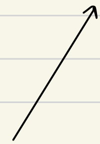
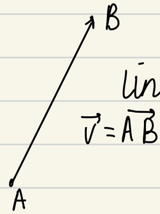
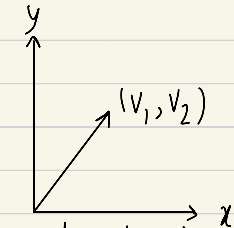
Geometric Vectors.



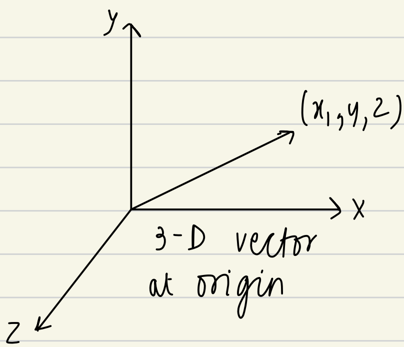
line



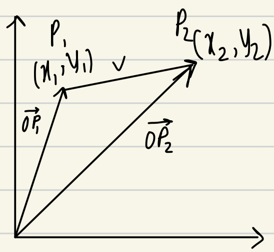
line w/ points



ordered pair
cartesian plane



3-D vector
at origin



$\vec{P_1 P_2}$: v : $(x_2 - x_1, y_2 - y_1)$ (origin)
: $P_2 - P_1 \rightarrow$ terminal - initial

vector as a tuple :
if n is a positive integer, an ordered n-tuple is a sequence of n real numbers $(v_1, \dots, v_n)$. Set of all n-tuples is called n-space and is donated by R^n

components of a vector : initial & final points

set v/s tuple : $(1,2,2,3) \neq (1,2,3)$ $\{1,2,2,3\} = \{1,2,3\}$
$(1,2,3) \neq (3,2,1)$ $\{1,3,2\} = \{2,1,3\}$

Euclidean Space

R: the field of real numbers

a set over which +, -, ÷, x
an "algebraic structure". eg. real, irration, complex

x : point / vector
x_i : coordinates of x
n : dimension of space (R^n)

$$v = (v_1, v_2, \dots v_n) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = [v_1 \ v_2 \ \dots \ v_n]$$

comma-delimited; column vector; row vector

R^n : real co-ord. space of dimension n
& set of all n-tuples of real numbers

C^n : complex co-ord space of dimension n
& set of all n-tuples of complex numbers

Norm

a function from a vector space over the real or complex numbers to the non-negative real numbers that satisfies $\underbrace{\text{certain}}_{\text{elements of } V}$ $\underbrace{\text{properties}}_{}$ ①.

elements of V
can be
$+, -, \div, \times,$
associative,
commutative,
zero vector;
multiply w/
scalar.

vec. space → over a field F of real/complex nos.

$$p : V \to R$$

f^n f^n output: real number

non-negatively valued.

V : normed vector space
↑
a vector space on which a norm is defined

① $\forall a \in F$ & $u, v \in V$

$p(u+v) \leq p(u) + p(v)$ (triangular inequality)

$p(av) = |a| p(v)$ (absolutely homogenious)

$p(v) = 0 \Rightarrow v = 0$ (positive definite / point-separating)

always gives positive value except when $v = 0$, then 0

extra info: Euclidean norm $= \sqrt{V \cdot V}$

Norm of a Vector

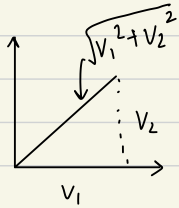
↓

gives length / magnitude of vector

euclidean definition

$v = (v_1, v_2)$

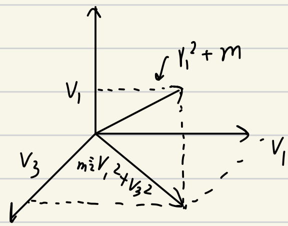
$\|v\| = \sqrt{v_1^2 + v_2^2}$



pythagorus.

$v = (v_1, v_2, v_3)$

$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$



theorem 6.1 if $v \in R^n$, k is any scalar :

$\|v\| \geq 0$ ← its a root.

