Introduction to Formal Language, Fall 2016 **Due: 05-May-2016 (Thursday)**

Homework 5 Solutions

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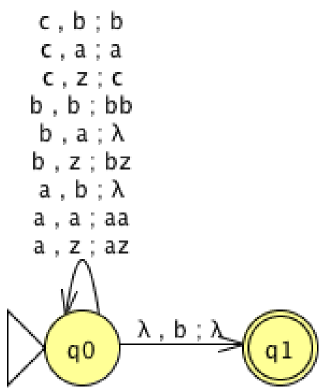
1. Construct npda that accepts the following language on $\Sigma = \{a, b, c\}$.

$$L = \{w : n_a(w) < n_b(w)\}.$$

Answer.
An NPDA that accepts L is $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$, where $Q = \{q_0, q_1\}$, $\Sigma = \{a, b, c\}$, $\Gamma = \{0, 1, z\}$, $F = \{q_1\}$, and the transition function δ is represented as the following graph



The idea is the same as an example in pushdown automata slide ($L = \{w : n_a(w) = n_b(w)\}$) which accepts the strings with equal number of as and bs, but now it only accepts if there is a b on top of the stack. □

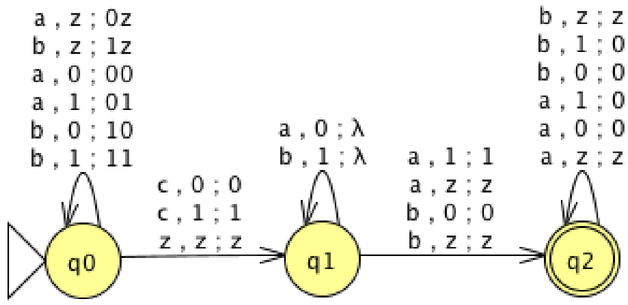
2. Find an npda on $\Sigma = \{a, b, c\}$ that accepts the language

$$L = \{w_1 c w_2 : w_1, w_2 \in \{a, b\}^*, w_1 \neq w_2^R\}.$$

Answer.

An NPDA that accepts L is $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$, where $Q = \{q_0, q_1, q_2\}$, $\Sigma = \{a, b, c\}$, $\Gamma = \{0, 1, z\}$, $F = \{q_2\}$, and the transition function δ is represented as the following graph

□



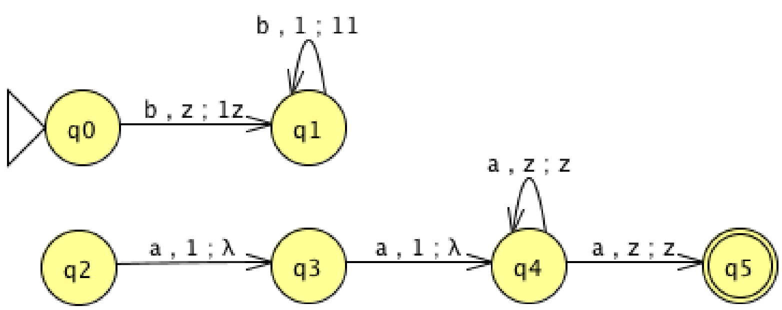
3. What language is accepted by the pda

$$M = (\{q_0, q_1, q_2, q_3, q_4, q_5\}, \{a, b\}, \{0, 1, a, z\}, \delta, z, q_0, \{q_5\}),$$

with

$$\begin{aligned}
\delta(q_0, b, z) &= \{(q_1, 1z)\}, \\
\delta(q_1, b, 1) &= \{(q_1, 11)\}, \\
\delta(q_2, a, 1) &= \{(q_3, \lambda)\}, \\
\delta(q_3, a, 1) &= \{(q_4, \lambda)\}, \\
\delta(q_4, a, z) &= \{(q_4, z), (q_5, z)\}?
\end{aligned}$$

Answer.



