- permutation matrix : unit matrix w/ row exchanges
- $\det(I_{n \times n}) = 1$
- row exchanges $\Rightarrow -\det$

ALL THESE PROPERTIES ALSO APPLY TO THE COLUMN

- $\det(\mathrm{Perm}(n)) \Rightarrow (-1)^n$

  n ↑ no. of row exchanges

- det is a linear function of each row separately
- $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ ($t = 0 \Rightarrow 0$ row $\Rightarrow \det = 0$)
- $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$ ← ofc other row is untouched
- subtracting a multiple of a row from another doesn't change the det
- A is $\Delta \Rightarrow |A|$ = product of diagonal elements

  A is diagonal $\Uparrow$

- A is singular $\Rightarrow |A| = 0$

  $A \to U$ (upper Δ) → U has a zero row, $|A| = |U| = 0$

- A is invertible $\Rightarrow U$ has pivots on dig $\Rightarrow |U|$ = product of non-zero elements

  $|A| = \pm |U|$ (row exchanges)

- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ & d-(c/a)b \end{vmatrix} = ad - a\left(\frac{c}{a}\right)b$

$= ad - bc$

- $|AB| = |A||B|$

proof 1 let $D(A) = |AB|/|B|$ if $D(A)$ satisfies rules 1,2,3, then it is a determinant

rule 1 : det of I

if $A = I$, $D(A) = |B|/|B| = 1$

rule 2 : sign reversal

2 rows of A are exchanged $\Rightarrow$ same two rows of $|AB|$ are exchanged $\Rightarrow$ $|AB|$ changes sign $\Rightarrow$ $D(A)$ changes sign

rule 3 : linearity

- 1 row of A is multiplied by t $\Rightarrow$ so 1 row of AB $\Rightarrow$ $|AB|$ is multiplied by t $\Rightarrow$ $D(A)$ is multiplied by t
- 1 row of A is added to 1 row of A' $\Rightarrow$ 1 row of AB is added to $A'B$ $\Rightarrow$ determinants add $\Rightarrow$ dividing by $|B|$, ratios add

AB $\begin{bmatrix} 1 & 1 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 8 & 18 \end{bmatrix}$ $|AB| = -4$

$A'B$ $\begin{bmatrix} 1 & 1 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 10 & 23 \end{bmatrix}$ $|A'B| = -4$

THIS PROOF WONT COME IN THE FINAL EXAM

proof 2

Case 1 $|B| = 0 \Rightarrow B$ is singular $\Rightarrow AB$ is singular

$|AB| = 0$

$|A||B| = 0 \quad \because |B| = 0$

$\therefore |AB| = |A||B|$

$\vec{B}\vec{x} = 0$ has ∞ soln ($\vec{x} \neq 0$) nontrivial

$(AB)\vec{x} = 0 \quad x$ can't be 0

$\therefore$ AB is not inv.

Case 2 $|B| \neq 0$

B is inv $\rightarrow$ $B\vec{x} = 0$ has trivial soln $\rightarrow$ AB is inv.

something that has to be proved before a theorem can be proved

Lemma: for elementary matrix E, $|EA| = |E||A|$

proof. EROs

i) row $\times$ constant $\leftarrow E_1 \quad k \neq 0$

ii) interchange 2 rows $\leftarrow E_2$

iii) add a multiple of one row to another $\leftarrow E_3$

$|E_1| = k|I| = k$

$|E_2| = -|I| = -1$

$|E_3| = |I| = 1$

$|E_1A| = |E_1||A| = k|A|$

$|E_2A| = -|A| = |E_2||A|$

$|E_3A| = |A| = |E_3||A|$ proves that $\det|EA| = |E||A|$

$\because$ B is inv $\Rightarrow$ B can be obtained from A by some EROs

$|B| = |E_n||E_{n-1}| \ldots\ldots |A|$ (by induction)

if $|A| = |E_n||E_{n-1}| \ldots. |I|$ $\leftarrow$ how?

