

Homework 2 Solutions

1. DFA $M = (Q, \Sigma, \delta, q_1, F)$, where

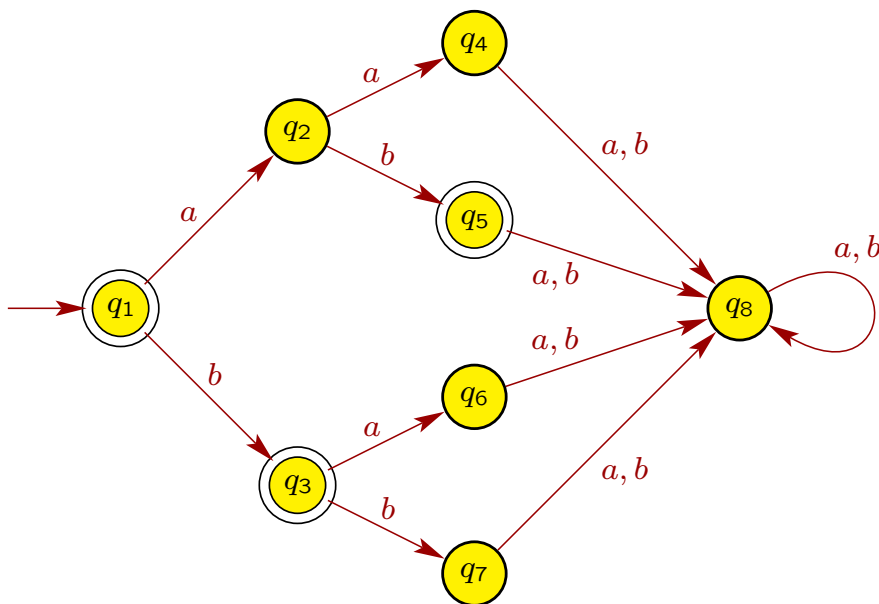
- $Q = \{q_1, q_2, q_3\}$
- $\Sigma = \{a, b\}$
- transition function δ is given by

	a	b
q_1	q_1	q_2
q_2	q_1	q_3
q_3	q_1	q_3

- q_1 is the start state
- $F = \{q_1, q_3\}$ is the set of accept states.

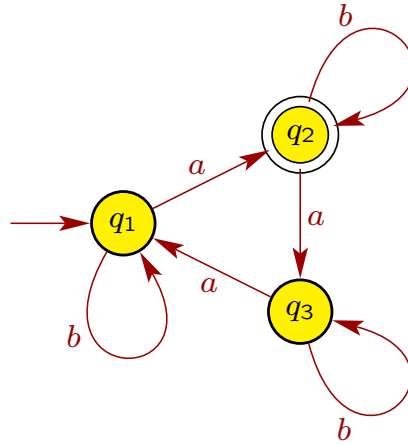
2. There are (infinitely) many correct DFAs for each part below.

(a) A DFA that recognizes the language $A = \{\varepsilon, b, ab\}$ is

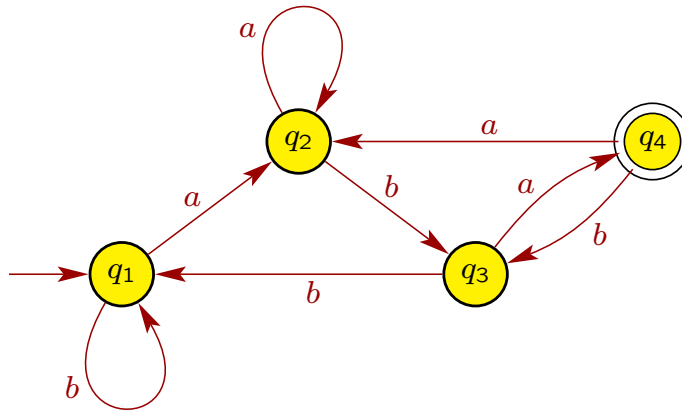


There are simpler DFAs that recognize this language. Can you come up with one with only 4 states?

