Homework-3

MA580H : Matrix Computations 2025 R. Alam

Stability of algorithms and pivot growth

1. Let $A \in \mathbb{C}^{m\times n}$ and suppose that an algorithm ALG computes the singular value decomposition $A = U\Sigma V^*$, where U and V are unitary, and Σ is diagonal with nonnegative diagonal entries appearing in descending order of their magnitudes. Describe what it means to say that the algorithm ALG is backward stable.

2. For each of the following problems state with justification whether the given algorithm is backward stable or not.

   (a) Data: $d \in F(\beta, t, L, U)$, Solution: $f(d) = 1 + d, \mathrm{ALG}(d) = \mathrm{fl}(1 + d)$.

   (b) Data: $d \in F(\beta, t, L, U)$, Solution: $f(d) = 2d, \mathrm{ALG}(d) = \mathrm{fl}(d + d)$.

   (c) Data: $d \in F(\beta, t, L, U)$, Solution: $f(d) = d^2, \mathrm{ALG}(d) = \mathrm{fl}(d * d)$.

   (d) Data: $d_1, \ldots, d_m \in F(\beta, t, L, U)$, Solution: $f(d) = \sum_{j=1}^{m} d_j$ and $\mathrm{ALG}(d_1, \ldots, d_m)$ is given by

      $s = d_1$
      for $j = 2{:}\, m$
          $s = \mathrm{fl}(s + d_j)$
      end

3. Let x and y be nonzero vectors in $\mathbb{R}^n$. Consider the rank-1 matrix $A := xy^\top$ also referred to as the outer product of x and y. Show that the computation of A in finite precision arithmetic is NOT backward stable.

4. Let $A = LU$ be the LU factorization of a matrix $A \in \mathbb{C}^{n\times n}$ with $L(i, j)| \leq 1$. Let A_i and U_i denote the i-th row of A and U, respectively. Show that $U_i = A_i - \sum_{j=1}^{i-1} L(i, j)U_j$ and use it to show that $\|U\|_\infty \leq 2^{n-1}\|A\|_\infty$. Define $PG(A) := \|L\|_\infty\|U\|_\infty/\|A\|_\infty$ and show that $PG(A) \leq n2^{n-1}$.

5. Suppose that $A \in \mathbb{R}^{n\times n}$ is SPD and that $A = GG^T$. Show that $\|G\|_2\|G^T\|_2 = \|A\|_2$ and that $\|G\|_\infty\|G^T\|_\infty \leq n^{3/2}\|A\|_\infty$. [Hint: Use relation between $\|x\|_\infty$ and $\|x\|_2$ for $x \in \mathbb{R}^n$.]

   Conclude that the spectral norm pivot growth $PG_2(A) := \frac{\|G\|_2\|G^T\|_2}{\|A\|_2} = 1$ and the ∞-norm pivot growth $PG_\infty(A) := \frac{\|G\|_\infty\|G^T\|_\infty}{\|A\|_\infty} \leq n^{3/2}$.

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