Indian Institute of Technology Guwahati
Statistical Foundation of Data Science (MA 589)
Problem Set 06

1. If X_1 and X_2 are independent random variables each having PDF $2xe^{-x^2}(0 < x < \infty)$, then find the PDF of the random variable $\sqrt{X_1^2 + X_2^2}$.

2. Let X_1, X_2, X_3 have the joint PDF

$$f(x_1,\, x_2,\, x_3) = \begin{cases} 48x_1x_2x_3 & \text{if } 0 < x_1 < x_2 < x_3 < 1 \\ 0 & \text{otherwise.} \end{cases}$$

   Find the marginal distributions of $Y_1 = \frac{X_1}{X_2}$, $Y_2 = \frac{X_2}{X_3}$, and $Y_3 = X_3$.

3. Let X_1, X_2, X_3 be i.i.d. $Exp(1)$ random variables. Find the joint PDF of $Y_1 = \frac{X_1}{X_1+X_2+X_3}$, $Y_2 = \frac{X_2}{X_1+X_2+X_3}$, and $Y_3 = X_1 + X_2 + X_3$. Also find the marginal PDF of Y_1, Y_2, and Y_3.

4. Let X_1, X_2, X_3 be i.i.d. $Exp(1)$ random variables. Find the joint PDF of $Y_1 = \frac{X_1}{X_1+X_2+X_3}$, $Y_2 = \frac{X_1+X_2}{X_1+X_2+X_3}$, and $Y_3 = X_1 + X_2 + X_3$. Also find the marginal PDF of Y_1, Y_2, and Y_3.

5. Let X and Y be two independent random variables having $Gamma(\alpha_1,\, \beta)$ and $Gamma(\alpha_2,\, \beta)$ distributions, respectively. Find the PDF of $\frac{X}{X+Y}$.

6. Let X be a random variable of continuous type. The integral part, Y, of X has a $P(\lambda)$ distribution and the fractional part, Z, has a $U(0,\, 1)$ distribution. Find the CDF of X, assuming that Y and Z are independent. Using the CDF find the PDF of X.

7. Let $X_1, X_2, \ldots, X_n$ be i.i.d. $U(0,\, 1)$ random variables. Define $X_{(n)} = \max\{X_1, \ldots, X_n\}$ and $X_{(1)} = \min\{X_1, \ldots, X_n\}$. Find the joint and marginal distributions of $X_{(1)}$ and $X_{(n)}$.

8. Let X_1 and X_2 be i.i.d. $P(\lambda)$ random variables. Find the PMF of $X_{(2)} = \max\{X_1,\, X_2\}$.

9. Let X_1 and X_2 be independent $N(0, 1)$ random variables and let $Y = X_1 + X_2$, $Z = X_1^2 + X_2^2$.

   (a) Show that the joint MFG of $(Y,\, Z)$ is $M_{Y,Z}(t_1, t_2) = (1 - 2t_2)^{-1}\, e^{\frac{t_1^2}{1-2t_2}}$ if $t_1 \in \mathbb{R}$ and $t_2 < \frac{1}{2}$.

   (b) Using (a), find $Corr(Y,\, Z)$.

10. Let X_1, X_2, X_3 be i.i.d. with common MGF $M(t) = ((3/4) + (1/4)e^t)^2$, for all $t \in \mathbb{R}$.

    (a) Determine the probabilities $P(X_1 = k)$ for $k \in \mathbb{R}$.

    (b) Find the MGF of $Y = X_1 + X_2 + X_3$, and then determine the probability $P(Y = k)$ for $k \in \mathbb{R}$.