Chapter 15: Minimax Lower Bounds:

15.1
$$\|\mathbb{P}_1 - \mathbb{P}_0\|_{TV} = \sup \int (p_1(x) - p_0(x)) f_1 \, dx$$

$$= 1 - (1 + \sup \int (p_1(x) - p_0(x)) f_1 \, dx$$

$$= 1 - (\int p_0(x) \, dx + \sup \int (p_1(x) - p_0(x)) f_1 \, dx$$

$$= 1 - \left(\int (1 - f_1) p_0(x) \, dx + \int p_1(x) f_1 \, dx \right)$$

$$= 1 - \inf_{f_0 + f_1 \geq 1} \left(\int f_0 \, p_0(x) \, dx + \int p_1(x) f_1 \, dx \right)$$

$$= 1 - \inf_{f_0 + f_1 \geq 1} \left(\mathbb{E}_0[f_0] + \mathbb{E}_1[f_1] \right)$$

15.2

a) $H(Q) = H(X)$

$$= -\sum q(x) \log q(x)$$

$H(x) \geq 0$: Proof by Induction

Base case $(k = \emptyset)$: $H(X) = -\sum_{i=0}^{k} q(x) \log q(x)$

$$= -\sum_{i=0}^{0} i \cdot \log i$$

$$= 0 \cdot \log 0$$

$$= 0$$

Books Definition

$0 \cdot \log 0 = 0$

Next Step ($k=n$): $H(X) = -\sum_{i=0}^{k} q(x) \log q(x)$

$= -\sum_{i=0}^{n} i \cdot \log i$

$\sum_{i=1}^{n} i \log i$

$\sum_{i=1}^{n} \log i^{i}$

≥ 0

Induction Step($k=n+1$): $H(X) = \sum_{i=0}^{k} q(x) \log q(x)$

$= \sum_{i=0}^{n+1} i \cdot \log i$

$= \sum_{i=1}^{n+1} i \log i$

$= (n+1) \log n+1 + \sum_{i=1}^{n} i \log i$

$= \log\left[(n+1)^{(n+1)}\right] + \sum_{i=1}^{n} \log i^{i}$

≥ 0

b) $H(x) \leq \log|X|$

<u>Proof by Deduction:</u>

If $p(x) = X$, then $H(x) = -\sum_{i=1}^{n} p(x) \log(p(x)) = -\sum_{i=1}^{n} X \log X \leq \log X$

15.3

a) $D(\mathbb{P} \| \mathbb{Q}) = \int \log \frac{p(x)}{q(x)} p(x)\, dx$

$D(\mathbb{P} \| \mathbb{Q}) \geq 0$: Proof by Deduction

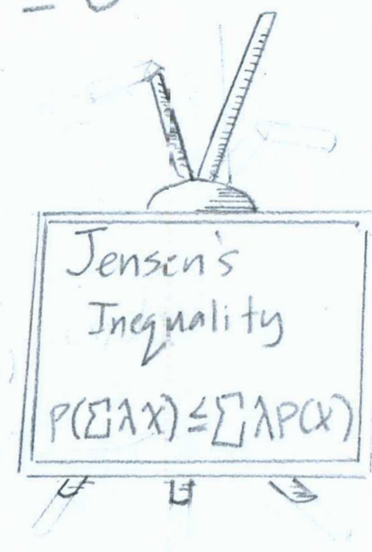
If $p(x) = q(x)$, then $D(\mathbb{P} \| \mathbb{Q}) = \int \log \frac{p(x)}{q(x)} p(x)\, dx$

$$= \int \log \frac{p(x)}{p(x)} p(x)\, dx$$

$$\geq 0$$

b) $D(\sum_{j=1}^{m} \lambda_j \mathbb{P}_j \| \mathbb{Q}) = \int \log \frac{\sum_{j=1}^{m} \lambda_j P(x)}{Q(x)} p(x)\, dx$

$$\leq \sum_{i=1}^{m} \lambda_i \int \log \frac{P(x)}{Q(x)} P(x)\, dx \quad \text{by}$$



$$\leq \sum_{i=1}^{m} \lambda_i D(\mathbb{P} | \mathbb{Q})$$

$$D(\mathbb{Q} \| \sum_{i=1}^{m} \lambda_i P) = \int \log \frac{Q(x)}{\sum_{i=1}^{m} \lambda_i P(x)} Q(x)\, dx$$

