

Review of Phasors

Phasors are essential tool in circuit analysis.

Definition 1. *Phasor transformation is defined as follows:*

$$v(t) = \Re\{|V|e^{j\theta_v}e^{j\omega t}\} \quad (1)$$

where $\tilde{V}|e^{j\theta_v}$ is the phasor of voltage $v(t)$. Phasor is a complex number in polar coordinate system and it is usually denoted with a capital letter with a tilde above it $\tilde{V} = |\tilde{V}|e^{j\theta_v}$. $|\tilde{V}|$ is the magnitude and θ_v is the phase of the complex number. Symbol \Re represents the real part of the expression in the curly brackets.

You may be wondering why would this specific equation be used. This is the case of inverse superposition. We add an addition voltage source to our circuit, denote it as imaginary by multiplying it with $j = \sqrt{-1}$, so that we are able to use the Euler's formula to write a sinusoidal signals in terms of complex exponentials. When we use complex exponentials, the time-domain differential equations transform to simple algebraic equations. Principle of superpositions tells us that the response of the circuit to the real part of the generator will be real, and the response of the imaginary part of the generator will be imaginary. Since we added additional source that wasn't there previously, we now need to take the response only from the real generator.

Example 1. *We will see how the phasor transformation is applied to an RC circuit shown in Figure 21. To solve this circuit in the time domain we apply Kirchhoff's voltage law as shown in Equation 2 -3.*

The circuit in Figure 1 is a simple RC circuit. KVL equation in the time domain is given in Equation 2.

$$v_s(t) = v_R(t) + v_C(t) \quad (2)$$

$$v_s(t) = Ri + \frac{1}{C} \int i(t)dt \quad (3)$$

Where

$$v_s(t) = A\cos(\omega t + \Theta) \quad (4)$$

Learning outcomes: Apply phasor transformation to a time-domain equation to obtain frequency-domain equation.

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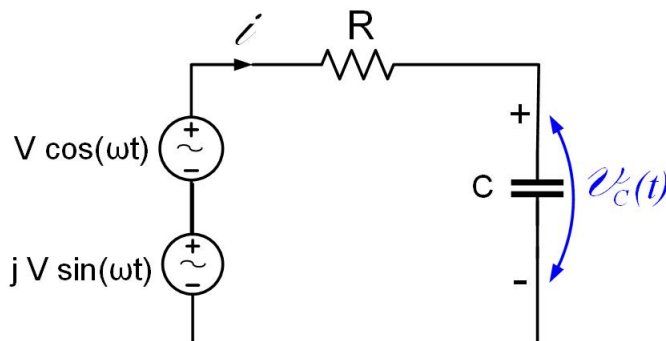


Figure 1: Using superposition to find phasors of voltages and currents in an RC circuit.

In order to solve this circuit we have to solve a differential equation. Use of phasors simplify the equations significantly. Differential equations become a set of linear equations. Phasor is another name for a complex number in polar coordinate system.

In order to use phasors, the circuit has to be linear. Circuits that have only capacitors, inductors and resistors are linear circuits. In a linear circuit, all currents and voltages are at the frequency of the generator. That means that we don't have to keep track of the frequency of voltages and currents when we are solving the circuit. We know the frequency of all currents and voltages once we know the frequency of the generator. The quantities that will differ for different currents and voltages are the amplitudes and phases. Phasors allow us to drop the information about the frequency of the signal, and only keep track of the magnitude and phase of the signal.

In order to remove $\cos(\omega t)$ term from the equations we have to use complex numbers. To write $\cos(\omega t)$ in a concise form as a complex number, we add to the forcing function a sinusoidal imaginary term.

$$A \cos(\omega t + \Theta) + j A \sin(\omega t + \Theta) \quad (5)$$

We have to make sure later when we are done with our calculations with complex number, that we only take the real part of the final expression. It seems that we made the above expression more complicated, however, if we remember Euler's identity, the expression becomes

It is customary to use $\cos(\omega t)$ for our time-domain signal. If signals in a circuit are given in terms of $\sin(\omega t)$, the sin function has to be converted to a cosine. To do that, subtract 90° from the phase of the sinusoid, because $\sin(\omega t) = \cos(\omega t - 90^\circ)$

$$\begin{aligned}
v_s(t) &= V \cos(\omega t + \Theta_V) = \Re\{A \cos(\omega t + \Theta) + j A \sin(\omega t + \Theta)\} = \\
&= \Re\{A e^{j(\omega t + \Theta)}\} = \Re\{A e^{j\Theta} e^{j\omega t}\}
\end{aligned} \tag{6}$$

