3,\,0.85} = 1.250$; Reject above p-value 0.15.

c) $\Lambda = \frac{\hat{p}^{\sum_i n_i p_i}(1-\hat{p})^{\sum_i n_i(1-\hat{p}_i)}}{\prod \hat{p}_i^{n_i p_i}(1-\hat{p}_i)^{n_i(1-\hat{p}_i)}}$; $-2\log\Lambda = \sum_i \frac{(X_i - n_i\hat{p})^2}{n_i\hat{p}(1-\hat{p})} = 2701$

$\boxed{\Lambda \cong \emptyset}$

41. $X_i = bin(n_i, p_i)$; $i = 1 \ldots m$

$H_0: p_1 = p_2 = \cdots = p_m$

$H_1: p_1 \neq p_2 \neq \cdots \neq p_m$: $\sum_i p_i = 1$

$$\Lambda = \frac{P(x|H_0)}{P(x|H_1)} = \frac{P(x|H_0)}{(1-P(x|H_0))} = \boxed{\frac{\sum_i \binom{n}{m} p^m (1-p)^{n-m}}{1 - \sum_i \binom{n}{m} p^m (1-p)^{n-m}}}$$

43. a) 9207 heads ; 8743 tails in 17,950 coin tosses. $\hat{p} = \frac{x}{n} = 8,975$; $\chi^2 = \frac{(9207-8975)^2 + (8743-8975)^2}{8975} = 11.99$

$\chi^2_{1,0.995} = 7.88$; $\boxed{\text{Reject } H_0 \text{ above } p = 0.005}$

b) $n = 5$

$T_4 = 3590$

# Heads	Freq.
0	100
1	524
2	1080
3	1126
4	655
5	105

$P(x) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{5}{x}(0.5)^x(0.5)^{5-x}$; $x = 0,1,2,3,4,5$

$X^2 = 21.6$

$\chi^2_{5,0.999} = 5.893$; $\boxed{\text{Reject } H_0 \text{ above } p\text{-value} = 0.001}$

c) $P(x|H_0) = P(\text{coin \#1}|H_0) = p(\text{coin \#2}|H_0) = \cdots = p(\text{coin \#5}|H_0) = \frac{1}{2}$

$X^2 = \sum \frac{[x_i - np(\theta)]^2}{np_i(\theta)} = 8.75$; $\chi^2_{5,0.9995} = 6.37$ $\boxed{\text{Reject } P(x|H_0) \text{ above } p\text{-value} = 0.0005}$

44.

	Haptoglobin Type		
	Hp1-1	Hp1-2	Hp2-2
O_i	10	68	112
E_i	12	72	106
$-2\ln\Lambda$	0.91		

$\hat{\theta} = 0.748$

Multinomial Distribution:

$f(\theta) = \frac{n!}{x_1! \cdots x_n!} P_1(\theta)^{x_1} \cdots P_n(\theta)^{x_n}$

Likelihood Ratio: $\Lambda = \prod_{i=1}^{m} \left(\frac{P_i(\theta)}{\hat{P}_i}\right)^{x_i}$; $-2\log\Lambda = 2\prod O_i \log\left(\frac{O_i}{E_i}\right)$

$P(\theta = 1/2, H_0)$; $P(\theta \neq 1/2, H_1)$; $\chi^2_{2,0.95} = 2.92$; $\boxed{\text{Accept } P(\theta = 1/2, H_0) \text{ at } p\text{-value of } 0.05}$

45. $n = 6115$ families

#	Frequency	Expected Freq.
0	7	4
1	45	38
2	181	180
3	478	511
4	829	980
5	1112	1336
6	1343	1327
7	1033	969
8	670	516
9	286	195
10	104	50
11	24	8
12	3	1

$\bar{X} = 5.5$ $P(X|H_0) = \text{Binomial}$

$P(\bar{x}) = 0.46$ $P(x|H_1) \neq \text{Binomial}$

$X^2 = 551$

$\chi^2_{12,0.9995} = 4.43$

$\boxed{\text{Reject } P(x|H_0) \text{ at } p\text{-value} = 0.0005}$

Binomial Random Variable

$P(1) = P$

$P(0) = 1-p$

$P(x) = 0$

$P(x) = \binom{n}{k} p^k (1-p)^{n-k}$

46. $Y = \sin^{-1}\sqrt{\hat{p}}$; $\hat{p} = \frac{x}{n}$; $X \sim \text{bin}(n,p)$

$\frac{dY}{d\hat{p}} = \frac{1}{\sqrt{1-\hat{p}^2}} = \frac{1}{\sqrt{(1-\hat{p})\hat{p}}}$

$Var(\sin^{-1}\sqrt{\hat{p}}) \approx n\sqrt{\hat{p}(1-\hat{p})} = \boxed{\sqrt{n}\, x(1-x)}$

47. $\lambda = \text{mean}$; $Y = \sqrt{X}$; $Y' = \frac{1}{2\sqrt{X}}$

Poisson Distribution

$P(x) = \frac{\lambda^x}{x!} e^{-\lambda x}$;

$\lim_{n \to \infty} \sqrt{n}(P(x) - \theta) \to N(\theta, \theta)$

$\left(2\sqrt{P(x)} - \frac{5}{\sqrt{n}}, 2\sqrt{P(x)} + \frac{5}{\sqrt{n}}\right)$

$\boxed{\left(\left(\sqrt{P(x)} - \frac{5}{2\sqrt{n}}\right)^2, \left(\sqrt{P(x)} + \frac{5}{2\sqrt{n}}\right)^2\right)}$

48. $E(X) = \mu$; $Var(X) = c\mu^2$

$Y = \int_0^x \frac{d\mu}{Var(x)} = \int_0^x \frac{d\mu}{c\mu^2} = \frac{1}{c}\ln(x)$; $\boxed{c = \frac{\ln(x)}{Y}}$

49.

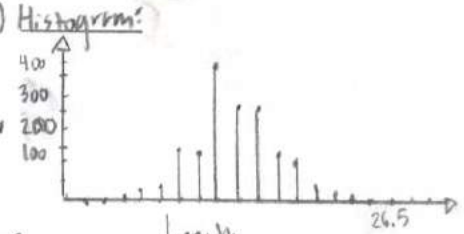
Len	Freq.
18.5	0
19.0	1
19.5	3
20.0	33
20.5	39
21.0	156
21.5	152
22.0	392
22.5	238
23.0	286
23.5	100
24.0	86
24.5	21
25.0	12
25.5	2
26.0	0
26.5	1

$n = 1572$

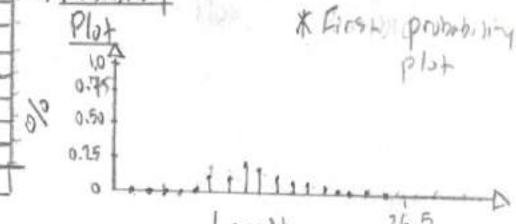
$E[x] = 22.3$

$\hat{n} = 87$

a) Histogram:

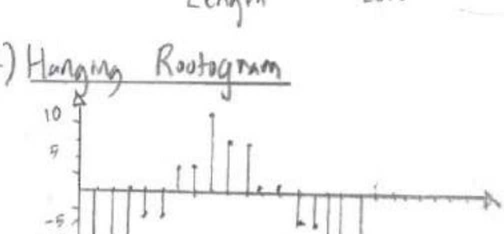


b) Probability Plot



* First probability plot

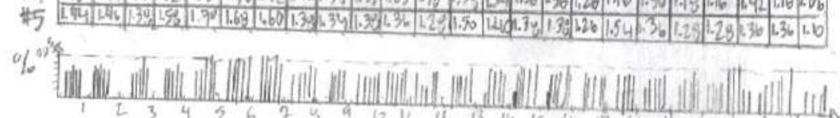
c) Hanging Rootogram

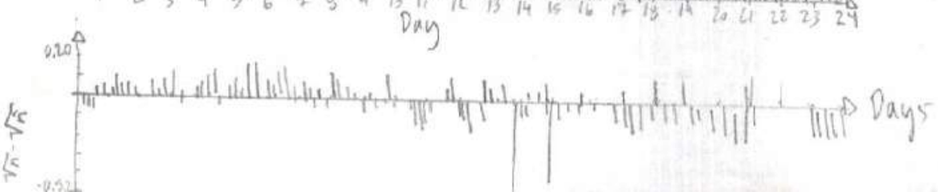


Note: Probability plots were correctly modelled by Problem #64

50.

	Days																							
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
#1	1.40	1.40	1.80	1.54	1.52	1.62	1.58	1.62	1.60	1.38	1.34	.50	1.26	1.52	1.50	1.40	1.32	1.16	1.24	1.30	1.30	1.45	1.32	1.44
#2	1.28	1.34	1.44	1.50	1.46	1.58	1.64	1.46	1.44	1.34	1.28	1.46	1.50	1.58	1.42	1.32	1.40	1.54	1.22	1.49	1.52	1.46	1.22	1.20
#3	1.36	1.54	1.46	1.48	1.42	1.62	1.62	1.38	1.41	1.36	1.08	1.28	1.52	1.48	1.38	1.48	1.40	1.40	1.20	1.28	1.76	1.48	1.72	1.10
#4	1.38	1.44	1.50	1.52	1.58	1.76	1.72	1.62	1.38	1.58	1.02	1.18	1.38	1.34	1.36	1.36	1.26	1.16	1.30	1.18	1.16	1.42	1.18	1.06
#5	1.44	1.46	1.38	1.58	1.70	1.68	1.60	1.38	1.34	1.38	1.36	1.28	1.50	1.40	1.34	1.38	1.26	1.54	1.36	1.25	1.28	1.30	1.36	1.10

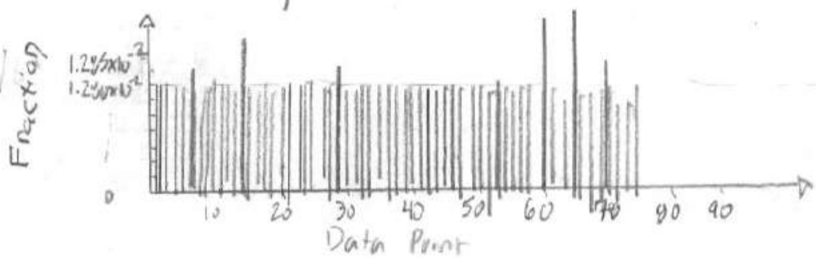




51. The horizontal bands of Figure 9.6 represent groupings of data with similar observations.

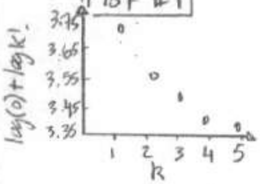
52. See chapter 9: Problem 52.

Probability Plot

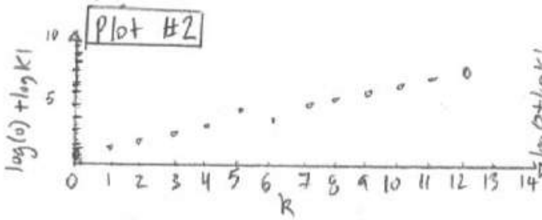


53. $E_K = nP(X=K) = ne^{-\lambda}\frac{\lambda^K}{K!}$; $\log E_K = \log n - \lambda + K\log\lambda - \log K!$

Plot #1



Plot #2



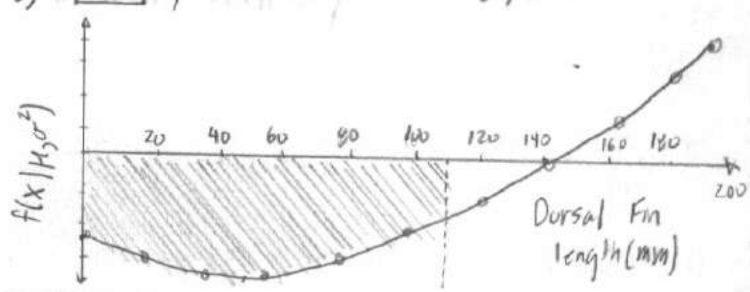
Plot #3

The plots contain regions of linearity.

54. $Y=\log(X)$ a) Log Normal Distribution

$$f(x|\mu,\sigma^2) = \ln p(x|\mu,\sigma^2) = \boxed{-\frac{(x-\mu)^2}{2\sigma^2} - \ln\sqrt{2\pi\sigma^2}}$$

b) Plot $\mu = 44.67\text{mm}$; $\sigma = 35.87\text{mm}$

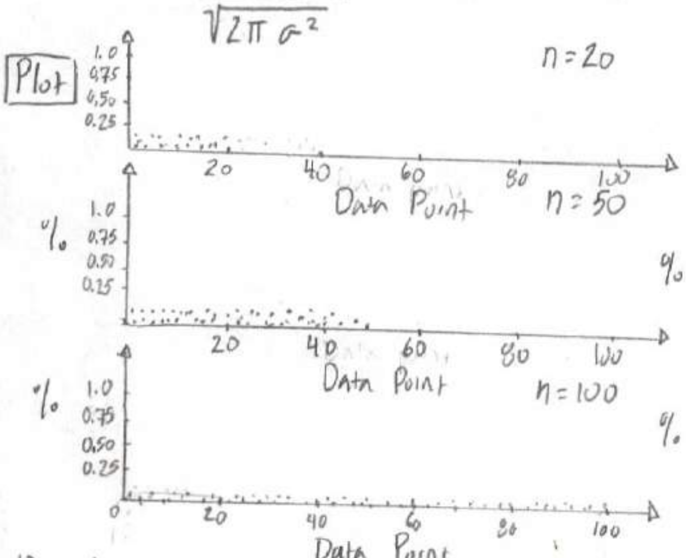


The data fits lognormal until length 115mm which is above a p-value of 0.05.

55. a) Normal Distribution:

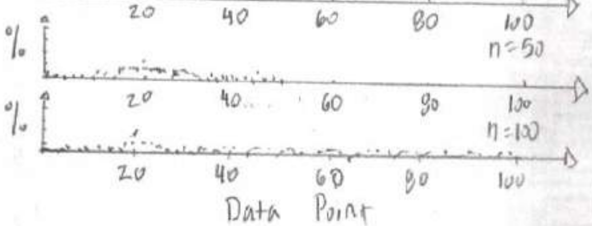
$$f(x|\mu,\sigma^2) = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}$$

Plot



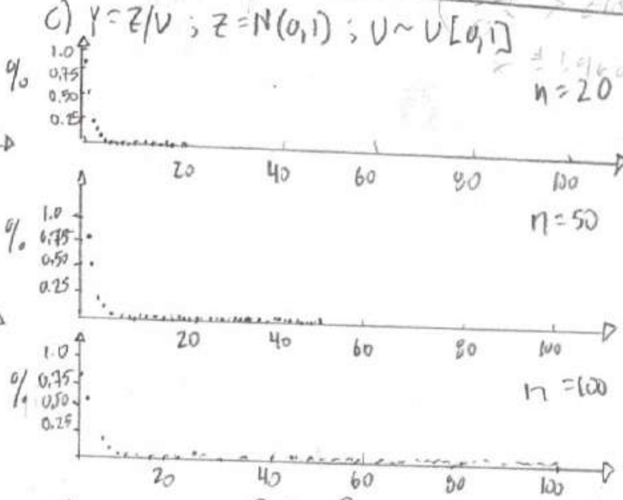
b) Chi-square Distribution

$$P(X|k) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} \cdot e^{-x/2}$$



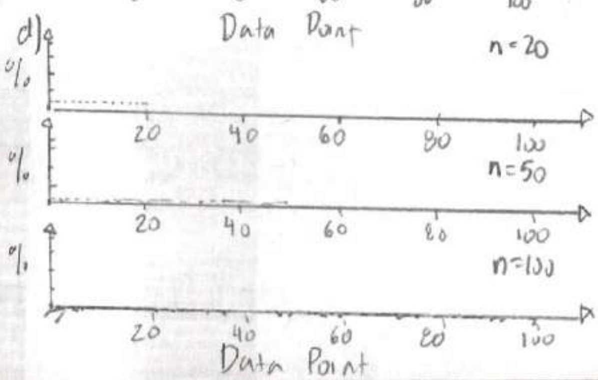
Degrees of Freedom = 10.

c) $Y = Z/U$; $Z = N(0,1)$; $U \sim U[0,1]$



Note: Probability plots were correctly modelled by problem #64

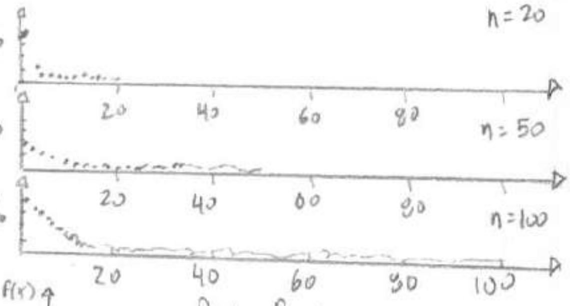
d)



Uniform Distribution

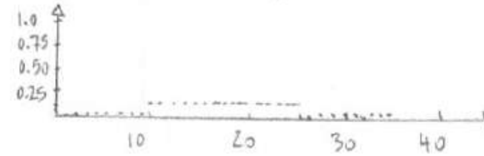
$$f(x) = \frac{1}{b-a}$$

e.

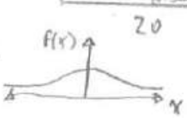


F. The distributions plotted are seperable from each other, and a normal distribution

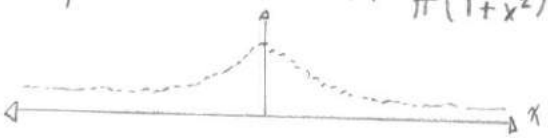
58. $F(\bar{x}) = 1 - e^{-\lambda x}$; $\lambda = \frac{1}{\bar{x}}$



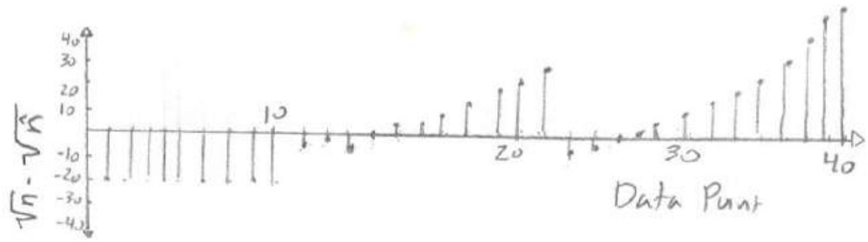
56.



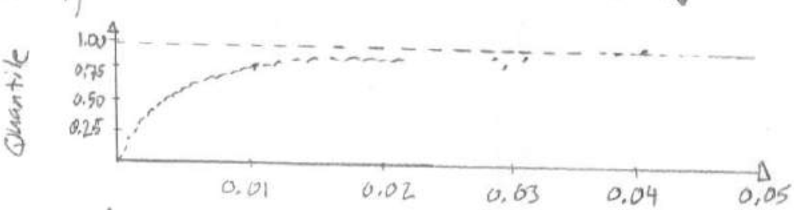
57. Cauchy Distribution $f(x) = \frac{1}{\pi}\left(\frac{1}{1+x^2}\right)$



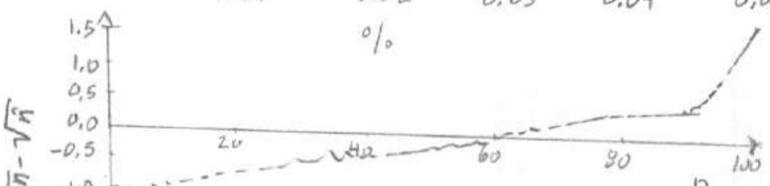
59.



60, $n = 76$ a)

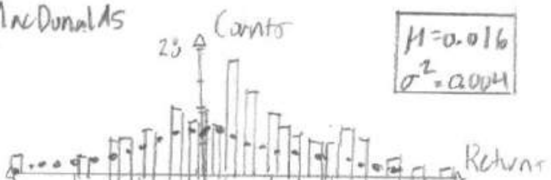


b)



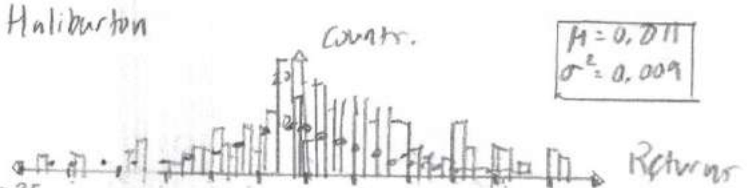
The appearence of the plot represents a convergence of % Stress failure for Kevlar 49/epoxy against quantile grouping of an exponential distribution.

61. a) MacDonalds



$\boxed{\mu = 0.016 \quad \sigma^2 = 0.004}$

Haliburton



$\boxed{\mu = 0.011 \quad \sigma^2 = 0.009}$

b) The more volatile stock company is Haliburton

Note: Probability plots were correctly modelled by problem #64

62. Poisson Dispersion Test

$$\Lambda = \prod \left(\frac{\bar{x}}{x_i}\right)^{x_i} e^{x_i - \bar{x}}$$

$$-2\log\Lambda = 2\sum x_i \log\left(\frac{x_i}{\bar{x}}\right)$$

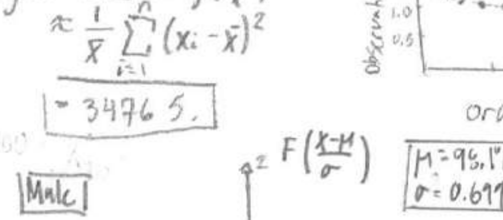
$$\approx \frac{1}{\bar{x}}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$\boxed{= 3476.5.}$

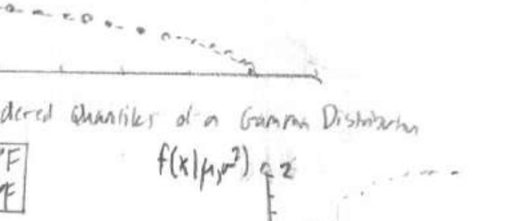
63.

$\boxed{a = 0.7997, b = 1.3188}$

64, a) Male

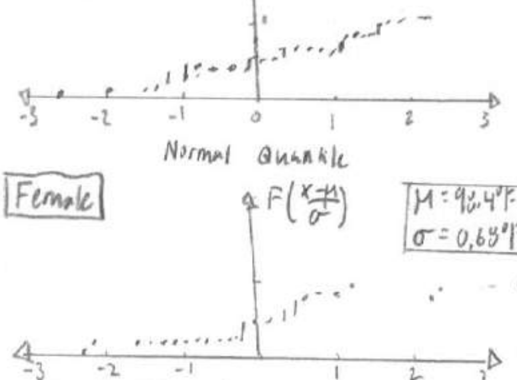


$\boxed{\mu = 98.1°F \quad \sigma = 0.69°F}$

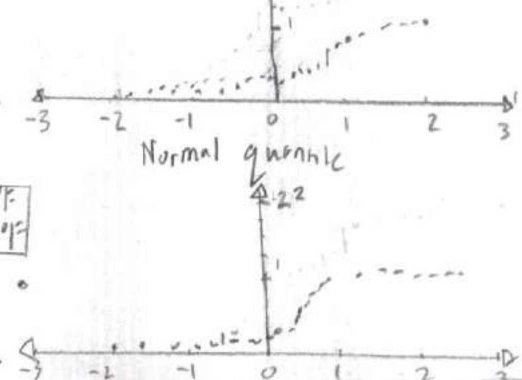


The assessment of body temperatures for both male and female demonstrate a higher proportion near the mean than normally distributed.

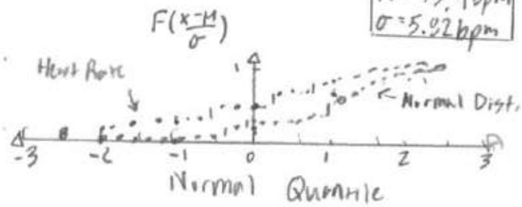
Female



$\boxed{\mu = 98.4°F \quad \sigma = 0.68°F}$

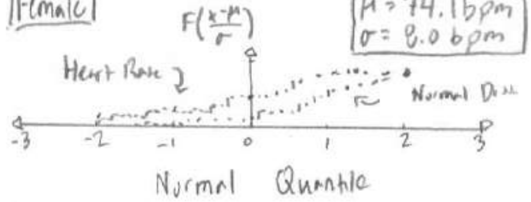


b $\boxed{\text{Male}}$



$\boxed{\mu = 73.4 bpm \\ \sigma = 5.92 bpm}$

$\boxed{\text{Female}}$



$\boxed{\mu = 74.1 bpm \\ \sigma = 8.0 bpm}$

Male and Female heart rate both contain a larger proportion of readining near the mean then a normal distribution.

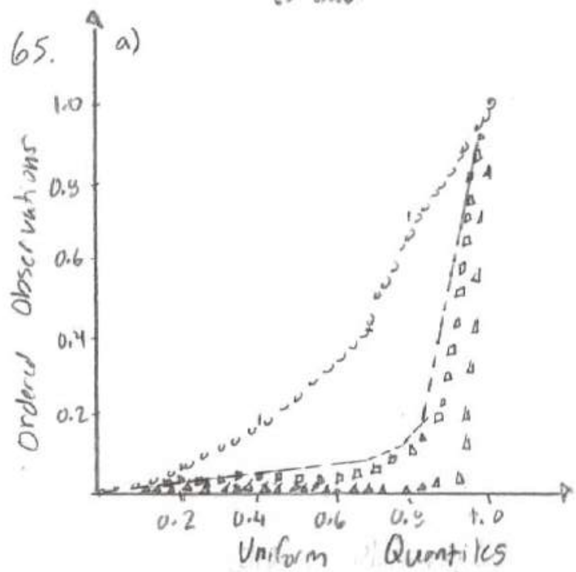
c. $P(x = 98.6°F | H_0)$ vs $P(x \neq 98.6°F | H_1)$ $\boxed{\text{Male}}$

$\bar{X}^2 = 0.3185$; $\chi^2_{65\ 0.005} = 39.38$; $\boxed{\text{Accept } P(x|H_0) \text{ at p-values} > 0.995}$

$P(x = 98.6°F | H_0)$ vs $P(x \neq 98.6°F | H_1)$

$\bar{X}^2 = 0.3956$; $\chi^2_{65\ 0.005} = 39.38$; $\boxed{\text{Accept } P(x|H_0) \text{ at p-values} > 0.995}$

65. a)



□ = Small Chromatin
$\mu =$ $\sigma =$
o = Medium Chromatin
$\mu = 426$ $\sigma = 1.94$
△ = Long Chromatin
$\mu = 260$ $\sigma = 1.38$

---- = Rayleigh Dist ($\sigma^2 = 1$)

b) $\bar{X}^2_{Short} = 24.43$ $\quad$ $X^2_{Med.} = 117.4$ $\quad$ $X^2_{long} = 182.0$

df = 95 $\quad$ df = 131 $\quad$ df = 248

$\chi^2_{95\ 0.995} = 63.25$ $\quad$ $\chi^2_{131\ 0.75} = 119.750$ $\quad$ $\chi^2_{248\ 0.995} = 194.391$

$\boxed{\text{Accept at p-value} > 0.005}$ $\quad$ $\boxed{\text{Reject at p-value} > 0.25}$ $\quad$ $\boxed{\text{Accept at p-value} > 0.005}$

28. $n=3$; $P(X_{(1)} < \eta < X_{(2)}) = 1 - P(\eta < X_{(1)} \text{ or } \eta > X_{(2)}) = 1 - P(\eta < X_{(1)}) - P(\eta > X_{(2)})$

$$= 1 - \frac{1}{2^3}\sum_{j=0}^{1}\binom{4}{j} = 1 - \frac{1}{8}(1+4) \boxed{= 37.50\%}$$

$P(X_{(1)} < \eta < X_{(3)}) = 1 - P(\eta < X_{(1)} \text{ or } \eta > X_{(3)}) = 1 - P(\eta < X_{(1)}) - P(\eta > X_{(3)}) = 1 - \frac{1}{2^{n-1}}\sum_{j=0}^{1}\binom{5}{0} = \boxed{62.50\%}$

29. a) The distribution is binomial because probability for success and failure exist when considering a outlier distribution.

b) $P(N \geq 10) = Bi(n \geq 10, p = 5/26) = \boxed{0.018}$

c) The probability of 1000 bootstrap samples would be $1000 \cdot Bi(n \geq 10, p\ 5/26)$ $\boxed{= 1.80}$

d) The probability that every sample is an outlier would be $(5/26)^{26}$ $\boxed{= 2.42 \times 10^{-19}}$

30. By sampling 1000 times without replacement the bootstrap standard deviation was 0.64 vs. the actual standard deviation of 0.87.

31. a) n = number of samples ; n = probability ; $\boxed{n^n}$

b) $n=3$, $X_1 = 1$, $X_2 = 3$, $X_3 = 4$; $\boxed{X \in \{(1,1,1), (1,1,3), (1,1,4), (1,3,1), (1,4,1), (3,1,1), (4,1,1), (1,3,4), (1,4,3), (3,3,3), (3,1,3), (3,4,3), (1,3,3), (4,3,3), (3,3,1), (3,3,4), (3,1,4), (3,4,1), (4,4,4), (4,4,1), (4,4,3), (4,1,4), (4,3,4), (1,4,4), (3,4,4), (4,1,3), (4,3,1)\}}$

c)

Sample	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
Mean	1.00	1.67	2.00	1.67	2.00	1.67	2.00	2.67	2.67	3	2.33	2.00	2.33	2.00	2.33	2.33	2.67	2.67	4.00	3.00	3.67	3.00	3.67	3.00	3.67	2.67	2.67

$\bar{x}$	1.00	1.67	2.00	2.33	2.67	3.00	3.33	3.67	4.00
P(x̄)	1/27	3/67	3/67	3/64	6/27	3/27	3/27	3/27	1/27

d) $S_{\hat{\theta}} = \sqrt{\frac{1}{n-1}(\bar{x}_i - \bar{\bar{x}})^2} = \boxed{0.5366 \text{ vs } 0.5469}$

32. The Median Absolute Deviation from the median (MAD) as defined by the median of $|X_i - \bar{x}|$ is approximated by a bootstrap through sampling the set produced by the definition.

33. The mean and standard deviation.

34. $\psi(x) = |x|$; $\psi'(x) = 0$ @ $x=0$; Median of $f(x) = |x|$ is 0.

35. The proportion of points marked by an asterisk would be <5% of sample because of the inner quartile range (IQR).

36. The IQR is divided by 1.35 because for a double sided distribution each tail is subdivided by 0.675σ, which each represent Q_1 and Q_3 of the sample.

Median Absolute Deviation from the median (MAD) contains 0.675 because of the quartile range being 0.675σ.

37. a) $\boxed{\text{Mean} = 14.58, \text{ Median} = 14.57, \text{ Mean}(10\%) = 14.59, \text{ Mean}(20\%) = 14.59}$

b) $\boxed{14.58 \pm 1.27\% \text{ wax}}$

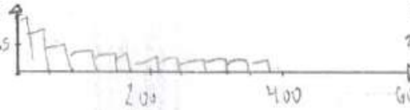
c) $\boxed{CI = (13.31, 15.85)}$

d) $SE_{10\%} = \frac{\sigma}{\sqrt{n}}\sqrt{1 - \left(\frac{n-1}{N-1}\right)} = \boxed{0.05\%}$

$SE_{20\%} = \boxed{0.03\%}$

22. Survival Function $S(t) = 1 - F(t)$; $S_n(t) = 1 - F(t)$

23. $X_{(k)}$ & $Y_{(k)} = \frac{k}{n+1}$;



Days Days

Linear Interpolation Function:

$$y = y_0\left(\frac{x_1 - x}{x_1 - x_0}\right) + y_1\left(\frac{x - x_0}{x_1 - x_0}\right) \; ; \; \frac{k}{(n+1)} \le p \le \frac{(k+1)}{(n+1)} \; ; \; \frac{-k}{(n+1)} \ge -p \ge -\frac{(k+1)}{(n+1)} \; ; \; 1 - \frac{k}{(n+1)} \ge 1 - p \ge 1 - \frac{(k+1)}{(n+1)}$$

$$\boxed{f(X) = X_{(k)}(n+1)\left(\frac{k+1}{n+1} - p\right) + X_{(k+1)}(n+1)\left(p - \frac{k}{n+1}\right)} \; ; 1, 2, 3, 4, 5, 6. ;$$

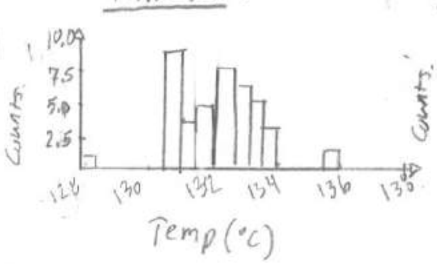
24. $\boxed{\text{Empirical Distribution } F_n = y\text{-axis}}$ $\boxed{\text{Theoretical Distribution } F = X\text{-axis}}$

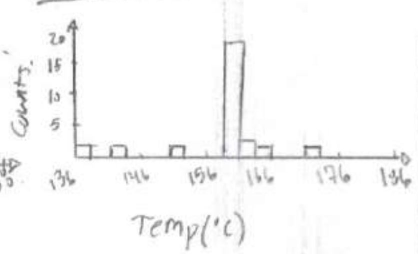
$\boxed{L.Q. = \frac{(n+1)}{4} = 2}$ $\boxed{\text{Median} = 3.5}$ $\boxed{U.Q. = \frac{3(n+1)}{4} = 5}$

25. $y_p = cx_p$; $F(x) = p$; $x = F^{-1}(p) = \frac{y_p}{c}$; $\boxed{F\left(\frac{y_p}{c}\right) = p = G(y)}$

26. Rhodium: Irridium:

a)



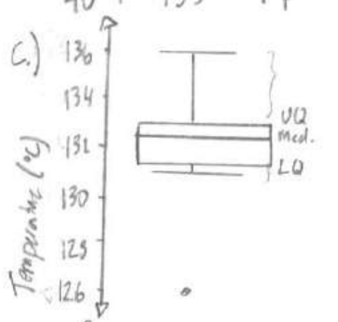


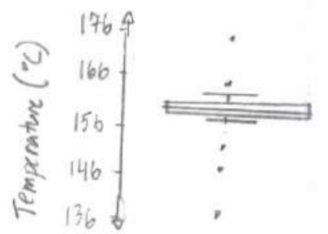
b.)

