Waves on Transmission Lines

Any wire, cable or line that guides energy from one point to another is a transmission line. Whenever you make a circuit on a breadboard, every wire you attach makes a transmission line with the ground wire. Whether we see the propagation (transmission line) effects on the line depends on the line length. At lower frequencies or very short line lengths we do not see any difference between the signal's phase at the generator and at the load, whereas at higher frequencies we do.

Types of transmission lines

(a) Coaxial Cable, Figure 1

(b) Microstrip, Figure 2

(c) Stripline, Figure 3

(d) Coplanar Waveguide, Figure 4

(e) Two-wire line, Figure 5

(f) Parallel Plate Waveguide, Figure 6

(g) Rectangular Waveguide, Figure 7

(h) Optical fiber, Figure 8

What are Transmission Line Effects?

Figure 9 shows a step voltage at the generator and the load of a circuit in Figure 11. The voltage needs T sec to appear at the load, once the switch closes. Figure 10 shows the step signal as it traveles on the transmission line at different time steps t=0, t=T/4, t=T/2 and t=T. How much time it takes for this signal to go from AA' end to BB' end? Since electromagnetic waves propagate with the constant speed, the speed of light, time that the signal needs to go from the generator to the load depends on the length of the transmission line l. $T = \frac{l}{c}$, where $c = 3\,10^8$. If the signal at the generator AA' is

Learning outcomes: Apply phasor transformation to a time-domain equation to obtain frequency-domain equation.

Author(s): Milica Markovic

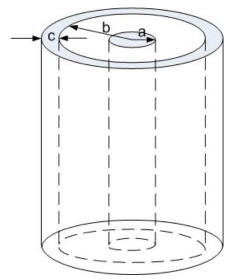

Figure 1: Coaxial Cable

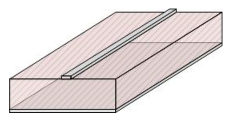

Figure 2: Microstrip

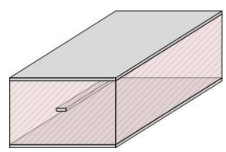

Figure 3: Stripline.

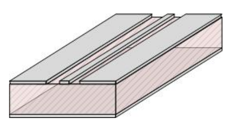

Figure 4: Coplanar Waveguide.

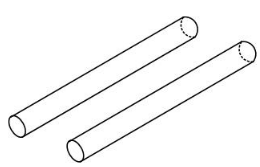

Figure 5: Two-wire line.

$$v_{AA'}(t) = Acos(\omega t) \tag{1}$$

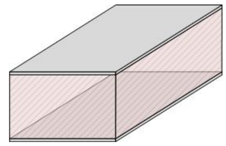

Figure 6: Parallel-Plate Waveguide.

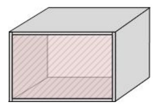

Figure 7: Rectangular Waveguide.

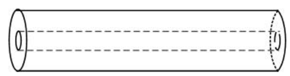

Figure 8: Optical Fiber.

Then at the other end the transmission line the signal is

$$v_{BB`}(t) = v_{AA`}(t - T) \tag{2}$$
$$v_{BB`}(t) = v_{AA`}(t - \frac{l}{c}) \tag{3}$$
$$v_{BB`}(t) = Acos(\omega(t - \frac{l}{c})) \tag{4}$$
$$v_{BB`}(t) = Acos(\omega t - \omega\frac{l}{c}) \tag{5}$$
$$v_{BB`}(t) = Acos(\omega t - \frac{\omega}{c}l) \tag{6}$$

Since we know that angular frequency is $\omega = 2\pi f$

$$v_{BB`}(t) = Acos(\omega t - \frac{2\pi f}{c}l) \tag{7}$$

The quantity $\frac{c}{f}$ is called the wavelength λ.

$$v_{BB`}(t) = A\,cos(\omega t - \frac{2\pi}{\lambda}l) \tag{8}$$

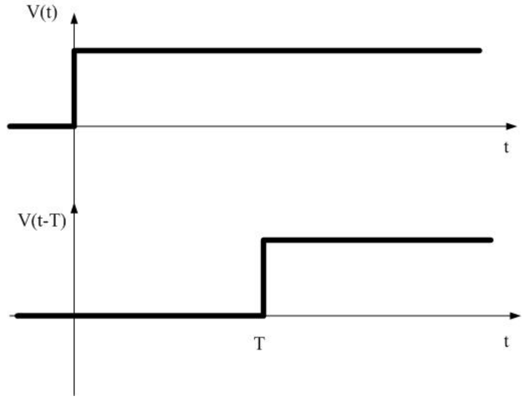

Figure 9: Voltage as a function of time at the generator side z=0 (top) and the load side z=l (bottom) of the transmission line in Figure 11, if the switch closes at t=0 the voltage arrives at t=l/c=T at the load. These graphs can be obtained by observing the voltage on an oscilloscope at the load and at the generator side.

