# MA579H Scientific Computing

# Newton's Method for Solving Systems of Nonlinear Equations

# Systems of Nonlinear Equations

Consider a nonlinear system of equations given by

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0 \end{aligned}$$

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Let $U \subset \mathbb{R}^n$. Define $F : U \longrightarrow \mathbb{R}^n$ by

$$F(x) := \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix} \text{ for } x := \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n.$$

Problem: Solve $F(x) = 0$.

# Jacobian matrix

If the function $F(x)$ is differentiable on $U$, that is, if the functions $f_1(x), \ldots, f_n(x)$ are differentiable on $U$ then the Jacobian matrix $J(c)$ of $F$ at $c \in U$ is given by

$$J(c) := \begin{bmatrix} \frac{\partial f_1(c)}{\partial x_1} & \cdots & \frac{\partial f_1(c)}{\partial x_n} \\ \frac{\partial f_2(c)}{\partial x_1} & \cdots & \frac{\partial f_2(c)}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n(c)}{\partial x_1} & \cdots & \frac{\partial f_n(c)}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

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Example: Let $F(x, y) = \begin{bmatrix} e^{x+y} - 2 & \sin x \end{bmatrix}^\top \in \mathbb{R}^2$. Then

$$J(x, y) = \begin{bmatrix} \frac{\partial (e^{x+y} - 2)}{\partial x} & \frac{\partial (e^{x+y} - 2)}{\partial y} \\ \frac{\partial (\sin x)}{\partial x} & \frac{\partial (\sin x)}{\partial y} \end{bmatrix} = \begin{bmatrix} e^{x+y} & e^{x+y} \\ \cos x & 0 \end{bmatrix}.$$

# Systems of Nonlinear Equations

**Example 1:** The system

$$e^{u+v} = 2$$
$$\sin u = 0$$

is the same as $F(u, v) = 0$ where $F(u, v) = [e^{u+v} - 2, \sin\ u]^\top$ and

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**Example 2:** The system

$$x^2 + 0.25 = y$$
$$y^2 + 0.25 = x$$

is the same as $F(x,y) = 0$ where $F(x) = [x^2 - y + .25, -x + y^2 + .25]^\top$ and

$$J(x,y) = \begin{bmatrix} 2x - 1 & -1 \\ -1 & 2y \end{bmatrix}.$$

# Newton's Method

If $F(x)$ is differentiable at $c \in U$ then $y = F(c) + J(c)(x - c)$ is a linear approximation of $y = F(x)$ at $c$ in the sense that

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This sets up **multivariate Newton's method** for solving $F(x) = 0$ with an initial guess $x_0$

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The multivariate Newton iterations can be rewritten as

for $k = 0, 1, 2, \ldots$

  Solve $J(x_k)s = -F(x_k)$ for $s$

  Set $x_{k+1} = x_k + s$

# Example

Consider the system

$$\begin{aligned} v - u^3 &= 0 \\ u^2 + v^2 &= 1 \end{aligned}$$

Here $F(u, v) = [v - u^3, u^2 + v^2 - 1]^\top$ and

$$J(u, v) = \begin{bmatrix} -3u^2 & 1 \\ 2u & 2v \end{bmatrix}.$$

Starting with $x_0 = [1, 2]^\top$, $F(x_0) = [1, 4]^\top$ and $s = [0, -1]^\top$ is a solution of $J(x_0)s = -F(x_0)$.

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Therefore the first iteration gives $x_1 := [1, 2]^\top + [0, -1]^\top = [1, 1]^\top$.

# Newton's Method

The iterations nearly converge after 7 steps to a solution $(0.8636, 0.5636)$.

| iteration | u | v |
|---|---|---|
| 1 | 1 | 2 |
| 2 | 1 | 1 |
| 3 | 0.875 | 0.625 |
| 4 | 0.829036348267l175 | 0.5643491124260355 |
| 5 | 0.8260401081706523 | 0.5636197735028443 |
| 6 | 0.8260313577324098 | 0.5636241621316300 |
| 7 | 0.826031357654187 | 0.5636241621612585 |

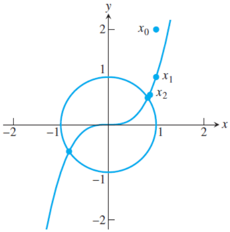


Figure : Figure shows the solutions of $F(x) = 0$ as dots on the circle and Newton iterations with $x_0 = (1, 2)$ converging to the solution $(0.8636, 0.5636)$.