Transmission line impedance and reflection coefficient

Relating forward and backward current and voltage waves on the transmission line

In equations 1-2 $\tilde{V}_0^+$ and $\tilde{V}_0^-$ are the phasors of forward and reflected going voltage waves, and $\tilde{I}_0^+$ and $\tilde{I}_0^-$ are the phasors of forward and reflected current waves. This section will relate the phasors of voltage and current waves with transmission-line impedance.

$$\tilde{V}(z) = \tilde{V}_0^+ e^{-\gamma z} + \tilde{V}_0^- e^{\gamma z} \tag{1}$$
$$I(z) = \tilde{I}_0^+ e^{-\gamma z} + \tilde{I}_0^- e^{\gamma z} \tag{2}$$

When substitute the voltage wave equation into Telegrapher's Equations **??**. The equation is repeated here Eq.3-4.

$$-\frac{\partial \tilde{V}(z)}{\partial z} = (R + j\omega L)I(z) \tag{3}$$
$$\gamma \tilde{V}_0^+ e^{-\gamma z} - \gamma \tilde{V}_0^- e^{\gamma z} = (R + j\omega L)I(z) \tag{4}$$

We now rearrange Eq.4

$$I(z) = \frac{\gamma}{R + j\omega L}(\tilde{V}_0^+ e^{-\gamma z} + \tilde{V}_0^- e^{\gamma z})$$
$$I(z) = \frac{\gamma \tilde{V}_0^+}{R + j\omega L} e^{-\gamma z} - \frac{\gamma \tilde{V}_0^-}{R + j\omega L} e^{\gamma z} \tag{5}$$

Now we compare Eq.5 with the Eq.2. For two transcendental equations to be equal, the coefficients next to exponential terms have to be the same.

Learning outcomes: Derive and calculate the transmission line impedance and reflection coefficient. Relate reflection coefficient to impedance.

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$$\tilde{I}_0^+ = \frac{\gamma \tilde{V}_0^+}{R + j\omega L}$$
$$\tilde{I}_0^- = -\frac{\gamma \tilde{V}_0^-}{R + j\omega L}$$

We can define the characteristic impedance of a transmission line as the ratio of the voltage to the current amplitude of the forward-going wave.

$$Z_0 = \frac{\tilde{V}_0^+}{\tilde{I}_0^+}$$
$$Z_0 = \frac{R + j\omega L}{\gamma}$$
$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Voltage Reflection Coefficient, Lossless Case

The equations for the voltage and current on the transmission line we derived so far are

$$\tilde{V}(z) = \tilde{V}_0^+ e^{-j\beta z} + \tilde{V}_0^- e^{j\beta z} \tag{6}$$
$$I(z) = \frac{\tilde{V}_0^+}{Z_0} e^{-j\beta z} - \frac{\tilde{V}_0^-}{Z_0} e^{j\beta z} \tag{7}$$

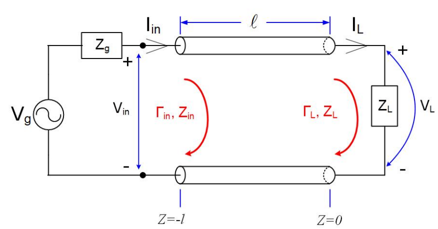

Figure 1: Transmission Line connects generator and the load.

At $z = 0$ the impedance of the load has to be

$$Z_L = \frac{V(0)}{I(0)}$$

Substitute the boundary condition in Eq.6

$$Z_L = Z_0 \frac{\tilde{V}_0^+ + \tilde{V}_0^-}{\tilde{V}_0^+ - \tilde{V}_0^-} \tag{8}$$

We can now solve the above equation for $\tilde{V}_0^-$

$$\begin{aligned}
\frac{Z_L}{Z_0}(\tilde{V}_0^+ - \tilde{V}_0^-) &= \tilde{V}_0^+ + \tilde{V}_0^- \\
(\frac{Z_L}{Z_0} - 1)\tilde{V}_0^+ &= (\frac{Z_L}{Z_0} + 1)\tilde{V}_0^- \\
\frac{\tilde{V}_0^-}{\tilde{V}_0^+} &= \frac{\frac{Z_L}{Z_0} - 1}{\frac{Z_L}{Z_0} + 1} = \frac{z_L - 1}{z_L + 1} \\
\Gamma = \frac{\tilde{V}_0^-}{\tilde{V}_0^+} &= \frac{Z_L - Z_0}{Z_L + Z_0} \qquad (9)
\end{aligned}$$

The quantity $\frac{\tilde{V}_0^-}{\tilde{V}_0^+}$ is called voltage reflection coefficient Γ. Γ relates the reflected and incident voltage phasor. The voltage reflection coefficient is, in general, a complex number, it has a magnitude and a phase.

Lowercase z_L is called the "normalized load impedance". It is the actual impedance divided by the transmission line impedance $z_L = \frac{Z_L}{Z_0}$. For example, if the load impedance is $Z_L = 100\Omega$, and the transmission-line impedance is $Z) = 50\Omega$, then the normalized impedance is $z_L = \frac{100\Omega}{50\Omega} = 2$. Normalized impedance is a unitless quantity.

Example

(a) $100\,\Omega$ transmission line is terminated in a series connection of a $50\,\Omega$ resistor and $10\,\text{pF}$ capacitor. The frequency of operation is $100\,\text{MHz}$. Find the voltage reflection coefficient.

(b) For purely reactive load $Z_L = jX_L$ find the reflection coefficient.