$$\leq \frac{1}{\sum_{i=1}^{m} \lambda_i} \int \log \frac{Q(x)}{P(x)} Q(x)\, dx$$

$$\leq \sum_{i=1}^{m} \lambda_i D(\mathbb{Q} \| \mathbb{P})$$

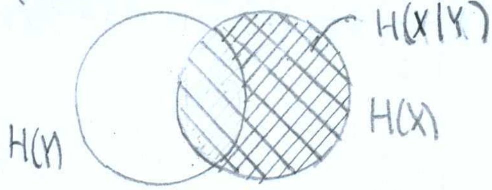
c) (Equation 15.11a)

$$D(\mathbb{P}^{1:n} \| \mathbb{Q}^{1:n}) = \sum_{i=1}^{n} D(\mathbb{P} \| \mathbb{Q}) = n\, D(\mathbb{P} \| \mathbb{Q})$$

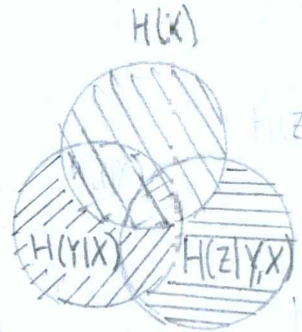
15.4 (Equation 15.59) $H(X|Y) := \mathbb{E}_Y[H(Q_{X:Y})]$

$$= \mathbb{E}_Y\left[\int_X q(x|Y)\log q(x|Y)\,\mu(dx)\right]$$

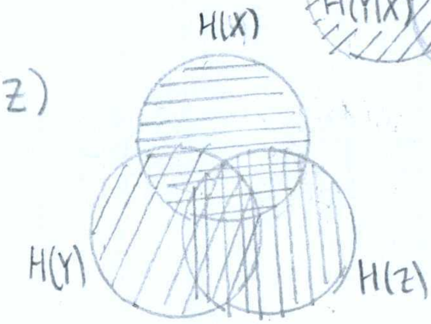
a) $H(X|Y) \leq H(X)$



b) $H(X,Y,Z) = H(X) + H(Y|X) + H(Z|Y,X)$



c) $H(X,Y,Z) \leq H(X) + H(Y) + H(Z)$



15.5 (Equation 15.10) $\|\mathbb{P}-\mathbb{Q}\|_{TV}^2 = \frac{1}{4}(p-q)^2$

$$= \frac{1}{4}(\sqrt{p}-\sqrt{q})^2(\sqrt{p}+\sqrt{q})^2$$

$$= \frac{1}{4}(\sqrt{p}-\sqrt{q})^2(2p+2q-(\sqrt{p}-\sqrt{q})^2)$$

$$= \frac{1}{4}H(P\|Q)^2(4-H(P\|Q)^2)$$

$$\|\mathbb{P}-\mathbb{Q}\|_{TV}^2 \leq H(\mathbb{P}\|\mathbb{Q})^2\sqrt{1-H(P\|Q)^2}$$

(Lemma 15.2) "Pinsker-Csiskar-Kullback Inequality"

For all disturbances $\mathbb{P}$ and $\mathbb{Q}$,

$$\|\mathbb{P}-\mathbb{Q}\| \leq \sqrt{\tfrac{1}{2}D(\mathbb{Q}\|\mathbb{P})}$$

"The Lemma models (or describes) the Euclidean Distance from a function to another function. An upper bound helps in a function's selection." A new connection between network nodes or web of nodes or branch in a tree across segments."

15.6 a) Bernouilli Distribution: $p(x) = P[X=x] = \begin{cases} q = 1-p & x=0 \\ p & x=1 \end{cases}$

