Our project treats the plasma as a single species (we combine electrons and ions) capable of generating currents, and thus also magnetic fields. These currents can also be influenced by magnetic fields, which means we can use imposed magnetic fields to confine the motions of the plasma/fluid.

The equations governing our plasma are those of "Ideal MHD" (MHD = magnetohydrodynamics" :

a). $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

b). $$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}\right) = -\nabla P + \vec{J} \wedge \vec{B}$$

c). $$E + \vec{u} \wedge \vec{B} = 0$$

d). $$\left(\frac{\partial}{\partial t} + \vec{u} \cdot \nabla\right)\left(P \rho^{-\gamma}\right) = 0$$

ρ = fluid density

P = scalar pressure

$\vec{u}$ = fluid velocity

$\vec{J}$ = current density

$\vec{B}$ = magnetic field

$\gamma = \frac{5}{3}$ = polytropic index.

To get these, you write a Boltzmann equation for the electrons and ions, then assume all interactions are mediated by long range $\vec{B}$ fields generated by the system, but act as collisions well enough for the equilibrium distribution to be maxwellian. You then take moments of the Boltzmann equations, combining the results to get equations in:

$$n_e = \int d^3v \, f_{electron} \qquad n_i = \int d^3v \, f_{ion}$$

p.de velocity

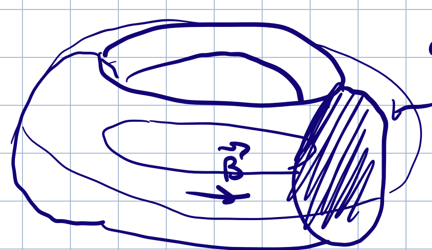
$$\rho = \frac{m_e n_e + m_i n_i}{m_e + n_e}, \qquad \vec{u} = \frac{m_i n_i \vec{v}_i + m_e n_e \vec{v}_e}{m_i n_i + m_e n_e}$$

$$\vec{J} = Z_i e n_i \vec{v}_i + e n_e \vec{v}_e$$

ion charge number

You then assume the plasma is a perfect conductor (valid for our short timescales), non-relativistic (also valid!). The assumption of being collisional/maxwellian is only valid in directions perpendicular to $\vec{B}$, but for equilibrium that's all we care about!

We are calculating equilibrium configurations for toroidally symmetric devices with closed magnetic fields, so we only care if each cross-section is stable:



equilibrium for a cross-section, equilibrium for the donut.

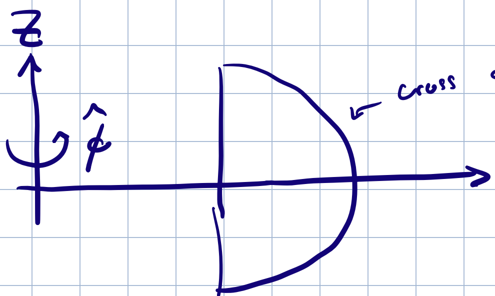
To work out if we have an equilibrium, we take the momentum equation b).:

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla p + \vec{J}\wedge\vec{B}$$

, assume no toroidal flow, and $\frac{\partial}{\partial t} \to 0$ (equilibrium)

$$\Rightarrow -\nabla p + \vec{J}\wedge\vec{B} = 0 \Rightarrow \nabla p = \vec{J}\wedge\vec{B}$$

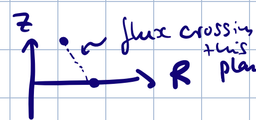
$\vec{J} = \frac{1}{\mu_0}\nabla\wedge\vec{B}$, so $\mu_0 \nabla p = (\nabla\wedge\vec{B})\wedge\vec{B}$



Toroidal symmetry $\Rightarrow \frac{\partial}{\partial \phi} = 0$

$$\vec{B} = \nabla\wedge\vec{A} = -\frac{\partial A_\phi}{\partial z}\hat{R} + B_z\hat{\phi} + \frac{1}{R}\frac{\partial(RA_\phi)}{\partial R}\hat{z}$$

defining $\Psi = RA_\phi$ (can be shown to be proportional to the $\vec{B}$ poloidal flux)



$$\Rightarrow \vec{B} = \nabla\Psi\wedge\nabla\phi + B_\phi\nabla\phi$$

$\vec{J} = \frac{1}{\mu_0}\nabla\wedge\vec{B}$, so after a few lines of algebra:

$$\mu_0\vec{J} = -\frac{1}{R}\Delta^*\Psi + \nabla(RB_\phi)\wedge\nabla\phi$$

for $\Delta^* = R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}\right) + \frac{\partial^2}{\partial z^2}$

↳ = "Stokes" or "Shafranov" operator.

$\vec{B}\cdot\nabla p = \vec{B}\cdot(\vec{J}\wedge\vec{B}) = 0$, so ∇p is perp. to $\vec{B}$, and thus can be written as a function of Ψ.

$$\therefore \nabla p = \vec{J}\wedge\vec{B} \Rightarrow \mu_0\frac{\partial p}{\partial\Psi}\nabla\Psi = -R^2\Delta^*\Psi\,\nabla\Psi - \underbrace{RB_\phi\frac{1}{R^2}\nabla(RB_\phi)}$$

$\Rightarrow \nabla(RB_\phi)$ parallel to $\nabla\Psi$, and thus RB_ϕ is a function of Ψ.