The quantity $\frac{2\pi}{\lambda}$ is the propagation constant β

Finally the expression for the voltage at BB end is

$$v_{BB^{`}}(t) = A\,cos(\omega t - \beta l) \tag{9}$$
$$v_{BB^{`}}(t) = A\,cos(\omega t - \Psi) \tag{10}$$

We see that at BB' the signal will experience a phase shift. We will derive this equation again later from the Telegrapher's equations. Now let's see how the length of the line l affects the voltage at the end BB'. Look at Equation 8. The signal will experience a phase shift of $2\pi\frac{l}{\lambda}$. If this phase shift is small, there will not be much difference between the phase of the signal at the generator and at the load. This means that we don't have to use transmission line theory to account for the effects of the line. If the phase shift is significant, then we do have to use the transmission line theory. Let's look at some numbers in the following example.

(a) If $\frac{l}{\lambda} < 0.01$ then the angle $2\pi\frac{l}{\lambda}$ is of the order of 0.0314 rad or about 2^0. In this case, the phase is obviosly something that we don't have to worry about. When the length of the transmission line is much smaller than λ, $l << \frac{\lambda}{100}$ the wave propagation on the line can be ingnored.

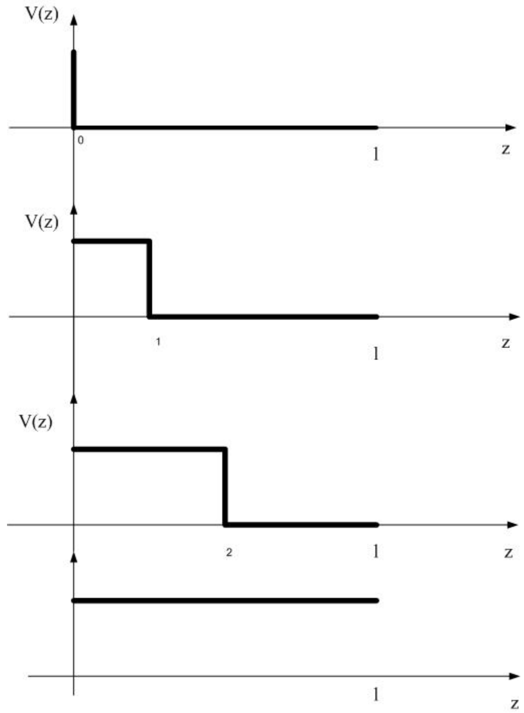

Figure 10: Voltage along the transmission line in Figure 11, for four different time intervals t=0, switch closes, t=T/4, t=T/2 and t=T. It is assumed that the length of the transmission line is equal to l=T/c.

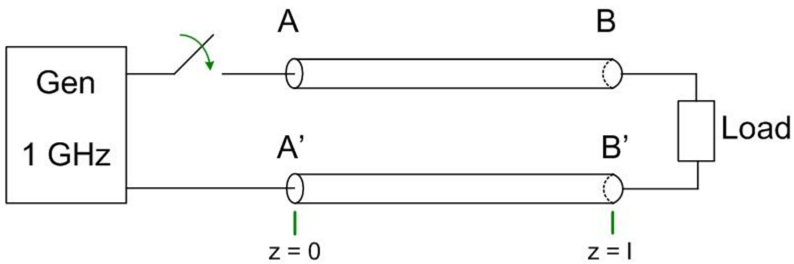

Figure 11: Electronic Circuit with an emphasis on cables that connect the generator and the load.

(b) If $\dfrac{l}{\lambda} > 0.01$, say $\dfrac{l}{\lambda} = 0.1$, then the phase is 20^0, which is a significant phase shift. In this case it may be necessary to account fro transmission line effects.

Propagation modes on a transmission line

Coax, two wire line, microstrip etc can be approximated as TEM up to the 30-40 GHz (unshielded), up to 140 GHz shielded.

(a) Transversal Electro-Magnetic Field (TEM). Electric (E), and Magnetic (M) fields are entirely transversal to the direction of propagation

(b) Transversal Electric field (TE), Transversal Magnetic Field (TM), M or E field is in the direction of propagation

Transmission lines that we are discussing here cary TEM fields.

Wave equation on a transmission line

In this section we will derive the expression for voltage and current along a transmission line. This expression will have two variables, time t and space z. So far we have only seen voltages and currents as a function of time. This is because all circuit elements seen so far were lumped elements. In distributed systems we want to derive the equations for voltage and current for the case when the transmission line is longer then the fraction of a wavelength. To make sure that we don't encounter any transmission line effects to start with, we can look at the piece of a transmission line that is much smaller then the fraction of a wavelength. In other words we cut the transmission line into small pieces to make sure there are no transmission line effects, as the pieces are shorter then the fraction of a wavelength. We then represent each piece with an equivalent circuit as shown in Figure 14 (a). To derive expressions for current and voltage on transmission line we will use the following five-step plan

(a) Look at an infinitesimal length of a transmission line Δz.

(b) Represent that piece with an equivalent circuit.

(c) Write KCL, KVL for the piece in the time domain (we get differential equations)

(d) Apply phasors (equations become linear)

(e) Solve the linear system of equations to get the expression for the voltage and current on the transmission line as a function of z.

Let's follow the plan now. Look at a small piece of a transmission line and represented it with an equivalent circuit. What is modeled by the circuit elements?

