SELECTED SOLUTIONS

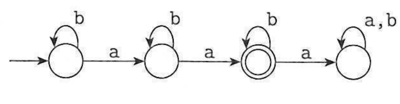
1.1 For M_1: **(a)** q_1; **(b)** $\{q_2\}$; **(c)** q_1, q_2, q_3, q_1, q_1; **(d)** No; **(e)** No
For M_2: **(a)** q_1; **(b)** $\{q_1, q_4\}$; **(c)** q_1, q_1, q_1, q_2, q_4; **(d)** Yes; **(e)** Yes

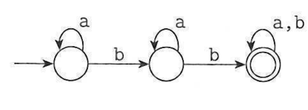
1.2 $M_1 = (\{q_1, q_2, q_3\}, \{\mathtt{a}, \mathtt{b}\}, \delta_1, q_1, \{q_2\})$.
$M_2 = (\{q_1, q_2, q_3, q_4\}, \{\mathtt{a}, \mathtt{b}\}, \delta_2, q_1, \{q_1, q_4\})$.
The transition functions are

δ_1	a	b
q_1	q_2	q_1
q_2	q_3	q_3
q_3	q_2	q_1

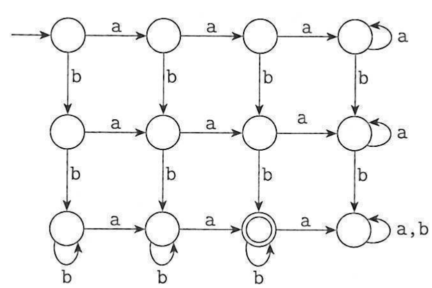
δ_2	a	b
q_1	q_1	q_2
q_2	q_3	q_4
q_3	q_2	q_1
q_4	q_3	q_4.

1.4 **(b)** The following are DFAs for the two languages $\{w \mid w \text{ has exactly two a's}\}$ and $\{w \mid w \text{ has at least two b's}\}$.

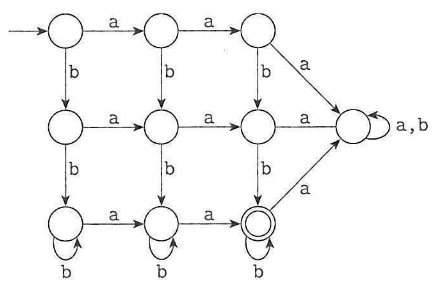




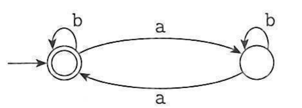
Combining them using the intersection construction gives the following DFA.

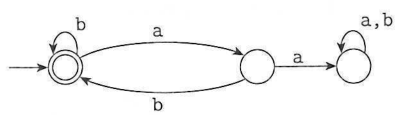


Though the problem doesn't request you to simplify the DFA, certain states can be combined to give the following DFA.



(d) These are DFAs for the two languages $\{w|\ w$ has an even number of a's$\}$ and $\{w|$ each a in w is followed by at least one b$\}$.





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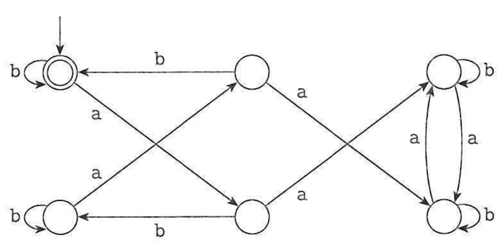
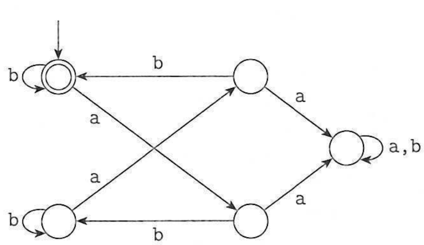
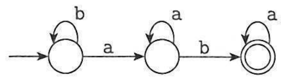
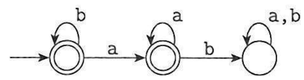


Though the problem doesn't request you to simplify the DFA, certain states can be combined to give the following DFA.

