MA579H Scientific Computing

Polynomial Interpolation-II

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Lecture outline

- Newton interpolating polynomial
- Divided difference

Newton interpolating polynomial

Example: For the data set $(x_0, f_0), (x_1, f_1), (x_2, f_2)$, consider Newton polynomial $p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$.

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$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & (x_1 - x_0) & 0 \\ 1 & (x_2 - x_0) & (x_2 - x_0)(x_2 - x_1) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix}.$$

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For the data set $(-2, -27), (0, -1), (1, 0)$, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -27 \\ -1 \\ 0 \end{bmatrix} \Longrightarrow [a_0, a_1, a_2] = [-27, 13, -4].$$

Hence $p(x) = -27 + 13(x + 2) - 4(x + 2)x = -1 + 5x - 4x^2$.

Newton interpolating polynomial

Define $\mathrm{N}_0(x) := 1$ and $\mathrm{N}_j(x) := (x - x_0) \cdots (x - x_{j-1})$ for $j = 1 : n$.
Then $\mathrm{N}_j(x_i) = 0$ for $i = 0 : j - 1$ and $j = 1 : n$.

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Fact: The Newton polynomials $\mathrm{N}_0(x), \ldots, \mathrm{N}_n(x)$ form a basis of $\mathcal{P}_n$ which is referred to as Newton basis.

Let $p(x) := a_0 \mathrm{N}_0(x) + \cdots + a_n \mathrm{N}_n(x)$. The interpolation conditions $p(x_j) = f_j$ for $j = 0 : n$ yield the lower triangular system

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$$\begin{bmatrix} \mathrm{N}_0(x_0) & 0 & 0 & \cdots & 0 \\ \mathrm{N}_0(x_1) & \mathrm{N}_1(x_1) & 0 & \cdots & 0 \\ \mathrm{N}_0(x_2) & \mathrm{N}_1(x_2) & \mathrm{N}_2(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{N}_0(x_n) & \mathrm{N}_1(x_n) & \mathrm{N}_2(x_n) & \cdots & \mathrm{N}_n(x_n) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}.$$

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Let $p(x) := a_0\mathrm{N}_0(x) + \cdots + a_n\mathrm{N}_n(x)$. The interpolation conditions $p(x_j) = f_j$ for $j = 0 : n$ yield the lower triangular system

$$\begin{bmatrix} \mathrm{N}_0(x_0) & 0 & 0 & \cdots & 0 \\ \mathrm{N}_0(x_1) & \mathrm{N}_1(x_1) & 0 & \cdots & 0 \\ \mathrm{N}_0(x_2) & \mathrm{N}_1(x_2) & \mathrm{N}_2(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{N}_0(x_n) & \mathrm{N}_1(x_n) & \mathrm{N}_2(x_n) & \cdots & \mathrm{N}_n(x_n) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}.$$

Solving the triangular system for $a_j, j = 0 : n$, we obtain the interpolating polynomial $p(x) = a_0\mathrm{N}_0(x) + \cdots + a_n\mathrm{N}_n(x)$ in Newton's form.

Newton interpolating polynomial

Remark:

- Solution of the lower triangular system requires $\mathcal{O}(n^2)$ flops.
- Computation of $p(x)$ requires $\mathcal{O}(n^2)$ operations.
- Can accommodate new data (x_{n+1}, f_{n+1}) with additional $\mathcal{O}(n)$ operations.
- Evaluation of $p(x)$ at a given x requires $\mathcal{O}(n^2)$ operations.

Computation of $\mathrm{N}_j(x_j)$ may be prone to overflow/underflow. However, $p(x)$ can be computed more efficiently using divided difference table.

Divided differences

Consider the data set $(x_0, f_0), \ldots, (x_n, f_n)$. Define

$$
\begin{aligned}
f[x_j] &:= f_j \text{ for } j = 0 : n, \\
f[x_0, x_1] &:= \frac{f[x_1] - f[x_0]}{x_1 - x_0}
\end{aligned}
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f[x_0, x_1, x_2] &:= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \\
&\vdots \\
f[x_0, x_1, \ldots, x_n] &:= \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}
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The divided differences can update new data (x_{n+1}, f_{n+1}) with $\mathcal{O}(n)$ additional operations.

Newton interpolating polynomial

Consider $(x_0, f_0), (x_1, f_1), (x_2, f_2)$ and the Newton polynomial $p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$. Set $f[x_j] := f_j$ for $j = 0, 1, 2$. Then $p(x_0) = f_0, p(x_1) = f_1$ and $p(x_2) = f_2$ yield

$$a_0 = f_0 = f[x_0]$$

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$$\begin{aligned} a_0 &= f_0 = f[x_0] \\ a_1 &= \frac{f[x_1] - f[x_0]}{x_1 - x_0} = f[x_0, x_1] \end{aligned}$$

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$$\begin{aligned} a_0 &= f_0 = f[x_0] \\ a_1 &= \frac{f[x_1] - f[x_0]}{x_1 - x_0} = f[x_0, x_1] \\ a_2 &= \left(\frac{f[x_2] - f[x_0]}{x_2 - x_0} - f[x_0, x_1] \right) / (x_2 - x_1) \end{aligned}$$

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&= \left(\frac{f[x_2] - f[x_1]}{x_2 - x_0} + \frac{f[x_1] - f[x_0]}{x_2 - x_0} - f[x_0, x_1] \right) / (x_2 - x_1)
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&= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = f[x_0, x_1, x_2]
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&= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = f[x_0, x_1, x_2]
\end{aligned}$$

This shows that
$p(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$.

