Prove that: Every integer $n > 1$ is the product of a square free integer and a perfect square.

Proof. 3.1.16b

1 is a square free integer. (1)

$1^2 = 1$ is a perfect square. (2)

1 divides all integers. (3)

By the FTA, $n = p_1 p_2 \cdots p_i n_i$.

We take $n_i = 1$.

$\forall\ p_{a_1} = p_{a_2} = \ldots = p_{a_k}$ we have $p_{a_1}^k$.

We take $k_m = 2s_m + r_m,\ r_m \in \{0, 1\}$ by the division algorithm.

We take

$$\begin{aligned} t =& p_1^{2s_1} p_2^{2s_2} \cdots p_i^{2s_i} \\ =& (p_1^{s_1} p_2^{s_2} \cdots p_i^{s_i})^2. \end{aligned}$$

This gives us

$$n = \frac{n}{t} t.$$

We have

$$\frac{n}{t} = p_1^{r_1} p_2^{r_2} \cdots p_i^{r_i}.$$

This gives us $\frac{n}{t}$ is a square free number because $r_m \in \{0, 1\}$.

t is a perfect square such that $\sqrt{t} = p_1^{s_1} p_2^{s_2} \cdots p_i^{s_i}$.

Therefore, we have $n = \frac{n}{t} \implies n$ is the product of a square free integer and a perfect square. □