MA579H Scientific Computing

Solutions of Nonlinear Equations I

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Outline

- Bisection method
- Regula-Falsi method
- Secant method

Zeros of Bessel's function

The Bessel function of the first kind

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$

is a solution of the Bessel's differential equation

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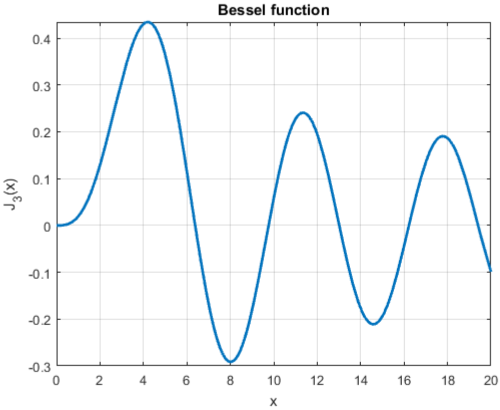
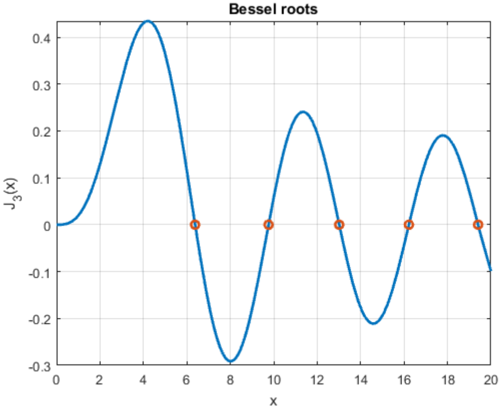
Plot of $J_3(x)$

```
J3 = @(x) besselj(3,x);
fplot(J3,[0 20]), grid on
xlabel('x'), ylabel('J_3(x)')
title('Bessel function')
```



Plot of zeros of $J_3(x)$

```
z = [];
for guess = [6,10,13,16,19]
z = [z;fzero(J3,guess)];
end
hold on, plot(z,J3(z),'o')
title('Bessel roots')
```



Zeros of $J_3(x)$

Zeros of $J_3(x)$ computed by `fzero` in $[0, 20]$.

```
z =

6.380161895923984e+00
9.761023129981670e+00
1.301520072169843e+01
1.622346616031877e+01
1.940941522643502e+01
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$$\text{Given } F : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n, \text{ solve } F(\mathbf{x}) = 0.$$

Example: Consider the system of nonlinear equations in two dimensions

$$\begin{aligned} x_1^2 - x_2 + 0.25 &= 0 \\ -x_1 + x_2^2 + 0.25 &= 0 \end{aligned}$$

for which $\mathbf{x} = [0.5 \;\; 0.5]^\top$ is a solution.

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- linear if there exists $0 < r < 1$ such that $\lim_{n\to\infty} \frac{\|x_{n+1} - \alpha\|}{\|x_n - \alpha\|} = r$. Then r is called the rate of convergence.

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- cubic if $\lim_{n\to\infty} \frac{\|x_{n+1} - \alpha\|}{\|x_n - \alpha\|^3} = r$ for some real number $r > 0$.

  Example: $10^{-1}, 10^{-8}, 10^{-24}, 10^{-128} \ldots$

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$$|x_n - \alpha| \leq \epsilon_n \quad \text{and} \quad \frac{\epsilon_{n+1}}{\epsilon_n} = \frac{1}{2} < 1.$$

Thus the convergence is linear and the rate of convergence is $1/2$.

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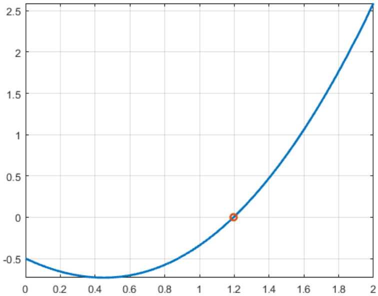
```
f =@(x) x.^2 -sin(x) -0.5;
fplot(f, [0, 2], 'LineWidth', 2), grid on
hold on
xc = bisect(f, 0, 2, 1e-12)
plot( xc, f(xc), 'o', 'LineWidth', 2)
```

