Dig-In:

Vectors

Vectors are lists of numbers that denote direction and magnitude.

The idea of vectors

The most successful textbook that was ever written was *Euclid's Elements* . While you are surely skeptical of this claim, and it is *good* to be skeptical, consider this: *Euclid's Elements* was used (in various editions) as a primary mathematics textbook for nearly 2000 years. There are few textbooks (if any) that can share this claim. However, *Euclid's Elements* does have its shortcomings. Euclid defines a point as "that which has no part." Many people (including this author) find this to be a pretty confusing definition. What does Euclid mean by this statement? However, from our modern viewpoint, a point is an ordered list of numbers, like

$$(1,1) \quad \text{or} \quad (4,2).$$

We have grown to see that a point should be thought of as *location*, and nothing but location. With this definition in mind, it doesn't really make sense to have operations between points like addition or subtraction.

When trying to understand the world around us, we are often concerned with quantities that denote both *direction* and *magnitude*. We can do this by starting with two points

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Learning outcomes: State the definition of a vector. Work with vectors in two or three dimensions. Multiply vectors by scalars. Add and subtract vectors. Calculate the magnitude of a vector. Find unit vectors. Use vectors in applied settings.

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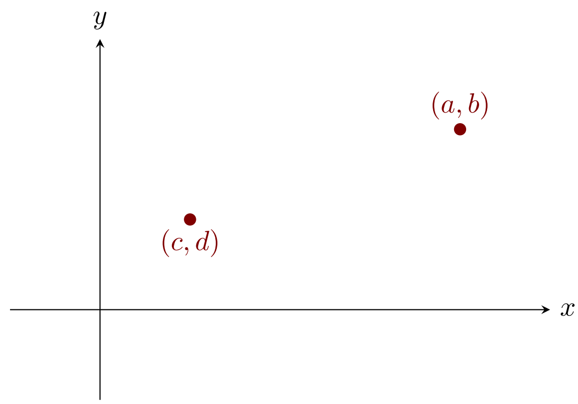
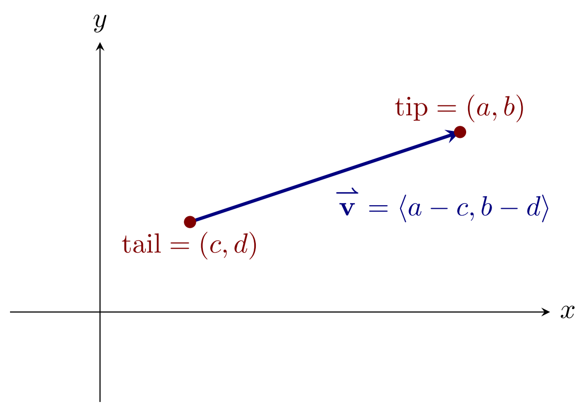
See *Euclid's Elements* at `https://mathcs.clarku.edu/~djoyce/java/elements/elements.html`



and thinking of the differences of their coordinates.



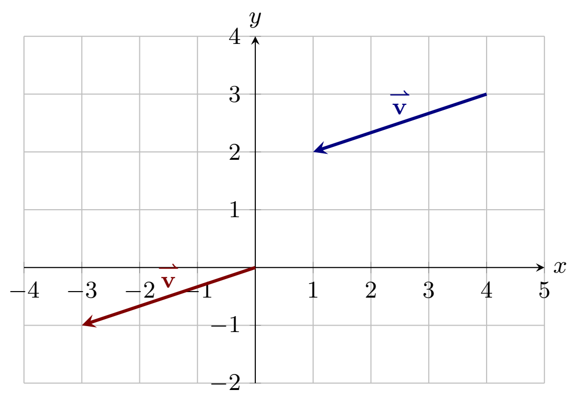
This object formed by the differences in the values of the coordinates of the points is called a *vector*. In the graph above, the vector is $\vec{\mathbf{v}} = \langle a - c, b - d \rangle$. We write vectors typographically in boldface, decorated with a harpoon (like $\vec{\mathbf{v}}$ or $\vec{\mathbf{w}}$). Other authors may simply use a boldface (like $\mathbf{v}$ or $\mathbf{w}$) or just a harpoon (like $\vec{v}$ or $\vec{w}$). We often visualize a vector (at least in two and three dimensions) as an arrow to explicitly show its direction and magnitude. This visualization leads us to our definition of a vector.

