Electromagnetics

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Contents

1 Sinusoidal signals

After completing this section, students should be able to do the following.

- Describe the basic properties of sinusoidal functions.
- Compare and contrast phase and time-delay.
- Recognize leading and lagging signals.
- Explain why a signal is leading or lagging.
- Remember eLi the iCe man and CIVIC mnemonics.
- Explain how a signal is different at the beginning and end of a transmission line.
- Derive equation of a wave on a transmission line by considering the finite propagation speed of electric signals.
- Determine whether the transmission-line theory should be used based on line length and signal frequency.

1.1 Basic Parameters of Sinusoidal Signals

Review of Sinusoidal Signals

Sinusoidal signal parameters

Sinusoidal signals are essential in electrical engineering as we use them to analyze and test circuit performance. All periodic signals can be represented with sinusoidal signals of different amplitudes and phases using the Fourier series.

A typical sinusoidal signal is shown in Figure 1. On the y-axis is the instantaneous value of the sinusoidal voltage, and on the x-axis is time. Instantaneous values of voltage change from -1V to 1V with time. We use the following parameters to characterize sinusoidal signals: peak amplitude, peak-to-peak, average, RMS, period, time-delay, and phase. We read peak amplitude, peak-to-peak, average, and RMS values, on the y-axis in Figure 1, whereas period, time delay, and phase on the x-axis.

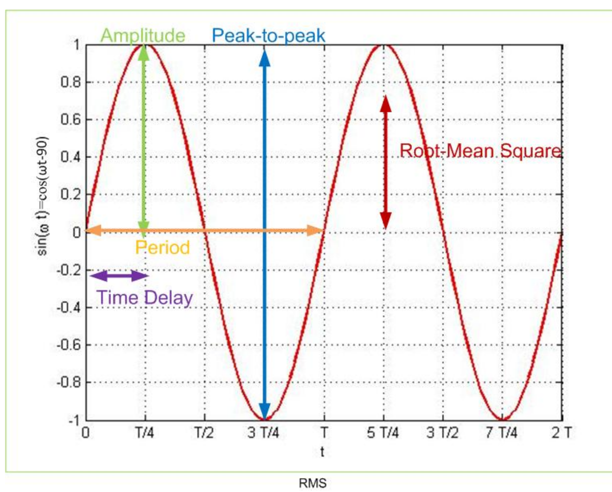


Figure 1: Vocabulary used in describing sinusoidal signals.

Parameters that are read on the y-axis.

Definition 1. *We read the peak amplitude on the y-axis from the average value of the signal (in this case, zero) to the signal's maximum value (in this case,* *1). For signal shown in Figure 1, the peak amplitude has a constant value of* $V_p = 1$*.*

Definition 2. *The sinusoidal signal's instantaneous value varies from -1 to 1V, and its value depends on the x-axis.*

Compared to the instantaneous value, the peak amplitude is always constant, and it does not vary with time.

Definition 3. *We measure peak-to-peak from the minimum value (in this case, -1) to the maximum value (in this case, 1). For signal shown in Figure 1, peak-to-peak voltage has a constant value of* $V_{pp} = 2$*.*

Definition 4. *RMS or root-mean-square is defined as* $v_{rms} = \frac{1}{T}\sqrt{\int_0^T v(t)^2 dt}$*. For signal shown in Figure 1, and other sinusoidal signals of this form,* $v_{rms} = \frac{V_p}{\sqrt{2}} = \frac{1}{\sqrt{2}} = 0.707$*. Root mean square value is important because it represents the equivalent amount of DC power.*

Definition 5. *Average value* $v_{ave1} = \frac{1}{T}\int_0^T v(t)dt$*. For the signal shown in Figure 1, the average value is* $V_{ave1} = 0$ *because the function has the same area under the function in the positive and negative cycle.*

Parameters that are read on the x-axis.

Definition 6. *We can represent sinusoidal signals as a function of time, Figure 2, or a function of angle, Figure 3. Take a few minutes to see how the graphs are the same and how they are different.*

Definition 7. *Period, T, is measured on the x-axis as the length of one full cycle of the sinusoidal signal. For signal shown in Figure 1, this value is* $period = T$*.*

Definition 8. *Frequency, f, is defined as a reciprocal value of the period T,* $f = \frac{1}{T}$*. It represents how fast the signal is changing in time. In Figure 4, sinusoidal signals of two different frequencies f are given.*

Definition 9. *Time delay and phase represent the lag (or lead) of one function with respect to another in the time domain and frequency domain. For example, in Figure 1, function* $\cos(\omega t - 90^o)$ *is time-delayed for* $\tau = \frac{T}{4}$ *with respect to* $\cos(\omega t)$*. To find the time delay from the phase, we look at how to represent the phase* 90^o *in terms of the product of frequency and time. Since in the sinusoidal signal expression* $\cos(\omega t + \Theta)$ *phase* Θ *is added to* ωt *term, the phase has the same units as* ωt*, and can be represented as the product of* $\omega\tau = \theta$*,* $\tau = \frac{\theta}{\omega}$*, where* τ *represents the time delay.*

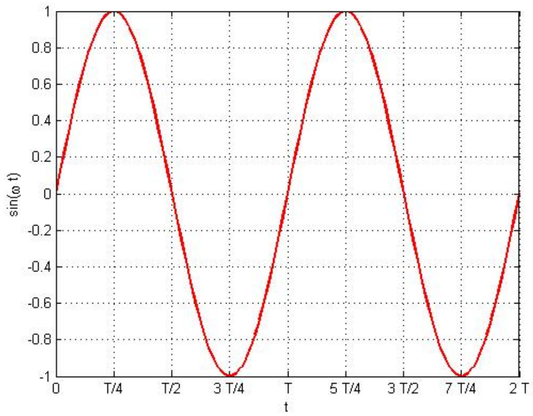


Figure 2: $sin(\omega t)$ as a function of time.

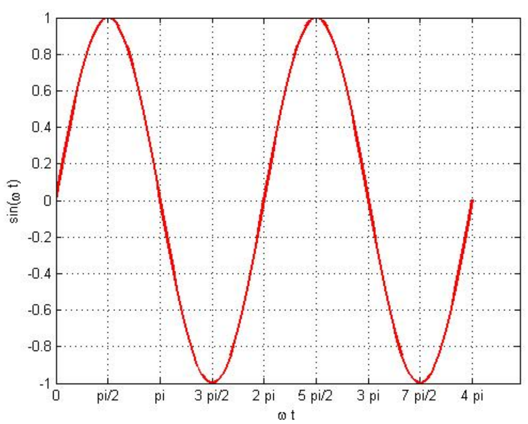


Figure 3: Sinusoidal signal as a function of angle ωt.

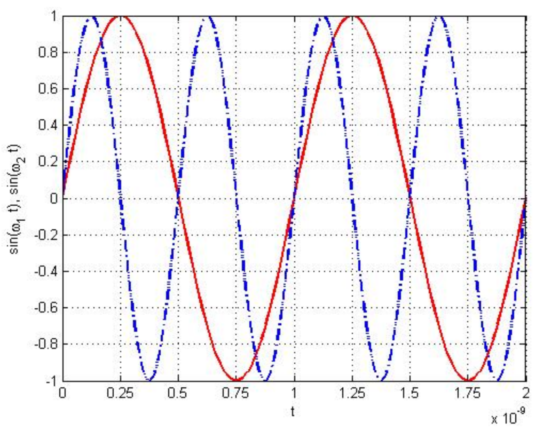


Figure 4: Sinusoidal signals of different frequencies $sin(\omega t)$

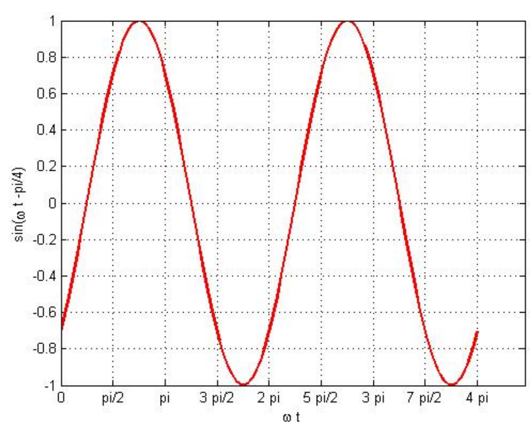


Figure 5: Sinusoidal signal as a function of angle ωt with a phase shift of $-\pi/4$

Question 1 *Calculate the time-delay in nanoseconds that you would observe on an oscilloscope if the frequency of the signal is f=0.159 GHz and the phase shift of the signal is $\theta = 10^o$.*

$$\frac{\theta}{2 * \pi * f} = \boxed{?}\ ns$$

Example 1. *Observe the three signals below, and change their amplitude and phase. Explain qualitatively how are the signals changing when we move the sliders?*

Geogebra link: `https://tube.geogebra.org/m/w8r4epyh`

1.2 Leading and Lagging Signals

Review of Sinusoidal Signals

Definition 10. *How do we recognize lagging and leading on a graph?*

In Figure 11 we observe two step functions, V(t) and V(t-T). Function V(t) step occurs at t=0, and V(t-T) step occurs at t=T. The function V(t-T) is shifted to the right, the step occurs later, at t=T, and is, therefore, lagging function V(t).

Similarly, if the step function is V(t+T), the function v(t) is shifted to the left. The step occurs earlier at t=-T, and therefore V(t+T) is leading V(t).

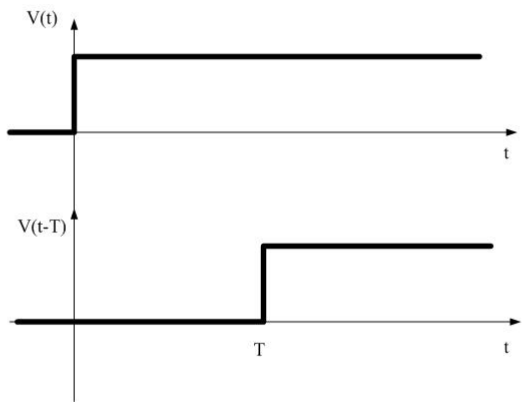


Figure 6: Voltage as a function of time at the generator side (top) and the load side (bottom) of a transmission line, if the switch closes at t=0 the voltage arrives at t=l/c=T at the load. These graphs can be obtained by observing the voltage on an oscilloscope at the load and at the generator side.

Example 2. *What if we have a sinusoidal signal? We will observe a specific point on the signal, such as the maximum value, and determine if it shifted left or right on the graph.*

When the phase of a signal is positive as in Figure 7 $\sin(\omega t + 45^o)$*, we say that the signal is leading with respect to the signal* $v(t) = \sin(\omega t)$*, because it is shifted to the left for* $45^o (pi/4)$*. The maximum of the function now occurs at t=-T, or* $\omega t = -45^o$*, and we can write the new function as the original sinusoidal function V(t) shifted left for a time T, V(t+T). The phase of the signal is* 45^o*, and the time-delay is T.*

Example 3.

When the phase of a signal is negative as in Figure 9, 8, $\sin(\omega t - 45^o)$*, we say that the signal is lagging with respect to the signal* $\sin(\omega t)$*, because it is shifted*

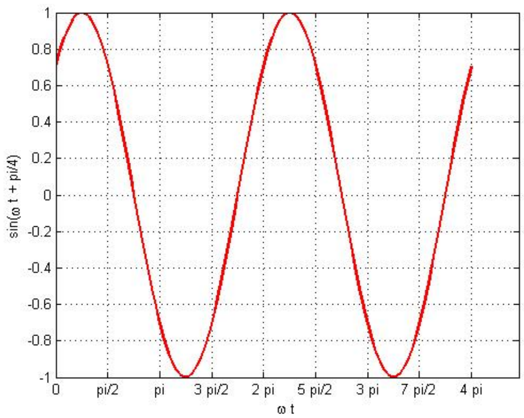


Figure 7: Sinusoidal signal as a function of angle ωt with a phase shift of $+\pi/4$

to the right for 45^o *(pi/4), or* $\tau = -\dfrac{pi/4}{\omega}$*. The lagging function's peak occurs later in time, and therefore it is lagging. The phase of the signal is* -45^o*.*

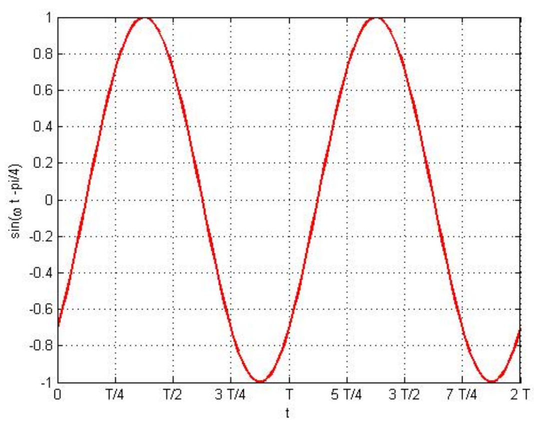


Figure 8: Sinusoidal signal shifted for time delay $-\dfrac{\pi/4}{\omega}$

Question 2 *Sinusoidal signal* $v_1 = \cos(\omega t - 25^o)$ *is given. Compared to* $v = \cos(\omega t)$*, signal* v_1

Multiple Choice:

(a) *Leads signal* v

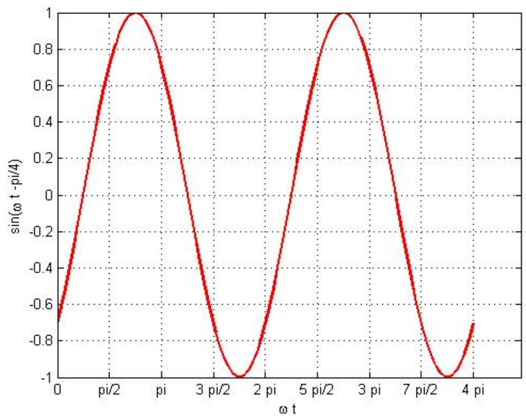
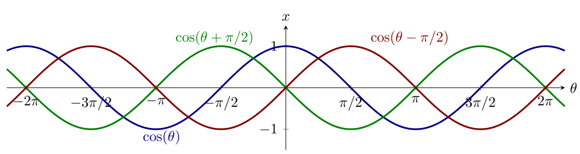


Figure 9: Sinusoidal signal with phase shift $-\pi/4$

(b) *Lags signal v*

Question 3 *Observe three signals in Figure below*



Which of the following functions leads $\cos(\omega t)$*?*

Multiple Choice:

(a) *The green signal.*

(b) *The red signal.*

(c) *The blue signal.*

1.3 eLi the iCe Man is CIVIL

Review of Sinusoidal Signals

In a resistor, the current and voltage maximums occur simultaneously, but in capacitors and inductors, maximums of current and voltage occur at different times.

“eLi the iCe man” and “CIVIL” are mnemonics to help us remember how the current and voltage lag or lead in inductors and capacitors.

On an inductor , maximum voltage occurs before the current maximum (sometimes voltage is labeled as ”e”, eLi, VIL). In an inductor, we say that the voltage leads the current, or the current lags the voltage.

In a capacitor, the current maximum occurs before the voltage maximum (iCe, CIV). In a capacitor, we say that the current leads the voltage, and voltage lags the current. By observing voltage and current maximums, we can tell if a capacitor or inductor produced the voltages and currents displayed.

Example 4. *Observe just the sinusoidal voltages and currents in the series RLC circuit below. Can you apply CIVIL or eLi the iCe man? You do not have to worry about the phasor vector diagrams; we'll talk about that later.*

Geogebra link: `https://tube.geogebra.org/m/dgftaeya`

1.4 Signal Delay on Transmission Lines

Review of Sinusoidal Signals

Lagging and leading is often used in circuits to describe signals.

The circuit in Figure 10 shows a generator and load connected with a cable, just as in the circuits lab. In the analysis and measurements in the circuit labs, the cable is usually ignored.

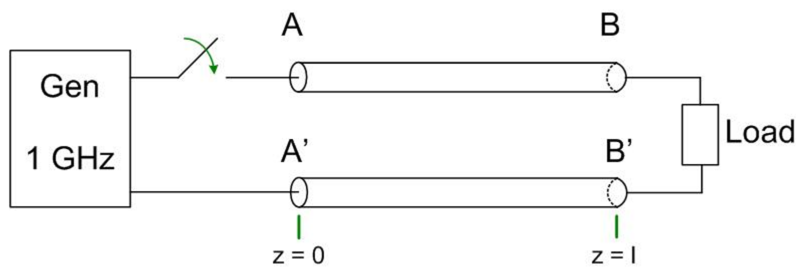


Figure 10: Electronic Circuit with an emphasis on cables that connect the generator and the load.

Let us assume that the switch closes, and the generator produces a step function at time t=0, as shown in Figure 11. Because the electrical signals propagate with light speed, the signal needs T sec to appear at the load after the switch closes. Figure 12 shows the step signal as it travels on the transmission line at different times t=0, t=T/4, t=T/2, and t=T.

How much time T does it take for this signal to go from AA' end to BB' end? Since electromagnetic waves propagate with the constant speed, the speed of light, the time that the signal needs to go from the generator to the load depends on the transmission line's length. If the transmission line is l, then the delay between the generator and the load will be $T = \dfrac{l}{c}$, where $c = 3 \times 10^8$. If the signal at the generator AA' is $v_g(t)$ =v(t), then the signal at the load end is $v_l(t)$ =v(t-T).

Figure 12 shows the signal on the transmission line at different times. Note that the horizontal axis shows distance, not time. Each graph shows a snapshot of the signal on the transmission line at different times. The top figure shows the moment when we turn the switch on, at t=0, and the signal shows up at the beginning of the transmission line, at the generator's end. A little later, at t=1=T/4, the front of the signal traveled a little farther along the line. At t=2=T/2, the signal traveled even further on the line. The bottom figure shows the signal that arrived at the load at some time t=4=T. Note that the load will not see the signal until t=4=T.

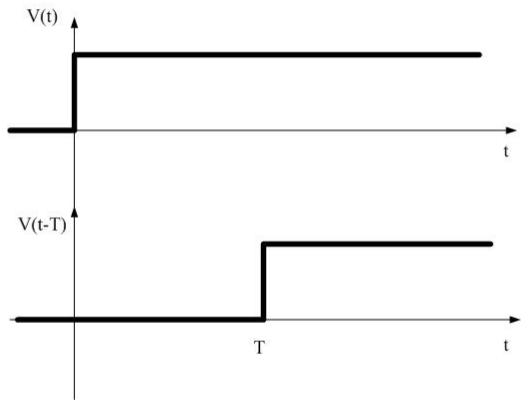


Figure 11: Voltage as a function of time at the generator side (top) and the load side (bottom) of a transmission line, if the switch closes at t=0 the voltage arrives at t=l/c=T at the load. These graphs can be obtained by observing the voltage on an oscilloscope at the load and at the generator side.

How much time will the signal need to arrive at the load? We know that the speed of light is $c = 3\,10^8$m/s. Since this is a constant speed, the time is $t = \frac{l}{c}$, where l is the transmission line's length.

Question 4 *How much time-delay will a l=1000km transmission line produce?*

Multiple Choice:

(a) *3.33 ns*

(b) *3.33 μs*

(c) *3.33 ms*

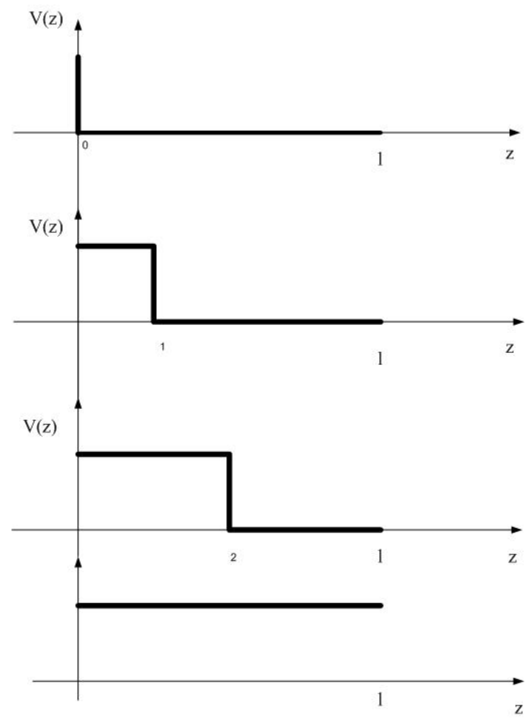


Figure 12: Voltage along the transmission line in Figure 10, for four different time intervals t=0, switch closes, t=T/4, t=T/2 and t=T. It is assumed that the length of the transmission line is equal to l=T/c.Note that the horizontal axis is the distance z from the generator to the cable, not time.

1.5 Engineering Design

The purpose of this section is to show one application of sinusoidal signals to engineering design.

When do we have to use the transmission-line theory? The answer to this question depends on the time-delay (or phase shift) that the line introduces. If the phase shift is small, for example, $\approx 2^0$, between the signal at the generator and the load, we don't have to use the transmission line theory, but if the phase between signals is much higher than $\approx 2^0$, then we do have to use it.

We will now derive an expression for the phase shift on a transmission line between the generator and the load.

If we have a sinusoidal generator at one end of the transmission line, what is the signal at the other end?

The signal at the generator is

$$v_{AA^`}(t) = Acos(\omega t) \tag{1}$$

At the other end, the transmission line the signal is delayed.

$$v_{BB^`}(t) = v_{AA^`}(t - T) \tag{2}$$

$$v_{BB^`}(t) = v_{AA^`}(t - \frac{l}{c}) \tag{3}$$

$$v_{BB^`}(t) = Acos(\omega(t - \frac{l}{c})) \tag{4}$$

$$v_{BB^`}(t) = Acos(\omega t - \omega\frac{l}{c}) \tag{5}$$

$$v_{BB^`}(t) = Acos(\omega t - \frac{\omega}{c}l) \tag{6}$$

Since we know that angular frequency is $\omega = 2\pi f$

$$v_{BB^`}(t) = Acos(\omega t - \frac{2\pi f}{c}l) \tag{7}$$

The quantity $\frac{c}{f}$ is called the wavelength λ, and it represents the distance between two maximums of a signal on a transmission line. λ is a period of the signal on a transmission line over distance, and the units are meters. It is analogous to the signal period $T = \frac{2\pi}{omega}$, but λ is a period in space, not time.

$$v_{BB^{`}}(t) = A\,cos(\omega t - \frac{2\pi}{\lambda}l) \tag{8}$$

The quantity $\frac{2\pi}{\lambda}$ is called the propagation constant β, and it is analogous to the angular frequency ω of a signal, however, it represents how fast the signal is changing over distance and not time.

Finally, the expression for the voltage at BB end is

$$v_{BB^{`}}(t) = A\,cos(\omega t - \beta l) \tag{9}$$

$$v_{BB^{`}}(t) = A\,cos(\omega t - \Psi) \tag{10}$$

We see that at BB' the signal will experience a phase shift. We will later show that the solution of the wave equation is the equation above. We will derive the wave equation from the Telegrapher's equations.

Now let's see how the length of the line l affects the voltage at the end BB'. Look at Equation 8. The signal will experience a phase shift of $2\pi\frac{l}{\lambda}$. If this phase shift is small, there will not be much difference between the signal's phase between the generator and the load. In this case, we don't have to use the transmission line theory to account for the line's effects. If the phase shift is significant, then we do have to use the transmission line theory. Let's look at some numerical examples.

(a) If $\frac{l}{\lambda} < 0.01$ then the angle $2\pi\frac{l}{\lambda}$ is of the order of 0.0314 rad or about 2^0. In this case, the phase is obviosly something that we don't have to worry about. When the length of the transmission line is much smaller than λ, $l << \frac{\lambda}{100}$ the wave propagation on the line can be ingnored.

(b) If $\frac{l}{\lambda} > 0.01$, say $\frac{l}{\lambda} = 0.1$, then the phase is 20^0, which is a significant phase shift. In this case it may be necessary to account fro transmission line effects.

Question 5 *Do we have to use transmission-line theory if the length of the line is 5000km, and the frequency of the signal is 60Hz?*

Multiple Choice:

(a) *Yes*

(b) *No*

Question 6 *Do we have to use transmission-line theory if the length of the line is 2 cm, and the frequency of the signal is 10 GHz?*

Multiple Choice:

(a) *Yes*

(b) *No*

2 Complex numbers

After completing this section, students should be able to do the following.

- Explain why complex numbers are important in circuits and electromagnetics.
- Sketch a complex number in rectangular and polar coordinates, and label magnitude, phase, real and imaginary parts.
- Derive the magnitude and phase from the real and imaginary parts of a complex number.
- Derive the real and imaginary parts of a complex number from the magnitude and phase.
- Explain how Euler's formula relates to sinusoidal signals and complex numbers.
- Describe which coordinate system to use when adding/subtracting and which one when multiplying/dividing two complex numbers.
- Apply complex numbers to solve a circuit element's impedance if the phasor of current through and voltage on it are known.
- Apply complex numbers to solve for voltage on a circuit element if phasor of current and impedance are known.
- Apply complex conjugate operation to a complex number in rectangular and polar coordinates.
- Derive magnitude of a complex number from a complex number and complex conjugate of the same number.
- Visualize the position of purely imaginary and purely real complex numbers on a unit circle.
- Convert visually purely imaginary and purely real complex numbers from rectangular to polar coordinates and vice versa.
- Prove that the magnitude of a complex number is a square root of the product of the number and its complex conjugate.

2.1 Review of Complex Numbers

This section aims to introduce two different ways we represent complex numbers: Cartesian coordinates, and Polar coordinates. We often use complex numbers in Cartesian coordinates when we discuss impedance or admittance. We often use complex numbers in polar coordinates to discuss magnitude and phase of voltages, currents, transfer functions, and Bode Plots. We can also represent sinusoidal signals with complex numbers with phasors. It is critically important that we understand this chapter.

Definition 11. *A complex number z can be represented in the Cartesian coordinate system, as shown in Equation 11, and Polar coordinate system, as shown in Equation 12.*

$$z = x + jy \tag{11}$$
$$z = |z|e^{j\Theta} \tag{12}$$

In Equation 11, a complex number z is represented in rectangular coordinate system, where x is the real part, y is the imaginary part, and $j = \sqrt{-1}$.

In Equation 12, we see a complex number z in the polar coordinate system, where $|z|$ is the magnitude, and Θ is the angle (aka phase) of the complex number.

The geometric interpretation of these two equations is shown in Figure 13. The magnitude is the length of the triangle's hypothenuse, and the angle is the angle that the hypothenuse makes with the x-axis.

In Figure 13, we represent a complex number with a "position vector." Position vectors are vectors that start at the center of the coordinate system and end at any point in the coordinate system. We see that phasor is a vector that represents a complex number in a polar coordinate system.

You may be wondering why we represent the phase of a complex number in the polar coordinate system as $e^{j\Theta}$ because in the circuits class, you used $\angle\theta$. Great question. That brings us to Euler's formula that we will discuss in section Euler's Formula.

Conversion between Cartesian and Polar coordinate systems

To find magnitude and angle when we know real and imaginary parts of a complex number, we use the Pythagorean Theorem to find the magnitude of

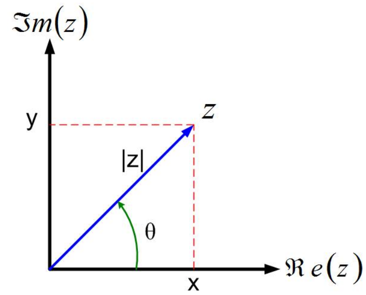


Figure 13: Visual representation of a complex number z in rectangular $z = x+jy$ and polar coordinates $z = |z|e^{j\theta}$.

the complex number as in Equation 13, and use the definition of the tangent to find the angle as in Equation 14.

$$|z| = \sqrt{x^2 + y^2} \tag{13}$$

$$\Theta = arctg\frac{y}{x} \tag{14}$$

To find the real and imaginary part of a complex number when we know magnitude and phase, we use trigonometry. To find the real part of the complex number as in Equation 15, use the definition of cosine and sine to find the imaginary part of the complex number as in Equation 16.

$$\Re\{z\} = x = r\cos(\Theta) \tag{15}$$

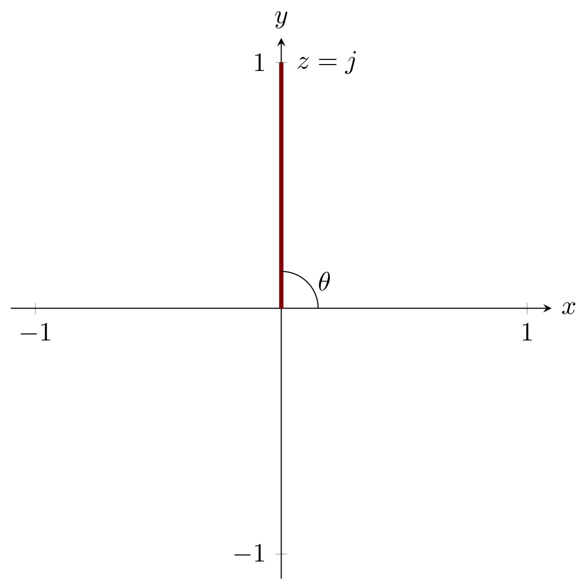
$$\Im\{z\} = y = r\sin(\Theta) \tag{16}$$

Question 7 *Explore the conversion of complex numbers between cartesian and polar coordinates.*

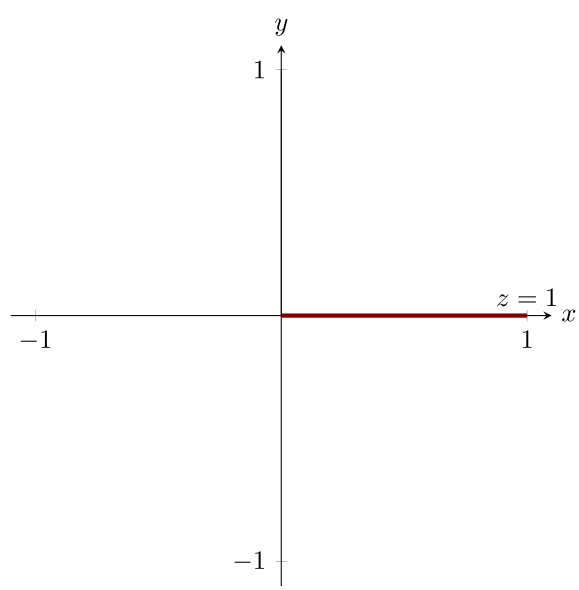
Geogebra link: `https://tube.geogebra.org/m/b8hu8ztx`

Example 5. *Find the magnitude and phase of complex numbers $z_1 = j$ and $z_2 = 1$.*

Explanation. *Complex number $z_1 = j$ is on the y-axis where y=1. By inspection, the magnitude of z_1 is $|z| = 1$, and the angle is $\theta = 90^o$.*



The complex plane is sketched below. Complex number $z_1 = 1$ is on the x-axis where x=1. By inspection, the magnitude of z_1 is $|z| = 1$, and the angle is $\theta = 0^o$.



Question 8 *Calculate magnitude and phase of complex number $z = -j$*

Multiple Choice:

(a) $|z| = 1, \theta = 180$

(b) $|z| = 1, \theta = -90$

(c) $|z| = -1, \theta = 180$

(d) $|z| = -1, \theta = -90$

Write magnitude and phase of complex number $z = -1$

Question 9 $-1 = \boxed{?}$

2.2 Operations with Complex Numbers

The purpose of this section is to review arithmetic operations with complex numbers. We use complex numbers to describe circuits. When we solve circuits to find voltages, currents, and power, we often encounter addition, subtraction, multiplication, division, and the complex conjugate of complex numbers.

Complex Conjugate

We will see complex conjugate when we discuss maximum power transfer.

$$z^* = (x + jy)^* = x - jy = |z|e^{-j\Theta} \tag{17}$$

Addition

In electrical engineering, we see complex number addition and subtraction when complex impedances are in series, and we are looking for the equivalent complex impedance. The easiest way to add two complex numbers is to find the Cartesian representation of both and then add the real parts separately and the imaginary part separately.

$$\begin{aligned} z_1 &= x_1 + jy_1 \\ z_2 &= x_2 + jy_2 \\ z_1 + z_2 &= x_1 + x_2 + j(y_1 + y_2) \end{aligned} \tag{18}$$

You can visually explore addition of two complex numbers with the app below.

Geogebra link: `https://tube.geogebra.org/m/yfvhfb8a`

Question 10 *Two impedances are given $Z_1 = 50 + j200\Omega$ and $Z_2 = 50 - j200\Omega$. If the two impedances are in series, what is the total impedance?*

$Z_1 + Z_2 = \boxed{?}$

Subtraction

$$\begin{aligned} z_1 &= x_1 + jy_1 \\ z_2 &= x_2 + jy_2 \\ z_1 - z_2 &= x_1 - x_2 + j(y_1 - y_2) \end{aligned} \tag{19}$$

You can visually explore subtraction of two complex numbers with the app below.

Geogebra link: https://tube.geogebra.org/m/ujsv2qkq

Multiplication and Division

We often see multiplication and division of complex numbers in Ohm's law or transfer function of a circuit. Two complex numbers can be multiplied or divided in either Cartesian or Polar forms. However, the easiest way to divide or multiply two complex numbers is to find the polar representation of both and then divide or multiply the amplitudes and subtract or add the phases, as shown in equations below.

$$\begin{aligned} z_1 &= |z_1|e^{j\Theta_1} \\ z_2 &= |z_2|e^{j\Theta_2} \\ \frac{z_1}{z_2} &= \frac{|z_1|}{|z_2|}e^{j\Theta_1 - \Theta_2} \end{aligned} \tag{20}$$

$$\begin{aligned} z_1 &= |z_1|e^{j\Theta_1} \\ z_2 &= |z_2|e^{j\Theta_2} \\ z_1 z_2 &= |z_1||z_2|e^{j\Theta_1 + \Theta_2} \end{aligned} \tag{21}$$

Explore the multiplication of two complex numbers visually with the app below. We see that the multiplication of numbers represent rotation.

Geogebra link: https://tube.geogebra.org/m/h34xreac

Question 11 *Three complex numbers are given* $Z_1 = 100 + j50$, $Z_2 = 3 * e^{j40^0}$ *and* $Z_3 = 20 + j100$. *Calculate* $\frac{Z_1 Z_2}{Z_2 + Z_3}$. *Present your answer as a complex* *number in Cartesian coordinates with two decimal places. For example* $0.11 + j0.23$ *:*

$$\frac{Z_1 Z_2}{Z_2 + Z_3} = \boxed{?}$$

2.3 Euler's Formula

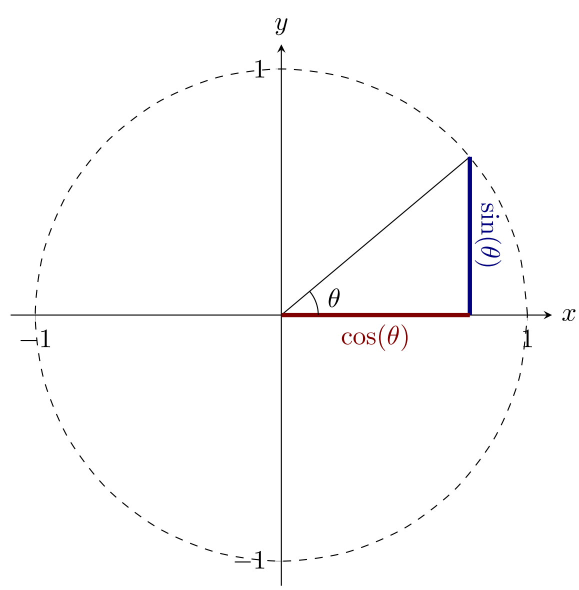
The purpose of this section is to relate sinusoidal signals and complex numbers using Euler's formula.

You may be wondering why we represent the phase of a complex number in the polar coordinate system as re^{jTheta} because, in the circuits class, we used $\angle\theta$. Great question. That brings us to Euler's formula.

Euler's formula relates Cartesian and Polar coordinates for complex numbers.

$$e^{j\Theta} = \cos\Theta + j\sin\Theta \tag{22}$$

Geometric interpretation of the Euler's formula is shown below. $z = r(\cos\theta + j\sin\theta)$, where $r\cos\theta = x$ and $r\sin\theta = y$. Euler's formula shows that number z given in Cartesian coordinates as $x+jy$ can be represented in Polar Coordinates as $e^{j\Theta}$. You have likely seen this proof in your Calculus class. **TIP: Your calculator may not know what $e^{j\theta}$ is. Check how to convert between polar and cartesian coordinates on your calculatorbefore the test.**



Voltages and currents that we represent have different amplitudes that sometimes change with time, as we will see in this course in the section on transmission lines. We can multiply Euler's formula with a constant, and get the general form that includes the amplitude of signals $e^{-\sigma t}$.

$$
\begin{aligned}
e^{-\sigma t}e^{j\Theta} &= e^{-\sigma t}\cos\Theta + je^{-\sigma t}\sin\Theta \\
e^{(-\sigma+j\omega)t} &= e^{-\sigma t}\cos\Theta + je^{-\sigma t}\sin\Theta
\end{aligned} \tag{23}
$$

If we further replace the angle Θ with ωt, we can use Euler's formula to represent sinusoidal signals that vary with time $\cos(\omega t)$ and $\sin(\omega t)$.

Observe the simulation below. On the left side is the complex number $e^{-\sigma+j\omega t} = e^{-\sigma t}\cos(\omega t) + je^{-\sigma t}\sin(\omega t)$ whose value on the real axis represents the real part of the complex number $e^{-\sigma t}\cos(\omega t)$, and imaginary axis represents the imaginary part of the complex number $e^{-\sigma t}\sin(\omega t)$. We see the real and imaginary parts of this complex number on the right side, where cosine and sine functions in time-domain. We use this relationship in circuit analysis. The complex number is called a "phasor". Phasors simplify circuit analysis as we will see in the section on Phasors.

Geogebra link: `https://tube.geogebra.org/m/tanbujyz`

3 Phasors

After completing this section, students should be able to do the following.

- Remember the cosine-referenced phasor transformation equation.
- Apply phasor transformation to transform time-domain circuit equations for a resistor, capacitor, and inductor to the frequency domain.
- Apply phasor transformation to transform time-domain sinusoidal signals to frequency-domain signals (phasors) and vice versa, transform phasors to time-domain.
- Explain the phasors theory in terms of a superposition of two signals.
- Solve a simple electric circuit using phasors.
- Sketch vector phasor diagram to visualize voltage and current addition.
- Visualize and sketch sinusoidal voltages and currents in time-domain for simple RC, RL circuits.
- Recognize a simple circuit from the visualized time-domain or phasor voltages and currents.

3.1 Review of Phasors

Phasors are essential tool in circuit analysis, used in many applications. Phasors are a special case of superposition, that simplifies circuit analysis.

Definition 12. *Phasor transformation is defined as follows:*

$$v(t) = \Re\{|V|e^{j\theta_v}e^{j\omega t}\} \tag{24}$$

where $|\widetilde{V}|e^{j\theta_v}$ is the phasor of voltage $v(t)$. Phasor is a complex number in polar coordinate system and it is usually denoted with a capital letter with a tilde above it $\widetilde{V} = |\widetilde{V}|e^{j\theta_v}$. $|\widetilde{V}|$ is the magnitude and θ_v is the phase of the complex number. Symbol $\Re$ represents the real part of the expression in the curly brackets.

In order to use phasors, the circuit has to be linear. Circuits that have only capacitors, inductors and resistors are linear circuits. In a linear circuit, all currents and voltages are at the frequency of the generator. That means that we do not have to keep track of the frequency of voltages and currents when we are solving the circuit. We know the frequency of all currents and voltages once we know the frequency of the generator. The quantities that will differ for different currents and voltages are the amplitudes and phases. Phasors allow us to drop the information about the signal's frequency and only keep track of the magnitude and phase of the signal. So $cos(\omega t)$ is an integral part of the circuit performance; however, it is not critical to analyze the circuit if all voltages and currents are at that frequency.

We have to use complex numbers to remove $\cos(\omega t)$ term from the generator and other circuit analysis equations. To transform the cosine function to a complex number, we add a sinusoidal imaginary term.

$$V\cos(\omega t + \Theta) + jV sin(\omega t + \Theta) \tag{25}$$

To simplify the math, we have added another generator to our circuit $jAsin(\omega t + \Theta)$, and all currents and voltages in the circuit will be a response to both cosine and sine generator. At the end of the analysis, we have to make sure that we only take the real part of the final expression after the complex number calculations. Using Euler's identity, the expression becomes

$$\begin{aligned} v_s(t) = V\cos(\omega t + \Theta_V) = \Re\{V cos(\omega t + \Theta) + jV sin(\omega t + \Theta)\} = \\ = \Re\{Ve^{j(\omega t + \Theta)}\} = \Re\{Ve^{j\Theta}e^{j\omega t}\} \end{aligned} \tag{26}$$

In Equation 52, we extracted the phase and amplitude information and separated it from the frequency. The amplitude and phase information is called phasor $V_S(j\omega) = Ae^{j\Theta}$.

$$v(t) = \Re\{V cos(\omega t + \Theta_V) + jV \sin(\omega t + \Theta_V)\} = \\ = \Re\{Ve^{j(\omega t + \Theta_V)}\} = \Re\{Ve^{j\Theta_V}e^{j\omega t}\} \tag{27}$$

If we look at the first and last expression in Equation 27, we see that the time domain signal is the real part of the product of phasor and the $e^{j\omega t}$ term.

$$v(t) = \Re\{Ve^{j\Theta_V}e^{j\omega t}\} \tag{28}$$

You still may be wondering why would this specific equation be used. This equation is an applied principle of superposition. In circuits classes, we use superposition to find voltages and currents from two sources. In this case, we add another imaginary source to simplify the math, and when we calculate our currents and voltages, we only take the current and voltage from the real part of the source, the cosine function.

We add a voltage source to our circuit and denote it as imaginary by multiplying it with $j = \sqrt{-1}$. Then we use Euler's formula to write a sinusoidal signal in terms of complex exponentials. When we use complex exponentials, the time-domain differential equations transform into simple algebraic equations. The principle of superposition and complex numbers allow us to separate the voltages and currents in the circuit. The circuit's response to the real part of the generator will be real, and the response of the imaginary part of the generator will be imaginary. Since we added a source that was not there previously, we now need to take the response only from the real generator.

In this section, we apply the phasor transformation to various circuit components, and why we can drop the exponential $e^{j\omega t}$ when analyzing circuits.

Phasor Transformation of Voltage

Sinusoidal voltage sources, or any other sinusoidal voltages are described as shown in Equation 29.

$$v(t) = V cos(\omega t + \Theta_V) \tag{29}$$

To convert the voltage source to a phasor, we add the imaginary sinusoidal voltage with the same amplitude and phase and write $\Re\{\}$ to select only the real part of this expression.

$$v(t) = \Re\{V\cos(\omega t + \Theta_V) + j\sin(\omega t + \Theta_V)\} \tag{30}$$

Using Euler's formula, we can then re-write the Equation 30 for voltage in time domain as:

$$v(t) = \Re\{|V|e^{j\theta_V}e^{j\omega t}\} \tag{31}$$

The phasor of the voltage source is therefore

$$\tilde{V} = |V|e^{j\theta_V} \tag{32}$$

Phasor Transformation of Current

Current source in the time domain

$$i(t) = Icos(\omega t + \Theta_I) \tag{33}$$

Using similar consideration as in the section above, we can write the equation for current in the time domain as:

$$i(t) = \Re\{|I|e^{j\theta_I}e^{j\omega t}\} \tag{34}$$

The phasor of the current is therefore

$$\tilde{I} = |I|e^{j\theta_I} \tag{35}$$

Ohm's Law for Resistor

For a resistor with resistance R, the relationship that describes voltage $v(t)$ on the resistor and current $i(t)$ through the resistor in the time domain is

$$v(t) = R\,i(t) \tag{36}$$

We can substitue the definition of phasors for voltage (Equation31) and current (Equation 34) above to get

$$\Re\{|V|e^{j\theta_V}e^{j\omega t}\} = R\,\Re\{|I|e^{j\theta_I}e^{j\omega t}\} \tag{37}$$

Since resistance R is a real number, we can place it inside the real part of the expression on the right.

$$\Re\{|V|e^{j\theta_V}e^{j\omega t}\} = \Re\{R|I|e^{j\theta_I}e^{j\omega t}\} \tag{38}$$

Now, on both the right and left side of the equation, we have the $\Re\{\}$, and if we want to be precise, we would have to keep this notation throughout our calculations. However, it is easier to drop the $\Re\{\}$, and remember that we have to only take the real part of the resulting voltage in the end. Also, both the left and right sides of the equation have the term $e^{j\omega t}$. We can cancel this term now, but again, when we complete the calculations, and before we take the real part of the voltage we have to multiply it with $e^{j\omega t}$, to get the correct voltage in the time domain.

The final equation for current and voltage in the phasor domain is

$$|V|e^{j\theta_V} = R|I|e^{j\theta_I} \tag{39}$$
$$\tilde{V} = R\tilde{I} \tag{40}$$

Ohm's Law for Capacitor

For a capacitor with capacitance C, the relationship that describes voltage $v(t)$ on the capacitor and current $i(t)$ through the capacitor in the time domain is

$$v_s(t) = \frac{1}{C}\int i(t)dt \tag{41}$$

In order to solve this circuit in the time-domain, we have to solve this integral. The use of phasors simplifies the equations significantly. Differential or integral equations become a set of linear equations. We can substitute the definition of phasors for voltage and current.

$$\Re\{|V|e^{j\theta_V}e^{j\omega t}\} = \frac{1}{C}\int \Re\{Ie^{j\Theta_I}e^{j\omega t}\}dt \tag{42}$$

The voltage on the capacitor is a bit more complicated. What is the integral of $i(t)$? We will look only at the right side of the Equation42. If we take all constants and integral inside the $\Re$, and take time-independent quantities in front of the integral, we get

$$\frac{1}{C}\int i(t)dt = \Re\{\frac{1}{C}\int Ie^{j\Theta_I}e^{j\omega t}\}dt = \Re\{\frac{1}{C}Ie^{j\Theta_I}\int e^{j\omega t}dt\} \tag{43}$$

$$\Re\{\frac{1}{C}Ie^{j\Theta_I}\frac{1}{j\omega}e^{j\omega t}\} = \Re\{\frac{1}{j\omega C}Ie^{j\Theta_I}e^{j\omega t}\} \tag{44}$$

We can now look at both sides of the equation:

$$\Re\{|V|e^{j\theta_V}e^{j\omega t}\} = \Re\{\frac{1}{j\omega C}Ie^{j\Theta_I}e^{j\omega t}\} \tag{45}$$

As in the case of resistors, we drop the common term in the previous equation $e^{j\omega t}$, and we can now drop Re, as long as we later remember to take only the real part of the expression for the phasor of voltage and current to get the time domain expression. We can now write the equation as

$$Ve^{j\Theta_V} = \frac{1}{j\omega C}Ie^{j\Theta_I} \tag{46}$$

$$\tilde{V} = \frac{1}{j\omega C}\tilde{I} \tag{47}$$

We now define expression $Z = \dfrac{1}{j\omega C}$ as the impedance of the capacitor. $X_C = \dfrac{1}{\omega C}$ is called the reactance of the capacitor.

Ohm's Law for Inductor

In case we have an inductor in the circuit, the voltage on an inductor can be derived, as shown in Equation 48.

$$\begin{aligned} v_L(t) = L\frac{\partial i(t)}{\partial t} = L\frac{Re\{Ie^{j\Theta_I}e^{j\omega t}\}}{\partial t}dt = \Re\{LIe^{j\Theta_I}\frac{\partial e^{j\omega t}}{\partial t}\} = \\ = \Re\{LIe^{j\Theta_I}j\omega e^{j\omega t}\} = \Re\{j\omega LIe^{j\Theta_I}e^{j\omega t}\} \end{aligned} \tag{48}$$

The voltage on the inductor in the phasor domain is then

$$\tilde{V} = j\omega L\tilde{I} \tag{49}$$

Where $Z = j\omega L$ is the impedance of inductor, and $X_c = \omega L$ is reactance of inductor.

Analysis of inductor and capacitor impedance

The following table analyzes the impedance of a capacitor and inductor as a function of frequency by replacing the angular frequency $\omega = 2\pi f$ with very small or large numbers. We see a capacitor acting as an open circuit at low frequencies and short circuit at high frequencies. An inductor acts as a short circuit at low frequencies and an open circuit at high frequencies. We often use this reasoning in filter analysis.

circuit element	impedance	low freq $f \to 0$	high freq $f \to \inf$
capacitor	$\frac{1}{j\omega C}$	∞	0
inductor	$j\omega L$	0	∞

Table 1: Impedance of the capacitor and inductor and their equivalent impedances at high and low frequencies.

3.2 Kirchoff's Laws

Phasors are essential tool in circuit analysis.

In this section, we apply the phasor transformation to an RC circuit shown in Figure 214. To solve this circuit in the time domain, we apply Kirchoff's voltage law, as shown in Equation 50 -51.

The circuit in Figure 14 is a simple RC circuit. Equation 50 shows the KVL the time domain.

$$v_s(t) = v_R(t) + v_C(t) \tag{50}$$

$$v_s(t) = Ri + \frac{1}{C}\int i(t)dt \tag{51}$$

As we discussed in the previous section, we will be using the principle of superposition, and add another generator to the circuit, as shown in Figure 14.

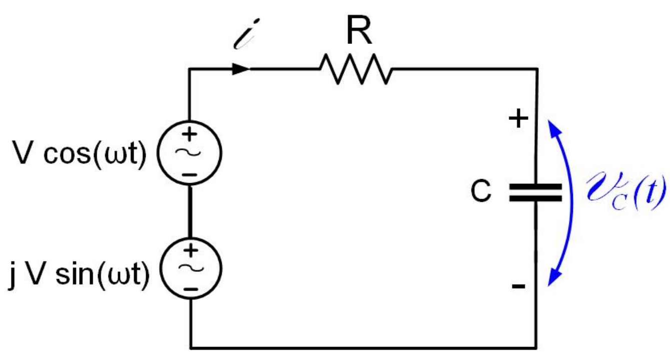


Figure 14: Using superposition to find phasors of voltages and currents in an RC circuit.

The generator we originally had in the circuit is now just the real part of the phasor expression shown in Equation 52.

$$\begin{aligned} v_s(t) = V\cos(\omega t + \Theta_V) = \Re\{V cos(\omega t + \Theta) + jV sin(\omega t + \Theta)\} = \\ = \Re\{V e^{j(\omega t + \Theta)}\} = \Re\{V e^{j\Theta} e^{j\omega t}\} \end{aligned} \tag{52}$$

We can now use the analysis from the previous section to replace the time-domain quantities in equation 51 with these newly developed expressions.

$$v_s(t) = v_R(t) + v_C(t) \tag{53}$$
$$Re\{V_s e^{j\Theta_V} e^{j\omega t}\} = \Re\{R\,I e^{j\Theta_I} e^{j\omega t}\} + \Re\{\frac{1}{j\omega C} I e^{j\Theta_I} e^{j\omega t}\} \tag{54}$$

A common term in the previous equation is $e^{j\omega t}$, and we can now drop Re, as long as we later remember to take only the real part of the expression for the voltage and current phasors to get the time domain expression. We can now write the equation as

$$V_s e^{j\Theta_V} = R\,I e^{j\Theta_I} + \frac{1}{j\omega C} I e^{j\Theta_I} \tag{55}$$
$$\tilde{V}_s = R\tilde{I} + \frac{\tilde{I}}{j\omega C} \tag{56}$$

Since this is a linear equation, we can easily solve it:

$$\tilde{I} = \frac{\tilde{V}_s}{R + \frac{1}{j\omega C}} \tag{57}$$

Converting the phasor back to the time domain

In general, if the phasor is $\tilde{V} = |V|e^{j\Theta_V}$, to find the time-domain signal, we first multiply the phasor with $e^{j\omega t}$ term, and then take the real part of it.

$$v(t) = \Re\{\tilde{V} e^{j\omega t}\} \tag{58}$$
$$v(t) = \Re\{|\tilde{V}| e^{j(\omega t + \Theta_V)}\} \tag{59}$$
$$v(t) = \Re\{|\tilde{V}| cos(\omega t + \Theta_V) + j|\tilde{V}| sin(\omega t + \Theta_V)\} \tag{60}$$
$$v(t) = V cos(\omega t + \Theta_V) \tag{61}$$

Example 6. *The phasor of current is $I = 3e^{j45^o}$. Obtain the signal in the time domain.*

Explanation. *To obtain the time-domain signal from the phasor:*

(a) *multiply the phasor $\tilde{I}$ with the $e^{j\omega t}$ term,*

(b) *use Euler's formula*

(c) *take the real part of the expression.*

$$\begin{aligned} i(t) &= \Re\{3e^{j45^o}e^{j\omega t}\} = \Re\{3e^{j\omega t+45^o}\} = \\ &= \Re\{3\cos(\omega t+45^o)+j3\sin(\omega t+45^o\} = 3\cos(\omega t+45^o) \end{aligned} \tag{62}$$

Question 12 *The phasor for the voltageis given as $\tilde{V} = 5e^{j\beta}$. Find the expression for the phasor in the time domain.*

Multiple Choice:

(a) $v(t) = 5\cos\beta t$

(b) $v(t) = 5\cos(\omega t+\beta)$

(c) $v(t) = 5\sin\beta t$

3.3 Example of circuit analysis with phasors

Phasors are essential tool in circuit analysis.

Step-by-step instructions on how to solve circuits using phasors:

(a) adopt cosine reference for generator voltage or current.

(b) replace all impedances with their phasor expressions,

(c) write KVL and KCL or use other Network Analysis techniques.

(d) find the phasor expression for the required current or voltage.

(e) calculate phasors

(f) sketch the phasor diagram

(g) multiply the phasor with $e^{j\omega t}$

(h) find the real part of the above expression to get the current in the time domain.

It is customary in Electrical Engineering to use $\cos(\omega t)$ for our time-domain signal, and we call these phasors "cosine reference phasors." If the generator in a circuit is $\sin(\omega t)$, we first have to convert the sin function to a cosine. To do that, subtract 90^o from the phase of the sinusoid, because $\sin(\omega t) = \cos(\omega t - 90^o)$.

