End Semester Examination

MA-581 : Numerical Computing Lab **Time : 3 hours** **50 marks**

November 11, 2025

Answer ALL question.

1. Suppose that t the running time of an algorithm of complexity $\mathcal{O}(n^p)$. Then $t \approx Cn^p$ for sufficiently large n. Hence $\log(t) \approx p\log(n) + \log(C)$. So, we expect that a graph of $\log(t)$ as a function of $\log(n)$ to be a straight line of slope p. Also, if t_n and t_N are running times with $n < N$ then $t_n/t_N \approx (n/N)^p \Longrightarrow t_n \approx t_N(n/N)^p$.

   Check the efficiency of LU factorization by plotting the timings on a log-log graph and comparing with $\mathcal{O}(n^3)$ flops. The result could vary significantly from machine to machine, but in theory the data should start to parallel the line as $n \to \infty$.

```
n = (200:100:2400)';
t = zeros(size(n));
for k = 1:length(n)
A = randn(n(k), n(k));
tic     % start a timer
for j = 1:6,  [L, U] = lu(A);  end
time = toc;
t(k) = time / 6;
end

clf
loglog(n, t, 'o-')
hold on
loglog(n,  t(end) * (n/n(end)).^3, 'k--')
axis tight
xlabel('size of matrix'), ylabel('time (sec)')
title('Timing of LU factorization')
legend('lu','O(n^3)','location','southeast');
```

   What is your observation? **5 marks**

2. (a) Wilkinson's matrix is defined as follows: 1 on the diagonal, -1 everywhere below the main diagonal, 1 in the last column, and 0 everywhere else. Write a MATLAB function `W = Wilkinson(n)` that generates Wilkinson's matrix W of size n using MATLAB functions **eye, tril** and **ones**.

   For $n = 32$, pick a random x and then compute $b := W * x$. Solve $Ax = b$ using MATLAB backslash command and compute the error $\|x - \hat{x}\|_\infty / \|x\|_\infty$. Does the size of the error confirm that GEPP is unstable for this system? Also compute $\text{cond}(A)$. Can the poor answer be attributed to ill-conditioning of the matrix W? Repeat the test for $n = 64$.

   (b) Pivot growth of Gaussian elimination with partial pivoting (GEPP) is given by $PG(A) = \max_{ij} |U(i,j)| / \max_{ij} |A(i,j)|$, which influences the accuracy of computed solution. Use MATLAB function `[L, U, p] = lu(A)` for computing LU decomposition of a nonsingular matrix A and compute the pivot growth `rho` $= PG(A)$.

It is well known that the pivot growth factor for GEPP satisfies $PG(A) \leq 2^{n-1}$ which is attained by the Wilkinson matrix. Verify this graphically by doing the following:

First plot the graph of 2^{n-1} in $\log 10$ scale for $n = 10 : .5 : 505$ by setting `X = 2.^(n-1)` and then typing `semilogy(n,X,'r')`. Hold this plot by typing `hold on` and type the following sequence of commands (which assumes that the Wilkinson matrix of size n is generated by the function `W = Wilkinson(n)`).

```
n = 10:20:500;
m = length(n); G = zeros(m,1);
for i = 1:m
W = Wilkinson(n(i)); [L,U,p] = lu(W);
G(i) = max(max(abs(U)))/max(max(abs(W)));
end
semilogy(n,G,'b*')
```

The second plot should come in the form of blue dots that fall on the red curve produced by the earlier plot.

However, statistics suggest that for most practical examples, $PG(A) \leq n^{2/3}$ for GEPP. In fact, for random matrices, the average pivot growth for GEPP is $PG(A) \leq \frac{1}{4}n^{0.71}$. You can verify this fact for random matrices as follows.

```
N=500;
n=[10 20 30 40 50 60 70 80 90 100];
for j=1:length(n)
m=0;
for i=1:N
A=rand(n(j));
[L,U,P]=lu(A);
rho = max( max( abs(U)))/max( max( abs(A)));
m=m+rho;
end;
g(j)=m/N;
end;
plot(n,g,'--', n,0.25*n.^(0.71),'-');
legend('average growth factor', '0.25*n^{0.71}','Location','NorthWest')
xlabel('matrix size'),ylabel('growth factor \rho')
```

Comment on your results. **10 marks**

3. The amount of waste (in millions of tons in a day) generated in a city from 19960 to 1995 was

year	1960	1965	1970	1975	1980	1985	1990	1995
amount	88	99.8	115.8	125	132.6	143.1	156.3	169.5

Find the equation of a straight line that best fits the data. Use the straight line to estimate the waste in the years 2000 and 2005.

