Ballooning in a levitated dipole field.

$$\delta W_2 = \frac{1}{2\mu_0}\int_{V_p} d^3r \left\{ \underbrace{|\vec{k}_\perp|^2 |\vec{b}\cdot\nabla X|^2}_{\text{stabilising line bending}} - \underbrace{\frac{2\mu_0}{B^2}\left((\vec{b}\wedge\vec{k}_\perp)\cdot\nabla p\right)\left((\vec{b}\wedge\vec{k}_\perp)\cdot\vec{\kappa}\right)|X|^2}_{\text{destabilising bad curvature}} \right\}$$

$$\text{for} \quad \vec{\xi} = \vec{\eta}\, e^{iS} = \Big(\overbrace{(\vec{\eta}_{0,\perp} + \vec{\eta}_{1,\perp} + \dots)}^{\vec{\eta}_\perp} + \vec{\eta}_\parallel\Big)\, e^{iS(\psi,\theta)}$$

where $\vec{\eta}_\parallel$ is such that: $\nabla\cdot\vec{\xi} = 0$, $\vec{\eta}_{0\perp} = \dfrac{X(r)}{B}\,\vec{k}_\perp\wedge\vec{b}$,

$$i\vec{k}_\perp\cdot\vec{\eta}_{1,\perp} = -2\vec{\eta}_{\perp,0}\cdot\vec{\kappa} - \nabla\cdot\vec{\eta}_{\perp,0}$$

and $\vec{k}_\perp = \nabla S$

For a dipole field, $\vec{B} = B_\theta \nabla\theta$ exclusively (we have no toroidal field) and we also have $j_\theta = j_\parallel = 0$. Thus we have no kink driving terms.

Since we have no B_θ, q is not a function that is defined here, so there is no shear. We thus don't need the ballooning transform, and the quasimodes are the real displacements.

Working in $(\nabla\psi, \nabla l, \nabla\phi)$ where ψ is the poloidal flux function from the GS equation: $\Delta^*\psi = -\mu_0 R^2 \dfrac{dp}{d\psi}$.

$l \simeq$ distance along a field line.

$$\vec{B} = \nabla\psi \wedge \nabla\phi, \quad \nabla p = \frac{\partial p}{\partial \psi}\nabla\psi, \quad \vec{\kappa} = (\vec{b}\cdot\nabla)\vec{b}$$

$$J^{-1} = \nabla\phi \cdot (\nabla\psi \wedge \nabla l)$$

$$\vec{B} = \underbrace{(\nabla l \cdot \nabla\psi)}_{=0 \text{ by definition}} \frac{J}{R^2}\nabla\psi - |\nabla\psi|^2 \frac{J}{R^2}\nabla l = -|\nabla\psi|^2 \frac{J}{R^2}\nabla l = \vec{B}$$

This implies the potentially useful conversion $\nabla l = -\dfrac{R^2}{J|\nabla\psi|^2}\nabla\psi\wedge\nabla\phi$

We can work out what J is right away, since:

$$\vec{B} = \nabla\psi\wedge\nabla\phi, \quad J^{-1} = \nabla\phi\cdot(\nabla\psi\wedge\nabla l) = \nabla l\cdot(\nabla\phi\wedge\nabla\psi)$$

$$= -\nabla l \cdot \vec{B}, \quad \text{but } \vec{B} = -|\nabla\psi|^2 \frac{J}{R^2}\nabla l$$

$$\Rightarrow J^{-1} = |\nabla l|^2 |\nabla\psi|^2 \frac{J}{R^2} \Rightarrow J^2 = \frac{R^2}{|\nabla l|^2|\nabla\psi|^2} \Rightarrow \boxed{J = \frac{R}{|\nabla l|\,|\nabla\psi|}}$$

$$\therefore \vec{B} = -|\nabla\psi|^2\frac{J}{R^2}\nabla l = -\frac{|\nabla\psi|}{|\nabla l| R}\nabla l = \boxed{-\frac{|\nabla\psi|}{R}\frac{\nabla l}{|\nabla l|} = \vec{B}}$$

Our displacement has the form:

$$\vec{\xi} = \vec{\eta}(\vec{r})e^{iS}, \quad \vec{B}\cdot\nabla S \overset{!}{=} 0, \quad S(\psi, l, \phi), \quad \text{and using } \nabla l\cdot\nabla\psi = 0:$$

$$\Rightarrow \vec{B}\cdot\nabla S = -\frac{|\nabla\psi|}{R}\frac{\nabla l}{|\nabla l|}\cdot\frac{\partial S}{\partial l}\nabla l = -\frac{|\nabla\psi||\nabla l|}{R}\frac{\partial S}{\partial l} \overset{!}{=} 0$$

$\therefore \dfrac{\partial S}{\partial l} = 0$, so we get no variation of the mode along the field line $\rightarrow$ only toroidal variation.

$$\therefore \vec{k}_\perp = \nabla S = \nabla\phi \frac{\partial S}{\partial \phi} + \nabla\psi \frac{\partial S}{\partial \psi}$$

(no ℓ dep)

