MA579H Scientific Computing

Interpolation with Chebyshev Nodes & Error Bounds

Rafikul Alam
Department of Mathematics
IIT Guwahati

Lecture outline

- Runge phenomenon
- Interpolation with Chebyshev nodes
- Interpolation error
- Hermite interpolation

Runge phenomenon

Consider the Runge function $f : [-1, 1] \longrightarrow \mathbb{R}$ given by

$$f(x) := \frac{1}{(1 + 25x^2)}.$$

Then for equally spaced nodes $x_0, \ldots, x_n$ and values $f_j := f(x_j)$ for $j = 0 : n$, the interpolant $p_n(x)$ does not converge to $f(x)$. In fact $\max_{|x| \leq 1} |f(x) - p_n(x)| \longrightarrow \infty$ as $n \to \infty$.

Runge phenomenon

Consider the Runge function $f : [-1, 1] \longrightarrow \mathbb{R}$ given by

$$f(x) := \frac{1}{(1 + 25x^2)}.$$

Then for equally spaced nodes $x_0, \ldots, x_n$ and values $f_j := f(x_j)$ for $j = 0 : n$, the interpolant $p_n(x)$ does not converge to $f(x)$. In fact $\max_{|x| \leq 1} |f(x) - p_n(x)| \longrightarrow \infty$ as $n \to \infty$.

What about other nodes? For the Chebyshev nodes $x_j := \cos\left(\frac{(2j+1)\pi}{2n+2}\right)$ for $j = 0 : n$, we have $\max_{|x| \leq 1} |f(x) - p_n(x)| \to 0$ as $n \to \infty$.

Runge phenomenon

Consider the Runge function $f : [-1, 1] \longrightarrow \mathbb{R}$ given by

$$f(x) := \frac{1}{(1 + 25x^2)}.$$

Then for equally spaced nodes $x_0, \ldots, x_n$ and values $f_j := f(x_j)$ for $j = 0 : n$, the interpolant $p_n(x)$ does not converge to $f(x)$. In fact $\max_{|x|\leq 1} |f(x) - p_n(x)| \longrightarrow \infty$ as $n \to \infty$.

What about other nodes? For the Chebyshev nodes $x_j := \cos\left(\frac{(2j+1)\pi}{2n+2}\right)$ for $j = 0 : n$, we have $\max_{|x|\leq 1} |f(x) - p_n(x)| \to 0$ as $n \to \infty$.

The Chebyshev nodes $x_j := \cos\left(\frac{(2j+1)\pi}{2n+2}\right)$ for $j = 0 : n$ are the zeros of the Chebyshev polynomial of degree $n + 1$.

Runge phenomenon

Consider the Runge function $f : [-1, 1] \longrightarrow \mathbb{R}$ given by

$$f(x) := \frac{1}{(1 + 25x^2)}.$$

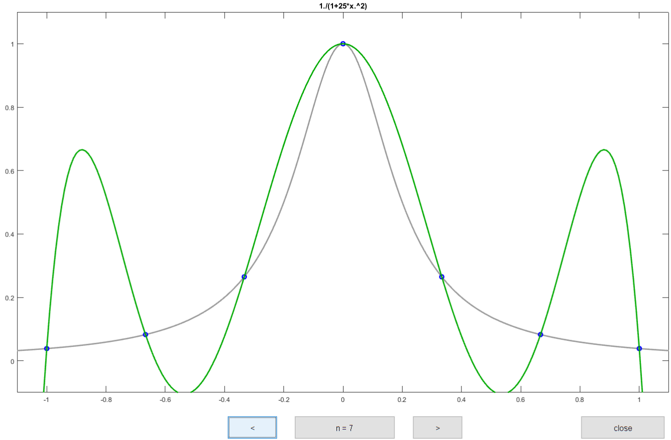
Then for equally spaced nodes $x_0, \ldots, x_n$ and values $f_j := f(x_j)$ for $j = 0 : n$, the interpolant $p_n(x)$ does not converge to $f(x)$. In fact $\max_{|x| \leq 1} |f(x) - p_n(x)| \longrightarrow \infty$ as $n \to \infty$.

What about other nodes? For the Chebyshev nodes $x_j := \cos\left(\frac{(2j+1)\pi}{2n+2}\right)$ for $j = 0 : n$, we have $\max_{|x| \leq 1} |f(x) - p_n(x)| \to 0$ as $n \to \infty$.

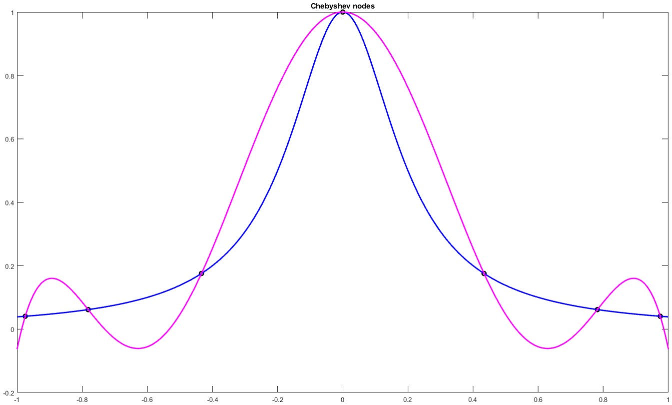
The Chebyshev nodes $x_j := \cos\left(\frac{(2j+1)\pi}{2n+2}\right)$ for $j = 0 : n$ are the zeros of the Chebyshev polynomial of degree $n + 1$.

The Runge phenomenon is eliminated by choosing Chebyshev nodes as interpolation points in $[-1, 1]$.

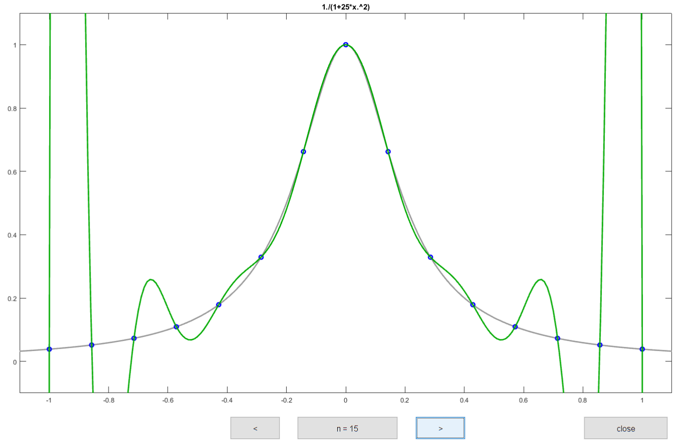
Runge phenomenon at equispaced nodes (n=7)