Example 7. *Using Vectors to Represent Phasors in an example*

A series RC circuit shown in Figure 15. The circuit is driven with a frequency of 1 GHz $v_s(t) = \cos\omega t$*, and R=1k*Ω*, C=* $\frac{1}{2\pi}10^{-12}$ *F.*

(a) *Identify magnitude, phase, and time-delay of the source voltage.*

(b) *Calculate magnitude, phase, and time-delay of the voltage across the resistor.*

(c) *Calculate magnitude, phase, and time-delay of the voltage across the capacitor.*

(d) *The simulated voltage magnitude across the resistor is about 0.5V, and the simulated voltage across the capacitor is about 0.85V. If we use KVL* $0.85V + 0.5V \neq 1V$*. Why? Look at Figure 18 to answer this question.*

(e) *Sketch the phasor diagram of voltages in the circuit.*

Explanation. *The magnitude of the voltage source is 1; the phase and time-delays are zero. To find the voltages and currents in the circuit, we will follow the process outlined above.*

Solving for voltages and currents in the circuit.

(a) *The generator is already given in terms of the cosine function. The phasor of this voltage is* $\tilde{V}_s = 1$

(b) *replace all impedances with their phasor expressions.* R *is not changed, and the impedance of a capacitor is* $Z_c = \frac{1}{j\omega C}$.

(c) *Use Kirchoff's Voltage law*

$$\tilde{V}_s = R\tilde{I} + \frac{\tilde{I}}{j\omega C} \tag{63}$$

(d) *find the phasor expression for the required current or voltage.*

$$\tilde{I} = \frac{\tilde{V}_s}{R + \frac{1}{j\omega C}} \tag{64}$$

$$\tilde{V_R} = R\tilde{I} = R\frac{\tilde{V}_s}{R + \frac{1}{j\omega C}} \tag{65}$$

$$\tilde{V_C} = Z_c\tilde{I} = \frac{1}{j\omega C}\frac{\tilde{V}_s}{R + \frac{1}{j\omega C}} \tag{66}$$

(e) *calculate the currents and voltages from the above expressions*

$$\tilde{I} = 0.54e^{j58^o}\ \mathrm{mA} \tag{67}$$

$$\tilde{V_R} = 0.54e^{j58^o}\ \mathrm{V} \tag{68}$$

$$\tilde{V_C} = 0.85e^{-j32^o}\ \mathrm{V} \tag{69}$$

(f) *When you calculate the magnitudes and phases of the voltages in the RC circuit, draw three complex numbers to represent all three vectors' magnitude and phases, as in Figure 17. The three points denote the three position-vectors, as shown in Figure 18. We must use vector addition to add the voltages on the resistor and capacitor to obtain the generator's voltage. The magnitudes of voltages do not add simply as real numbers, because the voltages have different phases. Complex numbers magnitudes are added only if their phases are the same; in other words, if the phasors point in the same direction, as is the case for purely resistive circuits. However, if we use vector addition (or complex number addition) in any circuit, we will get that the vectors of voltages add up to the source voltage.*

(g) *multiply the phasor with* $e^{j\omega t}$

$$0.54e^{j58^o}e^{j\omega t} = 0.54e^{j(\omega t+58^o)} \tag{70}$$
$$0.54e^{j58^o}e^{j\omega t} = 0.54e^{j(\omega t+58^o)} \tag{71}$$
$$0.85e^{-j32^o}e^{j\omega t} = 0.85e^{j(\omega t-32^o)} \tag{72}$$

(h) *find the real part of the above expression to get the current in the time domain. Time domain signals are shown in Figure 16.*

$$i(t) = \Re\{0.54e^{j(\omega t+58^o)}\} = 0.54\cos(\omega t + 58^o)\,\mathrm{mA} \tag{73}$$
$$v_R(t) = \Re\{0.54e^{j(\omega t+58^o)}\} = 0.54\cos(\omega t + 58^o)\,\mathrm{V} \tag{74}$$
$$v_C(t) = \Re\{0.85e^{j(\omega t-32^o)}\} = 0.85\cos(\omega t - 32^o)\,\mathrm{V} \tag{75}$$

To find time-delay for each signal, we will use the equation $T = \dfrac{\Theta}{2\pi f}$. *Since angles above are in degrees, we need to convert degrees to radians.* $\dfrac{\Theta_{deg} * 2\pi/360}{2\pi f} = \dfrac{\Theta_{deg}}{360f}$. *Time-delays are as follows:*

$$\tau_i = 160\,\mathrm{ns} \tag{76}$$
$$\tau_V r = 160\,\mathrm{ns} \tag{77}$$
$$\tau_V c = -88\,\mathrm{ns} \tag{78}$$

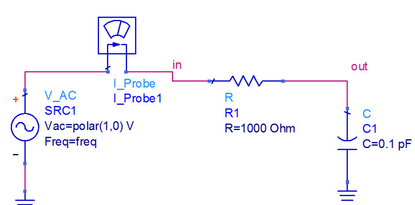


Figure 15: RC circuit in ADS.

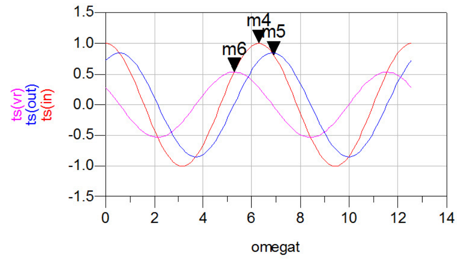


Figure 16: Sinusoidal signal as a function of angle ωt

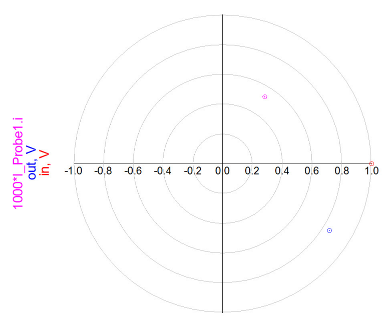


Figure 17: Points represent polar plot of complex voltages in RC circuit.

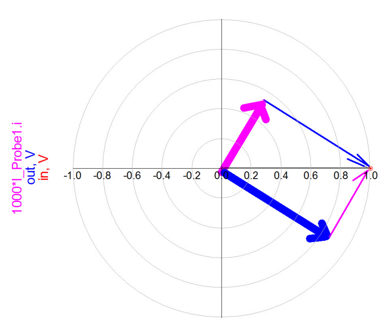


Figure 18: Vectors are drawn in the polar plot of complex voltages in RC circuit. Note how they add up to 1V with vector addition.

Example 8. *Observe the vector phasor diagram for an RC and an RL circuit below. How are the vector diagrams the same, and how are they different when the circuit's parameters change?*

Series RC Circuit

Geogebra link: *https://tube.geogebra.org/m/h4bxa4rk*

Series RL Circuit

Geogebra link: *https://tube.geogebra.org/m/xuq8wchs*

4 Waves on Transmission Lines

After completing this section, students should be able to do the following.

- Name several transmission lines.
- Explain the difference between phase shift and time delay of a sinusoidal signal.
- Evaluate whether the transmission line theory or circuit theory has to be used based on the length of the line and the frequency
- Students will explain the difference between lumped and distributed circuit elements.
- Derive the voltage on a transmission line from a consideration of a time-delay due to the finite speed of signals in a transmission-line circuit.
- Explain parts of propagation constant and what they represent.
- Calculate the phase and attenuation constant for specific transmission lines.
- Identify whether the wave travels in the positive or negative direction from the equation of a wave.
- Describe how signal flows on a transmission line
- Recognize and explain transmission line equivalent circuit model
- Derive the equations for voltage and current waves on a transmission line from the equivalent circuit model.
- Describe forward and reflected wave on a transmission line.
- Sketch forward and reflected wave as a function of distance, and explain how the graph changes as time passes.
- Derive phasor form of voltage and current wave from/to the time-domain form.
- Describe what wavelength represents on a graph of a wave vs. distance
- Explain how the wavelength is similar and different to waves period?
- Derive and calculate the transmission line impedance and reflection coefficient.
- Relate reflection coefficient to impedance.

- Derive and calculate the input impedance of a transmission line
- Calculate and visualize phasors of forward going voltage and current waves at various points on a transmission line.

4.1 Types of Transmission Lines

Any wire, cable, or line that guides energy from one point to another is a transmission line. Whenever we make a circuit on a breadboard, every wire attached forms a transmission line with the ground wire. Whether we see the propagation (transmission line) effects on the line depends on the line length, and the frequency of the signals used. At lower frequencies or short line lengths, we do not see any difference between the signal's phase at the generator and the load, whereas we do at higher frequencies.

Types of transmission lines

(a) Coaxial Cable, Figure 19

(b) Microstrip, Figure 20

(c) Stripline, Figure 21

(d) Coplanar Waveguide, Figure 22

(e) Two-wire line, Figure 23

(f) Parallel Plate Waveguide, Figure 24

(g) Rectangular Waveguide, Figure 25

(h) Optical fiber, Figure 26

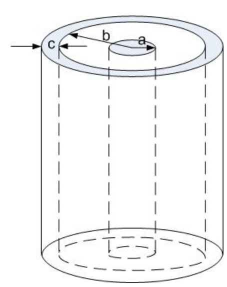


Figure 19: Coaxial Cable

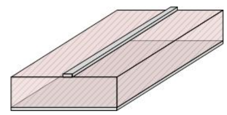


Figure 20: Microstrip

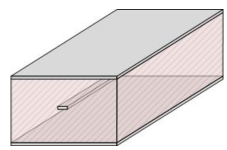


Figure 21: Stripline.

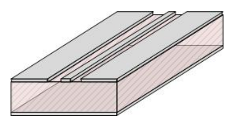


Figure 22: Coplanar Waveguide.

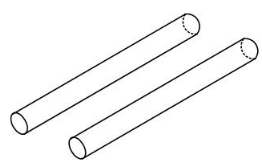


Figure 23: Two-wire line.

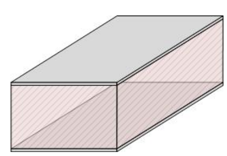


Figure 24: Parallel-Plate Waveguide.

Propagation modes on a transmission line

Coax, two-wire line, transmission line support TEM waves. Waves on microstrip lines can be approximated as TEM up to the 30-40 GHz (unshielded), up to 140 GHz shielded.

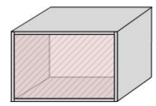


Figure 25: Rectangular Waveguide.

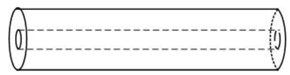


Figure 26: Optical Fiber.

(a) Transversal Electro-Magnetic Field (TEM). Electric (E), and Magnetic (M) fields are entirely transversal to the direction of propagation

(b) Transversal Electric field (TE), Transversal Magnetic Field (TM), M or E field is in the direction of propagation

Transmission lines we will discuss in this course carry TEM fields.

4.2 Wave Equation

Wave equation on a transmission line

In this section, we will derive the expression for voltage and current on a transmission line. This expression will have two variables, time t, and space z. So far, we have only seen voltages and currents as a function of time, because all circuit elements seen so far are lumped elements. In distributed systems, we want to derive the equations for voltage and current for the case when the transmission line is longer than the fraction of a wavelength. To make sure that we do not encounter any transmission line effects to start with, we can look at the piece of a transmission line that is much smaller than the fraction of a wavelength. In other words, we cut the transmission line into small pieces to make sure there are no transmission line effects, as the pieces are shorter than the fraction of a wavelength. We then represent each piece with an equivalent circuit, as shown in Figure 29 (a). To derive expressions for current and voltage on the transmission line, we will use the following five-step plan

(a) Look at an infinitesimal length of a transmission line Δz.

(b) Represent that piece with an equivalent circuit.

(c) Write KCL, KVL for the piece in the time domain (we get differential equations)

(d) Apply phasors (equations become linear)

(e) Solve the linear system of equations to get the expression for the voltage and current on the transmission line as a function of z.

Look at a small piece of a transmission line and represented it with an equivalent circuit. What is modeled by the circuit elements?

Write KVL and KCL equations for the circuit above.

KVL

$$-v(z,t) + R\,\Delta z\, i(z,t) + L\,\Delta z\,\frac{\partial i(z,t)}{\partial t} + v(z+\Delta z,t) = 0$$

KCL

$$i(z,t) = i(z+\Delta z) + i_{CG}(z+\Delta z,t)$$
$$i(z,t) = i(z+\Delta z) + G\,\Delta z\, v(z+\Delta z,t) + C\,\Delta z\frac{\partial v(z+\Delta z,t)}{\partial t}$$

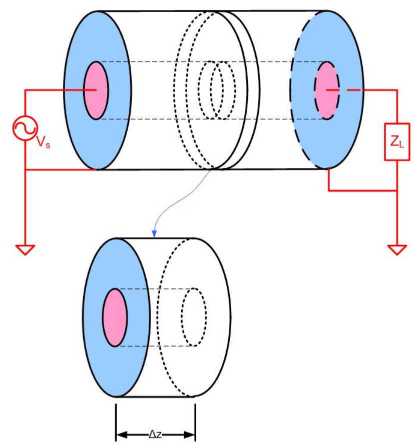


Figure 27: Coaxial cable is cut in short pieces.

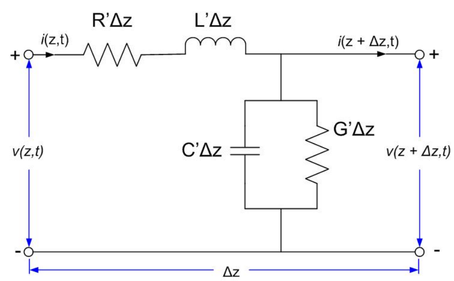


Figure 28: Equivalent circuit of a section of transmission line.

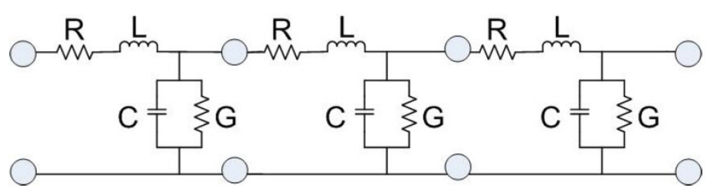


Figure 29: Equivalent circuit of transmission line.

Rearrange the KCL and KVL Equations 79, 83 and divide them with Δz. Equations 80, 84. let $\Delta z \to 0$ and recognize the definition of the derivative Equations, 81, 85.

KVL

$$-(v(z+\Delta z,t)-v(z,t)) = R\,\Delta z\, i(z,t) + L\,\Delta z \frac{\partial i(z,t)}{\partial t} \tag{79}$$

$$-\frac{v(z+\Delta z,t)-v(z,t)}{\Delta z} = R\, i(z,t) + L\,\frac{\partial i(z,t)}{\partial t} \tag{80}$$

$$\lim_{\Delta z \to 0}\{-\frac{v(z+\Delta z,t)-v(z,t)}{\Delta z}\} = \lim_{\Delta z \to 0}\{R\, i(z,t) + L\,\frac{\partial i(z,t)}{\partial t}\} \tag{81}$$

$$-\frac{v(z,t)}{\partial z} = Ri(z,t) + L\frac{\partial i(z,t)}{\partial t} \tag{82}$$

KCL

$$-(i(z+\Delta z,t)-i(z,t)) = G\,\Delta z\, v(z+\Delta z,t) + C\,\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} \tag{83}$$

$$-\frac{i(z+\Delta z,t)-i(z,t)}{\Delta z} = G\, v(z+\Delta z,t) + C\,\frac{\partial v(z+\Delta z,t)}{\partial t} \tag{84}$$

$$\lim_{\Delta z \to 0}\{-\frac{i(z+\Delta z,t)-i(z,t)}{\Delta z}\} = \lim_{\Delta z \to 0}\{G\, v(z+\Delta z,t) + C\,\frac{\partial v(z+\Delta z,t)}{\partial t}\} \tag{85}$$

$$-\frac{i(z,t)}{\partial z} = G\, v(z+\Delta z,t) + C\,\frac{\partial v(z+\Delta z,t)}{\partial t} \tag{86}$$

We just derived Telegrapher's equations in time-domain:

$$-\frac{v(z,t)}{\partial z} = R\, i(z,t) + L\,\frac{\partial i(z,t)}{\partial t}$$
$$-\frac{i(z,t)}{\partial z} = G\, v(z+\Delta z,t) + C\,\frac{\partial v(z+\Delta z,t)}{\partial t}$$

Telegrapher's equations are two differential equations with two unknowns, $i(z,t)$, $v(z,t)$. It is not impossible to solve them; however, we would prefer to have linear algebraic equations. We then express time-domain variables as phasors.

$$v(z,t) = Re\{\tilde{V}(z)e^{j\omega t}\}$$
$$i(z,t) = Re\{\tilde{I}(z)e^{j\omega t}\}$$

$\tilde{V}(z), \tilde{I}(z)$ are the voltage, and current anywhere on the line, and they depend on the position on the line z. The Telegrapher's equations in phasor form are

$$-\frac{\partial \tilde{V}(z)}{\partial z} = (R + j\omega L)\tilde{I}(z) \tag{87}$$
$$-\frac{\partial \tilde{I}(z)}{\partial z} = (G + j\omega C)\tilde{V}(z) \tag{88}$$

Two equations, two unknowns. To solve these equations, we first take a derivative of both equations with respect to z.

$$-\frac{\partial^2 \tilde{V}(z)}{\partial z^2} = (R + j\omega L)\frac{\partial \tilde{I}(z)}{\partial z} \tag{89}$$
$$-\frac{\partial^2 \tilde{I}(z)}{\partial z^2} = (G + j\omega C)\frac{\partial \tilde{V}(z)}{\partial z} \tag{90}$$

Rearange the previous equations:

$$-\frac{1}{(R + j\omega L)}\frac{\partial \tilde{I}(z)}{\partial z} = \frac{\partial^2 \tilde{V}(z)}{\partial z^2} \tag{91}$$
$$-\frac{1}{(G + j\omega C)}\frac{\partial \tilde{V}(z)}{\partial z} = \frac{\partial^2 \tilde{I}(z)}{\partial z^2} \tag{92}$$

Substitute Eq.91 into Eq.88 and Eq.92 into Eq.87 and we get

$$-\frac{\partial^2 \tilde{V}(z)}{\partial z^2} = (G + j\omega C)(R + j\omega L)\tilde{V}(z) \tag{93}$$
$$-\frac{\partial^2 \tilde{I}(z)}{\partial z^2} = (G + j\omega C)(R + j\omega L)\tilde{I}(z) \tag{94}$$

Or if we rearrange

$$\frac{\partial^2 \tilde{V}(z)}{\partial z^2} - (G + j\omega C)(R + j\omega L)\tilde{V}(z) = 0 \tag{95}$$
$$\frac{\partial^2 I(z)}{\partial z^2} - (G + j\omega C)(R + j\omega L)I(z) = 0 \tag{96}$$

The above Equations 95-96 are called wave equations, and they represent current and voltage wave on a transmission line. $\gamma^2 = (G+j\omega C)(R+j\omega L)$ is the complex propagation constant. This constant has a real and an imaginary part.

$$\gamma = \alpha + j\beta$$

where α is the attenuation constant and β is the phase constant.

$$\alpha = Re\{\sqrt{(G + j\omega C)(R + j\omega L)}\}$$
$$\beta = Im\{\sqrt{(G + j\omega C)(R + j\omega L)}\}$$

We can now write wave equations as:

$$\frac{\partial^2 \tilde{V}(z)}{\partial z^2} - \gamma^2 \tilde{V}(z) = 0 \tag{97}$$
$$\frac{\partial^2 I(z)}{\partial z^2} - \gamma^2 I(z) = 0 \tag{98}$$

The general solution of the second order differential equations with constant coefficients Equations 97 - 98 is:

$$\tilde{V}(z) = \tilde{V}_0^+ e^{-\gamma z} + \tilde{V}_0^- e^{\gamma z}$$
$$\tilde{I}(z) = \tilde{I}_0^+ e^{-\gamma z} + \tilde{I}_0^- e^{\gamma z}$$

Where $\tilde{V}(z)^+ = \tilde{V}_0^+ e^{-\gamma z}$ represents the forward-going voltage wave, $\tilde{V}(z)^- = \tilde{V}_0^- e^{\gamma z}$ represents the reflected voltage wave, $\tilde{I}(z)^+ = \tilde{I}_0^+ e^{-\gamma z}$ represents the forward going current wave, and $\tilde{I}(z)^- = \tilde{I}_0^- e^{\gamma z}$ represents the reflected current wave. We will see later that $\tilde{V}_0^+$ is the forward-going voltage wave at the load, $\tilde{V}_0^-$ is the reflected voltage wave at the load, $\tilde{I}_0^+$ is the forward going current at the load, and $\tilde{I}_0^-$ reflected current at the load.

In the next several sections, we will look at how to find the constants β, $\tilde{V}_0^+$, $\tilde{V}_0^-$, $\tilde{I}_0^+$, $\tilde{I}_0^-$. In order to find the constants, we will introduce the concepts of transmission line impedance Z_0, reflection coefficient $\Gamma(z)$, input impedance Z_{in}.

4.3 Visualization of waves on lossless transmission lines

$$\tilde{V}(z) = \tilde{V}_0^+ e^{-\gamma z} + \tilde{V}_0^- e^{\gamma z}$$
$$\tilde{I}(z) = \tilde{I}_0^+ e^{-\gamma z} + \tilde{I}_0^- e^{\gamma z}$$

In this equation $\tilde{V}(z)$ is the total voltage anywhere on the line (at any point z), $\tilde{I}(z)$ is the total current anywhere on the line (at any point z), $\tilde{V_0}^+$ and $\tilde{V_0}^-$ are the **phasors** of forward and reflected voltage waves at the load (where z=0), and $\tilde{I_0}^+$ and $\tilde{I_0}^-$ are the phasors of forward and reflected current wave at the load (where z=0).These voltages and currents are also phasors and have a constant magnitude and phase in a specific circuit, for example $\tilde{V_0}^+ = |\tilde{V_0}^+|e^{\Phi} = 4e^{25^0}$, and $\tilde{I_0}^+ = |\tilde{I_0}^+|e^{\Phi} = 5e^{-40^0}$. We can get the time-domain expression for the current and voltage on the transmission line by multiplying the phasor of the voltage and current with $e^{j\omega t}$ and taking the real part of it.

$$v(t) = Re\{(\tilde{V}_0^+ e^{(-\alpha - j\beta)z} + \tilde{V}_0^- e^{(\alpha + j\beta)z})e^{j\omega t}\}$$
$$v(t) = |\tilde{V}_0^+|e^{-\alpha z}\cos(\omega t - \beta z + \angle\tilde{V}_0^+) + |\tilde{V}_0^-|e^{\alpha z}\cos(\omega t + \beta z + \angle\tilde{V}_0^-) \tag{99}$$

If the signs of the ωt and βz terms are oposite the wave moves in the forward $+z$ direction. If the signs of ωt and βz are the same, the wave moves in the $-z$ direction.

In the next several sections, we will look at how to find the constants β, $\tilde{V}_0^+$, $\tilde{V}_0^-$, $\tilde{I}_0^+$, $\tilde{I}_0^-$. In order to find the constants, we will introduce the concepts of transmission line impedance Z_0, reflection coefficient $\Gamma(z)$, input impedance Z_{in}.

Example 9. *We will show next that if the signs of the ωt and βz have the opposite sign, as in Equation 100, the wave moves in the forward $+z$ direction. If the signs of ωt and βz are the same, as in Equation 101, the wave moves in the $-z$ direction. In order to see this, we will visualize Equations 100 and 101 using Matlab code below.*

$$v_f(t) = |\tilde{V}_0^+|e^{-\alpha z}\cos(\omega t - \beta z + \angle\tilde{V}_0^+) \tag{100}$$
$$v_r(t) = |\tilde{V}_0^-|e^{\alpha z}\cos(\omega t + \beta z + \angle\tilde{V}_0^-) \tag{101}$$

Explanation. *Figure 30 shows forward and reflected waves on a transmission line. z represents the length of the line, and on the y-axis is the magnitude of* *the voltage. The red line on both graphs is the voltage signal at a time .1 ns. We would obtain Figure 30 if we had a camera that can take a picture of the voltage, and we took the first picture at* $t_1 = .1$ *ns on the entire transmission line. The blue dotted line on both graphs is the same signal .1 ns later, at time* $t_2 = .2$ *ns. We see that the signal has moved to the right in 1 ns, from the generator to the load. On the bottom graph, we see that at a time .1 ns, the red line represents the reflected signal. The dashed blue line shows the signal at a time .2 ns. We see that the signal has moved to the left, from the load to the generator.*

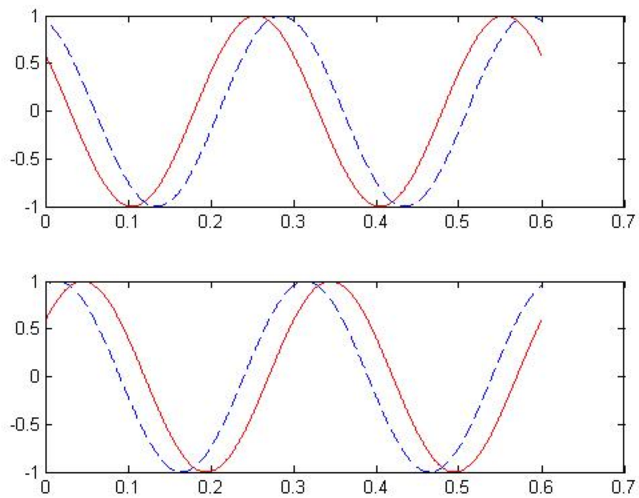


Figure 30: Forward (top) and reflected (bottom) waves on a transmission line.

```
clear all
clc
f = 10^9;
w = 2*pi*f
c=3*10^8;
beta=2*pi*f/c;
lambda=c/f;
t1=0.1*10^(-9)
t2=0.2*10^(-9)
x=0:lambda/20:2*lambda;

y1=sin(w*t1 - beta.*x);
y2=sin(w*t2 - beta.*x);
y3=sin(w*t1 + beta.*x);
y4=sin(w*t2 + beta.*x);
```

```
subplot (2,1,1),

    plot(x,y1,'r'),...
            hold on
    plot(x,y2,'--b'),...
    hold off
subplot (2,1,2),

    plot(x,y3,'r')
        hold on
    plot(x,y4,'--b')
    hold off
```

Using Matlab code above, repeat the visualization of signals in the previous section for a lossy transmission line. Assume that $\alpha = 0.1\,Np$, and all other variables are the same as in the previous section. How do the voltages compare in the lossy and lossless cases?

Question 13 *In the following simulation, we have three waves as a function of distance z. One is fixed $\cos(\beta z + 0^0)$ with a constant phase of 0^0, and for the other two signals the phase can be changed manually by changing the slider t that represents time. In the simulation, $\beta = 1$ and $\omega = 1$. This simulation is realistic only if time moves forward from 0 to 5. Observe how phase change ωt as the time increases from 0 to 5, then answer the question below.*

Geogebra link: `https://tube.geogebra.org/m/x5q7p7jx`

The sign in front of βz and ωt is opposite for the forward going wave.

Multiple Choice:

(a) *True*

(b) *False*

4.4 Propagation constant and loss

Lossless transmission line

In many practical applications, conductor loss is low $R \to 0$, and dielectric leakage is low $G \to 0$. These two conditions describe a lossless transmission line.

In this case, the transmission line parameters are

- Propagation constant

$$\begin{aligned} \gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} \\ \gamma &= \sqrt{j\omega L j\omega C} \\ \gamma &= j\omega\sqrt{LC} = j\beta \end{aligned}$$

- Transmission line impedance will be defined in the next section, but it is also here for completeness.

$$\begin{aligned} Z_0 &= \sqrt{\frac{R + j\omega L}{G + j\omega C}} \\ Z_0 &= \sqrt{\frac{j\omega L}{j\omega C}} \\ Z_0 &= \sqrt{\frac{L}{C}} \end{aligned}$$

- Wave velocity

$$\begin{aligned} v &= \frac{\omega}{\beta} \\ v &= \frac{\omega}{\omega\sqrt{LC}} \\ v &= \frac{1}{\sqrt{LC}} \end{aligned}$$

- Wavelength

$$\lambda = \frac{2\pi}{\beta}$$
$$\lambda = \frac{2\pi}{\omega\sqrt{LC}}$$
$$\lambda = \frac{2\pi}{\sqrt{\varepsilon_0\mu_0\varepsilon_r}}$$
$$\lambda = \frac{c}{f\sqrt{\varepsilon_r}}$$
$$\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_r}}$$

Voltage and current on lossless transmission line

On a lossless transmission line, where $\gamma = j\beta$ current and voltage simplify to

$$\tilde{V}(z) = \tilde{V}_0^+ e^{-j\beta z} + \tilde{V}_0^- e^{j\beta z}$$
$$\tilde{I}(z) = \tilde{I}_0^+ e^{-j\beta z} + \tilde{I}_0^- e^{j\beta z}$$

What does it mean when we say a medium is lossy or lossless?

In a lossless medium, electromagnetic energy is not turning into heat; there is no amplitude loss. An electromagnetic wave is heating a lossy material; therefore, the wave's amplitude decreases as $e^{-\alpha x}$.

medium	attenuation constant α [dB/km]
coax	60
waveguide	2
fiber-optic	0.5

In guided wave systems such as transmission lines and waveguides, the attenuation of power with distance follows approximately $e^{-2\alpha x}$. The power radiated by an antenna falls off as $1/r^2$. As the distance between the source and load increases, there is a specific distance at which the cable transmission is lossier than antenna transmission.

Low-Loss Transmission Line

This section is optional.

In some practical applications, losses are small, but not negligible. $R << \omega L$ [1]and $G << \omega C$[2].

In this case, the transmission line parameters are

- Propagation constant

  We can re-write the propagation constant as shown below. In somel applications, losses are small, but not negligible. $R << \omega L$ and $G << \omega C$, then in Equation 103, $RG << \omega^2 LC$.

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \tag{102}$$

$$\gamma = j\omega\sqrt{L\,C}\sqrt{1 \,-\, j\,\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right) - \frac{RG}{\omega^2 LC}} \tag{103}$$

$$\gamma \approx j\omega\sqrt{L\,C}\sqrt{1 \,-\, j\,\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right)} \tag{104}$$

  Taylor's series for function $\sqrt{1+x} = \sqrt{1 \,-\, j\,\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right)}$ in Equation 104 is shown in Equations 105-106.

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - ...\, for\, |x| < 1 \tag{105}$$

$$\gamma \approx j\omega\sqrt{L\,C}\sqrt{1 \,-\, j\,\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right)} = j\omega\sqrt{L\,C}\left(1 - \frac{j}{2}\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right)\right) \tag{106}$$

  The real and imaginary part of the propagation constant γ are:

$$\alpha = \frac{\omega\sqrt{L\,C}}{2}\left(\frac{R}{\omega L} + \frac{G}{\omega C}\right) \tag{107}$$

$$\beta = \omega\sqrt{L\,C} \tag{108}$$

  We see that the phase constant β is the same as in the lossless case, and the attenuation constant α is frequency independent. All frequencies

[1] metal resistance is lower than the inductive impedance

[2] dielectric conductance is lower than the capacitive impedance

of a modulated signal are attenuated the same amount, and there is no dispersion on the line. When the phase constant is a linear function of frequency, $\beta = const\,\omega$, then the phase velocity is a constant $v_p = \frac{\omega}{\beta} = \frac{1}{const}$, and the group velocity is also a constant, and equal to the phase velocity. In this case, all frequencies of the modulated signal propagate at the same speed, and there is no distortion of the signal.

Transmission-line parameters R, G, C, and L

To find the complex propagation constant γ, we need the transmission-line parameters R, G, C, and L. Equations for R, G, C, and L for a coaxial cable are given in the table below.

Transmission-line	R	G	C	L
Coaxial Cable	$\frac{R_{sd}}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$	$\frac{2\pi\sigma}{\ln b/a}$	$\frac{2\pi\varepsilon}{\ln b/a}$	$\frac{\mu}{2\pi}\ln b/a$

Where $R_{sd} = \sqrt{\pi f \mu_m/\sigma_m}$ is the resistance associated with skin-depth. f is the frequency of the signal, μ_m is the magentic permeability of conductors, σ_c is the conductivity of conductors.

Example 10. *Calculate capacitance per unit length of a coaxial cable if the inner radius is 0.02 m, the outer radius is 0.06 m, the dielectric constant is $\varepsilon_r = 2$. Use the applet below, Matlab, Matematica, or other software that you use.*

Geogebra link: `https://tube.geogebra.org/m/whkrg2pu`

4.5 Transmission Line Impedance

This section will relate the phasors of voltage and current waves through the transmission-line impedance.

In equations 109-110 $\tilde{V}_0^+ e^{-\gamma z}$ and $\tilde{V}_0^- e^{\gamma z}$ are the phasors of forward and reflected going voltage waves anywhere on the transmission line (for any z). $\tilde{I}_0^+ e^{-\gamma z}$ and $\tilde{I}_0^- e^{\gamma z}$ are the phasors of forward and reflected current waves anywhere on the transmission line.

$$\tilde{V}(z) = \tilde{V}_0^+ e^{-\gamma z} + \tilde{V}_0^- e^{\gamma z} \tag{109}$$

$$I(z) = \tilde{I}_0^+ e^{-\gamma z} + \tilde{I}_0^- e^{\gamma z} \tag{110}$$

To find the transmission-line impedance, we first substitute the voltage wave equation 109 into Telegrapher's Equation Eq.111 to obtain Equation 112.

$$-\frac{\partial \tilde{V}(z)}{\partial z} = (R + j\omega L) I(z) \tag{111}$$

$$\gamma \tilde{V}_0^+ e^{-\gamma z} - \gamma \tilde{V}_0^- e^{\gamma z} = (R + j\omega L) I(z) \tag{112}$$

We now rearrange Equation 112 to find the current I(z) and multiply through to get Equation 113.

$$I(z) = \frac{\gamma}{R + j\omega L} (\tilde{V}_0^+ e^{-\gamma z} + \tilde{V}_0^- e^{\gamma z})$$

$$I(z) = \frac{\gamma \tilde{V}_0^+}{R + j\omega L} e^{-\gamma z} - \frac{\gamma \tilde{V}_0^-}{R + j\omega L} e^{\gamma z} \tag{113}$$

We can now compare Equation 110 for current, a solution of the wave equation, with the Eq.113. Since both equations represent current, and for two transcendental equations to be equal, the coefficients next to exponential terms have to be the same. When we equate the coefficients, we get the equations below.

$$\tilde{I}_0^+ = \frac{\gamma \tilde{V}_0^+}{R + j\omega L} \tag{114}$$

$$\tilde{I}_0^- = -\frac{\gamma \tilde{V}_0^-}{R + j\omega L} \tag{115}$$

Definition 13. *We define the characteristic impedance of a transmission line as the ratio of the voltage to the current amplitude of the forward wave as shown in*

Equation 114, or the ratio of the voltage to the current amplitude of the reflected wave as shown in Equation 115.

$$Z_0 = \frac{\tilde{V}_0^+}{\tilde{I}_0^+} = \frac{R + j\omega L}{\gamma} \tag{116}$$

$$Z_0 = -\frac{\tilde{V}_0^-}{\tilde{I}_0^-} = \frac{R + j\omega L}{\gamma} \tag{117}$$

We can further simplify Equations 116-117 to obtain the final Equation 118 for the transmission line impedance. This equation is valid for both lossy and lossless transmission lines.

$$Z_0 = \frac{\tilde{V}_0^+}{\tilde{I}_0^+}$$
$$Z_0 = \frac{R + j\omega L}{\gamma}$$
$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

For lossless transmission line, where $R \to 0$ and $G \to 0$, the equation simplifies to

$$Z_0 = \sqrt{\frac{L}{C}} \tag{118}$$

Equations for voltage and current on a transmission-line

Using the definition of transmission-line impedance Z_0, we can now simplify the Equations 109-110 for voltage and current on the transmission line, by replacing the currents $\tilde{I}_0^+ = \frac{\tilde{V}_0^+}{Z_0}$, and $\tilde{I}_0^- = -\frac{\tilde{V}_0^-}{Z_0}$.

$$\tilde{V}(z) = \tilde{V}_0^+ e^{-j\beta z} + \tilde{V}_0^- e^{j\beta z} \tag{119}$$

$$I(z) = \frac{\tilde{V}_0^+}{Z_0} e^{-j\beta z} - \frac{\tilde{V}_0^-}{Z_0} e^{j\beta z} \tag{120}$$

4.6 Reflection Coefficient

In this section, we will derive the equation for the reflection coefficient. The reflection coefficient relates the forward-going voltage with reflected voltage.

Reflection coefficient at the load

Equations 121-122 represent the voltage and current on a lossless transmission line shown in Figure 31.

$$\tilde{V}(z) = \tilde{V}_0^+ e^{-j\beta z} + \tilde{V}_0^- e^{j\beta z} \tag{121}$$

$$I(z) = \frac{\tilde{V}_0^+}{Z_0} e^{-j\beta z} - \frac{\tilde{V}_0^-}{Z_0} e^{j\beta z} \tag{122}$$

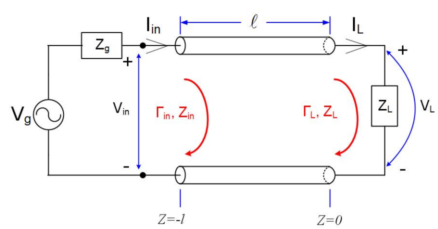


Figure 31: Transmission Line connects generator and the load.

We set up the z-axis so that the $z = 0$ is at the load, and the generator is at $z = -l$. At $z = 0$, the load impedance is connected. The definition of impedance is $Z = V/I$, therefore at the z=0 end of the transmission line, the voltage and current on the transmission line at that point have to obey boundary condition that the load impedance imposes.

$$Z_L = \frac{V(0)}{I(0)}$$

Substituting z=0, the boundary condition, in Equations 121-122, we get Equations 123-124.

$$\tilde{V}(0) = \tilde{V}_0^+ e^{-j\beta 0} + \tilde{V}_0^- e^{j\beta 0} = \tilde{V}_0^+ + \tilde{V}_0^- \tag{123}$$

$$I(0) = \frac{\tilde{V}_0^+}{Z_0} e^{-j\beta 0} - \frac{\tilde{V}_0^-}{Z_0} e^{j\beta 0} = \frac{\tilde{V}_0^+}{Z_0} - \frac{\tilde{V}_0^-}{Z_0} \tag{124}$$

Dividing the two above equations, we get the impedance at the load.

$$Z_L = Z_0 \frac{\tilde{V}_0^+ + \tilde{V}_0^-}{\tilde{V}_0^+ - \tilde{V}_0^-} \tag{125}$$

We can now solve the above equation for $\tilde{V}_0^-$

$$\begin{aligned}
\frac{Z_L}{Z_0}(\tilde{V}_0^+ - \tilde{V}_0^-) &= \tilde{V}_0^+ + \tilde{V}_0^- \\
(\frac{Z_L}{Z_0} - 1)\tilde{V}_0^+ &= (\frac{Z_L}{Z_0} + 1)\tilde{V}_0^- \\
\frac{\tilde{V}_0^-}{\tilde{V}_0^+} &= \frac{\frac{Z_L}{Z_0} - 1}{\frac{Z_L}{Z_0} + 1} = \frac{z_L - 1}{z_L + 1} \\
\Gamma_L &= \frac{\tilde{V}_0^-}{\tilde{V}_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} \qquad (126)
\end{aligned}$$

Lowercase z_L is called the "normalized load impedance". It is the actual impedance divided by the transmission line impedance $z_L = \frac{Z_L}{Z_0}$. For example, if the load impedance is $Z_L = 100\Omega$, and the transmission-line impedance is $Z) = 50\Omega$, then the normalized impedance is $z_L = \frac{100\Omega}{50\Omega} = 2$. Normalized impedance is a unitless quantity.

Definition 14. $\Gamma_L = \frac{\tilde{V}_0^-}{\tilde{V}_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}$ *is the voltage reflection coefficient at the load.* Γ_L *relates the reflected and incident voltage phasor and the load* Z_L *and transmission line impedance* Z_0*. The voltage reflection coefficient at the load is, in general, a complex number, it has a magnitude and a phase* $\Gamma_L = |\Gamma_L|e^{j\angle\Gamma_L}$.

Example

(a) $100\,\Omega$ transmission line is terminated in a series connection of a $50\,\Omega$ resistor and 10 pF capacitor. The frequency of operation is 100 MHz. Find the voltage reflection coefficient.

(b) For purely reactive load $Z_L = j50\Omega$, find the reflection coefficient.

Voltage and Current on a transmission line

Now that we related forward and reflected voltage on a transmission line with the reflection coefficient at the load, we can re-write the equations for the current and voltage on a transmission line as:

$$\tilde{V}(z) = \tilde{V}_0^+(e^{-j\beta z} + \Gamma_L e^{j\beta z}) \tag{127}$$

$$I(z) = \frac{\tilde{V}_0^+}{Z_0}(e^{-j\beta z} - \Gamma e^{j\beta z}) \tag{128}$$

We see that if we know the length of the line, line type, the load impedance, and the transmission line impedance, we can calculate all variables above, except for $\tilde{V}_0^+$. In the following chapters, we will derive the equation for the forward going voltage at the load, but first, we will look at little more at the various reflection coefficients on a transmission line.

Reflection coefficient anywhere on the line

Equations 121-122 can be concisely written as

$$\tilde{V}(z) = \tilde{V}(z)^+ + \tilde{V}(z)^- \tag{129}$$

$$\tilde{I}(z) = \tilde{I}(z)^+ + \tilde{I}(z)^- \tag{130}$$

Where $\tilde{V}(z)^+$ is the forward voltage anywhere on the line, $\tilde{V}(z)^-$ is reflected voltage anywhere on the line, $\tilde{I}(z)^+$ is the forward current anywhere on the line, and $\tilde{I}(z)^-$ is the reflected current anywhere on the line.

We can then define a reflection coefficient anywhere on the line as

Definition 15. $\Gamma(z) = \dfrac{\tilde{V}(z)^-}{\tilde{V}(z)^+} = \dfrac{\tilde{V}_0^- e^{j\beta z}}{\tilde{V}_0^+ e^{-j\beta z}} = \dfrac{\tilde{V}_0^-}{\tilde{V}_0^+} e^{2j\beta z}$ *is a voltage reflection coefficient anywhere on the line.* $\Gamma(z)$ *relates the reflected and incident voltage phasor at any z.*

Since we already defined $\Gamma_L = \dfrac{\tilde{V}_0^-}{\tilde{V}_0^+}$ *as the reflection coefficient at the load, we can now simplify the general reflection coefficient as*

$$\Gamma(z) = \Gamma_L e^{2j\beta z} \tag{131}$$

It is important to remember that we defined points between the generator and the load as the negative z-axis. If the line length is, for example, l m long, the generator is then at z=-l m, and the load at z=0. To find the reflection coefficient at some distance $l/2$ m away from the load, at $z = -l/2$ m, the equation for the reflection coefficient will be

$$\Gamma(z = -l/2) = \Gamma_L e^{-2j\beta l/2} \tag{132}$$

Since we already defined the reflection coefficient at the load, the reflection at any point on the line $z = -l$ is

$$\Gamma(z = -l) = \Gamma_L e^{-2j\beta l} \tag{133}$$

$$\Gamma(z = -l) = |\Gamma_L| e^{j(\angle\Gamma_L - 2\beta l)} \tag{134}$$

Reflection coefficient at the input of the transmission line

Using the reasoning above, the reflection coefficient at the input of the line whose length is l is

$$\Gamma(z = -l) = \Gamma_{in} = \Gamma_L e^{-2j\beta l} \tag{135}$$

Example 11. *The reflection coefficient at the load is* $\Gamma_L = 0.5e^{j60^0}$. *Find the input reflection coefficient if the electrical length of the line is* $\beta l = 45^0$.

Explanation. *The reflection coefficient at the input of the line is* $\Gamma(z = -l) = \Gamma_{in} = |\Gamma_L| e^{j(\angle\Gamma_L - 2\beta l)}$.

We substitute the expression for $\Gamma_L = 0.5e^{j60^0}$ *and* $\beta l = 90^0$, *we get the reflection coefficient at the input of the line* $\Gamma_{in} = 0.5e^{-j30}$.

4.7 Input impedance of a transmission line

Again, we will look at a transmission line circuit in Figure 32 to find the input impedance on a transmission line.

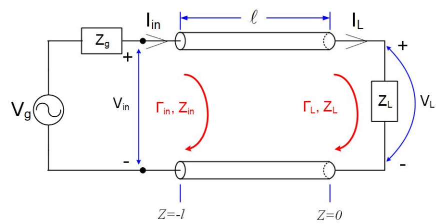


Figure 32: Transmission Line connects generator and the load.

The equations for the voltage and current anywhere (any z) on a transmission line are

$$\tilde{V}(z) = \tilde{V}_0^+(e^{j\beta z} + \Gamma_L e^{-j\beta z}) \tag{136}$$

$$I(z) = \frac{\tilde{V}_0^+}{Z_0}(e^{j\beta z} - \Gamma e^{-j\beta z}) \tag{137}$$

The voltage and current equations at the generator $z = -l$ are:

$$\tilde{V}_{in} = \tilde{V}(z = -l) = \tilde{V}_0^+(e^{j\beta l} + \Gamma_L e^{-j\beta l}) \tag{138}$$

$$\tilde{I}_{in} = I(z = -l) = \frac{\tilde{V}_0^+}{Z_0}(e^{j\beta l} - \Gamma e^{-j\beta l}) \tag{139}$$

Input impedance as a function of reflection coefficient

The input impedance is defined as $Z_{in} = \dfrac{V_{in}}{I_{in}}$. Since the line length is l, the input impedance is

$$Z_{in} = \frac{\tilde{V}_0^+(e^{-j\beta z} + \Gamma_L e^{j\beta z})}{\frac{\tilde{V}_0^+}{Z_0}(e^{-j\beta z} - \Gamma_L e^{j\beta z})} \tag{140}$$

If we cancel common terms, we get

$$Z_{in} = Z_0 \frac{(e^{-j\beta z} + \Gamma_L e^{j\beta z})}{(e^{-j\beta z} - \Gamma_L e^{j\beta z})} \tag{141}$$

Now we can take $e^{-j\beta z}$ in front of parenthesis from both numerator and denominator and then cancel it.

$$Z_{in} = Z_0 \frac{1 + \Gamma_L e^{2j\beta z}}{1 - \Gamma_L e^{2j\beta z}} \tag{142}$$

We have previously defined the reflection coefficient at the transmission line's input as $\Gamma_{in} = \Gamma_L e^{2j\beta z}$. The final equation for the input impedance is therefore

$$Z_{in} = Z_0 \frac{1 + \Gamma_{in}}{1 - \Gamma_{in}} \tag{143}$$

Input impedance as a function of load impedance

If we now look back at the Equation 141, here we can also use Euler's formula $e^{j\beta z} = \cos(\beta z) + j\sin(\beta z)$, and the equation for the reflection coefficient at the load $\Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0}$ we find the input impedance of the line as shown below.

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l} \tag{144}$$

This equation will be soon become obsolete when we learn how to use the Smith Chart.

Example 12. *Find the input impedance if the load impedance is $Z_L = 0\Omega$, and the electrical length of the line is $\beta l = 90^0$.*

Explanation. *Since the load impedance is a short circuit, and the angle is 90^0 the equation simplifies to $Z_{in} = jZ_0 \tan\beta l = \infty$.*

When we find the input impedance, we can replace the transmission line and the load, as shown in Figure 33. In the next section, we will use input impedance to find the forward going voltage on a transmission line.

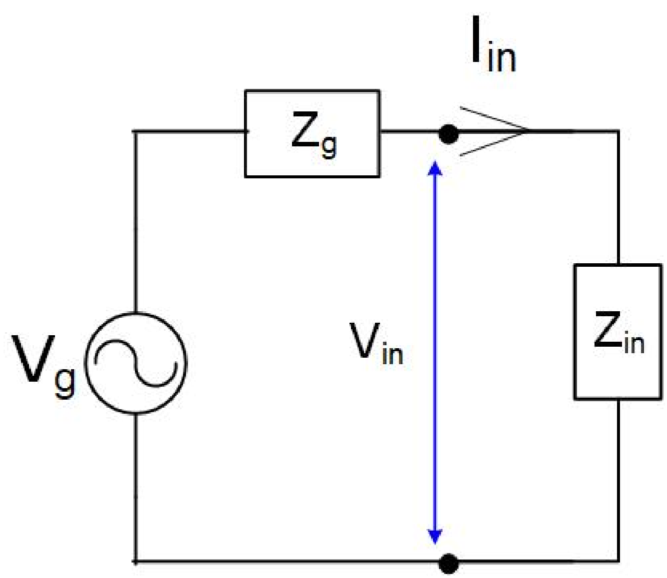


Figure 33: Transmission Line connects generator and the load.

4.8 Forward voltage on a transmission line

Again, we will look at a transmission line circuit in Figure 34 to find the input impedance on a transmission line.

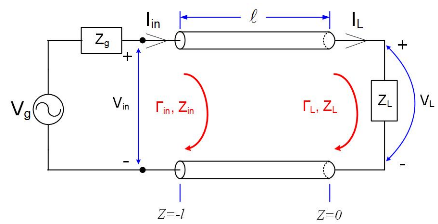


Figure 34: Transmission Line connects generator and the load.

The equations for the voltage and current anywhere (any z) on a transmission line are

$$\tilde{V}(z) = \tilde{V}_0^+(e^{-j\beta z} - \Gamma_L e^{j\beta z}) \tag{145}$$

$$I(z) = \frac{\tilde{V}_0^+}{Z_0}(e^{-j\beta z} - \Gamma_L e^{j\beta z}) \tag{146}$$

Using the equations from the previous section, we can replace the transmission line with its input impedance, Figure 35.

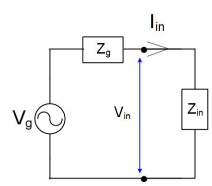


Figure 35: Transmission Line connects generator and the load.

Forward voltage phasor as a function of load impedance

From Figure 35, we can find the input voltage on a transmission line using the voltage divider.

$$\tilde{V}_{in} = \frac{Z_{in}}{Z_{in} + Z_g}\tilde{V}_g \tag{147}$$

Using Equation 146, we can also find the input voltage. The input voltage equation at the generator $z = -l$ is:

$$\tilde{V}_{in} = \tilde{V}(z = -l) = \tilde{V}_0^+(e^{j\beta l} + \Gamma_L e^{-j\beta l}) \tag{148}$$

Since these two equations represent the same input voltage we can make them equal.

$$\tilde{V}_0^+(e^{j\beta l} + \Gamma_L e^{-j\beta l}) = \frac{Z_{in}}{Z_{in} + Z_g}\tilde{V}_g \tag{149}$$

Rearranging the equation, we find $\tilde{V}_0^+$.

$$\tilde{V}_0^+ = \frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}} \frac{1}{e^{j\beta l} + \Gamma_L e^{-j\beta l}} \tag{150}$$

$$\tag{151}$$

Forward voltage phasor as a function of input reflection coefficient

There is another way to find the input impedance as a function of the input reflection coefficient.

We write KVL for the circuit in Figure 35.

$$\tilde{V}_g = Z_g \tilde{I}_{in} + \tilde{V}_{in} \tag{152}$$

Using Equations 146-146, we can also find the input voltage and current. Input voltage and current equation at the generator $z = -l$ are:

$$\tilde{V}_{in} = \tilde{V}(z = -l) = \tilde{V}_0^+(e^{j\beta l} + \Gamma_L e^{-j\beta l}) \tag{153}$$

$$\tilde{I}_{in} = I(z = -l) = \frac{\tilde{V}_0^+}{Z_0}(e^{j\beta l} - \Gamma e^{-j\beta l}) \tag{154}$$

Substituting these two equations in Equation 152 we get

$$\tilde{V}_g = Z_g \frac{\tilde{V}_0^+}{Z_0}(e^{j\beta l} - \Gamma_L e^{-j\beta l}) + \tilde{V}_0^+(e^{j\beta l} + \Gamma_L e^{-j\beta l}) \tag{155}$$

We can re-write this equation as follows.

$$\tilde{V}_g = Z_g e^{j\beta l} \frac{\tilde{V}_0^+}{Z_0}(1 - \Gamma_L e^{-2j\beta l}) + \tilde{V}_0^+ e^{j\beta l}(1 + \Gamma_L e^{-2j\beta l}) \tag{156}$$

Using that $\Gamma_{in} = \Gamma_L e^{-2j\beta l}$ is the input reflection coefficient, and multiplying through with Z_0.

$$\tilde{V}_g Z_0 = Z_g e^{j\beta l} \frac{\tilde{V}_0^+}{1}(1 - \Gamma_{in}) + \tilde{V}_0^+ Z_0 e^{j\beta l}(1 + \Gamma_{in}) \tag{157}$$

Rearranging the equation, we get $\tilde{V}_0^+$

$$\tilde{V}_0^+ = \tilde{V}_g e^{-j\beta l} \frac{Z_0}{Z_0(1 + \Gamma_{in}) + Z_g(1 - \Gamma_{in})} \tag{158}$$

$\Gamma_{in} = \Gamma_L e^{-2j\beta l}$ is the input reflection coefficient.

Special case - forward voltage when the generator and transmission-line impedance are equal

Because the generator's impedance is equal to the transmission line impedance, we will use the second equation. When $Z_g = Z_0$ we see that the denominator simplifies into $Z_0(1 + \Gamma_{in}) + Z_g(1 - \Gamma_{in}) = Z_0(1 + \Gamma_{in} + 1 - \Gamma_{in}) = 2$, and we can further simplify the fraction to get the final value of $\tilde{V}_0^+ = \frac{V_g}{2} e^{-j\beta l}$.

4.9 Traveling and Standing Waves

Standing Waves

In the previous section, we introduced the voltage reflection coefficient that relates the forward to reflected voltage phasor.

$$\tilde{V}(z) = \tilde{V}_0^+(e^{-j\beta z} + \Gamma_L e^{j\beta z}) \tag{159}$$

$$I(z) = \frac{\tilde{V}_0^+}{Z_0}(e^{-j\beta z} - \Gamma_L e^{j\beta z}) \tag{160}$$

Let us look at the physical meaning of these variables.

