A hybrid staggered/non-staggered formulation

For incompressible flows with block-structured mesh refinement

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Outlines

To-do List

For tracking progress and comments

- ☑ Debugging the code.
- ☒ Finish Wall Boundary Conditions feature. (HOLD)
- ☐ Benchmark the code.
  - ☐ Running 6 benchmark cases.
  - ☑ Fix L2-Norm.
  - ☐ Generate solution comparison at horizontal and vertical middle lines.
    - When running on one processor, the code prints out the solution. When running on multiple MPI processors, it does NOT.
- ☐ Prepare Beamer for presentation.
- ☐ Present at APS DFD 2023 on Nov 19th.

Introduction

Feature list

This is *an important text*.
This is an **alert text**.

Methodology I

The governing equations for imcompressible fluid consist of Navier-Stokes (momentum) equation and the continuity equation such that, in general coordinate system:

$$\frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_j} u_i u_j + \frac{1}{\mathrm{Re}} \frac{\partial^2 u_i}{\partial x_i \partial x_j} - \frac{\partial P}{\partial x_i} \tag{1}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2}$$

where: Re is the Reynolds number defined as:

$$\mathrm{Re} = \frac{UL}{\nu} \tag{3}$$

Dimensionless Equations

Eqs. (1) and (2) are non-dimenstionalized by a characteristic length L and velocity scale U.

Methodology II

Consider a two dimensional problem in Cartesian coordinate system, let us denote u and v so that they are velocity components in, respectively, x- and y-direction. Then the Navier-Stokes equation become:

$$\frac{\partial u}{\partial t} = -\left(\frac{\partial}{\partial x}uu + \frac{\partial}{\partial y}uv\right) + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\partial P_x}{\partial x} \tag{4}$$

$$\frac{\partial v}{\partial t} = -\left(\frac{\partial}{\partial x}vu + \frac{\partial}{\partial y}vv\right) + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{\partial P_y}{\partial y} \tag{5}$$

And the Continuity equation is as follow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6}$$

Methodology III

From Kim and Moin:

> *The fractional step, or time-split method, is in general a method of approximation of the evolution equations based on decomposition of the operators they contain. In application to the Navier-Stokes governing equations, one can interpret the role of pressure in the momentum equation equations as a projection operator which projects an arbitrary vector field into a divergence-free velocity field.*

Below are original formula: (see Eqs. (19) and (20) for the current implementation)

$$\frac{\hat{u}_i - u_i^n}{\triangle t} = \frac{1}{2}\left[3H_i^n - H_i^{n-1}\right] + \frac{1}{2}\frac{1}{\mathrm{Re}}\left[\frac{\delta^2}{\delta x_1^2} + \frac{\delta^2}{\delta x_2^2} + \frac{\delta^2}{\delta x_3^2}\right](\hat{u}_i + u_i^n) \quad (7)$$

$$\frac{u_i^{n+1} - \hat{u}_i}{\triangle t} = -G(\phi^{n+1}) \quad (8)$$

Methodology IV

From Le and Moin:

> *The modification has been proved to saved 49% CPU time in comparison to the original KM scheme in a flow over backward-facing step benchmark test.*

- Each time step is advanced in three sub-steps.
- Velocity field is advanced through a sub-step without the need to satisfy the continuity equation.
- The Poisson equation is used to project the predicted vector field into a divergence-free velocity field only at the final sub-step.
- Boundary conditions for the intermediate velocity field are derived using the method similar to that of LeVeque and Oliger.
- The method is implemented for a staggered grid.

Methodology V

$$\frac{u^k - u^{k-1}}{\Delta t} = \alpha_k L\left(u^{k-1}\right) + \beta_k L\left(u^k\right)$$
$$- \gamma_k N\left(u^{k-1}\right) - \zeta_k N\left(u^{k-2}\right) - (\alpha_k + \beta_k)\frac{\delta P_x^k}{\delta x}$$
$$\frac{v^k - v^{k-1}}{\Delta t} = \alpha_k L\left(v^{k-1}\right) + \beta_k L\left(v^k\right)$$
$$- \gamma_k N\left(v^{k-1}\right) - \zeta_k N\left(v^{k-2}\right) - (\alpha_k + \beta_k)\frac{\delta P_y^k}{\delta y}$$

where:

- $k = 1, 2, 3$ denote the sub-step ($k - 2$ is ignored for $k = 1$);
- u_i^0 and u_i^3 are the velocities at time step n and $n + 1$; and
- $\frac{\delta}{\delta x_i}$ is the finite difference operator.

