Introduction to Formal Language, Fall 2016 **Due: 29-Mar-2016 (Tuesday)**

Homework 3 Solutions

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1. The *symmetric difference* of two sets S_1 and S_2 is defined as

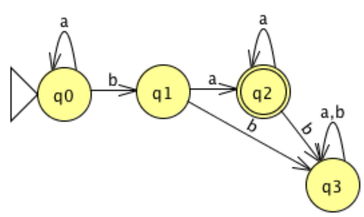
$$S_1 \ominus S_2 = \{x : x \in S_1 \text{ or } x \in S_2, \text{ but } x \text{ is not in both } S_1 \text{ and } S_2\}.$$

Show that the family of regular languages is closed under symmetric difference.
Answer.

From the definition of symmetric difference of two sets, we have that $S_1 \ominus S_2 = (S_1 \cap \overline{S}_2) \cup (S_2 \cap \overline{S}_1)$. Because of the closure of regular languages under intersection ($\cap$), complementation ($\overline{L}$), and union ($\cup$), the family of regular languages is closed under symmetric difference. □

2. Let $L_1 = L(a^*baa^*)$ and $L_2 = L(aba^*)$. Find L_1/L_2.
Answer.

We first construct a DFA that accepts L_1 as follows. We check each state q_0, q_1, q_2,



and q_3 to see whether there is a walk labeled aba^* to the final state q_2. We see that only q_0 qualifies. Thus, the result $L_1/L_2 = L(a^*)$. □

3. If L is a regular language, prove that $L_1 = \{uv : u \in L, |v| = 2\}$ is also regular.
Answer.

From the definition of L_1, we have that $L_1 = LL'$, where $L' = \{v : |v| = 2\}$. L' is regular since we can construct a DFA that accepts strings with two symbols. Thus, $L_1 = LL'$ is regular since the family of regular languages is closed under concatenation. □

4. For a string $a_1a_2 \cdots a_n$ define the operation $shift$ as

$$shift(a_1a_2 \cdots a_n) = a_2 \cdots a_na_1.$$

From this, we can define the operation on a language as

$$shift(L) = \{v : v = shift(w) \text{ for some } w \in L\}.$$

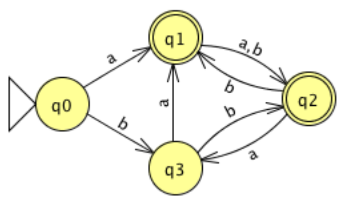
Show that the regularity is preserved under the *shift* operation.

Answer.

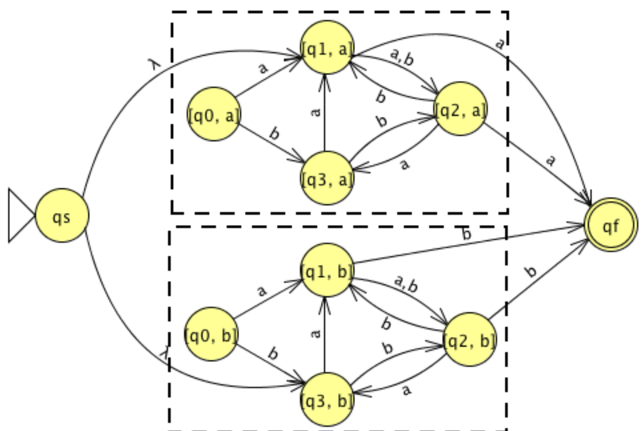
Assume that the language L is given in DFA $M = (Q, \Sigma, \delta, q_0, F)$. We construct an NFA $N = (Q', \Sigma, \delta', q_s, \{q_f\})$ satisfies that $shift(L) = L(N)$ with $Q' = Q \times \Sigma \cup \{q_s, q_f\}$ and δ' as follows. (Note that $Q \times \Sigma = \{[q, \sigma] : q \in Q, \sigma \in \Sigma\}$, where $[q, \sigma]$ is a state for the first symbol of the string in L to be σ.)

- $\delta'(q_s, \lambda) = \{[\delta(q_0, \sigma), \sigma] : \sigma \in \Sigma\}$ (here we guess the first symbol to be σ, we will verify this guess "in the end" when the last symbol appears);
- $\delta'([q, \sigma], \sigma') = \{[\delta(q, \sigma'), \sigma]\}, \forall q \in Q, \forall \sigma, \sigma' \in \Sigma$;
- Add q_f to $\delta'([r, \sigma], \sigma), \forall r \in F, \forall \sigma \in \Sigma$;

For example, the following is a DFA M such that $L = L(M)$. The construction of the



NFA N satisfies that $shift(L) = L(N)$ is shown as follows.



Thus, the regularity is preserved under the *shift* operation. $\square$

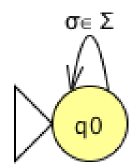
5. Exhibit an algorithm that, given any three regular language, L, L_1, L_2, determines whether or not $L = L_1 L_2$.