```
xc = 1.196082033297898e+00
```



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Now $p(x) = 0$ yields

$$x = a_n - \frac{(a_n - b_n)}{f(a_n) - f(b_n)} f(a_n) = a_n - \frac{f(a_n)}{f[a_n, b_n]}.$$

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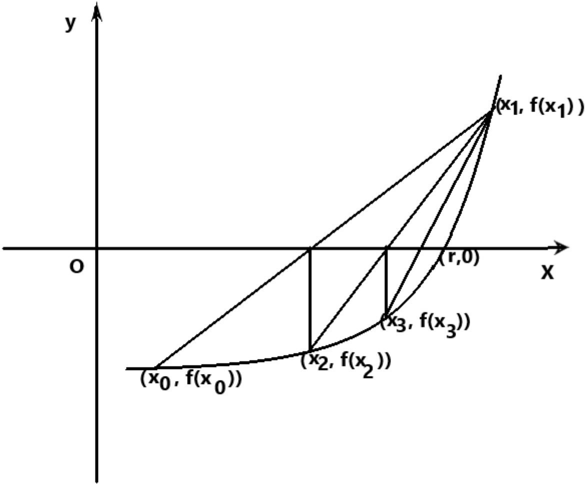

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Convergence analysis is easy when f is convex or concave. So assume that f is convex, say, $f''(x) > 0$ on $[a, b]$ with $f(a) < 0$ and $f(b) > 0$.

Then f has exactly one zero α in $[a, b]$. The secant joining $(a, f(a))$ and $(b, f(b))$ lies above the graph $y = f(x)$ and hence intersects x-axis to the left of α. This will be case for all subsequent secants.

Hence b remains fixed while a gets updated, producing increasing sequence with $x_1 = a$ and

$$x_{n+1} = x_n - \frac{x_n - b}{f(x_n) - f(b)} f(x_n), \; n = 1, 2, \ldots.$$

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Since (x_n) is increasing and bounded above by α and hence convergent, letting $n \to \infty$, we have $f(x_n) \to 0$ and hence $x_n \to \alpha$. (Why?)

Secant method

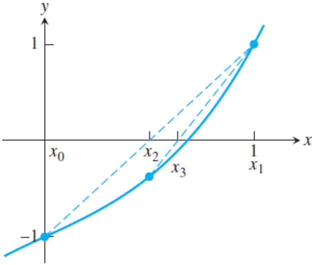

Figure : Secant method for finding the zero of $x^3 + x - 1$.

The secant method is a simple variant of regular-falsi method in which it is no longer required that f has opposite signs at the end points of each intervals generated.

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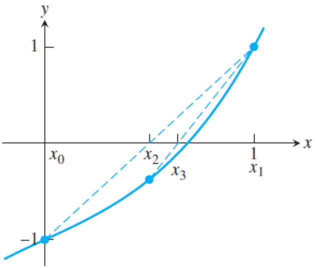

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One starts with two arbitrary $x_0 \neq x_1$ generates the iterates

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n), \; n = 1, 2, 3, \ldots .$$

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If α is a simple zero, that is, $f'(\alpha) \neq 0$, and if $x_n \to \alpha$ then

$$\frac{f[x_{n-1}, x_n, \alpha]}{f[x_{n-1}, x_n]} \longrightarrow \frac{f''(\alpha)}{2f'(\alpha)} \quad \text{as} \quad n \to \infty$$

when f is twice continuously differentiable near α.

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Then we have $\lim\limits_{n\to\infty} \dfrac{x_{n+1} - \alpha}{x_n - \alpha} = 0$ showing that the convergence is better than linear. The order of convergence is $p := \dfrac{1 + \sqrt{5}}{2}$.

Example

Consider $f(x) = \cos(x)\cosh(x) - 1$ with $a = 3\pi/2$ and $b = 2\pi$. Then f is convex on $[a, b]$ as $f''(x) = -2\sin(x)\sinh(x) > 0$ for $x \in [a, b]$.

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By contrast, secant method converges faster than both Bisection and Regula-Falsi methods. The rate of convergence of Regula-Falsi method is the slowest among all three methods.

Convergence of Regula-Falsi and Secant Methods

Both Secant and Regula-Falsi Methods can converge very slowly.

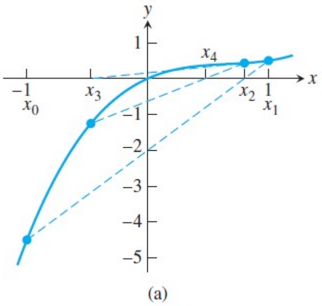

Figure : Slow convergence of (a) Secant Method and (b) Regula-Falsi Method to the root 0 of $x^3 - 2x^2 + \frac{3}{2}x$.