		Stem	Leaf
1	1	126	:4
13	12	131	:1234569
25	12	132	:123456789
37	12	133	:03458
39	2	134	:12
40	1	135	:7

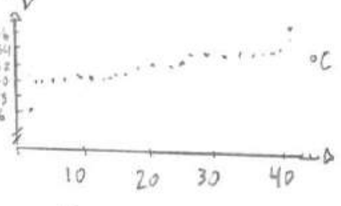
		Stem	Leaf
1	1	136	:6
2	1	145	:2
3	1	151	:5
14	11	159	:1235678
24	10	160	:123468
25	1	161	:7
26.	1	162	:9

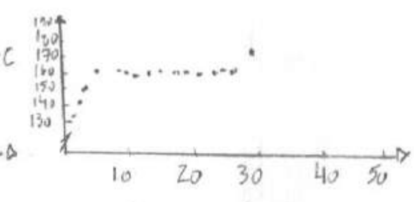
c.)





d)





e) $\boxed{\text{Yes, independendence and identically distributed fits the experimental measurement, in addition to plots of data.}}$

f. Rhodium Irridium:

Rhodium	Irridium
$\mu = 132.42\,°C$	$\mu = 158.81\,°C$
$\mu_{90\%} = 132.46\,°C$	$\mu_{90\%} = 159.86\,°C$
$\mu_{80\%} = 132.47\,°C$	$\mu_{80\%} = 159.84\,°C$

g. $\boxed{SE = 0.24\,°C}$ $\boxed{SE = 1.17\,°C}$

$\boxed{132.42 \pm 0.40\,°C}$ $\boxed{158.81 \pm 1.93\,°C}$

h. $\boxed{CI = (132.02\,°C, 132.82\,°C)}$ $\boxed{CI = (156.88\,°C, 160.74\,°C)}$

i. 10% trim $\boxed{SE = 0.25\,°C}$ 10% trim $\boxed{SE = 1.33\,°C}$

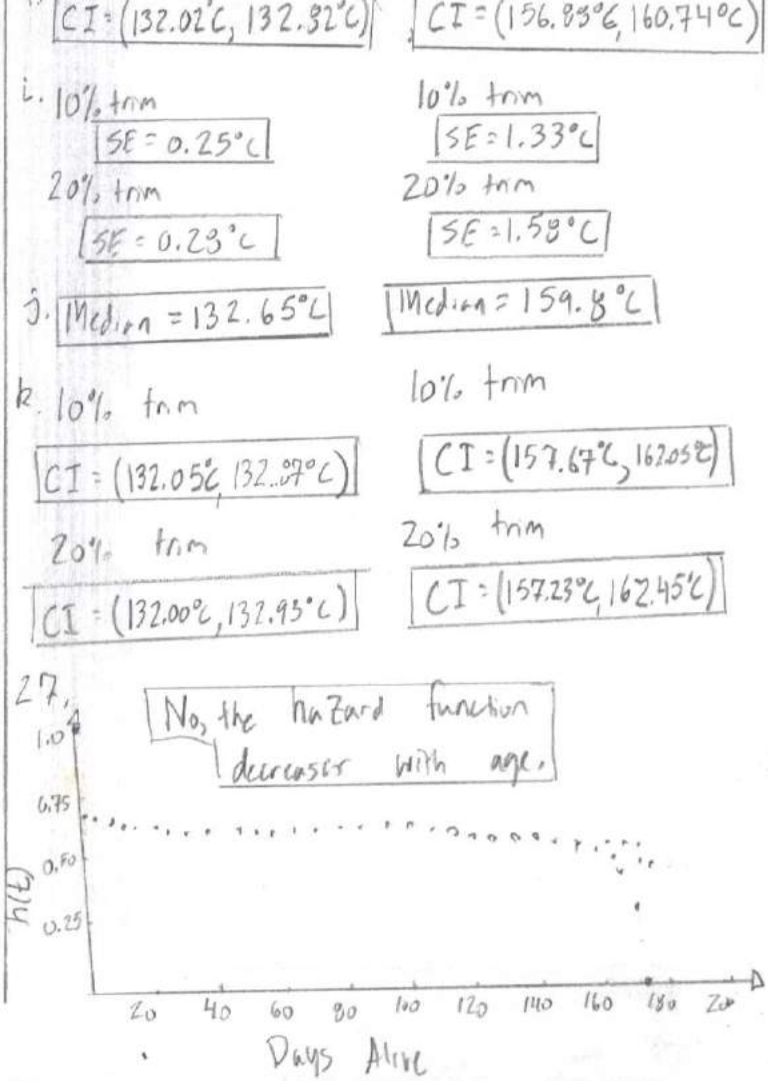
20% trim $\boxed{SE = 0.28\,°C}$ 20% trim $\boxed{SE = 1.58\,°C}$

j. $\boxed{Median = 132.65\,°C}$ $\boxed{Median = 159.8\,°C}$

k. 10% trim $\boxed{CI = (132.05\,°C, 132.87\,°C)}$ 10% trim $\boxed{CI = (157.67\,°C, 162.05\,°C)}$

20% trim $\boxed{CI = (132.00\,°C, 132.93\,°C)}$ 20% trim $\boxed{CI = (157.23\,°C, 162.45\,°C)}$

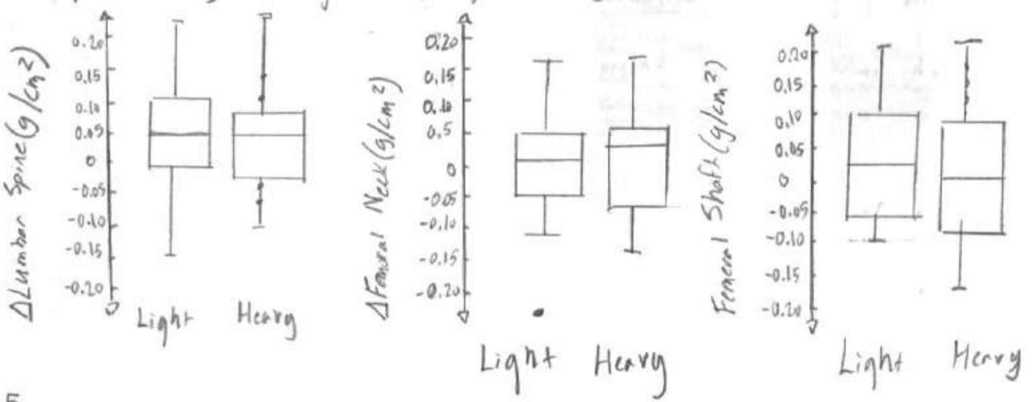
27.



$\boxed{\text{No, the hazard function decreases with age.}}$

43. When evaluating Kevlar [70%, 80%, 90%], the data's mean, median, and standard deviation demonstrate increasing skew for increasing stress levels.

44. Light Smoking (≤7 cigarettes/day) ; Heavy Smoking (>7 cigarettes/day)

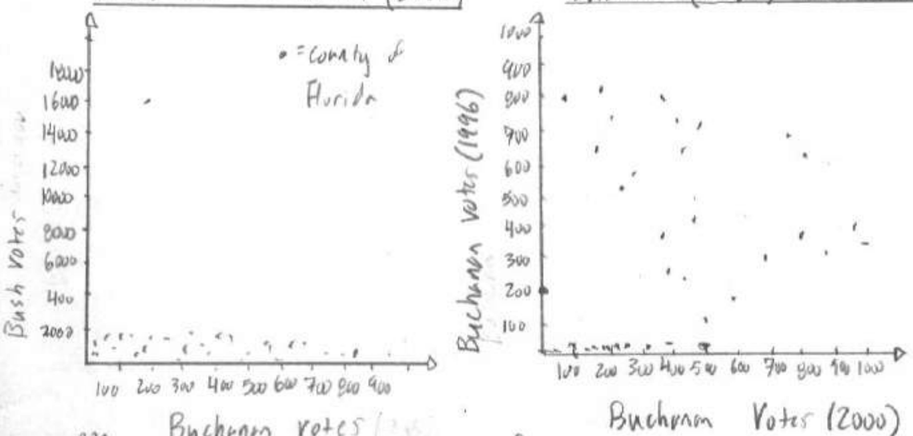


The bone density of light vs heavy smoking twins led to alternative conclusions. Although both light and heavy smoking twins had larger change to bone mass than the nonsmoking group. Data not shown.

45.

Bush vs. Buchanan (2000)

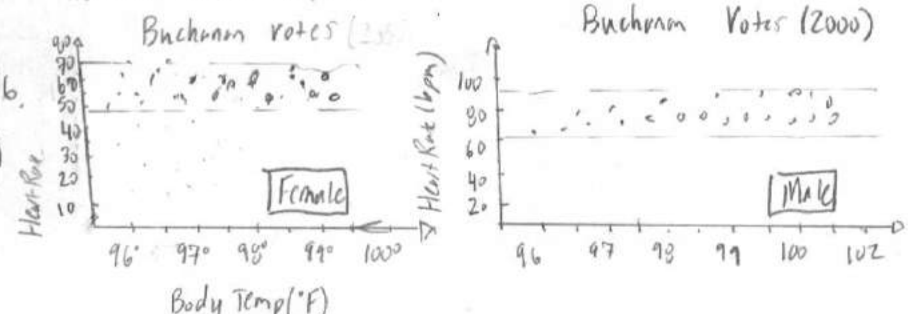
Buchanan (1996) vs Buchanan (2000)



The Buchanan (1996) vs. Buchanan (2000) shows similar amount of popular votes, while Bush's entry into presidency was led with 2x amount of votes, and in Monroe count 16,000+ votes

46.

a)



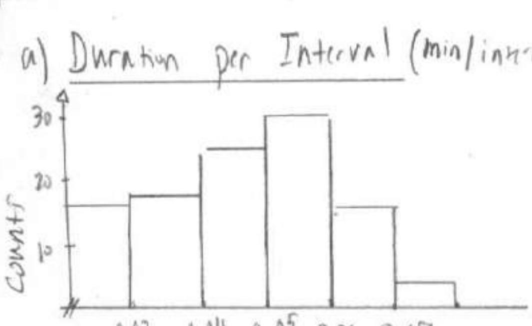
Heart rate for males and females remained within the same range, but male temperatures did contain maximum upwards of 100°F.

b) $r_{Male} = 1.72$; $r_{Female} = 2.61$; $\rho_{Male} = 0.034$ $\rho_{Female} = 0.077$

Heart rate and body temperature show positive correlation.

c) The female body temperature shows greater linear correlation than males.

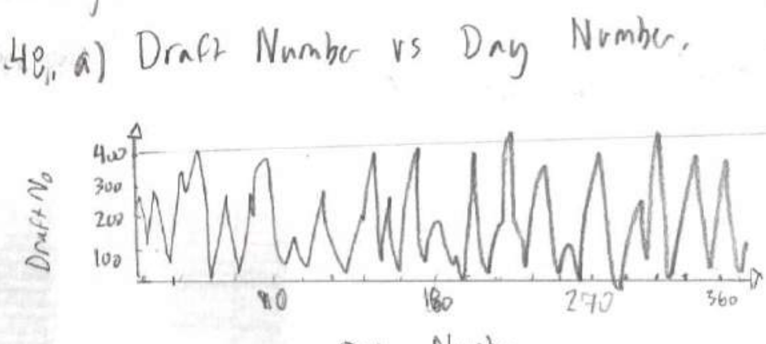
47. a) Duration per Interval (min/interval)



b)

Old Faithful when measured tends to erupt for 3.6 seconds per "gush".

48. a) Draft Number vs Day Number.



No trend

f. $\boxed{SD = 0.769\%\ \text{wax};\ IQR = 1.09\%\ \text{wax};\ MAD = 0.58\%\ \text{wax}}$

g. SE = 0.17% wax; Sampling Distribution

15.13, 15.15, 15.1?, 15.21, 15.2?, 15.31, 15.38, 15.4, 15.47, 15.47, 15.49, 15.56, 15.63, 15.91, 17.09.

38. Cauchy Distribution:

$$f(x) = \frac{1}{\pi}\left(\frac{1}{1+x^2}\right); -\infty < x < \infty$$

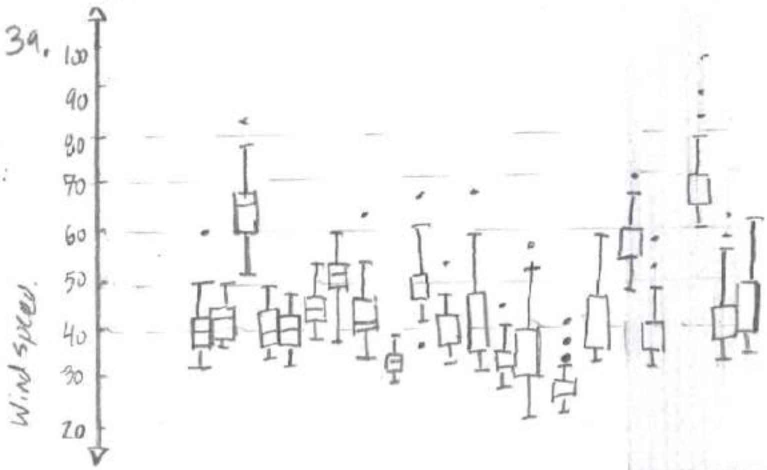
$\mu = 0.009813$; $SD = 0.039019$.
Median = 0

Distribution

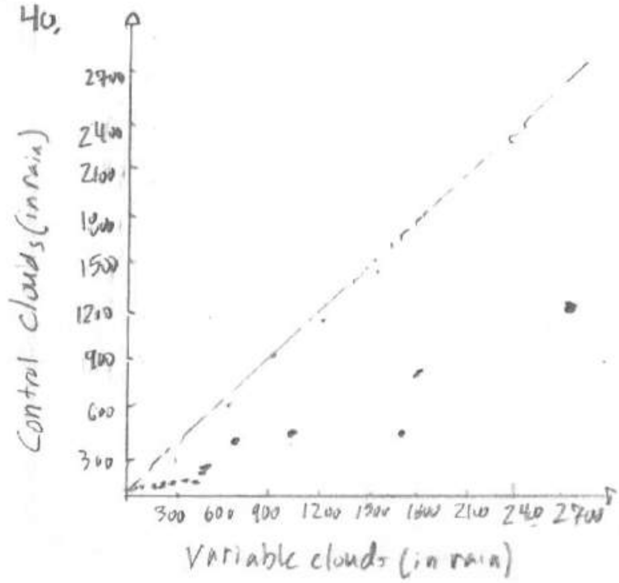
$\mu = 0.133712$; $SD = 16.52375$
Median = 0.000500

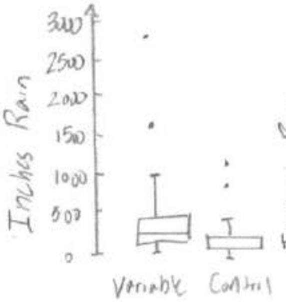
Sample.

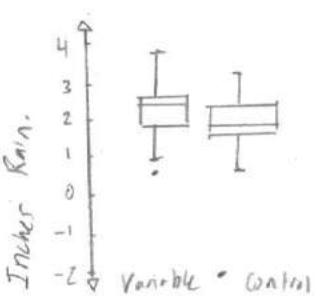
39.



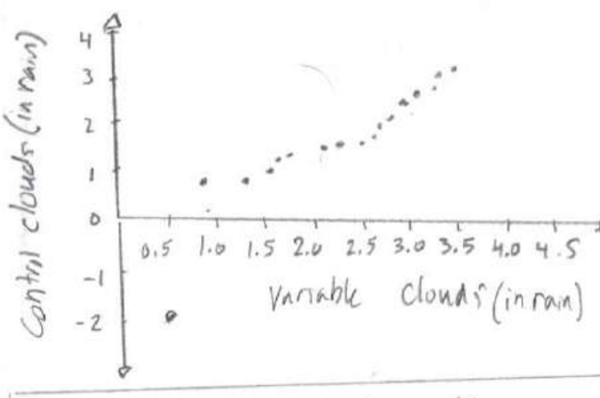
40.







The variable or "seeded" clouds produced more inches of rain than the control group of clouds



41. $P(X > x_p) = 1 - p$; $P(X < x_p) = p$;

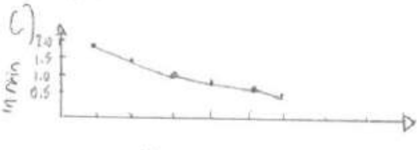
$$= 1 - P(X < p) = \boxed{1 - \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i}}$$

42. a) Skewed b)

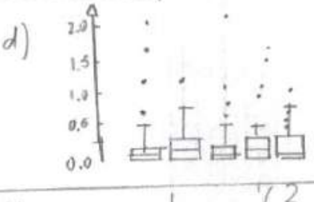
Year	'60	'61	'62	'63	'64
Average	0.22	0.27	0.18	0.26	0.19
Median	0.015	0.075	0.02	0.11	0.055

The median is different from average because of the large skew of the datasets.

c)



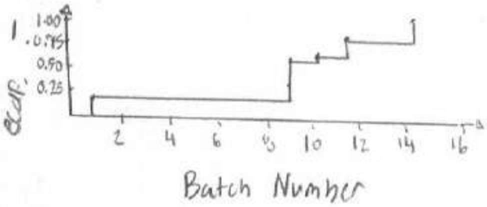
d)



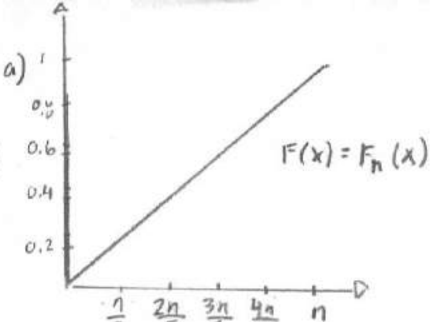
e) The wet years were '61 and '63, while dry years '62 and '64. The reason for wet years were because of storms not daily rainfall.

The above plots show the variable clouds produce more inches of rain than the control group of clouds. Seeded clouds produced more rain exponentially vs control set. The box plots of each graph would be much different because of mean and standard deviation differences.

Chapter 10:

1.


2. $X_1, X_2 \ldots X_n$ with Uniform Distribution. a)



b) $F_n(x)$ is a monotonically increasing function, while $F_n(x) - F(x)$ is a decreasing function

3. Lower quartile ($T < 63.4°F$)
Upper quartile ($T > 63.8°F$)

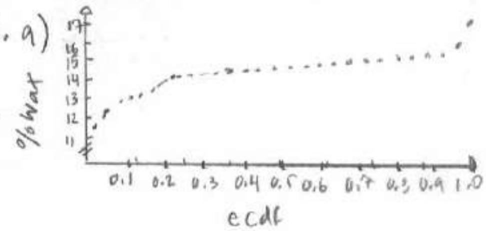
4. $I_{(-\infty, x]}(X_i)$ are independent random variables because the data are independent.

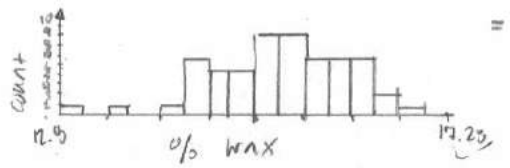
5. $Cov[F_n(u), F_n(v)] = \frac{1}{n}[F(m) - F(u)F(v)]$: $m = \min(u,v)$; $Cov(F_n(u), F_n(v)) = E(F_n(u)F_n(v)) - E(F_n(u))E(F_n(v))$

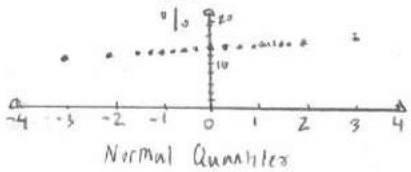
$$= \frac{1}{n}F(\min(u,v)) - F(u)F(v)$$

$$= \frac{1}{n}[F(m) - F(u)F(v)] = (+)$$

6. a)







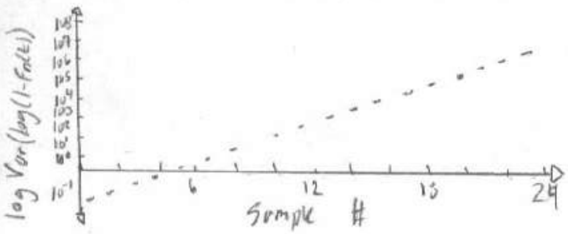
$q_{0.90} = 16.6\%$ $q_{0.50} = 14.7\%$ $q_{0.10} = 12.8\%$

$q_{0.75} = 15.9\%$ $q_{0.25} = 13.5\%$

b) Average (%) Wax = 85% ; A dilution of 1%, 3% and 5% microcrystalline wax are measurable quantities with a standardized average (%) wax.

7. The 10% weakest guinea pigs die within 200 days, and the 10% strongest survive until 400 days. While the median population live until 350 days.

8. $n = 100$; $\lambda = 1$ a) $S(t) = P(T > t) = 1 - F(t)$; $S_n(t) = 1 - F_n(t) \cdot 1 - e^{-t}$; $Var[\log(1 - F_n(t))] = \frac{1}{n}\left(\frac{F(t)}{1 - F(t)}\right)$



b) Several plots show exponentially growing standard deviation for the survival function.

9. Method of Propagation Error: $Y = g(X) \approx g(\mu_X) + (X - \mu_X)g'(\mu_X)$

$$E[Y] \approx g(\mu_X) + \frac{1}{2}\sigma_X^2 g''(\mu_X)$$

$$g(\mu_X) = \log S_n(\mu_X)$$

$$= \log(1 - F(\mu_X))$$

$$g'(\mu_X) = \frac{-F(\mu_X)}{1 - F(\mu_X)} = \frac{1}{n}\left(\frac{-F(X)}{1 - F(X)}\right)$$

$$E[Y] \approx \log(1 - F(\mu_X)) - \boxed{\frac{1}{2n}\frac{F(x)}{1 - F(x)}}$$

(-), Error

10. $X_1, \ldots, X_n = X_{(1)} \ldots X_{(n)}$ $\quad f_k(x) = n\binom{n-1}{k-1}[F(x)]^{k-1}[1 - F(x)]^{n-k} \cdot f(x)$

a) Uniform Distribution

$p(x) = \frac{1}{b-a}$, $[a = 0, b = 1]$

$$E(X_k) = \int_0^1 x f_k(x)\,dx = n\binom{n-1}{k-1}\int_0^1 x\left[\int_0^x f(x)\right]^{k-1}\left[1 - \int_0^x f(x)\right]^{n-k} \cdot 1\,dx$$

$$= n\binom{n-1}{k-1}\int_0^1 x^k(1-x)^{n-k}\,dx = n\binom{n-1}{k-1} \cdot Bet(k+1, n-k+1)$$

$$= n\frac{(n-1)!}{(k-1)!(n-k)!} \cdot \frac{\Gamma(k-1)\Gamma(n-k+1)}{\Gamma(n+2)} = \frac{n(n-1)!}{(k-1)!(n-k)!}$$

$$= \frac{(k!\,n-k)}{(n+1)!}\,\frac{1}{\,}\boxed{\frac{k}{n+1}}$$

$$\Gamma(p) = (p-1)\Gamma(p-1) = (p-1)!$$

$$E[X_k^2] = \int_0^1 x^2 f(x)\,dx = n\binom{n-1}{k-1}\int_0^1 x^{k+1}(1-x)^{n-k}\,dx = n\binom{n-1}{k-1} \times Bet(k+2, n-k+1)$$; $Var(X) = \frac{k(k+1)}{(n+1)(n+2)} - \frac{k^2}{(n+1)^2}$

$$= n\frac{(n-1)!}{(k-1)!(n-k)} \times \frac{\Gamma(k+2)\Gamma(n-k+1)}{\Gamma(n+3)} = n\frac{(n-1)!}{(k-1)!(n-k)!} \cdot \frac{(k+1)(n-k)!}{(n+2)!} = \frac{k(k+1)}{(n+1)(n+2)}$$

$$\boxed{= \frac{1}{(n+2)}\left(\frac{k}{n+1}\right)\left(1 - \frac{k}{n-1}\right)}$$

b. $X_i = F(Y_i)$; $Y_i = F^{-1}(X_i)$; $F(Y_i) = \int_0^y f(y)\,dy = y$; $\boxed{X_i = Y_i}$; $Y_{(k)} = F^{-1}(X_{(k)}) \approx F^{-1}\left(\frac{k}{k+1}\right) + \left(X_{(k)} - \frac{k}{n+1}\right)\frac{d}{dx}F^{-1}(x)\Big|_{k/n+1}$

$= E[Y_k] + \frac{1}{2}\sigma_{Y_k}^2 F''(Y_k)$

$\boxed{E[Y_k] = F^{-1}\left(\frac{k}{k+1}\right)}$

$\boxed{Var(Y_k) = \frac{k}{n+1}\left(1-\frac{k}{n+1}\right)\frac{1}{f(F^{-1}[k/(n+1)])^2}\left(\frac{1}{n+2}\right)}$

c. If $p = \frac{k}{n+1}$; $var(Y_k) = p(1-p)\cdot\frac{1}{f(x)^2}\cdot\frac{1}{n+2} \approx \boxed{\frac{p(1-p)}{n f(x)^2}}$

d. $N(\mu,\sigma^2)$; Median $= x = 1/2$

$Var(1/2) = \boxed{\frac{1}{n}\frac{p(1-p)}{f(1/2)^2}}$

$\frac{1}{n} \quad \frac{*}{n+2}$

11. $F(t) = 1 - e^{-\kappa t^\beta}$; Hazard Function: $P(t \le T \le t+\delta \mid T \ge t) = \frac{P(t \le T \le t+\delta)}{P(T \ge t)}$

$= \frac{F(t+\delta) - F(t)}{1 - F(t)} = \frac{-e^{-\kappa(t+\delta)^\beta} + e^{-\kappa(t)^\beta}}{e^{-\kappa(t)^\beta}}$

$\boxed{= 1 - e^{-\kappa(\delta)^\beta}}$ -or- $\frac{\frac{d}{dt}F(t)}{1-F(t)} = \frac{-\kappa\beta t^{\beta-1} e^{-\kappa t^\beta}}{-e^{-\kappa t^\beta}}$

12. $f(t) = h(t)e^{-\int_0^t h(s)ds}$; $h(t) = \frac{F(t)}{1-F(t)}$

$-\int_0^t \frac{F(s)}{1-F(s)}ds = -\int_0^t \frac{1-u}{u}du = -\int_0^t \frac{u+1}{u}du = -\int_0^t 1 + \frac{1}{u}du$

$= F(s)-1\Big|_0^t - \ln(|F(s)-1|)\Big|_0^t = F(t) - F(0) - \ln(|F(t)-1|) + \ln(|F(0)-1|)$

$= F(t) - \ln(F(t)-1)$; $f(t) = \frac{F(t)}{1-F(t)} e^{-F(t) - \ln(F(t)-1)} = \boxed{-F(t)e^{-F(t)}}$

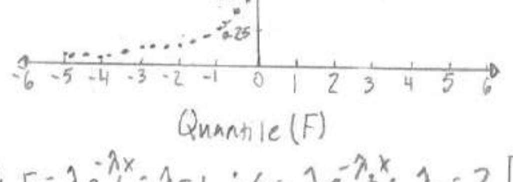
13. A probability distribution with increasing failure rate would be a Pareto distribution because $h(t)$ is positive. The uniform distribution has form e^{-1} too.

14. A probability distribution with decreasing failure rate would be the exponential distribution with the form $-F(t)e^{-F(t)}$.

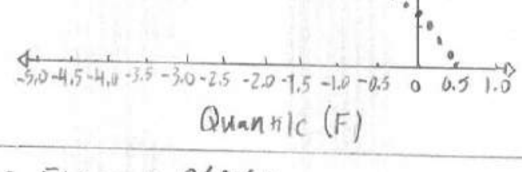
15. T = Time of release. $h(t) = \frac{f(t)}{1-F(t)} = \frac{\frac{1}{24}}{1-\frac{t}{24}} = \boxed{\frac{1}{24-t}}$; The smallest of t is 0 hours while largest 24 hours.

$h(5) = \frac{1}{19}$ vs $h(1) = \frac{1}{24}$: The prisoner has greater likelihoods.

16. $F = N(0,1)$; $G = N(1,1)$



$F: N(0,1)$; $G = N(1,4)$

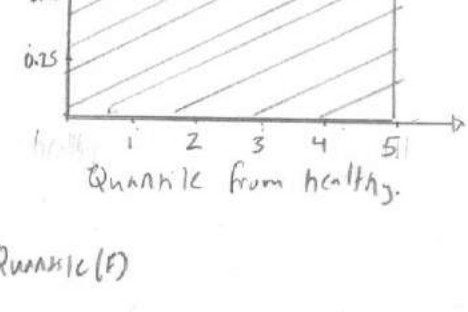


18. ill ∝ longer lifetime

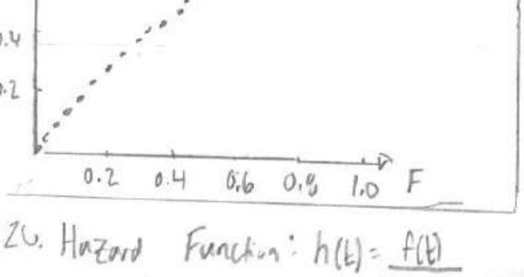
healthy ∝ shorter lifetime

$P(\text{healthy} \mid \text{Placebo}) = P(\text{healthy})$

$P(\text{ill} \mid \text{Placebo}) = P(\text{ill})$

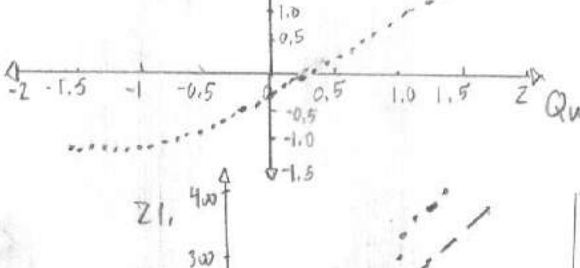


17. $F: \lambda e^{-\lambda x}$; $\lambda_1 = 1$; $G = \lambda_2 e^{-\lambda_2 x}$; $\lambda_2 = 2$.



19. $F(x) = x$; $0 \le x \le 1$

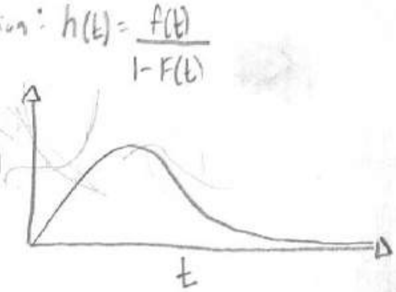
$G(x) = x^2$; $0 \le x \le 1$



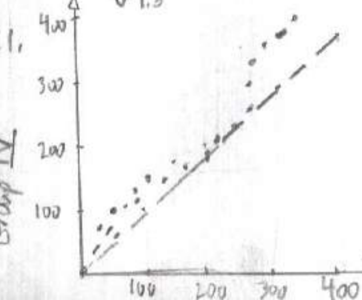
20. Hazard Function: $h(t) = \frac{f(t)}{1-F(t)}$

$f(t) = (-)$

$1 - F(t) = (-)$



21.

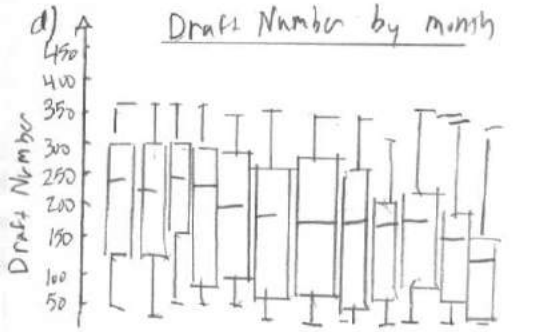


Group IV is living longer than Group V

b) Pearson Rank Coefficient (r) = -2.0×10^{-5} | No linear correlation or trend.