(a) Γ_L is the voltage reflection coefficient at the load (z=0),

(b) z is the axis in the direction of wave propagation,

(c) β is the phase constant,

(d) Z_0 is the impedance of the transmission line.

(e) $\tilde{V}_0^+$ is the phasor of the forward-going wave at the load,

(f) $\Gamma_L \tilde{V}_0^+$ is the phasor of the reflected going wave at the load,

(g) $\tilde{V}_0^+ e^{-j\beta z}$ is a forward voltage anywhere on the line,

(h) $\Gamma_L \tilde{V}_0^+ e^{j\beta z}$ is a reflected voltage anywhere on the line,

(i) $\tilde{V}(z)$ is the total voltage phasor on the line, the sum of forward and reflected voltage.

(j) $\tilde{I}_0^+$ is the phasor of the forward-going current at the load,

(k) $-\Gamma_L \dfrac{\tilde{V}_0^+}{Z_0}$ is the phasor of the reflected current at the load,

(l) $\tilde{I}_0^+ e^{-j\beta z}$ is the phasor of a forward current anywhere on the line,

(m) $-\Gamma_L \dfrac{\tilde{V}_0^+}{Z_0} e^{j\beta z}$ is the phasor of a reflected current anywhere on the line,

(n) $\tilde{I}(z)$ is the phasor of the total current on the line, the sum of forward and reflected voltage.

The magnitude of a complex number can be found as $|z| = \sqrt{zz^*}$.Therefore the magnitude of the voltage anywhere on the line is $|\tilde{V}(z)| = \sqrt{\tilde{V}(z)\tilde{V}(z)^*}$. We can simplify this equation as shown in Figure 161.

$$
\begin{aligned}
|\tilde{V}(z)| &= \sqrt{\tilde{V}(z)\tilde{V}(z)^*} \\
|\tilde{V}(z)| &= \sqrt{(\tilde{V}_0^+)^2(e^{-j\beta z} - |\Gamma_L|e^{j\beta z+\Theta_r})\tilde{V}_0^+(e^{j\beta z} - |\Gamma_L|e^{-(j\beta z+\angle\Gamma_L)})} \\
|\tilde{V}(z)| &= \tilde{V}_0^+\sqrt{(e^{-j\beta z} - |\Gamma_L|e^{j\beta z+|\angle\Gamma_L})(e^{j\beta z} - |\Gamma_L|e^{-(j\beta z+\angle\Gamma_L)})} \\
|\tilde{V}(z)| &= \tilde{V}_0^+\sqrt{1 + |\Gamma_L|e^{-(2j\beta z+|\angle\Gamma_L)} + |\Gamma_L|e^{j2\beta z+\angle\Gamma_L} + |\Gamma_L|^2)} \\
|\tilde{V}(z)| &= \tilde{V}_0^+\sqrt{1 + |\Gamma_L|^2 + |\Gamma_L|(e^{-(2j\beta z+\angle\Gamma_L)}\angle\Gamma_L + e^{(j2\beta z+\angle\Gamma_L)})} \\
|\tilde{V}(z)| &= \tilde{V}_0^+\sqrt{1 + |\Gamma_L|^2 + 2|\Gamma_L|\cos(2\beta z + \angle\Gamma_L)}
\end{aligned} \tag{161}
$$

The Equation 161 is written in terms of z. We set up the load at $z = 0$, and the generator at $z = -l$. The positions of the maximums and minimum total voltage on the line will be at some position at the negative part of z-axis $z_{max} = -l_{max}$, and the minimums will be at $z_{max} = -l_{min}$.

The magnitude of the total voltage on the transmission line is given by Eq.161. We will now visualize how the magnitude of the voltage looks on the transmission line.

Example 13. *Find the magnitude of the total voltage anywhere on the transmission line if* $\Gamma_L = 0$.

Explanation. *Let us start from a simple case when the voltage reflection coefficient on the transmission line is* $\Gamma_L = 0$ *and draw the magnitude of the total voltage as a function of z. Equation 161 shows the magnitude of the total voltage anywhere on the line is equal to the magnitude of the voltage at the load* $|\tilde{V}(z)| = |\tilde{V}_0^+|$. *The magnitude of the voltage is constant everywhere on the transmission line, and so the line is called "flat," and it represents a single, forward traveling wave from the generator to the load. The magnitude is the green line in Figure 36. To see the movie of this transmission line, go to the class web page under Instructional Videos. Forward voltage is shown in red, reflected voltage in pink, and the magnitude of the voltage is green.*

Example 14. *Find the magnitude of the total voltage anywhere on the transmission line if* $\Gamma_L = 0.5e^{j0}$.

Explanation.

Let's look at another case, $\Gamma_L = 0.5$ *and* $\angle\Gamma_L = 0$. *Equation 162 represents the magnitude of the voltage on the transmission line, and Figure 37 shows in green how this function looks on a transmission line. This case is shown in Figure 37.*

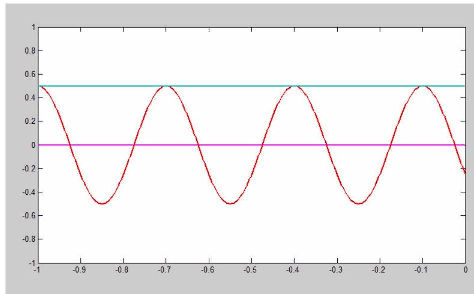


Figure 36: Flat line.

$$|\tilde{V}(z)| = \tilde{V}_0^+ \sqrt{\frac{5}{4} + cos2\beta z} \tag{162}$$

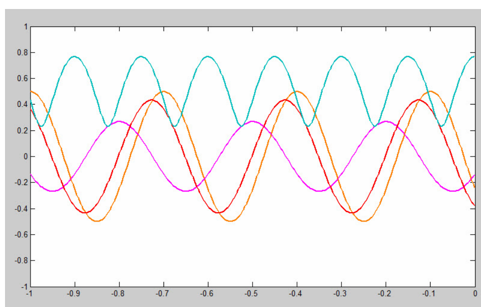


Figure 37: Voltage on a transmission line with reflection coefficient magnitude 0.5, and zero phase.

The function in Equation 162 is at its maximum when $cos(2\beta z + \angle\Gamma_L) = 1$ *or* $z = \frac{k}{2}\lambda$, *and the function value is* $\tilde{V}(z) = 1.5\tilde{V}_0^+$. *It is at its minimum when* $cos(2\beta z + \angle\Gamma_L) = -1$ *or* $z = \frac{2k+1}{4}\lambda$ *and the function value is* $\tilde{V}(z) = 0.5\tilde{V}_0^+$

The function that we see looks like a cosine with an average value of $\tilde{V}_0^+$, *but* **it is not a cosine**. *The minimums of the function are sharper than the maximums.*

Question 14 *Observe waves in the app below.*

Geogebra link: `https://tube.geogebra.org/m/bmr8euzu`

Is the black wave moving left or right?

Multiple Choice:

(a) *left*

(b) *right*

Is the black wave a forward, reflected or the sum of the two?

Multiple Choice:

(a) *Forward*

(b) *Reflected*

(c) *The sum of the two*

Example 15. *Find the magnitude of the total voltage anywhere on the transmission line if* $\Gamma_L = 1$.

Explanation. *Another case we will look at is when the reflection coefficient is at its maximum of* $\Gamma_L = 1$. *The function is shown if Figure 38. In this case, we have a pure standing wave on a transmission line.*

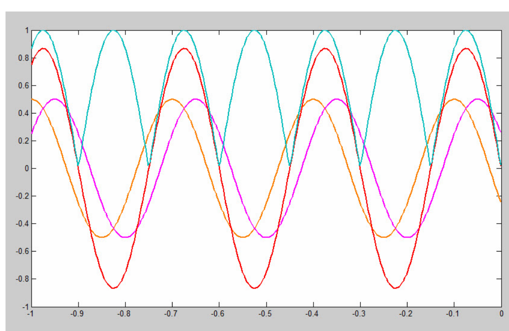


Figure 38: Shorted Transmission Line.

Question 15 *Observe waves in the app below.*

Geogebra link: `https://tube.geogebra.org/m/fwebfheh`

Is the black wave moving left or right? The wave is not moving left or right, it is standing in place, so it is called a standing wave. Is the black wave a forward, reflected or the sum of the two?

Multiple Choice:

(a) *Forward*

(b) *Reflected*

(c) *The sum of the two*

Example 16. *Find the magnitude of the total voltage anywhere on the transmission line for any Γ_L, and the position of voltage maximums on the line.*

Explanation. *The magnitude of the total voltage on the line is given in Equation161. In general the voltage maximums will occur when the cosine function is at its maximum $cos(2\beta z + \angle\Gamma_L) = 1$. In this case, the maximum value of the magnitude of total voltage on the line is shown in Equation 163.*

$$\begin{aligned} |\tilde{V}(z)_{max}| &= |\tilde{V}_0^+|\sqrt{1 + |\Gamma_L|^2 + 2|\Gamma_L|} \\ |\tilde{V}(z)_{max}| &= |\tilde{V}_0^+|\sqrt{(1 + |\Gamma_L|)^2} \\ |\tilde{V}(z)_{max}| &= |\tilde{V}_0^+|(1 + |\Gamma_L|) \end{aligned} \tag{163}$$

The Equation 164 shows position of voltage maximums z_{max} on the line.

$$\begin{aligned} cos(-2\beta l_{max} + \angle\Gamma_L) &= 1 \\ cos(2\beta l_{max} - \angle\Gamma_L) &= 1 \\ 2\beta z_{max} - \angle\Gamma_L &= 2n\pi \\ z_{max} &= \frac{2n\pi + \angle\Gamma_L}{2\beta} \\ z_{max} &= \lambda\frac{2n\pi + \angle\Gamma_L}{4\pi} \end{aligned} \tag{164}$$

In general the voltage minimums will occur when $cos(2\beta z) = -1$.

$$\begin{aligned} |\tilde{V}(z)_{min}| &= |\tilde{V}_0^+|\sqrt{1 + |\Gamma_L|^2 - 2|\Gamma_L|} \\ |\tilde{V}(z)_{min}| &= |\tilde{V}_0^+|\sqrt{(1 - |\Gamma_L|)^2} \\ |\tilde{V}(z)_{min}| &= |\tilde{V}_0^+|(1 - |\Gamma_L|) \end{aligned} \tag{165}$$

The Equation 166 shows the position of voltage minimums on the line.

$$
\begin{aligned}
cos(2\beta z_{max} - \angle\Gamma_L) &= -1 \\
2\beta z_{max} + \angle\Gamma_L &= (2n+1)\pi \\
z_{max} &= \frac{(2n+1)\pi + \angle\Gamma_L}{2\beta} \\
z_{max} &= \lambda\frac{(2n+1)\pi + \angle\Gamma_L}{4\pi} \qquad (166)
\end{aligned}
$$

Voltage Standing Wave Ratio (VSWR) - pron: "vee-s-uh-are"

The ratio of voltage minimum on the line over the voltage maximum is called the Voltage Standing Wave Ratio (VSWR) or just Standing Wave Ratio (SWR).

$$
\begin{aligned}
SWR &= \frac{\tilde{V}(z)_{max}}{\tilde{V}(z)_{min}} \\
SWR &= \frac{1+|\Gamma_L|}{1-|\Gamma_L|} \qquad (167)
\end{aligned}
$$

Note that the SWR is always equal or greater than 1.

4.10 Example Transmission Line Problem

Example 17. *A transmitter operated at 20MHz, Vg=100V with* $Z_g = 50\Omega$ *internal impedance is connected to an antenna load through l=6.33m of the line. The line is a lossless* $Z_0 = 50\Omega$, $\beta = 0.595rad/m$. *The antenna impedance at 20MHz measures* $Z_L = 36 + j20\Omega$. *Set the beginning of the z-axis at the load, as shown in Figure 39.*

(a) *What is the electrical length of the line?*

(b) *What is the input impedance of the line* Z_{in}*?*

(c) *What is the forward going voltage at the load* $\tilde{V}_0^+$*?*

(d) *Find the expression for forward voltage anywhere on the line.*

(e) *Find the expression for reflected voltage anywhere on the line.*

(f) *Find the total voltage anywhere on the line.*

(g) *Find the expression for forward current anywhere on the line.*

(h) *Find the expression for reflected current anywhere on the line.*

(i) *Find the total current anywhere on the line.*

(j) *Instead of the antenna, a load impedance* $Z_l = 50\Omega$ *is connected to this 50 Ohm line. How will that change the equations above?*

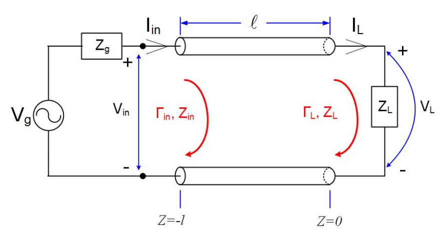


Figure 39: Transmission Line connects generator and the load.

Explanation. *The equations for the voltage and current anywhere (any z) on a transmission line are*

$$\tilde{V}(z) = \tilde{V}_0^+(e^{-j\beta z} + |\Gamma|e^{j\beta z + \Theta_r}) \tag{168}$$

$$I(z) = \frac{\tilde{V}_0^+}{Z_0}(e^{-j\beta z} - \Gamma e^{j\beta z}) \tag{169}$$

We are given phase constant β, and we have to find the other unknowns: phasor of voltage at the load $\tilde{V}_0^+$, and the reflection coefficient Γ.

Since we know the load impedance Z_L, and the transmission-line impedance Z_0, we can find the reflection coefficient Γ using Equation 170.

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = 0.279\, e^{j112^o} = 0.279\, e^{j1.95\,\mathrm{rad}} \tag{170}$$

To find the input impedance of the line, we use the equation

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l} = 70.8 + j27.1\Omega \tag{171}$$

We can use one of the following two equations to find the forward going voltage at the load:

$$\tilde{V}_0^+ = \frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}} \frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}} \tag{172}$$

$$\tilde{V}_0^+ = V_g e^{-j\beta l} \frac{Z_0}{Z_0(1+\Gamma_{in}) + Z_g(1-\Gamma_{in})} \tag{173}$$

Because the generator's impedance is equal to the transmission line impedance, we will use the second equation. When $Z_g = Z_0$ we see that the denominator simplifies into $Z_0(1+\Gamma_{in}) + Z_g(1-\Gamma_{in}) = Z_0(1+\Gamma_{in}+1-\Gamma_{in})$ and we can further simplify the fraction to get the final value of $\tilde{V}_0^+ = \frac{V_g}{2}e^{-j\beta l}$. Since $\beta l = \frac{2\pi}{\lambda}0.6\lambda$, the forward going voltage at the load is $\tilde{V}_0^+ = 50\, e^{-j1.2\pi} = 50\, e^{-j3.768} = 50\, e^{-j216^o}$.

The equations of the voltage and current anywhere on the line are therefore

$$\tilde{V}(z) = 50e^{-j3.768}(e^{-j0.595z} + 0.279e^{j0.595z+1.95}) \tag{174}$$

$$I(z) = e^{-j3.768}(e^{-j0.595z} - 0.279e^{j0.595z+1.95}) \tag{175}$$

Suppose we replace the antenna with another load of impedance 50Ω. In that case, the reflection coefficient from the load will be zero, and the reflected voltages will disappear, so the voltage and current will be equal to the forward-going voltage on the transmission line.

$$\tilde{V}(z) = \tilde{V}_0^+ e^{-j\beta z} \tag{176}$$

$$I(z) = \frac{\tilde{V}_0^+}{Z_0} e^{-j\beta z} \tag{177}$$

5 Smith Chart

After completing this section, students should be able to do the following.

- Describe Smith chart as a polar plot of reflection coefficient.
- Estimate the reflection coefficient from the Smith Chart.
- Read the reflection coefficient off the Smith Chart
- Mark the point on the Smith Chart given reflection coefficient.
- Calculate load impedance if reflection coefficient and transmission-line impedance are given
- Calculate reflection coefficient if load impedance and transmission-line impedance are given
- Explain how was Smith Chart developed
- Read normalized impedance and reflection coefficient on Smith Chart given a random point on the Smith Chart
- Given impedance find the normalized position of the impedance on the Smith Chart.
- Describe the reason for introducing admittance
- Write impedance and admittance of an inductor and capacitor.
- Explain why is the susceptance of an inductor negative and reactance is positive.
- Explain why is the admittance Smith Chart rotated 180 degrees.
- Distinguish between load impedance and normalized load impedance. Describe impedances on the Smith Chart as normalized impedances
- Given impedance, read admittance on combo Y/Z chart
- Given a random point on Y/Z chart, find admittance, impedance and the reflection coefficient.
- Explain electrical length
- Calculate the input impedance and input reflection coefficient
- Describe input reflection coefficient in terms of load reflection coefficient.

5.1 Smith Chart

Smith Chart is a handy tool that we use to visualize impedances and reflection coefficients. Lumped element and transmission line impedance matching would be challenging to understand without Smith Charts. Simulation software such as ADS and measurement equipment, such as Network Analyzers, use Smith Chart to represent simulated or measured data. Smith Chart first looks like Black Magic, but it is a straightforward and useful tool that will help us better understand impedance/admittance transformations and transmission lines.

Definition 16. *Smith Chart is a polar plot of the reflection coefficient.*

An example of a Smith Chart is given in Figure 40

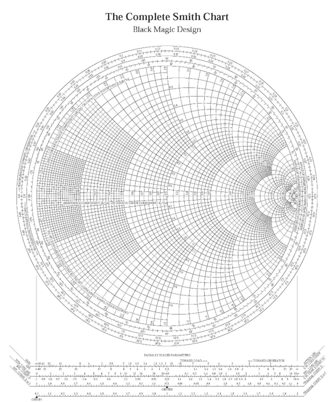
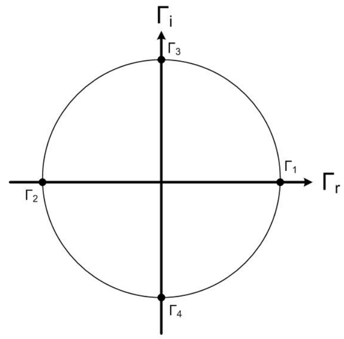


Figure 40: Smith Chart

In essence, Smith Chart is a unit circle centered at the origin with a radius of 1. Smith Chart is used to represent the reflection coefficient graphically in polar coordinates. The reflection coefficient's real and imaginary axis (the Cartesian coordinates) is not shown on the actual Smith Chart. However, the center of the Smith Chart is where the origin of the coordinate system would be. We usually represent the reflection coefficient in polar coordinates, with a magnitude and an angle. Magnitude is the distance between the point and the origin, and the angle is measured from the x-axis. An example location of several reflection coefficients is given in Figure 41. If you do not see why the points are positioned as shown, review the polar representation of complex numbers.

Figure 42, 43 circle and line represent all points on the Smith Chart that have constant magnitude or angle of the reflection coefficient. It is challenging to



$$\Gamma_1 = 1$$
$$\Gamma_2 = -1$$
$$\Gamma_3 = j = e^{j*90}$$
$$\Gamma_4 = -j = e^{-j*90}$$

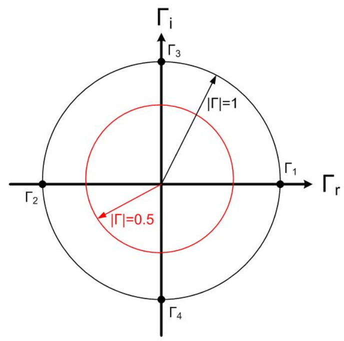
Figure 41: Examples of location of reflection coefficient on the Smith Chart.

measure impedances directly at high frequencies, as it is difficult to measure (or sometimes even define) voltage and current. To measure impedances, engineers use Network Analyzer shown in Figure 44.

Example 18. *We can estimate the reflection coefficient by looking at the position of a point on the Smith Chart. We know that the radius of the circle is 1. So, if the point is half-way between the center and the perimeter, we know the magnitude of the reflection coefficient is 0.5. We can also estimate the angle of the reflection coefficient. For example, if the magnitude is in the first quadrant, we know that the angle is between* 0^0 *and* 90^0*. If the point is on the positive y-axis, we know that the angle is equal to* 90^0*. Observe the simulation below and see if you can estimate the reflection coefficient's magnitude and angle by looking at the point's position.*

The simulation below shows only positive angles. In practice, if the point is below the x-axis, we use negative angles to describe the angle of the reflection coefficient. For example, if the point is at the negative y-axis, we would say that the reflection coefficient's angle is -90^0*.*

Geogebra link: `https://tube.geogebra.org/m/u9ehhbaj`



$$\Gamma_1 = 1$$
$$\Gamma_2 = -1$$
$$\Gamma_3 = j = e^{j*90}$$
$$\Gamma_4 = -j = e^{-j*90}$$

Figure 42: Points of constant magnitude of the reflection coefficient. |Γ|=0.5, |Γ|=1

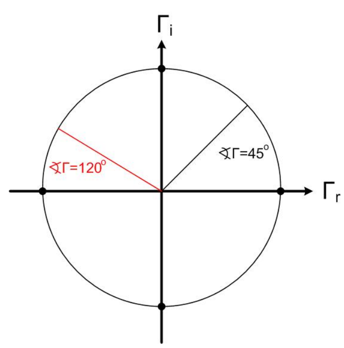


Figure 43: Points of constant phase of the reflection coefficient. $< \Gamma = 45°$, $< \Gamma = 120°$

Figure 44: HP8510 Network Analyzer in Microwave Laboratory measures impedances up to 26.5GHz. This piece of equipment is on permanent loan curtesy of Defense Micro Electronic Activity (DMEA), Sacramento

5.2 Impedance and admittance circles on the Smith Chart

Reflection Coefficient and Impedance

Reflection coefficient and impedance are related through Equation 178. We can find an impedance that corresponds to the reflection coefficient, Equation 179. Every point on the Smith Chart represents one reflection coefficient Γ and one impedance Z_L.

$$\Gamma = \frac{Z_L - Z_\circ}{Z_L + Z_\circ} \tag{178}$$

$$Z_L = Z_0 \frac{1+\Gamma}{1-\Gamma} \tag{179}$$

All impedances on the Smith Chart are normalized to the transmission line impedance Z_0. The normalized impedance is denoted in Equation 180 with lowercase z_L.

$$z_L = \frac{Z_L}{Z_0} = \frac{1+\Gamma}{1-\Gamma} \tag{180}$$

Derivation of Impedance and Admittance Circles on the Smith Chart

Impedance and reflection coefficient are complex numbers. The normalized impedance has a real and imaginary part $z_L = r_L + jx_L$, and the reflection coefficient can also be shown in Cartesian coordinates as $\Gamma = \Gamma_r + j\Gamma_i$. We can now substitute these equations into Equation 181.

$$r_L + jx_L = \frac{1 + \Gamma_r + j\Gamma_i}{1 - (\Gamma_r + j\Gamma_i)} \tag{181}$$

We can equate the real and imaginary parts on the left and right side of Equation 181 to get the equations of constant r_L and x_L.

$$\left(\Gamma_r - \frac{r_L}{1+r_L}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r_L}\right)^2 \tag{182}$$

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x_L}\right)^2 = \left(\frac{1}{x_L}\right)^2 \tag{183}$$

These are equations of a circle, with the constant resistance circle's center at $(\frac{r_L}{1+r_L}, 0)$ and radius $\frac{1}{1+r_L}$; and the constant reactance imaginary circle center at $(1, \frac{1}{x_L})$ and radius of $\frac{1}{x_L}$.

Figures 45- 46 show circles on the Smith Chart that represent constant (normalized) reactances, and resistances.

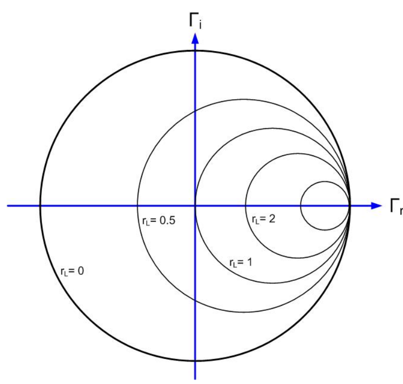


Figure 45: All points on the circle have the constant real part of the impedance (resistance). Normalized resistance circles.

Admittance circles can be similarly derived using the fact that $Y_L = \frac{1}{Z_L}$ and the Equation 180

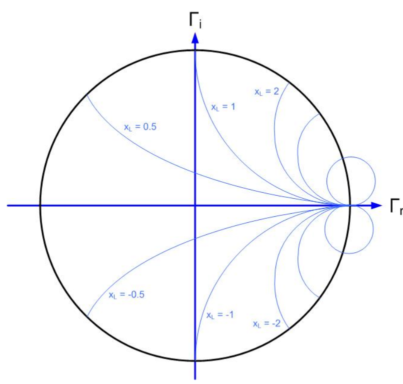


Figure 46: All points on the circle have the constant imaginary part of the impedance (reactance). Normalized reactance circles.

5.3 Impedance and Admittance on Smith Chart

Brief review of impedance and admittance

Impedance's $Z = R + jX$real part is called resistance R, and the imaginary part is called reactance X. It is easier to add impedances when elements are in series; see Figure 47. To find the total impedance, we add resistances and reactances separately.

The real part of an admittance $Y = G + jB$ is called conductance G, and the imaginary part is called susceptance B. It is easier to add admittances when elements are in parallel; see Figure 47. When adding two admittances, we add conductances and susceptances separately.

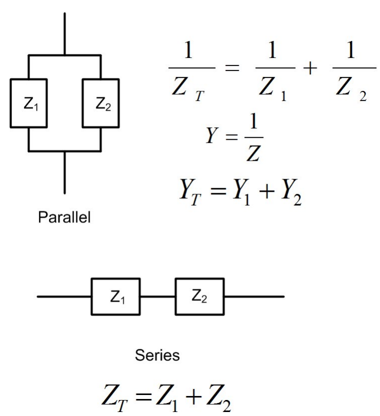


Figure 47: It is easier to use admittance when the circuit elements are in parallel and impedance when the circuit elements are in series.

Smith Chart in Figure 48 has impedance circles, and impedance coordinates on it. We can use this Smith Chart to read off the values for the impedance, and reflection coefficient. In the next section, we will learn to use impedance/admittance (Z/Y) Smith Chart, where both impedance and admittance circles are shown. We will use the Z/Y Smith Chart when we add impedances or admittances in parallel or series, which is useful in impedance matching that we will talk about in the next chapter.

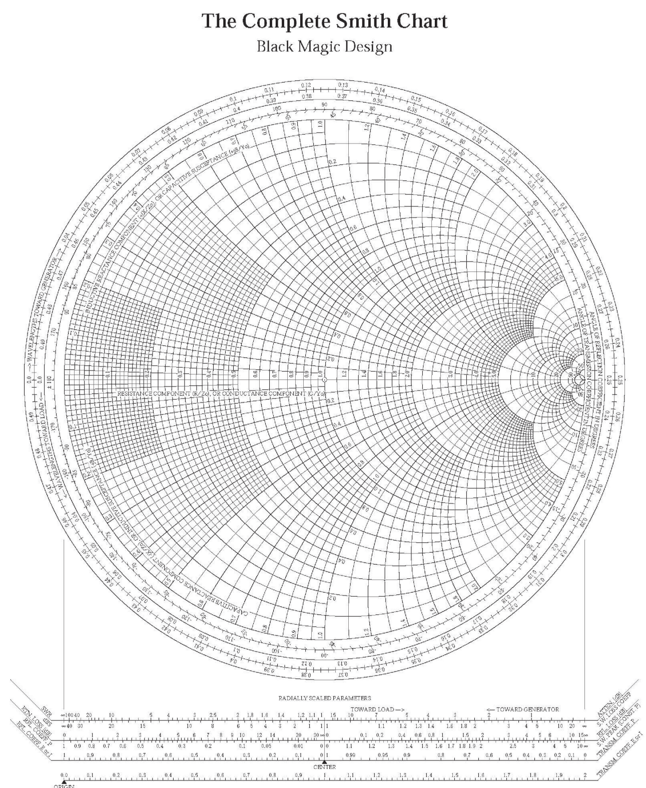


Figure 48: On Smith Chart impedances are shown in red, and admittances in green.

Impedance on the Smith Chart

Figure 49, shows how to find the location of normalized impedance $z_L = 1+j1$ on the Smith Chart. $z_L = 1+j1$ is at a point where the circle of constant resistance $r_L = 1$ crosses the circle of constant reactance $x_L = 1$. Figure 50 shows how to find the reflection coefficient if normalized load impedance $z_L = 1 + j1$ is given. Measure the distance between the origin and the point using the scale "Reflection Coefficient E or I" on the Smith Chart's bottom to find the reflection coefficient's magnitude. To find the reflection coefficient's angle, we read the scale "Angle of Reflection Coefficient" on the Smith Chart's perimeter, shown in green. The reflection coefficient is therefore $0.5e^{j62^o}$, which is close to the actual value $0.5e^{j64^o}$. If we use a ruler and compass, and a nicely sharpened pencil, we will get exactly the right answer. Try it out!

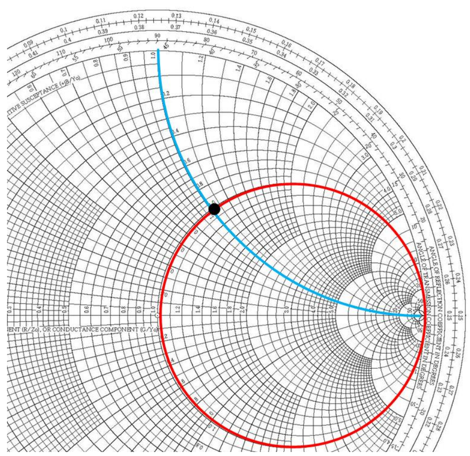


Figure 49: Red circle that represents all points on Smith Chart with normalized resistance $r = 1$ and blue circle that represents all points on Smith Chart with normalized reactance $x = 1$ cross at point where $Z_L = 1 + j1$.

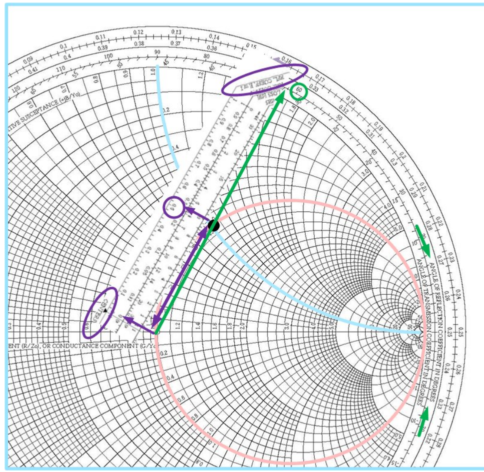


Figure 50: To find the magnitude of the reflection coefficient from impedance $Z_L = 1 + j1$, we measure the distance between the origin and the point using the scale "Reflection Coefficient E or I" on the bottom of the Smith Chart. To find the reflection coefficient's angle, we read the scale "Angle of Reflection Coefficient" on the perimeter of the Smith Chart.

Admittance on the impedance Smith Chart

The reflection coefficient can be found from impedance or admittance. To find the reflection coefficient from impedance, we use the formula that we previously derived, where Z_L is the load impedance, and $z_L = \frac{Z_L}{Z_0}$ is the normalized load impedance.

$$\Gamma_L = \frac{z_L - 1}{z_L + 1} \tag{184}$$

Admittance is defined as $Y_L = \frac{1}{Z_L}$, and the transmission-line admittance is defined as $Y_0 = \frac{1}{Z_0}$. If we now replace the impedances in the equation above with admittances, we get

$$\Gamma_L = \frac{\frac{1}{Y_L} - \frac{1}{Y_0}}{\frac{1}{Y_L} + \frac{1}{Y_0}} \tag{185}$$

$$\Gamma_L = \frac{Y_0 - Y_L}{Y_0 + Y_L} \tag{186}$$

$$\Gamma_L = -\frac{Y_L - Y_0}{Y_L + Y_0} \tag{187}$$

$$\Gamma_L = -\frac{y_L - 1}{y_L + 1} \tag{188}$$

$$\Gamma_L = -\Gamma_Y \tag{189}$$

Where Y_L is the load admittance and $y_L = \frac{Y_L}{Y_0}$, is the normalized load admittance. Note that the Smith Chart shown in Figure 49 shows circles for the impedance of the load. Therefore, the true reflection coefficient of the load is at the position of the load. We have defined Γ_Y above to see the position of admittance on the impedance Smith Chart.

From the above Equation 189, we can see that the reflection coefficient for the admittance (and the admittance) on the impedance Smith chart has the same magnitude as the load reflection coefficient. The Γ_Y is on the same SWR circle as the Γ_L, but it is 180^0 shifted with respect to the load reflection coefficient. 180^0 phase shift in reflection coefficient with the same magnitude can be found in two ways:

(a) by rotating the reflection coefficient Γ_L for 180^0 on the SWR circle to read the admittance of the load. Effectively, this means that you can draw a line through the load impedance and the center of the Smith Chart, and find the position of this line that crosses the SWR circle, 180^0 away. This method's disadvantage is that the point representing admittance changes place on the Smith Chart, depending on whether we are reading admittance or impedance. This method is used in some books, see, for example, Ulaby, Applied Electromagnetics.

(b) Another way to read the admittances on the Smith chart is to use combination of impedance/admittance (Z/I) chart, Figure 51. To make the Z/Y chart, we rotate the impedance Smith Chart for 180^0 to get the admittance point at the same position as the impedance point. Then, the impedance, admittance, and reflection coefficient are all at the same point, but we have to read the two scales, the impedance and admittance scale. The disadvantage of this method is that the Z/Y Smith Chart has more circles to read. However, this method is conceptually more straightforward, as the point on the Smith Chart does not change position, depending on whether we are reading admittance or impedance. The point is fixed, and we just read the different circles. We will use this method in our class. The example below discusses this method further.

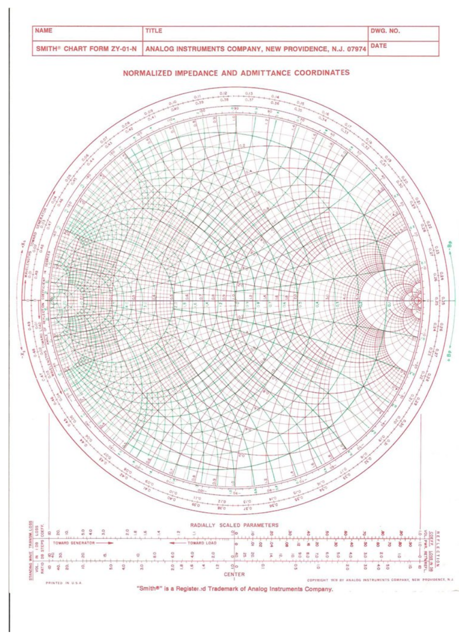


Figure 51: On Z/Y Smith Chart impedances are shown in red, and admittances in green.

Example 19. *The normalized impedance of the load is given $z_L = 1 + j1$. First, find the admittance of this impedance using your calculator. Then use two Smith Charts. On one, find the impedance position, and on the other, find the position of the admittance. Then rotate the admittance chart for 180^0 so that both points overlap. Observe the impedance and admittance circles on this combo Z/Y chart, and compare them to the Z/Y chart.*

The impedance $z_L = 1 + j1$ is shown on the Smith Chart in Figure 52.

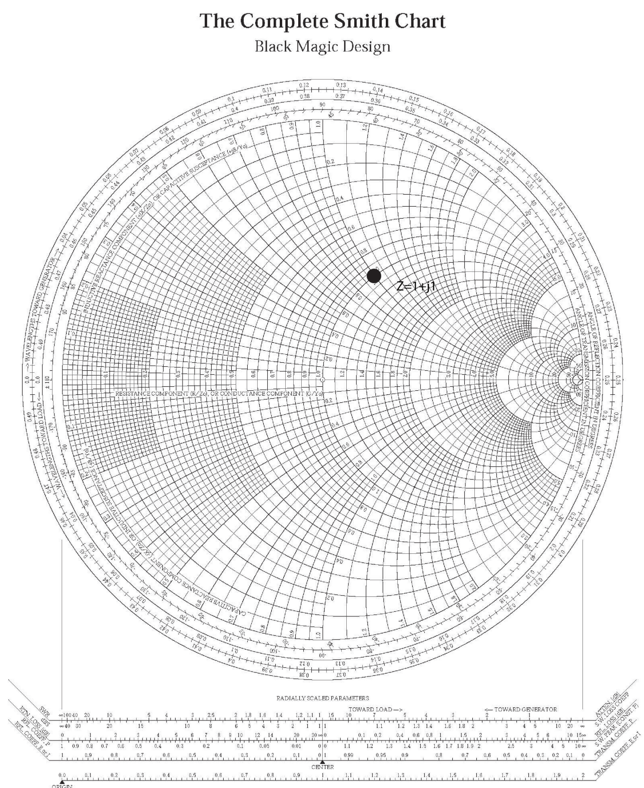


Figure 52: Impedance $z_L = 1 + j1$ on Smith Chart

The admittance $y_L = 0.5 - j0.5$ is shown on Smith Chart in Figure 53. If we now use this Smith Chart and rotate it for 180^0 (upside down), and place it over the chart above, we will see that the impedance and admittance point are in the same position on the Smith Chart. We call this Smith Chart a z/y Smith Chart.

Both impedance $z_L = 1 + j1$ and admittance $y_L = 0.5 - j0.5$ are at the same position on the z/y Smith Chart in Figure 54. We see two sets of circles on this

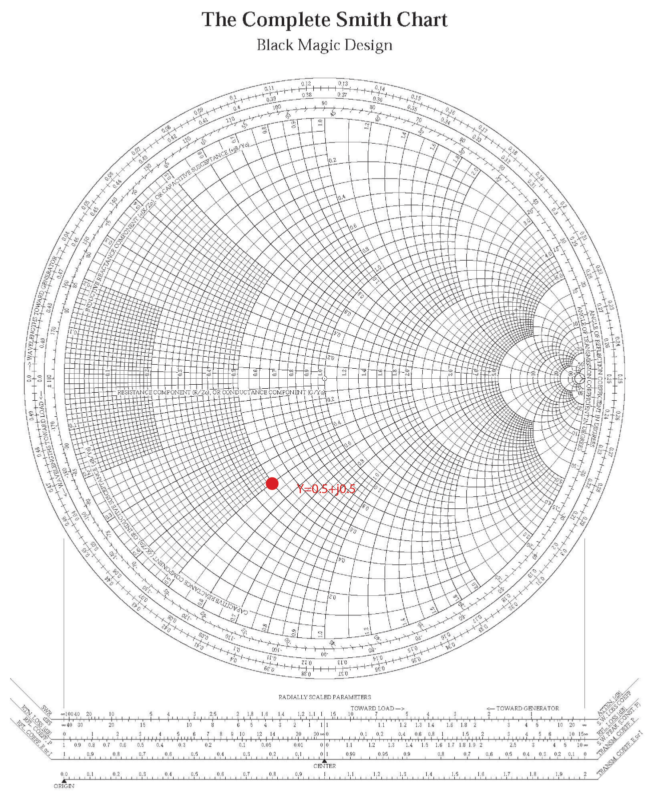


Figure 53: Admittance $y_L = 0.5 - j0.5$ on Smith Chart

chart, the red ones represent impedance circles, and the green ones represent admittance circles. To read the impedance of the point shown on the Smith Chart, we read the impedance circle scale, and to read the admittance, we read the admittance circle scales.

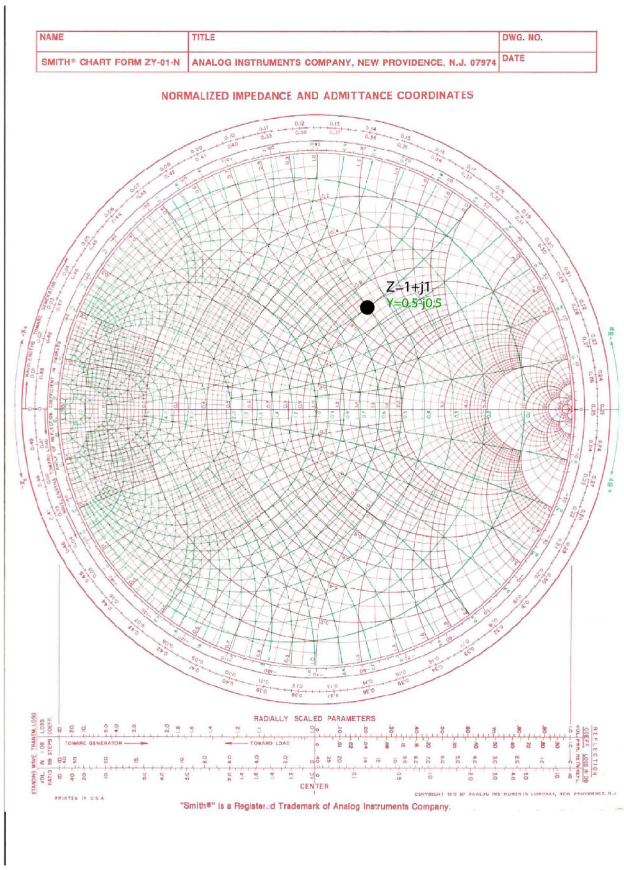


Figure 54: Both impedance $z_L = 1 + j1$ and admittance $y_L = 0.5 - j0.5$ are at the same position on the z/y Smith Chart

Converting Reflection Coefficient to Impedance and Admittance

Example 20. *Look at a Z/Y Smith Chart of your choice. Z/Y chart has both impedance and admittance circles. Pick a point on the Smith Chart, first estimate the reflection coefficient, and then read it from the Smith Chart. Then read the impedance and admittance. Use the app below to check your work.*

Geogebra link: `https://tube.geogebra.org/m/aswbwwx3`

5.4 Electrical Length

Electrical Length of the line in meters

We can express the physical length of the line in meters. However, in high-frequency electronics (microwave engineering), we usually convert this length to the fraction of a wavelength of a signal that is traveling on the line.

$$l = N\lambda \tag{190}$$

Typically, but not always, N is a fraction, for example, $N = \frac{1}{2} = 0.5$, $N = \frac{1}{4} = 0.25$ or $N = \frac{1}{8} = 0.125$; although it can be any The lenght of the line is then written as

$$l = \frac{\lambda}{2} \tag{191}$$

$$l\frac{\lambda}{4} \tag{192}$$

$$l = \frac{\lambda}{8} \tag{193}$$

If the physical length of the line is $l = \frac{\lambda}{4}$, we say: This line is quarter-wavelength long at 1GHz, meaning one-quarter of the wavelength fits on the line. We could also say that the line is 7.5cm long, as wavelength is $\lambda = 30\,\text{cm}$ at 1GHz.

When we say quarter-wavelength long, we refer to the lines physical length at a specific frequency.

Electrical length of the line in degrees

The phase shift between input and output signal on a transmission line is $\Theta = \beta * l$. *beta* is called the phase constant. It represents the spatial frequency of the signal. $\Theta = \beta * L$ is the phase in degrees or radians (related is a time delay in seconds). Θ can be, for example $\Theta = 45^0$, $\Theta = 90^0$, $\Theta = 180^0$. Θ is a function of frequency, because β is a function of frequency. If $\Theta = 90^0$, we say: The line is 90 degrees long at 1GHz, meaning the output signal at 1GHz will be shifted for 90^0 with respect to the input signal. When we say 90^0, we refer to the lines electrical length, representing the number of degrees that the line introduces between the input and the output signal.

Example 21. *What is the electrical length of a 30 cm line in terms of the fraction of wavelength at 1 GHz? What is the electrical length of the line at 1GHz?*

Explanation. *Wavelength at 1 GHz, assuming the wave is propagating in air is* $\lambda = \frac{c}{f} = 30\,cm$*. Since the line is also l=30 cm long, the length of the line in terms of wavelength is* $\frac{l}{\lambda} = 1$*, or* $l = \lambda$*.*

The electrical length of the line is $\theta = \beta l = \frac{2\pi}{\lambda}\lambda = 2\pi = 360^0$*.*

Example 22. *What is the electrical length of a 15 cm line in terms of the fraction of wavelength at 1 GHz? What is the electrical length of the line at 1GHz?*

Explanation. *Wavelenght at 1 GHz, assuming the wave is propagating in air is* $\lambda = \frac{c}{f} = 30\,cm$*. Since the line is 15 cm long, the length of the line in terms of wavelenth is* $\frac{l}{\lambda} = \frac{1}{2}$*, or* $l = \frac{\lambda}{2}$*.*

The electrical length of the line is $\theta = \beta l = \frac{2\pi}{\lambda}\frac{\lambda}{2} = \pi = 180^0$*.*

Example 23. *What is the electrical length of a 7.5 cm line in terms of the fraction of wavelength at 1 GHz? What is the electrical length of the line at 1GHz?*

Explanation. *Wavelength at 1 GHz, assuming the wave is propagating in air is* $\lambda = \frac{c}{f} = 30\,cm$*. Since the line is 7.5 cm long, the line's length in terms of wavelenth is* $l = \frac{\lambda}{4}$*.*

The electrical length of the line is $\theta = \beta l = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \pi/2 = 90^0$*.*

5.5 Input Reflection Coefficient and Impedance on Smith Chart

Input reflection coefficient and load reflection coefficient.

If the load reflection coefficient is $\Gamma_L = \dfrac{V_0^-}{V_0^+} = |\Gamma_L|e^{j\Theta_L}$, the input reflection coefficient Γ_{in} is:

$$\Gamma_{in} = \frac{V^-}{V^+} \tag{194}$$
$$\Gamma_{in} = \frac{V_0^- e^{-j\beta l}}{V_0^+ e^{j\beta l}} \tag{195}$$
$$\Gamma_{in} = \frac{V_0^-}{V_0^+} e^{-2j\beta l} \tag{196}$$
$$\Gamma_{in} = \Gamma_L e^{-j2\beta l} \tag{197}$$

Where V^+ is the reflected wave anywhere on the line, V^- is the forward wave anywhere on the line, V_0^+ is the forward wave at the load, V_0^- is the reflected wave at the load, βl is the electrical length of the line.

$$\Gamma_{in} = \Gamma_L e^{-j2\beta l} \tag{198}$$
$$\Gamma_{in} = |\Gamma_L| e^{j\Theta_L} e^{-j2\beta l} \tag{199}$$
$$\Gamma_{in} = |\Gamma_L| e^{j(\Theta_L - 2\beta l)} \tag{200}$$

From the above equations, we see that on a lossless transmission line, the magnitude of the reflection coefficient is the same anywhere on the line, but the phase differs for twice the electrical length of the line $-2\beta l$.

When we calculate input reflection coefficient, we can find input impedance:

$$Z_{in} = Z_0 \frac{1 + \Gamma_{in}}{1 - \Gamma_{in}} \tag{201}$$

From the previous equation, we see that we can find the input impedance if we know the input reflection coefficient. If we know the input impedance, we can find the input reflection coefficient as:

$$\Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} \tag{202}$$

Magnitude of the input reflection coefficient

We see that the input reflection coefficient and the load reflection coefficient have the same magnitude. All points on the Smith Chart that have the same magnitude of the reflection coefficient are on a circle centered at the center of the Smith Chart, with the radius determined by the load impedance or input impedance position. This circle is called the SWR circle, and it is shown in blue in Figure 55. The input impedance and load impedance are on the same SWR circle. If we know the load impedance, we know that the input impedance will be on the same SWR circle.

For example, if the load impedance is $Z_L = 100\Omega$, the transmission-line impedance is $Z_0 = 50\Omega$, the magnitude of the reflection coefficient is 0.33. Both the input reflection coefficient and the load reflection coefficient magnitudes will be the same, 0.33; however, their phases will differ depending on the line's length.

Phase of the input reflection coefficient

The input reflection coefficient angle will be decreased by twice the electrical length of the line $\Theta_L - 2\beta l$. On Smith Chart, decreasing the phase of the reflection coefficient means going clockwise on the SWR circle. For example, if the load impedance is $Z_L = 100\Omega$, the transmission-line impedance is $Z_0 = 50\Omega$, and the length of the line is $\frac{\lambda}{4}$, or $\Theta = \beta l = 90^0$, the angle between the load impedance and the transmission-line impedance will be $2\beta l = 180^0$. We will find the input reflection coefficient 180^0 away from the load reflection coefficient on the Smith Chart.

Reading the input reflection coefficient on the Smith Chart

We do not have to do calculations every time we want to find the input impedance or input reflection coefficient. To graphically find the input reflection coefficient or input impedance, we first identify the scale "WAVELENGTHS TOWARD GENERATOR" (WTG) on the Smith Chart's outer perimeter; see green oval in Figure 55. The WTG scale is labeled in terms of the transmission-line length in wavelengths (not electrical degrees). We see that 180^0 on the chart corresponds to $\lambda/4 = 0.25\lambda$, which is $2\beta l = 180$ electrical degrees.

If the WTG scale on the Smith Chart, shown in Figure 55, were a movable mechanical scale, we could move the reference position WTG=0 to the position of our load, and then read the input reflection coefficient from the known length of the line. Since we cannot move the position WTG=0 on the Smith Chart, we have to find a reference position of the load on the WTG scale, as shown in the black rectangle in Figure 55.

For example, if a normalized load impedance $z_L = 0.5 + j$ is given, using the Smith Chart find the input impedance and input reflection coefficient if the line is 0.145λ long.

To find the input impedance, we will start from the load impedance and read the reference position on the WTG scale for the load $ref = 0.135\lambda$, as shown in Figure 55. Then, we add the line length of $l = 0.145\lambda$ to the load impedance to find the input reflection coefficient phase. The point where this dashed line crosses the SWR circle is the point of input reflection coefficient and input impedance. Note that we must decrease the phase and go in the clockwise direction.

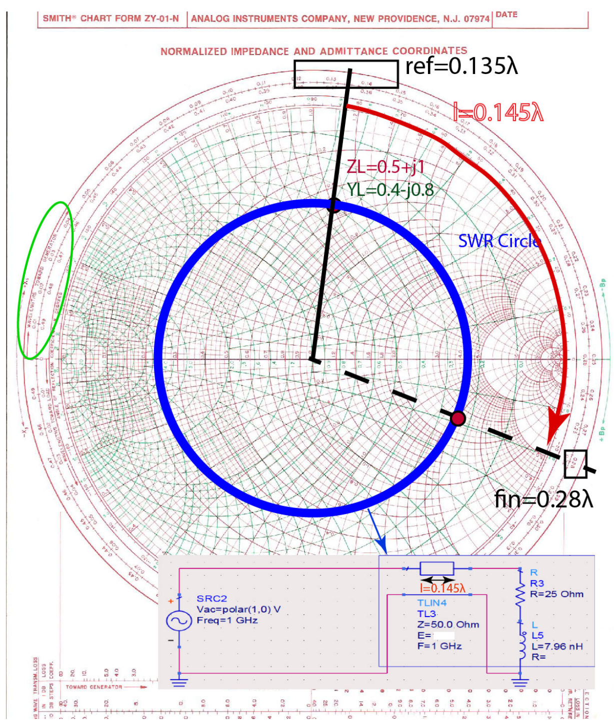


Figure 55: Finding input reflection coefficient from load reflection coefficient.

Finding load reflection coefficient on the Smith Chart if the input reflection coefficient is known

If the input reflection coefficient is given, then to find the load reflection coefficient, we can re-write the previous equations as

$$\Gamma_{in} = \Gamma_L e^{-j2\beta l} \tag{203}$$
$$\Gamma_L = \Gamma_{in} e^{j2\beta l} \tag{204}$$
$$\Gamma_L = |\Gamma_{in}| e^{j(\Theta_{in}+2\beta l)} \tag{205}$$

The above equation shows that when we are looking for a load reflection coefficient, we have to add $2\beta l$ to the input reflection coefficient's phase. We, therefore, have to go in a counter-clockwise direction on the Smith Chart. Identify the WAVELENGTHS TOWARD THE LOAD scale on the Smith Chart, and verify that the arrow points in the counter-clockwise direction.

Examples

Example 24. *The line is terminated with an impedance of $Z_L = 100 - j50\Omega$. Transmission line impedance is $Z_0 = 50\Omega$, and line length is $l = \lambda/4$ at 1 GHz. Calculate the impedance at the input of a transmission line, and the reflection coefficient at the input of a transmission line. Then, repeat this exercise using the Smith Chart.*

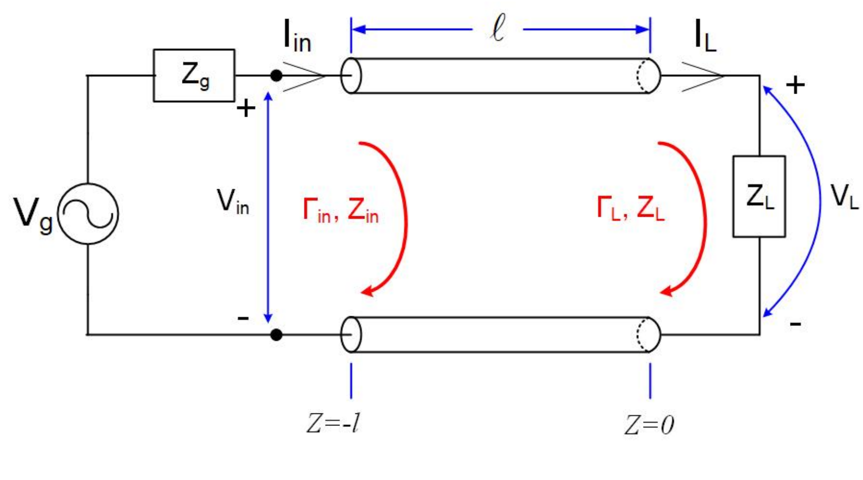


Figure 56: Finding input reflection coefficient from load reflection coefficient.

Explanation. *The input reflection coefficient* Γ_{in} *is*

$$\Gamma_{in} = \Gamma_L e^{-j2\beta l} \tag{206}$$
$$\Gamma_{in} = |\Gamma_L| e^{j\Theta_L} e^{-j2\beta l} \tag{207}$$
$$\Gamma_{in} = |\Gamma_L| e^{j(\Theta_L - 2\beta l)} \tag{208}$$

The load reflection coefficient is $\Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0} = 0.45 e^{-j26^0}$.

The electrical length of the line is $\beta l = 90^0$.

The input reflection coefficient is $\Gamma_{in} = |\Gamma_L| e^{j(\Theta_L - 2\beta l)} = 0.45 e^{j(-26^0 - 180^0} = 0.45 e^{-j206^0}$.