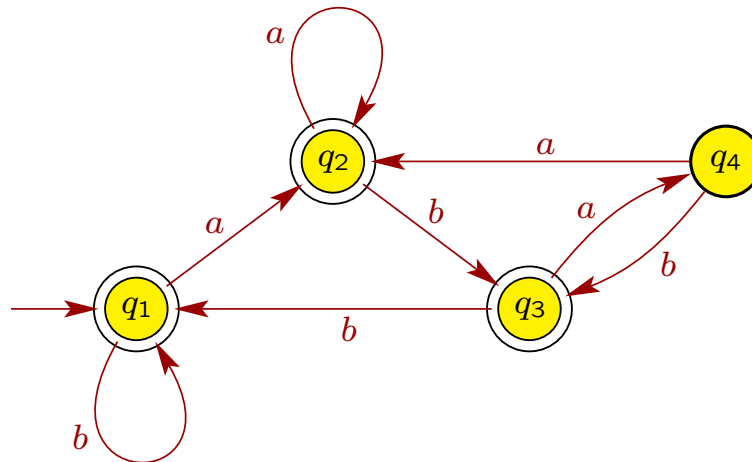
- (b) A DFA that recognizes the language $B = \{ w \in \Sigma^* \mid n_a(w) \bmod 3 = 1 \}$ is



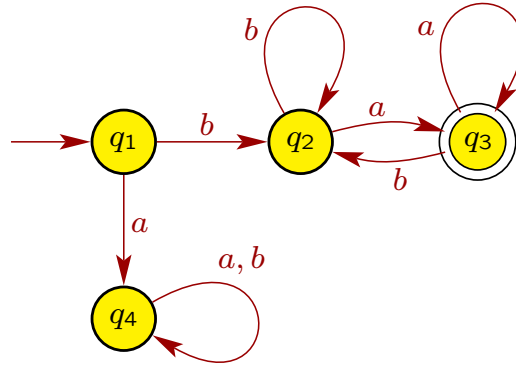
- (c) A DFA that recognizes the language $C = \{ w \in \Sigma^* \mid w = saba \text{ for some string } s \in \Sigma^* \}$ is



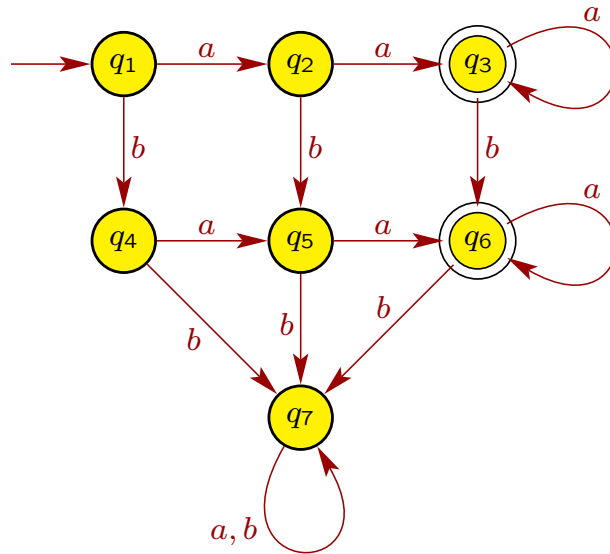
- (d) Since $D = \overline{C}$, the complement of C , we can convert the DFA for C into a DFA for D by swapping the accept and non-accept states:



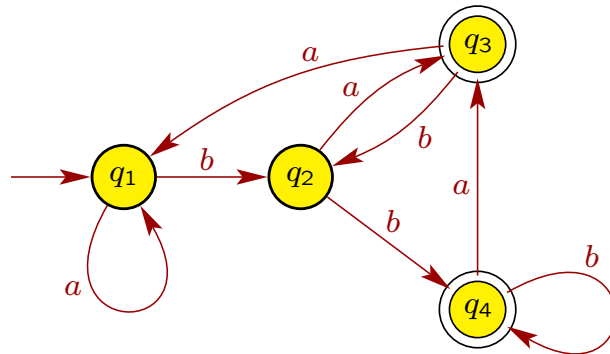
(e) A DFA for $E = \{ w \in \Sigma^* \mid w \text{ begins with } b \text{ and ends with } a \}$ is



(f) A DFA for $F = \{ w \in \Sigma^* \mid n_a(w) \geq 2, n_b(w) \leq 1 \}$ is



(g) A DFA for $G = \{ w \in \Sigma^* \mid |w| \geq 2, \text{ second-to-last symbol of } w \text{ is } b \}$ is



3. Show that, if M is a DFA that recognizes language B , swapping the accept and non-accept states in M yields a new DFA that recognizes \overline{B} , the complement of B . Conclude that the class of regular languages is closed under complement.

