4.1 Vector Spaces

- **vector space**: non-empty set V of vectors on which 2 operations are defined: ① addition and ② multiplication by scalars subjelt to the following axioms

  - $u+v$ is in V
  - $u+v = v+u$
  - $(u+v)+w = u+(v+w)$
  - there is a 0 in V : $0+u = u$
  - for each u in V, there is a vector $-u$ in V : $u+(-u) = 0$
  - $cu \in V$ (c = scalar)
  - $c(u+v) = cu + cv$
  - $(c+d)u = cu + cd$
  - $c(du) = cdu$
  - $0u = 0$

examples

- $R^n \rightarrow \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ each entry is a real number

- S: space of all $\underbrace{\text{doubly infinite}}_{\rightarrow \text{ both } \infty\text{'s}}$ sequences of numbers
  $\{y_k\} = (\ldots, y_{-1}, y_0, y_1, \ldots)$ ← 1 sequence
  $\{z_k\}$ ← another sequence

addition : if $\{y_k\}, \{z_k\} \in S$, $\{y_k + z_k\} \in S$
↳ result of adding corresponding terms of $\{y_k\}$ and $\{z_k\}$

scalar multiplication : $c\{y_k\} = \{cy_k\}$

- P_n : polynomials of degree n $(n > 0)$

  $p(t) = a_0 + a_1 t + a_2 t^2 \dots + a_n t^n$

  $\hookrightarrow \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}$ if all coeff $= 0$, "zero polynomial"

  $\downarrow$

  $\equiv$ zero vector

- V = set of real-valued functions on set D

  $(f+g)(t) = f(t) + f(g)$
  $c(f(t)) = f(ct)$

- set of 2x2 matrices

4.2 Subspaces

- subspace of a vector space V : subset H w/ 3 properties :

    - 0 vector of V is in H
    - H is closed under addition
    - H is closed under scalar multiplication

every vector space is a subspace of itself.

- subspace of R^n
  → set of solns of homogeneous linear eq^{n}s
  → set of all linear combinations of certain specified vectors

examples

- zero subspace : $\{0\}$

- P : set of all polynomials w/ real coefficients
  P_n is a subspace of P (for each $n \geq 0$)

- is R^m a subspace of R^n $(m<n)$
  no. R^m has m elements / unit vectors / variables while R^n has n. R^m is not even a subset of R^n

$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real} \right\}$ is a subset and a subspace

T.S.T H is a subspace of R^3

$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real} \right\}$ is a subset and a subspace

- $s, t = 0 \rightarrow$ zero vector . a fulfilled
- $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}, \begin{bmatrix} c \\ d \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} a+c \\ b+d \\ 0 \end{bmatrix}$ $a, c \in R \Rightarrow a+c \in R$ same for b, d b. fulfilled
- $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \rightarrow c \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} ca \\ cb \\ 0 \end{bmatrix}$ $ca, cb \in R$ c. fulfilled.

4.3 Subspace spanned by a set

· for v_1, v_2 in V, let $H = \text{span}\{v_1, v_2\}$

S.T. H is a subspace of V

let $\vec{v} \in H = av_1 + bv_2$

$a, b = 0 \rightarrow a$ ✓

theorem 4.1 If $v_1, \dots v_p$ are in vector space V, $\text{Span}\{v_1, \dots v_p\}$ is a subspace of V.

ex. 4.3.2 $H : \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix}$ $a, b \in \mathbb{R}$

S.T. H is a subspace of $\mathbb{R}^4$

$$H = \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{v_1} + b\underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{v_2} \in \mathbb{R}^4$$

clearly the span of v_1, v_2

$\therefore H$ is a subspace of $\mathbb{R}^4$ by 4.1

lecture ques : 8/2/23

$V = R^2$

$\hookrightarrow \left\{ \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}, \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}, \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} \dots \right\}$

$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

add : $\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$ mult. $k\vec{u} = \begin{bmatrix} ku_1 \\ 0 \end{bmatrix}$

is V a vector space?

$1\vec{u} = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} \neq \vec{u}$ $\therefore$ doesn't fulfill last axiom and is not a vector space.

4.4 Null Space of a Matrix

- Null space : set of x that satisfies $\overset{m \times n}{A}x = 0$
  aka all soln of the homogeneous eq. $Ax = 0$

$$N(A) = \{ \underset{n \times 1}{x} : x \text{ is in } R^n \text{ and } Ax = 0 \}$$

all x in R^n mapped into the 0 vector in R^m via linear transformation $x \mapsto Ax$

- for $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ 3x_1 + 6x_2 &= 0 \end{aligned} \rightarrow \left. \begin{aligned} x_1 + 2x_2 &= 0 \\ 0 & \end{aligned} \right]$$

$A' = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

aug : $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$x_1 + 2x_2 = 0 \rightarrow x_1 = -2x_2$$
$$x_2 = x_2$$

theorem 4.2 The null space of an $m \times n$ matrix A is a subspace of R^n. Equivalently, the set of all soln of the m homogeneous equations $Ax = 0$ in n unknowns is a subspace of R^n

proof.. t.p. subspace . satisfy a : has 0
b closed (+)
c closed (*)