Spearman Rank Correlation Coefficient (ρ) = -0.23. | Little negative correlation to justify

c) Statistical Significance via method listed in the book justified no correlation

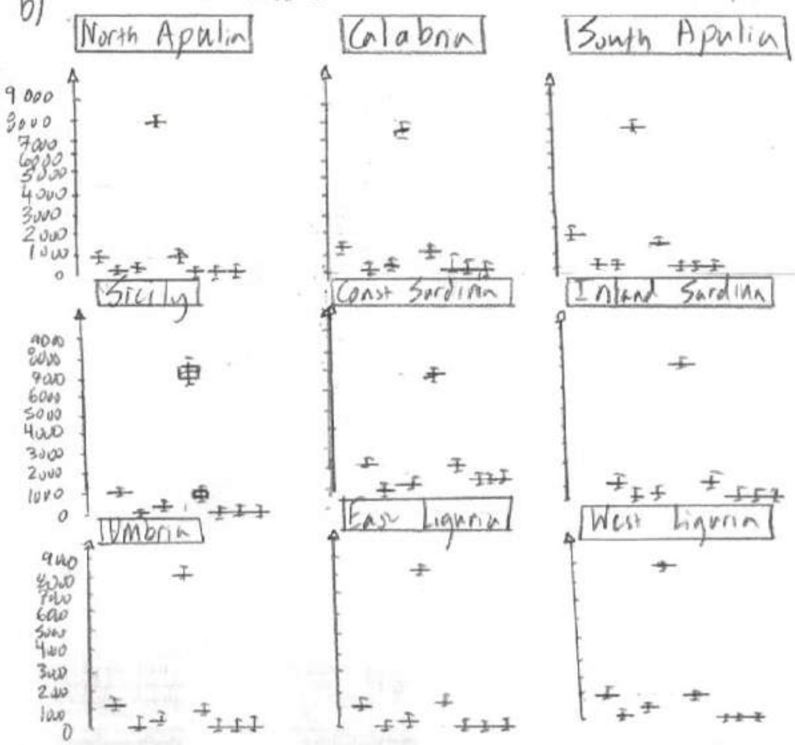
d) 

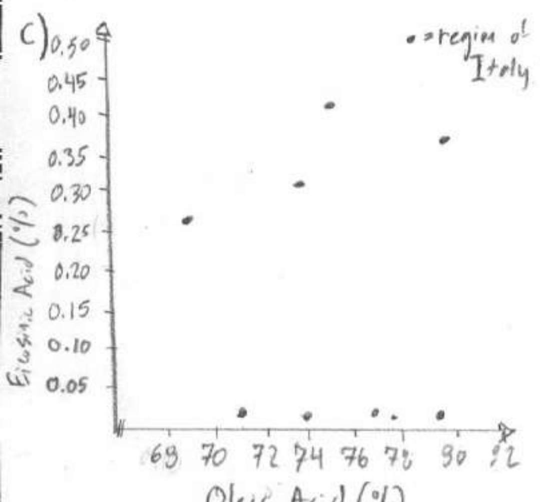
No pattern for 1970 lottery because max/min were equivalent and mean within 50 drafts.

49. A.

Oil Type		Palmitic	Palmitol	Stearic	Oleic	linoleic	linolenic	arachidic	eicosenoic
North Apulia	Mean	10.3%	0.6%	2.3%	78.2%	7.1%	0.4%	0.7%	0.3%
	Median	10.5%	0.6%	2.4%	78.2%	7.0%	0.5%	0.8%	0.3%
Calabria	Mean	13.0%	1.2%	2.6%	73.1%	8.2%	0.5%	0.6%	0.2%
	Median	13.0%	1.2%	2.6%	73.0%	8.3%	0.5%	0.7%	0.3%
South Apulia	Mean	14.0%	1.8%	2.1%	69.1%	11.7%	0.3%	0.6%	0.2%
	Median	13.7%	1.8%	2.1%	69.1%	11.7%	0.3%	0.6%	0.2%
Sicily	Mean	12.3%	1.0%	2.7%	73.6%	8.3%	0.4%	0.8%	0.3%
	Median	12.2%	1.0%	2.7%	73.6%	8.3%	0.4%	0.8%	0.4%
Coast Sardinia	Mean	11.4%	1.0%	2.4%	70.9%	13.4%	0.2%	0.7%	0.0%
	Median	11.4%	1.0%	2.4%	70.9%	13.4%	0.2%	0.7%	0.0%
Inland Sardinia	Mean	11.0%	0.9%	2.2%	73.6%	11.3%	0.3%	0.7%	0.0%
	Median	11.0%	1.0%	2.2%	73.7%	11.2%	0.3%	0.7%	0.0%
Umbria	Mean	10.9%	0.6%	1.9%	79.6%	6.0%	0.3%	0.4%	0.0%
	Median	10.9%	0.6%	2.0%	79.6%	6.0%	0.4%	0.4%	0.0%
East Liguria	Mean	11.5%	0.8%	2.4%	77.5%	6.9%	0.3%	0.6%	0.0%
	Median	11.6%	0.8%	2.4%	77.4%	6.8%	0.3%	0.7%	0.0%
West Liguria	Mean	10.5%	1.1%	2.6%	76.8%	9.0%	0.0%	0.0%	0.0%
	Median	10.4%	1.0%	2.5%	77.0%	9.13	0.0%	0.0%	0.0%

Note: Dataset inconsistent with %-values. A modification to demonstrate %-values was fraction of total.

b) 

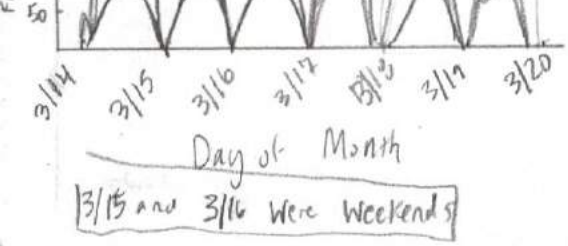
c) 

2% seperation as a pairwise analysis

d) The regions are distinguishable with simple tools and scale.

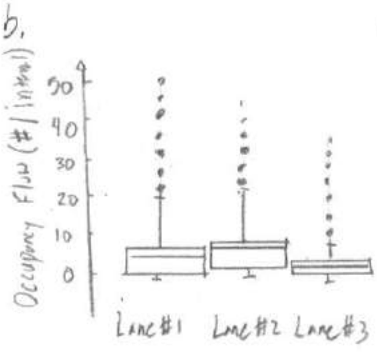
e) Completed.

c) Oleic and eicosenoic fatty acids demonstrate a paired measurement of Italy's regions with a composite of ±2% differences.

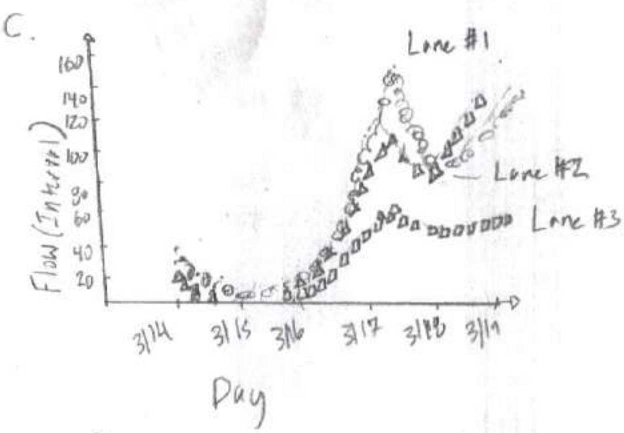


3/15 and 3/16 were weekends

b.



C.



In terms of solely flow and not occupancy, Lane #1 is the busiest lane.

The flow of lane #2 is twice that of lane #3.

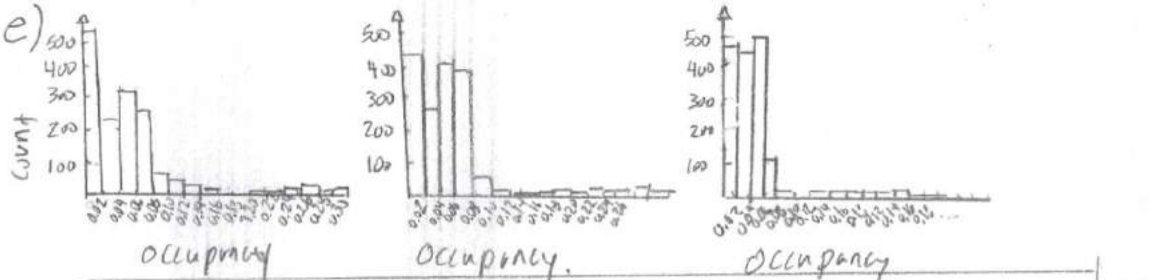
Lane #2 is busiest

d)

	Mean	Median
Lane #1	0.061	0.048
Lane #2	0.061	0.055
Lane #3	0.051	0.041

The distributions within each lane contribute to higher average because of the proportion of higher occupancy days.

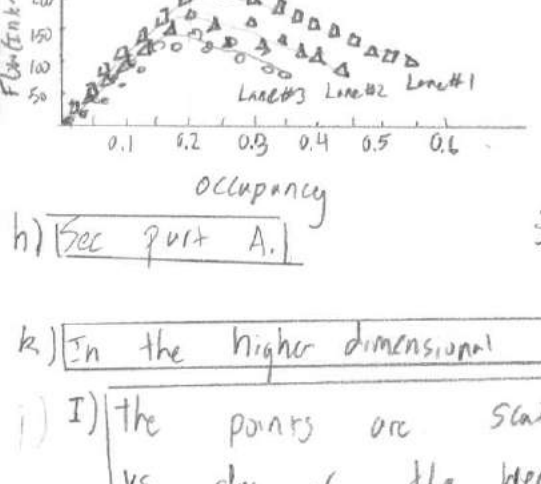
e)



The bins from 0-0.16 occupants seem to be good representation of the weekly activity. Although, a smaller distribution exists from 0.16-0.22 occupants.

f. The scatter plot of part c is evidence to argue "when one lane is busy the others are busy."

g. Throughput depends on congestion because of the scatterplot generated for occupancy vs. flow. Below 0.02 occupants flow increases proportionally, while more occupants decreases flow



i) The data is tabulated with ggobi.org.

h) See part A.

j) March 14th aided with finding the maximum of part g's plot of occupancy vs. flow.

k) In the higher dimensional scatterplots, the points generate a surface.

I) the points are scattered over three dimensions because occupancy vs flow vs day of the week is 3-dimensional.

II) Again, 3-dimensions because of triplet per datapoint.

III) The differences begin to occur near 3/17.

l) The right lane has lowest mean occupancy, so the taxi driver must merge right

Chapter 11: Comparing Two Samples:

1. $X \in \{1.1650, 0.6268, 0.0751, 0.3516\}$; $Y \in \{0.3035, 2.6961, 1.0591, 2.7971, 1.2641\}$

a) $\boxed{\mu_X = 0.55;\ \mu_Y = 1.62;\ \mu_X - \mu_Y = -1.07}$ b) Pooled Sample Variance: $s_p^2 = \frac{(n-1)s_x^2 + (m-1)s_y^2}{m+n-2}$ $\boxed{= 11.71}$; $s_x^2 = (n-1)\sum_{i=1}^{n}(X_i - \bar{X})^2$ $\boxed{= 1.95}$

$s_y^2 = (m-1)\sum_{i=1}^{m}(Y_i - \bar{Y})^2$ $\boxed{= 19.03}$

c) Pooled Standard Error: $s_{\bar{X}-\bar{Y}} = s_p\sqrt{\frac{1}{n} + \frac{1}{m}}$ $\boxed{= 2.30}$

d) Pooled Confidence Interval: $(\bar{X} - \bar{Y}) \pm t_{m+n-2}(\alpha/2)\, s_{\bar{X}-\bar{Y}}$

$1.07 \pm t_7(0.05) \cdot 2.30 = \boxed{1.07 \pm 3.25}$

e) A two sided test seems appropriate because of the statement "normal dist."

f) The p-value of a two-sided test of a null hypothesis represents the probability an alternative hypothesis is accepted.

g) Yes because the model for a 90% confidence interval is $90\% = 100\%(1-\alpha)$ when $\alpha = 0.1$.

h) The argument may change by refining the confidence interval to an $\alpha < 0.1$

2. The standard error of the mean $s_{\bar{X}-\bar{Y}} = s_p\sqrt{\frac{1}{n} + \frac{1}{m}}$ will halve by doubling the sample size and reduce difference of mean error of sampling.

3. $Var(\bar{X} - \bar{Y});\ s_p^2\left(\frac{1}{n} + \frac{1}{m}\right);\ \frac{s_x^2}{n} + \frac{s_y^2}{m} = \frac{(n-1)\sum_i (X - \bar{X})^2}{n} + \frac{(m-1)\sum_i (Y - \bar{Y})^2}{m} = s_x^2\left[\frac{1}{n} + \frac{1}{m}\right] = \boxed{s_p^2\left[\frac{1}{n} + \frac{1}{m}\right]}$

4. The t distribution is valid when sample sizes are small and standard deviation is not known, or both.

5. The expected measurement of any two methods can equal each other. By testing $\mu_X = \mu_Y$, a comparison of method accuracy occurs and is beneficial to scientists.

6. A test certifies foundational reasoning with others and when alone.

7. 1) $X_1, X_2, \ldots, X_n$ are independent random variables $\rightarrow$ when the normal distribution is drawn.
   2) $Y_1, Y_2, \ldots, Y_n$ are independent random variables $\rightarrow$ when the samples are drawn from a normal.
   3) X's and Y's are independent $\rightarrow$ when analyzing and making inferences about the data.

8. a) Yes, because the sample size is < 30 total.
   b) Yes, because the sample size is < 30 for each group.

9.

Concentration	$\bar{X}-\bar{Y}$	s_p	$s_{\bar{X}-\bar{Y}}$	df	t	t_{m+n-2}
10.2 mM	2.23	41.64	13.84	17	0.1612	-1.69
0.3 mM	0.89	111.72	37.24	17	0.0239	-1.69

Accept till significance level of $\alpha = 0.05$ for a one-sided distribution.
Accept till significance level of $\alpha = 0.05$ for a one-sided distribution.

10. $t = \frac{\mu_X - \mu_Y}{s_{\bar{X}-\bar{Y}}} = 0 < -t_{m+n-1}$ rejects 0 and $0 > t_{n+m-2}$ rejects

11. $H_0: \mu_X = \mu_Y + \Delta$ vs $H_A: \mu_X \neq \mu_Y + \Delta$

$t = \frac{\bar{X} - \bar{Y}}{s_{\bar{X}-\bar{Y}}} = \frac{\Delta}{s_{\bar{X}-\bar{Y}}}$: Reject $H_0 \begin{cases} |t| > t_{n+m-2}(\alpha/2) \\ t > t_{n+m-2}(\alpha) \\ t < -t_{n+m-2}(\alpha) \end{cases}$

12. $P(X_{(k)} \leq \eta_0 = \eta \leq X_{(n-k+1)}) = 100\%(1-\alpha)$

if $\eta_0 = 0$; $P(\eta_0 = 0 < k-1) = \frac{1}{2^n}\sum_{j=0}^{k-1}\binom{n}{j} = 1/2$

From section 11.3.3.

$S = \sum_i D_+ = 14 = Bin(n=24, p=0.5) = P(S \leq 14) = \boxed{0.8463}$

$P(S \geq 14) = 1 - P(S \leq 13) = 1 - Bin(n=24, p=0.5) = \boxed{0.2706}$

P-value $= \min(\boxed{0.2706}, 0.8463)$

13. $X_1, \ldots X_{25}$ i.i.d. $N(0.3, 1)$ $P(\mu = 0 | H_0)$ vs $P(\mu > 0 | H_A)$ at $\alpha = 0.05$

$H_1: \mu_X - \mu_Y = -\mu_Y = -0.3 = \Delta$; $\left|\frac{(\bar{X}=0) - \mu_X}{\sigma/\sqrt{n}}\right| > z(\alpha/2)$; $\frac{\mu_X}{1/\sqrt{25}} > 1.96$; $\mu_X > 0.392$; $1-\beta = P(\bar{T} \geq k) = 1 - \Phi\left(\frac{0.392 - 0.3}{1/5}\right)$ $\boxed{1 - 0.32}$

$S \sim Bin(25, 1/2) = \sum_k^{\wedge} \binom{25}{k} (1/2) \sim N\left(\frac{25}{2}, \frac{25}{4}\right)$; $0.05 = P(T \geq k-1) \approx 1 - \phi\left(\frac{k - 0.5 - 12.5}{4}\right) = 1 - \left(\frac{k-13}{2.5}\right)$

$k = 17.9$; $1-\beta = P(T \geq 17) = 1 - \Phi\left(\frac{17 - 0.5 - 15.5}{2.4}\right) = 1 - \Phi(0.41) = 0.34$

14. $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$; $H_0: \mu = \mu_0$, the test is often $t = \frac{\bar{X} - \mu_0}{S_X}$; $df = n-1$. $\Lambda = \prod N(\mu, \sigma^2) \prod N(\mu_0, \sigma^2)$

$= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(1/2)(X_i - \mu_0)^2/\sigma^2} \prod_{i=1}^m \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(1/2)(Y_i - \mu_0)^2/2\sigma^2}$

$l(\mu, \mu_0, \sigma^2) = -\frac{(m+n)}{2} \log 2\pi - \frac{(m+n)}{2} \log \sigma^2 - \frac{1}{2\sigma_0^2}\left[\sum (X_i - \mu_a)^2 + \sum (Y_i - \mu_0)^2\right]$

$\frac{dl(\mu, \mu_0, \sigma^2)}{d\mu_0} = \frac{1}{(m+n)}\left(\sum X_i + \sum Y_i\right) - \sum \mu_0$; $\frac{dl(\mu, \mu_0, \sigma^2)}{d\sigma^2} = \frac{1}{(m+n)}\left[\sum (X - \mu_0) + \sum (Y - \mu_0)^2\right] - \sum \sigma_0^2$

$\hat{\mu}_0 = \frac{1}{(m+n)}\left(\sum X_i + \sum Y_i\right)$; $\hat{\sigma}_0^2 = \frac{1}{(m+n)}\left[\sum (X - \mu_X) + \sum (Y - \mu_0)^2\right]$

$l(\hat{\mu}_0, \hat{\sigma}_0^2) = -\frac{(m+n)}{2} \log 2\pi - \frac{(m+n)}{2} \log \hat{\sigma}_0^2 - \frac{(m+n)}{2}$; $l(\mu_X, \mu_Y, \sigma_1^2) = -\frac{(m+n)}{2} \log 2\pi - \frac{(m+n)}{2} \log \hat{\sigma}_1^2 - \frac{m+n}{2}$

$-\Lambda = \frac{l(\mu_X, \mu_Y, \sigma^2)}{l(\hat{\mu}_0, \hat{\sigma}_0^2)} = \frac{m+n}{2} \log\left(\frac{\hat{\sigma}_1^2}{\hat{\sigma}_0^2}\right)$; $\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2} = \frac{\sum (\bar{X} - \mu_0)^2 + \sum (Y - \mu_0)^2}{\sum (X_i - \bar{X})^2 + \sum (Y_j - \bar{Y})^2} = \frac{\sum (\bar{X}_i - \mu_0)^2}{\sum (X_i - \bar{X})^2}$; if $Y = X$, $\bar{X} = \bar{Y}$

$\boxed{t = \Lambda = \frac{\sigma_0}{\sigma_1} = \frac{\bar{X} - \mu_0}{S_X}}$

15. $n = m$ = treatment = control

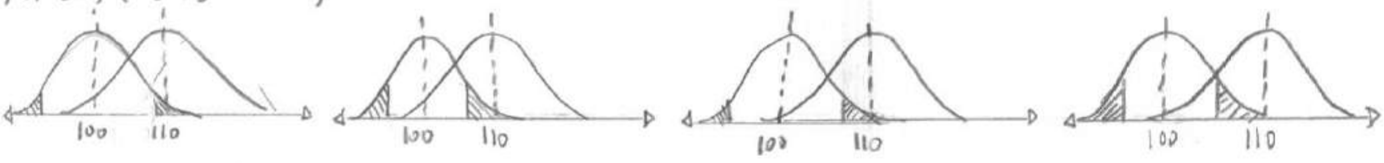
$\sigma_t = \sigma_c = 10$; n? 95% confidence Interval for $\mu_X - \mu_Y = 2$

$P\left(\left|\frac{\bar{X} - \bar{Y}}{\sigma\sqrt{\frac{2}{n}}}\right| > z(\alpha/2)\right) = P\left(\bar{X} - \bar{Y} > z(\alpha/2)\sqrt{\tfrac{2}{n}}\sigma\right) - P\left(\bar{X} - \bar{Y} < z(\alpha/2)\sqrt{\tfrac{2}{n}}\sigma\right) = 2$

$2 \cdot 1.96 \cdot \sqrt{\frac{2}{n}} \cdot 10 = 2$; $\boxed{n = 768}$

16. $H_0: \mu_X = \mu_Y$; $H_A: \mu_X > \mu_Y$; $\Delta = 0.5$ if $\mu_X - \mu_Y = 2$; $\sigma = 10$. $2 \cdot 1.64 \cdot \sqrt{\frac{2}{n}} \cdot 10 = 2$; $\boxed{n = 538}$

17. a) $n = 20$, $\alpha = 0.05$ b) $n = 20$, $\alpha = 0.10$ c) $n = 40$, $\alpha = 0.05$ d) $n = 40$, $\alpha = 0.10$



18. m = subjects a) $|\mu_x - \mu_y| \pm z(\alpha/2)\sigma\sqrt{\frac{2}{m}}$; The total subject allocation is independent of confidence interval and can be allocated in random proportions.

b) $H_o: \mu_x = \mu_y$; $\Delta = \mu_x - \mu_y$ is already as powerful as possible being $H_A: \Delta = \mu_x - \mu_y = 0$. The sample proportions are independent of the argument.

19. $n = 25$; $m = 25$; Normal Distribution
$$f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} ; \sigma = 5.$$

a) Pooled Standard Error: $S_{\bar{x}-\bar{y}} = S_p\sqrt{\frac{1}{n} + \frac{1}{m}}$; where $S_p = \sqrt{\frac{(n-1)S_x^2 + (m-1)S_y^2}{m+n-2}} = \sqrt{\frac{2\cdot 24\cdot 10^2}{48}} = 10$

$\boxed{S_{\bar{x}-\bar{y}} = 2\sqrt{2}}$

b) $\alpha = 0.05$; $H_o: \mu_x = \mu_y$ vs $H_A: \mu_y > \mu_x$; $\mu_y = \mu_x + 1$; $t = \frac{\bar{X} - \bar{Y}}{S_p\sqrt{\frac{1}{n} + \frac{1}{m}}} = \frac{1}{10\sqrt{\frac{2}{25}}} = \frac{1}{2\sqrt{2}} = 0.35$

${}_{0.975}t_{24} = 2.064$ is the rejection region.

c) Power of Test if $\mu_y = \mu_x + 1$

$1 - \beta = t/2 = 0.17$

d) p-value is 0.07 ; Would H_o reject if $\alpha = 0.10$; The test would reject because for a two-sided normal distribution $\alpha/2 = 0.05$ and the test arrived to a p-value greater than $\alpha/2$.

20. Example A : 11. 3.1 Bayes $= P(x|A) = \frac{P(x|B)\cdot P(B)}{P(A)}$
$$f(x|\mu,\xi^{-1}) = f(x|\theta,\xi)\cdot f(\theta) f(\xi) = \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{-\Sigma(x_i-\theta)^2}{2\sigma^2}\right)}_{\text{Likelihood}} \cdot \underbrace{\exp\left(-\frac{\xi_{prior}}{2}(\theta-\theta_0)^2\right)}_{\text{Priors}} \cdot \underbrace{\xi^{\alpha-1}\exp(-\lambda\xi)}_{\text{gamma prior}}$$

21. a) $t = \frac{\bar{X} - \bar{Y}}{S_{\bar{x}-\bar{y}}} = \frac{10.693 - 6.75}{1.80} = 4.98$; $\boxed{= 99.99\%\ \text{Positive}}$

1.05 133

If p-value = 0.05, $\alpha = 0.05$, for a one-sided distribution. $df = 9$; ${}_{9}t_{0.95} = 1.833$. The null hypothesis of $H_o: \mu_x = \mu_y$ is rejected.

b) Mann-Whitney Test [Nonparametric]

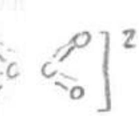
Type I	Rank	Type II	Rank
3.03	1	3.19	2
5.53	8	4.26	3
5.60	9	4.47	4
9.30	11	4.53	5
9.92	13	4.67	6
12.51	14	4.69	7
12.95	17	12.78	16
15.21	18	6.79	10
16.04	19	9.37	12
16.84	20	12.75	15
R	130	R	80
R'	80	R'	130
$R^*(0.05)$	78	$R^*(0.05)$	78

The null hypothesis (H_o) is rejected at a significance level of $\alpha = 0.05$, because $R' > R^*$.

c) Either the t-test or Mann-Whitney Test is applicable to determining the null hypothesis point of rejection. The key qualifiers for each test are $(m,n) < 30$, and are chosen by the case of extreme outliers which are not representative of this data set.

d) π is the probability that a component of one type will last longer than the component of another type (effect), or the probability that an observation from one distribution is smaller than the independent observation from another distribution.
$$\pi = \frac{1}{mn}\sum_{i}^{10}\sum_{j}^{10} Z_{ij} \text{ ; where } Z_{ij} = \begin{cases} 1 & \text{if } X_i < Y_j \\ 0 & \text{otherwise} \end{cases}$$
$$= \frac{1}{100}\cdot 25 = \boxed{\frac{1}{4}}$$

e) $\hat{\pi} = \frac{1}{mn}\sum_{i}^{10}\sum_{j}^{10} Z_{ij} = \boxed{\frac{1}{4}}$ f. $CI = \left\{\frac{9}{40}, \frac{5}{13}\right\}$

22. Calcium oxalate [CaC_2O_4]: Ca^{2+}  $^{2-}$

Parametric Test:

oxalate: $\mu = 2.39$, $\sigma = 1.45$

flame: $\mu = 2.35$, $\sigma = 1.42$

$s_p = 2.05$
$s_{\bar{x}-\bar{y}} = 0.19$
$t = 0.43$
$t_{117,0.9995} = 3.38$

Nonparametric Test:

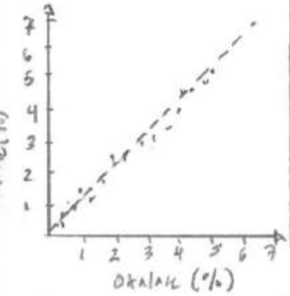
oxalate: $R = 14146.5$, $R' = 6798.5$

Flame: $R = 13819.5$, $R' = 7125.5$

Z-statistic $= 0.31$

p-value $= 0.62$

Graphical:



The parametric test t-statistic advises to accept the null hypothesis, Nonparametric advises to not reject the null hypothesis and the graphs do look similar.

23. $X_1, \dots X_n$ i.i.d. with cdf F; $Y_1, \dots Y_m$ i.i.d. with cdf G. $H_0: F = G$ vs $H_A: F \neq G$. $(m+n)\%2 = 0$

a) Hypergeometric Distribution

$$p(X=k) = \frac{\binom{r}{k}\binom{n-r}{m-k}}{\binom{n}{m}}$$

k = amount of total without replacement.

r = Total of type

n-r = Total of second type

m = total chosen

$$P(T=t) = \frac{\binom{(m+n)/2}{t}\binom{(m+n)/2}{n-t}}{\binom{m+n}{n}};$$

$r = (m+n)/2$ or total of values below median

$n-r = (m+n)/2$ or total of values greater than median.

$(m+n)/2 < \eta$

$(m+n)/2 > \eta$

k = amount of X's chosen without replacement below the median.

n = total X's chosen.

A rejection region would be discovered by $P(T<t) = 1-\alpha$.

b. CI for $G = P(X_{(i)} < \eta < X_{(j)})$ where η = median

The hypergeometric distribution is approximated by a binomial distribution.

$$P(T=t) = \frac{\binom{(m+n)/2}{t}\binom{(m+n)/2}{n-t}}{\binom{m+n}{n}} = \binom{n}{t}\prod_{k=1}^{t}\frac{\left(\frac{m+n}{2} - t + k\right)}{(m+n-x+k)} \cdot \prod_{j=1}^{n-t}\frac{(m+n-r-(j-k)-n)}{(m+j)}$$

$$\lim_{N\to\infty} P(T=t) = \binom{n}{t} p(1-p) \; ; \; \lim_{N\to\infty} P(X_i < \eta < X_j) = \sum_{k=i}^{j}\binom{n}{k}p^k(1-p)^{n-k}$$

$$= Bin(j-1, n, p) - Bin(i-1, n, p)$$

With $G(x) = F(x-\Delta)$, then $\boxed{Bin(j-1-\Delta, n, p) - Bin(i-1-\Delta, n, p)}$

c. $P(T<t) = 1 - P(T>t)$

$$= 1 - \sum_{i=1}^{T}\frac{\binom{(m+n)/2}{t}\binom{(m+n)/2}{n-t}}{\binom{m+n}{n}}$$

$$= 1 - \sum_{i=1}^{5}\frac{\binom{10}{t}\binom{10}{10-t}}{\binom{20}{10}} = \boxed{32.8\%}$$

24. Mann-Whitney Statistic

$R' = n(m+n+1) - R$

$E(U) = \frac{mn}{2}$

$Var(U) = \frac{mn(m+n+1)}{12}$

$m=3, n=2$

25. a) If 79.94 is arbitrarily small, say 0.001, then the new mean is 69.98J and standard deviation 26.45J. The pooled variance became 9.75J, with a standard error of 1.40J.
The final t-statistic was evaluated to 7.16.
At $t_{19}(0.005) = 2.861 < 7.16$ and would reject with a significance of $\alpha = 0.01$

$$\frac{U - E(U_y)}{\sqrt{Var(U_y)}} = N(0,1) \; ; \; U = N(0,1)\sqrt{Var(U_y)} + E(U_y) = \boxed{\left(\frac{e^{-x^2/2}}{\sqrt{2\pi}} + 1\right)3}$$

b. If method B (80.03 J) became 10,000 J. $\mu = 1319.93$, $\sigma = 3280.74$, $S_p^2 = 1208.69$, $S_{\bar{X}-\bar{Y}} = 15.62$

t-statistic $= -79.37$

The null hypothesis would reject

C.

Method A	Method B
79.93 (7.5)	80.03 (15)
80.04 (19)	79.94 (1)
80.02 (11)	79.98 (7.5)
80.04 (19)	79.97 (4.5)
80.03 (15)	79.97 (4.5)
80.03 (15)	80.03 (15)
80.04 (19)	79.95 (2)
79.97 (4.5)	79.97 (4.5)
80.05 (21)	
80.03 (15)	
80.02 (11)	
80 (9)	
80.02 (11)	

R 177 54

R' 232

Z-Statistic: 13.04

$t_{20}(0.005) = 2.85$

Null hypothesis would reject.

Modifying 79.94 to an arbitrarily low value had no effect on the data because 79.94 is the lowest rank. While raising 80.03 to a larger value of 10,000 changed the Z-Statistic to 12.64 with a null hypothesis also rejecting.

26. $X_1, \dots X_n$ be from $N(0,1)$: $Y_1 \dots Y_n$ be from $N(1,1)$.

a) $T_x = U_x + \frac{n(n+1)}{2}$; $U_x = \sum\sum Z_{ij}$; $Z_{ij} = \begin{cases} 1 & X_i < Y_i \\ 0 & \text{otherwise.} \end{cases}$

$$E[T_x] = E[U_x] + E\left[\frac{n(n+1)}{2}\right] = E\left[\sum\sum Z_{ij}\right] + \frac{n(n+1)}{2} = \sum\sum E[Z_{ij}] + \frac{n(n+1)}{2}$$

$$= \sum\sum P(X_i < Y_i) + \frac{n(n+1)}{2} = n^2 P(X_i - Y_i > 0) + \frac{n(n+1)}{2}$$

$$= n^2 P\left(\frac{X_1 - X_2 - E(X_1 - X_2)}{\sqrt{\sigma_{x_1}^2 + \sigma_{x_2}^2}} > \frac{0 - E(X_1 - X_2)}{\sqrt{\sigma_{x_1}^2 + \sigma_{x_2}^2}}\right) + \frac{n(n+1)}{2}$$

$$= n^2 P\left(Z > \frac{1}{\sqrt{2}}\right) + \frac{n(n+1)}{2} = n^2\left[1 - \Phi\left(\frac{1}{\sqrt{2}}\right)\right] + \frac{n(n+1)}{2} = n^2(1 - 0.7611) + \frac{n(n+1)}{2}$$

$$\boxed{= 0.2389n^2 + \frac{n(n+1)}{2}}$$

b. $Var(T_x) = Var\left(U_x + \frac{n(n+1)}{2}\right) = Var(U_x) = \sum\sum Var(Z_{ij}) + \sum Cov(Z_{ij}, Z_{kl})$

$$= n^2 p(1-p) + \sum\left(E(Z_{ij} Z_{kl}) - E[Z_{ij}]E[Z_{kl}]\right) = n^2 p(1-p) + \sum\left[E(\cancel{Z_{ij} Z_{kl}})^{0} - p^2\right] = \boxed{n^2 p(1-p)}$$

uncorrelated

27. Exact Null Distribution of W_+ where $n = 4$

$W_+ = \sum_{k=1}^{n} k I_k$ where $I_k = \begin{cases} 1 & \text{if the } k^{th} \text{ largest } |D_i| \text{ has } D_i > 0 \\ 0 & \text{Otherwise} \end{cases}$

$W_+ = \sum_{k=1}^{4} k I_k$: Total values : $\frac{n(n+1)}{2} = 10$; "Above Diagonal"

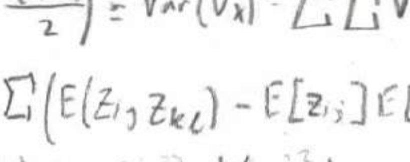
W	0	1	2	3	4	5	6	7	8	9	10
P(x)	1/16	1/16	1/16	2/16	2/16	2/16	2/16	2/16	1/16	1/16	1/16

	1	1	1	1	1	-1	-1	-1	1	1	1	-1	-1	1	-1	-1
	2	2	-2	2	2	-2	2	2	-2	-2	2	-2	2	-2	-2	-2
	3	3	3	-3	3	3	-3	3	-3	3	-3	3	-3	-3	-3	-3
	4	4	4	4	-4	4	4	-4	4	-4	4	-4	-4	-4	4	-4
Total (W_+)	10	9	8	7	6	7	6	5	5	4	3	3	2	1	4	0

28. $n = 10, 20, 30$ find $\alpha = 0.05$ and 0.01 for a two sided rank test

n	10	20	30
0.05	8	52	137
0.01	3	37	109
$E(W_+)$	27.5	105	233
$Var(W_+)$	96.25	718	2364
$Z(0.05)$	-1.98	-1.97	-1.97
$Z(0.01)$	-2.49	-2.54	-2.55

29. a)



b) The process of randomizing two samples, then comparing the difference is similar to the Mann-Whitney test, but more commonly the Wilcoxon signed rank test. Each of these methods compares to a randomized distribution.