Definition 1. *A* ***vector*** *is something that can be ascribed the qualities of direction and magnitude.*

Question 1 *What vector has its tip at* $(1, 2)$ *and its tail at* $(4, 3)$*?*

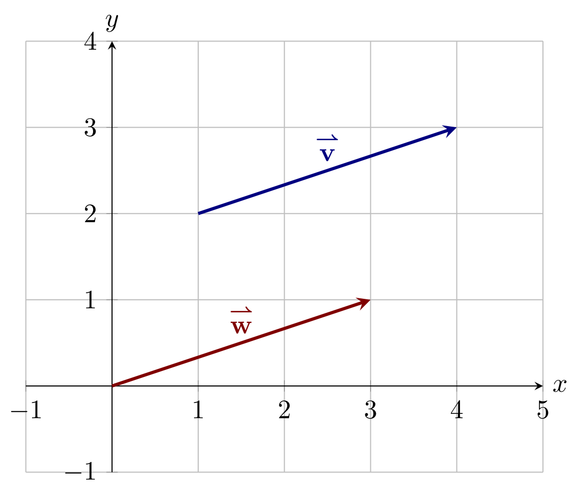
$$\vec{\mathbf{v}} = \left\langle \boxed{-3}, \boxed{-1} \right\rangle$$

Feedback(`correct`): *Note that since we are denoting the vector by a single pair of numbers, this pair of numbers represents the tip of the vector, and we assume that the tail of the vector is at the origin.*



Two vectors are equal when they have the same direction and magnitude.

Question 2 *True or False: Given vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ in the diagram below*



we have that $\vec{\mathbf{v}} = \vec{\mathbf{w}}$.

Multiple Choice:

(a) *true* ✓

(b) *false*

Vectors need not be limited to the (x, y)-plane. They can have any *dimension*.

Definition 2. *The **dimension** of a vector is the number of entries. Each individual entry of a vector is called a **component**.*

In $\mathbb{R}^2$ we usually label the first component the "x-component," and the second component the "y-component." In $\mathbb{R}^3$ we usually label the components "x," "y," and "z."

Question 3 *What is the dimension of the vector*

$$\langle 3, 4, 1, -4 \rangle?$$

$$\textit{Dimension} = \boxed{4}$$

Question 3.1 *What are the components of the vector* $\langle 1, 2, 3 \rangle$*?*

- *The x-component is* $\boxed{1}$.
- *The y-component is* $\boxed{2}$.
- *The z-component is* $\boxed{3}$.

So far, we have mostly studied functions which take single numbers as their inputs and output either individual numbers or ordered pairs (as in the case of parametric functions). Now, we set the stage for the study of functions that accept lists of numbers as inputs and give lists of numbers as outputs. When we want to keep track of more than one number at a time, especially when we have more than one output depending on the same input, we often use a vector.

Computing the direction and magnitude of vectors

Since vectors are determined only by their direction and magnitude, notation such as

$$\langle a, b, c \rangle$$

completely describes a vector, since we assume the tail is at the origin. We should point out that the following are other types of notation for vectors.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \begin{bmatrix} a & b & c \end{bmatrix}, \quad (a, b, c).$$

When dealing with a vector in 1, 2, or 3 dimensions, we can visualize the vector as a directed arrow, where the *magnitude* of the vector is the length of the arrow.

Question 4 *What is the magnitude of the vector $\langle 1, 1 \rangle$?*

$$\text{Magnitude} = \boxed{\sqrt{2}}$$

You were able to find the answer to the question above because you are used to working with 2 dimensional objects. We make the following definition in n dimensions.

Definition 3. *Let $\vec{\mathbf{v}} = \langle v_1, v_2, v_3, \ldots, v_n \rangle$ in $\mathbb{R}^n$ be an n-dimensional vector. Then the **magnitude** of $\vec{\mathbf{v}}$ is denoted by $|\vec{\mathbf{v}}|$ and is defined by:*

$$|\vec{\mathbf{v}}| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2}$$

Notice that the magnitude of the vector is just the distance between the origin and the point determined by the components of our vector!