$\|v\| = 0$ iff $v = 0$ ← its a root of squares (sum ≥ 0)

$\|kv\| = |k| \, \|v\|$

$k(v_1 \dots v_n) = (kv_1 \dots kv_n)$

$\|kv\| = \sqrt{k^2 v_1^2 + \dots + k^2 v_n^2}$

$= \sqrt{k^2 (v_1^2 + \dots v_n^2)}$

$= k \|v\|$

Unit Length Vector

- $k = \frac{1}{\|v\|} v$ k is a unit vector along

  "normalizing" a vector

- $\|v\|^2 = v \cdot v$

- distance b/w vectors

  $u = (7, 1) = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

  $v = (3, 2)$

  $q = (4, -1)$

  $q = u - v$

  $\|q\| = \|u - v\|$ = distance b/w vectors

  ↳ is the norm of their difference

euclidean inner product

Dot Product

- θ is the angle b/w $\vec{u}$ and $\vec{v}$: $0 \leqslant \theta \leqslant \pi$

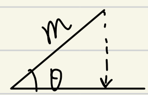
- $\cos\theta = \dfrac{u \cdot v}{\|u\| \, \|v\|}$

$u \cdot v > 0 \rightarrow$ acute
$u \cdot v < 0 \rightarrow$ obtuse
$u \cdot v = 0 \rightarrow$ right

- $u \cdot v = \|u\| \, \|v\| \cos\theta$

- $u \cdot v = \sum_{i=1}^{n} u_i v_i \qquad (u, v \in R^n)$

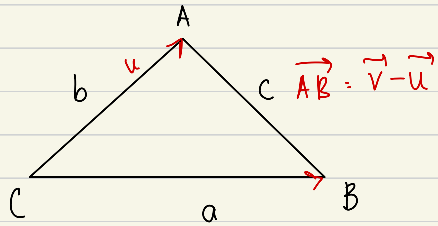
- $u \cdot v = \underbrace{\|u\| \times \cos\theta}_{\text{lengthprojecting } u \text{ on } v} \times \|v\|$ ← length of v



xcord.
$m\cos\theta$

- $u \cdot v = \underbrace{\|v\| \cos\theta}_{\text{length of projection of } v \text{ on } u} \times \|u\|$

- proving component form



$c^2 = a^2 + b^2 + 2ab\cos C$

law of cosines

$$\|\vec{AB}\|^2 = \|u\|^2 + \|v\|^2 + 2\|v\|\|u\|\cos\theta$$

$$\|u\|\|v\|\cos\theta = \frac{1}{2}\left(\|u\|^2 + \|v\|^2 - \|v-u\|^2\right)$$

$$u\cdot v = \frac{1}{2}\left(\sum u_i^2 + \sum v_i^2 - \sum (v_i - u_i)^2\right)$$

$$u\cdot v = \frac{1}{2}\left(\sum 2v_iu_i\right)$$

$u^2+v^2-u^2-v^2+2uv$

$$u\cdot v = \sum v_iu_i$$

- properties

→ $u\cdot v = v\cdot u$ (symmetric) (com)
→ $u\cdot(k+w) = u\cdot k + u\cdot w$ (dist.)
→ $k(u\cdot v) = ku\cdot v = kv\cdot u$ (homogen.) (ass.)
→ $v\cdot v \geq 0$ & $v\cdot v = 0$ iff $v = 0$

similar to norm

· assume u & v are n×1 matrices, A is an n×n matrix

$(Au)\cdot v = v^T(Au) = (A^Tv)^Tu = uA^Tv$

$u(Av) = (Av)^Tu = v^TA^Tu = v^T(A^Tu) = A^Tuv$

$Au\cdot v = u\cdot A^Tv$

$uAv = A^Tuv$

· $u\cdot v = u^Tv = v^Tu$

↳ transforming column vector to row so that multiplication is possible

Cauchy-Schwarz Inequality

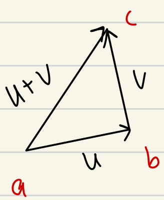
$$|u\cdot v| \leq \underbrace{\|u\|\|v\|}_{+ve}$$

$$u\cdot v = \|u\|\|v\|\cos\theta$$

$0<1$

$$-\|u\|\|v\| < u\cdot v < \|u\|\|v\|$$

triangle inequality



$$\|u+v\| \leq \|u\| + \|v\|$$

$$\|u+v\|^2 = (u+v)(u+v) = (u\cdot u) + 2(u\cdot v) + (v\cdot v)$$

$$\|u\|^2 + 2(u\cdot v) + \|v\|^2$$

$d(a,c)$

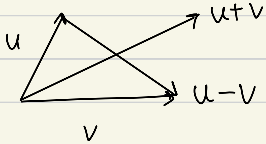
$$\leq \|u\|^2 + 2|u\cdot v| + \|v\|^2$$