30. $X_{\bar{A}} - X_{\bar{B}}$; Standard Error : $\sigma_y = \sqrt{Var(U_y)}/nm = \boxed{\sqrt{\frac{(m+n+1)}{12}}}$

31. Section 11.2.3: A Nonparametric Method: The Mann-Whitney Test ; $F=G$; $E(\hat{\pi}) = E\left[\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m} Z_{ij}\right]$

$= \frac{n \cdot m}{n \cdot m \cdot 2} = \boxed{\frac{1}{2}}$

$Var(\hat{\pi}) = \boxed{\frac{(m+n+1)}{12mn}}$

$n \neq m$ or $n = m$ does not influence outcomes of $E(\hat{\pi})$ or $Var(\hat{\pi})$.

32. $X \sim N(\mu_x, \sigma_x^2)$; $Y \sim N(\mu_y, \sigma_y^2)$: $\pi = P(X<Y)$

$\pi = P(X-Y) = sign(N(X_i \mid \mu_x, \sigma_x^2) - N(y_i \mid \mu_{y_i}, \sigma_y^2))$

$\boxed{= sign\left(\frac{1}{2\pi\sigma_x^2\sigma_y^2} e^{-(x_i-\mu_x)^2/2\sigma_x^2 - (y_i-\mu_y)^2/2\sigma_y^2}\right)}$

33. $N(\mu_x, \sigma_x^2)$; $N(\mu_y, \sigma_y^2)$; $H_0: \sigma_x = \sigma_y$; $\frac{S_x^2}{S_y^2} \sim F_{n-1,m-1}$

$F_{n-1,m-1} : \frac{X}{Y} = \frac{N(\mu_x, \sigma_x^2)}{N(\mu_y, \sigma_y^2)} = \frac{\sigma_x}{\sigma_y}$ a) $H_0: \sigma_x = \sigma_y$ $H_1: \sigma_x \neq \sigma_y$

One-sided: $\boxed{\frac{\sigma_x^2}{\sigma_y^2} > F_{n-1,m-1}(\alpha)}$

Two-sided: $\boxed{F_{n-1,m-1}(\alpha/2) < \frac{\sigma_x^2}{\sigma_y^2} < F_{n-1,m-1}(1-\frac{\alpha}{2})}$

b) CI : $\left(\frac{S_x^2}{S_y^2} F_{n-1,m-1}(\alpha/2) < \sigma_x^2/\sigma_y^2 < \frac{S_x^2}{S_y^2} F_{n-1,m-1}(1-\frac{\alpha}{2})\right)$

c) onesided Two-sided:

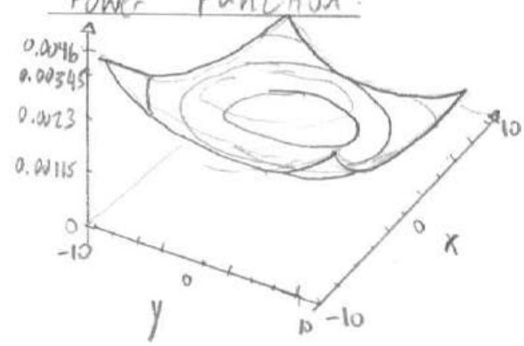
$0.59 > \frac{1}{F_{8,7}(0.05)}$ $\frac{\sigma_x^2}{\sigma_y^2}(F_{12,7}(0.025))^{-1} < 0.59 < F_{12,7}(0.975)\frac{\sigma_x^2}{\sigma_y^2}$

$0.59 > 0.29$ $\boxed{0.21 < 0.59 < 2.17}$

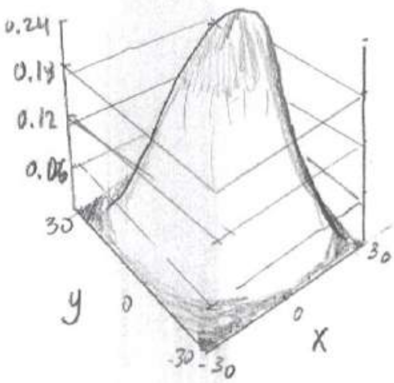
34. $H_0: \mu_x = \mu_y$ a) Paired: $Cov(X_i, Y_i) = 50$, $\sigma_x = \sigma_y = 10$, $i = 1 \ldots 25$ if $H_1: \mu_x \neq \mu_y$; $\rho = \frac{cov(X,Y)}{\sqrt{\sigma_x^2 \sigma_y^2}} = \frac{1}{2}$

$\Sigma = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix} = \begin{bmatrix} 10^2 & 50 \\ 50 & 10^2 \end{bmatrix}$; $\mu = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}$; $N(\mu, \Sigma)$

Power Function:

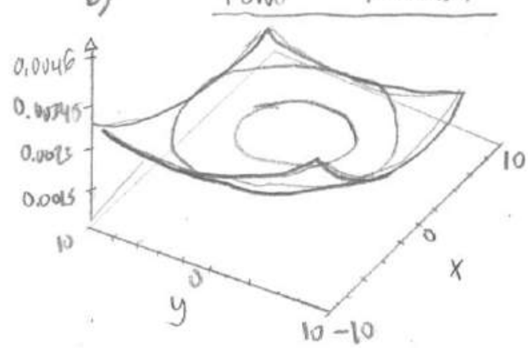


$N(\mu, \Sigma)$



$H_0: \mu_x = \mu_y$

b) Power Function



$N(\mu_x - \mu_y, \frac{\sigma_x^2}{n^2} + \frac{\sigma_y^2}{n^2}) = N(\mu_x - \mu_y, 8)$

$\boxed{\text{Power function is the only requirement.}}$

35. $n = 22$; $t = 70$ days, exposure time = 7 days to ozone

$n = 23$; $t = 70$ days, exposure time = 0 days to ozone.

$H_0: \mu_{control} = \mu_{ozone}$

$H_1: \mu_{control} \neq \mu_{ozone}$

	Mean	Median	SD
Control(23)	22.43g	22.7g	10.54g
Ozone(22)	11.01g	11.1g	18.60g

Parametric

S_p : 225.43g

$S_{\bar{X}-\bar{Y}}$ = 67.22g

t-statistic : 2.55

$t_{43}(0.05)$: 1.68

Nonparametric

R(control) = 661 ; R(Ozone) = 374

R'(Ozone) = 638

Z-statistic : 8.74

Z(0.05) : 1.96

For both parametric and nonparametric analysis, the null hypothesis (H_0) is rejected. The parametric pooled study shows a t-statistic greater than a significance of $\alpha = 0.05$. While the nonparametric test, Mann-Whitney, demonstrates a Z-statistic above the rejection level of 1.96, purporting a rejection of the null hypothesis.

36.

	E[X]	SD[X]	$\bar{X}-\bar{Y}$	S_p	$S_{\bar{X}-\bar{Y}}$
Microbiological Method	85.26%	20.48%	0.44%	20.65%	7.54%
Hydroxylamine Method	84.82%	20.82%			

$H_0: \bar{X} = \bar{Y}$; $H_1: \bar{X} \neq \bar{Y}$

Outcomes would be similar by randomizing the sample sets.
t-statistic : 0.06 ; $t_{30}(0.05)$: 2.04.
The null hypothesis is not rejected.

37. a)

	Word A ΔDose	Word A ΔPlacebo
	0.8	-0.4
	0.1	0.4
	0.55	-0.1
	0.6	-0.9
	0.34	0.2
	1.42	0.78
	1.74	0.3
	-0.29	0.64
	0.53	0.42
E[X]	0.64	0.15
SD[X]	0.59	0.51
Sp	0.55	
$S_{\bar{X}-\bar{Y}}$	0.26	
Z	1.92	

	Word B ΔDose	Word B ΔPlacebo
	-0.45	0.01
	0.15	0.44
	-0.19	-0.55
	0.12	-0.05
	0.03	0.04
	-0.15	-1.16
	0.48	-0.16
E[X]	-0.001	-0.25
SD[X]	0.28	0.43
Sp	0.36	
$S_{\bar{X}-\bar{Y}}$	0.19	
Z	1.28	

$H_{0A}: \mu_{\Delta Dose} = \mu_{\Delta Placebo}$
$H_{1A}: \mu_{\Delta Dose} \neq \mu_{\Delta Placebo}$
H_{0A} is below the significance level of $\alpha = 0.05$, and is not rejected.

$H_{0B}: \mu_{\Delta Dose} = \mu_{\Delta Placebo}$
$H_{1B}: \mu_{\Delta Dose} \neq \mu_{\Delta Placebo}$
H_{0B} is below the threshold of a Z(0.05) = 1.96 and accepted as a hypothesis.

b) Pooled Variance of ΔDose of Word A and Word B:

$$S_{Pd}^2 = \frac{(10-1)\,0.59^2 + (7-1)\,0.28^2}{(10+7-2)} = \boxed{0.24}$$

Pooled Variance of ΔPlacebo of Word A and Word B:

$$S_{Pp}^2 = \frac{(10-1)\cdot 0.51^2 + (7-1)\,0.43^2}{(10+7-2)} = \boxed{0.23}$$

Standard Error of ΔDose of Word A and Word B

$$S_{\bar{A}-\bar{B}} = S_P\sqrt{\left(\frac{1}{10} + \frac{1}{7}\right)} = \boxed{0.24}$$

Standard Error of ΔPlacebo of Word A and Word B.

$$S_{\bar{A}\cdot\bar{B}} = S_P\sqrt{\left(\frac{1}{10} + \frac{1}{7}\right)} = \boxed{0.24}$$

Z-statistic of ΔDose of Word A and Word B.

$$= \frac{0.64 + 0.001}{0.24} = \boxed{2.67}$$

Z-statistic of ΔPlacebo of Word A and Word B.

$H_0: \overline{\Delta Placebo}(A) = \overline{\Delta Placebo}(B)$
$H_1: \overline{\Delta Placebo}(A) \neq \overline{\Delta Placebo}(B)$

$$= \frac{0.15 + 0.25}{0.24} = \boxed{1.67.}$$

$\boxed{H_0 \text{ rejected} : Z(0.05) < Z = 1.67}$

Part B:

$H_0: \overline{\Delta Dose}(\text{Word A}) = \overline{\Delta Dose}(\text{Word B})$ Rejected

$H_1: \overline{\Delta Dose}(\text{Word A}) \neq \overline{\Delta Dose}(\text{Word B})$ Accepted

$\boxed{Z(0.05) < Z = 2.673}$

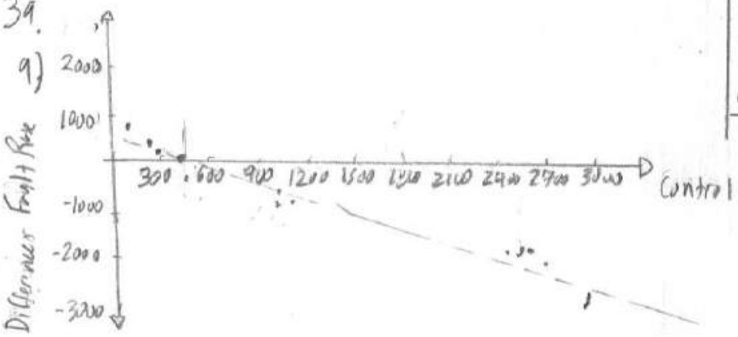
38. Δ = Added (%) - Found (%)

	E[x]	SD[x]	Sp	$S_{\bar{x}-\bar{y}}$	t
Δ Sulfonic Acid	0.007	0.015	0.014	0.006	0.19
Δ Pyrazolone-T	0.006	0.013			

Ho : $\overline{\Delta \text{Sulfonic Acid}} = \overline{\Delta \text{Pyrazolone-T}}$
H₁ : $\overline{\Delta \text{Sulfonic Acid}} \neq \overline{\Delta \text{Pyrazolone-T}}$ } Measurements on HPLC.

The null hypothesis (H₀) is not rejected because the t-statistic = 0.19 is below $t_{20}(0.005) = 2.845$. Bailey, Cox, and Spangers HPLC measurements were consistent for two sets of data.

39.
a)



b|c)

	E[x]	SD[x]	Median	Sp	$S_{\bar{x}-\bar{y}}$
Test	434.21	161.27	442.50	550.60	208.106
Control	395.50	761.78	616.00		

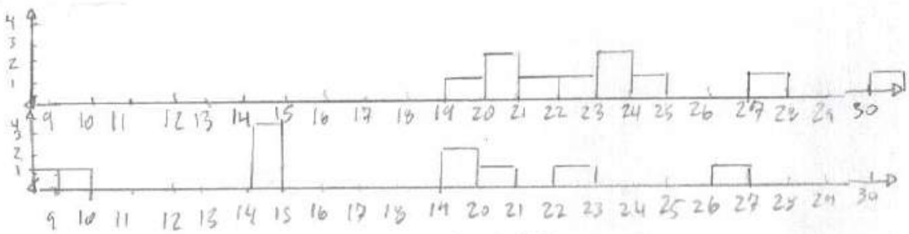
$\bar{X}-\bar{Y} = 523.3$ $\Delta\eta = -368.5$

CI = {115, 931} CI = {nq - z(0.05)$\sqrt{nq(1-q)}$, nq + z(0.05)$\sqrt{nq(1-q)}$}
= {R(4.46), R(12.54)}
= {-587, -163}

d) A t test seems more appropriate because of the sample size, but skew of the control demonstrates for one of the datasets suggests examination with a ranked test.

40. n = 10 cages, t = 30-day old mice Treated : ; T = 12 days 80 Oe/cm.
1cage = 3 rats Control : 30 mice.

a)



Field Absent [μ = 25g, σ = 5.4g

Field Present μ = 17g σ = 3.0g

Weight Gain (g)

b)

	E[x]	SD[x]	Sp	$S_{\bar{x}-\bar{y}}$	Lower CI	Upper CI
Field Absent	25g	5.4g	4.4g	1.9g	4.3g	11.7g
Field Present	17g	3.0g				

c) t_{18}-score = 3.93 H₀ : Mean Field Absent = Mean Field Present
p-value = 8×10^{-5} H₁ : Mean Field Absent ≠ Mean Field Present

H₀ is rejected at a significance of $\alpha = 0.05$ and alternative hypothesis accepted.

d)

	R	R'	U
Field Absent	143		
Field Present	67	153	6.07

p-value = 0.0 ; null hypothesis is rejected.

e)

	Median	ΔMedian
Field Absent	25g	5g
Field Present	17g	

f) SE of Mean = 1.9g : Bootstrap. $\sigma_{Median} = 1.253\sigma_{\bar{x}} = 2.4g$

g) CI = {R(1.9), R(8.1)} = {5, 11.8}

41. a) $E(\hat{\Delta}) = E(\text{median}(X_i - Y_j)) = \mu_x - \mu_y$

b) $\hat{\Delta}$ is robust to outliers because the method is rank based.

c) $\sigma_{Median} < \hat{\Delta} = 4g < \Delta$Median.

d) $\hat{\Delta} = 1.253\sigma_{\bar{x}}$; $\sigma_{\bar{x}} = 3.19g$. $f\left(x \mid \mu, \frac{1}{8f(\tilde{\mu})^2 \cdot 10}\right)$

e. CI = {-2.3g, 10.3g}

42. a) $\pi = \frac{1}{mn}\sum\sum Z_{ij}$; Where $Z_{ij} = \begin{cases} 1 & X_i > Y_i \\ 0 & \text{Otherwise} \end{cases}$; $\pi = \frac{229}{26 \cdot 9} = \boxed{0.3388}$

b) SE=0 For π because matrix was generated with exact values. Alternative methods include generating a sampling distribution For

$$S_x = \sqrt{\frac{1}{B}\sum(\bar{\pi}_i - \pi)^2}$$

c) The confidence interval = {0.3388}, but with a sampling distribution CI = $\{\pi - 1.96 S_x, \pi + 1.96 S_x\}$

43. $X_1 \dots X_n, Y_1 \dots Y_m$; $\mu_{20\%}$; A bootstrap could be used to estimate the standard error of a 20% trimmed mean vs. 0% trimmed mean by producing a sampling distribution $f(\bar{x} - \bar{Y} \mid \bar{x} - \bar{Y}, S_p^2)$, with those parameters.

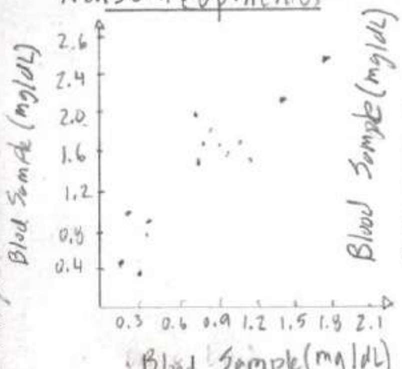
44 n=20 ; m=15 ; 2 months. Volume=2mL of blood at breakfast and urination.

a)

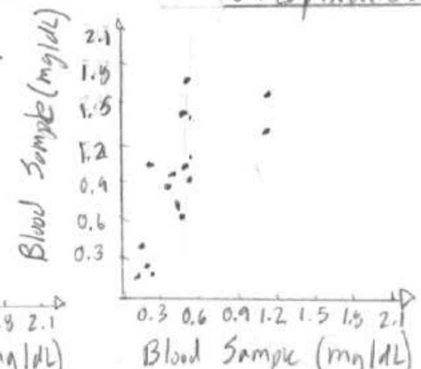
Post-urination: 1g ascorbic acid.

t_1 = 6 hours urine collection ; t_2 = 2 hours after dose of Vitamin C

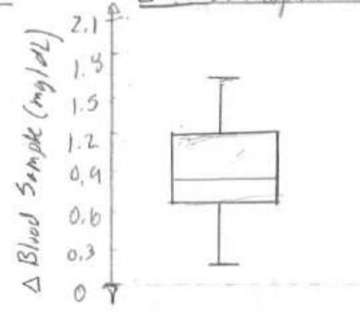
Non Schizophrenics:



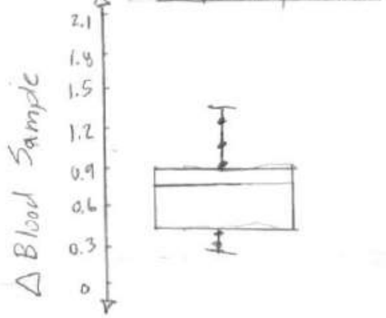
Schizophrenics:



Δ Non Schizophrenic



Δ Schizophrenic



b)

	E[X]	SD[X]	S_p	$S_{\bar{x}-\bar{Y}}$	t	df	t
Nonschizo (0hr)	0.62	0.42	0.34	0.12	2.45	15	1.69
Schizo (0hr)	0.33	0.26				20	
Nonschizo (2hr)	1.45	0.59	0.50	0.17	2.92	15	1.69
Schizo (2hr)	0.95	0.42				20	
Δ Nonschizophren	0.61	0.33	0.33	0.11	1.89	15	1.69
Δ Schizophrenic	0.33	0.34				20	

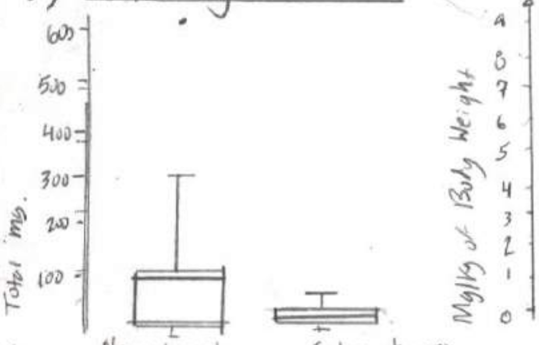
H_0: Nonschizophrenic (0hr) = Schizophrenic (0hr) — Reject at α=0.05
H_1: Nonschizophrenic (0hr) ≠ Schizophrenic (0hr) — Accept at α=0.05
H_0: Nonschizophrenic (2hr) = Schizophrenic (2hr) — Reject at α=0.05
H_1: Nonschizophrenic (2hr) ≠ Schizophrenic (2hr) — Accept at α=0.05
H_0: ΔNonschizophrenic = Schizophrenic (2hr) — Reject at α=0.05
H_1: ΔNonschizophrenic ≠ ΔSchizophrenic — Accept at α=0.05

c)

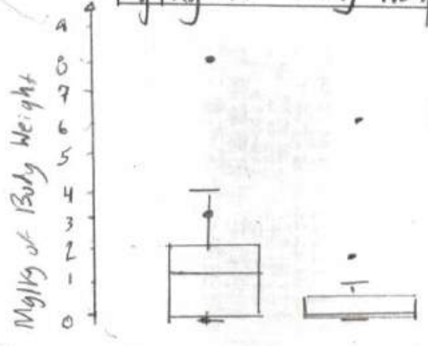
	R	R'	U
Nonschizo (0hr)	324		
Schizo (0hr)	306	234	2.8
Nonschizo (2hr)	345		
Schizo (0hr)	285	255	3.15
ΔNonschizophrenic	319		
ΔSchizophrenic	311	229	2.6

H_0: Nonschizophrenic (0hr) = Schizophrenic (0hr) — Reject at α=0.05
H_1: Nonschizophrenic (0hr) ≠ Schizophrenic (0hr) — Accept at α=0.05
H_0: Nonschizophrenic (2hr) = Schizophrenic (2hr) — Reject at α=0.05
H_1: Nonschizophrenic (2hr) ≠ Schizophrenic (2hr) — Accept at α=0.05
H_0: Nonschizophrenic = Schizophrenic — Reject at α=0.05
H_1: Nonschizophrenic ≠ Schizophrenic — Accept at α=0.05

d) Total mg Vitamin C



Mg/Kg of Body Weight



	E[X]	SD[X]	S_p	$S_{\bar{x}-\bar{Y}}$	df	t
Nonschizophrenic (Total mg)	122.7	153.7	116.9	39.9	33	-2.2
Schizophrenic (Total mg)	35.8	79.6				
Nonschizophrenic (mg/kg)	1.78	2.0	1.6	0.5	33	-1.2
Schizophrenic (mg/kg)	0.53	1.3				

The data shows a hypothesis of mean weights are not equivalent.

e) Assuming Normality is standard for Z-statistics, but the mean and median show normal distribution may not best fit the data.

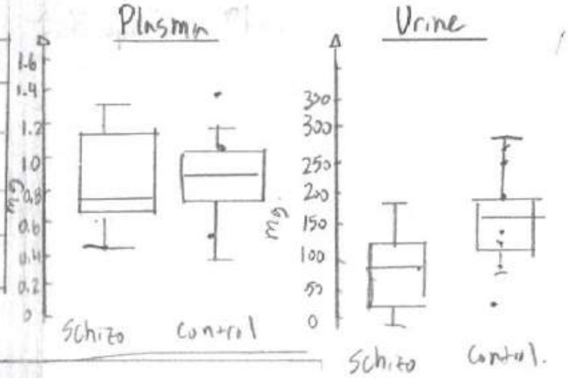
f)

	R	R'	U
Nonschizophrenic (Total)	344		
Schizophrenic (Total)	286	254	3.47
Nonschizophrenic (mg/kg)	312		
Schizophrenic (mg/kg)	233	207	2.34

If a hypothesis (H_{0A}: $\overline{\text{Nonschizophrenic (Total mg)}} = \overline{\text{Schizophrenic (Total mg)}}$) or ($H_{0B}$: $\overline{\text{Nonschizophrenic (mg/kg)}} = \overline{\text{Schizophrenic (mg/kg)}}$) were proposed, then they would be rejected at $\alpha = 0.05$.

g)

	E[X]	SD[X]	s_p	$s_{\bar{x}-\bar{y}}$	t	df	t(0.05)
Schizophrenic (Plasma)	0.89	0.26	0.24	0.09	0.14	28	2.04
Control (Plasma)	0.93	0.22					
Schizophrenic (Urine)	90.86	52.29	61.09	22.31	-3.56	28	2.04
Control (Urine)	160.31	65.54					



The hypothesis (H_0: $\overline{\text{Schizophrenic (Plasma)}} = \overline{\text{Control (Plasma)}}$) is accepted with the alternative (H_1: $\overline{\text{Schizophrenic (Plasma)}} \neq \overline{\text{Control (Plasma)}}$) because the t-statistic for 28 degrees of freedom is less than a standard curve at the significance level. While the urine samples show an argument or rejection of hypothesis (H_0: $\overline{\text{Schizophrenic (Urine)}} = \overline{\text{Control (Urine)}}$).

h) The normality is reasonable because the mean is within 10% of the median for each set of data.

i)

	R	R'	U
Schizo (Plasma)	161.5	303.5	7.92
Control (Plasma)	233.2		
Schizo (Urine)	96.0	369	10.63.
Control (Urine)	365.1		

Unlike part G), the Mann-Whitney shows reason to reject the hypothesis that the means are equivalent, which is argued against by the table of part G. Urine samples tested with Mann-Whitney list a similar outcome to part g.

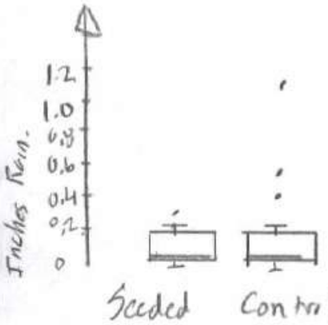
45. a)

Year	Experiment	E[X]	SD[X]	s_p	$s_{\bar{x}-\bar{y}}$	t	df	t(0.05)
1957	Seeded	0.07	0.07	0.08	0.03	0.31	30	2.04
	Unseeded	0.06	0.08					
1958	Seeded	0.06	0.08	0.09	0.03	0.55	30	2.04
	Unseeded	0.04	0.11					
1959	Seeded	0.02	0.04	0.16	0.06	-1.20	38	2.02
	Unseeded	0.09	0.22					
1960	Seeded	0.02	0.04	0.05	0.02	-1.13	30	2.04
	Unseeded	0.03	0.05					

Years	Experiment	E[X]	SD[X]	s_p	$s_{\bar{x}-\bar{y}}$	t	df	t(0.05)
57-'60	Seeded	0.04	0.06	0.11	0.02	-0.95	134	1.98
	Unseeded	0.06	0.14					

Tabulated data of each experiment individually demonstrate seeding did not influence the rainfall mid-day (H_0: $\mu_x = \mu_y$). The pooled years of seeding was tested to $t < t(0.05)$ and had an outcome of accepting the null hypothesis.

Pooled Years of Cloud Seeding Experiment:



b) The day on which seeding should be done should be chosen at random because daily parameter cycle throughout the month. Days are paired in the experiment because of similar conditions.

46.

Type	Experiment	E[X]	SD[X]	Sp	Sx-p	t	df	t(0.05)
I	Seeded	0.14	0.08	0.01	0.03	-0.38	33	2.03
I	Control	0.12	0.10					
II	Seeded	0.13	0.10	0.01	0.03	-0.96	33	2.03
II	Control	0.10	0.10					

Hypothesis:
$H_{0,\text{Type I}}$: Seeded Mean (Type I) = Control Mean (Type I)
$H_{1,\text{Type I}}$: Seeded Mean (Type I) ≠ Control Mean (Type I)
$H_{0,\text{Type II}}$: Seeded Mean (Type II) = Control Mean (Type II)
$H_{0,\text{Type II}}$: Seeded Mean (Type II) ≠ Control Mean (Type II)

The analysis accepts the null hypothesis for each type (I/II) at a significance level of $\alpha = 0.05$. ACN-cloud seeding project had no effect on outcomes of rain, and including analysis of cloud formations rather than years.

47. a)

Experiment	Variable	E[X]	SD[X]	Sp	Sx-p	t	df	t(0.05)
Seeded	Target	11.72	12.11	11.24	3.06	0.48	50	2.01
	Control	10.24	10.29					
Unseeded	Target	13.46	17.12	14.18	3.72	0.96	54	2.00
	Control	9.89	10.44					

$t < t(0.05)$ for each Experiment suggesting the null hypothesis cannot be rejected with the supplied information.

b) The square root transformation should play no effect on the data set or analysis because of the 1:1 relationship of input to output.

c) A control area provides a control variable to test against. Comparing seeded to unseeded limits experimental variability and requires a control variable to reference.

48. Parametric

Experiment	R	R'	U
Before	165	/	/
After	106	134	3.58

U > Z-statistic (0.05, Two-tailed)
3.58 > 1.96 : Reject $H_0: \mu_x = \mu_y$

Nonparametric:

Experiment	Mean (9/24)	S.d. (9/24)	Sp (9/24)	Sx-p (9/24)	t (9/24)	df	t
Before	9.27	4.14	4.26	1.74	2.69	22	2.07
After	4.58	2.64					

$t > t_{22}(0.05)$: Reject $H_0: \mu_x = \mu_y$
2.69 > 2.07

Chapter 12: Analysis of Variance:

1.

$$s_p^2 = \frac{(n-1)s_x^2+(m-1)s_Y^2}{m+n-2} = \frac{\sum_{i=1}^{2}(J-1)s_i^2}{2(J-1)} = \boxed{\frac{(J-1)s_1^2+(J-1)s_2^2}{2J-2}}$$

3. $I=2$ treatment groups; F-statistic

$$F = \frac{SS_B/(I-1)}{SS_W/[I(J-1)]} \text{ ; where } SS_B = J\sum_{i=1}^{I}\alpha_i^2+(I-1)\sigma^2;\ SS_W=\sum_{i=1}^{I}(J-1)s_i^2$$

t-statistic:

$$t = \frac{(\bar{X}-\bar{Y})-(\mu_x-\mu_y)}{s_p\sqrt{\frac{1}{n}+\frac{1}{m}}};$$

$$= \frac{J\sum_{i=1}^{2}\alpha_i^2+(1)\sigma^2}{(2-1)}\cdot\frac{2(J-1)}{\sum_{i=1}^{2}(J-1)s_i^2} = \frac{[J(\alpha_1^2+\alpha_2^2)+\sigma^2]\,2(J-1)}{(J-1)s_1^2+(J-1)s_2^2} = \frac{J(\alpha_1^2+\alpha_2^2)+\sigma^2}{s_p^2}$$

$$\sum\alpha_i^2=0$$

$$t^2 = \frac{(\bar{X}-\bar{Y})^2-2(\bar{X}-\bar{Y})(\mu_x-\mu_y)+(\mu_x+\mu_y)^2}{s_p^2\left(\frac{1}{n}+\frac{1}{m}\right)} = \boxed{\frac{\sigma^2}{s_p^2} \Longleftrightarrow = \frac{\sigma^2}{s_p^2}}$$

$$\boxed{t^2 = F} \ @\ I=2$$

4. Theorem A:

$$E(SS_W) = \sum_{i=1}^{I}\sum_{j=1}^{J}E(Y_{ij}-\bar{Y}_{i.})^2 = \sum_{i=1}^{I}\sum_{j=1}^{J}(\mu_i-\bar{\mu})^2+\frac{J-1}{J}\sigma^2$$

From section 12.2.1: $Y_{ij} = \mu+\alpha_i+\varepsilon_{ij}$

$$\boxed{= \sum_{i=1}^{I}\sum_{J}(\mu_i-\bar{\mu})^2+I(J-1)\sigma^2}$$

$$E(SS_B) = J\sum_{i=1}^{I}E(\bar{Y}_i-\bar{Y}_{..})^2 = J\sum_{i=1}^{I}(\mu_i-\bar{\mu})^2+\frac{(I-1)}{IJ}\sigma^2 = \boxed{J\sum_{i=1}^{I}\alpha_i^2+\frac{(I-1)\sigma^2}{IJ}}$$

Theorem B: $\mu=0;\ \sigma^2;\ SS_W/\sigma^2 = \chi^2_{I(J-1)}$

$$\boxed{= \frac{\sum_{i=1}^{I}(J-1)s_i^2}{\sigma^2} = (J-1)\sum^{I}\left(\frac{X_i-\mu}{\sigma}\right)^2 = \chi^2_{I(J-1)}}$$

$$SS_B/\sigma^2 = \frac{J\sum_{i=1}^{I}\alpha_i^2+(I-1)\sigma^2}{\sigma^2} = \frac{\sum\alpha_i^2}{Var(\bar{Y}_i)}+(I-1)\left(\frac{X-\mu}{\sigma}\right)^2 \ \boxed{\doteq \chi^2_{I-1}}$$

5. F-Statistic

$$F = \frac{SS_B/(I-1)}{SS_W/[I(J-1)]}$$

Null hypothesis:

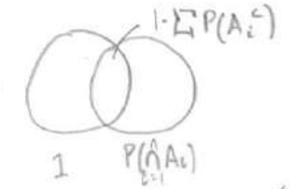
$H_0: \alpha_1=\alpha_2=\cdots\alpha_I=0$ means $F_1\approx 1$, while $H_1: \alpha_1\neq\alpha_2\neq\cdots\alpha_I\neq 0$ variation of groups and within groups exist.

