

# STATISTICAL FOUNDATION OF DATA SCIENCE (MA 589)

Lecture Slides

Topic 04: Limit Theorems

# Modes of Convergence

- If  $\{x_n\}$  is a sequence of real numbers, we know the meaning of  $x_n \rightarrow l$  as  $n \rightarrow \infty$ .
- If  $\{f_n\}$  be a sequence of functions, we have several modes of convergence.
- Here we will discuss four modes of convergence for a sequence of random variables  $\{X_n\}$ .
- These are quite useful concept in probability and statistics.

# Almost Sure Convergence

**Definition 4.1:** Let  $\{X_n\}$  be a sequence of random variables defined on a probability space  $(\mathcal{S}, \mathcal{F}, P)$ . Let  $X$  be a random variable defined on the same probability space  $(\mathcal{S}, \mathcal{F}, P)$ . We say that  $X_n$  converges almost surely or with probability 1 to a random variable  $X$  if

$$P(\omega : X_n(\omega) \rightarrow X(\omega)) = 1.$$

# Convergence in $r^{th}$ Mean

**Definition 4.2:** Let  $\{X_n\}$  be a sequence of random variables defined on a probability space  $(\mathcal{S}, \mathcal{F}, P)$ . Let  $X$  be a random variable defined on the same probability space  $(\mathcal{S}, \mathcal{F}, P)$ . We say that  $X_n$  converges in  $r^{th}$  mean to a random variable  $X$  if

$$E|X_n - X|^r \rightarrow 0.$$

# Convergence in Probability

**Definition 4.3:** Let  $\{X_n\}$  be a sequence of random variables defined on a probability space  $(\mathcal{S}, \mathcal{F}, P)$ . Let  $X$  be a random variable defined on the same probability space  $(\mathcal{S}, \mathcal{F}, P)$ . We say that  $X_n$  converges in probability to a random variable  $X$  if for any  $\epsilon > 0$ ,

$$P(|X_n - X| > \epsilon) \rightarrow 0.$$

# Convergence in Distribution

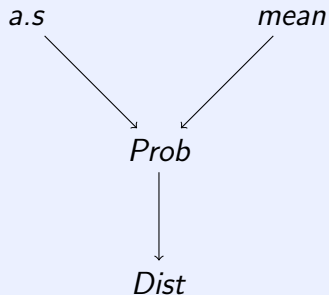
**Definition 4.4:** We say that  $X_n$  converges in distribution to a random variable  $X$  if

$$F_n(x) \rightarrow F(x)$$

for all  $x$  where  $F$  is continuous and where  $F_n$ s are the distribution functions of  $X_n$ s and  $F$  is the distribution function of  $X$ .

**Remark 4.1:** Unlike the first three modes of convergence, here  $X_n$ s can be defined on different probability spaces. We are only interested in the distribution functions. This flexibility makes this mode of convergence very useful.

# Relation between Modes of Convergence



# Strong Law of Large Numbers

**Theorem 4.1:** (Strong Law of Large Numbers) Let  $\{X_n\}$  be a sequence of i.i.d. RVs with finite mean  $\mu$ . Define  $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ . Then  $\{\bar{X}_n\}$  converges to  $\mu$  almost surely, i.e.,

$$P(\{\omega : \bar{X}_n(\omega) \rightarrow \mu\}) = 1.$$

We will use the notation  $\bar{X}_n \xrightarrow{a.s.} \mu$ .



# Some Examples

**Example 4.1:** Bernoulli proportion converges to success probability.

**Example 4.2:** Monte Carlo Integration.

**Example 4.3:** Let  $X_i$  and  $Y_i$ ,  $i = 1, 2, \dots$  are independently and identically distributed  $U(0, 1)$  random variables. Let  $N_n = \# \{k : 1 \leq k \leq n, X_k^2 + Y_k^2 \leq 1\}$ . Then  $\frac{4N_n}{n}$  converges to  $\pi$  with probability one.

# Central Limit Theorem

**Theorem 4.2:** (Central Limit Theorem) Let  $\{X_n\}$  be a sequence of i.i.d. RVs with mean  $\mu$  and variance  $\sigma^2 < \infty$ . Then, as  $n \rightarrow \infty$ ,

$$P\left(\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \leq a\right) \rightarrow \Phi(a) = \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt.$$

We will use the notation  $\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{\mathcal{D}} Z \sim N(0, 1)$ .

# Some Examples

**Example 4.4:**  $X_n \sim \text{Bin}(n, p)$ . Then

$$P\left(\frac{X_n - np}{\sqrt{np(1-p)}} \leq a\right) \rightarrow \Phi(a).$$

**Example 4.5:** The lifetimes of a special type of battery is a RV with mean 40 hours and standard deviation 20 hours. A battery is used until it fails, at which point it is replaced by a new one. Assume a stockpile of 25 such batteries, the lifetimes of which are independent, approximate the probability that over 1100 hours of use can be obtained. [ $\Phi(1) = 0.8413$ ]

**Example 4.6:** Show that

$$\lim_{n \rightarrow \infty} e^{-n} \sum_{k=0}^n \frac{n^k}{k!} = \frac{1}{2}.$$