Random Variables

- given a sample space Ω, set of events F, a probability function P, and countable set of real numbers D a discrete random variable is a function with domain Ω and range D
  ↳ for any outcome ω, $X(\omega)$ exists

- numbers from coin or dice ⟶ random variable
- flip a coin 32 times, record 1/0 depending on heads/tails
  ↳ 32 bit number ← also a random variable

- random variable is a function that quantifies the outcome of a random process

- random variables can take on many values with different probabilities and so it makes more sense to talk about the probability of a random variable relative to a certain value.

- for any $X = \underline{\underline{x}}$ theres a series of values associated with it
  set of outcomes $= \omega$
  $X(\omega) = x$ ⟿ $X = x$
  $P(\{\omega : X(\omega) = x\}) \approx P(\{X = x\}) \approx P(x)$
  $P(X(\omega) \leq x)$ can also be calced. etc.

DISCRETE

properties of probability distributions

expected values

- an estimate about what would happen over a large number of events, on a per-event basis.

- a weighted average of the outcomes and their probabilities.

- for outcomes $\{x_1, x_2 \dots x_n\}$ with respective probabilities $\{p_1, p_2 \dots p_n\}$

$$E(X) = x_1p_1 + x_2p_2 \dots + x_np_n = \sum_{i=1}^{n} x_ip_i = \mu_x$$

can take a value that the random variable doesn't take

f is a f^n on random variables

→ $E(0) = 0$

→ $E(kf) = kE(f)$

→ $E(f \pm g) = E(f) \pm E(g)$

→ $E[f(X)] = \sum f(x_i) \cdot p_i$

→ $E[XY] = E[X] \cdot E[Y]$

and when f = identity f^n $E(X)$ satisfies all these.

so if $f(X) = X^2$, $E[X^2] = \sum x_i^2 p_i$

or $f(X) = (X-\mu)^2$, $E[(X-\mu)^2) = Var[X] = \sum (x_i - \mu)^2 p_i$

- mean is the same as the expected value

- variance : $Var[X] = E[(X - E[X])^2] = E[X^2] - (E[X])^2$

- $E(X^2) = \sum x^2 \cdot p(x)$

$E(Y)$ where $Y = (aX + b)^2 = a^2E(X^2) + 2abE(X) + b^2$

- std deviation of random variable X

std({X}) = $\sqrt{var[X]}$

- variance:

for sample of n independent observations

$$s^2 = \frac{1}{n-1}\left[\sum x^2 - \frac{(\Sigma(x))^2}{n}\right]$$

variance of rv. X ⇝ s^2 as the sample size n becomes very large

the theoretical limit of the sample variance

$$var(X) = p_1(x_1-\mu)^2 + p_2(x_2-\mu)^2 + \dots + p_n(x_n-\mu)^2$$

or σ^2

$$= \sum_{i=1}^{n} p_i(x_i-\mu)^2 = E[(x-\mu)^2]$$

$$= \sum p_i(x_i^2 - 2x_i\mu + \mu^2)$$

$$= \sum p_i x_i^2 - 2\mu\underbrace{\sum p_i x_i}_{\mu} + \mu^2\underbrace{\sum p_i}_{1}$$

$$= E[X^2] - \mu^2$$

$$= E[X^2] - (E[X])^2$$

- if $Y = aX + b \longrightarrow E(Y) = aE(X) + b$

$var(Y) = a^2 var(X)$

$\sigma_y = \sqrt{a^2 var(X)} = a\sqrt{var(X)} = a\sigma_x$

same as independent else, you can't calc the variance of sum

$\sigma_{X\pm Y} = \sqrt{var(X\pm Y)}$

- and if $T = a + bX \pm cY$ (X, Y are uncorrelated)