Next, determine an exponential function $y = ce^{\alpha t}$, where c and α are constants to be determined, that best fits the data. Use the exponential model to estimate the waste in the years 2000 and 2005. Now plot the data points, the straight line and the function $y = ce^{\alpha t}$ in a single plot and comment on which model is better. **10 marks**

4. A compressed image of a grayscale image `A` is computed as follows. Compute the SVD $\mathtt{A} = U\Sigma V^\top$ and the best k rank approximation $A_k := U \begin{bmatrix} \mathrm{diag}(\sigma_1, \ldots, \sigma_k) & 0 \\ 0 & 0 \end{bmatrix} V^T$ for a chosen value of $k < \mathrm{rank}(A)$. Then A_k represents a compressed image. The compressed image can be displayed by the command `image(`A_k`)`.

   The storage required for A_k is $k(m+n)$ whereas the storage required for the full image is mn. Therefore, $\frac{(m+n)k}{mn}$ gives the compression ratio for the compressed image. Also the error in the representation is $\frac{\sigma_{k+1}}{\sigma_1}$.

   The MATLAB command `load clown` loads gray scale image of a clown in an array `X`. Use `X` in place of `A` and run the following commands for various choices of k and make a table that records the relative errors and compression ratios for each choice.

   ```
   load clown.mat; [U, S, V] = svd(X); colormap('gray');
   image(U(:, 1:k)*S(1:k, 1:k)*V(:,1:k)')
   ```

   How does the choice of approximating rank k affect the visual qualities of the images?

   5 marks

5 5. Determine the polynomial of degree 19 that best fits the function $f(t) = \sin(\frac{\pi}{5}t) + \frac{t}{5}$ for $t_1 = -5, t_2 = -4.5, \ldots, t_{23} = 6$. Setup the LSP $Ax = b$ and determine the polynomial p in two different ways:

   (a) By using the matlab command
   `>> A \ b`
   This uses QR factorization to solve the LSP $Ax = b$. Call this polynomial p_1.

   (b) By solving the normal equation $A^*Ax = A^*b$. Use `x = (A'*A)\(A'*b)`. Call this polynomial p_2.

   Compute the condition number (use the matlab command `cond(A)`) of the coefficient matrix associated with each of the systems that you are solving. Print the result to 16 digits (use `format long e`). Which one is the most ill conditioned?

   The norm of the residual $\|r\|_2 = \|Ax - b\|_2$ gives an idea of the goodness of the fit. Compute residual for each method.

   Finally, plot the polynomials p_1, p_2 and the function f on $[-5, 6]$. Use different colors to distinguish these plots. Do you observe any difference? If yes, which polynomial is a better approximation of f? **10 marks**

6 5. Write a MATLAB function $[\mathtt{x}, \mathtt{mu}] = \mathtt{powermethod}(\mathtt{A}, \mathtt{x0}, \mathtt{N})$ for performing `N` iterations of Power method ($x_j = Ax_{j-1}, x_j \leftarrow x_j/\|x_j\|_2, \mu_j = x_j^* A x_j$) for finding the dominant eigenvalue of a matrix A with starting vector `x0` such that $\|\mathtt{x0}\|_2 = 1$. Here iterates `x` is an $n \times N$ matrix such that `x(:, j)` is the vector obtained at the end of the j-th iteration and `mu` is a vector of size `N` such that `mu(j)` is the Rayleigh quotient $\mu_j = x_j^* A x_j$ obtained at the end of j-th iteration. We know that μ_j approximates the dominant eigenvalue of A and x_j approximates a corresponding eigenvector.