- System
  - consistent
    - 1
    - ∞
  - inconsistent
    - approx
      - QR
      - Normal Eqⁿ
      - Psuedo inv

lecture notes

- linear equation : $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$
- linear system : m eqns in n unkowns ← can have 0 or 1 or ∞ly many solutions

$$a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n = b_1$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

no of soln: 1, ∞

- homogeneous system : all $b_i = 0$
  - always consistent

trivial solution : $x_1 = 0, x_2 = 0, \dots x_n = 0$

Ques

1. Is the system consistent (does soln exist?)
2. If soln exists, is it the only one? (is soln unique?)

- EROs : elementary row operations (on matrices)
- pivot : first non-zero element of a row
- row echelon : all-zero rows should be at the bottom of the matrix, and every pivot should be on the right of the pivot in the previous row

- reduced row echelon (RRE): all that + every pivot should be part of a column that has only zeros (not including the pivot)

- gaussian elimination: tranform to row echelon
- gaussian jordan : —"— to reduced —"—

pivot perspective

(1) $m = n$ $\quad$ m = no. of eqn. , n = no of rows.

$\therefore$ square matrix (order $m \times n$)

(2) non-homogeneous system

RREF (f = form) of augmented matrix

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \end{array}\right] \\ \underbrace{x \quad y \quad z}_{} \quad b \end{array} \begin{array}{l} \longrightarrow x = \alpha \\ \longrightarrow y = \beta \\ \longrightarrow z = \gamma \end{array}$$

$\Rightarrow$ unique soln

$\Downarrow$

every column except the last is a pivot column

RREF

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \alpha = 0 \\ 0 & 1 & 0 & \beta = 0 \\ 0 & 0 & 0 & \gamma = 1 \end{array}\right] \rightarrow 0x + 0y + 0z = 1$$

no solution

$\Uparrow$

pivot in last colm

if $\gamma = 0$, ∞ soln $\Rightarrow$ last colm and some other col has no pivot

lams notes

- linear equation of n variables:

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

conditions: a_i, b are constants and not all 0

- linear system of m equations in n unknowns

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{matrix}$$

→ if all b's $= 0$, the system is homogeneous

- a solution of a linear system in n unknowns $x_1, x_2, \dots x_n$ is a sequence of n numbers $s_1, s_2 \dots s_n$ for which every substitution results in a true statement.

- consistent system: at least 1 solution
inconsistent system: no solutions

- $\underbrace{\text{linear systems}}_{\text{of 2 unknowns}}$ have zero, one or infinite solutions

of 2 unknowns ← is this a condition?

zero: parallel
one: once intersecting (skew)
infinite: same line

HOMOGENEOUS SYSTEM

- always consistent ie. the values = 0 "trivial solution"
- if other solutions exist, they are called nontrivial solutions
- it has 1 (trivial) or ∞ solutions → same line
- all lines intersect at the origin

MATRIX NOTATION

- $5x + y = 3$
  $2x - y = 4$

coeff. matrix : $\begin{bmatrix} 5 & 1 \\ 2 & -1 \end{bmatrix}$

augmented matrix : $\left[\begin{array}{cc|c} 5 & 1 & 3 \\ 2 & -1 & 4 \end{array}\right]$

$$\equiv$$

$$\begin{bmatrix} 5 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\equiv$$

linear combination of vectors

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} y = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

1.3 FUNDAMENTAL QUESTIONS

1. is it consistent aka. does at least 1 solution exist?
2. if a solution exists, is it unique?

answer by converting augmented matrix into a row equivalent matrix

⇓

done by using EROs (elementary row operations)
- multiply a row by a non-zero constant
- interchange two rows
- add a row to another

ROW ECHELON FORM

- all zero rows are at the bottom
- every pivot is on the right of the pivot in the previous row
  ↳ first non-zero number in the row
- can be obtained using gaussian elimination using EROs
  ↳ diff sequence ⇒ diff row echelon matrix

REDUCED ROW ECHELON FORM

pt is to eliminate every variable but one from each row

- matrix is in echelon form
- every column containing a pivot has pivot value = 1 and all other elements in the column are 0
- can be obtained by gaussian-jordan elimination using eros; a RRE is unique per matrix

aka convert into I matrix

- guassian elimination : basically find the pivot and make everything under it 0.