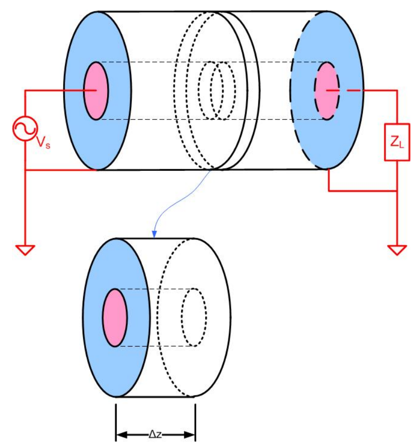

Figure 12: Coaxial cable is cut in short pieces.

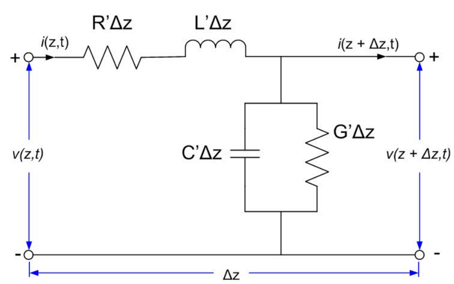

Figure 13: Equivalent circuit of a section of transmission line.

Write KVL and KCL equations for the circuit above.

KVL

$$-v(z,t) + R\Delta z i(z,t) + L\Delta z \frac{\partial i(z,t)}{\partial t} + v(z+\Delta z, t) = 0$$

KCL

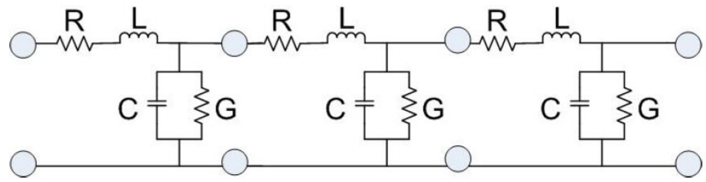

Figure 14: Equivalent circuit of transmission line.

$$i(z,t) = i(z+\Delta z) + i_{CG}(z+\Delta z,t)$$
$$i(z,t) = i(z+\Delta z) + G\Delta z v(z+\Delta z,t) + C\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t}$$

Rearrange the KCL and KVL Equations 11, 15 and divide them with Δz. Equations 12, 16. let $\Delta z \to 0$ and recognize the definition of the derivative Equations, 13, 17.

KVL

$$-(v(z+\Delta z,t) - v(z,t)) = R\Delta z i(z,t) + L\Delta z \frac{\partial i(z,t)}{\partial t} \tag{11}$$
$$-\frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = Ri(z,t) + L\frac{\partial i(z,t)}{\partial t} \tag{12}$$
$$\lim_{\Delta z \to 0}\{-\frac{v(z+\Delta z,t) - v(z,t)}{\Delta z}\} = \lim_{\Delta z \to 0}\{Ri(z,t) + L\frac{\partial i(z,t)}{\partial t}\} \tag{13}$$
$$-\frac{v(z,t)}{\partial z} = Ri(z,t) + L\frac{\partial i(z,t)}{\partial t} \tag{14}$$

KCL

$$-(i(z+\Delta z,t) - i(z,t)) = G\Delta z v(z+\Delta z,t) + C\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} \tag{15}$$
$$-\frac{i(z+\Delta z,t) - i(z,t)}{\Delta z} = Gv(z+\Delta z,t) + C\frac{\partial v(z+\Delta z,t)}{\partial t} \tag{16}$$
$$\lim_{\Delta z \to 0}\{-\frac{i(z+\Delta z,t) - i(z,t)}{\Delta z}\} = \lim_{\Delta z \to 0}\{Gv(z+\Delta z,t) + C\frac{\partial v(z+\Delta z,t)}{\partial t}\} \tag{17}$$
$$-\frac{i(z,t)}{\partial z} = Gv(z+\Delta z,t) + C\frac{\partial v(z+\Delta z,t)}{\partial t} \tag{18}$$

We just derived Telegrapher's equations in time-domain:

$$-\frac{v(z,t)}{\partial z} = Ri(z,t) + L\frac{\partial i(z,t)}{\partial t}$$
$$-\frac{i(z,t)}{\partial z} = Gv(z+\Delta z,t) + C\frac{\partial v(z+\Delta z,t)}{\partial t}$$

Telegrapher's equations are two differential equations with two unknowns, $i(z,t)$, $v(z,t)$. It is not impossible to solve them, however we would prefer to have linear equations. We then express time-domain variables as phasors.

$$v(z,t) = Re\{\tilde{V}(z)e^{j\omega t}\}$$
$$i(z,t) = Re\{\tilde{I}(z)e^{j\omega t}\}$$

Where $\tilde{V}(z), \tilde{I}(z)$ are the voltage and current anywhere on the line, and depend on the position on the line z. And we get the Telegrapher's equations in phasor form

$$-\frac{\partial\tilde{V}(z)}{\partial z} = (R+j\omega L)\tilde{I}(z) \tag{19}$$
$$-\frac{\partial\tilde{I}(z)}{\partial z} = (G+j\omega C)\tilde{V}(z) \tag{20}$$

Two equations, two unknowns. To solve these equations, we first take a derivative of both equations with respect to z.

$$-\frac{\partial^2\tilde{V}(z)}{\partial z^2} = (R+j\omega L)\frac{\partial\tilde{I}(z)}{\partial z} \tag{21}$$
$$-\frac{\partial^2\tilde{I}(z)}{\partial z^2} = (G+j\omega C)\frac{\partial\tilde{V}(z)}{\partial z} \tag{22}$$

