Proof. 1.2.4a
Prove $\binom{n}{r} < \binom{n}{r+1}$ if and only if $0 \leq r \leq \frac{1}{2}(n-1)$ for $n \geq 1$.
We have $\binom{n}{r} < \binom{n}{r+1}$
$\iff \frac{n!}{r!(n-r)!} < \frac{n!}{(r+1)!(n-r-1)!}$
$\iff (r+1)!(n-r-1)! < r!(n-r)!$
$\iff (r+1)r!(n-r-1)! < r!(n-r)(n-r-1)!$
$\iff r+1 < n-r$
$\iff 2r+1 < n$
$\iff 2r < n-1$
$\iff r < \frac{1}{2}(n-1)$.
We have $r \geq 0$ because of the binomial coefficient.
This gives us the inequality $0 \leq r \leq \frac{1}{2}(n-1) \iff \binom{n}{r} < \binom{n}{r+1}$. □

Proof. 1.2.4b
Prove $\binom{n}{r} > \binom{n}{r+1}$ if and only if $n-1 \geq r \geq \frac{1}{2}(n-1)$ for $n \geq 1$.
We have $\binom{n}{r} > \binom{n}{r+1}$
$\iff \frac{n!}{r!(n-r)!} > \frac{n!}{(r+1)!(n-r-1)!}$
$\iff (r+1)!(n-r-1)! > r!(n-r)!$
$\iff (r+1)r!(n-r-1)! > r!(n-r)(n-r-1)!$
$\iff r+1 > n-r$
$\iff 2r+1 > n$
$\iff 2r > n-1$
$\iff r > \frac{1}{2}(n-1)$.
We have $r+1 \leq n$ because of the binomial coefficient.
This gives us the inequality $n-1 \geq r \geq \frac{1}{2}(n-1) \iff \binom{n}{r} > \binom{n}{r+1}$. □

Proof. 1.2.4c
Prove $\binom{n}{r} = \binom{n}{r+1}$ if and only if $r = \frac{1}{2}(n-1)$ for $n \geq 1$ while n is odd.
We have $\binom{n}{r} = \binom{n}{r+1}$
$\iff \frac{n!}{r!(n-r)!} = \frac{n!}{(r+1)!(n-r-1)!}$
$\iff (r+1)!(n-r-1)! = r!(n-r)!$
$\iff (r+1)r!(n-r-1)! = r!(n-r)(n-r-1)!$
$\iff r+1 = n-r$
$\iff 2r+1 = n$
$\iff 2r = n-1$
$\iff r = \frac{1}{2}(n-1)$.
This gives us the equation $r = \frac{1}{2}(n-1) \iff \binom{n}{r} = \binom{n}{r+1}$. □

These three proofs show Pascal's triangle's relation to $\binom{r}{n}$ and the middle column.