$E(T) = a + bE(X) \pm cE(Y)$

$var(T) = b^2 var(X) + c^2 var(Y)$ ← note: always addition

↳ always > $b^2 var(X)$ or $c^2 var(Y)$

ex

x	0	1	2
$P(x)$	0.2	0.5	0.3

$Y = (X-1)^2$

$E[Y] = ?$

$var[Y] = ?$

y	1	0	1

$E[Y] = 1(0.2) + 0 + (0.3)(1)$

$E[Y] = 0.5$

$var[Y] = E[Y^2] - (E[Y])^2$

$\hookrightarrow [1^2(0.2) + 0^2 + 1^2(0.3)]$

$= 0.5 - (0.5)^2 = \underline{0.25}$

empirical quantity (in observed data)	theoretical quantity (mathematical d r v)	remarks
relative freq $(x_i) = \frac{f_i}{m}$	$P[X = x_i] = p_i$	$\frac{f_i}{n} \to p$ if $n \to \infty$
mean: $\bar{X} = \sum_{i=1}^{m} \frac{f_i}{m} \cdot x_i$	expectation $E[X] = \sum_{i=1}^{n} x_i p_i = \mu$	$\bar{X} \to \mu$ as $n \to \infty$
variance $s^2 = \sum_{i=1}^{m} \frac{(x_i - \bar{X})^2 f_i}{m-1}$	$var[X]$: $\sigma^2 = \sum_{i=1}^{n} (x_i - \bar{X})^2 p_i$	$s^2 \to \sigma^2$ as $n \to \infty$

Probability Distribution of $\underbrace{\text{Discrete Random Variable}}$

drv

- list of values of drv (X) and their probabilities is called
  Discrete Probability Distribution
  or Probability Mass Function (pmf)

3 special probability Distributions

1. Binomial
2. Poisson
3. Geometric

1. Binomial

- bernoulli trial : · each experiment has only 2 possible results
  ↳ success or failure
  $P(s) = 1 - P(f)$
  · probability of all trials stays the same
  · each trial is independent
  · finite number of trials
- let X be r.v = total number of successes in n trials

$$P(X = x) = \binom{n}{x} \cdot p^{x} \cdot (1-p)^{n-x}$$

number of arrangements

p(success)

p(failure)

X	0	1	n
$P(X=x)$	$({}^nC_0)p^0(1-p)^n$	$({}^nC_1)p^1(1-p)^{n-1}$	

↑ Binomial Distribution w/ parameters n and p

$P(X=x) = X \sim B(n,p)$

<u>mean</u>

$$E[X] = \sum x_i p_i = \sum x_i \binom{n}{x_1} p^{x_1} (1-p)^{n-x_1} = np$$

proof: $\displaystyle\sum_{x=1}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$

let $y = x-1$, $n = m-1$

$$E[X] = \sum_{y=0}^{m} \frac{(m+1)!}{y!(m-y)!} p^{y+1} (1-p)^{m-y}$$

$$= (m+1)p \underbrace{\sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^y (1-p)^{m-y}}$$

$$= (m+1)p \underbrace{\sum_{y=0}^{m} \binom{m}{y} p^y (1-p)^{m-y}}$$

$$= np \qquad (p+(1-p))^m = np$$

variance

- $Var[X] = \sigma_X^2 = E[X^2] - (E[X])^2 = np(1-p) = npq$

proof $= E[X(X-1)] + E[X] - (E[X])^2$

didn't get it.

- Binomial Random Variable :

- outcome can be success or failure
- each trial is independent of the others
- fixed number of trials
- probability p of success remains constant on each trial

is the discrete probability distribution of the number of events occuring in a given time period, given the average number of times the event occurs over that time period

2. Poisson

- rv. X is no of successes in a given time interval

- $P(X=x) = \dfrac{e^{-\mu}\mu^x}{x!} = X \sim Pois(\mu)$

$0 < x < \infty$

$\mu = E[X]$ (avg no of successes) → in the time interval we are calculating for

$e = 2.71828...$

$\mu = Var[X]$

- can use when:
  - → event can occur any number of times : $0 < x < \infty$
  - → events occur independently
  - → rate of occurance is constant
  - → p(event) ∝ length of time

use when:
$n \geq 100$
$p \leq 0.01$
$\mu = np$

Poisson as approximating Binomial

$n \to \infty$
$p \to 0$
$np = \mu$

$$P(X=k) = \lim_{n\to\infty} \binom{n}{k}\left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$\lim_{n\to\infty} \frac{n!}{(n-k)!\,k!} \frac{\lambda^k}{n^k} \left(1-\frac{\lambda}{n}\right)^n \cdot \left(1-\frac{\lambda}{n}\right)^{-k}$$

$$\lim_{n\to\infty} \underbrace{\frac{n(n-1)\cdots(n-k+1)}{n^k}}_{1} \times \frac{\lambda^k}{k!} \underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{e^{-\lambda}} \underbrace{\left(1-\frac{\lambda}{n}\right)^{-k}}_{1}$$

$$= \frac{\lambda^k}{k!} \cdot e^{-\lambda}$$

3 Geometric

- $P(m \geq 5)$ → 5 or more for success to occur

  $= P(\bar{m} \text{ for } 4)$ → prob of failure for first four

- $P(m \leq 2)$

  $= 1 - \underbrace{P(\bar{m} \text{ for } 2)}_{= P(\bar{m} \geq 3)}$

  $\geq = \rightarrow -1$

  $1- \rightarrow = \leq$

  $P(X \leq 3)$

  $= 1 - P(\bar{X} \text{ for } 3)$

  $= 1 - (0.1)^3$

- for a sequence of independent Bernoulli trials, where each trial is an 'experiment' with exactly 2 possible outcomes
- with no cap, the experiments are performed consecutively until the first success is obtained.