$\hookrightarrow$ toroidal symmetry $\Rightarrow$ $\underline{S = n\phi + S_p(\psi)}$

$$\boxed{\therefore \vec{k}_\perp = n\nabla\phi + \nabla\psi \frac{\partial S_p}{\partial \psi} \simeq n\nabla\phi \quad \text{for } n \gg 1.}$$

$$\vec{\kappa} = (\vec{b}\cdot\nabla)\vec{b}, \quad \vec{b} = \frac{\nabla \ell}{|\nabla \ell|}, \quad \nabla \ell = -\frac{R^2}{J|\nabla\psi|^2} \nabla\psi \wedge \nabla\phi$$

$$= -\frac{|\nabla \ell| R}{|\nabla \psi|} \nabla\psi\wedge\nabla\phi$$

$$\therefore \frac{\nabla \ell}{|\nabla \ell|} = -\frac{R}{|\nabla\psi|}\nabla\psi\wedge\nabla\phi = -\frac{R\vec{B}}{|\nabla\psi|}$$

$$\nabla = \nabla\psi \frac{\partial}{\partial\psi} + \nabla\ell \frac{\partial}{\partial \ell} + \nabla\phi \frac{\partial}{\partial\phi}$$

$$\Rightarrow \vec{b}\cdot\nabla = \frac{\nabla\ell}{|\nabla\ell|}\cdot\nabla = \frac{|\nabla\ell|^2}{|\nabla\ell|}\frac{\partial}{\partial\ell} = |\nabla\ell|\frac{\partial}{\partial\ell}$$

$$\therefore (\vec{b}\cdot\nabla)\vec{b} = -|\nabla\ell|\frac{\partial}{\partial\ell}\left(\frac{R\vec{B}}{|\nabla\psi|}\right)$$

$$\boxed{\vec{\kappa} = -|\nabla\ell|\frac{\partial}{\partial\ell}\left(\frac{R\vec{B}}{|\nabla\psi|}\right)}$$

- $\vec{b}\cdot\nabla X = |\nabla\ell|\dfrac{\partial X}{\partial \ell}$

- $\vec{b}\wedge\vec{k}_\perp = \dfrac{\nabla\ell}{|\nabla\ell|}\wedge\left\{n\nabla\phi + \nabla\psi\dfrac{\partial S_p}{\partial\psi}\right\} \simeq \dfrac{n\nabla\ell\wedge\nabla\phi}{|\nabla\ell|}$ for $n \gg 1$

$$\Rightarrow (\vec{b}\wedge\vec{k}_\perp)\cdot\nabla p = \frac{n}{|\nabla\ell|}\frac{\partial p}{\partial\psi}\left((\nabla\ell\wedge\nabla\phi)\cdot\nabla\psi\right) = \frac{n}{J|\nabla\ell|}\frac{\partial p}{\partial\psi}$$

$$\Rightarrow \; (\vec{b} \wedge \vec{\kappa}_\perp) \cdot \vec{\kappa} = \left(\frac{n \nabla \ell \wedge \nabla \phi}{|\nabla \ell|} \right) \cdot \left(- |\nabla \ell| \frac{\partial}{\partial \ell} \left(\frac{R\vec{B}}{|\nabla \psi|} \right) \right)$$

$$= - n \nabla \ell \wedge \nabla \phi \cdot \underbrace{\frac{\partial}{\partial \ell} \left(\frac{R\vec{B}}{|\nabla \psi|} \right)}_{\left(\frac{\partial}{\partial \ell} \left(\frac{R}{|\nabla \psi|} \right) \right) \vec{B} + \frac{\partial \vec{B}}{\partial \ell} \left(\frac{R}{|\nabla \psi|} \right)}$$

points along $\nabla \ell$, so $= 0$ when dotted with $\nabla \ell \wedge \nabla \phi$

$$= - \frac{nR}{|\nabla \psi|} (\nabla \ell \wedge \nabla \phi) \cdot \frac{\partial \vec{B}}{\partial \ell}$$

Plugged into δW_2:

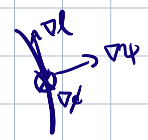
$$\delta W_2 = \frac{1}{2\mu_0} \int_{V_p} d^3 r \left\{ \kappa_\perp^2 | \vec{b} \cdot \nabla X |^2 - \frac{2\mu_0}{B^2} \left((\vec{b} \wedge \vec{\kappa}_\perp) \cdot \nabla p \right) \left((\vec{b} \wedge \vec{\kappa}_\perp) \cdot \vec{\kappa} \right) |X|^2 \right\}$$

$$= \frac{1}{2\mu_0} \int_{V_p} d^3 r \left\{ |n \nabla \phi|^2 \left| |\nabla \ell| \frac{\partial X}{\partial \ell} \right|^2 + \frac{2\mu_0}{B^2} \left\{ \left(\frac{n}{J |\nabla \ell|} \frac{\partial p}{\partial \psi} \right) \left(\frac{nR}{|\nabla \psi|} (\nabla \ell \wedge \nabla \phi) \cdot \frac{\partial \vec{B}}{\partial \ell} \right) |X|^2 \right\} \right.$$

Working in coords (ψ, l, ϕ) where $\vec{B} = B\nabla l = \nabla\psi \wedge \nabla\phi$, $\quad \nabla\psi \cdot \nabla l = 0$
$\nabla l \cdot \nabla\phi = 0$

Since the LDX field is <u>purely</u> poloidal $(B_\phi = 0)$

$$\mathcal{J}^{-1} = \nabla\phi \cdot (\nabla l \wedge \nabla\psi)$$



as always, $\nabla\psi \cdot \nabla\phi = 0$ for toroidal symmetry.