Assuming normal distribution, likelihood (Λ): $\dfrac{U/m}{V/n} = \dfrac{Z^2/m}{\sum V_i/n} = \dfrac{N(X|\mu,\sigma^2)^2/m}{\sum N(X_i|\mu,\sigma^2)^2/n}$

if $m=I(J-1),\ n=(I-1)$; $= \boxed{\dfrac{\chi^2_{I(J-1)}/I(J-1)}{\chi^2_{I-1}/(I-1)}}$

6. Bonferroni Inequality:

$P(\bigcap_{i=1}^{n} A_i) \geq 1 - \sum P(A_i^c)$;



; A_i is the input to the probability space. A_i^c is the complement input to the probability space.

7. Show Theorem B of Section 12.2.1: $SS_B/\sigma^2 \sim \chi^2_{I-1}$

$$SS_B/\sigma^2 = \frac{J\sum_{i=1}^{I} x_i^2 + (I-1)\sigma^2}{\sigma^2} = \frac{\sum x_i^2}{Var(\bar{Y}_i)} + (I-1) = \frac{\sum x_i^2}{var(\bar{Y}_i)} + (I$$

$$= (I-1)\sum_{i=1}^{I}\left(\frac{X-\mu}{\sigma}\right)^2 = \boxed{\chi^2_{I-1}}$$

8.

9. The columns of a t Distribution relate to studentized ranges because of similar derivation Tukey's method on the probability of random variable led from a max of mean differences per pooled standard deviation per degrees of freedom for a normal distribution to $\frac{sp}{\sqrt{J}}$; nevertheless, t distribution was closely derived from mean differences, but per standard deviation per degrees freedom of <u>two</u> normal distributions. Thus t distribution columns have a $\sqrt{2}$ for two distributions.

10. $I=7$, $J=10$. Tukeys Method: $(\bar{Y}_{i1}-\bar{Y}_{i2}) \pm q_{7,63}(\alpha)\frac{s_p}{\sqrt{J}} = 1 \pm \frac{q_{7,63}(\alpha)}{t_{63}(\alpha/n)}\sqrt{\frac{5}{J}} = 1 \pm \frac{4.31}{3.16}\sqrt{\frac{5}{10}}$

Bonferroni Method: $(\bar{Y}_{i1}-\bar{Y}_{i2}) \pm s_p \frac{t_{63}(\alpha/n)}{\sqrt{5}}$ $\boxed{= 1 \pm 0.96}$

Regular t-statistic: $(\bar{Y}_{i1}-\bar{Y}_{i2}) \pm t_{63}(0.025)s_p\sqrt{\frac{2}{J}} = 1 \pm 0.053$

11.

Factors (columns), Levels (rows):

I	A	B	C	D
II	A	B	C	D
III	A	B	C	D

Diagonal x

Factors (columns), Levels (rows):

$\bar{X}_1$	$\bar{X}_2$	$\bar{X}_3$	$\bar{X}_4$
$\bar{X}_2$	$\bar{X}_3$	$\bar{X}_4$	$\bar{X}_5$
$\bar{X}_3$	$\bar{X}_4$	$\bar{X}_5$	$\bar{X}_6$

Mean space:
$\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4, \bar{X}_5$

12.

49. $n = 126$ police officers ; $\bar{x}$ = Blood concentration of Lead ($\mu g/dL$) = $29.2 \mu g/dL$
$\sigma_{\bar{x}} = 7.5 \mu g/dL$.

$m = 50$ police officers ; $\bar{y} = 18.2 \mu g/dL$; $\sigma_{\bar{y}} = 5.8 \mu g/dL$.

$H_0: \bar{x} = \bar{y}$; $H_1: \bar{x} \neq \bar{y}$
$s_p = 7.1 \mu g/dL$; $s_{\bar{x}-\bar{y}} = 1.2 \mu g/dL$
$df: 174$; $t = 9.3 > t_{174}(1.05)$
Rejection of null hypothesis (H_0)

The officers from Cairo do not test to be of the same sample set as Abbasia

$CI_{Cairo-Abbasia} = \{8.69 \mu g/dL, 13.3 \mu g/dL\}$

50. a. $CI_{\overline{Male-Female\ Temp}} = \{-0.53°F, -0.04°F\}$
The use of normal approximation is reasonable because the mean ≃ median, and indicates low skew.

b. $CI_{\overline{Male-Female\ Heart\ Rate}} = \{-2.7 \text{ bpm}, 1.2 \text{ bpm}\}$
Application of a normal approximation better fits the male heart rate because of little skew, but the females mean heart rate (74 bpm) is low to the median (76 bpm) and may indicate the need for a parametric test.

c. Parametric:

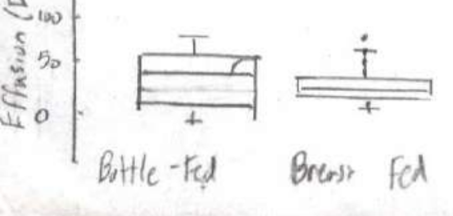
Experiment	R	R'	U
Temperature (Male)	3792	4733	12.2
Temperature (Female)	4733		
Heart Rate (Male)	4473	4442	10.8
Heart Rate (Female)	4423		

Nonparametric:

Experiment	E[x]	SD[x]	s_p	$s_{\bar{x}-\bar{y}}$	df	t
Temperature (Males)	98.1	0.69	0.71	0.13	128	-2.30
Temperature (Females)	98.4	0.77				
Heart Rate (Males)	73	5.9	7.02	1.23	128	-0.63
Heart Rate (Females)	74	8.00				

The nonparametric test is showing rejection of the null hypothesis for male and female mean temperatures, while acceptance of the alternative hypothesis for heart rate means. Although, since the sample size is large (>30), the parametric test should be established as the leading indicator, and demonstrates corresponding p-values greater than a significance of $\alpha = 0.05$

51. a.



b) A parametric test seems applicable because of the large skew between means and median values.

H_0: $\overline{\text{Bottle Fed (days)}} = \overline{\text{Breast Fed (Days)}}$
H_1: $\overline{\text{Bottle Fed (days)}} \neq \overline{\text{Breast Fed (Days)}}$

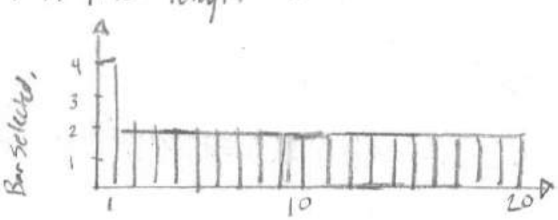
Experiment	R	R'	U
Bottle Fed	515	660	7.63
Breast Fed	660		

The model suggest rejection of the null hypothesis in favor of the alternative that the Breast Fed babies do not have prolonged Effusion

52. a) "Recover faster" does not indicate which disease.
b) Insufficient evidence to conclude the wife does or does not smoke.
c) How does breakfast relate to industrial accidents?
d) Did the student scores compare to the majority or minority school?
e) Would a questionnaire better be prepared if other days were tested?
f) A comparator would help determine if beer or alcohol should be reported.
g) Did the 15-year study have a controlled variable?
h) What about the other 35% of married couples?
i) Were the elderly of the same age group?

53. Both lettuce leaves and unlit cigarettes represent placebos to the experimental design. Lettuce leaves contain no amount of nicotine, while unlit cigarettes do, but are not inhaled and solely behaviorly considered.

54. The length of bar was not randomized, and yet the error would be detectable if not randomized over time because of selection bias being time independent.



	A	B	C
I	$\bar{X}_1$	$\bar{X}_2$	$\bar{X}_3$
II	$\bar{X}_2$	$\bar{X}_3$	$\bar{X}_4$

Sample Space
$\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4$

$\boxed{\text{Variance} = \frac{12}{N(N+1)} = K}$

13. Kruskal-Wallis test:

Mann-Whitney:
$\boxed{\text{Average Rank } \bar{R} = \frac{(N+1)}{2}}$

R_{ij} = the rank of Y_{ij} in the combined sample

$\bar{R}_i = \frac{1}{J_i}\sum_{j=1}^{J_i} R_{ij}$ = Average rank in i^{th} group

$\bar{R}_{..} = \frac{1}{N}\sum_{i=1}^{I}\sum_{j=1}^{J} R_{ij} = \frac{N+1}{2}$ = Average total rank

$SS_B = \sum_{i=1}^{I} J_i(\bar{R}_i - \bar{R}_{..})^2$ = Variance between ranks

$K = \frac{12}{N(N+1)} SS_B$ = Chi-squared Distribution with $(I-1)$ degrees of freedom.

$= \frac{12}{N(N+1)}\left(\sum_{i=1}^{I} J_i \bar{R}_i^2\right) - 3(N+1)$

14. Friedman's Test: $SS_A = J\sum_{i=1}^{I}(\bar{R}_i - \bar{R}_{..})^2$

$$Q = \frac{12J}{I(I+1)}\sum(\bar{R}_i - \bar{R}_{..})^2$$

Sign Test: $p = P(X > Y)$; $H_0: p = 0.50$

m pairs: $\{x_1, y_1\} \cdots \{x_n, y_n\}$

W = # pairs $y_i - x_i > 0 \sim \text{bin}(m, p)$

The tested probability of a signed rank test $(p = P(X>Y))$ is Kin to Friedman's probability $(P(X^2_{K-1} \geq Q))$ when two categories are presented $(k=2)$

15.

$$K = \frac{12}{N(N+1)}\sum_{i=1}^{I} J_i(\bar{R}_i - \bar{R}_{..})^2$$

$$= \frac{12}{N(N+1)}\sum_{i=1}^{I} J_i(\bar{R}_i^2 - 2\bar{R}_i\bar{R}_{..} + \bar{R}_{..}^2)$$

$$= \frac{12}{N(N+1)}\left[\sum_{i=1}^{I} J_i\bar{R}_i^2 - 2\sum_{i=1}^{I} J_i \frac{(N+1)}{2}\frac{1}{J_i}\sum_{j} R_{ij} + \sum_{i=1}^{I} J_i \frac{(N+1)^2}{2\cdot 2}\right]$$

$$= \frac{12}{N(N+1)}\sum_{i=1}^{I} J_i\bar{R}_i^2 - \frac{12\cdot\cancel{2}}{\cancel{N}(N+\cancel{1})}\cdot\frac{\cancel{N}(N+1)}{2}\frac{(N+\cancel{1})}{\cancel{2}} + \frac{12}{N(N+\cancel{1})}\frac{(N+1)^{\cancel{2}}}{4} = \boxed{\frac{12}{N(N+1)}\sum_{i=1}^{I} J_i \bar{R}_i^2 - 3(N+1)}$$

16. Prove $SS_{TOT} = SS_A + SS_B + SS_{AB} + SS_E = JK\sum^{I}(\bar{Y}_{i..} - \bar{Y}_{...})^2 + IK\sum^{J}(\bar{Y}_{.j.} - \bar{Y}_{...})^2 + K\sum^{I}\sum^{J}(\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...})^2 + \sum^{I}\sum^{J}\sum^{K}(Y_{ijk} - \bar{Y}_{ij})^2$

$$= \sum^{J}\sum^{K}\sum^{I}(\bar{Y}_{i..} - \bar{Y}_{...})^2 + \sum^{I}\sum^{J}\sum^{K}(\bar{Y}_{.j} - \bar{Y}_{...})^2 + \sum^{K}\sum^{I}\sum^{J}(\bar{Y}_{ij} - \bar{Y}_{i..} - Y_{.j.} - \bar{Y}_{...})^2 + \sum^{I}\sum^{J}\sum^{K}(Y_{ijk} - \bar{Y}_{ij})^2$$

$$= \sum^{I}\sum^{J}\sum^{K}\left[(\bar{Y}_{i..} - \bar{Y}_{...})^2 + (\bar{Y}_{.j} - \bar{Y}_{...})^2 + (\bar{Y}_{ij} - \bar{Y}_{i..} - Y_{.j.} - \bar{Y}_{...})^2 + (Y_{ijk} - \bar{Y}_{.j})^2\right]$$

$$\boxed{= \sum^{I}\sum^{J}\sum^{K}\left[(Y_{ijk} - \bar{Y}_{ij})^2\right]}$$

17. Find mle's of $\alpha_i, \beta_j, \delta_{ij}$, and μ.

$$l = -\frac{IJK}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum^{I}\sum^{J}\sum^{K}(Y_{ijk} - \mu - \alpha_i - \beta_j - \delta_{ij})^2$$

$$\frac{dl}{d\alpha_i} = \frac{\alpha_i}{\sigma^2}\sum^{I}\sum^{J}\sum^{K}(Y_{ijk} - \mu - \alpha_i - \beta_j - \delta_{ij}) = 0$$

$$\boxed{\hat{\alpha}_i = \bar{Y}_{i..} - \bar{Y}_{...}}$$

$$\frac{dl}{d\beta} = \frac{\beta_j}{\sigma^2}\sum^{I}\sum^{J}\sum^{K}(Y_{ijk} - \mu - \alpha_i - \beta_j - \delta_{ij}) = 0$$

$$\boxed{\beta_j = \bar{Y}_{.j.} - \bar{Y}_{...}}$$

$$\frac{dl}{d\delta_{ij}} = \frac{\delta_{ij}}{\sigma^2}\sum^{I}\sum^{J}\sum^{K}(Y_{ijk} - \mu - \alpha_i - \beta_j - \delta_{ij})$$

$$\boxed{\delta_{ij} = \bar{Y}_{ij} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...}}$$

$$\frac{dl}{d\mu} = \frac{\mu}{2\sigma^2}\sum^{I}\sum^{J}\sum^{K}(Y_{ijk} - \mu - \alpha_i - \beta_j - \delta_{ij}) = 0$$

$$\boxed{\mu = \bar{Y}_{...}}$$

18.

19. One-Way Layout

Y_{ij} = j^{th} observation of i^{th} treatment

$= \mu + \alpha_i + \varepsilon_{ij}$ ← random error of j^{th} treatment

overall mean; Differential Effect of i^{th} treatment

$\sum^{I} \alpha_i = 0$: Normalized

$\sum^{I}\sum^{J}(Y_{ij}-\bar{Y}_{..})^2 = \sum^{I}\sum^{J}(Y_{ij}-\bar{Y}_{i.})^2 + J\sum\sum(\bar{Y}_{i.}-\bar{Y}_{..})^2$

Where $\bar{Y}_i = \frac{1}{J}\sum^{J} Y_{ij}$: Average observation of i^{th} treatment

$\bar{Y}_{..} = \frac{1}{IJ}\sum^{I}\sum^{J} Y_{ij}$: Overall Average

$SS_{TOT} = SS_W + SS_B = \sum^{I}(J-1)s_i^2 + \sum^{I}\alpha_i^2 + \sigma^2$

Two-Way Layout

$\bar{Y}_{..} = \frac{1}{IJ}\sum^{I}\sum^{J} Y_{ij}$: Grand Average

$\bar{Y}_{i.}$: Average over rows

$\bar{Y}_{.j}$: Average over columns

$\hat{\alpha}_i = \bar{Y}_{i.} - \bar{Y}_{..}$: Differential row Average

$\hat{\beta}_i = \bar{Y}_{.j} - \bar{Y}_{..}$: Differential column Average.

$\hat{\delta}_{ij}$ = Interaction of rows × columns.

$Y_{ij}: \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_i + \hat{\delta}_{ij}$: Additive Model

$Y_{ijk} = \mu + \alpha_i + \beta_j + \delta_{ij} + \varepsilon_{ijk}$: Balanced.

$F(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\varepsilon_i)^2/2\pi^2$

$SS_{TOT} = SS_A + SS_B + SS_{AB} + SS_E$

Three-Way Layout

$Y_{ijkl} = \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j + \hat{\gamma}_k + \hat{\delta}_{ijkl} + \dots$

$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\varepsilon_{ijkl}^2/2\sigma^2)$

$l = -\frac{1}{2}\log 2\pi\sigma^2 - \varepsilon_i^2/2\sigma^2$

$\frac{dl}{d\hat{\mu}} = Y_{ijkl} - \hat{\alpha}_i - \hat{\beta}_j - \hat{\gamma}_k - \hat{\delta}_{ijkl} - \mu = 0$

$\boxed{\hat{\mu} = \bar{Y}_{...}}$

$\frac{dl}{d\alpha} = Y_{ijkl} - \alpha_i - \beta_j - \gamma_k - \delta_{ijkl} - \mu = 0$

$\boxed{\hat{\alpha}_i = Y_{i...} - \bar{Y}_{...}}$

$\frac{dl}{d\beta} = Y_{ijkl} - \mu - \alpha_i - \beta_j - \delta_{ijkl} = 0$

$\boxed{\hat{\beta}_j = \bar{Y}_{.j..} - Y_{....}}$

$\frac{dl}{d\gamma} = Y_{ijkl} - \mu - \alpha_i - \beta_i - \delta_{ijkl} = 0$

$\boxed{\gamma = \bar{Y}_{..k.} - \bar{Y}_{...}}$

$\frac{dl}{d\delta} = Y_{ijkl} - \mu - \alpha_i - \beta_i - \delta_{ijkl} = 0$

$\boxed{\hat{\delta}_{ijk} = \bar{Y}_{ijk.} - \bar{Y}_{i...} - \bar{Y}_{.j..} - \bar{Y}_{..k.} + \bar{Y}_{....}}$

The parametrization of a balanced three-way layout includes two-factor and three-factor interactions. A two-factor interaction $(\alpha_i, \beta_i, \gamma_i)$ represents the mean of a row, column, or depth of the matrix. While the three-factor interaction $(\hat{\delta}_{ijkl})$ is interpreted as the the residual within the cell.

20. $Y_{ij} = \mu + A_i + \varepsilon_{ij}$; A_i = Random ; $E(A_i) = 0$; $Var(A_i) = \sigma_A^2$; ε_{ij} independent of A_i

$E(\varepsilon_{ij}) = 0$; $Var(\varepsilon_{ij}) = \sigma_\varepsilon^2$

a) Show $E(MS_W) = \sigma_\varepsilon^2$; $E(MS_B) = \sigma_\varepsilon^2 + J\sigma_A^2$; $E(MS_W) = \frac{E(SS_{AB})}{(I-1)(J-1)} = \sigma^2 + \frac{1}{(I-1)(J-1)}\sum^{I}\sum^{J}\delta_{ij}^2 \stackrel{?}{=} \boxed{\sigma_\varepsilon^2}$ $= 0, \sum\sum\delta_{ij} = 0$

For dataset: Dye

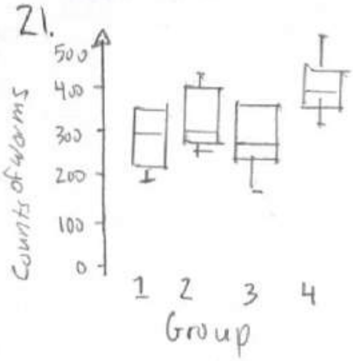
Source	dF	SS	MS	F
Samples	5	111.74	22.35	5.65
Error	5	113.65	22.73	
Total	10	225.39		

$E(MS_B) = \frac{E(SS_B)}{(J-1)} = \sigma_\varepsilon^2 + \frac{I}{(J-1)}\sum^{J}\beta_j^2 = \boxed{\sigma_\varepsilon^2 + J\sigma_A^2}$

$F = \frac{SS_B/(J-1)}{SS_W/[I(J-1)]} = 5.90.$

b) The parameters of the model are estimated by the F-statistic because the stat incorporates mean square error of multiple categories. The hypothesis $(H_0 = \alpha_1 = \alpha_2 = \dots = \alpha_I = 0)$ fails at a p-value of 0.05.

21.



Parametric Technique

Source	dF	SS	MS	F
Groups	3	27234	9078	2.27
Larvae	16	63953.	3997	
Total	19	91187		

$H_0: \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$

$H_1: \alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \alpha_4 \neq 0$

p-value > 0.10 $\quad F_{3,16} = 2.24$

Rejection of null hypothesis.

Nonparametric Technique:

Group	1	2	3	4
R	42	53	36	79

$H = 6.2,\ df = 3,\ \chi^2_{3,\,0.90} = 7.78$

Rejection of null hypothesis at p-value > 0.10.

Both parametric and nonparametric tests demonstrate unequal amounts of larvae per group.

22. Manufacturer #1

Source	df	SS	MS	F
Labs	6	0.125	0.021	5.66
Amount	63	0.231	0.004	
total	69	0.356		

$H_0: \alpha_1 = \alpha_{\dots} = \alpha_7 = 0$

$H_1: \alpha_1 \neq \alpha_{\dots} \neq \alpha_7 \neq 0$

$F_{6,60} = 3.12$

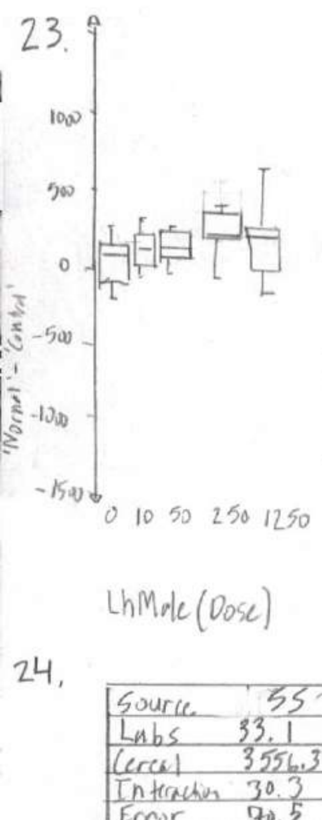
Rejection of the null hypothesis that lab data contains similar means and variances.

Manufacturer #2

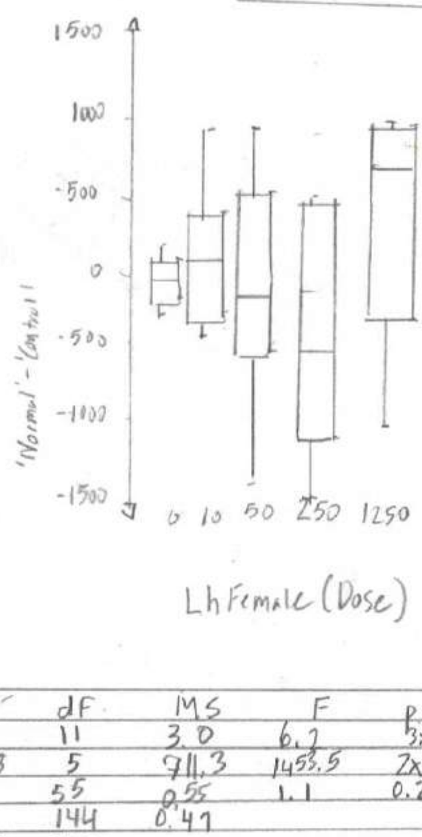
Source	df	SS	MS	F
Labs	6	0.153	0.026	11.90
Amount	63	0.135	0.002	
Total	69	0.288		

Also, rejection of the null hypothesis with a p-value > 0.01.

23.



LhMale (Dose)



LhFemale (Dose)

$_{23}\chi^2(0.95) = 41.337,\ \chi^2 = 1040$ Females 'Normal' - 'Constant'

$H_0: \bar{X} = \mu_0$; $H_1: \bar{X} \neq \mu_0$; Rejection of null hypothesis for females. There is indication of threshold dosage for luteinizing hormone.

Source	dF	SS	MS	F
Dosage	4	6639701	1659925	21.4
Light	1	9506947	9506947	122.4
Interaction	4	4641765	1160441	14.9
Error	50	3883247	77669	
Total	59	24671660		

The null hypothesis is rejected that the effect from luteinizing hormone generates a balanced result with increasing dosage for both men and women ($H_A: \alpha_i = \alpha_i$)

24.

Source	SS	dF	MS	F	p-value
Labs	33.1	11	3.0	6.2	3×10^{-8}
Cereal	3556.3	5	711.3	1453.5	2×10^{-121}
Interaction	30.3	55	0.55	1.1	0.29
Error	70.5	144	0.49		
Total	3690.2	215			

The first successful Two-Way ANOVA <u>by hand</u> presents inconsistent data per lab and bran vs. cereal because of a p-value < 0.05. The error is consistent between data sets at 0.29.

25. Null Hypothesis: $H_0: \alpha_1 = \cdots = \alpha_i = 0$

Alternative Hypothesis $H_1: \alpha_1 \neq \cdots \neq \alpha_i \neq 0$

Source	SS	df	MS	F	p-value	F crit
Between	201.8	9	22.4	2.84	0.005	1.99
Within	710.4	90	7.9			
Total	912.2	99				

The data from magnesium samples, at glance, presents large chemical content in column #1. An One-way Anova describes error between the groups and within at a p-value of 0.005, rejecting the null hypothesis of similar chemical content across samples and portions of the bar.

26. Parametric

Source	SS	df	MS	F	p-value	F-crit
Dogs	0.52	9	0.057	0.349	0.95	2.39
Chemical	3.29	20	0.16			
Total	3.81	29				

Nonparametric

Dog	1	2	3	4	5	6	7	8	9	10
R	37	60.5	67.5	38.5	45	43	48.5	55.5	44.5	25

$H = 0.87 > \chi^2_{9\ 0.005} = 1.73$

The F-statistic concludes no differences per dog or chemical exposed, with a p-value > 0.95. A nonparametric Kruskill-Wallace test confirmed no differences as compared to a 9-degree chi-squared distribution. Blood plasma did not change as compared to exposure.

27. Bonferroni Method:

Comparison	Mean Diff.	CI
C7-AJ	177.3	69.4
C7-F2	101.4	69.4
AJ-F2	-75.9	69.4

The Bonferroni method confirms each group is drastically differenc as represented by their means; While, confidence intervals show errors of nearly double. the comparator.

28. Parametric:

Source	SS	df	MS	F	p-value	F crit
Types (Between)	446.6	2	223.3	0.497	0.61	3.59
Types (Within)	7632.3	17	449.0			
Total	8078.9	19				

Nonparametric:

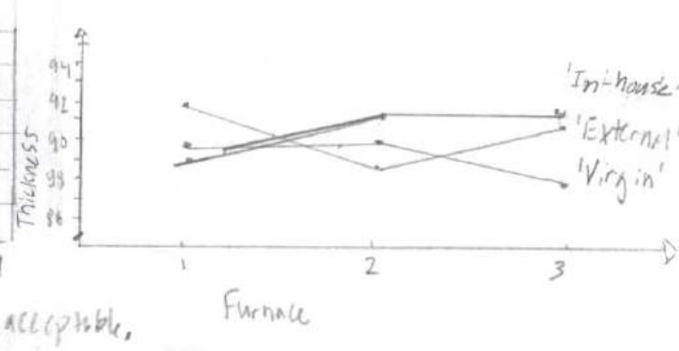
	Type I	Type II	Type III
R	76.5	78	55.5
n	9	6	5
H		2.15	

$\chi^2_{18\ 0.95} = 28.87$

Both the parametric and nonparametric analysis confirmed the hypothesis ($H_0: \Pi_1 = \Pi_2 = \Pi_3$).

29.

Source	SS	df	MS	F	p-value	F-crit
Furnace	4.1	2	2.05	1.45	0.26	3.55
Wafer Type	5.9	2	2.94	2.07	0.16	3.55
Furnace and Wafer	21.3	4	5.34	3.76	0.02	2.93
Residual	25.6	18	1.42			
Total	56.9	26				



Noticeable differences exist between the furnace and wafer thickness, but the type and furnace individually are acceptable.

30. Tukey's Method:

$q_{6,40}(0.05) = 4.23$

$Sp = 216.07$

$\sqrt{J} = 3.16$

Tukey $CI = 289.02$

Plot	Average
'3'	1268
'2'	1272.6
'5'	1306.4
'4'	1316.4
'6'	1339.4
'1'	1768.5

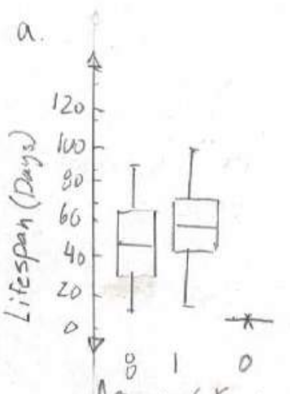
$Max(\bar{Y}_i - \bar{Y}_c) =$ Plot 3 - Plot 1

$= 500.5$

Tukey's method's comparision of maximal average differences is not Significant.

31. An additive model would provide a good fit to the dataset for secondary indication maximum windspeeds are variable

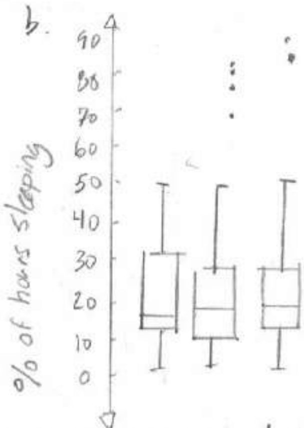
32. a.



Amount of Females	Mean	Variance
8	63.6	259.8
1	60.8	240.73
0	51.0	323.6

The lifespan without females was drastically for male fruitflies.

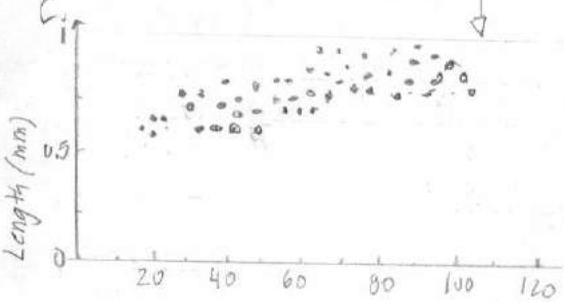
b.



Amount of Females	Mean	Variance
8	21.6	149.0
1	24.9	297.7
0	23.0	249.0

The hours slept were equivalent with or without a female fruitfly present.

c.



Lifespan of a fruitfly is correlated to the length of the thorax, with a positive relationship.

d. Thorax

Source	df	MS	F	p-value	F crit
Between	4	0.006	1.15	0.33	2.45
Within	120	0.006			
Total	124				

Lifespan

Source	df	MS	F	p-value	F crit
Between	4	2985	13.6	3.6×10^{9}	2.45
Within	120	219			
Total	124				

Sleep

Source	df	MS	F	p-value	F-crit
Between	4	121.7	0.47	0.75	2.45
Within	120	256.5			
Total	124				

Bonferroni Method Lifespan : CI = 0.02

Comparison	9/10	9/11	9/80	9/81	10/11	10/80	10/81	11/80	11/81	80/81
Mean Difference	-1.74	6.8	0.2	24.9	8.0	1.4	26.1	-6.6	18.0	24.6

Thorax : CI = 3.21

Comparison	9/10	9/11	9/80	9/81	10/11	10/80	10/81	11/80	11/81	80/81
Mean Difference	0.01	0.00	0.00	0.04	-0.01	-0.01	0.03	0.01	0.04	0.04

Comparison	9/10	9/11	9/80	9/81	10/11	10/80	10/81	11/80	11/81	80/81
Mean Difference	-2.5	-4.2	-3.6	0.8	-1.7	-1.1	3.3	0.6	5	4.4

Sleep : CI = 0.02.

Tukey Method:

Category	$q_{25,100}$	S_p	$\sqrt{J}$	CI	$max(\bar{Y}_{i.} - \bar{Y}_{i.})$
Thorax	5.32	3.05	2.3	7.2	0.04
Lifespan	5.32	0.02	2.3	0.04	26.08
Sleep	5.32	0.01	2.3	0.03	4.4

The lifespan is concluded as significant.

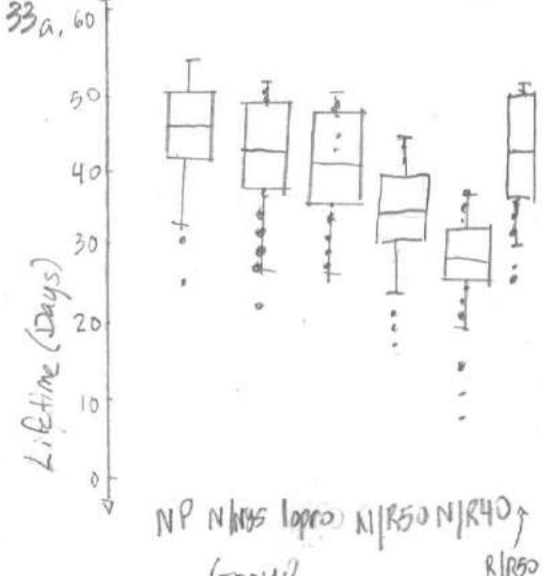
e. Kruskal-Wallis test:

Group	H	df	$\chi^2_{0.95}$
Lifespan	37.9	5	11.07
Thorax	5.6	5	11.07
Sleep	0.3	5	11.07

With another nonparametric analysis, lifespan of the fruitfly is significant, and an outlier to the information collected.

f. The dataset of one, eight (pregnant, or virgin) female provides no change on sleep.

