Model Answers of Quiz III (Total points:15)

1. (2 points) Let X and Y be independent random variables and both of them are uniformly distributed in $[0,1]$. If the smaller (of the two) is less than $\frac{1}{4}$, then what is the conditional probability that the larger is greater than $\frac{3}{4}$?

   (a) $\dfrac{1}{3}$ (b) $\dfrac{2}{7}$ (c) $\dfrac{1}{4}$ (d) $\dfrac{1}{7}$

   Solution: Option (b) is correct
   Let
   $$U = \min(X,Y), \quad V = \max(X,Y).$$
   Using the definition of conditional probability,
   $$P(V > 3/4 \mid U < 1/4) = \frac{P(V > 3/4,\, U < 1/4)}{P(U < 1/4)}. \tag{1}$$
   Now,
   $$\begin{aligned} P(U < 1/4) &= 1 - P(U \geq 1/4) \\ &= 1 - P(\min(X,Y) \geq 1/4) \\ &= 1 - P(X \geq 1/4,\, Y \geq 1/4) \\ &= 1 - P(X \geq 1/4)P(Y \geq 1/4) \quad \text{[Since } X, Y \text{ are independent]} \\ &= 1 - \left(\frac{3}{4}\right)^2 = \frac{7}{16}. \end{aligned}$$
   Again,
   $$\begin{aligned} P(V > 3/4,\, U < 1/4) &= P(\max(X,Y) > 3/4,\, \min(X,Y) < 1/4) \\ &= P(X > 3/4,\, Y < 1/4) + P(Y > 3/4,\, X < 1/4) \\ &= 2 \times \left(\frac{1}{4}\right)^2 = \frac{1}{8}. \end{aligned}$$
   From equation (1),
   $$P(V > 3/4 \mid U < 1/4) = \frac{\frac{1}{8}}{\frac{7}{16}} = \frac{2}{7}.$$

2. (2 points) Let (X,Y) be a continuous random vector such that
   $$f_{X|Y}(x|y) = \frac{|y| e^{-x^2 y^2}}{\sqrt{\pi}}, \qquad -\infty < x < \infty,$$
   and
   $$f_Y(y) = \frac{e^{-y^2}}{\sqrt{\pi}}, \qquad -\infty < y < \infty.$$
   Consider the following statements:

P: $E(X) = 0$.

Q: $P(-1 < X < 1) = 0.75$.

Which of the above statements is/are true?

(a) Q only (b) P only (c) Both P and Q (d) Neither P nor Q

Solution: Option (d) is correct.
The joint density function of (X, Y) is given by

$$f_{X,Y}(x, y) = f_{X|Y}(x|y) f_Y(y) = \frac{|y|e^{-x^2y^2}}{\sqrt{\pi}} \cdot \frac{e^{-y^2}}{\sqrt{\pi}} = \frac{|y|e^{-(x^2+1)y^2}}{\pi} \qquad \text{for } -\infty < x < \infty, \; -\infty < y < \infty.$$

the marginal density of X is,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy = \frac{1}{\pi} \int_{-\infty}^{\infty} |y|e^{-(x^2+1)y^2}\, dy.$$

Since the integrand is even,

$$\int_{-\infty}^{\infty} |y|e^{-ay^2}\, dy = 2\int_{0}^{\infty} ye^{-ay^2}\, dy = 2 \times \frac{1}{2a} = \frac{1}{a},$$

where $a = x^2 + 1 > 0$. Therefore,

$$f_X(x) = \frac{1}{\pi(1 + x^2)}, \quad x \in \mathbb{R}.$$

Thus, X follows a standard Cauchy distribution:

$$X \sim \text{Cauchy}(0, 1).$$

For a standard Cauchy random variable, the mean $E(X)$ does **not exist** because the integral

$$\int_{-\infty}^{\infty} |x| f_X(x)\, dx = \infty.$$

Hence, statement **P** is **false**.

