Common Subexpression Elimination with Subtree Isomporphisms

Robert King, Mentors: Hari Sundar, Milinda Fernando

University of Utah

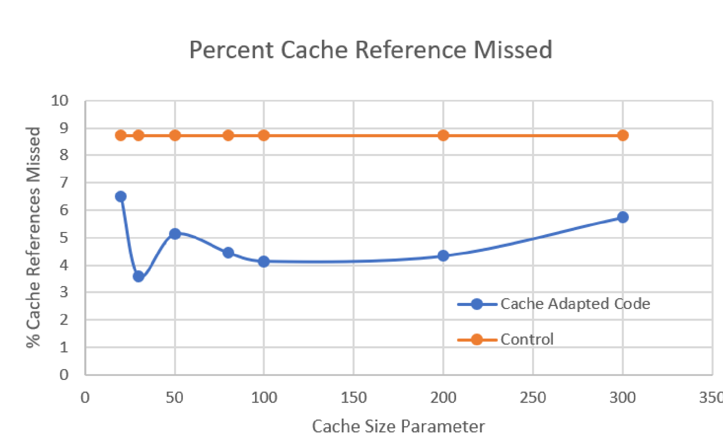
THE UNIVERSITY OF UTAH

Motivation

The purpose of this work is to improve the run time of the simulation of black hole collisions as seen in the Dendro-Gr framework. The BSSN equations and several other simulation codes consist of several complex partial differential equations and to model these equations the value of each variable is computed once per a time step in the model. The project presented aims to provide a method to increase cache utilization to increase runtime performance.

Contributions

- **Automatic code-generation**. Given the complexity of the Einstein equations, we have developed an automatic code generation framework for GR using `SymPy` that automatically generates architecture-optimized codes which helps to improve code portability.
- **Performance**. Developed a tree isomorphism algorithm for common subexpression elimination. The algorithm can be adapted to different architetures by specifying a cache size. The code is designed to be portability and easily add performance improvements to any code.
- **Visualization**. Code is able to represent expression trees with Graphiz and visualize expression tree staging
- **Proof of Concept** By providing the target size of the cache the code is able to generate high performance code that can utilize the cache effectively.



Symbolic code generation for BSSNKO equations

$$
\begin{aligned}
\partial_t \alpha &= \mathcal{L}_\beta \alpha - 2\alpha K, \\
\partial_t \beta^i &= \lambda_2 \beta^j \, \partial_j \beta^i + \frac{3}{4} f(\alpha) B^i \\
\partial_t B^i &= \partial_t \tilde{\Gamma}^i - \eta B^i + \lambda_3 \beta^j \, \partial_j B^i - \lambda_4 \beta^j \, \partial_j \tilde{\Gamma}^i \\
\partial_t \tilde{\gamma}_{ij} &= \mathcal{L}_\beta \tilde{\gamma}_{ij} - 2\alpha \tilde{A}_{ij}, \\
\partial_t \chi &= \mathcal{L}_\beta \chi + \frac{2}{3} \chi \left(\alpha K - \partial_a \beta^a \right) \\
\partial_t \tilde{A}_{ij} &= \mathcal{L}_\beta \tilde{A}_{ij} + \chi \left(-D_i D_j \alpha + \alpha R_{ij} \right)^{TF} + \\
&\quad \alpha \left(K \tilde{A}_{ij} - 2 \tilde{A}_{ik} \tilde{A}^k_j \right), \\
\partial_t K &= \beta^k \partial_k K - D^i D_i \alpha + \\
&\quad \alpha \left(\tilde{A}_{ij} \tilde{A}^{ij} + \frac{1}{3} K^2 \right), \\
\partial_t \tilde{\Gamma}^i &= \tilde{\gamma}^{jk} \partial_j \partial_k \beta^i + \frac{1}{3} \tilde{\gamma}^{ij} \partial_j \partial_k \beta^k + \beta^j \partial_j \tilde{\Gamma}^i - \\
&\quad \tilde{\Gamma}^j \partial_j \beta^i + \frac{2}{3} \tilde{\Gamma}^i \partial_j \beta^j - 2 \tilde{A}^{ij} \partial_j \alpha + \\
&\quad 2\alpha \left(\tilde{\Gamma}^i{}_{jk} \tilde{A}^{jk} - \frac{2}{3\chi} \tilde{A}^{ij} \partial_j \chi - \frac{2}{3} \tilde{\gamma}^{ij} \partial_j K \right)
\end{aligned}
$$

```
from DENDRO_sym import *
a_rhs = Dendro.Lie(b, a) - 2*a*K
b_rhs = [3/4 * f(a) * B[i] +
l2*vec_j_del_j(b, b[i]) for i in e_i]
l2*vec_j_del_j(b, b[i])
for i in e_i]
B_rhs = [Gt_rhs[i] - eta * B[i] +
l3 * vec_j_del_j(b, B[i]) -
l4 * vec_j_del_j(b, Gt[i])
for i in e_i]
gt_rhs =  Dendro.Lie(b, gt) - 2*a*At
chi_rhs = Dendro.Lie(b, chi) +
2/3*chi*(a*K - del_j(b))
At_rhs = Dendro.Lie(b, At) + chi *
Dendro.TF(-DiDj(a) +
a*Dendro.Ricci) +
a*(K*At -2*At_ikAtKj)
K_rhs = vec_k_del_k(K) - DIDi(a) +
a*(1/3*K*K + A_ij_A_IJ(At))
```

