Harmonic Analysis

Fourier Series and Beyond

K R Parthasarathy

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Henry Helson
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While investigating the properties of heat flow in 1807 the French scientist Jean Baptiste Joseph Fourier stumbled on the remarkably fruitful mathematical idea that the graph of any function in a bounded interval can be obtained as a linear superposition of sines and cosines. Since $\cos x + i\sin x = e^{ix}$ this led to the hypothesis that any integrable function f in the interval $[0, 2\pi]$ can be expanded as a Fourier series:

$$f(x) = \sum_n a_n e^{inx}, \qquad (1)$$

where $a_n = a_n(f)$ is the nth Fourier coefficient defined by

$$a_n = (2\pi)^{-1} \int_0^{2\pi} f(x) e^{-inx} dx,$$

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where

$$n = 0, \pm 1, \pm 2, \ldots . \qquad (2)$$

An infinite series can have several interpretations depending on the choice of the notion of its convergence. The investigation of convergence properties of the Fourier series (1) led to a vast amount of mathematical literature including the theory of the Lebesgue integral. The first chapter of Helson's little volume on Harmonic analysis provides a quick survey of these developments including the classical kernels of Dirichlet, Fejèr and Poisson, the general notion of an approximate identity in a convolution algebra and a result of the author and A Beurling on measures with bounded powers.

It is important to note that the set $\mathbf{Z}$ of all integers is a discrete abelian (or commutative) group under addition and discrete topology whereas the set T of all complex numbers of modulus unity is a compact abelian group under multiplication and the relative topology inherited from the complex plane. Furthermore, the function $B(n, z) = z^n$ on $\mathbf{Z} \times T$ has the properties

$$B(m+n, z) = B(m, z)B(n, z),$$
$$B(n, z_1 z_2) = B(n, z_1)B(n, z_2).$$

The map $z \to B(n, z)$ is a continuous homomorphism from T into itself for each n and every continuous homomorphism of T into itself is accounted for in this list. Similarly, the map $n \to z^n$ is a (continuous) homomorphism from $\mathbf{Z}$ into T and every homomorphism from Z into T is of this kind. The Fourier series (1) can now be expressed as

$$f(z) = \sum_{n \in \mathbf{Z}} a_n B(n, z), z = e^{ix} \in T \quad (3)$$

after identifying T with the interval $[0, 2\pi]$, where 0 and 2π on the line represent the same point 1 on T. Expressed in this way one has the following generalization of (3): Suppose G is any abelian group with the group operation denoted by $+$. Then there exists a discrete abelian group $\hat{G}$ whose operation is viewed as multiplication and a map $B(.,.)$ from $\hat{G} \times G$ into T such that

$$B(\chi_1\chi_2, x) = B(\chi_1, x)B(\chi_2, x),$$

$$B(\chi, x_1 + x_2) = B(\chi, x_1)B(\chi, x_2).$$

for all $\chi, \chi_1, \chi_2 \in \hat{G}$ and $x, x_1, x_2 \in G$. The group G admits a unique (group) translation invariant probability measure σ (called the Haar measure of G) and any σ-square integrable function f on G admits a Fourier-like expansion

$$f(x) = \sum_{\chi \in G} a(\chi)B(\chi, x), \quad (4)$$

where $a(\chi)$ is the Fourier coefficient of f given by

$$a(\chi) = \int_G \overline{B(\chi, x)} f(x) d\sigma(x) \quad (5)$$

and convergence of (4) is in the mean square sense. The classical Fourier series (1) is a special case of (4) when $G = T, \hat{G} = \mathbf{Z}$. However, there are no obvious analogues of the Dirichlet, Fejèr and Poisson kernels here owing to the lack of order in $\hat{G}$. $\hat{G}$ is called the *dual group* of G or the group of *characters* of G. Chapter 3 of this volume presents a very readable survey of this generalization which is easily accessible for our MSc students and college teachers who have the required curiosity to explore the possibilities outside their customary examination-oriented syllabi. As applications Helson provides new proofs of three old theorems: (1) Kolmogorov's extension theorem for a consistent family of finite dimensional probability measures;

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(2) Banach–Steinhaus' uniform boundedness principle for a sequence of linear operators;(3)Minkowski's theorem that any convex body in $\mathbf{R}^n$, which is symmetric about the origin and has volume $> 2^n$, has a lattice point other than the origin (the proof being due to C L Siegel and based on trigonometric sums).

It is to be noted that there exists a far reaching group-theoretic generalization of the expansion (4) when G is an arbitrary compact (but not necessarily abelian) group. This is known as the Peter–Weyl theory of which a glimpse of the abstract side is provided in the book *A Course on Topological Groups* by K Chandrasekharan which has recently appeared as TRIM 9 in the same series as the present volume. For the practical and computational aspects of this theory my favourite volume is *Group Theory and Physics* by S Sternberg (Cambridge University, Paperback Edition, 1995). Thanks to the contributions of Weyl, Wigner, Bargmann, Harish-Chandra, Gelfand and several other mathematicians and physicists, group-theoretic harmonic analysis is a flourishing industry today paving the way to new developments in the context of noncompact Lie groups as well as quantum groups.