The cumulative distribution function (CDF) of the standard Cauchy distribution is

$$F_X(x) = \frac{1}{2} + \frac{1}{\pi} \arctan(x).$$

Therefore,

$$\begin{aligned}
P(-1 < X < 1) &= F_X(1) - F_X(-1) \\
&= \left[\frac{1}{2} + \frac{1}{\pi}\arctan(1)\right] - \left[\frac{1}{2} + \frac{1}{\pi}\arctan(-1)\right] \\
&= \frac{1}{\pi}\big(\arctan(1) - \arctan(-1)\big) \\
&= \frac{1}{\pi}\left(\frac{\pi}{4} - (-\frac{\pi}{4})\right) = \frac{1}{2}.
\end{aligned}$$

Hence, $P(-1 < X < 1) = 0.5$, not 0.75.

So statement **Q** is also **false**.

3. (2 points) Suppose that X has an exponential distribution with mean $\frac{1}{2}$. Also, suppose that given $X = x$, Y has a Poisson distribution with mean x. Which one of the following statements is true?

(a) $\text{Var}(Y) = \frac{3}{4}$, (b) $\text{Var}(Y) = \frac{2}{3}$, (c) $\text{Var}(Y) = \frac{4}{5}$, (d) $\text{Var}(Y) = \frac{1}{2}$.

Solution: Option (a) is correct.
Given

$$X \sim \text{Exponential}(2) \quad \text{and} \quad Y \mid X = x \sim \text{Poisson}(x).$$

Recall

$$\text{Var}(Y) = \mathbb{E}\big[\text{Var}(Y \mid X)\big] + \text{Var}\big(\mathbb{E}[Y \mid X]\big). \tag{1}$$

Since $Y \mid X = x \sim \text{Poisson}(x)$,

$$\mathbb{E}[Y \mid X] = X, \qquad \text{Var}(Y \mid X) = X.$$

from equation(1)

$$\text{Var}(Y) = \mathbb{E}[X] + \text{Var}(X).$$

Since $X \sim \text{Exponential}(2)$,

$$\mathbb{E}[X] = \frac{1}{2}, \qquad \text{Var}(X) = \frac{1}{4}.$$

Thus,

$$\text{Var}(Y) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

4. (1 point) Let X be a continuous random variable and Y be a discrete random variable. Assume X and Y are independent. Which of the following statements is/are true?

(a) $X + Y$ is a continuous random variable.
(b) $X + Y$ is a discrete random variable.
(c) $X + Y$ is neither discrete nor continuous.
(d) $\Pr(X < Y) = \frac{1}{2}$.

Solution: Option (a) is correct.

Consider $X \sim U[0, 1]$ and Y be a degenerate distribution at $y = 2$. That is,

$$P[Y = y] = \begin{cases} 1, & y = 2 \\ 0, & \text{otherwise} \end{cases}$$

Then,

$$P[X < Y] = P[X < 2] = 1$$

Therefore, option (d) is not correct.

5. (2 points) Let the random vector (X, Y) has the joint cumulative distribution function

$$F(x,y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } y < 0 \\ \frac{1-e^{-x}}{4} & \text{if } x \geq 0,\ 0 \leq y < 1 \\ 1 - e^{-x} & \text{if } x \geq 0,\ y \geq 1. \end{cases}$$

Then $\text{Var}(X) + 32\text{Var}(Y)$ equals

(a) 7. (b) $\frac{1}{7}$. (c) 1. (d) 9.