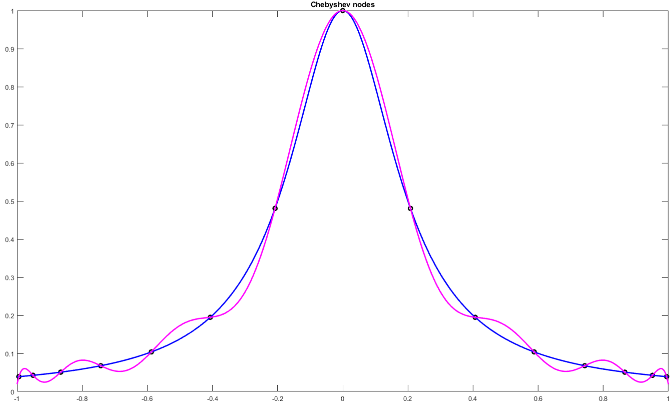
Runge phenomenon at Chebyshev nodes (n=7)



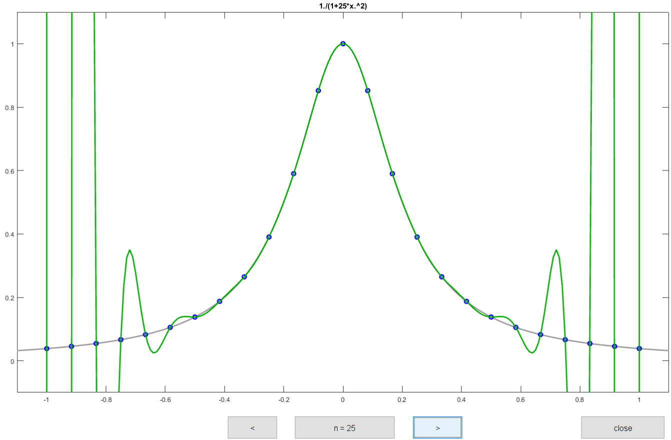
Runge phenomenon at equispaced nodes (n=15)



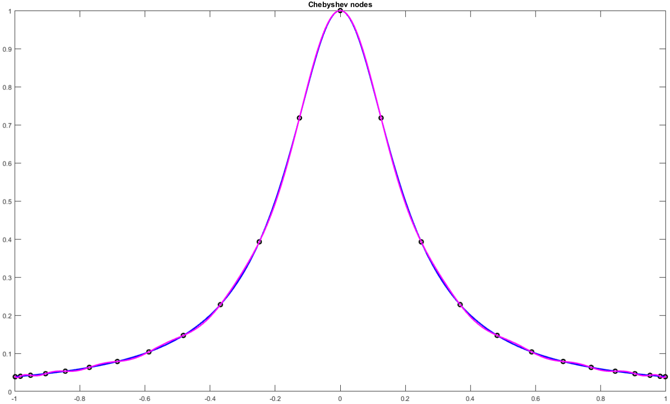
Runge phenomenon at Chebyshev nodes (n=15)



Runge phenomenon at equispaced nodes (n=25)



Runge phenomenon at Chebyshev nodes (n=25)



Chebyshev polynomials

Let $\theta \in [0, \pi]$ and $x \in [-1, 1]$. Define $T_n(x) := \cos(n \cos^{-1} x)$ for $n = 0, 1, \ldots$. Note that $T_0(x) = 1$ and $T_1(x) = x$. In fact, $T_n(x)$ is a polynomial of degree n and is called the Chebyshev polynomial.

Chebyshev polynomials

Let $\theta \in [0, \ \pi]$ and $x \in [-1, \ 1]$. Define $T_n(x) := \cos(n \cos^{-1} x)$ for $n = 0, 1, \ldots$. Note that $T_0(x) = 1$ and $T_1(x) = x$. In fact, $T_n(x)$ is a polynomial of degree n and is called the Chebyshev polynomial.

Also $T_n(x)$ satisfies a three term recurrence relation. Set $x = \cos\theta$. Then

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta$$

which gives

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \ n = 1, 2, \ldots.$$

Chebyshev polynomials

Let $\theta \in [0, \pi]$ and $x \in [-1, 1]$. Define $T_n(x) := \cos(n \cos^{-1} x)$ for $n = 0, 1, \ldots$. Note that $T_0(x) = 1$ and $T_1(x) = x$. In fact, $T_n(x)$ is a polynomial of degree n and is called the Chebyshev polynomial.

Also $T_n(x)$ satisfies a three term recurrence relation. Set $x = \cos\theta$. Then

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta$$

which gives

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \; n = 1, 2, \ldots.$$

For example,

$$T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \quad T_4(x) = 8x^4 - 8x^2 + 1.$$

Chebyshev polynomials

Let $\theta \in [0, \ \pi]$ and $x \in [-1, \ 1]$. Define $T_n(x) := \cos(n \cos^{-1} x)$ for $n = 0, 1, \ldots$. Note that $T_0(x) = 1$ and $T_1(x) = x$. In fact, $T_n(x)$ is a polynomial of degree n and is called the Chebyshev polynomial.

Also $T_n(x)$ satisfies a three term recurrence relation. Set $x = \cos\theta$. Then

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta \cos n\theta$$

which gives

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \ n = 1, 2, \ldots.$$

For example,

$$T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \quad T_4(x) = 8x^4 - 8x^2 + 1.$$

Note that

$$T_n(x) = 2^{n-1}x^n + \text{ lower degree terms.}$$

Chebyshev polynomials

The recursion $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \; n = 1, 2, \ldots$ can be written

$$\begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}$$

Chebyshev polynomials

The recursion $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),\ n = 1, 2, \ldots$ can be written

$$\begin{aligned} \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 2x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} \\ \begin{bmatrix} T_n(x) \\ T_{n+1}(x) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 2x \end{bmatrix} \begin{bmatrix} T_{n-1}(x) \\ T_n(x) \end{bmatrix} \text{ for } n = 2, 3, \ldots \end{aligned}$$

which shows that

$$\begin{bmatrix} T_n(x) \\ T_{n+1}(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2x \end{bmatrix}^n \begin{bmatrix} 1 \\ x \end{bmatrix} \text{ for } n = 1, 2, \ldots .$$

Chebyshev polynomials

The recursion $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \; n = 1, 2, \ldots$ can be written

$$\begin{aligned} \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 2x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} \\ \begin{bmatrix} T_n(x) \\ T_{n+1}(x) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 2x \end{bmatrix} \begin{bmatrix} T_{n-1}(x) \\ T_n(x) \end{bmatrix} \text{ for } n = 2, 3, \ldots \end{aligned}$$

which shows that

$$\begin{bmatrix} T_n(x) \\ T_{n+1}(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2x \end{bmatrix}^n \begin{bmatrix} 1 \\ x \end{bmatrix} \text{ for } n = 1, 2, \ldots.$$

The polynomial $T_n(x)$ can also be written as

$$T_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} x^{n-2j}(x^2 - 1)^j.$$