$\vec{j} \wedge \vec{B} = \nabla p$, $\Rightarrow \vec{B} \cdot \nabla p = 0$, and we have toroidal symmetry, so:

$$p = p(\psi) \Rightarrow \boxed{\nabla p = \frac{\partial p}{\partial \psi} \nabla\psi}$$

$\vec{B} = B\nabla l$ by definition, so $\boxed{\vec{b} = \dfrac{\nabla l}{|\nabla l|}}$

$$\nabla l = \gamma_1 \mathcal{J}\, \nabla\psi \wedge \nabla\phi + \gamma_2 \mathcal{J}\, \nabla\phi \wedge \nabla l + \gamma_3 \mathcal{J}\, \nabla l \wedge \nabla\psi$$

$$\Rightarrow \gamma_1 = |\nabla l|^2, \quad \gamma_2 = \nabla l \cdot \nabla\psi = 0, \quad \gamma_3 = \nabla l \cdot \nabla\phi = 0$$

$$\therefore \nabla l = \mathcal{J}(\nabla\psi \wedge \nabla\phi)|\nabla l|^2 = \mathcal{J}|\nabla l|^2 \vec{B} = \mathcal{J}|\nabla l|^2 B \nabla l$$

$$\Rightarrow \boxed{\mathcal{J} = \frac{1}{|\nabla l|^2 B}}$$

Since we have no B_ϕ, q is not a function that is defined here, so there is no shear. We thus don't need the ballooning transform, and the quasimodes <u>are</u> the real displacements:

$$\boxed{\vec{\varepsilon} = \vec{\eta}\, e^{iS}},$$

$\vec{B}\cdot\nabla\mathcal{E} \overset{!}{=} 0$ for interchanges: $e^{iS}\vec{B}\cdot\{\nabla\eta + i\eta\nabla S\}$

$\nabla S \gg \nabla\eta$ in ballooning.

$\Rightarrow i\eta e^{iS}\vec{B}\cdot\nabla S \overset{!}{=} 0 \Rightarrow \vec{B}\cdot\nabla S = 0$

$S = (n\phi + S_p(\psi,l)) \Rightarrow B\nabla l\cdot\left\{n\nabla\phi + \frac{\partial S_p}{\partial\psi}\nabla\psi + \frac{\partial S_p}{\partial l}\nabla l\right\} = 0$

$\Rightarrow \frac{\partial S_p}{\partial l} = \frac{\partial S}{\partial l} = 0$, $\boxed{S \text{ is constant along field lines.}}$

Now: $\vec{k}_\perp = \nabla S = \frac{\partial S}{\partial\psi}\nabla\psi + \frac{\partial S}{\partial\phi}\nabla\phi = n\nabla\phi + \frac{\partial S}{\partial\psi}\nabla\psi$

$n \gg 1 \Rightarrow \boxed{\vec{k}_\perp \simeq n\nabla\phi}$ ($\frac{\partial S}{\partial\psi}$ will be important in the next orders).

Looking at δW_2:

$$\delta W_2 = \frac{1}{2\mu_0}\int_{V_p} d^3r \left\{ k_\perp^2 |\vec{b}\cdot\nabla X|^2 - \frac{2\mu_0}{B^2}\left((\vec{b}\wedge\vec{k}_\perp)\cdot\nabla p\right)\left((\vec{b}\wedge\vec{k}_\perp)\cdot\vec{\kappa}\right)|X|^2 \right\}$$

The last things we need are $\vec{\kappa}$ and ∇X.

- $\vec{\kappa} = (\vec{b}\cdot\nabla)\vec{b}$, $\vec{b} = \frac{\nabla l}{|\nabla l|}$, $\nabla = \nabla\psi\frac{\partial}{\partial\psi} + \nabla l\frac{\partial}{\partial l} + \nabla\phi\frac{\partial}{\partial\phi}$

$\Rightarrow \vec{b}\cdot\nabla = |\nabla l|\frac{\partial}{\partial l} \Rightarrow (\vec{b}\cdot\nabla)\vec{b} = \boxed{|\nabla l|\frac{\partial}{\partial l}\left(\frac{\nabla l}{|\nabla l|}\right) = \vec{\kappa}}$

(I am sure we can simplify this further, but I'll do that later.)