- let X be number till first success
  so $X-1$ = no of failure

  $P(X = x) = (1-p)^{x-1} p$ ← $X \sim G(p)$ ← p(success)

- $E[X] = \dfrac{1}{p}$

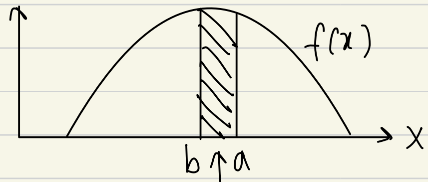
- $Var[X] = \dfrac{1-p}{p^2}$

CONTINUOUS

Probability Distribution of a Continuous R.V.

- Probability density function (pdf)

$f(x) \geq 0 \;\forall\; X$



and area under the curve $= 1$

$P(a > x > b) =$ area of $\int_a^b f(x)\,dx$

$P(X < x) \rightarrow$ cumulative distribution function (CDF)

- $\text{cdf of } X = \int_{-\infty}^{x} f(x)$

- $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \mu$

- $Var[X] = E[X^2] - E[X]^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

example!

$$f(x) = \begin{cases} \cos x & 0 \le x \le \pi/2 \\ 0 & \text{else.} \end{cases}$$

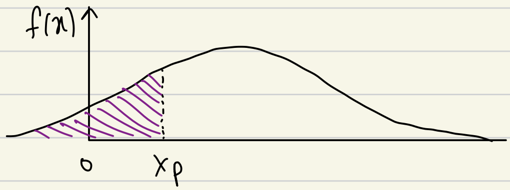
$$E[X] = \int_{-\infty}^{\infty} x f(x) = \int_{-\infty}^{0} x f(x)\,dx + \int_{0}^{\pi/2} x f(x)\,dx + \int_{\pi/2}^{\infty} x f(x)\,dx$$

$$= \int_{0}^{\pi/2} x\cos x\,dx = \left[x\sin x\right]_0^{\pi/2} - \int_{0}^{\pi/2} \sin x\,dx$$

$$= \frac{\pi}{2}\sin(90) + \left[\cos x\right]_0^{\pi/2}$$

$$= \frac{\pi}{2} - 1$$

Percentile



p^{th} percentile = value X_p

: $P(X < X_p) = p\%$

$\Rightarrow$ = $p\%$

$$\int_{-\infty}^{X_p} f(x)\,dx = \frac{p}{100}$$

eg.

$f(x) = e^{-x} \qquad x > 0$

$$\int_0^{X_p} e^{-x}\,dx = 0.25$$

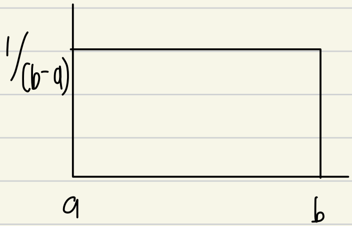
$$\left[-e^{-x}\right]_0^{X_p} = 0.25$$

$$1 - e^{-X_p} = 0.25$$

$$1 - e^{-X_p} = 0.25$$

$$\ln(0.75) = -X_p$$

Uniform Distribution



$$f(x) = \frac{1}{b-a}$$

uniform probability over this dis.

constant

$$P(X = c) = 0$$

$$X \sim U(a, b)$$

$$E[X] = \frac{b+a}{2}$$

$$Var[X] = \frac{(a-b)^2}{12}$$

Exponential Distribution

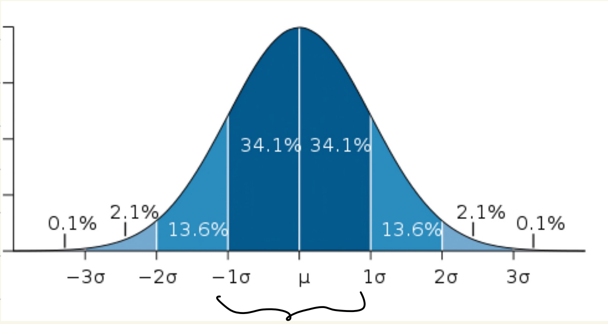
- no of success in time T $\rightarrow$ poisson dist
  then
  inter-success time X $\rightarrow$ exponential dist
- $f(x) = \lambda e^{-\lambda x} \qquad x \geq 0$
- λ = mean success
  $\frac{1}{\lambda}$ = mean inter-success time
- $X \sim Exp(\lambda)$
- $E[X] = 1/\lambda$
- $Var[X] = 1/\lambda^2$