$\leq d(a,b)+d(b,c)$

$$\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$$

$$= (\|u\|+\|v\|)^2$$

parallelogram



$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

$$\downarrow$$

$$(u+v)\cdot(u+v) + (u-v)\cdot(u-v)$$

$$2(u\cdot u) + 2(v\cdot v)$$

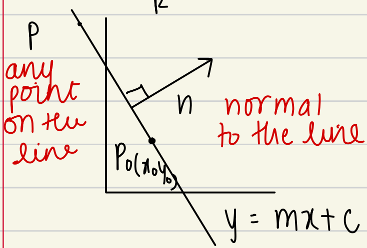
$$2\|u\|^2 + \|v\|^2$$

Chapter 7
Orthogonality

- 2 non-zero vectors u, v in R^n are orthogonal if $u \cdot v = 0$
- the zero vector in R^n is orthogonal to every other vector in R^n
- $\theta = \cos^{-1}\left(\frac{u \cdot v}{\|u\|\|v\|}\right) = 90°$ 0 ≠ 0 (non zero)

Lines and Planes by Points and Normals

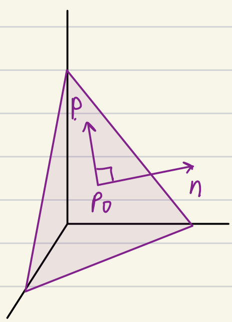
R^2



$(\overrightarrow{PP_0}) \cdot n = 0$

coefficients of a line or plane are the normal vector.

R^3



$(\overrightarrow{PP_0}) \cdot n = 0$

$$\begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\underbrace{ax + by + cz}_{D} = ax_0 + by_0 + cz_0$$

Orthogonal Complements

- subspace : W
  vector : z
  if z is orthogonal to every vector in W, z is orthogonal to w
  the set of all such z is the orthogonal complement of W → $W^\perp$

  $W^\perp$ → must be a subspace of R^n

  subspace → closed over +, mult w/ scalar, 0

if subspace $L = W^\perp$
$W = L^\perp$

- $A : m \times n \rightarrow m\left\{ \overbrace{\begin{bmatrix} \vdots & \cdots \\ \vdots & \\ \vdots & \end{bmatrix}}^{n} \right.$

  null space : $x : Ax = 0$

  $\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \rightarrow \begin{cases} \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \\ \begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \\ \begin{bmatrix} 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \end{cases}$

$(\text{Row space } (A))^\perp = \text{Null space } (A)$

$A^T = n \times m : \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \qquad A^T x = 0$

$(\text{col space } (A))^\perp = \text{Null space } (A^T)$

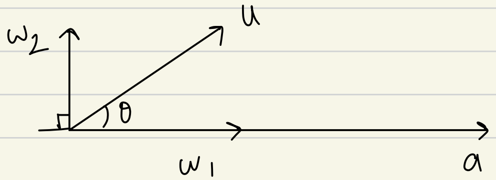
Orthogonal Projection

Decomposing vectors.
↓ into orthogonal vectors

- standard basis for R^n → the set of unit vectors that span R^n
- complex numbers also use orthogonal basis

projection theorem

u, a are vectors in R^n, $a \neq 0$: u can be expressed in exactly one way in the form $u = w_1 + w_2$
w_1 is a scalar multiple of a, w_2 is orthogonal to a



$$\text{proj}_a u = \|u\| \cos\theta = \left(\frac{u \cdot a}{\|a\|^2}\right) a$$

$$w_2 = u - w_1 = u - \text{proj}_a u = u - \frac{u \cdot a}{\|a\|^2} \cdot a$$

ex. $u = (2, -1, 3)$
$a = (4, -1, 2)$

$$\text{proj}_a u = \|u\|\cos\theta = \frac{u \cdot a}{\|a\|^2} a = \frac{5\cancel{15}}{7\cancel{21}}(a)$$