Rearange the previous equations:

$$-\frac{1}{(R+j\omega L)}\frac{\partial\tilde{I}(z)}{\partial z} = \frac{\partial^2\tilde{V}(z)}{\partial z^2} \tag{23}$$
$$-\frac{1}{(G+j\omega C)}\frac{\partial\tilde{V}(z)}{\partial z} = \frac{\partial^2\tilde{I}(z)}{\partial z^2} \tag{24}$$

Substitute Eq.23 into Eq.20 and Eq.24 into Eq.19 and we get

$$-\frac{\partial^2 \tilde{V}(z)}{\partial z^2} = (G + j\omega C)(R + j\omega L)\tilde{V}(z) \tag{25}$$

$$-\frac{\partial^2 \tilde{I}(z)}{\partial z^2} = (G + j\omega C)(R + j\omega L)\tilde{I}(z) \tag{26}$$

Or if we rearrange

$$\frac{\partial^2 \tilde{V}(z)}{\partial z^2} - (G + j\omega C)(R + j\omega L)\tilde{V}(z) = 0 \tag{27}$$

$$\frac{\partial^2 I(z)}{\partial z^2} - (G + j\omega C)(R + j\omega L)I(z) = 0 \tag{28}$$

The above Eq.27 and Eq.28 are called the wave equation, and they represent current and voltage wave on a transmission line. $\gamma = (G + j\omega C)(R + j\omega L)$ is the complex propagation constant. This constant has a real and an imaginary part.

$$\gamma = \alpha + j\beta$$

where α is the attenuation constant and β is the phase constant.

$$\alpha = Re\{\sqrt{(G + j\omega C)(R + j\omega L)}\}$$
$$\beta = Im\{\sqrt{(G + j\omega C)(R + j\omega L)}\}$$

The general solution of the second order differential equations with constant coefficients Equations 27 - 28 is:

$$\tilde{V}(z) = \tilde{V_0}^+ e^{-\gamma z} + \tilde{V_0}^- e^{\gamma z}$$
$$I(\tilde{z}) = \tilde{I_0}^+ e^{-\gamma z} + \tilde{I_0}^- e^{\gamma z}$$

In this equation $\tilde{V_0}^+$ and $\tilde{V_0}^-$ are the **phasors** of forward and reflected voltage waves at the load (where z=0), and $\tilde{I_0}^+$ and $\tilde{I_0}^-$ are the phasors of forward and reflected current wave at the load (where z=0).These voltages and currents are also phasors and have a constant magnitude and phase in a specific circuit, for example $\tilde{V_0}^+ = |\tilde{V_0}^+|e^{\Phi} = 4e^{25^0}$, and $\tilde{I_0}^+ = |\tilde{I_0}^+|e^{\Phi} = 5e^{-40^0}$. The time domain expression for the current and voltage on the transmission line we can get by multiplying the phasor of the voltage and current with $e^{j\omega t}$ and taking the real part of it.

$$v(t) = Re\{(\tilde{V}_0^+ e^{(-\alpha - j\beta)z} + \tilde{V}_0^- e^{(\alpha + j\beta)z}) e^{j\omega t}\}$$
$$v(t) = |\tilde{V}_0^+| e^{-\alpha z} \cos(\omega t - \beta z + \angle\tilde{V}_0^+) + |\tilde{V}_0^-| e^{\alpha z} \cos(\omega t + \beta z + \angle\tilde{V}_0^-) \quad (29)$$

We'll look at the Matlab program the next class to see that if the signs of the ωt and βz are the same the wave moves in the forward $+z$ direction. If the signs of ωt and βz are opposite, the wave moves in the $-z$ direction.

Visualization of Lossless Forward and Reflected Voltage Waves

We will show next that if the signs of the ωt and βz are the same, as in Equation 31, the wave moves in the forward $+z$ direction. If the signs of ωt and βz are opposite, as in Equation 30, the wave moves in the $-z$ direction. In order to see this, we will visualize Equations 30 and 31 using Matlab code below.

$$v_f(t) = |\tilde{V}_0^+| e^{-\alpha z} \cos(\omega t - \beta z + \angle\tilde{V}_0^+) \quad (30)$$
$$v_r(t) = |\tilde{V}_0^-| e^{\alpha z} \cos(\omega t + \beta z + \angle\tilde{V}_0^-) \quad (31)$$

Figure 15 shows forward and reflected waves on a transmission line. On x-axis is the spatial coordinate z from the generator to the load, where the transmission line is connected, and on y-axis is the magnitude of the voltage on the line. The red line on both graphs is the voltage signal at a time .1 ns. We would obtain Figure 15 if we had a camera that can take picture of voltages, and we took the first picture at t_1 = .1 ns on the entire transmission line. The blue dotted line on both graphs is the same signal .1 ns later, at time t_2 =.2 ns. We see that the signal has moved to the right in the time of 1 ns, or from the generator to the load. On the bottom graph we see that at a time .1 ns, the red line represents the reflected signal. Dashed blue line shows the signal at a time .2 ns. We see that the signal has moved to the left, or from the load to the generator.

```
clear all
clc
f = 10^9;
w = 2*pi*f
c=3*10^8;
beta=2*pi*f/c;
lambda=c/f;
t1=0.1*10^(-9)
t2=0.2*10^(-9)
x=0:lambda/20:2*lambda;
```