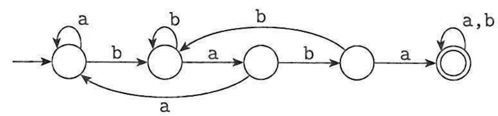


1.5 **(a)** The left-hand DFA recognizes $\{w \mid w \text{ contains } \texttt{ab}\}$. The right-hand DFA recognizes its complement, $\{w \mid w \text{ doesn't contain } \texttt{ab}\}$.

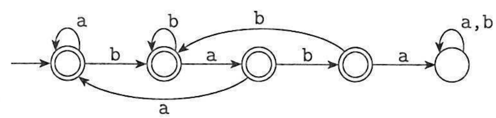




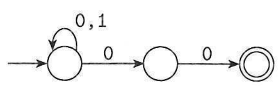
(b) This DFA recognizes $\{w \mid w \text{ contains } \texttt{baba}\}$.



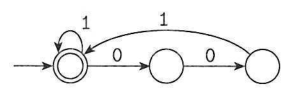
This DFA recognizes $\{w \mid w \text{ does not contain } \texttt{baba}\}$.



1.7 **(a)**



(f)



1.11 Let $N = (Q, \Sigma, \delta, q_0, F)$ be any NFA. Construct an NFA N' with a single accept state that recognizes the same language as N. Informally, N' is exactly like N except it has ε-transitions from the states corresponding to the accept states of N, to a new accept state, q_{accept}. State q_{accept} has no emerging transitions. More formally, $N' = (Q \cup \{q_{\text{accept}}\}, \Sigma, \delta', q_0, \{q_{\text{accept}}\})$, where for each $q \in Q$ and $a \in \Sigma_\varepsilon$

$$\delta'(q, a) = \begin{cases} \delta(q, a) & \text{if } a \neq \varepsilon \text{ or } q \notin F \\ \delta(q, a) \cup \{q_{\text{accept}}\} & \text{if } a = \varepsilon \text{ and } q \in F \end{cases}$$

and $\delta'(q_{\text{accept}}, a) = \emptyset$ for each $a \in \Sigma_\varepsilon$.

1.23 We prove both directions of the "iff."
($\rightarrow$) Assume that $B = B^+$ and show that $BB \subseteq B$.
For every language $BB \subseteq B^+$ holds, so if $B = B^+$, then $BB \subseteq B$.
($\leftarrow$) Assume that $BB \subseteq B$ and show that $B = B^+$.
For every language $B \subseteq B^+$, so we need to show only $B^+ \subseteq B$. If $w \in B^+$, then $w = x_1 x_2 \cdots x_k$ where each $x_i \in B$ and $k \geq 1$. Because $x_1, x_2 \in B$ and $BB \subseteq B$, we have $x_1 x_2 \in B$. Similarly, because $x_1 x_2$ is in B and x_3 is in B, we have $x_1 x_2 x_3 \in B$. Continuing in this way, $x_1 \cdots x_k \in B$. Hence $w \in B$, and so we may conclude that $B^+ \subseteq B$.

The latter argument may be written formally as the following proof by induction. Assume that $BB \subseteq B$.
Claim: For each $k \geq 1$, if $x_1, \ldots, x_k \in B$, then $x_1 \cdots x_k \in B$.
Basis: Prove for $k = 1$. This statement is obviously true.
Induction step: For each $k \geq 1$, assume that the claim is true for k and prove it to be true for $k + 1$.
If $x_1, \ldots, x_k, x_{k+1} \in B$, then by the induction assumption, $x_1 \cdots x_k \in B$. Therefore, $x_1 \cdots x_k x_{k+1} \in BB$, but $BB \subseteq B$, so $x_1 \cdots x_{k+1} \in B$. That proves the induction step and the claim. The claim implies that if $BB \subseteq B$, then $B^+ \subseteq B$.

1.29 **(a)** Assume that $A_1 = \{0^n 1^n 2^n \mid n \geq 0\}$ is regular. Let p be the pumping length given by the pumping lemma. Choose s to be the string $0^p 1^p 2^p$. Because s is a member of A_1 and s is longer than p, the pumping lemma guarantees that s can be split into three pieces, $s = xyz$, where for any $i \geq 0$ the string $xy^i z$ is in A_1. Consider two possibilities:

1. The string y consists only of 0s, only of 1s, or only of 2s. In these cases, the string $xyyz$ will not have equal numbers of 0s, 1s, and 2s. Hence $xyyz$ is not a member of A_1, a contradiction.
2. The string y consists of more than one kind of symbol. In this case, $xyyz$ will have the 0s, 1s, or 2s out of order. Hence $xyyz$ is not a member of A_1, a contradiction.