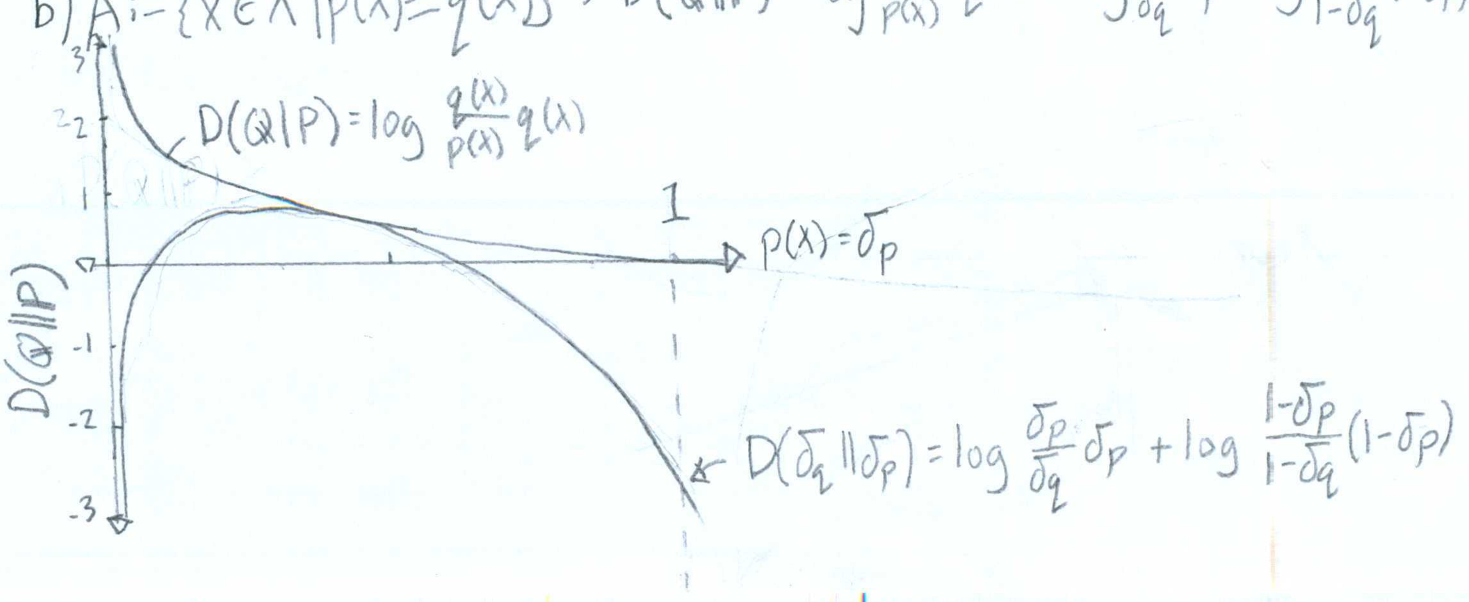
$$\|P-Q\| = (\delta_p - \delta_q) \leq \sqrt{\tfrac{1}{2} D(Q\|P)}$$

$$2(\delta_p - \delta_q)^2 \leq D(Q\|P)$$

$$\leq \sum_{x=0}^{1} \log \frac{q(x)}{p(x)} q(x)$$

$$\leq \log \frac{\delta_p}{\delta_q} \delta_p + \log \frac{(1-\delta_p)}{(1-\delta_q)} (1-\delta_p)$$

b) $A := \{x \in X \mid p(x) \geq q(x)\}$; $D(Q\|P) = \log \frac{q(x)}{p(x)} q(x) \geq \log \frac{\delta_p}{\delta_q} \delta_p + \log \frac{1-\delta_p}{1-\delta_q}(1-\delta_p)$



15.7 (Equation 15.12a) $\frac{1}{2}H^2(\mathbb{P}^{1:n} \| \mathbb{Q}^{1:n}) = 1 - \prod_{i=1}^{n}(1 - \frac{1}{2}H^2(\mathbb{P}_i \| \mathbb{Q}_i))$

Proof:

$$\frac{1}{2}H^2(\mathbb{P}^{1:n} \| \mathbb{Q}^{1:n}) = \frac{1}{2}\left(\int (\sqrt{p(x)} - \sqrt{q(x)})^2 \nu\, dx\right.$$

$$= \frac{1}{2}\int (p(x) + q(x) - 2\sqrt{p(x)q(x)})\nu\, dx$$

$$= \frac{1}{2}\int p(x)\nu\, dx + \frac{1}{2}\int q(x)\nu\, dx - \int \sqrt{p(x)q(x)}\nu\, dx$$

