2.1 Matrix Multiplication

- $Ax = T(x)$ is a "linear transformation"
- $A(B(x))$ is a composition of linear transformation
- $T_1 : \vec{x} \to \vec{y} = A\vec{x}$
  $T_2 : \vec{y} \to \vec{z} = B\vec{x}$
  $T : AB(x)$. $AB \neq BA$

2.2a) Inverse of a matrix

theorem 2.1 if A is an invertible $n \times n$ matrix, then for every b in R^n, the equation $Ax = b$ has the unique soln $x = A^{-1}b$

proof. $b \in R^n$

soln exists . sub $A^{-1}b$ in $Ax = b$
$\underline{\underline{x}}$

$A(A^{-1}b) = b = Ib = b$ LHS = RHS

unique soln : if u is the soln , $u = A^{-1}b$ ← S.T.

$Au = b$ $A^{-1}Au = A^{-1}b$
$u = A^{-1}b$

theorem 2.2

1. A is invertible, then A^{-1} is invertible and $(A^{-1})^{-1} = A$
2. if A and B are $n \times n$ inv. matrices, then so is AB and $(AB)^{-1} = B^{-1}A^{-1}$
3. $(A^T)^{-1} = (A^{-1})^T$

2.2b Inverse by EROs

- elementary matrix (E) is one obtained by performing a single ERO on an identity matrix
- for any matrix A, $E \times A = A'$
  $A' = A$ if the same operation was done on A as that on I to get E
- E is invertible, E^{-1} = matrix obtained by doing the inverse of the operation on I

$E: r_3' \rightarrow r_3 - 4r_1 \quad \longleftrightarrow \quad r_3 \rightarrow r_3' + 4r_1 \quad I$

$\therefore E^{-1} : r_3'' \rightarrow r_3 + 4r_1$

theorem 2.3 $n \times n$ matrix A is invertible iff A is row eq. to I_n and $A \xrightarrow{\text{EROs}} I_n$, $I_n \xrightarrow{\text{EROs}} A^{-1}$

same

inv of A: $\quad Ax = b \quad x = A^{-1}, \; b = I$

aug. matrix $= \begin{bmatrix} A & I \end{bmatrix} \xrightarrow{\text{EROs}} \begin{bmatrix} I & A^{-1} \end{bmatrix}$

proof. Part I: suppose A is inv.

RREA is I_n $\Rightarrow$ $A \equiv I_n$ (to show)

pivots along the diagonal

$\Rightarrow Ax = b$ has a soln $\forall \vec{b}$

$\Rightarrow$ pivot in every row (th. 1.2)

$\Rightarrow$ n-rows $\therefore$ n pivot positions has to be on the diag. since A is square

$\Rightarrow$ RRE of A is I ($\because$ RRE $\Rightarrow$ pivots are all 1)

values in upper triangular?

Part 2: suppose $A \equiv I_n$

$$A \sim E_1 A \rightsquigarrow E_2(E_1 A) \sim \cdots \cdot E_p(E_{p-1}(\cdots E_1 A)) = I_n$$

$\underset{\text{1 ERO}}{\Uparrow}$

all E_i are invertible. If A & B are inv, AB is too

$\because$ product of inv. matrices is inv,

$$(E_p \cdots E_1)^{-1}(E_p \cdots E_1 A) = (E_p \cdots E_1)^{-1} I_n$$

$$A = (E_p \cdots E_1)^{-1} I_n$$

$$A = (E_p \cdots E_1)^{-1}$$

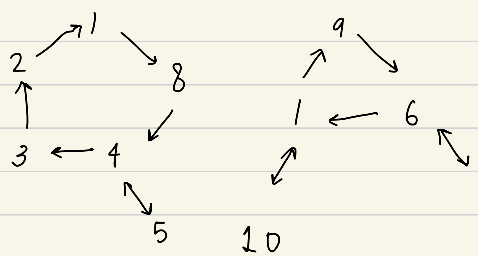
A is inv as it is the inv of an inv. matrix

$$A^{-1} = (E_p \cdots E_1 I_n)$$

theorem 2.4 The invertible matrix theorem

if A is a $n \times n$ square matrix, then the following are equivalent

1. A is invertible
2. A is row eq. to I
3. A has n pivot positions
4. $Ax = 0$ has only trivial solution
5. columns of A are linearly independent
6. $Ax = b$ has at least 1 soln $\forall\, b$ in R^n
7. columns of A span R^n
8. there exists $n \times n$ C : $CA = I$
9. there exists $n \times n$ D : $AD = I$
10. A^T is invertible



$T(Ax) = x$ A^{-1} is the transformer matrix proof. $8 \Rightarrow 4$ (inv $\Rightarrow$ trivial soln)

$\vec{A}\vec{x} = 0 \Rightarrow x = A^{-1}\vec{0} \Rightarrow \vec{x} = \vec{0} \Rightarrow$ trivial

$4 \Rightarrow 3$ (trivial soln $\Rightarrow$ n pivot position)

$\updownarrow$

unique soln $\Rightarrow$ no free variable $\Rightarrow$ every column has a pivot $\Rightarrow$ $n \times n$ $\therefore$ n columns have pivots $\Rightarrow$ pivot in every row

$3 \Rightarrow 2$ (n pivot $\Rightarrow$ row eq. to I_n)

A is square & has n pivot positions $\Rightarrow$ pivots must lie on diagonals $\Rightarrow$ RRE of A is I_n

$2 \Rightarrow 1$ (row eq to I_n $\Rightarrow$ invertible)

has been est.

