lec 3

- linear system of equations (m eq. in n unknowns) can be written in matrix notation

↓

each column is a vector (or row?)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \cdot & \cdot \\ a_{31} & \cdot & \cdot \end{bmatrix}$$

↓ in the variable matrix

~~in the matrix of coefficients~~

$$Ax = b$$

- vectors $\{v_1, \dots, v_p\}$ in R^m spans R^m if every vector in R^m can be represented as a linear combination of $v_1, \dots, v_p$.

$\equiv$ Span $\{v_1, \dots, v_p\} = R^m$

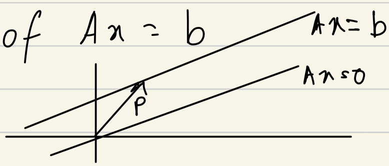
if A has a pivot in every row $\Rightarrow$ its echelon (normal or reduced) will have a pivot in every row too.

$Ax = 0 \;\equiv\; x = tv$

$Ax = b \;\equiv\; x = p + tv$

parametric vector form

p is a particular soln of $Ax = b$



Soln set of $Ax=b$ is $\vec{p}$

not trivial soln iff eq^n has 1 free variable

1.7 MATRIX EQUATION

$$Ax = b$$

coeff matrix $m \times n$; x matrix ; vector in R^m

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

let $\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{matrix} \equiv a_1$ ← a vector in R^m

↖ a column of A

$$Ax = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = x_1a_1 + x_2a_2 + x_3a_3 \ldots + x_na_n$$

$Ax = b$ is the linear combination of the columns of A using corresponding entries in x as weights

$Ax = b$ has a soln only if b is a linear combination of columns of a eg. $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & -2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ $A\vec{x} = b$ consistent?

$$x_1\vec{a_1} + x_2\vec{a_2} + x_3\vec{a_3} = \vec{b}$$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & -2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 10 & 6 & 4b_1 + b_2 \\ 0 & 7 & 5 & 3b_1 + b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 3 & 5 & b_1 + b_2 - b_3 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 10 & 6 & 4b_1 + b_2 \\ 1 & 0 & 3 & 4b_1 + b_3 - b_2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 - b_2 \\ 0 & 0 & 0 & -2b_1 + b_2 - 2b_3 \end{bmatrix}}$$

for $Ax = b$ to be consistent, $-2b_1 + b_2 - 2b_3 = 0$

Theorem 1.2

Let A be an $m \times n$ matrix, then the following statements are logically equivalent $\Rightarrow$ all true or all false

a. for each b in $\mathbb{R}^m$, the equation $Ax = b$ has a solution
b. each b in $\mathbb{R}^m$ is a linear combination of the columns of A
c. the columns of a span $\mathbb{R}^m$
d. A has a pivot in every row $\Longleftarrow$ what does this signify

proof + understanding

U is echelon for of some matrix A and b in $\mathbb{R}^m$

then $[A \;\; b] \xrightarrow{\text{reduced}} [U \;\; d]$ for some d in $\mathbb{R}^m$

assuming (d) is true (ie. very row has a pivot)
every row of U will also have a pivot
$\Rightarrow$ no pivot in augmented (d) column
$\Rightarrow$ $AX = b$ has a solution for any (b)
$\Rightarrow$ (a) is true
by definition, (b) and (c) are also true

Theorem 1.3

- A is an $m \times n$ matrix, u and v are vectors in R^m, and c is a scalar, then

$$A(u+v) = Au + Av$$
$$A(cu) = c(Au)$$

1·8 Solution sets of linear systems

- for $Ax = b$

$b = 0 \Rightarrow$ homogeneous solution → $x = 0$ trivial soln, $x \neq 0$ non-trivial

$b \neq 0 \Rightarrow$ non-homogeneous solution

EG 1 ·

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & +4 \\ 6 & 1 & -8 \end{bmatrix}$$

$R_2 + R_1$

$$\begin{bmatrix} 3 & 5 & -4 \\ 0 & 3 & 0 \\ 6 & 1 & -8 \end{bmatrix}$$

$R_3 - 2R_1$

$$\begin{bmatrix} 3 & 5 & -4 \\ 0 & 3 & 0 \\ 0 & -9 & 0 \end{bmatrix}$$

$R_3 + 3R_2$

$$\begin{bmatrix} 3 & 5 & -4 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

← row echelon

← no pivot

$R_1 - 5/3 R_2$

$R_1/3$, $R_2/3$

RRE

$$\begin{bmatrix} 1 & 0 & -4/3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_2 = 0$

$3x_1 + 5x_2 - 4x_3 = 0$

$3x_1 - 4x_3 = 0$

$x_1 = \frac{4}{3} x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

$x_2 = 0$

free variable → $x_3 = x_3$

∴ solution = scalar multiple of vector $v = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$

which is a line →

EG 2

$Ax = b \qquad b = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right]$$

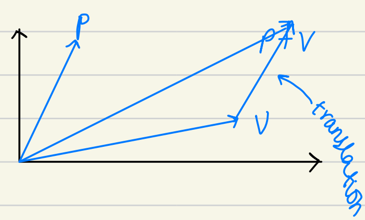
7	7	−3	−1
6	6	6	2
−18	0	0	0

$$\begin{bmatrix} 1 & 0 & 4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xleftrightarrow{\text{free}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{matrix} = -1 - 4/3\, x_3 \\ = 2 \\ = x_3 \end{matrix}$$

