

# Biot-Savart's Law

## Magnetic Field due to a Charge Distribution

We will first find the magnetic field due to a loop carrying current  $I$ , positioned in the X-Y plane, as shown in Figure 1. To solve this problem, we will first divide the loop into small pieces and label one of these pieces as  $dl$ . The magnetic field due to an infinitesimal current, can be found using Biot-Savart's law. Magnetic field is labeled in Figure 1 as  $dB$ . The infinitesimal current position is defined by a position vector  $\vec{r}_2$ . The position of point P, where the field will be calculated, is defined with the position vector  $\vec{r}_1$ . The distance between the current and the observation point is labeled as  $\vec{r}$ . The vector  $d\vec{B}$  is defined in Equation 1.

$$d\vec{B}_1 = \frac{\mu I}{4\pi r^3} (\hat{dl} \times \hat{r}) \quad (1)$$

The total magnetic field at a point P is then equal to the sum of all the fields due to the elemental currents, as shown in Figure 2. The equation for the total field is given in 2.

$$\vec{B} = \int_{all \text{ inf. currents}} d\vec{B} \quad (2)$$

The problem now is to represent all the variables in the Equation 1 (  $I$ ,  $dl$ ,  $\hat{r}$  and  $r$ ) using appropriate coordinate system and given current distribution. As seen in Figure 1,  $\vec{dl}$  is an arc length in the direction of theta (blue arrow next to  $dl$ )  $dl = a d\theta a_\theta$ , where  $a$  is the radius of the loop. The vector  $\vec{r}_2$  is the position vector of the arc length  $dl$ , and the vector  $\vec{r}_1$  is the position vector of the point P where we want the find the magnetic field. Point P is an arbitrary point in the Cartesian coordinate system, P(x,y,z), therefore its vector is shown in Equation3. The vector  $\vec{r}$  is the distance vector between the elemental current (the source) and the point at which we are calculating the electric field.

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Learning outcomes: Use Bio-Savart's Law to calculate magnetic field due to a current distribution.

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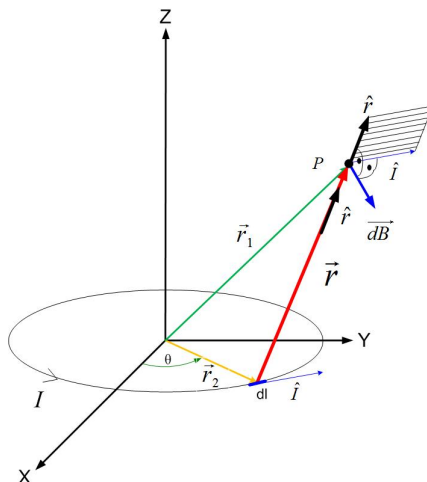


Figure 1: Loop of wire carrying current  $I$ . Magnetic field is shown due to a very small section (arc length) of the loop  $dl$ .

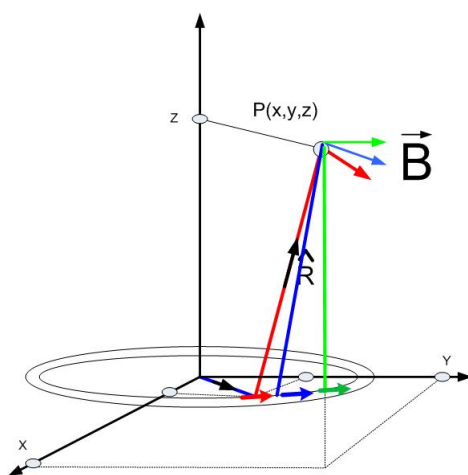


Figure 2: Loop of wire uniformly charged with line charge density  $\rho_l$ . Electric field is shown due to several very small sections (arc length) of the loop  $dl$ . Each section is modeled by a point charge  $dQ$ .

$$\vec{r}_1 = x\vec{a}_x + y\vec{a}_y + z\vec{a}_z \quad (3)$$

The vector  $\vec{r}_2$  can be written in Polar Coordinates as in Equation 4, where  $a$  is the radius of the loop. The equation 4 can be rewritten in Cartesian coordinate system as in Equation 6.