$L = \emptyset$, because the transition function would never reach the final state. □

4. Construct an npda corresponding to the grammar

$$\begin{aligned}
S &\to aABB|aAA, \\
A &\to aBB|a, \\
B &\to bBB|A.
\end{aligned}$$

Answer.
First, we convert the cdf to the corresponding GNF

$$S \to aABB|aAA,$$
$$A \to aBB|a,$$
$$B \to bBB|aBB|a.$$

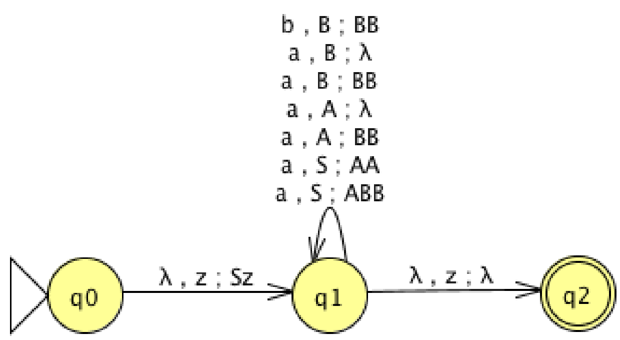
We construct npda $M = (\{q_0, q_1, q_2\}, \{a, b\}, \{S, A, B, Z\}, \delta, q_0, Z, \{q_2\})$ by the following steps:

(a) Initial: $\delta(q_0, \lambda, z) = \{(q_1, Sz)\}$

(b) Process the input:

- For $S \to aABB|aAA:\ \delta(q_1, a, S) = \{(q_1, ABB), (q_1, AA)\}$
- For $A \to aBB|a:\ \delta(q_1, a, A) = \{(q_1, BB), (q_1, \lambda)\}$
- For $B \to bBB|aBB|a:\ \delta(q_1, b, B) = \{(q_1, BB)\}, \delta(q_1, a, B) = \{(q_1, BB), (q_1, \lambda)\}$

(c) In the end: $\delta(q_1, \lambda, z) = \{(q_2, \lambda)\}$



□

5. Show that the following language is not context-free.

$$L = \{w \in \{a, b, c\}^* : n_a(w) = n_b(w) \leq n_c(w)\}.$$

Answer. Let $L' = L \cap L(a^*b^*c^*) = \{a^ib^ic^k | i \leq k\}$. Since context-free languages are closed under interaction with regular languages. We only show L' to be not context-free. We pick a string $w = a^mb^mc^m \in L'$, $m \in N$. There are many ways to decompose w as $w = uvxyz$ with $|vxy| \leq m$ and $|vy| \geq 1$. However, for all of them have a winning countermove such that $uv^ixy^iz \notin L$:

- $v = a^k, y = a^s : uv^0xy^0z = a^{m-k-s}b^mc^m \notin L.$
- $v = a^k, y = a^rb^s : uv^2xy^2z = a^{m+2k+r}b^{m+s}a^rb^sc^m \notin L.$
- $v = a^k, y = b^l : uv^2xy^2z = a^{m+k}b^{m+l}c^m \notin L.$

- $v = a^r b^s, y = b^t : uv^0xy^0z = a^{m-r}b^{m-s-t}c^m \notin L$.
- $v = b^r, y = b^s : uv^0xy^0z = a^m b^{m-r-s}c^m \notin L$.
- $v = b^s, y = b^r c^t : uv^0xy^0z = a^m b^{m-s-r}c^{m-t} \notin L$.
- $v = b^s, y = c^r : uv^0xy^0z = a^m b^{m-s}c^{m-r} \notin L$.
- $v = b^s c^r, y = c^t : uv^0xy^0z = a^m b^{m-s}c^{m-r-t} \notin L$.
- $v = c^r, y = c^s : uv^0xy^0z = a^m b^m c^{m-r-s} \notin L$.

Therefore, by the pumping lemma for context-free languages, L is not context-free. □

6. Show that the following language on $\Sigma = \{a, b, c\}$ is not context-free.

$$L = \{a^n b^j c^k : k = jn\}.$$

Answer.
We pick a string $w = a^m b^m c^{m^2} \in L$, $m \in N$. There are many ways to decompose w as $w = uvxyz$ with $|vxy| \leq m$ and $|vy| \geq 1$. However, for all of them have a winning countermove such that $uv^ixy^iz \notin L$:

- $v = a^s, y = a^r : uv^0xy^0z = a^{m-r-s}b^m c^{m^2} \notin L$.
- $v = a^s, y = a^r b^t : uv^0xy^0z = a^{m-s-r}b^{m-t}c^{m^2} \notin L$.
- $v = a^s, y = b^r : uv^0xy^0z = a^{m-s}b^{m-r}c^{m^2} \notin L$.
- $v = a^s b^r, y = b^t : uv^0xy^0z = a^{m-s}b^{m-r-t}c^{m^2} \notin L$.
- $v = b^s, y = b^r : uv^0xy^0z = a^m b^{m-r-s}c^{m^2} \notin L$.
- $v = b^s, y = b^r c^t : uv^0xy^0z = a^m b^{m-s-r}c^{m^2-t} \notin L$.
- $v = b^s, y = c^r : uv^0xy^0z = a^m b^{m-s}c^{m^2-r} \notin L$.
- $v = b^s c^r, y = c^t : uv^0xy^0z = a^m b^{m-s}c^{m^2-r-t} \notin L$.
- $v = c^r, y = c^s : uv^0xy^0z = a^m b^m c^{m^2-r-t} \notin L$.

Therefore, by the pumping lemma for context-free languages, L is not context-free. □

7. Determine whether or not the following languages is context free, and prove your answer.

$$L = \{a^n b^n c^j : n \leq j\}.$$

Answer.
We pick a string $w = a^m b^m c^m \in L$, $m \in N$. There are many ways to decompose w as $w = uvxyz$ with $|vxy| \leq m$ and $|vy| \geq 1$. However, for all of them have a winning countermove such that $uv^ixy^iz \notin L$:

- $v = a^r, y = a^s : uv^0xy^0z = a^{m-r-s}b^m c^m \notin L$.
- $v = a^r, y = a^s b^t : uv^2xy^2z = a^{m+2r+s}b^{m+t}a^s b^t c^m \notin L$.
- $v = a^r, y = b^s : uv^2xy^2z = a^{m+r}b^{m+s}c^m \notin L$.
- $v = a^r b^s, y = b^t : uv^2xy^2z = a^{m+r}b^{m+s+t}c^m \notin L$.

- $v = b^r, y = b^s : uv^0xy^0z = a^m b^{m-r-s} c^m \notin L$.
- $v = b^r, y = b^s c^t : uv^0xy^0z = a^m b^{m-s-r} c^{m-t} \notin L$.
- $v = b^r, y = c^s : uv^0xy^0z = a^m b^{m-r} c^{m-s} \notin L$.
- $v = b^r c^s, y = c^t : uv^0xy^0z = a^m b^{m-r} c^{m-s-t} \notin L$.
- $v = c^r, y = c^s : uv^0xy^0z = a^m b^m c^{m-r-t} \notin L$.

Therefore, by the pumping lemma for context-free languages, L is not context-free. □

8. Show that the family of context-free languages is not closed under difference in general, but is closed under regular difference, that is, if L_1 is context-free and L_2 is regular, then $L_1 - L_2$ is context-free.

Answer.

The answer contains the following two parts:

- If $L_3 = L_1 - L_2$ is context-free for context-free L_1 and L_2, then $L_1 \cap L_2 = L_1 - (L_1 - L_2) = L_1 - L_3$ is also context-free. This is a contradiction. Thus, $L_1 - L_2$ is not necessarily context-free for context-free L_1 and L_2.
- Let L_1 be a context-free language and L_2 a regular language. By the closure property of regular languages, we know that $\bar{L_2}$ is also regular. By theorem, we know that $L_1 \cap L_2$ is context-free by the closure property under regular intersection. Therefore, $L_1 - L_2 = L_1 \cap \bar{L_2}$ is also context-free by the closure property under regular intersection.

□

9. Show that the following language is context-free.

$$L = \{w \in \{a,b\}^* : n_a(w) = n_b(w); w \text{ does not contain a substring } aab\}.$$

Answer.

Let $L_1 = \{w \in \{a,b\}^* : n_a(w) = n_b(w)\}$ and $L_2 = \{w \in \{a,b\}^* :$ w contains 'aab' as a string$\}$. Since L_1 is context-free and L_2 is regular, $L = L_1 - L_2$ is context-free by the result in the previous problem. □