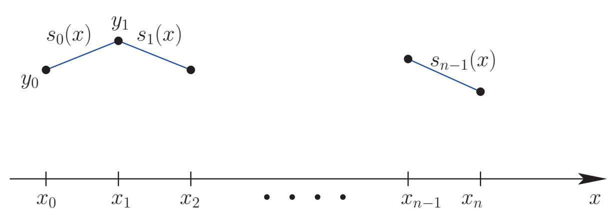
Motivation

- It turns out that high order interpolation using a global polynomial often exhibit these oscillations hence it is “dangerous” to use (in particular on equidistant grids).
- Another strategy is to use piecewise interpolation. For instance, piecewise linear interpolation.



Motivation

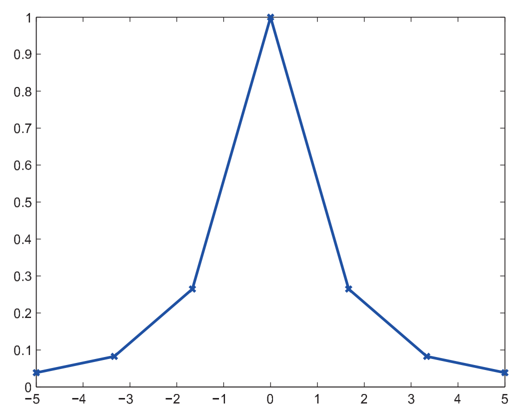
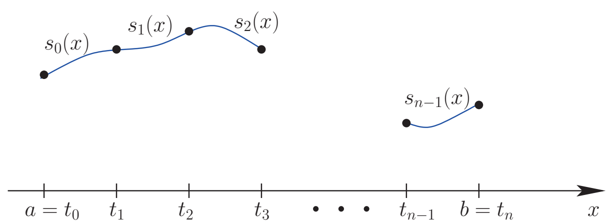


Figure: Runge's example interpolated using piecewise linear interpolation. We have used 7 points to interpolate the function in order to ensure that we can actually see the discontinuities on the plot.

A better strategy - spline interpolation

- We would like to avoid the Runge phenomenon for large datasets $\Rightarrow$ we cannot do higher order interpolation.
- The solution to this is using piecewise polynomial interpolation.
- However piecewise linear is not a good choice as the regularity of the solution is only C^0.
- These desires lead to splines and spline interpolation.



Splines - definition

A function $S(x)$ is a spline of degree k on $[a, b]$ if

- $S \in C^{k-1}[a, b]$.
- $a = t_0 < t_1 < \cdots < t_n = b$ and

$$S(x) = \begin{cases} S_0(x), & t_0 \leq x \leq t_1 \\ S_1(x), & t_1 \leq x \leq t_2 \\ \vdots & \\ S_{n-1}(x), & t_{n-1} \leq x \leq t_n \end{cases}$$

where $S_i(x) \in \mathbb{P}^k$.

Cubic spline

$$S(x) = \begin{cases} S_0(x) = & a_0x^3 + b_0x^2 + c_0x + d_0, \qquad t_0 \le x \le t_1 \\ \quad \vdots & \\ S_{n-1}(x) = & a_{n-1}x^3 + b_{n-1}x^2 + c_{n-1}x + d_{n-1}, \ \ t_{n-1} \le x \le t_n. \end{cases}$$

which satisfies

$$S(x) \in C^2[t_0, t_n] : \left. \begin{array}{r} S_{i-1}(x_i) = S_i(x_i) \\ S'_{i-1}(x_i) = S'_i(x_i) \\ S''_{i-1}(x_i) = S''_i(x_i) \end{array} \right\}, i = 1, 2, \cdots, n-1.$$

Cubic spline - interpolation

Given $(x_i, y_i)_{i=0}^n$. Task: Find $S(x)$ such that it is a cubic spline interpolant.

- The requirement that it is to be a cubic spline gives us $3(n-1)$ equations.
- In addition we require that

  $$S(x_i) = y_i, \qquad i = 0, \cdots, n$$

  which gives $n+1$ equations.
- This means we have $4n-2$ equations in total.
- We have $4n$ degrees of freedom $(a_i, b_i, c_i, d_i)_{i=0}^{n-1}$.
- Thus we have 2 degrees of freedom left.

Cubic spline - interpolation

We can use these to define different subtypes of cubic splines:

- $S''(t_0) = S''(t_n) = 0$ - natural cubic spline.
- $S'(t_0), S'(t_n)$ given - clamped cubic spline.
- $$\left.\begin{aligned} S_0'''(t_1) &= S_1'''(t_1) \\ S_{n-2}''(t_{n-1}) &= S_{n-1}'''(t_{n-1}) \end{aligned}\right\} \text{ - Not a knot condition (MATLAB)}$$

Natural cubic splines

Task: Find $S(x)$ such that it is a natural cubic spline.

- Let $t_i = x_i, i = 0, \cdots, n$.
- Let $z_i = S''(x_i), i = 0, \cdots, n$. This means the condition that it is a natural cubic spline is simply expressed as $z_0 = z_n = 0$.
- Now, since $S(x)$ is a third order polynomial we know that $S''(x)$ is a linear spline which interpolates (t_i, z_i).
- Hence one strategy is to first construct the linear spline interpolant $S''(x)$, and then integrate that twice to obtain $S(x)$.

