Expected Values of Normal Order Statistics

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Expected values of normal order statistics

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1. HISTORY

The problem of order statistics has received a great deal of attention from statisticians dating at least as far back as a paper by Karl Pearson (1902) giving a solution of a generalization of a problem proposed by Galton (1902). The generalized problem is that of finding the average difference between the pth and the $(p+1)$th individuals in a sample of size n when the sample is arranged in order of magnitude. The result is

$$\frac{n}{(n-p)!\,p!}\int_{-\infty}^{\infty}\alpha^{n-p}(1-\alpha)^p\,dx, \tag{1·1}$$

where $\alpha = \int_{-\infty}^{x}\phi(x)\,dx$ and $\phi(x)$ is the probability density function of the variable x. Pearson stated a theorem, which he attributed to W. F. Sheppard, that the average differences between successive individuals are the successive terms in the binomial expansion of

$$\int_{-\infty}^{\infty}\{\alpha+(1+\alpha)\}^n\,dx. \tag{1·2}$$

In a footnote, Pearson remarked, 'Clearly a knowledge of the average difference in character of two adjacent individuals involves also a knowledge of the average difference in character between any two individuals'. For a symmetric population, such knowledge also involves a knowledge of the expected values of all the order statistics, since for odd sample sizes $n = 2k+1$, where k is an integer, $E(x_{k+1}) = \mu$ (the population mean), while for even sample sizes $n = 2k$, $\frac{1}{2}[E(x_k)+E(x_{k+1})] = \mu$.

Irwin (1925) gave expressions for the mean difference between the pth and qth individuals in order of magnitude and for the moments of the frequency distribution of differences between consecutive individuals. Tippett (1925) published a seven-decimal-place table of the probability integral of the largest individual in samples of size n from a normal population for

$$n = 3, 5, 10 \text{ and } x = -2·6\,(0·2)\,5·8;$$
$$n = 20, 30, 50 \text{ and } x = -0·1\,(0·1)\,6·0; \quad n = 100\,(100)\,1000 \text{ and } x = 1·0\,(0·1)\,6·5.$$

The same paper included a five-decimal-place table of the mean range of samples of size $n = 2\,(1)\,100$ from a normal population from which the expected values of the largest and smallest individuals could of course be derived. The expected values of normal order statistics other than the first and last were not computed until somewhat later.

Karl Pearson & Margaret V. Pearson (1931) obtained an expansion in Taylor series for $E(x_i)$, accurate to 5 or 6 decimal places for $|E(x_i)|$ not too large (say < 1). Fisher & Yates (1938, Table XX) published a two-decimal-place table of the expected values of all normal order statistics for samples of size $n = 2\,(1)\,50$. Their values are correct except for four errors of a unit in the last place, due to rounding. Hastings, Mosteller, Tukey & Winsor (1947) published a five-decimal-place table of the means and standard deviations of all order statistics for samples of size $n = 2\,(1)\,10$ from a normal population, also from a uniform population and from a selected long-tailed population. Their values for the means of normal order statistics are correct except for $n = 10$, where there are errors of from 1 to 7 units in the last place.

Wilks (1948) published an expository paper summarizing work on order statistics up to that time and listing 90 references.

Godwin (1949*a*) published a table of the expected values of rank differences in normal samples, to 10 decimal places for $n = 2$; 9 decimal places for $n = 3, 4$; 8 decimal places for $n = 5$; 7 decimal places for $n = 6, 7$; 6 decimal places for $n = 8$; and 5 decimal places for $n = 9, 10$. Godwin (1949*b*) also published a seven-decimal-place table of the means and standard deviations of all normal order statistics for samples of size $n = 2\,(1)\,10$. His values for the means of the first-order statistics are accurate to 7 decimal places, and his other values are probally equally accurate, since they were computed by the same method. Cadwell (1953) published a table of moments (mean, variance, β_1 and β_2) and selected percentage points of the first quasi-range for samples of size $n = 10\,(1)\,30$. His values of the means are correct except for one error of a unit in the last place, due to rounding. E. S. Pearson & Hartley (1954, Table 28) published a table of expected values of normal order statistics, to 3 decimal places for $n = 2\,(1)\,20$ and to 2 decimal places for

$$n = 21\,(1)\,26\,(2)\,50;$$

values for $n = 2\,(1)\,10$ were compiled from Godwin's table, those for $n = 11\,(1)\,20$ were freshly computed by Jean H. Thompson, while those for $n > 20$ were taken from the table by Fisher and Yates. These values are correct except for three errors of a unit in the last place, due to rounding. Harter (1959) published a six-decimal-place table (accurate to within a unit in the last place) of the expected values of the range and of the first 8 quasi-ranges for samples of size $n = 2\,(1)\,100$ taken from a normal population. By dividing these values by two, the expectations of the absolute values of the nine largest and the nine smallest normal deviates can be obtained.

Federer (1951) used a somewhat different approach than did most of the aforementioned authors, who depended largely on numerical integration for the determination of tabular values. Federer made use of the recurrence formula

$$E(x_{m,\,i+1}) = \frac{1}{i}\{mE(x_{m-1,\,i}) - (m-i)\,E(x_{m,\,i})\}, \tag{1·3}$$

where $x_{m,i}$ is the ith largest deviate from a sample of size m. Starting from Tippett's table of expected values of the range, Federer computed three-decimal-place values of the three largest normal deviates for samples of size $n = 41\,(1)\,200$ and two-decimal-place values of the fourth largest normal deviate for $n = 41\,(1)\,200$ and of the fifth largest normal deviate for $n = 41\,(1)\,100$. Because of loss of accuracy with repeated application of the recurrence formula, some of Federer's values are in error by from 1 to 3 units in the last place, and it is evident that the form of the recurrence formula given by (1·3) is of little value in computation. The author is indebted to the Editor for pointing out that, if written in the form

$$E(x_{m-1,\,i}) = \frac{1}{m}\{iE(x_{m,\,i+1}) + (m-i)\,E(x_{m,\,i})\}, \tag{1·4}$$

the recurrence formula can be used for working *downwards* with no serious accumulation of rounding errors. Similar recurrence formulae for the variance and covariance of order statistics have recently been obtained by Govindarajulu (1959).

2. Method of computation

The expected value of the kth largest observation in a sample of size n from a standard normal population (mean zero and variance one) is given by the equation

$$E(x_{k|n}) = \frac{n!}{(n-k)!\,(k-1)!}\int_{-\infty}^{\infty} x[\tfrac{1}{2}-\Phi(x)]^{k-1}\,[\tfrac{1}{2}+\Phi(x)]^{n-k}\,\phi(x)\,dx, \qquad (2\cdot1)$$

where $\phi(x) = (2\pi)^{-\frac{1}{2}}\,e^{-\frac{1}{2}x^2}$ and $\Phi(x) = \int_0^x \phi(x)\,dx$. The expected value of the kth smallest observation is given by the same expression preceded by a minus sign, so that for a given value of n it is necessary only to compute the expected values for $k = 1\,(1)\,[\frac{1}{2}n]$. This was done by numerical integration on the Univac Scientific (ERA 1103 A) computer, for $n = 2\,(1)\,100$ and for values of n, none of whose prime factors exceeds seven, up through $n = 400$. Values of $\log_{10} n!$ for $n = 1\,(1)\,400$ from a table by Pearson & Hartley (1954) and values of $\phi(x) = \phi(-x)$ for $x = 0\,(0{\cdot}05)\,7{\cdot}60$, $2\Phi(x) = -2\Phi(-x)$ for $x = 0\,(0{\cdot}05)\,5{\cdot}95$, and $1-2\Phi(x) = 1+2\Phi(-x)$ for $x = 6{\cdot}00\,(0{\cdot}05)\,7{\cdot}60$ from tables by the National Bureau of Standards (1953, Tables I and II) were read into the computer. For each pair of values of n and k, the product $I(n,k,x)$ of the multiplicative constant and the integrand was determined for $x = -7{\cdot}60\,(0{\cdot}05)\,7{\cdot}60$ by computing $e^{\log_e I(n,k,x)}$, where

$$\log_e I(n,k,x) = \log_e n! - \log_e (n-k)! - \log_e (k-1)! + \log_e x + (k-1)\log_e[\tfrac{1}{2}-\Phi(x)] + (n-k)\log_e[\tfrac{1}{2}+\Phi(x)] + \log_e \phi(x). \qquad (2\cdot2)$$

Fixed-point binary arithmetic was used, and the numbers were scaled so as to retain as much accuracy as possible. Since $I(n,k,x)$ is zero (to the number of places carried in the computer) for all values of n and k when $|x| > 7{\cdot}60$, the resulting value of $E(x_{k|n})$, obtained by using either the trapezoidal rule or the seven-point Lagrangian integration formula, is found by summing $I(n,k,x)$ for $x = -7{\cdot}60\,(h)\,7{\cdot}60$ and multiplying by the interval, h. Results were computed and printed out (to seven decimal places) for $h = 0{\cdot}05$ and $h = 0{\cdot}10$, and agreement is sufficiently close to guarantee that the values of $E(x_{k|n})$ for $h = 0{\cdot}05$ are accurate to within a unit in the fifth decimal place. Accordingly, the values for $h = 0{\cdot}05$ were rounded to five decimal places, and the five-decimal-place values were punched on cards and printed on the IBM 407 tabulator. The results for $n = 2\,(1)\,100\,(25)\,250\,(50)\,400$ are shown in Table 1.

