7.2.10) If every prime that divides n also divides m then $\phi(mn) = n\phi(m)$.

Proof. Take p_i as a prime $\forall\ i$.
Take $R = \{r : p_r|mn\}\,; S = \{s : p_s|m\}\,; T = \{t : p_t|n\}\,.$
Take $k_r = a_r + b_r$ where

$$\begin{aligned} &p_r^{a_r}|m && p_r^{a_r+1} \not| \ m \\ &p_r^{b_r}|n && p_r^{b_r+1} \not| \ n \\ &p_r^{k_r}|mn && p_r^{k_r+1} \not| \ mn. \end{aligned}$$

By the FTA

$$\prod_{j\in T} p_j^{b_j} = n.$$

By Theorem 7.3

$$\phi(m) = \prod_{j\in S}\left(p_j^{a_j}\left(1 - \frac{1}{p_j}\right)\right).$$

By Theorem 7.3

$$\begin{aligned} \phi(mn) &= \prod_{j\in R}\left(p_j^{k_j}\left(1 - \frac{1}{p_j}\right)\right) \\ &= \prod_{j\in R}\left(p_j^{a_j+b_j}\left(1 - \frac{1}{p_j}\right)\right) \\ &= \prod_{j\in R} p_j^{b_j} \prod_{j\in R}\left(p_j^{a_j}\left(1 - \frac{1}{p_j}\right)\right). \end{aligned}$$

By the condition, we know $R = S$ and $T \subset S$ and $i \notin T \implies b_i = 0$. Take

$$\begin{aligned} \phi(mn) &= \prod_{j\in T} p_j^{b_j} \prod_{j\in S}\left(p_j^{a_j}\left(1 - \frac{1}{p_j}\right)\right) \\ &= n\phi(m). \end{aligned}$$

Taking $m = n$ we get $\phi(n^2) = \phi(mn) = n\phi(m) = n\phi(n)$. □