33a.



Kruskal-Wallis Test:

$H = 159.01$; $\chi^2_{0.95} = 11$ $\quad H_0: \pi_1 = \pi_2 = \pi_3 = \pi_4 = \pi_5 = \pi_6$

$df = 5$ $\quad H_1: \pi_1 \neq \pi_2 \neq \pi_3 \neq \pi_4 \neq \pi_5 \neq \pi_6$

The Kruskall-Wallis test rejected the null hypothesis at a p-value = 0.05, and accepted diet effects lifespan.

Preweaning (R/R50) does not lower lifespan of a mouse when the life is of restricted calories.

b. The reduction of protein does effect lifespan.

c. Reduction to 40 kcal per week does lower lifespan.

34. a.

Source	SS	df	MS	F	p-value	F-crit
Treatment	91.9	3	30.6	11.8	5.92×10^{-5}	3.0
Poison	105.1	5	21.0	8.1	0.0001	2.6
Treatment x Poison	39.2	15	2.6	1.01	0.48	2.1
Error	62.1	24	2.6			
Total	298.4	47				

The treatment and poison category show significance, but the interaction of the treatment x poison does not accept alternatives.

b.

Source	SS	df	MS	F	p-value	Fcrit
Treatment	0.20	3	0.07	23.6	2.41×10^{-7}	3.0
Poison	0.40	5	0.07	24.9	3.99×10^{-9}	2.6
Treatment x Poison	0.02	15	0.00	0.5	0.39	2.1
Error	0.07	24				
Total	0.65	47				

Bob and Cox (1964) analysis of reciprical data conclude similar outcomes to part a. Treatments x Poisons is not significant, and individual treatments or poisons are significant.

35. The interaction of dosage and serum were not important, indicating the culture produced similar amounts of estrogen relative to primary categories. Although, estrogen production did have significant differences for type of serum, in addition to increased dosage.

Source	SS	df	MS	F	p-value	Fcrit
Dosage	2.79×10^{9}	11	2.54×10^{8}	23.85	0.00	2.06
Treatment	6.30×10^{8}	2	3.15×10^{8}	29.60	0.00	3.26
Dosage x Treatment	3.98×10^{8}	22	1.81×10^{7}	1.70	0.07	1.85
Error	3.83×10^{8}	36	1.1×10^{7}			
Total	4.21×10^{9}	71				

e. $H_0: \beta_{1,Male} = \beta_{2,Female}$

$H_1: \beta_{1,Male} \neq \beta_{2,Female}$

$t = \dfrac{\beta_{1,Male} - \beta_{2,Female}}{S_{\beta_1 - \beta_1}}$; $S_{\beta_M - \beta_F} = S_p \sqrt{\dfrac{1}{n} + \dfrac{1}{m}} = \sqrt{\dfrac{(n-1)S_x^2 + (m-1)S_Y^2}{m+n-2}\left[\dfrac{1}{n} + \dfrac{1}{m}\right]}$

$= \sqrt{\dfrac{(65-1)1.039^2 + (64-1)1.316^2}{65+64-2}\left[\dfrac{1}{65} + \dfrac{1}{64}\right]}$

$\boxed{= 0.209}$

$= \dfrac{3.13 - 1.64}{0.209} = 7.13$

Degrees of Freedom $= \boxed{127}$

$t_{120}(0.95) < t$; An alternative hypothesis is acceptable; including: male and female temperature vs heart rate are not relatable.

F. $H_0: \beta_{0,Male} = \beta_{0,Female}$

$H_1: \beta_{0,Male} = \beta_{0,Female}$.

$t = \dfrac{|\beta_{0,Male} - \beta_{0,Female}|}{S_{\beta_M - \beta_F}}$; $S_{\beta_M - \beta_F} = \sqrt{\dfrac{(65-1)101.9^2 + (64-1)129.52}{65+64-2}\left[\dfrac{1}{65} + \dfrac{1}{64}\right]}$

$\boxed{= 20.4}$

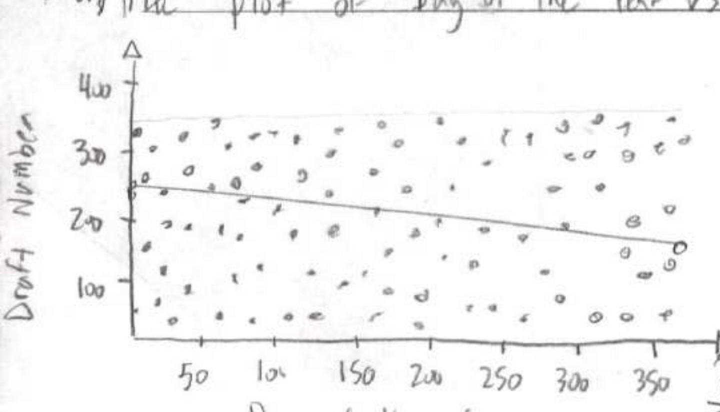
$= \dfrac{|-233+87|}{20.4} = \boxed{7.16}$

$t_{120}(0.95) < t$; An alternative hypothesis is acceptable.

53. a) Old faithfuls interval between explosion is related to duration by a linear fit having an R^2 of 0.73

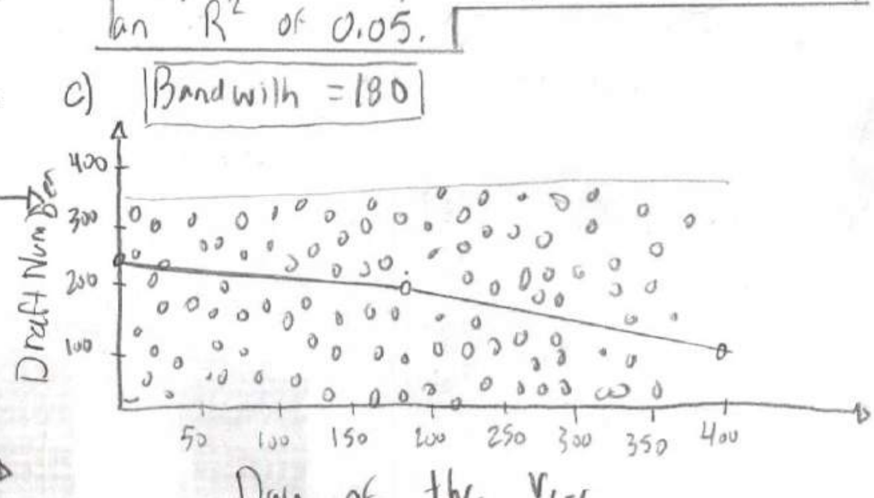
b) If the duration is 2 minutes, to predict a time until next eruption would require a function. Linear regression of $Y = 10.74 \cdot \text{Duration} + 33.83$ minutes is best fit and describes an interval till next eruption to be 55 min.

54. a) The plot of 'Day of the Year' vs 'Draft Number' shows lack of trend

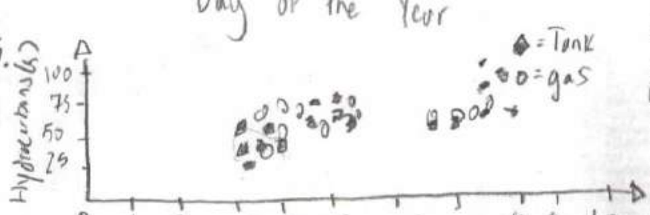


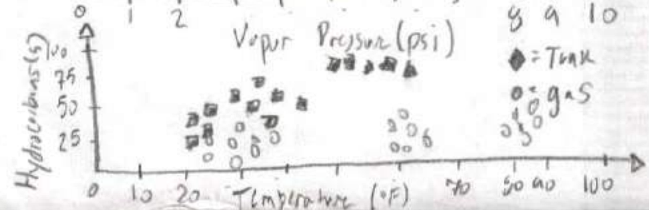
b) The linear regression model $Y = -0.22X + 225$ is plotted in part a), but has an R^2 of 0.05.

c) Bandwith = 180



55. a)





The groupings (vapor pressure, temperature) are associated to hydrocarbon production by visual observations, especially vapor pressure.

b) The best fits are tank vapor pressure vs. hydrocarbon weight and gas vapor pressure vs hydrocarbon weight.

c) Root mean square Prediction: $RMSPE = \sqrt{\frac{1}{40}\sum_{i=1}^{40}(Y_i - \hat{Y}_i)^2}$

Model: Gas Vapor Pressure vs. Hydrocarbon Weight

Coefficient	Estimate
β_0	6.62
β_1	5.56

$RMSPE = \sqrt{\frac{1}{40}\sum^{40}(Y_i - 6.62 - 5.56X_i)}$

$\boxed{= 0.667}$

Model: Tank Vapor Pressure vs Hydrocarbon weight

Coefficient	Estimate
β_0	3.59
β_1	6.34

$RMSPE = \sqrt{\frac{1}{40}\sum^{40}(Y_i - 3.59 - 6.34X_i)}$

$\boxed{= 0.634}$

56. a) After examination of different combinations of data, insolation fits the upper ranges of oxidation, while wind speed, temperature, and humidity better fit oxidation when less than 30 ppm. Oxidation level maximums are modeled with insolation to a regression of $Y = 0.23X - 1.43$, but do have a squared multiple correlation coefficient (R^2) of 0.22.

b) Serial correlation (or experimental drift) appears in time-resolved data as increasing (or decreasing) residual error.

ABO System

Severity	O	A	AB	B
Moderate-Advanced	7	5	3	13
Minimal	27	32	8	18
Not Present	55	50	7	24

$df = 6, X^2 = 20.20, X^2(0.995) = 18.55$

O	A	AB	B
1.13	2.64	0.38	11.33
0.31	0.81	0.98	1.07
0.37	0.05	1.03	0.10

MN System

Severity	MM	MN	NN
Moderate Advanced	21	6	1
Minimal	54	27	5
Not Present	74	51	11

$df = 4, X^2 = 42.05, X^2(0.995) = 14.86$

MM	MN	NN
1.11	1.23	0.14
0.02	0.23	0.18
0.61	38.15	0.33

$H_0: \pi_{ij} = \pi_{i.} \pi_{.j}$ (Independent Datasets) $H_1 = \pi_{ij} \neq \pi_{i.} \cdot \pi_{.j}$ (Dependent)

Each of the datasets test $X^2 > X^2_{0.995}$, which implies disease severity is dependent on blood type.

4.

	Male	Female	
Mentioned	86	55	141
Not Mentioned	283	360	643
	369	415	784

$df = 1 ; X^2 = 13.38 ; X^2(0.995) = 7.88$

5.81	5.17
1.27	1.13

$H_0: \pi_{i1} = \pi_{i2}$; $H_1: \pi_{i1} \neq \pi_{i2}$

The null hypothesis is rejected at a p-value < 0.005, that implies males and females do attribute differently.

5.

Ethnic Origin	Yes	No	
Italian	78	47	125
Northern European	56	29	85
Other European	43	29	72
English	53	32	85
Irish	43	30	73
French Canadian	36	22	58
French	42	23	65
Portuguese	29	7	36
	380	219	599

0.02	0.04
0.03	0.01
0.02	0.01
0.02	0.03
0.02	0.03
0.02	0.03
0.01	0.03
0.02	0.03

$df = 7, X^2 = 0.42$

${}_{7}X^2_{0.005} = 0.989$

$H_0 = \pi_{i1} = \pi_{i2}$

$H_1 = \pi_{i1} \neq \pi_{i2}$

The hypothesis is accepted at a p-value > 0.95, suggesting each ethnic origin equally responded 'yes' and 'no' to the questionnaire

Chapter 13: The Analysis of Categorical Data

1.

	Diabetic	Normal	
Bb bb	12	4	16
BB	39	49	88
	51	53	104

$$p(Bb \text{ or } bb \mid \text{Diabetic}) = \frac{\binom{P(\text{Diabetic})}{P(Bb\text{ or }bb \mid \text{Diabetic})}\binom{P(BB)}{P(BB \mid \text{Diabetic})}}{\binom{\text{Total}}{P(\text{Diabetic})}}$$

Fishers Exact Test

$$= \frac{\binom{51}{12}\binom{88}{39}}{\binom{101}{51}} = 7.48 \times 10^{-5}$$

Yes, the probability of alleles (BB or bb) is less than a significance of $\alpha = 0.05$, which suggests an imbalance between the categories and acceptance of the alternative hypothesis ($p(Bb \text{ or } bb) \neq p(BB)$).

Chi-Square Test

	Diabetic	NonDiabetic
BB bb	2.99	2.11
BB	0.399	0.384

$$\chi^2 = \sum^{I}\sum^{J} \frac{(O_{ij} - E_{ij})^2}{E_{ij}} = \sum^{I}\sum^{J} \frac{(n_{ij} - n_{i.}n_{.j}/n_{..})^2}{n_{i.}n_{.j}/n_{..}} = 5.08, \; df = 1$$

$$\chi^2_{1,\,0.975} = 5.02$$

The outcome of comparing χ^2-statistic was a p-value < 0.025, which agrees with Fishers exact statistic.

2

Week	Chinese	Jewish	
-2	55	141	196
-1	33	145	178
1	70	139	209
2	49	161	210
	207	586	793

$$\chi^2 = 0.29 + 0.10 + 3.90 + 1.38 + 4.37 + 0.58 + 0.62 + 0.22$$
$$= 11.46, \; df = 3; \; \chi^2_{3,\,0.995} = 11.34$$

-2	0.29	0.10
-1	3.90	1.38
1	4.37	0.58
2	0.62	0.22

By leaving the data alone, a p-value < 0.005 was evaluated as a rejection of the null hypothesis. Although, by combining the information before ceremony and weeks after into two groups. χ^2 became 2.55 with $df = 1$, an acceptance of the null hypothesis (H_0: Weeks Prior to Ceremony = Weeks after Ceremony) with a p-value > 0.10.

-2/-1	88	286	374
1/2	119	300	419
	119	300	

-2/-1	0.949	0.458
1/2	0.947	0.299

793

0.949 0.458

3.

Severity
Moderate-Advanced
Minimal

0.947 0.299

χ^2 2.55

6.

Father's Activity	Female Offspring	Male Offspring	
Flying Fighters	51	38	89
Flying Transports	14	16	30
Not Flying	38	46	84
	103	100	203

0.756	0.779
0.931	0.101
0.501	0.526

$X^2 = 3.64$, $df = 2$, $X^2_{0.90} = 4.61$

$H_0: \pi_{i1} = \pi_{i2}$; $H_1: \pi_{i1} \neq \pi_{i2}$; The null hypothesis was accepted with this information at a p-value > 0.10, implying the offspring genders were independent of Father's activity.

7.

Grade	Psychology	Biology	Other	
A	8	15	13	36
B	14	19	15	48
C	15	4	7	26
D-F	3	1	4	8
	40	39	39	118

1.45	0.81	0.10
0.32	0.62	0.05
4.34	2.45	0.30
0.03	1.02	0.695

$df = 6$, $X^2 = 13.19$

${}_6X^2_{0.975} = 14.45$

$H_0: \pi_{i1} = \pi_{i2} = \pi_{i3}$; $H_1: \pi_{i1} \neq \pi_{i2} \neq \pi_{i3}$; The analysis of the data suggests a rejection of the null hypothesis in favor of the alternative, and grades dependendent to major.

8.

	# Patients	Incidence	
Placebo	165	95	260
Chlorpromazine	152	52	204
Dimenhydrinate	85	52	137
Pentobarbital (100mg)	67	35	102
Pentobarbital (150mg)	85	37	122
	554	271	825

Tabulated Afimation tests to be a null hypothesis positive. The expected cases of patients is independent of Incidence, implying none of the medications were effective to Placebo.

0.53	1.09
1.64	3.36
0.53	1.09
0.03	0.07
0.12	0.24

$df = 4$

$X^2 = 8.69$

${}_4X^2(0.95) = 9.49$

9.

Austen	Imitator
24	2
273	81
26	1
424	153
14	17
496	204
1247	458

1.31	3.56
0.77	2.09
1.98	5.39
0.009	0.03
3.32	9.03
0.69	1.89

$df = 5$, $X^2 = 30.04$

${}_5X^2(0.995) = 16.75$

$H_0: \pi_{i1} = \pi_{i2}$

$H_1: \pi_{i1} \neq \pi_{i2}$

The null hypothesis is rejected at a p-value of 0.005. Austen's Imitator did not write to similar standards as her original works.

10. Chi-Square Test of Independence:

Contingency Table:

$\pi_{i.} = \sum^{J} \pi_{ij}$; $\pi_{ij} = \pi_{i.}\pi_{.j}$; $H_0 : \pi_{ij} = \pi_i \pi_j$; MLE H_0 : $\hat{\pi}_{ij} = \hat{\pi}_{i.}\hat{\pi}_{.j} = \frac{n_{i.}}{n} \times \frac{n_{.j}}{n}$

$\pi_{.j} = \sum^{I} \pi_{ij}$

McNemars Test: $\hat{\pi}_{12} = \hat{\pi}_{21} = \frac{n_{12} + n_{21}}{2n}$

$$X^2 = \frac{(n_{12} - n_{21})^2}{n_{12} + n_{21}}$$

Derived From Multinomial:

$$lik(\pi_1, \pi_2 \ldots \pi_\ell) = \prod_{j=1}^{J} \binom{n_{.j}}{n_{1j} n_{2j} \ldots n_{Ij}} \pi_1^{n_{1j}} \pi_2^{n_{2j}} \cdots \pi_\ell^{n_{\ell j}}$$

$$l(\pi, \lambda) = \sum_i \log \binom{n_{.j}}{n_{1j} n_{2j} \cdots n_{Ij}} + \sum_i^{\ell} n_i \log \pi_i + \lambda\left(\sum_i^{\ell} \pi_i - 1\right)$$

$\frac{dl}{d\pi_i} = \frac{n_{i.}}{\pi_i} + \lambda$; $i = 1 \ldots \ell$

$\hat{\pi}_i = \frac{-n_i}{\lambda}$; $\sum \pi_i = 1$; $\sum_i n_i = n$

$\lambda = -n$

$\boxed{\pi_i = \frac{n_i}{n}}$

11. a) Likelihood Ratio Test for Homogeneity:

$$\Lambda = \frac{P(\theta | H_0)}{P(\theta | H_1)} = \frac{P(\hat{\theta})}{\hat{P}(\theta)} = \frac{\frac{n!}{x_1! \cdots x_m!} P_1(\hat{\theta})^{x_1} \cdots P_m(\hat{\theta})^{x_m}}{\frac{n!}{x_1! \cdots x_m!} \hat{p}_1^{x_1} \cdots p_m^{x_m}}$$

$$= \prod^{m} \left(\frac{P_i(\hat{\theta})}{\hat{p}_i}\right)^{x_i} ; \quad -2\log\Lambda = -2n \sum_i p_i \log\left(\frac{P_i(\hat{\theta})}{\hat{p}_i}\right) = 2\sum_i\sum_j O_i \log\left(\frac{O_i}{E_i}\right)$$

b) $-2\log\Lambda = 2\sum_i\sum_j O_i \log\left(\frac{O_i}{E_i}\right) = 2\sum_i\sum_j n_{ij} \cdot \log\left(\frac{n_{ij}^2}{n_{.j} \cdot n_{i.}}\right) =$

c) Likelihood Ratio Test for Independence:

$$\Lambda_{ij} = \Lambda_i \cdot \Lambda_j = \frac{P(\theta_i | H_0)}{P(\theta_i | H_1)} \cdot \frac{P(\theta_j | H_0)}{P(\theta_j | H_1)} = \frac{\frac{n!}{x_1! \cdots x_m!} P_1(\hat{\theta})^{x_1} \cdots P_m(\hat{\theta})^{x_m}}{\frac{n!}{x_1! \cdots x_m!} \hat{p}_1^{x_1} \cdots p_m^{x_m}} \cdot \frac{\frac{n!}{y_1! \cdots y_m!} P_1(\hat{\theta}_j)^{y_1} \cdots P_m(\hat{\theta}_j)^{y_m}}{\frac{n!}{y_1! \cdots y_m!} \hat{p}_1^{y_1} \cdots p_m^{y_m}}$$

$$\Lambda_{ij} = \prod^{m}\left(\frac{P_i(\hat{\theta})}{\hat{p}_i}\right)^{x_i} \cdot \prod^{m}\left(\frac{P_i(\hat{\theta})}{\hat{p}_i}\right)^{y_i}$$

$-2\log\Lambda = 2\sum_i\sum_j O_i \cdot \log\frac{O_i}{E_i} \, O_j \log\frac{O_j}{E_j}$; $\hat{\pi}_{ij} = \frac{n_i}{n} \cdot \frac{n_j}{n} = \boxed{\frac{n_{ij}}{n^2}}$

d) $2\log\Lambda = 16.52$; $\boxed{X^2 = 16.52}$

12. McNemars Test:

$\hat{\pi}_{11} = \frac{n_{11}}{n}$; $\hat{\pi}_{22} = \frac{n_{22}}{n}$; $\hat{\pi}_{12} = \hat{\pi}_{21} = \frac{n_{12} + n_{21}}{2n}$

$X^2 = \frac{(n_{12} - n_{21})^2}{n_{12} + n_{21}}$; $H_0 : \pi_{12} = \pi_{21}$

Example Dataset representing marks of 0/1

0	1
1	0

Test	value
McNemars	0
t-test	0

13. a) A test of independence would entail a hypothesis ($H_0 : \pi_{ij} = \pi_{ij}\frac{n_{ij}}{n}$) and a chi-squared statistic of columns of younger vs older with rows of children. If $X^2 < \chi^2_{6, .05}$ then the age of the sister is independent to number of children.

b) A test of family size distribution for sisters is described by ($H_0 : \pi_{i1} = \pi_{i2}$) with column of potentially number of children.

14. No high school Education.

Degrees of Interest	Under 45	Over 45
Great	71	217
Little	305	652

Some high school or more

Degrees of Interest	Under 45	Over 45
Great	305	180
Little	869	259

a. No highschool Education: $X^2 = 5.4$; $\chi^2_{0.95} = 3.84$

Some high school Education: $X^2 = 34.3$; $\chi^2_{1,\,0.95} = 3.84$

Each analysis of political interest generated results to reject the null hypothesis and accept the alternative that age is an indicator for political interests.

b. H_1: Given educational level, age, and degree of interest are unrelated.

H_2: Given age, educational level and degree of interest are unrelated.

By adding the columns of educational level,

	No school	Some school
Great	288	485
Little	957	1128

$X^2 = 17.12$
Reject $H_0: \pi_1 = \pi_2$
Accept H_1.

By adding the rows of age,

	Under 45	over 45
No School	376	869
Some School	1174	439

$X^2 = 513$
Reject H_0, Accept H_2.

15.

Source	X^2	df	$\chi^2_{.95}$	p-value
Incidence Among Males	13.54	4	9.49	0.00
Incidence Among Females	4.47	4	9.49	0.12

The number of females who grew a tumor was not significant because the expected values of predicted tumors correlated to the observed outcomes. Male tumor data has a p-value of ~0.00 and does present significant investigation.

16. A Chi-squared analysis ($X^2 = 27.29$, $df = 4$) confirms rejection of the null hypothesis ($H_0: \pi_{1.} = \pi_{2.} = \pi_{3.}$) and accepts the alternative hypothesis ($H_1: \pi_{1.} \neq \pi_{2.} \neq \pi_{3.}$) at a significance level of α 0.995. There is a relationship between personality type and attitude toward small cars, explorers have an unfavorable attitude.

17. London ($X^2 = 42.4$, $df = 1$) and Manchester ($X^2 = 5.5$, $df = 1$) data conclude different results upon the relationship between blood type and peptic ulcers. The city of London contains a relationship between blood type to disease, while Manchester does not. A p-value of <0.00 for London is reason for investigation to the source.

18.

Test	X^2	df	$\chi^2_{0.95}$	H_0	H_1	Suggestion	p-value
Matched Pair Design	75.03	1	3.84	$\pi_{12} = \pi_{21}$	$\pi_{12} \neq \pi_{21}$	Reject H_0	0.14
Homogeneity	0.03	1	3.84	$\pi_{.1} = \pi_{.2}$	$\pi_{.1} \neq \pi_{.2}$	Accept H_0	1.00

Estrogen is correlated to cancer.

19. Fischer's Exact Test:

$$p(n_{11}) = \frac{\binom{n_{1.}}{n_{11}}\binom{n_{2.}}{n_{21}}}{\binom{n_{..}}{n_{.1}}}$$

n_{11}	$p(n_{11})$
5	0.03
6	0.06
7	0.09
8	0.12
9	0.12
10	0.10
11	0.06
12	0.03
13	0.01
14	0.000
15	0.000

$\boxed{\text{The Fisher Exact Test presents no significant difference between the high and low anxiety groups at } n_{12} = 12.}$

20. Random Sample — Prospective Study — Retrospective Study

a) Random Sample:

$$P(D|X) = \frac{\pi_{11}}{\pi_{10}+\pi_{11}}$$

$$P(D|\bar{X}) = \frac{\pi_{01}}{\pi_{00}+\pi_{01}}$$

$$\Delta = \frac{odds(D|X)}{odds(D|\bar{X})} = \frac{P(D|X)}{1-P(D|X)}\,\frac{1-P(D|\bar{X})}{P(D|\bar{X})}$$

$$\boxed{= \frac{\pi_{11}\pi_{00}}{\pi_{01}\pi_{10}}}$$

Prospective Study:

$$P(D|X) = \frac{n_{11}}{n_{10}+n_{11}}$$

$$P(D|\bar{X}) = \frac{n_{01}}{n_{00}+n_{01}}$$

$$\Delta = \frac{odds(D|\bar{x})}{odds(D|\bar{x})} = \frac{P(D|X)}{1-P(D|X)}\,\frac{1-P(D|\bar{X})}{P(D|X)}$$

$$\boxed{= \frac{n_{11}}{n_{01}}\left(\frac{n_{00}}{n_{10}}\right)}$$

Retrospective Study:

$$P(X|D) = \frac{n_{10}}{n_{0\phi}}$$

$$P(X|\bar{D}) = \frac{n_{10}}{n_{00}}$$

$$\Delta = \frac{odds(D|X)}{odds(D|\bar{X})} = \frac{P(D|X)}{1-P(D|X)}\,\frac{1-P(D|\bar{X})}{P(D|\bar{X})}$$

$$\boxed{= \frac{n_{11}}{n_{01}}\cdot\frac{n_{00}}{n_{10}}}$$

b) Method of Propagation Error: $Y = g(x) \approx g(\mu_x) + (X-\mu_x)g'(\mu_x)$

$$\mu_y = g(\mu_x)$$

$$\sigma_Y^2 = \sigma_x^2\left[g'(\mu_x)\right]^2$$

Random Sample — Prospective Study — Retrospective Study

Random Sample:

$$\log\Delta = \log\frac{\pi_{11}\pi_{00}}{\pi_{01}\pi_{10}}$$

$$Var\log(\Delta) = \frac{\sum(x-\mu_x)^2}{n}\circ\log'(\Delta)^2$$

$$= \frac{\sum\sum(\pi_{ij}-\mu)^2}{m\cdot n}\circ\frac{1}{\Delta^2}$$

$$\boxed{= \frac{\sigma^2}{\Delta^2}} = \sigma^2\left(\frac{\pi_{01}\pi_{10}}{\pi_{11}\pi_{00}}\right)^2$$

Prospective Study:

$$\log\Delta = \log\frac{n_{11}}{n_{01}}\cdot\frac{n_{00}}{n_{10}}$$

$$Var\log\Delta = \frac{\sum(X-\mu_x)^2}{n}\circ\log'\Delta$$

$$= \frac{\sum\sum(\pi_{ij}-\mu)^2}{m\cdot n}\circ\frac{1}{\Delta}$$

$$\boxed{= \frac{\sigma^2}{\Delta^2}} = \sigma^2\left(\frac{n_{01}n_{10}}{n_{11}n_{00}}\right)^2$$

Retrospective Study:

$$\log\Delta = \log\frac{n_{11}}{n_{01}}\cdot\frac{n_{00}}{n_{10}}$$

$$Var\log\Delta = \frac{\sum(x-\mu_x)^2}{n}\log\Delta$$

$$= \frac{\sum\sum(\pi_{ij}-\mu)^2}{m\cdot n}\circ\frac{1}{\Delta}$$

$$\boxed{= \frac{\sigma^2}{\Delta} = \sigma^2\left(\frac{n_{01}n_{10}}{n_{11}n_{00}}\right)^2}$$

21.

	Diabetic	Normal
Bb or bb	12	4
BB	39	49

$P(D|X) = \frac{49}{39+49}$; $odds(D|X) = \frac{49}{39}$; $\Delta = \frac{49\cdot 12}{39\cdot 4}$

$P(D|\bar{X}) = \frac{12}{12+4}$; $odds(D|\bar{X}) = \frac{4}{12}$; $\boxed{= 3.77}$

$\boxed{\text{The odds of being normal (Dominant) to Diabetic (Dominant recessive) is } 3.77:1.}$

22. a

	Advised	Not Advised
Male	48	47
Female	80	136

Proportion of males advised to quit: $\frac{48}{48+47} = 50.6\%$

Proportion of females advised to quit: $\frac{80}{80+136} = 37.7\%$

Standard error of male proportion: 20.05

Standard error of female proportion: 1.91

Standard error of their difference: ∞ ; $\boxed{\text{Significant with t-test}}$

	Advised	Not Advised
White	26	34
African-American	102	149

Proportion of Whites asked to quit: 43%

Proportion of African-Americans asked to quit: 41%

Standard error of White proportion: 0.6

Standard error of African-American proportion: ∞ ; $\boxed{\text{Significant}}$ t-test

	Advised	Not Advised
<15	64	112
15-25	39	54
>25	25	16

Proportion of smokers who smoke <15, advised to quit: 36%
Proportion of smokers, 15-25, advised to quit: 42%
Proportion of smokers who smoke >25, advised to quit: 61%
Standard error of smokers (<15): 1.8
Standard error of smokers (15-25): 0.77
Standard error of smokers (>25): 0.70

The difference in proportion are significant to an F-stat, one-sided.

b)

	Advised	Not Advised
Male	78	94
Females	50	89

Proportion of male physicians who advised: 45%
Proportion of female physicians who advised: 36%
Standard error of male physicians who advised: 0.61
Standard error of female physicians who advised: 1.65
Standard error of the difference: ∞ ; $t < t(0.95)$
$F < F_{crit}$ | Significant results |

	Advised	Not Advised
Smoker	13	37
Nonsmoker	115	146

Proportion of smokers advised: 26%
Proportion of nonsmokers advised: 44%
Standard error of smokers advised: 1.70
Standard error of smokers not advised: 0.96
Standard error of the difference: ∞ ; $t < t(0.95)$
$F < F_{crit}$ | Significant results |

	Advised	Not Advised
<30	88	128
30-39	28	37
>39	12	18

Proportion of physicians age (<30): 41%
Proportion of physicians age (30-39): 43%
Proportion of physicians age (>39): 40%
Standard error of physicians (<30): 1.36
Standard error of physicians (30-39): 0.56
Standard error of physicians (>39): 0.55
Standard error of the difference: F-statistic - 17.07
F-crit : 9.55
| Not significant |

23.

Previous Day	Day of Infarction		
	Exertion	No Exertion	Total
Exertion	4	9	13
No exertion	50	1165	1215
Total	54	1174	1228

McNemar's Test:
$H_0: \pi_{12} = \pi_{21}$
$H_1: \pi_{12} \neq \pi_{21}$
$$X^2 = \frac{(n_{12} - n_{21})^2}{n_{12} + n_{21}} = 28.49, \; df = 1; \; \chi^2_{0.95} = 3.84$$

| The results by McNemar's Test are significant, Exertion is associated to infarction. |

24.

Sport	Red	Blue
Boxing	148	120
Freestyle Wrestling	27	24
Greco Roman Wrestling	25	23
Tae Kwon Do	45	35

a. π_R = contestant in red wins
$= \frac{n_{1i}}{n} = 0.55$ $\quad X^2 = 0.30$
df = 3
$H_0: \pi_R = 1/2$
$H_1: \pi_R \neq 1/2$ $\quad \chi^2 = 7.81$

There appears to be no significant relationship between sport and color. The means of red and blue clothes are roughly equivalent

b. $H_0: \pi_R = 1/2$ $\quad X^2 = 0.30; X^2 = 7.81$

$H_1: \pi_{R} \neq 1/2$ $\quad df = 3$ $\boxed{\text{Not significant between sports}}$

c. Both of the tests are an equivalent argument with a chi-squared statistic.

d. There is little evidence to suggest red is favorable to the wins of sports.

e.