$$= 1 - \int \sqrt{p(x)q(x)}\nu\, dx$$

$$= 1 - \prod_{i=1}^{n}(1 - \frac{1}{2}H^2(\mathbb{P}_i \| \mathbb{Q}_i))$$

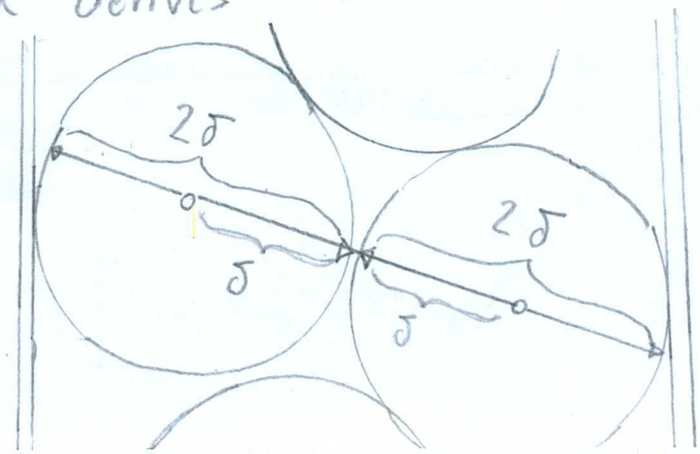
(Lemma 15.9) "Le Cam"

For any 2δ-separated classes of distributions $\mathbb{P}_0$ and $\mathbb{P}_1$ within P, any estimators worst-case:

$$\sup_{\mathbb{P}\in P} \mathbb{E}\left[\rho(\hat{\theta}, \theta(\mathbb{P}))\right] \geq \frac{\delta}{2} \sup_{P_0, P_1}\{1 - \|\mathbb{P}_0 - \mathbb{P}_1\|_{TV}\}$$

"A take on the lemma, average error's minimum is half at minimum."

"The book derives the lemma from "ball-packing" error"



$\rho(\theta(\mathbb{P}_0), \theta(\mathbb{P}_1)) \geq 2\delta$

"The metric presumes "the lowest error" in the space of other radial errors."

$$\|P_0^n - P_0^n\|_{TV} \leq \sqrt{\frac{1}{4}\left\{e^{n\theta^2/\sigma^2} - 1\right\}}$$

$$= \sqrt{\frac{1}{4}\left\{e^{4n\delta^2/\sigma^2} - 1\right\}} \quad \text{where } \theta = 2\delta$$

$$\inf_{\hat{\theta}} \sup_{\theta \in \mathbb{R}} \mathbb{E}_\theta\left[(\hat{\theta} - \theta)^2\right] \geq \frac{\delta^2}{2}\left\{1 - \frac{1}{2}\sqrt{e-1}\right\} \quad \text{"Lemma 15.9"}$$

$$\geq \frac{\delta^2}{4} \quad \text{where } \delta = \frac{1}{2}\frac{\sigma}{\sqrt{n}}$$

$$= \frac{1}{16}\frac{\sigma}{\sqrt{n}}$$

$$\inf_{\hat{\theta}} \sup_{\theta \in \mathbb{R}} \mathbb{E}_\theta\left[(\hat{\theta} - \theta)^2\right] \geq \frac{\delta^2}{2}\left\{1 - \frac{1}{2}\sqrt{e-1}\right\}$$

$$\geq \frac{\delta^2}{4} \quad \text{where } \delta = \frac{1}{2}\frac{\sigma}{\sqrt{n}}$$

$$= \frac{1}{16}\frac{\sigma^2}{n}$$

15.9 $\hat{\theta} = \min[Y_1, \ldots, Y_n]$

$$\sup_{\theta \in \mathbb{R}} \mathbb{E}\left[(\hat{\theta} - \theta)^2\right] = \int_0^1 \left(1 - \|P_1 - P_n\|_{TV}\right)^n dt \quad \text{"Equation 15.14"}$$

$$= \int_0^1 \left(1 - \mathbb{P}[Z^2 \geq t]\right)^n dt$$

$$= \int_0^1 \left(1 - \mathbb{P}[Z \geq \sqrt{t}]\right)^n dt$$

$$= \int_0^1 \left(1 - \sqrt{t}\right)^n dt$$

$$= \frac{2}{n^2 + 3n + 2}$$

$$\leq \frac{2}{n^2}$$

15.10

a) $\|P - Q\|_{TV}^2 \leq \frac{1}{2} D(P \| Q)$ "Lemma 15.2"

$$\leq \frac{1}{2} \int \log \frac{p(x)}{q(x)} \, p(x) \, v(dx) \qquad \log(x) \leq x - 1$$

$$\leq \frac{1}{2}\left(\int \frac{p^2(x)}{q(x)} v(dx) - 1\right)$$

b) $\| \mathbb{P}_\theta^n - \mathbb{P}_0^n \|_{TV}^2 \leq \frac{1}{2}\left[\left(\int \frac{p^2(x)}{q(x)} v(dx)\right)^n - 1\right]$

