Hilbert Space

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The field of mathematics known as functional analysis has a core in linear spaces. German mathematician David Hilbert (1862 - 1943) formulated what we call Hilbert space, an infinite dimensional generalization of Euclidean space [4]. Euclidean space, or $\mathbb{R}^n$, is relatively easy to understand. People with minimal background in mathematics can easily identify with a line $\mathbb{R}$, a plane $\mathbb{R}^2$, and the real world $\mathbb{R}^3$. If one wants to visualize Euclidean space of more than three dimensions, it becomes quite a bit more difficult. David Hilbert provided a mathmatical definition that encompasses Euclidean space in any number of dimensions, including infinite, now known as Hilbert space [3]. David Hilbert and his connection to Hilbert space are important topics to explore, as well as what it means to be a complete inner product space, which will assist in understanding why this is fundamentally important to topics such as functional analysis.

David Hilbert was one of the most influential and important mathematicians of the early 20th century. He was a pure mathematician who would turn out to assist in applying rigor to the mathematics of physics, as he worked alongside Albert Einstein. Hilbert gained his doctorate in 1885. His contributions to mathematics increased significantly in pace around the year 1900. Around 1909 Hilbert began work in the area of functional analysis; it was around this time that Hilbert wanted a way to use infinite dimensional Euclidean space, and so he defined what is now known as Hilbert space [4]. In 1912, Hilbert turned his eye toward physics, which, it so happens, made great use of the notion of Hilbert space in the area of quantum mechanics.

Hilbert space has a rather simple definition with less simple implications. The natural ordering is to understand first what an inner product space is, alternatively known as a pre-Hilbert space, then to progress on to what it means for one to be complete.

Definition (Inner product)**.** Let V be a complex vector space. A mapping $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is called an *inner product* if $\forall x, y, z \in V, a \in \mathbb{C}$

$$\langle x, x \rangle \geq 0$$

$$\langle x, x \rangle = 0 \iff x = \vec{0}$$

$$\langle x, y \rangle = \overline{\langle y, x \rangle}$$

$$\langle ax + y, z \rangle = a\langle x, z \rangle + \langle y, z \rangle$$

where $\overline{a}$ is the complex conjugate of a [3].

The main example of an inner product is a dot product in $\mathbb{R}^n$ or $\mathbb{C}^n$. For vectors $\vec{u} = [u_1 u_2 \dots u_n]^T$ and $\vec{v} = [v_1 v_2 \dots v_n]^T$ in $\mathbb{R}^n$, the dot product is defined as $\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_1 v_1$ and gives an idea of the lengths of $\vec{v}$ and $\vec{u}$ as well as the angle made between them.

Definition (Inner product space)**.** A vector space V is an *inner product space* $(V, \langle \cdot, \cdot \rangle)$ if it has inner product $\langle \cdot, \cdot \rangle$ defined on it [3].

An inner product space $(V, \langle \cdot, \cdot \rangle)$ allows us to define a norm $||x|| = \sqrt{\langle x, x \rangle}$ (thus making it a *norm space* $(V, || \cdot ||)$) and distance function $d(x, y) = ||x - y||$ on V (thus making it a *metric space* (V, d)) [3]. Here, $\mathbb{R}^n$ and $\mathbb{C}$ are good examples of inner product spaces. The inner product of $\mathbb{R}^n$ is the dot product, as mentioned above, so the distance between vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ is $d(\vec{u}, \vec{v}) = ||\vec{u} - \vec{v}|| = \sqrt{(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})} = \sqrt{\sum_{i=1}^{n} (u_i - v_i)^2}$. The inner product of a complex number $c = \alpha + i\beta$ with itself is $\langle c, c \rangle = (\alpha + i\beta)(\alpha - i\beta)$. This results in the norm of c, $||c|| = \sqrt{\alpha^2 + \beta^2}$ equal to the modulus, or absolute value of c, which in turn is the distance of c from 0.

Definition (Cauchy sequence)**.** A sequence $\{a_n\} \in V$ is a *Cauchy sequence* if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n, m > N \implies d(a_n, a_m) < \epsilon$ [3].

