

STATISTICAL FOUNDATION OF DATA SCIENCE (MA 589)

Lecture Slides

Topic 04: Limit Theorems

Modes of Convergence

- If $\{x_n\}$ is a sequence of real numbers, we know the meaning of $x_n \rightarrow l$ as $n \rightarrow \infty$.
- If $\{f_n\}$ be a sequence of functions, we have several modes of convergence.
- Here we will discuss four modes of convergence for a sequence of random variables $\{X_n\}$.
- These are quite useful concept in probability and statistics.

Almost Sure Convergence

Definition 4.1: Let $\{X_n\}$ be a sequence of random variables defined on a probability space $(\mathcal{S}, \mathcal{F}, P)$. Let X be a random variable defined on the same probability space $(\mathcal{S}, \mathcal{F}, P)$. We say that X_n converges almost surely or with probability 1 to a random variable X if

$$P(\omega : X_n(\omega) \rightarrow X(\omega)) = 1.$$

Convergence in r^{th} Mean

Definition 4.2: Let $\{X_n\}$ be a sequence of random variables defined on a probability space $(\mathcal{S}, \mathcal{F}, P)$. Let X be a random variable defined on the same probability space $(\mathcal{S}, \mathcal{F}, P)$. We say that X_n converges in r^{th} mean to a random variable X if

$$E|X_n - X|^r \rightarrow 0.$$

Convergence in Probability

Definition 4.3: Let $\{X_n\}$ be a sequence of random variables defined on a probability space $(\mathcal{S}, \mathcal{F}, P)$. Let X be a random variable defined on the same probability space $(\mathcal{S}, \mathcal{F}, P)$. We say that X_n converges in probability to a random variable X if for any $\epsilon > 0$,

$$P(|X_n - X| > \epsilon) \rightarrow 0.$$

Convergence in Distribution

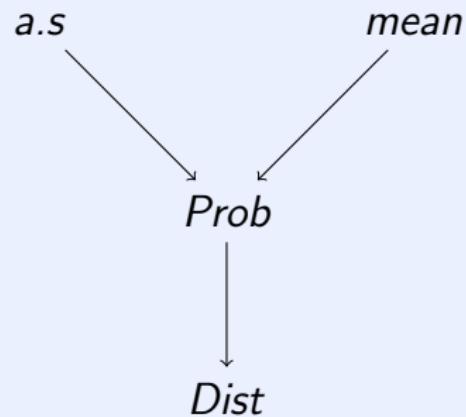
Definition 4.4: We say that X_n converges in distribution to a random variable X if

$$F_n(x) \rightarrow F(x)$$

for all x where F is continuous and where F_n s are the distribution functions of X_n s and F is the distribution function of X .

Remark 4.1: Unlike the first three modes of convergence, here X_n s can be defined on different probability spaces. We are only interested in the distribution functions. This flexibility makes this mode of convergence very useful.

Relation between Modes of Convergence



Strong Law of Large Numbers

Theorem 4.1: (Strong Law of Large Numbers) Let $\{X_n\}$ be a sequence of i.i.d. RVs with finite mean μ . Define $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$. Then $\{\bar{X}_n\}$ converges to μ almost surely, i.e.,

$$P(\{\omega : \bar{X}_n(\omega) \rightarrow \mu\}) = 1.$$

We will use the notation $\bar{X}_n \xrightarrow{a.s.} \mu$.

Some Examples

Example 4.1: Bernoulli proportion converges to success probability.

Example 4.2: Monte Carlo Integration.

Example 4.3: Let X_i and Y_i , $i = 1, 2, \dots$ are independently and identically distributed $U(0, 1)$ random variables. Let

$N_n = \#\{k : 1 \leq k \leq n, X_k^2 + Y_k^2 \leq 1\}$. Then $\frac{4N_n}{n}$ converges to π with probability one.

Central Limit Theorem

Theorem 4.2: (Central Limit Theorem) Let $\{X_n\}$ be a sequence of i.i.d. RVs with mean μ and variance $\sigma^2 < \infty$. Then, as $n \rightarrow \infty$,

$$P\left(\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \leq a\right) \rightarrow \Phi(a) = \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt.$$

We will use the notation $\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{\mathcal{D}} Z \sim N(0, 1)$.

Some Examples

Example 4.4: $X_n \sim Bin(n, p)$. Then

$$P\left(\frac{X_n - np}{\sqrt{np(1-p)}} \leq a\right) \rightarrow \Phi(a).$$

Example 4.5: The lifetimes of a special type of battery is a RV with mean 40 hours and standard deviation 20 hours. A battery is used until it fails, at which point it is replaced by a new one. Assume a stockpile of 25 such batteries, the lifetimes of which are independent, approximate the probability that over 1100 hours of use can be obtained. $[\Phi(1) = 0.8413]$

Example 4.6: Show that

$$\lim_{n \rightarrow \infty} e^{-n} \sum_{k=0}^n \frac{n^k}{k!} = \frac{1}{2}.$$