$$\leq \frac{1}{2}\left[\left(\frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} \frac{e^{\frac{-n(x-2\theta)^2}{2\sigma^2}}}{e^{\frac{-2n\theta^2}{2\sigma^2}}}\right)^n - 1\right]$$

$$\leq \frac{1}{2}\left[\frac{1}{\sqrt{2\pi\sigma^2}} \sqrt{\frac{\pi}{2n}} \, \sigma \, e^{n\theta^2/\sigma^2} \operatorname{erf}\left(\frac{\sqrt{n}(x-2\theta)}{\sqrt{2}\sigma}\right) - 1\right]$$

$$\leq \frac{1}{4}\left[\frac{e^{n\theta^2/\sigma^2}}{\sqrt{n}} - 1\right]$$

$$\leq \frac{1}{4}\left[e^{n\theta^2/\sigma^2} - 1\right]$$

c) $\|P - P_0\|_{TV}^2 \leq \frac{1}{2}\left[\left(\int \frac{p^2(x)}{q(x)} v(dx)\right)^n - 1\right]$

$$\leq \frac{1}{2}\left[\left(\frac{1}{2}\int \frac{p_\theta^2(x)}{q(x)} v(dx) + \frac{1}{2}\int \frac{p_\theta^2(x)}{q(x)} v(dx)\right)^n - 1\right]$$

$$\leq \frac{1}{2}\left[\left(\int_{-\infty}^{\infty} \frac{e^{\frac{-2n(x+2\theta)^2}{2\sigma^2}}}{e^{\frac{-4n\theta^2}{2\sigma^2}}} v(dx)\right)^n - 1\right]$$

$$\leq \frac{1}{4}\left[e^{2n\theta^2/n^2} - 1\right]$$

15.11 $\bar{Q} = \frac{1}{M} \sum_{j=1}^{M} \mathbb{P}_j$

$$\frac{1}{M}\sum_{j=1}^{M} D(\mathbb{P}_j \| \bar{Q}) = \frac{1}{M}\sum_{j=1}^{M} \int \log \frac{\mathbb{P}(x)}{Q(x)} \mathbb{P}(x)\, \nu(dx)$$

$$= \frac{1}{M}\sum_{j=1}^{M} \int \log \frac{\mathbb{P}(x)}{\frac{1}{M}\sum_{j=1}^{M} \mathbb{P}_j} \mathbb{P}(x)\, \nu(dx)$$

$$= \frac{1}{M}\sum_{j=1}^{M} \int \log \frac{\mathbb{P}(x)}{Q_1(x)} \mathbb{P}(x)\nu(dx) + \int \log \frac{\mathbb{P}(x)}{Q_2(x)} \mathbb{P}_2(x)\nu(dx)$$

$$+ \cdots$$

$$= \frac{1}{M}\sum_{j=1}^{M} D(\mathbb{P} \| Q)$$

15.12 $D_f(\mathbb{P} \| Q) = \int q(x) f(p(x)/q(x))\, \nu(dx)$

a) $f = t\log t$; $D_f(\mathbb{P}\|Q) = \int q(x) \frac{p(x)}{q(x)} \log \frac{p(x)}{q(x)} \nu(dx)$

$$= \int p(x) \log \frac{p(x)}{q(x)} \nu(dx)$$

$$= D(\mathbb{P} \| Q)$$

b) $f = -\log t$; $D_f(\mathbb{P}\|Q) = \int q(x)\left(-\log \frac{p(x)}{q(x)}\right) \nu(dx)$

$$= -D(\mathbb{P} \| Q)$$

c) $H^2(\mathbb{P}\|Q) = \int (\sqrt{p} - \sqrt{q})^2 \nu(dx)$

$$= \int q \left(\frac{\sqrt{p}}{\sqrt{q}} - 1\right)^2 \nu(dx)$$

$= D(\mathbb{P} \| \mathbb{Q})$ where $f(t) = t - 1$

d) $f(t) = 1 - \sqrt{t}$: $D(\mathbb{P} \| \mathbb{Q}) = \int q(x) \left(1 - \frac{\sqrt{p(x)}}{\sqrt{q(x)}}\right) v(dx)$