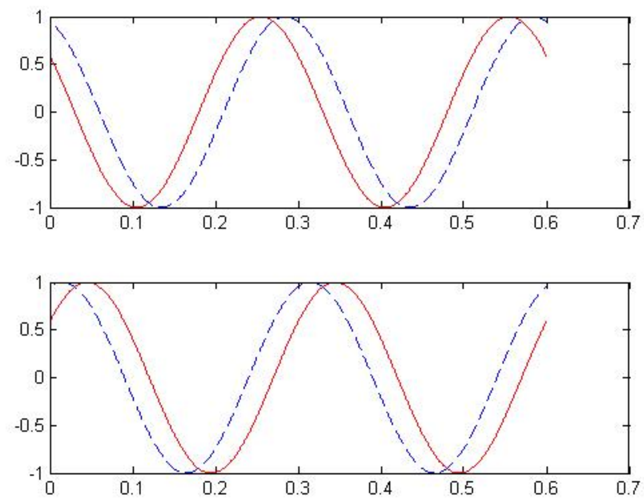

Figure 15: Forward (top) and reflected (bottom) waves on a transmission line.

```
y1=sin(w*t1 - beta.*x);
y2=sin(w*t2 - beta.*x);
y3=sin(w*t1 + beta.*x);
y4=sin(w*t2 + beta.*x);

subplot (2,1,1),

    plot(x,y1,'r'),...
            hold on
    plot(x,y2,'--b'),...
    hold off
subplot (2,1,2),

    plot(x,y3,'r')
        hold on
    plot(x,y4,'--b')
    hold off
```

Using matlab code above, repeat the visualization of signals in the previous section for a lossy transmission line. Assume that $\alpha = 0.1\,\mathrm{Np}$, and all other variables are the same as in the previous section. How do the voltages compare in the lossy and lossless cases?

Relating forward and backward current and voltage waves on the transmission line

In equations 32-33 $\tilde{V}_0^+$ and $\tilde{V}_0^-$ are the phasors of forward and reflected going voltage waves, and $\tilde{I}_0^+$ and $\tilde{I}_0^-$ are the phasors of forward and reflected current waves. In this section, we will see how the phasors of forward and reflected voltage and current waves are related to transmission line impedance.

$$\tilde{V}(z) = \tilde{V}_0^+ e^{-\gamma z} + \tilde{V}_0^- e^{\gamma z} \tag{32}$$
$$I(z) = \tilde{I}_0^+ e^{-\gamma z} + \tilde{I}_0^- e^{\gamma z} \tag{33}$$

When substitute the voltage wave equation into Telegrapher's Equations 19. The equation is repeated here Eq.34-35.

$$-\frac{\partial \tilde{V}(z)}{\partial z} = (R + j\omega L)I(z) \tag{34}$$
$$\gamma \tilde{V}_0^+ e^{-\gamma z} - \gamma \tilde{V}_0^- e^{\gamma z} = (R + j\omega L)I(z) \tag{35}$$

We now rearrange Eq.35

$$I(z) = \frac{\gamma}{R + j\omega L}(\tilde{V}_0^+ exp^{-\gamma z} + \tilde{V}_0^- exp^{\gamma z})$$
$$I(z) = \frac{\gamma \tilde{V}_0^+}{R + j\omega L} e^{-\gamma z} - \frac{\gamma \tilde{V}_0^-}{R + j\omega L} e^{\gamma z} \tag{36}$$

Now we compare Eq.36 with the Eq.33. In order for two transcendental equations to be equal, the coefficients next to exponential terms have to be the same.

$$\tilde{I}_0^+ = \frac{\gamma \tilde{V}_0^+}{R + j\omega L}$$
$$\tilde{I}_0^- = -\frac{\gamma \tilde{V}_0^-}{R + j\omega L}$$

We can define the characteristic impedance of a transmission line as the ratio of the voltage to current amplitude of the forward going wave.

$$Z_0 = \frac{\tilde{V}_0^+}{\tilde{I}_0^+}$$
$$Z_0 = \frac{R + j\omega L}{\gamma}$$
$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Lossless transmission line

In many practical applications $R \to 0$ and $G \to 0$. This is a lossless transmission line.

In this case the transmission line parameters are

- Propagation constant

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$
$$\gamma = \sqrt{j\omega L j\omega C}$$
$$\gamma = j\omega\sqrt{LC} = j\beta$$

- Transmission line impedance

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$
$$Z_0 = \sqrt{\frac{j\omega L}{j\omega C}}$$
$$Z_0 = \sqrt{\frac{L}{C}}$$

- Wave velocity

$$v = \frac{\omega}{\beta}$$
$$v = \frac{\omega}{\omega\sqrt{LC}}$$
$$v = \frac{1}{\sqrt{LC}}$$

metal resistance is low
dielectric conductance is low

- Wavelength

$$\lambda = \frac{2\pi}{\beta}$$
$$\lambda = \frac{2\pi}{\omega\sqrt{LC}}$$
$$\lambda = \frac{2\pi}{\sqrt{\varepsilon_0\mu_0\varepsilon_r}}$$
$$\lambda = \frac{c}{f\sqrt{\varepsilon_r}}$$
$$\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_r}}$$

What does it mean when we say a medium is lossy or lossless?

