# MA579H Scientific Computing

# Numerical Integration III

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# Lecture outline

- Computation of nodes and weights for Gaussian quadrature

# Construction of orthogonal polynomials

Let $\langle p, q\rangle$ be an inner product on $\mathcal{P}_n = \mathrm{span}(1, x, \ldots, x^n)$. For example,

$$\text{Legendre:} \quad \langle p, q\rangle \quad := \quad \int_{-1}^{1} p(x)q(x)dx$$

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Given an inner product on $\mathcal{P}_n$, there exist orthogonal polynomials $\phi_0(x), \phi_1(x), \ldots, \phi_n(x)$ such that $\deg(\phi_\ell(x)) = \ell$ for $\ell = 0 : n$ and

$$\mathrm{span}(1, x, \ldots, x^j) = \mathrm{span}(\phi_0(x), \phi_1(x), \ldots, \phi_j(x)), \quad j = 0 : n.$$

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Recurrence relation (monic Legendre):

$$P_{n+1}(x) = xP_n(x) - \frac{n^2}{4n^2-1}P_{n-1}(x).$$

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$$L_n(x) := \frac{e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^n) = \frac{1}{n!}\left(\frac{d}{dx}-1\right)^n x^n.$$

Recurrence relation (monic Laguerre):

$$L_{n+1}(x) = (x-(2n+1))L_n(x) - n^2L_{n-1}(x).$$

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$$\begin{aligned} T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x), \\ \widehat{T}_{n+1}(x) &= x\widehat{T}_n(x) - \frac{1}{4}\widehat{T}_{n-1}(x), \end{aligned}$$

where $\widehat{T}_n(x) = T_n(x)/2^{n-1}$ is a monic Chebyshev polynomial.

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$$\alpha_n = \frac{\langle x\phi_n,\, \phi_n\rangle}{\langle \phi_n,\, \phi_n\rangle}, \quad \beta_n = \frac{\langle \phi_n,\, \phi_n\rangle}{\langle \phi_{n-1},\, \phi_{n-1}\rangle} > 0.$$

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Proof: Note that $\phi_{n+1} - x\phi_n \in \mathcal{P}_n$, hence

$$\phi_{n+1} - x\phi_n = \sum_{j=0}^{n} \frac{\langle \phi_{n+1} - x\phi_n, \, \phi_j \rangle}{\langle \phi_j, \, \phi_j \rangle} \phi_j$$

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&= -\frac{\langle x\phi_n,\, \phi_n \rangle}{\langle \phi_n,\, \phi_n \rangle} \phi_n - \frac{\langle x\phi_n,\, \phi_{n-1} \rangle}{\langle \phi_{n-1},\, \phi_{n-1} \rangle} \phi_{n-1}
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because $\displaystyle\sum_{j=0}^{n-2} \frac{\langle x\phi_n,\, \phi_j \rangle}{\langle \phi_j,\, \phi_j \rangle} \phi_j = 0$ as $x\phi_j \in \mathcal{P}_{n-1}$ for $j = 0, \ldots, n-2$.

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Indeed $x\phi_{n-1} = \phi_n +$ lower order terms, we have $\langle \phi_n,\, x\phi_{n-1}\rangle = \langle \phi_n,\, \phi_n\rangle$ which yields $\beta_n$. ■

# Zeros of orthogonal polynomials

Theorem: Let $\phi_0(x), \ldots, \phi_{n+1}(x)$ be orthogonal polynomials such that

$$\phi_{n+1}(x) = (x - \alpha_n)\phi_n(x) - \beta_n \phi_{n-1}(x).$$

Define the Jacobi matrix

$$A = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \ddots & \ddots & \\ & & \ddots & \ddots & \sqrt{\beta_n} \\ & & & \sqrt{\beta_n} & \alpha_n \end{bmatrix}.$$

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```
[V, D] = eig(A); x = diag(D); w = transpose(inp*V(1, :).^2);
```

where `inp` = $\langle 1, 1\rangle$

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Observe that $\mathbf{u}_j := \begin{bmatrix} \psi_0(x_j) & \cdots & \psi_n(x_j) \end{bmatrix}^\top$ is an eigenvector of $A$ corresponding to the eigenvalue $x_j$. Hence $\mathbf{u}_j = \mathbf{v}_j d_j$ for some $d_j \in \mathbb{R}$.

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Set $U := \begin{bmatrix} \mathbf{u}_0 & \cdots & \mathbf{u}_n \end{bmatrix}$ and $V := \begin{bmatrix} \mathbf{v}_0 & \cdots & \mathbf{v}_n \end{bmatrix}$. Then $U = VD$, where $D = \mathrm{diag}(d_0, \ldots, d_n)$.

# Proof

Observe that $\mathbf{u}_j := \begin{bmatrix} \psi_0(x_j) & \cdots & \psi_n(x_j) \end{bmatrix}^\top$ is an eigenvector of $A$ corresponding to the eigenvalue $x_j$. Hence $\mathbf{u}_j = \mathbf{v}_j d_j$ for some $d_j \in \mathbb{R}$.

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By exactness of $G_n(f)$, we have

$$\begin{aligned} w_0\psi_0(x_0) + \cdots + w_n\psi_0(x_n) &= G_n(\psi_0) = \int_a^b \psi_0(x)\mu(x)dx = \langle 1, 1 \rangle / \|\phi_0\| \\ w_0\psi_j(x_0) + \cdots + w_n\psi_j(x_n) &= G_n(\psi_j) = \int_a^b \psi_j(x)\mu(x)dx = \langle \psi_j, 1 \rangle = 0 \end{aligned}$$

for $j = 1 : n$.

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# Proof

Since $V$ is orthogonal, we have $U^{-1} = (VD)^{-1} = D^{-1}V^\top$. Hence

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Now using the fact that $d_j = 1/(\|\phi_0\|\mathbf{e}_1^\top\mathbf{v}_j)$, we have

$$\begin{aligned} w_j &= \mathbf{e}_j^\top\mathbf{w} = (\langle 1, 1\rangle/\|\phi_0\|)\mathbf{e}_j^\top D^{-1}V^\top\mathbf{e}_1 \\ &= (\langle 1, 1\rangle/(\|\phi_0\|d_j))\mathbf{e}_j^\top V^\top\mathbf{e}_1 = \langle 1, 1\rangle(\mathbf{e}_1^\top V\mathbf{e}_j)^2 \\ &= \langle 1, 1\rangle(\mathbf{e}_1^\top\mathbf{v}_j)^2. \quad \blacksquare \end{aligned}$$

Thus the weights `w` and nodes `x` can be computed in MATLAB as

```
[V, D] = eig(A); x = diag(D); w = transpose(inp*V(1, :).^2);
```

where `inp` = $\langle 1, 1\rangle$

# Gauss Legendre quadrature

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$$A = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & & & \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{15}} & & \\ & \frac{2}{\sqrt{15}} & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{n}{\sqrt{4n^2-1}} \\ & & & \frac{n}{\sqrt{4n^2-1}} & 0 \end{bmatrix}.$$

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```
[V, D] = eig(A); x = diag(D); w = transpose(pi*V(1, :).^2);
```