$\to \vec{x}$

soln : $\underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{p} + x_3 \underbrace{\begin{bmatrix} -4/3 \\ 0 \\ 1 \end{bmatrix}}_{v}$

$p + x_3 v = p + \underbrace{t}_{\text{scalar multiple}} v = x$



when $t = 0$, $x = \vec{p}$

and thats the particular solution

adding p to each point on $x = tv$ gives a line parallel to $x = tv$, through p

Theorem 1.4

let $Ax = b$ is consistent for some b and p is a solution
then solution set of $Ax = b$ is $w = p + v_h$
where v_h is any solution of $Ax = 0$

1.9 Linear independence

- a set of vectors is linearly dependent when every (/any) vector within it can be represented as a linear combination of all the other vectors.

$S = \{ v_1, v_2, v_3, v_4 \}$

$V_3 = aV_1 + bV_2 + cV_4$

$aV_1 + bV_2 + cV_4 + dV_3 = 0$ $(d = -1)$

now, if all coefficients are zero, and that is the only solution that exists, trivial solution, then and only then are the vectors linearly independent

so, if $aV_1 + bV_2 + cV_4 + dV_3 = 0$

$Ax = 0$ has $\xrightarrow{\text{a trivial soln}}$ only for $a = b = c = d = 0$, linearly independent

$Ax = 0$ has ∞ soln. else " dependent

↳ if aug. matrix has all zeroes in a row or if theres a free variable $\Rightarrow \infty$ solution

columns of matrix A are linearly dependent iff. the equation $Ax = 0$ has only the trivial solution

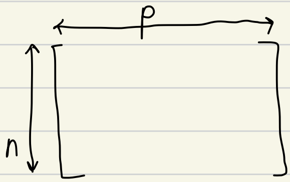
linearly dependent vectors exist in the same space, ie it they are linearly dependent, they are in each other's span

$w \in \text{Span}\{u, v\}$ iff $\{u, v, w\}$ are linearly dependent

Theorem 1.5

if a set contains more vectors than there are entries in each vector, then the set is said to be linearly dependent, i.e. any set $\{v_1, \dots, v_p\}$ in R^n is linearly dependent if $p > n$

if $v_1 \in R^n \Rightarrow v_1 = \begin{bmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{bmatrix}$

if 

$p > n$, then it's linearly dependent
$(n > m)$
"the columns"

if theres more variables (n) than equations $\Rightarrow$ there must be a free variable because there will be columns without pivots (making them free) $\Rightarrow$ $Ax = 0$ has a non-trivial solution $\Rightarrow$ columns are linearly dependent

Theorem 1.6

if a set S has a zero vector, it is linearly dependent.

$1v_1 + 0v_2 + 0v_3 + \dots + 0v_p = 0$

coeffs $\neq 0$, $Ax = 0$ has a solution lecture ques (16/01/2023)

1. could a set of 3 vectors in R^4 span all of R^4

n $\qquad$ m

$$\begin{array}{ccc} a_1 & a_2 & a_3 \end{array}$$
$$\begin{bmatrix} \boxed{\cdot} & \cdot & \cdot \\ \cdot & \boxed{\cdot} & \cdot \\ \cdot & \cdot & \boxed{\cdot} \\ \cdot & \cdot & \cdot \end{bmatrix}$$

4×3

theorem 1.2, all equivalent
(c) columns of A span R^m
iff (d) A has a pivot in every row

not possible
$\therefore$ (c) false
$\therefore$ no

2. solutions of $Ax = 0$ in parametric form $\rightarrow a\vec{v} + \vec{p}$

A is row eq to $\begin{bmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \end{bmatrix}$

$$\equiv \begin{bmatrix} 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 3x_2 - 4x_4 = 0$

$x_1 = 4x_4 - 3x_2$

gen soln: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3$

ans: $\vec{x} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} t_1 + \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} t_4 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} t_3$

3. A is a 3×2 matrix w/ 2 pivot positions.

$$\left[\begin{array}{ccc} \boxed{a_1} & a_2 & 0 \\ a_1 & \boxed{a_2} & 0 \\ a_1 & a_2 & 0 \end{array}\right]$$

a) non-trivial soln?

$\Downarrow$

infinite soln $\Rightarrow$ free variable

no free variables

$\hookrightarrow$ when columns don't have pivots, the variable of that column is free

b) does $Ax = b$ have at least 1 soln for every b?