→ $N(A) \in R^n$ ∵ A has n columns
→ $\vec{0}$ is in $N(A)$ ∵ $A\vec{0} = \vec{0}$
→ $\vec{u}, \vec{v} \in N(A)$, $A(\vec{u} + \vec{v}) = 0$ ∴ ✓
→ $cA(\vec{u}) = 0$

∴ $N(A)$ is a subspace of Rn

· Spanning set of a null space of a matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

$\downarrow$ RRE

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{aligned} x_1 - 2x_2 \qquad - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

x_2, x_4, x_5 are free variables

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underset{u}{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + x_4 \underset{v}{\begin{bmatrix} +1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}} + x_5 \underset{w}{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}$$

$\downarrow$

linear combo of u, v, w

Spanning set : Span$\{u, v, w\}$

4.5 The Column Space of a Matrix

- Column space of $m \times n$ matrix A : $C(A)$, set of all linear combinations of the columns of A.
  if $A = [a_1 \dots a_n]$ then $C(A) = \text{Span}\{a_1, \dots a_n\}$

  $C(A) = \{\underbrace{b}_{m \times 1} : b = Ax \text{ for some } x \text{ in } R^n\}$

theorem 4.3 The column space of an $m \times n$ matrix A is a subspace of R^m

proof Span$\{a_1, \dots a_n\}$ is a subspace (theorem 4.1)
Cols of $A \in R^m$ (m entries in A)
$\therefore$

$\updownarrow$

cols of A span R^m iff $Ax = b$ has a solution for each b in R^m

$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$ $b = .4\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + .3\begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$ $x = \begin{bmatrix} .4 \\ .3 \end{bmatrix}$

forms a plane

$\downarrow$

$C(A)$ = linear combo of cols of A

example sums

R^m

$m \times n$

$C(A) \quad N(A)$

1. $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}_{3 \times 4}$

- $C(A)$ is a subspace of R^k $\quad k = ? = 3$

  → $b : Ax = b$

  $Ax = b$

  $4 \times 1 \quad 3 \times 1$

- $N(A)$ is a subspace of R^k $\quad k = ? = 4$

  → $x : Ax = 0$

  $Ax = 0$

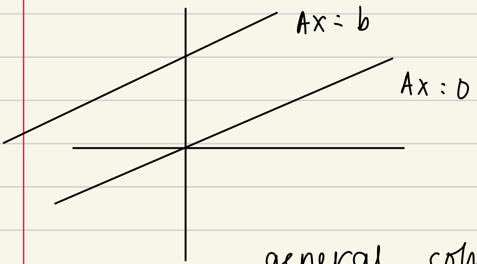
  $3 \times 4 \quad 4 \times 1 \quad 3 \times 1$

- nonzero vector in $C(A)$ = linear combo of cols = $\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$

- ——— " ——— $N(A)$ → solve for $Ax = 0$

N(A)	**C**(A)
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

extra lecture knowledge



or parallel planes

general soln : sum of particular soln
+ soln of homogeneous soln

1. $T : R^3 \to R^3$ projects pts in R^3 onto x-y plane

kernel = x in domain ST $T(x) = 0$

↑ all points on z axis

range = all b in codomain ST. $T(x) = b$

$\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$

↑ all pts in x-y plane

2. T : rotation through angle θ

ker (T) = 0   rotate 0 to zero, nothing else

range (T) = x, y plane aka R^2 ya baby.

3. Det whether given set spans P_2

$S = \{1+x+x^2,\ -1-x,\ 2+2x+x^2\}$ vector space of poly $\leq$ deg2

✓ x^2

✓ $1+x$ no it doesnt.

assume $a+bx+cx^2 \in P_2$

$$a+bx+cx^2 = k_1(1+x+x^2) + k_2(-1-x) + k_3(2+2x+x^2)$$

$$\begin{aligned} k_1 - k_2 + 2k_3 &= a \\ k_1 - k_2 + 2k_3 &= b \\ k_1 + k_3 &= c \end{aligned} \longrightarrow \begin{bmatrix} 1 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

solving for k_1, k_2, k_3

↳ $|A| = 0$

↳ not inv

↳ no span

4.6 Kernel and range of a Linear Transf.

- linear transformation T from a vector space V into a vector space W is a rule that assigns to each vector x in V, a unique vector $T(x)$ in W:

$T(u+v) = T(u) + T(v)$
$T(cu) = cT(u) \quad \forall\, u \in V$ & all scalars c

kernel $\equiv$ null space of T = set of all u in $V : T(u) = 0$

range $\equiv C(A)$ = column space = set of all $W = T(X)$ for some x in V

4.7 Basis

- Linear independence :