Normal Distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$ ← no need to remember

$$-\infty < x < \infty$$

$X \sim N(\mu, \sigma^2)$ (mean, var)



68%

95%

99.7%

$X = \mu + Z\sigma$

$$Z = \frac{X-\mu}{\sigma}$$

$E[Z] = 0$
$Var[Z] = 1$

E(constant) = constant.

$Z \sim N(0, 1)$ ← standard normal distribution
↓
because all bellcurves can be transformed into such bell curves

$m + n = 80$

$3(80 - n) + 8(n) = 490$

$240 - 3n + 8n = 490$

$5n = 250$

3R 7G

odd : 1, 3, 1, 3, 5, 7 → 6

even : 4

even green : 2, 4, 6

odd red : 2

10	even	else	
(2)(4)	(2)(3)	(1)(4) + (1)(3)	7
8	3		

odd r odd — 9 odd r even — 9

$\left(\frac{6}{10}\right)\left(\frac{5}{9}\right)\left(\frac{4}{8}\right) = \frac{1}{6}$ (500)

$\frac{6}{10} - \frac{1}{6} = \frac{13}{30}$ (100)

$1 - \frac{1}{6} - \frac{13}{30} = \frac{2}{5}$ (0)

$0.09 \times 50 \longrightarrow 4.5$

$0.42 \times 20 \longrightarrow 8.4$

0.49×0

5	3	−10
0.5	0.12	0.36

small

10 mil   -1 mil

$(0.4)^5$ $1-(0.4)^5$

102400 → 989760

0

large

15 mil   -5 mil

$(0.3)^3$ $1-(0.3)^3$

405000   486500

$\mu = 3$

$$P(X=0) = \frac{e^{-3} \cdot 3^0}{0!} = e^{-3}$$

$ans = 1 - e^{-3}$

$$\mu_{hr} = \frac{3}{8}$$

$$P(X=1) = \frac{e^{-3/8} (3/8)^1}{1!}$$

$\mu_w = 3$ $\mu_{month} = 12$

$P(X \geq x) = 0.5$

$P(X \geq 12) = P(X=0) + P(X=1) + P(X=2) \ldots$

$$e^{-12} \left(\frac{12^0}{0!} + \frac{12^1}{1!} + \frac{12^2}{2!} + \frac{12^3}{3!} + \frac{12^4}{4!} \cdots \right)$$

$\mu_{candle} = 500$
$\sigma_C = 15\,g$
$\mu_S = 200\,g$
$\sigma_S = 8\,g$

$\mu_T = 700$
$\sigma_T = 17$

$P(S) = 0.4$
$n = 4$
$P(X = 3) = \binom{n}{x} p^x (1-p)^{n-x}$
${}^4C_3\,(0.4)^3(0.6)$

$var[X] = (0.75)^2(0.5) + 0.015625 + 0.390625$
$0.2815 \quad = 0.6875$

$\sigma_X = \sqrt{0.6875} = 0.83$

0	0.4	0
700	0.2	500
1000	0.15	500
1000	0.15	500
1000	0.1	500

0 10 20 30

$P = 10X - 1$
$\sigma = 10\sigma_X$

$P(D \geq 4)$
$= P(\bar{D} \text{ for } 3)$

D : forgets
$\bar{D}$ = rem : 0.3

$= \rightarrow \frac{81}{1000}$
$1 - \rightarrow \frac{81}{10000}$
0.0081

1. $P(X = 19) = \binom{20}{19}(0.95)^{19}(0.05)^{5}$

2.

$+15$	-10
0.3	0.7

3.