Acknowledgment, with thanks, is made to Eugene H. Guthrie, who programmed the problem for computation on the ERA 1103 A.

3. Blom's approximation

In 1954 Blom became interested in the problem of plotting points on normal probability paper and, after reading a paper by Chernoff & Lieberman (1954), in the related problem of estimating parameters by means of linear functions of order statistics, Blom (1958) proposed approximating the ith normal order statistic (ith smallest normal deviate) for a sample of size n by means of the relation

$$E(x_i) = \Phi^{-1}\left(\frac{i-\alpha}{n-2\alpha+1}\right), \qquad (3\cdot1)$$

where $\Phi(x) = \int_{-\infty}^{x} \phi(x)\,dx$, with $\phi(x) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2}$. Note that $\Phi(x)$ is defined differently here than in § 2. It should be mentioned that there has been an argument of long-standing between advocates of the approximations corresponding to $\alpha = 0$ and $\alpha = 0{\cdot}5$, neither of which is correct. Blom tabulated the value of α required to yield the correct value of $E(x_i)$ for $i = 1\,(1)\,[\frac{1}{2}n]$ when $n = 2\,(2)\,10\,(5)\,20$. The values of α increase as n increases, the lowest value being 0·330 for $n = 2$, $i = 1$. For a given n, α is least for $i = 1$, rises quickly to a peak

Table 2. *Values of $\alpha_{i,n}$ such that $E(x_i) = \Phi^{-1}[(i-\alpha_{i,n})/(n-2\alpha_{i,n}+1)]$*

i	$n = 25$	$n = 50$	$n = 100$	$n = 200$	$n = 400$	i	$n = 100$	$n = 200$	$n = 400$
1	0·377	0·384	0·391	0·396	0·401	30	0·394	0·404	0·414
2	·394	·403	·412	·419	·426	35	·393	·402	·412
3	·395	·405	·415	·423	·430	40	·392	·400	·410
4	·394	·405	·415	·424	·431	45	·391	·398	·408
5	·392	·403	·414	·423	·431	50	·391	·397	·407
6	0·391	0·402	0·412	0·422	0·430	55	—	0·396	0·405
7	·390	·400	·411	·421	·429	60	—	·395	·404
8	·389	·399	·410	·420	·429	65	—	·394	·403
9	·388	·398	·408	·418	·428	70	—	·394	·402
10	·388	·397	·407	·417	·427	75	—	·393	·401
11	0·387	0·396	0·406	0·416	0·426	80	—	0·393	0·400
12	·387	·395	·405	·415	·425	85	—	·392	·399
13	—	·394	·404	·414	·424	90	—	·392	·399
14	—	·393	·403	·414	·423	95	—	·391	·398
15	—	·393	·402	·413	·423	100	—	·391	·398
16	—	0·392	0·402	0·412	0·422	110	—	—	0·396
17	—	·392	·401	·411	·421	120	—	—	·396
18	—	·391	·400	·410	·420	130	—	—	·395
19	—	·391	·399	·410	·420	140	—	—	·394
20	—	·391	·399	·409	·419	150	—	—	·394
21	—	0·390	0·398	0·408	0·419	160	—	—	0·393
22	—	·390	·398	·408	·418	170	—	—	·393
23	—	·390	·397	·407	·417	180	—	—	·392
24	—	·390	·397	·407	·417	190	—	—	·392
25	—	·390	·396	·406	·416	200	—	—	·391

for a relatively small value of i, and then drops off slowly; as an example, for $n = 20$, $\alpha = 0{\cdot}374$ for $i = 1$, the peak value of α is 0·391 for $i = 3$, and α drops to 0·386 for $i = 8, 9, 10$. Blom conjectured that α always lies in the interval (0·33, 0·50). He suggested the use of $\alpha = \frac{3}{8}$ as a compromise value.

If one solves (3·1) for the value of α required to yield the correct value of $E(x_i)$ for given i and n, one obtains

$$\alpha_{i,n} = \frac{i-(n+1)\,\Phi[E(x_i)]}{1-2\Phi[E(x_i)]}. \tag{3·2}$$

Values of $\alpha_{i,n}$ for $i = 1\,(1)\,[\frac{1}{2}n]$ when $n = 25, 50, 100, 200, 400$ have been computed on the Burroughs E 101–3 computer, and the results, rounded to three decimal places, are shown in Table 2. For brevity, results have been given only for values of i which are multiples of 5 for i between 25 and 100 and multiples of 10 for i between 100 and 200. A glance at the values in Table 2 is sufficient to show that the compromise value of $\frac{3}{8}$ or 0·375 for α proposed by Blom is too low except for small values of n. If, however, one wishes to minimize the maximum error in estimating $E(x_i)$ for $n \leqslant 400$, one is led to choose a value of α even small than $\frac{3}{8}$, since the estimate of $E(x_i)$ is much more sensitive to changes in α for small values of n (and i) than for large values. The maximum error in estimating $E(x_i)$ is minimized by choosing $\alpha = 0{\cdot}363$. This gives a maximum error of 0·018, which is hardly satisfactory. It is possible, however, to do a fairly good job of estimating $E(x_i)$ by choosing one or two compromise values of α for each n. One can choose a single compromise value, α_n, for each n, to be used for all values of i, and simultaneously insure that the error in $E(x_i)$ does not exceed four units in the third decimal place. If one uses $\alpha_{1,n}$ to estimate $E(x_1)$ and $\alpha_{2,n}$ to estimate $E(x_i)$ for $i \neq 1$, the error in $E(x_i)$ will not exceed one unit in the third decimal place. Values of α_n, $\alpha_{1,n}$ and $\alpha_{2,n}$ are given in Table 3 for $n = 2\,(2)\,10\,(5)\,25, 50, 100, 200, 400$ along with regression equations

Table 3. *Compromise values of* α

n	α_n	$\alpha_{1,n}$	$\alpha_{2,n}$
2	0·330	0·330	—
4	·349	·347	0·359
6	·359	·355	·368
8	·364	·360	·374
10	0·368	0·364	0·378
15	·374	·370	·385
20	·378	·374	·390
25	·381	·377	·394
50	0·389	0·384	0·403
100	·396	·391	·412
200	·402	·396	·419
400	·407	·401	·426

To estimate α for intermediate values of n, use the following equations:

$$\begin{aligned} \alpha_n &= 0{\cdot}314195 + 0{\cdot}063336X - 0{\cdot}010895X^2, \\ \alpha_{1,n} &= 0{\cdot}315065 + 0{\cdot}057974X - 0{\cdot}009776X^2, \\ \alpha_{2,n} &= 0{\cdot}327511 + 0{\cdot}058212X - 0{\cdot}007909X^2, \end{aligned}$$

where $X = \log_{10} n$.

to be used for intermediate values of n. Values of α found by substituting tabular values of n in these regression equations do not differ from the corresponding tabular values of α by more than two units in the third decimal place, and this error in α does not increase the error in $E(x_i)$ by more than one unit in the third decimal place. There is reason to believe that results for intermediate values of n will be equally good, but use of these equations for $n > 400$ is emphatically discouraged. Thus, if one wishes to interpolate for intermediate values of n, the maximum errors are two units in the third decimal place for the approximation based on a single compromise value of α and five units in the third decimal place for the approximation based on two compromise values of α. These errors compare with a maximum error of between one and two units in the third decimal place for linear interpolation between successive value of n for a given $i(k)$ in Table 1. Comparison of the maximum errors might lead to the conclusion that interpolation in Table 1 is always more accurate than interpolation using Blom's approximation This would be erroneous, since the maximum error for the former occurs for large values of i (near $\frac{1}{2}n$), while the maximum error for the latter occurs for small values of i. Interpolation using Blom's approximation for large values of i, especially when the desired value of n lies about midway between widely separated successive tabular values of n (for example, when $n = 232$), and interpolation in Table 1 otherwise will limit the error to no more than a unit in the third decimal place. If more accurate values are required, they should be computed in the same way that Table 1 was computed, as should values for $n > 400$, or else they should be computed by working downwards from the next higher tabular value of n, using the recurrence formula (1·4). Table 4 summarizes the above results, giving maximum errors in determining $E(x_i)$ by various methods.