In Equation 6 we extracted the phase and amplitude information and separated it from the frequency. The amplitude and phase information is called phasor $V_S(j\omega) = A e^{j\Theta}$. Why is this expression better than the one with a $\cos(\omega t)$ and how can we remove t from Equation 3? To answer this question, we first have to write general expressions for voltage $v(t)$ and current $i(t)$ in the Equation 3:

$$\begin{aligned}
v(t) &= \Re\{V \cos(\omega t + \Theta_V) + j V \sin(\omega t + \Theta_V)\} = \\
&= \Re\{V e^{j(\omega t + \Theta_V)}\} = \Re\{V e^{j\Theta_V} e^{j\omega t}\}
\end{aligned} \tag{7}$$

If we look at the first and last expression in Equation 7 we see that the time domain signal is the real part of the product of phasor and the $e^{j\omega t}$ term.

$$v(t) = \Re\{V e^{j\Theta_V} e^{j\omega t}\} \tag{8}$$

Similarly for current

$$\begin{aligned}
i(t) &= I \cos(\omega t + \Theta_I) = \Re\{I \cos(\omega t + \Theta_I) + j I \sin(\omega t + \Theta_I)\} = \\
&= \Re\{I e^{j(\omega t + \Theta_I)}\} = \Re\{I e^{j\Theta_I} e^{j\omega t}\}
\end{aligned} \tag{9}$$

If we look at the first and last expression in Equation 9 we get a similar expression for current.

$$i(t) = \Re\{I e^{j\Theta_I} e^{j\omega t}\} \tag{10}$$

In Equations 7-9 V and I are voltage and current amplitudes, and Θ_V and Θ_I are voltage and current phases. The voltage on the resistor is then given in Equation 11

$$v_R(t) = R \times i(t) = R \times \Re\{I e^{j\Theta_I} e^{j\omega t}\} = \Re\{R \times I e^{j\Theta_I} e^{j\omega t}\} \tag{11}$$

The voltage on the capacitor is a bit more complicated. We know that $i(t) = \Re\{I e^{j\Theta_I} e^{j\omega t}\}$, but what is the integral of $i(t)$?

$$v_C(t) = \frac{1}{C} \int \Re\{Ie^{j\Theta_I} e^{j\omega t}\} dt \quad (12)$$

Now if integral and Re exchange places, and if we take all time-independent quantities in front of the integral,

$$v_C(t) = \frac{1}{C} \int i(t) dt = \Re\left\{\frac{1}{C} \int Ie^{j\Theta_I} e^{j\omega t}\right\} dt = \Re\left\{\frac{1}{C} Ie^{j\Theta_I} \int e^{j\omega t} dt\right\} \quad (13)$$

$$v_C(t) = \Re\left\{\frac{1}{C} Ie^{j\Theta_I} \frac{1}{j\omega} e^{j\omega t}\right\} = \Re\left\{\frac{1}{j\omega C} Ie^{j\Theta_I} e^{j\omega t}\right\} \quad (14)$$

We now replace the time-domain quantities in equation 3 with these newly developed expressions.

$$v_s(t) = v_R(t) + v_C(t) \quad (15)$$

$$\Re\{Ae^{j\Theta} e^{j\omega t}\} = \Re\{R \times Ie^{j\Theta_I} e^{j\omega t}\} + \Re\left\{\frac{1}{j\omega C} Ie^{j\Theta_I} e^{j\omega t}\right\} \quad (16)$$

A common term in the previous equation is $e^{j\omega t}$, and we can now drop Re, as long as we later remember to take only the real part of the expression for the phasor of voltage and current to get the time domain expression. We can now write the equation as

$$V_S e^{j\Theta_{V_S}} = R \times Ie^{j\Theta_I} + \frac{1}{j\omega C} Ie^{j\Theta_I} \quad (17)$$

$$V_S(j\omega) = RI(j\omega) + \frac{I(j\omega)}{j\omega C} \quad (18)$$

Since this is a linear equation, we can easily solve it:

$$I(j\omega) = \frac{V_S(j\omega)}{R + \frac{1}{j\omega C}} \quad (19)$$

Let's say that the values are given for R, C, ω and V_S such that the phasor of the current is $I = 3e^{j45^\circ}$. To obtain the signal in the time domain, we multiply the phasor I with the $e^{j\omega t}$ term, and then we find the real part of the expression to obtain its current in the time domain, as shown in Figure 20

$$\begin{aligned} i(t) &= \Re\{3e^{j45^\circ} e^{j\omega t}\} = \Re\{3e^{j\omega t + 45^\circ}\} = \\ &= \Re\{3\cos(\omega t + 45^\circ) + j3\sin(\omega t + 45^\circ)\} = 3\cos(\omega t + 45^\circ) \end{aligned} \quad (20)$$

circuit element	impedance	low frequencies $f \rightarrow 0$	high frequencies $f \rightarrow \infty$
capacitor	$\frac{1}{j\omega C}$	∞	0
inductor	$j\omega L$	0	∞

Table 1: Impedance of the capacitor and inductors and their equivalent impedances at high and low frequencies.

