Homework-4

MA-579H : Scientific Computing R. Alam **Date:** September 03, 2025

Solution of nonlinear equations

1. What is the definition of order of convergence p of an iterative method? Suppose that the errors at successive iterations of an iterative method are as follows:

   (a) $10^{-2}, 10^{-4}, 10^{-8}, 10^{-16}, \ldots$

   (b) $10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}, \ldots$

   How would you characterize the order of convergence?

2. If the bisection method for finding a zero of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ starts with an initial interval of length 1, what is the length of the interval containing the root after six iterations? Does the convergence of bisection method dependent on whether the solution sought is a simple zero or a multiple zero?

3. Is Newton's method for finding a zero of a smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$ an example of a fixed-point iteration scheme? If so, what is the iteration function ϕ in this case such that $\phi(\alpha) = \alpha$, where $f(\alpha) = 0$? If not, then explain why not. Suppose that f is twice continuously differentiable. If $|f(x)f''(x)| < |f'(x)|^2$ for all $x \in [a, b]$ then show that the Newton method converges for any starting iterate $x_0 \in [a, b]$.

4. Show that the iterative method

$$x_{j+1} = \frac{x_{j-1}f(x_j) - x_j f(x_{j-1})}{f(x_j) - f(x_{j-1})}, \quad j = 0, 1, 2 \ldots,$$

   is mathematically equivalent to the secant method for solving the equation $f(x) = 0$.

5. If the Newton's method for solving $f(x) = 0$ with starting guess $x_0 := 4$ and $f(x_0) = 1$ yields $x_1 = 3$ then determine the derivative of f at x_0.

6. For the secant method for solving $f(x) = 0$, we have $x_0 = 2, x_1 = -1$ and $x_2 = -2$ with $f(x_1) = 4$ and $f(x_2) = 3$. Determine $f(x_0)$.

7. Can the bisection method be used to find the zeros of $f(x) := \sin x + 1$? Why or why not? Can Newton's method be used to find the zeros of f? If so, what will be its order of convergence?

8. Consider the function $\phi(x) := (x^2 + 4)/5$ for $x \in \mathbb{R}$. Find the fixed points of $\phi(x)$. Would the fixed point iteration $x_{n+1} = \phi(x_n)$ converge to a fixed point in the interval $[0, 2]$ for all initial guesses $x_0 \in [0, 2]$?

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