Table 4. *Maximum errors in determining $E(x_i)$ by various methods*

Method	Values of n in Table 3	Intermediate values of n
Blom's approximation:		
$\alpha = 0{\cdot}363$ for all values of n	0·018	0·018
One value of α for each n	·004	·005
Two values of α for each n	·001	·002
Interpolation in Table 1	—	·002
Recurrence formula (1·4)	·00001	·00001
Numerical integration ($h = 0{\cdot}05$)	$<{\cdot}00001$	$<{\cdot}00001$

4. Applications

Pearson & Hartley (1954, p. 56) have given two examples of applications of tables of expected values of normal order statistics. The first of these is concerned with estimating the weight of the five heaviest of 30 lambs at age $2\frac{1}{2}$ months, given the mean and standard deviation of the population, which is assumed to be normal. The second deals with the use of order statistics in estimating the population standard deviation. Pearson & Hartley and also Fisher & Yates (1953, p. 76) mention the use of expected values of normal order statistics in the analysis of variance of ranked data. The potential use of expected values of normal order statistics for transformation to standard normal scores preliminary to the analysis of variance was the principal motivation for the present study. In cases where only the rank of the observations is known, there is no reasonable alternative to transformation to standard normal scores, but the usefulness of this method is not restricted to such cases. When the data are known to have come from a population which does not satisfy the assumptions underlying the analysis of variance, of which normality is one, or when the data themselves give a strong indication to that effect, the experimenter seeks a transformation which will minimize or eliminate departures from the assumptions. One transformation which should be considered is the transformation to standard normal scores, and a preliminary investigation has shown that this transformation has some very desirable properties; in some cases it reduces both non-additivity and non-homogeneity of variance to lower levels than does any transformation of the form $(x+c)^p$. It has, of course, the obvious disadvantage of not being reversible.

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Addendum

An account of methods of computing expected values of normal order statistics would be incomplete without mention of the series expansions worked out by David & Johnson (1954) and by Plackett (1958). Saw (1960) has made a comparison of the David–Johnson series and the Plackett series. Neither series seems particularly well adapted to the computation of tables of the sort included in this paper, though either would be quite useful in obtaining very accurate expected values for isolated cases. The author wishes to thank Dr F. N. David for drawing his attention to these papers.

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Table 1. *Expected values of normal order statistics* (see overleaf)

$$E(x_{k|n}) = \frac{n!}{(n-k)!\,(k-1)!}\int_{-\infty}^{\infty} x[\tfrac{1}{2}-\Phi(x)]^{k-1}\,[\tfrac{1}{2}+\Phi(x)]^{n-k}\,\phi(x)\,dx,$$

where $$\phi(x) = (2\pi)^{-\frac{1}{2}}\,e^{-\frac{1}{2}x^2} \quad \text{and} \quad \Phi(x) = \int_0^x \phi(x)\,dx.$$

[Tabular values are the expected values of the kth largest normal deviate for a sample of size n from $N(0, 1)$; or when preceded by a minus sign, they are the expected values of the kth smallest normal deviate.]

k \ n	2	3	4	5	6	7	8	9
1	0·56419	0·84628	1·02938	1·16296	1·26721	1·35218	1·42360	1·48501
2	—	·00000	0·29701	0·49502	0·64176	0·75737	0·85222	0·93230
3	—	—	—	·00000	·20155	·35271	·47282	·57197
4	—	—	—	—	—	·00000	·15251	·27453
5	—	—	—	—	—	—	—	·00000

k \ n	10	11	12	13	14	15	16	17	18	19
1	1·53875	1·58644	1·62923	1·66799	1·70338	1·73591	1·76599	1·79394	1·82003	1·84448
2	1·00136	1·06192	1·11573	1·16408	1·20790	1·24794	1·28474	1·31878	1·35041	1·37994
3	0·65606	0·72884	0·79284	0·84983	0·90113	0·94769	0·99027	1·02946	1·06573	1·09945
4	·37576	·46198	·53684	·60285	·66176	·71488	·76317	0·80738	0·84812	0·88586
5	·12267	·22489	·31225	·38833	·45557	·51570	·57001	·61946	·66479	·70661
6	—	0·00000	0·10259	0·19052	0·26730	0·33530	0·39622	0·45133	0·50158	0·54771
7	—	—	—	·00000	0·08816	·16530	·23375	·29519	·35084	·40164
8	—	—	—	—	—	·00000	·07729	·14599	·20774	·26374
9	—	—	—	—	—	—	—	·00000	·06880	·13072
10	—	—	—	—	—	—	—	—	—	·00000

k \ n	20	21	22	23	24	25	26	27	28	29
1	1·86748	1·88917	1·90969	1·92916	1·94767	1·96531	1·98216	1·99827	2·01371	2·02852
2	1·40760	1·43362	1·45816	1·48137	1·50338	1·52430	1·54423	1·56326	1·58145	1·59888
3	1·13095	1·16047	1·18824	1·21445	1·23924	1·26275	1·28511	1·30641	1·32674	1·34619
4	0·92098	0·95380	0·98459	1·01356	1·04091	1·06679	1·09135	1·11471	1·13697	1·15822
5	·74538	·78150	·81527	0·84697	0·87682	0·90501	0·93171	0·95705	0·98115	1·00414
6	0·59030	0·62982	0·66667	0·70115	0·73354	0·76405	0·79289	0·82021	0·84615	0·87084
7	·44833	·49148	·53157	·56896	·60399	·63690	·66794	·69727	·72508	·75150
8	·31493	·36203	·40559	·44609	·48391	·51935	·55267	·58411	·61385	·64205
9	·18696	·23841	·28579	·32965	·37047	·40860	·44436	·47801	·50977	·53982
10	·06200	·11836	·16997	·21755	·26163	·30268	·34105	·37706	·41096	·44298
11	—	0·00000	0·05642	0·10813	0·15583	0·20006	0·24128	0·27983	0·31603	0·35013
12	—	—	—	·00000	·05176	·09953	·14387	·18520	·22389	·26023
13	—	—	—	—	—	·00000	·04781	·09220	·13361	·17240
14	—	—	—	—	—	—	—	·00000	·04442	·08588
15	—	—	—	—	—	—	—	—	—	·00000

k \ n	30	31	32	33	34	35	36	37	38	39
1	2·04276	2·05646	2·06967	2·08241	2·09471	2·10661	2·11812	2·12928	2·14009	2·15059
2	1·61560	1·63166	1·64712	1·66200	1·67636	1·69023	1·70362	1·71659	1·72914	1·74131
3	1·36481	1·38268	1·39985	1·41637	1·43228	1·44762	1·46244	1·47676	1·49061	1·50402
4	1·17855	1·19803	1·21672	1·23468	1·25196	1·26860	1·28466	1·30016	1·31514	1·32964
5	1·02609	1·04709	1·06721	1·08652	1·10509	1·12295	1·14016	1·15677	1·17280	1·18830
6	0·89439	0·91688	0·93841	0·95905	0·97886	0·99790	1·01624	1·03390	1·05095	1·06741
7	·77666	·80066	·82359	·84555	·86660	·88681	0·90625	0·92496	0·94300	0·96041
8	·66885	·69438	·71875	·74204	·76435	·78574	·80629	·82605	·84508	·86343
9	·56834	·59545	·62129	·64596	·66954	·69214	·71382	·73465	·75468	·77398
10	·47329	·50206	·52943	·55552	·58043	·60427	·62710	·64902	·67009	·69035
11	0·38235	0·41287	0·44185	0·46942	0·49572	0·52084	0·54488	0·56793	0·59005	0·61131
12	·29449	·32686	·35755	·38669	·41444	·44091	·46620	·49042	·51363	·53592
13	·20885	·24322	·27573	·30654	·33582	·36371	·39032	·41576	·44012	·46348
14	·12473	·16126	·19572	·22832	·25924	·28863	·31663	·34336	·36892	·39340
15	·04148	·08037	·11695	·15147	·18415	·21515	·24463	·27272	·29954	·32520
16	—	0·00000	0·03890	0·07552	0·11009	0·14282	0·17388	0·20342	0·23159	0·25849
17	—	—	—	·00000	·03663	·07123	·10399	·13509	·16469	·19292
18	—	—	—	—	—	·00000	·03461	·06739	·09853	·12817
19	—	—	—	—	—	—	—	·00000	·03280	·06395
20	—	—	—	—	—	—	—	—	—	·00000