- jordan : $\underbrace{\text{get echelon,}}_{\text{forward phase}}$ $\underbrace{\text{then make rest 0}}_{\text{backward phase}}$

forward phase: 0 under pivot

backward phase: 0 above pivot, pivot = 1

ANSWERING THE QUESTIONS

- system → augmented matrix → row echelon → new system → infer

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ 0 & 0 & 0 & i \end{bmatrix}$$

if any row has all except last element = 0 ⇒ no solution

impossible $0x_1 + 0x_2 + 0x_3 = i$ ∴ no soln

- reduced row form gives solution

pivoted variables are called leading variables

others are called free variables → can take any value ∴ ∞ sol

↓

get $eq^n s$ with the leading as subjects

↓

replace rhs variables w/ letters / number

↓

then rep $\boxed{\text{all}}$ as $eq^n s$ w/ those variables

those are the solutions

stop there? → yes

1.5 LINEAR COMBINATION OF VECTORS

- $y = c_1 v_1 + \dots + c_p v_p$

vector

linear combination of $\underbrace{v_1, v_2 \dots v_p}_{\text{vectors}}$; with weights $\underbrace{c_1, c_2, \dots c_p}_{\text{scalars}}$

eg. $\vec{a_1} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$ $\vec{a_2} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix}$

- same as asking if b exists in the span of a_1, a_2

can b be repped as a linear combo of a_1, a_2

$x_1 a_1 + x_2 a_2 = b \longrightarrow$ is a linear system

make the augmented matrix

$\underbrace{\text{iff it has a soln}}_{\text{answer}} \Rightarrow \vec{b}$ can be repped

1.6 SPAN

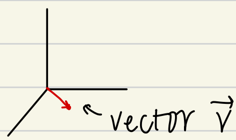
- R^n : n dimensional space of real vectors

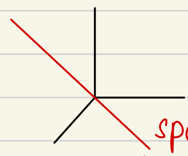
- all linear combinations of $v_1, v_2, \dots v_p$ is denoted by Span $\{v_1, v_2, v_3 \dots v_p\}$ and is called the subset of R^n spanned (or generated) by $v_1, v_2 \dots v_p$.

- Span $\{v_1, v_2 \dots v_p\}$ = collection of all the vectors that can be written as $c_1v_1 + c_2v_2 + \dots c_pv_p$ with scalars $c_1, c_2 \dots c_p$

- geometric view

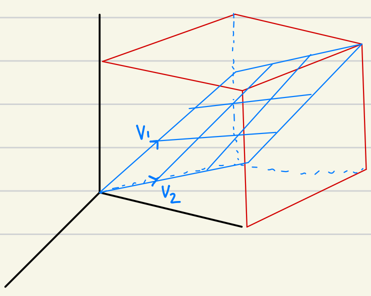
Span $\{v\}$
$\downarrow$
scalar multiples of v = set of points in the line in R^3 through v and 0





Span $\{u, v\}$
$\downarrow$
plane in R^3 containing u, v and 0

if they arent collinear, in 2-D, the span is the entire plane. if they are collinear, the span is still the line they lie on



very important

lec 2

Pivot's Perspective

a) unique : every column except the last is a pivot column

b) no soln : pivot in the last column ← aug.