Question 5 *What is the magnitude of the vector*

$$\vec{\mathbf{v}} = \langle 2, -1, 4, -2 \rangle?$$

$$|\vec{\mathbf{v}}| = \boxed{5}$$

Operations on vectors

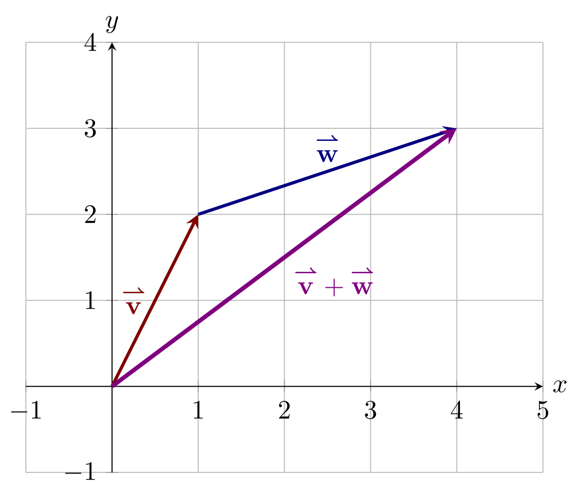
We can add vectors of the same dimension together by component-wise addition. Here, it is useful to write vectors vertically.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix}.$$

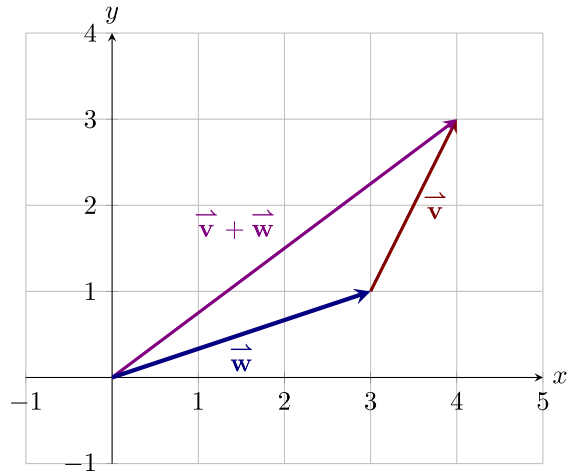
Question 6

$$\langle 1,2,3\rangle + \langle -1,2,2\rangle = \left\langle \boxed{0}, \boxed{4}, \boxed{5} \right\rangle$$

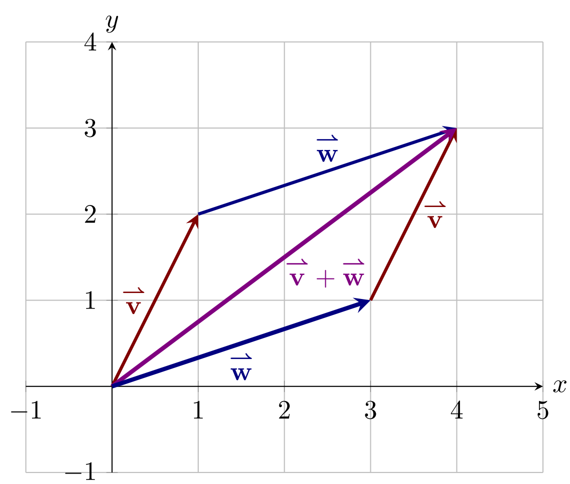
Now, let us investigate the geometry of addition of vectors. Let $\vec{\mathbf{v}} = \langle 1,2\rangle$ and $\vec{\mathbf{w}} = \langle 3,1\rangle$. If we place the tail of the vector $\vec{\mathbf{w}}$ at the tip of the vector $\vec{\mathbf{v}}$, like this:



or like this:



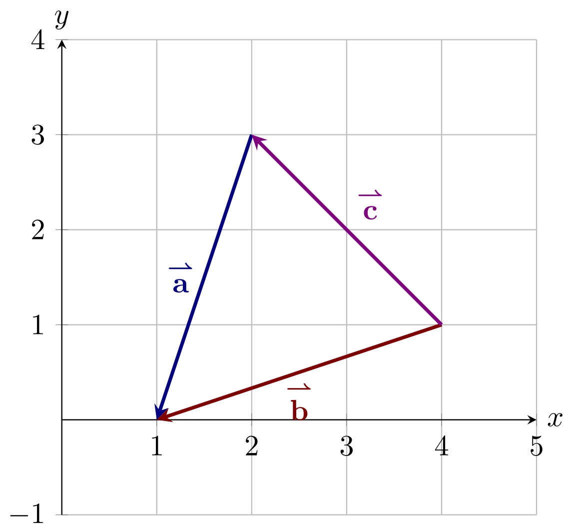
then the sum $\vec{\mathbf{v}}+\vec{\mathbf{w}}$ connects the tail of $\vec{\mathbf{v}}$ to the tip of $\vec{\mathbf{w}}$. In fact, you can think of the sum of two vectors as being the diagonal of the parallelogram formed by the two vectors.