k \ n	40	41	42	43	44	45	46	47	48	49
1	2·16078	2·17068	2·18032	2·18969	2·19882	2·20772	2·21639	2·22486	2·23312	2·24119
2	1·75312	1·76458	1·77571	1·78654	1·79707	1·80733	1·81732	1·82706	1·83655	1·84582
3	1·51702	1·52964	1·54188	1·55377	1·56533	1·57658	1·58754	1·59820	1·60860	1·61874
4	1·34368	1·35728	1·37048	1·38329	1·39574	1·40784	1·41962	1·43108	1·44224	1·45312
5	1·20330	1·21782	1·23190	1·24556	1·25881	1·27170	1·28422	1·29641	1·30827	1·31983
6	1·08332	1·09872	1·11364	1·12810	1·14213	1·15576	1·16899	1·18186	1·19439	1·20658
7	0·97722	0·99348	1·00922	1·02446	1·03924	1·05358	1·06751	1·08104	1·09420	1·10701
8	·88114	·89825	0·91480	0·93082	0·94634	0·96139	0·97599	0·99018	1·00396	1·01737
9	·79259	·81056	·82792	·84472	·86097	·87673	·89201	·90684	0·92125	0·93525
10	·70988	·72871	·74690	·76448	·78148	·79795	·81391	·82939	·84442	·85902
11	0·63177	0·65149	0·67052	0·68889	0·70666	0·72385	0·74049	0·75663	0·77228	0·78748
12	·55736	·57799	·59788	·61707	·63561	·65353	·67088	·68768	·70397	·71978
13	·48591	·50749	·52827	·54830	·56763	·58631	·60438	·62186	·63881	·65523
14	·41688	·43944	·46114	·48204	·50220	·52166	·54046	·55865	·57625	·59331
15	·34978	·37337	·39604	·41784	·43885	·45912	·47868	·49759	·51588	·53360
16	0·28423	0·30890	0·33257	0·35533	0·37723	0·39833	0·41868	0·43834	0·45734	0·47573
17	·21988	·24569	·27043	·29418	·31701	·33898	·36016	·38060	·40034	·41942
18	·15644	·18345	·20931	·23411	·25792	·28081	·30285	·32410	·34460	·36441
19	·09362	·12192	·14897	·17488	·19972	·22358	·24652	·26862	·28992	·31049
20	·03117	·06085	·08917	·11625	·14219	·16707	·19097	·21396	·23610	·25746
21	—	0·00000	0·02969	0·05803	0·08513	0·11109	0·13600	0·15993	0·18296	0·20514
22	—	—	—	·00000	·02835	·05546	·08144	·10637	·13033	·15338
23	—	—	—	—	—	·00000	·02712	·05311	·07805	·10203
24	—	—	—	—	—	—	—	·00000	·02599	·05095
25	—	—	—	—	—	—	—	—	—	·00000

k \ n	50	51	52	53	54	55	56	57	58	59
1	2·24907	2·25678	2·26432	2·27169	2·27891	2·28598	2·29291	2·29970	2·30635	2·31288
2	1·85487	1·86371	1·87235	1·88080	1·88906	1·89715	1·90506	1·91282	1·92041	1·92786
3	1·62863	1·63829	1·64773	1·65695	1·66596	1·67478	1·68340	1·69185	1·70012	1·70822
4	1·46374	1·47409	1·48420	1·49407	1·50372	1·51315	1·52237	1·53140	1·54024	1·54889
5	1·33109	1·34207	1·35279	1·36326	1·37348	1·38346	1·39323	1·40278	1·41212	1·42127
6	1·21846	1·23003	1·24132	1·25234	1·26310	1·27361	1·28387	1·29391	1·30373	1·31334
7	1·11948	1·13162	1·14347	1·15502	1·16629	1·17729	1·18804	1·19855	1·20882	1·21886
8	1·03042	1·04312	1·05550	1·06757	1·07934	1·09083	1·10205	1·11300	1·12371	1·13419
9	0·94887	0·96213	0·97504	0·98762	0·99988	1·01185	1·02352	1·03493	1·04607	1·05695
10	·87321	·88701	·90045	·91354	·92629	0·93873	0·95086	0·96271	0·97427	0·98557
11	0·80225	0·81661	0·83058	0·84417	0·85742	0·87033	0·88292	0·89520	0·90719	0·91890
12	·73513	·75004	·76455	·77866	·79240	·80578	·81883	·83155	·84397	·85609
13	·67117	·68666	·70170	·71633	·73057	·74444	·75794	·77111	·78396	·79649
14	·60986	·62592	·64152	·65668	·67143	·68578	·69976	·71337	·72665	·73960
15	·55077	·56742	·58358	·59928	·61455	·62940	·64385	·65793	·67164	·68502
16	0·49354	0·51080	0·52755	0·54380	0·55960	0·57495	0·58989	0·60444	0·61860	0·63241
17	·43789	·45578	·47312	·48995	·50629	·52217	·53761	·55263	·56725	·58150
18	·38357	·40211	·42007	·43749	·45439	·47080	·48675	·50226	·51736	·53205
19	·33036	·34957	·36818	·38621	·40369	·42065	·43713	·45314	·46872	·48388
20	·27807	·29799	·31726	·33592	·35400	·37154	·38856	·40510	·42117	·43681
21	0·22653	0·24719	0·26716	0·28648	0·30518	0·32331	0·34090	0·35797	0·37456	0·39068
22	·17559	·19702	·21772	·23772	·25708	·27583	·29400	·31163	·32875	·34538
23	·12511	·14735	·16880	·18953	·20957	·22896	·24774	·26595	·28362	·30078
24	·07494	·09803	·12029	·14177	·16252	·18259	·20201	·22082	·23906	·25677
25	·02496	·04896	·07206	·09434	·11584	·13661	·15669	·17614	·19498	·21325
26	—	0·00000	0·02400	0·04712	0·06940	0·09091	0·11170	0·13180	0·15127	0·17013
27	—	—	—	·00000	·02312	·04541	·06693	·08773	·10785	·12733
28	—	—	—	—	—	·00000	·02229	·04382	·06463	·08476
29	—	—	—	—	—	—	—	·00000	·02153	·04234
30	—	—	—	—	—	—	—	—	—	·00000