   Answer.

   Check if L xor $L_1 L_2$ is empty or not. If yes, then $L = L_1 L_2$. Otherwise, $L \neq L_1 L_2$.

   The algorithm for the check of L xor $L_1 L_2$: Assume that the three languages are given in DFAs M, M_1, and M_2.

- Construct an NFA N for $L(M_1)L(M_2)$ by the algorithm in the textbook;
- Convert N into DFA M';
- Construct $M'' = M \times M'$ for $L(M)$ xor $L(M_1)L(M_2)$ by the algorithm given in the class;
- Minimize M'' as M''' and determine whether M''' is equivalent to the following DFA:



□

6. Describe an algorithm which, when given a regular grammar G, can tell us whether or not $L(G) = \Sigma^*$.

Answer.

[Solution 1] Check if $L(G)$ xor Σ^* is empty or not. If yes, then $L(G) = \Sigma^*$. Otherwise, $L(G) \neq \Sigma^*$. (This can be done by the method in Problem 5.)

[Solution 2] A desirable algorithm is described as follows.

- Convert the regular grammar G to an NFA M';
- Convert M to a DFA M;
- Construct the complement DFA $\overline{M}$ of M (change the non-final states in M to final states and final states in M to non-final states);
- Check if $\overline{M}$ accepts any string in Σ^* or not. If no, that means $L(\overline{M}) = \emptyset$, i.e., $L(G) = \Sigma^*$. Otherwise, $L(G) \neq \Sigma^*$.

□

7. Prove that the following language is not regular.

$$L = \{a^n b^\ell a^k : n = \ell \text{ or } \ell \neq k\}.$$

Answer.

Let m be the constant in the pumping lemma. We choose $w = a^m b^m a^m \in L$, $|w| = 3m \geq m$. For all possible x, y, z with $w = xyz$, $|xy| \leq m$, $|y| \geq 1$, there are following cases:

- Case 1: $x = a^r$, $y = b^s$, $z = a^{m-r-s} b^m a^m$, $r + s \leq m$, $s \geq 1$. We let $i = 0$. $xy^0 z = a^r (b^s)^0 z = a^{m-s} b^m a^m \notin L$.
- Case 2: no other cases.

Thus, L is not regular. $\square$

8. Prove that the following language is not regular.

$$L = \{ww : w \in \{a, b\}^*\}.$$

Answer.

Let m be the constant in the pumping lemma. We choose $w = a^m b^m a^m b^m \in L$, $|w| = 4m \geq m$. For all possible x, y, z with $w = xyz$, $|xy| \leq m$, $|y| \geq 1$, there are following cases:

- Case 1: $x = a^r$, $y = b^s$, $z = a^{m-r-s} b^m a^m b^m$, $r + s \leq m$, $s \geq 1$. We let $i = 0$. $xy^0 z = a^r (b^s)^0 z = a^{m-s} b^m a^m b^m \notin L$.
- Case 2: no other cases.

Thus, L is not regular. $\square$

9. Determine whether or not the following language on $\Sigma = \{a\}$ is regular

$$L = \{a^n : n = 2^k \text{ for some } k \geq 0\}.$$

Answer.

Let m be the constant in the pumping lemma. We choose $w = a^{2^m} \in L$, $|w| = 2^m \geq m$. For all possible x, y, z with $w = xyz$, $|xy| \leq m$, $|y| \geq 1$, there are following cases:

- Case 1: $x = a^r$, $y = b^s$, $z = a^{2^m - r - s}$, $r + s \leq m$, $s \geq 1$. We let $i = 2$. $xy^2 z = a^r (b^s)^2 z = a^{2^m + s} \notin L$, because

$$2^m < 2^m + s \leq 2^m + m < 2^m + 2^m = 2^{m+1}, 2^m + s \neq 2^k \text{ for any } k$$

- Case 2: no other cases.

Thus, L is not regular. $\square$

10. Make a conjecture whether or not the following language is regular. Then prove your conjecture.

$$L = \{a^n b^\ell : |n - \ell| = 2\}.$$

Answer.

Let m be the constant in the pumping lemma. We choose $w = a^m b^{m+2} \in L$, $|w| = 2m + 2 \geq m$. For all possible x, y, z with $w = xyz$, $|xy| \leq m$, $|y| \geq 1$, there are following cases:

- Case 1: $x = a^r$, $y = b^s$, $z = a^{m-r-s} b^{m+2}$, $r + s \leq m$, $s \geq 1$. We let $i = 0$. $xy^0 z = a^r (b^s)^0 z = a^{m-s} b^{m+2} \notin L$, because

$$|(m - s) - (m + 2)| = |-s - 2| \neq 2$$

- Case 2: no other cases.

Thus, L is not regular. $\square$