- $\boxed{\vec{b}\cdot\nabla X = |\nabla l|\frac{\partial X}{\partial l}}$

Plugging into δW_2:

$$\delta W_2 = \frac{1}{2\mu_0}\int_{V_p} d^3\vec{r}\,\Big\{ |n\nabla\phi|^2 \Big|\,|\nabla l|\frac{\partial X}{\partial l}\Big|^2 + $$
$$ - \frac{2\mu_0}{B^2}\left[\left(n\frac{\nabla l}{|\nabla l|}\wedge\nabla\phi\right)\cdot\nabla\psi\frac{\partial p}{\partial\psi}\right]\left[\left(n\frac{\nabla l}{|\nabla l|}\wedge\nabla\phi\right)\cdot|\nabla l|\frac{\partial}{\partial l}\left(\frac{\nabla l}{|\nabla l|}\right)\right]|X|^2\Big\}$$

$$= \frac{1}{2\mu_0}\int_{V_p} d^3\vec{r}\,\Big\{ |n\nabla\phi|^2\Big|\,|\nabla l|\frac{\partial X}{\partial l}\Big|^2 - \frac{2\mu_0}{B^2}\left[-\frac{n}{|\nabla l|\mathcal{J}}\frac{\partial p}{\partial\psi}\right]\left[n(\nabla l\wedge\nabla\phi)\cdot\frac{\partial}{\partial l}\left(\frac{\nabla l}{|\nabla l|}\right)\right]|X|^2\Big\}$$

$d^3r = \mathcal{J}\, dl\, d\phi\, d\psi$, $\mathcal{J} = \dfrac{1}{|\nabla l|^2 B}$, $X \in$ Real by definition.

$$\Rightarrow \delta W_2 = \frac{1}{2\mu_0}\int_{V_p} dl\, d\phi\, d\psi\,\Big\{ \frac{n^2}{BR^2}\left(\frac{\partial X}{\partial l}\right)^2 + \frac{2\mu_0}{B^2}\left[\left(\frac{n}{|\nabla l|}\frac{\partial p}{\partial\psi}\right)\left(n\,\nabla l\wedge\nabla\phi\cdot\frac{\partial}{\partial l}\left(\frac{\nabla l}{|\nabla l|}\right)\right)\right]X^2\Big\}$$

↳ instabilities only possible for $\partial p/\partial\psi < 0$.

Which minimises at marginal stability ($\delta W_2 = 0$) to give the Euler-Lagrange equation:

$$\frac{\partial}{\partial l}\left(\frac{2n^2}{BR^2}\frac{\partial X}{\partial l}\right) - \frac{4\mu_0}{B^2}\left[\left(\frac{n}{|\nabla l|}\frac{\partial p}{\partial\psi}\right)\left(n\,\nabla l\wedge\nabla\phi\cdot\frac{\partial}{\partial l}\left(\frac{\nabla l}{|\nabla l|}\right)\right)\right]X = 0$$

$$\Rightarrow \boxed{\frac{\partial}{\partial l}\left(\frac{1}{BR^2}\frac{\partial X}{\partial l}\right) - \frac{2\mu_0}{B^2|\nabla l|}\frac{\partial p}{\partial\psi}\left(\nabla l\wedge\nabla\phi\cdot\frac{\partial}{\partial l}\left(\frac{\nabla l}{|\nabla l|}\right)\right)X = 0}$$

Once I have an equilibrium, I can solve the above for X.

↳ Still have to look at the literature for how to solve this via shooting.

for $\vec{\mathcal{E}} = \vec{\eta}\, e^{iS} = \Big(\big(\underbrace{\vec{\eta}_{0,\perp} + \vec{\eta}_{1,\perp} + \dots}_{\vec{\eta}_\perp} \big) + \vec{\eta}_{\parallel} \Big) e^{iS(\psi,\theta)}$

where $\vec{\eta}_{\parallel}$ is such ~~that~~ that: $\nabla \cdot \vec{\mathcal{E}} = 0$, $\vec{\eta}_{0\perp} = \dfrac{X(r)}{B} \vec{\kappa}_\perp \wedge \vec{b}$,

$$i \vec{\kappa}_\perp \cdot \vec{\eta}_{1,1} = -2 \vec{\eta}_{1,0} \cdot \vec{\varkappa} - \nabla_0\, \vec{\eta}_{1,0}$$

and $\vec{\kappa}_\perp = \nabla S$

$$\delta W_f = \frac{1}{2\mu_0}\int_{V_p} dV \Big\{ |\vec{Q}_\perp|^2 + B^2 |\nabla\cdot\vec{\xi}_\perp + 2\vec{\xi}_\perp\cdot\kappa|^2 + \mu_0 \gamma p |\nabla\cdot\vec{\xi}|^2$$

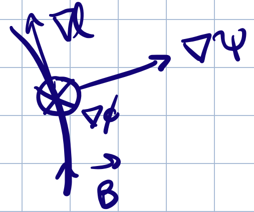
$$- 2\mu_0 (\vec{\xi}_\perp\cdot\nabla p)(\vec{\xi}_\perp^{\,*}\cdot\vec{\kappa}) - \mu_0 j_\parallel \left((\vec{\xi}_\perp^{\,*}\wedge\hat{b})\cdot\vec{Q}_\perp \right) \Big\}$$

Dipole has some big simplifications:

- $B_\phi = 0 \Rightarrow \vec{B} = B\nabla l = \nabla f(\psi) \wedge \nabla\phi$, $\nabla\psi\cdot\nabla l = 0$, $\nabla l\cdot\nabla\phi = 0$
- $j_\parallel = 0$, since we have no poloidal flows.
- $B_\phi = 0 \Rightarrow$ no shear, so no periodicity issues, and thus quasimodes are modes.

