MA589 | Quiz II
Statistical Foundation for Data Science | 10:30–10:55 IST
IIT GUWAHATI | August 26, 2024

Model Answers of Quiz II

1. (5 points) Let X be a random variable with probability density function

$$f(x)=\begin{cases}\frac{1}{x^2} & \text{if } x\geq 1\\ 0 & \text{otherwise.}\end{cases}$$

If $Y=\ln X$, then find $P(Y<1 \mid Y\leq 2)$.

Solution:

$$P(Y<1 \mid Y\leq 2)=\frac{P(Y<1)}{P(Y\leq 2)}=\frac{P(\ln X<1)}{P(\ln X\leq 2)}=\frac{P(X<e)}{P(X\leq e^2)}=\frac{\displaystyle\int_1^e \frac{1}{x^2}dx}{\displaystyle\int_1^{e^2}\frac{1}{x^2}dx}=\frac{e}{1+e}.$$

2. (5 points) Let X be a random variable having the cumulative distribution function

$$F(x)=\begin{cases}0 & \text{if } x<1\\ \frac{a}{2} & \text{if } 1\leq x<2\\ \frac{c}{6} & \text{if } 2\leq x<3\\ 1 & \text{if } x\geq 3,\end{cases}$$

where a and c are appropriate real constants. Moreover, assume that $P(X\leq 1)=\frac{1}{2}$ and $E(X^2-2X+3)=3$. Then find $E(X)$.

Solution: Here, $P(X\leq 1)=\frac{1}{2} \implies F(1)=\frac{1}{2} \implies \frac{a}{2}=\frac{1}{2} \implies a=1$. Therefore, the set of discontinuity of the CDF $F(\cdot)$ is a subset of $\{1,\, 2,\, 3\}$, which is a finite set. Moreover,

$$P(X=1)=\frac{1}{2},\ P(X=2)=\frac{c}{6}-\frac{1}{2},\ \text{ and } P(X=3)=1-\frac{c}{6}.$$

As the sum these probabilities is one, X is a discrete random variable. Therefore,

$$E\left(X^2-2X+3\right)=3 \implies 2\times\frac{1}{2}+3\times\left(\frac{c}{6}-\frac{1}{2}\right)+6\times\left(1-\frac{c}{6}\right)=3 \implies c=5.$$

Thus,

$$E(X)=1\times\frac{1}{2}+2\times\left(\frac{5}{6}-\frac{1}{2}\right)+3\times\left(1-\frac{5}{6}\right)=\frac{5}{3}.$$

3. (5 points) Let X be a random variable having uniform distribution on the interval $\left(-\frac{\pi}{2},\, \frac{\pi}{2}\right)$. Then show that, for $t<1$, the moment generating function of the random variable $Y=-\ln\left(\frac{1}{2}+\frac{X}{\pi}\right)$ is $M(t)=(1-t)^{-1}$.

Solution: The MGF of Y is

$$\begin{aligned}
M(t) &= E\left(e^{tY}\right) \\
&= E\left(\exp\left\{-t\ln\left(\frac{1}{2}+\frac{X}{\pi}\right)\right\}\right) \\
&= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left\{-t\ln\left(\frac{1}{2}+\frac{x}{\pi}\right)\right\} dx.
\end{aligned}$$

Take $y = -\ln\left(\frac{1}{2}+\frac{x}{\pi}\right)$, then $dx = -\pi e^{-y}dy$. Thus,

$$M(t) = \int_0^\infty e^{-(1-t)y}dy = \frac{1}{1-t}$$

if $t < 1$.