Hence,

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle 4, 3 \rangle .$$

Question 7 *Consider the following diagram.*



Which equation is represented by the diagram above?

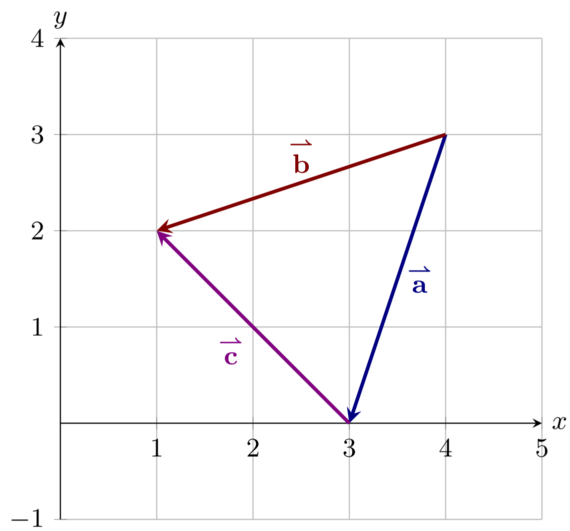
Multiple Choice:

(a) $\overrightarrow{\mathbf{a}} + \overrightarrow{\mathbf{b}} = \overrightarrow{\mathbf{c}}$

(b) $\overrightarrow{\mathbf{a}} + \overrightarrow{\mathbf{c}} = \overrightarrow{\mathbf{b}}$ ✓

(c) $\overrightarrow{\mathbf{b}} + \overrightarrow{\mathbf{c}} = \overrightarrow{\mathbf{a}}$

Feedback(`correct`): *Notice that we also could have drawn the diagram above like this.*



We can also multiply vectors by a **scalar** (a number), by multiplying each component by the scalar:

Question 8

$$4 \cdot \langle 2, 4, 0, 1\rangle = \left\langle \boxed{8}, \boxed{16}, \boxed{0}, \boxed{4} \right\rangle$$

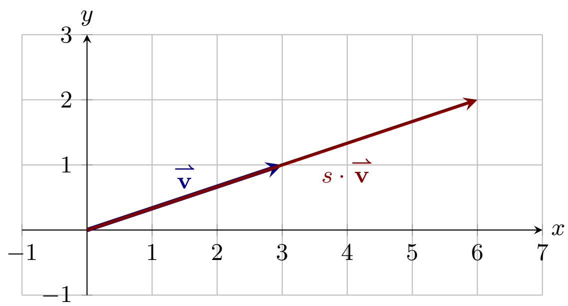
Question 9 *True or False: Multiplying a vector by a nonzero scalar will not change the direction of the vector.*

Multiple Choice:

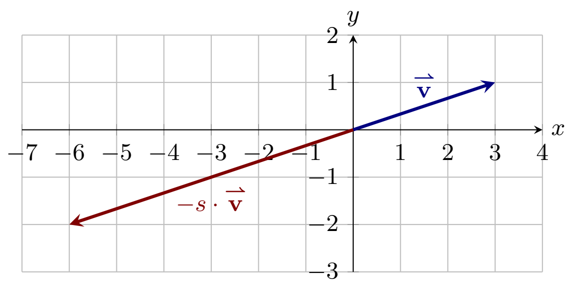
(a) *true*

(b) *false* ✓

Feedback(`attempt`): *Multiplying a vector by a positive scalar s will not change the direction of the vector.*



However, if we multiply a vector by a negative scalar $-s$, then the direction will change.



Thinking about how the magnitude of a vector changes when we multiply by a scalar reveals why scalars are called *scalars*.

You can use this interactive to see how scalars affect vectors.

