Statistical Foundation of Data Science (MA 589)

Lecture Slides

Topic 04: Limit Theorems

Modes of Convergence

- If $\{x_n\}$ is a sequence of real numbers, we know the meaning of $x_n \to l$ as $n \to \infty$.
- If $\{f_n\}$ be a sequence of functions, we have several modes of convergence.
- Here we will discuss four modes of convergence for a sequence of random variables $\{X_n\}$.
- These are quite useful concept in probability and statistics.

Almost Sure Convergence

Definition 4.1: Let $\{X_n\}$ be a sequence of random variables defined on a probability space $(\mathcal{S}, \mathcal{F}, P)$. Let X be a random variable defined on the same probability space $(\mathcal{S}, \mathcal{F}, P)$. We say that X_n converges almost surely or with probability 1 to a random variable X if

$$P(\omega : X_n(\omega) \to X(\omega)) = 1\,.$$

Convergence in r^{th} Mean

Definition 4.2: Let $\{X_n\}$ be a sequence of random variables defined on a probability space $(\mathcal{S}, \mathcal{F}, P)$. Let X be a random variable defined on the same probability space $(\mathcal{S}, \mathcal{F}, P)$. We say that X_n converges in r^{th} mean to a random variable X if

$$E|X_n - X|^r \to 0\,.$$

Convergence in Probability

Definition 4.3: Let $\{X_n\}$ be a sequence of random variables defined on a probability space $(\mathcal{S}, \mathcal{F}, P)$. Let X be a random variable defined on the same probability space $(\mathcal{S}, \mathcal{F}, P)$. We say that X_n converges in probability to a random variable X if for any $\epsilon > 0$,

$$P(|X_n - X| > \epsilon) \to 0\,.$$

Convergence in Distribution

Definition 4.4: We say that X_n converges in distribution to a random variable X if

$$F_n(x) \to F(x)$$

for all x where F is continuous and where F_ns are the distribution functions of X_ns and F is the distribution function of X.

Remark 4.1: Unlike the first three modes of convergence, here X_ns can be defined on different probability spaces. We are only interested in the distribution functions. This flexibility makes this mode of convergence very useful.

Relation between Modes of Convergence

$$\begin{tikzcd}
a.s \arrow[dr] & & mean \arrow[dl] \\
& Prob \arrow[d] & \\
& Dist &
\end{tikzcd}$$

Strong Law of Large Numbers

Theorem 4.1: (Strong Law of Large Numbers) Let $\{X_n\}$ be a sequence of i.i.d. RVs with finite mean μ. Define $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then $\{\overline{X}_n\}$ converges to μ almost surely, *i.e.*,

$$P\left(\left\{\omega : \overline{X}_n(\omega) \to \mu\right\}\right) = 1.$$

We will use the notation $\overline{X}_n \stackrel{a.s.}{\to} \mu$.

Some Examples

Example 4.1: Bernoulli proportion converges to success probability.

Example 4.2: Monte Carlo Integration.

Example 4.3: Let X_i and Y_i, $i = 1, 2, \ldots$ are independently and identically distributed $U(0, 1)$ random variables. Let $N_n = \#\{k : 1 \leq k \leq n, X_k^2 + Y_k^2 \leq 1\}$. Then $\frac{4N_n}{n}$ converges to π with probability one.

Central Limit Theorem

Theorem 4.2: (Central Limit Theorem) Let $\{X_n\}$ be a sequence of i.i.d. RVs with mean μ and variance $\sigma^2 < \infty$. Then, as $n \to \infty$,

$$P\left(\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} \leq a\right) \to \Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt.$$

We will use the notation $\dfrac{\sqrt{n}\left(\overline{X}_n - \mu\right)}{\sigma} \xrightarrow{\mathcal{D}} Z \sim N(0, 1)$.

Some Examples

Example 4.4: $X_n \sim Bin(n, p)$. Then

$$P\left(\frac{X_n - np}{\sqrt{np(1-p)}} \leq a\right) \to \Phi(a).$$

Example 4.5: The lifetimes of a special type of battery is a RV with mean 40 hours and standard deviation 20 hours. A battery is used until it fails, at which point it is replaced by a new one. Assume a stockpile of 25 such batteries, the lifetimes of which are independent, approximate the probability that over 1100 hours of use can be obtained. $[\Phi(1) = 0.8413]$

Example 4.6: Show that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}.$$