Chebyshev nodes

Since $T_n(x) = \cos(n \cos^{-1} x)$, we have $|T_n(x)| \leq 1$ for $x \in [-1, 1]$. Further,

$$\begin{aligned} T_n(x_j) &= 0 \quad \text{for} \quad x_j := \cos\left(\frac{2j+1}{2n}\pi\right),\ j = 0, 1, n-1; \\ T_n(y_j) &= (-1)^j \quad \text{for} \quad y_j := \cos\left(\frac{j\pi}{n}\right),\ j = 0, 1, \ldots, n. \end{aligned}$$

Chebyshev nodes

Since $T_n(x) = \cos(n \cos^{-1} x)$, we have $|T_n(x)| \leq 1$ for $x \in [-1, 1]$. Further,

$$\begin{aligned} T_n(x_j) &= 0 \quad \text{for} \quad x_j := \cos\left(\frac{2j+1}{2n}\pi\right), \ j = 0, 1, n-1; \\ T_n(y_j) &= (-1)^j \quad \text{for} \quad y_j := \cos\left(\frac{j\pi}{n}\right), \ j = 0, 1, \ldots, n. \end{aligned}$$

Chebyshev nodes in $[-1, 1]$: The zeros $x_j := \cos\left(\frac{2j+1}{2n+2}\pi\right)$, $j = 0 : n$, of $T_{n+1}(x)$ are called Chebyshev nodes.

Gauss-Lobatto nodes in $[-1, 1]$: The extremal points $y_j := \cos\left(\frac{j\pi}{n}\right)$, $j = 0 : n$, of $T_n(x)$ are called Gauss-Lobatto nodes.

Chebyshev nodes

Since $T_n(x) = \cos(n\cos^{-1} x)$, we have $|T_n(x)| \leq 1$ for $x \in [-1, 1]$. Further,

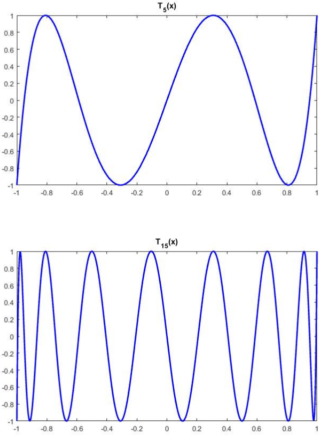
$$\begin{aligned} T_n(x_j) &= 0 \quad \text{for} \quad x_j := \cos\left(\frac{2j+1}{2n}\pi\right), \ j = 0, 1, n-1; \\ T_n(y_j) &= (-1)^j \quad \text{for} \quad y_j := \cos\left(\frac{j\pi}{n}\right), \ j = 0, 1, \ldots, n. \end{aligned}$$

Chebyshev nodes in $[-1, 1]$: The zeros $x_j := \cos\left(\frac{2j+1}{2n+2}\pi\right)$, $j = 0 : n$, of $T_{n+1}(x)$ are called Chebyshev nodes.

Gauss-Lobatto nodes in $[-1, 1]$: The extremal points $y_j := \cos\left(\frac{j\pi}{n}\right)$, $j = 0 : n$, of $T_n(x)$ are called Gauss-Lobatto nodes.

The Chebyshev nodes in $[-1, 1]$ for constructing an interpolating polynomial of degree at most n passing through $n+1$ data points $(x_0, f_0), \ldots, (x_n, f_n)$ are the zeros $x_j := \cos\left(\frac{2j+1}{2n+2}\pi\right)$, $j = 0 : n$, of $T_{n+1}(x)$.

Chebyshev polynomials



Barycentric Lagrange interpolation with Chebyshev nodes

The Lagrange interpolating polynomial for the nodes $(x_0, f_0), \ldots, (x_n, f_n)$

$$p(x) = f_0\ell_0(x) + \cdots + f_n\ell_n(x)$$

can be rewritten in barycentric form

$$p(x) = \frac{\sum_{j=0}^{n} \frac{w_j f_j}{x - x_j}}{\sum_{j=0}^{n} \frac{w_j}{x - x_j}}$$

where $w_j = 1/\prod_{i \neq j}(x_j - x_i)$ for $j = 0 : n$.

Barycentric Lagrange interpolation with Chebyshev nodes

The Lagrange interpolating polynomial for the nodes $(x_0, f_0), \ldots, (x_n, f_n)$

$$p(x) = f_0\ell_0(x) + \cdots + f_n\ell_n(x)$$

can be rewritten in barycentric form

$$p(x) = \frac{\sum_{j=0}^{n} \frac{w_j f_j}{x - x_j}}{\sum_{j=0}^{n} \frac{w_j}{x - x_j}}$$

where $w_j = 1/\prod_{i \neq j}(x_j - x_i)$ for $j = 0 : n$.

Remark: For the Chebyshev nodes $x_j = \cos(j\pi/n), j = 0 : n$, in $[-1, 1]$, the barycentric interpolation is given by

$$p(x) = \frac{{\sum'}_{j=0}^{n} \frac{(-1)^j f_j}{x - x_j}}{{\sum'}_{j=0}^{n} \frac{(-1)^j}{x - x_j}}, \tag{1}$$

where the primes on the summation signs signify that the terms $j = 0$ and $j = n$ are multiplied by $1/2$.

Barycentric Lagrange interpolation with Chebyshev nodes

The Lagrange interpolating polynomial for the nodes $(x_0, f_0), \ldots, (x_n, f_n)$

$$p(x) = f_0\ell_0(x) + \cdots + f_n\ell_n(x)$$

can be rewritten in barycentric form

$$p(x) = \frac{\sum_{j=0}^{n} \frac{w_j f_j}{x - x_j}}{\sum_{j=0}^{n} \frac{w_j}{x - x_j}}$$

where $w_j = 1/\prod_{i \neq j}(x_j - x_i)$ for $j = 0 : n$.

Remark: For the Chebyshev nodes $x_j = \cos(j\pi/n), j = 0 : n$, in $[-1, 1]$, the barycentric interpolation is given by

$$p(x) = \frac{{\sum'}_{j=0}^{n} \frac{(-1)^j f_j}{x - x_j}}{{\sum'}_{j=0}^{n} \frac{(-1)^j}{x - x_j}}, \qquad (1)$$

where the primes on the summation signs signify that the terms $j = 0$ and $j = n$ are multiplied by $1/2$. The barycentric interpolation formula (1) remains valid for Chebyshev nodes in $[a, b]$.

Approximation

Let $C[a,b] := \{f : [a,b] \longrightarrow \mathbb{R} \mid f \text{ is continuous}\}$. For $f \in C[a,b]$, define

$$\|f\|_\infty := \max\{|f(x)| : \; x \in [a,b]\}.$$

Then $\|f\|_\infty = 0 \iff f = 0$, $\|\alpha f\|_\infty = |\alpha| \, \|f\|_\infty$ for $\alpha \in \mathbb{R}$ and $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$. Thus $\|\cdot\|_\infty$ is a norm.