V is linearly independent if : $x_1v_1 + x_2v_2 + \dots + x_pv_p = 0$

has the trivial soln

all weights must $= 0$ for eqn $= 0$

- $V = \{v\}$: linearly independent iff. $v \neq 0$
- $V = \{v_1, v_2\}$: linearly dependent iff $v_1 = cv_2$ scalar multiples
- $V = \{\vec{0}, v_1, v_2 \dots\}$: linearly dependent

$$c\vec{0} + c_1v_1 + c_2v_2 \dots = 0 \quad \text{if } c \in R \ \& \ c_i = 0$$

not zero

theorem 4.4 An indexed set $\{v_1 \dots v_p\}$ of 2/+ vectors, with $v_1 \neq 0$ is linearly dependent iff some v_j $(j > 1)$ is a linear combination of preceding vectors $v_1, \dots v_{j-1}$

has $\vec{0}$ vect, subset, closed over addition, closed over multiplication

- H is a subspace of vector space V.
An indexed set of vectors $\beta = \{b_1, \dots b_p\}$ in V is a basis for H if
i) β is linearly independent ($\beta x = 0$ has only trivial soln)
ii) subspace spanned by β coincides with H
↳ $H = \text{span}\{b_1 \dots b_p\}$
↳ every linear combination of $\vec{b}_1 \dots \vec{b}_p$

- true when $H = V$
basis of V is a linearly independent set that spans V

eg.
- $\{\vec{i}, \vec{j}\}$: basis for R^2

- inv. matrix $A = [a_1 \dots a_n]$
↳ cols are linearly independent
↳ they span R^n

- cols of I_n → I_n is inv. ∴ yes ofc.
↳ $e_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix}$ $e_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \end{bmatrix} \dots$
set $\{e_1, e_2 \dots e_n\}$ = standard basis for R^n

- $S = \{1, t, t^2, \dots, t^n\}$: standard basis for P_n

- to see if a 3x3 matrix's cols can form the basis of R^3, check if it is inv. ⇒ $|A| \neq 0$

4.8 Spanning Set Theorem

theorem 4.5 let $S = \{v_1, \dots, v_p\} \in V$, let $H = \text{Span}\{v_1 \dots v_p\}$

a) If one of the vectors in S, say v_k, is a linear combination of the remaining vectors in S, then the set formed from S by removing v_k still spans H.

b) If $H \neq \{0\}$, some subset of S is a basis for H

4.9 Bases for N(A) & C(A)

N(A)
- → all x that satisfy $Ax = 0$
- → these x are linearly independent (provided $N(A) \neq \vec{0}$)
- → basis for $N(A) \equiv$ spanning set of $N(A)$ = soln of $Ax=0$

C(A)
- → all b : $Ax = b$
  - ↳ linear combo of cols of A
- → span$\{a_1 \dots a_n\}$ where $a_i \neq$ linear combo of rest
  - ↳ = basis for C(A)

theorem 4.6 pivot columns of A form a basis for C(A)

- B is RREF of A
- pivot cols are linearly independent (of A & B)
- A ≡ B ∵ R operations do not change row relations
- every non-pivot column of A is a linear combo the the pivot cols
  ↑
  so remove.
- ∴ pivot cols are basis for C(A)

lecture

1. Basis for $C(A)$?

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & 6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

$$RREF(A) = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

solve to get $k_1, k_2 = 0$
$\Rightarrow$ they are linearly independent

$$\begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$k_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

after you check

2. $N(A)$ basis. $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & 4 \end{bmatrix}$

$$RREF(A) = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

aug: $\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$x_1 - x_2 + 2x_3 = 0$$
$$x_1 = x_2 - 2x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$N(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

3. Basis of N(A) & C(A)

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 7 & 1 \\ -2 & 1 & 7 \end{bmatrix} \qquad RREF(A) = \begin{bmatrix} 1 & 3 & -2 \\ 0 & -2 & 7 \\ 0 & 0 & 55/2 \end{bmatrix}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3$

basis $C(A) = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$

provided a_1, a_2, a_3 are linearly independent

$\Rightarrow |A| \neq 0$ (A has to be inv.)

basis N(A) $\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$

$x_1 = -3x_2 + 2x_3$

$x_2 = \frac{-7}{2} x_3$

$x_3 = 0$

all cols have pivot

$\Rightarrow$ only trivial soln

4. $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ ← linearly independent?

$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$

can check by det. $\quad |v_1\, v_2\, v_3| = 0 \rightarrow$ not inv.

↑ only for square

$\Rightarrow$ non trivial soln. to $Ax = 0$

$\Rightarrow$ linearly dependent.

past year questions

PYQ

Nov-Dec '21

2(b) $B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ $B^T = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 0 \end{bmatrix}$ basis of $N(B^TB)$

$B^TB = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 5 \\ 4 & 5 & 10 \end{bmatrix} \xrightarrow{\text{EROs}} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}$

basis of $N(A)$

Dimension = $n(N(A))$ here, 1

T/F 1. col vector of a 3×5 matrix are linearly dependent : T

2. $T: R^n \to R^n$: $T(x) = 0 \Rightarrow A = \vec{0}_{n \times n}$ (T)

3. A $n \times n$ matrix,

$A\vec{x} = 4\vec{x}$ has a unique soln iff $A - 4I$ is inv.

$A\vec{x} - 4\vec{x} = 0$

$(A - 4I)\vec{x} = 0$

unique soln for homo $\Rightarrow$ trivial

true if inv is existent