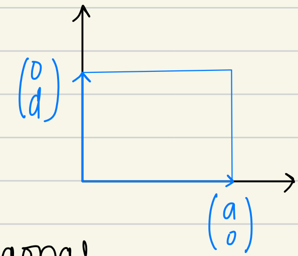
$|AB| = |E_n E_{n-1} \ldots. E_1 I B| = |A||B|$

geometric interpretation

theorem 3.1 A is a 2×2 matrix, area of parallelogram described by the columns of A is $|A|$. If A is a 3x3 matrix, the volume of the parallelopiped determined by the columns of A is $|A|$

proof. $\left\| \begin{matrix} a & 0 \\ 0 & d \end{matrix} \right\| = |ad| =$ area of rect

absolute value

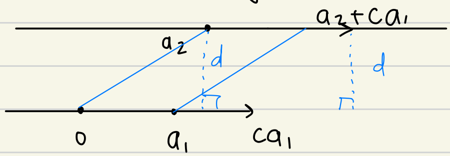


more general matrix A : $[\vec{a}_1\ \vec{a}_2]_{2\times 2}$

ques: can we change A into a diagonal matrix w/ no change in the area of the associated llgram

can! → by ① exchanging 2 columns, ② adding a multiple of one column to another

②. assume $a_2 \neq ca_1$, $a_2, a_1 \neq 0$
then area of llg a_1, a_2 = area of llg $a_1, a_2 + ca_1$



a_2, $a_2 + ca_1$ have the same ⊥ distance to A
∴ (base × height) is the same

linear transformations

"how does the area of the transformed set compare with that of the original"

theorem 3.2 let $T : R^2 \to R^2$ be the linear transformation determined by A. If S is a llgm in R^2,

area of $T(S)$ = abs(|A|) x area of S

if $R^3 \to R^3$ area → volume

proof consider the 2x2 case, $A = [\vec{a}_1, \vec{a}_2]$

llgm at origin

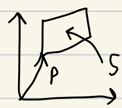
a llg at the origin in R^2 det by vectors b_1, b_2 has the form $S = \{ s_1 b_1 + s_2 b_2 : 0 \leq s_1 \leq 1, 0 \leq s_2 \leq 1 \}$

$T(s_1 b_1 + s_2 b_2) = s_1 T(b_1) + s_2 T(b_2) = s_1 A b_1 + s_2 A b_2$

$\therefore$ $T(S)$ is the llgm determined by columns of $[Ab_1\ Ab_2] = AB$ where $B = [b_1\ b_2]$

area $T(S)$ = abs(|AB|) = abs(|A|) abs(|B|) = abs(|A|) (area of S)

general case. any llgm : $\underset{\text{vector}}{p} + S \to$ llgm at origin



$T(p+S) = T(p) + T(S)$

area of $T(p+S)$ = area(T(p) + T(S))

= area (T(S)) ← translation doesn't affect area

= abs(|A|) (area of S)

= abs(|A|) (area of p+S)

☆☆ theorem 3·2 is applicable ∀ shapes since its just a set of points

lecture quest.

1. Nov / Dec '22 :

T / F. 1 (d) (iii)

If E is the elementary matrix which adds 3 times row 1 to row 2, then E^2 adds 9 times row 1 to row 2.

false $E^2 = E(E)$

add 3x, add 3x

$a + 3 \times 1 + 3 \times 1 = a + 6x$

2. Nov / Dec '22

2 (c) $A_{n \times n} \Rightarrow$ nilpotent if $A^m = 0$ ← zero matrix for some positive integer m

show $I - A$ is not singular

[Hint : Use $I - A^m = (I - A)(I + A + A^2 + \dots + A^{m-1})$]

$A^m = 0$

$$I = \underbrace{(I - A)}_{B} \underbrace{(I + A + A^2 + \dots + A^{m-1})}_{B^{-1}}$$

$\Rightarrow I - A$ is inv

$\Rightarrow A$ is not singular.

P.T. $I - A$ is inv $\Rightarrow A$ is inv ??

$I - A \rightarrow$ EROs $\rightarrow I$