Assuming pertubations of the form $\vec{\xi} = \vec{\eta}e^{iS}$, where $\vec{\eta}$ is slow varying, and $\vec{k} = \nabla S$ has all the rapid variation. For equilibrium length scale L, we can order: $|\nabla S| \gg \frac{1}{L}$, $\vec{\eta} = \vec{\eta}_0 + \vec{\eta}_1 + \dots$ where the successive terms are smaller by: $\frac{|\eta_{i+1}|}{|\eta_i|} = \frac{1}{|\nabla S| L}$. For clarity, let's call $|\nabla S| \sim \frac{1}{\delta}$.

Let's work in the coordinates (ψ, l, ϕ)



$\Rightarrow J^{-1} = \nabla\phi\cdot(\nabla l \wedge \nabla\psi)$

In equilibrium, $\vec{j}\wedge\vec{B} = \nabla p \Rightarrow \vec{B}\cdot\nabla p = 0$

$\Rightarrow p = p(\psi)$ if we have toroidal symmetry (which we always do!).

For some flux function $f(\psi)$:

$\vec{B} = B\nabla l = \nabla f \wedge \nabla\phi, \quad \hat{b} = \frac{\nabla l}{|\nabla l|}$

$\nabla l = \gamma_1 J \nabla\psi\wedge\nabla\phi + \gamma_2 J \nabla\phi\wedge\nabla l + \gamma_3 J \nabla l \wedge \nabla\psi$

$\Rightarrow \gamma_1 = |\nabla l|^2, \; \gamma_2 = \nabla l \cdot \nabla\psi = 0, \; \gamma_3 = \nabla\phi\cdot\nabla l = 0$

$\therefore \nabla l = J(\nabla\psi\wedge\nabla\phi)|\nabla l|^2$

$\Rightarrow B\nabla l = BJ|\nabla l|^2(\nabla\psi\wedge\nabla\phi) = \nabla f\wedge\nabla\phi = \frac{\partial f}{\partial\psi}\nabla\psi\wedge\nabla\phi$

$\therefore \underline{BJ|\nabla l|^2 = \frac{\partial f}{\partial\psi}}$. w.l.g we can set $f(\psi) = \psi$, which just pushes the information about the geometry into l, J.

$$\Rightarrow \boxed{J = \frac{1}{B|\nabla l|^2}}$$

Let's now start tailoring δW to ballooning modes.

$\vec{\varepsilon} = \vec{\eta} e^{iS} = \{\vec{\eta}_0 + \vec{\eta}_1 + \dots\} e^{iS(\psi, l, \phi)}$

$\vec{Q}_\perp = \nabla\wedge(\vec{\varepsilon}\wedge\vec{B}) = \nabla\wedge(\vec{\eta}e^{iS}\wedge\vec{B}) = e^{iS}\{\nabla\wedge(\vec{\eta}\wedge\vec{B}) + i\nabla S\wedge(\vec{\eta}\wedge\vec{B})\}$

$$i\nabla S\wedge(\vec{\eta}\wedge\vec{B}) = i\vec{k}\wedge(\vec{\eta}\wedge\vec{B}) = i\underbrace{(\vec{k}\cdot\vec{B})}_{=0 \text{ for interchanges.}}\vec{\eta} - i\underbrace{(\vec{k}\cdot\vec{\eta})\vec{B}}_{\text{is } \parallel \text{ so ignored by } \perp \text{ operator on the bracket.}}$$

$\Rightarrow \vec{Q}_\perp = \left(e^{iS}\nabla\wedge(\vec{\eta}\wedge\vec{B})\right)_\perp$

$\therefore$ • $|\vec{Q}_\perp|^2 = |(\nabla\wedge(\vec{\eta}\wedge\vec{B}))_\perp|^2$

• $B^2|\nabla\cdot\vec{\varepsilon}_\perp + 2\vec{\varepsilon}_\perp\cdot\varkappa|^2 = B^2|\nabla\cdot\vec{\eta}_\perp + i\vec{\eta}_\perp\cdot\vec{k}_\perp + 2\vec{\eta}_\perp\cdot\vec{\varkappa}|^2$

• $-2\mu_0(\vec{\varepsilon}_\perp\cdot\nabla p)(\vec{\varepsilon}_\perp^{\,*}\cdot\vec{\varkappa}) = -2\mu_0\{(\vec{\eta}_\perp\cdot\nabla p)(\vec{\eta}_\perp^{\,*}\cdot\vec{\varkappa})\}$

$$\Rightarrow \delta W = \frac{1}{2\mu_0}\int_{V_p} dV \Big\{|(\nabla\wedge(\vec{\eta}\wedge\vec{B}))_\perp|^2 + B^2|\nabla\cdot\vec{\eta}_\perp + i\vec{\eta}_\perp\cdot\vec{k}_\perp + 2\vec{\eta}_\perp\cdot\vec{\varkappa}|^2 - 2\mu_0\{(\vec{\eta}_\perp\cdot\nabla p)(\vec{\eta}_\perp^{\,*}\cdot\vec{\varkappa})\}\Big\}$$

($\varepsilon_\parallel$ picked to make $\nabla\cdot\vec{\varepsilon}=0$ for maximum instability).