$16 + 1 + 4 = \sqrt{21}$

$$w_2 = (2, -1, 3) - \frac{5}{7}(4, -1, 2)$$

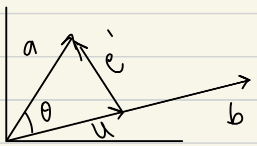
proof:

$u = \text{proj}_b\, a$ (proj a on b)

$= \dfrac{a \cdot b}{\|b\|^2}\, b$

try

one way: use cosine law for θ



$e = a - u$ "the residual"

e is ortho to b

$e \cdot b = 0$

$(a - u) \cdot b = 0$

$a \cdot b - u \cdot b = 0$

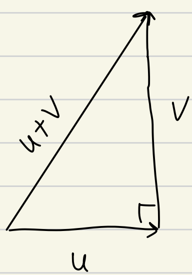
$a \cdot b = u \cdot b$

$(a \cdot b) b = (u \cdot b) b$

$(a \cdot b) b = u (b \cdot b)$

$u = \dfrac{(a \cdot b)}{b \cdot b}\, b = \dfrac{(a \cdot b) \cdot b}{\|b\|^2}$

pythagoreus theorem



$\|u+v\|^2 = \|u\|^2 + \|v\|^2$

$+ 2\|u\|\|v\|\cos\theta = 0$

$+ 2u \cdot v$

$(u+v) \cdot (u+v)$

Orthogonal Sets

- a set of vectors in R^n is an orthogonal set iff every pair $u_i \cdot u_j = 0$ $i \neq j$, $u_i \neq 0$

  linearly independent

  basis for subset

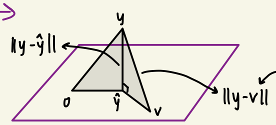
- ex. standard basis

proof

$$0 = c_1 u_1 + \cdots + c_p u_p \quad \text{(linearly independent)}$$
$$0 = 0 \cdot u_1 = (c_1 u_1 + \cdots c_p u_p) u_1$$
$$0 = c_1(u_1 \cdot u_1) + \underbrace{c_2 u_2 \cdot u_1 \cdots + c_p u_p \cdot u_1}_{0}$$
$$0 = \underbrace{c_1 (\|u_1\|^2)}_{\|u_1\|^2 \neq 0}$$

$\therefore c_1 = 0$

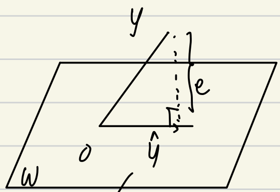
do $\forall u \in S$



hypo has to be > side by pythagorus

Projecting a vector on a subspace

Span by an orthogonal basis $\rightarrow \|u_i\| \neq 1 \ \forall u_i$



$\hat{y} = \text{proj}_W y$

W is a subspace

$\|y - \hat{y}\| < \|y - v\|$

$v \varepsilon (W - \hat{y})$

closest point to y in W

$e = y - \hat{y} \quad , \quad e \perp W$

$y = e + \hat{y}$

example R^5 : orthogonal basis of R^5 : $\{u_1 \dots u_5\}$

orthogonal set → linearly independent

$y = c_1 u_1 + \dots + c_5 u_5 \qquad (y \in R^5)$

$c_i = \dfrac{y \cdot u_j}{u_j \cdot u_j}$

let $W = \text{span}\{u_1, u_2\}$

called orthogonal decomposition

$y \longrightarrow \text{proj}_W y$ & orthogonal to W

$\underbrace{c_1 u_1 + c_2 u_2}_{} + \underbrace{c_3 u_3 + c_4 u_4 + c_5 u_5}_{}$