Geogebra link: `https://tube.geogebra.org/m/WYNdzcGP`

Question 10 *Consider a vector*

$$\vec{\mathbf{v}} = \langle a, b, c \rangle .$$

What is the magnitude of $\vec{\mathbf{v}}$?

$$|\vec{\mathbf{v}}| = \boxed{\sqrt{a^2 + b^2 + c^2}}$$

Question 10.1 *What is the magnitude of $6 \cdot \vec{\mathbf{v}}$?*

$$|6 \cdot \vec{\mathbf{v}}| = \boxed{6} \cdot \sqrt{a^2 + b^2 + c^2}$$

Question 10.1.1 *What is the magnitude of $-6 \cdot \vec{\mathbf{v}}$?*

$$|-6 \cdot \vec{\mathbf{v}}| = \boxed{6} \cdot \sqrt{a^2 + b^2 + c^2}$$

Feedback(`attempt`): *If s is a positive constant, and $\vec{\mathbf{v}}$ is a vector, then vector $s \cdot \vec{\mathbf{v}}$ points in the same direction as $\vec{\mathbf{v}}$, but its length is scaled by a factor of s. If s is negative, then $s \cdot \vec{\mathbf{v}}$ points in the opposite direction of $\vec{\mathbf{v}}$, and its length is scaled by a factor of $|s|$.*

Unit vectors

Vectors with magnitude 1 are particularly important.

Definition 4. *A **unit vector** is a vector of magnitude 1. In this text, unit vectors will wear hats: $|\hat{\mathbf{u}}| = 1$.*

Theorem 1. *If $\vec{\mathbf{v}}$ is a nonzero vector, then the unit vector which points in the same direction as $\vec{\mathbf{v}}$ is $\dfrac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}$.*

Question 11 *Find a unit vector $\hat{\mathbf{u}}$ which points in the same direction as the vector $\vec{\mathbf{v}} = \langle 2, 1, 3, 7, 1 \rangle$.*

$$\hat{\mathbf{u}} = \left\langle \boxed{2/8}, \boxed{1/8}, \boxed{3/8}, \boxed{7/8}, \boxed{1/8} \right\rangle$$

Hint: *Scaling the vector $\vec{\mathbf{v}}$ by the reciprocal of its magnitude should result in a magnitude 1 vector which points in the same direction.*

Hint: $|\vec{\mathbf{v}}| = \sqrt{2^2 + 1^2 + 3^2 + 7^2 + 1^2} = \sqrt{64} = 8$

Now consider any vector $\vec{\mathbf{v}}$. We can extract its direction and magnitude in the following way.

$$\vec{\mathbf{v}} = \underbrace{|\vec{\mathbf{v}}|}_{\text{magnitude}} \cdot \underbrace{\frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|}}_{\text{direction}}$$

This equation illustrates the fact that a vector has both magnitude and direction, where we view a unit vector as supplying *only* direction information. Identifying unit vectors with direction allows us to define *parallel vectors.*

Definition 5. *Unit vectors* $\hat{\mathbf{u}}$ *and* $\hat{\mathbf{v}}$ *are* ***parallel*** *if*

$$\hat{\mathbf{u}} = \pm\hat{\mathbf{v}}.$$

Nonzero vectors $\vec{\mathbf{a}}$ *and* $\vec{\mathbf{b}}$ *are* ***parallel*** *if their respective unit vectors are parallel.*

It is equivalent to say that vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are parallel if there is a scalar $s \neq 0$ such that $\vec{\mathbf{a}} = s \cdot \vec{\mathbf{b}}$.

Question 12 *Let* $\vec{\mathbf{v}} = \langle 1, -4, 2\rangle$. *Find all unit vectors parallel to* $\vec{\mathbf{v}}$. *Write your answers in the order of increasing* x*-coordinates:*

$\hat{\mathbf{u}}_1 = \left\langle \boxed{-1/\sqrt{21}}, \boxed{4/\sqrt{21}}, \boxed{-2/\sqrt{21}} \right\rangle$ $\hat{\mathbf{u}}_2 = \left\langle \boxed{1/\sqrt{21}}, \boxed{-4/\sqrt{21}}, \boxed{2/\sqrt{21}} \right\rangle$

Note that the zero vector $\vec{\mathbf{0}}$ is directionless, because there is no unit vector in the "direction" of $\vec{\mathbf{0}}$. Different authors have different conventions regarding the zero vector. Some even say the zero vector is "parallel to every vector." We prefer to simply say that the zero vector has no direction, as this statement is grounded in the fact that **unit vectors provide direction information**. So, in our case, the zero vector is not parallel to any vector. Check for yourself using our definition of parallel vectors!

