4.3.10) Prove that any integer whose digits add up to 15 can not be a square or a cube.

Proof. Take $a \in \mathbb{Z}$ and $r \in [-4,4] \cap \mathbb{Z} : a \equiv r \mod 9$. This is a complete set of residues that gives us

$$\begin{aligned} a^2 &\equiv r^2 \mod 9 \\ \text{and } a^3 &\equiv r^3 \mod 9 \end{aligned}$$

Take $c = |b|$. The sum of the digits of a negative number is clearly equal to the sum of the digits of its absolute value. Also, it is clear that for every negative cube, there is a corresponding positive cube such that the sum of the digits of b equals the sum of the digits of $|b|$. It is also clear that the sum of the digits of 0 is not 15.
Take digits $d_n \in [0,9] \cap \mathbb{Z} : \forall\ c$,

$$c = \sum_{k=0}^{+\infty} d_k 10^k.$$

We have

$$\forall\ c\ \exists\ h : \forall\ n \in [h, +\infty) \cap \mathbb{N}, d_n = 0 \implies c = \sum_{k=0}^{+\infty} d_k 10^k = \sum_{k=0}^{h} d_k 10^k.$$

Take the sum of the digits. We have

$$\sum_{k=0}^{h} d_k.$$

Take $b - 6$. We get

$$\sum_{k=0}^{h} d_k 10^k - 6 = d_0 - 6 + \sum_{k=1}^{h} d_k 10^k$$

If $d_0 < 6$, we should instead take $j = \min\{i : d_i > 0\}$:

$$\sum_{k=0}^{h} d_k 10^k - 6 = d_0 + 4 - 10^j + \sum_{k=1}^{h} d_k 10^k.$$

Take digits $d_n \in [0,9] \cap \mathbb{Z} : \forall\ b$,

$$c - 6 = \sum_{k=0}^{h} d_k 10^k - 6 \implies$$

$$\sum_{k=0}^{h} d'_k = \sum_{k=0}^{h} d_k - 6 \quad \forall\ d_0 \geq 6,$$

$$\sum_{k=0}^{h} d'_k = \sum_{k=0}^{h} d_k + 4 - 1 \quad \forall\ d_0 < 6.$$

Take $c \mod 9$. We get $0 \equiv \sum_{k=0}^{h} d'_k \mod \leftrightarrow 9 \mid \sum_{k=0}^{h} d_k - 6$. We have $-3 \equiv 6 \mod 9 \therefore 9 \nmid b - 6 \implies$ the digits of b do not add up to 15. Take cases of $b = a^2$ or $b = a^3$. We take r^2 and $r^3 \mod 9$.

$$\begin{aligned} 4^2 \equiv (-4)^2 \equiv 7 \mod 9 \qquad & 3^2 \equiv (-3)^2 \equiv 0 \mod 9 \\ 2^2 \equiv (-2)^2 \equiv 4 \mod 9 \qquad & 1^2 \equiv (-1)^2 \equiv 1 \mod 9 \\ 0^2 \equiv 0 \mod 9 \qquad & \end{aligned}$$

$$\begin{aligned} (-4)^3 \equiv 8 \mod 9 \qquad & (-3)^3 \equiv 0 \mod 9 \qquad & (-2)^3 \equiv 1 \mod 9 \\ (-1)^3 \equiv 8 \mod 9 \qquad & 0^3 \equiv 0 \mod 9 \qquad & 1^3 \equiv 1 \mod 9 \\ 2^3 \equiv 8 \mod 9 \qquad & 3^3 \equiv 0 \mod 9 \qquad & 4^3 \equiv 1 \mod 9 \end{aligned}$$

It is clear that $6 \not\equiv x \ \forall \ x \in \{0, 1, 4, 7, 8\} \therefore 9 \nmid b - 6 \therefore$ the digits of a^2 and a^3 never add up to 15 $\forall \ a \in \mathbb{Z}$. $\square$