Normalized input impedance is

$$z_{in} = \frac{1 + \Gamma_{in}}{1 - \Gamma_{in}} = 0.4 + j0.2 \tag{209}$$

The input impedance is $Z_L = 50\,(0.4 + j0.2) = (20 + j10)\,\Omega$

To find the input reflection coefficient and impedance on the Smith Chart, we first normalize the given load impedance $Z_L = 2 - j$, *then find its position on the Smith Chart and draw the SWR circle. We know that the input reflection coefficient will be on this circle because the magnitude of the input reflection coefficient and the load reflection coefficient has to be the same.*

The reference position of the load on the WAVELENGTHS TOWARD GENERATOR scale is 0.285λ. *Next, we add the length of the line to the reference position* $0.285\lambda + 0.25\lambda = 0.535\lambda$. *If we look at the scale, we see that it resets at* 0.5λ. *To find* 0.535λ, *we have to continue in the direction of the arrow for another* 0.035λ. *We now found the phase of the input reflection coefficient. To find the input reflection coefficient, we find the line that starts at the center of the Smith Chart and ends on the* 0.035λ, *and then find the point where this line crosses the SWR circle. Figure 57 shows these steps graphically.*

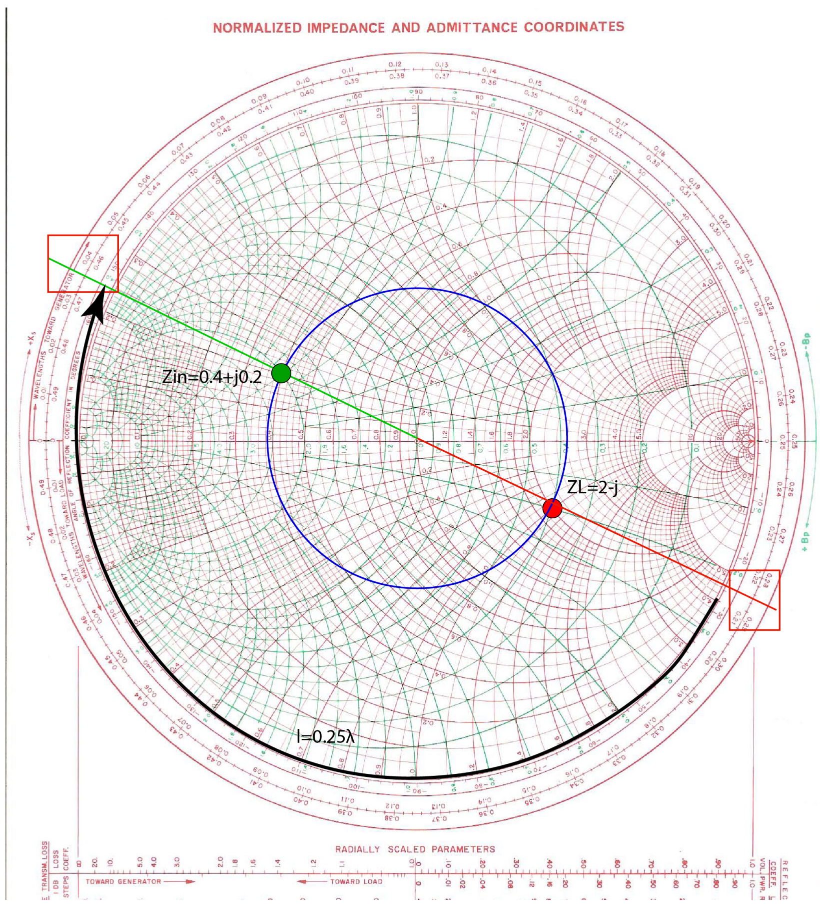


Figure 57: Example finding input impedance.

6 Impedance Matching

After completing this section, students should be able to do the following.

- Explain why is the impedance matching needed
- How real power changes from the input to the output of a lossless transmission line
- Explain how maximum available power from the generator affects power on transmission line.
- Explain how minimum reflection from the generator affects power on a transmission line.
- Explain how minimum reflection from the load affects power on a transmission line.
- Design a simple impedance matching network for any load impedance and discuss pros and cons of various designs.
- Design a mixed impedance matching network for any load impedance and discuss pros and cons of various designs.
- Design a transmission-line impedance matching network for any load impedance and discuss pros and cons of various designs.
- Design a lumped element impedance matching network for any load impedance and discuss pros and cons of various designs.

6.1 Power

The current and voltage phasor transformation is defined as

$$v(t) = \Re\{|V|e^{j\theta_v}e^{j\omega t}\} \tag{210}$$
$$i(t) = \Re\{|I|e^{j\theta_i}e^{j\omega t}\} \tag{211}$$

Where $|V|e^{j\theta_v}$ and $|I|e^{j\theta_i}$ are phasors of voltage and current, and usually denoted with a tilde $\tilde{}$ over capital letters $\tilde{V}$, $\tilde{I}$.

$$v(t) = \Re\{\tilde{V}e^{j\omega t}\} \tag{212}$$
$$i(t) = \Re\{\tilde{I}e^{j\omega t}\} \tag{213}$$

The real part of a complex number can also be found as $\Re\{z\} = \frac{1}{2}(z + z^*)$, so the above two equations can be re-written as

$$v(t) = \frac{1}{2}(\tilde{V}e^{j\omega t} + \tilde{V}^*e^{-j\omega t}) \tag{214}$$
$$i(t) = \frac{1}{2}(\tilde{I}e^{j\omega t} + \tilde{I}^*e^{-j\omega t}) \tag{215}$$

Power is defined as a product of voltage and current.

$$p(t) = v(t)i(t) \tag{216}$$

If we replace voltage and current in Equation 236 in the time domain with Equations 214 and 215 we get

$$p(t) = \frac{1}{4}(\tilde{V}e^{j\omega t} + \tilde{V}^*e^{-j\omega t})(\tilde{I}e^{j\omega t} + \tilde{I}^*e^{-j\omega t}) \tag{217}$$

Multiplying the terms above, and rearanging, we get:

$$p(t) = \frac{1}{4}(\tilde{V}\tilde{I}^* + (\tilde{V}\tilde{I}^*)^* + \tilde{V}\tilde{I}^*e^{2j\omega t} + (\tilde{V}\tilde{I}^*e^{2j\omega t})^*) \tag{218}$$

We can again apply equation $\Re\{z\} = \frac{1}{2}(z + z^*)$ to simplify the above equation to

$$p(t) = \frac{1}{2}(\Re\{\tilde{V}\tilde{I}^*\} + \Re\{\tilde{V}\tilde{I}e^{2j\omega t}\}) \tag{219}$$

This can also be re-written as

$$p(t) = \frac{1}{2}\Re\{|V|e^{j\theta_v}|I|e^{-j\theta_i}\} + \frac{1}{2}\Re\{|V|e^{j\theta_v}|I|e^{j\theta_i}e^{2j\omega t}\} \tag{220}$$

$$p(t) = \frac{1}{2}\Re\{|V||I|e^{j(\theta_v-\theta_i)}\} + \frac{1}{2}\Re\{|V||I|e^{j(\theta_v+\theta_i)}e^{2j\omega t}\} \tag{221}$$

p(t) above is instantaneous power, $S = \tilde{V}\tilde{I}^* = |V||I|e^{j(\theta_v-\theta_i)}$ is complex power. Complex power has real and reactive parts S=P+jQ. The first part of the equation represents the average real power P delivered to the load $P = \frac{1}{2}\Re\{\tilde{V}\tilde{I}^*\}$, and the second part represents the fluctuating power. We are usually interested in the average real power P delivered to the load.

To find the real power delivered to the load, one would take the real part of the complex power. If we know that the impedance of the load is $Z = R + jX$, the voltage is $\tilde{V} = Z\tilde{I}$ and we remember that $\tilde{I}\tilde{I}^* = |I|^2$ then the real power is

$$P = \frac{1}{2}\Re\{\tilde{V}\tilde{I}^*\} \tag{222}$$

$$P = \frac{1}{2}\Re\{(R + jX)\tilde{I}\tilde{I}^*\} \tag{223}$$

$$P = \frac{1}{2}\Re\{(R + jX)|I|^2\} \tag{224}$$

$$P = \frac{1}{2}|I|^2\Re\{(R + jX)\} \tag{225}$$

$$P = \frac{1}{2}R|I|^2 \tag{226}$$

Example 25. *A transmitter operated at 20MHz, Vg=100V with* 50Ω *internal impedance is connected to an antenna load through 6.33m of the line. The line is a lossless* 50Ω, $\beta = 0.595 rad/m$. *The antenna impedance at 20MHz measures* $Z_L = 36 + j20\Omega$.

(a) *What is the electrical length of the line? (answer: length=0.6λ)*

(b) *How much power is delivered to the line? Hint: Find the input impedance, then find the input power as* $P_{ave,in} = \frac{1}{2}R_{in}|I_{in}|^2$

(c) *What is the time-average power absorbed by* Z_L. $P_L = \frac{1}{2}R_L|I_L|^2$

(d) *If now we match load impedance Z_l to 50 Ohm line, what is the input impedance of the line, and how much average power is delivered to the line and load?*

6.2 Power Transfer on a transmission line

Power Transfer on a Transmission Line

Power on a transmission line can be found similarly to the derivation in the previous section. It is just that now, the total voltage is the sum of the forward-going and reflected voltage, and the total current is the sum of the forward and reflected current.

$$\tilde{V}(z) = \tilde{V}_0^+ e^{-\gamma z} + \tilde{V}_0^- e^{\gamma z} \tag{227}$$
$$I(z) = \tilde{I}_0^+ e^{-\gamma z} + \tilde{I}_0^- e^{\gamma z} \tag{228}$$

Where $\tilde{V}^+(z) = \tilde{V}_0^+ e^{-\gamma z}$ is the phasor of forward voltage anywhere on the line, and $\tilde{V}^-(z) = \tilde{V}_0^- e^{\gamma z}$ is the phasor of reflected voltage anywhere on the line. $\tilde{V}_0^+$ is the phasor of forward voltage at the load, where z=0, and $\tilde{V}_0^-$ is the phasor of reflected voltage at the load, where z=0. The currents are similarly named. If the line is lossless, then the attenuation coefficient $\alpha = 0$ and $\gamma = \alpha + j\beta = j\beta$, so the equations become

$$\tilde{V}(z) = \tilde{V}_0^+ e^{-j\beta z} + \tilde{V}_0^- e^{j\beta z} \tag{229}$$
$$I(z) = \tilde{I}_0^+ e^{-j\beta z} + \tilde{I}_0^- e^{j\beta z} \tag{230}$$

If we define a phasor of reflection coefficient at the load as $\Gamma = \dfrac{\tilde{V}_0^-(z)}{\tilde{V}_0^+(z)}$, and define the transmission-line impedance as $Z_0 = \dfrac{\tilde{V}_0^+}{\tilde{I}_0^+} = -\dfrac{\tilde{V}_0^-}{\tilde{I}_0^-}$ then the equations become

$$\tilde{V}(z) = \tilde{V}_0^+ e^{-j\beta z} + \Gamma \tilde{V}_0^+ e^{j\beta z} \tag{231}$$
$$I(z) = \frac{\tilde{V}_0^+}{Z_0} e^{-j\beta z} - \Gamma \frac{\tilde{V}_0^+}{Z_0} e^{j\beta z} \tag{232}$$

By multiplying phasors above, we get the phasor of the average real incident (forward) P_i and reflected power P_r anywhere on the line.

$$P_i = \frac{|\tilde{V}_0^+|^2}{2Z_0} \tag{233}$$

$$P_r = |\Gamma|^2 \frac{|\tilde{V}_0^+|^2}{2Z_0} \tag{234}$$

Total power delivered to the load is then $P_L = P_i - P_r$

$$P_L = \frac{|\tilde{V}_0^+|^2}{2Z_0} - |\Gamma|^2 \frac{|\tilde{V}_0^+|^2}{2Z_0} \tag{235}$$

$$P_L = \frac{|\tilde{V}_0^+|^2}{2Z_0}(1 - |\Gamma|^2) \tag{236}$$

Maximizing power transfer on a transmission line

Looking at Equation 236, to maximize power delivered to the load P_L , we have to maximize $|\tilde{V}_0^+|$, or minimize Z_0 and $|\Gamma|$.

(a) Z_0, the transmission-line impedance, is fixed if we are using a specific type of a coaxial cable as we have seen previously, typical impedances of coaxial cables are 50Ω, 75Ω, 300Ω. However, if we are using microstrip lines, transmission line impedance can be part of the circuit design, and we can make Z_0 lower.

(b) To minimize $|\Gamma| = \dfrac{\tilde{V}_0^-}{\tilde{V}_0^+} = \dfrac{Z_L - Z_0}{Z_L + Z_0} = 0$ we have to make the reflected voltage (and power) zero by making the load impedance equal to the transmission line impedance $Z_L - Z_0 = 0$, or $Z_L = Z_0$.

(c) To maximize $|\tilde{V}_0^+|$, according to the maximum power transfer theorem, the input impedance to the transmission line has to be equal to the conjugate of the generator's impedance $Z_{in} = Z_g^*$. If the impedance of the generator is complex $Z_g = R_g + jX_g$, then the input impedance has to be $Z_{in} = R_g - jX_g$.

There are two requirements, $Z_L = Z_0$ that minimizes reflected power, and $Z_{in} = Z_g^*$ that maximizes forward power. The incident power on a transmission line is given as

$$P_i = \frac{|\tilde{V}_0^+|^2}{2Z_0} \tag{237}$$

The forward going voltage on a transmission line depends on the input reflection coefficient, as shown in Equation 238.

$$\tilde{V}_0^+ = \frac{Z_0}{Z_0(1+\Gamma_{in}) + Z_g(1-\Gamma_{in})} V_g e^{-j\beta l} \tag{238}$$

$\Gamma_{in} = \dfrac{Zg - Z_0}{Zg + Z_0}$ is the reflection coefficient looking into the generator. It is interesting to see that when the generator impedance and transmission line impedance are the same (the reflection coefficient looking into the generator Γ_g is zero), the forward-going voltage does not depend on the load impedance Z_L. Forward-going voltage is equal to $\tilde{V}_0^+ = \dfrac{1}{2} V_g e^{j\beta z}$!

When we turn on the generator, the generator sees a voltage divider: the generator's impedance Z_g and only the transmission-line impedance Z_0 at the input of the line, not Z_{in}, because in the transient, the reflected voltage is not there to make the total impedance Z_{in}. The signal initially starts traveling toward the load, reflects off the load, and the reflected part of the voltage travels back to the generator. When the reflected part of the voltage arrives at the generator if $\Gamma_g \neq 0$, then the voltage wave reflects again from the generator, and the forward going voltage $\tilde{V}_0^+$ will change for the amount of that new reflection. However, if $\Gamma_g = 0$, the reflected wave will be absorbed by the generator, and no wave will be reflected. When input reflection coefficient is zero, $\Gamma_g = \dfrac{Zg - Z_0}{Zg + Z_0} = 0$. The generator impedance and the transmission-line impedance are the same.

Suppose we also select the reflected power to be zero by setting the load impedance to the transmission line impedance $Z_L = Z_0$. In that case, we have the best of both worlds, maximum power transfer and minimum reflection!

Why we do impedance matching?

We perform impedance matching to remove the reflected wave on the transmission line and maximize the power available from the generator.

Impedance matching is a technique that

(a) ensures maximum power transfer between the generator V_g and the load Z_L, and

(b) minimizes the reflected power from the load.

To ensure that the voltage does not reflect from the generator, the generator's impedance should be the same as the transmission-line impedance.

When we insert the impedance matching circuit between the load Z_L and the transmission line $Z_\circ$, as shown in Figure 58 the input impedance presented to the generator and the transmission line from the impedance matching circuit is $Z_\circ$, and the impedance presented to the load impedance Z_L will be Z_L^*. Our task is to add capacitors and inductors to the load Z_L to make the input impedance equal to $Z_\circ$. We **do not** want to use resistors in an impedance matching circuit because they will use some of the power that we need at Z_L.

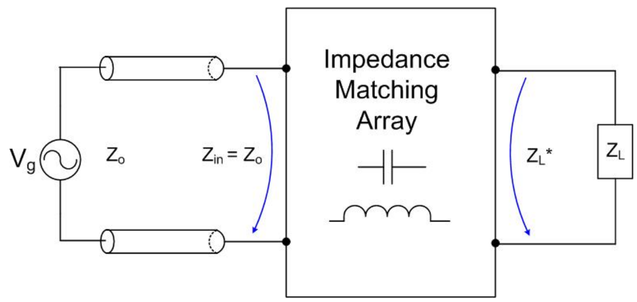


Figure 58: The result of impedance matching.

6.3 Simple impedance matching case

Simple impedance matching case

The simplest impedance matching case is when the real part of the load impedance is already equal to the transmission line impedance.

Let's say that the load impedance is $Z_L = R_L + j\omega L = 50 + j80\Omega$ and needs to be matched to a $Z_0 = 50\Omega$ line. This impedance represents a resistor of 50Ω connected in series with a 80Ω-reactance inductor. The reactance of 80Ω is an inductor of inductance 12.73nH at 1GHz.

To make this impedance look like a 50Ω impedance, we have to add an element with $Z_{add} = -j80\Omega$ impedance, so that $Z_L + Z_{add} = 50\Omega$. Since Z_{add} impedance is negative, we know that we have to add a capacitor. Because we are adding two impedances, we know that they must be in series.

To look at this solution on the Smith Chart, we first normalize the impedance of the load $\overline{Z}_L = \dfrac{Z_L}{Z_0} = 1 + j1.6$, and then place it on the Smith Chart as shown in Figure 59 as a red dot. To get to the center of the Smitch Chart, we use only the resistance/conductance circles. On these circles, centered on x-axis, only the reactance/susceptance of the impedance changes, and we don't change the resistance. We only add capacitors or inductors in matching circuits, and so we keep constant the real part of the impedance or admittance. We see that the circle we have to use is the $Z = 1$ circle, where the real part of the load impedance is constant. To get to the center of the chart, where $Z = 1$, we find the reactance to add to the load impedance. To make the final, matched impedance equal to one, we have to add impedance $Z_{add} = -j1.6$. Since this impedance is negative, it's a capacitor. Another way to see that this is a capacitor is to notice that by adding this element, we are moving from the inductive (upper half) to capacitive part (lower part) of the Smith chart.

Finally, we have to find the capacitance of a $Z_{add} = -j1.6$. The added impedance is first multiplied by 50Ω to re-normalize the impedance. The impedance of any capacitor is $Z = \dfrac{1}{j\omega C} = -j\dfrac{1}{\omega C}$. To find the capacitance of this capacitor we set $-j\dfrac{1}{\omega C} = -j1.6 \times 50$. From this equation, and keeping in mind that $\omega = 2\pi f$, we calculate that the capacitance is $C \approx 2\text{pF}$.

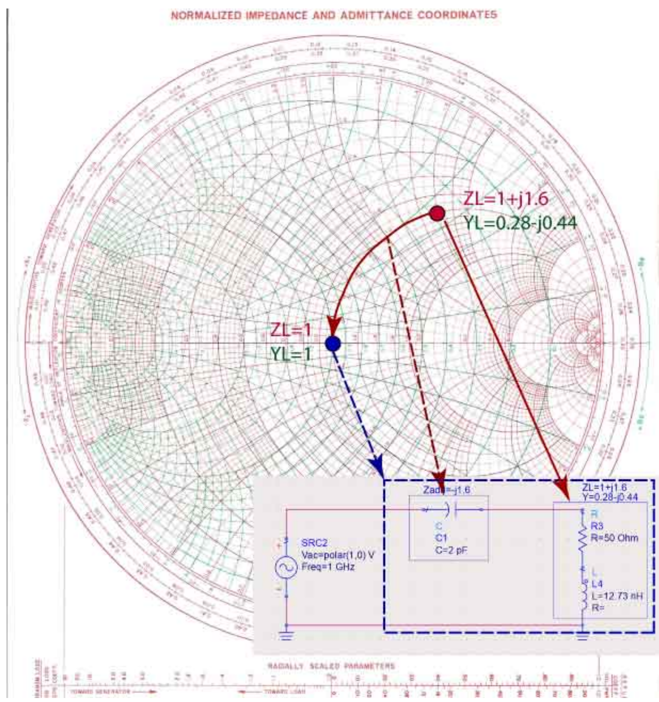


Figure 59: Impedance matching circuit for impedance $Z_L = R_L + j\omega L = 50 + j80\Omega$.

6.4 Mixed Impedance Matching

The real part of the load impedance is rarely equal to the transmission-line impedance. In most cases, we have to transform the real part of the impedance as well.

For example, load impedance $Z_L = 25 + j50\Omega$ represents a series connection of a 25Ω resistor and a $7.96\,\text{nH}$ inductor at $1\,\text{GHz}$. To match $Z_L = 25 + j50\Omega$ impedance to the transmission-line impedance $Z_0 = 50\Omega$, we first normalize the load impedance to transmission-line impedance.

$$\overline{Z}_L = \frac{Z_L}{Z_0} = 0.5 + j1 \tag{239}$$

This impedance is shown in Figure 60.

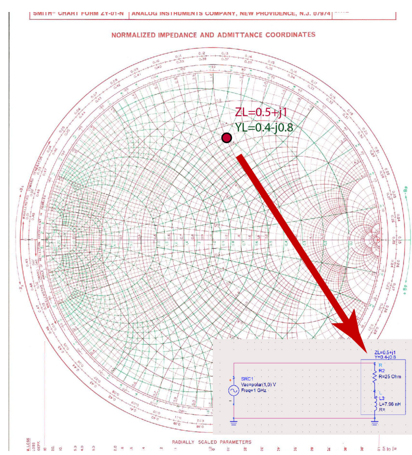


Figure 60: Load impedance $Z_L = 0.5 + j1$ on Smith Chart.

SWR circle

Then, we identify an SWR circle that this impedance is on, as shown in Figure 61. The point where the SWR circle intersects the green circle is of interest because the real part of the input admittance is equal to one $Y_1 = 1$. This second point, where $Y = 1 + j1.6$, will give us the length of the line that we have to add to the load impedance.

Example 26. *Explain how the position of the load impedance on the Smith Chart changes the SWR circle.*

Explanation. *To see how SWR circle changes depending on where the load impedance is use the following simulation. Click on point A, to change its position. Observe how SWR circle changes.*

Geogebra link: https://tube.geogebra.org/m/ugy2wcxc

Length of the line that will transform the real part of load impedance

To find the length of the line that will transform the real part of z_L to $y_1 = 1$, we identify the position of the load impedance $Z_L = 0.5 + j1$, and the input impedance $Z_1 = 0.3 - j0.45$ at the *Wavelengths Towards Generator* (WTG) scale. The reason we picked impedance $Z_1 = 0.3 - j0.45$ is because the real part of the admittance $y_1 = 1/z_1$ is equal to one $\Re(y_1) = 1$. Load impedance Z_L is at 0.135λ, and the input impedance Z_1 is at 0.425λ. The difference between these two positions gives us the length of the line 0.29λ. In electrical degrees, this length is approximately 105^0. The input admittance to the line is now $Y_L = 1 + 1.6$.

Adding a lumped-element to remove the susceptance

The final step is to add a susceptance that will remove the imaginary part of the input admittance $Y_1 = 1 + j1.6$. We see that to get the final admittance of $Y_M = 1$, numerically, we have to add an admittance of $Y_{add} = -j1.6$. This represents an inductance. Since we are adding the two admittances $Y_1 + Y_{add}$, they have to be in parallel (as we know that when elements are in parallel, we add their admittances).

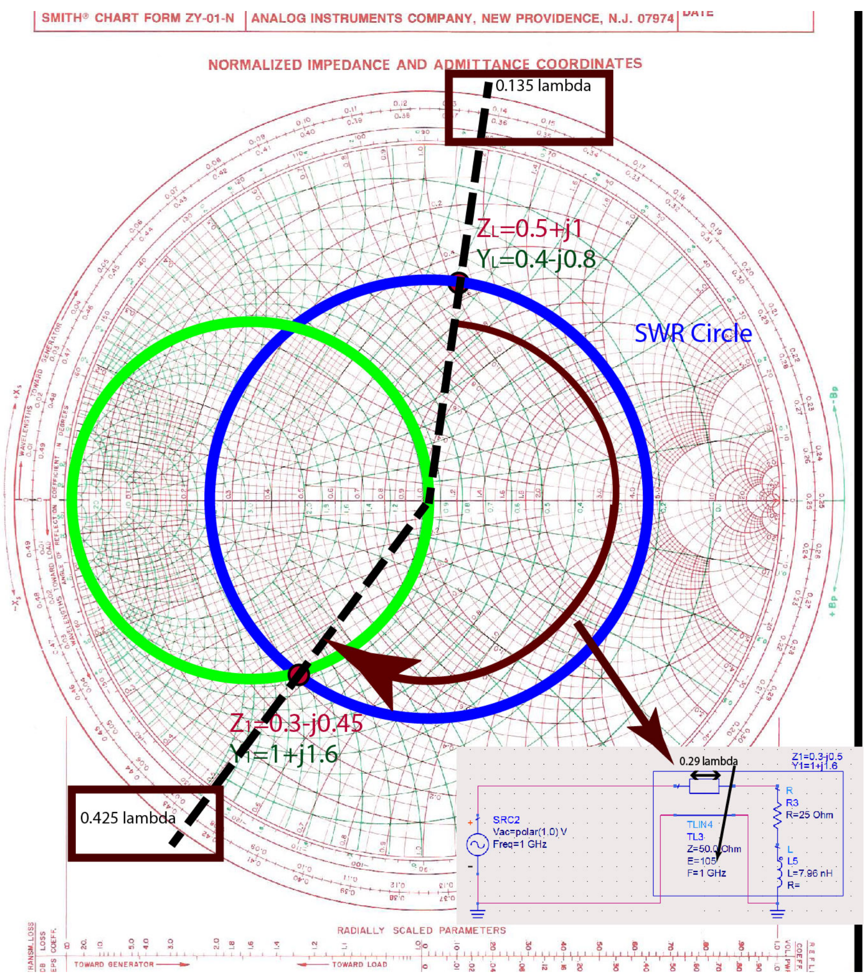


Figure 61: SWR circle for impedance $Z_L = 0.5 + j1$.

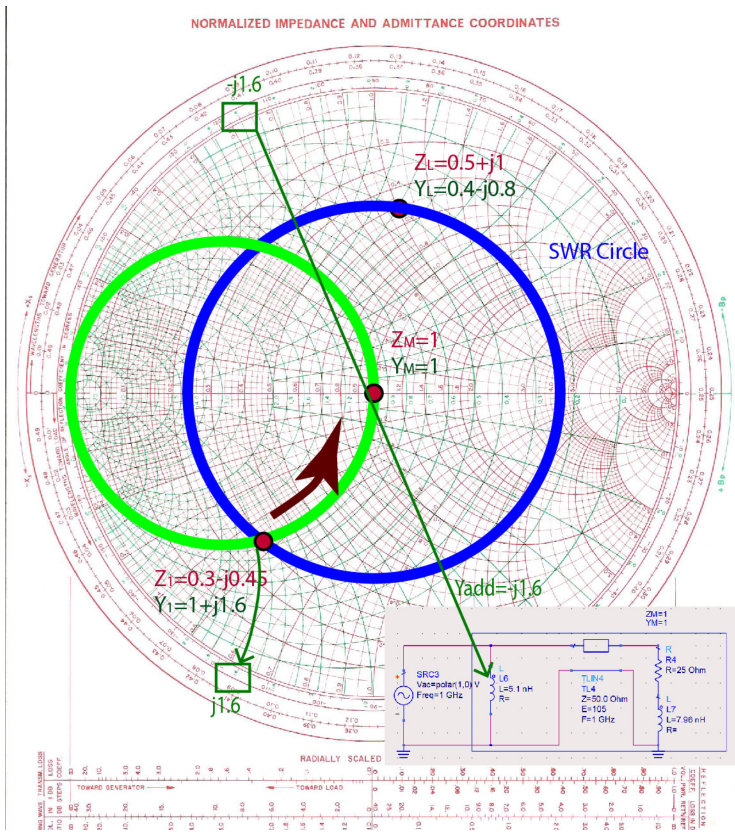


Figure 62: The result of impedance matching.

Other possible solutions

Graphically, there are several different mixed or transmission-line impedance matching circuits that we can make for a specific impedance. For example, for impedance $Z_L = 25 + j50\Omega$, $z_L = 0.5 + j1$, there are four different circuits that we can make, as shown in Figure 63. In this paragraph, we used the green path on the Smith Chart, with intermediate admittance Y_2.

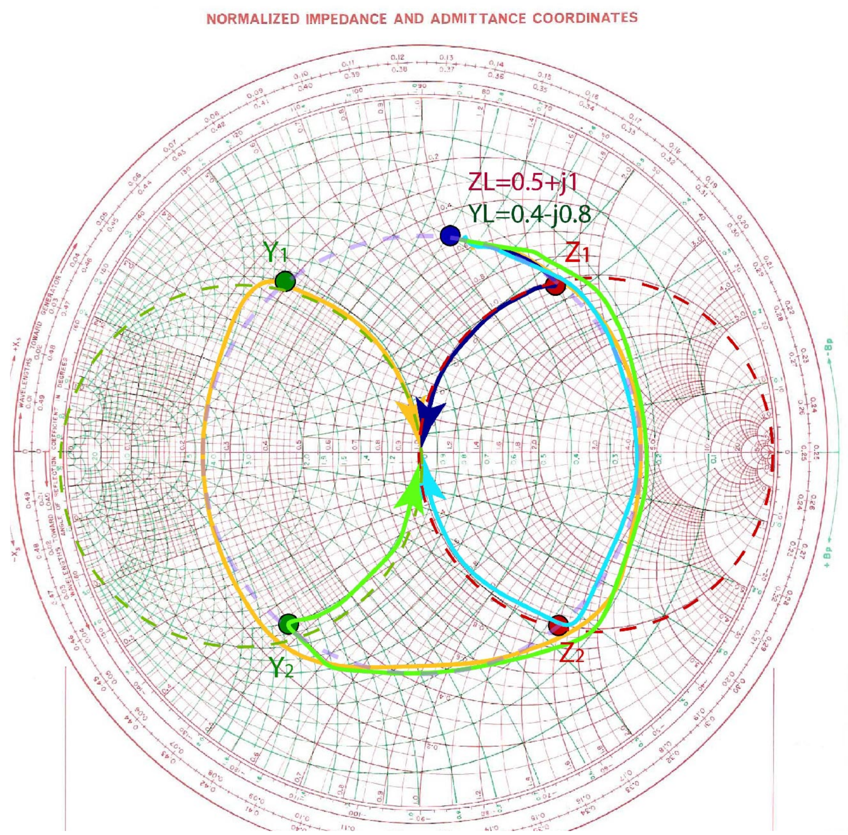


Figure 63: A variety of possible impedance matching circuits for impedance $Z_L = 25 + j50\Omega$.

6.5 Transmission-line impedance matching

Design of a transmission-line impedance matching circuits

Transmission-line impedance matching circuits are used at higher frequencies where the lumped elements become very small and impractical to use.

To design fully transmission-line matching circuits, we have to first learn how to replace the lumped element in the matching circuit from the last step in the previous section with a transmission line.

Replacing lumped-elements with equivalent transmission-line stubs

To make fully transmission line impedance matching circuits, we can replace capacitors and inductors with "stubs", which are shorted or open transmission lines.

The input impedance of shorted or open transmission lines can be made purely inductive or capacitive, as shown in Figures 64-66. SWR circle of an open or shorted stub is the outer perimeter of the Smith Chart. The length of the transmission line will determine the input impedance of the stub. The input impedance is always purely reactive.

To gain intuition of how the input impedance changes, as the length of the line changes, for a transmission-line terminated in open circuit, use the following simulation. In the simulation, change the length of the line, and observe how the input impedance changes.

Geogebra link: `https://tube.geogebra.org/m/w5fkzhvj`

Example 27. *Find the input impedance of a* $\frac{\lambda}{8}$ *long transmission line terminated in an open circuit.*

Explanation. *In Figure 64, the length of the line is* $\frac{\lambda}{8}$*. The "load" connected to the output of this line is an open circuit* $Z_L = \infty$*.*

On the Smith Chart, we start from the load impedance and add a 0.125λ *line, following the WTG scale. The input impedance is purely capacitive* $Z_{in} = -j$*.*

The input impedance is equal to a capacitor's impedance $Z_{in} = -j$ *at a single frequency, where the electrical length of the line is* 0.125λ*. At other frequencies,*

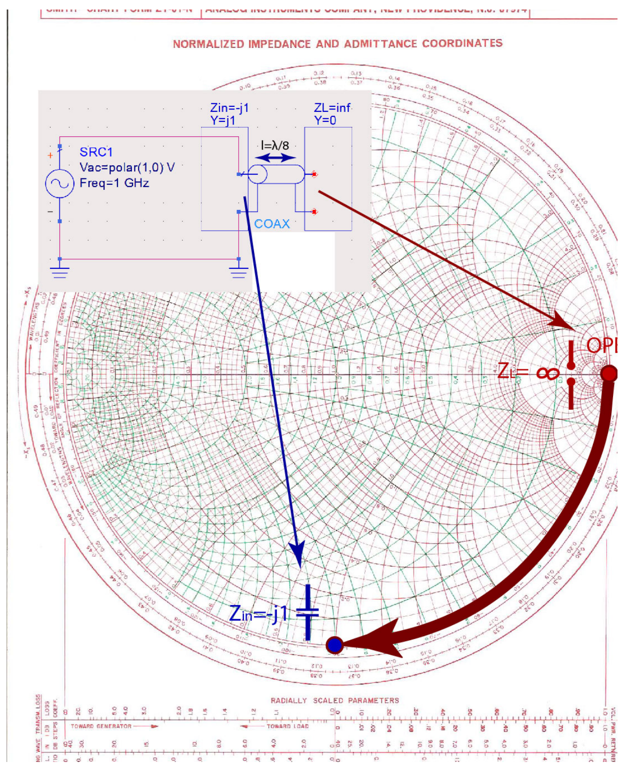


Figure 64: Input impedance $Z_{in} = -j$ of a $\dfrac{\lambda}{8}$ open transmission line.

the electrical length of the line is different, and therefore the input impedance is lower or higher than $Z_{in} = -j$.

Example 28. *Find the input impedance of a $\frac{3\lambda}{8}$ long transmission line terminated in an open circuit.*

Explanation. *Another example is shown in Figure 65, where the length of the line is $\frac{3\lambda}{8}$. The load connected to this line is again an open circuit $Z_L = \infty$. On the Smith Chart, we start from the load impedance and add a 0.375λ line, following the WTG scale. The input impedance is purely inductive $Z_{in} = j$.*

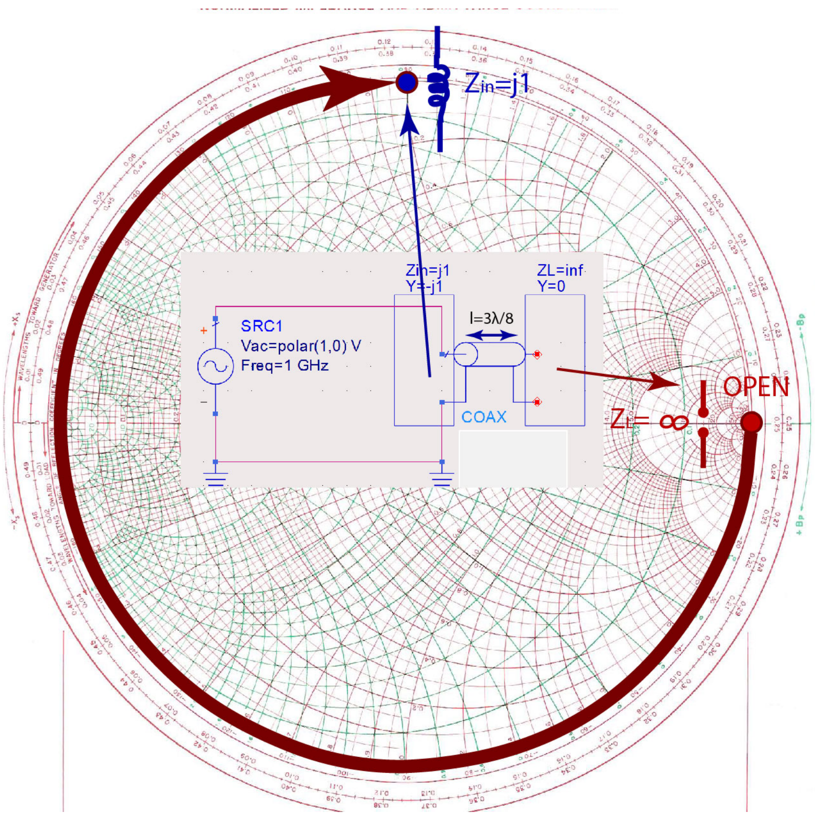


Figure 65: Input impedance $Z_{in} = j$ of a $\frac{3\lambda}{8}$ open transmission line.

Simulation of the input impedance of shorted stub

So far we discussed the open circuit at the end of the line. To see how input impedance the input impedance changes, as the length of the line changes, for a transmission-line terminated in a **short** circuit, use the following simulation. In the simulation, change the length of the line, and observe how the input impedance changes. How is this case different than the previous one?

Geogebra link: `https://tube.geogebra.org/m/yutfjed9`

Example 29. *Find the input impedance of a* $\frac{\lambda}{8}$ *long transmission line terminated in a short circuit.*

Explanation. *The last example is shown in Figure 66, where the length of the line is* $\frac{3\lambda}{8}$*. The load connected to this line is now a short circuit* $Z_L = 0\Omega$*. On the Smith Chart, we start from the load impedance and add a* 0.375λ *line, following the WTG scale. The input impedance is purely inductive* $Z_{in} = j$*.*

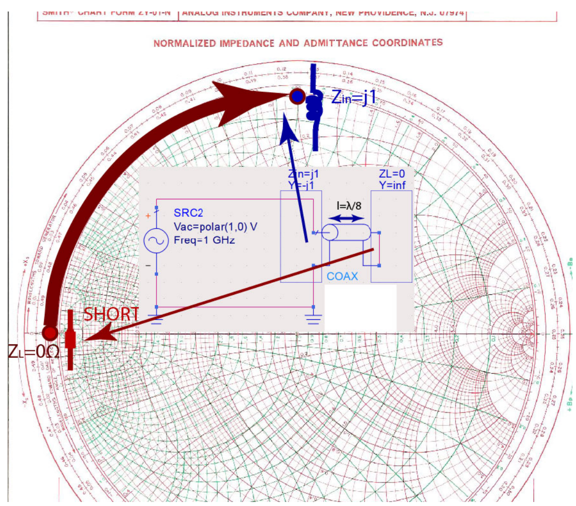


Figure 66: Input impedance $Z_{in} = j$ of a $\frac{\lambda}{8}$ shorted transmission line.

Fully transmission-line matching circuits.

The first few steps in the fully transmission-line matching circuit are the same as in the mixed-impedance matching. We find the point on the Smith Chart that represents the normalized impedance of the load $Z_L = 0.5+j1$, as in Figure 67.

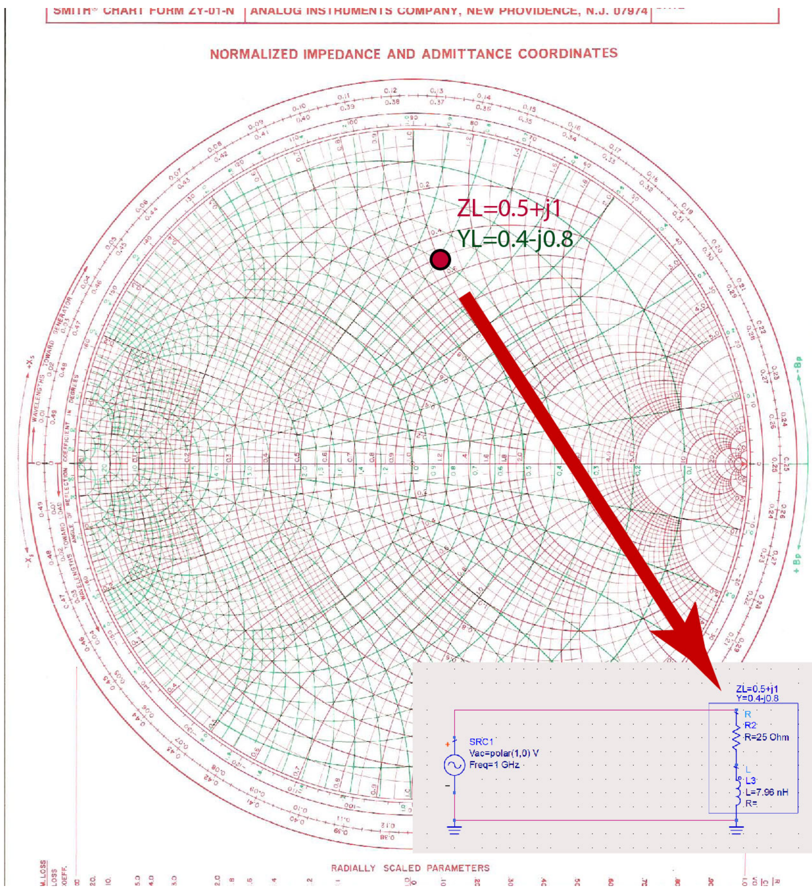


Figure 67: Position of normalized impedance $Z_L = 0.5 + j1$ on Smith Chart.

A section of transmission line has to be added to impedance $Z_L = 0.5 + j1$ to transform the load impedance to input admittance $Y_{in} = 1 + 1.6$, as shown in Figure 68. The length of the line is calculated by reading the position of the load impedance and input impedance on the WTG circle. Load impedance is at 0.135λ, and the input impedance is at 0.425λ. The difference between these two positions gives us the length of the line 0.29λ. In electrical degrees, this length is approximately 105^0. The input admittance to the line is now $Y_L = 1 + 1.6$.

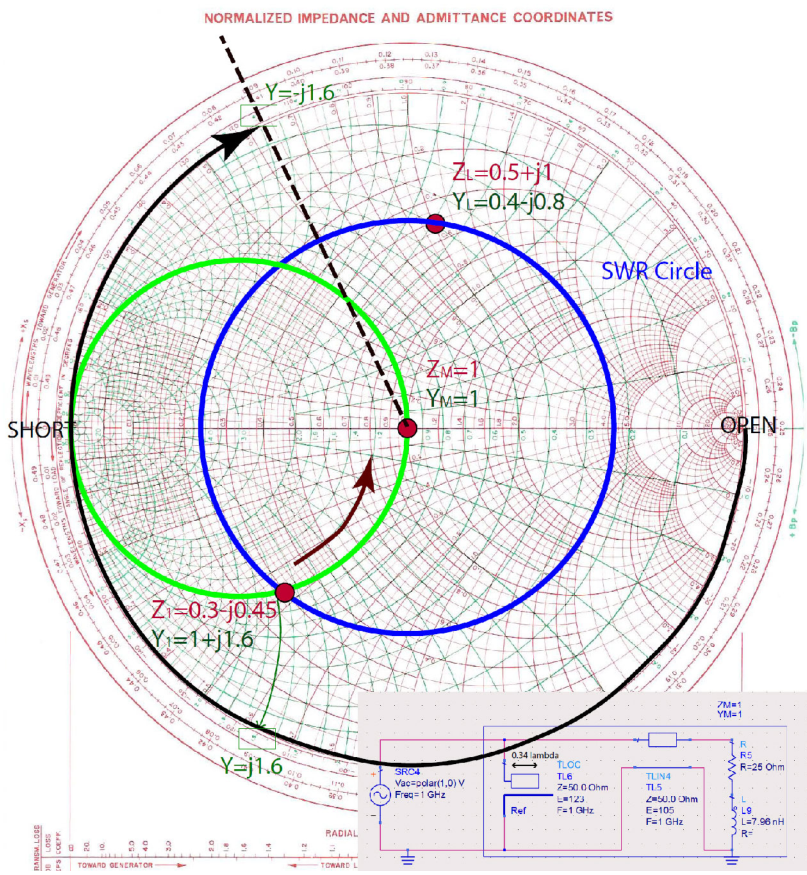


Figure 68: The result of impedance matching.

The last step is to determine the shunt reactance that will remove the reactive part of the admittance $Y_L = 1 + 1.6$, and make the total input impedance and admittance $Z_M = Y_M = 1$. By inspection, we have to add $Y_{add} = -j1.6$. A lumped element in parallel (because we are adding two admittances), as in the previous mixed-matching problem, is an inductor (because admittance of an inductor is negative) $Y_{add} = \frac{1}{j\omega L} = -j1.6$. However, we want to make a fully transmission line impedance matching circuit. Therefore, we have to replace the inductor with an equivalent transmission line, a stub.

We can pick whether the stub is open or shorted. Usually, we want to pick the shortest possible stub, because the smaller the stub, the smaller the circuit will be. In some other cases, it is easier to leave the transmission line open, because making a short circuit requires drilling the PC Board and making a via-hole to connect the line to the ground. In this case, we picked an open stub to replace the lumped element, as shown in Figure 69.

To find the stub that will have the same input admittance as an inductor $Y_{add} = -j1.6$, we first identify the position of admittance $-j1.6$ on the Smith Chart. This is the input impedance of the stub. We picked an open circuit for the load of the stub. To find the length of the stub, we start on the position of WTG scale from the load $Z_L = \infty$, WTG=0.25λ, going in the WTG direction, we pass the short circuit position WTG=0, and then another WTG=0.09λ to reach the input admittance of $-j1.6$. The length of the stub is $0.25\lambda + 0.09\lambda = 0.34\lambda$. In electrical degrees, this length is $\Theta = \beta l = 2\pi 0.34 \approx 123^0$.

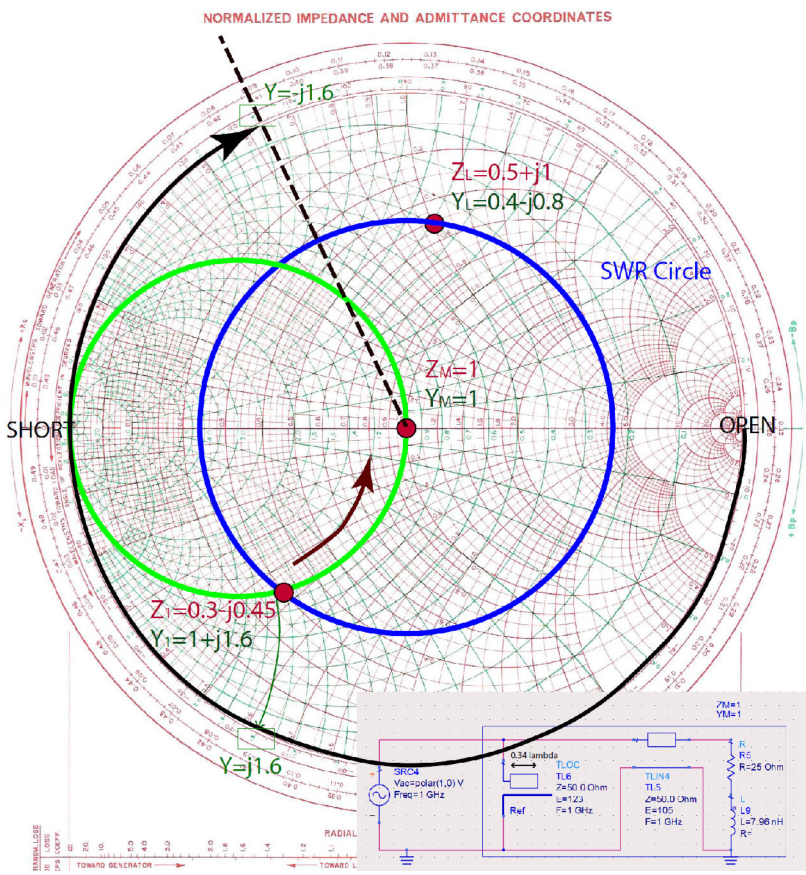


Figure 69: Transmission-line impedance matching circuit.

6.6 Lumped element impedance matching

In lumped-element impedance matching, we use only capacitors and inductors. Fully lumped-impedance matching circuits are used at lower frequencies, where transmission lines are too long to be practical for matching. Note that when we are matching the load with lumped elements, there are no SWR circles on the Smith Chart because we are not using transmission lines.

Graphically, there are several different lumped-element impedance matching circuits that we can make for a specific impedance. For example, for impedance $Z_L = 25 + j50\Omega$, there are four different 2-element circuits that we can make, as shown in Figure 70. In the next paragraph, we will use the orange path on the Smith Chart, with intermediate impedance Z_2.

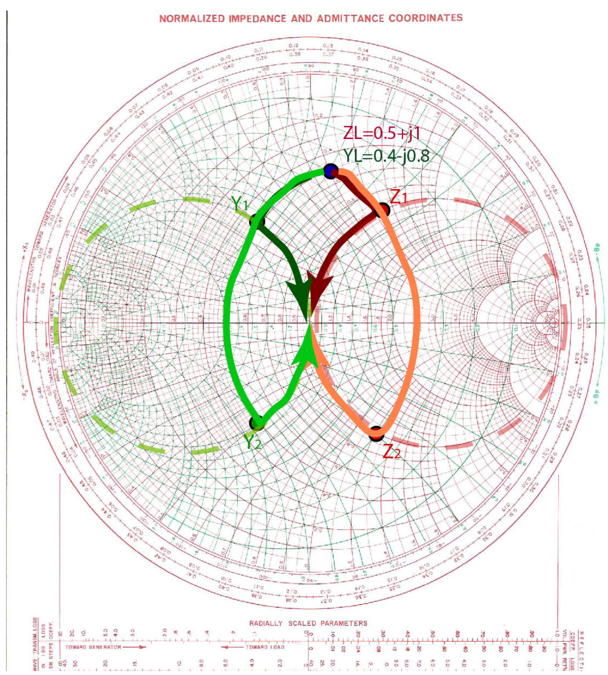


Figure 70: A variety of lumped-element impedance matching circuits for impedance $Z_L = 25 + j50\Omega$.

We start from the load impedance $Z_L = 25 + j100\Omega$, and we normalize it to the 50Ω to get $\overline{Z}_L = 0.5 + j1$. The equivalent admittance of this impedance is $\overline{Y}_L = 0.4 - j0.8$. We identify the position of this impedance on the Smith Chart. Figure 71 shows the green admittance line we follow to impedance $\overline{Z}_1 = 1 - j1.2$. The equivalent admittance of this impedance is $\overline{Y}_1 = 0.4 + j0.5$.

The green admittance line shows that we have to add $\overline{Y}_{add} = +j1.3$ to the load admittance $\overline{Y}_L = 0.4 - j0.8$ to get the total admittance of $\overline{Y}_1 = 0.4 + j0.5$. $\overline{Y}_L + \overline{Y}_{add} = \overline{Y}_1$. Since we are adding two admittances, the elements have to be in parallel. Because the added admittance is positive, the added element is a capacitor, as shown in Figure 71. To find the capacitance of the capacitance we need to add in parallel, we first have to re-normalize the additional admittance $\overline{Y}_{add}$ by multiplying it with the 0.02Si. The admittance of a capacitor is $j\omega C$ has to be equal to this added admittance. $2\pi f C = 1.3 * 0.02$. From this equation, we find that the capacitance is $C \approx 4.1$pF.

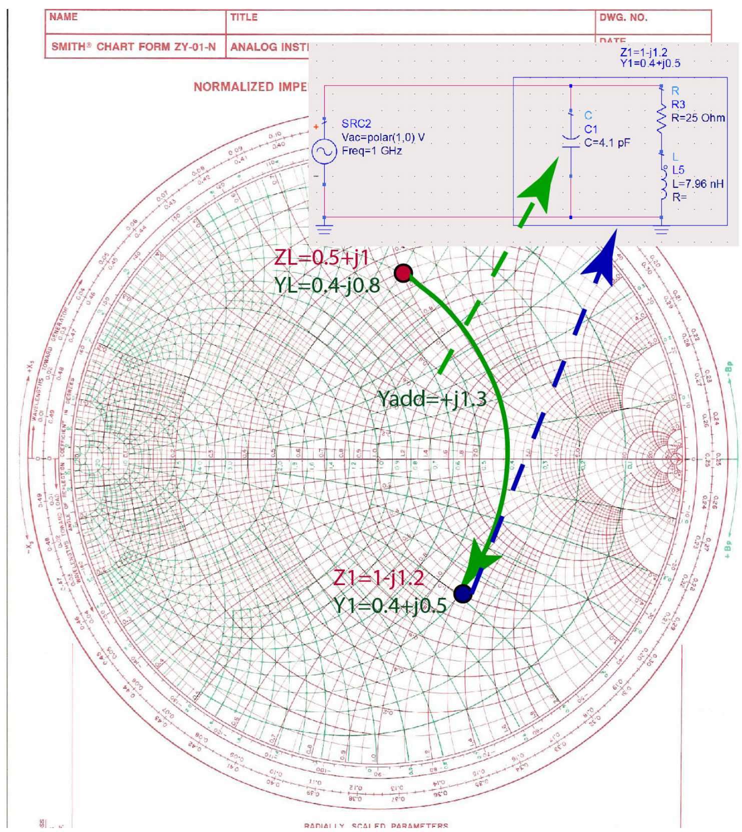


Figure 71: Adding an element in parallel to load impedance to get the equivalent real part of the impedance equal to one.

When we add 4.1pF capacitor in parallel with the load $\overline{Z}_L = 0.5 + j1$, the total impedance of the two elements in parallel is $\overline{Z}_1 = 1 - j1.2$.The real part of this impedance is already 50Ω. The final step is to remove the reactive part of the impedance by adding an additional impedance of $\overline{Z}_{add} = j1.2$, as shown in Figure 72. The total impedance is then $\overline{Z}_M = \overline{Z}_1 + \overline{Z}_{add} = 1$. Since we are adding two impedances, the elements must be in series. Because the added impedance is positive, it must be an inductor. To find the inductance of the inductor, $\overline{Z}_{add}50 = \omega L$. From this equation, we get that the inductance is $L \approx 9.86$nH.