In a lossless medium electromagnetic wave power is not turning into heat, there is no loss of amplitude. In lossy medium electromagnetic wave is heating up the medium, therefore its power is decreasing as $e^{-\alpha x}$.

medium	attenuation constant α [dB/km]
coax	60
waveguide	2
fiber-optic	0.5

In guided wave systems such as transmission lines and waveguides the attenuation of power with distance follows approximately $e^{-2\alpha x}$. The power radiated by an antenna falls off as $1/r^2$.

Low-Loss Transmission Line

In some practical applications, losses are small, but not negligible. $R << \omega L$ and $G << \omega C$.

In this case the transmission line parameters are

metal resistance is lower than the inductive impedance
dielectric conductance is lower than the capacitive impedance

- Propagation constant

  We can re-write the propagation constant as shown below. In somel applications, losses are small, but not negligible. $R << \omega L$ and $G << \omega C$, then in Equation 38, $RG << \omega^2 LC$.

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \tag{37}$$

$$\gamma = j\omega\sqrt{LC}\sqrt{1 - j\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right) - \frac{RG}{\omega^2 LC}} \tag{38}$$

$$\gamma \approx j\omega\sqrt{LC}\sqrt{1 - j\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right)} \tag{39}$$

Taylor's series for function $\sqrt{1+x} = \sqrt{1 - j\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right)}$ in Equation 39 is shown in Equations 40-41.

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - ... \, for \, |x| < 1 \tag{40}$$

$$\gamma \approx j\omega\sqrt{LC}\sqrt{1 - j\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right)} = j\omega\sqrt{LC}\left(1 - \frac{j}{2}\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right)\right) \tag{41}$$

The real and imaginary part of the propagation constant γ are:

$$\alpha = \frac{\omega\sqrt{LC}}{2}\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right) \tag{42}$$

$$\beta = \omega\sqrt{LC} \tag{43}$$

We see that the phase constant β is the same as in the lossless case, and the attenuation constant α is frequency independent. This means that all frequencies of a modulated signal are attenuated the same amount, and there is no dispersion on the line. When the phase constant is a linear function of frequency, $\beta = const\,\omega$, then the phase velocity is a constant $v_p = \frac{\omega}{\beta} = \frac{1}{const}$, and the group velocity is also a constant, and equal to the phase velocity. In this case, all frequencies of the modulated signal are propagated with the same speed, and there is no distortion of the signal. This is the case only when the losses in the transmission line are small.

We usually represent phase and group velocity on $\omega - \beta$ diagrams, shown in Figure 16. At a frequency ω_1, the ratio of $\frac{\omega}{\beta}$ gives the phase velocity (graphically, this is the slope of the red line on the graph), whereas the slope of the $\omega - \beta$ curve (blue line on the graph) gives the group velocity at this frequency. These diagrams are useful, as they show how phase constant β varies with frequency, and it also shows how phase and group velocities vary with frequency. We can see that in this case, both group and phase velocity for this line are positive quantities, which is a representation of what is called a forward wave. In a forward wave, both signal and the energy propagate in the forward direction. Backward waves are waves where the signal propagates forward, however energy propagates backwards. In backward waves, phase and group velocities have opposite signs.

When the phase constant is a linear function of frequency, then, phase velocity and group velocity are equal, and do not depend on frequency. Both velocities are equal to the slope of the line in Figure 17.

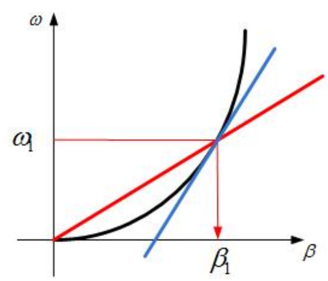

Figure 16: Omega-Beta diagrams, Phase constant β is a **nonlinear** function of frequency.

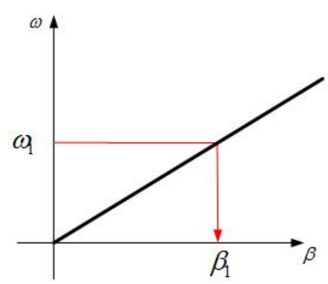

Figure 17: Omega-Beta diagrams, Phase constant β is a **linear** function of frequency.

Voltage Reflection Coefficient, Lossless Case

The equations for the voltage and current on the transmission line we derived so far are

$$\tilde{V}(z) = \tilde{V}_0^+ e^{-j\beta z} + \tilde{V}_0^- e^{j\beta z} \tag{44}$$
$$I(z) = \frac{\tilde{V}_0^+}{Z_0} e^{-j\beta z} - \frac{\tilde{V}_0^-}{Z_0} e^{j\beta z} \tag{45}$$

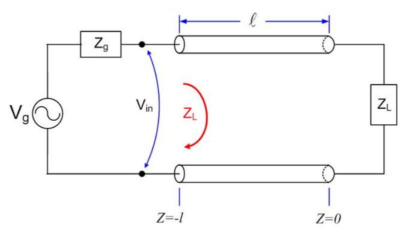

Figure 18: Transmission Line connects generator and the load.