Approximation

Let $C[a, b] := \{f : [a, b] \longrightarrow \mathbb{R} \mid f \text{ is continuous}\}$. For $f \in C[a, b]$, define

$$\|f\|_\infty := \max\{|f(x)| : \; x \in [a, b]\}.$$

Then $\|f\|_\infty = 0 \iff f = 0$, $\|\alpha f\|_\infty = |\alpha| \, \|f\|_\infty$ for $\alpha \in \mathbb{R}$ and $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$. Thus $\| \cdot \|_\infty$ is a norm.

Weierstrass approximation theorem: Let $f \in C[a, b]$ and $\epsilon > 0$. Then there is a polynomial $p(x)$ such that $\|f - p\|_\infty \leq \epsilon$. In other words,

$$\max\{|f(x) - p(x)| : x \in [a, b]\} \leq \epsilon.$$

Approximation

Let $C[a, b] := \{f : [a, b] \longrightarrow \mathbb{R} \mid f \text{ is continuous}\}$. For $f \in C[a, b]$, define

$$\|f\|_\infty := \max\{|f(x)| : \ x \in [a, b]\}.$$

Then $\|f\|_\infty = 0 \iff f = 0$, $\|\alpha f\|_\infty = |\alpha| \, \|f\|_\infty$ for $\alpha \in \mathbb{R}$ and $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$. Thus $\| \cdot \|_\infty$ is a norm.

Weierstrass approximation theorem: Let $f \in C[a, b]$ and $\epsilon > 0$. Then there is a polynomial $p(x)$ such that $\|f - p\|_\infty \leq \epsilon$. In other words,

$$\max\{|f(x) - p(x)| : x \in [a, b]\} \leq \epsilon.$$

Let $p_n(x)$ be the interpolating polynomial for $(x_0, f_0), \ldots, (x_n, f_n)$. Suppose that $f_j = f(x_j)$, $j = 0 : n$, for some continuous function f.

Question: Does $p_n(x)$ approximate $f(x)$ for large enough n? In other words, does $\|p_n - f\|_\infty \longrightarrow 0$ as $n \to \infty$?

Interpolation error

Let $f \in C[a, b]$ and $[x_0, \ldots, x_n]$ be distinct nodes in $[a, b]$. Set $f_j = f(x_j)$ for $j = 0 : n$. Consider the Lagrange interpolating polynomial

$$p_n(x) := f_0\ell_0(x) + \cdots + f_n\ell_n(x).$$

Interpolation error

Let $f \in C[a,b]$ and $[x_0, \ldots, x_n]$ be distinct nodes in $[a,b]$. Set $f_j = f(x_j)$ for $j = 0 : n$. Consider the Lagrange interpolating polynomial

$$p_n(x) := f_0\ell_0(x) + \cdots + f_n\ell_n(x).$$

Define $\lambda_n(x) := |\ell_0(x)| + \cdots + |\ell_n(x)|$ and $\Lambda_n := \|\lambda_n\|_\infty$. Then $\lambda_n(x)$ is called the Lebesgue function and Λ_n is called the Lebesgue constant.

Interpolation error

Let $f \in C[a, b]$ and $[x_0, \ldots, x_n]$ be distinct nodes in $[a, b]$. Set $f_j = f(x_j)$ for $j = 0 : n$. Consider the Lagrange interpolating polynomial

$$p_n(x) := f_0\ell_0(x) + \cdots + f_n\ell_n(x).$$

Define $\lambda_n(x) := |\ell_0(x)| + \cdots + |\ell_n(x)|$ and $\Lambda_n := \|\lambda_n\|_\infty$. Then $\lambda_n(x)$ is called the Lebesgue function and Λ_n is called the Lebesgue constant.

Set $\mathrm{E}_n(f) := \min\{\|f - p\|_\infty : p \in \mathcal{P}_n\}$. Then $|p_n(x)| \leq \Lambda_n\|f\|_\infty$ and

$$\|f - p_n\|_\infty \leq (1 + \Lambda_n)\mathrm{E}_n(f).$$

Interpolation error

Let $f \in C[a, b]$ and $[x_0, \ldots, x_n]$ be distinct nodes in $[a, b]$. Set $f_j = f(x_j)$ for $j = 0 : n$. Consider the Lagrange interpolating polynomial

$$p_n(x) := f_0\ell_0(x) + \cdots + f_n\ell_n(x).$$

Define $\lambda_n(x) := |\ell_0(x)| + \cdots + |\ell_n(x)|$ and $\Lambda_n := \|\lambda_n\|_\infty$. Then $\lambda_n(x)$ is called the Lebesgue function and Λ_n is called the Lebesgue constant.

Set $\mathrm{E}_n(f) := \min\{\|f - p\|_\infty : p \in \mathcal{P}_n\}$. Then $|p_n(x)| \leq \Lambda_n\|f\|_\infty$ and

$$\|f - p_n\|_\infty \leq (1 + \Lambda_n)\mathrm{E}_n(f).$$

Fact: For equispaced nodes $\Lambda_n \sim \dfrac{2^n}{en\log n}$ and for Chebyshev nodes $\Lambda_n \leq \dfrac{2}{\pi}\log(n+1) + 1.$

Interpolation error

Let $f \in C[a,b]$ and $[x_0, \ldots, x_n]$ be distinct nodes in $[a,b]$. Set $f_j = f(x_j)$ for $j = 0 : n$. Consider the Lagrange interpolating polynomial

$$p_n(x) := f_0\ell_0(x) + \cdots + f_n\ell_n(x).$$

Define $\lambda_n(x) := |\ell_0(x)| + \cdots + |\ell_n(x)|$ and $\Lambda_n := \|\lambda_n\|_\infty$. Then $\lambda_n(x)$ is called the Lebesgue function and Λ_n is called the Lebesgue constant.

Set $\mathrm{E}_n(f) := \min\{\|f - p\|_\infty : p \in \mathcal{P}_n\}$. Then $|p_n(x)| \leq \Lambda_n\|f\|_\infty$ and

$$\|f - p_n\|_\infty \leq (1 + \Lambda_n)\mathrm{E}_n(f).$$

Fact: For equispaced nodes $\Lambda_n \sim \dfrac{2^n}{en\log n}$ and for Chebyshev nodes $\Lambda_n \leq \dfrac{2}{\pi}\log(n+1) + 1.$

For equispaced nodes, the Runge function $f(x) := 1/(1 + 25x^2)$, $x \in [-1,\ 1]$, shows the worst growth of Λ_n.

Error term for smooth function

Let $C^n[a,b]$ denote the set of n times continuously differentiable functions on $[a,b]$.

Theorem: If $f \in C^{n+1}[a,b]$ and $p_n(x)$ be the unique polynomial of degree at most n passing through $(x_0, f(x_0)), \ldots, (x_n, f(x_n))$. Then

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\theta_x)}{(n+1)!} w(x)$$

for some $\theta_x \in [x_{\min},\ x_{\max}]$, where $x_{\min}$ and $x_{\max}$ are the largest and the smallest nodes in $[x_0, \ldots, x_n, x]$ and $w(x) := (x - x_0) \cdots (x - x_n)$.

