`RMT_Tokamak` background information

1 Grad-Shafranov equation derivation

Our project treats the plasma as a single species (we combine electrons and ions) capable of generating currents, and thus also magnetic fields. These currents can also be influenced by magnetic fields, which means we can use imposed magnetic fields to confine the motions of the plasma/fluid.

The equations governing our plasma are those of "Ideal Magnetohydrodynamics (MHD)":

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla P + \mathbf{J} \times \mathbf{B} \tag{2}$$

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0 \tag{3}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \left(P \rho^{-\gamma} \right) = 0 \tag{4}$$

where ρ is the fluid density, P is the scalar pressure, $\mathbf{u}$ is the fluid velocity, $\mathbf{J}$ is the current density, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, and $\gamma := \frac{5}{3}$ is the polytropic index. To obtain these, you write a Boltzmann equation for the electrons and the ions, then assume all interactions are mediated by long-range $\mathbf{B}$ fields generated by the system, but act as collisions well enough for the equilibrium distribution to be maxwellian. You then take moments of the Boltzmann equations, combining the results, to get equations in:

$$n_e = \int d^3\mathbf{v}\, f_{\text{electrons}} \tag{5}$$

$$n_i = \int d^3\mathbf{v}\, f_{\text{ions}} \tag{6}$$

$$\rho = \frac{m_e n_e + m_i n_i}{m_e + m_i} \tag{7}$$

$$\mathbf{u} = \frac{m_e n_e \mathbf{v}_e + m_i n_i \mathbf{v}_i}{m_e n_e + m_i n_i} \tag{8}$$

$$\mathbf{J} = e n_e \mathbf{v}_e + Z_i e n_i \mathbf{v}_i \tag{9}$$

where Z_i is the ion charge number. You then assume the plasma is a perfect conductor, which is valid for our short timescales, and non-relativistic, which is also valid. The assumption that being collisional/maxwellian is only valid in directions perpendicular to $\mathbf{B}$, but for equilibrium that is all we care about!

We are calculating equilibrium configurations for toroidally symmetric devices with closed magnetic fields, so we only care if each cross-section is stable; this is shown pictorially in Figure 1.

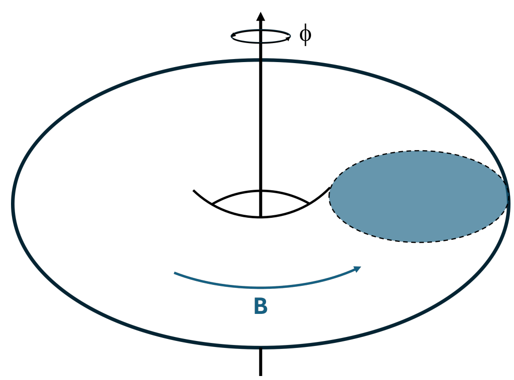

Figure 1: Toroidally symmetric system (such as a tokamak). This has our prescribed toroidal magnetic field $\mathbf{B}_\phi$ about $\hat{\boldsymbol{\phi}}$, where an equilibrium poloidal magnetic field configuration at a fixed angle (blue ellipse with dashed outline) is an equilibrium configuration for the entire system by toroidal symmetry.

To work out if we have an equilibrium configuration, we take Eqn. 2 then assume: there is no toroidal flow and that we are in equilibrium ($\partial_t = 0$).

$$-\nabla P + \mathbf{J} \times \mathbf{B} = 0 \implies \nabla P = \mathbf{J} \times \mathbf{B}. \tag{10}$$

Since we know that $\mathbf{J} = \frac{1}{\mu_0} \nabla \times \mathbf{B}$, we have

$$\mu_0 \nabla P = (\nabla \times \mathbf{B}) \times \mathbf{B}. \tag{11}$$

Since we are solving this within a tokamak, we know that our system is toroidally symmetric: $\partial_\phi = 0$. This allows us to write our magnetic field as

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_\phi}{\partial z}\hat{\mathbf{R}} + B_z \hat{\boldsymbol{\phi}} + \frac{1}{R}\frac{\partial (R A_\phi)}{\partial R}\hat{\mathbf{Z}}. \tag{12}$$

If we define $\psi := RA_\phi$ (which is valid as it can be shown to be proportional to the $\mathbf{B}$ poloidal flux) we have

$$\mathbf{B} = \nabla\psi \times \nabla\phi + B_\phi \nabla\phi \tag{13}$$

$$\mathbf{J} = \frac{1}{\mu_0}\nabla \times \mathbf{B}. \tag{14}$$

After a bit of arithmetic, we arrive at

$$\mu_0 \mathbf{J} = -\frac{1}{R}\Delta^*\psi\nabla\phi + \nabla(RB_\phi) \times \nabla\phi \tag{15}$$

$$\Delta^* := R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}\right) + \frac{\partial^2}{\partial Z^2} \tag{16}$$

where Δ^* is called the Shafranov operator (sometimes called the Stoke's operator). It is important to see that we can write

$$\mathbf{B} \cdot \nabla P = \mathbf{B} \cdot (\mathbf{J} \times \mathbf{B}) = 0 \tag{17}$$

so ∇P is perpendicular to $\mathbf{B}$. This implies that it can be written as a function of ψ. Therefore, recalling that $\nabla P = \mathbf{J} \times \mathbf{B}$, we have

$$\mu_0 \frac{\partial P}{\partial \psi}\nabla\psi = -R^2\Delta^*(\psi\nabla\psi) - RB_\phi\frac{1}{R^2}\nabla(RB_\phi). \tag{18}$$