In case we have an inductor in the circuit, the voltage on an inductor can be derived as shown in Figure 21.

$$\begin{aligned}
 v_L(t) &= L \frac{\partial i(t)}{\partial t} = L \frac{\text{Re}\{I e^{j\Theta_I} e^{j\omega t}\}}{\partial t} dt = \text{Re}\{L I e^{j\Theta_I} \frac{\partial e^{j\omega t}}{\partial t}\} = \\
 &= \text{Re}\{L I e^{j\Theta_I} j\omega e^{j\omega t}\} = \text{Re}\{j\omega L I e^{j\Theta_I} e^{j\omega t}\} \quad (21)
 \end{aligned}$$

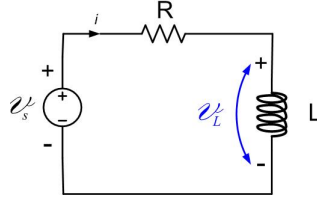


Figure 2: Complex number z in rectangular and polar coordinates.

Step-by-step instructions on how to solve circuits using phasors is given as follows:

- Adopt cosine reference for generator voltage or current.
- replace all impedances with their phasor expressions,
- write KVL and KCL, or use other Network Analysis techniques.
- find the phasor expression for the required current or voltage.
- multiply the phasor with $e^{j\omega t}$
- find the real part of the above expression to get the current in the time domain.

Example 2. *Using Vectors to Represent Phasors in an example*

Calculate on paper the magnitude and phase of the current and voltages in a series RC circuit shown in Figure 6 (a) if the circuit is driven with a frequency of 1 GHz (phase is zero) and $R=1k\Omega$, $C=\frac{1}{2\pi}10^{-12} F$.

Answer the following questions:

- What are the magnitude, phase and time delay of the source voltage?
- What are the magnitude, phase and time delay of the voltage across the resistor?
- What are the magnitude, phase and time delay of the voltage across the capacitor?
- The simulated voltage magnitude across the resistor is about $0.5V$ and the simulated voltage across the capacitor is about $0.85V$. If we use KVL $0.85V + 0.5V \neq 1V$. Why? Look at Figure 6 to help you with answer this question.

Explanation. When you calculate the magnitudes and phases of the voltages in the RC circuit, draw three points, to represent magnitude and phases of all three vectors, as in Figure 5. The three points denote the three position vectors as shown in Figure 6. Vector addition must be used to add the voltages on the resistor and capacitor to obtain the voltage of the generator.

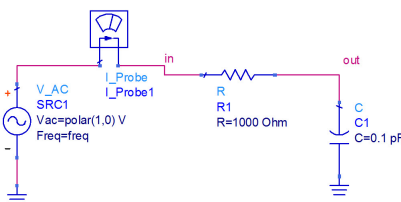


Figure 3: RC circuit in ADS.

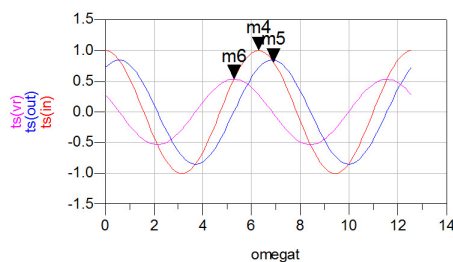


Figure 4: Sinusoidal signal as a function of angle ωt

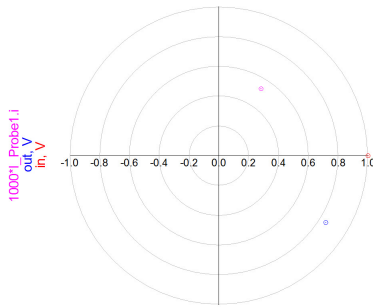


Figure 5: Points represent polar plot of complex voltages in RC circuit.

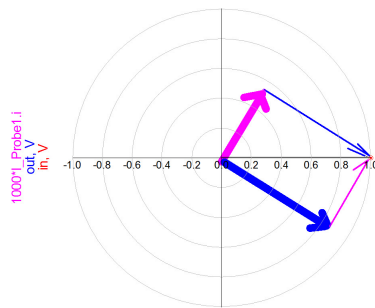


Figure 6: Vectors are drawn in the polar plot of complex voltages in RC circuit. Note how they add up to 1V.

```
1 var('s t')
2 x(t) = 3*cos(t)
3 y(t) = 3*sin(t)
4 c(t) = (x(t),y(t))
5 circle=parametric_plot(c(t),(t,0,2*pi),color="black")
6 circle
```

```
_____ SAGE-Output _____
1 var('s t')
2 x(t) = 3*cos(t)
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4 c(t) = (x(t),y(t))
5 circle=parametric_plot(c(t),(t,0,2*pi),color="black")
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```
