Problem Set 1

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Problem 1.a Assuming an equally weighted die.

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{6} P(x_i) \cdot x_i \\
&\because P(x_1) = P(x_2) = ... = P(x_6) = \frac{1}{6} \\
&= \sum_{i=1}^{6} \frac{1}{6} \cdot x_i \\
&= \frac{1}{6} \cdot \sum_{i=1}^{6} x_i
\end{aligned}
$$

$$
\begin{aligned}
E[Y] &= \sum_{i=1}^{6} P(y_i) \cdot y_i \\
&\because P(y_1) = P(y_2) = ... = P(y_6) = \frac{1}{6} \\
&= \sum_{i=1}^{6} \frac{1}{6} \cdot y_i \\
&= \frac{1}{6} \cdot \sum_{i=1}^{6} y_i
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
E[X] + E[Y] &= \frac{1}{6} \cdot \sum_{i=1}^{6} x_i + \frac{1}{6} \cdot \sum_{i=1}^{6} y_i \\
&= \frac{1}{6} \sum_{i=1}^{6} (x_i + y_i) \\
&= E[X + Y] \\
&= \frac{1}{6} \cdot ((1 + 2 + 3 + 4 + 5 + 6) + \\
&\quad (1 + 2 + 3 + 4 + 5 + 6)) \\
&= \frac{1}{6} \cdot (21 + 21) \\
&= \frac{42}{6} = 7
\end{aligned}
$$

Problem 1.b The numbers 2 through 12 represent the possible values for Z if $Z = X + Y$. However, the probability of each one of these sums occurring differ. The *fundamental principal of counting* states that if one thing can be done N_1 ways while a subsequent thing can be done N_2 ways, the two things can be done in succession $N_1 \cdot N_2$ ways. Therefore, for subsequent rolls, there are $6 \cdot 6 = 36$ possible roles but several will have identical sums. The probabilities for each number have been calculated and listed in the table below.

Sum	Possible Combinations	Probability
2	1	$\frac{1}{36}$
3	2	$\frac{2}{36} = \frac{1}{18}$
4	3	$\frac{3}{36} = \frac{1}{12}$
5	4	$\frac{4}{36} = \frac{1}{9}$
6	5	$\frac{5}{36}$
7	6	$\frac{6}{36} = \frac{1}{6}$
8	5	$\frac{5}{36}$
9	4	$\frac{4}{36} = \frac{1}{9}$
10	3	$\frac{3}{36} = \frac{1}{12}$
11	2	$\frac{2}{36} = \frac{1}{18}$
12	1	$\frac{1}{36}$

Therefore:

$$\begin{aligned}
E[Z] &= \sum_{i=1}^{11} P(z_i) \cdot z_i \\
&= \frac{1}{36} \cdot 2 + \frac{1}{18} \cdot 3 + \frac{1}{12} \cdot 4 + \frac{1}{9} \cdot 5 + \frac{5}{36} \cdot 6 + \frac{1}{6} \cdot 7 + \\
&\quad \frac{5}{36} \cdot 8 + \frac{1}{9} \cdot 9 + \frac{1}{12} \cdot 10 + \frac{1}{18} \cdot 11 + \frac{1}{36} \cdot 12 \\
&= \frac{1}{18} + \frac{1}{6} + \frac{1}{3} + \frac{5}{9} + \frac{5}{6} + \frac{7}{6} + \frac{10}{9} + 1 + \frac{5}{6} + \frac{11}{18} + \frac{1}{3} \\
&= 7
\end{aligned}$$

Problem 2.a Assuming an equally weighted die.

$$\begin{aligned}
E[Y] &= \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 \\
&= \frac{1}{6} \cdot (1 + 2 + 3 + 4 + 5 + 6) \\
&= 3.5
\end{aligned}$$

$$\begin{aligned}
E[Y^2] &= \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 9 + \frac{1}{6} \cdot 16 + \frac{1}{6} \cdot 25 + \frac{1}{6} \cdot 36 \\
&= \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) \\
&= 15.106 \approx 15.1
\end{aligned}$$

Even though neither 3.5 nor 15.106 are possible values for one roll of a dice but 3.5 could be round up or down to a possible roll while 15.106 is well beyond the possible range.

Problem 2.b

$$
\begin{aligned}
\mathrm{E}[(\mathrm{Y}-a)^2] &= \mathrm{E}[\mathrm{Y}^2 - 2a\mathrm{Y} + a^2] \\
&= \mathrm{E}[\mathrm{Y}^2] = 2a\mathrm{E}[\mathrm{Y}] + a^2 \\
&= 15.1 - 2(3.5)a + a^2
\end{aligned}
$$

Problem 2.c

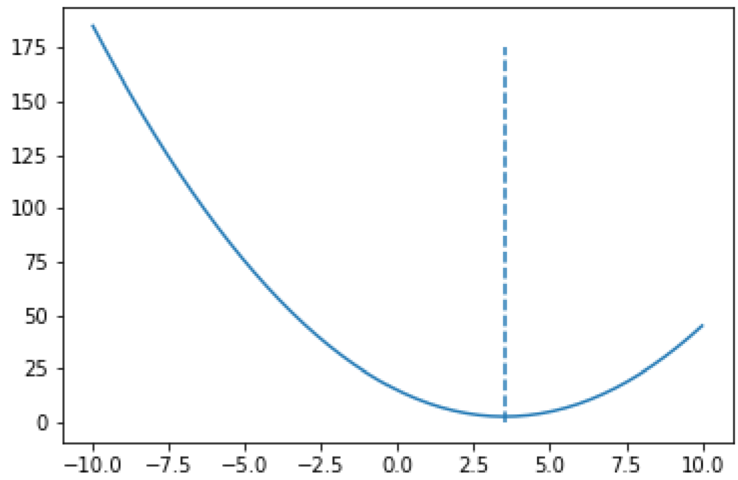

Figure 1: Plot of the function $\mathrm{E}[(\mathrm{Y}-a)^2] = 15.1 - 7 \cdot a + a^2$. The dotted line represents $a = 3.5$.

Problem 3 Note: The expected value of an expectation value is the expectation value $\mathrm{E}[\mathrm{E}[\mathrm{X}]] = \mathrm{E}[\mathrm{X}]$.

$$
\begin{aligned}
Var(\mathrm{Z}) &= \mathrm{E}[(\mathrm{Z} - \mathrm{E}[\mathrm{Z}])^2] && (1) \\
&= \mathrm{E}[\mathrm{Z}^2 - 2 \cdot \mathrm{Z} \cdot \mathrm{E}[\mathrm{Z}] + \mathrm{E}[\mathrm{Z}]^2] && (2) \\
&= \mathrm{E}[\mathrm{Z}^2] - 2 \cdot \mathrm{E}[\mathrm{Z}] \cdot \mathrm{E}[\mathrm{Z}] + \mathrm{E}[\mathrm{Z}]^2 && (3) \\
&= \mathrm{E}[\mathrm{Z}^2] - 2 \cdot \mathrm{E}[\mathrm{Z}]^2 + \mathrm{E}[\mathrm{Z}]^2 && (4) \\
&= \mathrm{E}[\mathrm{Z}^2] - \mathrm{E}[\mathrm{Z}]^2 && (5)
\end{aligned}
$$