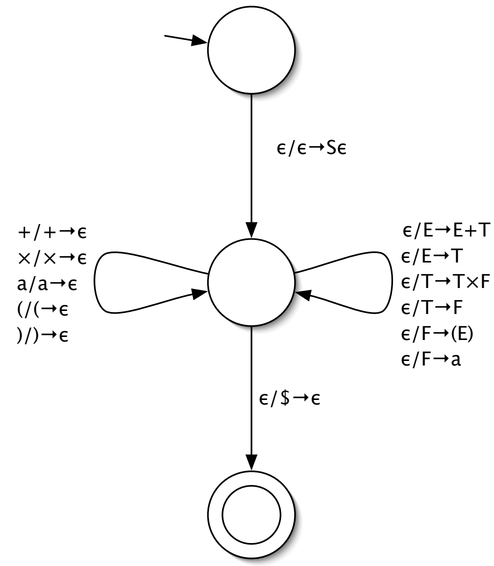
CSE105 - HOMEWORK 5 SOLUTIONS

2.11 We use the shorthand notation for pushing multiple symbols onto the stack at once that is described in the proof of theorem 2.20 (page 117).



2.30 **a.** $L = \{0^n1^n0^n1^n |\ n \geq 0\}$. Strings in L consist of four consecutive *segments*: a zero segment followed by a one segment followed by another zero segment followed by another one segment. Assume that L is context-free and let p denote the pumping length of L. Consider the string $0^p1^p0^p1^p \in L$. We can write this string as $uvxyz$ where $|vxy| \leq p$ and $|vy| > 1$. We look at two cases:

<u>case 1:</u> Either v or y contains more than one type of character (*i.e.* at least one of v and y contains both zeros and ones). In this case, pumping up once to uv^2xy^2z inserts an extra 0^i1^j sequence into s—*i.e.* the resulting string will contain at least 3 zero segments and 3 one segments. For instance, if $v = 0011$, then pumping once would result in an additional 00 segment followed by an additional 11 segment. But, strings in L contain only four segments.

<u>case 2:</u> Both v and y contain only one type of character (*i.e.* they are both all zeros or all ones). Since $vxy \leq p$, vxy contains letters from at most 2 segments of s. Pumping once to uv^2xy^2z increases the number of characters in one or two of the segments, but leaves the remaining segments untouched. Thus, the number of characters in each segment will be unbalanced.

b. answer in text

c. answer in text

d. $L = \{t_1\#t_2\# \cdots \#t_k |\ k \geq 2$, each $t_i \in \{a, b\}^*$, and $t_i = t_j$ for some $i \neq j\}$. Assume L is context-free and let p denote its pumping length. Consider $s = 0^p1^p\#0^p1^p \in L$. By the pumping lemma, we can write s as $uvxyz$ where $|vxy| \leq p$ and $|vy| > 1$.

Suppose vxy lies entirely on one side of the # symbol. Then, pumping once to uv^2xy^2z results in a string where $t_1 \neq t_2$, so $uv^2xy^2z \notin L$.

Suppose that vxy contains the # symbol. If either v or y contains the # symbol, than we can pump s down to uv^0xy^0z, which will not contain the # symbol and hence will not be in L. Otherwise, the # symbol is contained in x, v is a substring of 1^p, and y is a substring of 0^p (since $|vxy| \leq p$). Pumping s down to uv^0xy^0z reduces either the number of ones in t_1 or the number of zeros in t_2 or both. As a result, $t_1 \neq t_2$ for uv^0xy^0z, so the string is not in L.

2.31 Let $n_0(x)$ denote the number of zeros in x and $n_1(x)$ the number of ones in x. Let

$$B = \{s \in \{0,1\}^* | n_0(s) = n_1(s) \text{ and } s = ww^R \text{ for some } w \in \{0,1\}^*\}.$$

Assume that B is context-free and let p be its pumping length. Let $s = 0^p1^p1^p0^p \in B$. By the pumping lemma, we can write s as $uvxyz$ where $|vxy| \leq p$, $|vy| > 1$, and $uv^ixy^iz \in B$ for all $i \geq 0$.

Since $|vy| \geq 0$, we know that there is at least one character in either v or y. Moreover, the pumping lemma guarantees that $uv^0xy^0z \in B$, so $n_0(uv^2xy^2z) = n_1(uv^2xy^2z)$. This in turn implies that the string vy must contain equal numbers of zeros and ones. Since in addition $|vxy| \leq p$, we conclude that vxy must either be substring of 0^p1^p or 1^p0^p (since vy contains both zeros and ones). But then pumping s down once to uv^0xy^0z leaves us with a string that is not a palindome, since we have changed one side of s but not the other.

2.39 For two languages A and B over the alphabet Σ, let the shuffle of A and B be

$$\{w \mid w = a_1b_1 \cdots a_kb_k, \text{ where } a_1 \cdots a_k \in A, \;\; b_1 \cdots b_k \in B \text{ and each } a_i, b_i \in \Sigma^*\}.$$

Define A and B as follows

$$A = \{w \in \{0,1\}^* \mid n_0(w) = n_1(w)\}$$
$$B = \{w \in \{a,b\}^* \mid n_a(w) = n_b(w)\},$$

where $n_0(w)$ denotes the number of zeros in w, $n_a(w)$ denotes the number of a's in w and so on.

The shuffle of A and B is the language

$$S = \{w \in \{0,1,a,b\}^* \mid n_0(w) = n_1(w) \text{ and } n_a(w) = n_b(w)\},$$

which is not context-free, as we now show. Assume that S is context-free and let p be its pumping length. Let $s = 0^pa^p1^pb^p$. By the pumping lemma, s can be written as $uvxyz$, where $|vy| > 1$ and $|vxy| \leq p$. Since $|vxy| \leq p$, it cannot contain both zeros and ones or both a's and b's. Hence, pumping once to uv^2xy^2z will result in a string with either an unbalanced number of zeros and ones or an unbalanced number of a's and b's or both. Thus, $uv^2xy^2z \notin S$, so S is not context-free.

We showed in class that A is context-free and B is similarly context-free. Since the shuffle of A and B is not context-free, we conclude that the class of CFLs is not closed under the shuffle operation.