$\therefore$ To lowest order in $\frac{1}{L}$ $\left(\nabla p \sim \nabla\eta \sim \vec{\varkappa} \sim \frac{1}{L}\right)$,

$$\delta W_0 = \frac{1}{2\mu_0}\int_{V_p} dV\left\{B^2|\vec{\eta}_{\perp,0}\cdot\vec{k}_\perp|^2\right\} \overset{!}{=} 0$$

$$\Rightarrow \boxed{\vec{\eta}_{\perp,0} = Y\,\hat{b}\wedge\vec{k}_\perp\,,\quad |\nabla Y| \sim \frac{1}{L}}$$

$$\delta W_2 = \frac{1}{2\mu_0}\int_{V_p} dV\Big\{|(\nabla\wedge(\vec{\eta}_0\wedge\vec{B}))_\perp|^2 + B^2|\nabla\cdot\vec{\eta}_{0,\perp} + i\vec{\eta}_{1,\perp}\cdot\vec{k}_\perp + 2\vec{\eta}_{\perp,0}\cdot\vec{\varkappa}|^2 - 2\mu_0\{(\vec{\eta}_{\perp,0}\cdot\nabla p)(\vec{\eta}_{\perp,0}^{\,*}\cdot\vec{\varkappa})\}\Big\}$$

, let's start plugging in and hacking away at terms.

• $\underline{(\nabla\wedge(\vec{\eta}_0\wedge\vec{B}))_\perp} = \Big\{\nabla\wedge\{(Y\,\hat{b}\wedge\vec{k}_\perp)\wedge(B\hat{b})\}\Big\}_\perp$

$= \{\nabla\wedge\{\Gamma\,\{(\hat{b}\wedge\vec{k}_\perp)\wedge\hat{b}\}\}\}_\perp$, $\boxed{\Gamma = BY}$

$(\hat{b}\wedge\vec{k}_\perp)\wedge\hat{b} = \varepsilon_{ijk}(\varepsilon_{jlm} b_l k_m) b_k = (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il}) b_l k_m b_k$

$= k_i - b_i \underbrace{k_k b_k}_{0} = \vec{k}_\perp \Rightarrow (\nabla\wedge(\Gamma\vec{k}_\perp))_\perp = (\nabla\Gamma\wedge k_\perp + \Gamma\,\nabla\wedge\vec{k}_\perp)_\perp$

$= (\nabla\Gamma\wedge k_\perp)_\perp = \left(\nabla_\perp\Gamma\wedge k_\perp + (\hat{b}\cdot\nabla\Gamma)\,\hat{b}\wedge\vec{k}_\perp\right)_\perp = (\hat{b}\cdot\nabla\Gamma)\,\hat{b}\wedge\vec{k}_\perp$

$$\therefore \left(\nabla \wedge (\vec{\eta}_0 \wedge \vec{B})\right)_\perp = (\hat{b}\cdot\nabla\Gamma)\,\hat{b}\wedge\vec{\kappa}_\perp$$

- $\nabla\cdot\vec{\eta}_{0,\perp} + i\vec{\eta}_{1,\perp}\cdot\vec{\kappa}_\perp + 2\vec{\eta}_{\perp,0}\cdot\vec{\varkappa} = 0$ if we set $i\vec{\eta}_{1,\perp}\cdot\vec{\kappa}_\perp = -\{\nabla\cdot\vec{\eta}_{0,\perp} + 2\vec{\eta}_{0,\perp}\cdot\vec{\varkappa}\}$

so we can ignore the whole middle term.

- $-2\mu_0\{(\vec{\eta}_{\perp,0}\cdot\nabla p)(\vec{\eta}_{\perp,0}^{*}\cdot\vec{\varkappa})\}$, $\vec{\eta}_{0,\perp} = Y\ \hat{b}\wedge\vec{\kappa}_\perp$

$$= -2\mu_0\left\{\left(Y(\hat{b}\wedge\vec{\kappa}_\perp)\cdot\nabla p\right)\left(Y^*(\hat{b}\wedge\vec{\kappa}_\perp)\cdot\vec{\varkappa}\right)\right\} = -\frac{2\mu_0|\Gamma|^2}{B^2}\left((\hat{b}\wedge\vec{\kappa}_\perp)\cdot\nabla p\right)\left((\hat{b}\wedge\vec{\kappa}_\perp)\cdot\vec{\varkappa}\right)$$