k \ n	60	61	62	63	64	65	66	67	68	69
1	2·31928	2·32556	2·33173	2·33778	2·34373	2·34958	2·35532	2·36097	2·36652	2·37199
2	1·93516	1·94232	1·94934	1·95624	1·96301	1·96965	1·97618	1·98260	1·98891	1·99510
3	1·71616	1·72394	1·73158	1·73906	1·74641	1·75363	1·76071	1·76767	1·77451	1·78122
4	1·55736	1·56567	1·57381	1·58180	1·58963	1·59732	1·60487	1·61228	1·61955	1·62670
5	1·43023	1·43900	1·44760	1·45603	1·46430	1·47241	1·48036	1·48817	1·49584	1·50338
6	1·32274	1·33195	1·34097	1·34982	1·35848	1·36698	1·37532	1·38351	1·39154	1·39942
7	1·22869	1·23832	1·24774	1·25698	1·26603	1·27490	1·28360	1·29213	1·30051	1·30873
8	1·14443	1·15445	1·16427	1·17388	1·18329	1·19252	1·20157	1·21044	1·21915	1·22769
9	1·06760	1·07802	1·08821	1·09819	1·10797	1·11754	1·12693	1·13613	1·14516	1·15401
10	0·99662	1·00742	1·01799	1·02833	1·03846	1·04838	1·05810	1·06762	1·07696	1·08612
11	0·93034	0·94153	0·95247	0·96317	0·97365	0·98391	0·99395	1·00380	1·01345	1·02291
12	·86793	·87950	·89081	·90187	·91270	·92329	·93367	0·94383	0·95379	0·96355
13	·80873	·82068	·83237	·84379	·85496	·86590	·87660	·88708	·89735	·90741
14	·75224	·76459	·77665	·78843	·79996	·81123	·82226	·83306	·84364	·85400
15	·69807	·71081	·72324	·73540	·74727	·75889	·77025	·78138	·79226	·80293
16	0·64587	0·65901	0·67183	0·68436	0·69659	0·70856	0·72025	0·73170	0·74290	0·75387
17	·59538	·60893	·62214	·63504	·64764	·65996	·67200	·68377	·69529	·70657
18	·54637	·56033	·57395	·58723	·60020	·61288	·62526	·63737	·64921	·66080
19	·49864	·51303	·52705	·54073	·55408	·56712	·57985	·59230	·60447	·61638
20	·45202	·46685	·48129	·49537	·50911	·52252	·53561	·54841	·56091	·57314
21	0·40637	0·42164	0·43652	0·45101	0·46515	0·47894	0·49240	0·50555	0·51839	0·53095
22	·36155	·37729	·39260	·40752	·42207	·43625	·45009	·46360	·47680	·48969
23	·31745	·33366	·34944	·36480	·37976	·39435	·40857	·42245	·43601	·44925
24	·27396	·29066	·30691	·32272	·33812	·35312	·36775	·38201	·39594	·40953
25	·23098	·24820	·26494	·28122	·29706	·31249	·32753	·34219	·35649	·37045
26	0·18842	0·20618	0·22343	0·24019	0·25650	0·27237	0·28784	0·30290	0·31759	0·33192
27	·14621	·16452	·18230	·19957	·21636	·23269	·24859	·26408	·27917	·29389
28	·10425	·12315	·14148	·15927	·17656	·19337	·20973	·22565	·24116	·25627
29	·06248	·08198	·10089	·11923	·13704	·15435	·17118	·18755	·20349	·21902
30	·02081	·04096	·06047	·07938	·09774	·11556	·13288	·14972	·16611	·18207
31	—	0·00000	0·02014	0·03966	0·05858	0·07694	0·09478	0·11211	0·12896	0·14536
32	—	—	—	·00000	·01952	·03844	·05681	·07465	·09199	·10885
33	—	—	—	—	—	·00000	·01893	·03730	·05514	·07249
34	—	—	—	—	—	—	—	·00000	·01837	·03622
35	—	—	—	—	—	—	—	—	—	·00000

k \ n	70	71	72	73	74	75	76	77	78	79
1	2·37736	2·38265	2·38785	2·39298	2·39802	2·40299	2·40789	2·41271	2·41747	2·42215
2	2·00120	2·00720	2·01310	2·01890	2·02462	2·03024	2·03578	2·04124	2·04662	2·05191
3	1·78783	1·79432	1·80071	1·80699	1·81317	1·81926	1·82525	1·83115	1·83696	1·84268
4	1·63373	1·64063	1·64742	1·65410	1·66067	1·66714	1·67350	1·67976	1·68592	1·69200
5	1·51078	1·51805	1·52520	1·53223	1·53914	1·54594	1·55263	1·55921	1·56569	1·57207
6	1·40717	1·41478	1·42226	1·42961	1·43684	1·44395	1·45094	1·45782	1·46459	1·47125
7	1·31680	1·32473	1·33252	1·34017	1·34770	1·35510	1·36237	1·36953	1·37657	1·38350
8	1·23608	1·24431	1·25240	1·26034	1·26815	1·27583	1·28338	1·29080	1·29810	1·30529
9	1·16270	1·17123	1·17961	1·18784	1·19592	1·20387	1·21168	1·21936	1·22691	1·23434
10	1·09511	1·10393	1·11259	1·12110	1·12945	1·13766	1·14572	1·15365	1·16145	1·16912
11	1·03220	1·04130	1·05024	1·05902	1·06764	1·07610	1·08442	1·09260	1·10063	1·10854
12	0·97313	0·98252	0·99173	1·00078	1·00966	1·01838	1·02695	1·03537	1·04364	1·05178
13	·91728	·92695	·93644	0·94576	0·95490	0·96387	0·97269	0·98135	0·98986	0·99822
14	·86416	·87412	·88388	·89346	·90286	·91209	·92115	·93005	·93880	·94739
15	·81338	·82362	·83366	·84351	·85317	·86265	·87196	·88110	·89008	·89890
16	0·76462	0·77514	0·78546	0·79558	0·80550	0·81524	0·82480	0·83418	0·84339	0·85244
17	·71761	·72843	·73903	·74942	·75960	·76960	·77940	·78903	·79848	·80776
18	·67214	·68325	·69413	·70480	·71526	·72551	·73557	·74544	·75512	·76463
19	·62803	·63943	·65060	·66155	·67227	·68279	·69310	·70322	·71314	·72289
20	·58510	·59681	·60827	·61950	·63050	·64128	·65185	·66222	·67239	·68237
21	0·54323	0·55525	0·56701	0·57852	0·58980	0·60085	0·61168	0·62230	0·63272	0·64294
22	·50230	·51463	·52669	·53850	·55006	·56138	·57248	·58336	·59403	·60449
23	·46219	·47484	·48721	·49932	·51117	·52277	·53414	·54528	·55621	·56692
24	·42281	·43579	·44848	·46089	·47304	·48493	·49657	·50798	·51917	·53013
25	·38404	·39739	·41041	·42313	·43558	·44777	·45970	·47138	·48283	·49404

k \ n	70	71	72	73	74	75	76	77	78	79
26	0·34591	0·35958	0·37292	0·38597	0·39873	0·41122	0·42343	0·43540	0·44711	0·45859
27	·30825	·32227	·33596	·34934	·36242	·37521	·38772	·39997	·41196	·42371
28	·27102	·28540	·29945	·31317	·32657	·33968	·35250	·36504	·37731	·38934
29	·23416	·24893	·26333	·27740	·29114	·30457	·31770	·33055	·34311	·35542
30	·19762	·21277	·22756	·24199	·25608	·26984	·28329	·29645	·30931	·32190
31	0·16134	0·17690	0·19208	0·20688	0·22133	0·23543	0·24922	0·26269	0·27586	0·28875
32	·12527	·14125	·15683	·17202	·18684	·20130	·21543	·22923	·24272	·25591
33	·08936	·10579	·12178	·13737	·15257	·16740	·18188	·19602	·20983	·22334
34	·05357	·07045	·08688	·10289	·11848	·13370	·14854	·16303	·17718	·19101
35	·01785	·03520	·05209	·06852	·08453	·10014	·11536	·13021	·14471	·15888
36	—	0·00000	0·01736	0·03424	0·05068	0·06670	0·08231	0·09754	0·11240	0·12691
37	—	—	—	·00000	·01689	·03333	·04935	·06497	·08020	·09507
38	—	—	—	—	—	·00000	·01644	·03247	·04809	·06333
39	—	—	—	—	—	—	—	·00000	·01602	·03165
40	—	—	—	—	—	—	—	—	—	·00000