$\hat{y} \ \varepsilon \ W \qquad\qquad e \ \varepsilon \ W^\perp$

prove accurate : $\hat{y} \cdot (e) = 0$

$\rightarrow e \perp \text{to } \hat{y} \rightsquigarrow e(u_1) = e(u_2) = 0$

ex

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad S' = \begin{array}{ccc} u_1 & u_2 & u_3 \\ \end{array} \begin{bmatrix} 3 & -1 & -1/2 \\ 1 & 2 & -2 \\ 1 & 1 & 7/2 \end{bmatrix}$$

$$y = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$$

y orthognally decomposed:

$$y = c_1 u_1 + c_2 u_2 + c_3 u_3$$

$$c_i = \frac{y \cdot u_i}{\|u_i\|^2} = \frac{11}{11}$$ and so on

Orthonormal Set and Orthonormal Basis

$\searrow \|u_i\| = 1 \ \forall \ u_i$

- an orthogonal set with all unit length vectors

- a matrix U $m \times n$ has orthonormal columns iff $U^T U = I$

$n \times m \quad m \times n$

$I_{n \times n}$

- or orthonormal
  orthogonal matrix : $UU^T = U^T U = I$
  must be square $\therefore U^T = U^{-1}$

- if $U\ (m \times n) \xrightarrow{\text{has}} n$ orthogonal cols $(m > n)$

$$\begin{bmatrix} u_1 \cdot u_1 & 0 & 0 \\ 0 & u_2 \cdot u_2 & 0 \\ 0 & 0 & u_3 \cdot u_3 \end{bmatrix}$$

$U^T U = I_{n \times n}$

$UU^T = m \times m = P$ $\nearrow$ rank n

$\searrow$ spans columnspace of U $\rightarrow Ux = b$

$\searrow$ all linear combos of columns of U

$\searrow$ cols are linearly independent and span set of b

property of projection matrix: $P \times P = P$

projection matrix

$\hat{y} = UU^T y = Py$

$\uparrow$ $\text{proj}_u y$

properties of orthonormal columns

- $\|Ux\| = \|x\|$
- $(Ux) \cdot (Uy) = x \cdot y$
- $(Ux) \cdot (Uy) = 0$ iff $x \cdot y = 0$

mapping $x \mapsto Ux$ preserves length and orthogonality and angle theorem 10 if $\{u_1, u_2 \dots u_p\}$ is an orthonormal basis for subspace W of R^n

$$\text{proj}_W y = (y \cdot u_1)u_1 + (y \cdot u_2)u_2 + (y \cdot u_3)u_3 + \dots$$

$U = [u_1 \; u_2 \; \cdots \; u_p]$ ($u_1, u_2 \dots$ are column vectors of U)

then $\text{proj}_W y = U U^T y$

$U \to u$, $U^T y \to c$

$$\begin{bmatrix} u_1 \rightarrow \\ \vdots \\ u_p \rightarrow \end{bmatrix}_{p \times n} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n \times 1} = \begin{bmatrix} u_1 \cdot y \\ \vdots \\ u_p \cdot y \end{bmatrix}_{p \times 1}$$

$$U \cdot (U^T y) = \underbrace{[u_1 \dots u_p]}_{1 \times p} \begin{bmatrix} u_1 \cdot y \\ \vdots \\ u_p \cdot y \end{bmatrix}_{p \times 1} = (y \cdot u_1)u_1 + \dots$$

1 value scalar

economy decomp
3x2 → Q can be rectangle or same as A

full decomp
Q is square

QR decomposition

- $A_{m\times n} = Q_{m\times n} R_{n\times n}$

properties

Q
- $C(Q) == C(A)$
- $Q^T Q = I \Rightarrow$ Q has orthogonal columns
- QQ^T = projection matrix into col(A)

- R is an upper triangular matrix
  ↳ if A cols are dependent, R is not invertible
  if A cols are independent, R is invertible

theorem for independent column only

$A_{m\times n}$ w/ linearly independent columns, then A can be factored as $A = QR$, where Q is an $m\times n$ matrix whose columns form an orthogonal basis for column space of A and R is a $n\times n$ upper triangular invertible matrix with positive entries on its diagonal

use:

$A = QR$

$$Ax = y$$
$$QRx = y$$
$$Q^TQRx = Q^Ty$$
$$Rx = Q^Ty$$

upper Δ ∴ easy to solve for x cause back substitution why is $C(Q) =: C(A)$

columns span the same vector space

col of q are orthonormal $\Rightarrow \|q_1\| = \|q_2\| = 1$

and $q_1 \cdot q_2 = 0$

orthonormal basis for $\underline{\text{range}(A)}$

same as column space

$$Ax = b$$
$$QRx = b$$
$$Q^T Q Rx = Q^T b$$
$$Rx = Q^T b$$

$$A \xrightarrow{\text{orthogonalise}} Q$$

projecting y on orthogonal v/s orthonormal basis defined space

orthogonal $\rightarrow$ $u_i \cdot u_j = 0$ $i \neq j$ $|u_i| \neq 1$

$$\hat{y} = \frac{y \cdot u_1}{u_1 u_1} u_1 + , \ldots + \frac{y \cdot u_p}{u_p u_p} u_p$$

$$y = \underset{\downarrow}{\hat{y}} + \underset{\searrow}{z}$$

W $\quad$ $W^{\perp}$

$$W = \text{Span}\{\underbrace{u_1 \ldots u_p}_{\text{orthogonal basis of } W}\}$$

orthonormal $\rightarrow$ $u_i \, u_j = 0$ $i \neq j$ $|u_i| = 1$

$$\hat{y} = y \cdot u_1 \cdot u_1 + \ldots + (y \cdot u_p) \cdot u_p$$

$$U = [u_1 \ldots\ldots u_p]$$

UU^T = projection matrix

$$\hat{y} = UU^T y$$

Gram - Schmidt
QR decomposition

$A = QR$

doesnt matter if cols are dependent or independent. they are simply orthonormal

R invertibility depends on A's columns dependency

confirm difference — If A is square, Q is ~~orthonormal~~ / orthogonal matrix

if Q $m \times n$ $(m \neq n)$ $\longrightarrow$ economy QR factorisation

$Ax = \hat{y}$ — $Q^TQ = I_{n \times n}$

$x = (QR)^{-1} QQ^T y$ — QQ^T = proj matrix

$R^{-1}Q^{-1}QQ^Ty$ — $\hat{y} = QQ^Ty$ $\leftarrow$ " least squares error approximation of y in the column space of A "

$= R^{-1}Q^Ty$

nearest pt to y in the col space of A and Q

A — basis $\{x_1, \dots, x_p\} \rightarrow$ non-zero subspace W of R^n

$v_1 = x_1$

$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} \cdot v_1 \quad \leftarrow \in x_1^\perp$

$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} \cdot v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} \cdot v_2$

$\vdots$

Q — then $\{v_1 \dots v_p\} \rightarrow$ orthogonal basis for W

forming on orthonormal basis from orthogonal ones:
just normalize all the vectors

$$A = [x_1, x_2 \dots x_n]$$

$$Q = [\text{normalized vectors}] = [u_1 \dots\dots u_n]_{m \times n}$$

$$R = \begin{bmatrix} u_1 x_1 & u_1 x_2 & u_3 x_3 & \cdots \\ 0 & u_2 x_2 & u_2 x_3 & \cdots \\ 0 & 0 & u_3 x & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$x_1 = u_1 x_1 u_1$$
$$x_2 = u_1 x_2 u_1 + u_2 x_2 u_2$$
$$x_3 = u_1 x_3 u_1 + u_2 x_3 u_2 + u_3 x_3 u_3$$
$$x_n = \sum_{j=1}^{n} u_j \cdot x_n\, y_j$$

when a has dependent columns
if $q_j = 0 \Rightarrow a_j$ is dependent on $a_1, \dots a_{j-1}$

R diagonal may not be all +ve, can be 0

??

- full factorisation when A is tall

$A_{m \times n}$ $\quad m > n$

$A_{m \times n} = \underset{m \times m}{Q} \; \underset{m \times n}{R}$ ← rect.