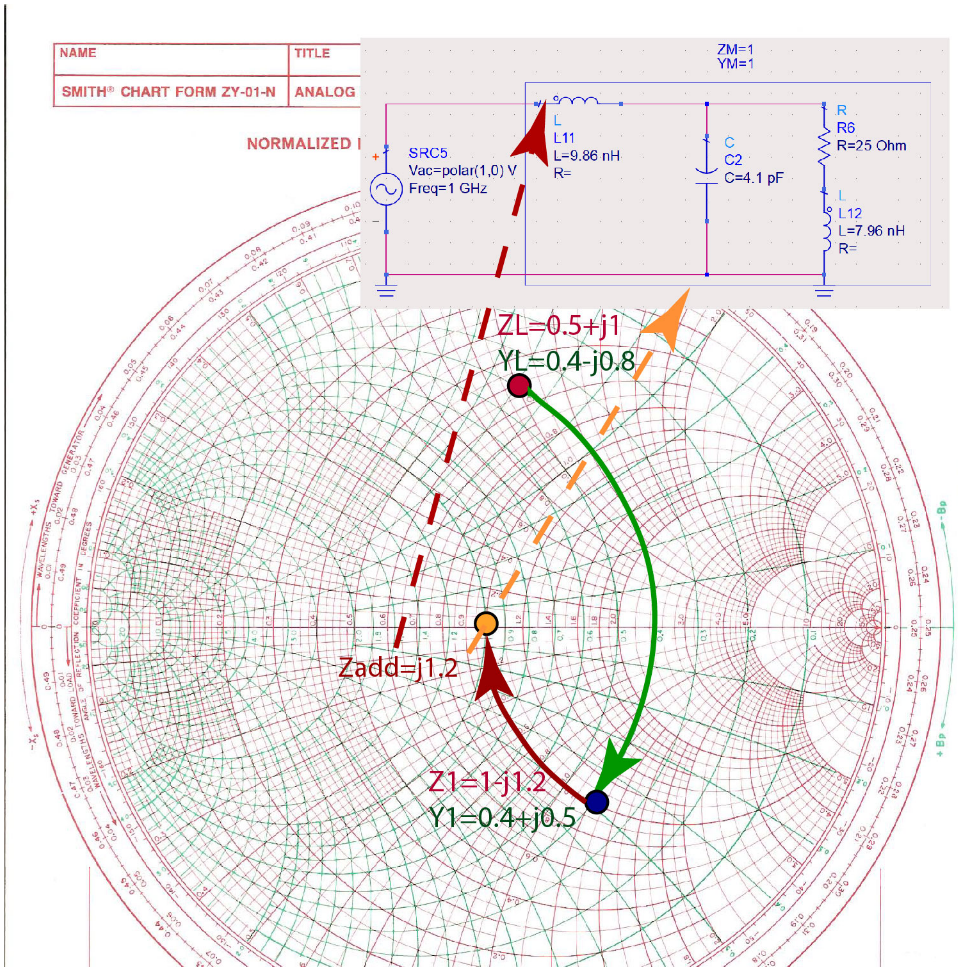


Figure 72: Finalized lumped-element impedance-matching circuit.

7 Electrostatics

After completing this section, students should be able to do the following.

- distinguish between the distance vector and
- vector at a point whose length describes strength of the field
- Identify charge of electron, neutron and proton
- State and explain Coulombs law
- Apply Coulombs law using vector algebra to find electrostatic force at a point due to several point charges
- Explain definition of electric field
- Derive and calculate electric field due to several point charges using vector algebra
- Define conditions under which Gausss law can be used.
- Outline steps necessary to apply Gausss law
- Derive electric field due to line and sphere of charge
- Discuss analogy of circuits with water pipes
- Define potential as difference in potential energy per charge
- Derive potential of point charge
- Find electric field from potential and vice versa
- Define capacitance
- Derive capacitance of simple symetrical structures.
- Explain displacement current
- Describe how materials affect electric fields.

7.1 Electrostatic Force

Electic Charges

Electric charges observed in nature are multiples of a charge of an electron $e = -1.610^{-19}$ C. JJ Thomson discovered the electron in his cathode ray tube experiments in 1897. R. Millikan measured the mass to charge ratio of the electron in 1909 through his oil-drop experiment. In 1960, J. G. King proved experimentally that one proton carries a positive charge of $e = 1.610^{-19}$ C.

Electrostatic Force

The electrostatic force acts between electric charges in the following way:

- two positive charges repel each other.
- two negative charges repel each other.
- a positive and a negative charge attract each other.
- the force between two charges decreases inversely proportional to the square of the distance.
- the force acts along the line that connects the charges.
- in nature, positive and negative charges are balanced, and the net result is electrical neutrality! Balance is formed by tight fine mixtures of positive and negative charges.

A demonstration of the electric force by the MIT professor emeritus Walter Lewin.

YouTube link: `https://www.youtube.com/watch?v=FKqWmnem3M4`

Coulomb's Law

What we described is exactly the electrostatic force. All matter is a mixture of positive protons and negative electrons in a perfect balance. Coulomb described the strength and direction of the electrostatic force through his torsion-balance experiment in 1785. We can represent this electrostatic force visually. Figure 73 shows a stationary charge $+q_1$, repelling charge $+q_2$ with a force F_2. The figure shows the unit vector $\hat{r}_{21}$, and the distance vector $\overrightarrow{\mathbf{R}}_{21} = r_{21}\hat{r}_{21}$, where $r_{21} = |\overrightarrow{\mathbf{R}}_{21}|$ is the distance between the two point charges. The equation that describes the electrostatic (Coulomb) force $\overrightarrow{\mathbf{F}}_2$ is given in Equation 240.

$$\overrightarrow{\mathbf{F_2}} = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{21}^2}\hat{r}_{21} \tag{240}$$

In the above equation, $\varepsilon_0 = 8.81\,10^{-12}\,\mathrm{F/m}$ is the electrical permittivity of air, r_{21} is the distance between charges, vector $\hat{r}_{21}$ is a unit vector oriented from charge 1 to charge 2. The unit vector is on the line that connects charges 1 and 2, and therefore the electrostatic force is also on the line that connects the two charges. The force will either point in the direction of the unit vector if the force is repulsive (charges have the same sign), or in the opposite direction when the force is attractive (charges have the opposite sign). Note that we need at least two charges to find the electrostatic force.

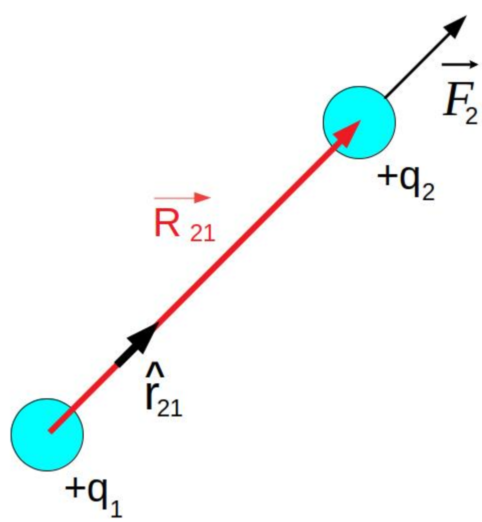


Figure 73: Vector representation of Coulomb's force between two static charges.

If the total net charge of an object is q, and if that object has n_e electrons and n_p protons, then the total charge is $q = n_p e - n_e e$.

Example 30. *Two positive unit charges $q_1 = 1\,nC$ and $q_2 = 1\,nC$ are fixed in air in Cartesian coordinate system at points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, see*

Figure 74. Find the electric force that a charge at point A exerts on charge at point B, and the force that charge at point B exerts on charge at point A. How would your answer change if one charge becomes negative?

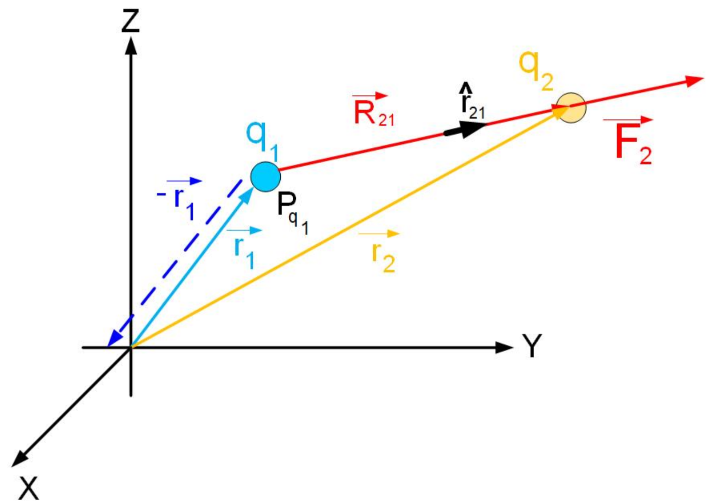


Figure 74: Electric Field due to a unit charge in Rectangular coordinate system.

Explanation. *The electrostatic force F_2 on charge q_2 is given by*

$$\overrightarrow{\mathbf{F_2}} = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{21}^2}\hat{r}_{21} \tag{241}$$

To find the force, we have to calculate

(a) *The magnitude of the force $|\overrightarrow{\mathbf{F_2}}| = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{21}^2}$. To find the magnitude, we need to find the distance between the two charges r_{21}. To find the distance, we need to find the distance vector $\overrightarrow{\mathbf{R}}_{21}$.*

(b) *To find the unit vector, we need to find the distance vector first. $\hat{r}_{21} = \frac{\overrightarrow{\mathbf{R}}_{21}}{r_{21}}$.*

The distance vector is a particular type of vector that starts at one point in the coordinate system and ends at another point. To find the distance vector $\overrightarrow{\mathbf{R}}_{21}$, we first have to know the position of charge q_1, where the distance vector starts, and the position of charge q_2 where the distance vector ends. In this problem, *the location of two charges is given by two points in the coordinate system A and B.*

The next step is finding the position vector r_1 *of point A, and the position vector* r_2 *of point B. Position vectors are special vectors that start in the coordinate system origin and point to various points in the coordinate system. Charge* q_1 *is at point* $A(x_1, y_1, z_1)$*, therefore the position vector* $\overrightarrow{\mathbf{r_1}}$ *of this point is shown in Equation 242.*

$$\overrightarrow{\mathbf{r_1}} = x_1 \overrightarrow{\mathbf{x}} + y_1 \overrightarrow{\mathbf{y}} + z_1 \overrightarrow{\mathbf{z}} \tag{242}$$

The position vector of charge q_2 *is*

$$\overrightarrow{\mathbf{r_2}} = x_2 \overrightarrow{\mathbf{x}} + y_2 \overrightarrow{\mathbf{y}} + z_2 \overrightarrow{\mathbf{z}} \tag{243}$$

The two vectors mark the beginning and the end of the distance vector $\overrightarrow{\mathbf{R}}_{21}$ *between charges* q_1 *and* q_2*. The vector* $\overrightarrow{\mathbf{R}}_{21}$ *is the sum of vectors* $-\overrightarrow{\mathbf{r_1}}$ *and* $\overrightarrow{\mathbf{r_2}}$*.*

$$\overrightarrow{\mathbf{R}}_{21} = \overrightarrow{\mathbf{r_2}} + (-\overrightarrow{\mathbf{r_1}}) \tag{244}$$

When we substitute position vectors r_1 *and* r_2*:*

$$\overrightarrow{\mathbf{R}}_{21} = (x_2 - x_1)\overrightarrow{\mathbf{x}} + (y_2 - y_1)\overrightarrow{\mathbf{y}} + (z_2 - z_1)\overrightarrow{\mathbf{z}} \tag{245}$$

The magnitude of vector $\overrightarrow{\mathbf{R}}_{21}$ *is*

$$|\overrightarrow{\mathbf{R}}_{21}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{246}$$

Unit vector in the direction of vector $\overrightarrow{\mathbf{R}}_{21}$ *is:*

$$\hat{r}_{21} = \frac{\overrightarrow{\mathbf{R}}_{21}}{|\overrightarrow{\mathbf{R}}_{21}|} \tag{247}$$

$$\hat{r}_{21} = \frac{\overrightarrow{\mathbf{R}}_{21}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}} \tag{248}$$

Substituting expressions for $\hat{r}_{21}$, and $|\overrightarrow{\mathbf{R}}_{21}|$ in equation for the electrostatic force

$$\overrightarrow{\mathbf{F_2}} = \frac{q_1 q_2}{4\pi\varepsilon_0 {r_a}^2}\hat{r}_{21} \tag{249}$$

We get

$$\overrightarrow{\mathbf{F_2}} = \frac{q_1 q_2}{4\pi\varepsilon_0 \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}^3}\overrightarrow{\mathbf{R}}_{21} \tag{250}$$

What if the charge is in an insulator (aka dielectric) other than air?

If the charge is within a dielectric material, then we need to account for that by changing this ε_0 somehow. If we place the charge inside a dielectric material, what do you think will happen with the atoms in the material? The atoms will get distorted and polarized. Such a polarized atom we call an electric dipole. The distortion process is called polarization. Because the material polarizes, the electric field around this point charge is different than if there was no material. To compensate for this new polarization, we multiply the dielectric permittivity of free space ε_0 with a unitless quantity of ε_r. ε_r is called a relative dielectric constant. ε_r values for different materials can found on the internet. Some examples of dielectric constants are ε_r: air ε_r=1, Teflon ε_r=2.2, glass ε_r=4.4, Silicon ε_r= 11, GaAs ε_r=12, distilled water ε_r= 80. Equation 251 is the definition of the electrostatic force between two charges. Sometimes, the product of $\varepsilon_0\varepsilon_r$ is written as ε.

$$\overrightarrow{\mathbf{F_e}} = \frac{q_1 q_2}{4\pi\varepsilon_0\varepsilon_r r^2}\hat{r}_{12} \tag{251}$$

$$\varepsilon = \varepsilon_0\varepsilon_r \tag{252}$$

Principle of Superposition

The principle of superposition states that in linear systems, we can calculate contributions of forces individually from different charges, then add them all up to get the total force on a charge.

If we have three or more charges, the total force from two charges to one charge is equal to the vector sum of the forces due to individual charges, see Figure 75.

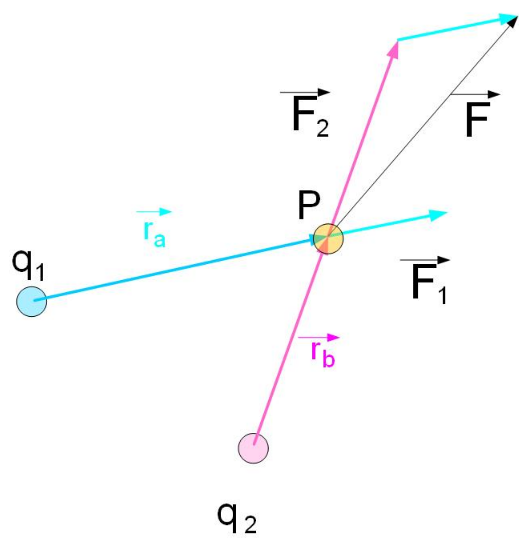


Figure 75: Electric Field due to two charges.

The force on the yellow charge below from charges q_1 and q_2 are:

$$\overrightarrow{\mathbf{F_1}} = \frac{q_1 q_y}{4\pi\varepsilon_0 {r_a}^2}\hat{r}_a \tag{253}$$

$$\overrightarrow{\mathbf{F_2}} = \frac{q_2 q_y}{4\pi\varepsilon_0 {r_b}^2}\hat{r}_b \tag{254}$$

Where $\hat{r_a}$ and $\hat{r_b}$ are unit vectors in the direction of r_a and r_b. The total field due to both charges is

$$\overrightarrow{\mathbf{F}} = \overrightarrow{\mathbf{F_1}} + \overrightarrow{\mathbf{F_2}} \tag{255}$$

Example 31. *Calculate the total force on a positive charge q_2 at a point (x_2, y_2, z_2) due to two other postive charges q_1 at a point (x_1, y_1, z_1) and charge q_3 at a point (x_3, y_3, z_3)*

Explanation. *Figure 76 shows the charges, distance vectors, position vectors and forces on charge q_2. The total force is equal to the sum of two individual forces from charges q_1 and q_2.*

$$\vec{\mathbf{F}} = \frac{q_1 q_2}{4\pi\varepsilon_0 \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}^3}\vec{\mathbf{r}}_{21} + \frac{q_2 q_3}{4\pi\varepsilon_0 \sqrt{(x_2-x_3)^2+(y_2-y_3)^2+(z_2-z_3)^2}^3}\vec{\mathbf{r}}_{23} \quad (256)$$

Where $\vec{\mathbf{r}}_{21}$ ist the distance vector from charge q_1 to q_2 and , $\vec{\mathbf{r}}_{23}$ is the distance vector from charge q_3 to q_2.

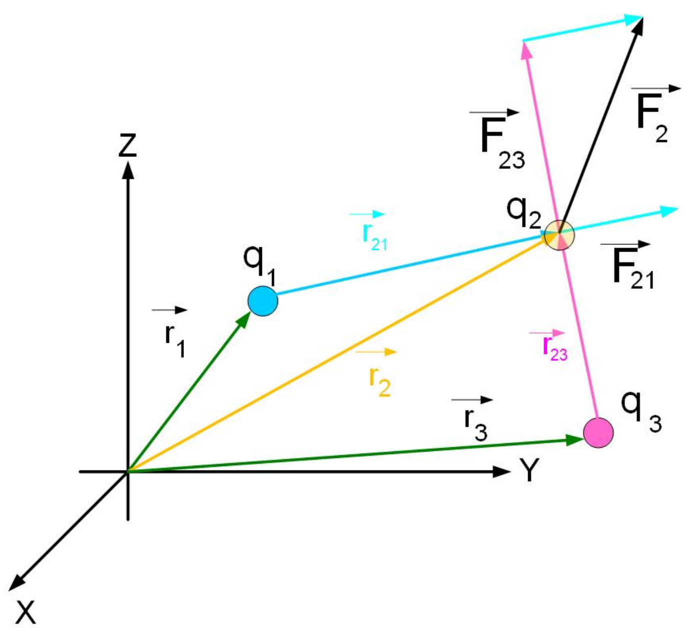


Figure 76: Electric field due to two charges in Rectangular coordinate system.

Problem 16 *Three positive charges, each $q = 100\mu C$ are placed at A(0,0), B(8,0), and C(4,4). Calculate the magnitude and direction of total force exerted on B due to charges A and C. Check your result with the calculator below. Explore with the calculator below how would the direction of the net force on B change if the charges A and C become negative.*

Geogebra link: `https://tube.geogebra.org/m/xqytpecf`

Question 17 *Four negative charges Q are distributed at (-1,0), (1,0), (0,1) and (0, -1). If we place the fifth charge Q at the origin (0,0), what will be the total force on this charge regardless of it's polarity?*

Multiple Choice:

(a) 0

(b) *Not enough information*

(c) $\dfrac{4Q^2}{4\pi\varepsilon_0}$

(d) $\dfrac{-4Q^2}{4\pi\varepsilon_0}$

Now, take a look at the simulation of the Baloon experiment. Charge the baloon by rubbing it on the sweater, then bring it to the wall. What happens? Observe how the neutral balloon is not attracted to the wall or sweater. When you rub it on the sweater, it will become attracted to the neutral wall. Why?

Geogebra link: `https://tube.geogebra.org/m/NcNVtwAQ`

A live demonstration of the electrostatic force between charged and charged, and charged and neutral body.Charging by induction. Observe the types of materials: metals and dielectrics (insulators). Towards the end of the video is a live demonstration of the balloon experiment you worked on in Geogebra app. The demonstration is by the MIT professor emeritus Walter Lewin.

YouTube link: `https://www.youtube.com/watch?v=3x0SuvuWe8Q`

7.2 Electrostatic Field

More about the Electrical force and field

We talked about the Electric Force in the previous section. To find the force on one charge, we had to know the position of all charges and what is their charge. To introduce Electric Field, we'll call charge q_1 the source of the force and charge q_2 the charge that feels the force. Charge one produces the force, and charge two feels the force.

By introducing the concept of the field, we can separate the cause of the field and the effect that the field has on other charges. We can find a field due to a charge or charge distribution, and once we know this field, we don't have to keep track of the source charge. We can just find the field's effect on other charges or charge distributions.

In the Equation 257 q_1 and q_2 are charges, r is the distance between the two charges, and $\hat{r}$ is the unit vector directed from charge one to charge two. The electric field of a source charge q_1 is defined as the force that a charge q_1 would impress on a positive charge q_2, divided by the amount of charge q_2, as shown in Equation 258. One thing to remember is that in the definition of the electric field, we always assume that the charge q_2 is positive! This way, we remove the ambiguity of the field direction. The direction of the electric field at a point P from charge q_1 is always in the direction of the force that would act on a positive charge q_2 placed at that point, as shown in Figure 77.

$$\overrightarrow{\mathbf{F_2}} = \frac{q_1 q_2}{4\pi\varepsilon r^2}\hat{r} \tag{257}$$

$$\overrightarrow{\mathbf{E}} = \frac{\overrightarrow{\mathbf{F_2}}}{q_2} \tag{258}$$

$$\overrightarrow{\mathbf{E}} = \frac{q_1}{4\pi\varepsilon_0 r^2}\hat{r} \tag{259}$$

If you carefully look at the Equation 259, you see that the electric field depends only on the source charge q_1.

If the electric charge is in a medium other than air, the electric field becomes

$$\overrightarrow{\mathbf{E}} = \frac{q_1}{4\pi\varepsilon_0\varepsilon_r r^2}\hat{R_{12}} \tag{260}$$

$$\varepsilon = \varepsilon_0\varepsilon_r \tag{261}$$

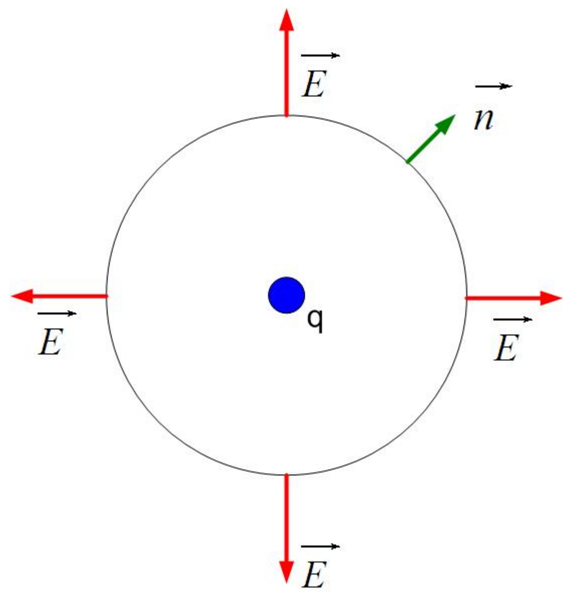


Figure 77: Electric field due to a unit charge q.

We added a unitless quantity, ε_r, called the relative dielectric constant, or relative permittivity, of a material. ε_r values for different materials can be found online. For example, you can see its values for different materials here `https://en.wikipedia.org/wiki/Relative_permittivity.`

Problem 18 *Find the electric field at a center of the Cartesian Coordinate system if positive charges q are placed at points (0,1) and (-1,0).*

positive charge that is in an electric field experiences a force that is

Problem 19 *A positive charge that is in an electric field E experiences a force that is*

Principle of Superposition

What is the electric field if we have more than one charge?

The total electric field at a point in space from the two charges is equal to the sum of the electric fields from the individual charges at that point.

If we have two charges, the total field due to both charges is equal to the vector sum of the fields due to individual charges, see Figure 78. The field at

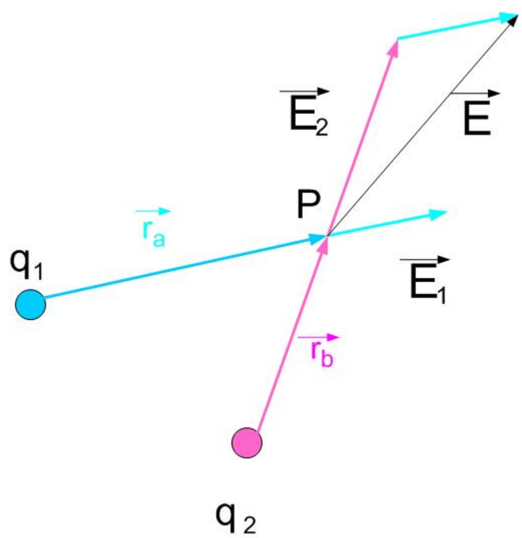


Figure 78: Electric Field due to two charges.

The fields or charges q_1 and q_2 are:

$$\overrightarrow{\mathbf{E_1}} = \frac{q_1}{4\pi\varepsilon_0 {r_a}^2}\hat{r_a} \tag{262}$$

$$\overrightarrow{\mathbf{E_2}} = \frac{q_1}{4\pi\varepsilon_0 {r_b}^2}\hat{r_b} \tag{263}$$

Where $\hat{r_a}$ and $\hat{r_b}$ are unit vectors in the direction of r_a and r_b. The total field due to both charges is

$$\overrightarrow{\mathbf{E}} = \overrightarrow{\mathbf{E_1}} + \overrightarrow{\mathbf{E_2}} \tag{264}$$

Electric Field in Rectangular Coordinates

The general equation for the electric field is given as

$$\overrightarrow{\mathbf{E}} = \frac{q_1}{4\pi\varepsilon_0 {r_a}^2}\hat{r_a} \tag{265}$$

The electric field at a point $P(x, y, z)$ due to a charge q_1 positioned at a point $P_{q_1}(x_1, y_1, z_1)$ in the rectangular coordinate system is shown in Figure 79. The position vector of the point P_{q_1} is

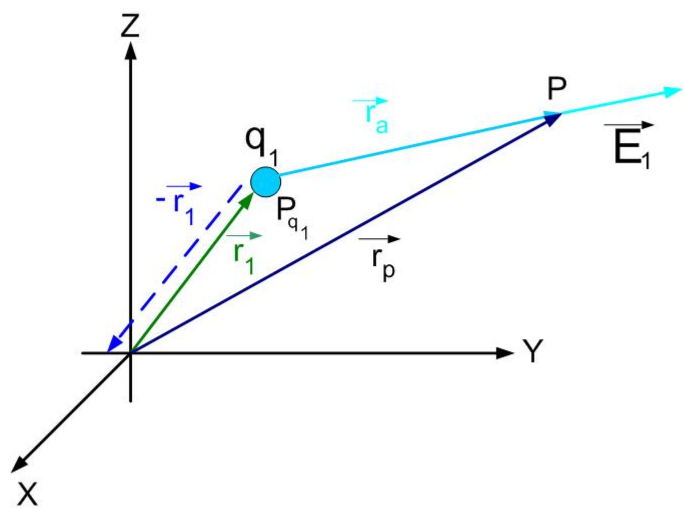


Figure 79: Electric Field due to a unit charge in Rectangular coordinate system.

$$\overrightarrow{\mathbf{r_1}} = x_1\overrightarrow{\mathbf{x}} + y_1\overrightarrow{\mathbf{y}} + z_1\overrightarrow{\mathbf{z}} \tag{266}$$

The position vector of point P is equal to

$$\overrightarrow{\mathbf{r_p}} = x\overrightarrow{\mathbf{x}} + y\overrightarrow{\mathbf{y}} + z\overrightarrow{\mathbf{z}} \tag{267}$$

The two vectors mark the beginning and the end of the distance vector $\overrightarrow{\mathbf{r_a}}$ between points P_{q_1} and P. The vector $\overrightarrow{\mathbf{r_a}}$ is the sum of vectors $-\overrightarrow{\mathbf{r_p}}$ and $\overrightarrow{\mathbf{r_1}}$

$$\overrightarrow{\mathbf{r_a}} = \overrightarrow{\mathbf{r_p}} + (-\overrightarrow{\mathbf{r_1}}) \tag{268}$$

When we substitute position vectors r_1 and r_p:

$$\overrightarrow{\mathbf{r_a}} = (x - x_1)\overrightarrow{\mathbf{x}} + (y - y_1)\overrightarrow{\mathbf{y}} + (z - z_1)\overrightarrow{\mathbf{z}} \tag{269}$$

Vector $\overrightarrow{\mathbf{r_a}}$ has the magnitude of:

$$|\overrightarrow{\mathbf{r_a}}| = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} \tag{270}$$

Unit vector in the direction of vector $\overrightarrow{\mathbf{r_a}}$ is:

$$\hat{r_a} = \frac{\overrightarrow{\mathbf{r_a}}}{|\overrightarrow{\mathbf{r_a}}|} \tag{271}$$

$$\hat{r_a} = \frac{\overrightarrow{\mathbf{r_a}}}{\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}} \tag{272}$$

$$\overrightarrow{\mathbf{E_1}} = \frac{q_1}{4\pi\varepsilon_0 {r_a}^2}\hat{r_a} \tag{273}$$

Where r_a is the distance between the charge q_1 and the point P. Substituting expressions for $\hat{r_a}$, and $|\overrightarrow{\mathbf{r_a}}|$ in equation 265 we get

$$\overrightarrow{\mathbf{E_1}} = \frac{q_1}{4\pi\varepsilon_0 \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}^3}\overrightarrow{\mathbf{r_a}} \tag{274}$$

Substituting

For two charges, as shown in Figure 80 equation 274 becomes

$$\begin{aligned}\overrightarrow{\mathbf{E}} = &\frac{q_1}{4\pi\varepsilon_0 \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}^3}\overrightarrow{\mathbf{r_a}} + \\ &\frac{q_2}{4\pi\varepsilon_0 \sqrt{(x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2}^3}\overrightarrow{\mathbf{r_b}}\end{aligned} \tag{275}$$

Example 32. *Find the point P where total electric field is zero inside of an equilateral triangle, if the three charges of magnitude $3\,nC$, $3\,nC$, and $3\,nC$ are placed in the corners of equilateral triangle of side $2\,m$. Use the app below to confirm your result.*

Geogebra link: `https://tube.geogebra.org/m/kupge9gc`

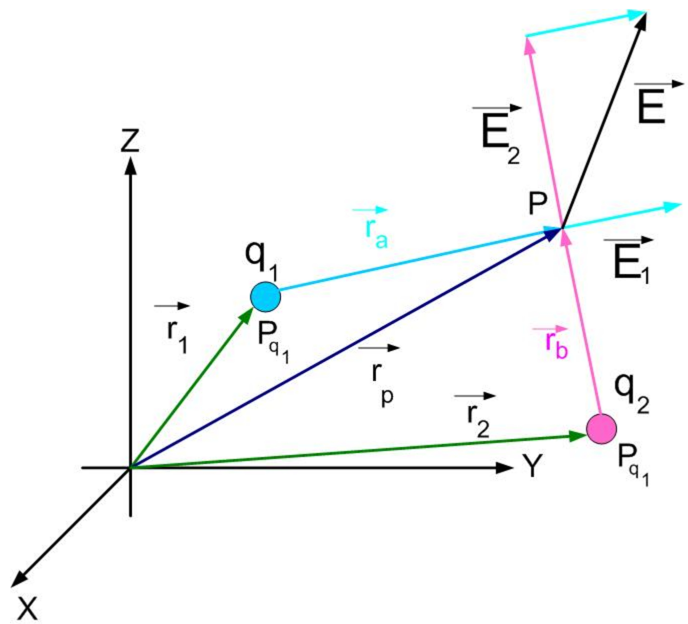


Figure 80: Electric field due to two charges in Rectangular coordinate system.

7.3 Electrostatic Potential

Electrostatic Potential Energy

When charges are pushed around in an electric field, the energy is not lost. Electrical forces are conservative.

How much energy do we have to bring into the system to bring two far-away positive charges at a distance r?

To answer that question, we can look at Figure 81. Bringing the first charge to a certain position in space would require no work since there are no other charged particles around, no electric field, and therefore no force. To bring the second charge at a distance r from the first charge, we need to overcome the repulsive force between the charges. How much work does this take? The work that we need to do to bring charges together is equal to the work that the first charge has to do to repel the second charge. The only difference is that we have to move the charges closer together, from infinity to some distance r, and the repulsive force does the work (pushes the charge q_2 away) from the distance r to infinity.

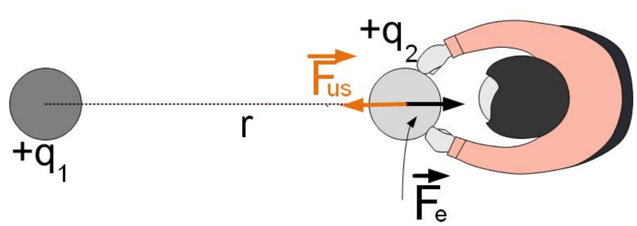


Figure 81: Moving the charge q_2 with force $\overrightarrow{\mathbf{F}}_{us}$ against the electrostatic force $\overrightarrow{\mathbf{F}}_e$ due to charge q_1 to find the electrostatic potential energy

$$W_{us} = We \tag{276}$$

$$W_{us} = \int_{\infty}^{r} \overrightarrow{\mathbf{F}}_{us} \cdot \overrightarrow{\mathbf{dr}} \tag{277}$$

$$W_e = \int_{r}^{\infty} \overrightarrow{\mathbf{F}}_e \cdot \overrightarrow{\mathbf{dr}} \tag{278}$$

The electric force is $\overrightarrow{\mathbf{F}}_e = \frac{q_1 q_2}{4\pi\varepsilon_o r^2}\overrightarrow{\mathbf{r}}$, so we can calculate the total work necessary to bring the two charges at distance r.

$$W_{us} = \int_r^{\infty} \overrightarrow{\mathbf{F}}_e \cdot \overrightarrow{\mathbf{dr}} \tag{279}$$

$$W_{us} = \int_r^{\infty} \frac{q_1 q_2}{4\pi\varepsilon_o r^2}\overrightarrow{\mathbf{r}}\,\overrightarrow{\mathbf{dr}} \tag{280}$$

Since the charge is moved in the direction of the force, we can drop vectors and calculate the work we have to do.

$$W_{us} = \frac{q_1 q_2}{4\pi\varepsilon_o}\int_r^{\infty} \frac{dr}{r^2} \tag{281}$$

$$W_{us} = \frac{q_1 q_2}{4\pi\varepsilon_o}(-\frac{1}{r}\Big|_r^{\infty})W_{us} = \frac{q_1 q_2}{4\pi\varepsilon_o r} \tag{282}$$

This work is the same no matter what path we take to move charge q_2 from infinity to a distance r from charge q_1, see Figure 82. It depends only on the initial and end position of the charge.

Definition of Potential and Voltage

The work needed to move the two charges closer together depends on both charges q_1 and $q2$, just like the electrostatic force depends on both charges. It would be easier to define another variable that will separate the cause, charge q_1, and the effect, the work that we have to do to move charge q_2. This variable is Electric Potential. Electric potential is defined as the work we need to do to move the charge divided by the amount of charge q_2.

$$V = \frac{W_{us}}{q_2} \tag{283}$$

$$V = \frac{q_1}{4\pi\varepsilon_o r} \tag{284}$$

Observe that in Equation 284 the potential is a function of the "source" charge q_1. We again separated the source and the effect, but this time of the potential energy. The source is a charge q_1 that produces potential V. If we now want to see what is the potential energy or work that we need to do to move another charge, we don't have to know which charge produced it. We only need to know the potential in an area, from which we can find the potential energy change of charge q_2.

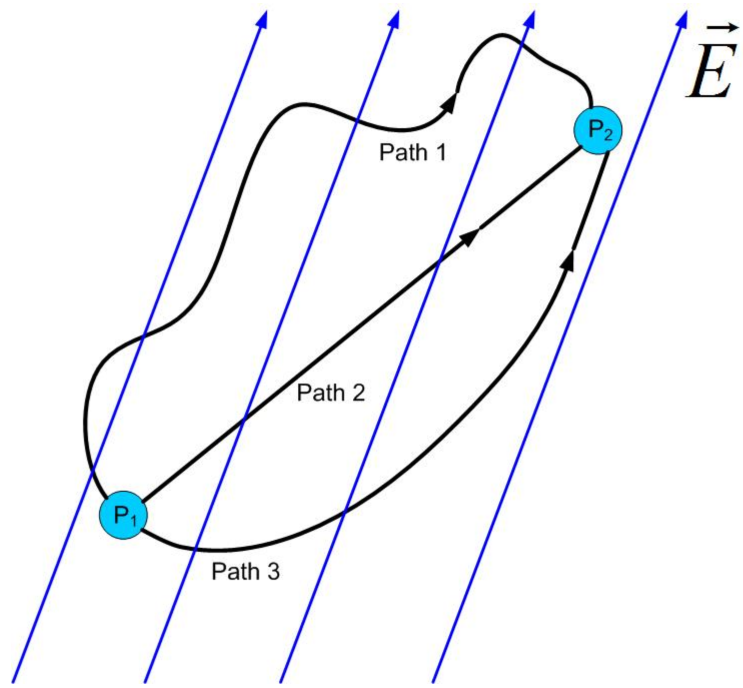


Figure 82: Potential is not dependent on the specific path.

We can find potential energy and potential of any number of charges using the above expression and the principle of superposition.

Question 20 *The difference in electric potential is most closely associated with*

Multiple Choice:

(a) *Work per unit charge*

(b) *Number of electrons in an atom*

(c) *Mechanical force on a charge*

The general relationship between the electric field and potential

In the first section, we defined the work necessary to bring two charges togeter at a distance r as

$$W_{us} = \int_r^\infty \overrightarrow{\mathbf{F}}_e \cdot \overrightarrow{\mathbf{dr}} \tag{285}$$

Electric force can be expressed in terms of electric field as

$$\overrightarrow{\mathbf{F}} = q\overrightarrow{\mathbf{E}} \tag{286}$$

If we substitute the electric force from the above equation into Equation 285, we get

$$W_{us} = \int_r^\infty q\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dr}} \tag{287}$$

The potential of a point R with respect to infinity is then defined as

$$V_R = \frac{W_e}{q} = \int_r^\infty \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dr}} \tag{288}$$

The potential at a point due to a unit positive charge is found to be V. If the distance between the charge and the point is tripled, the potential becomes

Question 21 *The potential at a point due to a unit positive charge is V. If the distance between the charge and the point is doubled, the potential becomes*

Multiple Choice:

(a) V^2

(b) $2V$

(c) $V/2$

(d) *need more information*

Voltage - the potential difference

The potential difference between two points is defined as $V_{AB} = V_A - V_B$. We defined the potential at a point in the equation above as

$$V_A = \int_A^\infty \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dr}} \tag{289}$$

$$V_B = \int_B^\infty \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dr}} \tag{290}$$

The potential difference, or voltage is then defined as

$$\Delta V = V_A - V_B = \int_A^\infty \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dr}} - \int_B^\infty \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dr}} \tag{291}$$

$$\Delta V = \int_A^\infty \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dr}} + \int_\infty^B \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dr}} \tag{292}$$

$$\Delta V = \int_A^B \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dr}} \tag{293}$$

Demonstration of potential difference in radial electric field of a VanDenGraaf generator by Prof. Emeritus at MIT, Walter Lewin.

YouTube link: `https://www.youtube.com/watch?v=cYOBSzh2SLE`

Now you can enjoy shocking John Travoltage in this PhET simulation.

Geogebra link: `https://tube.geogebra.org/m/zm6qzDwt`

Electric field calculation from potential difference

From the equation above, if we look at the potenital of two points that are very close on the x-axis, V_A, and $V_B = V_A + dV$, the voltage is equal to $V_A - V_B = V_A - (V_A + dV) = -dV = -E_x dx$. Therefore the electric field in the x-direction is

$$\overrightarrow{\mathbf{E_x}} = -\frac{\partial V}{\partial x} \tag{294}$$

The above equations states that the field is proportional to the change of potential over distance. The larger the change of potential over distance, the stronger the field.

To find the electric field in 3 dimensions, we use the gradient function, which represents a 3-dimensional derivative.

$$\overrightarrow{\mathbf{E}} = -\boldsymbol{\nabla} V = -(\frac{\partial V}{\partial x}\overrightarrow{\mathbf{x}} + \frac{\partial V}{\partial y}\overrightarrow{\mathbf{y}} + \frac{\partial V}{\partial z}\overrightarrow{\mathbf{z}}) \tag{295}$$

Example 33. *Figure 83 shows an electric field, and equipotential lines in an x-y plane. Equipotential lines are the lines of the same potential. We can observe that the electric field is always normal to the equipotential lines. The difference in potential between two points is $dV = \overrightarrow{\mathbf{E}}\overrightarrow{\mathbf{dr}}$. The lines, or surfaces, normal to the electric field vector are always equipotential surfaces, as the angle between the electric field vector and a perpendicular vector is 90^0, and the dot product between them is zero. In other words, the work that we have to do to move the charge in the direction perpendicular to the electric force is zero.*

Example 34. *Two charges, a positive and a negative one are placed on x-y plane. The potential around the charges is shown as a red surface. Observe the simulation below.*

(a) *Where is the potential positive, and where is it negative?*

(b) *What is the direction of the electric field?*

(c) *If we place a positive point charge closer to the positive charge Q1, which way will it go?*

(d) *What if we have a negative charge?*

Now click on the 2D view.

(a) *Where is the field the strongest?*

(b) *Can you tell the direction of the field from equipotential lines?*

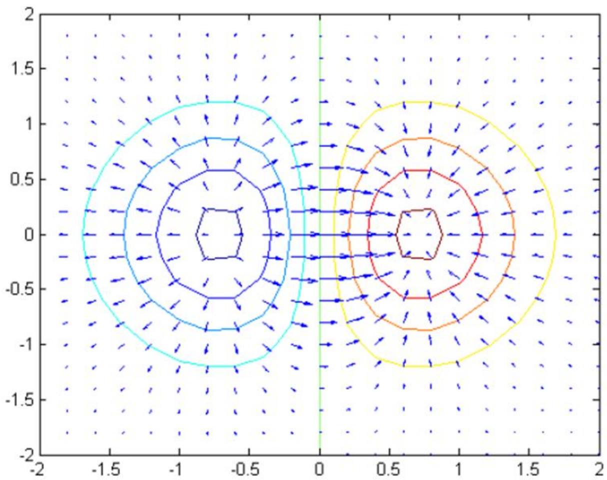


Figure 83: Electric field and equipotential lines.

Geogebra link: https://tube.geogebra.org/m/tmkauhfc

Problem 22 *Rougly adjust the direction and length of the vectors shown in the graph below. When done, click on "Grade Update " to check your answer.*

Geogebra link: https://tube.geogebra.org/m/dqvffmrh

7.4 Electrostatic fields from distributed charges

Electric Field due to a Charge Distribution

This is an optional section.

Derive Analytical Solution for the Electric field Due to a Loop of Charge

We will first find the electric field due to a loop of charge. The loop of charge is charged with the line charge density ρ_l and is in the X-Y plane, as shown in Figure 84. To solve this problem, we first divide the loop into small pieces. The small (blue) arc obtained in this way can be considered a point charge. The electric field due to a point charge is shown in Figure 84(the blue arrow). The position of the point charge is defined by a position vector $\overrightarrow{\mathbf{r_2}}$. The position of point P is defined with the position vector $\overrightarrow{\mathbf{r_1}}$. The vector $\overrightarrow{\mathbf{dE}}$ is defined in Equation 296.

$$\overrightarrow{\mathbf{dE}} = \frac{dQ}{4\pi\varepsilon_0 r^2}\hat{r} \tag{296}$$

The total electric field at a point P is then equal to the sum of all the fields due to the point charges, as shown in Figure 85. The equation for the total field is given in Equation 297.

$$\overrightarrow{\mathbf{E}} = \int\limits_{all\ point\ charges} \overrightarrow{\mathbf{dE}} \tag{297}$$

The problem now is to represent all the variables in the Equation 296 (dQ, dl, $\hat{r}$ and r) using appropriate coordinate system and given charge distribution. The total charge on the segment dl is equal to $dQ = \rho_l dl$. As seen in Figure 85, dl is an arc length in the direction of theta, $dl = a\,d\theta$, where a is the radius of the loop. The vector $\overrightarrow{\mathbf{r_2}}$ is the position vector of the arc length dl, and the vector $\overrightarrow{\mathbf{r_1}}$ is the position vector of the point be of the electric field calculation. Point P is an arbitrary point in the Cartesian coordinate system, P(x,y,z), therefore its vector is shown in Equation 298. The vector $\overrightarrow{\mathbf{r}}$ is the distance vector between the point charge (the source) and the point at which we are calculating the electric field.

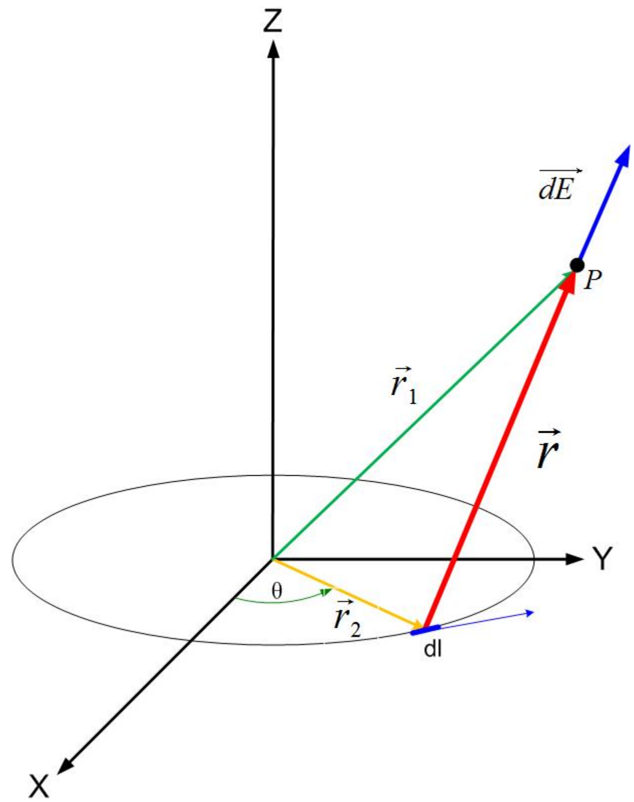


Figure 84: Loop of wire uniformly charged with line charge density ρ_l. Electric field is shown due to a very small section (arc length) of the loop dl.

$$\overrightarrow{\mathbf{r_1}} = x\overrightarrow{\mathbf{a_x}} + y\overrightarrow{\mathbf{a_y}} + z\overrightarrow{\mathbf{a_z}} \tag{298}$$

The vector $\overrightarrow{\mathbf{r_2}}$ can be written in Polar Coordinates as in Equation 299,where a is the radius of the loop. The equation 299 can be rewritten in Cartesian coordinate system as shown in Equation 301.

$$\overrightarrow{\mathbf{r_2}} = a\,\overrightarrow{\mathbf{a_r}} \tag{299}$$

$$\overrightarrow{\mathbf{a_r}} = \ cos\theta\,\overrightarrow{\mathbf{a_x}} + sin\theta\,\overrightarrow{\mathbf{a_y}} \tag{300}$$

$$\overrightarrow{\mathbf{r_2}} = a\,cos\theta\,\overrightarrow{\mathbf{a_x}} + a\,sin\theta\,\overrightarrow{\mathbf{a_y}} \tag{301}$$

The two vectors mark the beginning and the end of the distance vector $\overrightarrow{\mathbf{r}}$. The

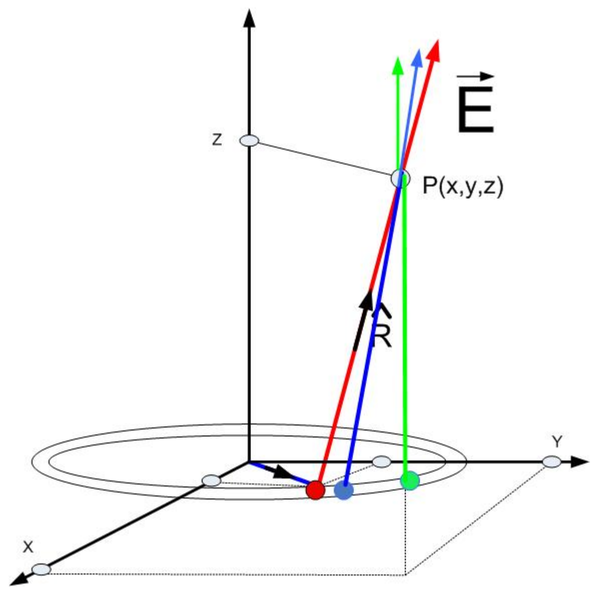


Figure 85: Loop of wire uniformly charged with line charge density ρ_l. Electric field is shown due to several very small sections (arc lengths) of the loop dl.

vector $\overrightarrow{\mathbf{r}}$ is the sum of vectors $-\overrightarrow{\mathbf{r_2}}$ and $\overrightarrow{\mathbf{r_1}}$.

$$\overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{r_1}} + (-\overrightarrow{\mathbf{r_2}}) \tag{302}$$

Or:

$$\overrightarrow{\mathbf{r}} = (x - a\,cos\theta)\overrightarrow{\mathbf{a_x}} + (y - a\,sin\theta)\overrightarrow{\mathbf{a_y}} + z\overrightarrow{\mathbf{a_z}} \tag{303}$$

Vector $\overrightarrow{\mathbf{r}}$ has the magnitude of:

$$|\overrightarrow{\mathbf{r}}| = \sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2} \tag{304}$$

Unit vector in the direction of vector $\overrightarrow{\mathbf{r}}$ is:

$$\hat{r} = \frac{\overrightarrow{\mathbf{r}}}{|\overrightarrow{\mathbf{r}}|} \tag{305}$$

$$\hat{r} = \frac{\overrightarrow{\mathbf{r}}}{\sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}} \tag{306}$$

Replacing other variables in the Equations 298-306 to the Equation 296, we get the Equation 307 for the electric field $\overrightarrow{\mathbf{dE}}$ at a point P.

$$\overrightarrow{\mathbf{dE}} = \frac{\rho_l\, a\, d\theta}{4\pi\varepsilon_0 \sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}^3} \cdots$$
$$\cdots (x - a\,cos\theta)\overrightarrow{\mathbf{a_x}} + (y - a\,sin\theta)\overrightarrow{\mathbf{a_y}} + z\overrightarrow{\mathbf{a_z}} \tag{307}$$

Components of the electric field are given in Equations 308-310.

$$\overrightarrow{\mathbf{dE_x}} = \frac{\rho_l\, a\, d\theta}{4\pi\varepsilon_0 \sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}^3}(x - a\,cos\theta)\overrightarrow{\mathbf{a_x}} \tag{308}$$
$$\overrightarrow{\mathbf{dE_y}} = \frac{\rho_l\, a\, d\theta}{4\pi\varepsilon_0 \sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}^3}(y - a\,sin\theta)\overrightarrow{\mathbf{a_y}} \tag{309}$$
$$\overrightarrow{\mathbf{dE_z}} = \frac{\rho_l\, a\, d\theta}{4\pi\varepsilon_0 \sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}^3} z\overrightarrow{\mathbf{a_z}} \tag{310}$$

Each component of the field can be integrated separately, as shown in Equations 311-313.

$$\overrightarrow{\mathbf{E_x}} = \int_0^{2\pi} \frac{\rho_l\, a\,(x - a\,cos\theta) d\theta}{4\pi\varepsilon_0 \sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}^3}\overrightarrow{\mathbf{a_x}} \tag{311}$$
$$\overrightarrow{\mathbf{E_y}} = \int_0^{2\pi} \frac{\rho_l\, a\,(y - a\,sin\theta) d\theta}{4\pi\varepsilon_0 \sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}^3}\overrightarrow{\mathbf{a_y}} \tag{312}$$
$$\overrightarrow{\mathbf{E_z}} = \int_0^{2\pi} \frac{\rho_l\, a\, z d\theta}{4\pi\varepsilon_0 \sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}^3}\overrightarrow{\mathbf{a_z}} \tag{313}$$

The integrals in Equations 311-313 can be integrated analytically in some special cases only. In general, this is in an elliptical integral and cannot be solved analytically. However, this integral can be sold numerically.

Derive Numerical solution to the Electric Field due to a Loop of Charge

The integrals in equations 311-313 can be represented as infinite sums using simple numerical integration as shown in Equations 314-316. Here we see that the continuous function of θ was replaced with the discrete values of θ.

$$\overrightarrow{\mathbf{E_x}} = \sum_{i=0}^{n} \frac{\rho_l \, a \, (x - a\, cos\theta_i)\Delta\theta}{4\pi\varepsilon_0 \sqrt{(x - a\, cos\theta_i)^2 + (y - a\, sin\theta_i)^2 + z^2}^3} \overrightarrow{\mathbf{a_x}} \tag{314}$$

$$\overrightarrow{\mathbf{E_y}} = \sum_{i=0}^{n} \frac{\rho_l \, a \, (y - a\, sin\theta_i)\Delta\theta}{4\pi\varepsilon_0 \sqrt{(x - a\, cos\theta_i)^2 + (y - a\, sin\theta_i)^2 + z^2}^3} \overrightarrow{\mathbf{a_y}} \tag{315}$$

$$\overrightarrow{\mathbf{E_z}} = \sum_{i=0}^{n} \frac{\rho_l \, a \, z\Delta\theta}{4\pi\varepsilon_0 \sqrt{(x - a\, cos\theta_i)^2 + (y - a\, sin\theta_i)^2 + z^2}^3} \overrightarrow{\mathbf{a_z}} \tag{316}$$

$\Delta\theta$ represents the length of the interval that the line is divided into, θ_i represents the value of angle at a certain point, and i designates different points on the loop. It should be noted here that the error due to the trapezoidal rule can be improved by using more advanced numerical integration techniques.