Solution: The marginal CDF of X is

$$F_X(x) = \lim_{y \to \infty} F(x,y) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \geq 0. \end{cases}$$

The marginal CDF of Y is

$$F_Y(y) = \lim_{x \to \infty} F(x,y) = \begin{cases} \frac{1}{4} & \text{if } 0 \leq y < 1 \\ 1 & \text{if } y \geq 1. \end{cases}$$

Therefore, $Var(X) = 1$. Now,

$$P(Y = 0) = P(Y \leq 0) - P(Y < 0) = \frac{1}{4}$$

and

$$P(Y = 1) = P(Y \leq 1) - P(Y < 1) = 1 - \frac{1}{4} = \frac{3}{4}.$$

Thus,

$$E(Y) = \sum_y y \cdot P(Y = y) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{3}{4} \implies E(Y) = \frac{3}{4},$$

and

$$\text{Var}(Y) = E(Y^2) - (E(Y))^2 = (0 + \frac{3}{4}) - (\frac{3}{4})^2 = \frac{3}{16}.$$

Therefore,

$$\text{Var}(X) + 32 \cdot \text{Var}(Y) = 1 + 32 \cdot \frac{3}{16} \implies \text{Var}(X) + 32 \cdot \text{Var}(Y) = 7.$$

Hence, **option(a)** is the **correct** one.

6. (2 points) Let the joint probability density function of the random variables X_1, X_2, and X_3 be

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = \begin{cases} \frac{1}{8\pi^3}(1 - \sin x_1 \sin x_2 \sin x_3) & \text{if } 0 < x_1, x_2, x_3 < 2\pi \\ 0 & \text{otherwise.} \end{cases}$$

Then which of the following statements is/are true?

(a) X_1, X_2, and X_3 are pairwise independent.
(b) Variance of $X_1 + X_2$ is $\frac{2}{3}\pi^2$.
(c) X_1, X_2, and X_3 are independent.
(d) $\mathrm{E}(X_1|X_3 = \frac{\pi}{2}) = \frac{\pi}{2}$.

Solution: For $x_1,\ x_2 \in (0,\ 2\pi)$, the joint PDF of (X_1, X_2) is

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= \int_0^{2\pi} \frac{1}{8\pi^3}(1 - \sin x_1 \sin x_2 \sin x_3)\ dx_3 \\ &= \frac{1}{8\pi^3}\left[x_3|_0^{2\pi} - sinx_1 sinx_2 \int_0^{2\pi} sinx_3\ dx_3\right] \\ &= \frac{1}{8\pi^3}\left[2\pi - sinx_1 sinx_2 (-cosx_3)_0^{2\pi}\right] \\ &= \frac{1}{8\pi^3}\left[2\pi - 0\right] \\ &= \frac{1}{4\pi^2} \end{aligned}$$

Thus, the joint PDF of $(X_1,\ X_2)$ is

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} \frac{1}{4\pi^2} & 0 < x_1,\ x_2 < 2\pi \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the joint probability densities of (X_2, X_3) and (X_1, X_3) will be:

$$f_{X_2,X_3}(x_2,x_3) = \frac{1}{4\pi^2}\ ;\ 0 < x_2, x_3 < 2\pi,$$

and

$$f_{X_1,X_3}(x_1,x_3) = \frac{1}{4\pi^2}\ ;\ 0 < x_1, x_3 < 2\pi.$$

For $0 < x_1 < 2\pi$, the probability density function of X_1 is obtained as:

$$\begin{aligned} f_{X_1}(x_1) &= \int_0^{2\pi}\int_0^{2\pi} \frac{1}{8\pi^3}(1 - \sin x_1 \sin x_2 \sin x_3)\ dx_3\ dx_2 \\ &= \frac{1}{8\pi^3}\int_0^{2\pi}\left[x_3|_0^{2\pi} - sinx_1 sinx_2 \int_0^{2\pi} sinx_3\ dx_3\right]\ dx_2 \\ &= \frac{1}{8\pi^3}\int_0^{2\pi}\left[2\pi - sinx_1 sinx_2 (-cosx_3)_0^{2\pi}\right]\ dx_2 \\ &= \frac{1}{4\pi^2}\int_0^{2\pi} dx_2 \\ &= \frac{1}{2\pi} \end{aligned}$$

