4.3.8) Prove $\forall a \in \mathbb{Z}, a^2 - a + 7 \equiv x \mod 10, \quad x \in \{3, 7, 9\}$.

Proof. Take $r \in [-4, 5] \cap \mathbb{Z} : a \equiv r \mod 10$. This gives us

$$a^2 \equiv r^2 \mod 10$$
$$a^2 - a \equiv r^2 - r \mod 10$$
$$a^2 - a + 7 \equiv r^2 - r + 7 \mod 10.$$

Take cases of $r^2 - r + 7 \equiv x \mod 10$

$$\begin{aligned} 1^2 - 1 + 7 &\equiv 7 & 0^2 - 0 + 7 &\equiv 7 \\ 2^2 - 2 + 7 &\equiv 9 & (-1)^2 - (-1) + 7 &\equiv 9 \\ 3^2 - 3 + 7 &\equiv 3 & (-2)^2 - (-2) + 7 &\equiv 3 \\ 4^2 - 4 + 7 &\equiv 9 & (-3)^2 - (-3) + 7 &\equiv 9 \\ 5^2 - 5 + 7 &\equiv 7 & (-4)^2 - (-4) + 7 &\equiv 7 \end{aligned}$$

Therefore

$$a^2 - a + 7 \equiv 3, 7 \text{ or } 9 \mod 10.$$

$\square$