Matlab Code to Find the Electric Field due to a Loop of Charge

Equations 314-316 can be implemented in Matlab as shown below. Cut-and-paste the program below in Matlab editor. Play with values for the size of the ring and meshgrid values. For some values of the step, the vectors on the graph will completely disappearcomment on why that may be.

```
clear all
%Specify the extents of x,y,z axes
rad=-2:1.9965:2;
%Make X,Y,Z matrices
[X Y Z]=meshgrid(rad);
%Define constants epsilon, Q. They are obviously not physical.
%They are set
%to small constants so that we get smaller numbers.
eps=1;
Q=1;
a=1;
const=Q/(2*pi*eps);
%Set the initial electric field components to zero.
Ex=0;
Ey=0;
Ez=0;
%Here we find the fieldl at all X,Y,Z points defined previously
% from each of the "unit" charges on the ring.
%th variable starts from 0.1
%to avoid the infinite value of V at the point r=1,theta=0.
for th=0.1:pi/20:2*pi
```

```
t=const./ (sqrt((X-cos(th)).^2 + (Y-sin(th)).^2 + (Z).^2)).^(3);
Ex = Ex+t.*(X-cos(th));
Ey = Ey+t.*(Y-sin(th));
Ez = Ez+t.*Z;
end
%Plot the field components Ex,Ey, Ez at points X,Y,Z.
%Scale the vectors by factor 3.
quiver3(X,Y,Z,Ex,Ey,Ez,3)
hold on;
%Plot the ring of charge where it is positioned in x-y plane.
t = linspace(0,2*pi,1000);
r=1;
x = r*cos(t);
y = r*sin(t);
plot(x,y,'b');
hold off
```

Short line of charge

Problem 23 *Derive electric field integral, numerical solution for integral, then implement the code in MATLAB, to find the electric field from a 1 m stick of charge, charged with a uniform line charge density of $\rho_l = 1nC/m$. The stick is positioned as in Figure 86.*

In the simulation below, observe how changing the section of the selected piece of charge (a red square) changes the field at a point P.

Geogebra link: `https://tube.geogebra.org/m/fmwgw66c`

Then click on the play button in the lower left corner of the simulation below to watch how each piece of charge contributes to the total electric field.

Geogebra link: `https://tube.geogebra.org/m/akfavjkx`

Potential

Derive Analytical Solution for the Potential Due to a Loop of Charge

Derive the potential due to a ring of charge, charged with a line charge density ρ_l.

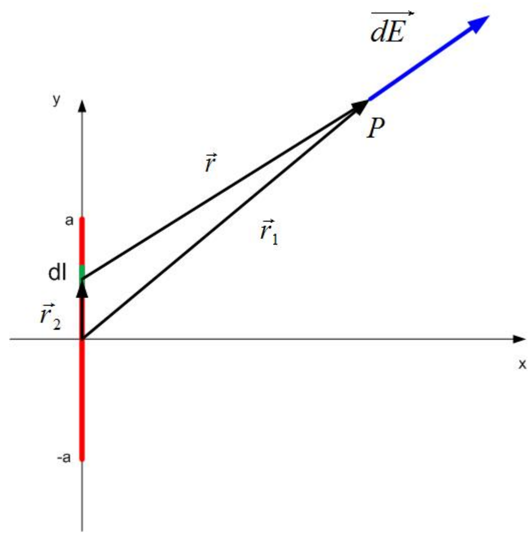


Figure 86: Stick of wire uniformly charged with line charge density ρ_l. Electric field is shown due to a very small section of the line dl.

Potential due to a point charge is given in Equation 317. We will first find the potential due to a loop of charge. Assume that the loop of charge is charged with the line charge density ρ_l. The loop of charge is in the X-Y plane, as shown in Figure 87. First, we will divide the loop into small pieces. We will assume that the small arc obtained in this way can be considered a point charge. The potential due to a point charge at a point P is labeled in Figure 87 as dV. The position of the point charge is defined by a position vector $\overrightarrow{\mathbf{r_2}}$. The position of point P is defined with the position vector $\overrightarrow{\mathbf{r_1}}$. The electric scalar potential dV is defined in Equation 317.

$$dV = \frac{dQ}{4\pi\varepsilon_0 r} \tag{317}$$

The total potential at a point P is then equal to the sum of all the potentials due to the point charges, as shown in Figure 88. The equation for the total potential is given in 318.

$$V = \int\limits_{all\ point\ charges} dV \tag{318}$$

The problem now is to represent all the variables in the Equation 317 (dQ, dl, $\hat{r}$ and r) using appropriate coordinate system and given charge distribution. The

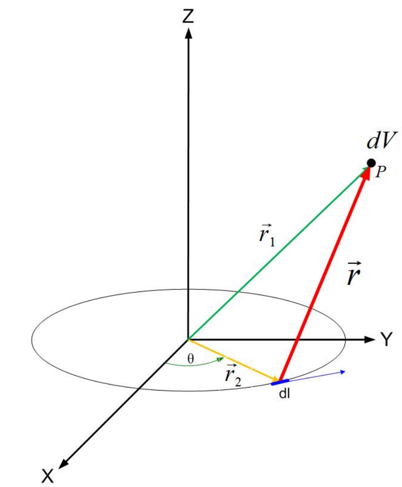


Figure 87: Loop of wire uniformly charged with line charge density ρ_l. Electric field is shown due to a very small section (arc length) of the loop dl.

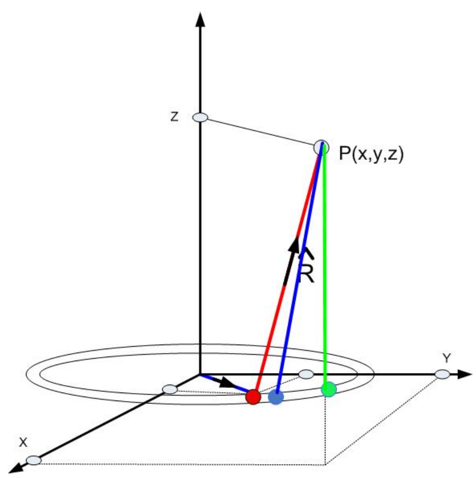


Figure 88: Loop of wire uniformly charged with line charge density ρ_l. Electric field is shown due to several very small sections (arc lengths) of the loop dl. Each section is modeled by a point charge dQ.

total charge on the segment dl is equal to $dQ = \rho_l dl$. As seen in Figure 87, dl is an arc length in the direction of theta (blue arrow next to dl) $dl = a\, d\theta$, where a is the radius of the loop. The vector $\overrightarrow{\mathbf{r_2}}$ is the position vector of the arc length dl, and the vector $\overrightarrow{\mathbf{r_1}}$ is the position vector of the point be of the electric field calculation. Point P is an arbitrary point in the Cartesian coordinate system, P(x,y,z), therefore its vector is shown in Equation 319. The vector $\overrightarrow{\mathbf{r}}$ is the distance vector between the point charge (the source) and the point at which we are calculating the electric field.

$$\overrightarrow{\mathbf{r_1}} = x\overrightarrow{\mathbf{a_x}} + y\overrightarrow{\mathbf{a_y}} + z\overrightarrow{\mathbf{a_z}} \tag{319}$$

The vector $\overrightarrow{\mathbf{r_2}}$ can be written in Polar Coordinates as in equation 320,where a is the radius of the loop. The Equation 320 can be rewritten in Cartesian coordinate system as in Equation 322.

$$\overrightarrow{\mathbf{r_2}} = a\,\overrightarrow{\mathbf{a_r}} \tag{320}$$

$$\overrightarrow{\mathbf{a_r}} = \ cos\theta\,\overrightarrow{\mathbf{a_x}} + sin\theta\,\overrightarrow{\mathbf{a_y}} \tag{321}$$

$$\overrightarrow{\mathbf{r_2}} = a\,cos\theta\,\overrightarrow{\mathbf{a_x}} + a\,sin\theta\,\overrightarrow{\mathbf{a_y}} \tag{322}$$

The two vectors mark the beginning and the end of the distance vector $\overrightarrow{\mathbf{r}}$. The vector $\overrightarrow{\mathbf{r}}$ is the sum of vectors $-\overrightarrow{\mathbf{r_2}}$ and $\overrightarrow{\mathbf{r_1}}$.

$$\overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{r_1}} + (-\overrightarrow{\mathbf{r_2}}) \tag{323}$$

Therefore the vector's $\overrightarrow{\mathbf{r}}$ magnitude is shown in Equations 325.

$$\overrightarrow{\mathbf{r}} = (x - a\,cos\theta)\overrightarrow{\mathbf{a_x}} + (y - a\,sin\theta)\overrightarrow{\mathbf{a_y}} + z\overrightarrow{\mathbf{a_z}} \tag{324}$$

Vector $\overrightarrow{\mathbf{r}}$ has the magnitude of:

$$|\overrightarrow{\mathbf{r}}| = \sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2} \tag{325}$$

Replacing other variables in the Equations 319-325 to the Equation 317, we get the Equation 326 for the potential dV at a point P.

$$dV = \frac{\rho_l\,a\,d\theta}{4\pi\varepsilon_0\sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}} \tag{326}$$

$$V = \int_0^{2\pi} \frac{\rho_l\,a\,d\theta}{4\pi\varepsilon_0\sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}} \tag{327}$$

The integral in Equation 327 can be integrated analytically in some special cases; however, this equation can be solved numerically, as shown in the next section.

Derive Numerical solution to the Electric Field due to a Loop of Charge

The integrals in equations 327 can be represented with infinite sum using simple numerical integration (trapezoidal rule), as shown in Equations 328. Here we see that the continuous function of θ was replaced with the discrete values of θ.

$$V = \sum_{i=0}^{n} \frac{\rho_l \, a \Delta\theta}{4\pi\varepsilon_0 \sqrt{(x - a\,cos\theta_i)^2 + (y - a\,sin\theta_i)^2 + z^2}} \tag{328}$$

Where $\Delta\theta$ represents the length of the interval that the line is divided into, θ_i represents the value of angle at a certain point, and i designates different points on the loop. It should be noted here that the error due to the trapezoidal rule can be improved by using more advanced numerical integration techniques.

Matlab Code to Find the Electric Potential due to a Loop of Charge

Plot the cross-section of the potential and the equipotential lines on the planes x=0, y=0, z=0.

```
clear all
%Specify the extents of x,y,z axes
rad=-1.5:0.1:1.5

%Make X,Y,Z matrices
[X Y Z]=meshgrid(rad,rad,rad)
%Define constants epsilon, Q. They are obviously not physical.
%They are set
%to small constants so that we get smaller numbers.
eps=1
Q=1
const=Q/(2*pi*eps)
%Set the initial potential value to zero. Initialize V.
V=0
%Here we find the potential at all X, Y, Z points defined
% previously from
%each of the "unit" charges on the ring.
```

```
%The th variable starts from 0.1
%to avoid the infinite value of V at the point r=1,theta=0.

 for th=0.1:pi/20:2*pi
     t=const./ sqrt((X-cos(th)).^2 + (Y-sin(th)).^2 + (Z).^2)
V= V+t;
 end
%Plot the volume distribution of potential at planes
% x=0, y=0, z=0
slice(X,Y,Z,V,[0],[0],[0])
%Keep the same figure
hold on;
%Plot contours of potential at planes x=0, y=0, z=0. See
%more on contours in appendix.
h=contourslice(X,Y,Z,V,[0],[0],[0])
set(h,'EdgeColor','k','LineWidth',1.5)
```

Plot several equipotential surfaces.

```
clear all
%Specify the extents of x,y,z axes
rad=-1.5:0.1:1.5

%Make X,Y,Z matrices
[X Y Z]=meshgrid(rad,rad,rad)
%Define constants epsilon, Q. They are obviously not physical.
%They are set
%to small constants so that we get smaller numbers.
eps=1
Q=1
const=Q/(2*pi*eps)
%Set the initial potential value to zero. Initialize V.
V=0
%Here we find the potential at all X, Y, Z points defined
%previously from
%each of the "unit" charges on the ring. The th variable
% starts from 0.1
%to avoid the infinite value of V at the point r=1,theta=0.

 for th=0.1:pi/20:2*pi
     t=const./ sqrt((X-cos(th)).^2 + (Y-sin(th)).^2 + (Z).^2)
V= V+t;
 end
%Plot the volume distribution of potential at planes
```

```
% x=0, y=0, z=0
p = patch(isosurface(X,Y,Z,V,7));
isonormals(X,Y,Z,V,p)
set(p,'FaceColor','red','EdgeColor','none');
daspect([1 1 1])
view(3); axis tight
camlight
lighting gouraud
```

Observe the equipotential surfaces. Explain why the surfaces doughnut are shaped? What is the electric field direction on the surface? Change the isosurface from 7 to 10 or some larger number. How does the isosurface look now? How does the potential of a ring of charge looks at the distances far away from the ring? What about the field?

Visualizing Scalar Fields in Matlab

To visualize scalar fields in Matlab, we can use the following functions: slice, contourslice, patch, isonormals, camlight, and lightning. Please note that a more detailed explanation about these functions shown here can be found in Matlab's help.

slice

Slice is a command that shows the magnitude of a scalar field on a plane that slices the volume where the potential field is visualized. The format of this command is as shown below.

```
slice(x,y,z,v,xslice,yslice,zslice)
```

Where X, Y, and Z are coordinates of points where the scaler function is calculated, V is v the scalar function at those points, and the last three vectors xslice, yslice, and zslice are showing where will the volume will be sliced.

An example of slice command is given below. In the example below, there is an additional command colormap that colors the volume with a specific palette. To see more about different color maps, see Matlab's help. xslice has three points at which the x-axis will be slice. They are -1.2, .8, 2. This means that the volume will be slice with a plane that is perpendicular to the x-axis, and it crosses the x-axis at points -1.2, .8, and 2.

```
clc
clear all
```

```
[x,y,z] = meshgrid(-2:.2:2,-2:.25:2,-2:.16:2);
v = x.*exp(-x.^2-y.^2-z.^2);
xslice = [-1.2,.8,2]; yslice = 1; zslice = [-2,0];
slice(x,y,z,v,xslice,yslice,zslice)
colormap hsv
```

contourslice

Contourslice command will display equipotential lines on a plane being the volume where the potential field is visualized. An example of a contourslice function is shown below.

```
[x,y,z] = meshgrid(-2:.2:2,-2:.25:2,-2:.16:2);
v = x.*exp(-x.^2-y.^2-z.^2); % Create volume data
[xi,yi,zi] = sphere; % Plane to contour
contourslice(x,y,z,v,xi,yi,zi)
view(3)
```

patch

Patch command creates a patch of color.

isonormals

Command isonormals create equipotential surfaces.

camlight

```
camlight('headlight') creates a light at the camera position.

camlight('right') creates a light
right and up from camera.

camlight('left') creates a light
left and up from camera.camlight with no arguments is the
same as camlight('right').

camlight(az,el) creates a light
at the specified azimuth (az) and elevation (el)
with respect to the camera position.
```

```
The camera target is the center of rotation,
and az and el are in degrees.
```

lighting

```
Lighting flat selects flat lighting.

Lighting gouraud selects gouraud lighting.

Lighting phong selects phong lighting.

lighting none turns off lighting.
```

7.5 Calculation of electric field using Gauss's Law

Field Visualization

There are several ways of visualizing fields:

(a) vectors of different lengths represent the strength and direction of the field at different points.

(b) streamlines show the field flow. The field vector direction is tangential to a flow line. Streamlines are lines that a particle would follow in a field.

(c) Streamlines above can be drawn to show that the vector's strength is proportional to the density of field lines (number of field lines per unit area). Vector lines are usually represented as a fixed number of streamlines, not individual vectors. The lines bunch up where the field is stronger and diverge from each other where the field is weak.

Flux

Let's assume that in this example, we'll visualize a field with a density of field lines per area, as we have shown in 3 above. Rivers are symbols of flux. For example, if we imagine a river flow and a small rectangular frame submerged in it, we can qualitatively explain the "amount" of the field going through a surface. A simple way to understand flux is to count how many field lines poke a surface. If many field lines poke the surface, the flux is strong; otherwise, the flux is weak. The flux is the rate of flow of water through the frame.

Mathematically, the flux of any vector $\overrightarrow{\mathbf{A}}$ through a surface $\overrightarrow{\mathbf{S}}$ is defined as

$$\Phi = \int_S \overrightarrow{\mathbf{A}} \cdot \overrightarrow{\mathbf{dS}} \tag{329}$$

In the equation above, the surface is a vector so that we can define the direction of the flow of the vector. The surface vector $\overrightarrow{\mathbf{dS}}$ is defined as a surface of the frame dS multiplied with a vector perpendicular to the surface $dS\overrightarrow{\mathbf{n}}$. The flux is the highest if the normal to the surface and the vector field point in the same direction.

Example 35. *Observe the simulation below, change the angle between the frame and the field, then change the strenght of the field. How does the flux change?*

Geogebra link: `https://tube.geogebra.org/m/CdSZEQcW`

Gauss's Law

Gauss's Law states that the flux of electric field through a **closed** surface is equal to the charge enclosed divided by a constant.

$$\oint_S \vec{\mathbf{E}} \cdot \vec{\mathbf{dS}} = \frac{Q_{inS}}{\varepsilon} \tag{330}$$

It can be shown that no matter the shape of the closed surface, the flux will always be equal to the charge enclosed. This proof is beyond the scope of these lectures.

Gauss's Law is used to find the electric field when a charge distribution is given. We can apply Gauss's Law using analytical expressions only to a specific set of symmetric charge distributions.

The key to finding the Electric field from Gauss's Law is selecting the simplest surface to perform the integration in Equation 330. If the simplest surface is found, the above equation simplifies to $E\,S = \frac{Q_{inS}}{\varepsilon}$, where S is the surface (or sum of surfaces) where the flux exists, E is the electric field on that surface, and Q is the charge enclosed.

Example 36. *Observe the 2D flux from electric charge through a square. Set all charges to zero, except for one. Move the charge inside and outside of the square. What can you conclude about the flux when the charge is inside the square, and what when it is outside?*

Geogebra link: `https://tube.geogebra.org/m/r7Ue9Nac`

Question 24 *An positive Q and a negative charge -Q separated by a distance d (a dipole) are enclosed in a cube. What is the flux through the cube?*

Multiple Choice:

(a) Q/ε

(b) *zero*

(c) $2Q/\varepsilon$

(d) $Q/2\varepsilon$

Applying Gauss's Law to special, symmetric charge distributions

The key to applying Gauss's Law to find the field from a symmetric charge distribution is to find the surface S so that the normal to the surface is either perpendicular or parallel to the electric field. Practically speaking, this means that if the charge distribution has spherical symmetry, we'll choose a sphere for the surface. If the charge density has cylindrical symmetry, we'll choose a cylinder for the surface. If the charge density is an infinite plane, we'll choose a box (or, as we'll see later, a cylinder again). As you will see, before we apply Gauss's Law to find the electric field, we have to know how the electric field looks from a particular charge distribution so that we can carry out the integration.

(a) The parts of the surface where the electric field $\overrightarrow{\mathbf{E}}$ is perpendicular to the normal on the surface $dS\overrightarrow{\mathbf{n}}$ have zero flux through it, as the field does not poke the surface. Mathematically this can be explained through the dot product. Since the angle between the electric field and the normal to the surface is $\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}}) = 90^o$, the dot product becomes zero $\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = E\,dS\, \cos(\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}}))$ and so the integral $\int_S \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}}$ through that surface is zero.

(b) The parts of the surface where the electric field $\overrightarrow{\mathbf{E}}$ is parallel with the normal on the surface $dS\overrightarrow{\mathbf{n}}$ the dot product between the two quantities $\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}}$ becomes $EdS\cos(\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}})) = EdS$, because $\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}}) = 0$, and $\cos(\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}})) = 1$. If we select a surface in such a way that all points of this surface are the same distance from the charge, the electric field is constant on the surface, so it can be taken out of the integral, and Gauss's law then simplifies to:

$$E \oint_S dS = \frac{Q_{inS}}{\varepsilon} \tag{331}$$

In the equation above, you can see that the integral $\int_S dS$ is just the surface area of the chosen surface, so the equation further simplifies to $E\,S = \frac{Q_{inS}}{\varepsilon}$

Electric Field of a Point Charge

Our first example is to find the electric field of a point charge +Q.

To start this problem, we have to know the direction of the field. The electric field lines from a point charge are pointed radially outward from the charge

(Figure 89). Mathematically we can write that the field direction is $\overrightarrow{\mathbf{E}} = E\,\hat{r}$. We have to know the direction and distribution of the field if we want to apply Gauss's Law to find the electric field.

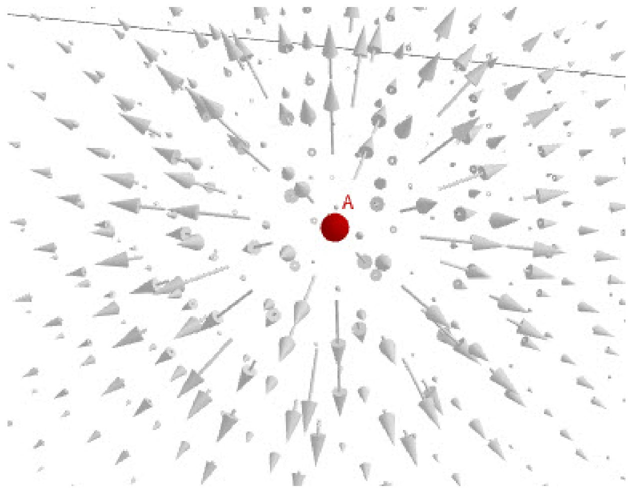


Figure 89: Electric field of a point charge

We want to find the magnitude of the electric field as a function of distance r from the charge. To do this, we will enclose the charge with an imaginary surface. We first have to decide on what kind of imaginary surface we are going to use in this case. Since the field has spherical symmetry, we will use a sphere, as shown in Figure 90. There are multiple reasons why we should use a sphere:

(a) Symmetry of the charge dictates using a sphere.

(b) The field on the sphere is in the same direction as the outward normal to the sphere.

(c) Electric field is constant everywhere on the sphere.

$$\oint_S \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = \frac{Q_{inS}}{\varepsilon} \tag{332}$$

Because the electric field is in the same direction with the outward normal to the sphere as shown in Figure 90, the dot product becomes just the product of E and dS $\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = E\,dS\cos(\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}})) = E\,dS\cos(\angle(0^o)) = EdS$, and the Gauss's law equation becomes

$$\oint_S E\,dS = \frac{Q_{inS}}{\varepsilon} \tag{333}$$

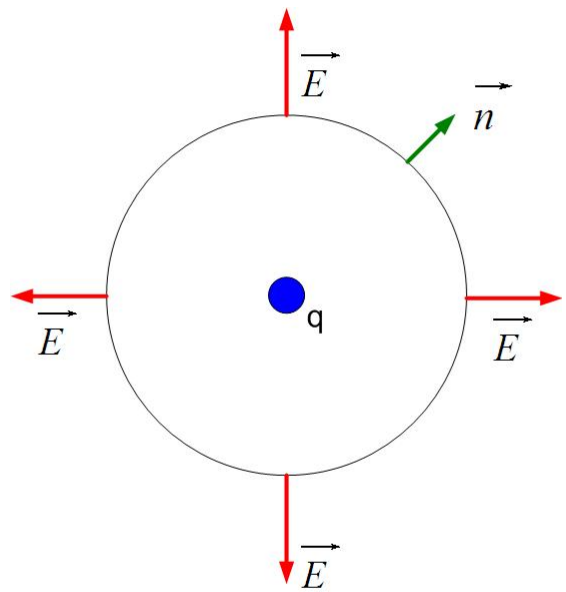


Figure 90: Application of Gauss's Law to find Electric Field of a point charge.

Since the electric field is constant everywhere on the surface, we can take it out of the integral.

$$E \oint_S dS = \frac{Q_{inS}}{\varepsilon} \tag{334}$$

In the above equation, the $\oint_S dS$ is just the surface area of the sphere, $4\pi r^2$.

$$E\,4\pi r^2 = \frac{Q_{inS}}{\varepsilon} \tag{335}$$

The final expression for the magnitude of the field is:

$$E = \frac{Q_{inS}}{4\pi\varepsilon\, r^2} \tag{336}$$

Note that we only found the magnitude of the field in the above equation. We started this problem with the knowledge of field radial direction. So the final expression for the field is

$$\overrightarrow{\mathbf{E}} = \frac{Q_{inS}}{4\pi\varepsilon\, r^2}\,\hat{r} \tag{337}$$

Note that this equation can also be obtained from Coulomb's Law. One last step in finding the electric field is to find the total charge enclosed by the imaginary surface S. Depending on whether we're looking for the field inside or outside of a charge distribution, and whether the charge distribution is a volume charge distribution ρ, surface charge distribution σ or line charge distribution ρ_l, the whole or fraction of volume V, surface S or line l of the charge distribution will be enclosed.

$$Q_{inS} = \int_V \rho \, dV \tag{338}$$

$$Q_{inS} = \int_S \sigma \, dS \tag{339}$$

$$Q_{inS} = \int_l \rho_l \, dl \tag{340}$$

Electric Field of a Spherical Charge Distribution

A spherical region of radius a is charged with uniform volume charge density ρ=const. Find the field inside the spherical region of charge at a distance r from the center of the charge density and the field outside of the spherical region of charge at (another) distance r away from the center of the charge.

Electric Field outside of the sphere

We will first look at the field outside of the spherical charge distribution. This process is the same as in the previous problem, where we found the field from a point charge. Following the reasoning in the previous problem, we select a sphere for the integration surface. It is important to mention that we pick a point outside of the distributed charge at the distance r from the center, and that will be one point on the sphere's surface. We picked a point at random distance r, not at r=a because we want to find out electric field anywhere outside of the charge distribution E(r), and r could be any point $r > a$.

$$\oint_S \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = \frac{Q_{inS}}{\varepsilon} \tag{341}$$

Because the electric field is in the same direction with the outward normal to the sphere as shown in Figure 91, the dot product becomes just the product of scalars E and dS $\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = E\, dS \cos(\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}})) = E\, dS \cos(\angle(0^o)) = EdS$, and the Gauss's law equation becomes

$$\oint_S E\, dS = \frac{Q_{inS}}{\varepsilon} \tag{342}$$

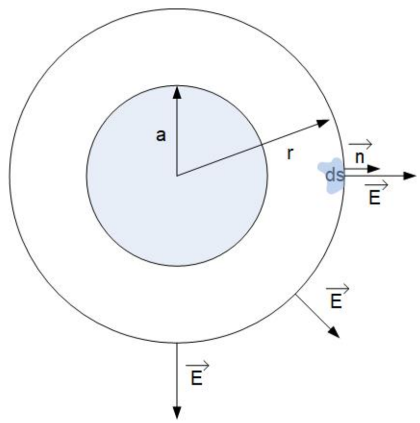


Figure 91: Application of Gauss's Law to find Electric Field of a spherical volume charge density.

Since the electric field is constant everywhere on the surface, we can take it out of the integral.

$$E \oint_S dS = \frac{Q_{inS}}{\varepsilon} \tag{343}$$

In the above equation, the $\oint_S dS$ is just the surface area of the sphere, $4\pi r^2$.

$$E\, 4\pi r^2 = \frac{Q_{inS}}{\varepsilon} \tag{344}$$

The final expression for the magnitude of the field is:

$$E = \frac{Q_{inS}}{4\pi\varepsilon\, r^2} \tag{345}$$

Note that we only found the magnitude of the field in the above equation. We started this problem with the knowledge of field radial direction. So the final expression for the field is

$$\overrightarrow{\mathbf{E}} = \frac{Q_{inS}}{4\pi\varepsilon\, r^2}\, \hat{r} \tag{346}$$

Note that this equation can also be obtained from Coulomb's Law. The charge Q_{inS} is the total charge in the spherical volume where the charge is located. If the total charge in the volume was known, then this is the solution. However, in this problem, the charge density is known, so we have to find the total charge.

$$Q_{inS} = \int_V \rho \, dV \tag{347}$$

Since the charge density ρ is constant, it can be taken outside of the integral.

$$Q_{inS} = \rho \int_V dV \tag{348}$$

The integral above is then just the entire volume of the charge distribution, which is $V = \dfrac{4\pi a^3}{3}$. The final expression for the electric field is

$$\overrightarrow{\mathbf{E}} = \frac{\rho \, 4\pi \, a^3}{12 \, \pi\varepsilon \, r^2} \, \hat{r} \tag{349}$$

$$\overrightarrow{\mathbf{E}} = \frac{\rho \, a^3}{3 \, \varepsilon \, r^2} \, \hat{r} \tag{350}$$

Note that now the electric field is inversely proportional to the square of the distance from the center of the sphere, $E \sim 1/r^2$. The electric field decreases as we move away from the sphere.

Electric field inside the sphere

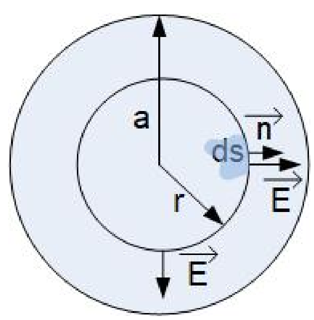


Figure 92: Application of Gauss's Law to find electric field of a spherical charge distribution.

Inside the spherical charge distribution, we'll again use a spherical imaginary surface S to enclose charge because of the spherical symmetry of the problem. The normal to the imaginary surface S is in the same direction with the electric field inside the spherical charge distribution as shown in Figure 92, and therefore the same analysis can be applied as above to get to the conclusion that:

$$E\,4\pi r^2 = \frac{Q_{inS}}{\varepsilon} \tag{351}$$

The difference in the analysis here is if we look at the right side of Gauss's law equation because we now have to determine the amount of charge enclosed by the imaginary surface we created. The charge enclosed in the imaginary surface is not the total charge Q. It is a fraction of the total charge. What fraction of the charge is enclosed? It is the fraction of the charge in the volume enclosed by the surface S.

$$Q_{inS} = \int_V \rho\, dV \tag{352}$$

Since the charge density ρ is constant, it can be taken outside of the integral.

$$Q_{inS} = \rho \int_V dV \tag{353}$$

The integral above is then just the fraction of the volume of the charge distribution enclosed by the surface S, which is $V = \dfrac{4\pi r^3}{3}$. The final expression for the electric field is

$$\overrightarrow{\mathbf{E}} = \frac{\rho\, 4\pi\, r^3}{12\,\pi\varepsilon\, r^2}\,\hat{r} \tag{354}$$

$$\overrightarrow{\mathbf{E}} = \frac{\rho\, r}{3\,\varepsilon}\,\hat{r} \tag{355}$$

Note that now the electric field is proportional to the distance from the center of the sphere, $E \sim r$, the field increases from the center of the spere out until $r = a$. At $r = a$, the entire charge has been enclosed, and the field is maximum at that point.

Electric field due to an infinite line of charge

A cylindrical region of radius a and infinite length is charged with uniform volume charge density ρ=const and centered on the z-axis. Find the field inside the cylindrical region of charge at a distance r from the axis of the charge density and the field outside of the spherical region of charge at (another) distance r away from the z-axis.

We will use Gauss's Law to solve this problem.

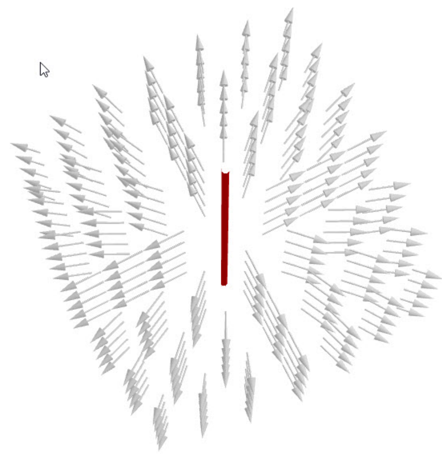


Figure 93: 3-dimensional electric field of a wire.

Electric field outside the line of charge (a wire)

We first look at the field outside of the cylindrical charge distribution. The wire is shown as a blue line in the direction of the z-axis, as shown in Figure 94. Because of the symmetry of the problem, we select a cylinder for the closed surface S. It is important to mention that we pick a point outside of the distributed charge at the distance r from the z-axis. The point that we picked is one point on the sphere's surface. We picked a point from the z-axis at a distance r, not at r=a, because we want to find out electric field anywhere outside of the charge distribution E(r), and r could be any point $r > a$. Gauss's Law states that

$$\oint_S \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = \frac{Q_{inS}}{\varepsilon} \tag{356}$$

We will now apply Gauss's law to the three surfaces shown in Figure 94. The cylindrical surface consists of three surfaces: two bases S1 and S3 and the side surface S2. The normals to the bases are S1 and S3 are $\hat{n}_1 = -\hat{z}$, $\hat{n}_3 = \hat{z}$ and the normal to the side surface is $\hat{n}_2 = \hat{r}$. We can now split the flux of the electric field vector (the left-side of Gauss's law) through this closed cylindrical surface intro three integrals:

$$\oint_S \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = \int_{S1} \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} + \int_{S2} \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} + \int_{S3} \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} \tag{357}$$

The electric field shown in Figure 94 is in the radial direction. The electric field lines only poke the side surface S2 and not surfaces S1 and S3. We, therefore, see that the flux through surfaces S1 and S3 is zero. This can also be shown

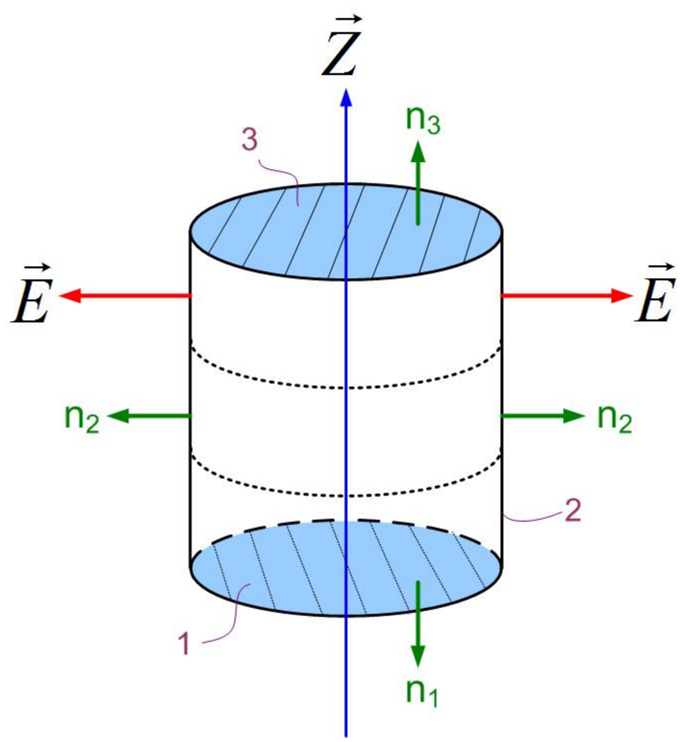


Figure 94: Application of Gauss's Law to find electric field of a line charge distribution oriented in the z-direction.

mathematically. For the bases of the cylinder, the dot product between the top surface vector and the electric field vector in Gauss's law becomes $\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = E\,dS\cos(\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}})) = E\,dS\cos(\angle(90^o)) = 0$, and the Gauss's law equation becomes

$$\int_{S2} E\,dS = \frac{Q_{inS}}{\varepsilon} \tag{358}$$

Since the electric field is constant everywhere on the side surface, we can take it out of the integral.

$$E\int_{S2} dS = \frac{Q_{inS}}{\varepsilon} \tag{359}$$

In the above equation, the $\displaystyle\int_{S2} dS$ is just the rectangular surface area of the side of the cylinder, $2\pi rl$. Where r is the radius of the cylinder, and l is the length of the cylinder.

$$E\,2\pi rl = \frac{Q_{inS}}{\varepsilon} \tag{360}$$

From here we can find the magnitude of the field:

$$E = \frac{Q_{inS}}{2\pi\varepsilon\,l\,r} \tag{361}$$

Note that we only found the magnitude of the field in the above equation. We started this problem with the knowledge of the field's radial direction.

$$\overrightarrow{\mathbf{E}} = \frac{Q_{inS}}{2\pi\varepsilon\,l\,r}\,\hat{r} \tag{362}$$

The charge Q_{inS} is the total charge in the spherical volume where the charge is located. If the total charge in the volume was known, then this is the solution. However, in this problem, the charge density is known, so we have to find the total charge.

$$Q_{inS} = \int_V \rho\,dV \tag{363}$$

Since the charge density ρ is constant, it can be taken outside of the integral.

$$Q_{inS} = \rho\int_V dV \tag{364}$$

The integral above is then just the entire volume of the charge distribution, which is $V = a^2 \pi l$. The final expression for the electric field is

$$\overrightarrow{\mathbf{E}} = \frac{\rho\, a^2 \pi l}{2\, \pi \varepsilon\, l\, r}\, \hat{r} \tag{365}$$

$$\overrightarrow{\mathbf{E}} = \frac{\rho\, a^2}{2\, \varepsilon\, r}\, \hat{r} \tag{366}$$

Note that now the electric field is inversely proportional to the distance from the z-axis, $E \sim 1/r$. The electric field decreases as we move away from the sphere, but slower than in the case of the sphere of charge.

Electric field inside the line of charge

To find the electric field inside the cylindrical charge distribution, we zoom in on the wire in the previous figure and select a cylindrical imaginary surface S inside the wire, as shown in Figure 95. Since the electric field is in the same direction inside the wire, and the flux of the electric field is the same, we conclude that the left side of Gauss's Law is the same as in the previous case.

$$E\, S = \frac{Q_{inS}}{\varepsilon} \tag{367}$$

$$E\, 2\pi r l = \frac{Q_{inS}}{\varepsilon} \tag{368}$$

The difference here is the amount of charge enclosed by the surface S.

$$Q_{inS} = \int_V \rho\, dV \tag{369}$$

Since the charge density ρ is constant, it can be taken outside of the integral.

$$Q_{inS} = \rho \int_V dV \tag{370}$$

The integral above is then just the fraction of the volume of the charge distribution enclosed by the surface S, which is $V = r^2 \pi\, l$. The final expression for the electric field is

$$\overrightarrow{\mathbf{E}} = \frac{\rho\, r^2\, \pi l}{2\pi l \varepsilon\, r}\, \hat{r} \tag{371}$$

$$\overrightarrow{\mathbf{E}} = \frac{\rho\, r}{2\, \varepsilon}\, \hat{r} \tag{372}$$

The electric field is proportional to the distance from the center of the sphere, $E \sim r$, the field increases from the center of the spere out until $r = a$. At $r = a$, the entire charge has been enclosed, and the field is maximum at that point.

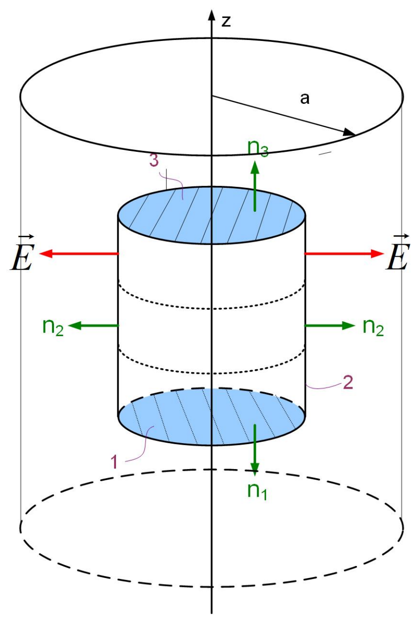


Figure 95: Applying Gauss's law to find the electric field inside the cylindrical charge distribution ρ_V of infinite length, and radius a.

Electric Field due to an infinite plane of Charge

An infinite plane of charge has an electric field in the direction away from it, as shown in Figure 96. In this figure, the light blue plane represents the charged plane, and only electric field vectors are represented above the plane. Below the plane, the vectors will be in the opposite direction, away from the plane.

The flux of the electric field vector is zero for any frame that is perpendicular to the plane of charge. For our imaginary surface, we can then use a box, where the flux will only exist through the top and bottom surfaces that are perpendicular to the field, or we can use a cylinder whose bases are parallel to the plane, as shown in Figure 97.

We can now separate the integral around the closed cylindrical surface to three surfaces, two that are parallel to the plane of charge, where the flux is not zero, and one side surface where the flux is zero.

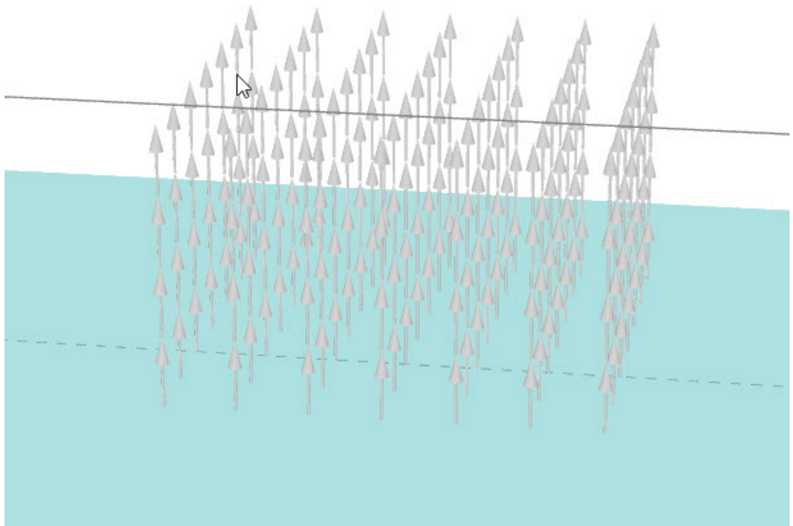


Figure 96: Electric field above an infinite plane charged with positive surface charge density ρ_S.

$$\oint_S \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = \int_{S1} \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} + \int_{S2} \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} + \int_{S3} \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} \tag{373}$$

Mathematically the dot product between the electric field and the normal to the cylindrical top (S3) and bottom surface (S1) is just the product of the magnitudes of E and dS $\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = E\,dS\cos(\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}})) = E\,dS\cos(\angle(0^o)) = EdS$. The dot product on the side surface (S2) is zero, because the angle between the normal to the surface and the electric field is 90^o, therefore the $\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = E\,dS\cos(\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}})) = E\,dS\cos(\angle(90^o)) = 0$. We can subsequently take the electric field outside of the integral, and the integral around the closed surface then becomes

$$E\int_{S1} dS + E\int_{S2} dS = \frac{Q_{inS}}{\varepsilon} \tag{374}$$

The integral of surface S1 and S1 is just the surface area of two surfaces.

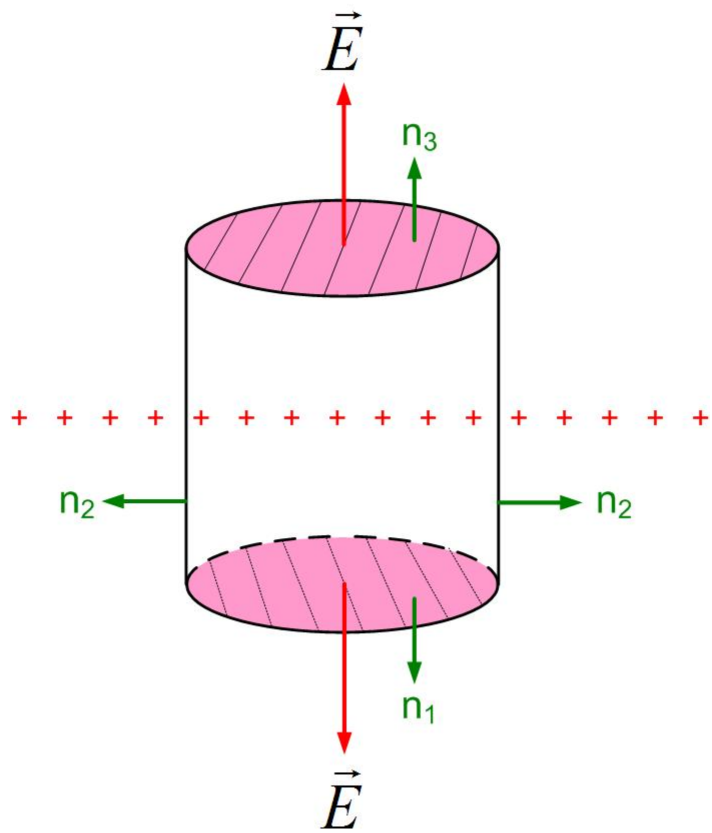


Figure 97: Applying Gauss's Law to an infinite plane charged with positive surface charge density ρ_S.

$$E\,S + E\,S = \frac{Q_{inS}}{\varepsilon} \tag{375}$$

$$2E\,S = \frac{Q_{inS}}{\varepsilon} \tag{376}$$

$$E = \frac{Q_{inS}}{2\varepsilon S} \tag{377}$$

In the above equation, the ration $\frac{Q_{inS}}{S}$ is just the surface charge density σ. The final electric field expression for the infinite sheet of charge becomes

$$\overrightarrow{\mathbf{E}} = \pm\frac{\sigma}{2\varepsilon}\,\hat{z} \tag{378}$$

Note that in the above expression, we added $\pm$ because we knew ahead of time the direction of the electric field. Below the z-axis, it is in the negative z-direction $-\hat{z}$, and above the z-axis, it is in the positive z-direction $\hat{z}$. Further, the electric field does not depend on the distance from the infinite plane. It is a constant that only depends on the surface charge density σ and the dielectric permittivity of surrounding space ε.

Two Infinite Planes

A parallel-plate capacitor can be modeled with two infinite parallel plates of opposite charge densities. To find the total field, we can use the principle of superposition, as shown in Figure 98. The Figure shows first just the negative sheet of charge (the first region to the left), then only the positive sheet of charge (the region between two red vertical lines), and finally both sheets of charge in the region to the right. We first find the field separately for the negative sheet of charge, then the positive sheet of charge, finally, we sum them up to get the total field.

In the previous section, we found that the electric field from an infinite sheet of charge is constant. The electric field anywhere around the negative sheet of charge is

$$\overrightarrow{\mathbf{E}_{-}} = \mp\frac{\sigma}{2\varepsilon}\,\hat{z} \tag{379}$$

The electric field, due to the positive sheet of charge, is

$$\overrightarrow{\mathbf{E}_{+}} = \pm\frac{\sigma}{2\varepsilon}\,\hat{z} \tag{380}$$

If you look at the direction of the field, the fields from individual plates between the plates of the capacitor add up, and the fields above and below the capacitor subtract. The final field is only between the plates of the capacitor, and it is equal to double the value of the one sheet of charge.

$$\overrightarrow{\mathbf{E}} = \frac{\sigma}{\varepsilon}\,\hat{z} \tag{381}$$

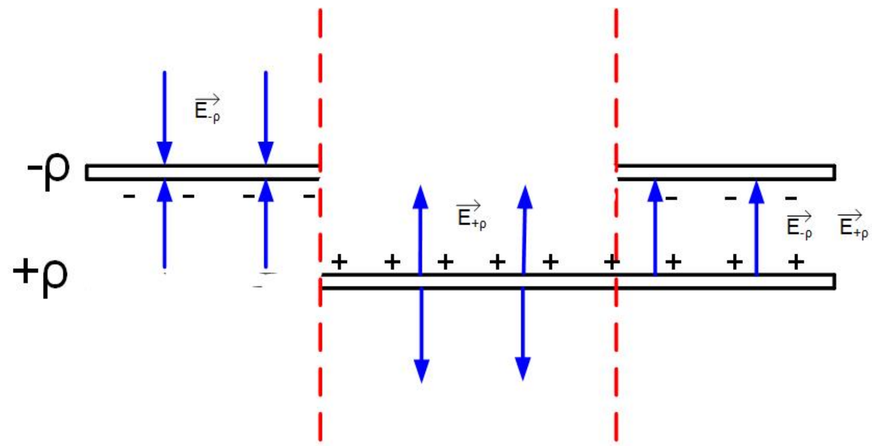


Figure 98: Two infinite planes charged with positive surface charge density ρ_S and $-\rho_S$.

7.6 Electrostatic Boundary Conditions

Conductors in the electrostatic field

Conductors conduct current well because the atoms of good conductors have many loosely bound electrons that can leave the atoms in the presence of an external electric field. In the absence of an external electric field, a piece of metal is shown in Figure 99 on the left. When there is no electric field, the electrons are close to the nucleus. On the right, the same metal piece is placed in an electric field of the battery. Under the influence of the external field, E_v, electrons can freely move away from the atoms in the direction opposite to the direction of the external field. This type of current is called conduction current. The point form of Ohm's law states that $E = J/\sigma$, where J is the current density, E is the electric field, and σ is the conductivity of the material. Conductors have very high conductivity, and so the electric field inside the conductors is zero.

In electrostatics, we assume that the charges are not moving, so there is no conduction current. The electric field inside the conductors is zero. As we will see later in this section, the charges on the conductor in electrostatic fields can exist only on its surface, and the vector of the electric field must be perpendicular to the surface of the metal. The tangential electric field is zero. All points on a conductor in electrostatic fields have the same potential, and so the conductor is an equipotential surface.

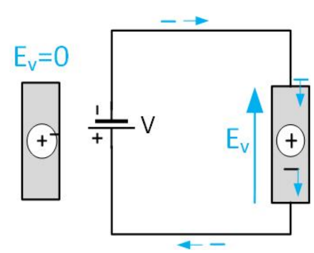


Figure 99: Conductor in Electric Field.

Dielectrics in the electrostatic field

As shown in Figure 100, in dielectrics, in the absence of an electric field, the electrons are close to the nucleus. The difference here is that the electrons are tightly bound to the nucleus, and they cannot escape in the presence of an electric field. When a battery establishes electric field E_v inside the dielectric, the atoms of the dielectric stretch because the nucleus is pulled in the direction of the field, and electrons in the opposite direction and the atom can be represented by a dipole. On the other hand, the free electrons in the wire connected to the dielectric start bunching up on top of the dielectric piece, and the dipole's positive charge is attracted to electrons. The negative dipole's bound charge pushes electrons away from the bottom conductor. Looking from the outside, the current flows, but the electrons are not flowing through the dielectric. This type of current is called a displacement current. If the battery is removed, the free negative and positive charges are trapped on the top and bottom of the dielectric piece.

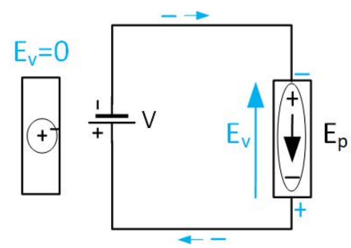


Figure 100: Dielectric in Electric Field.

The electrons in the metal on top of the dielectric establish an electric field across it, as shown in Figure 101. This field, in turn, produces electric dipoles in the dielectric, as explained above. The internal positive and negative charges cancel each other, and the positive bound charge from the dielectric on top and negative on the bottom produce their own field, which is in the opposite direction from the external field, as shown in Figure 102.

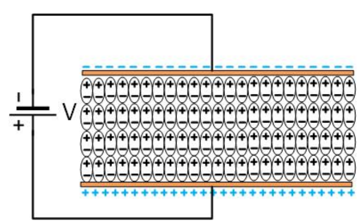


Figure 101: Polarization of a dielectric in external electric Field. Each oval represents one atom.

The total field in the dielectric is the sum of the electric fields from free charges on top and bottom metal pieces E_v, and the electric field from the separated polarization charges of the dielectric E_p, as shown in Equation 382. The induced field E_v is a fraction of the external field, and we can represent it in terms of the

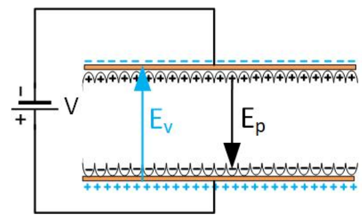


Figure 102: Two fields acting inside the dielectric. The external field E_v from the free charges in the metal on top and bottom, and the polarized dielectric field E_p. The inner part of the dielectric is removed to show clearly the fields.

external field as $E_p = mE_v$, where m is some constant. We can then express the total field as a fraction of the external field in Equation 384.

$$E_{total} = E_v - E_p \tag{382}$$
$$E_{total} = E_v - mE_v \tag{383}$$
$$E_{total} = E_v(1 - m) \tag{384}$$

Relative dielectric permittivity of the material ε is defined as $1 - m = \dfrac{1}{\varepsilon_r}$. Therefore the total field inside the dielectric is lower than if no dielectric is present.

$$E_{total} = \frac{E_v}{\varepsilon_r} \tag{385}$$

Dielectric permittivity of a material is defined as the relative permittivity multiplied by the permittivity of free space $\varepsilon_0 = 8.85\ 10^{-12}$ F/m.

Relative dielectric constant

Relative dielectric constant is in general a complex number $\varepsilon_r = \varepsilon_r{}' + j\varepsilon_r{}''$.

$\varepsilon_r{}'$ in data sheets is called a dielectric constant, or design dielectric constant, and it varies from 1 in the air to 13 in GaAs. An outlier is a dielectric constant of distilled water, $\varepsilon_r{}' = 80$.

We can sketch the complex relative dielectric constant on a complex plane. The angle between the magnitude of the dielectric constant and the x-axis is called $\tan\delta$, and it is used to describe the losses in the dielectric material. In datasheets for PC boards, you can see that $\tan\delta$ is from about 0.001 for microwave substrates, such as Rogers Duroid, to 0.02 for low-frequency FR4 substrates.

Boundary conditions at a dielectric-dielectric boundary

In many electrical structures, more than one dielectric is used so that the electric field exists in different dielectrics. In such cases, we are interested in how will the electric field change from one dielectric to the other. Figure 103 shows the boundary between the two dielectrics with permittivities ε_1 and ε_2, and the electric fields E_1 in material 1 and E_2 in material 2. At the boundary between two materials, we may have surface charge density ρ_s.

At the boundary between any two dielectrics, the tangential components of the electric field E_{1t}, E_{2t} are continuous, and the normal components E_{1n}, E_{2n} are discontinuous and equal to the surface charge density.

$$E_{1t} = E_{2t} \tag{386}$$

$$\varepsilon_1 E_{1z} - \varepsilon_2 E_{2z} = \rho_s \tag{387}$$

If the free surface charge density at the boundary is zero, then the normal components of the electric field at the boundary are

$$E_{1t} = E_{2t} \tag{388}$$

$$\varepsilon_1 E_{1z} = \varepsilon_2 E_{2z} \tag{389}$$

We can also write electric flux density vectors at the boundary. Since $D_1 = \varepsilon_1\, E_1$ and $D_2 = \varepsilon_2\, E_2$, the above equations can be re-written as

$$\varepsilon_2 D_{1t} = \varepsilon_1 D_{2t} \tag{390}$$

$$D_{1z} = D_{2z} \tag{391}$$

Question 25 *The four equations below show the tangential and normal electric field at the boundary of two dielectrics. Dielectric 1 is a Teflon with a relative dielectric constant of 2.2, and dielectric 2 is Silicon with a relative dielectric constant of 11.2. Which set of equations represents a possible electric field?*

Multiple Choice:

(a) $2.2\, E_{1t} = 11.2\, E_{2t}$ *and* $E_{1z} = E_{2z}$

(b) $E_{1t} = E_{2t}$ *and* $2.2\, E_{1z} = 11.2\, E_{2z}$

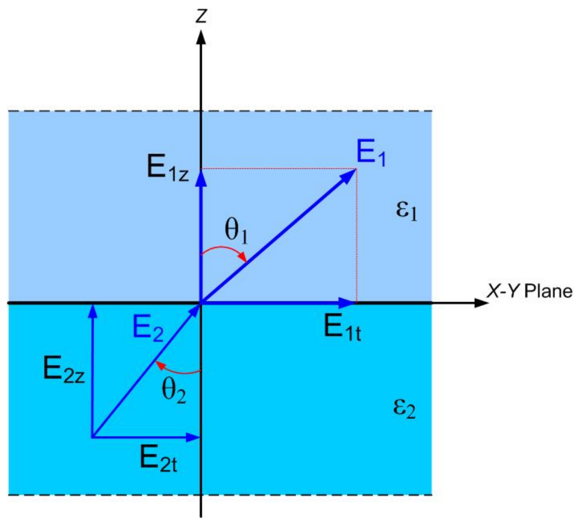


Figure 103: Boundary Conditions for Electric Field.