So our equation for δW_2 is:

$$\delta W_2 = \frac{1}{2\mu_0}\int_{V_p} dV\left\{\left|\left(\nabla\wedge(\vec{\eta}\wedge\vec{B})\right)_\perp\right|^2 + B^2\left|\nabla\cdot\vec{\eta}_\perp + i\vec{\eta}_\perp\cdot\vec{\kappa}_\perp + 2\vec{\eta}_\perp\cdot\vec{\varkappa}\right|^2 - 2\mu_0\{(\vec{\eta}_\perp\cdot\nabla p)(\vec{\eta}_\perp^{*}\cdot\vec{\varkappa})\}\right\}$$

$$\delta W_2 = \frac{1}{2\mu_0}\int_{V_p} dV\left\{\left|(\hat{b}\cdot\nabla\Gamma)\,\hat{b}\wedge\vec{\kappa}_\perp\right|^2 - \frac{2\mu_0|\Gamma|^2}{B^2}\{(\hat{b}\wedge\vec{\kappa}_\perp)\cdot\nabla p\}\{(\hat{b}\wedge\vec{\kappa}_\perp)\cdot\vec{\varkappa}\}\right\}$$

for $\Gamma = BY(\vartheta)$ where $\vec{\eta}_{\perp,0} = Y\ \hat{b}\wedge\vec{\kappa}_\perp$

and we use: $\cdot\ \varepsilon_\parallel$ to set $\nabla\cdot\vec{\varepsilon} = 0$

$\cdot\ \eta_{1,\perp}$ to set $i\vec{\eta}_{1,\perp}\cdot\vec{\kappa}_\perp = -\{\nabla\cdot\vec{\eta}_{0,\perp} + 2\vec{\eta}_{0,\perp}\cdot\vec{\varkappa}\}$

From our geometry: $\hat{b} = \dfrac{\nabla l}{|\nabla l|} \Rightarrow \hat{b}\cdot\nabla\Gamma = \dfrac{\nabla l}{|\nabla l|}\cdot\left\{\nabla l\dfrac{\partial\Gamma}{\partial l}\right\}$

$$= |\nabla l|\frac{\partial\Gamma}{\partial l}$$

Since $\vec{B}\cdot\nabla S \overset{!}{=} 0$, $S(\psi,\phi)$ and has no component/variation along the field.

$\kappa_\perp = \nabla S = \dfrac{\partial S}{\partial\psi}\nabla\psi + \dfrac{\partial S}{\partial\phi}\nabla\phi$, will be handy later.

For now, let's use euler-lagrange to work out the marginal stability condition.

$$\left\{ \left|(\hat{b}\cdot\nabla\Gamma)\hat{b}\wedge\vec{k}_\perp\right|^2 - \frac{2\mu_0|\Gamma|^2}{B^2}\{(\hat{b}\wedge\vec{k}_\perp)\cdot\nabla p\}\{(\hat{b}\wedge\vec{k}_\perp)\cdot\vec{\kappa}\} \right\} =$$

$$\left(\left|\frac{\partial\Gamma}{\partial l}\hat{b}\wedge\vec{k}_\perp\right|^2 , \quad \Gamma\in\mathrm{Real} \Rightarrow \left(\frac{\partial\Gamma}{\partial l}\right)^2 |\hat{b}\wedge\vec{k}_\perp|^2 \right)$$

$$= |\hat{b}\wedge\vec{k}_\perp|^2\left(\frac{\partial\Gamma}{\partial l}\right)^2 - \frac{2\mu_0}{B^2}\{(\hat{b}\wedge\vec{k}_\perp)\cdot\nabla p\}\{(\hat{b}\wedge\vec{k}_\perp)\cdot\vec{\kappa}\}\Gamma^2 = 0$$

$$\Rightarrow \frac{\partial}{\partial l}\left\{2|\hat{b}\wedge\vec{k}_\perp|^2\frac{\partial\Gamma}{\partial l}\right\} + \frac{4\mu_0}{B^2}\{(\hat{b}\wedge\vec{k}_\perp)\cdot\nabla p\}\{(\hat{b}\wedge\vec{k}_\perp)\cdot\vec{\kappa}\}\Gamma = 0$$

$$\Rightarrow \boxed{\frac{\partial}{\partial l}\left\{|\hat{b}\wedge\vec{k}_\perp|^2\frac{\partial\Gamma}{\partial l}\right\} + \frac{2\mu_0}{B^2}\{(\hat{b}\wedge\vec{k}_\perp)\cdot\nabla p\}\{(\hat{b}\wedge\vec{k}_\perp)\cdot\vec{\kappa}\}\Gamma = 0}$$

$$\hat{b} = \frac{\nabla l}{|\nabla l|}, \quad \vec{k}_\perp = \nabla S = \frac{\partial S}{\partial\psi}\nabla\psi + \frac{\partial S}{\partial\phi}\nabla\phi = \frac{\partial S}{\partial\psi}\nabla\psi + in\nabla\phi$$

$$\Rightarrow \hat{b}\wedge\vec{k}_\perp = \frac{\partial S}{\partial\psi}\frac{\nabla l\wedge\nabla\psi}{|\nabla l|} + in\frac{\nabla l\wedge\nabla\phi}{|\nabla l|}$$ (units work).

