$\boxed{\text{Model Answers of Quiz I}}$

1. (5 points) Let a fair 6-sided die is rolled n times. What is the probability that all faces have appeared?

    Solution: Note that the required probability is zero for $n < 6$. For $n \geq 6$ we can proceed as follows. Let A_i denote the event that i has not appeared in n rolls, $i = 1, 2 \ldots, 6$. We need to find

$$P\left(\cap_{i=1}^{6} A_i^c\right) = 1 - P\left(\cup_{i=1}^{6} A_i\right) = \quad 1 - \sum_{i=1}^{6} P(A_i) + \sum_{i<j} P(A_i \cap A_j) - \ldots + P\left(\cap_{i=1}^{6} A_i\right).$$

    Now

$$P(A_i) = \left(\frac{5}{6}\right)^n \quad \text{for all } i$$

$$P(A_i \cap A_j) = \left(\frac{4}{6}\right)^n \quad \text{for all } i \neq j$$

$$P(A_i \cap A_j \cap A_k) = \left(\frac{3}{6}\right)^n \quad \text{for all } i \neq j \neq k$$

$$P(A_i \cap A_j \cap A_k \cap A_l) = \left(\frac{2}{6}\right)^n \quad \text{for all } i \neq j \neq k \neq l$$

$$P(A_i \cap A_j \cap A_k \cap A_l \cap A_m) = \left(\frac{1}{6}\right)^n \quad \text{for all } i \neq j \neq k \neq l \neq m$$

$$P\left(\cap_{i=1}^{6} A_i\right) = 0.$$

    Hence the required probability, for $n \geq 6$, is

$$1 - 6\left(\frac{5}{6}\right)^n + 15\left(\frac{4}{6}\right)^n - 20\left(\frac{3}{6}\right)^n + 15\left(\frac{2}{6}\right)^n - 6\left(\frac{1}{6}\right)^n.$$

2. (5 points) Consider four coding machines M_1, M_2, M_3, and M_4 producing binary codes 0 and 1. The machine M_1 produces codes 0 and 1 with respective probabilities $\frac{1}{4}$ and $\frac{3}{4}$. The code produced by machine M_k is fed into machine M_{k+1}, $(k = 1, 2, 3)$, which may either leave the received code unchanged or may change it. Suppose that each of the machines M_2, M_3, and M_4 change the code with probability $\frac{3}{4}$. Given that the machine M_4 has produced code 1, find the conditional probability that the machine M_1 produced code 0.

    Solution: Let A_{1j} denote the event that j^{th} machine produces code i, where $i = 0, 1, j = 1, 2, 3, 4$. Note that $\{A_{01}, A_{11}\}$ are mutually exclusive and exhaustive events. Now, using Bayes Theorem, the required probability is

$$P\left(A_{01} \mid A_{14}\right) = \frac{P\left(A_{14} \mid A_{01}\right) P\left(A_{01}\right)}{P\left(A_{14} \mid A_{01}\right) P\left(A_{01}\right) + P\left(A_{14} \mid A_{11}\right) P\left(A_{11}\right)}.$$

Now each code gets changed maximum 3 times. If M_1 produces code 0 and M_4 produces code 1 , the code thees changed odd number of times in between. Similarly, if both M_1 and M_4 produce code 1, then the code has changed even number of times. Thus,

$$P\left(A_{14} \mid A_{01}\right) = \binom{3}{1}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 + \binom{3}{3}\left(\frac{3}{4}\right)^3$$

and

$$P\left(A_H \mid A_{11}\right) = \binom{3}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) + \binom{3}{0}\left(\frac{1}{4}\right)^3.$$

Therefore $P\left(A_{01} \mid A_{14}\right) = \frac{3}{10}$.

3. (5 points) Let E_1, E_2, ... be a collection of mutually exclusive events such that $P(E_i) > 0$ for all $i = 1, 2, \ldots$ and $P\left(\cup_{i=1}^{\infty} E_i\right) = 1$. For any event B, show that

$$P\left(B\right) = \sum_{i=1}^{\infty} P\left(B|E_i\right) P(E_i).$$

Solution: Let $E = \cup_{i=1}^{\infty} E_i$. Then $P\left(E\right) = 1 \implies P\left(E^c\right) = 0$. Then using monotonicity property of probability, $P\left(B \cap E^c\right) = 0$. As, $B \cap E$ and $B \cap E^c$ are disjoint and $B = \left(B \cap E\right) \cup \left(B \cap E^c\right)$,

$$
\begin{aligned}
P\left(B\right) &= P\left(B \cap E\right) + P\left(B \cap E^c\right) \\
&= P\left(B \cap E\right), \quad \text{as } P\left(B \cap E^c\right) = 0 \\
&= P\left(B \cap \left(\cup_{i=1}^{\infty} E_i\right)\right), \quad \text{as } E = \cup_{i=1}^{\infty} E_i \\
&= P\left(\cup_{i=1}^{\infty}\left(B \cap E_i\right)\right) \\
&= \sum_{i=1}^{\infty} P\left(B \cap E_i\right), \quad \text{as } E_i\text{'s are disjoint, so are } B \cap E_i\text{'s} \\
&= \sum_{i=1}^{\infty} P\left(B \mid E_i\right) P\left(E_i\right), \text{as } P\left(E_i\right) > 0.
\end{aligned}
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Hence proved.