(c) $2.2\,E_{1t} = 11.2\,E_{2t}$ *and* $E_{1z} = E_{2z}$

(d) $E_{1t} = E_{2t}$ and $11.2\,E_{1z} = 2.2\,E_{2z}$

Boundary conditions at a conductor-dielectric boundary

The electric field inside perfect conductors $\sigma \to \infty$ is zero. Ohm's law states that

$$E = \frac{J}{\sigma} \tag{392}$$

When $\sigma \to \infty$, from the above equation, we see that the electric field is zero. This means that at the boundary of the dielectric and metal, the tangential field in the dielectric must be zero as well, and the only field at the boundary of a metal is the normal electric field D_n, and it is equal to the induced charge at the surface of the conductor.

$$D_n = \rho_s \tag{393}$$

Figure 104 shows the field at the boundary of the metallic sphere. Watch this demonstration of separation of charges on a metallic sphere in the electric field of the VanDenGraaff generator.

YouTube link: `https://www.youtube.com/watch?v=hOD2TOfYuM8`

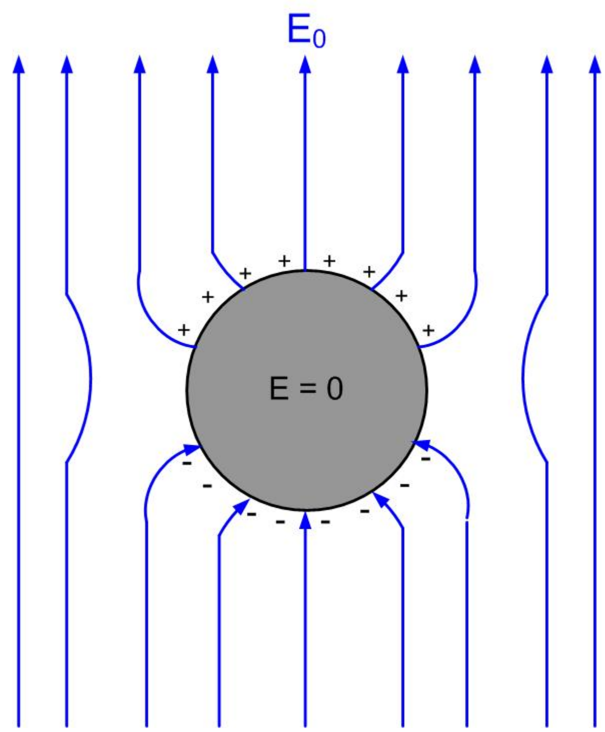


Figure 104: Metallic sphere in an external electric field.

Shielding with Faraday's Cage

The electric field is zero inside the closed metallic conductor, even if the conductor is hollow, as shown in Figure 105, and no charge is induced inside a metallic shield. This is Faraday's cage.

Watch a demonstration of zero electric fields, and no charge, inside a hollow conductor by Prof Emeritus of MIT Walter Lewin.

YouTube link: `https://www.youtube.com/watch?v=W_NelWgJ3LY`

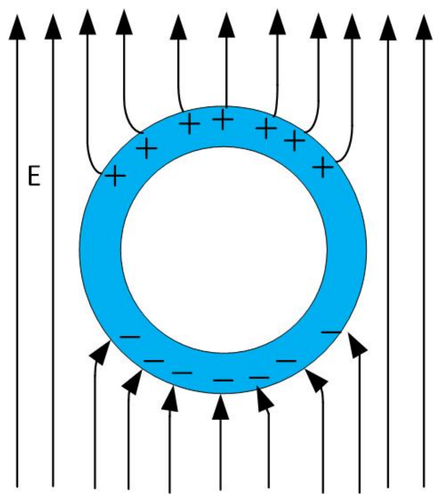


Figure 105: Electric field inside hollow metallic conductor (Faraday's Cage).

Grounding

In Figure 106, we introduce a charge inside a hollow conductor, and the electric field forms inside the conductor. The charge in the metallic shell will redistribute so that the field is zero inside the metal. The charge on the surface of the conductor will be uniformly distributed, regardless of the position of the charge inside the hollow part.

Figure 107 shows a grounded hollow conductor with a charge inside it. In this case, the positive charge on the outside of the conductor will attract negative charges from the ground that neutralize the positive charge inside the shell, so there will be no field outside the shell.

Watch a demonstration of Faraday's Cage by Prof Emeritus of MIT Walter Lewin. He will enter the Faraday's Cage with a tinsel, a transmitter (his wireless microphone that likely works at a frequency of a few GHz), and a receiver (a radio that works at a couple of hundred Megahertz frequency). The Faraday's cage is likely not grounded. He cannot receive the radio signal as the outside radio waves cannot enter the Faraday's cage, but the waves his microphone transmitter generates inside the cage still reach the receiver that is placed somewhere in the classroom since the cage is not grounded.

YouTube link: `https://www.youtube.com/watch?v=t5uHzCfiXp4`

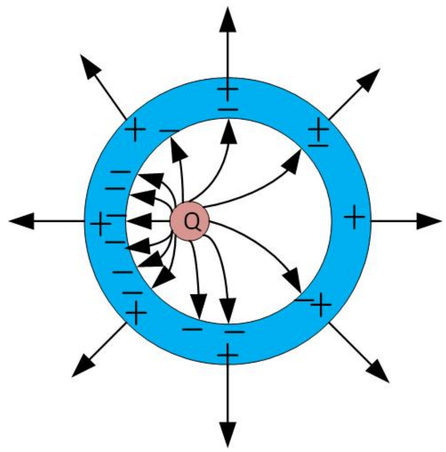


Figure 106: Hollow conductor with a charge inside.

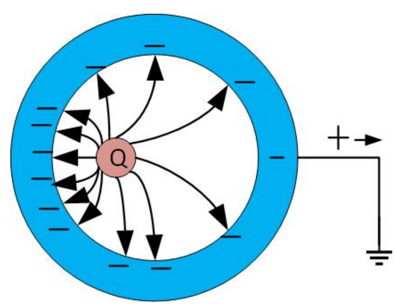


Figure 107: Grounded hollow conductor with a charge inside.

Proof of boundary conditions

We will now use Maxwell's equations to derive the electrostatic boundary conditions.

First, we will use Gauss's law to find the normal component of the fields at the boundary between two dielectrics, as shown in Figure 108. As we can see from the figure, the flux of the electric field exists through both bases and the side of the cylinder. We can find the components of the fields in both dielectrics, one parallel to the boundary x and one perpendicular to the boundary, in the direction of y.

$$\overrightarrow{\mathbf{E_1}} = \overrightarrow{\mathbf{E_{1x}}} + \overrightarrow{\mathbf{E_{1y}}} = E_{1x}\overrightarrow{\mathbf{x}} + E_{1y}\overrightarrow{\mathbf{y}} \tag{394}$$

$$\overrightarrow{\mathbf{E_2}} = \overrightarrow{\mathbf{E_{2x}}} + \overrightarrow{\mathbf{E_{2y}}} = E_{2x}\overrightarrow{\mathbf{x}} + E_{2y}\overrightarrow{\mathbf{y}} \tag{395}$$

The tangential components of the fields produce flux through the sides, and normal components produce flux through the bases. Since we are interested in what happens at the boundary, we will let the height of the cylinder be infinitesimally small $h \to 0$. Because the height of the cylinder is zero, and therefore the surface area is zero, the flux through the side surface S3 is zero. The flux through the top and bottom surfaces will only exist due to the normal components of the field.

$$\oint_S \overrightarrow{\mathbf{D}} \cdot \overrightarrow{\mathbf{dS}} = Q_{inS} \tag{396}$$

$$\int_{S1} \overrightarrow{\mathbf{D}} \cdot \overrightarrow{\mathbf{dS}} + \int_{S2} \overrightarrow{\mathbf{D}} \cdot \overrightarrow{\mathbf{dS}} + \int_{S3} \overrightarrow{\mathbf{D}} \cdot \overrightarrow{\mathbf{dS}} = Q_{inS} \tag{397}$$

$$\int_{S1} \overrightarrow{\mathbf{D}} \cdot \overrightarrow{\mathbf{dS}} + \int_{S2} \overrightarrow{\mathbf{D}} \cdot \overrightarrow{\mathbf{dS}} + 0 = Q_{inS} \tag{398}$$

$$\int_{S1} (\varepsilon_1 E_{1n}\overrightarrow{\mathbf{y}} + \varepsilon_1 E_{1t}\overrightarrow{\mathbf{x}}) \cdot dS\overrightarrow{\mathbf{y}} + (\varepsilon_2 E_{2n}\overrightarrow{\mathbf{y}} + \varepsilon_2 E_{2t}\overrightarrow{\mathbf{x}}) \cdot dS\overrightarrow{\mathbf{y}} = Q_{inS} \tag{399}$$

$$-\varepsilon_1 E_{1n} S + \varepsilon_2 E_{2n} S = Q_{inS} \tag{400}$$

$$\varepsilon_2 E_{2n} - \varepsilon_1 E_{1n} = \frac{Q_{inS}}{S} \tag{401}$$

The tangential components of the field can be obtained from the equation for Faraday's law for static fields, as shown in Figure 109. We choose a rectangular contour, as shown in the figure with length l and width w. Since again we are interested in the boundary, we will let the width of the contour go to zero. The integral along the w-pieces will then be zero. The integral along the l-pieces of contour will depend on the orientation of contour, and we will pick a counter-clockwise path. Because of the counter-clockwise path, the x-component of the

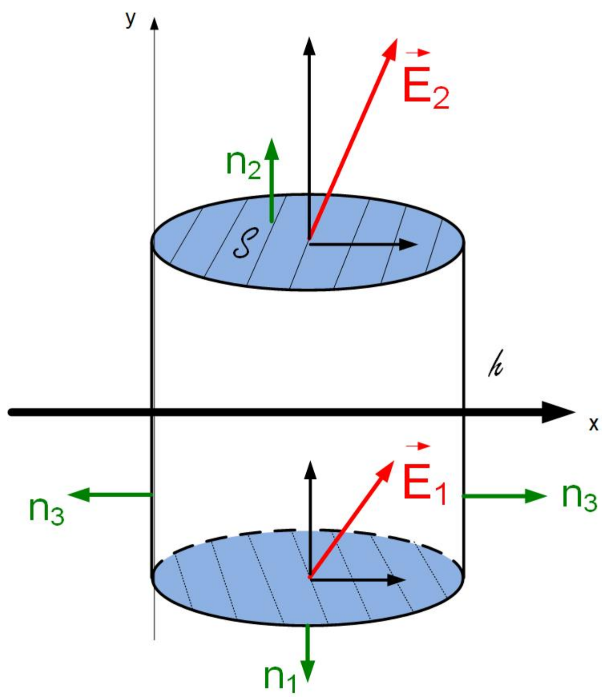


Figure 108: Derivation of equation for normal components of electric field on the boundary of two dielectrics.

E_1 field will be negative, and we have that the x-components of the fields in two dielectrics have to be the same.

$$\oint_C \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} = 0 \tag{402}$$

$$\int_{l1} (E_{1x}\overrightarrow{\mathbf{x}} + E_{1y}\overrightarrow{\mathbf{y}}) \cdot dy\overrightarrow{\mathbf{y}} + \int_{l2} (E_{2x}\overrightarrow{\mathbf{x}} + E_{2y}\overrightarrow{\mathbf{y}}) \cdot dy\overrightarrow{\mathbf{y}} = 0 \tag{403}$$

$$-E_{1x}l + E_{2x}l = 0 \tag{404}$$

$$E_{1x} = E_{2x} \tag{405}$$

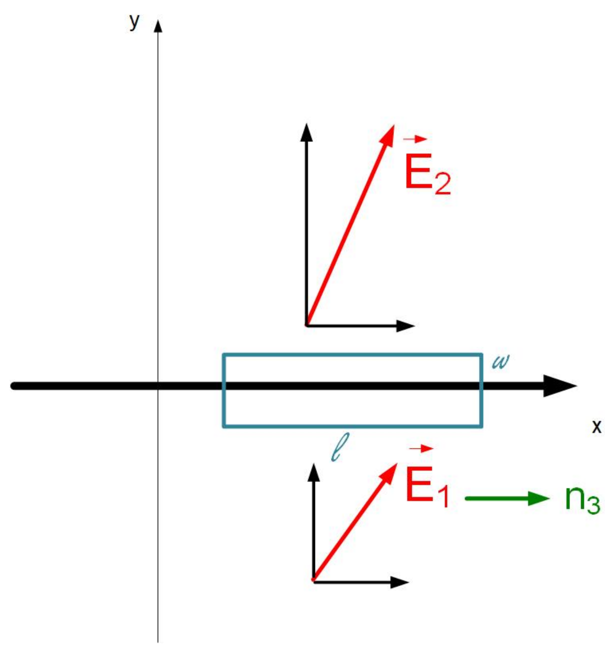


Figure 109: Derivation of equation for tangential components of electric field on the boundary of two dielectrics.

Charge distribution around sharp edges

Theshape of the conductive material impacts the chargde distribution and charge density, as shown in Figure 110. We can see that the charge distribution and electric field on round objects are uniform. The highest charge density and strongest electric fields are produced on sharp edges of condutive bodies.

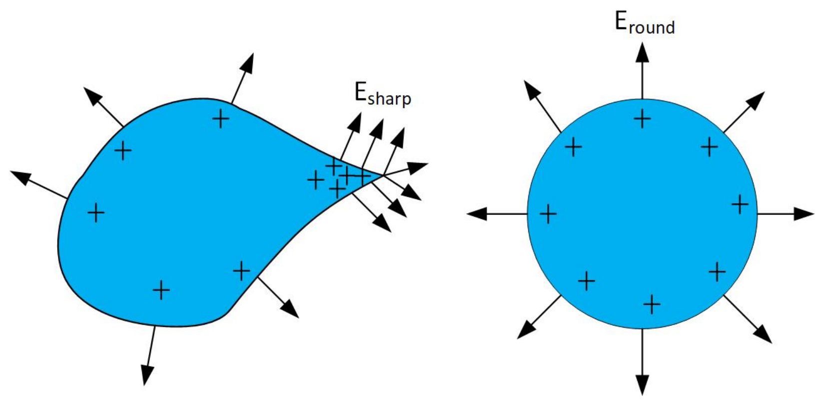


Figure 110: Electric field and charge distribution close to sharp edges.

Demonstration of higher charge density near sharp edges by Prof. Emeritus at MIT, Walter Lewin.

YouTube link: `https://www.youtube.com/watch?v=cYOBSzh2SLE`

7.7 Capacitance

What is Capacitance?

Capacitance is a constant that relates the amount of charge on a conductor with a potential difference between conductors. Whenever we have two conductors at different potentials, separated by an insulator (dielectric), there will be electric field between them, and we can define the ability for those two conductors to store charge. This is called a capacitance.

$$C = \frac{Q}{V} \tag{406}$$

From Gauss's law, we found that the electric field is proportional to the charge enclosed $Q \sim E$. The capacitance therefore represents the strenght of the electric field around a conductor for a fixed voltage between the conductors. If we for example have two sets of conductors, for the same voltage between them the electric field will be stronger around the conductor set with higher capacitance, and this conductor set will have higher ability to store charge. It will store more charge for the same voltage difference than the other conductor set.

Capacitance relates the current with the change in voltage. The current is defined as the change of charge in time $i = \frac{dq}{dt}$. Since charge is from the above equation equal to the product of capacitance and voltage, Q=CV, then the current is proportional to the change in voltage. If the voltage changes twice as fast, the current doubles. The voltage V is the potential difference between the positive and negative plate of the capacitor.

$$i = C\frac{dv}{dt} \tag{407}$$

Computing Capacitance

Capacitance can be computed:

(a) from its definition $C = \frac{Q}{V}$. You can first apply Gauss's law to find the electric field, then find the potential difference between the conductors. The potential difference depends on charge, and so, the charges will cancel.

(b) from the expression for the electrostatic energy $W_e = \frac{1}{2}\int_V \varepsilon_o \varepsilon_r E^2 \, dV$. First apply Gauss's law to find the electric field E, square it, and integrate throughout the volume enclosed by the two conductors.

Capacitors

Capacitors are components that provide capacitance in electronic circuits. They are used in filters, PCB power-distribution networks, matching circuits, delay lines etc. Typical capacitors are shown in Figure 111. Earth is a conductor, so any conductor either on it's own or in pair with another conductor can be modeled as a capacitor. For a single conductor, for example a metallic conductor in air, the other "conductor" is earth, or another nearby metallic object. Transmission lines have two conductors, a "signal" and "ground" conductor, so they have capacitance per unit lenght, and we can model them with a distributed capacitors. In this section, using Maxwell's equations, we will derive equations for capacitance for typical capacitors and transmission lines.

Figure 111: Various commercially available capacitors. Reference: Wikipedia

Analysis of a Parallel-Plate Capacitor with a homogeneous dielectric

A parallel plate capacitor consists of two conductive plates, as shown in Figure 112. The surface area of the plates is S and the charge on each plate is Q. The plates are separated by a distance d ($d << S$). Homogeneous dielectric with dielectric constant of ε_r fills the space between electrodes.

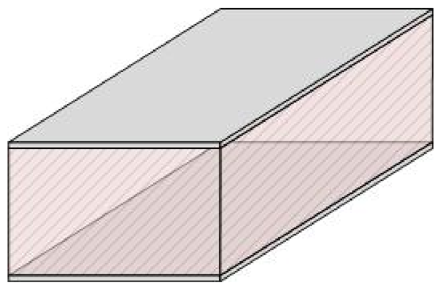


Figure 112: Two infinite planes charged with positive surface charge density ρ_S and $-\rho_S$.

The electric field in an actual parallel-plate capacitor is shown in Figure

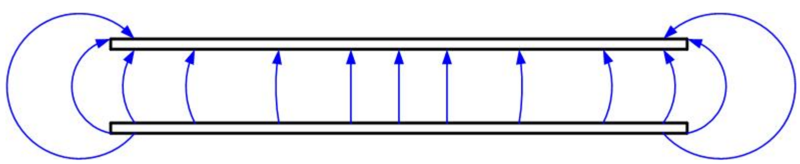


Figure 113: Fringing fields in a parallel-plate capacitor. The bottom plate is positively charged, and the top plate is negatively charged.

The condition that d¡¡S allows us to ignore fringing fields at the edge of the capacitor, and to assume that the field is equal to the field of two infinite parallel plates of charge. This means that the field is constant in between the plates, oriented from positive to negative plate and zero outside of the plates, as shown in Figure 114.

To find the electric field inside the capacitor, we apply Gauss's law to a cylinder whose bottom half is in the dielectric, with the bottom base at the point where we want to find the field, and the top half is in the air above one of the charged plates as shown in Figure 115. We see that the flux through the capacitor exists only through the bottom base. The flux through the top base is zero because the field outside the capacitor is zero, and the flux through the side of the cylinder is zero because the electric field does not not go through that surface

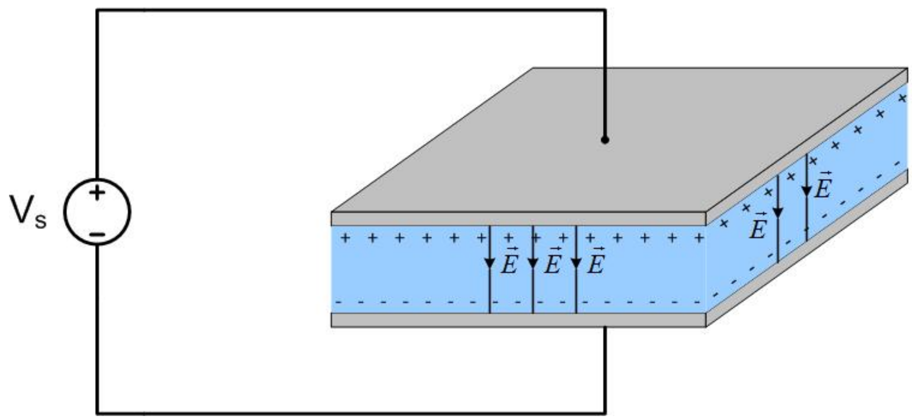


Figure 114: Two infinite planes charged with positive surface charge density ρ_S and $-\rho_S$.

(in other words the angle between the field and the normal to the surface is 90^o, and therefore the dot product is zero.) Mathematically, we start from the Gauss's law, assuming that the dielectric permeability is $\varepsilon = \varepsilon_o \varepsilon_r$, where ε_r is the dielectric constant of the material between capacitor's plates.

$$\oint_S \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = \frac{Q_{inS}}{\varepsilon} \tag{408}$$

We split the cylindrical surface S into two base surfaces and a side surface.

$$\oint_S \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = \int_{S1} \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} + \int_{S2} \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} + \int_{S3} \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} \tag{409}$$

The dot product between the electric field and the normal to the cylindrical top (S3) surface is zero, because electric field is zero outside of the capacitor. The dot product is zero on the side surface because there is no flux through it, the normal to the surface and the electric field are perpendicular $\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = E\,dS\cos(\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}})) = E\,dS\cos(\angle(90^o)) = 0$. The flux through the bottom surface (S1) is just the product of the magnitudes of E and dS $\overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dS}} = E\,dS\cos(\angle(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{dS}})) = E\,dS\cos(\angle(0^o)) = EdS$, because the normal to the surface and the electric field are in the same direction. We can now take the electric field outside of the integral, and the integral around the closed surface then becomes

$$E \int_{S1} dS = \frac{Q_{inS}}{\varepsilon} \tag{410}$$

The integral of surface S1 is just the surface area of the bottom surface.

$$E\,S = \frac{Q_{inS}}{\varepsilon} \tag{411}$$

$$E = \frac{Q_{inS}}{\varepsilon S} \tag{412}$$

In the above equation, the ration $\frac{Q_{inS}}{S}$ is just the surface charge density σ. The final electric field expression for the infinite sheet of charge should include the unit vector of the direction of the field. We will assume that the z-axis is in the up direction from the bottom to the top plate. The field is then

$$\overrightarrow{\mathbf{E}} = \frac{\sigma}{\varepsilon}(-\overrightarrow{\mathbf{z}}) \tag{413}$$

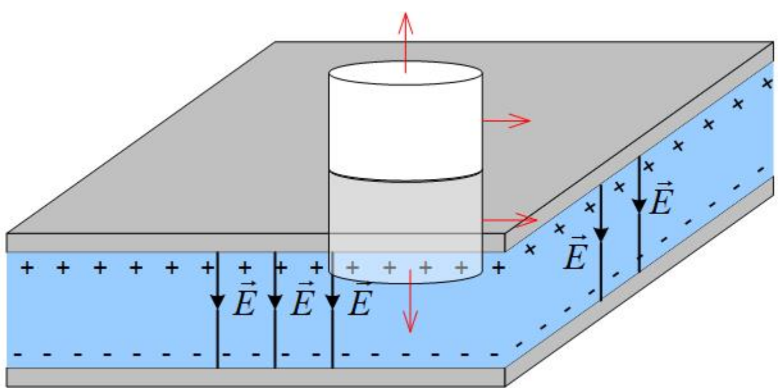


Figure 115: Two infinite planes charged with positive surface charge density ρ_S and $-\rho_S$.

Now that we found the electric field of the paralle plate capacitor, we need to find the potential difference between the two plates, from positive to negative plate. The positive plate in this case is the top plate, so we integrate from d to 0. The potential is defined as

$$V_{ab} = \int_d^0 \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} \tag{414}$$

We will find the potential from the bottom plate at z=0 to the top plate at z=d. In the equation above $\overrightarrow{\mathbf{dl}}$ will always

$$V_{ab} = \int_d^0 \frac{\sigma}{\varepsilon}(\text{-}\overrightarrow{\mathbf{z}}) \cdot dz\,\overrightarrow{\mathbf{z}} \tag{415}$$

$$V_{ab} = \text{-}\frac{\sigma}{\varepsilon}\int_d^0 dz \tag{416}$$

$$V_{ab} = \frac{\sigma}{\varepsilon}d \tag{417}$$

This potential should be positive, because we integrated from the higher to the lower potential. The potential at a lower plate is lower than the potential at the higher plate. If you make a mistake and get a negative potential, it just means you should have started from the positive plate and integrated to the negative plate to get the positive potential difference. However, the $V_{ba} = -Vab$, so we can just take the magnitude of the negative potential difference. *If the potential difference is a $\log x$ function, make sure that the $x > 1$.* The total charge on one plate is $Q = \sigma S$, where S is the area of the plates. The capacitance is then

$$C = \frac{Q}{V} \tag{418}$$

$$C = \frac{\sigma S}{\frac{\sigma}{\varepsilon}d} \tag{419}$$

$$C = \varepsilon\frac{S}{d} \tag{420}$$

Question 26 *The potential difference between the plates of a capacitor is V. The distance between the plates is d. Ignore fringe effects. The magnitude of the electric field between the plates is:*

Multiple Choice:

(a) *not enough information*

(b) V^2/d

(c) d/V

(d) V/d

(e) d/V^2

Question 27 *The area of the plates of a parallel-plate capacitor is doubled, the capacitance is*

Multiple Choice:

(a) *not enough information*

(b) *stays the same*

(c) *quartered*

(d) *halved*

(e) *doubled*

If the area of the plates of a parallel-plate capacitor is doubled, the capacitance is

Coaxial-Cable Capacitance

Coaxial cable consists of a solid inner conductor and a conductive outer shell, Figure 118.

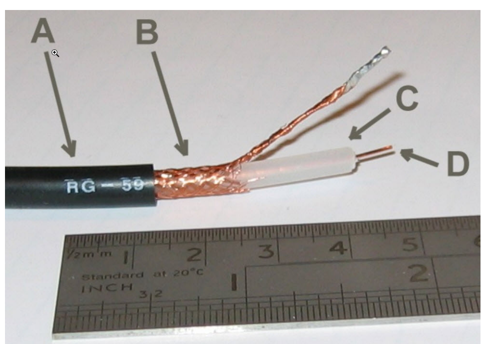


Figure 116: A coaxial cable. A - insulator, B - outer conductor, C - dielectric, D - inner conductor. Click to see the Wikipedia page on coaxial cables.

If we apply Gauss's law to a cylinder outside of the outer conductor, as shown in Figure 117, we see that the field is zero, because the total charge enclosed is zero.

To find the capacitance we need to find the electric field between the inner and outer conductor of a coax. We apply Gauss's law to the green closed cylindrical surface shown in Figure 118. Following the steps in the section on finding the field around the infinite line of charge, we find the field inside the capacitor.

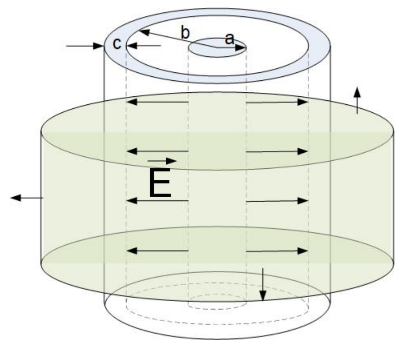


Figure 117: A coaxial cable, with inner conductor charged with positive scharge Q and outer conductor charged with $-Q$.

$$E\,S = \frac{Q_{inS}}{\varepsilon} \tag{421}$$

$$E = \frac{Q_{inS}}{2\pi h \varepsilon r} \tag{422}$$

To find the potential, we integrate from the inner to the outer conductor.

$$V = \int_a^b \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dr}} \tag{423}$$

$$V = \int_a^b \frac{Q_{inS}}{2\pi h \varepsilon r} \hat{r}\; dr \hat{r} \tag{424}$$

$$V = \frac{Q_{inS}}{2\pi h \varepsilon} \log \frac{b}{a} \tag{425}$$

The capacitance is then

$$C = \frac{Q}{V} \tag{426}$$

$$C = \frac{Q}{\frac{Q_{inS} \log \frac{b}{a}}{2\pi h \varepsilon}} \tag{427}$$

$$C = \frac{2\pi h \varepsilon}{\log \frac{b}{a}} \tag{428}$$

$$C' = \frac{C}{h} \tag{429}$$

$$C' = \frac{2\pi \varepsilon}{\log \frac{b}{a}} \tag{430}$$

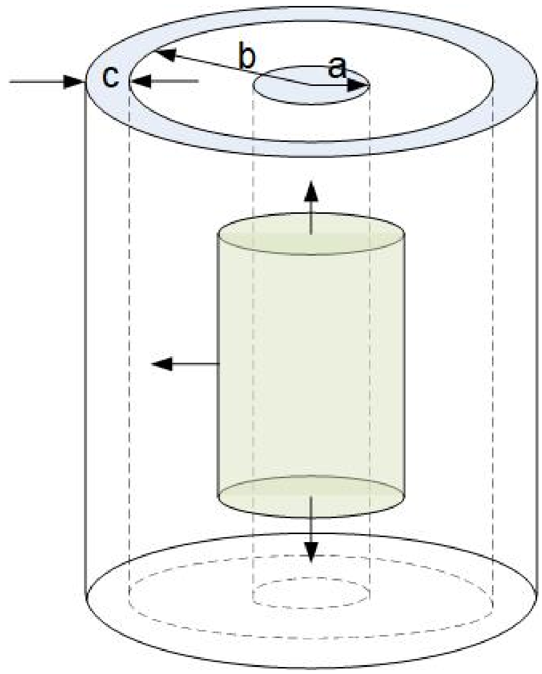


Figure 118: A coaxial cable, with inner conductor charged with positive scharge Q and outer conductor charged with $-Q$.

Use the calculator below to find the capacitance of RG6 Coaxial Cable, made by Mouser, part number 40001. See data sheet here. Assume PE dielectric constant is 2.25. Compare with the value in the data sheet.

Geogebra link: `https://tube.geogebra.org/m/whkrg2pu`

Spherical Capacitance

Our earth is a giant spherical capacitor, with ground as one electrode and the ionosphere as another. Spherical capacitor in Figure 119 consists of two concentric shells with radii a and b. The thickness of the outer shell is c-b. The inner shell is charged with charge Q and outer shell charged with charge -Q. If we apply Gauss's Law to a sphere larger than c, we see that the total charge enclosed is zero, and the field must be zero as well. Between the plates, the field is oriented radially from the positive to the negative charge. Application of Gauss's law to a sphere of a radius r between radii a and b leads to the following equation

$$E\,S = \frac{Q}{\varepsilon} \tag{431}$$

The surface area of the sphere is $4\pi r^2$.

$$E 4\pi r^2 = \frac{Q}{\varepsilon} \tag{432}$$

$$\overrightarrow{\mathbf{E}} = \frac{Q}{4\pi\varepsilon r^2}\hat{r} \tag{433}$$

The potential difference between the inner and outer shell is

$$V = \int_a^b \overrightarrow{\mathbf{E}} \cdot \hat{dr} \tag{434}$$

$$V = \int_a^b \frac{Q}{4\pi\varepsilon r^2}\hat{r}\ dr\hat{r} \tag{435}$$

$$V = -\frac{Q}{4\pi\varepsilon}\frac{1}{r}|_a^b \tag{436}$$

$$V = \frac{Q}{4\pi\varepsilon}\frac{b-a}{ab} \tag{437}$$

The capacitance is then

$$C = \frac{Q}{V} \tag{438}$$

$$C = 4\pi\varepsilon\frac{ab}{b-a} \tag{439}$$

Electrostatic energy of a charged capacitor

Electrostatic energy is defined as

$$W_e = \frac{1}{2}\int_V \varepsilon E^2 dv \tag{440}$$

Energy of a charged capacitor can be expressed by any of the following equations

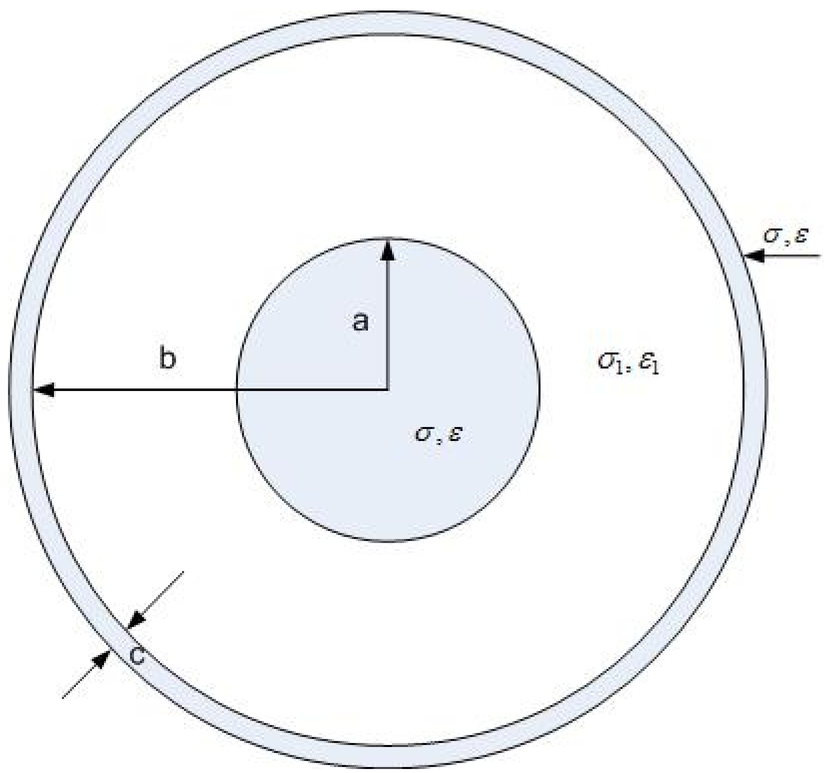


Figure 119: A spherical capacitor, with inner conductor charged with positive scharge Q and outer conductor charged with $-Q$.

$$W_e = \frac{1}{2}CV^2 \tag{441}$$

$$W_e = \frac{1}{2}QV \tag{442}$$

$$W_e = \frac{1}{2}\frac{Q^2}{C} \tag{443}$$

From the definition of electrostatic energy, and the expression for energy of a charged capacitor, we can find the capacitance:

$$C = \frac{Q^2}{\int_V \varepsilon E^2 dv} \tag{444}$$

You can try and solve the capacitance for the above problems again, by using the Equation 444.

Now, watch the videos below to experience the energy stored in a charged capacitor, and see how a fuse and camera flash work by discharging a capacitor.

Do not attempt these demonstrations at home, touching the capacitor's electrodes can be fatal, as humans are good conductors of electricity. Demonstrations were performed at MIT by Prof. Emeritus Walter Lewin.

YouTube link: `https://www.youtube.com/watch?v=ED-Aelxm13A`

YouTube link: `https://www.youtube.com/watch?v=_cx-PUh2Q7A`

7.8 Method of images

Method of images states that we can replace a charge above an infinite conducting plane with an equivalent configuration of the charge and its image. For example, if we have a positive charge Q, the field above the conducting plane will be the same if we replace the plane with an equal and opposite charge -Q as shown in Figure 120.

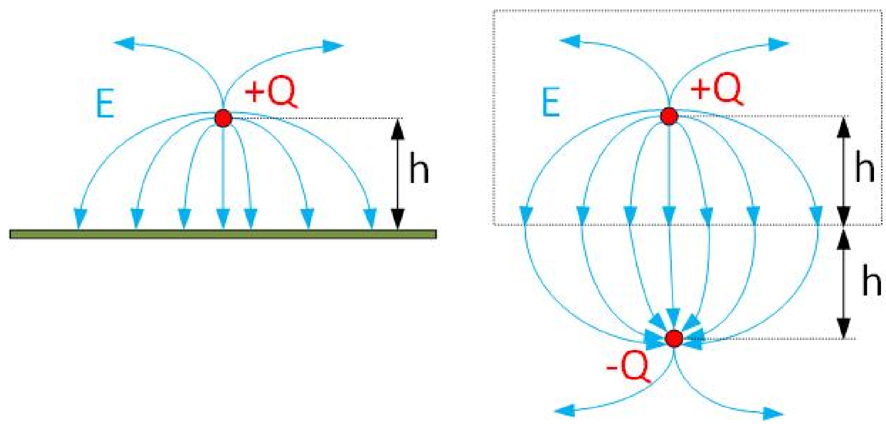


Figure 120: Electric Field due to a point charge above infinite conducting plane.

8 Magnetostatics

After completing this section, students should be able to do the following.

- Apply cross product (right-hand rule) to two vectors.
- Visualize magnetic field from a straight conductor
- Calculate the magnetic force
- Explain the operation of a simple DC-motor
- Apply Ampere's Law to derive a magnetic field from given currents.
- Calculate the magnetic energy of simple symmetrical structures carrying currents.
- Describe inductance through a flux of the magnetic field
- Conceptually explain what inductance is
- Calculate the inductance for various conductor configurations.

8.1 Charged particles in static electric and magnetic fields

Cross product

Cross product is defined as

$$\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}} \tag{445}$$

To find the direction of the vector $\vec{\mathbf{C}}$, use the right-hand rule.

(a) Place the first vector in the cross product $\vec{\mathbf{A}}$ in the palm of your right hand, and the second vector $\vec{\mathbf{B}}$ sticking out perpendicularly to your palm.

(b) Sweep the fingers from vector $\vec{\mathbf{A}}$ to $\vec{\mathbf{B}}$ and your thumb will be pointing to the direction of the vector $\vec{\mathbf{C}}$.

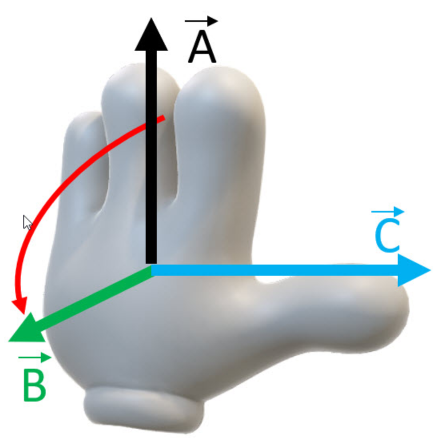


Figure 121: Using right-hand rule to find the direction of cross product.

Magnetic Force on a charged particle

Magnetic force on a charged particle $F = q\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{B}}$ will change the direction of the particle's path, perpendicularly to the direction of the magnetic field and the velocity vector. However, it will not change the speed of the particle. The magnetic field cannot accelerate a charged particle.

Example 37. *Use the applet below to change the charge, mass, and initial velocity of the particle to see how the path of the particle changes. Note that the initial velocity of the particle is in the horizontal direction.*

Geogebra link: *https://tube.geogebra.org/m/HXP8xUC3*

Example 38. *Now, use the applet below to see how the particle changes path if the initial velocity has a component in the direction of the magnetic field.*

Geogebra link: *https://tube.geogebra.org/m/xpRMzPgc*

Question 28 *An electron moves with a constant speed $\overrightarrow{\mathbf{v}}$ in vacuum in a presence of a magnetic field with flux density $\overrightarrow{\mathbf{B}}$. The magnetic field can change*

Multiple Choice:

(a) *both the direction of $\overrightarrow{\mathbf{v}}$, and $|\overrightarrow{\mathbf{v}}|$ magnitude.*

(b) *neither the direction of $\overrightarrow{\mathbf{v}}$, nor $|\overrightarrow{\mathbf{v}}|$ magnitude.*

(c) *the direction of $\overrightarrow{\mathbf{v}}$, but not $|\overrightarrow{\mathbf{v}}|$ magnitude.*

(d) *not enough information*

Lorenz Force

The force on a charged particle in an electric and magnetic field, a Lorentz force, is

$$\overrightarrow{\mathbf{F}} = q\overrightarrow{\mathbf{E}} + q\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{B}} \tag{446}$$

q is the charge of the particle, $\overrightarrow{\mathbf{E}}$ is the external electric field, v is the particle's velocity, and $\overrightarrow{\mathbf{B}}$ is the magnetic field.

Here is an interesting applet that lets you see the charged particle's path by setting the electric and magnetic fields' strength, charge and mass of the particle, and the speed direction and magnitude.

Example 39. *Now, use the applet below to explore the path of the particle changes in the presence of both electric and magnetic fields. Note that this applet is a little slow, and the browser will likely ask you what to do with the page. Select "wait" for the page to load.*

Geogebra link: `https://tube.geogebra.org/m/J9nrrpQH`

8.2 Force on Conductors

Force on a conductor due to external magnetic field

Force $\vec{\mathbf{F}}$ on a conductor of length $\vec{\mathbf{l}}$ due to an external magnetic field $\vec{\mathbf{B}}$ is

$$\vec{\mathbf{F}} = I\,\vec{\mathbf{l}} \times \vec{\mathbf{B}} \tag{447}$$

Vector $\vec{\mathbf{l}}$ is in the direction of the current. To find the force's direction, we apply the right-hand rule shown in Figure 122. We place the current vector in the palm of our right hand, then orient the palm so that the fingers can sweep towards the magnetic field vector.

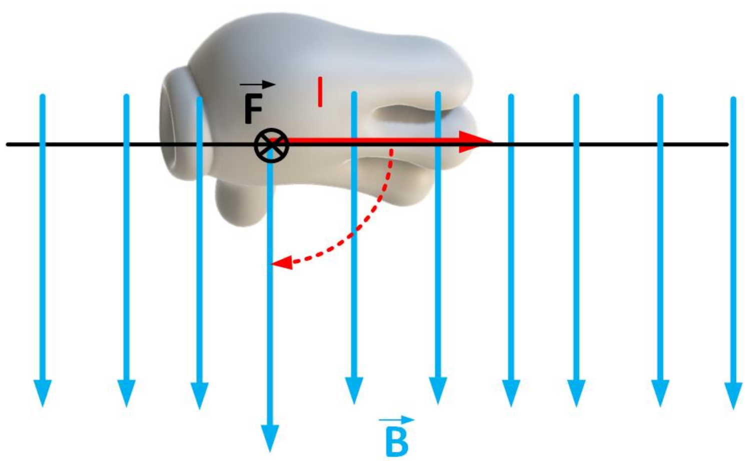


Figure 122: Applying right-hand rule to a current-carrying wire in external magnetic field.

Direction of a magnetic field of a conductor

The direction of a magnetic field of a conductor can be determined using the right-hand rule. If you point the thumb of your right hand in the direction of the current in a straight conductor, your fingers will curl in the direction of the magnetic field.

Observe how the magnetic field changes around a straight conductor. Change the strength and the direction of the current in the applet below. The distance between the magnetic field rings represents the strength of the magnetic field. When the rings are closer, the magnetic field is stronger, and vice versa. The vertical arrow represents the direction of the current. The small horizontal arrow shows the direction of the magnetic field.

Geogebra link: https://tube.geogebra.org/m/Xr65rR3b

Force between two conductors

When a conductor of length l_1, carrying current I_1, is in the vicinity of another conductor l_2, carrying current I_2, the force acts between them.

$$\overrightarrow{\mathbf{F_1}} = I_1 \overrightarrow{\mathbf{l_1}} \times \overrightarrow{\mathbf{B_2}} \tag{448}$$

$$\overrightarrow{\mathbf{F_2}} = I_2 \overrightarrow{\mathbf{l_2}} \times \overrightarrow{\mathbf{B_1}} \tag{449}$$

Where F_1 is the force on conductor l_1 B_1 is the magnetic field due to current I_1 in conductor l_1, and B_2 is the magnetic field due to current I_2.

To find the direction of the force, you have to use the right-hand rule. Here is a simulation that shows the force when the currents are in the same direction. What is the direction of the force?

YouTube link: https://www.youtube.com/watch?v=fz385mDNj84

This video shows a simulation that shows the force when the currents are in the opposite direction. What is the direction of the force now?

YouTube link: https://www.youtube.com/watch?v=P49PHzqIS8s

Magnetic field of a loop of current

The right-hand rule can be used to determine the magnetic field's direction from a loop of current. Point the thumb in the direction of the current, and fingers will curl in the direction of the magnetic field. See the video below that shows a simulation of magnetic field lines around straight and circular conductors.

YouTube link: https://www.youtube.com/watch?v=EUG23qVZUgc

Bar magnet and a loop of current

Permanent magnets have similar magnetic fields to loops of current. The source of the magnetic field of a permanent bar magnet is at the north pole, and the sink is at the south pole. The current loop magnetic field is similar to a bar magnet. The magnetic field lines circulate around the loop. The force between two current loops can be visualized by looking at two permanent magnets. The north pole of a magnet points in the direction of the loop's magnetic field, as shown in Figure 124. We determine the force by looking at the repulsive or attractive force of two bar magnets. We can again see that the force between two loops with currents in the opposite directions is repulsive.

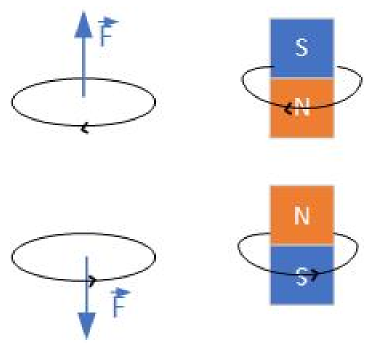


Figure 123: Repulsion between the permanent magnets and a loops of current.

Question 29 *Two coils carrying positive currents I_1 and I_2 are positioned as shown in Figure 124.*

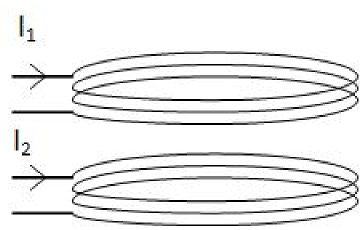


Figure 124: Two coils carrying currents I_1 and I_2.

Will the force between them be:

Multiple Choice:

(a) *attractive*

(b) *repulsive*

(c) *not enough information*

8.3 Force on Conductors

Force on a loop of current due to external magnetic field

Figure 125 shows a rectangular loop of wire carrying current I in the presence of an external magnetic field B. The loop is fixed along the horizontal pivot axis and can rotate only clockwise or counterclockwise. If we observe the loop from the left, we see the current flows toward us on top, indicated by a circle with a dot on the side view. The current flows away from us on the bottom, indicated by the circle with an x.

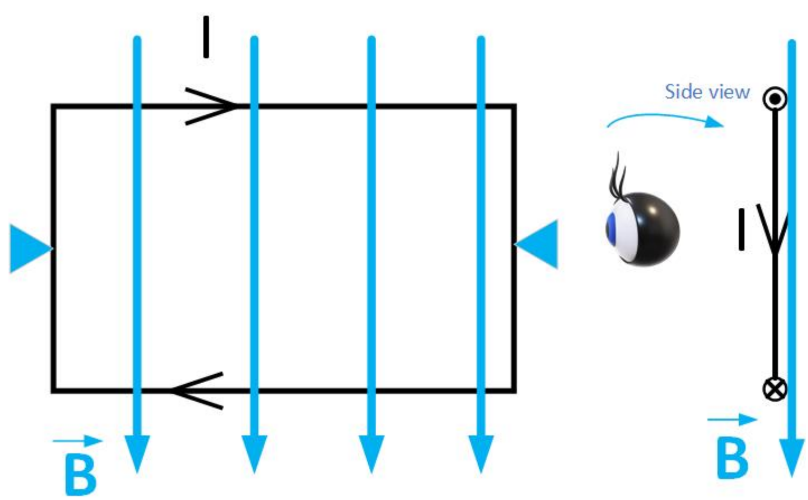


Figure 125: Current loop in external magnetic field.

Force $\vec{\mathbf{F}}$ on a conductor of length $\vec{\mathbf{l}}$ due to an external magnetic field $\vec{\mathbf{B}}$ is

$$\vec{\mathbf{F}} = I\,\vec{\mathbf{l}} \times \vec{\mathbf{B}} \tag{450}$$

Vector $\vec{\mathbf{l}}$ is in the direction of the current.

Which way will the loop turn? We use the right-hand rule to determine the direction of rotation, as shown in Figure 126. Imagine that the current is in the palm of your right hand. Orient the palm so that you can sweep your fingers toward the magnetic field vector. The thumb shows the direction of the force. Which way will the loop rotate? Think of pushing the top of the loop into the page in the force's direction and pulling the bottom of the loop towards you. The loop will rotate clockwise in the direction of the orange arrow.

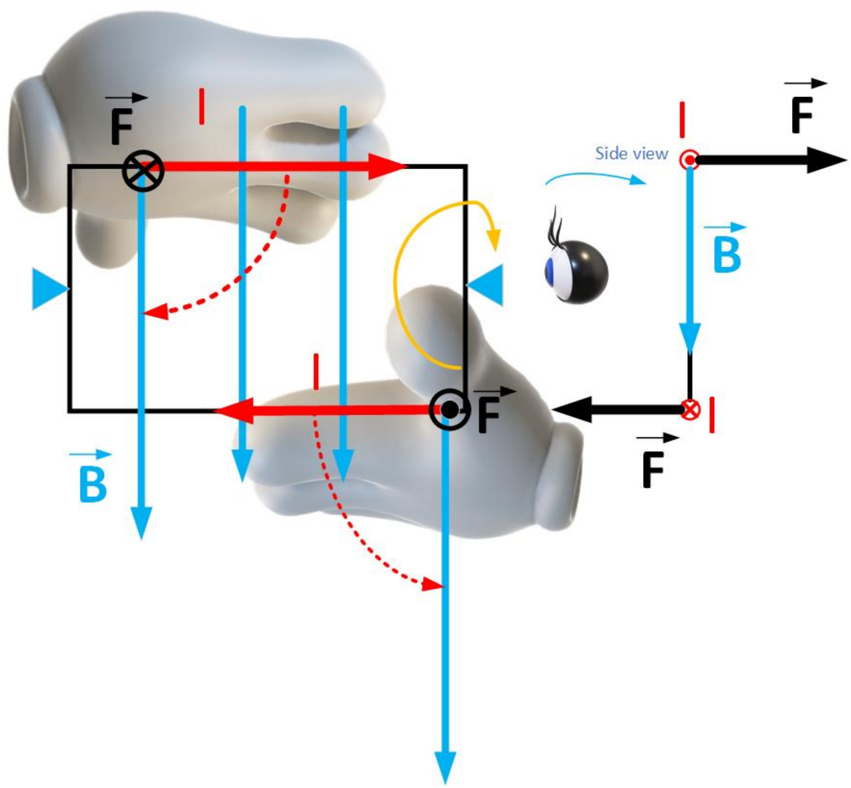


Figure 126: Torque on a loop carrying current in a magentic field.

When the loop moves from the original position, the magnetic force will continue to rotate the loop, as shown in Figure 127.

When the loop surface is perpendicular to the vector of the magnetic field, as shown in Figure 128, the magnetic force becomes zero. We may think that the loop will stop moving because the force becomes zero. However, because the loop has mass, the loop will continue rotating because of the inertia.

The magnetic forces will start working to move the loop back to this stable point, as shown in Figure 129. The loop will then move back and forth until it stops.

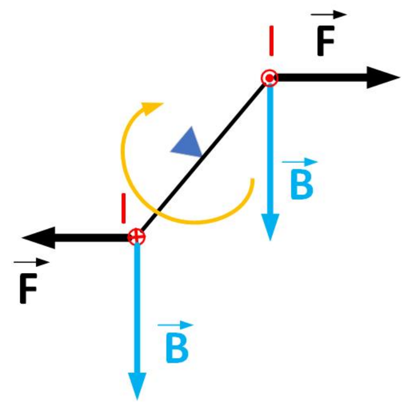


Figure 127: Torque on a loop carrying current in a magentic field.

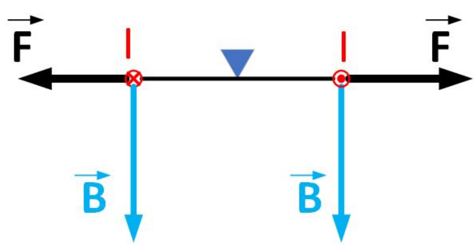


Figure 128: Torque on a loop carrying current in a magentic field.

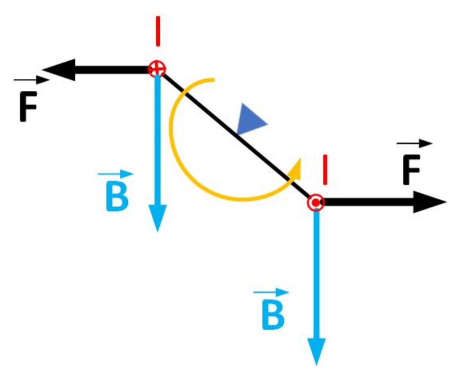


Figure 129: Torque on a loop carrying current in a magentic field.

If we want continuous rotation, we need to change the current direction at this stable point or the magnetic field. That brings us to DC motors.

DC Motor

DC motors convert electrical energy to mechanical energy. A battery supplies DC voltage to a current loop that produces a current in the loop. DC motors have a special split-ring piece that connects the loop to a voltage source called a commutator. When the magnetic field rotates the loop to the stable position, as we discussed in the previous section, the commutator's gap stops the current in the same direction for a moment, then connects the opposite pole of the battery, making the current go in the opposite direction. The orientation of the current and the magnetic field produce force in such a direction to continue the loop rotation.