Error term for smooth function

Let $C^n[a, b]$ denote the set of n times continuously differentiable functions on $[a, b]$.

Theorem: If $f \in C^{n+1}[a, b]$ and $p_n(x)$ be the unique polynomial of degree at most n passing through $(x_0, f(x_0)), \ldots, (x_n, f(x_n))$. Then

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\theta_x)}{(n+1)!} w(x)$$

for some $\theta_x \in [x_{\min},\ x_{\max}]$, where $x_{\min}$ and $x_{\max}$ are the largest and the smallest nodes in $[x_0, \ldots, x_n, x]$ and $w(x) := (x - x_0) \cdots (x - x_n)$.

Proof: Define $F(t) := f(t) - p_n(t) - (f(x) - p_n(x))w(t)/w(x)$. Then

Error term for smooth function

Let $C^n[a, b]$ denote the set of n times continuously differentiable functions on $[a, b]$.

Theorem: If $f \in C^{n+1}[a, b]$ and $p_n(x)$ be the unique polynomial of degree at most n passing through $(x_0, f(x_0)), \ldots, (x_n, f(x_n))$. Then

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\theta_x)}{(n+1)!} w(x)$$

for some $\theta_x \in [x_{\min},\ x_{\max}]$, where $x_{\min}$ and $x_{\max}$ are the largest and the smallest nodes in $[x_0, \ldots, x_n, x]$ and $w(x) := (x - x_0) \cdots (x - x_n)$.

Proof: Define $F(t) := f(t) - p_n(t) - (f(x) - p_n(x))w(t)/w(x)$. Then $F(x) = 0$ and $F(x_j) = 0$ for $j = 0 : n$. By Rolle's theorem $F^{(n+1)}(t)$ has at least one zero in $[x_{\min},\ x_{\max}]$.

Error term for smooth function

Let $C^n[a, b]$ denote the set of n times continuously differentiable functions on $[a, b]$.

Theorem: If $f \in C^{n+1}[a, b]$ and $p_n(x)$ be the unique polynomial of degree at most n passing through $(x_0, f(x_0)), \ldots, (x_n, f(x_n))$. Then

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\theta_x)}{(n+1)!} w(x)$$

for some $\theta_x \in [x_{\min},\ x_{\max}]$, where $x_{\min}$ and $x_{\max}$ are the largest and the smallest nodes in $[x_0, \ldots, x_n, x]$ and $w(x) := (x - x_0) \cdots (x - x_n)$.

Proof: Define $F(t) := f(t) - p_n(t) - (f(x) - p_n(x))w(t)/w(x)$. Then $F(x) = 0$ and $F(x_j) = 0$ for $j = 0 : n$. By Rolle's theorem $F^{(n+1)}(t)$ has at least one zero in $[x_{\min},\ x_{\max}]$.

Thus $F^{(n+1)}(\theta_x) = 0$ for some $\theta_x \in [x_{\min},\ x_{\max}]$ which yields the desired result. ■

Error term for smooth function

Let $C^n[a, b]$ denote the set of n times continuously differentiable functions on $[a, b]$.

Theorem: If $f \in C^{n+1}[a, b]$ and $p_n(x)$ be the unique polynomial of degree at most n passing through $(x_0, f(x_0)), \ldots, (x_n, f(x_n))$. Then

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\theta_x)}{(n+1)!} w(x)$$

for some $\theta_x \in [x_{\min},\ x_{\max}]$, where $x_{\min}$ and $x_{\max}$ are the largest and the smallest nodes in $[x_0, \ldots, x_n, x]$ and $w(x) := (x - x_0) \cdots (x - x_n)$.

Proof: Define $F(t) := f(t) - p_n(t) - (f(x) - p_n(x))w(t)/w(x)$. Then $F(x) = 0$ and $F(x_j) = 0$ for $j = 0 : n$. By Rolle's theorem $F^{(n+1)}(t)$ has at least one zero in $[x_{\min},\ x_{\max}]$.

Thus $F^{(n+1)}(\theta_x) = 0$ for some $\theta_x \in [x_{\min},\ x_{\max}]$ which yields the desired result. ■

Rolle's theorem: Let $f \in C[a, b]$ be differentiable on (a, b). If $f(a) = f(b)$ then there exists $a < c < b$ such that $f'(c) = 0$.

Error term for smooth function

Thus, if $f \in C^{n+1}[a, b]$ then the interpolation error

$$|f(x) - p_n(x)| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}|w(x)| = \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}\prod_{j=0}^{n}|x - x_j|.$$

Error term for smooth function

Thus, if $f \in C^{n+1}[a, b]$ then the interpolation error

$$|f(x) - p_n(x)| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}|w(x)| = \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}\prod_{j=0}^{n}|x - x_j|.$$

Example: Let $p_4(x)$ be the interpolating polynomial of $f(x) = e^x$ at the nodes $-1, -1/2, 0, 1/2, 1$. Then

$$|e^x - p_4(x)| \leq \frac{|(x+1)(x+1/2)x(x-1/2)(x-1)|}{5!}e.$$

Error term for smooth function

Thus, if $f \in C^{n+1}[a, b]$ then the interpolation error

$$|f(x) - p_n(x)| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}|w(x)| = \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}\prod_{j=0}^{n}|x - x_j|.$$

Example: Let $p_4(x)$ be the interpolating polynomial of $f(x) = e^x$ at the nodes $-1, -1/2, 0, 1/2, 1$. Then

$$|e^x - p_4(x)| \leq \frac{|(x+1)(x+1/2)x(x-1/2)(x-1)|}{5!}e.$$

x	$\lvert e^x - p_4(x)\rvert$	Upper bound
-0.8	0.0008	0.0025
-0.25	0.0004	0.001
0.25	0.0004	0.001
0.8	0.0011	0.0025

Chebyshev's theorem

Goal: Choose nodes $x_0, \ldots, x_n$ in $[a, b]$ that minimize $\max_{x \in [a,b]} \prod_{j=0}^{n} |x - x_j|$. Equivalently, solve the min-max problem

$$\min_{x_0,\ldots,x_n} \max_{x \in [a,b]} \prod_{j=0}^{n} |x - x_j|.$$

Chebyshev's theorem

Goal: Choose nodes $x_0, \ldots, x_n$ in $[a, b]$ that minimize $\max_{x\in[a,b]} \prod_{j=0}^{n} |x - x_j|$. Equivalently, solve the min-max problem

$$\min_{x_0,\ldots,x_n} \max_{x\in[a,b]} \prod_{j=0}^{n} |x - x_j|.$$

Theorem (Chebyshev): We have

$$\min_{x_0,\ldots,x_n} \max_{x\in[-1,1]} \prod_{j=0}^{n} |(x - x_j)| = 2^{-n}$$

and the minimum is attained when

$$w(x) := \prod_{j=0}^{n} (x - x_j) = 2^{-n} T_{n+1}(x).$$

Thus, the minimum is attained when $x_0, \ldots, x_n$ are Chebyshev nodes in $[-1, 1]$.

