$x^T A x = 0$

End Semester Examination

MA-580H : Matrix computations **Time : 2 hours** **80 marks**

November 17, 2025

Answer ALL questions

1. Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite. Show that all the leading principal submatrices of A are symmetric positive definite. State why LU decomposition of A exists. Use LU decomposition of A to show that A can be decomposed uniquely as $A = LDL^T$, where L is unit lower triangular and D is diagonal. Deduce Cholesky factorization of A. **12 marks**

2. Consider the system $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ is nonsingular and $b \in \mathbb{R}^n$. Suppose that $A+\Delta A$ is nonsingular. Consider the perturbed system $(A+\Delta A)\widehat{x} = b + \Delta b$. Show that
$$\frac{\|x-\widehat{x}\|}{\|\widehat{x}\|} \leq \mathrm{cond}(A)\left(\frac{\|\Delta A\|}{\|A\|} + \frac{\|\Delta b\|}{\|A\|\,\|\widehat{x}\|}\right),$$
where $\mathrm{cond}(A) := \|A\|\,\|A^{-1}\|$. **5 marks**

3. Compute QR factorization of $A := \begin{bmatrix} 1 & 0 & 1 \\ -2 & -1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$ using Householder reflectors. **10 marks**

4. Let $y \in \mathbb{R}^n$ and $\lambda > 0$. Show that the regularized least squares problem (LSP)
$$\min_{x\in\mathbb{R}^n}\left(\|x-y\|_2^2 + \lambda\sum_{i=1}^{n-1}(x_i - x_{i+1})^2\right)$$
can be written as $\min_{x\in\mathbb{R}^n}\left(\|x-y\|_2^2 + \lambda\|Dx\|_2^2\right)$ for an appropriate matrix D. Determine the matrix D and derive the normal equation of the regularized LSP. Show that the regularized LSP has a unique solution. **10 marks**

5. Let $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. Suppose that $\mathrm{rank}(A) = r < \min(m,n)$. Describe an algorithm (outline only the steps with justification) that uses SVD of A to compute all solutions of the LSP $Ax \approx b$. Define Moore-Penrose pseudo-inverse of A. Deduce that $x = A^+b$ is the minimum norm solution of the LSP, where A^+ is the Moore-Penrose pseudo-inverse of A. **15 marks**

6. Describe (outline the steps) shifted inverse power method for computing an eigenvalue and a corresponding eigenvector of a matrix $A \in \mathbb{C}^{n\times n}$. Determine the flop count for performing ℓ steps of shifted inverse power method. Show that any eigenvalue of A can be computed by shifted inverse power method by choosing an appropriate shift. **8 marks**

7. Let $A \in \mathbb{R}^{n\times n}$ be upper Hessenberg. Consider the QR step $A = QR$ and $A_1 = RQ$. Show that $A_1 = Q^*AQ$. Show that A_1 is upper Hessenberg when A is nonsingular. If the QR decomposition of A is computed using rotations then show that A_1 is upper Hessenberg even if A is singular. **10 marks** 6

8. Let $T \in \mathbb{R}^{n\times n}$ be symmetric and tridiagonal and given by $T(j,j) = \alpha_j$ for $j = 1:n$ and $T(j,j+1) = T(j+1,j) = \beta_j \neq 0$ for $j = 1:n-1$. Show that
$$\mu := \alpha_n + \delta - \mathrm{sgn}(\delta)\sqrt{\delta^2 + \beta_{n-1}^2}$$
is the Wilkinson's shift for T, where $\delta := (\alpha_{n-1} - \alpha_n)/2$. Describe one step of implicit QR algorithm for T with Wilkinson shift μ. **10 marks**

$A^T A x = A^T b$

$x = (A^T A)^{-1} A^T b$

*** End ***

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