Sport	Red	Blue
Boxing	123	105
Free Wrestling	28	39
GR Wrestling	25	34
Taekwon Do	27	26

$X^2 = 8.59$ $\quad df = 3$ $\quad X^2_{0.95} = 7.81$

Examination of larger datasets show a different conclusion; more specifically, color does effect the outcome of a match. The means are significant across wrestling.

25. a. Individual analysis of aspirin's effects were conducted for myocardial infarction and stroke. Myocardial disease ($X^2 = 1.38, df = 1$) results were not significant, but aspirin did effect a stroke ($X^2 = 30.65, df = 1$).

b. Aspirin analysis significant results for lowering myocardial mortalities, but overall had no influence on death ($X^2 = 12.61; df = 4$).

26. McNemars Test evaluated DKA relationship to therapy. A chi-squared of 5.53, with one degree of freedom lists significant results for the relationship between side effects before and after therapy.

27. Upon examination, the defendants race does not influence death penalty. By including the victims race, the results could change the outcome, but not the analysis. The chi-square statistic ($X^2 = 15.92, df = 3$) does list significant results for a relationship of race to death penalty.

28. The chi-square test is independent to counts, frequency, or percents, and presents similar outcomes

29. Model: Satisfied Somewhat Dissatisfied Very Dissatisfied

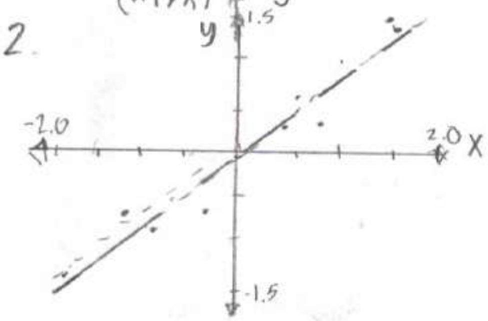
A statistician should carry out a test of homogeneity, rather than a test for independence because of the workers being already independent to each other.

Chapter 14: Linear Least Squares

1. a. $y = \frac{a}{(b+cx)}$; $\log y = \log a - \log(b+cx)$ b. $y = ae^{-bx}$; $\log y = -bx + \log a$ c. $y = ab^x$; $\log y = x \log b + \log a$

d. $y = \frac{x}{(a+bx)}$; $\frac{1}{y} = \frac{a}{x} + b$; $\frac{1}{y} = k$; $\frac{1}{x} = j$; $k = aj + b$ e. $y = \frac{1}{1+e^{bx}}$; $\log k - 1 = bx$ where $k = \frac{1}{y}$

2.



a. $\beta_0 = \frac{\sum x^2 \sum y - \sum x \sum xy}{n\sum x^2 - (\sum x)^2} = -0.03$

$\beta_1 = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2} = 0.904$

b. $x = c + dy$; $c = -\frac{\beta_0}{\beta_1}$, $d = \frac{1}{\beta_1}$; c. The lines are equivalent graphically because the inverse is being plotted.

3. $y = \mu + e_i$; $i = 1 \ldots n$; where e_i = independent errors $[\mu_e = 0, \sigma_e^2]$

$= \beta_0 + \cancel{\beta_1 x}^{0} + e_i = \beta_0 + e_i$; $\beta_0 = \frac{(\sum x^2)(\sum y) - (\sum x)(\sum xy)}{n\sum x^2 - (\sum x)^2} = \frac{\mu(\sum x^2 - (\sum x)^2)}{\sum x^2 - (\sum x)^2}\frac{n}{n} = \mu$

4. GPA: $Y_i = \beta_0 + \beta_1 X_i + e_i$, $i = 1, 2 \ldots n$

Model: $Y_F = I_F(i)\beta_F + I_M(i)\beta_M + \beta_i x_i + e_i$; where $I_F(i)$ and $I_M(i)$ are indicator variables that are 0/1 for male or female.

Design Matrix: $\boxed{\mathbb{X} = \begin{pmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_n \end{pmatrix}}$

5. $P_1 < P_2 < P_3$ a. $\mathbb{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$ b. $\mathbb{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{bmatrix}$ c. $\mathbb{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{bmatrix}$ $\mathbb{X} = \begin{array}{c} (P_1 \quad P_2 \quad P_3)^T \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \end{array}$ $\beta = (\mathbb{X}\mathbb{X})^{-1}\mathbb{X}^T Y$

=

6. w_1 and w_2

1) Object 1 Weighed by itself, 3g ; a) $\mathbb{X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ $\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$ $\mathbb{Y} = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 7 \end{bmatrix}$ $e = Y - \hat{Y} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$

2) Object 2 Weighed by itself, 3g

3) Objects 1 and object 2, $\Delta w = 1g$.

4) $\sum w = 7g$

b) $S(\beta_0, \beta_1) = \sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x)^2$; $S(w_1, w_2) = \sum_{i=1}^{4}(y_i - w_1 - w_2 x_i)^2 = (3-w_1)^2 + (3-w_2)^2 + (1-w_1+w_2)^2 + (7-w_1-w_2)^2$

$\frac{dS}{dw_1} = -2(3-w_1) - 2(1-w_1+w_2) - 2(7-w_1-w_2) = 0$; $11 - 3w_1 = 0$; $w_1 = \boxed{\frac{11}{3}}$

$\frac{dS}{dw_2} = -2(3-w_2) + 2(1-w_1+w_2) - 2(7-w_1-w_2) = 0$; $9 - 3w_2 = 0$; $w_2 = \boxed{3}$

c) $s^2 = \frac{1}{n-2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2$; $s^2 = \frac{1}{4-2}\left[(3-\frac{11}{3})^2 + (3-3)^2 + (1-\frac{11}{3}+3)^2 + (7-\frac{11}{3}-3)^2\right]$; $\boxed{s^2 = 1/3}$

d. $\sigma_s^2 = (w_1 - w_2) s^2 = \frac{2}{3} \cdot \frac{1}{3} = \boxed{\frac{2}{9}}$

e. $t = \frac{w_1 - w_2}{\sqrt{\sigma_s^2}} = \frac{2/3}{\sqrt{2/9}} = \boxed{\sqrt{2}}$

F. $H_0: w_1 = w_2$; $H_1: w_1 \neq w_2$; ${}_3t_{0.95} > t$; $2.353 > \sqrt{2}$; Accept null hypothesis.

7. $y_i = \beta_0 + \beta_1 x_i + e_i$; $Var(e_i) = \rho_i^2 \sigma^2$; $\rho_i^{-1} y_i = \rho_i^{-1}\beta_0 + \rho_i^{-1}\beta_1 x_i + \rho_i^{-1} e_i$ or $z_i = u_i\beta_0 + v_i\beta_1 + \delta_i$

where $u_i = \rho_i^{-1}$; $v_i = \rho_i^{-1} x_i$; $\delta_i = \rho_i^{-1} e_i$

a. $z_i = u_i\beta_0 + v_i\beta_1 + \delta_i$; $\frac{y_i}{\rho_i} = \frac{\beta_0}{\rho_i} + \frac{x\beta_1}{\rho_i} + \frac{e_i}{\rho_i}$; $y_i = \beta_0 + x\beta_1 + e_i$

b. $S(\beta_0, \beta_1) = \sum_{i=1}^{n} (z_i - u_i\beta_0 - v_i\beta_1 - \delta_i)^2$; $\frac{dS}{d\beta_0} = -2\sum u_i (z_i - u_i\beta_0 - v_i\beta_1 - \delta_i) = 0$; $\sum z_i = \beta_0 \sum u_i + \beta_1 \sum v_i$

$\frac{dS}{d\beta_1} = -2\sum v_i (z_i - u_i\beta_0 - v_i\beta_1 - \delta_i) = 0$; $\sum v_i z_i = (\beta_0 \sum u_i + \beta_1 \sum v_i) v_i$

$\sum z_i = \frac{\beta_0 n}{\rho_i} + \frac{\beta_1}{\rho_i}\sum x_i$; $\rho_i \sum z_i = n\beta_0 + \beta_1 \sum x_i$

$\sum v_i z_i = \beta_0 \sum v_i + \beta_1 \sum v_i^2$; $\rho_i \sum v_i z_i = \beta_0 \sum x + \beta_1 \sum x^2$

$$\boxed{\beta_0 = \frac{\rho_i \sum x_i^2 \cdot \sum y_i - \rho_i \sum x_i \sum x_i y_i}{n\sum x_i^2 - (\sum x_i)^2}}$$

$$\boxed{\beta_1 = \frac{n\rho_i \sum x_i y_i - \rho_i(\sum x_i)(\sum y_i)}{n\sum x_i^2 - (\sum x_i)^2}}$$

c. $\frac{d}{d\beta_0} \sum (y_i - \beta_0 - \beta_1 x_i)^2 \rho_i^{-2} = -2\sum (y_i - \beta_0 - \beta_1 x_i) \cdot \rho_i^{-2} = 0$; $\beta_0 = \frac{\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i}{n\sum x_i^2 - (\sum x_i)^2}$

$\beta_1 = \frac{n\sum x_i y_i - (\sum x_i)(\sum y_i)}{n\sum x_i^2 - (\sum x_i)^2}$

d. $\boxed{Var(\beta_0) = \frac{\sigma^2 \sum x_i^2 \cdot \rho_i^2}{n\sum x_i^2 - (\sum x_i)^2}}$; $\boxed{Var(\beta_1) = \frac{n\sigma^2 \rho_i^2}{n\sum x_i^2 - (\sum x_i)^2}}$

8. $\underset{(n\times p)}{\mathbb{X}} = \underset{(n\times p)}{\mathbb{Q}} \cdot \underset{(p\times p)}{\mathbb{R}}$ where $\mathbb{Q}^T\mathbb{Q} = I$; $\mathbb{R} = (r_{ij} = 0, i > j)$; Show $\hat{\beta} = (\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T\mathbb{Y}$ is $\hat{\beta} = \mathbb{R}^{-1}\mathbb{Q}^T\mathbb{Y}$

$\mathbb{Y} = \mathbb{X}\beta$; $\mathbb{X}^T\mathbb{Y} = (\mathbb{X}^T\mathbb{X})\beta$; $\boxed{(\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T\mathbb{Y} = \hat{\beta}}$

$= (\mathbb{Q}^T\mathbb{R}^T\mathbb{Q}\mathbb{R})^{-1}\mathbb{Q}^T\mathbb{R}^T\mathbb{Y}$

$= (\mathbb{Q}^T\mathbb{R}^T\mathbb{Q})^{-1} \cdot \mathbb{R}^{-1}\mathbb{Q}^T \cdot \mathbb{R}^T\mathbb{Y}$

$\boxed{= \mathbb{R}^{-1}\mathbb{Q}^T\mathbb{Y}}$

The last equation ($\mathbb{R}\hat{\beta} = \mathbb{Q}^T\mathbb{Y}$) may be solved by back-substitution because there are p variables for p equations

9. Cholesky Decomposition $\mathbb{X}^T\mathbb{X} = \mathbb{R}^T\mathbb{R}$ where $\mathbb{R} = (r_{ij} = 0, i > j)$

Show $\mathbb{R}^T v = \mathbb{X}^T\mathbb{Y}$; $\mathbb{R}\hat{\beta} = v$; Show $\mathbb{R}^T v = \mathbb{X}^T\mathbb{Y}$ is solved with Back-substitution because of the dimensions $(p\times p)(p\times 1)$, indicating p variables and p equations, which is similar to $\mathbb{R}\hat{\beta} = v$.

$\mathbb{R}^T\mathbb{R}\hat{\beta} = \mathbb{X}^T\mathbb{Y}$

$\boxed{\hat{\beta} = \cancel{\mathbb{X}}^T\mathbb{Y}}$

10. $\hat{\beta}_0 = \bar{y} - \beta_1 \bar{x}$ and $\beta_1 = \dfrac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$; $\bar{y} = \beta_0 + \beta_1 \bar{x}$; $\boxed{\beta_0 = \bar{y} - \beta_1 \bar{x}}$

11. $Corr(\beta_0, \beta_1) = \dfrac{Cov(\beta_0, \beta_1)}{\sqrt{Var(\beta_1) \cdot Var(\beta_0)}}$

$$= \frac{\dfrac{-\sigma^2 \sum x_i}{n\sum x_i^2 - (\sum x_i)^2}}{\sqrt{\dfrac{\sigma^2 \sum x_i^2}{n\sum x_i^2 - (\sum x_i)^2} \; \dfrac{n\sigma^2}{n\sum x_i^2 - (\sum x_i)^2}}}$$

@ $\bar{x} = 0$; $Corr(\beta_0, \beta_1) = 0$

$\beta_1 = \dfrac{n\sum x_i y_i - (\sum x_i)(\sum y_i)}{n\sum x_i^2 - (\sum x_i)^2}$

$\boxed{= \dfrac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}}$

$(xy - \bar{x}y + y$

$xy + x_i y_i + \bar{x}\bar{y} - \bar{x}y$

12. $\hat{\beta}_0 = \bar{y} - \beta_1 \bar{x}$

$\hat{\beta}_1 = \dfrac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$

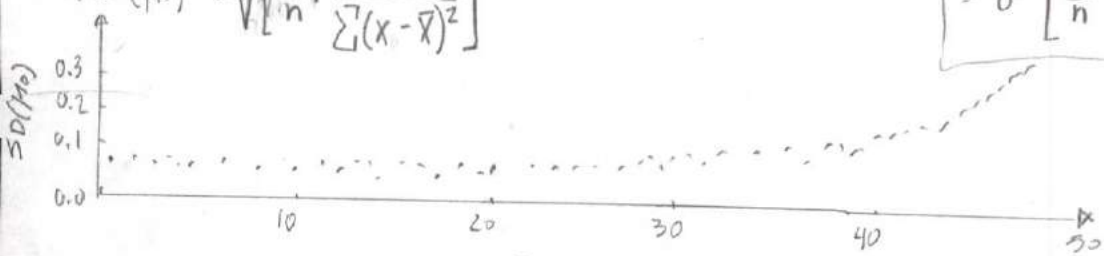
@ $(\bar{x}, \bar{y})$; $\boxed{y = \beta_0 + \beta_1 \bar{x} = \beta_0 + \beta_1 \bar{x} = \bar{y} - \beta_1 \bar{x} + \beta_1 \bar{x} = \bar{y}}$

13. $\mu_0 = \hat{\beta}_0 + \hat{\beta}_1 x_0$; $Var(\mu_0) = Var(\hat{\beta}_0 + \hat{\beta}_1 x_0) = Var(\beta_0) + Var(\hat{\beta}_1 x_0) = \dfrac{\sigma^2 \sum (x_0 - \bar{x}_0)^2}{n\sum (x_0 - \bar{x})^2} + \dfrac{\sigma^2 (x_0 - \bar{x})^2}{\sum (x_0 - \bar{x})^2}$

a)

$\boxed{= \sigma^2 \left[\dfrac{1}{n} + \dfrac{(x_0 - \bar{x})^2}{\sum (x_0 - \bar{x})^2} \right]}$

b) $SD(\mu_0) = \sigma \sqrt{\left[\dfrac{1}{n} + \dfrac{(x_0 - \bar{x})^2}{\sum (x - \bar{x})^2} \right]}$



c. 95% confidence interval: $\boxed{\mu_0 \pm SD(\mu_0) \cdot t_{n-2}(\alpha/2)}$

14. $Y_0 = \beta_0 + \beta_1 x_0 + e_0$ by the estimate $\hat{Y}_0 = \hat{\beta}_0 + \hat{\beta}_1 x_0$

a) $Var(\hat{Y}_0 - Y_0) = Var(\hat{\beta}_0 + \hat{\beta}_1 x_0 - \beta_0 - \beta_1 x_0 - e_0) = Var(\hat{\beta}_0 - \beta_0) + Var[(\hat{\beta}_1 - \beta_0) x_0] - Var(e_i)$

$= Var(\hat{\beta} - \beta_0) - Var[(\hat{\beta}_1 - \beta_0) x_0]$

$\boxed{= (\hat{\sigma}^2 - \sigma_0^2) \left[\dfrac{1}{n} + \dfrac{(x_0 - \bar{x})^2}{\sum (x_0 - \bar{x})^2} \right]}.$

b) $\boxed{(\hat{Y} - Y_0) \pm \sqrt{(\hat{\sigma}^2 - \sigma_0^2) \left[\dfrac{1}{n} + \dfrac{(x_0 - \bar{x})^2}{\sum (x_0 - \bar{x})^2} \right.} \cdot t_{n-2}(\alpha/2)}$

15. $y = \beta x$ to points (x_i, y_i) where $i = 1 \ldots n$

$\beta = (X^T X)^{-1} X^T y$ -or- $S = \sum (y_i - \beta x)^2$; $\dfrac{dS}{d\beta} = -2\sum x_i (y - \beta x) = 0$; $\sum x_i y_i - \beta \sum x_i^2 = 0$; $\boxed{\beta = \dfrac{\sum x_i y_i}{\sum x_i^2}}$

16. a $y = \beta_0 x + \beta_1 x^2$; $X^T X = \begin{pmatrix} x \\ x^2 \end{pmatrix} (x \;\; x^2)$ $= \begin{pmatrix} x^2 & x^3 \\ x^3 & x^4 \end{pmatrix}$ $= \begin{pmatrix} \sum x^2 & \sum x^3 \\ \sum x^3 & \sum x^4 \end{pmatrix}$

$(X^T X)^{-1} = \dfrac{1}{\sum x^2 \sum x^4 - (\sum x^3)^2} \begin{bmatrix} \sum x^4 & -\sum x^3 \\ -\sum x^3 & \sum x^2 \end{bmatrix}$

$Y = \beta X^2$

$= \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} (x \;\; x^2)$

$\sum x$

$(X^T Y) = \begin{pmatrix} \sum x^2 y & \sum x^3 y \\ \sum x^3 y & \sum x^4 y \end{pmatrix}$

$\beta = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \dfrac{\begin{pmatrix} \sum x^4 & -\sum x^3 \\ -\sum x^3 & \sum x^2 \end{pmatrix} \begin{pmatrix} \sum x^2 y & \sum x^3 y \\ \sum x^3 y & \sum x^4 y \end{pmatrix}}{\sum x^2 \sum x^4 - (\sum x^3)^2}$

$$\frac{\begin{pmatrix} \sum x^4\sum x^3y-\sum x^3y\sum x^3 & \sum x^4\sum x^3y-\sum x^3\sum x^4y \\ \sum x^4\sum x^3y-\sum x^3\sum x^4y & \sum x^3\sum x^3y+\sum x^2\sum x^4y \end{pmatrix}}{\sum x^2\sum x^4-(\sum x^3)^2} = \begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}$$

$$\beta_0 = \frac{\sum x^4\sum x^3y-\sum x^3y\sum x^3+\sum x^4\sum x^3y-\sum x^3\sum x^4y}{\sum x^2\sum x^4-(\sum x^3)^2} = \boxed{\frac{2\sum x^4\sum x^3y-\sum x^3(\sum x^3y-\sum x^4y)}{\sum x^2\sum x^4-(\sum x^3)^2}}$$

$$\beta_1 = \frac{\sum x^4\sum x^3y-\sum x^3\sum x^4y+\sum x^3\sum x^3y+\sum x^2\sum x^4y}{\sum x^2\sum x^4-(\sum x^3)^2} = \boxed{\frac{2\sum x^3y(\sum x^4+\sum x^3)+(\sum x^2-\sum x^3)\sum x^4y}{\sum x^2\sum x^4-(\sum x^3)^2}}$$

b. $\boxed{Cov(\beta_0,\beta_1)=\frac{\sum x^4\sum x^3y-\sum x^3\sum x^4y}{\sum x^2\sum x^4-(\sum x^3)^2}\sigma^2}$

17. $E(X)=\mu_x$ $\quad E(Y)=\mu_y$

$Var(X)=\sigma_x^2$ $\quad Var(Y)=\sigma_y^2$

a) $E(Y-\hat{Y})^2=E(Y^2)-2E(Y\hat{Y})+E(\hat{Y})^2=\mu_y^2-2E(Y\cdot(\alpha+\beta X))+E((\alpha+\beta X)^2)$

$=\mu_Y^2-2\alpha\mu_y-2\beta Cov(X,Y)+E(\alpha)^2+2E(\alpha\beta X)+E(\beta^2X^2)$

$=\mu_Y^2-2\alpha\mu_y-2\beta\sigma_{xy}+\alpha^2+2\alpha\beta\mu_x+\beta^2\sigma_x^2$

$\frac{dE(Y-\hat{Y})^2}{d\alpha}=-2\mu_y+2\alpha+2\beta\mu_x=0;$ $\boxed{\alpha=\mu_y-\beta\mu_x}$

$\frac{dE(Y-\hat{Y})^2}{d\beta}=-2\sigma_{XY}+\beta\sigma_x^2;$ $\beta=\frac{\sigma_{xy}}{\sigma_x^2}$

b. Prove $\frac{Var(Y)-Var(Y-\hat{Y})}{Var(Y)}=r_{xy}^2$; $\frac{Var(Y)-Var(Y-\hat{Y})}{Var(Y)}=\frac{\sigma_y^2-\sigma_y^2+\emptyset+\beta^2\sigma_x^2}{Var(Y)}=\boxed{\frac{\sigma_{xy}^2}{Var(Y)Var(X)}}$

18. $Y_i=\beta_0+\beta_1x_i+e_i$ where $i=1\ldots n$, $e_i=N(\mu=0|\sigma^2)$; Find the mle's of β_0 and β_1.

$S(\beta_0,\beta_1)=\sum(y_i-\beta_0-\beta_1x_i+e_i)^2$; $\frac{dS}{d\beta_1}=-2\sum(y_i-\beta_0-\beta_1x_i+e_i)\sum x_i=0$; $\sum x_iy_i-\beta_0\sum x_i-\beta_1(\sum x_i^2)=0$

$\sum x_iy_i=\beta_0\sum x_i+\beta_1(\sum x_i^2)$

$\frac{dS}{d\beta_0}=-2\sum(y_i-\beta_0-\beta_1x_i+e_i)=0$; $\sum y_i=n\beta_0+\beta_1\sum x_i$

$n\sum x_iy_i=\sum x\sum y-\beta_1(\sum x_i)^2+n\beta_1\sum x_i^2=\sum x\sum y+\beta_1(\sum x_i^2-(\sum x_i)^2)$

$\boxed{\beta_1=\frac{n\sum x_iy_i-\sum x\sum y}{n\sum x_i^2-(\sum x_i)^2}}$ $\quad \sum y_i=n\beta_0+\frac{n\sum x\sum x_iy_i-(\sum x_i)^2\sum y_i}{n\sum x_i^2-(\sum x_i)^2}$

$\sum y_i=\frac{n\beta_0[n\sum x_i^2-(\sum x_i)^2]+n\sum x\sum xy-\sum x_i^2\sum y_i}{n\sum x_i^2-(\sum x_i)^2}$

$\frac{n\sum y_i\sum x_i^2-\sum y_i(\sum x_i)^2}{}=\beta_0[\sum x_i\sum x_i^2-(\sum x)^3]+n\sum x\sum xy-\sum x^2\sum y_i$

$\boxed{\frac{n\sum x_i^2\sum y_i-n\sum x\sum xy}{n\sum x_i^2-(\sum x)^2}=\beta_0}$

19. a. Vector of Residuals: $\hat{e} = Y - \hat{Y} = (I-P)Y$

b. $\sum_{i=1}^{2} e_i = Y_1 - X_1\beta + Y_2 - X_2\beta = Y - \hat{Y} = 0$

$\hat{e}XX^T = \begin{vmatrix} Y_1 - X_1\beta \\ Y_2 - X_2\beta \end{vmatrix}^T [X_1\ X_2]^T = Y_1X_1 - X_1^TX_1\beta + Y_2X_2 - X_2^TX_2\beta$
$= Y_1X_1 - Y_1X_1 + Y_2X_2 - Y_2X_2$
$\boxed{=0}$

20. Assume X, X_1, X_p are orthogonal; that is $X_i^TX_j = 0$ for $i \neq j$

If $X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$; Covariance matrix of a least squares estimate:

$$\Sigma_{\beta\beta} = \sigma^2(X^TX)^{-1} = \sigma^2\left(\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}\right)^{-1} = \sigma^2\begin{bmatrix} X_1^2 & 0 \\ 0 & X_2^2 \end{bmatrix}^{-1}$$

$$\boxed{= \sigma^2\begin{bmatrix} \frac{1}{X_1^2} & 0 \\ 0 & \frac{1}{X_2^2} \end{bmatrix}}$$

$AI = A$

$\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [$

21. $Var(\beta_1) = \dfrac{\sigma^2}{\sum(X_i - \bar{X})^2}$

To minimize the variance of β_1, the average value should be close to zero.

22. Family Income $\propto$ consumption; $X_{90\%}$ 0% 50% 100%

less than 50% of families would be at the 90% percentile of consumption when the total proportion is above the 90% of family income.

23a. $\rho = \dfrac{cov(Exam, midterm)}{\sqrt{Var(Exam)}\sqrt{Var(Midterm)}} = 0.5$; $\bar{X} = \dfrac{Exam + midterm}{2} = 75$; $\sigma = \sqrt{\dfrac{(Exam-75)^2 + (midterm-75)^2}{2}} = 10$

If student scores 95 on the midterm, what is the final exam?

$\dfrac{95+75}{2} = \boxed{85}$

b. If a student scored 85 on the final, what would you guess her score on the midterm was?

$\dfrac{85+75}{2} = \boxed{80}$

24. $\hat{Y} = X\beta = KX\beta = [K_1\ K_2 \cdots K_n]\begin{bmatrix} X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & \ddots X_n \end{bmatrix}[\beta_1\ \beta_2 \cdots \beta_n] = \begin{bmatrix} K_1X_1 & 0 & 0 \\ 0 & K_2X_2 & 0 \\ 0 & 0 & \ddots K_nX_n \end{bmatrix}[\beta_1\ \beta_2 \cdots \beta_n] = U\beta$

The linear least squares constant β does not change.

25. Y_{i1}, Y_{i2} are regressive mean responses to a linear least squares because the process is independent of grouping data points.

26. Z_1, Z_2, Z_3, Z_4 are random variables with $Var(Z_i) = 1$ and $Cov(Z_i, Z_j) = \rho$ for $i \neq j$

Prove $Z_1 + Z_2 + Z_3 + Z_4$ is uncorrelated with $Z_1 + Z_2 - Z_3 - Z_4$

If $Y_1 = Z_1 + Z_2 + Z_3 + Z_4$ and $Y_2 = Z_1 + Z_2 - Z_3 - Z_4$

$cov(Y_1, Y_2) = var(Z_1) + \cancel{cov(Z_1, Z_2)} - \cancel{cov(Z_1, Z_3)} - \cancel{cov(Z_1, Z_4)} + var(Z_2) - \cancel{cov(Z_2, Z_3)}$
$- cov(Z_2, Z_4) - var(Z_3) - cov(Z_3, Z_4) + \cancel{cov(Z_4, Z_1)} + \cancel{cov(Z_4, Z_2)}$
$- cov(Z_4, Z_3) - var(Z_4) + \cancel{Cov(Z_2, Z_1)} + \cancel{cov(Z_3, Z_1)} + \cancel{cov(Z_3, Z_2)}$
$= 1 + 1 - 1 - 2\rho - 1 = \boxed{2\rho}$

27. $\sigma^2 I = \Sigma_{\hat{Y}\hat{Y}} + \Sigma_{\hat{e}\hat{e}}$; Prove $n\sigma^2 = \sum_{i=1}^{n} Var(Y_i) + \sum_{i=1}^{n} Var(e_i)$

$\Sigma_{YY} = \sigma^2 I$; $\Sigma_{ee} = \sigma^2 I$; $\Sigma_{YY} + \Sigma_{ee} = \sigma^2 I$

$n\sigma^2 = n(\Sigma_{YY} + \Sigma_{ee}) = \boxed{\sum_{i=1}^{n} Var(Y_i) + \sum_{i=1}^{n} var(e_i)}$

28. $Y = \sum a_i X_i$; $var(Y) = \sigma^2$; $\bar{X} = \mu_i$

a) If $Z = \sum b_i X_i$; Use theorem D ; $Y = AX$; $Z = BX$; $\Sigma_{YZ} = A\Sigma_{XX}B^T$

Find $Cov(Y,Z) = \Sigma_{YZ} = A\Sigma_{XX}B^T = \sum a_i \Sigma_{XX} \sum b_i = \boxed{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} b_{ij} \sigma_{ij}}$

b.) Theorem C of 14.4.1

$E(X^T A X) = trace(A\Sigma) + \mu^T A \mu$

$E(X_i X_j) = \sigma_{ij} + \mu_i \mu_j$

$E(\sum\sum X_i X_j a_{ij}) = \sum\sum \sigma_{ij} a_{ij} + \sum\sum \mu_i \mu_j a_{ij}$

$= \boxed{trace(A\Sigma) + \mu^T A \mu}$

$E(\sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j) = \sigma_{ij} + \mu_i \mu_j$
$= \sigma + \mu^2$

29. $Cov(X_1, X_2) = 0$; $Var(X_1) = Var(X_2) = \sigma^2$; Use matrix methods to show $Y = X_1 + X_2$
$Z = X_1 - X_2$

Find $\Sigma_{YZ} = Cov(Y, Z) = Cov(X_1 + X_2, X_1 - X_2)$

$= Var(X_1) - Cov(X_1, X_2) + Cov(X_1, X_2) - Var(X_2) = \boxed{0}$

30. $X_1 \ldots X_n$; $Var(X_i) = \sigma^2$; $Cov(X_i, X_j) = \rho\sigma^2$ for $i \neq j$; Find $Var(\bar{X})$;

$Var(\bar{X}) = Var\left(\frac{X_1 + \cdots + X_n}{n}\right) = \frac{1}{n^2} Var(X_1 + \cdots + X_n) = \frac{1}{n^2}[Var(X_1) + \cdots + Var(X_n)] = \boxed{\frac{\sigma^2}{n}}$

31. $Z_{1\times 4}$; $\Sigma_{ZZ} = \sigma^2 I$; Let $V = Z_1 + Z_2 + Z_3 + Z_4$ and $Z = (Z_1 + Z_2) - (Z_3 + Z_4)$

Find $Cov(V, Z) = cov(Z_1 + Z_2 + Z_3 + Z_4, (Z_1 + Z_2) - (Z_3 + Z_4))$

$= var(Z_1) + cov(Z_1, Z_2) - cov(\cancel{Z_1}, Z_3) - cov(\cancel{Z_1, Z_4}) + var(Z_2) + cov(Z_1, Z_2)$

$- cov(\cancel{Z_2}, Z_3) - cov(\cancel{Z_2}, Z_4) + cov(\cancel{Z_3}, Z_1) + cov(\cancel{Z_3}, Z_2) - var(Z_3)$

$- cov(Z_3, Z_4) + cov(\cancel{Z_4}, Z_1) + cov(\cancel{Z_4}, Z_2) - cov(Z_4, Z_3) - var(Z_4)$

$= 2Cov(Z_1, Z_2) - 2Cov(Z_3, Z_4) = \boxed{0}$

32. $Y_1 = X_1$, $Y_i = X_i - X_{i-1}$, $i = 1, 2, \ldots n$

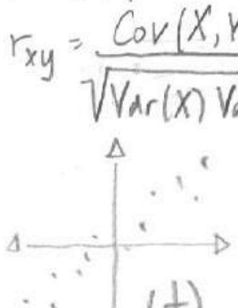
a. $\Sigma_{YY} = \sum_{i=1}^{n} X_i - X_{i-1} = X_n = \boxed{\sigma^2 I}$ b. $\Sigma_{XX} = \sum Y_i = \boxed{\sigma^2 I}$

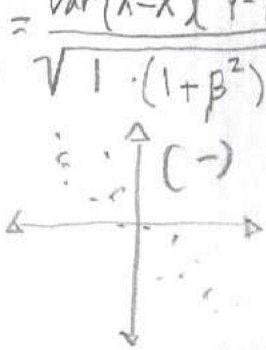
33. $X \sim N(0,1)$ and $E \sim N(0,1)$, let $Y = X + \beta E$

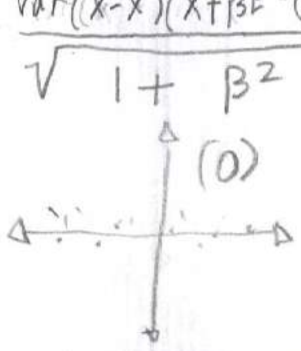
a. $r_{xy} = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{var(X - \bar{X})(Y - \bar{Y})}{\sqrt{1 \cdot (1 + \beta^2)}} = \frac{var((X - \bar{X})(X + \beta E - 0 + 0)}{\sqrt{1 + \beta^2}} = \boxed{\frac{var(X) - \beta^2 var(XE) - Var(\bar{X}X) - Var(\bar{X}\beta E)}{\sqrt{1 + \beta^2}}}$

$= \frac{1}{\sqrt{1 + \beta^2}}$

b.