Therefore, marginal PDF of X_1 is

$$f_{X_1}(x_1) = \begin{cases} \frac{1}{2\pi} & 0 < x_1 < 2\pi \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the probability density functions of X_2 and X_3 are

$$f_{X_2}(x_2) = \frac{1}{2\pi} \ ; 0 < x_2 < 2\pi,$$

and

$$f_{X_3}(x_3) = \frac{1}{2\pi} \ ; 0 < x_3 < 2\pi.$$

Hence,

$$f_{X_1}(x_1)f_{X_2}(x_2)f_{X_3}(x_3) \neq f_{X_1,X_2,X_3}(x_1, x_2, x_3)$$

for all x_1, x_2, $x_3 \in (0, 2\pi)$. Therefore, X_1, X_2 and X_3 are not independent and thus, **option (c)** is **false**.

On the other hand,

$$f_{X_1}(x_1) \cdot f_{X_2}(x_2) = \frac{1}{4\pi^2} = f_{X_1,X_2}(x_1, x_2)$$

Hence, X_1 and X_2 are independent. Similarly, X_2 and X_3 are independent while X_1 and X_3 are also independent. Thus, **option (a)** is **true**.

Now,

$$\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2) + 2\ \text{Cov}(X_1, X_2)$$

To find $\text{Var}(X_1)$, we calculate first $E(X_1)$. Thus,

$$E(X_1) = \int_0^{2\pi} x_1 f_{X_1}(x_1)\ dx_1 = \frac{1}{2\pi} \int_0^{2\pi} x_1\ dx_1 = \pi.$$

Also,

$$E(X_1^2) = \int_0^{2\pi} x_1^2 f_{X_1}(x_1)\ dx_1 = \frac{1}{2\pi} \int_0^{2\pi} x_1^2\ dx_1 = \frac{4\pi^2}{3}.$$

Hence,

$$\text{Var}(X_1) = E(X_1^2) - (E(X_1))^2 = \frac{\pi^2}{3}.$$

Similarly, $\text{Var}(X_2) = \frac{\pi^2}{3}$. As, X_1 and X_2 are independent,

$$\text{Cov}(X_1, X_2) = 0.$$

Alternatively, X_1 and X_2 are pairwise independent; hence, $\text{Cov}(X_1, X_2) = 0$. Therefore,

$$\text{Var}(X_1 + X_2) = \frac{\pi^2}{3} + \frac{\pi^2}{3} = \frac{2\pi^2}{3}.$$

Hence, **option(b)** is **true**.

As X_1 and X_3 are independent,

$$E\left(X_1 \mid X_3 = \frac{\pi}{2}\right) = E(X_1) = \int_0^{2\pi} x_1\ \frac{1}{2\pi}\ dx_1 = \pi \neq \frac{\pi}{2}.$$

Therefore, **option(d)** is **false**. Thus, **options (a) and (b)** are **correct**.

7. (2 points) At a party 101 men throw their hats into the center of a room. The hats are mixed up and each men randomly selects one. Then variance of number of men who select their own hats equals

(a) 1. (b) 101. (c) 2. (d) 50.

Solution: P(a man selects his own hat) $= \frac{1}{101}$. Thus, P(a man does not select his own hat) $= \frac{100}{101}$.

Let

$$X_i = \begin{cases} 1 & \text{if } i^{th}\text{man chooses own hat} \\ 0 & \text{otherwise.} \end{cases}$$

for $i = 1, 2, \ldots, 101$. Let X be the number of men selecting their own hats. Then,

$$X = \sum_{i=1}^{101} X_i$$

We have to calculate Var(X). Now,

$$E(X) = \sum_{i=1}^{101} E(X_i) = \sum_{i=1}^{101} \frac{1}{101} \implies E(X) = 1.$$

