Common Subexpression Elimination with Subtree Isomporphisms

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Motivation

The purpose of this work is to improve the run time of the simulation of black hole collisions as seen in the Dendro-Gr framework. The BSSN equations and several other simulation codes consist of several complex partial differential equations and to model these equations the value of each variable is computed once per a time step in the model. The project presented aims to provide a method to increase cache utilization to increase runtime performance.

Contributions

- **Wavelet Adaptive GR**. This is the first highly adaptive computational fully relativistic—i.e., including the full Einstein equations—code with an arbitrary localized unstructured wavelet based adaptive mesh.
- **Automatic code-generation**. Given the complexity of the Einstein equations, we have developed an automatic code generation framework for GR using `SymPy` that automatically generates architecture-optimized codes which helps to improve code portability.
- **Performance**. We developed a new parallel search algorithm, TreeSearch to improve the efficiency of octree meshing. We also developed efficient `unzip` and `zip` operations to allow the application of the core computational kernels on small process-local regular blocks which leads to greater performance and code portability (i.e. GPU architectures).
- **Visualization**. Code is able to represent expression trees with Graphiz and visualize expression tree staging
- **Implementation** Dendro is implemented in Python using `SymPy` for the autocode generation for the partial differntial equations. The Code is freely available at `https://github.com/paralab/Dendro-GR` under the MIT license.

Symbolic code generation for BSSNKO equations

$$
\begin{aligned}
\partial_t \alpha &= \mathcal{L}_\beta \alpha - 2\alpha K, \\
\partial_t \beta^i &= \lambda_2 \beta^j \partial_j \beta^i + \frac{3}{4} f(\alpha) B^i \\
\partial_t B^i &= \partial_t \tilde{\Gamma}^i - \eta B^i + \lambda_3 \beta^j \partial_j B^i - \lambda_4 \beta^j \partial_j \tilde{\Gamma}^i \\
\partial_t \tilde{\gamma}_{ij} &= \mathcal{L}_\beta \tilde{\gamma}_{ij} - 2\alpha \tilde{A}_{ij}, \\
\partial_t \chi &= \mathcal{L}_\beta \chi + \frac{2}{3}\chi\left(\alpha K - \partial_a \beta^a\right) \\
\partial_t \tilde{A}_{ij} &= \mathcal{L}_\beta \tilde{A}_{ij} + \chi\left(-D_i D_j \alpha + \alpha R_{ij}\right)^{TF} + \\
&\quad \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j\right), \\
\partial_t K &= \beta^k \partial_k K - D^i D_i \alpha + \\
&\quad \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right), \\
\partial_t \tilde{\Gamma}^i &= \tilde{\gamma}^{jk}\partial_j\partial_k\beta^i + \frac{1}{3}\tilde{\gamma}^{ij}\partial_j\partial_k\beta^k + \beta^j\partial_j\tilde{\Gamma}^i - \\
&\quad \tilde{\Gamma}^j\partial_j\beta^i + \frac{2}{3}\tilde{\Gamma}^i\partial_j\beta^j - 2\tilde{A}^{ij}\partial_j\alpha + \\
&\quad 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{jk} - \frac{2}{3\chi}\tilde{A}^{ij}\partial_j\chi - \frac{2}{3}\tilde{\gamma}^{ij}\partial_j K\right)
\end{aligned}
$$

```
from DENDRO_sym import *
a_rhs = Dendro.Lie(b, a) - 2*a*K
b_rhs = [3/4 * f(a) * B[i] +
l2*vec_j_del_j(b, b[i]) for i in e_i]
l2*vec_j_del_j(b, b[i])
for i in e_i]
B_rhs = [Gt_rhs[i] - eta * B[i] +
l3 * vec_j_del_j(b, B[i]) -
l4 * vec_j_del_j(b, Gt[i])
for i in e_i]
gt_rhs =  Dendro.Lie(b, gt) - 2*a*At
chi_rhs = Dendro.Lie(b, chi) +
2/3*chi*(a*K - del_j(b))
At_rhs = Dendro.Lie(b, At) + chi *
Dendro.TF(-DiDj(a) +
a*Dendro.Ricci) +
a*(K*At -2*At_ikAtKj)
K_rhs = vec_k_del_k(K) - DIDi(a) +
a*(1/3*K*K + A_ij_A_IJ(At))
```

The left panel shows the BSSNKO formulation of the Einstein equations. These are tensor equations, with indices $i, j, \ldots$ taking the values $1, 2, 3$. On the right we show the `DENDRO_sym` code for these equations. `DENDRO_sym` uses `SymPy` and other tools to generate optimized C++ code to evaluate the equations. Note that $\mathcal{L}_\beta$, D, ∂ denote Lie derivative, covariant derivative and partial derivative respectively, and we have excluded $\partial_t \Gamma^i$ from `DENDRO_sym` to save space.