c) infinite soln : last column and some other column is not a pivot column

m : no of eq ; n : no of variables

$m = n$ coeff : square

$m < n$ coeff : FAT

$M > n$ coeff : TALL

2 planes in 3D

eqⁿ of a plane : $a_1 x + b_1 y + c_1 z = d_1$
$a_2 x + b_2 y + c_2 z = d_2$

solution is along the line of intersection and ∴ there are infinite

- pivot in every row ⇒ consistent. (+ spans)
- non-pivot col ⇒ its a linear combo of pivot cols.
- pivot in every col → no free variable

ex 12 $\left(\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 2 & 8 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right)$

- if an equation is a multiple of the other, the system is that of redundant equations.

ex 19. $\left(\begin{array}{cccc|c} 1 & 1 & 0 & 5 & 1 \\ 0 & 1 & 0 & 2 & 6 \\ 0 & 0 & 1 & 4 & 8 \end{array}\right)$

$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & -5 \\ 0 & 1 & 0 & 2 & 6 \\ 0 & 0 & 1 & 4 & 8 \end{array}\right)$

$$\begin{aligned} x + 3w &= -5 \\ y + 2w &= 6 \\ z + 4w &= 8 \\ w + 0 &= w \\ x &= -5 - 3w \\ y &= 6 - 2w \\ z &= 8 - 4w \\ w &= w \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -5 \\ 6 \\ 8 \\ 0 \end{pmatrix} + w \begin{pmatrix} -3 \\ -2 \\ -4 \\ 1 \end{pmatrix}$$

soln set : $\left\{ \begin{pmatrix} -5 \\ 6 \\ 8 \\ 0 \end{pmatrix} + a \begin{pmatrix} -3 \\ -2 \\ -4 \\ 1 \end{pmatrix} : a \in \mathbb{R} \right\}$

solution: put all pivot variables on one side, and $\boxed{\text{non-pivot}}$ on the other-side, add a trivial equation to obtain a system where its easy to read off the solution

free variables

ex20

$$\left(\begin{array}{cccc|c} 1 & 0 & 7 & 0 & 4 \\ 0 & 1 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$x + 7z = 4$$
$$y + 3z + 4w = 1$$

$$x = 4 - 7z$$
$$y = 1 - 3z - 4w$$
$$z = z$$
$$w = w$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \underbrace{\left\{ \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 7 \\ 3 \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} 0 \\ -4 \\ 0 \\ 1 \end{pmatrix} \right\}}_{\uparrow \text{ soln set}} \quad z, w \in \mathbb{R}$$

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1. $w + 3a = 1$
$x + a = 2$
$y + a = 3$
$z + 2a = 0$
$a = a$

$$\begin{pmatrix} w \\ x \\ y \\ z \\ a \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} + a \begin{pmatrix} -3 \\ -1 \\ -1 \\ -2 \\ 1 \end{pmatrix}$$

2. homogeneous.
trivial soln : all 0

3. $a + b + d + f = 1$
$c + 2d + 2f = -1$
$e + 3f = 1$
$2f = -2$
$g = 1$

$a + b + d = 2$
$c + 2d = 1$
$e = 4$
$f = -1$

$a = 2 - b - d$
$c = 1 - 2d$
$d = d$
$e = 4$
$f = -1$
$g = 1$
$b = b$

$$= \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 4 \\ -1 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + d \begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

2.

$$\left(\begin{array}{ccccc|c} 2 & 5 & -8 & 2 & 2 & 0 \\ 6 & 2 & -10 & 6 & 8 & 6 \\ 3 & 6 & +2 & 3 & 5 & 6 \\ 3 & 1 & -5 & 3 & 4 & 3 \\ 6 & 7 & -3 & 6 & 9 & 9 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} -1 & -1 & -10 & -1 & -3 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 7 & 0 & 1 & 3 \\ 3 & 1 & -5 & 3 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 10 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 7 & 0 & 1 & 3 \\ 0 & -2 & -20 & 0 & -5 & -15 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 10 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 7 & 0 & 1 & 3 \\ 0 & 0 & 0 & +2 & +11 & +27 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$