34,

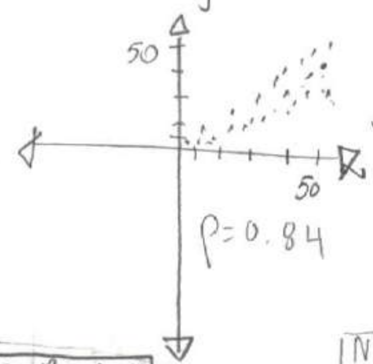
x	y
1	0.94
2	2.09
3	2.05
4	3.13
5	7.03
6	5.96
7	4.09
8	9.78
9	6.78
10	5.84
11	19.42
12	11.77
13	13.80
14	13.77
15	24.25
16	9.72
17	11.28
18	18.36
19	18.04
20	10.88
21	23.99
22	15.37
23	19.51
24	11.56
25	21.99
26	32.89
27	20.71
28	13.53
29	25.76
30	20.54
31	40.26
32	30.81
33	36.15
34	25.18
35	24.26
36	53.2:
37	23.70
38	23.65
39	36.69
40	37.53
41	40.62
42	19.44
43	63.42
44	36.12
45	46.04
46	4743
47	79.14
48	43.81
49	29.17762
50	56.30
X	1275
y	1205
X^2	42925
xy	41073

β_0	-1.22
β_1	0.99
ρ_{xy}	0.84

$y = -1.22 + 0.99x$

X	$Y-\hat{Y}$
1	0.16
2	1.76
3	2.80
4	1.02
5	3.20
6	7.39
7	4.54
8	9.48
9	11.76
10	10.01
11	13.98
12	10.06
13	10.04
14	9.84
15	19.00
16	16.56
17	4.45
18	20.68
19	26.74
20	32.59
21	14.77
22	16.37
23	16.48
24	21.67
25	32.37
26	14.22
27	24.29
28	35.09
29	29.17
30	20.33
31	18.95
31	18.95
33	22.19
34	55.78
35	25.27
36	21.97
37	28.99
38	34.31
39	22.23
40	20.97
41	34.78
42	36.51
43	48.14
44	5855
45	42.29
46	49.99
47	2706
48	28.55
49	62.49
50	41.57

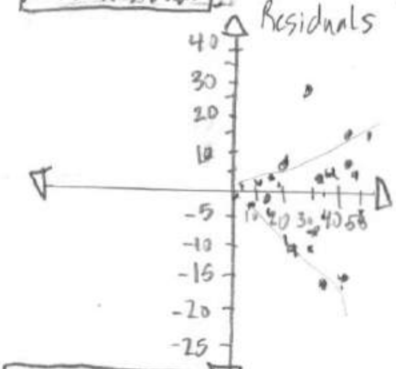
Scatterplot



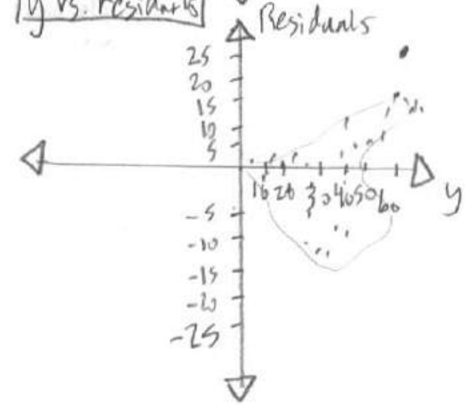
$\rho = 0.84$

X vs. Residuals

Not to scale
-Lost my ruler-



y vs. residuals

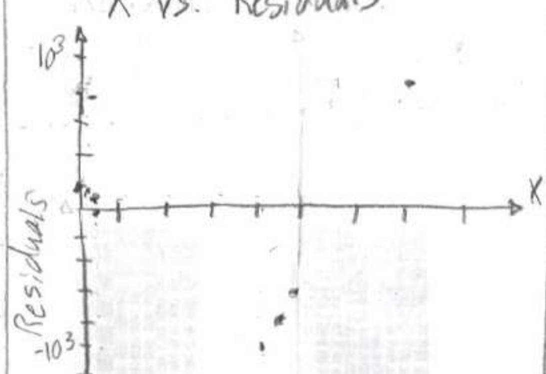


35. $X_3 = X_1 + X_2$; $\mathbb{X} = \begin{pmatrix}1 & X_1\\ 1 & X_2\\ 1 & X_3\end{pmatrix} = \begin{pmatrix}1 & X_1\\ 1 & X_2\\ 1 & X_2 - X_1\end{pmatrix} \xrightarrow{R_3+R_1} \begin{pmatrix}1 & X_1\\ 1 & X_2\\ 2 & X_2\end{pmatrix} \xrightarrow{R_3-R_2} \begin{pmatrix}1 & X_1\\ 1 & X_2\\ 1 & 0\end{pmatrix} \xrightarrow{-1} \begin{pmatrix}0 & X_1\\ 0 & X_2\\ 0 & 0\end{pmatrix} \to \boxed{\begin{pmatrix}0X_1\\ 0X_2\end{pmatrix}}$

36.

$\sum x$	14383
$\sum y$	448764
$\sum xy$	19512852
$\sum x^2$	39764174
β_0	24818
β_1	-8.49

X vs. Residuals



The residuals demonstrate the fit wasn't an exact approximation. Separately, an analysis of variance demonstrated an F-statistic that accepted the alternative.

37. $\ln(\text{Pressure}) = A + \frac{B}{T}$; $\beta_0 = 6.51$; $\beta_1 = 91.66$; $A = 6.51$; $B = 91.66$

$var(\beta_0) = Var(A) = \frac{\sigma^2 \sum X_i^2}{n\sum X_i^2 - (\sum X_i)^2} = 0.2$; $var(\beta_1) = var(B) = \frac{n\sigma^2}{n\sum X_i^2 - (\sum X_i)^2}$

$\hat{A} = 6.51 \pm t\left(\frac{0.05}{2}\right)_{23-2} \cdot \sigma$

$= 6.51 \pm 2.056 \cdot 0.4$

$\boxed{= 6.51 \pm 0.822}$

$\hat{B} = 91.66 \pm t\left(\frac{0.05}{2}\right)_{23-1} \cdot \sigma$ $\boxed{= 180.4}$

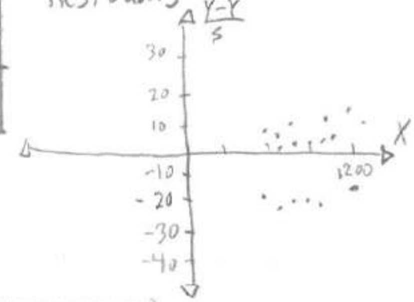
$= 91.66 \pm 2.056 \cdot 13.4$

$\boxed{= 91.66 \pm 27.551}$

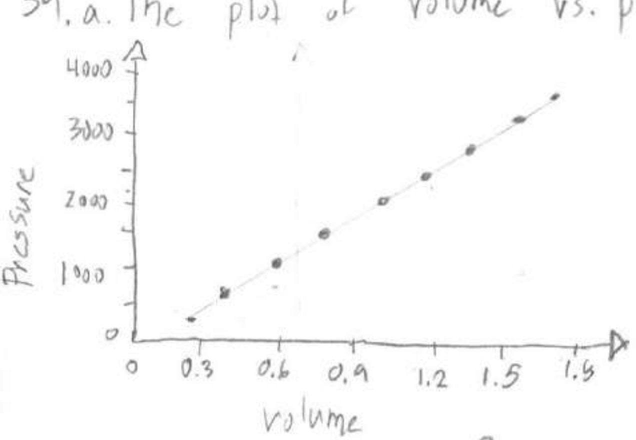
38.

Coefficient	Estimate	Standard Error	Confidence Interval
β_0	628.63	13.423	$\pm$ 27.78
β_1	-0.61	0.018	$\pm$ 0.037

Residuals



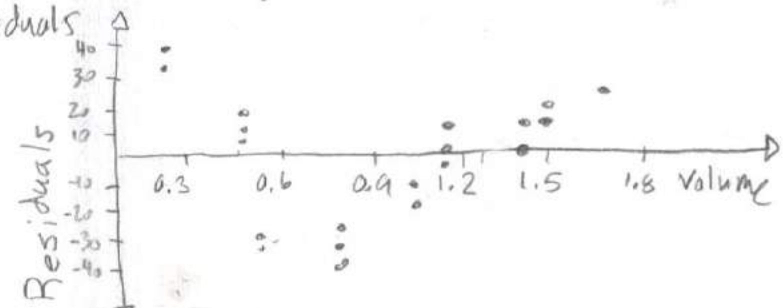
39. a. The plot of volume vs. pressure is linear.



b.

Coefficient	Estimate	Standard Error
β_0	-257.301	9.43
β_1	2316.469	9.24

Residuals



The residual plot is multiple measurement at a given volume.

c. $S = \sum(y - \beta_0 - \beta_1 X - \beta_2 X^2)^2$

$\frac{dS}{d\beta_0} = -2\sum(y - \beta_0 - \beta_1 X - \beta_2 X^2) = 0$

$\sum y = n\beta_0 + \beta_1 \sum X + \beta_2 \sum X^2$

$\frac{dS}{d\beta_1} = -2\sum X(y - \beta_0 - \beta_1 X - \beta_2 X^2) = 0$

$\sum Xy = \beta_0 \sum X + \beta_1 (\sum X)^2 + \beta_2 \sum X^3$

$\frac{dS}{d\beta_2} = -2\sum X^2(y - \beta_0 - \beta_1 X - \beta_2 X^2) = 0$

$\sum X^2 y = \beta_0 \sum X^2 + \beta_1 \sum X^3 + \beta_2 (\sum X)^4$

$$\beta_0 = \frac{D_0}{D} = \frac{\begin{bmatrix} \sum y & \sum X & \sum X^2 \\ \sum Xy & (\sum X)^2 & \sum X^3 \\ \sum X^2 y & \sum X^3 & \sum X^4 \end{bmatrix}}{\begin{bmatrix} n & \sum X & \sum X^2 \\ \sum X & (\sum X)^2 & \sum X^3 \\ \sum X^2 & \sum X^3 & \sum X^4 \end{bmatrix}} \quad \beta_1 = \frac{D_1}{D} = \frac{\begin{bmatrix} n & \sum y & \sum X^2 \\ \sum X & \sum Xy & \sum X^3 \\ \sum X^2 & \sum X^2 y & \sum X^4 \end{bmatrix}}{\begin{bmatrix} n & \sum X & \sum X^2 \\ \sum X & (\sum X)^2 & \sum X^3 \\ \sum X^2 & \sum X^3 & \sum X^4 \end{bmatrix}} \quad \beta_2 = \frac{D_2}{D} = \frac{\begin{bmatrix} n & \sum X & \sum y \\ \sum X & (\sum X)^2 & \sum Xy \\ \sum X^2 & \sum X^3 & \sum X^2 y \end{bmatrix}}{\begin{bmatrix} n & \sum X & \sum X^2 \\ \sum X & (\sum X)^2 & \sum X^3 \\ \sum X^2 & \sum X^3 & \sum X^4 \end{bmatrix}}$$

Cramer's Rule did not work

$$\beta_2 = \frac{[\sum x^2y - \frac{1}{n}\sum x^2 \sum y][\sum x^2 - \frac{1}{n}(\sum x)^2] - [\sum xy - \frac{1}{n}\sum x \sum y][\sum x^3 - \sum x \sum x^2/n]}{[\sum x^2 - (\sum x)^2/n][\sum x^4 - \frac{1}{n}(\sum x^2)^2] - (\sum x^3 - \sum x \sum x^2/n)^2}$$

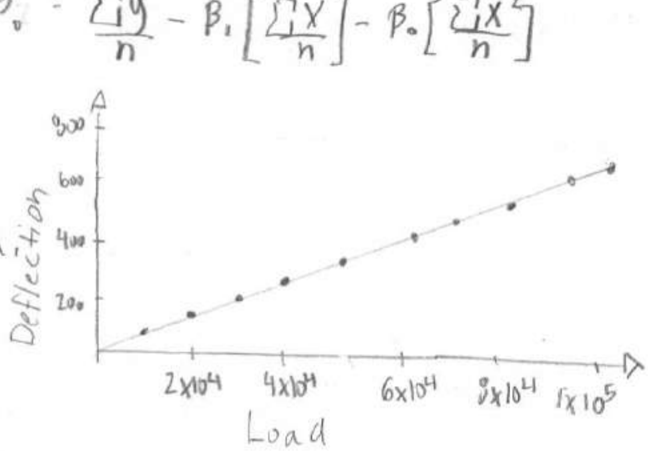
$$\beta_1 = \frac{[\sum xy - \sum x \sum y/n][\sum x^4 - (\sum x^2)^2/n] - [\sum x^2y - \sum x^2 \sum y/n][\sum x^3 - \sum x \sum x^2/n]}{[\sum x^2 - (\sum x)^2/n][\sum x^4 - \frac{1}{n}(\sum x^2)^2] - (\sum x^3 - \sum x \sum x^2/n)^2}$$

$$\beta_0 = \frac{\sum y}{n} - \beta_1\left[\frac{\sum x}{n}\right] - \beta_0\left[\frac{\sum x^2}{n}\right]$$

Coefficient	Estimate
β_0	-205
β_1	2164
β_2	83.19

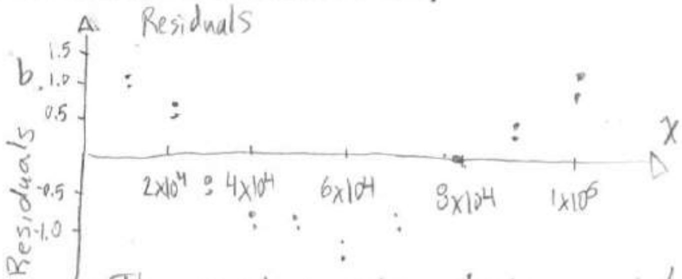
The fit is slightly better but negligible to linear.

40. a.



The plot resembles a linear function.

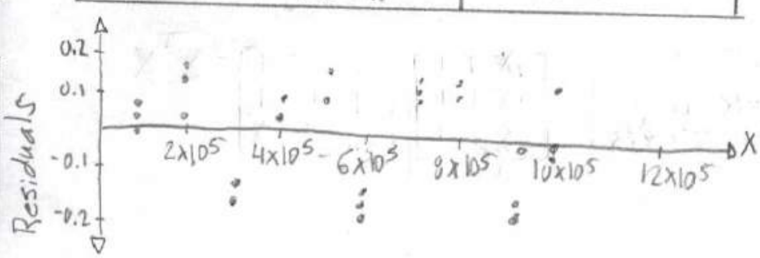
b.



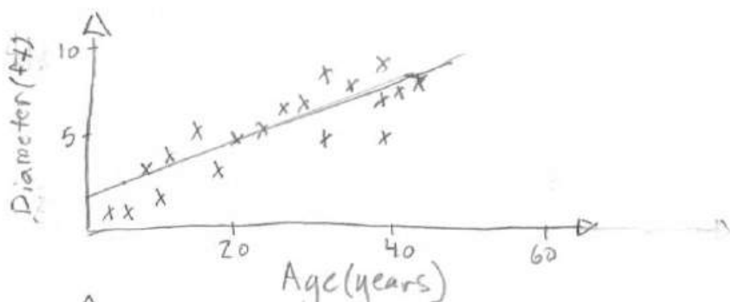
The residual plot helps aid the recognition of regions that over or under fit.

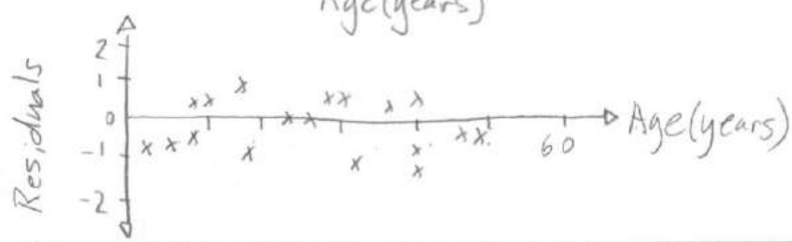
c.

Coefficient	Estimate	Standard Error
β_0	0.1362	0.073
β_1	0.0068	3.04×10^{-6}
β_2	7.29×10^{10}	2.69×10^{-11}



41.





I thought a better fit of Age vs Diameter required Age vs. $(\text{Diameter})^2$, but the book reasons $\sqrt{\text{Age}}$ vs Diameter.

42. The residuals of velocity vs $\sqrt{\text{Distance}}$ were 10x smaller than residuals extracted from velocity vs Distance. A reason to explain the difference is the second-order differential equation $m\ddot{x} = mg - kv^2$, in which re-arranged is

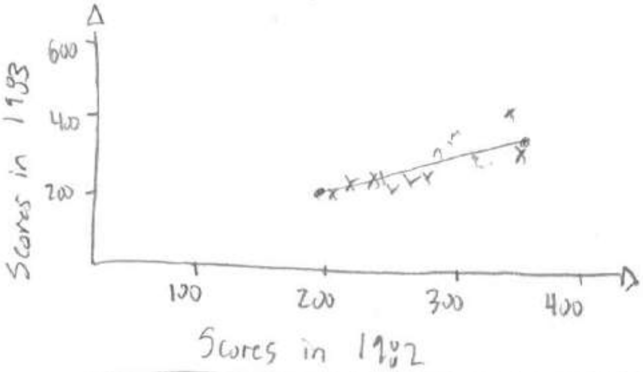
$$v = \sqrt{\frac{m}{k}\left(g - \frac{\Delta\Delta \text{Distance}}{\Delta\Delta \text{time}}\right)}$$

43. The information provided of cyst diameters as a function of temperature looks to be quadratic.

Coefficient	Estimate	Standard Error
β_0	23.86	0.83
β_1	-0.17	0.13
β_2	0.07	0.15
β_3	-0.45	0.18

44. A correlation coefficient ($\rho = -0.37$) signifies weak correlation for the asthma dataset, and similar insignificance for the cystfibr dataset having a correlation of ($\rho = -0.27$). The F-statistic for asthma ($F_{0.95}(40,1) = 0.01 < 6.57$) and cystfibr ($F_{0.95}(22,1) = 0.20 < 1.75$) also say no significance. A plot of either datasets fits were unresolved, having no confident, "yes!"

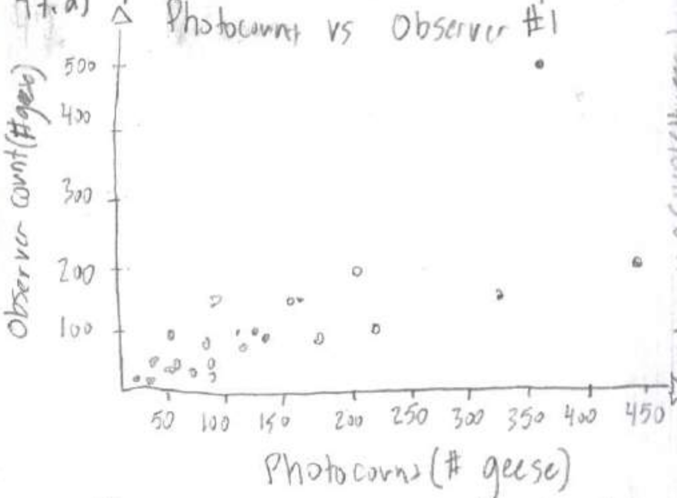
45. The relationship of student reading scores exists from year-to-year.

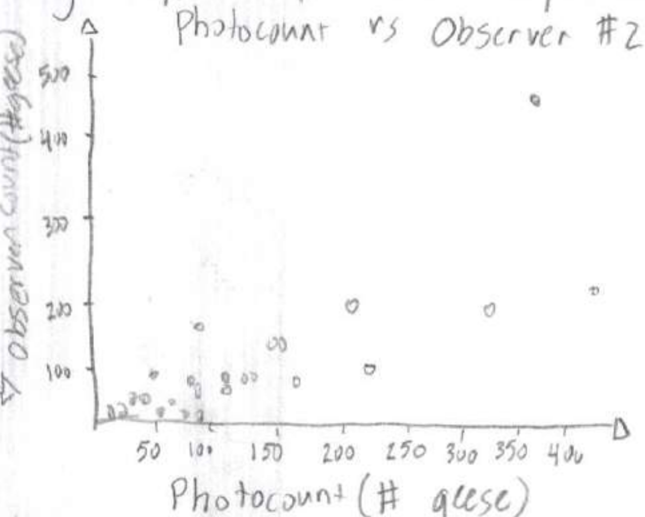


Coefficient	Estimate	Standard Error
β_0	57.54	24.39
β_1	0.80	0.09.

46. Phase Contrast Microscopy (PCM) Fit Scanning Electron Microscopy (SEM) with an R^2 of 0.37. The residuals of data points were as large as the magnitude of the datapoint, but generally, do present a positive correlation.

47. a)





Each plot comparison of the true photocount to the observer looks to be linear.

b)

Coefficient	Estimate	Standard Error
β_0	-4.93	9.31
β_1	0.85	0.07

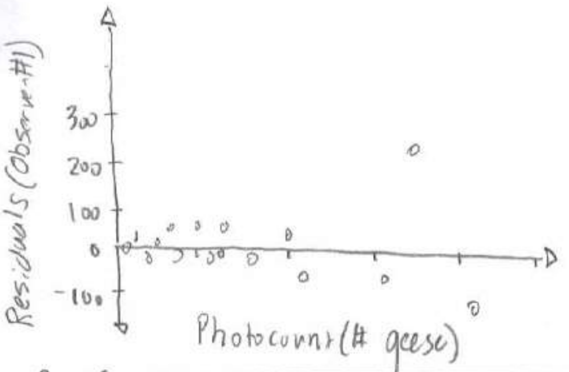
Standard Residual Error: 1815

Coefficient	Estimate	Standard Error
β_0	-4.15	8.71
β_1	1.11	0.07

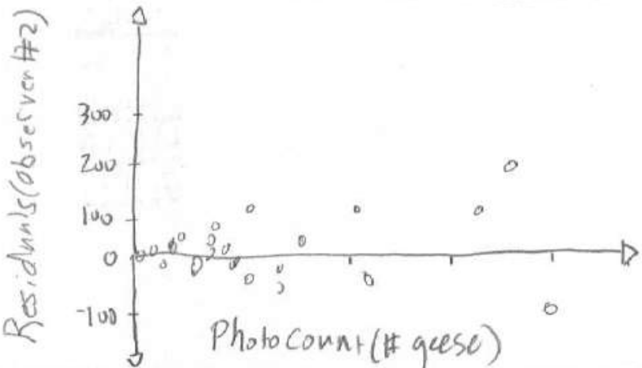
Standard Residual Error: 1584

The coefficients and standard residual error for each plot demonstrate similarities.

Residuals of Observer #1:



Residuals of Observer #2:



c. Square rooting the photocounts did not stabilize variance, and increased the errors across sampled datasets.

d. Again, the original fit was appropriate to the analysis.

e. Both observers either over-, or under-estimated the number of geese by 15%.

48.

Coefficients	Estimate	Standard Error
β_0	-57.99	8.64
β_1	4.71	0.26
β_2	0.34	0.13

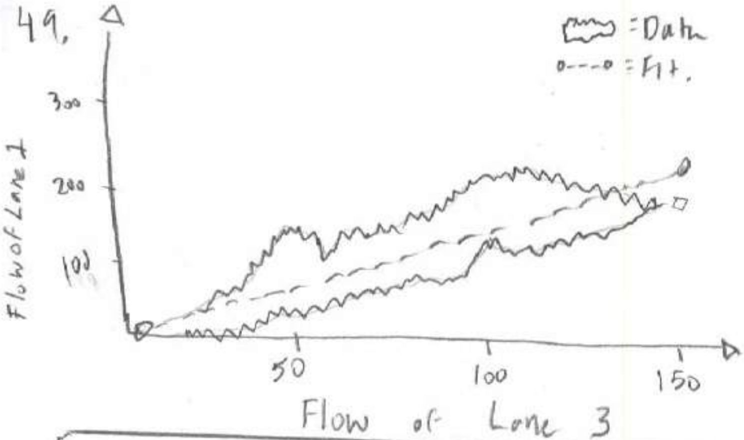
Model: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2$

where X_1 = height, X_2 = Diameter.

Y = Volume.

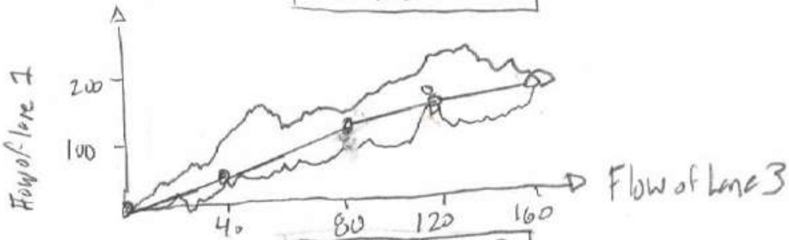
The relationship volume $= \pi \left(\frac{\text{Diameter}}{2}\right)^2 \times \text{Height}$ did not fit well.

49.



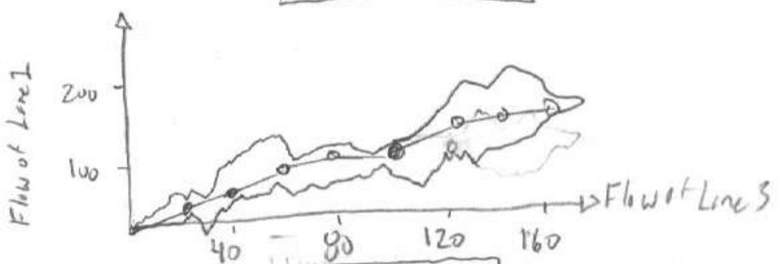
A linear regression of lane 1 vs lane 3 flow is justified by observation and tabulated data.

Bandwidth = 40

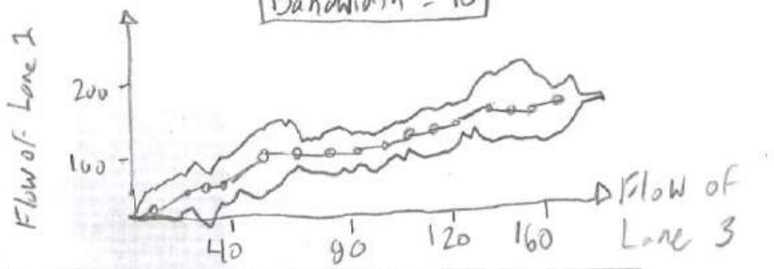


The best balance between a small bandwidth (10) and large (160) became a $w(x_i - \bar{x}) = 40$. My perspective is only by observation and not numerical determinations.

Bandwidth = 20

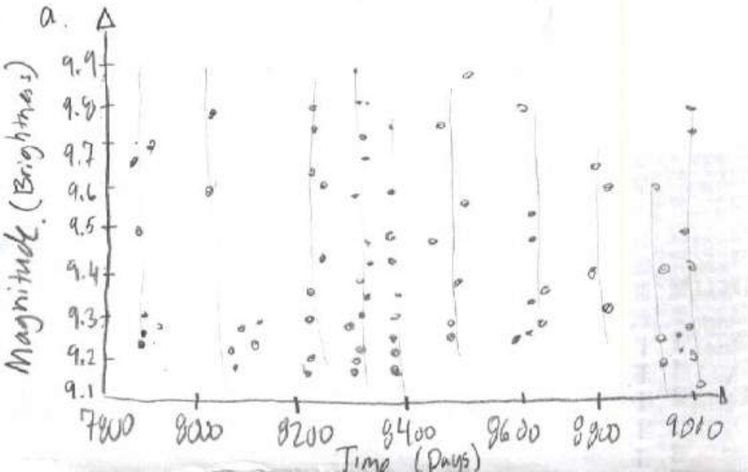


Bandwidth = 10



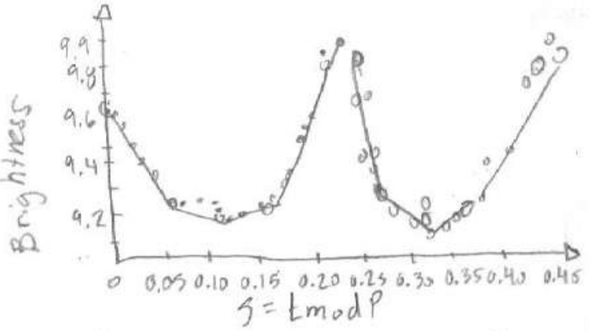
50.

a.



The structure of time vs brightness seems to be measurements near the same time.

b. $P = 0.407528$ days ; $S = t \bmod P$ c. see part b)



The structure seems to be periodic from the $S = t \bmod P$ transformation.

d) As the modular period changes, the graph becomes less periodic. The true period could be estimated with a sinusoidal wave.

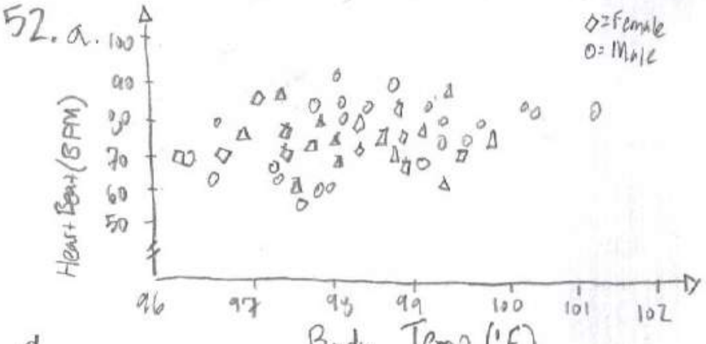
51. Model : Disney $= \beta_0 + \beta_1 \cdot$ MacDonalds $+ \beta_2 \cdot$ Schumberger $+ \beta_3 \cdot$ Haliburton.

Coefficient	Estimate	Standard Error
β_0	0.0938	0.0981
β_1	-0.8812	1.1045
β_2	1.3151	1.9877
β_3	-0.1717	1.3442

Standard Deviation of Residuals = 0.056
R^2 of Fit = 0.6298.
The fitted errors of multiple linear regression
$$S = \sum_i (y_i - \beta_0 - \beta_1 X_1 - \beta_2 X_2 - \beta_3 X_3)^2$$
has a standard deviation of 0.002.

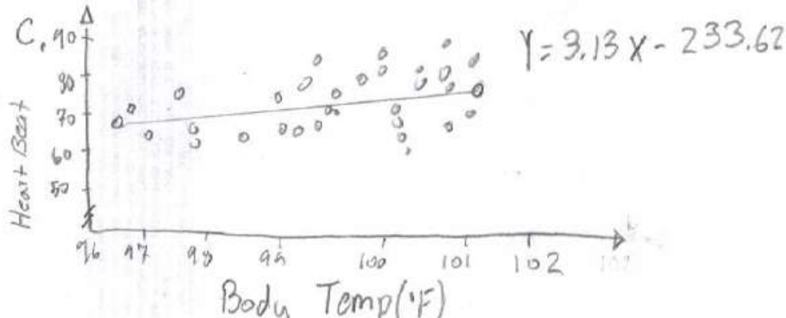
The comparison of monthly returns from 1998 to 1999 coefficient estimates of different values. The fundamental model has three independent variables that represent individual stock price.

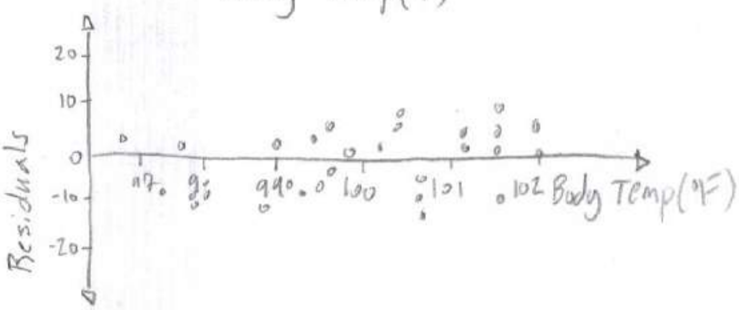
52. a.



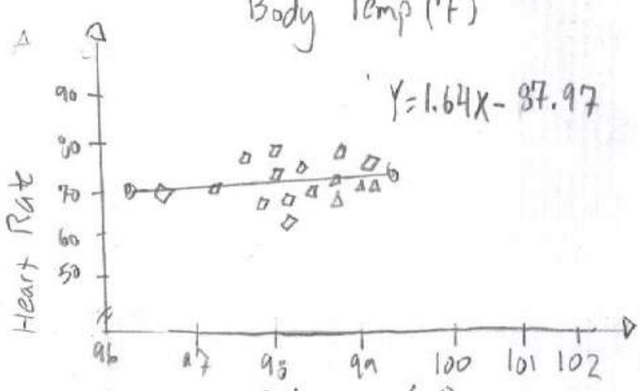
b. Male vs female data do relate through a positive correlation. value ($\rho_{male} = 0.28$, $\rho_{female} = 0.20$).

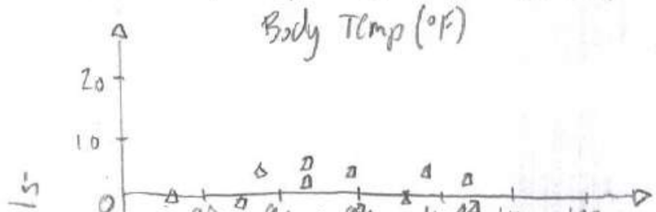
C.





d.





The R^2 of 0.02 demonstrates increasingly minor correlation to a line.

Male body temperature is regressed to a linear fit with a standard error of 7.70 for residuals. The R^2 of 0.08 may indicate a slightly positive relationship to body measurements.