k \ n	80	81	82	83	84	85	86	87	88	89
1	2·42677	2·43133	2·43582	2·44026	2·44463	2·44894	2·45320	2·45741	2·46156	2·46565
2	2·05714	2·06228	2·06735	2·07236	2·07729	2·08216	2·08696	2·09170	2·09637	2·10099
3	1·84832	1·85387	1·85935	1·86475	1·87007	1·87532	1·88049	1·88560	1·89064	1·89561
4	1·69798	1·70387	1·70968	1·71540	1·72104	1·72660	1·73209	1·73750	1·74283	1·74810
5	1·57836	1·58455	1·59065	1·59665	1·60258	1·60841	1·61417	1·61984	1·62544	1·63096
6	1·47781	1·48428	1·49064	1·49691	1·50309	1·50918	1·51518	1·52110	1·52693	1·53269
7	1·39032	1·39704	1·40366	1·41017	1·41659	1·42292	1·42915	1·43529	1·44135	1·44732
8	1·31236	1·31932	1·32617	1·33292	1·33957	1·34611	1·35257	1·35893	1·36520	1·37138
9	1·24165	1·24884	1·25593	1·26290	1·26977	1·27653	1·28320	1·28976	1·29624	1·30262
10	1·17666	1·18409	1·19139	1·19859	1·20567	1·21264	1·21951	1·22628	1·23295	1·23952
11	1·11631	1·12396	1·13148	1·13889	1·14618	1·15336	1·16043	1·16740	1·17426	1·18102
12	1·05978	1·06764	1·07539	1·08300	1·09050	1·09788	1·10515	1·11231	1·11936	1·12631
13	1·00644	1·01453	1·02249	1·03031	1·03802	1·04560	1·05306	1·06041	1·06765	1·07478
14	0·95584	0·96414	0·97231	0·98034	0·98825	0·99603	1·00369	1·01122	1·01865	1·02596
15	·90757	·91609	·92447	·93271	·94082	·94880	0·95665	0·96437	0·97198	0·97948
16	0·86134	0·87007	0·87867	0·88711	0·89542	0·90360	0·91164	0·91956	0·92735	0·93502
17	·81687	·82583	·83464	·84329	·85180	·86017	·86841	·87651	·88449	·89234
18	·77398	·78315	·79217	·80103	·80975	·81832	·82675	·83504	·84320	·85123
19	·73246	·74186	·75109	·76016	·76908	·77785	·78647	·79496	·80330	·81152
20	·69217	·70179	·71124	·72053	·72965	·73862	·74744	·75611	·76465	·77304
21	0·65297	0·66282	0·67249	0·68199	0·69133	0·70050	0·70952	0·71838	0·72710	0·73568
22	·61476	·62484	·63473	·64445	·65399	·66337	·67259	·68165	·69056	·69932
23	·57742	·58773	·59785	·60779	·61755	·62714	·63656	·64581	·65492	·66387
24	·54088	·55143	·56178	·57193	·58191	·59171	·60133	·61079	·62009	·62923
25	·50504	·51583	·52641	·53680	·54700	·55701	·56684	·57650	·58600	·59533
26	0·46985	0·48088	0·49170	0·50232	0·51274	0·52297	0·53301	0·54288	0·55258	0·56210
27	·43522	·44651	·45757	·46842	·47907	·48952	·49979	·50986	·51976	·52949
28	·40111	·41265	·42397	·43506	·44594	·45662	·46710	·47739	·48750	·49743
29	·36747	·37927	·39084	·40218	·41330	·42421	·43491	·44542	·45574	·46587
30	·33423	·34630	·35813	·36972	·38108	·39223	·40316	·41389	·42443	·43477
31	0·30136	0·31371	0·32580	0·33765	0·34926	0·36065	0·37182	0·38278	0·39353	0·40409
32	·26881	·28144	·29381	·30592	·31779	·32943	·34084	·35203	·36300	·37378
33	·23655	·24947	·26212	·27450	·28664	·29852	·31018	·32161	·33281	·34381
34	·20453	·21775	·23069	·24335	·25576	·26790	·27981	·29148	·30292	·31415
35	·17272	·18625	·19949	·21244	·22512	·23753	·24970	·26162	·27330	·28476
36	0·14108	0·15493	0·16848	0·18172	0·19469	0·20738	0·21981	0·23199	0·24392	0·25562
37	·10959	·12377	·13763	·15118	·16444	·17741	·19012	·20256	·21475	·22669
38	·07820	·09272	·10691	·12078	·13434	·14761	·16059	·17330	·18576	·19796
39	·04689	·06177	·07629	·09049	·10436	·11793	·13121	·14420	·15692	·16938
40	·01562	·03087	·04575	·06028	·07448	·08836	·10193	·11521	·12821	·14094
41	—	0·00000	0·01524	0·03013	0·04466	0·05886	0·07275	0·08633	0·09961	0·11262
42	—	—	—	·00000	0·01488	·02942	·04362	·05751	·07110	·08439
43	—	—	—	—	—	·00000	·01454	·02874	·04263	·05622
44	—	—	—	—	—	—	—	·00000	·01421	·02810
45	—	—	—	—	—	—	—	—	—	·00000

Table 1 (*cont.*)

k \ n	90	91	92	93	94	95	96	97	98	99
1	2·46970	2·47370	2·47764	2·48154	2·48540	2·48920	2·49297	2·49669	2·50036	2·50400
2	2·10554	2·11004	2·11448	2·11887	2·12321	2·12749	2·13172	2·13590	2·14003	2·14411
3	1·90052	1·90536	1·91015	1·91487	1·91953	1·92414	1·92869	1·93318	1·93763	1·94201
4	1·75329	1·75842	1·76348	1·76848	1·77341	1·77828	1·78309	1·78784	1·79254	1·79718
5	1·63641	1·64178	1·64709	1·65232	1·65749	1·66259	1·66763	1·67261	1·67752	1·68238
6	1·53836	1·54396	1·54949	1·55494	1·56033	1·56564	1·57089	1·57607	1·58118	1·58624
7	1·45321	1·45903	1·46476	1·47042	1·47600	1·48151	1·48695	1·49232	1·49762	1·50286
8	1·37747	1·38348	1·38941	1·39526	1·40103	1·40673	1·41235	1·41790	1·42338	1·42879
9	1·30891	1·31511	1·32123	1·32726	1·33321	1·33909	1·34489	1·35061	1·35626	1·36183
10	1·24600	1·25239	1·25869	1·26491	1·27104	1·27708	1·28305	1·28894	1·29475	1·30049
11	1·18769	1·19426	1·20073	1·20712	1·21342	1·21964	1·22577	1·23182	1·23779	1·24368
12	1·13316	1·13990	1·14656	1·15311	1·15958	1·16596	1·17226	1·17847	1·18459	1·19064
13	1·08181	1·08873	1·09555	1·10228	1·10891	1·11546	1·12191	1·12827	1·13455	1·14075
14	1·03316	1·04026	1·04726	1·05415	1·06095	1·06765	1·07426	1·08078	1·08721	1·09356
15	0·98686	0·99413	1·00129	1·00835	1·01531	1·02217	1·02894	1·03561	1·04219	1·04868
16	0·94258	0·95002	0·95735	0·96458	0·97170	0·97872	0·98564	0·99246	0·99919	1·00583
17	·90007	·90769	·91519	·92258	·92986	·93704	·94411	·95109	·95797	0·96475
18	·85914	·86693	·87460	·88215	·88959	·89693	·90416	·91129	·91831	·92524
19	·81960	·82756	·83540	·84312	·85072	·85822	·86560	·87288	·88006	·88713
20	·78131	·78944	·79745	·80533	·81310	·82075	·82829	·83572	·84305	·85027
21	0·74412	0·75243	0·76061	0·76866	0·77659	0·78441	0·79210	0·79968	0·80716	0·81452
22	·70795	·71643	·72478	·73300	·74110	·74907	·75692	·76466	·77228	·77980
23	·67267	·68134	·68986	·69825	·70651	·71464	·72266	·73055	·73832	·74598
24	·63822	·64706	·65576	·66432	·67275	·68105	·68922	·69727	·70519	·71301
25	·60451	·61353	·62241	·63115	·63974	·64821	·65654	·66474	·67282	·68079
26	0·57147	0·58068	0·58974	0·59865	0·60742	0·61605	0·62454	0·63291	0·64115	0·64926
27	·53905	·54845	·55769	·56678	·57572	·58452	·59318	·60170	·61010	·61837
28	·50718	·51677	·52620	·53547	·54459	·55356	·56239	·57108	·57963	·58805
29	·47582	·48561	·49522	·50468	·51398	·52312	·53212	·54097	·54969	·55827
30	·44493	·45491	·46472	·47436	·48384	·49316	·50233	·51136	·52024	·52898
31	0·41445	0·42463	0·43464	0·44447	0·45414	0·46364	0·47299	0·48218	0·49123	0·50013
32	·38436	·39474	·40495	·41498	·42483	·43452	·44404	·45341	·46263	·47170
33	·35461	·36520	·37561	·38584	·39588	·40576	·41547	·42501	·43440	·44364
34	·32517	·33598	·34660	·35702	·36727	·37733	·38722	·39695	·40652	·41593
35	·29601	·30704	·31787	·32850	·33895	·34921	·35929	·36920	·37895	·38853
36	0·26710	0·27835	0·28940	0·30025	0·31090	0·32136	0·33163	0·34173	0·35166	0·36142
37	·23841	·24990	·26117	·27223	·28309	·29375	·30423	·31452	·32464	·33458
38	·20991	·22164	·23314	·24443	·25550	·26637	·27705	·28754	·29785	·30797
39	·18159	·19356	·20530	·21681	·22810	·23919	·25008	·26077	·27127	·28159
40	·15341	·16563	·17761	·18936	·20088	·21219	·22328	·23418	·24488	·25539
41	0·12536	0·13783	0·15006	0·16205	0·17380	0·18533	0·19665	0·20776	0·21866	0·22937
42	·09740	·11014	·12262	·13486	·14685	·15861	·17015	·18148	·19259	·20351
43	·06952	·08253	·09528	·10777	·12001	·13201	·14378	·15533	·16666	·17778
44	·04169	·05499	·06801	·08076	·09325	·10550	·11750	·12928	·14083	·15217
45	·01389	·02748	·04078	·05381	·06656	·07906	·09131	·10332	·11510	·12666
46	—	0·00000	0·01359	0·02689	0·03992	0·05267	0·06518	0·07743	0·08944	0·10123
47	—	—	—	·00000	·01330	·02633	·03909	·05159	·06385	·07586
48	—	—	—	—	—	·00000	·01303	·02579	·03829	·05055
49	—	—	—	—	—	—	—	·00000	·01276	·02527
50	—	—	—	—	—	—	—	—	—	·00000