Interpolation with Chebyshev nodes

Hence if $f \in C^{n+1}[-1,1]$ then for Chebyshev nodes

$$|f(x) - p_n(x)| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}\|w\|_\infty = \frac{\|f^{(n+1)}\|_\infty}{2^n(n+1)!}$$

where $w(x) := (x - x_0)\cdots(x - x_n)$ and $\|w\|_\infty = \max_{x\in[-1,1]} |w(x)|$.

Interpolation with Chebyshev nodes

Hence if $f \in C^{n+1}[-1, 1]$ then for Chebyshev nodes

$$|f(x) - p_n(x)| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}\|w\|_\infty = \frac{\|f^{(n+1)}\|_\infty}{2^n(n+1)!}$$

where $w(x) := (x - x_0)\cdots(x - x_n)$ and $\|w\|_\infty = \max_{x\in[-1,1]}|w(x)|$.

Example: Let $p_4(x)$ be the unique Chebyshev interpolating polynomial of degree at most 4 that approximates $f(x) := e^x$ with Chebyshev nodes $x_0 := \cos\left(\frac{\pi}{10}\right), x_1 = \cos\left(\frac{3\pi}{10}\right), x_2 := \cos\left(\frac{5\pi}{10}\right), x_3 := \cos\left(\frac{7\pi}{10}\right)$ and $x_4 := \cos\left(\frac{9\pi}{10}\right)$ in $[-1, 1]$.

Interpolation with Chebyshev nodes

Hence if $f \in C^{n+1}[-1,1]$ then for Chebyshev nodes

$$|f(x) - p_n(x)| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}\|w\|_\infty = \frac{\|f^{(n+1)}\|_\infty}{2^n(n+1)!}$$

where $w(x) := (x - x_0)\cdots(x - x_n)$ and $\|w\|_\infty = \max\limits_{x\in[-1,1]} |w(x)|$.

Example: Let $p_4(x)$ be the unique Chebyshev interpolating polynomial of degree at most 4 that approximates $f(x) := e^x$ with Chebyshev nodes $x_0 := \cos\left(\frac{\pi}{10}\right), x_1 = \cos\left(\frac{3\pi}{10}\right), x_2 := \cos\left(\frac{5\pi}{10}\right), x_3 := \cos\left(\frac{7\pi}{10}\right)$ and $x_4 := \cos\left(\frac{9\pi}{10}\right)$ in $[-1,1]$. We have

$$|e^x - p_4(x)| \leq \frac{|(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)|}{5!}e \leq \frac{e}{2^4 5!} \approx 0.00142.$$

Chebyshev nodes in $[a, b]$

Chebyshev nodes in $[a, b]$: $x_j := \frac{a+b}{2} + \frac{b-a}{2} \cos\left(\frac{2j+1}{2n+2}\pi\right), \; j = 0 : n.$

Chebyshev nodes in $[a, b]$

Chebyshev nodes in $[a, b]$: $x_j := \frac{a+b}{2} + \frac{b-a}{2} \cos\left(\frac{2j+1}{2n+2}\pi\right), \; j = 0 : n.$

Change of interval: The function $\phi : x \longmapsto \dfrac{a+b}{2} + \dfrac{(b-a)}{2}x$ carries Chebyshev nodes in $[-1, 1]$ to Chebyshev nodes in $[a, b]$.

Chebyshev nodes in $[a, b]$

Chebyshev nodes in $[a, b]$: $x_j := \frac{a+b}{2} + \frac{b-a}{2} \cos\left(\frac{2j+1}{2n+2}\pi\right), \ j = 0 : n.$

Change of interval: The function $\phi : x \longmapsto \dfrac{a+b}{2} + \dfrac{(b-a)}{2}x$ carries Chebyshev nodes in $[-1, 1]$ to Chebyshev nodes in $[a, b]$.

The function ϕ does two things:

(i) Scales the points in $[-1, 1]$ by the factor $(b-a)/2$.

Chebyshev nodes in $[a, b]$

Chebyshev nodes in $[a, b]$: $x_j := \frac{a+b}{2} + \frac{b-a}{2} \cos\left(\frac{2j+1}{2n+2}\pi\right),\ j = 0 : n.$

Change of interval: The function $\phi : x \longmapsto \dfrac{a+b}{2} + \dfrac{(b-a)}{2} x$ carries Chebyshev nodes in $[-1, 1]$ to Chebyshev nodes in $[a, b]$.

The function ϕ does two things:

(i) Scales the points in $[-1, 1]$ by the factor $(b-a)/2$.

(ii) Translates the points in $[-1, 1]$ by $(a+b)/2$ to move the midpoint 0 of $[-1, 1]$ to the midpoint of $[a, b]$.

Chebyshev nodes in $[a, b]$

Chebyshev nodes in $[a, b]$: $x_j := \frac{a+b}{2} + \frac{b-a}{2} \cos\left(\frac{2j+1}{2n+2}\pi\right), \; j = 0 : n.$

Change of interval: The function $\phi : x \longmapsto \dfrac{a+b}{2} + \dfrac{(b-a)}{2} x$ carries Chebyshev nodes in $[-1, 1]$ to Chebyshev nodes in $[a, b]$.

The function ϕ does two things:

(i) Scales the points in $[-1, 1]$ by the factor $(b-a)/2$.

(ii) Translates the points in $[-1, 1]$ by $(a+b)/2$ to move the midpoint 0 of $[-1, 1]$ to the midpoint of $[a, b]$.

Thus for Chebyshev nodes in $[-1, 1]$ we have

$$\prod_{j=0}^{n} |(\phi(x) - \phi(x_j))| = \left(\frac{b-a}{2}\right)^{n+1} \prod_{j=0}^{n} |(x - x_j)| \leq \frac{1}{2^n} \left(\frac{b-a}{2}\right)^{n+1}.$$

Chebyshev interpolation error

Theorem: Let $f \in C^{n+1}[a,b]$. Then for the Chebyshev nodes $x_j := \frac{b+a}{2} + \frac{b-a}{2}\cos\left(\frac{2j+1}{2n+2}\pi\right)$ for $j = 0 : n$, we have

$$|f(x) - p_n(x)| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}\|w\|_\infty = \left(\frac{b-a}{2}\right)^{n+1} \frac{\|f^{(n+1)}\|_\infty}{2^n(n+1)!},$$

where $w(x) := (x - x_0)\cdots(x - x_n)$.

