

MODEL ANSWERS OF QUIZ II (TOTAL POINTS:15)

1. (2 points) Let  $X$  be a random variable having the cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{a}{2} & 1 \leq x < 2 \\ \frac{c}{6} & 2 \leq x < 3 \\ 1 & x \geq 3, \end{cases}$$

where  $a$  and  $c$  are appropriate constants, such that  $P(X \leq 1) = \frac{1}{2}$  and  $E(X) = \frac{5}{3}$ . If  $p_A$  denotes  $P(X \in A)$ , then which of the following statements is/are true?

1.  $p_{(1,3)} = \frac{1}{3}$
2.  $p_{(1,3]} = \frac{1}{2}$
3.  $p_{\{2\}} = \frac{1}{3}$
4.  $p_{[2,3]} = \frac{1}{3}$

**Solution:**

$$P(X \leq 1) = \frac{1}{2} \implies \frac{a}{2} = \frac{1}{2} \implies a = 1.$$

Therefore,  $P(X = 1) = \frac{1}{2}$ ,  $P(X = 2) = \frac{c}{6} - \frac{1}{2}$  and  $P(X = 3) = 1 - \frac{c}{6}$ . For any appropriate value of  $c$ ,  $P(X = 1) + P(X = 2) + P(X = 3) = 1$ . Thus,  $X$  is a DRV with support  $S = \{1, 2, 3\}$ . Now,

$$E(X) = \frac{5}{3} \implies 1 \times \frac{1}{2} + 2 \times \left(\frac{c}{6} - \frac{1}{2}\right) + 3 \times \left(1 - \frac{c}{6}\right) = \frac{5}{3} \implies c = 5.$$

Hence,

$$\begin{aligned} p_{(1,3)} &= P(1 < X < 3) = F(3-) - F(1) = \frac{1}{3}, \\ p_{(1,3]} &= P(1 < X \leq 3) = F(3) - F(1) = \frac{1}{2}, \\ p_{\{2\}} &= P(X = 2) = F(2) - F(2-) = \frac{1}{3}, \\ p_{[2,3]} &= P(2 \leq X \leq 3) = F(3) - F(2-) = \frac{1}{2}. \end{aligned}$$

2. (2 points) Let  $X$  be a random variable with uniform distribution on the interval  $(-1, 1)$  and  $Y = (X + 1)^2$ . Then the probability density function  $f(y)$  of  $Y$ , over the interval  $(0, 4)$ , is (A)  $\frac{3\sqrt{y}}{16}$  (B)  $\frac{1}{4\sqrt{y}}$  (C)  $\frac{1}{6\sqrt{y}}$  (D)  $\frac{1}{\sqrt{y}}$

**Solution:** Here, the support of  $X$  is  $S_X = (-1, 1)$ . Take  $g(x) = (x + 1)^2$ . Then,  $g(\cdot)$  is differentiable function on  $S_X$  and  $g'(x) > 0$  for all  $x \in S_X$ . Moreover,  $g^{-1}(y) = \sqrt{y} - 1$  on  $S_Y = \{g(x) : x \in S_X\} = (0, 4)$ . Therefore,  $Y = g(X) = (X + 1)^2$  is a CRV with PDF

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & \text{if } 0 < y < 4 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{4\sqrt{y}} & \text{if } 0 < y < 4 \\ 0 & \text{otherwise.} \end{cases}$$

3. (2 points) Let  $X$  be a random variable with the probability density function

$$f(x) = \begin{cases} c(x - [x]), & 0 < x < 3 \\ 0, & \text{elsewhere} \end{cases}$$

where  $c$  is an appropriate constant and  $[x]$  denotes the greatest integer less than or equal to  $x$ . If  $p = P(0.5 \leq X \leq 2)$ , then which of the following statements is/are true?

1.  $p \in (0.1, 0.3)$
2.  $p \in (0.2, 0.4)$
3.  $p \in (0.3, 0.5)$
4.  $p \in (0.4, 0.6)$

**Solution:** As  $f(\cdot)$  is a PDF,  $c$  satisfies

$$c \int_0^3 (x - [x]) dx = 1 \implies c \left[ \int_0^1 x dx + \int_1^2 (x - 1) dx + \int_2^3 (x - 2) dx \right] = 1 \implies c = \frac{2}{3}.$$

Therefore,

$$P(0.5 \leq X \leq 2) = \frac{2}{3} \int_{0.5}^2 (x - [x]) dx = \frac{7}{12}.$$

4. (1 point) Let  $X$  be a random variable with probability density function

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

If  $Y = \log_e X$ , then  $P(Y < 1 \mid Y < 2)$  equals (A)  $\frac{e}{1+e}$  (B)  $\frac{e-1}{e+1}$  (C)  $\frac{1}{1+e}$  (D)  $\frac{1}{e-1}$

**Solution:**

$$P(Y < 1 \mid Y < 2) = \frac{P(Y < 1)}{P(Y < 2)} = \frac{P(X < e)}{P(X < e^2)} = \frac{\int_1^e \frac{1}{x^2} dx}{\int_1^{e^2} \frac{1}{x^2} dx} = \frac{e}{1+e}.$$

5. (2 points) Let  $X$  be a random variable having Poisson distribution with mean  $\lambda > 0$  such that  $P(X = 4) = 2P(X = 5)$ . If  $p_k = P(X = k)$ ,  $k = 0, 1, 2, \dots$ , and  $p_\alpha = \max_k p_k$ , then find the value of  $\alpha$ .