The left panel shows the BSSNKO formulation of the Einstein equations. These are tensor equations, with indices $i, j, \ldots$ taking the values $1, 2, 3$. On the right we show the `DENDRO_sym` code for these equations. `DENDRO_sym` uses `SymPy` and other tools to generate optimized C++ code to evaluate the equations. Note that $\mathcal{L}_\beta$, D, ∂ denote Lie derivative, covariant derivative and partial derivative respectively, and we have excluded $\partial_t \Gamma^i$ from `DENDRO_sym` to save space.

For additional details please refer to **Massively Parallel Simulations of Binary Black Hole Intermediate-Mass-Ratio Inspirals**, Milinda Fernando, Hari Sundar https://arxiv.org/abs/1807.06128 and the Dendro project.

dendro

Methods

This research for consists of two main projects. The first is the subtree isomorphism problem that will focus on the common subexpression elimination and the second is a lower bound analysis for the number of temporary variables needed to solve the partial differential equations. The goal is to create an algorithm that will be able to analyze the different partial differential equations and reorder the temporary variable calculations to maximize cache effectiveness and variable reuse.

Staging

The expression DAG contains the desired target variables that solve the Einstein equations, the sources, their corresponding dependencies, the internal nodes, which are calculate from derivates and constants, the sinks. The targets have a significant number of dependecies, the BSSN equation have dependecies on the order of 100s, such that the cache misses occur while calculating the target at each grid point in the mesh. This approach mitigates cache misses by reducing the original expression DAG into smaller sub graphs such that the number of dependencies does not exceed the cache size of the specified machine. In order to maintain correctness some of the dependencies of the original expression graph must be duplicated into multiple subgraphs. Despite computing the expression tree multiple times, the goal is to reduce runtime through increased cache efficiency.