Divided differences table

Divided differences can be generated by using the table of divided differences:

x	**f**			
x_0	f_0			
x_1	f_1	$f[x_0, x_1]$		
x_2	f_2	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$	
x_3	f_3	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$
.	.	$\cdots$	$\cdots$	$\cdots$

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x_3	f_3	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$
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The $n+1$ diagonal entries in the table of divided differences are the coefficients of the Newton interpolating polynomial $p(x)$.

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x_3	f_3	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$
.	.	$\dots$	$\dots$	$\dots$

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Adding new data (x_{n+1}, f_{n+1}) means adding more row at the bottom of the table. As each divided difference is calculated in 3 flops, this can be done with $\mathcal{O}(n)$ additional operations.

Divided differences table

Example: The divided difference table for the data points $(0,3), (1,4), (2,7)$ and $(4,19)$ is

x	f			
0	3			
1	4	$\frac{4-3}{1-0} = 1$		
2	7	$\frac{7-4}{2-1} = 3$	$\frac{3-1}{2-0} = 1$	
4	19	$\frac{19-7}{4-2} = 6$	$\frac{6-3}{4-1} = 1$	$\frac{1-1}{4-0} = 0$

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The Newton interpolating polynomial is given by

$$p(x)=3+1\cdot(x-0)+1\cdot(x-0)(x-1)+0\cdot(x-0)(x-1)(x-2)=3+x^2.$$

Divided differences table

If the point $(5, 22)$ is added to the data, then the updated table is:

x	f				
0	3				
1	4	$\frac{4-3}{1-0} = 1$			
2	7	$\frac{7-4}{2-1} = 3$	$\frac{3-1}{2-0} = 1$		
4	19	$\frac{19-7}{4-2} = 6$	$\frac{6-3}{4-1} = 1$	$\frac{1-1}{4-0} = 0$	
5	22	$\frac{22-19}{5-4} = 3$	$\frac{3-6}{5-2} = -1$	$\frac{-1-1}{5-1} = -\frac{1}{2}$	$\frac{-\frac{1}{2}-0}{5-0} = -\frac{1}{10}$

The Newton interpolating polynomial is given by

$$q(x) = p(x) - \frac{1}{10}(x-0)(x-1)(x-2)(x-4) = x^4 - 7x^3 + 15x^2 - 8x + 3.$$

Newton polynomial via divided differences

Theorem: Let $p(x) = a_0\mathrm{N}_0(x) + \cdots + a_n\mathrm{N}_n(x)$ be such that $p(x_j) = f_j$ for $j = 0 : n$. Then

$$a_j = f[x_0, x_1, \ldots, x_j] \text{ for } j = 0 : n.$$

Proof: The result is true for $n = 0$. Assume that the result is true for Newton interpolating polynomials of degree $\leq n - 1$.

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Let $q(x) = \sum_{j=0}^{n-1} b_j\mathrm{N}_j(x)$ be the interpolating polynomial for the data $(x_1, f_1), \ldots, (x_n, f_n)$ and $s(x) = \sum_{j=0}^{n-1} c_j\mathrm{N}_j(x)$ be the interpolating polynomial for the data $(x_0, f_0), \ldots, (x_{n-1}, f_{n-1})$.

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By induction hypothesis $b_{n-1} = f[x_1, \ldots, x_n]$ and $c_{n-1} = f[x_0, \ldots, x_{n-1}]$.

Note that $r(x) := q(x) + \frac{x-x_n}{x_n-x_0}(q(x) - s(x))$ is the interpolating polynomials for the data $(x_0, f_0), \ldots, (x_n, f_n)$. (Check)

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By uniqueness, $r(x) = p(x)$. Equating the coefficients of x^n, we have

$$a_n = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0} = f[x_0, x_1, \ldots, x_n]. \blacksquare$$

Summary

For the data $(x_0, f_0), \ldots, (x_n, f_n)$, there is a unique $p(x) \in \mathcal{P}_n$ given by

- Vandermode interpolating polynomial: $p(x) = \sum_{j=0}^n a_j x^j$.
- Lagrange interpolating polynomial: $p(x) = \sum_{j=0}^n \ell_j(x) f_j$.
- Barycentric Lagrange: $p(x) = \dfrac{\sum_{j=0}^n \frac{w_j f_j}{x - x_j}}{\sum_{j=0}^n \frac{w_j}{x - x_j}}$.
- Newton interpolating polynomial:

$$p(x) = \sum_{j=0}^{n} f[x_0, \ldots, x_j](x - x_0) \cdots (x - x_{j-1}).$$

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- Newton interpolating polynomial:

$$p(x) = \sum_{j=0}^{n} f[x_0, \ldots, x_j](x - x_0) \cdots (x - x_{j-1}).$$

Example: For the data set $(-2, -27), (0, -1), (1, 0)$, we have the interpolating polynomials in various forms:

Vandermode:	$p(x) = -1 + 5x - 4x^2.$
Lagrange:	$p(x) = -\frac{9}{2}x(x-1) + \frac{1}{2}(x+2)(x-1).$
Newton:	$p(x) = -27 + 13(x+2) - 4(x+2)x$

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