At $z = 0$ the impedance of the load has to be

$$Z_L = \frac{V(0)}{I0}$$

Substitute the boundary condition in Eq.44

$$Z_L = Z_0 \frac{\tilde{V}_0^+ + \tilde{V}_0^-}{\tilde{V}_0^+ - \tilde{V}_0^-} \tag{46}$$

We can now solve the above equation for $\tilde{V}_0^-$

$$\frac{Z_L}{Z_0}(\tilde{V}_0^+ - \tilde{V}_0^-) = \tilde{V}_0^+ + \tilde{V}_0^-$$
$$(\frac{Z_L}{Z_0} - 1)\tilde{V}_0^+ = (\frac{Z_L}{Z_0} + 1)\tilde{V}_0^-$$
$$\frac{\tilde{V}_0^-}{\tilde{V}_0^+} = \frac{\frac{Z_L}{Z_0} - 1}{\frac{Z_L}{Z_0} + 1}$$
$$\frac{\tilde{V}_0^-}{\tilde{V}_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{47}$$

The quantity $\frac{\tilde{V}_0^-}{\tilde{V}_0^+}$ is called voltage reflection coefficient Γ. It relates the reflected to incident voltage phasor. Voltage reflection coefficient is in general a complex number, it has a magnitude and a phase.

Examples

(a) $100\,\Omega$ transmission line is terminated in a series connection of a $50\,\Omega$ resistor and 10 pF capacitor. The frequency of operation is 100 MHz. Find the voltage reflection coefficient.

(b) For purely reactive load $Z_L = jX_L$ find the reflection coefficient.

The end of this lecture is spent in the lab making a Matlab program to make a movie of a wave moving left and right.

Standing Waves

In the previous section we introduced the voltage reflection coefficient that relates the forward to reflected voltage phasor.

$$\Gamma = \frac{\tilde{V}_0^-}{\tilde{V}_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{48}$$

If we substitute Equation 48 to Eq.49 we get for the voltage wave

$$\begin{aligned} \tilde{V}(z) &= \tilde{V}_0^+ e^{-j\beta z} + \tilde{V}_0^- e^{j\beta z} && (49) \\ \tilde{V}(z) &= \tilde{V}_0^+ e^{-j\beta z} + \Gamma \tilde{V}_0^+ e^{j\beta z} \\ \tilde{V}(z) &= \tilde{V}_0^+ (e^{-j\beta z} - \Gamma e^{j\beta z}) && (50) \end{aligned}$$

since $\Gamma = |\Gamma| e^{j\Theta_r}$ Eq.50 becomes

$$\tilde{V}(z) = \tilde{V}_0^+ (e^{-j\beta z} - |\Gamma| e^{j\beta z + \Theta_r}) \tag{51}$$

and for the current wave

$$I(z) = \frac{\tilde{V}_0^+}{Z_0}e^{-j\beta z} - \frac{\tilde{V}_0^-}{Z_0}e^{j\beta z} \tag{52}$$

$$I(z) = \frac{\tilde{V}_0^+}{Z_0}e^{-j\beta z} + \Gamma\frac{\tilde{V}_0^+}{Z_0}e^{j\beta z}$$

$$I(z) = \frac{\tilde{V}_0^+}{Z_0}(e^{-j\beta z} - \Gamma e^{j\beta z}) \tag{53}$$

The voltage and the current waveform on a transmission line are therefore given by Eqns.50, 53. Now we have two equations and one unknown $\tilde{V}_0^+$! We will solve these two equations in Lecture 7. Now let's look at the physical meaning of these equations.

In Eq.50, Γ is the voltage reflection coefficient, $\tilde{V}_0^+$ is the phasor of the forward going wave, z is the axis in the direction of wave propagation, β is the phase constant, Z_0 is the impedance of the transmission line. $\tilde{V}(z)$ is a complex number, phasor. We will find the magnitude and phase of the voltage on the transmission line.

The magnitude of a complex number can be found as $|z| = \sqrt{zz^*}$.

$$|\tilde{V}(z)| = \sqrt{\tilde{V}(z)\tilde{V}(z)^*}$$

$$|\tilde{V}(z)| = \sqrt{\tilde{V}_0^+(e^{-j\beta z} - |\Gamma|e^{j\beta z+\Theta_r})\tilde{V}_0^+(e^{j\beta z} - |\Gamma|e^{-(j\beta z+\Theta_r)})}$$

$$|\tilde{V}(z)| = \tilde{V}_0^+\sqrt{(e^{-j\beta z} - |\Gamma|e^{j\beta z+|\Theta_r})(e^{j\beta z} - |\Gamma|e^{-(j\beta z+\Theta_r)})}$$

$$|\tilde{V}(z)| = \tilde{V}_0^+\sqrt{1 + |\Gamma|e^{-(2j\beta z+|\Theta_r)} + |\Gamma|e^{j2\beta z+\Theta_r} + |\Gamma|^2)}$$

$$|\tilde{V}(z)| = \tilde{V}_0^+\sqrt{1 + |\Gamma|^2 + |\Gamma|(e^{-(2j\beta z+\Theta_r)} + e^{(j2\beta z+\Theta_r)})}$$

$$|\tilde{V}(z)| = \tilde{V}_0^+\sqrt{1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta z + \Theta_r)} \tag{54}$$

The magnitude of the total voltage on the transmission line is given by Eq.54. It seems like a complicated function.