Answer:

Suppose language B over alphabet Σ has a DFA

$$M = (Q, \Sigma, \delta, q_1, F).$$

Then, a DFA for the complementary language \overline{B} is

$$\overline{M} = (Q, \Sigma, \delta, q_1, Q - F).$$

The reason why \overline{M} recognizes \overline{B} is as follows. First note that M and \overline{M} have the same transition function δ . Thus, since M is deterministic, \overline{M} is also deterministic. Now consider any string $w \in \Sigma^*$. Running M on input string w will result in M ending in some state $r \in Q$. Since M is deterministic, there is only one possible state that M can end in on input w . If we run \overline{M} on the same input w , then \overline{M} will end in the same state r since M and \overline{M} have the same transition function. Also, since \overline{M} is deterministic, there is only one possible ending state that \overline{M} can be in on input w .

Now suppose that $w \in B$. Then M will accept w , which means that the ending state $r \in F$, i.e., r is an accept state of M . But then $r \notin Q - F$, so \overline{M} does not accept w since \overline{M} has $Q - F$ as its set of accept states. Similarly, suppose that $w \notin B$. Then M will not accept w , which means that the ending state $r \notin F$. But then $r \in Q - F$, so \overline{M} accepts w . Therefore, \overline{M} accepts string w if and only if M does not accept string w , so \overline{M} recognizes language \overline{B} . Hence, the class of regular languages is closed under complement.

4. We say that a DFA M for a language A is *minimal* if there does not exist another DFA M' for A such that M' has strictly fewer states than M . Suppose that $M = (Q, \Sigma, \delta, q_0, F)$ is a minimal DFA for A . Using M , we construct a DFA \overline{M} for the complement \overline{A} as $\overline{M} = (Q, \Sigma, \delta, q_0, Q - F)$. Prove that \overline{M} is a minimal DFA for \overline{A} .

Answer:

We prove this by contradiction. Suppose that \overline{M} is not a minimal DFA for \overline{A} . Then there exists another DFA D for \overline{A} such that D has strictly fewer states than \overline{M} . Now create another DFA D' by swapping the accepting and non-accepting states of D . Then D' recognizes the complement of \overline{A} . But the complement of \overline{A} is just A , so D' recognizes A . Note that D' has the same number of states as D , and \overline{M} has the same number of states as M . Thus, since we assumed that D has strictly fewer states than \overline{M} , then D' has strictly fewer states than M . But since D' recognizes A , this contradicts our assumption that M is a minimal DFA for A . Therefore, \overline{M} is a minimal DFA for \overline{A} .

5. Suppose A_1 and A_2 are defined over the same alphabet Σ . Suppose DFA M_1 recognizes A_1 , where $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$. Suppose DFA M_2 recognizes A_2 , where $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$. Define DFA $M_3 = (Q_3, \Sigma, \delta_3, q_3, F_3)$ for $A_1 \cap A_2$ as follows:

- Set of states of M_3 is

$$Q_3 = Q_1 \times Q_2 = \{ (x, y) \mid x \in Q_1, y \in Q_2 \}.$$

- The alphabet of M_3 is Σ .
- M_3 has transition function $\delta_3 : Q_3 \times \Sigma \rightarrow Q_3$ such that for $x \in Q_1, y \in Q_2$, and $\ell \in \Sigma$,

$$\delta_3((x, y), \ell) = (\delta_1(x, \ell), \delta_2(y, \ell)).$$

- The initial state of M_3 is $s_3 = (q_1, q_2) \in Q_3$.
- The set of accept states of M_3 is

$$F_3 = \{ (x, y) \in Q_1 \times Q_2 \mid x \in F_1 \text{ and } y \in F_2 \} = F_1 \times F_2.$$

Since $Q_3 = Q_1 \times Q_2$, the number of states in the new DFA M_3 is $|Q_3| = |Q_1| \cdot |Q_2|$. Thus, $|Q_3| < \infty$ since $|Q_1| < \infty$ and $|Q_2| < \infty$.