Thus,

$$\begin{aligned} \text{Var}(X) &= \text{Var}(\sum_{i=1}^{101} X_i) = \sum_{i=1}^{101} \text{Var}(X_i) + 2\sum_{i}\sum_{j}_{i<j} \text{Cov}(X_i, X_j) \\ &= \sum_{i=1}^{101}[\frac{1}{101} - (\frac{1}{101})^2] + 2\sum_{i}\sum_{j}_{i<j} [\frac{1}{101 \times 100} - (\frac{1}{101})^2] \\ &= \sum_{i=1}^{101}[\frac{100}{(101)^2}] + 2\sum_{i}\sum_{j}_{i<j} [\frac{1}{(101)^2 \times 100}] \\ &= \frac{100}{101} + 2 \cdot \frac{101 \times 100}{2} \cdot \frac{1}{(101)^2 \times 100} \\ &= \frac{100}{101} + \frac{1}{101} = 1. \implies \text{Var}(X) = 1. \end{aligned}$$

Hence, **option(a)** is the **correct** one.

8. (2 points) Let the joint probability density function of the random variables X_1 and X_2 be

$$f(x_1, x_2) = \begin{cases} 8x_1x_2 & \text{if } 0 < x_1 < x_2 < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then which of the following statements is/are true?

Staement P: X_1 and X_2 are independent.

Staement Q: $\dfrac{X_1}{X_2}$ and X_2 are independent.

(a) Only Q. (b) Both P and Q. (c) Only P. (d) Neither P nor Q.

Solution: Given:

$$f(x_1, x_2) = \begin{cases} 8x_1x_2 & \text{if } 0 < x_1 < x_2 < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Here, we need to calculate the probability density functions of X_1, X_2 and $\frac{X_1}{X_2}$. Now,

$$f(x_1) = \int_{x_1}^{1} 8x_1x_2 \, dx_2 \implies f(x_1) = 4x_1(1 - x_1^2) \quad \text{for } 0 < x_1 < 1.$$

and

$$f(x_2) = \int_{0}^{x_2} 8x_1x_2 \, dx_1 \implies f(x_2) = 4x_2^3 \quad \text{for } 0 < x_2 < 1.$$

Therefore,

$$f(x_1)f(x_2) = 16x_1x_2^3 - 16x_1^3x_2^3 \neq f(x_1, x_2)$$

Hence, X_1 and X_2 are not independent and thus, **Statement P** is **not true**.
Again, let $Z = \frac{X_1}{X_2}$ and $W = X_2$. Then, $X_1 = WZ$ and $X_2 = W$. Now, to get the joint probability density function of (Z, W), which is $f(z, w)$, from $f(x_1, x_2)$, we need to calculate the Jacobian of the transformation $(x_1, x_2) \to (z, w)$.
Hence, the Jacobian **J** of the transformation is

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial z} & \frac{\partial x_1}{\partial w} \\ \frac{\partial x_2}{\partial z} & \frac{\partial x_2}{\partial w} \end{vmatrix} \implies J = \begin{vmatrix} w & z \\ 0 & 1 \end{vmatrix} \implies J = w.$$

Thus, for $0 < z, w < 1$,

$$f(z, w) = f(x_1, x_2)|_{x_1 = zw,\ x_2 = w} \cdot |J| = 8zw^3$$

Now, we find out the probability densities of Z and W.

$$f(z) = \int_0^1 8zw^3 dw = 8z \cdot \frac{1}{4} \cdot 1 \implies f(z) = 2z \ ; \ 0 < z < 1.$$

and

$$f(w) = \int_0^1 8zw^3 dw = 8w^3 \cdot \frac{1}{2} \cdot 1 \implies f(w) = 4w^3 \ ; \ 0 < w < 1.$$

Hence,

$$f(z) \cdot f(w) = 8zw^3 = f(z, w) \implies \text{The random variables } Z \text{ and } W \text{ are independent.}$$

Thus, **Statement Q** is correct. Hence, **option (a)** is the **correct option**.