Homework-1

MA580H : Matrix Computations 2025 R. Alam

Block matrices, outer product, orthogonal vectors

1. Let $X = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 2 & 3 \end{bmatrix}$ and $Y := \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \end{bmatrix}$.

   (a) Compute the outer product expansion of $XY^\top$.

   (b) Compute the outer product expansion of $YX^\top$. How is the outer product expansion of $YX^\top$ related to the outer product expansion of $XY^\top$?

2. Let $U := \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_m \end{bmatrix} \in \mathbb{R}^{m\times m}$ and $V := \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} \in \mathbb{R}^{n\times n}$. Let

$$S := \left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_p & \\ \hline & 0 & & 0 \end{array}\right] \in \mathbb{R}^{m\times n}$$

   be a diagonal matrix, where $\sigma_1, \ldots, \sigma_p$ are nonzero real numbers. Show that $A = USV^\top$ can be expressed as an outer product expansion of the form

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^\top + \cdots + \sigma_p \mathbf{u}_p \mathbf{v}_p^\top.$$

3. Let $A := \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ be a block upper triangular matrix where each block is an $n \times n$ matrix. If A_{11} and A_{22} are nonsingular, then show that A must also be nonsingular and that A^{-1}must be of the form $A^{-1} = \begin{bmatrix} A_{11}^{-1} & C \\ 0 & A_{22}^{-1} \end{bmatrix}$. Determine C.

4. Let A and B be $n \times n$ matrices and define $2n \times 2n$ matrices S and M by

$$S = \begin{bmatrix} I_n & A \\ 0 & I_n \end{bmatrix} \text{ and } M := \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}.$$

   Determine the block form of S^{-1} and use it to compute the block form of the product $S^{-1}MS$.

5. Let $A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ be such that A_{11} is an $m \times m$ nonsingular matrix and A_{22} is an $n \times n$ matrix. Show that A can be factored as a product of block matrices

$$A = \begin{bmatrix} I_m & 0 \\ B & I_n \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & S \end{bmatrix}$$

   where $B := A_{21}A_{11}^{-1}$ and $S := A_{22} - A_{21}A_{11}^{-1}A_{12}$. The matrix S is called the Schur complement of A_{11} in A. Show that A is nonsingular $\iff$ S is nonsingular. Also show that $\det(A) = \det(A_{11})\det(S)$.

6. Let B be an $n \times n$ matrix such that $B^2 = 0$. Consider the block matrix $A = \begin{bmatrix} 0 & I_n \\ I_n & B \end{bmatrix}$. Determine the block form of $A^{-1} + A^2 + A^3$.

7. Let $\mathbf{u}$ and $\mathbf{v}$ be linearly independent vectors in $\mathbb{R}^n$ and $S := \text{span}(\mathbf{u}, \mathbf{v})$. Define $A := \mathbf{u}\mathbf{v}^\top + \mathbf{v}\mathbf{u}^\top$. Show that A is symmetric and $N(A) = S^\perp$. Further, show that $\text{rank}(A) = 2$.

8. Use Gram Schmidt procedure to transform the basis $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -0 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right\}$ to an orthogonal basis. If the order in which the vectors are taken changes, will the elements of the orthogonal basis remain the same?

9. Extend the orthonormal set $S := \{\frac{1}{2}[1, 1, 1, -1]^\top, \frac{1}{6}[1, 1, 3, 5]^\top\} \subset \mathbb{R}^4$ to an orthonormal basis of $\mathbb{R}^4$.

10. Find the orthogonal complement of the subspace $\mathcal{V} := \text{span}([1, 2, 1]^\top, [1, -1, 2]^\top)$.

11. Let $B := \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be an orthonormal subset of $\mathbb{C}^n$.

    (a) Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. Show that $\mathbf{x} = \mathbf{y} \iff \langle \mathbf{x}, \mathbf{w} \rangle = \langle \mathbf{y}, \mathbf{w} \rangle$ for all $\mathbf{w} \in \mathbb{C}^n$.

    (b) Show that $|\langle \mathbf{v}, \mathbf{u}_1 \rangle|^2 + \cdots + |\langle \mathbf{v}, \mathbf{u}_m \rangle|^2 \le \|\mathbf{v}\|^2$ for all $\mathbf{v} \in \mathbb{C}^n$.

    (c) Show that $|\langle \mathbf{v}, \mathbf{u}_1 \rangle|^2 + \cdots + |\langle \mathbf{v}, \mathbf{u}_m \rangle|^2 = \|\mathbf{v}\|^2$ for all $\mathbf{v} \in \mathbb{C}^n \iff B$ is an orthonormal basis of $\mathbb{C}^n$.

    (d) Suppose that $B := \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is an orthonormal basis of $\mathbb{C}^n$. For any $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ prove that
$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{u}_1 \rangle \overline{\langle \mathbf{y}, \mathbf{u}_1 \rangle} + \cdots + \langle \mathbf{x}, \mathbf{u}_n \rangle \overline{\langle \mathbf{y}, \mathbf{u}_n \rangle}.$$

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