Defining $f(\Psi) = RB_\phi$:

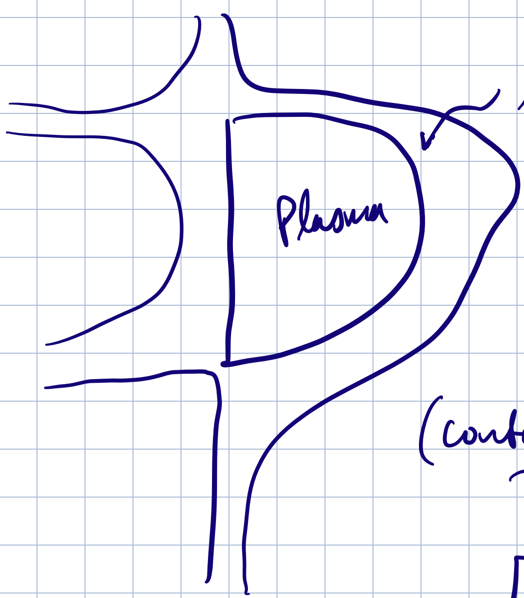
$$\boxed{\mu_0 R^2 \frac{\partial p}{\partial \psi} + f(\psi)\frac{\partial f}{\partial \psi} = -\Delta^* \psi}$$ Grad-Shafranov.

plasma currents.

$$\boxed{\text{Or: } \Delta^* \psi = -\mu_0 R j_\phi \quad \text{for} \quad j_\phi = r\frac{\partial p}{\partial \psi} + \frac{1}{R} f \frac{\partial f}{\partial \psi}}$$

No plasma can confine itself, (Virial Theorem), so we use coils to crush the plasma into shape.

Travelling along /around flux surfaces is fast, so plasma is confined to regions with closed flux surfaces:



last closed flux surface.

By fixing location of lcfs, we fix where the plasma is.

(contours of ψ).

We are thus solving:

$$\boxed{\begin{aligned} \Delta^* \psi &= \begin{cases} -\mu_0 R j_\phi & \text{(within lcfs)} \\ 0 & \text{(outside lcfs)} \end{cases} \\ \psi &= \psi_X \text{ on lcfs} \end{aligned}}$$

we fix this Dirichlet BC using coils.

$$\Psi(R,Z) = \Psi_p(R,Z) + \Psi_H(R,C)$$

flux due to plasma currents

flux due to coils.

Δ^* has green's function:

$$G(R,Z;R',Z') = \frac{\sqrt{RR'}}{2\pi}\left\{\frac{(2-\kappa^2)\overbrace{\int_0^{\frac{\pi}{2}} d\theta\,(1-\kappa^2\sin^2(\theta))^{-1/2}}^{K(\kappa)} - 2\overbrace{\int_0^{\pi/2} d\theta\,(1-\kappa^2\sin^2(\theta))^{1/2}}^{E(\kappa)}}{\kappa}\right\}$$

for $\kappa = \dfrac{\sqrt{4RR'}}{[(R+R')^2 + (Z-Z')^2]^{1/2}}$

So $\boxed{\Psi_p(R,Z) = \int_{\{R',Z'\}\in \text{lcfs}} dR'dZ'\; G(R,Z;R',Z')\, j_\phi(R',Z')}$

$$\boxed{\Psi_H(R,Z) = \sum_{n=0}^{N_p} a_n \underbrace{\sum_{m=0}^{n-2} A_{nm} R^{n-m} Z^m}_{\lambda_n(R,Z),\ \text{flux of pole } n} \quad \text{for } A_{nm} = -\frac{(n+2-m)(n-m)}{m(m-1)} A_{n,m-2}}$$

Reusch paper.

multipole expansion of the coils, $\Delta^*\Psi_H = 0$.

a_n: contribution of each multipole, chosen to fit the lcfs.

So the code does the following: (flowchart on next page :)

Pick lcfs

$\downarrow$

Find all grid points $<$ ∂(lcfs.)   = boundary of "

$\downarrow$

To fit N_p poles, pick N_p points on lcfs. Calculate ψ_p at those points: $\psi_{p,i}$ for $i \in [1, N_p]$.

$\downarrow$

$\Delta_i = \psi_X - \psi_{p,i}$. This is the gap at each point that the coils need to make up.

$\downarrow$

(skip n=1 pole)

$$\overleftrightarrow{w} \cdot \vec{a} = \vec{\Delta}, \quad \overleftrightarrow{w} = \begin{pmatrix} \lambda_{i=1,n=0} & \lambda_{i=1,n=2} & \cdots & \lambda_{i=1,n=N_p} \\ \lambda_{i=2,n=0} & \lambda_{i=2,n=2} & \cdots & \lambda_{i=2,n=N_p} \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

$\vec{a} = \overleftrightarrow{w}^{-1} \vec{\Delta}$, gives weight/contribution of each pole.

$\downarrow$

Calculate $\psi = \psi_p(R,Z) + \vec{a} \cdot \vec{\lambda}(R,Z)$ for all grid points.