Table 1 (*cont.*)

k \ n	100	125	150	175	200	225	250	300	350	400
1	2·50759	2·58634	2·64925	2·70148	2·74604	2·78485	2·81918	2·87777	2·92651	2·96818
2	2·14814	2·23630	2·30638	2·36434	2·41365	2·45649	2·49431	2·55867	2·61207	2·65761
3	1·94635	2·04090	2·11578	2·17755	2·22999	2·27547	2·31555	2·38365	2·44004	2·48806
4	1·80176	1·90146	1·98019	2·04500	2·09991	2·14746	2·18932	2·26033	2·31904	2·36897
5	1·68718	1·79137	1·87341	1·94081	1·99783	2·04713	2·09050	2·16397	2·22462	2·27615
6	1·59123	1·69947	1·78448	1·85419	1·91308	1·96395	2·00864	2·08427	2·14663	2·19955
7	1·50803	1·62002	1·70777	1·77959	1·84019	1·89247	1·93837	2·01595	2·07985	2·13402
8	1·43414	1·54966	1·63997	1·71376	1·77594	1·82953	1·87654	1·95592	2·02122	2·07654
9	1·36734	1·48623	1·57896	1·65462	1·71828	1·77310	1·82115	1·90220	1·96882	2·02521
10	1·30615	1·42828	1·52333	1·60075	1·66583	1·72182	1·77084	1·85348	1·92133	1·97871
11	1·24950	1·37477	1·47206	1·55118	1·61760	1·67470	1·72466	1·80879	1·87781	1·93614
12	1·19661	1·32493	1·42438	1·50514	1·57287	1·63103	1·68189	1·76746	1·83758	1·89681
13	1·14687	1·27819	1·37975	1·46210	1·53109	1·59027	1·64199	1·72894	1·80013	1·86021
14	1·09982	1·23409	1·33771	1·42161	1·49182	1·55200	1·60455	1·69283	1·76504	1·82595
15	1·05509	1·19226	1·29791	1·38333	1·45472	1·51588	1·56923	1·65880	1·73201	1·79371
16	1·01238	1·15243	1·26007	1·34697	1·41953	1·48163	1·53577	1·62659	1·70076	1·76324
17	0·97145	1·11435	1·22396	1·31232	1·38602	1·44904	1·50395	1·59599	1·67109	1·73432
18	·93208	1·07783	1·18937	1·27917	1·35399	1·41792	1·47359	1·56681	1·64283	1·70678
19	·89411	1·04268	1·15616	1·24738	1·32330	1·38812	1·44452	1·53891	1·61582	1·68048
20	·85739	1·00879	1·12417	1·21680	1·29381	1·35950	1·41663	1·51216	1·58994	1·65530
21	0·82179	0·97601	1·09330	1·18731	1·26540	1·33195	1·38980	1·48645	1·56508	1·63112
22	·78720	·94426	1·06344	1·15883	1·23798	1·30539	1·36393	1·46169	1·54116	1·60786
23	·75353	·91342	1·03449	1·13126	1·21146	1·27971	1·33895	1·43780	1·51809	1·58544
24	·72070	·88344	1·00639	1·10452	1·18577	1·25485	1·31478	1·41470	1·49580	1·56379
25	·68863	·85423	0·97907	1·07855	1·16084	1·23074	1·29135	1·39233	1·47423	1·54285
26	0·65725	0·82573	0·95245	1·05329	1·13661	1·20733	1·26861	1·37063	1·45332	1·52257
27	·62651	·79789	·92650	1·02868	1·11303	1·18457	1·24651	1·34957	1·43303	1·50289
28	·59635	·77065	·90115	1·00469	1·09005	1·16240	1·22500	1·32908	1·41332	1·48378
29	·56672	·74398	·87638	0·98125	1·06763	1·14079	1·20405	1·30914	1·39414	1·46520
30	·53758	·71782	·85212	·95835	1·04574	1·11970	1·18361	1·28971	1·37546	1·44711
31	0·50890	0·69215	0·82836	0·93594	1·02434	1·09909	1·16365	1·27076	1·35725	1·42948
32	·48062	·66692	·80506	·91399	1·00340	1·07895	1·14415	1·25225	1·33947	1·41228
33	·45273	·64212	·78219	·89247	0·98290	1·05923	1·12507	1·23415	1·32211	1·39550
34	·42518	·61770	·75973	·87135	·96279	1·03992	1·10640	1·21646	1·30515	1·37910
35	·39796	·59365	·73764	·85062	·94307	1·02098	1·08810	1·19914	1·28854	1·36306
36	0·37102	0·56993	0·71590	0·83025	0·92371	1·00241	1·07016	1·18217	1·27229	1·34736
37	·34436	·54653	·69450	·81022	·90469	0·98418	1·05256	1·16553	1·25637	1·33199
38	·31793	·52343	·67341	·79051	·88599	·96626	1·03528	1·14921	1·24076	1·31693
39	·29173	·50061	·65261	·77110	·86760	·94866	1·01830	1·13320	1·22544	1·30216
40	·26572	·47804	·63210	·75197	·84950	·93134	1·00161	1·11746	1·21041	1·28767
41	0·23990	0·45571	0·61185	0·73312	0·83167	0·91429	0·98520	1·10200	1·19565	1·27344
42	·21423	·43361	·59184	·71453	·81410	·89751	·96905	1·08680	1·18114	1·25947
43	·18870	·41172	·57208	·69618	·79678	·88098	·95314	1·07185	1·16688	1·24574
44	·16330	·39002	·55253	·67806	·77969	·86469	·93748	1·05713	1·15285	1·23225
45	·13800	·36851	·53319	·66016	·76283	·84862	·92204	1·04264	1·13904	1·21897
46	0·11279	0·34717	0·51405	0·64247	0·74619	0·83277	0·90682	1·02836	1·12545	1·20590
47	·08765	·32598	·49509	·62498	·72975	·81712	·89180	1·01429	1·11207	1·19304
48	·06257	·30494	·47632	·60768	·71350	·80168	·87699	1·00042	1·09888	1·18037
49	·03753	·28403	·45770	·59056	·69744	·78642	·86236	0·98674	1·08587	1·16789
50	·01251	·26325	·43925	·57361	·68156	·77134	·84792	·97324	1·07305	1·15559