Natural cubic splines

- The linear spline is simply expressed as

$$S_i''(x) = z_i \frac{x - t_{i+1}}{t_i - t_{i+1}} + z_{i+1} \frac{x - t_i}{t_{i+1} - t_i}.$$

- We introduce $h_i = t_{i+1} - t_i, i = 0, \cdots, n$ which leads to

$$S''(x) = z_{i+1} \frac{x - t_i}{h_i} + z_i \frac{t_{i+1} - x}{h_i}.$$

- We now integrate twice

$$S_i(x) = \frac{z_{i+1}}{6h_i} (x - t_i)^3 + \frac{z_i}{6h_i} (t_{i+1} - x)^3 + C_i (x - t_i) + D_i (t_{i+1} - x).$$

Natural cubic splines

- Interpolation gives:

$$S_i(t_i) = y_i \Rightarrow \frac{z_i}{6} h_i^2 + D_i h_i = y_i, i = 0, \cdots, n.$$

- Continuity yields:

$$S_i\left(t_{i+1}\right) = y_{i+1} \Rightarrow \frac{z_{i+1}}{6} h_i^2 + C_i h_i = y_{i+1}.$$

Natural cubic splines

- We insert these expressions to find the following form of the system

$$
\begin{aligned}
S_i(x) = {} & \frac{z_{i+1}}{6h_i}(x - t_i)^3 + \frac{z_i}{6h_i}(t_{i+1} - x)^3 \\
& + \left(\frac{y_{i+1}}{h_i} - \frac{z_{i+1}}{6}h_i\right)(x - t_i) \\
& + \left(\frac{y_i}{h_i} - \frac{h_i}{6}z_i\right)(t_{i+1} - x)\,.
\end{aligned}
$$

- We then take the derivative.

Natural cubic splines

- The derivative reads

$$S_i'(x) = \frac{z_{i+1}}{2h_i}(x - t_i)^2 - \frac{z_i}{2h_i}(t_{i+1} - x)^2 + \underbrace{\frac{1}{h_i}(y_{i+1} - y_i)}_{b_i} - \frac{h_i}{6}(z_{i+1} - z_i).$$

- In our abscissas this gives

$$\begin{aligned} S_i'(t_i) &= -\frac{1}{2}z_i h_i + b_i - \frac{h_i}{6}z_{i+1} + \frac{1}{6}h_i z_i \\ S_i'(t_{i+1}) &= \frac{z_{i+1}}{2}h_i + b_i - \frac{h_i}{6}z_{i+1} + \frac{1}{6}h_i z_i \\ S_{i-1}(t_i) &= \frac{1}{3}z_i h_{i+1} + \frac{1}{6}h_{i-1}z_{i-1} + b_{i-1} \\ S_i'(t_i) &= S_{i-1}(t_i) \Rightarrow \\ 6(b_i - b_{i-1}) &= h_{i-1}z_{-1} + 2(h_{i-1} + h_i)z_i + h_i z_{i+1}. \end{aligned}$$

Natural cubic splines - algorithm

This means that we can find our solution using the following procedure:

- First do some precalculations

$$
\begin{aligned}
h_i &= t_{i+1} - t_i, & i &= 0, \cdots, n-1 \\
b_i &= \frac{1}{h_i}\left(y_{i+1} - y_i\right), & i &= 0, \cdots, n-1 \\
v_i &= 2\left(h_{i-1} + h_i\right), & i &= 1, \ldots, n-1 \\
u_i &= 6\left(b_i - b_{i-1}\right), & i &= 1, \cdots, n-1 \\
z_0 &= z_n = 0
\end{aligned}
$$

Natural cubic splines - algorithm

- Then solve the tridiagonal system

$$\begin{bmatrix} v_1 & h_1 & & & & \\ h_1 & v_2 & h_2 & & & \\ & h_2 & v_3 & h_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & h_{n-2} \\ & & & & h_{n-2} & v_{n-1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_{n-2} \\ z_{n-1} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{bmatrix}.$$

Natural cubic splines - example

- Given the dataset

i	0	1	2	3
x_i	0.9	1.3	1.9	2.1
y_i	1.3	1.5	1.85	2.1
$h_i = x_{i+1} - x_i$	0.4	0.6	0.2	
$b_i = \frac{1}{h_i}(y_{i+1} - y_i)$	0.5	0.5833	1.25	
$v_i = 2(h_{i-1} + h_i)$		2.0	1.6	
$u_i = 6(b_i - b_{i-1})$		0.5	4	

- The linear system reads

$$\begin{bmatrix} 2.0 & 0.4 \\ 0.4 & 1.6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.4 \end{bmatrix}$$

Natural cubic splines - example

- We find $z_0 = 0.5, z_1 = 0.125$. This gives us our spline functions

$$S_0(x) = 0.208\,(x - 0.9)^3 + 3.78\,(x - 0.9) + 3.25\,(1.3 - x)$$
$$S_1(x) = 0.035\,(x - 1.3)^3 + 0.139\,(1.9 - x)^3 + 0.664 - 0.62x$$
$$S_2(x) = 0.104\,(x - 1.9)^3 + 10.5\,(x - 1.9) + 9.25\,(2.1 - x)$$