Homework Set 3

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Problem 1 - a

Suppose $\exists$ a maximal matching M and a maximum matching $M*$ in G=(V,E). This implies that for each matched vertex in M, there must be at least two matched vertices in $M*$, and that there must be at least one matched vertex in M for which $\exists$ 4 matched vertices (4 because only an even number of matched vertices may exist)in $M*$

This results in a contradiction: Take these four vertices v_1,v_2,v_3,v_4 and at least 2 edges e_1 and e_2 which must connect either $v_1 \to v_2$ and
$v_3 \to v_4$ or $v_1 \to v_3$ and $v_2 \to v_4$ in $M*$. In M, only one of these 4 vertices is matched (thus using a third edge by necessity), leaving one of e_1 or e_2 as a free edge (with neither of its vertices included in the M matching).

$\therefore$ $|M|$ cannot be less than $\frac{1}{2}$ $|M*|$. Thus, $|M| \geq \frac{1}{2}$ $|M*|$.

Then, to compute a maximal matching, simply iterate over all edges in G and match each free edge found. This yields a running time of $O(E)$.
Pseudocode:

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for each edge e = (u,v) in E:
    if u and v are free, mark u and v, using e.
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Problem 2

If a graph is bipartite, then every path is an alternating path between odd and even vertices, and no vertex can be both even and odd (by definition).
Assume $\exists$ some odd cycle $v_1 \to v_2$ ...v_n where n is odd. Then, assign v_1 to be odd and walk the cycle. v_2 must be even, v_2 odd, and so forth, with v_n being odd. then, walk from v_n to v_1. This path is not an alternating path since v_1 and v_n are both odd.
$\therefore$ if G is bipartite, there can be no odd cycle.
Now, any path that is not a cycle can be an alternating path. An even cycle can be drawn as an alternating path with v_1 as odd, v_2 even ... v_n even since n is even (even cycle). Therefore, $v_1 \to v_n \to v_1$ is an alternating path. so, if a graph is made up of only non-cyclical paths and/or even cycles then it is a bipartite graph.
$\therefore$ if G has no odd cycles, it is bipartite

We may conclude: G has no cycles $\Leftrightarrow$ G is bipartite.