Input impedance of a transmission line

Again, we will look at a transmission line circuit in Figure 1 to find the input impedance on a transmission line.

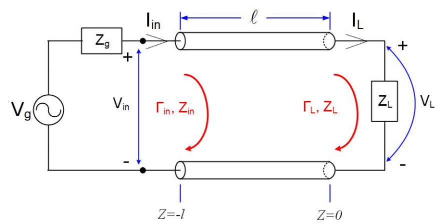

Figure 1: Transmission Line connects generator and the load.

The equations for the voltage and current anywhere (any z) on a transmission line are

$$\tilde{V}(z) = \tilde{V}_0^+(e^{j\beta z} + \Gamma_L e^{-j\beta z}) \tag{1}$$

$$I(z) = \frac{\tilde{V}_0^+}{Z_0}(e^{j\beta z} - \Gamma e^{-j\beta z}) \tag{2}$$

The voltage and current equations at the generator $z = -l$ are:

$$\tilde{V}_{in} = \tilde{V}(z = -l) = \tilde{V}_0^+(e^{j\beta l} + \Gamma_L e^{-j\beta l}) \tag{3}$$

$$\tilde{I}_{in} = I(z = -l) = \frac{\tilde{V}_0^+}{Z_0}(e^{j\beta l} - \Gamma e^{-j\beta l}) \tag{4}$$

Input impedance as a function of reflection coefficient

The input impedance is defined as $Z_{in} = \dfrac{V_{in}}{I_{in}}$. Since the line length is l, the input impedance is

Learning outcomes: Derive and calculate the input impedance of a transmission line.
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$$Z_{in} = \frac{\tilde{V}_0^+ (e^{-j\beta z} + \Gamma_L e^{j\beta z})}{\frac{\tilde{V}_0^+}{Z_0} (e^{-j\beta z} - \Gamma_L e^{j\beta z})} \tag{5}$$

If we cancel common terms, we get

$$Z_{in} = Z_0 \frac{(e^{-j\beta z} + \Gamma_L e^{j\beta z})}{(e^{-j\beta z} - \Gamma_L e^{j\beta z})} \tag{6}$$

Now we can take $e^{-j\beta z}$ in front of parenthesis from both numerator and denominator and then cancel it.

$$Z_{in} = Z_0 \frac{1 + \Gamma_L e^{2j\beta z}}{1 - \Gamma_L e^{2j\beta z}} \tag{7}$$

We have previously defined the reflection coefficient at the transmission line's input as $\Gamma_{in} = \Gamma_L e^{2j\beta z}$. The final equation for the input impedance is therefore

$$Z_{in} = Z_0 \frac{1 + \Gamma_{in}}{1 - \Gamma_{in}} \tag{8}$$

Input impedance as a function of load impedance

If we now look back at the Equation 6, here we can also use Euler's formula $e^{j\beta z} = \cos(\beta z) + j \sin(\beta z)$, and the equation for the reflection coefficient at the load $\Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0}$ we find the input impedance of the line as shown below.

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \tag{9}$$

This equation will be soon become obsolete when we learn how to use the Smith Chart.

Example 1. *Find the input impedance if the load impedance is $Z_L = 0\Omega$, and the electrical length of the line is $\beta l = 90^0$.*

Explanation. *Since the load impedance is a short circuit, and the angle is 90^0 the equation simplifies to $Z_{in} = jZ_0 \tan \beta l = \infty$.*

When we find the input impedance, we can replace the transmission line and the load, as shown in Figure 2. In the next section, we will use input impedance to find the forward going voltage on a transmission line.

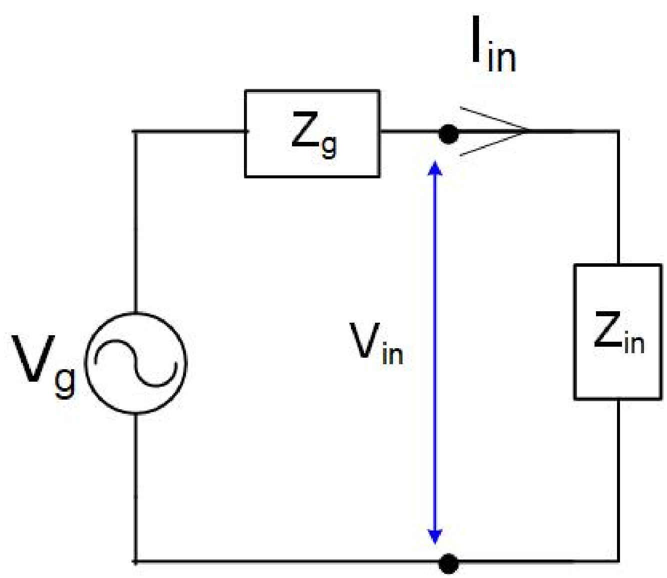

Figure 2: Transmission Line connects generator and the load.