

MODEL ANSWERS OF QUIZ IV

1. (5 points) Let X_1, X_2, \dots, X_n be a random sample on the lifetime of an integrated circuit. Let the lifetime of the integrated circuit has the probability density function

$$f(x, \theta) = \begin{cases} 2\lambda x e^{-\lambda x^2} & \text{if } x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda > 0$ is an unknown parameter. Let α denote the mean number of integrated circuits (out of n circuits) that fail within the first unit time. Find maximum likelihood estimator of α .

Solution: Let X be a random variable that denotes the number of circuits that fail within the first unit time. Then $X \sim \text{Bin}(n, p(\lambda))$, where

$$p(\lambda) = P(X_1 \leq 1) = \int_0^1 2\lambda x e^{-\lambda x^2} dx = 1 - e^{-\lambda}.$$

Therefore, we need to find the MLE of $\alpha = n(1 - e^{-\lambda})$. Now, the likelihood function of λ based on observed data is

$$L(\lambda) = 2^n \lambda^n \left(\prod_{i=1}^n x_i \right) e^{-\lambda \sum_{i=1}^n x_i} \quad \text{if } \lambda > 0.$$

Therefore, the log-likelihood function of λ is

$$l(\lambda) = \ln L(\lambda) = c + n \ln \lambda - \lambda \sum_{i=1}^n x_i^2 \quad \text{for } \lambda > 0.$$

Then

$$\frac{dl}{d\lambda} = 0 \implies \lambda = \frac{n}{\sum_{i=1}^n x_i}.$$

Moreover,

$$\frac{d^2 l}{d\lambda^2} = -\frac{n}{\lambda^2} < 0 \quad \text{for all } \lambda > 0.$$

Thus, the likelihood function attains its' maximum at $\lambda = \frac{n}{\sum_{i=1}^n x_i}$. Therefore, the MLE of λ is

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^n X_i^2}.$$

Now, using invariance property of MLE, the MLE of α is

$$\hat{\alpha} = n(1 - e^{-\hat{\lambda}}) = n \left(1 - \exp \left\{ -\frac{n}{\sum_{i=1}^n X_i^2} \right\} \right).$$

2. (5 points) Let X_1, X_2, \dots, X_n be a random sample from a population having $Uniform(\theta_1, \theta_2)$ distribution, where $-\infty < \theta_1 < \theta_2 < \infty$. Also, assume that both θ_1 and θ_2 are unknown parameters. Is $T = \frac{1}{2} (X_{(1)} + X_{(n)})$ an unbiased estimator of population mean $\mu = \frac{1}{2} (\theta_1 + \theta_2)$? Justify your answer.

Solution: Here, we want to check if $E(T) = \mu$ for all θ_1 and θ_2 . Note that

$$E(T) = \frac{1}{2} (E(X_{(1)}) + E(X_{(n)})) .$$

Now, the CDF of $X_{(1)}$ is

$$F_{X_{(1)}}(x) = \begin{cases} 0 & \text{if } x < \theta_1 \\ 1 - \left(1 - \frac{x - \theta_1}{\theta_2 - \theta_1}\right)^n & \text{if } \theta_1 \leq x \leq \theta_2 \\ 1 & \text{if } x > \theta_2. \end{cases}$$

Therefore, the PDF of $X_{(1)}$ is

$$f_{X_{(1)}}(x) = \begin{cases} \frac{n(\theta_2 - x)^{n-1}}{(\theta_2 - \theta_1)^n} & \text{if } \theta_1 \leq x \leq \theta_2 \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{aligned} E(\theta_2 - X_{(1)}) &= \frac{n}{(\theta_2 - \theta_1)^n} \int_{\theta_1}^{\theta_2} (\theta_2 - x)^{n+1} dx = \frac{n}{n+1} (\theta_1 + \theta_2) \\ \implies E(X_{(1)}) &= \frac{1}{n+1} \theta_2 + \frac{n}{n+1} \theta_1. \end{aligned}$$

Now, the CDF of $X_{(n)}$ is

$$F_{X_{(n)}}(x) = \begin{cases} 0 & \text{if } x < \theta_1 \\ \left(\frac{x - \theta_1}{\theta_2 - \theta_1}\right)^n & \text{if } \theta_1 \leq x \leq \theta_2 \\ 1 & \text{if } x > \theta_2. \end{cases}$$

Therefore, the PDF of $X_{(n)}$ is

$$f_{X_{(n)}}(x) = \begin{cases} \frac{n(x - \theta_1)^{n-1}}{(\theta_2 - \theta_1)^n} & \text{if } \theta_1 \leq x \leq \theta_2 \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{aligned} E(X_{(n)} - \theta_1) &= \frac{n}{(\theta_2 - \theta_1)^n} \int_{\theta_1}^{\theta_2} (x - \theta_1)^{n+1} dx = \frac{n}{n+1} (\theta_2 - \theta_1) \\ \implies E(X_{(n)}) &= \frac{n}{n+1} \theta_2 + \frac{1}{n+1} \theta_1. \end{aligned}$$

Therefore,

$$E(\hat{\mu}) = \frac{1}{2} (E(X_{(1)}) + E(X_{(n)})) = \frac{\theta_1 + \theta_2}{2} = \mu,$$

for all $-\infty < \theta_1 < \theta_2 < \infty$. Hence, T is an unbiased estimator of population mean μ .

3. (5 points) Let X_1, X_2 be a random sample of size 2 from a population having cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{if } x < \theta \\ 1 - (1 - x + \theta)^2 & \text{if } \theta \leq x \leq \theta + 1 \\ 1 & \text{if } x > \theta + 1, \end{cases}$$

where $\theta \in \mathbb{R}$ is an unknown parameter. Is $[X_{(2)} - 1, X_{(2)}]$ a 95% confidence interval for θ ? Justify your answer.

Solution: The CDF of $X_{(2)} = \max\{X_1, X_2\}$ is

$$F_{X_{(2)}}(x) = \begin{cases} 0 & \text{if } x < \theta \\ (1 - (1 - x + \theta)^2)^2 & \text{if } \theta \leq x \leq \theta + 1 \\ 1 & \text{if } x \geq \theta + 1. \end{cases}$$

Therefore,

$$\begin{aligned} P(X_{(2)} - 1 \leq \theta \leq X_{(2)}) &= P(-1 \leq \theta - X_{(2)} \leq 0) \\ &= P(\theta \leq X_{(2)} \leq \theta + 1) \\ &= F_{X_{(2)}}(\theta + 1) - F_{X_{(2)}}(\theta) \\ &= 1 \geq 0.95. \end{aligned}$$

Thus, $[X_{(2)} - 1, X_{(2)}]$ is a 95% confidence interval for θ .