1. Given, $n = 30$, $\mu = \frac{\sum_{x}^{n} x_i}{n} = \frac{17}{30}$

$\mu_{TRUE} = 0.56$

$$P(x) = \begin{cases} q & , x = 0 \\ 1-q & , x = 1 \end{cases}$$

we can also write this as,

$$P(x) = q^{1-x}(1-q)^{x}$$

$$L(q) = \prod_{i=1}^{n} q^{1-x_i} \cdot (1-q)^{x_i}$$

$$L(q) = q^{1-x_1} \cdot (1-q)^{x_1} \times q^{1-x_2}(1-q)^{x_2} \times \dots \times q^{1-x_n}(1-q)^{x_n}$$

$$\Rightarrow L(q) = q^{\sum_{i=1}^{n}(1-x_i)} \times (1-q)^{\sum_{i=1}^{n} x_i}$$

$$L(q) = q^{n - \sum_{i=1}^{n} x_i} \times (1-q)^{\sum_{i=1}^{n} x_i}$$

2. To find MLE, we'll take log on both side

$$\ln(L(q)) = (n - \sum_{i=1}^{n} x_i)\ln(q) + \sum_{i=1}^{n} x_i \ln(1-q)$$

Taking derivative w.r.t q & equating to 0,

$$\frac{d\,L(q)}{dq} = \frac{d}{dq}\left(n - \sum_{i=1}^{n} x_i\right) \ln(q) + \frac{d}{dq} \sum_{i=1}^{n} x_i \ln(1-q)$$

$$0 = \left(n - \sum_{i=1}^{n} x_i\right) \frac{1}{q} + \sum_{i=1}^{n} x_i \left(\frac{1}{1-q}\right)$$

$$(1-q)\left(n - \sum_{i=1}^{n} x_i\right) + q \sum_{i=1}^{n} x_i = 0$$

$$n - \sum_{i=1}^{n} x_i - qn + q \sum_{i=1}^{n} x_i + q \sum_{i=1}^{n} x_i = 0$$

$$2q \sum_{i=1}^{n} x_i - qn = \sum_{i=1}^{n} x_i - n$$

$$q = \frac{\left(\sum_{i=1}^{n} x_i - n\right)}{\left(2 \sum_{i=1}^{n} x_i - n\right)}$$

Taking derivative w.r.t q & equating to 0, we get $\rightarrow$

$$\frac{d}{dq}L(q) = \frac{d}{dq}\left(n - \sum_{i=1}^{n} x_i\right)\ln(q) + \frac{d}{dq}\left(\sum_{i=1}^{n} x_i \ln(1-q)\right)$$

$$0 = \left(n - \sum_{i=1}^{n} x_i\right)\frac{1}{q} + \sum_{i=1}^{n} x_i \left(\frac{-1}{(1-q)}\right)$$

$$\Rightarrow (1-q)\left(n - \sum_{i=1}^{n} x_i\right) + q \sum_{i=1}^{n} x_i (-1) = 0$$

$$\Rightarrow n - \sum_{i=1}^{n} x_i - qn + q\sum_{i=1}^{n} x_i - q\sum_{i=1}^{n} x_i = 0$$

$$\hat{q} = \frac{n - \sum_{i=1}^{n} x_i}{n}$$

$\hat{q} \Rightarrow$ q-hat represents estimated q

3o Given the same, $n = 30$

and $\sum_{i=1}^{n} x_i = 17$

$$\hat{q} = \frac{30-17}{30}$$

$$\hat{q} = 0.43$$

4. To prove that MLE is unbiased, we need to show if $\hat{q} = q$

$$\hat{q} = \frac{n - \sum_{i=1}^{n} x_i}{n}$$

$$= 1 - \frac{1}{n} \sum_{i=1}^{n} x_i$$

Putting x_i in terms of q,

$$= 1 - \frac{1}{n} \Big(\underbrace{\sum_{i=1}^{n} \big(1 \cdot p(x=1) + 0 \cdot p(x=0) \big)}_{\text{sum based on sample}} \Big)$$

$$= 1 - \frac{1}{n} \Big(\sum_{i=1}^{n} (1 - q + 0) \Big)$$

$$= 1 - \frac{1}{n} (n - nq)$$

$$= 1 - 1 + q$$

$$= q$$

hence our estimate is unbiased