Either way we arrive at a contradiction. Therefore, A_1 is not regular.

(c) Assume that $A_3 = \{\mathtt{a}^{2^n} \mid n \geq 0\}$ is regular. Let p be the pumping length given by the pumping lemma. Choose s to be the string $\mathtt{a}^{2^p}$. Because s is a member of A_3 and s is longer than p, the pumping lemma guarantees that s can be split into three pieces, $s = xyz$, satisfying the three conditions of the pumping lemma.

The third condition tells us that $|xy| \leq p$. Furthermore, $p < 2^p$ and so $|y| < 2^p$. Therefore, $|xyyz| = |xyz| + |y| < 2^p + 2^p = 2^{p+1}$. The second condition requires $|y| > 0$ so $2^p < |xyyz| < 2^{p+1}$. The length of $xyyz$ cannot be a power of 2. Hence $xyyz$ is not a member of A_3, a contradiction. Therefore, A_3 is not regular.

1.40 **(a)** Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA recognizing A, where A is some regular language. Construct $M' = (Q', \Sigma, \delta', q_0', F')$ recognizing $NOPREFIX(A)$ as follows:

1. $Q' = Q$.
2. For $r \in Q'$ and $a \in \Sigma$, define $\delta'(r, a) = \begin{cases} \{\delta(r, a)\} & \text{if } r \notin F \\ \emptyset & \text{if } r \in F. \end{cases}$
3. $q_0' = q_0$.
4. $F' = F$.

1.44 Let $M_B = (Q_B, \Sigma, \delta_B, q_B, F_B)$ and $M_C = (Q_C, \Sigma, \delta_C, q_C, F_C)$ be DFAs recognizing B and C, respectively. Construct NFA $M = (Q, \Sigma, \delta, q_0, F)$ that recognizes $B \overset{1}{\leftarrow} C$ as follows. To decide whether its input w is in $B \overset{1}{\leftarrow} C$, the machine M checks that $w \in B$, and in parallel nondeterministically guesses a string y that contains the same number of 1s as contained in w and checks that $y \in C$.

1. $Q = Q_B \times Q_C$.
2. For $(q, r) \in Q$ and $a \in \Sigma_\varepsilon$, define

$$\delta((q,r),a) = \begin{cases} \{(\delta_B(q,\mathtt{0}), r)\} & \text{if } a = \mathtt{0} \\ \{(\delta_B(q,\mathtt{1}), \delta_C(r,\mathtt{1}))\} & \text{if } a = \mathtt{1} \\ \{(q, \delta_C(r,\mathtt{0}))\} & \text{if } a = \varepsilon. \end{cases}$$

3. $q_0 = (q_B, q_C)$.
4. $F = F_B \times F_C$.

1.46 **(b)** Let $B = \{\mathtt{0}^m\mathtt{1}^n \mid m \neq n\}$. Observe that $\overline{B} \cap \mathtt{0}^*\mathtt{1}^* = \{\mathtt{0}^k\mathtt{1}^k \mid k \geq 0\}$. If B were regular, then $\overline{B}$ would be regular and so would $\overline{B} \cap \mathtt{0}^*\mathtt{1}^*$. But we already know that $\{\mathtt{0}^k\mathtt{1}^k \mid k \geq 0\}$ isn't regular, so B cannot be regular.

Alternatively, we can prove B to be nonregular by using the pumping lemma directly, though doing so is trickier. Assume that $B = \{\mathtt{0}^m\mathtt{1}^n \mid m \neq n\}$ is regular. Let p be the pumping length given by the pumping lemma. Observe that $p!$ is divisible by all integers from 1 to p, where $p! = p(p-1)(p-2)\cdots 1$. The string $s = \mathtt{0}^p\mathtt{1}^{p+p!} \in B$, and $|s| \geq p$. Thus the pumping lemma implies that s can be divided as xyz with $x = \mathtt{0}^a$, $y = \mathtt{0}^b$, and $z = \mathtt{0}^c\mathtt{1}^{p+p!}$, where $b \geq 1$ and $a+b+c = p$. Let s' be the string $xy^{i+1}z$, where $i = p!/b$. Then $y^i = \mathtt{0}^{p!}$ so $y^{i+1} = \mathtt{0}^{b+p!}$, and so $s' = \mathtt{0}^{a+b+c+p!}\mathtt{1}^{p+p!}$. That gives $s' = \mathtt{0}^{p+p!}\mathtt{1}^{p+p!} \notin B$, a contradiction.