$$\nabla l\wedge\nabla\psi = \lambda\nabla\phi \Rightarrow \nabla\phi\cdot(\nabla l\wedge\nabla\psi) = \lambda|\nabla\phi|^2 = \frac{\lambda}{R^2} = J^{-1} = B|\nabla l|^2$$

$$\Rightarrow \lambda = BR^2|\nabla l|^2 \Rightarrow \nabla l\wedge\nabla\psi = BR^2|\nabla l|^2\nabla\phi$$

$$\nabla l\wedge\nabla\phi = \lambda\nabla\psi \Rightarrow -J^{-1} = \lambda|\nabla\psi|^2 \Rightarrow \lambda = -\frac{B|\nabla l|^2}{|\nabla\psi|^2} \Rightarrow \nabla l\wedge\nabla\phi = -\frac{B|\nabla l|^2}{|\nabla\psi|^2}\nabla\psi$$

$$\therefore \hat{b}\wedge\vec{k}_\perp = \frac{\partial S}{\partial\psi}BR^2|\nabla l|\nabla\phi + in\frac{B|\nabla l|\nabla\psi}{|\nabla\psi|^2}$$

$$\therefore (\hat{b}\wedge\vec{k}_\perp)^2 = \left(\frac{\partial S}{\partial\psi}\right)^2 J^{-1}BR^2 + \frac{n^2 B J^{-1}}{|\nabla\psi|^2}$$

$$\frac{\partial}{\partial l}\left\{|\hat{b}\wedge\vec{\kappa}_\perp|^2\frac{\partial\Gamma}{\partial l}\right\}+\frac{2\mu_0}{B^2}\{(\hat{b}\wedge\vec{\kappa}_\perp)\cdot\nabla p\}\{(\hat{b}\wedge\vec{\kappa}_\perp)\cdot\vec{\kappa}\}\Gamma=0$$

I picked $|\nabla l|=1$, l is the physical length along the B field.

$$\hat{b}=\nabla l,\quad \vec{\kappa}_\perp=n\nabla\phi \Rightarrow \hat{b}\wedge\vec{\kappa}_\perp=n\nabla l\wedge\nabla\phi=-n\nabla\psi$$

$$\Rightarrow 0=n^2\frac{\partial}{\partial l}\left\{|\nabla\psi|^2\frac{\partial\Gamma}{\partial l}\right\}+\Gamma\frac{2\mu_0}{B^2}\left\{(-n|\nabla\psi|^2)(-n\nabla\psi\cdot\vec{\kappa})\right\}$$

$$\Rightarrow 0=\frac{\partial}{\partial l}\left\{|\nabla\psi|^2\frac{\partial\Gamma}{\partial l}\right\}+\frac{2\mu_0}{B^2}|\nabla\psi|^2\frac{\partial P}{\partial\psi}(\nabla\psi\cdot\vec{\kappa})\Gamma=0$$

$[\psi]=BL^2$, $[l]=L$, $[\Gamma]=$ doesn't matter.

$$\frac{2\mu_0}{B^2}=\frac{1}{\text{energy density}}\qquad [\text{energy density}]=\text{kg}\,L^2T^{-2}\times\frac{1}{L^3}\Rightarrow\left[\frac{2\mu_0}{B^2}\right]=\frac{LT^2}{M}$$

$$\left[\frac{\partial}{\partial l}\left\{|\nabla\psi|^2\frac{\partial\Gamma}{\partial l}\right\}\right]=\frac{B^2L^2}{L^2}[\Gamma]=B^2$$

$$\left[\frac{2\mu_0}{B^2}|\nabla\psi|^2\frac{\partial P}{\partial\psi}(\nabla\psi\cdot\vec{\kappa})\Gamma\right]=B^2L^2\times\frac{1}{BL^2}\times BL\times\frac{1}{L}\Gamma$$

$$=B^2\Gamma$$

! units work!

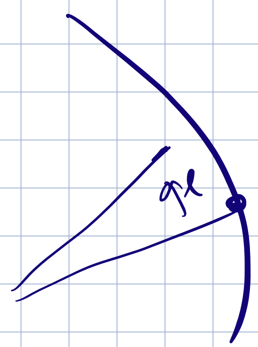
$$\boxed{\frac{\partial}{\partial l}\left\{|\nabla\psi|^2\frac{\partial\Gamma}{\partial l}\right\}+\frac{2\mu_0}{B^2}|\nabla\psi|^2\frac{\partial P}{\partial\psi}(\nabla\psi\cdot\vec{\kappa})\Gamma=0}$$

↳ at least units work.

$$\vec{B}=B\nabla l,\qquad \nabla l=\gamma\nabla\psi\wedge\nabla\phi,\quad \gamma=\mathcal{J}|\nabla l|^2=\mathcal{J}$$

$$J = \frac{1}{B} \Rightarrow \vec{B} = \frac{B}{B} \nabla\psi \wedge \nabla\phi \Rightarrow \vec{B} = \nabla\psi \wedge \nabla\phi$$

My code spits out $\psi(R, Z)$, allows me to find $\nabla\psi$ and thus $\vec{B}$, need some parametrisation of things in terms of l.



$$B(l=0) \stackrel{!}{=} B(l=L)$$

$$\nabla\psi(l=0) \stackrel{!}{=} \nabla\psi(l=L), \quad |\nabla l| = 0$$

$L =$

(Nash-Moser Iteration).