Question 13 *True or False: If two vectors are parallel, then they point in the same direction.*

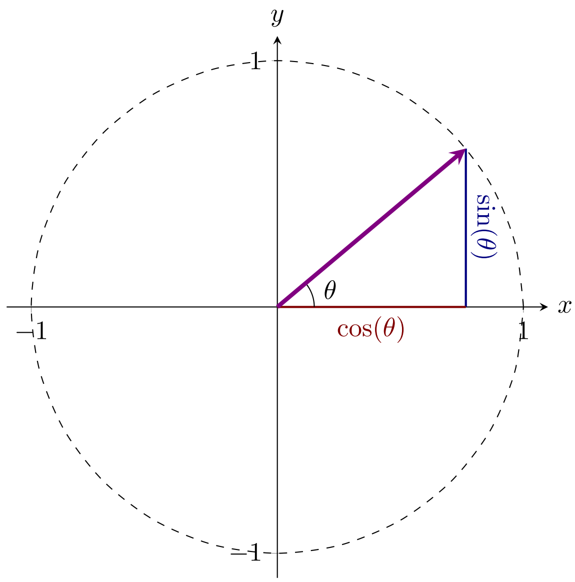
Multiple Choice:

(a) *true*

(b) *false* ✓

Angles and vectors

Sometimes you want to specify a vector with an angle relative to a given line. If we graphed all of the unit vectors in $\mathbb{R}^2$ with their tails at the origin, then the tips would all lie on the unit circle.



Based on what we know from trigonometry, we can then say that the components of any unit vector in $\mathbb{R}^2$ can be expressed as $\langle\cos(\theta), \sin(\theta)\rangle$ for some angle θ.

Theorem 2. *Unit vectors that make an angle of θ radians with the x-axis are given by*

$$\hat{\mathbf{u}} = \langle\cos(\theta), \sin(\theta)\rangle .$$

Question 14 *What vector has magnitude 6 and makes an angle of $\pi/6$ radians with the x-axis?*

$$\vec{\mathbf{a}} = \left\langle \boxed{6 \cdot \sqrt{3}/2}, \boxed{6/2} \right\rangle .$$

Question 14.1 *What vector has magnitude 5 and makes an angle of $2\pi/3$ radians with the x-axis?*

$$\vec{\mathbf{b}} = \left\langle \boxed{-5/2}, \boxed{5 \cdot \sqrt{3}/2} \right\rangle .$$

Question 14.1.1 *What vector has magnitude 4 and makes an angle of $\pi/3$ radians with the y-axis?*

$$\vec{\mathbf{c}} = \left\langle \boxed{-2 \cdot \sqrt{3}}, \boxed{2} \right\rangle .$$

Famous unit vectors

There are three famous unit vectors: $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{k}}$. Typically when working in two dimensions

$$\hat{\boldsymbol{\imath}} = \langle 1, 0 \rangle \qquad \text{and} \qquad \hat{\boldsymbol{\jmath}} = \langle 0, 1 \rangle$$

and in three dimensions

$$\hat{\boldsymbol{\imath}} = \langle 1, 0, 0 \rangle , \qquad \hat{\boldsymbol{\jmath}} = \langle 0, 1, 0 \rangle , \qquad \hat{\boldsymbol{k}} = \langle 0, 0, 1 \rangle .$$

Any two- or three-dimensional vector can be expressed in terms of these vectors.

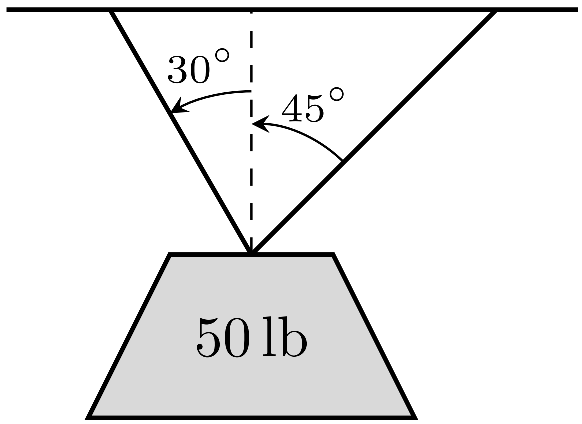
Question 15 *Express $\langle 1, 2, 3 \rangle$ in terms of $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$.*

$$\langle 1, 2, 3 \rangle = \boxed{1} \cdot \hat{\boldsymbol{\imath}} + \boxed{2} \cdot \hat{\boldsymbol{\jmath}} + \boxed{3} \cdot \hat{\boldsymbol{k}}$$