1.50 Assume to the contrary that some FST T outputs $w^{\mathcal{R}}$ on input w. Consider the input strings 00 and 01. On input 00, T must output 00, and on input 01, T must output 10. In both cases, the first input bit is a 0 but the first output bits differ. Operating in this way is impossible for an FST because it produces its first output bit before it reads its second input. Hence no such FST can exist.

1.52 **(a)** We prove this assertion by contradiction. Let M be a k-state DFA that recognizes L. Suppose for a contradiction that L has index greater than k. That means some set X with more than k elements is pairwise distinguishable by L. Because M has k states, the pigeonhole principle implies that X contains two distinct strings x and y, where $\delta(q_0, x) = \delta(q_0, y)$. Here $\delta(q_0, x)$ is the state that M is in after starting in the start state q_0 and reading input string x. Then, for any string $z \in \Sigma^*$, $\delta(q_0, xz) = \delta(q_0, yz)$. Therefore, either both xz and yz are in L or neither are in L. But then x and y aren't distinguishable by L, contradicting our assumption that X is pairwise distinguishable by L.

(b) Let $X = \{s_1, \ldots, s_k\}$ be pairwise distinguishable by L. We construct DFA $M = (Q, \Sigma, \delta, q_0, F)$ with k states recognizing L. Let $Q = \{q_1, \ldots, q_k\}$, and define $\delta(q_i, a)$ to be q_j, where $s_j \equiv_L s_i a$ (the relation $\equiv_L$ is defined in Problem 1.51). Note that $s_j \equiv_L s_i a$ for some $s_j \in X$; otherwise, $X \cup s_i a$ would have $k+1$ elements and would be pairwise distinguishable by L, which would contradict the assumption that L has index k. Let $F = \{q_i \mid s_i \in L\}$. Let the start state q_0 be the q_i such that $s_i \equiv_L \varepsilon$. M is constructed so that for any state q_i, $\{s \mid \delta(q_0, s) = q_i\} = \{s \mid s \equiv_L s_i\}$. Hence M recognizes L.

(c) Suppose that L is regular and let k be the number of states in a DFA recognizing L. Then from part (a), L has index at most k. Conversely, if L has index k, then by part (b) it is recognized by a DFA with k states and thus is regular. To show that the index of L is the size of the smallest DFA accepting it, suppose that L's index is *exactly* k. Then, by part (b), there is a k-state DFA accepting L. That is the smallest such DFA because if it were any smaller, then we could show by part (a) that the index of L is less than k.

1.55 **(a)** The minimum pumping length is 4. The string 000 is in the language but cannot be pumped, so 3 is not a pumping length for this language. If s has length 4 or more, it contains 1s. By dividing s into xyz, where x is 000 and y is the first 1 and z is everything afterward, we satisfy the pumping lemma's three conditions.

(b) The minimum pumping length is 1. The pumping length cannot be 0 because the string ε is in the language and it cannot be pumped. Every nonempty string in the language can be divided into xyz, where x, y, and z are ε, the first character, and the remainder, respectively. This division satisfies the three conditions.

(d) The minimum pumping length is 3. The pumping length cannot be 2 because the string 11 is in the language and it cannot be pumped. Let s be a string in the language of length at least 3. If s is generated by $0^*1^+0^+1^*$ and s begins either 0 or 11, write $s = xyz$ where $x = \varepsilon$, y is the first symbol, and z is the remainder of s. If s is generated by $0^*1^+0^+1^*$ and s begins 10, write $s = xyz$ where $x = 10$, y is the next symbol, and z is the remainder of s. Breaking s up in this way shows that it can be pumped. If s is generated by 10^*1, we can write it as xyz where $x = 1$, $y = 0$, and z is the remainder of s. This division gives a way to pump s.