- Let's start from a simple case when the voltage reflection coefficient on the tranmission line is $\Gamma = 0$ and draw the magnitude of the total voltage. The magnitude is the green line in Figure 19. To see the movie of this transmission line go to the class web page under Instructional Videos. Forward voltage is shown in red, reflected voltage in pink, and the magnitude of the voltage is green. Magnitude of the voltage is constant everywhere on the transmission line, and so the line is called flat.

imaginary part of the complex propagation constant
defined as the ratio of forward going voltage and current

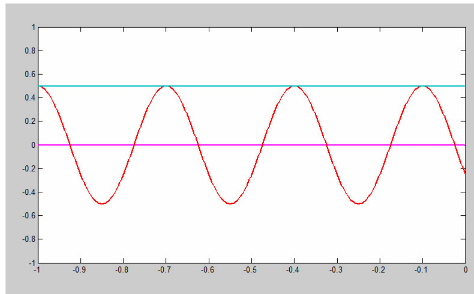

Figure 19: Flat line.

- Let's look at another case, $\Gamma = 0.5$ and $\Theta_r = 0$. Equation 55 represents the magnitude of the voltage on the transmission line, and Figure 20 shows in green how this function looks on a transmission line. This case is shown in Figure 20.

$$|\tilde{V}(z)| = \tilde{V}_0^+ \sqrt{\frac{5}{4} + cos 2\beta z} \tag{55}$$

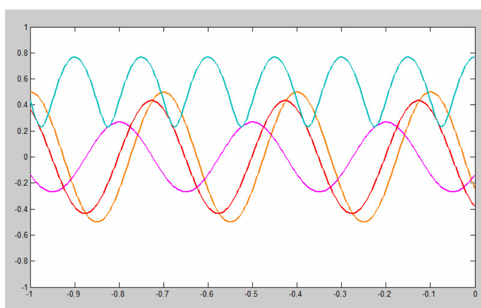

Figure 20: Voltage on a transmission line with reflection coefficient magnitude 0.5, and zero phase.

The function 55 is at it's maximum when $cos(2\beta z) = 1$ or $z = \frac{k}{2}\lambda$, and the function value is $\tilde{V}(z) = 1.5\tilde{V}_0^+$. It is at it's minimum when $cos(2\beta z) = -1$ or $z = \frac{2k+1}{4}\lambda$ and the function value is $\tilde{V}(z) = 0.5\tilde{V}_0^+$

It is important to mention here that the function that we see looks like a cosine with an average value of $\tilde{V}_0^+$, but **it is not**. The minimums of the function are sharper then the maximums, so when the reflection coefficient is at it's maximum of $\Gamma = 1$ the function looks like this:

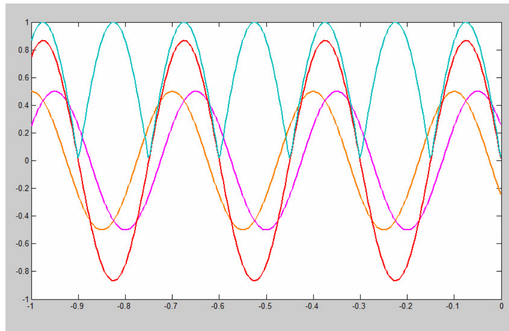

Figure 21: Shorted Transmission Line.

- General Case.

  In general the voltage maximums will occur when $cos(2\beta z) = 1$

$$
\begin{aligned}
|\tilde{V}(z)_{max}| &= \tilde{V}_0^+ \sqrt{1 + |\Gamma|^2 + 2|\Gamma|} \\
|\tilde{V}(z)_{max}| &= \tilde{V}_0^+ \sqrt{(1 + |\Gamma|)^2} \\
|\tilde{V}(z)_{max}| &= \tilde{V}_0^+ (1 + |\Gamma|)
\end{aligned} \tag{56}
$$

  In general the voltage minimums will occur when $cos(2\beta z) = -1$,

$$
\begin{aligned}
|\tilde{V}(z)_{min}| &= \tilde{V}_0^+ \sqrt{1 + |\Gamma|^2 - 2|\Gamma|} \\
|\tilde{V}(z)_{min}| &= \tilde{V}_0^+ \sqrt{(1 - |\Gamma|)^2} \\
|\tilde{V}(z)_{min}| &= \tilde{V}_0^+ (1 - |\Gamma|)
\end{aligned} \tag{57}
$$

  The ratio of voltage minimum on the line over the voltage maximum is called the Voltage Standing Wave Ratio (VSWR) or just Standing Wave Ratio (SWR).

$$
\begin{aligned}
SWR &= \frac{\tilde{V}(z)_{max}}{\tilde{V}(z)_{min}} \\
SWR &= \frac{1 + |\Gamma|}{1 - |\Gamma|}
\end{aligned} \tag{58}
$$

The voltage maximum position on the line is where

$$\begin{aligned} cos(2\beta z) &= 1 \\ 2\beta z + \Theta_r &= 2n\pi \\ z &= \frac{2n\pi - \Theta_r}{2\beta} \\ z &= \frac{2n\pi - \Theta_r}{4\pi} \end{aligned} \tag{59}$$

Display image.