An applied problem

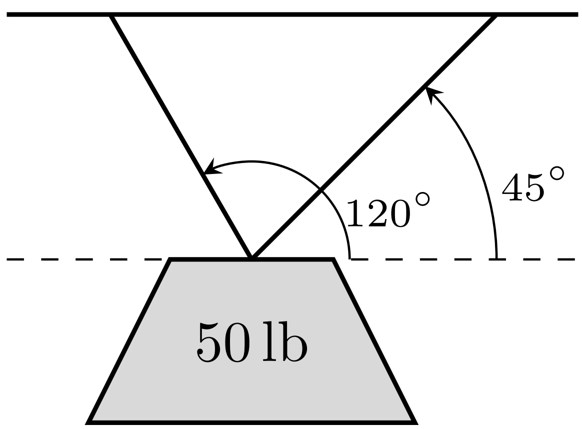
Vectors are a great tool for applied mathematics.

Example 1. *Consider a weight of* 50 lb *hanging from two chains.*

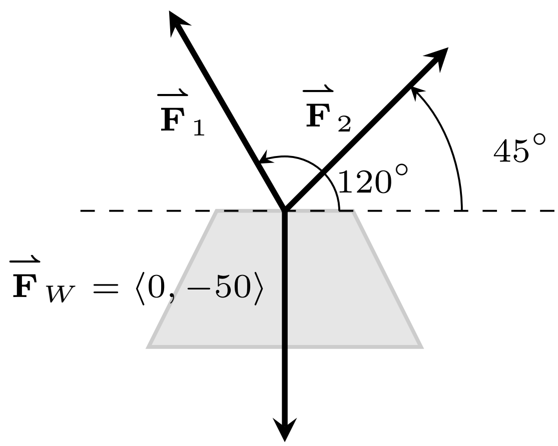


One chain makes an angle of 30° *with the vertical, and the other an angle of* 45°*. Find the magnitude of the force applied to each chain.*

Explanation. *Start by converting all angles to be measured from the horizontal.*



We can view each chain as "pulling" the weight up, preventing it from falling. We can represent the force from each chain with a vector. First, we'll let $\overrightarrow{\mathbf{F}}_W$ *be the force of the weight due to gravity. Since pounds are a unit of force, this is just* $\left\langle \underbrace{\boxed{0}}_{\text{given}}, \underbrace{\boxed{-50}}_{\text{given}} \right\rangle$*. Let* $\overrightarrow{\mathbf{F}}_1$ *represent the force from the chain making an angle of* 120° *with the horizontal axis, and let* $\overrightarrow{\mathbf{F}}_2$ *represent the force from the other chain.*



Thus,

$$\vec{\mathbf{F}}_1 = m_1 \langle \cos(120^\circ), \sin(120^\circ) \rangle$$

and

$$\vec{\mathbf{F}}_2 = m_2 \langle \cos(45^\circ), \sin(45^\circ) \rangle .$$

As the weight is not moving, we know the sum of the forces is $\vec{\mathbf{0}}$*:*

$$\vec{\mathbf{F}}_W + \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 = \vec{\mathbf{0}}.$$

Writing our vectors vertically, we have the following.

$$\begin{bmatrix} 0 \\ \underset{\text{given}}{\boxed{-50}} \end{bmatrix} + m_1 \begin{bmatrix} \cos\left(\underset{\text{given}}{\boxed{120}}^\circ\right) \\ \sin\left(\underset{\text{given}}{\boxed{120}}^\circ\right) \end{bmatrix} + m_2 \begin{bmatrix} \cos\left(\underset{\text{given}}{\boxed{45}}^\circ\right) \\ \sin\left(\underset{\text{given}}{\boxed{45}}^\circ\right) \end{bmatrix}$$
$$= \begin{bmatrix} \underset{\text{given}}{\boxed{0}} \\ \underset{\text{given}}{\boxed{0}} \end{bmatrix}$$

The sum of the entries in the first component is 0, and the sum of the entries in the second component is also 0. This leads us to the following system of equations.

$$m_1 \cos(120^\circ) + m_2 \cos(45^\circ) = 0$$
$$m_1 \sin(120^\circ) + m_2 \sin(45^\circ) = 50$$