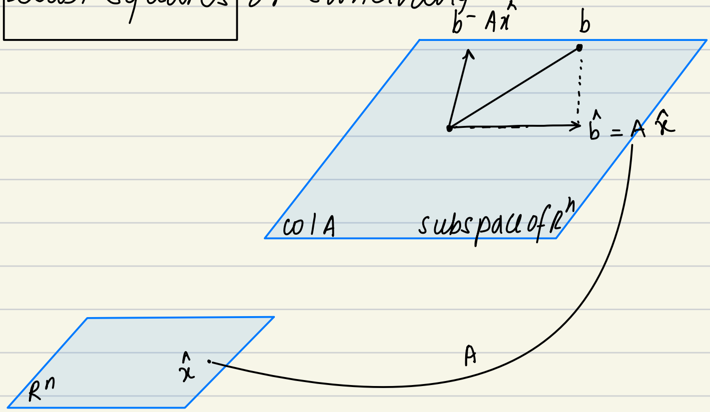
$A = Q_1 R_1$

$$A = \underbrace{\begin{bmatrix} Q_1 & Q_2 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} R_1 \\ 0 \end{bmatrix}}_{R}$$

Q_1 : spans col space of A
Q_2 : span⊥ subspace of A

???

$\boxed{\text{least squares}}$ or something



$\hat{x} = (A^TA)^{-1}A^Tb$ ← least squares soln $\hat{x}$

$\hat{b} = A\hat{x}$

$\hat{b} = \underbrace{A(A^TA)^{-1}A^T}_{QQ^T} b$

Lecture Notes

ortho
"vectors orthogonal to a subspace"
dot product is imp for deep learning

least sq.
least square gives approx soln.
↳ helps w/ regression

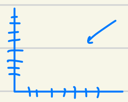
Eigenvalue helps w/ dimension reduction
from what ive understood, cuts down vector we use to split data in decision trees and such.

—x—

$\|v\| \geq 0$ ←
$\|v\| = 0 \Leftrightarrow v = 0$
$\|kv\| = k\|v\|$

$\|v\|$ = norm = length / mag : $\sqrt{v_1^2 + v_2^2 + v_3^2 \dots + v_n^2}$

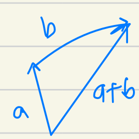
euclidenspace : R^n · n dimensional



equally divided

· addition result
tail → head tail → head
tail ——→ head

triangle rule

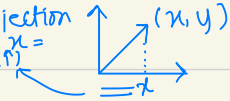


· $u \cdot v = \|u\| \|v\| \cos\theta$ ← geometric formula

· $\cos\theta = \dfrac{u \cdot v}{\|u\| \cdot \|v\|}$

· $u \cdot v = \sum_{i}^{n} u_i v_i$

projection on x = $v_x(\hat{i})$

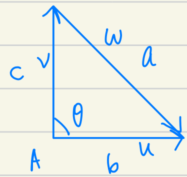


$\therefore$ projection of v on any vector p

$$= v \cdot \frac{p}{\|p\|}$$

- projection : $u \cdot v = \|u\| (\|v\| \cos\theta) = \|v\| (\|u\| \cos\theta)$

- cosine law : $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$



$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$\|w\|^2 = \|u\|^2 + \|v\|^2 - 2\|v\|\|u\|\cos\theta$$
$$\|w\|^2 = \|u\|^2 + \|v\|^2 - 2\, u \cdot v$$

$\cos\theta = 1 \rightarrow \theta = 0 \Rightarrow$ same line

$\cos\theta = 0 \rightarrow \theta = 90 \Rightarrow \perp$

- $uv = vu$

$u(v + w) = uv + uw$

$R(u\, v) = (Ru) \cdot v$

$v \cdot v \geq 0$ & $v\, v = 0$ iff. $v = 0$

$v \cdot v = v_1^2 + v_n^2 \cdots$

$\hookrightarrow = \|v\|^2$

transformation on dot product.

$Au \cdot v = u \cdot A^T v$

$u \cdot Av = A^T(u) \cdot v$

$u \cdot v = u^T v$

$(AB)^T = B^T A^T$

Additional notes:

- Differences between LU and QR factorization
  - LU is applied to any square matrix, QR is applied to a matrix with independent columns
  - LU factorization produces an upper-triangle and a lower-triangle matrix
  - QR factorization produces an orthonormal matrix and an upper-triangle
  - Find LU factorization through Gaussian elimination
  - Find QR factorization using the Gram-Schmidt algorithm
  - Different use cases:
    - LU factorization is used to find solutions of systems of linear equations, matrix inversion, and matrix determinant.
    - QR factorization is used in least-squares, eigenvalue, and signal processing.