Eigen and Singular values.

Eigenvectors and Eigenvalues

- eigenvector of an nxn matrix A is a nonzero vector x :
$Ax = \lambda x$, λ is a scalar

- λ is the eigenvalue of A and can be 0
it exists only if $Ax = \lambda x$ has a nontrivial solution
→ set of all eigenvalues = spectrum of A

- x is the eigenvector corresponding to λ

- multiple eigenvalues can exist for a matrix (up to n for nxn matrix)

ex. $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ $u = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$Av = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2v$ $\therefore$ v is an eigenvector
2 is its eigenvalue

- basically Ax must lie on the same line as x

ex. given A, st. 7 is an eigenvalue of A, + find the corr. eigenvector

$Ax = 7x$
$(A - 7I)x = 0$
↳

$$(A - 7I) = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

linearly dependent. $\Rightarrow$ non trivial soln.

7 is an eigenvalue

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\therefore$ eigenvectors $= x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$: $x_2 \neq 0$

this is a line $(x = y)$

that line is the eigenspace of $\lambda = 7$

Eigen Space

- λ is the eigenvalue of A iff $(A - \lambda I)x = 0$ has a non-trivial soln.

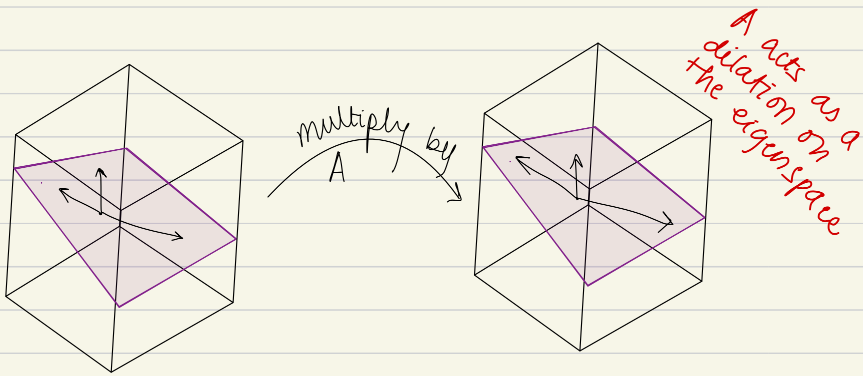
- this set of non-trivial solns = null $((A - \lambda I))$
  = subspace of R^n = eigenspace of A corresponding to λ

- eigenspace contains the zero vector + all eigenvectors

- finding basis for eigenspace:

  if space is repped by say: $x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

  then any random pair x_2, x_3 can be chosen for the basis



if $x \in$ eigenspace of A, Ax also belongs to it

Characteristic Equation

- x, λ are both unknowns.

- $Ax = \lambda x$

  $\vdots$

  $(A - \lambda I)x = 0$

  $(\lambda I - A)x = 0$ ← must be singular non inv.

  dependent cols

  so that soln is non-trivial

  characteristic eqn → $\underline{\det(A - I\lambda) = 0}$

- a square matrix is invertible iff. $\det A \neq 0$

  $\Rightarrow$ if matrix is not inv, $\det = 0$

when we open, ← solve to get λ

we get a polynomial

↓

characteristic polynomial

↓

degree n ← size of A

theorem xyz

given A and scalar λ : $Ax = \lambda x$

→ λ is an eigenvalue of A
→ $N(A - \lambda I) \neq \{0\}$
→ matrix $A - \lambda I$ is singular
→ $\det(A - \lambda I) = 0$

1. eigenvalues of Δ matrices

- $\det(\Delta)$ = product of diagonal terms

- $$A - \lambda I = \begin{bmatrix} a_1-\lambda & a_2 & a_3 \\ 0 & a_4-\lambda & a_5 \\ 0 & 0 & a_6-\lambda \end{bmatrix}$$

- if $\det(A - \lambda I) = 0 \Rightarrow (A - \lambda I)x = 0$ has a nontrivial soln,

  λ is an eigenvalue

  $\therefore (a_1 - \lambda)(a_4 - \lambda)(a_6 - \lambda) = 0$ ← characteristic equation
  $\therefore \lambda = \{$ diagonal entries $\}$

2. eigenvalue = 0

$\Rightarrow$ $Ax = 0x$ has a nontrivial soln

$Ax = 0$ " "

$\therefore$ A is singular, non-inv

3. eigenvalue and algebraic multiplicity

if $A = \begin{bmatrix} 5 & a & b \\ 0 & 3 & c \\ 0 & 0 & 5 \end{bmatrix}$

char eq (A) :

$(5-\lambda)(3-\lambda)(5-\lambda) = 0$

$(5-\lambda)^2(3-\lambda) = 0$

$\uparrow$

eigenvalue 5 has multiplicity = 2

· sum of all multiplicities = N

Similarity and Diagonalization

- if A & P are both square, and P is inv.

$B = P^{-1}AP$
$PBP^{-1} = A$

A and B are similar matrixes and the transformation from A to B $= P^{-1}AP$ is called similarity transformation

if B is a diagonal matrix, A is said to be diagonalizable

- similarity invariant : properties preserved by a similarity transformation

→ det
→ invertibility
→ rank
→ nullity
→ trace
→ characteristic poly
→ eigenvalues
→ eigenspace dimensions
→ dim(space) = min no of vectors in basis

· 6 reasons why (we love diagonal matrix)

1. eigenvalues = diagonal entries
2. det = product of diagonal entries
3. rank = = no of non-zero diag. entries
4. multiplication :
   AD : rows of A x diag. elements
   DA : cols of A x diag. elements
5. D^{-1} = reciprocal of diag elements
6. $D_1 D_2$ is easily computable

· when is $A_{n\times n}$ diagonalizable

↓

A has n eigenvectors that are linearly dependent
and those vectors form the matrix P
and then D becomes the eigenvalues for each vector

↳ A is diagonalizable iff there are enough eigen-
vectors to form the basis of R^n
∴ P becomes a basis of R^n

· assume P is inv ⇒ P^{-1} exist

$P^{-1}AP = B$

$PP^{-1}AP = PB$

if P contains distinct & independent vectors : $PB = P\underline{D}$
↳ diagonal matrix of eigen values

$AP = A[v_1\ v_2\ \cdots\ \cdots\ v_n] = [Av_1\ Av_2 \cdots Av_n]$

$= [\lambda_1 v_1\ \lambda_2 v_2\ \lambda_3 v_3\ \cdots]$

$= P \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & \cdots & & \\ 0 & 0 & \ddots & \lambda_n \\ 0 & 0 & & \end{bmatrix}$

$\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} [\lambda_1\ 0\ 0\ 0\ 0]$

✓ check out.

n distinct eigenvalues $\Rightarrow$ n distinct eigenvectors
at least

- steps to diagonalizes :

1. find eigenvalues
2. find eigenvectors
3. form P, D

- D matrix : diag = eigenvalues

- if a matrix has repeating eigenvalues

i) algebraic multiplicity : multiciplity of eigenvalue
ii) geometric multiplicity : dimension of eigenspace corr. to λ_k

$geo(\lambda_k) \leq al(\lambda_k)$ always
if $geo(\lambda_k) = al(\lambda_k) \rightarrow$ diag $\forall \lambda$

- Power of A

- $A^k = P D^k P^{-1} = P \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix} P^{-1}$