Chebyshev interpolation error

Theorem: Let $f \in C^{n+1}[a,b]$. Then for the Chebyshev nodes $x_j := \frac{b+a}{2} + \frac{b-a}{2}\cos\left(\frac{2j+1}{2n+2}\pi\right)$ for $j = 0:n$, we have

$$|f(x) - p_n(x)| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}\|w\|_\infty = \left(\frac{b-a}{2}\right)^{n+1} \frac{\|f^{(n+1)}\|_\infty}{2^n(n+1)!},$$

where $w(x) := (x - x_0)\cdots(x - x_n)$.

Example: Consider $f(x) := \sin(x)$ for $x \in [0, \pi/2]$. For Chebyshev nodes

$$|\sin(x) - p_n(x)| \leq \left(\frac{\pi/2 - 0}{2}\right)^{n+1} \frac{\|f^{(n+1)}\|_\infty}{2^n(n+1)!} \leq \left(\frac{\pi}{4}\right)^{n+1} \frac{1}{2^n(n+1)!}$$

For $n = 8$ the error bound is $\approx 0.1224 \times 10^{-8}$. For $n = 9$ the error is $\approx 0.4807 \times 10^{-10}$. Thus for Chebyshev nodes $p_9(x)$ approximates $\sin(x)$ correct to 10 decimal places.

Error term via divided difference

Suppose that $f \in C[a, b]$ and $f(x_j) = f_j$ for $j = 0 : n$. Consider the Newton interpolating polynomial $p_n(x) := \sum_{j=0}^{n} f[x_0, \ldots, x_j] \mathrm{N}_j(x)$.

Error term via divided difference

Suppose that $f \in C[a, b]$ and $f(x_j) = f_j$ for $j = 0 : n$. Consider the Newton interpolating polynomial $p_n(x) := \sum_{j=0}^{n} f[x_0, \ldots, x_j]\mathrm{N}_j(x)$.

For fixed $x \in [a, b]$, let $q(t) \in \mathcal{P}_{n+1}$ be the Newton interpolating polynomial that interpolates $(x_0, f_0), \ldots, (x_n, f_n),(x, f(x))$. Then

Error term via divided difference

Suppose that $f \in C[a, b]$ and $f(x_j) = f_j$ for $j = 0 : n$. Consider the Newton interpolating polynomial $p_n(x) := \sum_{j=0}^{n} f[x_0, \ldots, x_j]\mathrm{N}_j(x)$.

For fixed $x \in [a, b]$, let $q(t) \in \mathcal{P}_{n+1}$ be the Newton interpolating polynomial that interpolates $(x_0, f_0), \ldots, (x_n, f_n), (x, f(x))$. Then

$$q(t) = p_n(t) + f[x_0, \ldots, x_n, x]\mathrm{N}_{n+1}(t).$$

Error term via divided difference

Suppose that $f \in C[a, b]$ and $f(x_j) = f_j$ for $j = 0 : n$. Consider the Newton interpolating polynomial $p_n(x) := \sum_{j=0}^{n} f[x_0, \ldots, x_j]\mathrm{N}_j(x)$.

For fixed $x \in [a, b]$, let $q(t) \in \mathcal{P}_{n+1}$ be the Newton interpolating polynomial that interpolates $(x_0, f_0), \ldots, (x_n, f_n), (x, f(x))$. Then

$$q(t) = p_n(t) + f[x_0, \ldots, x_n, x]\mathrm{N}_{n+1}(t).$$

Setting $t = x$ and using the fact that $q(x) = f(x)$, we have

Error term via divided difference

Suppose that $f \in C[a, b]$ and $f(x_j) = f_j$ for $j = 0 : n$. Consider the Newton interpolating polynomial $p_n(x) := \sum_{j=0}^{n} f[x_0, \ldots, x_j]\mathrm{N}_j(x)$.

For fixed $x \in [a, b]$, let $q(t) \in \mathcal{P}_{n+1}$ be the Newton interpolating polynomial that interpolates $(x_0, f_0), \ldots, (x_n, f_n), (x, f(x))$. Then

$$q(t) = p_n(t) + f[x_0, \ldots, x_n, x]\mathrm{N}_{n+1}(t).$$

Setting $t = x$ and using the fact that $q(x) = f(x)$, we have

$$E_n(x) := f(x) - p_n(x) = f[x_0, \ldots, x_n, x]\mathrm{N}_{n+1}(x).$$

Error term via divided difference

Suppose that $f \in C[a,b]$ and $f(x_j) = f_j$ for $j = 0 : n$. Consider the Newton interpolating polynomial $p_n(x) := \sum_{j=0}^{n} f[x_0, \ldots, x_j]\mathrm{N}_j(x)$.

For fixed $x \in [a,b]$, let $q(t) \in \mathcal{P}_{n+1}$ be the Newton interpolating polynomial that interpolates $(x_0, f_0), \ldots, (x_n, f_n), (x, f(x))$. Then

$$q(t) = p_n(t) + f[x_0, \ldots, x_n, x]\mathrm{N}_{n+1}(t).$$

Setting $t = x$ and using the fact that $q(x) = f(x)$, we have

$$E_n(x) := f(x) - p_n(x) = f[x_0, \ldots, x_n, x]\mathrm{N}_{n+1}(x).$$

If $f \in C^{n+1}[a,b]$ then comparing with error term of Lagrange interpolating polynomial we have

$$f[x_0, \ldots, x_n, x] = \frac{f^{(n+1)}(\theta_x)}{(n+1)!}$$

for some θ_x between the smallest and the largest nodes $[x_0, \ldots, x_n, x]$.

Divided differences at repeated nodes

Theorem: If $f \in C^n[a, b]$ and $[x_0, \ldots, x_n]$ are distinct nodes then

$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\theta)}{n!}$$

for some θ between the smallest and the largest nodes $[x_0, \ldots, x_n]$.

Divided differences at repeated nodes

Theorem: If $f \in C^n[a, b]$ and $[x_0, \ldots, x_n]$ are distinct nodes then

$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\theta)}{n!}$$

for some θ between the smallest and the largest nodes $[x_0, \ldots, x_n]$.

If x_j tends to x_0 for $j = 1 : n$, then θ, being trapped between the nodes, must tend to x_0. Hence taking limit as $x_j \to x_0$ for $j = 1 : n$, we have

$$f[x_0, \ldots, x_0] = \frac{f^{(n)}(x_0)}{n!}.$$

Divided differences at repeated nodes

Theorem: If $f \in C^n[a, b]$ and $[x_0, \ldots, x_n]$ are distinct nodes then

$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\theta)}{n!}$$

for some θ between the smallest and the largest nodes $[x_0, \ldots, x_n]$.

If x_j tends to x_0 for $j = 1 : n$, then θ, being trapped between the nodes, must tend to x_0. Hence taking limit as $x_j \to x_0$ for $j = 1 : n$, we have

$$f[x_0, \ldots, x_0] = \frac{f^{(n)}(x_0)}{n!}.$$

This defines the n-th order divided difference of f at $n + 1$ times repeated node x_0.

