

7.3.10) Show that the units digit of any number a is the same as the units digit of a^{4n+1} .

Proof. Important note, $4n + 1$ is odd. Odd powers do not change the sign. The only way to find an integer $b \equiv c \pmod{10}$: the units digit of b is not the units digit of c is to have either b or c negative but not both.

By theorem 7.1 we have $\phi(5) = 4$ and $\phi(2) = 1$.

By multiplicity of ϕ , we have $\phi(10) = \phi(2)\phi(5) = 4$.

Take $d = \gcd(10, a)$.

This implies $\gcd(\frac{10}{d}, a) = 1$ because 10 is a square free number.

Take (by Euler's theorem)

$$\begin{aligned}
 & a^{\phi(\frac{10}{d})} \equiv 1 \pmod{\frac{10}{d}} \\
 \implies & a^{\phi(\frac{10}{d})\phi(d)n} \equiv 1^{\phi(d)n} \equiv 1 \pmod{\frac{10}{d}} \\
 \implies & a^{\phi(10)n} \equiv a^{4n} \equiv 1 \pmod{\frac{10}{d}} \\
 \implies & da^{4n} \equiv d \pmod{10} \\
 \implies & \frac{a}{d}da^{4n} \equiv \frac{a}{d}d \pmod{10} \\
 \implies & a^{4n+1} \equiv a \pmod{10}.
 \end{aligned}$$

Therefore, by the note, the units digit of a is equal to the units digit of a^{4n+1} for any a . \square