$\downarrow$ th 1.2 (a)

(d) is not fulfilled (pivot in every row)

hence no

1.10 Linear transformation

$Ax = b$

A : object $\xrightarrow[\text{on}]{\text{acts}}$ vector x (by multipication), to produce a new vector called Ax

basically a function

transformation T from R^n to R^m ($T: R^n \to R^m$) is a rule that assigns to each vector x in R^n, a vector $T(x)$ in R^m

R^n : domain of T
R^m : codomain of T
$T(x)$ = image of x
↑
set of : range of T

—x—

$A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$ $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$ $c = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

$T: R^2 \to R^3$ by $T(x) = Ax$ $A = \begin{bmatrix} \hat{i} & \hat{j} \end{bmatrix}$
$T(u) = Au$
$T(x) = Ax = b$ find x (soln by guassian)

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -3 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -3 \\ 0 & 2 \\ 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

if c is in the range of T : $c = T(x)$ for some x, or is $Ax = c$ consistent

—x—

fuck this

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ 1 & 7 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ 0 & 10 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 10 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 3 \\ 1 & 1 & 2 \\ 0 & 5 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 3 \\ 1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

???

1. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$A(x)$ eliminates the third variable, and maps to plane $x_1 - x_2$ $\therefore$ projection of 3D vector to a 2-d plane

2. $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$Ax = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} b+3a \\ a \end{bmatrix}$ ← changed

↑ base fixed

shear transformation

1.11 Linear Transformation

T is linear if

$T(u+v) = T(u) + T(v)$ for all u, v in domain
$T(cu) = cT(u)$

$$T(c_1 v_1 + \dots + c_p v_p) = c_1 T(v_1) + \dots + c_p T(v_p)$$

$T(x) = rx$ line contracts/expands
$T: R^2 \to R^2$ contraction when $0 \le r \le 1$
dilation when $r > 1$

· ST A reps a linear transformation by finding images of $\underset{\begin{bmatrix}4\\1\end{bmatrix}}{u}$ and $\underset{\begin{bmatrix}2\\3\end{bmatrix}}{v}$

if $A(u+v) = Au + Av$ ✓

$$\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}6\\4\end{bmatrix} = \begin{bmatrix}-4\\6\end{bmatrix} = \begin{bmatrix}-1\\4\end{bmatrix} + \begin{bmatrix}-3\\2\end{bmatrix}$$

1.12 Matrix of linear transformation

- transformations : shear, projection, dilation

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \left.\begin{matrix} T(e_1) \\ T(e_2) \end{matrix}\right\} \qquad T = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$$

$$\begin{matrix} \uparrow & \uparrow \\ \vec{e}_1 & \vec{e}_2 \end{matrix}$$

Theorem 1.7

let $T : R^n \to R^m$ be a linear transformation ; then there is a unique matrix A such that

$$T(x) = Ax \quad \forall\, x \in R^n$$

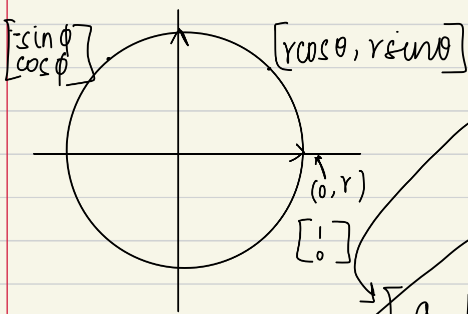
A is the $m \times n$ matrix whose j^{th} column is the vector $T(e_j)$ (e_j is the j^{th} column of the identity matrix in R^n)

$$A = [\, T(e_1) \ldots T(e_n) \,]$$

$\therefore$ A is the standard matrix for the linear transformation T

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle Φ, with counterclockwise rotation for a positive angle. Find the standard matrix A for this transformation.

$$A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} \qquad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$$T(e_1) = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$$

$$T(e_2) = \begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix}$$

lecture ques

4. how many pivot columns must a 7×5 matrix have, if the columns are linearly independent

$Ax = 0$ → only trivial → no free variable → 5 pivot cols.

5. how many pivot cols must a 5×7 matrix have, if its colms span R^5

every row must have one ⇒ 5 columns

6. $Ax = 0$ non-trivial soln :

$$\begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$$
$a_1 \quad a_2 \quad a_3$

obs : $a_3 = a_1 + a_2$
∴ columns aren't linearly independent

$Ax = 0$
$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$
$(1) a_1 + (1) a_2 + (-1) a_3 = 0$

∴ soln : $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

7. $T(e_1) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $a = 2$, $c = 5$ $\begin{bmatrix} 2 & -1 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \end{bmatrix}$

$T(e_2) = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$ $d = 6$, $b = -1$

$$\begin{bmatrix} 5 \\ -3 \end{bmatrix} = 5e_1 - 3e_2$$

8. $T : R^2 \to R^2$

$$T(\underbrace{x_1, x_2}_{x^T}) = (x_1 + x_2, \; 4x_1 + 5x_2)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 + x_2 \\ 4x_1 + 5x_2 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$ $T(x) = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 3 \\ 4 & 5 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -4 \end{bmatrix} \quad \begin{bmatrix} x_1 = 7 \\ x_2 = -4 \end{bmatrix}$$