Since $B(n,z) = z^n$ the expansion (3) suggests a link between Fourier series and the theory of analytic functions of a complex variable. This leads to the notion of the Hardy spaces $H^p(T)$, $1 \leq p < \infty$. $H^p(T) \subset L^p(T)$ is the subspace consisting of functions f for which the Fourier coefficients a_n in (2) vanish for $n < 0$. If $B(1,z) = \chi(z)$, $z \in T$, then for any $f \in L^2(T)$ denote by M_f the closed linear span of $\{f, \chi f, \chi^2 f, \ldots\}$. Then $M_f \subset L^2(T)$ is invariant under multiplication by χ. If $M_f = L^2(T)$ then f is called an *outer function.* It is a theorem of Beurling that a subspace $M \subset L^2(T)$ invariant under multiplication by χ belongs to one of two types. Either it consists of all functions in $L^2(T)$ with support in a fixed measurable subset of T or it is $qH^2(T)$ for some function q of modulus unity. The first kind of subspace is called a *Wiener subspace* and the second, a *Beurling subspace.* Any nonnull element f of $H^2(T)$ can be factorized as $f = qg$, where q and g belong to $H^2(T)$, g is outer and q is of modulus unity. Elements of $H^2(T)$ which are of unit modulus are called *inner functions.* This factorization into an inner and an outer function is unique up to a constant factor of modulus unity. This implies that every nonnull element of $H^1(T)$ can be factorized as qg^2, where q is inner and g is outer in $H^2(T)$. A function f in $H^2(T)$ is outer if and only if

$$\int_0^{2\pi} \log \mid f(\mathrm{e}^{ix} \mid \mathrm{d}x > -\infty\,.$$

From these results of Beurling it is possible to deduce the following theorem due to G Szegö: If w is a nonnegative integrable function on T then

$$\exp\frac{1}{2\pi}\int_0^{2\pi} \log w(\mathrm{e}^{ix})\mathrm{d}x =$$

$$\inf_P \frac{1}{2\pi}\int_0^{2\pi} \mid (1+P(\mathrm{e}^{ix})) \mid^2 w(\mathrm{e}^{ix})\mathrm{d}x\,;$$

where P ranges over all polynomials. The densely packed chapter on Hardy spaces covers all these and much more. It may be noted that this last theorem of Szegö is at the heart of the theory of prediction of discrete time one-dimensional stationary stochastic processes developed by N Wiener in the US and A N Kolmogorov in the former USSR during the Second World war. (A multidimensional version of Szegö's theorem when w is a positive definite matrix-valued function on T with summable entries was obtained by N Wiener and P Masani when they met at the Indian Statistical Institute in Calcutta during 1955–56.) By exploiting the standard conformal map from the unit disk to the upper half plane the author indicates how a theory of Hardy spaces $H^p(\mathbf{R})$ could be built. (This can be used to develop the prediction theory of one dimensional continuous time stationary stochastic processes.)

Thanks to the contributions of Weyl, Wigner, Bargmann, Harish-Chandra, Gelfand and several other mathematicians and physicists, group-theoretic harmonic analysis is a flourishing industry today paving the way to new developments in the context of noncompact Lie groups as well as quantum groups.

A fairly extensive discussion of the theory of conjugate functions in a whole chapter is followed by a brief account of ($\mathbf{R}$ and $\mathbf{R}_+$) translation invariant subspaces of $L^2(\mathbf{R})$ and $L^1(\mathbf{R})$ covering the results of Wiener, Beurling and Titchmarsh.

If $\varphi \in L^{\inf}(\mathbf{R})$ then its Fourier transform $\hat{\varphi}$ is a tempered distribution in $\mathbf{R}$ and the support of $\hat{\varphi}$ is called the *spectral set* of φ. The spectral set of a bounded bilateral sequence, i.e. an element of $l^\infty(\mathbf{Z})$ can be similarly defined

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as a subset of T. An element $\varphi \in L^{\infty}(\mathbf{R})$ has exactly one point λ in its spectral set if and only if $\varphi(x) = \exp i\lambda x$. If $\{\alpha_n\}$ is a bilateral sequence whose terms are drawn from a finite set of complex numbers then its spectral set is the whole of T unless $\{\alpha_n\}$ is periodic. A bilateral sequence of 0's and 1's is the Fourier–Stieltjes transform of a complex measure on T if and only if it is periodic after dropping a finite number of terms. Pretty surprises of this kind are strewn around in several places in this flower garden of harmonic analysis.

Helson concludes with a little chapter on equidistribution theorems originating in the work of H Weyl. A sequence $\{u_k\}$, $k \geq 1$ in $[0, 1]$ is said to be *equidistributed* if for any interval $[a, b] \subset [0, 1]$

$$\lim_{n\to\infty} \frac{1}{n} \#\{j \mid u_j \in [a, b], 1 \leq j \leq n\} = b - a\,,$$

where $\#$ denotes cardinality. To verify the equidistribution of a real sequence $\{u_k\}$ modulo 1 it is enough to check that

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \mathrm{e}^{2\pi i j u_k} = 0$$

for every $j \neq 0$. Thus equidistribution and trigonometric sums are closely related. It is a theorem of van der Corput that a sequence $\{u_k\}$ is equidistributed modulo 1 if for every positive integer p the sequence $\{u_{k+p} - u_k\}$, $k \geq 1$ is equidistributed modulo 1. Equidistribution theorems and uniquely ergodic transformations are intimately connected as pointed out by H Furstenberg. Exploiting these relations it is shown that for any real polynomial $P(x)$ with at least one term of the form ux^n, where u is irrational and $n \geq 1$, the sequence $\{P(k)\}$, $k \geq 1$ is equidistributed modulo 1.

With its well punctuated historical comments and instructive exercises this little but very rich volume offers an enjoyable guided tour of classical harmonic analysis with some scope in trimming its price for the Indian market.

K R Parthasarathy is with the Indian Statistical Institute, New Delhi 110 016.