15.14. $\sigma > 0$, $\int_{-\infty}^{\infty} x q(x) = 0$

$$\int_{-\infty}^{\infty} x^2 q(x) \leq \sigma^2$$

$$H(p) = \int p \log p \, dv$$

$$= \int_{-\infty}^{\infty} \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi}\,\sigma} \log \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi}\,\sigma} \, dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \frac{-x^2}{2\sigma^2} \cdot e^{-x^2/2\sigma^2} dx - \frac{\log \sqrt{2\pi}\,\sigma}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} dx$$

$$= \frac{-1}{4\sigma^2} - \frac{\log(\sqrt{2\pi}\,\sigma)}{2} \cdot \operatorname{erf}\left(\frac{x}{\sqrt{2}\,\sigma}\right)$$

$$= \frac{1}{2}\left[\frac{1}{2\sigma^2} - \log(\sqrt{2\pi}\,\sigma)\right]$$

$$= \frac{1}{2}\left[\log \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi}\,\sigma}\right]$$

$$= \frac{1}{2}\left[\log N(x, \sigma^2)\right]$$

15.13. $Q = \frac{1}{(2\pi)^{k/2}\sqrt{\Sigma_i}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma_i^{-1} (x-\mu)\right)$

$$D(Q_1 \| Q_2) = \int \log \frac{Q_1}{Q_2} Q_1 \, dx$$

$$= \int_{-\infty}^{\infty} \log \frac{\frac{1}{(2\pi)^{k/2}\sqrt{\Sigma_1}} \exp(-\frac{1}{2}(X-\mu_1)^T \Sigma_1^{-1} (X-\mu_1))}{\frac{1}{(2\pi)^{k/2}\sqrt{\Sigma_2}} \exp(-\frac{1}{2}(X-\mu_2)^T \Sigma_2^{-1} (X-\mu_2))}$$

$$\times \frac{1}{(2\pi)^{k/2}\sqrt{\Sigma_1}} \exp(-\frac{1}{2}(X-\mu_1)^T \Sigma_1^{-1} (X-\mu_1))\, dx$$

$$= \log \frac{\sqrt{\Sigma_1}}{\sqrt{\Sigma_2}} \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{k/2}\sqrt{\Sigma_1}} \circ \exp(-\frac{1}{2}(X-\mu_1)^T \Sigma_1^{-1} (X-\mu_1))\, dx$$

$$+ \frac{1}{(2\pi)^{k/2}\sqrt{\Sigma_1}} \int_{-\infty}^{\infty} \exp(-\frac{1}{2}(X-\mu_1)^T \Sigma_1^{-1} (X-\mu_1)) \circ (-\frac{1}{2}(X-\mu_1)^T \Sigma_1^{-1} (X-\mu_1))\, dx$$

$$+ \frac{1}{(2\pi)^{k/2}\sqrt{\Sigma_1}} \int_{-\infty}^{\infty} \exp(-\frac{1}{2}(X-\mu_1)^T \Sigma_1^{-1} (X-\mu_1))(-\frac{1}{2}(X-\mu_2) \Sigma_2^{-1} (X-\mu_2))\, dx$$

$$= \frac{1}{2}\left[\log \frac{\sqrt{\Sigma_1}}{\sqrt{\Sigma_2}} - d + tr\left(\frac{\Sigma_1}{\Sigma_2}\right) + (\mu_2-\mu_1)^T \Sigma^{-1} (\mu_2-\mu_1)\right]$$

$$= \frac{1}{2}\left[(\mu_1-\mu_2) \Sigma^{-1} (\mu_1-\mu_2)\right] \quad \text{when } \Sigma_1 = \Sigma_2$$

b) See the previous two lines.

15.15.

$$[\Theta^j(\varepsilon)] = \begin{cases} \sqrt{(1-\theta^2)/(s-1)} & l \in S \\ \Theta_j \varepsilon_l = \Theta_j \varepsilon_j & l = j \\ 0 & \text{otherwise} \end{cases}$$

$$T_{SS} = I_{S-1} + v\,\frac{1-\theta^2}{S-1} \qquad T_{S^cS^c} = I_{d-S+1} + \frac{\theta^2}{d-S+1}$$

$$\mathbb{E}\left[\log(\mathrm{cov}(\mathcal{E}(\varepsilon)))\right] = \log\left[1 + \frac{v(S-1)(1-\theta^2)}{S-1}\right] + (d-S+1)\log\left(1+\frac{v\theta^2}{(d-S+1)}\right) - \log(1+v)$$

$$= \log\left[\frac{1+v(1-\theta^2)}{1+v}\right] + (d-S+1)\log\left[1+\frac{v\theta^2}{(d-S+1)}\right]$$