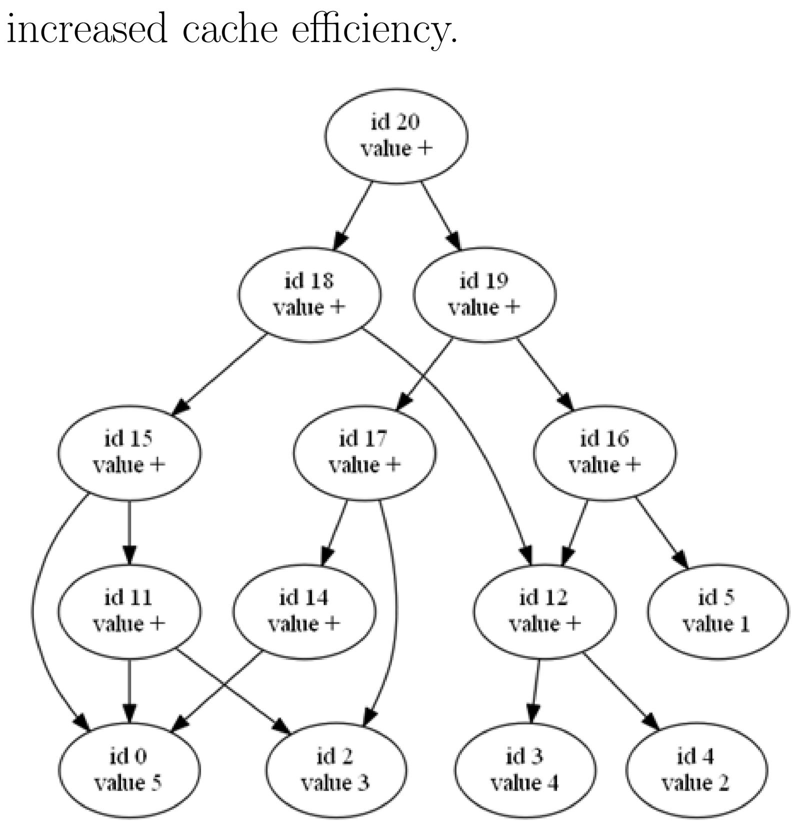


Figure 3: Staged Expression Tree

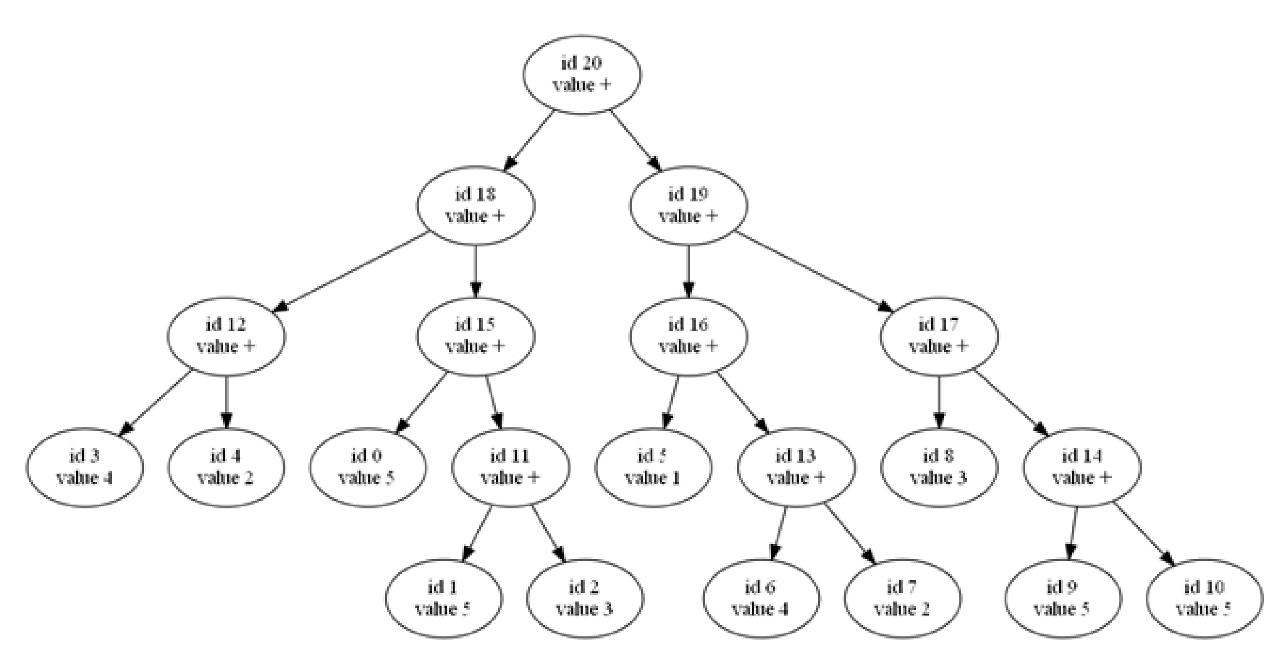


Figure 1: Initial Expression Tree

Rebuilding

Once the subgraph expressions are created the goal is to order the evaluation of the sub graphs to maintain correctness and maximize cache locality. Expression subgraphs are that are a dependency of another subgraph must be computed first. If several subgraphs have no dependencies, then the subgraphs are order such that graphs with the largest Jaccard Similarity are computed one after another. In doing so variables within the cache can increase usability.

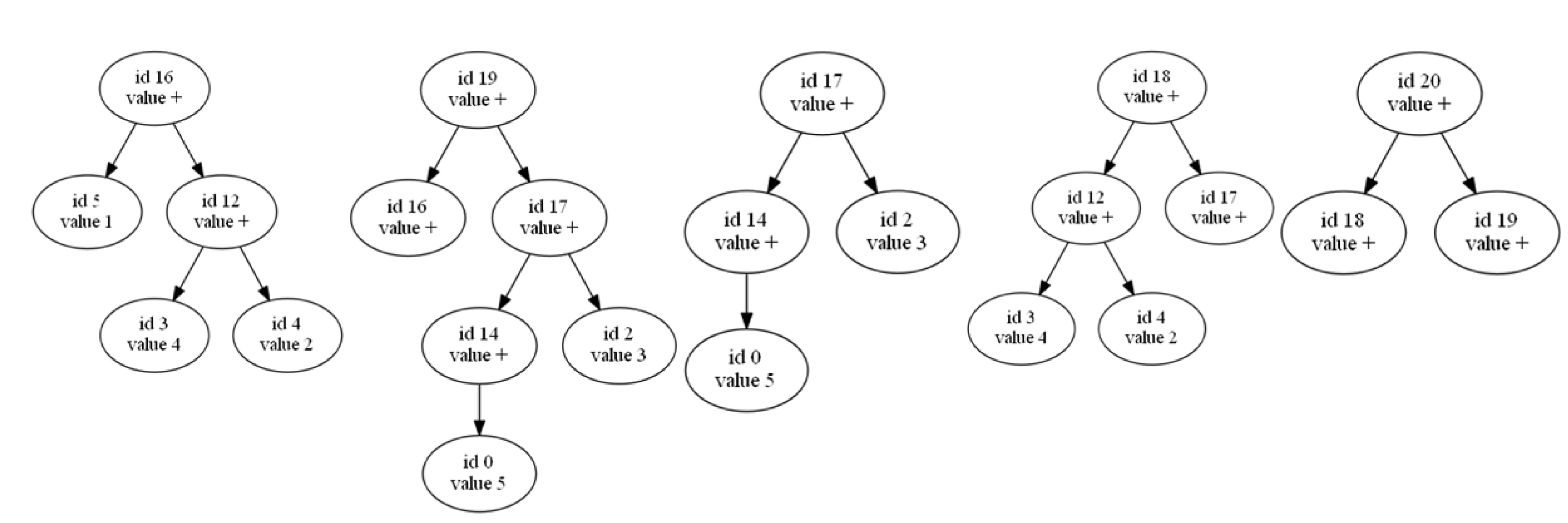


Figure 2: Rebuilt Expression Tree