MA579H Scientific Computing

Some Reminders of Basic Calculus

Reminders from Calculus

Intermediate Value Theorem: Let f be a continuous function between $f(a)$ and $f(b)$. Then f attains every value between $f(a)$ and $f(b)$, i.e., for every real number say, y between $f(a)$ and $f(b)$, there exists $a \leq c \leq b$, such that $f(c) = y$.

Continuous Limits: Let f be a continuous function on some interval containing x_0. Then for any sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x_0$,

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x_0).$$

Mean Value Theorem: Let f be a continuously differentiable function on an interval $[a, b]$. Then there exists c such that $a < c < b$ and

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Reminders from Calculus

Rolle's Theorem: Let f be a continuously differentiable function on an interval $[a, b]$ such that $f(b) = f(a)$. Then there exists c such that $a < c < b$ and $f'(c) = 0$.

Taylor's Theorem with Remainder: Let f be a continuously $k+1$-times differentiable function on an interval between x and x_0. Then there exists a number c between x and x_0 such that

$$\begin{aligned} f(x) = & f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \cdots \\ & + \frac{(x - x_0)^k}{k!} f^{(k)}(x_0) + \frac{(x - x_0)^{k+1}}{(k+1)!} f^{(k+1)}(c) \end{aligned}$$

Mean Value Theorem for Integrals: Let f be a continuous function on $[a, b]$ and g be an integrable function that does not change sign on $[a, b]$. Then there exists c between a and b such that

$$\int_a^b f(x) g(x) dx = f(c) \int_a^b g(x) dx.$$

Reminders from Calculus

Generalized Intermediate Value Theorem: Let f be a continuous function on an interval $[a, b]$. Let $x_1, \ldots, x_n$ be points in $[a, b]$ and $a_1, \ldots, a_n > 0$. Then there exists a number c between a and b such that

$$(a_1 + \cdots + a_n)f(c) = a_1 f(x_1) + \cdots + a_n f(x_n).$$

Proof: Let $f(x_j) = \min_{1 \leq i \leq n} f(x_i)$ and $f(x_k) = \max_{1 \leq j \leq n} f(x_i)$. Then,

$$(a_1 + \cdots + a_n)f(x_j) \leq a_1 f(x_1) + \cdots + a_n f(x_n) \leq (a_1 + \cdots + a_n)f(x_k),$$

so that

$$f(x_j) \leq \frac{a_1 f(x_1) + \cdots + a_n f(x_n)}{a_1 + \cdots + a_n} \leq f(x_k).$$

As f is continuous on (a, b), by the Mean Value Theorem, there exists c belonging to the interval between x_j and x_k (and hence to (a, b)) such that

$$f(c) = \frac{a_1 f(x_1) + \cdots + a_n f(x_n)}{a_1 + \cdots + a_n}.$$