$$\leq \frac{v(1-\theta^2)}{1+v} + v\theta^2 \quad \text{when } \log(1+x) \cong x$$

$$\leq \frac{v - v\theta^2 + v\theta^2 + v^2\theta^2}{1+v}$$

$$\leq \frac{v\theta^2}{1+v}$$

$$\frac{v\theta^2}{1+v} \leq \frac{c_0}{n}\log(d-S+1)$$

$$n \leq c_0\frac{(1+v)}{v\theta^2}\log(d-S+1)$$

15.16 Sup $\theta^* \in \mathbb{B}_0(S) \cap S^{d-1}$ $\mathbb{E}\left[\|\hat{\theta} - \theta^*\|_2^2\right] \leq tr(\sigma^2)\, S \log\frac{ed}{S}$ "Chapter 6"

$$\leq \frac{c_0(v+1)}{nv^2}\, S\log\frac{ed}{S} \quad \text{where } \sigma^2 = \frac{c_0}{n}\frac{v+1}{v^2}$$

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15.17. $\theta^* \in \mathbb{R}^d$; $\mathbb{P}_\theta(y_1, \dots y_n) = \prod_{i=1}^{n}\left[h(y_i)\exp\left(\frac{y_i\langle x_i, \theta\rangle - \Phi\langle x_i, \theta\rangle}{s(\sigma)}\right)\right]$

a) $D(\mathbb{P}_0 \| \mathbb{P}_{0'}) = \int \log \frac{\mathbb{P}_0}{\mathbb{P}_{0'}} \mathbb{P}_0 \, dx$

$$= \int \log P_0 \cdot P_0 \, dx - \int \log P_{0'} \cdot P_0 \, dx$$

$$= \int n \log h(y_i) P_0 \, dx + n \int \frac{h(y_i)^2 [y_i \langle X_i, \theta_1 \rangle - \Phi \langle X, \theta_1 \rangle]}{s(\sigma)} P_0 \, dx$$

$$- \int n \log h(y_i) \, dx - n \int \frac{h(y_i)^2 [y_i \langle x_i, \theta_1 \rangle - \Phi \langle X_i, \theta_2 \rangle - \langle X_i, \theta \rangle]}{s(\sigma)} \times P_0 \, dx$$

$$= n \int \frac{h(y_i)^2 [y_i [(X_i, \theta_1) - (X_i, \theta_2)] - \Phi [\langle X_i, \theta_2 \rangle - \langle X_i, \theta_1 \rangle] - \Phi \langle X, \theta_1 \rangle] P_0}{s(\sigma)}$$

b) $L = \frac{n \, h(y_i)^2}{s(\sigma)}$

c) ???

d) The Lipschitz constant in part b represents the lowest regression error because the smallest distance, information distance between two columns, functions or rows.

15.18

a) $\inf_{\hat{f}} \sup_{f \in \mathcal{F}_{\text{hom}}} \mathbb{E}[\|\hat{f}-f\|_2^2] = d\left(\frac{\sigma^2}{n}\right)^{\frac{2\alpha}{2\alpha+1}}$ (Equation 15.55)

"Yang-Barron Technique"

b) $\inf_{\hat{f}} \sup_{f \in \mathbb{F}_{\text{add}}} \mathbb{E}[\|\hat{f}-f\|_2^2] = \frac{1}{M}\sum_{j=1}^{M} D(\mathbb{Q}\|\mathbb{P})$

$$= \frac{1}{M}\left\{\log N + \frac{n}{2}\mathbb{E}[\|\hat{f}-f\|_n^2]\right]$$

$$= s\left(\frac{\sigma^2}{n}\right)^{\frac{2\alpha}{2\alpha+1}} + \frac{s\log(ed/s)}{n}$$

"Raskutti, Wainwright, Yu, (2012)"

These guys argue minimax optimal rates for sparse matrices. The derivation stems in Chapter 2 with "Martingale" exponential-cutoffs in datasets dependent upon number of measurements (n), observations (d), sparsity (s), standard deviation (σ) and n^{th} derivatives (α). The two term output is a traditional exponential-cutoff about data dimensions with an additive n^{th} derivative term for extraneous change. An application invades a two row (or column) "Microsoft Excel spreadsheet."

A normal person in "Microsoft Excel" writes numbers in two rows. The expected difference between columns and then rows is predictable.

Average information.