Table 1 (*cont.*)

k \ n	125	150	175	200	225	250	300	350	400
51	0·24258	0·42094	0·55682	0·66585	0·75644	0·83365	0·95991	1·06041	1·14346
52	·22201	·40278	·54019	·65030	·74170	·81955	·94676	1·04793	1·13149
53	·20154	·38475	·52371	·63490	·72712	·80561	·93376	1·03561	1·11969
54	·18115	·36684	·50737	·61966	·71270	·79183	·92093	1·02345	1·10804
55	·16084	·34904	·49116	·60456	·69842	·77819	·90824	1·01144	1·09654
56	0·14059	0·33136	0·47508	0·58959	0·68428	0·76470	0·89570	0·99957	1·08518
57	·12040	·31378	·45913	·57476	·67028	·75135	·88329	·98784	1·07396
58	·10026	·29630	·44329	·56005	·65641	·73812	·87102	·97624	1·06287
59	·08016	·27891	·42756	·54546	·64267	·72503	·85888	·96478	1·05192
60	·06009	·26160	·41193	·53099	·62904	·71206	·84687	·95344	1·04108
61	0·04005	0·24437	0·39641	0·51663	0·61553	0·69921	0·83498	0·94222	1·03037
62	·02002	·22721	·38098	·50237	·60213	·68647	·82320	·93112	1·01978
63	·00000	·21012	·36564	·48822	·58884	·67384	·81154	·92013	1·00930
64	—	·19309	·35039	·47416	·57566	·66132	·79998	·90925	0·99893
65	—	·17612	·33521	·46020	·56257	·64891	·78854	·89848	·98866
66	—	0·15919	0·32012	0·44632	0·54958	0·63659	0·77719	0·88782	0·97850
67	—	·14232	·30510	·43253	·53668	·62437	·76595	·87725	·96844
68	—	·12548	·29014	·41882	·52386	·61224	·75480	·86678	·95848
69	—	·10868	·27525	·40519	·51114	·60020	·74374	·85640	·94861
70	—	·09191	·26042	·39164	·49850	·58824	·73277	·84612	·93883
71	—	0·07516	0·24565	0·37816	0·48593	0·57637	0·72189	0·83592	0·92914
72	—	·05844	·23093	·36474	·47344	·56458	·71110	·82581	·91954
73	—	·04173	·21626	·35139	·46103	·55287	·70039	·81579	·91002
74	—	·02503	·20164	·33811	·44869	·54124	·68976	·80584	·90058
75	—	·00834	·18706	·32488	·43641	·52967	·67920	·79598	·89122
76	—	—	0·17252	0·31171	0·42420	0·51818	0·66872	0·78619	0·88194
77	—	—	·15802	·29859	·41205	·50676	·65831	·77648	·87274
78	—	—	·14355	·28553	·39997	·49540	·64798	·76684	·86361
79	—	—	·12911	·27251	·38794	·48410	·63771	·75727	·85455
80	—	—	·11470	·25954	·37596	·47287	·62751	·74777	·84556
81	—	—	0·10031	0·24661	0·36404	0·46169	0·61738	0·73833	0·83663
82	—	—	·08594	·23373	·35218	·45058	·60730	·72896	·82778
83	—	—	·07159	·22088	·34036	·43952	·59729	·71966	·81899
84	—	—	·05725	·20807	·32859	·42851	·58734	·71041	·81026
85	—	—	·04293	·19529	·31686	·41755	·57745	·70123	·80159
86	—	—	0·02862	0·18254	0·30518	0·40665	0·56761	0·69211	0·79298
87	—	—	·01431	·16983	·29354	·39579	·55783	·68304	·78443
88	—	—	·00000	·15714	·28194	·38498	·54810	·67403	·77594
89	—	—	—	·14448	·27038	·37421	·53842	·66507	·76750
90	—	—	—	·13184	·25885	·36349	·52879	·65617	·75912
91	—	—	—	0·11922	0·24736	0·35280	0·51922	0·64732	0·75079
92	—	—	—	·10662	·23590	·34216	·50968	·63852	·74252
93	—	—	—	·09404	·22447	·33156	·50020	·62976	·73429
94	—	—	—	·08147	·21307	·32099	·49076	·62106	·72611
95	—	—	—	·06891	·20170	·31046	·48136	·61240	·71798
96	—	—	—	0·05637	0·19035	0·29997	0·47201	0·60379	0·70990
97	—	—	—	·04383	·17903	·28951	·46269	·59522	·70186
98	—	—	—	·03130	·16773	·27907	·45342	·58670	·69387
99	—	—	—	·01878	·15645	·26867	·44419	·57822	·68593
100	—	—	—	·00626	·14520	·25830	·43499	·56978	·67802

Table 1 (*cont.*)

$k \backslash n$	225	250	300	350	400	$k \backslash n$	350	400
101	0·13396	0·24796	0·42583	0·56138	0·67016	**151**	0·17626	0·31517
102	·12274	·23764	·41670	·55302	·66234	**152**	·16900	·30860
103	·11153	·22735	·40761	·54470	·65456	**153**	·16174	·30203
104	·10034	·21708	·39856	·53641	·64682	**154**	·15450	·29548
105	·08916	·20683	·38953	·52817	·63912	**155**	·14726	·28895
106	0·07799	0·19661	0·38054	0·51996	0·63145	**156**	0·14003	0·28242
107	·06683	·18641	·37158	·51178	·62383	**157**	·13280	·27591
108	·05568	·17622	·36265	·50364	·61624	**158**	·12558	·26941
109	·04453	·16606	·35375	·49553	·60868	**159**	·11837	·26292
110	·03340	·15591	·34487	·48745	·60116	**160**	·11117	·25644
111	0·02226	0·14577	0·33602	0·47941	0·59367	**161**	0·10397	0·24998
112	·01113	·13566	·32720	·47139	·58622	**162**	·09678	·24352
113	·00000	·12555	·31841	·46341	·57880	**163**	·08959	·23707
114	—	·11546	·30963	·45545	·57141	**164**	·08240	·23064
115	—	·10538	·30089	·44753	·56405	**165**	·07522	·22421
116	—	0·09531	0·29216	0·43963	0·55672	**166**	0·06805	0·21779
117	—	·08526	·28346	·43176	·54942	**167**	·06088	·21138
118	—	·07520	·27478	·42392	·54215	**168**	·05371	·20498
119	—	·06516	·26612	·41610	·53491	**169**	·04654	·19859
120	—	·05513	·25748	·40831	·52770	**170**	·03938	·19220
121	—	0·04510	0·24885	0·40054	0·52051	**171**	0·03221	0·18583
122	—	·03507	·24025	·39280	·51335	**172**	·02505	·17946
123	—	·02505	·23167	·38508	·50622	**173**	·01789	·17310
124	—	·01503	·22310	·37738	·49911	**174**	·01074	·16674
125	—	·00501	·21455	·36970	·49203	**175**	·00358	·16040
126	—	—	0·20601	0·36205	0·48497	**176**	—	0·15406
127	—	—	·19749	·35442	·47794	**177**	—	·14772
128	—	—	·18898	·34681	·47093	**178**	—	·14139
129	—	—	·18049	·33922	·46394	**179**	—	·13507
130	—	—	·17201	·33164	·45698	**180**	—	·12875
131	—	—	0·16354	0·32409	0·45004	**181**	—	0·12244
132	—	—	·15508	·31656	·44312	**182**	—	·11613
133	—	—	·14664	·30904	·43622	**183**	—	·10983
134	—	—	·13820	·30154	·42934	**184**	—	·10353
135	—	—	·12978	·29406	·42248	**185**	—	·09723
136	—	—	0·12136	0·28659	0·41564	**186**	—	0·09094
137	—	—	·11296	·27914	·40883	**187**	—	·08465
138	—	—	·10456	·27171	·40203	**188**	—	·07837
139	—	—	·09617	·26429	·39524	**189**	—	·07209
140	—	—	·08778	·25689	·38848	**190**	—	·06581
141	—	—	0·07940	0·24950	0·38174	**191**	—	0·05954
142	—	—	·07103	·24212	·37501	**192**	—	·05326
143	—	—	·06266	·23475	·36830	**193**	—	·04699
144	—	—	·05430	·22740	·36160	**194**	—	·04072
145	—	—	·04594	·22006	·35492	**195**	—	·03445
146	—	—	0·03758	0·21274	0·34826	**196**	—	0·02819
147	—	—	·02923	·20542	·34161	**197**	—	·02192
148	—	—	·02088	·19812	·33498	**198**	—	·01566
149	—	—	·01252	·19082	·32836	**199**	—	·00939
150	—	—	·00417	·18354	·32176	**200**	—	·00313