**Solution:** Note that the PMF of  $X \sim Poi(\lambda)$  is

$$f(k) = \begin{cases} \frac{e^{-\lambda} \lambda^k}{k!} & \text{if } k = 0, 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$P(X = 4) = 2P(X = 5) \implies \lambda = \frac{5}{2}.$$

Therefore, for  $k = 0, 1, 2, \dots$ ,

$$\frac{f(k+1)}{f(k)} = \frac{5}{2(k+1)}.$$

Now,

$$\frac{f(k+1)}{f(k)} > 1 \implies k < \frac{3}{2} \quad \text{and} \quad \frac{f(k+1)}{f(k)} < 1 \implies k > \frac{3}{2}.$$

Thus,  $f(k)$  attains its maximum at  $k = 2$ , and hence,  $\alpha = 2$ .

6. (2 points) Suppose that 10 coins are tossed and each one shows heads with probability  $p$ , independent of each of the others. Each coin which shows heads is tossed again. Then find the probability of getting  $x$  heads after the second round of tosses.

**Solution:** The number  $X$  of heads after second round is same as if we toss the coins twice and count the number of coins that show heads on both occasions. Each coin shows heads twice with probability  $p^2$ . Thus,  $X \sim Bin(10, p^2)$ , and hence, for  $x = 0, 1, 2, \dots, 10$ ,

$$P(X = x) = \binom{10}{x} p^{2x} (1 - p^2)^{10-x}.$$

7. (2 points) Consider the function  $F : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$F(x) = \begin{cases} 1 - \left(\frac{2}{3}\right)^{[x]+1} & \text{if } x \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where  $[x]$  denotes the largest integer not exceeding  $x$ . Then which of the following statements is/are correct?

P:  $F(\cdot)$  is a cumulative distribution function of a random variable having continuous distribution.

Q:  $F(\cdot)$  is a cumulative distribution function of a random variable  $X$  and

$$P(X > n + m \mid X \geq n) = P(X > m)$$

for all positive integer  $n$  and  $m$ .

1. Only P
2. Only Q

3. Both P and Q
4. Neither P nor Q

**Solution:** As for  $x < y$ ,  $(2/3)^{[x]+1} \geq (2/3)^{[y]+1}$ ,  $F(\cdot)$  is a non-decreasing function. Moreover,  $\lim_{x \rightarrow -\infty} F(x) = 0$  and  $\lim_{x \rightarrow \infty} F(x) = 1 - \lim_{x \rightarrow \infty} (2/3)^{[x]+1} = 1$ . For all  $x \notin \{0, 1, 2, \dots\}$ ,  $F(\cdot)$  is continuous at  $x$ . For  $x \in \{0, 1, 2, \dots\}$ ,  $\lim_{h \downarrow 0} F(x+h) = 1 - \lim_{h \downarrow 0} (2/3)^{[x+h]+1} = 1 - (2/3)^{x+1} = F(x)$ . Thus,  $F(\cdot)$  is a right continuous function. Hence,  $F(\cdot)$  is a CDF of a RV  $X$ , say. Now,  $P(X = 0) = F(0) - F(0-) = \frac{1}{3}$ . Thus,  $X$  is not continuous random variable.

$$\begin{aligned}
 P(X > n+m \mid X \geq n) &= \frac{P(X > n+m)}{P(X \geq n)} \\
 &= \frac{1 - F(n+m)}{1 - F(n-)} \\
 &= \frac{\left(\frac{2}{3}\right)^{n+m+1}}{\left(\frac{2}{3}\right)^n} \\
 &= \left(\frac{2}{3}\right)^{m+1} \\
 &= P(X > m).
 \end{aligned}$$

8. (2 points) Let  $f(\cdot)$  be a probability density function with corresponding cumulative distribution function  $F(\cdot)$ . Then, which of the following statements is/are correct?

P:  $g(x) = 2f(x)F(x)$  is a probability density function.

Q:  $h(x) = f(x) \left\{1 + \frac{1}{2}(1 - 2F(x))\right\}$  is a probability density function.

1. Only P
2. Only Q
3. Both P and Q
4. Neither P nor Q

**Solution:** Note that  $g(x) \geq 0$  for all  $x \in \mathbb{R}$ . Again

$$\int_{-\infty}^{\infty} g(x) dx = 2 \int_{-\infty}^{\infty} f(x)F(x) dx = 2 \int_0^1 u du = 1.$$

The second inequality is obtained by taking  $u = F(x)$ . Thus,  $g(\cdot)$  is a PDF.

For all  $x \in \mathbb{R}$ ,

$$-1 \leq 1 - 2F(x) \leq 1 \implies \frac{1}{2} \leq 1 + \frac{1}{2}(1 - 2F(x)) \leq \frac{3}{2} \implies h(x) \geq 0.$$

Moreover,

$$\begin{aligned}\int_{-\infty}^{\infty} h(x)dx &= \int_{-\infty}^{\infty} f(x) \left\{ 1 + \frac{1}{2} (1 - 2F(x)) \right\} dx \\ &= \int_{-\infty}^{\infty} f(x)dx + \frac{1}{2} \int_{-\infty}^{\infty} f(x)(1 - 2F(x))dx \\ &= 1 + \int_0^1 (1 - 2u)du \\ &= 1.\end{aligned}$$

Therefore,  $h(\cdot)$  is also a PDF.