Homework-1

MA-579H : Scientific Computing R. Alam **Date:** August 07, 2025

Polynomial Interpolation

1. True or false: When interpolating a continuous function by a polynomial at equally spaced points on a given interval, the polynomial interpolant always converges to the function as the number of interpolation points increases.

2. Is it ever possible for two distinct polynomials to interpolate the same n data points? If so, under what conditions, and if not, why?

3. For Lagrange polynomial interpolation of $n+1$ data points $(x_0, f_0), \ldots, (x_n, f_n)$,

   (a) What is the degree of each polynomial function $\ell_j(x)$ in the Lagrange basis?

   (b) What function results if we sum the $n+1$ functions in the Lagrange basis, that is, if we take $g(x) = \sum_{j=0}^{n} \ell_j(x)$, then determine $g(x)$.

4. How does Hermite interpolation differ from ordinary interpolation? How many times is a Hermite cubic interpolant continuously differentiable?

5. Given the data set $(-1, 1), (0, 1), (1, 2), (2, 0)$, determine the interpolating polynomial of least degree using (a) Lagrange basis and (b) Newton basis. Show that the two representations give the same polynomial.

6. Find the polynomial of lowest degree that passes through the points $(-2, -9), (-1, -1), (1, -9), (3, -9)$, and $(4, 9)$. Also, find a degree 6 polynomial that passes through the data points.

7. Determine the degree 25 polynomial that passes through the points $(1, -1)$, $(2, -2), \ldots,$ $(25, -25)$ and has constant term equal to 25.

8. Let $[x_0, \ldots, x_n]$ be distinct nodes in $[a, b]$. Consider the Lagrange basis $\ell_0(x), \ldots, \ell_n(x)$ of $\mathcal{P}_n$. Define $L_n : C[a, b] \longrightarrow \mathcal{P}_n$ by $L_n(f) = f(x_0)\ell_0 + \cdots + f(x_n)\ell_n$. Consider the Lebesgue function $\lambda_n(x) := |\ell_0(x)| + \cdots + |\ell_n(x)|$.

   (a) Set $p_n := L_n(f)$. Show that $f(x) - p_n(x) = \sum_{j=0}^{n}[f(x) - f(x_j)]\ell_j(x)$ for $f \in C[a, b]$.

   (b) Let $\hat{f}_j = f_j + \epsilon_i$, where $|\epsilon_i| \leq \epsilon$. Let $\hat{p}_n(x)$ be the Lagrange interpolating polynomial of the data set $(x_0, \hat{f}_0), \ldots, (x_n, \hat{f}_n)$. Show that $|p_n(x) - \hat{p}_n(x)| \leq \epsilon\, \lambda_n(x)$ for $x \in [a, b]$. What does this tell you about propagation of errors in the function values of the data points?

9. Let $p_n(x)$ be the interpolating polynomial in $\mathcal{P}_n$ that interpolates e^x at the nodes $x_j := j/n$, $j = 0 : n$. Derive an upper bound of the error $E_n := \max_{0 \leq x \leq 1} |e^x - p_n(x)|$ and determine the smallest n guaranteeing $E_n \leq 10^{-6}$.

10. Assume that Chebyshev interpolation is used to find a fifth degree interpolating polynomial $p(x)$ on the interval $[-1, 1]$ for the function $f(x) = e^x$. Use the interpolation error formula to find a worst-case estimate for the error $|e^x - p(x)|$ that is valid for x throughout the interval $[-1, 1]$. How many digits after the decimal point will be correct when $p(x)$ is used to approximate e^x?

11. For a function f, the divided differences are given by

x	**f**			
5	f_0			
5	f_1	$f[x_0, x_1]$		
6	4	5	-3	
4	2	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$

Determine the unknown entries in the table.

12. Consider the Chebyshev polynomial $T_n(x) := \cos(n \cos^{-1} x)$. For the nodes $[x_0, \ldots, x_n]$ in $[-1, 1]$, determine the n-th order divided difference $T_n[x_0, x_1, \ldots, x_0]$.

13. Prove that if p is a polynomial of degree m then for $n > m$, $p[x_0, x_1, \ldots, x_n] = 0$.

14. Determine the Hermite interpolating polynomial p that interpolates the data $p(0) = 1, p(1) = 2, p'(1) = 3, p(2) = 6, p'(2) = 4, p(3) = 5, p'(3) = 8$.

*************End************