Definition (Complete space)**.** If every Cauchy sequence in a space V converges to an element $v \in V$, then V is *complete* [3].

A complete space is analogous to a continuous function. A continuous function can be defined as a function $f : D \to R$ such that for all sequences $\{x_n\} \subset D, n \in \mathbb{N}$ that converge to a point $c \in D$, the corresponding sequence $\{f(x_n)\}$ converges to $f(c)$. Every Euclidean space is a Hilbert space, as well as every complex space $\mathbb{C}^n$.

Hilbert space is fundamental in functional analysis. It allows for any function to be represented as a linear combination of simpler functions, such as polynomials or trigonometric functions. Fourier series are a famous functional decomposition technique that have an important interpretation in Hilbert space. The Fourier expansion of a function $f : [-\pi, \pi] \to \mathbb{R}$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \sin(nx) + b_n \cos(nx)).$$

The coefficients a_0, a_n, b_n need to be defined for all $n \in \mathbb{N}$, so not all functions have a Fourier expansion. However, it can be shown that a Fourier expansion of f exists if f is square integrable.

Definition (Square integrable)**.** A function $f : V \to \mathbb{C}$ such that the integral

$$\int_V |f|^2 d\mu < \infty$$

is called *square integrable* [5]. The collection of all square integrable functions is called $L^2(\mu)$ where μ is a measure. $L^2 = L^2(\mu)$ for μ being an appropriate measure for the situation.

Square integrable functions are an area where Hilbert space becomes extremely convenient. The Fourier expansion of any square integrable function is identical to a Hilbert space expansion of the same function in terms of an orthogonal basis. This is because L^2

is a Hilbert space, and the terms of a Fourier series form an orthogonal basis to L^2 [5]. In fact, all Hilbert spaces are isometric to L^2, meaning there exists a bijective $f : L^2 \to H$ for any Hilbert space H such that $d_{L^2}(x, y) = d_H(f(x), f(y))$.

A recent example of the use of Hilbert space is in the area of machine learning techniques, called RKHS, or reproducing kernel Hilbert space.

Definition (RKHS)**.** A Hilbert space H of functions $f : X \to R$ is an RKHS if for all $f, g \in H, x \in X, \epsilon > 0$, there exists δ such that $d_H(f, g) < \delta \implies |f(x) - g(x)| < \epsilon$ [2].

In other words, in an RKHS, $\forall f \in H, \{f_n\} \subset H$,

$$\lim_{n\to\infty} ||f_n - f|| = 0 \implies \lim_{n\to\infty} f_n = f,$$

which is not necessarily true in all Hilbert spaces.

Definition. A reproducing kernel is a function $k : X \times X \to \mathbb{R}$ such that $\forall y \in X, g \in H$,

$$f(x) = k(x, y) \implies f \in H,$$

and

$$f(x) = k(x, y) \implies \langle g, f\rangle_H = g(y).$$

The second condition means the kernel can reproduce, as $a, b \in X, f(x) = k(x, a), g(y) = k(y, b) \implies k(a, b) = \langle f, g\rangle_H$. It turns out that this kernel k exists and is unique given H [2]. In addition, given a reproducing kernel k, one can derive an RKHS H with k as its reproducing kernel by the Moore-Aronszajn theorem. The use of an RKHS in machine learning applications can smooth noise in features [1]. This would be effective if the application is identifying the subject of a dirty image, or identifying a speaker with significant background noise. While there are other simpler methods to smooth noise, an RKHS can do so for an infinite number of features, with features being the observed variables used to predict or classify the outcome.

Hilbert space is a powerful tool of functional analysis that assists in making generalized statements that apply to important vector spaces, such as $\mathbb{R}^n$ and L^2. Hilbert himself was a pure mathematician who defined the complete inner product space. He was motivated to generalize Euclidean space with this powerful tool that allows for vector algebra and calculus in infinite dimensions. Hilbert space additionally helped make strides in quantum mechanics where Hilbert himself did work later on, and in the more recent field of machine learning. Powerful theorems that can be proven in Hilbert space become even more powerful due to the breadth of their reach into real, complex, and functional spaces.

References

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