

Homework-2

MA-580H : Matrix Computations

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Linear Systems

1. Let $A := \begin{bmatrix} 2 & 2 & -4 \\ 1 & 1 & 5 \\ 1 & 3 & 6 \end{bmatrix}$ and $b := \begin{bmatrix} 10 \\ -2 \\ -5 \end{bmatrix}$. Compute LU factorization with partial pivoting $PA = LU$ by hand and solve the system $Ax = b$.
2. Let $A = LU$ be the LU factorization of a non-singular matrix $A \in \mathbb{R}^{n \times n}$, where L is unit lower triangular and U is upper triangular. Let A_k denote the $k \times k$ leading principal submatrix of A for $k = 1 : n - 1$. Show that $u_{11} = a_{11}$ and $u_{kk} = \frac{\det(A_k)}{\det(A_{k-1})}$ for $k = 2 : n$.
3. Let $A \in \mathbb{R}^{n \times n}$ be nonsingular. Let $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$ be such that $A + uv^\top$ is nonsingular.

(a) **Sherman-Morrison formula:** Show that $1 + v^\top A^{-1}u \neq 0$ and that

$$(A + uv^\top)^{-1} = A^{-1} - \frac{A^{-1}uv^\top A^{-1}}{1 + v^\top A^{-1}u}.$$

- (b) Suppose that LU factorization $A = LU$ is given so that solving $Ax = b$ for any $b \in \mathbb{R}^n$ costs just $\mathcal{O}(n^2)$ flops. Use Sherman-Morrison formula and describe an efficient algorithm (outline only the steps) for solving $(A + uv^\top)x = b$ and determine the total flop count of the algorithm.
4. Let $A \in \mathbb{R}^{n \times n}$. Show that $\text{rank}(A) = r$ if and only if at the $(r + 1)$ -th step of Gaussian elimination with complete pivoting (GECP), the largest entry found in the submatrix $A(r + 1 : n, r + 1 : n)$ is zero (in exact arithmetic).
 5. Suppose that $A \in \mathbb{R}^{n \times n}$ is nonsingular. Describe Gaussian elimination based efficient algorithms for solving the following problems.
 - (a) Compute $c^\top A^{-1}b$, for $c, b \in \mathbb{R}^n$.
 - (b) Solve $A^m x = b$, where $b \in \mathbb{R}^n$ and m is a natural number.
- [**Hint:** Use LU factorization of A]

You should describe your algorithm (outline only the steps) and determine the flop count.

6. Suppose that $A \in \mathbb{R}^{n \times n}$ is symmetric positive definite. Partition A as 2-by-2 block matrix as given below and use the fact that

$$A = \begin{bmatrix} a_{11} & w^\top \\ w & B \end{bmatrix} = \begin{bmatrix} \sqrt{a_{11}} & 0 \\ w/\sqrt{a_{11}} & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & B - ww^\top/a_{11} \end{bmatrix} \begin{bmatrix} \sqrt{a_{11}} & w^\top/\sqrt{a_{11}} \\ 0 & I \end{bmatrix}$$

to prove the existence of Cholesky factorization of A , where $w \in \mathbb{R}^{n-1}$ and $B \in \mathbb{R}^{(n-1) \times (n-1)}$.

Also show that $|a_{ij}| < \sqrt{a_{ii}a_{jj}}$ for all $i \neq j$. What does this statement say about $\max_{ij} |a_{ij}|$?

7. Let $A \in \mathbb{R}^{n \times n}$ be symmetric positive definite. Partition A as 2-by-2 block matrix

$$A = \begin{bmatrix} A_{n-1} & b \\ b^\top & a_{nn} \end{bmatrix}, \text{ where } A_{n-1} \in \mathbb{R}^{(n-1) \times (n-1)} \text{ and } b \in \mathbb{R}^{n-1},$$

and prove the existence and uniqueness of Cholesky factorization of A .

8. Let $A := \begin{bmatrix} 2 & 4 & 4 & 2 \\ 4 & 5 & 8 & -5 \\ 4 & 8 & 6 & 2 \\ 2 & -5 & 2 & -26 \end{bmatrix}$. Compute LDL^\top decomposition of A and check positive definiteness of A . Compute Cholesky decomposition of A , if it exists.

9. Let $A := \begin{bmatrix} 4 & -2 & 4 & 2 \\ -2 & 10 & -2 & -7 \\ 4 & -2 & 8 & 4 \\ 2 & -7 & 4 & 7 \end{bmatrix}$ and $b := \begin{bmatrix} 8 \\ 2 \\ 16 \\ 6 \end{bmatrix}$. Compute Cholesky decomposition of A by hand using outer product form and solve the system $Ax = b$.

10. Let $x \in \mathbb{C}^n$. Show that

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2 \leq n\|x\|_\infty.$$

11. Let $D := \text{diag}(d_1, \dots, d_n)$ be a diagonal matrix. Show that

$$\|D\|_2 = \|D\|_1 = \|D\|_\infty = \max(|d_1|, \dots, |d_n|).$$

12. Let $A \in \mathbb{R}^{n \times n}$ be nonsingular and $\text{cond}(A) := \|A\| \|A^{-1}\|$ be the condition number of A . If \hat{x} is a computed solution of $Ax = b$ and $r := b - A\hat{x}$ is the residual then show that

$$\frac{\|r\|}{\text{cond}(A)\|b\|} \leq \frac{\|x - \hat{x}\|}{\|x\|} \leq \text{cond}(A) \frac{\|r\|}{\|b\|}.$$

Can small residual guarantee small error in the solution?

*****End*****