Operations with Complex Numbers

The purpose of this section is to review arithmetic operations with complex numbers. We use complex numbers to describe circuits. When we solve circuits to find voltages, currents, and power, we often encounter addition, subtraction, multiplication, division, and the complex conjugate of complex numbers.

Complex Conjugate

We will see complex conjugate when we discuss maximum power transfer.

$$z^* = (x + jy)^* = x - jy = |z|e^{-j\Theta} \tag{1}$$

Addition

In electrical engineering, we see complex number addition and subtraction when complex impedances are in series, and we are looking for the equivalent complex impedance. The easiest way to add two complex numbers is to find the Cartesian representation of both and then add the real parts separately and the imaginary part separately.

$$\begin{aligned} z_1 &= x_1 + jy_1 \\ z_2 &= x_2 + jy_2 \\ z_1 + z_2 &= x_1 + x_2 + j(y_1 + y_2) \end{aligned} \tag{2}$$

You can visually explore addition of two complex numbers with the app below.

Geogebra link: https://tube.geogebra.org/m/yfvhfb8a

Learning outcomes: Describe which coordinate system to use when adding/subtracting and which one when multiplying/dividing two complex numbers. Apply complex numbers to solve a circuit element's impedance if the phasor of current through and voltage on it is known. Apply complex numbers to solve for voltage on a circuit element if phasor of current and impedance are known. Apply complex conjugate operation to a complex number in rectangular and polar coordinates. Derive magnitude of a complex number from a complex number and complex conjugate of the same number.

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Question 1 *Two impedances are given $Z_1 = 50 + j200\Omega$ and $Z_2 = 50 - j200\Omega$. If the two impedances are in series, what is the total impedance?*

$Z_1 + Z_2 = \boxed{100}$

Subtraction

$$\begin{aligned} z_1 &= x_1 + jy_1 \\ z_2 &= x_2 + jy_2 \\ z_1 - z_2 &= x_1 - x_2 + j(y_1 - y_2) \end{aligned} \tag{3}$$

You can visually explore subtraction of two complex numbers with the app below.

Geogebra link: https://tube.geogebra.org/m/ujsv2qkq

Multiplication and Division

We often see multiplication and division of complex numbers in Ohm's law or transfer function of a circuit. Two complex numbers can be multiplied or divided in either Cartesian or Polar forms. However, the easiest way to divide or multiply two complex numbers is to find the polar representation of both and then divide or multiply the amplitudes and subtract or add the phases, as shown in equations below.

$$\begin{aligned} z_1 &= |z_1|e^{j\Theta_1} \\ z_2 &= |z_2|e^{j\Theta_2} \\ \frac{z_1}{z_2} &= \frac{|z_1|}{|z_2|}e^{j\Theta_1 - \Theta_2} \end{aligned} \tag{4}$$

$$\begin{aligned} z_1 &= |z_1|e^{j\Theta_1} \\ z_2 &= |z_2|e^{j\Theta_2} \\ z_1 z_2 &= |z_1||z_2|e^{j\Theta_1 + \Theta_2} \end{aligned} \tag{5}$$

Explore the multiplication of two complex numbers visually with the app below. We see that the multiplication of numbers represent rotation.

Geogebra link: https://tube.geogebra.org/m/h34xreac

Question 2 *Three complex numbers are given* $Z_1 = 100 + j50$, $Z_2 = 3 * e^{j40^0}$ *and* $Z_3 = 20 + j100$. *Calculate* $\frac{Z_1 Z_2}{Z_2 + Z_3}$. *Present your answer as a complex number in Cartesian coordinates with two decimal places. For example* $0.11 + j0.23$ *:*

$$\frac{Z_1 Z_2}{Z_2 + Z_3} = \boxed{0.36 + j3.16}$$