For additional details please refer to **Massively Parallel Simulations of Binary Black Hole Intermediate-Mass-Ratio Inspirals**, Milinda Fernando, David Neilsen, Hyun Lim, Eric Hirschmann, Hari Sundar, `https://arxiv.org/abs/1807.06128`.

Adaptability

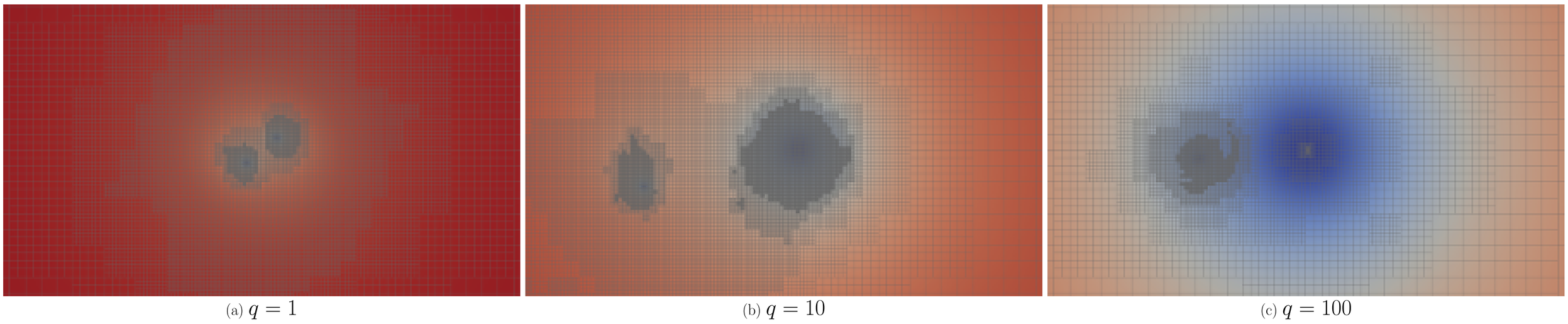
In a black hole binary, the mass of the smallest black hole effectively sets the minimum length scale for the simulation. The total mass of the binary $M = m_1 + m_2$ is a global scaling parameter and is typically fixed to a constant value. Then the mass of the smaller black hole can be written $m_2 = M/(q+1)$, showing that the minimum resolution scale for the binary is inversely proportional to the mass ratio. In $3d$ the number of points required to resolve the smallest black hole grows as q^3.

Staging

Staging is focused on finding variable reuse within the partial differential equations. The first problem is to create an expression tree from the partial differential equations generated from the SymPy auto generated code. Once the expression tree is created, subtree isomorphism analysis can begin. This will be a bottom up approach that considers the values used in each leaf node in addition to finding similar tree structure. Each node within the tree will keep track of all the leaf values that it depends on. Set similarity will be used as a precondition before calculating the more expensive tree isomorphism. By the end of the Staging Process the most common subexpressions will be identified. Each expression will be valued depending on the number of leaf node dependents, to mitigate the total number of temporary variables, and the number of times each expression appears, to maximize data reuse.

Rebuilding

The rebuilding phase takes the results from the staging process and rebuilds the expression tree. In order of importance the rebuilding phase must preserve the correct final answers, maximize cache effectiveness, and minimize the total number of temporary variables used. The current strategy is to use a dynamic programming approach. Dynamic programming will also allow for the rebuilding process to scale effectively. Once complete, these results will be measured experimentally. Once a proof of concept has been completed the rebuilding process will take in parameters such as memory architecture, and L1 cache size to optimize the calculations for each machine. Once the rebuilding process is completed a parser will be created to transform the original SymPy autogenerated code to reflect the updated expression tree.



(a) $q = 1$ (b) $q = 10$ (c) $q = 100$