Lab Session 6

MA581: Numerical Computations Lab R. Alam September 11, 2025

1. Solve the ODE $y' = 2y + 2\cos(t)\sin(2t)$ for multiple values of initial condition $y(0)$ as follows.

```
y0 = -5:5;  % vector of initial conditions
inter = [0 3];
yprime = @(t,y) -2*y + 2*cos(t).*sin(2*t);
[t,y] = ode45(yprime,inter,y0);
plot(t,y, 'LineWidth', 2)
grid on
xlabel('t')
ylabel('y')
title('Solutions of y'' = -2y + 2 cos(t) sin(2t), y(0) = -5,-4,...,4,5', ...
'interpreter','latex')
```

2. The van der Pol equation is a second-order ODE $y'' - \mu(1-y^2)y' + y = 0$ which can be reduced to first order systems of ODE and solved (for $\mu = 1$) as follows.

```
yprime = @(t, y) [y(2); (1-y(1)^2)*y(2)-y(1)];
[t,y] = ode45(yprime,[0 20],[2; 0]);
plot(t,y(:,1),'-o',t,y(:,2),'-o')
title('Solution of van der Pol Equation (\mu = 1) with ODE45');
xlabel('Time t');
ylabel('Solution y');
legend('y_1','y_2')
```

3. Let $y(x)$ be a solution curve of the IVP $\frac{dy}{dx} = f(x,y)$, $y(u) = v$, where $f : [a,b] \to \mathbb{R}$ is continuous and $u \in [a,b]$.

   (a) Write a MATLAB function program `[x,y] = Eulerforward(f,int,ics,m)` to solve the IVP via the forward-Euler method. The outputs should be a column $x = [x_1 \ \cdots \ x_m]^T$ of m equidistant points $x_i \in [a,b]$ such that $x_1 = a$ and $x_m = b$ and a column vector $y = [y_1 \ \cdots \ y_m]^T$ such that $y_i \approx y(x_i)$. The input `f` should be the function handle, `int` = [a b] and `ics` = [u, v].

   (b) Write a MATLAB function program `[x,y] = RungeKutta2(f,int,ics,m)` to generate the same output as in part (a) from the same inputs via the second order Runge-Kutta method with trapezoid steps.

   (c) Write a MATLAB function program `[x,y] = RungeKutta4(f,int,ics,m)` to generate the same output as in part (a) from the same inputs via the fourth order Runge-Kutta method.

4. The problems $y' = \cos x,\ y(0) = 0$ and $y' = \sqrt{1-y^2},\ y(0) = 0$ have the same solution on the interval $[0, \pi/2]$. Solve the problems using MATLAB command `ode45`.

   (a) Use second and fourth order Runge-Kutta codes written above to compute the solutions to both the problems and plot them graphically. Make one plot of solutions for both problems via second order Runge-Kutta method and another plot for the solutions via fourth order Runge-Kutta method.

(b) What happens to the computed solutions if the interval is changed to $[0, \pi]$?

(c) What happens on the interval $[0, \pi]$ if the second problem is changed to $y' = \sqrt{|1 - y^2|}$, $y(0) = 0$?

5. The ODE problem $y' = -1000(y - \sin x) + \cos x,\ \ y(0) = 1$ on the interval $[0, 1]$ is mildly stiff. Roughly speaking, a differential equation is stiff when certain numerical methods for solving the equation are numerically unstable, unless the step size is taken to be extremely small. Use `Eulerforward`, `RungeKutta2` and `RungeKutta4` to solve the problem. Plot the solutions in a single plot and comment on the results. Use step size for a better result.

6. The error function $\mathrm{erf}(x)$ is defined $\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ but it can also be defined as the solution to the differential equation

$$y' = \frac{2}{\sqrt{\pi}} e^{-x^2}, \ \ y(0) = 0.$$

Use `RungeKutta4` to solve this differential equation on the interval $[0, 2]$. Compare the results with the built-in MATLAB function $\mathrm{erf}(x)$. Solve the problem again using `ode45` and compare with built-in MATLAB function $\mathrm{erf}(x)$.

7. Consider the n-by-n system of first order ODEs $\frac{d\mathbf{y}}{dx} = \mathbf{f}(x, \mathbf{y})$, where $\mathbf{f} : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $u \in [a, b]$. Let $\mathbf{y}(x) := [y_1(x), \ldots, y_n(x)]^\top$ where $y_i : [a, b] \to \mathbb{R}$, $i = 1, \ldots, n$, is a unique solution of the associated IVP satisfying the initial condition $\mathbf{y}(u) = \mathbf{v}$, for $u \in [a\ b]$.

Write a MATLAB function program `[x,Y] = Eulersystem(f,int,ics,n,m)` to solve the above IVP using vector version of the forward-Euler method. It should generate a column vector $x = [x_1 \ \cdots \ x_m]^T$ of m equidistant points in $[a, b]$ such that $x_1 = a$ and $x_m = b$ and an $n \times m$ matrix Y such that $Y(:, i) \approx \mathbf{y}(x_i)$. The input `f` should be the function handle, `int` $=$ [`a b`] and `ics` $=$ [`u`, $\mathbf{v}$].

Solve the IVP $y'' + \sin xy' + x^2 y = x^2 \sin x$, $y(0) = 0, y'(0) = 1$, numerically by converting it into an IVP associated with a 2-by-2 first order ODE and using `[x,Y] = Eulersystem(f,int,ics,2,m)` with `int` $=$ [`0, pi/2`] and $m = 1000$. Plot the solution curve. Solve the problem again using `ode45` and plot the solution curve.

8. Consider the system $\frac{dv}{dt} = 0.2(1 - v) - 3vw$, $\frac{dw}{dt} = (3v - 1)w$ with initial condition $v(0) = 0.95$, $w(0) = 0.05$. Use `ode45` to find the long-term steady values of $v(t)$ and $w(t)$. Plot $v(t)$ and $w(t)$ as functions of time.

*** End ***