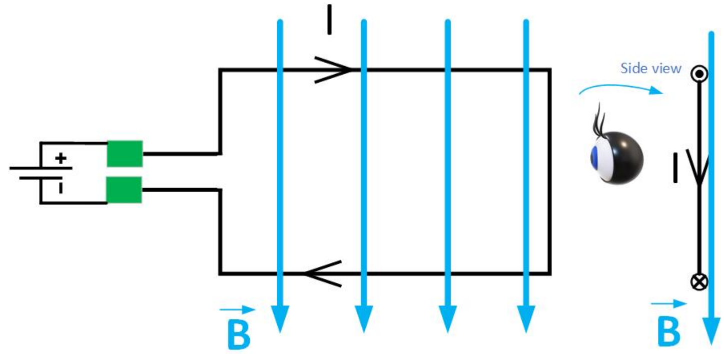


Figure 130: Simplified schematic of a DC motor.

To see the simulation of a DC motor, take a look at the following simulation

Geogebra link: `https://tube.geogebra.org/m/PN2YrxBb`

8.4 Biot-Savart's Law

Magnetic Field due to a Charge Distribution

We will first find the magnetic field due to a loop carrying current I, positioned in the X-Y plane, as shown in Figure 131. To solve this problem, we will first divide the loop into small pieces and label one of these pieces as dl. The magnetic field due to an infinitesimal current, can be found using Biot-Savart's law. Magnetic field is labeled in Figure 131 as dB. The infinitesimal current position is defined by a position vector $\overrightarrow{\mathbf{r_2}}$. The position of point P, where the field will be calculated, is defined with the position vector $\overrightarrow{\mathbf{r_1}}$. The distance between the current and the observation point is labeled as $\overrightarrow{\mathbf{r}}$. The vector $\overrightarrow{\mathbf{dB}}$ is defined in Equation 451.

$$\overrightarrow{\mathbf{dB_l}} = \frac{\mu I}{4\pi r^3}(\hat{dl} \times \hat{r}) \tag{451}$$

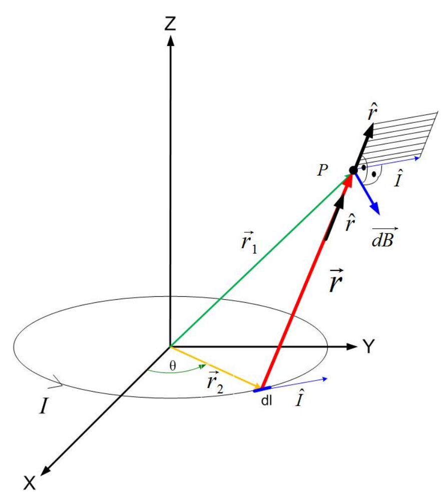


Figure 131: Loop of wire carrying current I. Magnetic field is shown due to a very small section (arc length) of the loop dl.

The total magnetic field at a point P is then equal to the sum of all the fields due to the elemental currents, as shown in Figure 132. The equation for the total field is given in 452.

$$\overrightarrow{\mathbf{B}} = \int_{all\ inf.\ currents} \overrightarrow{\mathbf{dB}} \tag{452}$$

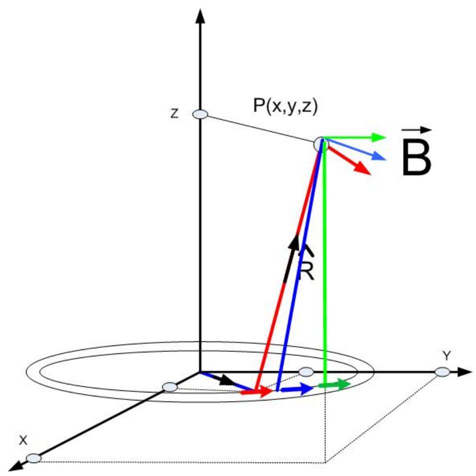


Figure 132: Loop of wire uniformly charged with line charge density ρ_l. Electric field is shown due to several very small sections (arc length) of the loop dl. Each section is modeled by a point charge dQ.

The problem now is to represent all the variables in the Equation 451 (I, dl, $\hat{r}$ and r) using appropriate coordinate system and given current distribution. As seen in Figure 131, $\overrightarrow{\mathbf{dl}}$ is an arc length in the direction of theta (blue arrow next to dl) $dl = a\, d\theta a_\theta$, where a is the radius of the loop. The vector $\overrightarrow{\mathbf{r_2}}$ is the position vector of the arc length dl, and the vector $\overrightarrow{\mathbf{r_1}}$ is the position vector of the point P where we want the find the magnetic field. Point P is an arbitrary point in the Cartesian coordinate system, P(x,y,z), therefore its vector is shown in Equation453. The vector $\overrightarrow{\mathbf{r}}$ is the distance vector between the elemental current (the source) and the point at which we are calculating the electric field.

$$\overrightarrow{\mathbf{r_1}} = x\overrightarrow{\mathbf{a_x}} + y\overrightarrow{\mathbf{a_y}} + z\overrightarrow{\mathbf{a_z}} \tag{453}$$

The vector $\overrightarrow{\mathbf{r_2}}$ can be written in Polar Coordinates as in Equation 454,where a is the radius of the loop. The equation 454 can be rewritten in Cartesian coordinate system as in Equation 456.

$$\overrightarrow{\mathbf{r_2}} = a\,\overrightarrow{\mathbf{a_r}} \tag{454}$$
$$\overrightarrow{\mathbf{a_r}} = \;cos\theta\,\overrightarrow{\mathbf{a_x}} + sin\theta\,\overrightarrow{\mathbf{a_y}} \tag{455}$$
$$\overrightarrow{\mathbf{r_2}} = a\,cos\theta\,\overrightarrow{\mathbf{a_x}} + a\,sin\theta\,\overrightarrow{\mathbf{a_y}} \tag{456}$$

The two vectors mark the beginning and the end of the distance vector $\overrightarrow{\mathbf{r}}$. The vector $\overrightarrow{\mathbf{r}}$ is the sum of vectors $-\overrightarrow{\mathbf{r_2}}$ and $\overrightarrow{\mathbf{r_1}}$.

$$\overrightarrow{\mathbf{r}} = \overrightarrow{\mathbf{r_1}} + (-\overrightarrow{\mathbf{r_2}}) \tag{457}$$

Therefore the vector's $\overrightarrow{\mathbf{r}}$ magnitude and the unit vector are shown in Equations 459-461.

$$\overrightarrow{\mathbf{r}} = (x - a\,cos\theta)\overrightarrow{\mathbf{a_x}} + (y - a\,sin\theta)\overrightarrow{\mathbf{a_y}} + z\overrightarrow{\mathbf{a_z}} \tag{458}$$

Vector $\overrightarrow{\mathbf{r}}$ has the magnitude of:

$$|\overrightarrow{\mathbf{r}}| = \sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2} \tag{459}$$

Unit vector in the direction of vector $\overrightarrow{\mathbf{r}}$ is:

$$\hat{r} = \frac{\overrightarrow{\mathbf{r}}}{|\overrightarrow{\mathbf{r}}|} \tag{460}$$
$$\hat{r} = \frac{\overrightarrow{\mathbf{r}}}{\sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}} \tag{461}$$

Cross product between the distance vector $\overrightarrow{\mathbf{r}}$ and the vector of the direction of current $\hat{I}$ is found in Equations 465-466.

$$\overrightarrow{\mathbf{dl}} = a\,d\theta\,\overrightarrow{\mathbf{a_\theta}} \tag{462}$$
$$a_\theta = -\,sin\theta\,a_x \;+\; cos\theta\overrightarrow{\mathbf{a_y}} \tag{463}$$
$$\overrightarrow{\mathbf{dl}} = -\,a\,sin\theta\,d\theta\,\overrightarrow{\mathbf{a_x}} \;+ a\,cos\theta\,d\theta\,\overrightarrow{\mathbf{a_y}} \tag{464}$$

$$\overrightarrow{\mathbf{dl}} \times \overrightarrow{\mathbf{r}} = \cdots$$
$$\Big(-\,a\,sin\theta\,d\theta\,\overrightarrow{\mathbf{a_x}} \;+ a\,cos\theta\,d\theta\,\overrightarrow{\mathbf{a_y}}\Big) \times \Big((x - a\,cos\theta)\overrightarrow{\mathbf{a_x}} + (y - a\,sin\theta)\overrightarrow{\mathbf{a_y}} + z\overrightarrow{\mathbf{a_z}}\Big) \tag{465}$$

$$\overrightarrow{\mathbf{dl}} \times \overrightarrow{\mathbf{r}} = \begin{vmatrix} \overrightarrow{\mathbf{a_x}} & \overrightarrow{\mathbf{a_y}} & \overrightarrow{\mathbf{a_z}} \\ -\,a\,sin\theta\,d\theta & a\,cos\theta\,d\theta & 0 \\ (x - a\,cos\theta) & (y - a\,sin\theta) & z \end{vmatrix} \tag{466}$$

$$dl \times \overrightarrow{\mathbf{r}} = \cdots$$
$$(za\,cos\theta\,d\theta)\,\overrightarrow{\mathbf{a_x}} + (a\,z\,sin\theta\,d\theta)\,\overrightarrow{\mathbf{a_y}} + (a^2\,-\,a\,(y\,sin\theta\,+\,x\,cos\theta))\,d\theta\,\overrightarrow{\mathbf{a_z}} \tag{467}$$

Replacing other variables in the Equations 453-461, we get the Equation 468 for the magnetic field $\overrightarrow{\mathbf{dB}}$ at a point P.

Components of the magnetic field are given in Equations 468-470.

$$\overrightarrow{\mathbf{dB}}_x = \frac{\mu I}{4\pi\sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}^3}(z\,a\,cos\theta\,d\theta)\overrightarrow{\mathbf{a_x}} \tag{468}$$

$$\overrightarrow{\mathbf{dB}}_y = \frac{\mu I}{4\pi\sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}^3}(a\,z\,sin\theta\,d\theta)\overrightarrow{\mathbf{a_y}} \tag{469}$$

$$\overrightarrow{\mathbf{dB}}_z = \frac{\mu I}{4\pi\sqrt{(x - a\,cos\theta)^2 + (y - a\,sin\theta)^2 + z^2}^3}\cdots$$
$$\cdots(a^2\,-\,a\,(y\,sin\theta\,+\,x\,cos\theta))\,d\theta\,\overrightarrow{\mathbf{a_z}} \tag{470}$$

Each field component can be integrated separately, as shown in Equations 471-473.

$$\overrightarrow{\mathbf{B_x}} = \int_0^{2\pi} \frac{\mu\,I\,z\,a\,cos\theta\,d\theta}{4\,\pi\,\sqrt{(x\,-\,a\,cos\theta)^2 + (y\,-\,a\,sin\theta)^2\,+\,z^2}^3}\,\overrightarrow{\mathbf{a_x}} \tag{471}$$

$$\overrightarrow{\mathbf{B_y}} = \int_0^{2\pi} \frac{\mu\,I\,a\,z\,sin\theta\,d\theta}{4\,\pi\,\sqrt{(x\,-\,a\,cos\theta)^2 + (y\,-\,a\,sin\theta)^2\,+\,z^2}^3}\,\overrightarrow{\mathbf{a_y}} \tag{472}$$

$$\overrightarrow{\mathbf{B_z}} = \int_0^{2\pi} \frac{\mu\,I(a^2\,-\,a\,(y\,sin\theta\,+\,x\,cos\theta))\,d\theta}{4\,\pi\,\sqrt{(x\,-\,a\,cos\theta)^2 + (y\,-\,a\,sin\theta)^2\,+\,z^2}^3}\overrightarrow{\mathbf{a_z}} \tag{473}$$

The magnetic field above is shown in Figure 133

Visualizing Scalar Fields in Matlab

To visualize scalar fields in Matlab, we can use the following functions: slice, contourslice, patch, isonormals, camlight, and lightning. Please note that a more detailed explanation about these functions can be found in Matlab help.

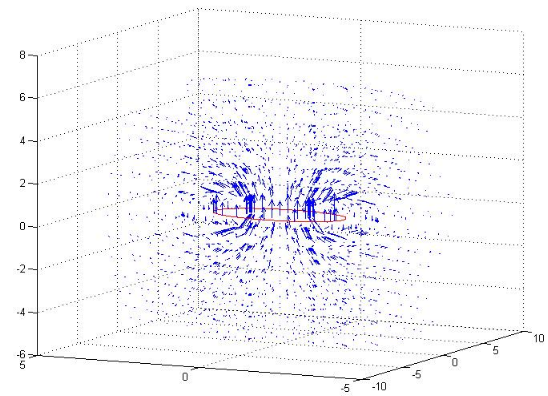


Figure 133: Magnetic Field of a Loop of Current

slice

Slice is a command that shows the magnitude of a scalar field on a plane that slices the volume where the potential field is visualized. The format of this command is as shown below.

```
slice(x,y,z,v,xslice,yslice,zslice)
```

Where X, Y, and Z are coordinates of points where the scalar function is calculated, V is v the scalar function at those points, and the last three vectors xslice, yslice, and zslice are showing where will the volume will be sliced.

An example of a slice command is given below. There is an additional command colormap that colors the volume with a specific palette in the example below. To see more about different color maps, see Matlab help. xslice has three points at which the x-axis will be slice. They are -1.2, .8, 2. The volume will be sliced with a plane perpendicular to the x-axis, and it crosses the x-axis at points -1.2, .8, and 2.

```
clc
clear all
[x,y,z] = meshgrid(-2:.2:2,-2:.25:2,-2:.16:2);
v = x.*exp(-x.^2-y.^2-z.^2);
xslice = [-1.2,.8,2]; yslice = 1; zslice = [-2,0];
slice(x,y,z,v,xslice,yslice,zslice)
colormap hsv
```

contourslice

Contourslice command will display equipotential lines on a plane being the volume where the potential field is visualized. An example of contourslice function is shown below.

```
[x,y,z] = meshgrid(-2:.2:2,-2:.25:2,-2:.16:2);
v = x.*exp(-x.^2-y.^2-z.^2); % Create volume data
[xi,yi,zi] = sphere; % Plane to contour
contourslice(x,y,z,v,xi,yi,zi)
view(3)
```

patch

Patch command creates a patch of color.

isonormals

Command isonormals creates equipotential surfaces.

camlight

```
camlight('headlight') creates a light at the camera
 position.camlight('right') creates a light
right and up from camera.

camlight('left') creates a light
left and up from camera.camlight with no arguments is the
same as camlight('right').

camlight(az,el) creates a light
at the specified azimuth (az) and elevation (el)
with respect to the camera position. The camera target
is the center of rotation
and az and el are in degrees.
```

lighting

```
lighting flat selects flat lighting.
```

`Lighting gouraud` selects gouraud lighting.

`Lighting phong` selects phong lighting.

`Lighting none` turns off lighting.

8.5 Ampere's Law

Static Magnetic Field

The sources of static magnetic fields are direct currents (DC) and magnets. The direction of the magnetic field from a DC current is determined through the right-hand rule. The magnetic field's direction from the magnet is outward-oriented from the north pole and inward into the south pole.

It is interesting to see that the magnetic field from a current loop looks very much like a magnetic field of a magnet placed in the center of the loop, perpendicularly to it.

Geographical earth's south pole is earth's magnetic north pole.

Ampere's Law

Ampere's law for static magnetic fields states that the magnetic field's integral H around closed contour is equal to the current enclosed by the contour.

$$\oint_c \overrightarrow{\mathbf{H}} \cdot \overrightarrow{\mathbf{dl}} = I \tag{474}$$

If the contour encloses only part of the current, whose current density is J, then to find the total current, we have to integrate through the cross-section of the conductor carrying current.

$$\oint_c \overrightarrow{\mathbf{H}} \cdot \overrightarrow{\mathbf{dl}} = \int_S \overrightarrow{\mathbf{J}} \cdot \overrightarrow{\mathbf{dS}} \tag{475}$$

The vector $\overrightarrow{\mathbf{dS}}$ is the surface area enclosed by the contour. The vector of the surface area is determined from the direction of the contour with the right-hand rule. If we orient our fingers in the direction of the contour, the thumb will point in the direction of the surface vector.

We can also re-write the above equations in terms of magnetic flux density $B = \mu H$.

$$\oint_c \overrightarrow{\mathbf{B}} \cdot \overrightarrow{\mathbf{dl}} = \mu I \tag{476}$$

Application of Ampere's law

Example 40. *The magnetic field of an infinite wire with a circular cross-section*

Find the magnetic field inside and around an infinitely long wire carrying current I. The current is uniform through the wire, with a constant current density of J. The radius of the wire is a. The wire is made from a material with relative magnetic permeability $\mu_r = 1$.

Explanation. *To find the magnetic field H(r) around the wire when* $r > a$, *we will start from Ampere's law.*

$$\oint_c \overrightarrow{\mathbf{H}} \cdot \overrightarrow{\mathbf{dl}} = I \tag{477}$$

To apply this equation to an infinite wire, we first have to know how the magnetic field looks. Using the right-hand rule, we determine that the magnetic field lines curl around the wire $\overrightarrow{\mathbf{H}} = H\overrightarrow{\mathbf{\Theta}}$. *The magnetic field is constant on a circle centered at the wire. The magnetic field is constant because all points on this circle are the same distance away from the source (current). We then choose a contour c that is a circle of radius r because this will simplify the integral solution.*

The current in the above equation is the current enclosed by the contour c, or in other words, the total current that pokes the surface of the contour. We see that in this case, this is the total current I flowing through the wire.

We pick the direction of the countour in the direction of the magnetic field, as shown in Figure 134. Since $\overrightarrow{\mathbf{dl}} = dl\overrightarrow{\mathbf{\Theta}}$, *the dot product* $\overrightarrow{\mathbf{H}} \cdot \overrightarrow{\mathbf{dl}} = Hdl\cos(\overrightarrow{\mathbf{H}}, \overrightarrow{\mathbf{dl}}) = Hdl$, *because the angle between the magnetic field and the contour vector is zero.*

Ampere's law then becomes:

$$\oint_c Hdl = I \tag{478}$$

Since the magnetic field is presumed constant on this contour, it can be taken in front of the integral.

$$H \oint_c dl = I \tag{479}$$

The integral above is equal to the circumference of the contour $l = 2\pi r$.

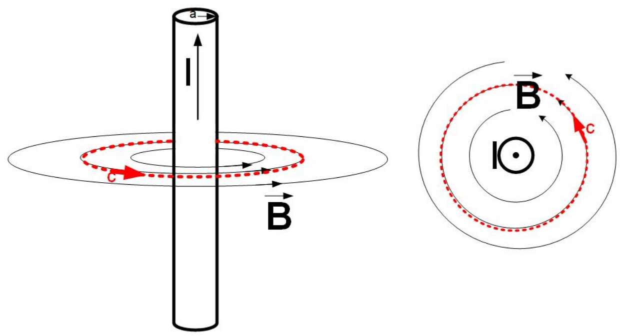


Figure 134: Application of Ampere's law outside an infinite straight conductor

$$Hl = I \tag{480}$$

$$H = \frac{I}{l} \tag{481}$$

$$H(r) = \frac{I}{2\pi r} \tag{482}$$

Equation 482 describes the magnetic field's behavior anywhere around the wire, for $r > a$.

We can now apply Ampere's law to a contour inside the wire, as shown in Figure 135. We then choose a contour c that is a circle of radius r, but this time, $r < a$.

$$\oint_c \overrightarrow{\mathbf{H}} \cdot \overrightarrow{\mathbf{dl}} = I_{part} \tag{483}$$

The current in the above equation is the current enclosed by the contour c. In this case, only the portion of the total current I is flowing through the contour.

We can find the current flowing through the contour by using a proportion since the current density is constant through the wire cross-section.

$$\frac{I_{part}}{r^2\pi} = \frac{I}{a^2\pi} \tag{484}$$

$$I_{part} = I\frac{r^2}{a^2} \tag{485}$$

Using the same reasoning as in the previous part of the problem, we find that the left side of Ampere's law is $H2\pi r$:

$$H2\pi r = I\frac{r^2}{a^2} \tag{486}$$

$$H(r) = I\frac{r}{2\pi a^2} \tag{487}$$

We see that when $r < a$, the magnetic field increases linearly from zero at the axis of the wire to the maximum value of $H(a) = I\frac{1}{2\pi a}$

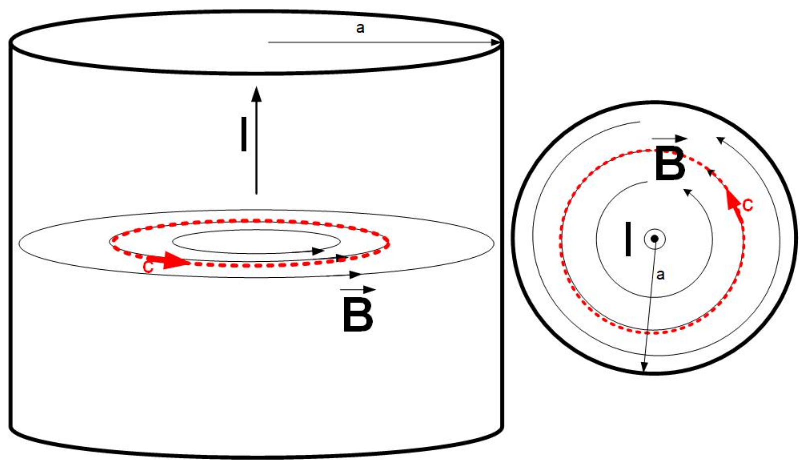


Figure 135: Application of Ampere's law inside an infinite straight conductor

Example 41. *The magnetic field of a solenoid*

Find the magnetic field of a solenoid with N turns of uniformly, densely wound wire around an air core, as shown in Figure136. The length of the solenoid is L, and the current flowing through the wire is I.

Explanation. *To find the magnetic field inside the solenoid, we apply Ampere's law to a contour shown in Figure 136. Using a similar discussion as before, we find that the line integral of the magnetic field along the square loop l is*

$$Hl = N'I \tag{488}$$

Where N' represents the number of windings enclosed by the contour. Since the solenoid is uniformly wound, $N'/l = N/L$, $N' = Nl/L$. l is the length of the contour, L is the solenoid's length, and N is the total number of windings.

$$Hl = \frac{NlI}{L} \tag{489}$$

$$H = \frac{NI}{L} \tag{490}$$

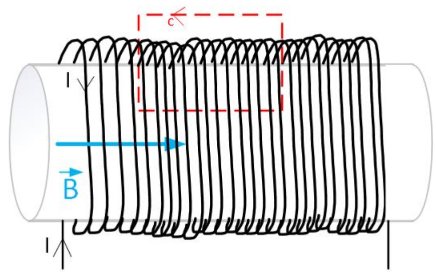


Figure 136: Application of Ampere's law to a solenoid.

Example 42. *The magnetic field of a toroid*

Find the magnetic field inside and outside of a toroid with inner radius a and outer radius b. A toroid is a donut-shaped form, as shown in Figure 137. The wire wound on the toroid carries a current I, and there are N dense, uniform windings on the air core.

Explanation. *If we chose a contour outside of the toroid, we see that N windings of current penetrate the contour's surface twice. Once down into the surface, and the other time going up out of the surface. Therefore, the right side of Ampere's law has 2N currents that cancel each other, and the magnetic field is zero.*

If we choose a contour inside the toroid's hollow part, there is no current enclosed by the contour. Therefore, the current and the magnetic field are zero.

If we choose a contour inside the toroid, we see that the current penetrates the surface N times. Therefore, Ampere's law states:

$$\oint_c \overrightarrow{\mathbf{H}} \cdot \overrightarrow{\mathbf{dl}} = NI \tag{491}$$

We again evaluate the left side of the above equation along a circular contour in the magnetic field direction.

$$H 2\pi r = NI \tag{492}$$

$$H(r) = \frac{NI}{2\pi r} \tag{493}$$

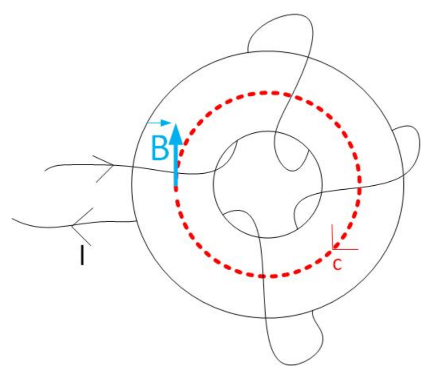


Figure 137: Application of Ampere's law to a toroid.

8.6 Inductance

Magnetic flux, review of electric flux

A conceptual definition of magnetic flux is the number of magnetic field lines penetrating a surface.

$$\Phi_B = N_B \tag{494}$$

Mathematically, magnetic flux is defined through magnetic flux density vector B as

$$\Phi_B = \int_S \overrightarrow{\mathbf{B}} \cdot \overrightarrow{\mathbf{dS}} \tag{495}$$

If the angle between the surface dS and B vector is the same, and the B vector is constant on the surface, we have that $\Phi_B = BS$, or $B = \dfrac{\Phi_B}{S}$. Vector $\overrightarrow{\mathbf{B}}$ is called magnetic flux density, just like in electrostatics, vector $D = \dfrac{\Phi_E}{S}$ was called electric flux density vector.

Magnetic flux density vector $\overrightarrow{\mathbf{B}}$ and magnetic field vector $\overrightarrow{\mathbf{H}}$ are related through the magnetic permeability of the material μ.

$$\overrightarrow{\mathbf{B}} = \mu \overrightarrow{\mathbf{H}} \tag{496}$$

Similarly, in electrostatics, the electric flux density vector $\overrightarrow{\mathbf{D}}$, and electric field vector $\overrightarrow{\mathbf{E}}$ are related through the electric permittivity of the material ε.

$$\overrightarrow{\mathbf{D}} = \varepsilon \overrightarrow{\mathbf{E}} \tag{497}$$

Definition of inductance

A simplified, conceptual definition of inductance is the number of magnetic field lines around a conductor, divided by the current in a wire.

$$L = \frac{N_B}{I} \tag{498}$$

For example, if we have five magnetic field lines around the wire that carries the current 1A, then the inductance is L=5/1=5 H. If we have another wire that makes ten magnetic field lines around it for the same current flowing through it of 1A, then the inductance of this wire is L=10 H. This type of inductance is called self-inductance. We will talk about mutual inductance in the section below.

The scientific definition of inductance is the ratio of magnetic flux to the current that produced it.

$$L = \frac{\Phi}{I} \tag{499}$$

$$L = \frac{\int_S \overrightarrow{\mathbf{B}} \cdot \overrightarrow{\mathbf{dS}}}{I} \tag{500}$$

Magnetostatic energy

Inductors store magnetic energy. Capacitors store electric energy. The magnetic energy stored in an inductor is

$$W_m = \frac{1}{2} L I^2 \tag{501}$$

The total magnetic energy stored in a volume is

$$W_m = \frac{1}{2} \int_v \overrightarrow{\mathbf{B}} \cdot \overrightarrow{\mathbf{H}}\, dv \tag{502}$$

B is the magnetic flux density, H is the magnetic field, and v is the volume in which the energy is stored.

By equating the above two equations, we can find the inductance of an inductor as

$$L = \frac{1}{I^2} \int_v \overrightarrow{\mathbf{B}} \cdot \overrightarrow{\mathbf{H}}\, dv \tag{503}$$

In linear, homegenous materials $\overrightarrow{\mathbf{B}} = \mu \overrightarrow{\mathbf{H}}$, the equation simplifies to

$$L = \frac{\mu}{I^2} \int_v H^2 dv \tag{504}$$

In the electrostatics section, we found that the capacitance is found as

$$C = \frac{\varepsilon}{V^2} \int_v E^2 dv \tag{505}$$

V is the potential difference, v is the volume where electric energy is stored, and E is the electric field.

Example 43. Deriving inductance for a coaxial cable

Derive the inductance of a length h of a coaxial cable carrying current I. The inner radius of the coaxial cable is a, and the outer radius is b.

Types of inductance

(a) Internal self-inductance. The definition is the number of magnetic field lines inside a wire, divided by the current that flows through the same wire.

(b) External self-inductance. The definition is the number of magnetic field lines outside of a wire, divided by the current that flows through the same wire.

(c) Mutual inductance. The definition of this inductance is the number of magnetic field lines around one wire produced by the current flowing through another wire.

(d) Partial inductance. This inductance is defined for a portion of a current or a wire because we do not know (or we are not interested in) how the current returns back to the source.

Self Inductance

A simple conceptual definition of self-inductance is the number of magnetic field lines around the wire, divided by the current in the same wire (that produced the magnetic field around the wire).

$$L_s = \frac{N_s}{I_s} \tag{509}$$

For example, if we have five magnetic field lines around the wire that carries the current 1A, then the inductance is L=5/1=5 H. If we have another wire that makes ten magnetic field lines around it for the same current flowing through it of 1A, then the inductance of this wire is L=10 H. This type of inductance is called (external) self-inductance.

Mutual Inductance

Mutual inductance is defined for two or more wires. It is defined as the number of magnetic field lines around one conductor, divided by the current produced through another conductor.

$$L_m = \frac{N}{I_2} \tag{510}$$

For example, in Figure 138, we have two currents, $I_1 = 1\,A$ and $I_2 = 1\,A$ flowing in the same direction. Their magnetic fields, therefore, rotate in the same direction. Current I_1 produces four magnetic field lines, and current I_2 produces three magnetic field lines. Two of the magnetic field lines from conductor 1 encircle wire 2 as well. The total number of magnetic field lines around the second conductor is therefore 5. Two magnetic field lines from conductor 1, and three from conductor 2 itself is 2+3=5. Therefore mutual inductance is $L_m = 5/1 = 5\,H$. When the magnetic field from two wires is in the same direction, the fluxes add.

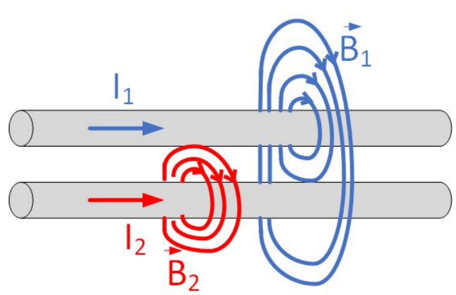


Figure 138: Mutual Inductance: Increasing the magnetic field and therefore current in one wire due to another wire in vicinity.

In another example, in Figure 139, we have two currents, $I_1 = 1\,A$ and $I_2 = 1\,A$ flowing in the opposite directions. Current I_1 again produces four magnetic field lines, and current I_2 produces three magnetic field lines. Two of the magnetic field lines from conductor 1 encircle wire 2 as well. However, this time, the magnetic field lines flow in opposite directions. Therefore, the total number of magnetic field lines around conductor 2 is 3-2=1. Therefore mutual inductance is $L_m = 1/1 = 1\,H$. When the magnetic field from two wires is in the opposite direction, fluxes subtract.

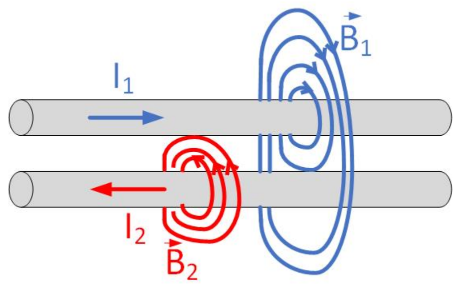


Figure 139: Mutual Inductance: Increasing the magnetic field and therefore current in one wire due to another wire in vicinity.

9 Changing electromagnetic fields

After completing this section, students should be able to do the following.

- Explain original Faraday's experiment.
- Name two types of electromotive Force (EMF)
- Apply Faraday's Law to simple structures.
- State Lenz's law.
- Aply Lenz's law to simple circuits (find the direction of current).
- Explain how transformer works.
- Describe transformer EMF.
- Observe a flying ring and falling magnet experiments and explain observations using Faraday's law, Lenz's law and electromagnetic forces.

9.1 Faraday's Law

Changing electromagnetic fields

Static electric fields are independent of static magnetic fields. In dynamic electromagnetic fields changing electric field induces changing electric field and vice versa.

Faraday's law

Faraday's law of electromagnetic induction states that the electromotive force (voltage) induced in a loop is equal to the change of magnetic flux through the loop in time. The flux can be changed by physically moving the loop or changing the current through the loop. The minus in the equation below has to do with the polarity of induced voltage (and the direction of induced current) that we will discuss in the next section on Lenz's law.

$$e = -\frac{d\Phi}{dt} \tag{511}$$

The electromotive force can be found by taking a line integral of the induced electric field (by the flux) around the closed loop.

$$e = \oint_C \overrightarrow{\mathbf{E}}_{ind} \cdot \overrightarrow{\mathbf{dl}} \tag{512}$$

Equating the above two equations, we get Faraday's law of induction.

$$e = \oint_C \overrightarrow{\mathbf{E}}_{ind} \cdot \overrightarrow{\mathbf{dl}} = -\frac{d\Phi}{dt} \tag{513}$$

Example 44. Faraday's experiment

Michael Faraday observed that the current in a loop is established only if the flux through the loop is changing. One way to change the flux is by using a permanent magnet and move it through coils of current. Observe how the lightbulb in the simulation below turns on and off as the magnet position is changed. You can select to move the magnet yourself or to have the magnet moved sinusoidally by a spring. Select the radio button to see the magnetic field from the magnet.

Geogebra link: https://tube.geogebra.org/m/JPFxyhtA

Example 45. *Demonstration of eddy-current induced levitation of a coil over a metallic plate by MIT Prof. Emeritus Walter Lewin.*

YouTube link: https://www.youtube.com/watch?v=XsinTqi66n8

Example 46. *Demonstration of "spark plug" by MIT Prof. Emeritus Walter Lewin.*

YouTube link: https://www.youtube.com/watch?v=h0k79HhJr7Q

9.2 Lenz's Law

Transformer emf polarity and induced current direction are somewhat challenging to find directly from Faraday's law. However, Lenz's law gives us a simple way to find the emf polarity and induced current direction.

Lenz's law states that the induced current will be in such a direction that its induced magnetic field opposes the external magnetic field's change. We can say that nature dislikes the change of flux.

Example 47. Example of application of Lenz's law

An infinite line carries a current I in the positive z-direction. The current increases linearly with time $I{=}t\,A$, where t is time. Determine the direction of the induced current through and the voltage on the resistor R in loop shown in Figure 140

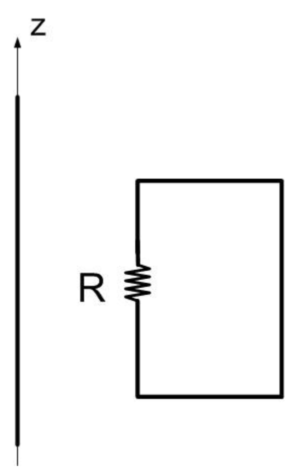


Figure 140: Example problem for Lenz's law.

Explanation. *The magnetic field due to an infinite wire carrying current I is $\overrightarrow{\mathbf{H}} = \dfrac{I}{2\pi r}$ as shown in Figure 141. The flux through the loop from the infinite wire's magnetic field is then directed into the paper.*

Since the current is increasing, $I{=}t\,A$, the magnetic field is increasing as well, $\overrightarrow{\mathbf{H}}(t) = \dfrac{t}{2\pi r}$. Increasing magnetic field will induce a current in the loop so that its direction opposes the increase in the wire's magnetic field flux. The

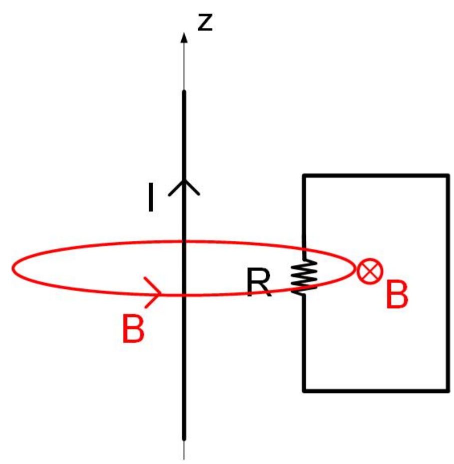


Figure 141: Magnetic field direction of an infinite wire carrying current I.

direction of the countering flux from the loop will then be out of the page, and the current direction will be counter-clockwise (CCW). CCW direction of the current induces positive voltage on the top of the resistor and negative on the bottom, as shown in Figure 142.

Example 48. *In the previous problem, if the current in the infinite wire decreases with time, the direction of the induced current, induced magnetic flux density, and induced voltage are shown in Figure 143.*

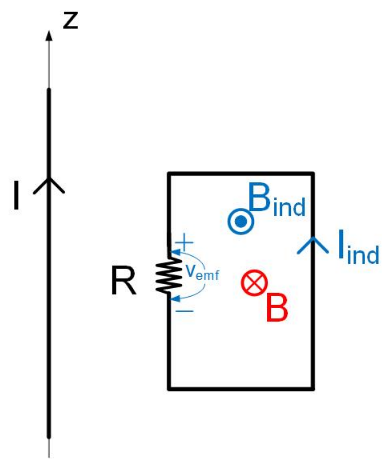


Figure 142: Direction of current, induced magnetic field and voltage for increasing current in the infinite conductor.

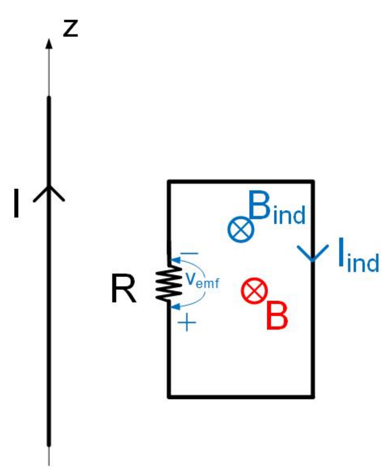


Figure 143: The direction of the induced magnetic field and current from the decreasing magnetic field from the infinite wire.

9.3 Transformers

Ideal Transformer

Transformers consist of two coils of wire on a ferromagnetic core, as shown in Figure 144. The left coil with an AC generator is called a primary, and the right coil with the resistor is called a secondary. There is no voltage source in the secondary. Transformers are used in many applications, some of which are to electrically decouple circuits, match impedances, and increase or decrease the primary voltage. In an ideal transformer, it is assumed that the total flux produced by the primary will be circulating through the core, and therefore the secondary as well. The induced voltage on the primary and secondary are

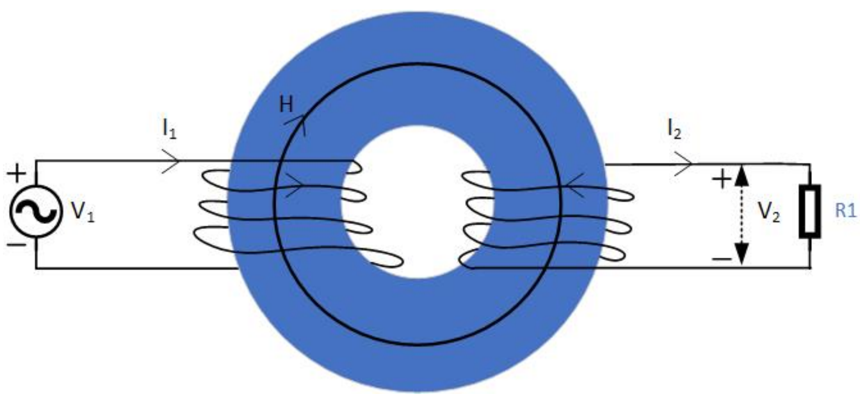


Figure 144: Transformer.

$$e_1 = -N_1 \frac{d\Phi}{dt} \tag{514}$$

$$e_2 = -N_2 \frac{d\Phi}{dt} \tag{515}$$

If we divide these equations, the ratio of the emf and voltages is

$$\frac{V_1}{V_2} = \frac{N_1}{N_2} \tag{516}$$

Since we are assuming an ideal transformer with no loss, the input power generated by the primary will be equal to the output power delivered to the secondary

$P_1 = V_1 I_1 = P_2 = V_2 I_2$, we can replace the ratio of voltages in the equation above with currents to get

$$\frac{I_1}{I_2} = \frac{N_2}{N_1} \tag{517}$$

The more turns we have on the secondary, the higher the voltage be. Watch the following video to see a demonstration of a transformer.

Example 49. *Demonstration of a transformer by MIT Prof. Emeritus Walter Lewin.*

YouTube link: `https://www.youtube.com/watch?v=sRcev0_Ilv4`

9.4 Magnetic Coupling

Applications of Faraday's law to AC circuits

We will now look at an example of two coils carrying alternating current, as shown in Figure 145. The coils are connected to two AC signal generators, and have internal resistance R.

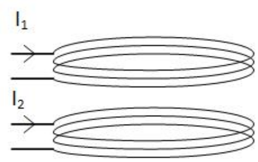


Figure 145:

Coils with no coupling

We will first look at the case when the coils are not coupled. We will see that these are two simple RL circuits. If the coils are not coupled, the mutual flux is zero, and the fluxes through the coils are

$$\Phi_{1s} = L_1 I_1 \tag{518}$$
$$\Phi_{2s} = L_2 I_2 \tag{519}$$

The above equation states that the currents in the coils are in phase with flux Φ. This equation is similar to two other equations that define resistance and capacitance. V=R I means that the current is in phase with the voltage on a resistor. Q=C V, the charge is in phase with the voltage on a capacitor.

Since mutual flux is zero, according to Faraday's law, the induced voltage in each coil is

$$e_1 = -\frac{d\Phi_{1s}}{dt} = -L_1 \frac{dI_1}{dt} \tag{520}$$
$$e_2 = -\frac{d\Phi_{2s}}{dt} = -L_2 \frac{dI_2}{dt} \tag{521}$$

The induced voltage and the voltage in the rest of the circuit are related through Faraday's law:

$$\oint_c \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} = -\frac{d\Phi}{dt} \tag{522}$$

Induced current flows as a current would inside the generator, from negative to positive voltage terminal, as shown in Figure 146.

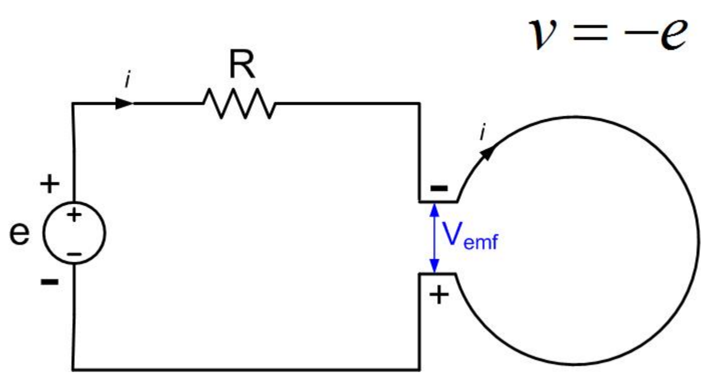


Figure 146: Voltage at the end of the coil and the induced emf.

Therefore, the voltage at the output of the coils is $v_1 = -e_1$ and $v_2 = -e_2$

$$v_1 = L_1 \frac{dI_1}{dt} \tag{523}$$

$$v_2 = L_2 \frac{dI_2}{dt} \tag{524}$$

The above equation states that the voltage at the coils' output leads the current and flux, by 90^0, if we assume that the resistance in Figure 146 is small.

If the resistance is not small, then the voltage will be leading with angle $\arctan(\frac{\omega L}{R})$. We can derive this equation from Faraday's law.

$$\oint_c \overrightarrow{\mathbf{E}} \cdot \overrightarrow{\mathbf{dl}} = -\frac{d\Phi}{dt} \tag{525}$$

If we integrate the electric field along the closed loop and assume that the resistance of the coil and other losses are included in R

$$-e + RI = -L\frac{dI}{dt} \tag{526}$$

We see that this is a time-domain equation for an RL circuit. If we use the definition of phasors, we get

$$\tilde{E} = (R + j\omega L)\tilde{I} \tag{527}$$

Therefore the voltage will be leading current by an angle $\arctan(\frac{\omega L}{R})$.

Coils with coupling

When two coils are close together, their magnetic fields couple and some of the flux from coil 1 will pass through the second coil. This additional flux produces induced voltage in the second coil, as we discussed before. The amount of interaction between the coils is called mutual flux. Associated with mutual flux is mutual inductance $M = L_{12} = \frac{\Phi_1}{I_2} = L_{21} = \frac{\Phi_2}{I_1}$. Mutual inductance can be positive or negative, as we have seen before because the fluxes can add or subtract. The mutual flux through coil 1 can be in phase or out of phase with the current that produced it.

If the coils are coupled, the mutual flux is not zero, and the fluxes through the coils are

$$\Phi_1 = \Phi_{1s} + \Phi_{21} = L_{1s}I_1 \pm L_{12}I_2 \tag{528}$$
$$\Phi_2 = \Phi_{2s} + \Phi_{12} = L_{2s}I_2 \pm L_{21}I_1 \tag{529}$$

The induced voltages in each coil are

$$e_1 = -L_1\frac{dI_1}{dt} \mp L_{12}\frac{dI_2}{dt} \tag{530}$$
$$e_2 = -L_2\frac{dI_2}{dt} \mp L_{21}\frac{dI_1}{dt} \tag{531}$$

The voltages at the end of the coils are therefore

$$v_1 = L_1\frac{dI_1}{dt} \pm L_{12}\frac{dI_2}{dt} \tag{532}$$
$$v_2 = L_2\frac{dI_2}{dt} \pm L_{21}\frac{dI_1}{dt} \tag{533}$$

It is difficult in circuit notation to describe how fluxes are adding or subtracting. Therefore a "dot" notation is developed to explain flux interaction on a circuit diagram. For example, in Figure 147, two coupled coils connected in series are shown. If the current flows from left (or right), it will encounter a dot (or no dot) at each inductor. The fluxes will therefore add, and the equations that we write are

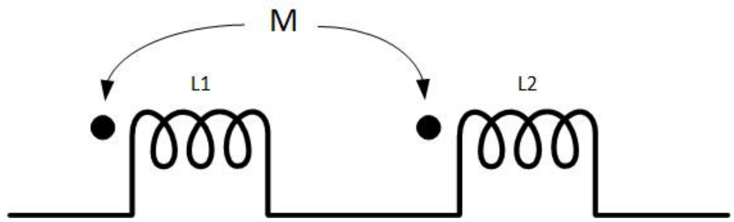


Figure 147: Dot notation.

$$v_1 = L_1 \frac{dI}{dt} + L_{12} \frac{dI}{dt} \tag{534}$$

$$v_2 = L_2 \frac{dI}{dt} + L_{21} \frac{dI}{dt} \tag{535}$$

The above equations show that the self and mutual inductance add.

$$v_1 = (L_1 + L_{12}) \frac{dI}{dt} \tag{536}$$

$$v_2 = (L_2 + L_{21}) \frac{dI}{dt} \tag{537}$$

Figure 148 shows two coupled coils connected in series, but this time the dots are on the opposite sides of the coils. If the current flows from the left, it will encounter a dot at the first inductor but no dot on the second inductor. The fluxes will therefore subtract, the mutual inductance is negative, and the equations that we write are

$$v_1 = L_1 \frac{dI}{dt} - L_{12} \frac{dI}{dt} \tag{538}$$

$$v_2 = L_2 \frac{dI}{dt} - L_{21} \frac{dI}{dt} \tag{539}$$

Or

$$v_1 = (L_1 - L_{12}) \frac{dI}{dt} \tag{540}$$

$$v_2 = (L_2 - L_{21}) \frac{dI}{dt} \tag{541}$$

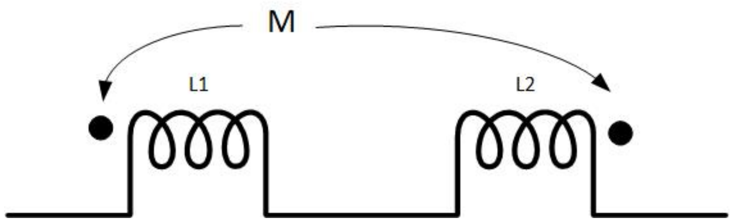


Figure 148: Dot notation.

The above equations show that the self and mutual inductance subtract.

In phasor notation and a circuit like the one in Figure 146, the only difference is the addition of the mutual inductance. So the equations will be

$$\tilde{E}_1 = (R + j\omega L_1)\tilde{I}_1 \pm j\omega L_{12} I_2 \tag{542}$$
$$\tilde{E}_1 = (R + j\omega L_2)\tilde{I}_2 \pm j\omega L_{21} I_1 \tag{543}$$

9.5 Flying ring

Flying Ring

Flying ring is an experiment shown in Figure 149. A fixed coil of N-turns is wound on a ferromagnetic core and energized by an AC generator. On top of the ferromagnetic core is a ring. Since the current is initially increasing in the coil, the magnetic field through the ring is increasing. The current in the ring will then produce an induced magnetic field to counteract the increase of the coil's magnetic field. The current associated with the induced magnetic field will flow in the opposite direction from the coil's current. When two currents are opposite, the wires repel each other so that the ring will fly off the electromagnetic core.

It is interesting to note that no matter whether the coil's current is increasing or decreasing, the ring's current will always be in the opposite direction. We will look here at this problem conceptually. Both ring and the coil represent two RL circuits. In the coil, the current lags voltage for 90^0. The flux in the coil is in phase with the current. The induced voltage in the ring will be 90^0 lagging with respect to the coil's flux. The induced voltage in the ring will establish a current lagging 90^0 with the induced voltage. Therefore the currents in the coil and ring will always be 180^0 out of phase. This is, of course, in cases where the internal resistance of the ring is minimal. In practice, the higher the resistance, the currents may not precisely be 180^0 out of phase, but the force will still be repulsive on average.

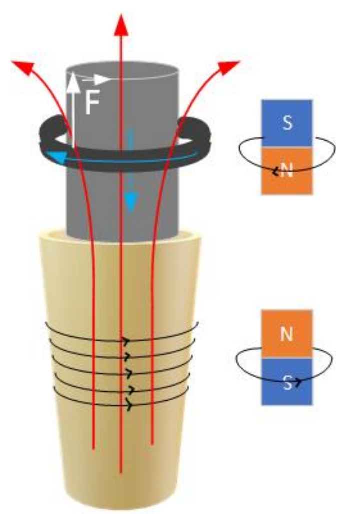


Figure 149: Flying ring experiment.

9.6 Falling Magnet

Falling magnet

Figure 150 shows a magnet falling through a copper tube. The magnet will be falling a lot slower than if the tube is not there. We will assume that the magnet falls with its south pole pointing down, as shown in Figure 150, to simplify the conceptual explanation, although this assumption is not necessary. We will also use the equivalence of a magnetic field between a magnet and a loop of current. A clockwise current in a loop is equivalent to a magnet whose north pole is pointing down, and vice versa; a counterclockwise current is equivalent to a magnet whose north pole is pointing up.

When the magnet is falling, its north pole's magnetic field points up in the positive z-direction. The copper tube sees increasing flux underneath the magnet and the decrease of flux above the magnet. Underneath the magnet, a current will form in the tube in such a direction to stop the magnetic field from increasing. In this case, this is the "down" direction. The current induced in the copper tube will therefore be in a clockwise direction, looking from above. This current has a magnetic field that looks like a magnet with its south pole pointing up, repelling the falling magnet.

Above the magnet, because the magnet is falling, the flux is decreasing, and the induced magnetic field from the current in copper will be in the direction of the magnetic field to prevent it from decreasing. Therefore, the current in copper will flow counterclockwise, looking from above. This current attracts the magnet because its magnetic field looks like a magnet with a south pole pointing down, attracting the falling magnet.

The magnet, therefore, slows down when falling in a copper tube.

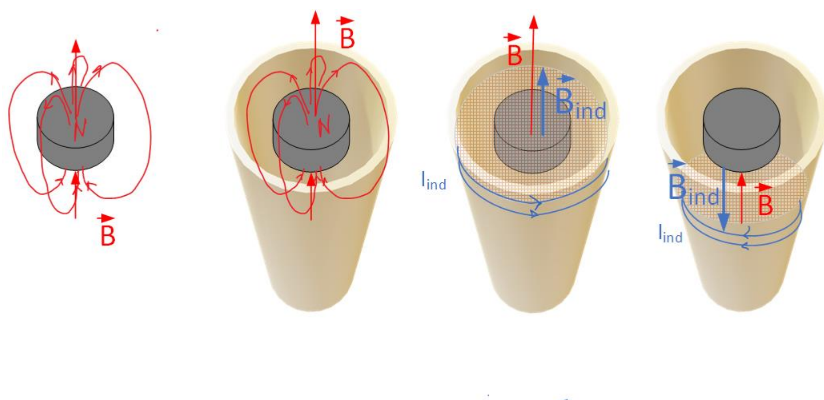


Figure 150: Falling magnet experiment.

9.7 Voltage droop

Voltage droop in electronic circuits

We looked previously at two somewhat prosaic applications of Faraday's law. In electronic circuits, we often see capacitors on the PCB DC power-rails. These are the lines that travel on a PCB and feed ICs with the necessary DC voltage. An example of power rails is shown in Figure 151. When the IC starts working, the power is drawn from the power rails. This current will be constant in a steady-state, but the current starts increasing before it reaches the constant value. Since the current is changing, there is a voltage drop inducted in the power rail.

$$v = L\frac{di}{dt} \tag{544}$$

The voltage at the IC pin will be lower than the DC voltage at the 5V battery. The voltage will drop until the steady-state is reached. The capacitors at the power rail supply reduce the voltage droop. The decoupling capacitance can be calculated from the power dissipation of the IC, the allowed percentage of droop p, and the time necessary to increase the voltage on the IC pin Δt, and the DC rail voltage V, as

$$C = \frac{P\Delta t}{pV^2} \tag{545}$$

The decoupling capacitance provides current for time Δt until the regulator can provide the appropriate current.

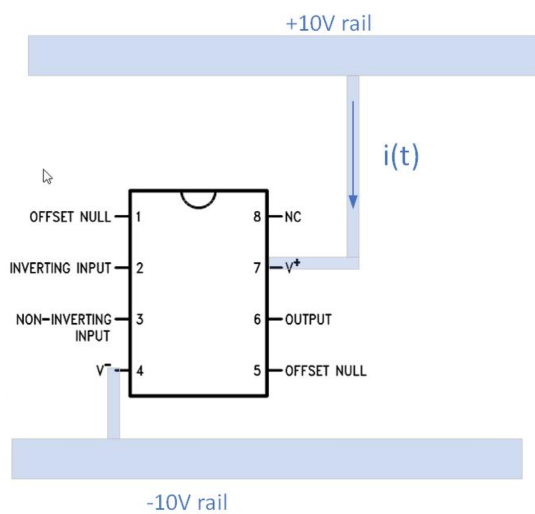


Figure 151: IC circuit on a PCB, connected to DC rail.