Lab Session 4

MA581: Numerical Computations Lab R. Alam August 26, 2025

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1. The MATLAB command `integral(fun, a, b)` computes $\int_a^b f(x)dx$, where `fun` is a function handle. For example, $\int_0^\infty e^{-x^2}(\log x)^2 dx$ can be computed as follows.

```
fun = @(x) exp(-x.^2).*log(x).^2;
I = integral(fun,0,Inf)
```

   Compute the integral $\int_0^\infty x^5 e^{-x} \sin x dx$.

2. Planck's theory of blackbody radiation leads to the integral $\displaystyle\int_0^\infty \frac{x^3}{e^x - 1} dx$. Compute the integral using the command `integral`.

3. Write MATLAB function program `I = trapezoid(f,a,b,n)` to evaluate $\int_a^b f(x)dx$ via composite trapezoid rule with n equi-spaced nodes on $[a, b]$. Your function will look like

```
function I = trapezoid(f, a, b, n)
 h = (b-a)/n; x = a+h*(0:n)';
 y = f(x); I = h*( sum(y(2:n))+ 0.5(y(1)+y(n+1)) );
```

   Now compute $\int_0^2 e^{\sin(7x)} dx$ accurately using the command

```
I = integral(f,a,b,'abstol',1e-14,'reltol',1e-14);
fprintf('Integral = %.15f\n',I)
```

   Next, compute `T = trapezoid(f,a,b,40); error = abs( I-T)`

   In order to check the order of accuracy, double n a few times and observe how the error decreases as follows

```
n = 40*2.^(0:5)';
err = zeros(size(n));
for k = 1:length(n)
T = trapezoid(f,a,b,n(k));
err(k) = abs(I - T);
end
table(n,err)
```

   Each doubling of n should cut the error by a factor of about 4, which is consistent with second-order convergence. Alternatively, the slope on a log-log graph should be 2

```
loglog(n,err,'.-')
hold on, loglog(n,3e-3*(n/n(1)).^(-2),'--')
xlabel('n'), ylabel('error'), axis tight
title('Convergence of trapezoidal quadrature')
legend('observed','2nd order')
```

4. The period of a simple pendulum is determined by the complete elliptic integral of the first kind
$$K(x) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - x^2 \sin^2 \theta}}.$$
Use MATLAB function `integral` to evaluate this integral for enough values of x to draw a smooth plot of $K(x)$ over the range $0 \le x \le 1$.

5. Use composite midpoint and trapezoid quadrature rules for $n = 10 \times 2^k$ for $k = 1, 2, \ldots, 10$ to verify or refute the following equality

   (a) $\displaystyle\int_0^1 \frac{e^{-9x^2} + e^{-1024(x-1/4)^2}}{\sqrt{\pi}} dx = 0.2$

   (b) $\displaystyle\int_0^1 \sqrt{x} \log(x) dx = -\frac{4}{9}.$

   (c) $\int_0^{\pi/2} e^x \cos(x) dx = \frac{e^{\pi/2}-1}{2}.$

   (d) $\int_0^1 x^2 \tan^{-1}(x) dx = \frac{\pi-2+2\log 2}{12}.$

   Note that as $\log x$ is not defined at $x = 0$, you will need to replace the interval $[0, 1]$ by [`realmin`, 1] when approximating the integral in (b) by any quadrature rule. Also, verify the equality using the command `integral`.

6. Write a MATLAB function program `y = simpson(f,a,b,n)` to evaluate $\int_a^b f(x)dx$ using composite Simpson one-third rule with n equi-spaced nodes on $[a, b]$.

7. Since $\displaystyle\int_0^1 \frac{4}{1+x^2} dx = \pi$ one can compute an approximate value for π using numerical integration of the given function.

   Use the composite midpoint and trapezoid quadrature rules and the command `integral` to compute the approximate value of π for various stepsizes h. Display the accuracy of the rules (based on the known value of π) by plotting the errors for each method for $n = 1, 2, \ldots, 200$ in semilogy scale. Show all plots in a single figure using `hold on`. Is there any point beyond which decreasing h yields no further improvement?

8. Write a MATLAB function program `[w,x] = gausslegendre(n)` to generate the weight vector `w` and the node vector `r` of $n+1$ weights and nodes respectively for Gauss quadrature rule in the interval $[-1, 1]$ from the roots of the Legendre polynomial.

   (a) Use the following commands to find roots of the Legendre polynomial
   ```
   syms y
   roots = vpasolve(legendreP(n+1,y) == 0);
   ```
   and generate the weights by the method of undetermined coefficients.

   (b) Next, compute nodes and weights from the eigenvalues and eigenvectors of the matrix
$$A = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & & & \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{15}} & & \\ & \frac{2}{\sqrt{15}} & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{n}{\sqrt{4n^2-1}} \\ & & & \frac{n}{\sqrt{4n^2-1}} & 0 \end{bmatrix}.$$

Then the weights `w` and nodes `x` can be computed in MATLAB as

```
[V, D] = eig(A); x = diag(D); w = transpose(2*V(1, :).^2);
```

Write a MATLAB function program `y = gaussquad(f,a,b,n)` that computes $\int_a^b f(x)dx$ using Gauss quadrature rule with $n+1$ Legendre nodes. Your program should call `w = gausslegendre(n)`.

The intensity of diffracted light near a straight edge is determined by the values of the Fresnel integrals

$$C(x) := \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt \quad \text{and} \quad S(x) := \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt.$$

Use Gauss quadrature and MATLAB function `integral` to evaluate these integrals for enough values of x to draw a smooth plot of $C(x)$ and $S(x)$ over the range $0 \leq x \leq 5$. Do you observe any difference if the nodes and weights are computed as (a) and (b) in Problem 7?

*** End ***