We leave it to the reader to verify that the solution is

$$m_1 = \underset{\text{given}}{\boxed{50(\sqrt{3}-1)}}\,\text{lb} \qquad m_2 = \underset{\text{given}}{\boxed{\frac{50\sqrt{2}}{1+\sqrt{3}}}}\,\text{lb}.$$

Look again at the big picture for our example with the weight. We knew from the physical situation that we had three vectors which should together sum to the zero vector. Instead of having to somehow describe this situation with a single equation, we used the components of the vectors to form a system of equations, which was much easier to solve! Modeling the problem with vectors helped us to apply our mathematical tools smartly.

The difference between a point and a vector

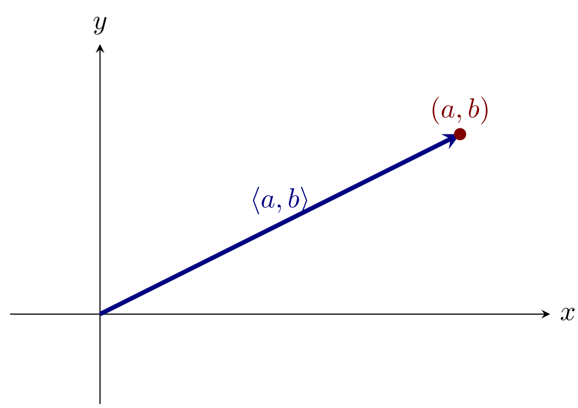
You may be wondering, "What's the difference between a point and a vector?" Here's the deal: A point P specifies **location** alone. This location is denoted by an n-tuple

$$P = (p_1, p_2, \ldots, p_n).$$

A vector $\overrightarrow{\mathbf{v}}$ is also represented by an n-tuple

$$\overrightarrow{\mathbf{v}} = \langle v_1, v_2, \ldots, v_n \rangle \,,$$

but the **interpretation** of this n-tuple is quite different than that of a point. With a vector, the tuple represents the location of the "tip" of the vector when the "tail" of the vector is at the origin. By thinking of this tuple, $\overrightarrow{\mathbf{v}}$, as a vector, we can perform many arithmetic and algebraic calculations as we discussed above. Remember that being able to do these operations helped us to distinguish vectors from points! However, as long as it makes sense to do so, we can also denote points with vectors.



Since a vector can be represented by an n-tuple, we denote a point with a vector by imagining the tail of the vector as being at the origin. Placing a vector with its tail at the origin is sometimes called *standard position.*

We summarize the arithmetic and algebraic properties of vectors below.

Theorem 3 (Properties of Vectors)**.** *The following are true for all scalars s and t, and for all vectors $\overrightarrow{\mathbf{u}}$, $\overrightarrow{\mathbf{v}}$ and $\overrightarrow{\mathbf{w}}$ in $\mathbb{R}^n$.*

Commutativity: $\overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{v}} = \overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{u}}$.

Associativity: $(\overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{v}}) + \overrightarrow{\mathbf{w}} = \overrightarrow{\mathbf{u}} + (\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{w}})$.

Additive identity: $\overrightarrow{\mathbf{v}} + \overrightarrow{\mathbf{0}} = \overrightarrow{\mathbf{v}}$.

Compatibility with scalars: $(st)\overrightarrow{\mathbf{v}} = s(t\overrightarrow{\mathbf{v}}) = (ts)\overrightarrow{\mathbf{v}}$.

Distributivity of scalars over vectors: $s(\overrightarrow{\mathbf{u}} + \overrightarrow{\mathbf{v}}) = s\overrightarrow{\mathbf{u}} + s\overrightarrow{\mathbf{v}}$.

Distributivity of vectors over scalars: $(s+t)\overrightarrow{\mathbf{v}} = s\overrightarrow{\mathbf{v}} + t\overrightarrow{\mathbf{v}}$.

Scaling of magnitude: $|s\overrightarrow{\mathbf{v}}| = |s| \cdot |\overrightarrow{\mathbf{v}}|$.

Definiteness: $|\overrightarrow{\mathbf{u}}| = 0$ *if and only if* $\overrightarrow{\mathbf{u}} = \overrightarrow{\mathbf{0}}$.

Note, you should not memorize these properties (yet!). Rather, you should be able to work with vectors, and use these properties when appropriate.