Methodology VI

Equation dump:

$L(u_i)$ represents second-order finite difference approximation to the viscous terms, such that:

$$L(u_i) = \frac{1}{\mathrm{Re}} \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{9}$$

Meanwhile, $N(u_i)$ represents second-order finite difference approximation to the convective terms, such that:

$$N(u_i) = \frac{\partial}{\partial x_j} u_i u_j \tag{10}$$

From Eq.(1), at an abitrary step $n+1$ in which time $t_n + \triangle t$:

$$\frac{\partial u_i^{n+1}}{\partial t} = -\frac{\partial}{\partial x_j} u_i^{n+1} u_j^{n+1} + \frac{1}{\mathrm{Re}} \frac{\partial^2 u_i^{n+1}}{\partial x_i \partial x_j} - \frac{\partial P^{n+1}}{\partial x_i} \tag{11}$$

Methodology VII

$$\frac{3u_i^{n+1} - 4u_i^n + u_i^{n-1}}{2\triangle t} = -N(u_i^{n+1}) + L(u_i^{n+1}) - \frac{\delta P^{n+1}}{\delta x_i} = \text{RHS}(u_i^{n+1}) \tag{12}$$

$$\frac{3u_i^{n+1} - 3u_i^n - u_i^n + u_i^{n-1}}{2\triangle t} = \text{RHS}(u_i^{n+1}) \tag{13}$$

$$\frac{3}{2\triangle t}\left(u_i^{n+1} - u_i^n\right) - \frac{1}{2\triangle t}\left(u_i^n - u_i^{n-1}\right) = \text{RHS}(u_i^{n+1}) \tag{14}$$

Define a function F such that:

$$F(u_i^{n+1}) = \text{RHS}(u_i^{n+1}) - \frac{3}{2\triangle t}\left(u_i^{n+1} - u_i^n\right) + \frac{1}{2\triangle t}\left(u_i^n - u_i^{n-1}\right) \tag{15}$$

Notice that:

$$F(u_i^{n+1}) = 0 \tag{16}$$

Methodology VIII

Consider an approximation $\hat{u}_i^{n+1}$ to u_i^{n+1} such that:

$$\hat{u}_i^{n+1} = u_i^*(\mathbf{x}, t_n + \triangle t) \quad (17)$$

$$u_i^*(\mathbf{x}, t_n) = u_i(\mathbf{x}, t_n) \quad (18)$$

where u_i^* is the immediate velocity in the Runge-Kutta advance scheme. Then, the two-step advancement of the fractional step method can be applied for Eq. (11):

$$\frac{3\hat{u}_i^{n+1} - 4u_i^n + u_i^{n-1}}{2\triangle t} = -N(\hat{u}_i^{n+1}) + L(\hat{u}_i^{n+1}) - \frac{\delta P^n}{\delta x_i} = \mathrm{RHS}(\hat{u}_i^{n+1}) \quad (19)$$

$$\frac{u_i^{n+1} - \hat{u}_i^{n+1}}{\triangle t} = -\mathcal{G}(\phi^{n+1}) \quad (20)$$

with

$$\mathcal{D}(u_i^{n+1}) = 0 \quad (21)$$

Methodology IX

$\mathcal{G}$ and $\mathcal{D}$ represent discrete gradient and divergence operators, respectively.

ϕ^{n+1} and P^{n+1} is related by:

$$P^{n+1} = P^n + \phi^{n+1} \tag{22}$$

Taking the divergence for both sides in Eq. (20) yields us a Poisson equation:

$$\nabla \phi^{n+1} = \frac{3}{2\triangle t} \Delta \cdot \hat{u}_i^{n+1} \tag{23}$$

After solving the Poisson equation (23) and thus, obtaining ϕ^{n+1}, the update of velocity at next timestep is from Eq. (20):

$$u_i^{n+1} = \hat{u}_i^{n+1} - \frac{2\triangle t}{3} \Delta \phi^{n+1} \tag{24}$$

Methodology X

4th-order Runge-Kutta scheme with pseudo time τ:

1. *Start by assigning:*

$$u_i^*(\mathbf{x}, t_n) = u_i(\mathbf{x}, t_n) \tag{25}$$

2. *The following loop is repeated until:*

$$\mid u_i^*(\mathbf{x}, t_n) - \hat{u}_i \mid < \epsilon, \qquad \epsilon << 1 \tag{26}$$

2.1 Let us use a notation $\hat{u}_i^k$, such that:

$$\hat{u}_i^1 = u_i^*(\mathbf{x}, t_n) \tag{27}$$

$$\hat{u}_i^4 = u_i^*(\mathbf{x}, t_n + \triangle t) \tag{28}$$

2.2 Calculate $F(\hat{u}_i^k)$:

$$F(\hat{u}_i^k) = \mathrm{RHS}(\hat{u}_i^k) - \frac{3}{2\triangle t}\left(\hat{u}_i^k - u_i^{n-1}\right) + \frac{1}{2\triangle t}\left(u_i^{n-1} - u_i^{n-2}\right) \tag{29}$$

Methodology XI

2.3 Runge-Kutta advance:

$$\hat{u}_i^k = u_i^*(\mathbf{x}, t_n) + \alpha\,(\kappa\triangle t)\,F(\hat{u}_i^k) \qquad k = 2, 3, 4 \tag{30}$$

$$\tag{31}$$

Viscous Terms I

CDS2 Scheme

Now let us find the expression for the finite difference operators in Eq.(9) and Eq.(10).

First, consider the Viscous term (9) as follows:

$$L(u_i) = \frac{1}{\mathrm{Re}} \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{32}$$

For a 2 dimensional coordinate system, one has the expression such that:

In x-direction:

$$L_x(u) = \frac{1}{\mathrm{Re}} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{33}$$

In y-direction:

$$L_y(v) = \frac{1}{\mathrm{Re}} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \tag{34}$$

Viscous Terms II

CDS2 Scheme

The second derivative at an arbitrary point (i, j) is approximated by the second central difference:

$$\frac{\partial^2 u_{i,j}}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} \tag{35}$$

$$\frac{\partial^2 u_{i,j}}{\partial y^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} \tag{36}$$

and:

$$\frac{\partial^2 v_{i,j}}{\partial x^2} = \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{\Delta x^2} \tag{37}$$

$$\frac{\partial^2 v_{i,j}}{\partial y^2} = \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{\Delta y^2} \tag{38}$$

Convection Terms I

QUICK Scheme

For inner nodes:

$$
\begin{aligned}
F = {} & um\left[1/8*(-U_{i+2}-2U_{i+1}+3U_i)+U_{i+1}\right] \\
& + up\left[1/8*(-U_{i-1}-2U_i+3U_{i+1})+U_i\right]
\end{aligned} \tag{39}
$$

For begin node ($i = 1$):

$$
\begin{aligned}
F = {} & um\left[1/8*(-U_{i+2}-2U_{i+1}+3U_i)+U_{i+1}\right] \\
& + up\left[1/8*(-U_i-2U_i+3U_{i+1})+U_i\right]
\end{aligned} \tag{40}
$$

For end node ($i = N$):

$$
\begin{aligned}
F = {} & um\left[1/8*(-U_{i+1}-2U_{i+1}+3U_i)+U_{i+1}\right] \\
& + up\left[1/8*(-U_{i-1}-2U_i+3U_{i+1})+U_i\right]
\end{aligned} \tag{41}
$$

Methodology I

Benchmark Problems

2D Taylor-Green vortex is a classic benchmark problem which solution is a periodic array of vortices, that repeats itself in the x-ăandăy-directions with a periodic length L:

$$u(x, y, t) = u_0 \sin(kx) \cos(ky) f(t) \tag{42}$$

$$v(x, y, t) = -u_0 \cos(kx) \sin(ky) f(t) \tag{43}$$

$$p = \frac{\rho u_0^2}{4} f(t)^2 \left[\cos(2kx) + \cos(2ky)\right] \tag{44}$$

$$f(t) = \exp\left(-2\nu k^2 t\right) \tag{45}$$

$$k = \frac{2\pi}{L} \tag{46}$$

Try it out!

Get the source of code from

`https://github.com/milk-white-way/AMRESSIF`