$$\vec{r}_2 = a \vec{a}_r \quad (4)$$

$$\vec{a}_r = \cos\theta \vec{a}_x + \sin\theta \vec{a}_y \quad (5)$$

$$\vec{r}_2 = a \cos\theta \vec{a}_x + a \sin\theta \vec{a}_y \quad (6)$$

The two vectors mark the beginning and the end of the distance vector  $\vec{r}$ . The vector  $\vec{r}$  is the sum of vectors  $-\vec{r}_2$  and  $\vec{r}_1$ .

$$\vec{r} = \vec{r}_1 + (-\vec{r}_2) \quad (7)$$

Therefore the vector's  $\vec{r}$  magnitude and the unit vector are shown in Equations 9-11.

$$\vec{r} = (x - a \cos\theta)\vec{a}_x + (y - a \sin\theta)\vec{a}_y + z\vec{a}_z \quad (8)$$

Vector  $\vec{r}$  has the magnitude of:

$$|\vec{r}| = \sqrt{(x - a \cos\theta)^2 + (y - a \sin\theta)^2 + z^2} \quad (9)$$

Unit vector in the direction of vector  $\vec{r}$  is:

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} \quad (10)$$

$$\hat{r} = \frac{\vec{r}}{\sqrt{(x - a \cos\theta)^2 + (y - a \sin\theta)^2 + z^2}} \quad (11)$$

Cross product between the distance vector  $\vec{r}$  and the vector of the direction of current  $\hat{I}$  is found in Equations 15-16.

$$\vec{dl} = a d\theta \vec{a}_\theta \quad (12)$$

$$a_\theta = -\sin\theta a_x + \cos\theta \vec{a}_y \quad (13)$$

$$\vec{dl} = -a \sin\theta d\theta \vec{a}_x + a \cos\theta d\theta \vec{a}_y \quad (14)$$

$$\vec{dl} \times \vec{r} = \dots$$

$$\left( -a \sin\theta d\theta \vec{a}_x + a \cos\theta d\theta \vec{a}_y \right) \times \left( (x - a \cos\theta) \vec{a}_x + (y - a \sin\theta) \vec{a}_y + z \vec{a}_z \right) \quad (15)$$

$$\vec{dl} \times \vec{r} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ -a \sin\theta d\theta & a \cos\theta d\theta & 0 \\ (x - a \cos\theta) & (y - a \sin\theta) & z \end{vmatrix} \quad (16)$$

$$dl \times \vec{r} = \dots$$

$$(za \cos\theta d\theta) \vec{a}_x + (a z \sin\theta d\theta) \vec{a}_y + (a^2 - a(y \sin\theta + x \cos\theta)) d\theta \vec{a}_z \quad (17)$$

Replacing other variables in the Equations 3-11, we get the Equation 18 for the magnetic field  $\vec{dB}$  at a point P.

Components of the magnetic field are given in Equations 18-20.

$$\vec{dB}_x = \frac{\mu I}{4\pi \sqrt{(x - a \cos\theta)^2 + (y - a \sin\theta)^2 + z^2}^3} (z a \cos\theta d\theta) \vec{a}_x \quad (18)$$

$$\vec{dB}_y = \frac{\mu I}{4\pi \sqrt{(x - a \cos\theta)^2 + (y - a \sin\theta)^2 + z^2}^3} (a z \sin\theta d\theta) \vec{a}_y \quad (19)$$

$$\vec{dB}_z = \frac{\mu I}{4\pi \sqrt{(x - a \cos\theta)^2 + (y - a \sin\theta)^2 + z^2}^3} \dots$$

$$\dots (a^2 - a(y \sin\theta + x \cos\theta)) d\theta \vec{a}_z \quad (20)$$

Each field component can be integrated separately, as shown in Equations 21-23.