100	10
0.1	0.9

$(81)^2(0.1) + (9)^2(0.9)$

4.

$$\begin{array}{r} 1819 \\ \underline{1497} \\ 322 \end{array}$$

$\geq = -1$

$1 - = \leq$

5. $P(V \leq 3)$

$= 1 - P(\bar{V} \text{ for } 3)$

$= 1 - (0.94)^3$

$=$

6. binomial

→ sucess / failure

→ trials are independente

→ fixed no of trials

→ p(succ) doesnt change

8. $E[X] = 3.37$

$\frac{1160 + a + 6b}{500} = 3.37$

$a + b = 150$

$a + 6b = 525$

$5b = 375$

$b = 75$

9. $P' = p + 10$

10

B	NB
0.3	0.7

binomial var

$n = 15$

$p = 0.3$

$q = 0.7$

$E[X] = np$

$\sigma_x = \sqrt{npq}$

11

no	w	c	m
0	10	15	25
0.4	0.6	0.3	0.12

$1 - 0.6 \leftarrow$ un.

$0.6 \times 0.5 \leftarrow w, c'$

$0.6 \times 0.5 \times 0.6 \leftarrow w, c, m'$

$0.6 \times 0.5 \times 0.4 \leftarrow w, c, m$

$\sigma_x + \sigma_y = 12.7379$

0.121×2

$p = 0.7$

$n = 3$

1000	0	-1000
0.5^2	2×0.5^2	$(0.5)^2$
0.25	0.5	0.25

$P(X = 2) = \binom{3}{2}(0.7)^2(0.3)$

$P(X = 6) = P(f)^5 P(\gamma)$

$P(f) \to 7/8$, $P(\gamma) \to 1/8$

1	2	3	4
0·2	(0·8×0·2)	(0·8×0·8×0·2)	1 − rest (0·8×0·8×0·8)

geometric.

P(succ) is const
no of trial undef.

$P(P<3) = 1 - P(\bar{X} \text{ for } 2) = 1 - 0.512$

is the set of numbers $P(\{X = x\})$ for each value x that X can take

probability distribution of discrete rv

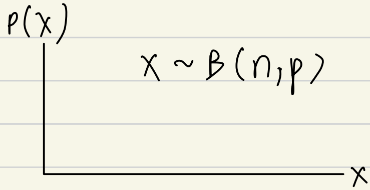
lecture notes

X = total no of successes in n trials
prob of x successes

$$P(X = x) = \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x}$$

${}^nC_x$

binomial distribution
(n, p) parameters
$X \sim B(n,p)$



mean and variance — proof on mulearn

mean $E[X] = np$ $\quad \left(E[X] = \sum x\, p(x) \right)$

var. $Var[X] = E[X^2] - E[X]^2 = npq$ $\quad (q = (1-p))$

$ab \quad \bar{a}\bar{b} \quad a\bar{b} + b\bar{a}$

$\left(\frac{1}{2} \times \frac{1}{2}\right) \quad \left(\frac{1}{2} \times \frac{1}{2}\right) \quad 2\left(\frac{1}{2} \times \frac{1}{2}\right)$

$\frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{2}$

$2^3 \to 8$

1 HHH
2 HHT , HTH , THH
3 HTT , THT ; THH
4 TTT

$n = 3$, $p = 0.5$

$P(X = x)$: $X \sim B(3, 0.5)$ $= \binom{3}{2} p^2 (1-p)^{3-2}$

$x = 2$

$= 3 \cdot 0.5^2 \cdot 0.5$

$= 3(0.125)$

$= 0.375$

10B , 30G $\quad P(B) = 1/4 = 0.25$

$n = 10$ $\quad \because$ only 2 outcomes: bernoulli.

$E[X] = np = 10(0.25) = 2.5$

$Var[X] = npq = 10(0.25)(0.75)$

$= 1.875$

$P(X=2) = \binom{10}{2} p^2 (1-p)^8$

$P(\text{rej}) = P(X \geq 2)$

$= 1 - p(X < 2)$

at least 1 telephone order in first 4

$P(C < 5) = 1 - P(C \geq 5)$

$P(C=1) + P(C=2) + P(C=3) + P(C=4)$

$0.1 + (0.9)(0.1) + (0.9)^2(0.1) + (0.9)^3(0.1)$

$\cancel{(0.1)}(1)\dfrac{(0.9^4 - 1)}{\cancel{0.9-1}}$

$1 - P\left(\text{no telephone orders in first 4}\right) \rightarrow 1 - (0.9)^4$

p

Khan academy

- $E(X \pm Y) = \mu_x \pm \mu_y$
- $\sigma_{x \pm y} = \sqrt{var(X) + var(Y)}$
- $Y = AX + b \Rightarrow E[Y] = AE[X]$, $Var[Y] =$