Hermite interpolation

Problem: Given distinct nodes $\mathbf{x} := [x_0, \ldots, x_n]$, positive integers $[m_0, \ldots, m_n]$ and $f \in C^{m-1}[a, b]$ with $m := \max(m_0, \ldots, m_n)$, find a polynomial $p(x)$ of least degree such that for $j = 0 : n$,

$$p^{(i)}(x_j) = f^{(i)}(x_j) \quad \text{for} \quad i = 0, 1, \ldots, m_j - 1.$$

Hermite interpolation

Problem: Given distinct nodes $\mathbf{x} := [x_0, \ldots, x_n]$, positive integers $[m_0, \ldots, m_n]$ and $f \in C^{m-1}[a, b]$ with $m := \max(m_0, \ldots, m_n)$, find a polynomial $p(x)$ of least degree such that for $j = 0 : n$,

$$p^{(i)}(x_j) = f^{(i)}(x_j) \text{ for } i = 0, 1, \ldots, m_j - 1.$$

The problem can be thought of as a limiting case of polynomial interpolation if we consider x_j to be a repeated node of multiplicity m_j for $j = 0 : n$, that is, $\underbrace{x_0, \ldots, x_0}_{m_0}, \underbrace{x_1, \ldots, x_1}_{m_1}, \ldots, \underbrace{x_n, \ldots, x_n}_{m_n}$.

Hermite interpolation

Problem: Given distinct nodes $\mathbf{x} := [x_0, \ldots, x_n]$, positive integers $[m_0, \ldots, m_n]$ and $f \in C^{m-1}[a, b]$ with $m := \max(m_0, \ldots, m_n)$, find a polynomial $p(x)$ of least degree such that for $j = 0 : n$,

$$p^{(i)}(x_j) = f^{(i)}(x_j) \ \text{ for } \ i = 0, 1, \ldots, m_j - 1.$$

The problem can be thought of as a limiting case of polynomial interpolation if we consider x_j to be a repeated node of multiplicity m_j for $j = 0 : n$, that is, $\underbrace{x_0, \ldots, x_0}_{m_0}, \underbrace{x_1, \ldots, x_1}_{m_1}, \ldots, \underbrace{x_n, \ldots, x_n}_{m_n}$.

Set $N := m_0 + \cdots + m_n$. Then the Hermite interpolating polynomial $p(x)$ of degree $N - 1$ can be constructed from the divided differences table for repeated nodes.

Hermite interpolation

Problem: Given distinct nodes $\mathbf{x} := [x_0, \ldots, x_n]$, positive integers $[m_0, \ldots, m_n]$ and $f \in C^{m-1}[a, b]$ with $m := \max(m_0, \ldots, m_n)$, find a polynomial $p(x)$ of least degree such that for $j = 0 : n$,

$$p^{(i)}(x_j) = f^{(i)}(x_j) \text{ for } i = 0, 1, \ldots, m_j - 1.$$

The problem can be thought of as a limiting case of polynomial interpolation if we consider x_j to be a repeated node of multiplicity m_j for $j = 0 : n$, that is, $\underbrace{x_0, \ldots, x_0}_{m_0}, \underbrace{x_1, \ldots, x_1}_{m_1}, \ldots, \underbrace{x_n, \ldots, x_n}_{m_n}$.

Set $N := m_0 + \cdots + m_n$. Then the Hermite interpolating polynomial $p(x)$ of degree $N - 1$ can be constructed from the divided differences table for repeated nodes.

For $j = 0 : n$, set $f_j^{(i)} := f^{(i)}(x_j), i = 0, 1, \ldots, m_j - 1$.

Hermite interpolation

For example, if $m_j = 4$, then the divided differences are given by

$$
\begin{array}{c|ccccc}
\mathbf{x} & \mathbf{f} & & & \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot \\
x_j & f_j & & & \\
x_j & f_j & f_j' & & \\
x_j & f_j & f_j' & f_j''/2 & \\
x_j & f_j & f_j' & f_j''/2 & f_j'''/6 \\
\cdot & \cdot & \cdots & \cdots & \cdots
\end{array}
$$

Hermite interpolation

For example, if $m_j = 4$, then the divided differences are given by

$\mathbf{x}$	$\mathbf{f}$			
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
x_j	f_j			
x_j	f_j	f_j'		
x_j	f_j	f_j'	$f_j''/2$	
x_j	f_j	f_j'	$f_j''/2$	$f_j'''/6$
$\cdot$	$\cdot$	$\cdots$	$\cdots$	$\cdots$

Example: Find $p \in \mathcal{P}_3$ such that
$p(x_0) = f_0, p(x_1) = f_1, p'(x_1) = f_1', p(x_2) = f_2$, where $x_0 < x_1 < x_2$.

Hermite interpolation

For example, if $m_j = 4$, then the divided differences are given by

$\mathbf{x}$	$\mathbf{f}$			
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
x_j	f_j			
x_j	f_j	f_j'		
x_j	f_j	f_j'	$f_j''/2$	
x_j	f_j	f_j'	$f_j''/2$	$f_j'''/6$
$\cdot$	$\cdot$	$\cdots$	$\cdots$	$\cdots$

Example: Find $p \in \mathcal{P}_3$ such that
$p(x_0) = f_0, p(x_1) = f_1, p'(x_1) = f_1', p(x_2) = f_2$, where $x_0 < x_1 < x_2$.

$\mathbf{x}$	$\mathbf{f}$			
x_0	f_0			
x_1	f_1	$f[x_0, x_1]$		
x_1	f_1	f_1'	$f[x_0, x_1, x_1]$	
x_2	f_2	$f[x_1, x_2]$	$f[x_1, x_1, x_2]$	$f[x_0, x_1, x_1, x_2]$

Hermite interpolation

Example: Determine a polynomial $p(x)$ such that
$p(1) = 2, p'(1) = 3, p(2) = 6, p'(2) = 7, p''(2) = 8.$

Hermite interpolation

Example: Determine a polynomial $p(x)$ such that
$p(1) = 2, p'(1) = 3, p(2) = 6, p'(2) = 7, p''(2) = 8.$

x	**f**				
1	2				
1	2	3			
2	6	4	1		
2	6	7	3	2	
2	6	7	4	1	-1

Hermite interpolation

Example: Determine a polynomial $p(x)$ such that
$p(1) = 2, p'(1) = 3, p(2) = 6, p'(2) = 7, p''(2) = 8.$

x	f				
1	2				
1	2	3			
2	6	4	1		
2	6	7	3	2	
2	6	7	4	1	-1

Thus

$$p(x) = 2 + 3\cdot(x-1) + 1\cdot(x-1)^2 + 2\cdot(x-1)^2(x-2) - 1\cdot(x-1)^2(x-2)^2.$$

***