$$\vec{B}_x = \int_0^{2\pi} \frac{\mu I z a \cos\theta d\theta}{4\pi \sqrt{(x - a \cos\theta)^2 + (y - a \sin\theta)^2 + z^2}^3} \vec{a}_x \quad (21)$$

$$\vec{B}_y = \int_0^{2\pi} \frac{\mu I a z \sin\theta d\theta}{4\pi \sqrt{(x - a \cos\theta)^2 + (y - a \sin\theta)^2 + z^2}^3} \vec{a}_y \quad (22)$$

$$\vec{B}_z = \int_0^{2\pi} \frac{\mu I (a^2 - a(y \sin\theta + x \cos\theta)) d\theta}{4\pi \sqrt{(x - a \cos\theta)^2 + (y - a \sin\theta)^2 + z^2}^3} \vec{a}_z \quad (23)$$

The magnetic field above is shown in Figure 3

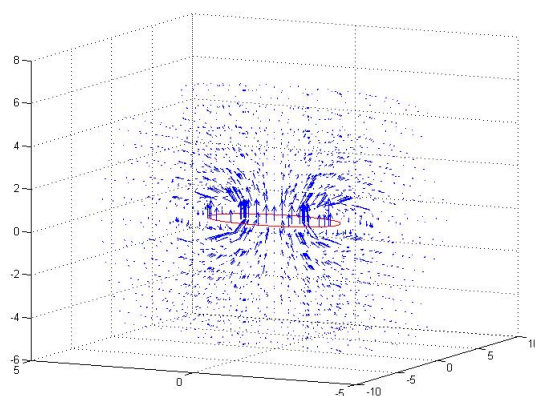


Figure 3: Magnetic Field of a Loop of Current

## Visualizing Scalar Fields in Matlab

To visualize scalar fields in Matlab, we can use the following functions: `slice`, `contourslice`, `patch`, `isonormals`, `camlight`, and `lightning`. Please note that a more detailed explanation about these functions can be found in Matlab help.

### `slice`

`Slice` is a command that shows the magnitude of a scalar field on a plane that slices the volume where the potential field is visualized. The format of this command is as shown below.

```
slice(x,y,z,v,xslice,yslice,zslice)
```

Where `X`, `Y`, and `Z` are coordinates of points where the scalar function is calculated, `V` is `v` the scalar function at those points, and the last three vectors `xslice`, `yslice`, and `zslice` are showing where will the volume will be sliced.

An example of a `slice` command is given below. There is an additional command `colormap` that colors the volume with a specific palette in the example below. To see more about different color maps, see Matlab help. `xslice` has three points at which the `x`-axis will be slice. They are `-1.2`, `.8`, `2`. The volume will be sliced with a plane perpendicular to the `x`-axis, and it crosses the `x`-axis at points `-1.2`, `.8`, and `2`.

```
clc
clear all
[x,y,z] = meshgrid(-2:.2:2,-2:.25:2,-2:.16:2);
v = x.*exp(-x.^2-y.^2-z.^2);
xslice = [-1.2,.8,2]; yslice = 1; zslice = [-2,0];
slice(x,y,z,v,xslice,yslice,zslice)
colormap hsv
```

## **contourslice**

Contourslice command will display equipotential lines on a plane being the volume where the potential field is visualized. An example of contourslice function is shown below.

```
[x,y,z] = meshgrid(-2:.2:2,-2:.25:2,-2:.16:2);
v = x.*exp(-x.^2-y.^2-z.^2); % Create volume data
[xi,yi,zi] = sphere; % Plane to contour
contourslice(x,y,z,v,xi,yi,zi)
view(3)
```

## **patch**

Patch command creates a patch of color.

## **isonormals**

Command isonormals creates equipotential surfaces.

## **camlight**

camlight('headlight') creates a light at the camera position.  
camlight('right') creates a light right and up from camera.

camlight('left') creates a light left and up from camera.  
camlight with no arguments is the same as camlight('right').

camlight(az,el) creates a light at the specified azimuth (az) and elevation (el) with respect to the camera position. The camera target

is the center of rotation  
and az and el are in degrees.

## lighting

lighting flat selects flat lighting.

Lighting gouraud selects gouraud lighting.

Lighting phong selects phong lighting.

Lighting none turns off lighting.