$\nabla(RB_\phi)$ is parallel to $\nabla\psi$, and thus RB_ϕ is a function of ψ. If we define $f(\psi) := RB_\phi$, we have

$$\mu_0 R^2 \frac{\partial P}{\partial \psi} + f(\psi)\frac{\partial f}{\partial \psi} = -\Delta^*\psi. \tag{19}$$

In a more familiar (famous) form, we have

$$\boxed{\begin{aligned} \Delta^*\psi &= -\mu_0 R j_\phi && (20)\\ j_\phi &:= R\frac{\partial P}{\partial \psi} + \frac{1}{R}f\frac{\partial f}{\partial \psi} && (21) \end{aligned}}$$

where j_ϕ is the *Grad-Shafranov plasma current*. No plasma can confine itself due to the Virial Theorem, so we use external coils to crush the plasma into shape. Traveling along/around flux surfaces is fast, so the plasma is confined to regions with closed flux surfaces.

2 Application to our system

By fixing the location of our last closed flux surface (LCFS), we can fix where the plasma is. We are thus solving:

$$\Delta^*\psi = \begin{cases} -\mu_0 R j_\phi & \text{within LCFS} \\ 0 & \text{outside LCFS} \end{cases} \tag{22}$$

$$\psi := \psi_X \text{ on LCFS} \tag{23}$$

such that we fix ψ_X as the Dirichlet boundary condition using our coils. In solving this, we will separate ψ into two contributions

$$\psi(R, Z) = \psi_p(R, Z) + \psi_H(R, Z) \tag{24}$$

where ψ_p is the flux due to the plasma currents and ψ_H is the flux due to the coils. It is important to note that the Green's function of Δ^* has a closed form:

$$G(R, Z; R', Z') = \frac{\sqrt{RR'}}{2\pi} \cdot \frac{(2-\kappa^2)K(\kappa) - 2E(\kappa)}{\kappa} \tag{25}$$

$$\kappa := \sqrt{\frac{4RR'}{(R+R')^2 + (Z-Z')^2}} \tag{26}$$

where K and E are the complete elliptic integrals of the first and second kind, respectively. Since we know this (can solve this numerically), we have

$$\boxed{\begin{aligned} \psi_p(R, Z) &= \int_{(R',Z')\in\text{LCFS}} dR'dZ' \\ &\quad \times G(R, Z; R', Z')j_\phi(R', Z') && (27)\\ \psi_H(R, Z) &= \sum_{n=0}^{N_p} a_n \sum_{m=0}^{n-2} A_{nm}R^{n-m}Z^m && (28)\\ A_{nm} &:= -\frac{(n+2-m)(n-m)}{m(m-1)}A_{n,m-2}. && (29) \end{aligned}}$$

Our sums within ψ_H are from a multipole expansion of the coils on $\Delta^*\psi_H = 0$, where N_p is the number of these poles. Then a_n represents the contribution of each multipole, chosen to fit our LCFS.

3 Code workflow

1. Pick LCFS
2. Find all grid points in (R, Z) plane within boundary of LCFS
3. To fit N_p poles, pick N_p points on LCFS. Calculate ψ_p at those points. $\psi_{p,i}$ for $i \in [1, N_p] \subseteq \mathbb{N}$ satisfy
$$\Delta_i = \psi_X - \psi_{p,i}. \tag{30}$$
This is the gap at each point that the coils must make up.
4. $W\mathbf{a} = \boldsymbol{\Delta} \implies \mathbf{a} = W^{-1}\boldsymbol{\Delta}$ gives the contribution of each pole a_n determined by
$$W := \begin{pmatrix} \lambda_{i=1,n=0} & \lambda_{i=1,n=2} & \cdots & \lambda_{i=1,n=N_p} \\ \lambda_{i=2,n=0} & \lambda_{i=2,n=2} & \cdots & \lambda_{i=2,n=N_p} \\ \vdots & \vdots & & \vdots \end{pmatrix} \tag{31}$$

which defines $\lambda_n(R, Z) := \sum_{m=0}^{n-2} A_{nm} R^{n-m} Z^m$ as the flux of pole n. Note that we skip the $n = 1$ pole.

5. Calculate ψ_p by numerically integrating our Green's function.

6. Calculate $\psi(R, Z) = \psi_p(R, Z) + \mathbf{a} \cdot \boldsymbol{\lambda}(R, Z)$ for all grid points.