$9 \Rightarrow 6$ ($AD = I$ $\Rightarrow$ soln for any b)

$AD = I_n \Rightarrow AD\vec{b} = I_n\vec{b} = \vec{b}$

$A(\underbrace{D\vec{b}}_{\vec{X}}) = \vec{b}$

matrix factorization

- matrix multiplication ⇒ synthesis of data.
- A expressed as a product of 2/+ matrices ⇒ analysis of data
  ↓
  matrix factorisation

LU factorisation

- $Ax = b_1$
  $Ax = b_2$
  $\vdots$
  } solving by computing $A^{-1}b_i$ is inefficient.

- factorise $A_{m \times n} = L_{m \times n} \times U_{m \times n}$

  A: assuming A can be reduced to echelon form w/o row interchanges

  L: unit lower Δ

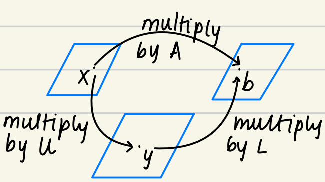
  U: upper Δ

$Ax = b \rightarrow LUx = b \rightarrow Ly = b$ ← ① solve for y by forward substitution

$y = Ux$

↑ ② solve for x by backward substitution

easy because triangular



eg. solve for $Ax = b$

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{U}$$

$$b = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix} \qquad LUx = b \qquad Ly = b$$

$$Ux = y$$

$Ly = b$: $$[L \ \ b] = \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{bmatrix} \xrightarrow{ERO} \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= [I \ \ y]$$

$Ux = y$

$$[U \ \ y] \xrightarrow{ERO} [I \ \ x]$$

LU procedure

- row reduction of A to U produces L w/o extra work

→ assuming A can be reduced to echelon form w/o row interchanges

$E_p \ldots E_1 A = U$ (E_i = unit lower Δ elementary matrices)

$A = (E_p \ldots E_1)^{-1} U = LU$

$\therefore L = (E_p \ldots E_1)^{-1}$ (products & invs of unit lower Δ is unit lower Δ)

$(E_p \ldots E_1) L = I$

$A \xrightarrow{ERO} U$

$L \xrightarrow{ERO} I$

eg. $A = \begin{bmatrix} 2 & -2 & 3 \\ 6 & -7 & 14 \\ 4 & -8 & 30 \end{bmatrix}$

$E_{21} A = \begin{bmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 4 & -8 & 30 \end{bmatrix}$ $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2 - 3R_1$

$E_{31} E_{21} A = \begin{bmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & -4 & 24 \end{bmatrix}$ $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$E_{32} E_{31} E_{21} A = \begin{bmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & 0 & 4 \end{bmatrix}$ $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix}$

↑ U

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 6 & -7 & 14 \\ 4 & -8 & 30 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & -4 & 24 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & 0 & 4 \end{bmatrix} = U$$

$$\begin{array}{ccc} 1 & & 2 \end{array}$$

$$L = \begin{bmatrix} 2 & & \\ 6 & -1 & \\ 4 & -4 & 4 \end{bmatrix} \rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix}$$

divide by pivot in each column

$\uparrow$

$$L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}$$

lecture questions (1st Feb)

Q1 col of B are linearly dep $\Rightarrow$ col of AB are also linearly dependendent

$\vec{B}\vec{x} = \vec{0}$ trivial $\Rightarrow$ independent

dep $\Rightarrow$ $\vec{x} \neq \vec{0}$: $Bx = 0$

$AB\vec{x} = \vec{0}$
$(AB)\vec{x} = \vec{0}$
$\Rightarrow$ AB is not linearly independent

Q2 $D_{7\times7}$: Col are linearly independentent
$Dx = b$ has a unique soln

True $\because$ LI $\Rightarrow$ trivial $\Rightarrow$ 0 $\Rightarrow$ unique soln.
D is inv. $\Leftarrow$ col are lineary indep
$\Leftarrow$ unique soln $\forall$ $b \in R^n$

Q3. col of $A_{n\times n}$ are linearly independent $\Rightarrow$ col of A^2 span R^n

square $\rightarrow$ A is inv

A^2 . product of inv matrix $\Rightarrow$ it is inv.

Q4. if AB is inv, so is B $\leftarrow$ ST

$CAB = I$
$\Rightarrow (CA) = B^{-1}$ associativity
$\Rightarrow MB = I$
$\Rightarrow B^{-1}$ exists

Q. Nov/Dec '22

2 a) $A_{3\times 4}$ soln $A\vec{x}=\vec{0}$ is $\vec{x} = c_1\begin{bmatrix}1\\1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}-2\\-1\\0\\1\end{bmatrix}$

if $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$ find RREF of A

Pivot = 1
rest 0
RRE

$$\underset{3\times 4}{\begin{bmatrix}\boxed{1} & 0 & -1 & 2\\ 0 & \boxed{1} & -1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}}\underset{4\times 1}{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix}0\\0\\0\end{bmatrix}}$$

$x_1 = c_1 - 2c_2$ $\quad$ $x_1 = x_3 - 2x_4$ $\quad$ $x_1 - x_3 + 2x_4$

$x_2 = c_1 - c_2$ $\quad$ $x_2 = x_3 - x_4$ $\quad$ $x_2 - x_3 + x_4$

$x_3 = c_1$ $\quad$ $x_3 = x_3$

$x_4 = c_2$ $\quad$ $x_4 = x_4$

Nov/Dec '21

1 d) $T(R^2 \to R^3)$ $\quad$ $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}$

$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\3\\1\end{bmatrix}$

$\begin{bmatrix}1\\2\end{bmatrix} = 2\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix}$

T

$2\begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}0\\3\\1\end{bmatrix} = \begin{bmatrix}2\\1\\5\end{bmatrix}$

2 (a) $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ $A^2 = kA - 2I$

$\uparrow$

$= ?$

as per usual.